Slido: **#ADA2021**

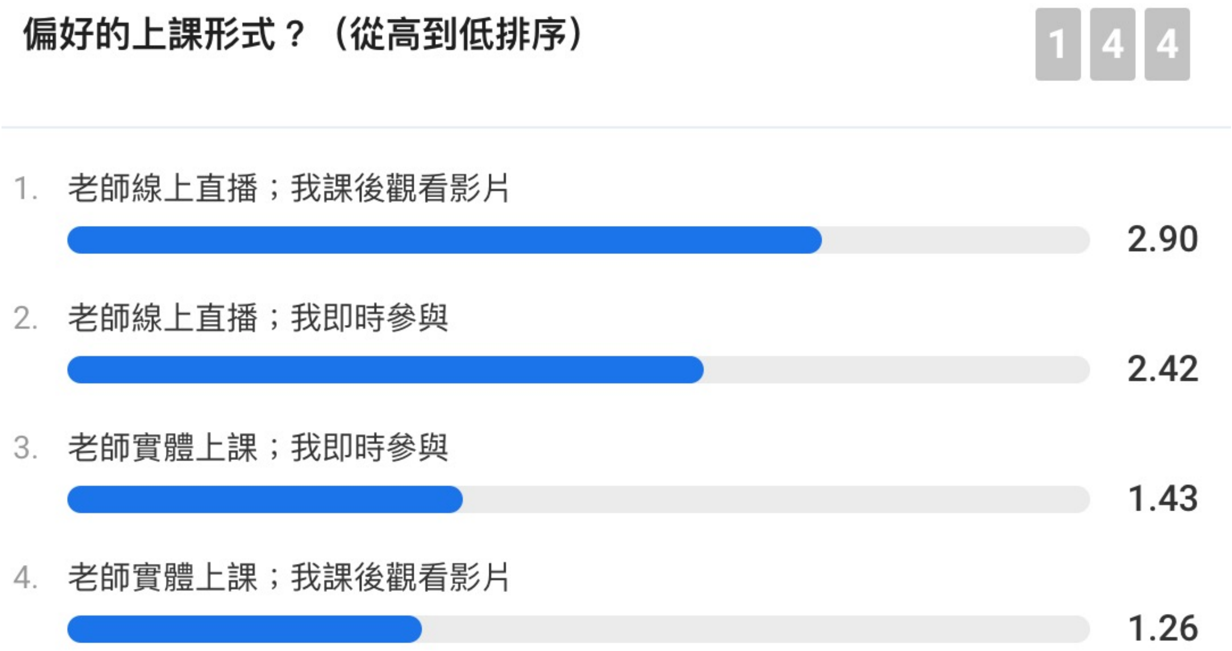
CSIE 2136 Algorithm Design and Analysis, Fall 2021

Graph Algorithms - II

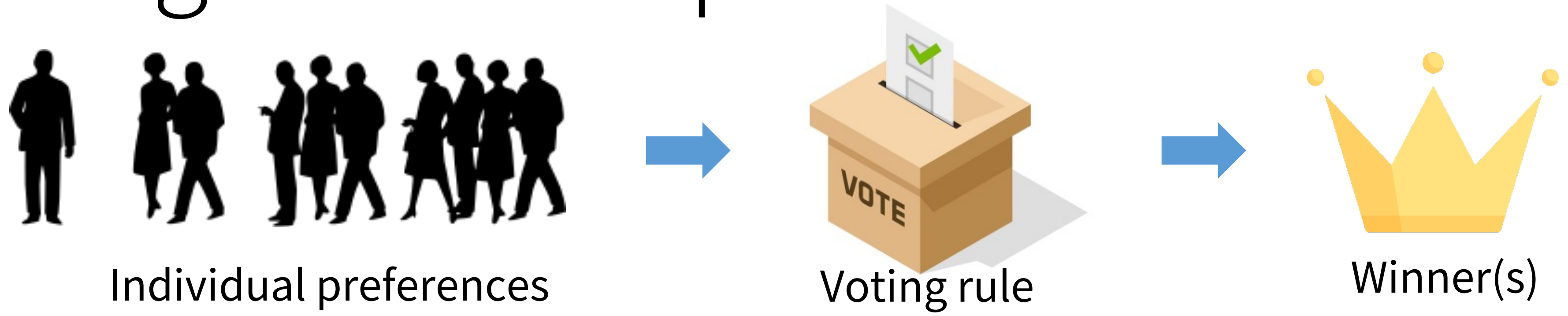
Hsu-Chun Hsiao

課堂小調查

- 維持線上直播



Voting rule examples



- **Plurality (多數決)**: each voter awards one point to top candidate, and the candidate with the most points wins
- **Veto (否決制)**: each voter vetos least preferred candidate, and the candidate with the least vetoes wins
- **Borda (計數法)**: each voter awards $m - k$ points to k^{th} ranked candidate, and the candidate with the most points wins

Q: Which voting rule did we use on slido?

- Borda

Manipulation: Borda as an example

Borda: each voter awards $m - k$ points to k^{th} ranked candidate, and the candidate with the most points wins

Voter 1	B	A	C	D
Voter 2	B	A	C	D
Voter 3	A	B	C	D

A: 7
$\boxed{\text{B: 8}}$
C: 3
D: 0

Q: Can voter 3 benefit from lying about his or her preferences? (Assume others' preferences are known)

- Yes

Manipulation: Borda as an example

Borda: each voter awards $m - k$ points to k^{th} ranked candidate, and the candidate with the most points wins

Voter 1	B	A	C	D
Voter 2	B	A	C	D
Voter 3	A	B	C	D

A: 7
B: 8
C: 3
D: 0

Can voter 3 benefit from lying about his or her preferences?
(Assume others' preferences are known)

Voter 1	B	A	C	D
Voter 2	B	A	C	D
Voter 3	A	C	D	B

A: 7
B: 6
C: 4
D: 1

My scheme is intended only for honest men

Intersted in voting theory & algorithms?

- Checkout the old video in 2019 (I may update it if time permitted…)

- Which voting rules are “better”?
  - Condorcet winner criterion
  - Strategyproof
- Preventing manipulation (and achieve Strategyproofness)?
  - Gibbard-Satterthwaite theorem

Today's Agenda

- Finish last week's slides⋯
- DFS applications
  - Topological sort [Ch. 22.4]
  - Strongly-connected components [Ch. 22.5]
- Minimum spanning trees [Ch. 23]
  - Kruskal's algorithm
  - Prim's algorithm
- Shortest paths: terminology and properties
  - Edge relaxation
  - Shortest-paths properties

