Tower of Hanoi Problem with Arbitrary Number of Pegs and Present a Solution

Article *in* World Applied Sciences Journal · January 2010

World Applied Sciences Journal 10(11): 1351-1353, 2010
ISSN 1818-4952

Tower of Hanoi Problem with Arbitrary Number of Pegs and Present a Solution

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Abstract: Suppose that we havem pegs and n disks of distinct sizes such that initially disks are stacked on the first peg ordred by size with the smallest at the top and the largest at the bottom. In this Paper we want to find a solution for transition disks to one of another pegs usingthe well-known movements of Tower of Hanoi such that there are no solution better than it yet.

Key words: Tower of Hanoi % Recursive formula % Pegs

INTRODUCTION

In the well-known Tower of Hanoi problem, composed over a hundred years ago by Lucas [1], a player is given 3 pegs and a certain number $n$of disks of distinct sizes and is required to transfer them from one peg to another.

Initially all disks are stacked (composing a tower) on the first peg (the source) ordered by size, with the smallest at the top and the largest at the bottom. The goal is to transfer them to the third peg (the destination), moving only topmost disks and never placing a disk on top of a smaller one. The known recursive algorithm, which may be easily shown to be optimal, takes $h_n = 2^n - 1$ steps to accomplish the task. Since there is not much mathematical mystery left about the originalgame, its lovers developed various versions of it [2-6]. In this paper we want to solve Tower of Hanoi problem with arbitrary number of pegs.

Tower of Hanoi with M Pegs: Supposethatwe have $n$disks ofdistinct sizes $d_1, d_2, ..., d_n$ suchthat $d_1 > d_2 > ... > d_n$ and pegs (m>3). Initially disks are stacked on the irst peg ordred by size (Figure 1). The goal is to transfer all disks to one of another pegs.

Algorithm of Solution: For transfer of disksfrom peg 1 to one of another pegs sush as peg m, at the first d_1 must be placed in the bottom of the peg m. It is possible when all other disks be stacked in another m-2 pegs. Then d_2 must be transfered to peg m, top of d_1. But it is possible when d_2 be in one peg alone, for example peg 2 and all disks except d_1 and d_2 be stacked in nother m-3 pegs. In this manner for transition of d_{m-2} to peg m top of $d_1, d_2, ..., d_{m-3}$ disks must have beena position as following (Figure 2).

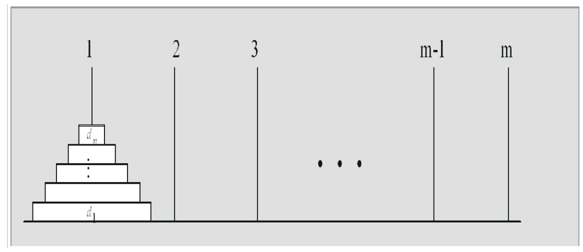

Fig. 1: The board of the Tower of Hanoi problem with m pegs

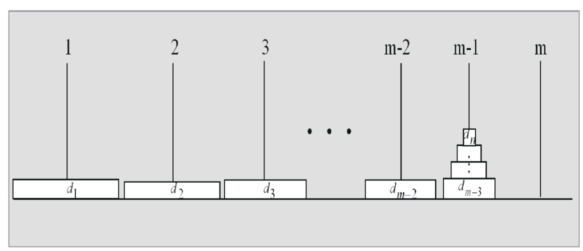

Fig. 2: steps of transition of disks

That means $n - (m - 2)$ remainder disks must be stacked on one peg such as peg $m - 1$ in Figure, 2. certainly. In fact we want to find a recursive formula using this idea for solving this problem.

If we show the number of motions for transition of n disks to the peg m with H_n^m, it is obvious that if $n<m$ then $H_n^m = 2n - 1$.

Now with this introductions we want to find H_n^m for $n\ \$\ m$, usingfollowing steps:

Step 1: Transition of disks $d_{m-3}, ..., d_n$ with $H^m_{n-(m-2)}$ motions from peg 1 to peg $m - 1$.

Step 2: Transition of disks $d_2, d_3, ..., d_{m-2}$ 2 from peg 1 to pegs $2, 3, ..., m-2$, with $m-3$ motions.

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Step 3: Transition of disks $d_1, d_2, d_3, ..., d_{m-2}$ to peg m with $m-2$ motions.

