Traveling waves with continuous profile for hyperbolic Keller-Segel equation

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Abstract

This work describes a hyperbolic model for cell-cell repulsion with population dynamics. We consider the pressure produced by a population of cells to describe their motion. We assume that cells try to avoid crowded areas and prefer locally empty spaces far away from the carrying capacity. Here, our main goal is to prove the existence of traveling waves with continuous profiles. This article complements our previous results about sharp traveling waves. We conclude the paper with numerical simulations of the PDE problem, illustrating such a result.

Keywords: Traveling waves; hyperbolic equation; continuous-wave profiles; discontinuous wave profiles, sharp traveling waves

AMS Subject Classification: 92C17, 35L60, 35D30

1 Introduction

The model and its motivation: In this paper, we mainly consider the following equation:

$$\begin{cases} \partial_t u(t,x) = \underbrace{\chi\, \partial_x \left(u(t,x)\partial_x p(t,x)\right)}_{\text{Cell-cell repulsion}} + \underbrace{\lambda\, u(t,x)(1-\frac{u(t,x)}{\kappa})}_{\text{Vital dynamic}}, \quad t>0, x\in\mathbb{R}, \\ p(t,x) - \sigma^2\, \partial_{xx} p(t,x) = u(t,x), \quad t>0, x\in\mathbb{R}, \end{cases} \tag{1.1}$$

with the initial distribution

$$u(0,x) = u_0(x) \in \mathrm{L}^\infty(\mathbb{R}), \tag{1.2}$$

where $\lambda>0$ is the growth rate, $\kappa>0$ is the carrying capacity, $\chi>0$ is the dispersion coefficient, $\sigma>0$ is a sensing coefficient, $x\to u(t,x)$ is the density of population, and $p(t,x)$ is an external pressure.

Here the term density of population means that

$$\int_{x_1}^{x_2} u(t,x)dx$$

is the number of individuals between x_1 and x_2 (when $x_1<x_2$).

In the model the term $\chi\partial_x\left(u(t,x)\partial_x p(t,x)\right)$ describes the cell-cell repulsion, and a logistic term $u(t,x)(1-u(t,x))$ corresponds the cell division, cell mortality, and the quadratic term $u(t,x)^2/\kappa$ corresponds to growth limitations due to quorum sensing (for short slow down the process of cell division) and due to competition for resources.

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Replacing $u(t,x)$ and $p(t,x)$ by $\widehat{u}(t,x) = u(\dfrac{t}{\lambda},x)/\kappa$ and $\widehat{p}(t,x) = p(\dfrac{t}{\lambda},x)/\kappa$, we obtain (dropping the hat notation)

$$\begin{cases} \partial_t u(t,x) = \chi\, \partial_x \left(u(t,x)\partial_x p(t,x)\right) + u(t,x)(1-u(t,x)), & t>0, x\in\mathbb{R}, \\ p(t,x) - \sigma^2\, \partial_{xx} p(t,x) = u(t,x), & t>0, x\in\mathbb{R}. \end{cases} \tag{1.3}$$

Therefore, through the paper we will assume that

$$\lambda = 1 \text{ and } \kappa = 1.$$

Our original motivation comes from the description of cells motion in a Petri dish. In a previous paper [8], we derived a two-dimensional version of (1.3) to model the cell-cell repulsion in a Petri dish. We considered that cells grow in a circular domain (the Petri dish) and generate a repulsive gradient that pushes back neighboring cells. We built a numerical simulation framework to study the solutions of the partial differential equation and compared the results to some real experiments realized by Pasquier and collaborators [16]. When starting from an isolated disk-like islet, the solution of the PDE looks like an expanding disk whose radius seems to be growing at a constant speed. We can study the shape of an expanding islet by considering traveling waves for (1.3). Previously, we studied the well-posedness of the problem (1.3) in [9] and proved the existence of an asymptotic propagation discontinuous profile — a traveling wave — in [10], corresponding to an initial data that is equal to 0 outside of some bounded region. In other words, in [10], we considered the case of an initial population of cells with compact support: no cell exists initially outside of the islet. The traveling waves constructed in [10] are called *sharp* because the transition between the occupied space (the area where $u(t,x) > 0$) and the empty space (when $u(t,x) = 0$) occurs at some finite position. We also proved in [10] that sharp traveling waves are necessarily discontinuous. Our model is related to the study of Ducrot et al. [5] who introduced a complete model of in-vitro cell dynamics with many different behaviors at the cellular level. Other features of closely related models have been investigated in [4, 6, 7, 11, 12].

In the previous paper, we proved the existence of sharp traveling waves for (1.3). Our goal here is to complete the description of existing traveling waves that are not sharp. Formally, our work relates to the result of de Pablo and Vazquez [17], who studied the existence of sharp and not sharp traveling waves for a porous medium equation. The porous medium equation corresponds (formally) to the case $\sigma \to 0$. So far, the convergence of the traveling waves when $\sigma \to 0$ has been observed only numerically in [10]. Such a question remains open.

The notion of solution: In order to give a sense of the solution (1.3), we first assume that $x \to p(t,x)$ is regular enough. Then the nonlinear diffusion can be understood as

$$\chi \partial_x \left(u(t,x)\partial_x p(t,x)\right) = \chi \partial_x u(t,x) \partial_x p(t,x) + \chi u(t,x) \partial_{xx} p(t,x)$$

and by using the second equation of (1.3) we obtain

$$\chi \partial_x \left(u(t,x)\partial_x p(t,x)\right) = \chi \partial_x u(t,x) \partial_x p(t,x) + \frac{\chi}{\sigma^2} u(t,x) \left[p(t,x) - u(t,x)\right].$$

Therefore, the system (1.3) is understood for $t \geq 0$ and $x \in \mathbb{R}$ as

$$\begin{cases} \partial_t u(t,x) = \chi \partial_x u(t,x)\partial_x p(t,x) + u(t,x)\left(\left(1+\frac{\chi}{\sigma^2} p(t,x)\right) - \left(1+\frac{\chi}{\sigma^2}\right) u(t,x)\right), \\ p(t,x) - \sigma^2 \partial_{xx} p(t,x) = u(t,x), \end{cases} \tag{1.4}$$

with the initial distribution

$$u(0,x) = u_0(x) \in \mathrm{L}^\infty(\mathbb{R}). \tag{1.5}$$

The existence and uniqueness of solutions of (1.4) in $\mathrm{L}^\infty(\mathbb{R})$ have been considered as a subset of the weighted space $L^1_\eta(\mathbb{R})$ (with $\eta > 0$) with the norm

$$\|u\|_{L^1_\eta} = \int_{\mathbb{R}} e^{-\eta|x|} |u(x)| dx.$$

The existence and uniqueness of solutions for (1.4) has been studied by Fu, Griette, and Magal [9, Theorem 2.2].

Notion of traveling wave:

Definition 1.1 *A **traveling wave** is a special solution having the specific form*

$$u(t,x) = U(x-ct), \text{ for a.e. } (t,x) \in \mathbb{R}^2,$$

*where the **profile** U has the following behavior at $\pm\infty$:*

$$\lim_{z\to-\infty} U(z) = 1, \quad \lim_{z\to\infty} U(z) = 0.$$

*A traveling wave is **sharp** if there exists $x_0 \in \mathbb{R}$, such that*

$$U(x) = 0, \text{ for all } x > x_0.$$

*A traveling waves is **not sharp** if*

$$U(x) > 0, \text{ for all } x \in \mathbb{R}.$$

In Fu, Griette, and Magal [10, Proposition 2.4], we proved that the sharp traveling waves must be discontinuous. That is to say that $x \to U(x)$ the traveling wave profile of (1.3) can be either continuous or discontinuous.

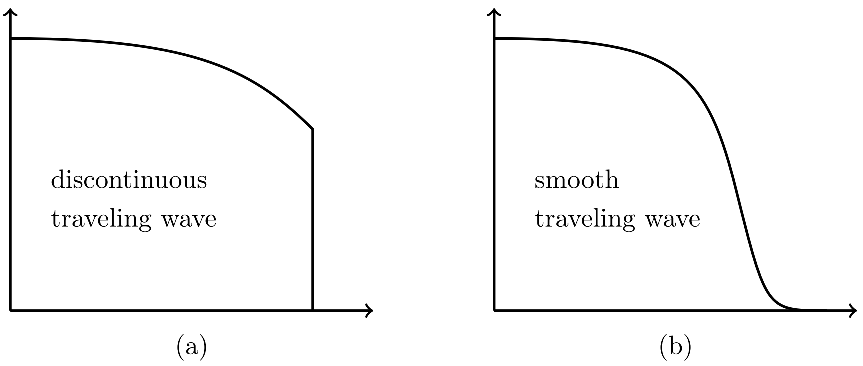

Figure 1: *An illustration of two types of traveling wave solutions.*

Estimations on the traveling speed for the discontinuous profile: Under a technical assumption on $\hat{\chi} = \frac{\chi}{\sigma^2}$, we can prove the existence of sharp traveling waves which present a jump at the vanishing point.