Application of DFS: Topological Sort

Textbook chapter 22.4

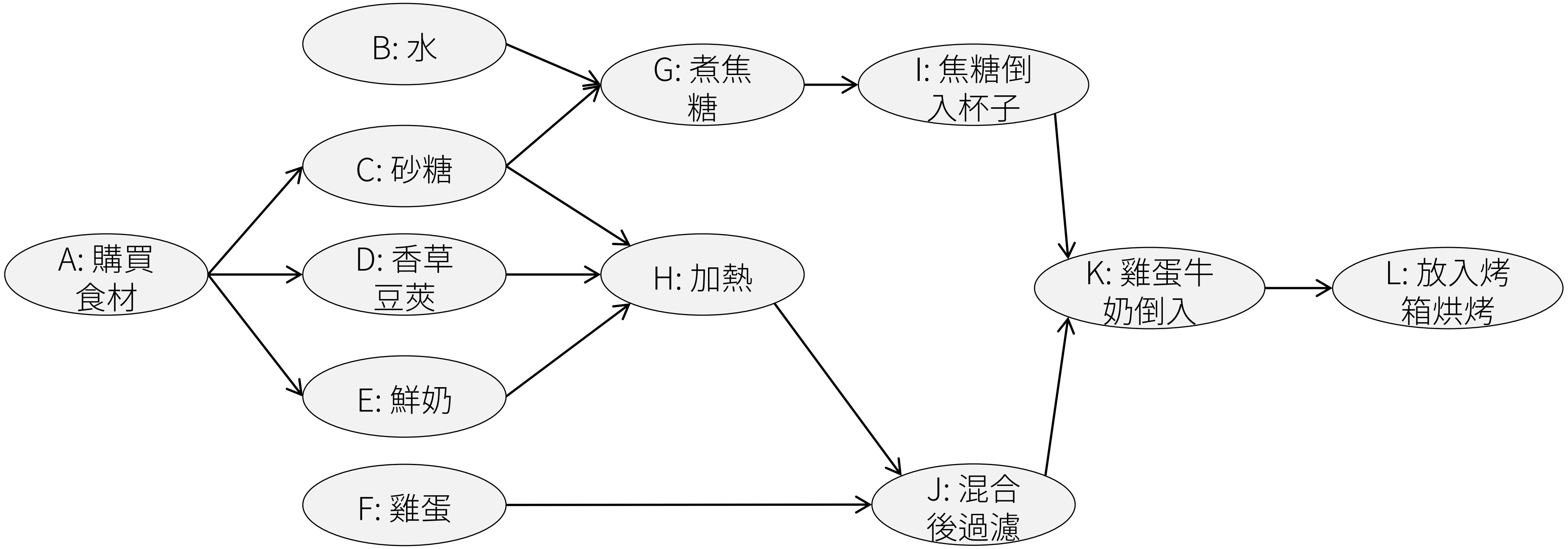
MasterChef: 布丁篇

Q: 新手一次只能做一件事，用什麼順序才能順利做出布丁？

- One valid order is: A C D E H B G I F J K L

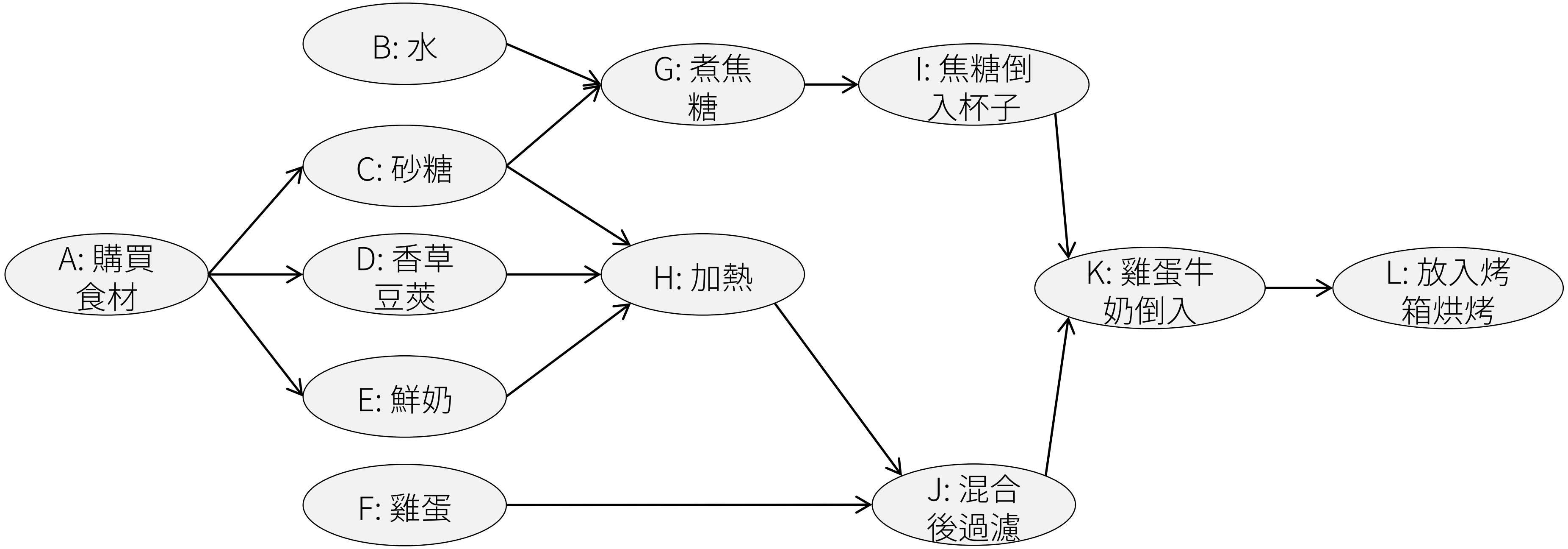
A->B: 要先處理完 A 才能處理 B

Intuition: 前置作業要先完成，才能做後面的步驟



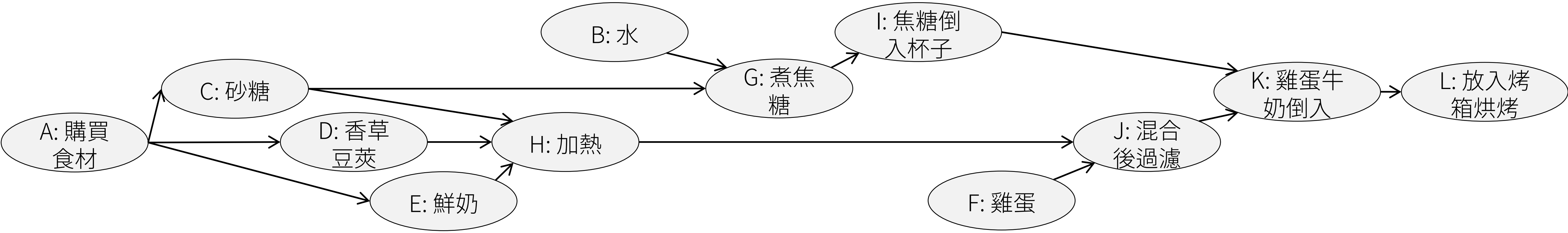
Topological Sort

- Input: a directed acyclic graph (DAG) $G = (V, E)$
  - Often indicates precedence among events (X must happen before Y)
- Output: a linear ordering of all its vertices such that for all edges (u, v) in E, u precedes v in the ordering



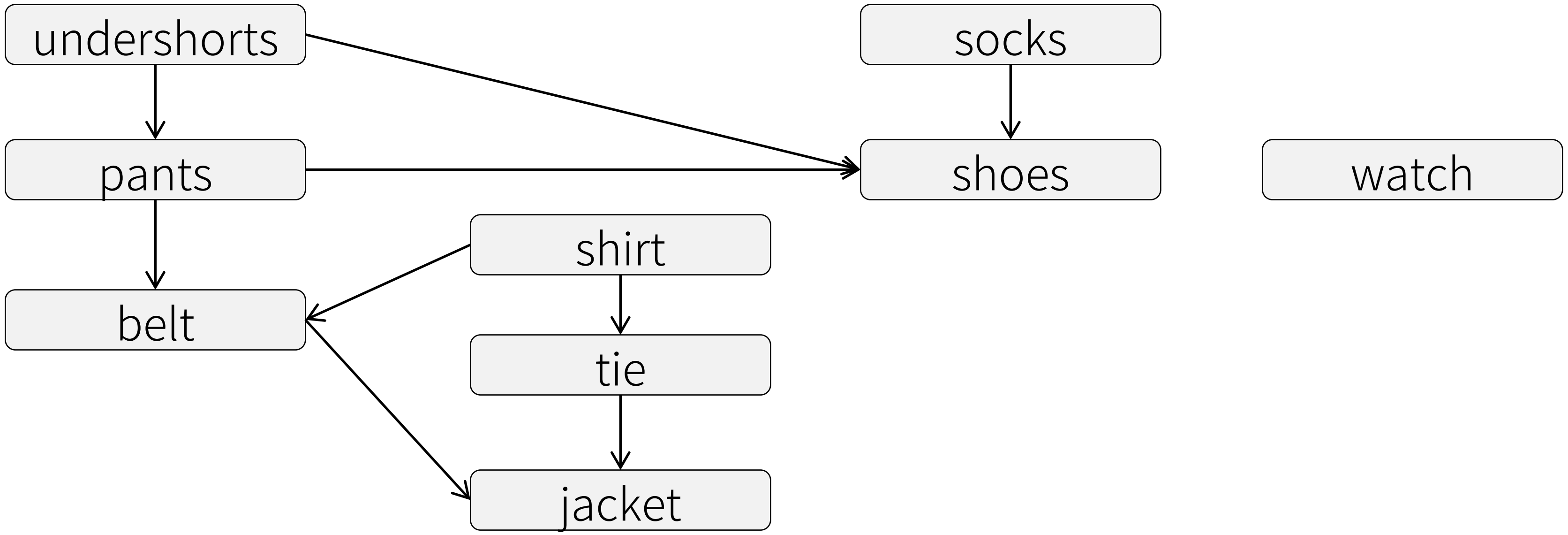
Topological Sort

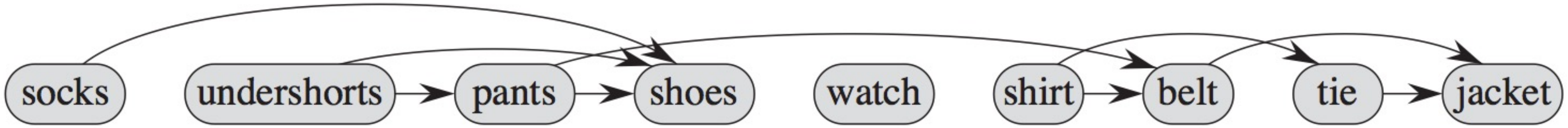
- <u>Input</u>: a directed acyclic graph (DAG) $G = (V, E)$
  - Often indicates precedence among events (X must happen before Y)
- <u>Output</u>: a linear ordering of all its vertices such that for all edges (u, v) in E, u precedes v in the ordering
- <u>Alternative view</u>: a vertex ordering along a horizontal line so that all directed edges go from left to right



Topological Sort

- Alternative view: a vertex ordering along a horizontal line so that all directed edges go from left to right