Step 4: Transition of disks $d_{m-3}, ..., d_n$ with $H^m{}_{n-(m-2)}$ motions from peg $m-1$ to peg m.

Therefore for $n \geq m$,we have:

$$H_n^m = H_{n-(m-2)}^m + (m-3) + (m-2) + H_{n-(m-2)}^m$$
$$\Rightarrow H_n^m = 2H_{n-(m-2)}^m + 2m - 5$$

Upper relation is a non-homogeneous recursive formula of degree $m-2$ and we want to solve it with primal conditions:

$$H_0^m = 0,\ \ H_1^m = 1,\ \ H_2^m = 3, ..., H_{m-3}^m = 2(m-3) - 1$$

We must solve this recursive formula in $m-2$ cases:

Case 1: $n = (m-2)k$

Case 2: $n = (m-2)k+1$

Case 3: $n = (m-2)k+2$

$\vdots \qquad\qquad \vdots$

Case m-2: $n = (m-2)k + (m-3)$

In all upper cases k is a non-negative integer number.

Studing of Case 1:

In this case $n = (m-2)k$ and we have:

$$\begin{aligned} H_n^m &= 2H_{n-(m-2)}^m + 2m - 5 \\ &= 2(2H_{n-2(m-2)}^m + 2m-5) + 2m - 5 = 2^2 H_{n-2(m-2)}^m + 3(2m-5) \\ &\quad\vdots \qquad\qquad\qquad\qquad \vdots \\ &= 2^k H_{n-k(m-2)}^m + (2^k - 1)(2m-5) = 2^k H_0^m + (2^k - 1)(2m-5) \end{aligned}$$
$$\Rightarrow H_n^m = (2^{\frac{n}{m-2}} - 1)(2m-5)$$

Studing of Case 2:

In this case $n = (m-2)k+1$ and we have:

$$H_n^m = 2^k H_1^m + (2^k - 1)(2m-5) = 2^k + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = 2^{\frac{n-1}{m-2}} + (2^{\frac{n-1}{m-2}} - 1)(2m-5)$$

Studing of Case 3:

In this case $n = (m-2)k+2$ and we have:

$$H_n^m = 2^k H_2^m + (2^k - 1)(2m-5) = (2^k \times 3) + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = 3 \times 2^{\frac{n-2}{m-2}} + (2^{\frac{n-2}{m-2}} - 1)(2m-5)$$

Studing of Case m-2:

In this case $n = (m-2)k + (m-3)$ and we have:

$$H_n^m = 2^k H_{m-3}^m + (2^k - 1)(2m-5) = 2^k (2(m-3) - 1) + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = (2(m-3) - 1)2^{\frac{n-(m-3)}{m-2}} + (2^{\frac{n-(m-3)}{m-2}} - 1)(2m-5)$$

Thereforein the abstractwe have:

If $n = (m-2)k$ then

$$H_n^m = (2^{\frac{n}{m-2}} - 1)(2m-5)$$

If $n = (m-2)k + 1$ then

$$H_n^m = 2^{\frac{n-1}{m-2}} + (2^{\frac{n-1}{m-2}} - 1)(2m-5)$$

If $n = (m-2)k + 2$ then

$$H_n^m = 3 \times 2^{\frac{n-2}{m-2}} + (2^{\frac{n-2}{m-2}} - 1)(2m-5)$$

$\vdots$

If $n = (m-2)k + (m-3)$ then

$$H_n^m = (2(m-3) - 1)2^{\frac{n-(m-3)}{m-2}} + (2^{\frac{n-(m-3)}{m-2}} - 1)(2m-5)$$

CONCLUSION

We found a solution for transfer of n disks useingthe well-known movementsofTower of Hanoi with m pegs from one peg to one of another pegs. We did not findoptimal solution, but there are no solution better thanthis solution yet.

In general case, $H^m{}_n$ can be obtained as following:

$$H_n^m = \begin{cases} (2m-5)(2^{\frac{n}{m-2}} - 1) & i = 0 \\ (2i-1)\ 2^{\frac{n-i}{m-2}} + (2m-5)(2^{\frac{n-i}{m-2}} - 1) & i \neq 0 \end{cases}$$

Such that i isremainder of division of n by $m-2$ ($i = n \bmod (m-2)$).

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