Assumption 1.2 (Bounds on $\hat{\chi}$) *Let $\chi > 0$ and $\sigma > 0$ be given and define $\hat{\chi} := \frac{\chi}{\sigma^2}$. We assume that $0 < \hat{\chi} < \bar{\chi}$, where $\bar{\chi}$ is the positive unique root of the function*

$$\hat{\chi} \mapsto \ln\left(\frac{2-\hat{\chi}}{\hat{\chi}}\right) + \frac{2}{2+\hat{\chi}}\left(\frac{\hat{\chi}}{2}\ln\left(\frac{\hat{\chi}}{2}\right) + 1 - \frac{\hat{\chi}}{2}\right).$$

The existence of traveling waves with discontinuous profile has been studied in Fu, Griette, and Magal [10, Theorem 2.4].

Theorem 1.3 (Existence of a sharp discontinuous traveling wave) *Let Assumption 1.2 be satisfied. There exists a traveling wave $u(t,x) = U(x-ct)$ traveling at speed*

$$c \in \left(\frac{\sigma\hat{\chi}}{2+\hat{\chi}}, \frac{\sigma\hat{\chi}}{2}\right),$$

where

$$\hat{\chi} = \frac{\chi}{\sigma^2}.$$

Moreover, the profile U satisfies the following properties (up to a shift in space):

(i) *U is* sharp *in the sense that $U(x) = 0$ for all $x \geq 0$; moreover, U has a discontinuity at $x = 0$ with $U(0^-) \geq \frac{2}{2+\hat{\chi}}$.*

(ii) *U is continuously differentiable and strictly decreasing on $(-\infty, 0]$, and satisfies*

$$-cU' - \chi(UP')' = U(1-U) \text{ on } (-\infty, 0),$$

and

$$U = 0 \text{ on } (0, \infty),$$

and

$$P - \sigma^2 P'' = U \text{ on } \mathbb{R}.$$

In this article, we focus on the existence of traveling waves with continuous profiles. The main result of this paper is the following theorem.

Theorem 1.4 (Existence of a continuous traveling wave) *We assume that*

$$c \geq 2\sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)}.$$

There exists a traveling wave $u(t,x) = U(x-ct)$ with a continuous profile $x \to U(x)$ is continuously differentiable and strictly decreasing, and

$$\lim_{x\to-\infty} U(x) = 1, \text{ and } \lim_{x\to+\infty} U(x) = 0, \tag{1.6}$$

and satisfies traveling wave problem

$$-cU' - \chi(UP')' = U(1-U), \text{ on } \mathbb{R}, \tag{1.7}$$

where

$$P - \sigma^2 P'' = U, \text{ on } \mathbb{R}. \tag{1.8}$$

Estimations on the traveling speed: We obtain the following condition for the existence of a traveling wave with a continuous profile for all-speed

$$c \geq c^*_{\text{cont}} := 2\sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)}.$$

For the Fisher-KPP equation [2, 14, 20], traveling waves only exist for half-line of positive traveling speeds. Moreover, there is a minimum speed $c_* > 0$ below which no traveling wave exists. Moreover, we can construct traveling waves for any values c above c_*. The existence of minimum speed is also true for porous medium equations with logistic dynamics [17]. By analogy with the porous medium equations, we expect that the minimal speed of the traveling waves corresponds to the sharp traveling wave constructed in [10]. In contrast, the continuous traveling waves constructed in the present paper correspond to higher velocities. Recall from [10, Theorem 2.4] that the sharp traveling wave is expected to travel at a speed $c_{\text{sharp}} \in \left(\frac{\chi/\sigma}{2+\chi/(\sigma^2)}, \frac{\chi}{2\sigma}\right)$ and indeed we have that

$$c^*_{\text{cont}} = 2\sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)} \geq 2\frac{\chi}{\sigma} \geq \frac{1}{2}\frac{\chi}{\sigma} \geq c_{\text{sharp}}.$$

Further analysis will be necessary to connect the gap between c^*_{cont} and c_{sharp} and to possibly prove the non-existence of traveling waves slower than sharp waves. Understanding the relationships between the profiles and the traveling speeds is still an open problem.

The paper is organized as follows. Section 2 is devoted to preliminary results. Section 3 presents the fixed point problem and its properties. Section 4 is devoted to the proof of Theorem 1.4. In Section 5, we present some numerical simulations. In Section 6, we present an application to wound healing.

2 Preliminary

Definition 2.1 *We will say that system* (1.3) *has a **traveling wave** (with continuous profile) if we can find a bounded decreasing continuous function* $U : \mathbb{R} \to \mathbb{R}$*, such that*

$$u(t,x) = U(x-ct), \forall (t,x) \in \mathbb{R}^2,$$

is a solution of (1.4) *which satisfies in addition*

$$\lim_{x \to -\infty} U(x) = 1, \quad \lim_{x \to \infty} U(x) = 0.$$

The function U *is called a **traveling wave profile**.*

In Figure 2, we illustrate the continuous traveling wave profiles.

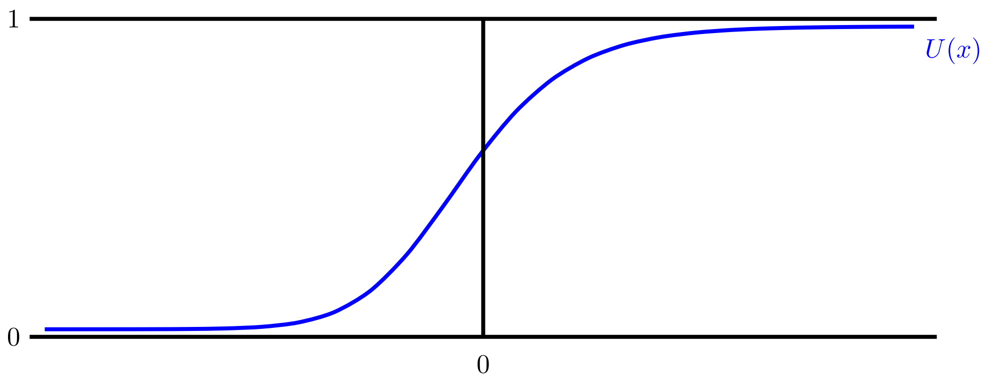

Figure 2: *In this figure, we plot the traveling wave profile* $x \to U(x)$*.*

Lemma 2.2 *Assume that system* (1.4) *has a traveling wave* $u(t,x) = U(x-ct)$*. Then we must have*

$$p(t,x) = P(x-ct),$$

where $P : \mathbb{R} \to \mathbb{R}$ *is the unique bounded continuous function satisfying the elliptic equation*

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}.$$

Proof.

$$p(t,x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} u(t,x-y)\mathrm{d}y = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U(x-y-ct)\mathrm{d}y, \tag{2.1}$$

therefore $p(t,x) = P(x-ct)$ where

$$P(x) = \frac{1}{2\sigma} \int_{\mathbb{R}} e^{-\frac{|y|}{\sigma}} U(x-y)\mathrm{d}y \Leftrightarrow P(x) - \sigma^2 P''(x) = U(x), x \in \mathbb{R}.$$

∎

The following proposition was proved in Fu, Griette and Magal [10, Proposition 2.4].

Proposition 2.3 *Assume that* U *is a continuous profile of traveling wave. Then* $U : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable, and*

$$c + \chi P'(x) > 0, \forall x \in \mathbb{R} \left(\Leftrightarrow 0 \leq -P'(x) < \frac{c}{\chi}, \forall x \in \mathbb{R} \right). \tag{2.2}$$

Transforming $U(x)$ **into** $\widehat{U}(x) = U(-x)$**:** By Definition 2.1 and Lemma 2.2, we get the following traveling wave problem

$$-\left(c + \chi P'(x)\right) U'(x) = U(x) \left(\left(1 + \frac{\chi}{\sigma^2} P(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) U(x) \right), \forall x \in \mathbb{R}, \tag{2.3}$$

where

$$P(x)-\sigma^2 P''(x)=U(x),\forall x\in\mathbb{R}. \tag{2.4}$$

Equation (2.3) has the following behavior at $\pm\infty$:

$$\lim_{x\to-\infty} U(x)=1, \quad \lim_{x\to+\infty} U(x)=0.$$

Now, let us perform the change of variables to reverse the time direction. Setting $\widehat{U}(x)=U(-x)$, and $\widehat{P}(x)=P(-x)$, then equations (2.3) and (2.4) become

$$\left(c-\chi\widehat{P}'(x)\right)\widehat{U}'(x)=\widehat{U}(x)\left(\left(1+\frac{\chi}{\sigma^2}\widehat{P}(x)\right)-\left(1+\frac{\chi}{\sigma^2}\right)\widehat{U}(x)\right),\forall x\in\mathbb{R}, \tag{2.5}$$

where

$$\widehat{P}(x)-\sigma^2\widehat{P}''(x)=\widehat{U}(x),\forall x\in\mathbb{R}.$$

Assume that $c+\chi P'(x)>0,\forall x\in\mathbb{R}$, then we have $c-\chi\widehat{P}'(x)>0,\forall x\in\mathbb{R}$ by using $\widehat{P}(x)=P(-x)$.