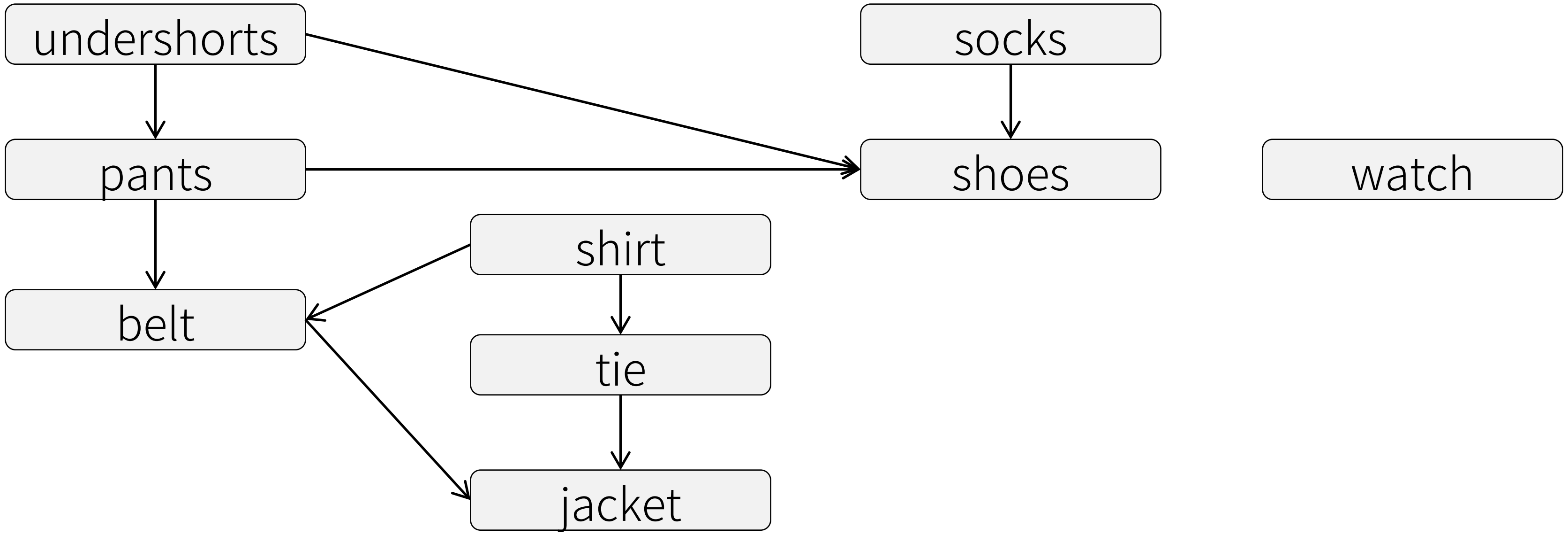


Topological sort algorithm

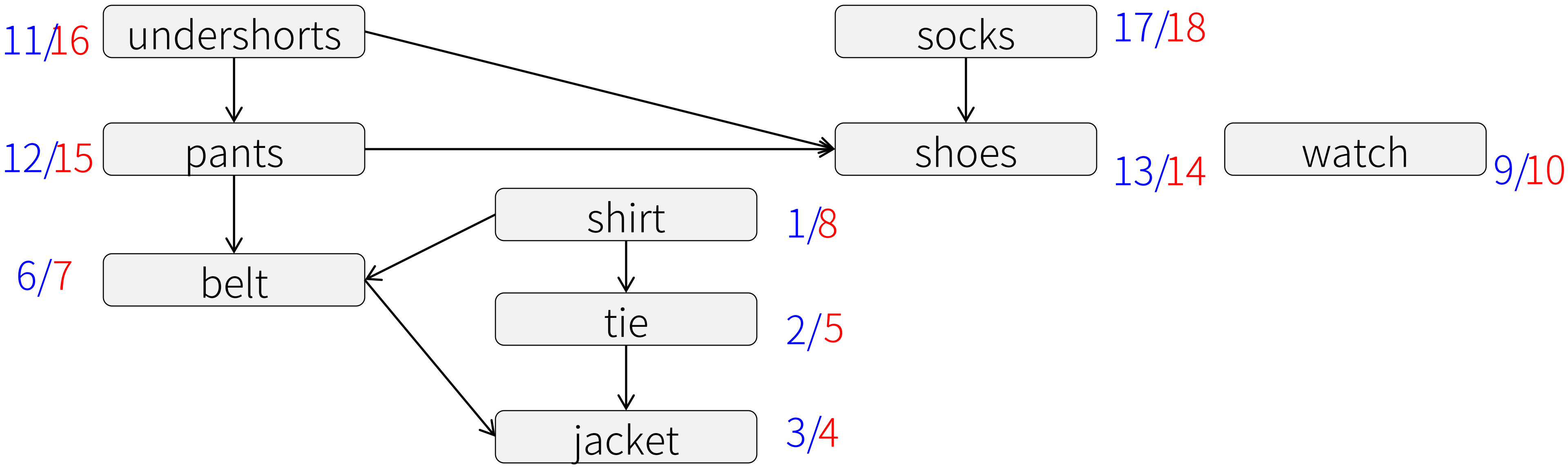
```
TOPOLOGICAL-SORT(G) //G is a DAG
    Call DFS(G) to compute finishing times v.f for each vertex v
    As each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
```

- Vertices are ordered by their DFS finishing times (in a descending order)
- We will prove this linked list comprises a topological ordering

Topological sort using DFS



Topological sort using DFS



socks	undershorts	pants	shoes	watch	shirt	belt	tie	jacket

Running time analysis

```
TOPOLOGICAL-SORT(G) //G is a DAG
    Call DFS(G) to compute finishing times v.f for each vertex v
    As each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
```

- DFS with adjacency lists: $\Theta(V + E)$ time
- Insert each vertex to the linked list: $\Theta(V)$ time
- => total running time is $\Theta(V + E)$

Theorem 22.12 Correctness of topological sort algorithm

The algorithm produces a topological sort of the input DAG

對所有的 edge (u, v)，證明在此 vertex list 中 u 一定在 v 前面（也就是 $u.f > v.f$ 成立）

Proof

- When (u, v) is explored, u is gray.
- Consider three cases of v: gray, white, black

Theorem 22.12 Correctness of topological sort algorithm

The algorithm produces a topological sort of the input DAG

Proof (cont.)

- v = gray
  => (u, v) = back edge
  => G is cyclic (by Lemma 22.11)
  => Contradiction, so v cannot be gray
- v = white
  => v becomes descendant of u (by white-path theorem)
  => v will be finished before u (by parenthesis theorem)
  => $v.f < u.f$
- v = black
  => v is already finished
  => $v.f < u.f$

Q: Is there a DFS forest for a cyclic graph?

Yes

Q: Is there a topological order for a cyclic graph?

No

Q: Given a topological order, is there always a DFS traversal (ordered by the discovery times) that produces the same order?

Yes. One possible construction is running DFS from the rightmost vertex in the topological order.

Another topological sort algorithm: Kahn's algorithm

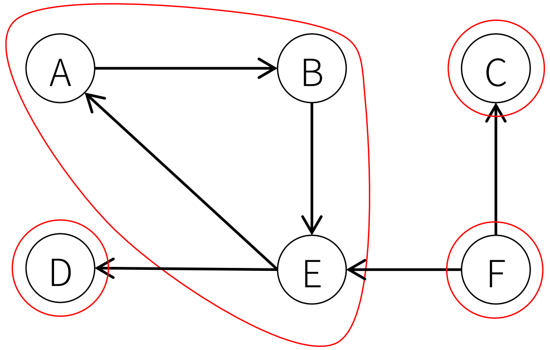
- Intuition: removing "source vertices" one by one and updating in-degree values
  - Source vertices: vertices with in-degree = 0

- Correctness: why is there always a vertex with zero in-degree?
- Running time is $\Theta(V+E)$
  - Need to maintain in-degree values and a queue of current source vertices

Strongly Connected Components (SCC)

Strongly connected components of a directed graph

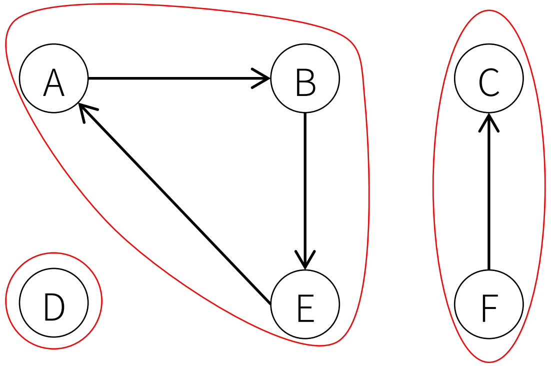
The strongly connected components of a directed graph are the equivalence classes of vertices under the "mutually reachable" relation.
That is, a strong component is a maximal subset of mutually reachable nodes.



4 strongly connected components: {A,B,E}, {C}, {D}, {F}

Weakly connected components of a directed graph

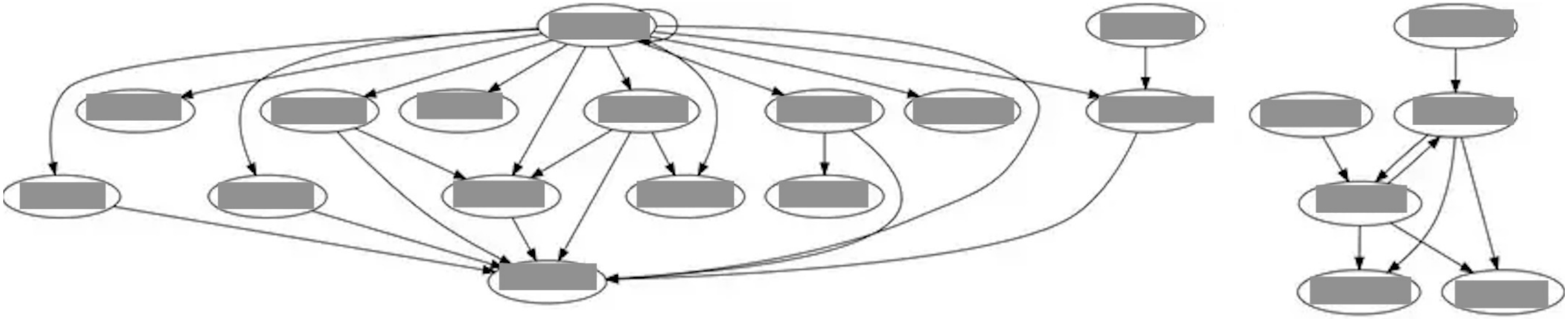
The weakly connected components of a directed graph are the equivalence classes of vertices under the "is reachable from" relation if all directed edges are replaced by undirected ones.



