Problem 1 (Convolution)

$$f_i = \begin{cases} \frac{1}{2}, & i=0 \\ \frac{1}{2}, & i=1 \\ 0, & \text{else} \end{cases}$$

1st Convolution:

	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0
0	0	0	0

$$y_1[0] = \frac{1}{4}$$
$$y_1[1] = \frac{1}{2}$$
$$y_1[2] = \frac{1}{4}$$

2nd Convolution:

	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	0

$$y_2[0] = \frac{1}{8}$$
$$y_2[1] = \frac{3}{8}$$
$$y_2[2] = \frac{3}{8}$$
$$y_2[3] = \frac{1}{8}$$

3rd Convolution:

	$\frac{1}{2}$	$\frac{1}{2}$	0
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	0
$\frac{3}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	0
$\frac{3}{8}$	$\frac{3}{16}$	$\frac{3}{16}$	0
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	0

$$y_3[0] = \frac{1}{16}$$
$$y_3[1] = \frac{4}{16}$$
$$y_3[2] = \frac{6}{16}$$
$$y_3[3] = \frac{4}{16}$$
$$y_3[4] = \frac{1}{16}$$

Overall:

$$(f * f * f * f) = \begin{cases} \frac{1}{16} & \text{for } i = 0 \wedge 4; \\ \frac{4}{16} & \text{for } i = 1 \wedge 3; \\ \frac{6}{16} & \text{for } i = 2 \\ 0 & \text{else}. \end{cases}$$

Problem 2 (Properties of the Convolution)

$$(f * \omega)_i = \sum_{k=-\infty}^{\infty} f_{i-k} \cdot \omega_k$$

a) Linearity $(a \cdot f + \beta \cdot g) * \omega = a(f * \omega) + \beta(g * \omega)$

$\forall\, a, \beta \in \mathbb{R}$

$$= \sum_{k=-\infty}^{\infty} (a \cdot f_{i-k} + \beta\, g_{i-k}) \cdot \omega_k$$

$$= a \sum_{k=-\infty}^{\infty} f_{i-k}\, \omega_k + \beta \sum_{k=-\infty}^{\infty} g_{i-k} \cdot \omega_k$$

$$= a\,(f * \omega) + \beta\,(g * \omega)$$

b) Commutativity $(f * \omega) = (\omega * f)$

$$(f * \omega) = \sum_{k=-\infty}^{\infty} f_{i-k} \cdot \omega_k \quad | \text{ reindexing} \ldots \quad m = i-k$$

$$= \sum_{m=-\infty}^{\infty} f_m \cdot \omega_{i-m} = (\omega * f)$$

c) Identity: $(f * e)_i = \sum_{k=-\infty}^{\infty} (f_{i-k} \cdot e_k) \overset{!}{=} f_i$

therefore e_k must be 1 only at $k=0$

and 0 everywhere else;

$e_k|_{k=0} = 1$ and $e_k|_{k=\text{else}} = 0$ (sifting property)

Problem 3 (2D continuous Fourier Transform)

$$g(x,y) := \frac{1}{8}\Big[-f(x-1,y-1) + f(x+1,y-1) - 2f(x-1,y) + 2f(x+1,y) - f(x-1,y+1) + f(x+1,y+1)\Big]$$

$$F[g(x,y)] = \frac{1}{8}\Big[-F(f(x-1,y-1)) + F(f(x+1,y-1)) - 2F(f(x-1,y)) + 2F(f(x+1,y)) - F(f(x-1,y+1)) + F(f(x+1,y+1))\Big]$$

shifting

$$= \frac{1}{8}\Big[-F(f(x,y))\,e^{-j\omega(u+v)} + F(f(x,y))\,e^{-j\omega(-u+v)} - 2F(f(x,y))\,e^{-j\omega u} + 2F(f(x,y))\,e^{-j\omega(-u)} - F(f(x,y))\,e^{-j\omega(u-v)} + F(f(x,y))\,e^{-j\omega(-u-v)}\Big]$$

$$= \frac{1}{8}F(f(x,y))\Big[-e^{-j\omega(u+v)} + e^{-j\omega(-u+v)} - 2e^{-j\omega u} + 2e^{j\omega u} - e^{-j\omega(u-v)} + e^{-j\omega(-u-v)}\Big]$$

$$= \frac{1}{8}F(f(x,y))\Big[2e^{+j\omega u} - 2e^{-j\omega u} + e^{+j\omega(u+v)} - e^{-j\omega(u+v)} + e^{+j\omega(u-v)} - e^{-j\omega(u-v)}\Big]$$

$$= \frac{1}{8}F(f(x,y))\Big[4j\sin(\omega u) + 2j\sin(\omega(u+v))\; 2j\sin(\omega(u-v))\Big]$$