For convenience, we drop the hat notation, and system (2.5) becomes a logistic equation

$$U'(x)=\lambda(x)\,U(x)-\kappa(x)\,U(x)^2,\forall x\in\mathbb{R}, \tag{2.6}$$

where

$$\lambda(x):=\frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(x)},\forall x\in\mathbb{R}, \tag{2.7}$$

and

$$\kappa(x):=\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)},\forall x\in\mathbb{R}, \tag{2.8}$$

with $P(x)$ is the unique solution of the elliptic equation

$$P(x)-\sigma^2 P''(x)=U(x),\forall x\in\mathbb{R}. \tag{2.9}$$

System (2.6) has the following behavior at $\pm\infty$

$$\lim_{x\to-\infty} U(x)=0, \quad \lim_{x\to\infty} U(x)=1. \tag{2.10}$$

Lemma 2.4 *Assume that $U:\mathbb{R}\to\mathbb{R}$ is an increasing C^1 function. Then the map $x\to P(x)$ solving the elliptic equation*

$$P(x)-\sigma^2 P''(x)=U(x),\forall x\in\mathbb{R},$$

is an increasing C^3 function, and we have the following estimation of the first derivative of $P(x)$

$$\sup_{x\in\mathbb{R}} P'(x)\le\sup_{x\in\mathbb{R}} U'(x). \tag{2.11}$$

Proof. The result follows the following inequality

$$0\le P'(x)=\frac{1}{2\sigma}\int_{\mathbb{R}} \mathrm{e}^{-\frac{|y|}{\sigma}}U'(x-y)\mathrm{d}y\le\sup_{x\in\mathbb{R}} U'(x). \tag{2.12}$$

∎

Lemma 2.5 *Assume that*

$$0\le U'(x)\le\frac{c}{2\chi},\forall x\in\mathbb{R},$$

and

$$0<u_0<\frac{\sigma^2}{2\left(\sigma^2+\chi\right)}.$$

Then any $U:\mathbb{R}\to\mathbb{R}$ is an increasing C^1 function satisfying (2.6), (2.10) and

$$U(0)=u_0, \tag{2.13}$$

which is given by the following formula

$$U(x)=\frac{u_0\mathrm{e}^{\int_0^x\lambda(s)\mathrm{d}s}}{1+u_0\int_0^x\kappa(s)\mathrm{e}^{\int_0^s\lambda(\tau)\mathrm{d}l}\mathrm{d}s},\forall x\in\mathbb{R},$$

where $\lambda(x)$, $\kappa(x)$, and $P(x)$ are given by equations (2.7), (2.8), and (2.9) above.

Proof. Let us prove that the formula

$$\frac{u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}}{1+u_0\int_0^x \kappa(s)\mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l}\mathrm{d}s}, \tag{2.14}$$

is well define for all $x \in \mathbb{R}$. So let us prove that if $0 < u_0 < \frac{\sigma^2}{2(\sigma^2+\chi)}$, then we have

$$1-u_0\int_{-\infty}^0 \kappa(s)\mathrm{e}^{-\int_s^0 \lambda(l)\mathrm{d}l}\mathrm{d}s > 0.$$

Indeed, since by assumption

$$0 \le U'(x) \le \frac{c}{2\chi}, \ \forall x \in \mathbb{R},$$

and Lemma 2.4, we deduce that

$$0 \le P'(x) \le \sup_{x\in\mathbb{R}} U'(x) \le \frac{c}{2\chi}, \ \forall x \in \mathbb{R}, \tag{2.15}$$

hence

$$c-\chi P'(x) \ge \frac{c}{2}, \ \forall x \in \mathbb{R},$$

and since $P'(x) \ge 0$, we deduce that

$$\frac{1}{c} \le \frac{1}{c-\chi P'(x)} \le \frac{2}{c}, \ \forall x \in \mathbb{R}. \tag{2.16}$$

Now by combining (2.16), and $0 \le P(x) \le 1$, for any $x \in \mathbb{R}$, we deduce that

$$\frac{1}{c} \le \lambda(x) \le \frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right) \text{ and } 0 < \frac{1}{c}\left(1+\frac{\chi}{\sigma^2}\right) \le \kappa(x) \le \frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right), \forall x \in \mathbb{R}. \tag{2.17}$$

Define

$$G := 1-u_0\int_{-\infty}^0 \kappa(s)\mathrm{e}^{-\int_s^0 \lambda(l)\mathrm{d}l}\mathrm{d}s.$$

Using (2.17), we have that

$$\begin{aligned} G &\ge 1-u_0\int_{-\infty}^0 \frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right)\mathrm{e}^{-\int_s^0 \frac{1}{c}\mathrm{d}l}\mathrm{d}s \\ &= 1-\frac{2u_0}{c}\left(1+\frac{\chi}{\sigma^2}\right)\int_{-\infty}^0 \mathrm{e}^{\frac{s}{c}}\mathrm{d}s \\ &= 1-2u_0\left(1+\frac{\chi}{\sigma^2}\right) > 0, \end{aligned} \tag{2.18}$$

by using the assumption

$$0 < u_0 < \frac{\sigma^2}{2\left(\sigma^2+\chi\right)}.$$

■

3 The relationship between the fixed point and traveling waves

Definition 3.1 *Let $\mathcal{A}$ be the set of all admissible function $U : \mathbb{R} \to [0, 1]$ satisfying*

(i) $U \in C^1(\mathbb{R})$;

(ii) $0 \le U(x) \le 1, \forall x \in \mathbb{R}$;

(iii) $0 \le U'(x) \le \frac{c}{2\chi}, \forall x \in \mathbb{R}$.

For each $U \in \mathcal{A}$, we define

$$\mathcal{T}(U)(x) := V(x), \forall x \in \mathbb{R}, \tag{3.1}$$

where

$$V(x) = \frac{u_0 e^{\int_0^x \lambda(s)ds}}{1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds}, \forall x \in \mathbb{R}, \tag{3.2}$$

with

$$0 < u_0 < \frac{\sigma^2}{2(\sigma^2 + \chi)}, \tag{3.3}$$

and

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)}, \forall x \in \mathbb{R}, \tag{3.4}$$

and

$$\kappa(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)}, \forall x \in \mathbb{R}, \tag{3.5}$$

and $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}. \tag{3.6}$$

Assumption 3.2 *We assume that*

$$c \geq 2\sqrt{\chi\left(1 + \frac{\chi}{\sigma^2}\right)} \text{ and } 0 < u_0 < \frac{\sigma^2}{2(\sigma^2 + \chi)}.$$

Lemma 3.3 (Invariance of $\mathcal{A}$ by $\mathcal{T}$) *Let Assumption 3.2 be satisfied. Let $\mathcal{T}$ be the map defined by (3.1). Then*

$$\mathcal{T}(\mathcal{A}) \subset \mathcal{A}.$$

Proof. We divide the proof in four steps.
Step 1. We prove that $V = \mathcal{T}(U) \in C^1(\mathbb{R})$. Indeed, V is continuously differentiable and

$$V'(x) = \frac{\lambda(x) u_0 e^{\int_0^x \lambda(s)ds} (1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds) - \kappa(x) \left(u_0 e^{\int_0^x \lambda(s)ds}\right)^2}{(1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds)^2}, \tag{3.7}$$

hence

$$V'(x) = \lambda(x) V(x) - \kappa(x) V(x)^2, \forall x \in \mathbb{R}. \tag{3.8}$$

It follows from the definitions of $\lambda(x)$ and $\kappa(x)$ (see (3.4) and (3.5)), that $\lambda(x)$ and $\kappa(x)$ are continuously differentiable. Therefore, we have

$$V \in C^1(\mathbb{R}).$$

Step 2. We prove that $0 < V(x) \leq 1$, $\forall x \in \mathbb{R}$. By (2.11) and $U \in \mathcal{A}$, we have that

$$0 \leq \sup_{x \in \mathbb{R}} P'(x) \leq \sup_{x \in \mathbb{R}} U'(x) \leq \frac{c}{2\chi}.$$

Therefore, we have $c - \chi P'(x) > 0$. Recall that

$$V(x) = \frac{u_0 e^{\int_0^x \lambda(s)ds}}{1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds}, \forall x \in \mathbb{R}.$$

By using (3.3) we know that

$$1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds > 0, \forall x \in \mathbb{R}. \tag{3.9}$$

Therefore, by definition of $V(x)$, we have that $V(x) > 0, \forall x \in \mathbb{R}$. On the other hand, by using (3.8), we have that

$$V'(x) = \frac{1}{c - \chi P'(x)} V(x) \left(\left(1 + \frac{\chi}{\sigma^2} P(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) V(x)\right), \forall x \in \mathbb{R}, \tag{3.10}$$

and

$$P(x)=\frac{1}{2\sigma}\int_{\mathbb{R}}\mathrm{e}^{-\frac{|x-y|}{\sigma}}U(y)\mathrm{d}y=\frac{1}{2\sigma}\int_{\mathbb{R}}\mathrm{e}^{-\frac{|y|}{\sigma}}U(x-y)\mathrm{d}y,\ \forall x\in\mathbb{R}.$$

Since $0\le U(x)\le 1$, $\forall x\in\mathbb{R}$, we have

$$0\le P(x)\le 1,\forall x\in\mathbb{R}. \tag{3.11}$$

Define the map $\tau:\mathbb{R}\to\mathbb{R}$ satisfies

$$\tau'(x)=c-\chi P'(\tau(x)),\ \text{for } x\in\mathbb{R},\ \text{and } \tau(0)=0. \tag{3.12}$$

Let $W(x)=V(\tau(x))$, then by (3.10), we have that

$$\begin{aligned}W'(x)&=V'(\tau(x))(c-P'(\tau(x)))\\&=V(\tau(x))\left(\left(1+\frac{\chi}{\sigma^2}P(\tau(x))\right)-\left(1+\frac{\chi}{\sigma^2}\right)V(\tau(x))\right)\\&=W(x)\left(1+\frac{\chi}{\sigma^2}P(\tau(x))-\left(1+\frac{\chi}{\sigma^2}\right)W(x)\right).\end{aligned} \tag{3.13}$$