3 weakly connected components: {A,B,E}, {C,F}, {D}

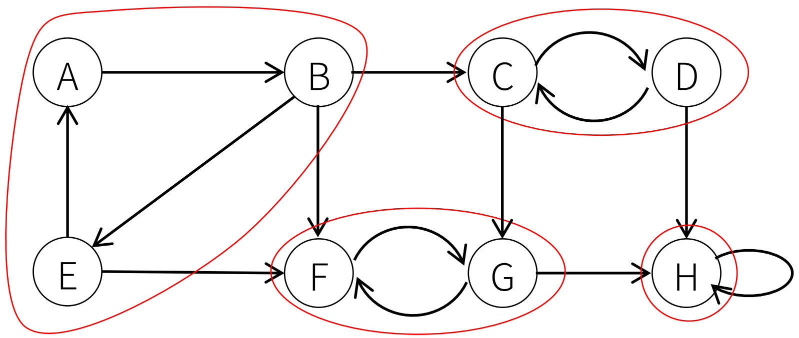
Example: Homework-reference graph

- A directed graph in which vertices are students and edges represent "acknowledgments"
- To ease the grading process, the TAs want to cluster the answers by identifying weakly and strongly connected components



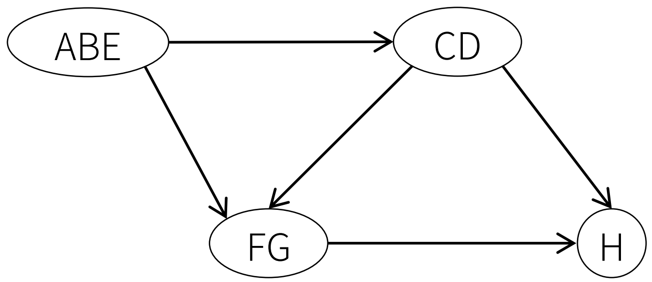
Decomposing a directed graph

- A directed graph is a DAG of its SCC



$G = (V, E)$

Contract each SCC
into one vertex



Component graph $G^{scc} = (V^{scc}, E^{scc})$

Q: Show that a component graph must be a DAG

If there were a cycle on the component graph, vertices on the cycle are mutually reachable and should have belonged to a bigger SCC.

Q: Does the following algorithm **determine** whether a graph G is strongly connected in $O(V + E)$ time?

```
Run BFS in G from any node s
Run BFS in the transpose of G, from the same source node s
If both BFS executions found all nodes, return true; otherwise, return false
```

Yes

Note: we denote a transpose or reverse graph of a directed graph $G = (V, E)$ as G^T, and $G^T = (V, E^T)$ where $E^T = \{(v, u) \mid (u, v) \in E\}$

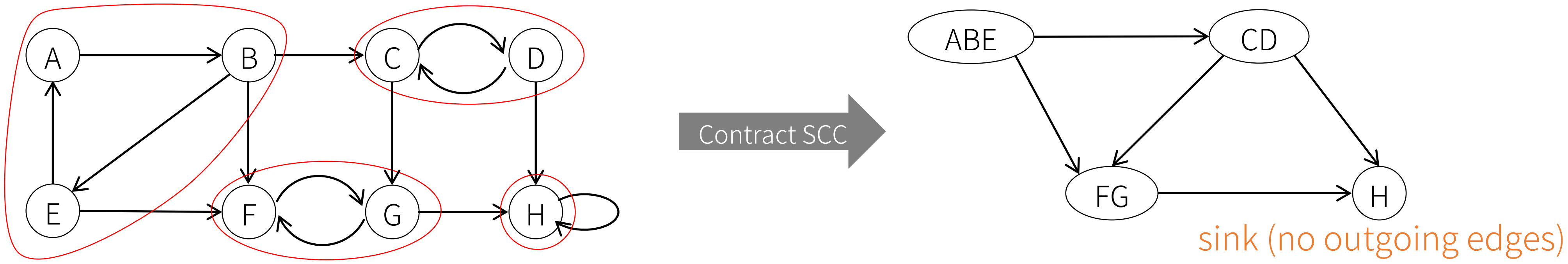
Finding SCC: the Kosaraju-Sharir algorithm

```
Strongly-Connected-Components(G)
1   call DFS(G) to compute finishing times u.f for each vertex u
2   compute G^T
3   call DFS(G^T), but in the main loop of DFS, consider the vertices in order of
        decreasing u.f (as computed in line 1)
4   output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component
```

- Input: a directed graph $G = (V, E)$
- Output: strongly connected components
- Time complexity
  - 2 DFS executions
  - $\Theta(V + E)$ using adjacency lists

Finding SCC

- Observation 1: Starting from s, DFS finds all reachable nodes from s. Hence, if we can select a vertex in a sink SCC as the starting vertex for DFS, then DFS will discover all (and only) vertices in the sink SCC.
  - => we can find SCCs one by one in a reverse topological order of G^{scc}!
  - However, how to identify a vertex in a sink SCC?



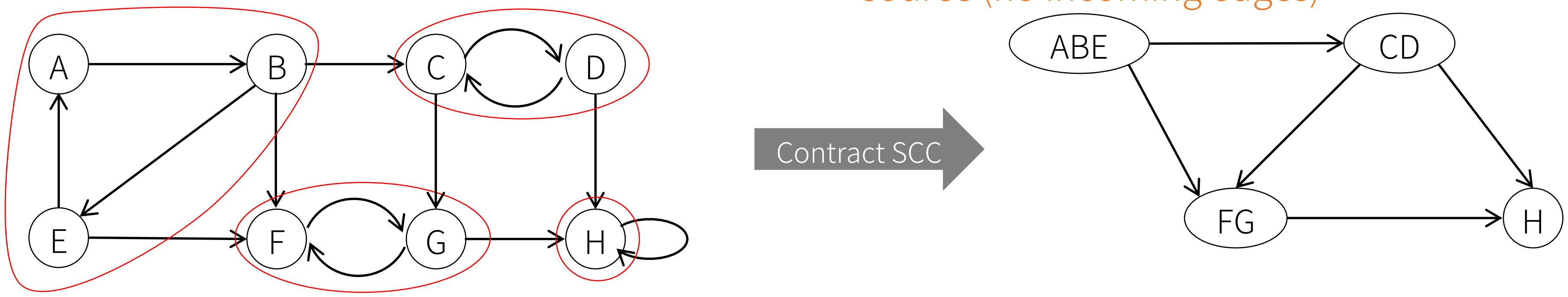
$G = (V, E)$

Component graph $G^{scc} = (V^{scc}, E^{scc})$

Finding SCC

- Observation 2 (Exercises 22.5-4): An SCC in G is also an SCC in G^T. Also, a source SCC in G is a sink SCC in G^T.
- Observation 3: Finding a vertex in a source SCC is easy. The vertex with the highest finishing time (found by running DFS in G) must be in a source SCC.
  - Implied by Lemma 22.14 (will prove it in a few slides)

source (no incoming edges)



$G = (V, E)$

Component graph $G^{scc} = (V^{scc}, E^{scc})$

Finding SCC: the Kosaraju-Sharir algorithm

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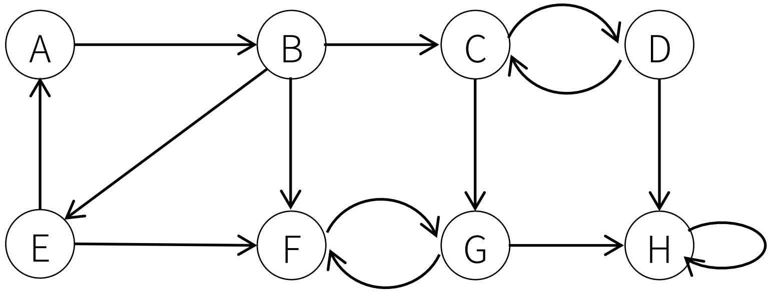
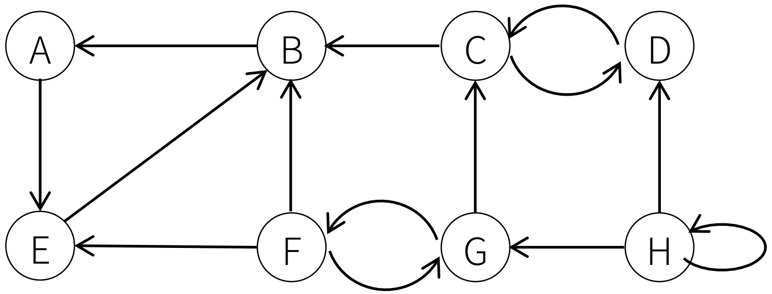
- Observation 3: Finding a vertex in a source SCC is easy. The vertex with the highest finishing time (found by running DFS in G) must be in a source SCC.

Let's try it!

```
1   call DFS(G) to compute u.f
```




```
2   compute G^T
3   call DFS(G^T), in decreasing
        order of u.f
```



Lemma 22.14

Let C and C' be distinct strongly connected components in directed graph $G = (V, E)$. Suppose that there is an edge (u, v) where u in C and v in C'. Then $f(C) > f(C')$.
Here we define $f(U) = max_{u \in U}\{u.f\}$, and $d(U) = min_{u \in U}\{u.d\}$

Proof: Consider two cases: $d(C) < d(C')$ and $d(C) > d(C')$

- If $d(C) < d(C')$:
  - Let x be the first vertex discovered in C
  - => At $t = x.d$, all vertices in C and C' are WHITE
  - => At $t = x.d$, there is a white path from x to every vertex in C and C'
  - => By the white-path theorem, they are all x's decendants in the DFS tree
  - => By the parenthesis theorem, $x.f$ is the largest
  - => $f(C) = x.f > f(C')$

Lemma 22.14

Let C and C' be distinct strongly connected components in directed graph $G = (V, E)$. Suppose that there is an edge (u, v) where u in C and v in C'.
Then $f(C) > f(C')$.
Here we define $f(U) = max_{u \in U}\{u.f\}$, and $d(U) = min_{u \in U}\{u.d\}$

Proof (cont'd) If $d(C) > d(C')$:

- Let y be the first vertex discovered in C'

=> At $t = y.d$, all vertices in C' are white

=> At $t = y.d$, there is a white path from y to every vertex in C'

=> By the white-path theorem and the parenthesis theorem, all other vertices in C' are y's descendants and $y.f$ is the largest among them

=> $f(C') = y.f$

- Moreover, because there is no path from C' to C (why?), no vertex in C is reachable from y

=> At $t = y.f$, all vertices in C are still WHITE

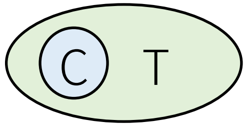
=> $f(C) > d(C) > y.f = f(C')$

Theorem 22.16 Correctness of the Kosaraju-Sharir algorithm

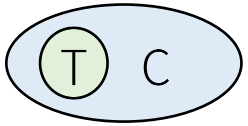
The Kosaraju-Sharir algorithm correctly computes the strongly connected components of the directed graph G provided as its input

Proof by induction on the number of DFS trees in line 3

- Inductive hypothesis: the first k trees produced are SCC
  - Base case: when $k = 0$, trivially correct
- Inductive step: assume the first k trees are SCC, consider the $(k+1)$th tree T
  - Let u be the first vertex of T, and let u be in SCC C
  - We will show that the vertices of T are the same as vertices in C



All vertices in C are in T:



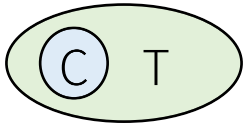
All vertices in T are in C:

Theorem 22.16 Correctness of the Kosaraju-Sharir algorithm

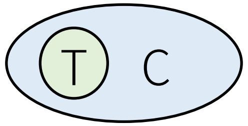
The Kosaraju-Sharir algorithm correctly computes the strongly connected components of the directed graph G provided as its input

Proof by induction (cont'd)

- Inductive step: assume the first k trees are SCC, consider the $(k+1)$th tree T
  - Let u be the first vertex of T, and let u be in SCC C
  - We will show that the vertices of T are the same as vertices in C
  - All vertices in C are in T:
    By the inductive hypothesis, at $t = u.d$, all other vertices of C are white.
    By the white-path theorem, all vertices in C are descendants of u in T.



  - All vertices in T are in C:
    By construction, $u.f$ is the largest among vertices that have yet to be visited in line 3.
    That is, $u.f = f(C) > f(C')$, where C' is any SCC other than C that has yet to be visited.
    Lemma 22.4 implies that there is no edge from C' to C in G (thus no edge from C to C' in G^T), so T will not contain any vertices in any C'.



Q: Can the following algorithms correctly find SCCs?

Strongly-Connected-Components-1(G)

1 compute G^T
2 call $DFS(G^T)$ to compute finishing times $u.f$ for each vertex u
3 call $DFS(G)$, but in the main loop of DFS, consider the vertices in order of
    decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component

Strongly-Connected-Components-2(G)

1 call $DFS(G)$ to compute finishing times $u.f$ for each vertex u
2 call $DFS(G)$, but in the main loop of DFS, consider the vertices in order of
    increasing $u.f$ (as computed in line 1)
3 output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component

Strongly-Connected-Components-1(G) Yes.

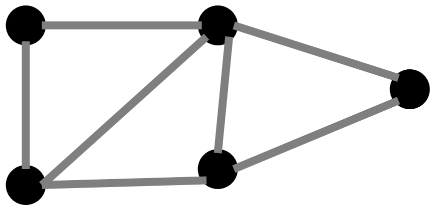
Strongly-Connected-Components-2(G) No.

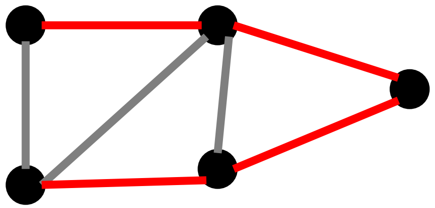
Minimum Spanning Trees

Textbook Chapter 23

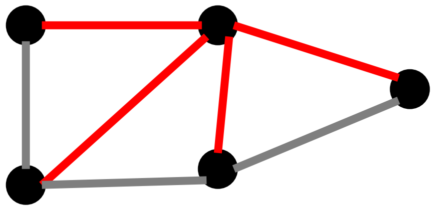
Spanning tree

- Spanning tree of a connected undirected graph G = a subgraph that is a tree and connects all the vertices
  - Exactly $|V| - 1$ edges
  - Acyclic
- There can be many spanning trees of a graph





Spanning tree 1



Spanning tree 2

Spanning tree

- BFS and DFS also generate spanning trees
  - BFS tree is typically “short and bushy”
  - DFS tree is typically “long and stringy”

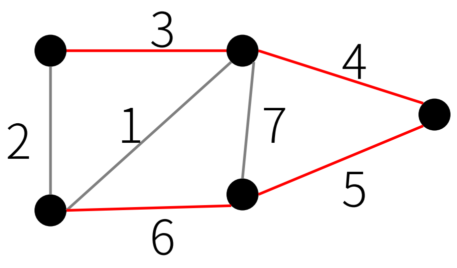
Q: Can these spanning trees be generated from BFS or DFS?

Left: can be BFS or DFS

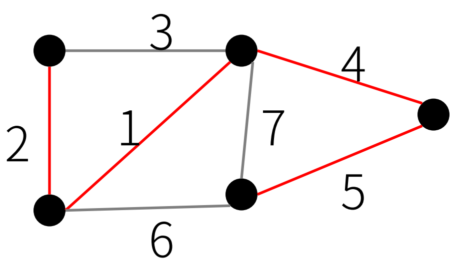
right: can be BFS but not DFS

Minimum spanning tree (MST)

- A minimum spanning tree of a graph G is a spanning tree with minimal weight
- Weight of a tree T = the sum of weights of all edges in T



Weight = 18



Weight = 12, MST

Q: How to find an MST in an unweighted graph (i.e., edges have equal weights)?

Any spanning tree is an MST in an unweighted graph

Q: Given a weighted graph G, can there be more than one MST?

Yes, consider an unweighted graph: every spanning tree is an MST.

But we will show that MST is unique if all edge weights are distinct.

Q: If the edge weights of G are all increased by the same constant, does an MST of the old graph remain an MST in the re-weighted graph?

Yes

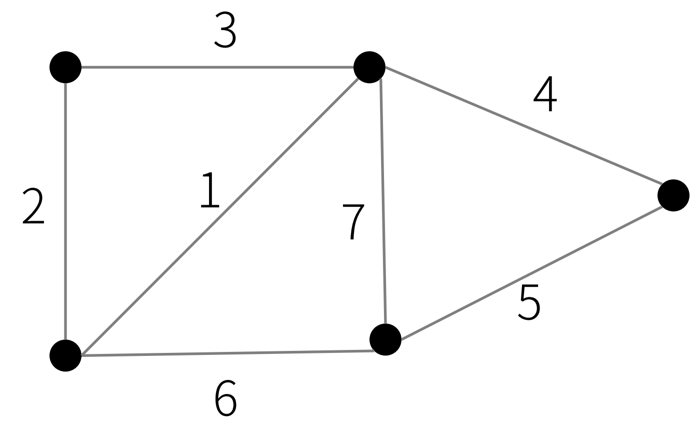
Minimum spanning tree (MST)

- Finding an MST is an optimization problem

- Two greedy algorithms compute an MST:
  - Kruskal's algorithm: consider edges in ascending order of weight. At each step, select the next edge as long as it does not create cycle.
  - Prim's algorithm: start with any vertex s and greedily grow a tree from s. At each step, add the edge of the least weight to connect an isolated vertex.

Kruskal’s algorithm

```
Kruskal(G)
    start with T = V (no edges)
    for each edge in increasing order by weight
        if adding edge to T does not create a cycle
            then add edge to T
```



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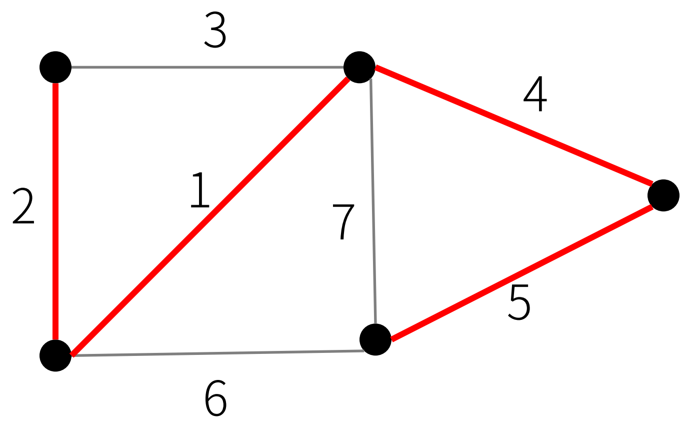


Weight = 12
MST

Implementation of Kruskal's algorithm

```
MST-KRUSKAL(G,w)  // w = weights
    A = empty // edge set of MST
    for v in G.V
        MAKE-SET(v)
    sort the edges of G.E into non-decreasing order by weight
    for (u,v) in G.E, taken in non-decreasing order by weight
        if FIND-SET(u) ≠ FIND-SET(v) // cycle test
            A = A U {u, v}
            UNION(u,v)
    return A
```