Therefore, by using (3.11), we deduce that for $x\in\mathbb{R}$

$$\begin{aligned}W'(x)&=W(x)\left(1+\frac{\chi}{\sigma^2}P(\tau(x))-\left(1+\frac{\chi}{\sigma^2}\right)W(x)\right)\\&\le W(x)\left(1+\frac{\chi}{\sigma^2}-\left(1+\frac{\chi}{\sigma^2}\right)W(x)\right).\end{aligned} \tag{3.14}$$

By (3.14) and the comparison principle, we have that

$$V(\tau(x))=W(x)\le 1,\forall x\in\mathbb{R}.$$

Therefore, we have

$$0\le V(x)\le 1,\ \forall x\in\mathbb{R}. \tag{3.15}$$

Step 3. Let us prove that

$$0<V'(x)\le\frac{c}{2\chi},\ \forall x\in\mathbb{R}.$$

Since

$$0\le U'(x)\le\frac{c}{2\chi},\ \forall x\in\mathbb{R},$$

then we have

$$0\le P'(x)=\frac{1}{2\sigma}\int_{\mathbb{R}}e^{-\frac{|y|}{\sigma}}U'(x-y)\mathrm{d}y\le\sup_{x\in\mathbb{R}}U'(x)\le\frac{c}{2\chi},\ \forall x\in\mathbb{R}. \tag{3.16}$$

By (3.16), we have that

$$c-\chi P'(x)\ge\frac{c}{2},\ \forall x\in\mathbb{R}.$$

Therefore we deduce

$$0<\frac{1}{c-\chi P'(x)}\le\frac{2}{c},\ \forall x\in\mathbb{R}. \tag{3.17}$$

Since $0\le U(x)\le 1$, and

$$P(x)=\frac{1}{2\sigma}\int_{\mathbb{R}}\mathrm{e}^{-\frac{|y|}{\sigma}}U(x-y)\mathrm{d}y,\ \forall x\in\mathbb{R},$$

we deduce that

$$0\le P(x)\le 1,\ \forall x\in\mathbb{R}. \tag{3.18}$$

Therefore, by using the fact that $0\le V(x)\le 1$, $\forall x\in\mathbb{R}$, (3.10), (3.17), and (3.18), we deduce that for $x\in\mathbb{R}$

$$V'(x)\le\frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right),\forall x\in\mathbb{R}. \tag{3.19}$$

Since $c\ge 2\sqrt{\chi\left(1+\frac{\chi}{\sigma^2}\right)}$, it follows that

$$\frac{\chi}{c^2}\left(1+\frac{\chi}{\sigma^2}\right)\le\frac{1}{4},$$

hence

$$\frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right) \leq \frac{c}{2\chi}. \tag{3.20}$$

Using (3.19), and (3.20), we obtain on one hand,

$$V'(x) \leq \frac{c}{2\chi}, \forall x \in \mathbb{R}. \tag{3.21}$$

On the other hand, by using (3.7), we have

$$\begin{aligned} V'(x) &= \frac{\lambda(x) u_0 e^{\int_0^x \lambda(s)ds} \left(1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds\right) - \kappa(x) \left(u_0 e^{\int_0^x \lambda(s)ds}\right)^2}{\left(1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds\right)^2} \\ &= \frac{\lambda(x) u_0 e^{\int_0^x \lambda(s)ds} + \Lambda(x)}{\left(1 + u_0 \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds\right)^2}, \end{aligned} \tag{3.22}$$

where

$$\Lambda(x) := u_0^2 e^{\int_0^x \lambda(s)ds} \left[\lambda(x) \int_0^x \kappa(s) e^{\int_0^s \lambda(l)dl} ds - \kappa(x) e^{\int_0^x \lambda(s)ds}\right].$$

From (3.22), to prove $V'(x) > 0$ for $x \in \mathbb{R}$, we only need to prove that

$$\Lambda_1(x) := \lambda(x) u_0 e^{\int_0^x \lambda(s)ds} + \Lambda(x) > 0.$$

Indeed, we have

$$\Lambda(x) = \kappa(x) \left(u_0 e^{\int_0^x \lambda(s)ds}\right)^2 \left(\frac{\lambda(x) \int_0^x \kappa(s) e^{-\int_s^x \lambda(l)dl} ds}{\kappa(x)} - 1\right). \tag{3.23}$$

By using the definitions $\lambda(x)$ and $\kappa(x)$ in (3.4) and (3.5), and the formula (3.23), we deduce that

$$\Lambda(x) = \kappa(x) \left(u_0 e^{\int_0^x \lambda(s)ds}\right)^2 \left(\int_0^x \frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}dl} ds - 1\right). \tag{3.24}$$

Since by assumption U is increasing, it follows from (3.16) that P is increasing. Then, for any $s < x$, we have $P(s) \leq P(x)$. We deduce that

$$\begin{aligned} \int_0^x \frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}dl} ds &\geq \int_0^x \frac{1+\frac{\chi}{\sigma^2}P(s)}{c-\chi P'(s)} e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}dl} ds \\ &= \int_0^x \frac{d}{ds}\left(e^{-\int_s^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}dl}\right) ds \\ &= 1 - e^{-\int_0^x \frac{1+\frac{\chi}{\sigma^2}P(l)}{c-\chi P'(l)}dl} \\ &= 1 - e^{-\int_0^x \lambda(s)ds}, \end{aligned}$$

therefore by combining (3.24) and the above inequality, we obtain

$$\begin{aligned}\Lambda(x) &\geq \kappa(x)\left(u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\right)^2\left(1-\mathrm{e}^{-\int_0^x \lambda(s)\mathrm{d}s}-1\right)\\ &= -\kappa(x)u_0^2 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}.\end{aligned} \tag{3.25}$$

By using (3.25) and the definitions of $\lambda(x)$ and $\kappa(x)$ for $x \in \mathbb{R}$, we have that

$$\begin{aligned}\Lambda_1(x) &= \lambda(x)u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s} + \Lambda(x)\\ &\geq \lambda(x)u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s} - \kappa(x)u_0^2 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\\ &= u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\left[\frac{1+\frac{\chi}{\sigma^2}P(x)}{c-\chi P'(x)} - u_0\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)}\right]\\ &= \frac{u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}}{c-\chi P'(x)}\left[1+\frac{\chi}{\sigma^2}P(x) - u_0\left(1+\frac{\chi}{\sigma^2}\right)\right].\end{aligned} \tag{3.26}$$

To conclude it remains to recall that by assumption we have

$$u_0 < \frac{1}{2}\frac{\sigma^2}{\sigma^2+\chi} < \frac{\sigma^2}{\sigma^2+\chi},$$

therefore since $P(x) \geq 0$ and $c - \chi P'(x) > 0$, we have that

$$\Lambda_1(x) \geq \left(u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}\right)\frac{1+\frac{\chi}{\sigma^2}}{c-\chi P'(x)}\left[\frac{\sigma^2}{\sigma^2+\chi} - u_0\right] > 0,\ \forall x \in \mathbb{R},$$

which implies

$$V'(x) > 0,\ \forall x \in \mathbb{R}. \tag{3.27}$$

The conclusion of the Step 3 now follows from (3.21) and (3.27). The proof is completed. ■

Let $\eta > 0$. Let $\mathrm{BUC}(\mathbb{R})$ be the space of bounded and uniformly continuous maps from $\mathbb{R}$ to itself. Define the weighted space of continuous functions

$$BUC_\eta(\mathbb{R}) = \left\{U \in C(\mathbb{R}) : x \to e^{-\eta|x|}U(x) \in \mathrm{BUC}(\mathbb{R})\right\},$$

and the weighted space n-times continuous differentiable functions

$$BUC_\eta^n(\mathbb{R}) = \left\{U \in C^n(\mathbb{R}) : x \to e^{-\eta|x|}U^{(k)}(x) \in \mathrm{BUC}(\mathbb{R}), \forall \mathrm{k} = 0, \ldots, \mathrm{n}\right\},$$

which is a Banach space endowed with the norm

$$\|U\|_{n,\eta} := \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U(x)| + \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U'(x)| + \ldots + \sup_{x\in\mathbb{R}} \mathrm{e}^{-\eta|x|}|U^{(n)}(x)|. \tag{3.28}$$

Here we will use the above weighted space of $\mathrm{BUC}_\eta^1(\mathbb{R})$ maps to ensure that

$$\mathcal{A} = \left\{U \in C^1(\mathbb{R}) : 0 \leq U(x) \leq 1,\ \text{and } 0 \leq U'(x) \leq \frac{c}{2\chi}, \forall x \in \mathbb{R}\right\},$$

is a closed subset of $\mathrm{BUC}_\eta^1(\mathbb{R})$. As a consequence, the subset $\mathcal{A}$ equipped with the distance

$$d_\eta(U_1, U_2) = \|U_1 - U_2\|_{1,\eta}.$$

is a complete metric space.