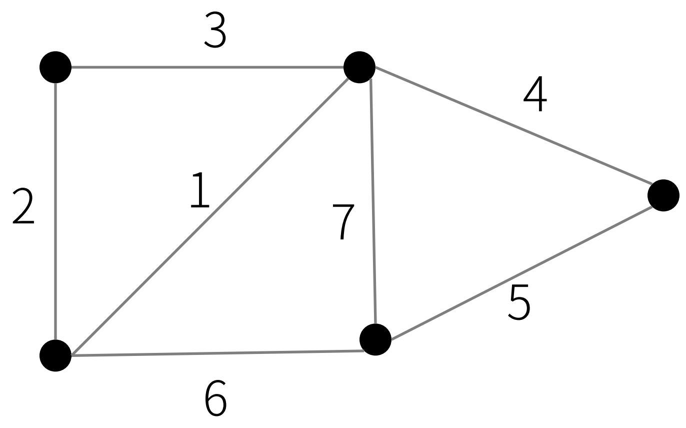
- Disjoint-set data structure: MAKE-SET, FIND-SET, UNION
- Each set contains the vertices in one tree of the current forest

Running time analysis

- Using disjoint-set-forest implementation with union-by-rank and path compression, Kruskal's algorithm can run in $O(E \lg V)$
- [Ch. 21] The disjoint-set-forest implementation with union-by-rank and path compression for m operations on n elements is $O(m\,\alpha(n))$, where $\alpha(n)$ is a very slow growing function.
- Kruskal's running time = sorting edge + disjoint-set operations
  - Sorting edge = $O(E \lg E)$ = $O(E \lg V)$
  - Disjoint-set operations = $O(m\,\alpha(n)) = O((2V + E - 1)\,\alpha(V)) = O(E\,\alpha(V))$
    - $m = 2V + E - 1, n = V$
  - Note that $V^2 \geq E \geq V - 1$ on a connected graph without multi-edges

Prim’s Algorithm

```
Prim(G)
    Start with a tree T with one vertex (any vertex)
    while T is not a spanning tree
        Find least-weight edge that connects T to a new vertex
        Add this edge to T
```



Prim's Algorithm

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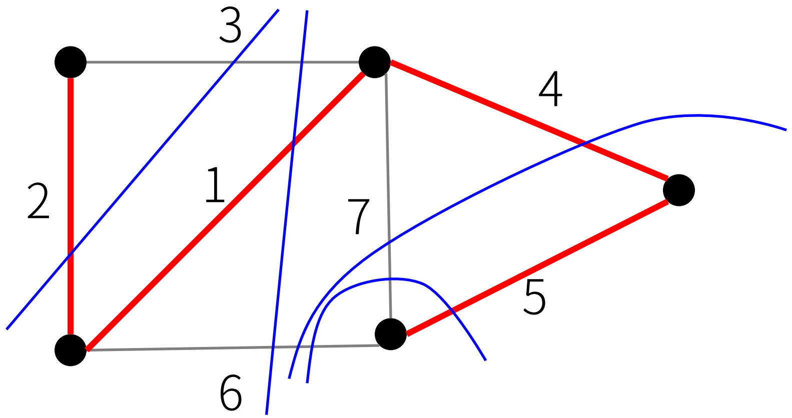


Weight = 12
MST

Implementation of Prim's algorithm

```
MST-PRIM(G, w, r) //w = weights, r = root
    for u in G.V
        u.key = ∞
        u.π = NIL
    r.key = 0
    Q = G.V //BUILD-MIN-QUEUE
    while Q ≠ empty
        u = EXTRACT-MIN(Q)
        for v in G.adj[u]
            if v ∈ Q and w(u,v) < v.key
                v.π = u
                v.key = w(u,v) //DECREASE-KEY
```

- Q = min-priority queue, containing vertices not yet in the tree
- $v.key$ = minimum weight of any edge connecting v to the tree
- $v.\pi$ = the parent of v in the tree

Running time analysis

- Binary min-heap [Ch. 6]
  - BUILD-MIN-HEAP = $O(V)$
  - EXTRACT-MIN = $O(\lg V)$
  - DECREASE-KEY = $O(\lg V)$
- Running time of Prim = $O(V \lg V + E \lg V)$
  = $O(E \lg V)$, because $V = O(E)$ in a connected graph
- Can be improved to $O(E + V \lg V)$ using Fibonacci heaps [Ch. 19]

MST properties

MST Uniqueness

MST is unique if all edge weights are distinct

Cycle property

For simplicity, assume all edge weights are distinct, thus an unique MST. Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.

Cut property

For simplicity, assume all edge weights are distinct, thus an unique MST. Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.

MST Uniqueness

MST is unique if all edge weights are distinct

Proof by contradiction

- Suppose there are two MSTs T_A and T_B on the same graph
- Let e be the least-weight edge in $T_A \cup T_B$ and e is not in both
- WLOG, assume e is in T_A
- Add e to T_B

=> $\{e\} \cup T_B$ contains a cycle C

=> C includes at least one edge e' that is not in T_A

=> In T_B, replacing e' with e yields a MST with less cost

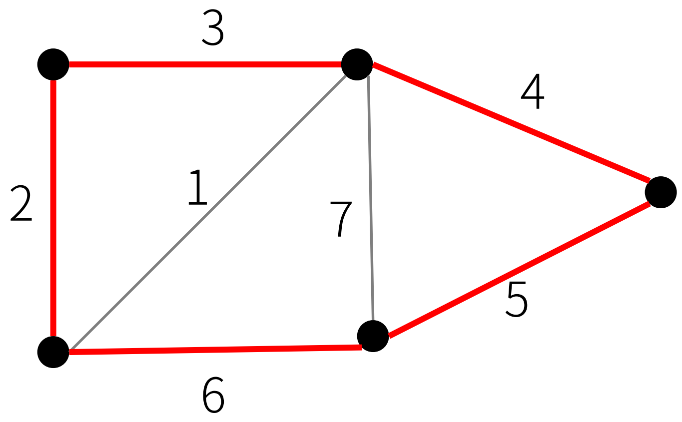
=> Contradiction!

MST uniqueness when edge weights are not distinct

- We can still break tie and ensure a unique MST by applying a lexicographical order of edges
- Let's define a new weight function w' over edges such that
  - $w'(e_i) < w'(e_j)$ if $w(e_i) < w(e_j)$ or ($w(e_i) = w(e_j)$ and $i < j$)
  - $w'(S_i) < w'(S_j)$ if $w(S_i) < w(S_j)$ or ($w(S_i) = w(S_j)$ and $S_i \backslash S_j$ has a lower indexed edge than $S_j \backslash S_i$)
- Hence, there is a unique MST w.r.t. to this new weight function w'
- Note: Having a unique edge order (and a unique MST) is useful for proving the correctness of Prim's and Kruskal's algorithms. However, the two algorithms **DO NOT** require the weights to be distinct.

Cycle property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.



No MST contains the edge of cost 6

Cycle property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.

Proof by contradiction

- Suppose e is in the MST

=> Removing e disconnects the MST T into two components T_1 and T_2

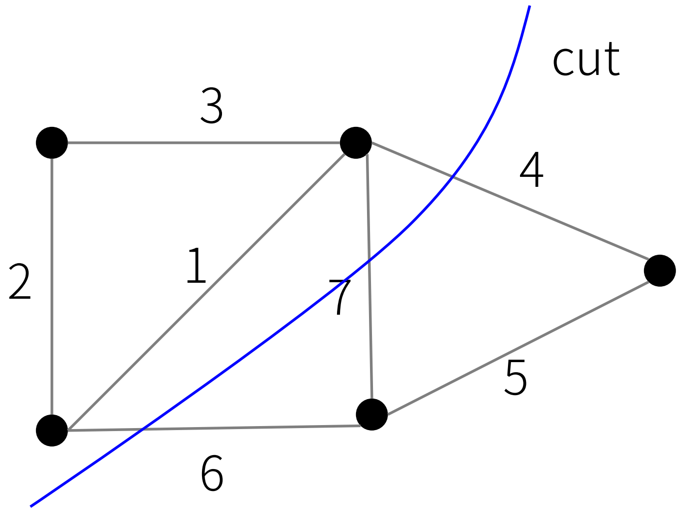
=> There exists another edge e' in C that can reconnect T_1&T_2 into T'

=> Since weight(e') < weight(e), the new tree T' has a lower weight than T

=> Contradiction!

Cut property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.



There is an MST containing the edge of cost 4

Cut property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.

Proof by contradiction

- Suppose e is not in the current MST T

=> Adding e creates a cycle in the MST T

=> There exists another edge e' in the cut C that can break the cycle; removing e' to generate a new tree T'

=> Since weight(e') > weight(e), the new tree has a lower weight

=> Contradiction!

Correctness of Kruskal’s algorithm

Kruskal's algorithm computes the MST

Proof

- Consider whether adding e creates a cycle:

1. If adding e to T creates a cycle C
   - Then e is the max weight edge in C
   - The cycle property ensures that e is not in the MST
2. If adding $e = (u, v)$ to T does not create a cycle
   - Before adding e, the current set contains at least two trees T_1 and T_2 such that u in T_1 and v in T_2
   - e is the minimum cost edge on the cut of T_1 and $V \backslash T_1$
   - The cut property ensures that e is in the MST