Lemma 3.4 (Compactness of $\mathcal{T}$) *Let Assumption 3.2 be satisfied. Then the set $\overline{\mathcal{T}(\mathcal{A})}$ is a compact subset of the metric space $\mathcal{A}$ equipped with the distance d_η.*

Proof. Let $\{U_n\}_{n\geq 0} \subset \mathcal{A}$ be a sequence, and define the corresponding sequence $\{P_n\}_{n\geq 0}$ solution of equation (3.6) where U is replaced by U_n. Define the corresponding sequences $\{\lambda_n\}_{n\geq 0}$ and $\{\kappa_n\}_{n\geq 0}$ by using (3.4) and (3.5) where $P(x)$ is replaced by $P_n(x)$. Denote $V_n = \mathcal{T}(U_n), \forall n \in \mathbb{N}$. By Lemma 3.3, we know that $\mathcal{T}(\mathcal{A}) \subset \mathcal{A}$. Therefore we have

$$0 < V_n(x) \leq 1 \text{ and } 0 < V_n'(x) \leq \frac{c}{2\chi}, \ \forall x \in \mathbb{R}. \tag{3.29}$$

Similarly to equation (3.8) in the proof of Lemma 3.3, we have that

$$V_n'(x) = \lambda_n(x)V_n(x) - \kappa_n(x)V_n^2(x), \ \forall x \in \mathbb{R}, \tag{3.30}$$

where

$$\lambda_n(x) = \frac{1 + \frac{\chi}{\sigma^2}P_n(x)}{c - \chi P_n'(x)}, \ \forall x \in \mathbb{R},$$

and

$$\kappa_n(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P_n'(x)}, \ \forall x \in \mathbb{R},$$

and $P_n(x)$ is the unique solution of the elliptic equation

$$P_n(x) - \sigma^2 P_n''(x) = U_n(x), \forall x \in \mathbb{R}.$$

It follows from Lemma 2.4 that the map $x \to P_n(x)$ solving the above equation is an increasing C^3 function. By (3.16), we have that $c - \chi P_n'(x) > 0$. Since $U_n(x) \in C^1(\mathbb{R})$, we have that $\lambda_n(x), \kappa_n(x) \in C^2(\mathbb{R})$, and $V_n'(x) \in C^1(\mathbb{R})$. Then, we obtain that $V_n(x) \in C^2(\mathbb{R})$. Therefore by using (3.29) and (3.30), we deduce that the families $V_n'|_{[-k,k]}$ and $V_n''|_{[-k,k]}$ are uniformly Lipschitz continuous on $[-k,k]$ for each $k \in \mathbb{N}$. Applying Ascoli-Arzelà theorem, we have that the sets $\{V_n|_{[-k,k]}\}_{n>0}$, and $\{V_n'|_{[-k,k]}\}_{n>0}$ are relatively compact on $[-k,k]$ for each $k \in \mathbb{N}$.

Using a diagonal extraction process, there exists a sub-sequence n_p and a bounded continuous function V such that $V_{n_p} \to V$ uniformly on every compact subset of $\mathbb{R}$ as $p \to \infty$. Indeed, recall that $0 < V_n(x) \leq 1$ and $0 < V_n'(x) \leq \frac{c}{2\chi}$, $\forall x \in \mathbb{R}$. By the Ascoli-Arzelà theorem, there exists a sub-sequence $\{V_{m_p^1}\}_{p\geq 0}$ of V_n and a function $V_1 \in C^1([-1,1])$ such that

$$\lim_{p\to\infty} \|V_{m_p^1} - V_1\|_{C^1([-1,1])} = 0.$$

Now we can extract $\{V_{m_p^2}\}_{p\geq 0}$ a sub-sequence of $\{V_{m_p^1}\}_{p\geq 0}$, and function $V_2 \in C^1([-2,2])$

$$\lim_{p\to\infty} \|V_{m_p^2} - V_2\|_{C^1([-2,2])} = 0.$$

By construction, we will have

$$V_2(x) = V_1(x), \forall x \in [-1,1].$$

Replacing eventually m_1^2 by m_1^1, we can assume that $m_p^2 = \{m_1^1, m_2^2, \cdots, m_p^2, \cdots\}$. Proceeding by induction, we can find a $\{V_{m_p^k}\}_{p\geq 0}$ a sub-sequence $\{V_{m_p^{k-1}}\}_{p\geq 0}$, such that

$$m_p^k = m_p^{k-1}, \forall p = 1, \ldots, k-1,$$

and a function $V_k \in C^1([-k,k])$ such that

$$\lim_{p\to\infty} \|V_{m_p^k} - V_k\|_{C^1([-k,k])} = 0.$$

Set

$$n_p = m_p^p,$$

and

$$V(x) = V_n(x), \text{ for any } x \in [-k,k], \forall k \geq 1.$$

Then the sub-sequence $\{V_{n_p}\}_{p\geq 0}$ converges locally uniformly with respect to the C^1-norm, so we can define

$$V(x) = \lim_{p\to\infty} V_{n_p}(x), \forall x \in \mathbb{R},$$

and

$$V'(x) = \lim_{p\to\infty} V'_{n_p}(x), \forall x \in \mathbb{R}.$$

By construction, we will have

$$0 < V(x) \leq 1, \text{ and } 0 < V'(x) \leq \frac{c}{2\chi}, \forall x \in \mathbb{R}. \tag{3.31}$$

Now, we are ready to show that $\|V_{n_p} - V\|_{1,\eta} \to 0$ as $p \to +\infty$. Let $\varepsilon > 0$ be given. Let k be large enough to satisfy

$$\mathrm{e}^{-\eta k} \leq \min\left\{\frac{\varepsilon}{4}, \frac{\varepsilon\chi}{2c}\right\}. \tag{3.32}$$

For all k large enough, since $0 < V_{n_p}(x) \leq 1$, $0 < V'_{n_p}(x) \leq \frac{c}{2\chi}$, $\forall x \in \mathbb{R}$ and (3.31), we deduce that

$$\begin{aligned}\sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)| &\leq \mathrm{e}^{-\eta k}\sup_{x\in\mathbb{R}}(|V_{n_p}(x)| + |V(x)|)\\ &\leq 2\mathrm{e}^{-\eta k}\\ &\leq \frac{\varepsilon}{2},\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)| &\leq \mathrm{e}^{-\eta k}\sup_{x\in\mathbb{R}}(|V'_{n_p}(x) + |V'(x)|)\\ &\leq \frac{c}{\chi}\mathrm{e}^{-\eta k}\\ &\leq \frac{\varepsilon}{2}.\end{aligned} \tag{3.34}$$

Moreover, since V_{n_p} converges locally uniformly to V and V'_{n_p} converges locally uniformly to V', for any fixed $x \in [-k, k]$, there exists an integer $p_0 > 0$ such that

$$\sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)| \leq \frac{\varepsilon}{2}, \forall p \geq p_0, \tag{3.35}$$

and

$$\sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)| \leq \frac{\varepsilon}{2}, \forall p \geq p_0. \tag{3.36}$$

It follows from (3.33), (3.34), (3.35) and (3.36) that for $p \geq p_0$,

$$\begin{aligned}\|V_{n_p} - V\|_{1,\eta} =& \Bigg\{ \max\left\{\sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)|, \sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V_{n_p}(x) - V(x)|\right\}\\ &+ \max\left\{\sup_{x\in\mathbb{R}\setminus[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)|, \sup_{x\in[-k,k]} \mathrm{e}^{-\eta|x|}|V'_{n_p}(x) - V'(x)|\right\}\Bigg\}\\ \leq& \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since the above inequality is true for any $\varepsilon > 0$, this completes the proof of lemma. ∎

The most difficult part of the proof of existence of traveling waves is the continuity of the map $\mathcal{T} : \mathcal{A} \to \mathcal{A}$. To consider this problem, we decompose the real line into several intervals $(-\infty, -K]$, $[-K, K]$, and $[K, \infty)$. This decomposition is illustrated in Figure 3.

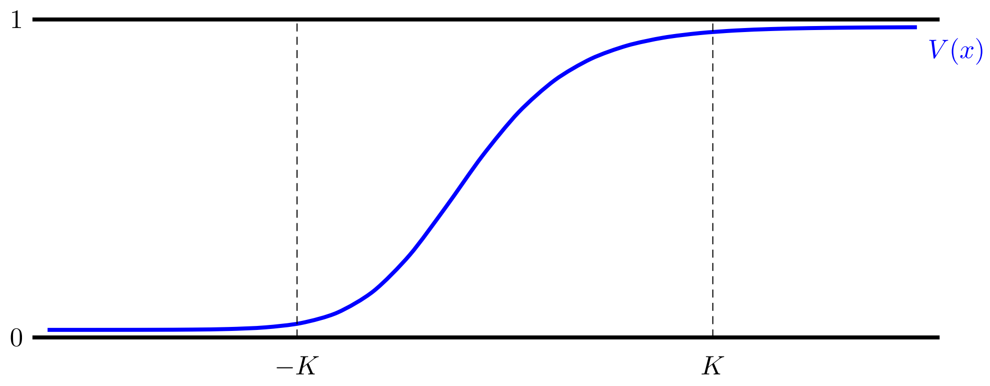

Figure 3: *In this figure, we plot $x \to V(x)$ the function obtained from $\mathcal{T}(U)$.*

Before proving the continuity of $\mathcal{T}$, we establish the continuity of its components separately.

Lemma 3.5 (Continuity of P, P', λ and κ) . *Let Assumption 3.2 be satisfied. Assume that $0 < \eta < \frac{1}{\sigma}$. Let $U_1, U_2 \in \mathcal{A}$ and define, for $i = 1, 2$,*

$$P_i(x) = \frac{1}{2\sigma}\int_{\mathbb{R}} e^{-\frac{|x-y|}{\sigma}} U_i(y)\mathrm{d}y, \qquad P_i'(x) = \frac{1}{2\sigma^2}\int_{\mathbb{R}} -\mathrm{sign}(x-y) e^{-\frac{|x-y|}{\sigma}} U_i(y)\mathrm{d}y,$$
$$\lambda_i(x) = \frac{1 + \frac{\chi}{\sigma^2} P_i(x)}{c - \chi P_i'(x)}, \qquad \kappa_i(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P_i'(x)}.$$

There exist continuous functions of $x \in \mathbb{R}$, $C_P(x)$, $C_\lambda(x)$ and $C_\kappa(x)$ such that, for all $x \in \mathbb{R}$,

$$|P_1(x) - P_2(x)| \le C_P(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.37}$$
$$|P_1'(x) - P_2'(x)| \le \frac{1}{\sigma} C_P(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.38}$$
$$|\lambda_1(x) - \lambda_2(x)| \le C_\lambda(x)\|U_1 - U_2\|_{0,\eta}, \tag{3.39}$$
$$|\kappa_1(x) - \kappa_2(x)| \le C_\kappa(x)\|U_1 - U_2\|_{0,\eta}. \tag{3.40}$$

Proof. **Step 1:** We show (3.37). We have, for $x > 0$:

$$\begin{aligned}
|P_1(x) - P_2(x)| &= \frac{1}{2\sigma}\left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\dfrac{|x-y|}{\sigma}} (U_1(y) - U_2(y))\mathrm{d}y\right| \\
&= \frac{1}{2\sigma}\left|\int_{-\infty}^{+\infty} \mathrm{e}^{-\dfrac{|x-y|}{\sigma} + \eta|y|} \mathrm{e}^{-\eta|y|}(U_1(y) - U_2(y))\mathrm{d}y\right| \\
&\le \frac{1}{2\sigma}\int_{\mathbb{R}} \mathrm{e}^{-\dfrac{|x-y|}{\sigma} + \eta|y|}\mathrm{d}y \|U_1 - U_2\|_{0,\eta} \\
&= \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\int_{-\infty}^{0} e^{-\frac{x-y}{\sigma} - \eta y}\mathrm{d}y + \int_0^x e^{-\frac{x-y}{\sigma} + \eta y}\mathrm{d}y + \int_x^{+\infty} \mathrm{e}^{\frac{x-y}{\sigma} + \eta y}\mathrm{d}y\right) \\
&= \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\frac{1}{\frac{1}{\sigma} - \eta} e^{-\frac{x}{\sigma}} + \frac{1}{\frac{1}{\sigma} + \eta}\left(e^{\eta x} - e^{-\frac{x}{\sigma}}\right) + e^{\eta x}\frac{1}{\frac{1}{\sigma} - \eta}\right),
\end{aligned} \tag{3.41}$$

and similarly for $x < 0$:

$$|P_1(x) - P_2(x)| \le \frac{1}{2\sigma}\|U_1 - U_2\|_{0,\eta}\left(\frac{1}{\frac{1}{\sigma} - \eta} e^{-\frac{|x|}{\sigma}} + \frac{1}{\frac{1}{\sigma} + \eta}\left(e^{\eta|x|} - e^{-\frac{1}{\sigma}|x|}\right) + e^{\eta|x|}\frac{1}{\frac{1}{\sigma} - \eta}\right). \tag{3.42}$$

Rearranging the terms in (3.41) and (3.42) we have

$$|P_1(x) - P_2(x)| \le \frac{1}{2\sigma}\left[\left(\frac{1}{\sigma} - \eta\right)^{-1}\left(\mathrm{e}^{-\dfrac{|x|}{\sigma}} + \mathrm{e}^{\eta|x|}\right) + \left(\frac{1}{\sigma} + \eta\right)^{-1}\left(\mathrm{e}^{\eta|x|} - \mathrm{e}^{-\dfrac{|x|}{\sigma}}\right)\right]\|U_1 - U_2\|_{0,\eta} \tag{3.43}$$

and (3.37) is proved.

Step 2: We show (3.38). We have

$$|P_1'(x)-P_2'(x)| = \frac{1}{2\sigma^2}\left|\int_{-\infty}^{+\infty} -\mathrm{sign}(x-y)\mathrm{e}^{-\dfrac{|x-y|}{\sigma}}(U_1(y)-U_2(y))\mathrm{d}y\right|$$
$$\leq \frac{1}{2\sigma^2}\int_{-\infty}^{+\infty} \mathrm{e}^{-\dfrac{|x-y|}{\sigma}+\eta|y|}\mathrm{e}^{-\eta|y|}\,|U_1(y)-U_2(y)|\,\mathrm{d}y,$$

so that the exact computations leading to (3.43) can be reproduced, and we have

$$|P_1'(x)-P_2'(x)| \leq \frac{1}{\sigma}C_P(x)\|U_1-U_2\|_{0,\eta}.$$

(3.38) is proved.

Step 3: We show (3.39). It follows from the definitions of $\lambda_1(x)$ and $\lambda_2(x)$ that, for all $x\in\mathbb{R}$,

$$\begin{aligned}
&|\lambda_2(x)-\lambda_1(x)|\\
&=\left|\frac{1+\frac{\chi}{\sigma^2}P_2(x)}{c-\chi P_2'(x)}-\frac{1+\frac{\chi}{\sigma^2}P_1(x)}{c-\chi P_1'(x)}\right|\\
&=\frac{\chi}{|c-\chi P_2'(x)||c-\chi P_1'(x)|}\left|\frac{c}{\sigma^2}(P_2(x)-P_1(x))+P_2'(x)-P_1'(x)+\frac{\chi}{\sigma^2}(P_1(x)P_2'(x)-P_2(x)P_1'(x))\right|.
\end{aligned} \tag{3.44}$$

Since by Definition 3.1 we have $U_i'(x)\leq \frac{c}{2\chi}$ $(i=1,2)$, then $P_i'(x)=\int_{\mathbb{R}}\frac{1}{2\sigma}e^{-\frac{|x-y|}{\sigma}}U'(y)\mathrm{d}y\leq\frac{c}{2\chi}$ $(i=1,2)$, therefore

$$c-\chi P_i'(x)\geq c-\chi\frac{c}{2\chi}=\frac{c}{2},\qquad i=1,2. \tag{3.45}$$

It follows from (3.44) and (3.45) that

$$\begin{aligned}
|\lambda_2(x)-\lambda_1(x)| &\leq \frac{4\chi}{c\sigma^2}|P_2(x)-P_1(x)|+\frac{4\chi}{c^2}|P_2'(x)-P_1'(x)|+\frac{4\chi^2}{c^2\sigma^2}|P_1(x)P_2'(x)-P_2(x)P_1'(x)|\\
&\leq \frac{4\chi}{c\sigma^2}|P_2(x)-P_1(x)|+\frac{4\chi}{c^2}|P_2'(x)-P_1'(x)|\\
&\quad+\frac{4\chi^2}{c^2\sigma^2}\left(P_1(x)|P_2'(x)-P_1'(x)|+|P_1'(x)||P_1(x)-P_2(x)|\right).
\end{aligned}$$

Using the fact that $0\leq P_1(x)\leq 1$ and $0\leq P_1'(x)\leq\frac{c}{2\chi}$ for $x\in\mathbb{R}$, then (3.39) is a consequence of (3.37) and (3.38).

Step 4: We show (3.40). We have:

$$\begin{aligned}
\frac{1}{1+\frac{\chi}{\sigma^2}}|\kappa_1(x)-\kappa_2(x)| &= \left|\frac{1}{c-\chi P_1'(x)}-\frac{1}{c-\chi P_2'(x)}\right| = \left|\frac{c-\chi P_1'(x)-c+\chi P_2'(x)}{\left(c-\chi P_2'(x)\right)\left(c-P_1'(x)\right)}\right|\\
&=\chi\left|\frac{P_2'(x)-P_1'(x)}{\left(c-\chi P_2'(x)\right)\left(c-P_1'(x)\right)}\right|\leq\frac{4\chi}{c^2}|P_2'(x)-P_1'(x)|.
\end{aligned}$$

Thus (3.40) is a consequence of (3.38). Lemma 3.5 is proved. ■

Lemma 3.6 (Continuity of $\mathcal{T}$) *Let Assumption 3.2 be satisfied. Assume that $0<\eta<\frac{1}{\sigma}$. Then the map $\mathcal{T}:\mathcal{A}\to\mathcal{A}$ is continuous on $\mathcal{A}$ endowed with distance $d(U_1,U_2)=\|U_1-U_2\|_{1,\eta}$.*