Prim's algorithm computes the MST

Proof

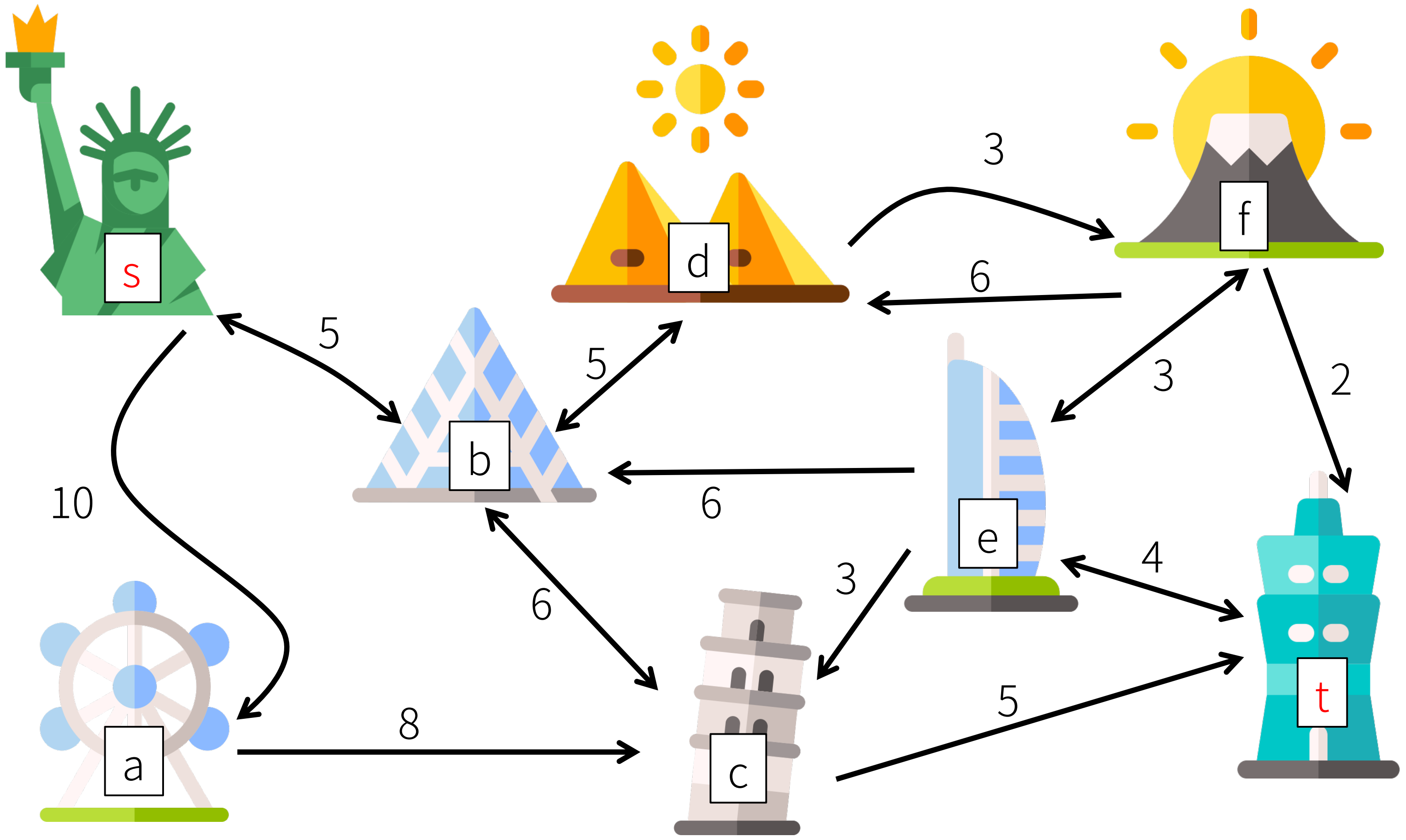
1. Prove that all edges found by Prim's are in the MST:
   - Prim's algorithm adds the cheapest edge e with exactly one endpoint in the current tree T
   - The cut property ensures that e is in the MST
2. Because Prim's outputs a spanning tree, |edges found by Prim's| = $V - 1$

- => Edges found by Prim's = edges on the MST

Shortest Paths: Terminology and Properties

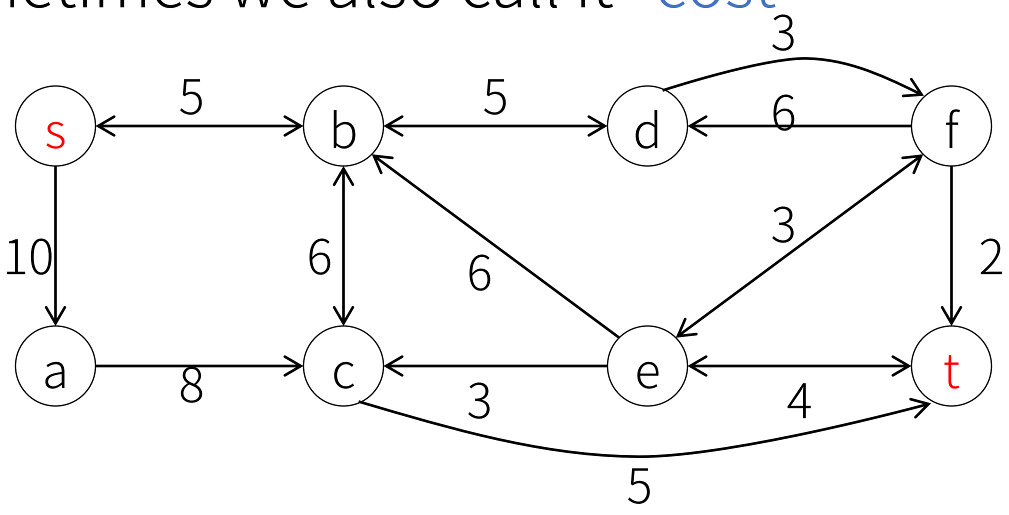
Textbook Chapter 24

Example



Definitions

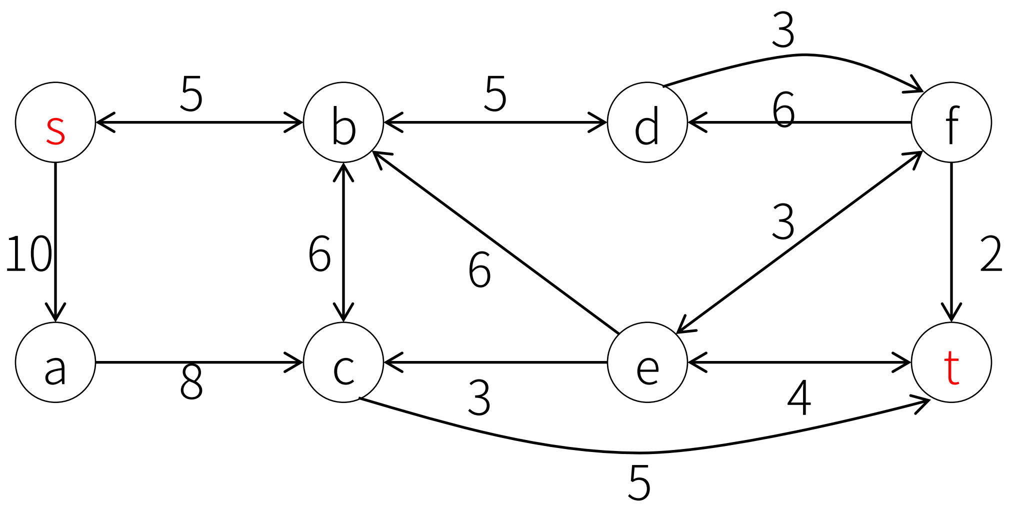
- Given a weighted, directed graph $G = (V, E)$
- Given a weight function w mapping an edge to a weight
  - Note that weights are arbitrary numbers, not necessarily distances
  - Weight function needs not satisfy triangle inequality (think about airline fares)
- Weight of path $p = w(p)$ = sum of weights of edges on p
  - Sometimes we also call it "cost"



The weight of path s->a->c->t is 23

Definitions

- Shortest-path weight $\delta(s, t)$ = minimum weight of path from s to t
- A shortest path from s to t = any path with weight $\delta(s, t)$



$\delta(s, t)$ = ?

Shortest path from s to t = ?

Q: Can a shortest path contain a negative-weight edge?

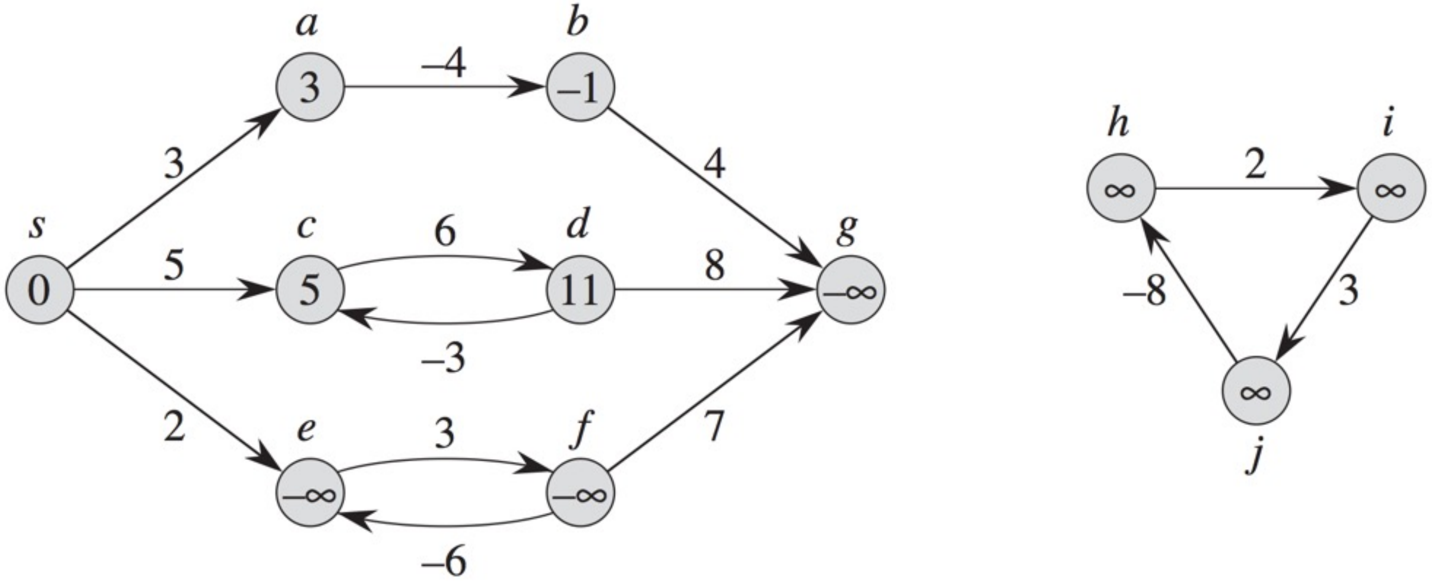
Yes.