Proof. Let $U_0\in\mathcal{A}$ be fixed, and $U\in\mathcal{A}$, and define

$$V_0=\mathcal{T}(U_0)\text{ and }V=\mathcal{T}(U).$$

Part A: We prove that for each admissible profile $U_0\in\mathcal{A}$ and $\varepsilon>0$, there is a $\delta_1>0$ such that

$$\|V-V_0\|_{0,\eta}\leq\frac{\varepsilon}{2}, \tag{3.46}$$

whenever

$$\|U - U_0\|_{0,\eta} \le \delta_1.$$

Let $K > 0$ be such that

$$e^{-\eta K} \le \frac{\varepsilon}{12}.$$

Then since $V \in \mathcal{A}$ by Lemma 3.3, we have $0 \le V(x) \le 1$ and $0 \le V_0(x) \le 1$ for all $x \in \mathbb{R}$, therefore

$$\begin{aligned}
\|V - V_0\|_{0,\eta} &= \sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x) - V_0(x)| \\
&\le \sup_{x\le -K} e^{-\eta|x|}|V(x) - V_0(x)| + \sup_{|x|\le K} e^{-\eta|x|}|V(x) - V_0(x)| + \sup_{x\ge K} e^{-\eta|x|}|V(x) - V_0(x)| \\
&\le e^{-\eta K}\left(\sup_{x\le K}|V(x) - V_0(x)| + \sup_{x\ge K}|V(x) - V_0(x)|\right) + \sup_{|x|\le K} e^{-\eta|x|}|V(x) - V_0(x)| \\
&\le 4e^{-\eta K} + \sup_{|x|\le K} e^{-\eta|x|}|V(x) - V_0(x)| \le \frac{2\varepsilon}{6} + \sup_{|x|\le K} e^{-\eta|x|}|V(x) - V_0(x)|.
\end{aligned}$$

Thus there remains only to establish that

$$\sup_{|x|\le K} e^{-\eta|x|}|V(x) - V_0(x)| \le \frac{\varepsilon}{6}, \tag{3.47}$$

if $\|U - U_0\|_{1,\eta} \le \delta_1$, for $\delta_1 > 0$ sufficiently small. Recall that

$$V(x) = \frac{u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)}{1 + u_0\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s},$$

wherein

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2}P(x)}{c - \chi P'(x)},$$

and

$$\kappa(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)},$$

and $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}.$$

By using the definitions of $V_0(x)$ and $V(x)$ for $x \in \mathbb{R}$, we find that

$$\begin{aligned}
|V(x) - V_0(x)| &= \left|\frac{u_0 \exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)}{1 + u_0\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s} - \frac{u_0 \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)}{1 + u_0\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s}\right| \\
&= \frac{u_0}{\left|1 + u_0\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right|\left|1 + u_0\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s\right|} \\
&\quad \times \Bigg|\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right) - \exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \\
&\quad + u_0\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s \\
&\quad - u_0\exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s\Bigg|.
\end{aligned} \tag{3.48}$$

Since

$$\frac{1}{1+u_0\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s} = \frac{V_0(x)}{u_0\exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)},$$

and similarly

$$\frac{1}{1+u_0\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s} = \frac{V(x)}{u_0\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)},$$

we have that

$$\begin{aligned}
&|V(x)-V_0(x)| \\
&= \left|\frac{V(x)V_0(x)}{u_0\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)\exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)}\right| \bigg|\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)-\exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right) \\
&\quad + u_0\exp\left(\int_0^x \lambda(s)\mathrm{d}s\right)\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s \\
&\quad - u_0\exp\left(\int_0^x \lambda_0(s)\mathrm{d}s\right)\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s\bigg| \\
&= \frac{|V(x)V_0(x)|}{u_0}\bigg|\exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right)-\exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right) \\
&\quad + u_0\exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right)\int_0^x \kappa_0(s)\exp\left(\int_0^s \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s \\
&\quad - u_0\exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right)\int_0^x \kappa(s)\exp\left(\int_0^s \lambda(l)\mathrm{d}l\right)\mathrm{d}s\bigg|,
\end{aligned}$$

hence

$$|V(x)-V_0(x)| \le \frac{1}{u_0}H(x)+I(x), \tag{3.49}$$

where

$$H(x) := \left|\exp\left(-\int_0^x \lambda_0(s)\mathrm{d}s\right)-\exp\left(-\int_0^x \lambda(s)\mathrm{d}s\right)\right|, \tag{3.50}$$

and

$$I(x) := \left|\int_0^x \kappa_0(s)\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s - \int_0^x \kappa(s)\exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s\right|. \tag{3.51}$$

We divide the rest of the proof of Part A into two steps, to estimate $H(x)$ and $I(x)$.

Step 1: We show that

$$H(x) \le C_H(x)\|U-U_0\|_{0,\eta},\ \forall x\in\mathbb{R}, \tag{3.52}$$

for some continuous function $C_H(x)$ independent of ε, U, U_0.

By Taylor's theorem, we have that

$$|\mathrm{e}^A-\mathrm{e}^B| \le |A-B|\mathrm{e}^{\max\{A,B\}}, \quad \forall A,B\in\mathbb{R}. \tag{3.53}$$

Now we use (3.53) to estimate $H(x)$ defined in (3.50).

$$\begin{aligned}
H(x) &\le \left|\int_0^x \lambda(s)-\lambda_0(s)\mathrm{d}s\right|\exp\left(\max\left\{-\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s\right\}\right) \\
&\le \exp\left(\max\left\{-\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s\right\}\right)\int_0^x |\lambda(s)-\lambda_0(s)|\,\mathrm{d}s.
\end{aligned} \tag{3.54}$$

Recall from (3.39) in Lemma 3.5 that there is a continuous function $C_\lambda(x)$ such that

$$|\lambda(x)-\lambda_0(x)| \le C_\lambda(x)\|U-U_0\|_{0,\eta},\ \forall x\in\mathbb{R}.$$

Thus we can rewrite (3.54) as

$$H(x) \leq \exp\left(-\int_0^x \lambda(s)\mathrm{d}s, -\int_0^x \lambda_0(s)\mathrm{d}s\right) \int_0^x C_\lambda(s)\mathrm{d}s \|U - U_0\|_{0,\eta}. \tag{3.55}$$

Next recall the definition of $\lambda(x)$:

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)}, \ \forall x \in \mathbb{R}.$$

Since by Definition 3.1 we have $U'(x) \leq \frac{c}{2\chi}$, then $P'(x) = \int_{\mathbb{R}} \frac{1}{2\sigma} e^{-\frac{|x-y|}{\sigma}} U'(y)\mathrm{d}y \leq \frac{c}{2\chi}$, therefore

$$c - \chi P'(x) \geq c - \chi \frac{c}{2\chi} = \frac{c}{2}, \ \forall x \in \mathbb{R},$$

and therefore

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)} \leq \frac{2}{c}\left(1 + \frac{\chi}{\sigma^2}\right), \ \forall x \in \mathbb{R}.$$

Clearly, we have the same upper bound for $\lambda_0(x)$ and $\lambda(x)$, and (3.55) becomes

$$H(x) \leq \exp\left(\frac{2}{c}\left(1 + \frac{\chi}{\sigma^2}\right)|x|\right) \int_{[0,x]} C_\lambda(s)\mathrm{d}s \|U - U_0\|_{0,\eta} = C_H(x)\|U - U_0\|_{0,\eta}, \ \forall x \in \mathbb{R},$$

where $C_H(x)$ is a continuous function. Therefore, (3.52) is proved.

Step 2: We show that

$$I(x) \leq C_I(x)\|U - U_0\|_{0,\eta}, \ \forall x \in \mathbb{R}, \tag{3.56}$$

for some continuous function $C_I(x)$ independent from ε, U, U_0.

Indeed we have

$$\begin{aligned} I(x) \leq & \left| \int_0^x \kappa_0(s) \exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right)\mathrm{d}s - \int_0^x \kappa_0(s) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s \right| \\ & + \left| \int_0^x \kappa_0(s) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s - \int_0^x \kappa(s) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s \right| \\ = & \left| \int_0^x \kappa_0(s) \left(\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right) - \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\right)\mathrm{d}s \right| \\ & + \left| \int_0^x (\kappa_0(s) - \kappa(s)) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s \right| \\ =: & \ I_1(x) + I_2(x), \end{aligned} \tag{3.57}$$

where

$$I_1(x) := \left| \int_0^x \kappa_0(s) \left(\exp\left(-\int_s^x \lambda_0(l)\mathrm{d}l\right) - \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\right)\mathrm{d}s \right|, \tag{3.58}$$

and

$$I_2(x) := \left| \int_0^x (\kappa_0(s) - \kappa(s)) \exp\left(-\int_s^x \lambda(l)\mathrm{d}l\right)\mathrm{d}s \right|. \tag{3.59}$$

Using (3.53), we rewrite (3.58) as

$$I_1(x) \leq \int_0^x |\kappa_0(s)| \exp\left(-\max\left\{-\int_s^x \lambda(l)\mathrm{d}l, -\int_s^x \lambda_0(l)\mathrm{d}l\right\}\right) \int_s^x |\lambda(l) - \lambda_0(l)|\mathrm{d}l\mathrm{d}s. \tag{3.60}$$

Since by Definition 3.1 we have $U'(x) \leq \frac{c}{2\chi}$, then $P'(x) = \int_{\mathbb{R}} \frac{1}{2\sigma} e^{-\frac{|x-y|}{\sigma}} U'(y)\mathrm{d}y \leq \frac{c}{2\chi}$, therefore

$$c - \chi P'(x) \geq c - \chi \frac{c}{2\chi} = \frac{c}{2},$$

and finally

$$|\lambda(x)| \leq \frac{2}{c}\left(1 + \frac{\chi}{\sigma^2}\right) \text{ and } |\kappa(x)| \leq \frac{2}{c}\left(1 + \frac{\chi}{\sigma^2}\right). \tag{3.61}$$