$\delta(s, v)$ remains well defined for all v, if G contains no negative-weight cycles reachable from the source s.

Q: Can a shortest path contain a negative-weight cycle?

Doesn't make sense.

If there is a negative-weight cycle on some path from s to v, we define $\delta(s, v) = -\infty$.

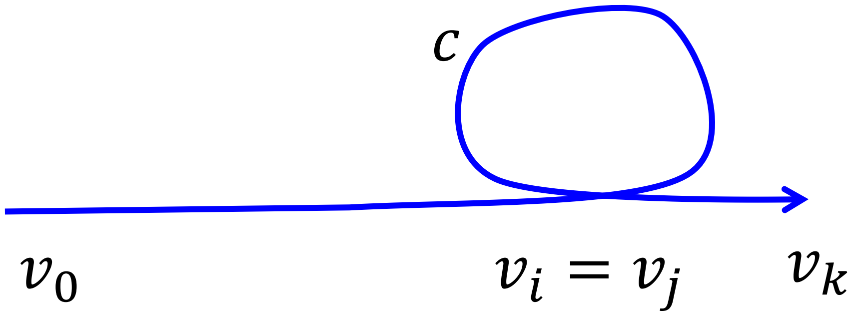


Q: Can a shortest path contain a positive-weight cycle?

No

Q: Can a shortest path contain a zero-weight cycle?

It may contain a zero-weight cycle, but then there must exist a simple path of the same weight.



$p =< v_0, v_1, \dots, v_k >$

$c =< v_i, v_{i+1}, \dots, v_j >$

Let $p' = < v_0, v_1, \dots v_i, v_{j+1}, v_{j+2}, \dots, v_k >$

$w(p') \leq w(p)$ if $w(c) \geq 0$

Q: Can a shortest path contain a cycle?

We safely assume shortest paths have no cycles

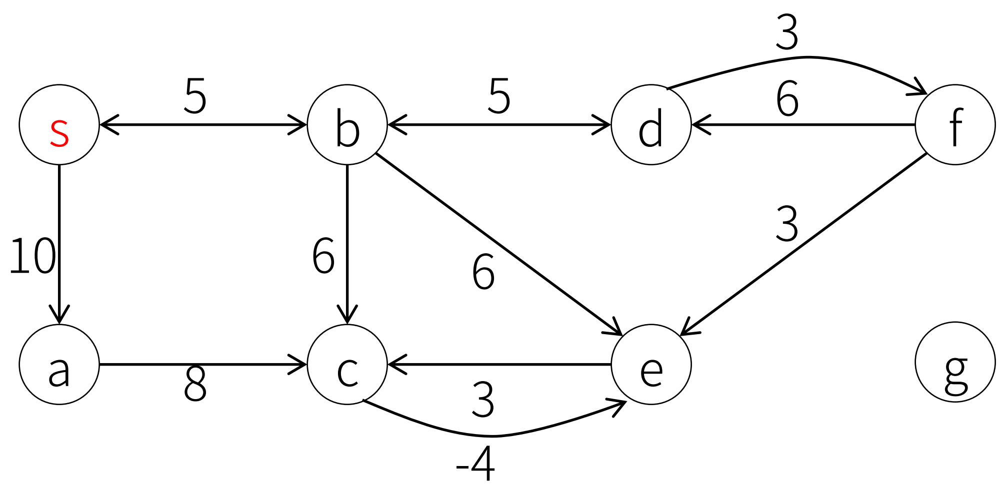
- Define $\delta(u, v) = \infty$ if v is unreachable from u
- Define $\delta(u, v) = -\infty$ if there exists a negative cycle on a path from u to v

Q: Is it correct that a shortest path has at most $|V| - 1$ edges?

Yes.

Having no cycle implies that a shortest path has at most $|V| - 1$ edges.

Practice



Destination v	Shortest path from s to v	Shortest path weight
a	s a	10
b		
c	NIL	$-\infty$
d		
e		
f	s b d f	13
g	NIL	∞

Single-source shortest-path algorithms

- Given a graph $G = (V, E)$ and a source vertex s in V, find the minimum cost paths from s to every vertex in V
- Dijkstra algorithm
  - Greedy
  - Requiring that all edge weights are nonnegative
- Bellman-Ford algorithm
  - Dynamic programming
  - General case, edge weights may be negative
- Both on a weighted, directed graph
- We'll introduce them next week

A very important technique: Relaxation

A common workflow for single-source shortest-path algorithms:

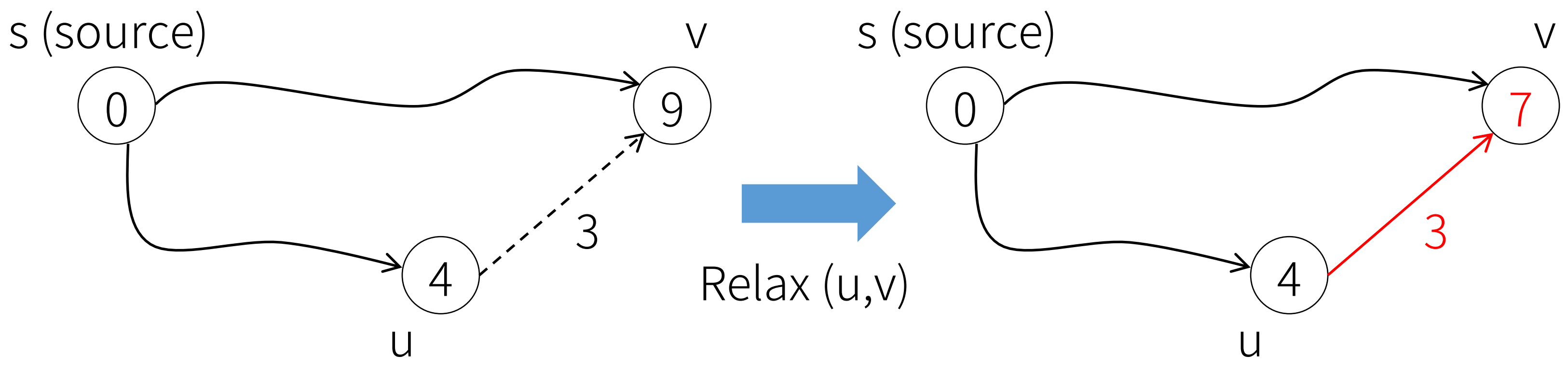
```
INITIALIZE-SINGLE-SOURCE(G,s)
    for v in G.V
        v.d = ∞ //estimate
        v.π = NIL //predecessor
    s.d = 0
```

Take a **sequence of** relaxation **steps** to update `v.d` and `v.π`

Output `v.d` and reconstruct shortest-paths from `v.π`

A very important technique: Relaxation

- The process of relaxing an edge (u, v)
  = testing whether the shortest path weight of v found so far can be reduced by traveling over u
- 試試看經過 u 會不會比較好（更短的 $s \rightsquigarrow v$ 路徑）



A very important technique: Relaxation

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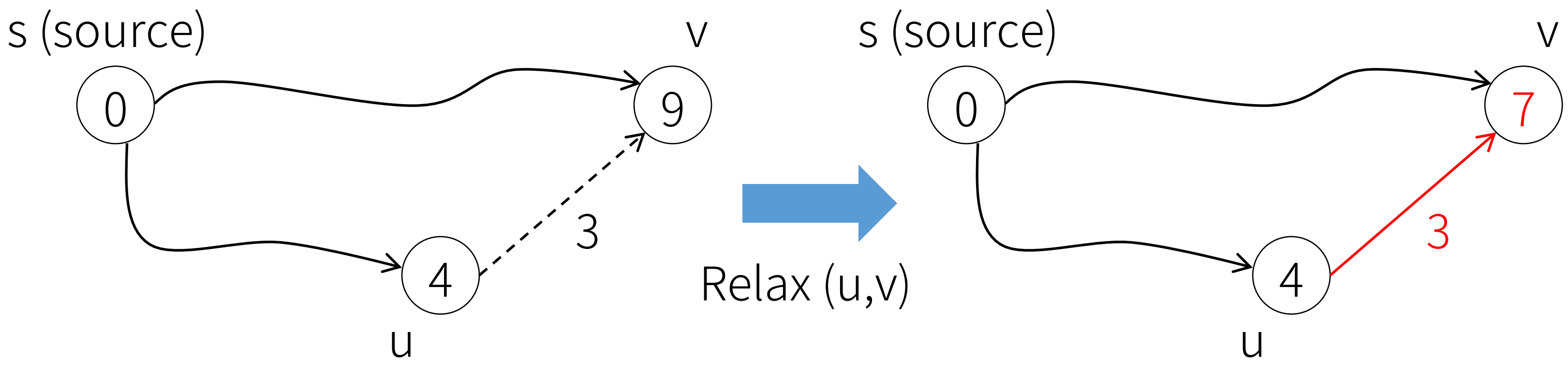


```
RELAX(u, v)
    if v.d > u.d + w(u, v)
        v.d = u.d + w(u, v)
        v.π = u
```

$v.d$ = shortest-path estimate

- An upper bound on $\delta(s, v)$ (Lemma 24.11)
- $v.d$ never increases during relaxation

$v.\pi$ = predecessor attribute