By using (3.60), (3.61) and (3.39) in Lemma 3.5 we rewrite as

$$I_1(x) \leq \frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right)\int_{[0,x]} \exp\left(\frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right)|x-s|\right)\int_{[s,x]} C_\lambda(l)\mathrm{d}l\mathrm{d}s\|U-U_0\|_{0,\eta}.$$

Thus there exists a continuous function $C_{I_1}(x)$ such that

$$I_1(x) \leq C_{I_1}(x)\|U-U_0\|_{0,\eta}. \tag{3.62}$$

Next we estimate $I_2(x)$ in (3.59). By using (3.40) and (3.61), we have

$$\begin{aligned} I_2(x) &\leq \int_{[0,x]} |\kappa_0(s)-\kappa(s)| \exp\left(\frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right)|x-s|\right)\mathrm{d}s \\ &\leq \int_{[0,x]} C_\kappa(s) \exp\left(\frac{2}{c}\left(1+\frac{\chi}{\sigma^2}\right)|x-s|\right)\mathrm{d}s\|U-U_0\|_{0,\eta}, \end{aligned}$$

thus there exists a continuous function $C_{I_2}(x)$ such that

$$I_2(x) \leq C_{I_2}(x)\|U-U_0\|_{0,\eta}. \tag{3.63}$$

Combining (3.57), (3.62) and (3.63), there exists a continuous function $C_I(x) := C_{I_1}(x) + C_{I_2}(x)$ such that (3.56) holds. Step 2 is completed.

Conclusion of Part A: By choosing δ_1 such that

$$\delta_1 := \frac{\varepsilon}{6}\left(\frac{1}{\sup\limits_{x\in[-K,K]} \frac{1}{u_0}C_H(x) + C_I(x)}\right), \tag{3.64}$$

we conclude from (3.42), (3.52) and (3.56) that indeed

$$\begin{aligned} \sup_{x\in[-K,K]} e^{-\eta|x|}|V(x)-V_0(x)| &\leq \sup_{x\in[-K,K]} |V(x)-V_0(x)| \leq \sup_{x\in[-K,K]} \frac{1}{u_0}H(x) + I(x) \\ &\leq \sup_{x\in[-K,K]}\left(\frac{1}{u_0}C_H(x)+C_I(x)\right)\|U-U_0\|_{0,\eta} \\ &\leq \frac{\varepsilon}{6}, \end{aligned}$$

whenever $\|U-U_0\|_{0,\eta} \leq \delta_1$. Thus (3.47) holds, and this concludes Part A.

Part B: We prove that for each admissible profile $U_0 \in \mathcal{A}$ and $\varepsilon > 0$, there is $\delta > 0$ such that whenever

$$\|U-U_0\|_{0,\eta} \leq \delta,$$

we have

$$\|V-V_0\|_{1,\eta} \leq \varepsilon.$$

By Lemma 3.3 we know that $V = \mathcal{T}(U) \in \mathcal{A}$ and $V_0 = \mathcal{T}(U_0) \in \mathcal{A}$. Therefore

$$|V'(x)| \leq \frac{c}{2\chi} \text{ and } |V_0'(x)| \leq \frac{c}{2\chi},\ \forall x \in \mathbb{R}.$$

Let $K > 0$ be such that

$$\frac{c}{2\chi}e^{-\eta K} \leq \frac{\varepsilon}{12}.$$

We have

$$\begin{aligned} \sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0'(x)| \leq &\sup_{x\leq -K} e^{-\eta|x|}|V'(x)-V_0'(x)| + \sup_{|x|\leq K} e^{-\eta|x|}|V'(x)-V_0'(x)| \\ &+ \sup_{x\geq K} e^{-\eta|x|}|V'(x)-V_0'(x)| \end{aligned}$$

$$\leq e^{-\eta K}\left(\sup_{x\leq -K}|V'(x)-V_0'(x)|+\sup_{x\geq K}|V'(x)-V_0'(x)|\right)$$
$$+\sup_{|x|\leq K} e^{-\eta|x|}|V'(x)-V_0'(x)|$$
$$\leq \frac{2\varepsilon}{6}+\sup_{|x|\leq K} e^{-\eta|x|}|V'(x)-V_0'(x)|. \tag{3.65}$$

Thus there remains only to establish that

$$\sup_{|x|\leq K} e^{-\eta|x|}|V'(x)-V_0'(x)|\leq \frac{\varepsilon}{6}. \tag{3.66}$$

We note that V and V_0 satisfy (3.2), therefore

$$V'(x)=\lambda(x)V(x)-\kappa(x)V^2(x),\ \text{ and } V_0'(x)=\lambda_0(x)-\kappa_0(x),\ \text{ for all } x\in\mathbb{R}. \tag{3.67}$$

Then we have that

$$\begin{aligned}|V'(x)-V_0'(x)| &= |\lambda(x)V(x)-\kappa(x)V^2(x)-\lambda_0(x)V_0(x)+\kappa_0(x)V_0^2(x)|\\ &= |(\lambda(x)-\lambda_0(x))V(x)+\lambda_0(x)(V(x)-V_0(x))\\ &\quad +(\kappa_0(x)-\kappa(x))V_0^2(x)+\kappa(x)(V_0^2(x)-V^2(x))|\\ &\leq |\lambda(x)-\lambda_0(x)||V(x)|+\lambda_0(x)|V(x)-V_0(x)|\\ &\quad +|\kappa_0(x)-\kappa(x)|V_0^2(x)+\kappa(x)|V_0^2(x)-V^2(x)|\\ &= |\lambda(x)-\lambda_0(x)||V(x)|+\lambda_0(x)|V(x)-V_0(x)|\\ &\quad +|\kappa_0(x)-\kappa(x)|V_0^2(x)+\kappa(x)|V_0(x)-V(x)||V_0(x)+V(x)|.\end{aligned}$$

By using the fact that $0\leq V(x)\leq 1$ and $0\leq V_0(x)\leq 1$ for $x\in\mathbb{R}$, we have that

$$|V'(x)-V_0'(x)|\leq|\lambda(x)-\lambda_0(x)|+\lambda_0(x)|V(x)-V_0(x)|+|\kappa_0(x)-\kappa(x)|+2\kappa(x)|V_0(x)-V(x)|. \tag{3.68}$$

It follows from (3.49), (3.61), (3.56), (3.52) and (3.39) and (3.40) that

$$|V'(x)-V_0'(x)|\leq\left(C_\lambda(x)+\frac{2}{cu_0}\left(1+\frac{\chi}{\sigma^2}\right)(C_H(x)+u_0C_I(x))+C_\kappa(x)+\frac{4}{c}\left(1+\frac{\chi}{\sigma^2}\right)\right)\|U-U_0\|_{0,\eta}. \tag{3.69}$$

Let

$$\delta_2:=\frac{\varepsilon}{3}\left[\sup_{x\in[-K,K]}\left(C_\lambda(x)+\frac{2}{cu_0}\left(1+\frac{\chi}{\sigma^2}\right)(C_H(x)+u_0C_I(x))+C_\kappa(x)+\frac{4}{c}\left(1+\frac{\chi}{\sigma^2}\right)\right)e^{-\eta|x|}\right]^{-1}, \tag{3.70}$$

then whenever $\|U-U_0\|_{0,\eta}\leq\delta_2$ we have

$$\sup_{x\in[-K,K]} e^{-\eta|x|}|V'(x)-V_0'(x)|\leq\frac{\varepsilon}{3},$$

therefore, recalling (3.65),

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0'(x)|\leq\frac{\varepsilon}{2}. \tag{3.71}$$

Conclusion of Part B: Let $\delta:=\min(\delta_1,\delta_2)$ where δ_1 is defined in (3.64) and δ_2 is defined in (3.70). Then if $\|U-U_0\|_{0,\eta}\leq\delta$ we know from Part A (3.46) that

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x)-V_0(x)|\leq\frac{\varepsilon}{2},$$

and from (3.71) that

$$\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0'(x)|\leq\frac{\varepsilon}{2}.$$

So finally

$$\|V-V_0\|_{1,\eta}=\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V(x)-V_0(x)|+\sup_{x\in\mathbb{R}} e^{-\eta|x|}|V'(x)-V_0'(x)|\leq\varepsilon.$$

Part B is proved. Since we always have $\|U-U_0\|_{0,\eta}\leq\|U-U_0\|_{1,\eta}$, the continuity holds for the norm $\|\cdot\|_{1,\eta}$. Lemma 3.5 is proved. ∎

4 Proof of Theorem 1.4

From the definition of admissible functions $\mathcal{A}$, it is a nonempty, closed, convex, bounded subset of the Banach space $BUC^1_\eta(\mathbb{R})$. By Lemmas 3.4 and 3.6, we obtain that $\mathcal{T}$ is a continuous compact operator on $\mathcal{A}$. Therefore, by the Schauder fixed-point theorem, there exists U in $\mathcal{A}$ such that

$$\mathcal{T}(U) = U.$$

Applying Lemma 3.3, we have that $U \in C^1(\mathbb{R})$ and $0 < U'(x) < \frac{c}{2\chi}$ for any $x \in \mathbb{R}$. Therefore, we have that

$$U(x) = \frac{u_0 \mathrm{e}^{\int_0^x \lambda(s)\mathrm{d}s}}{1 + u_0 \int_0^x \kappa(s) \mathrm{e}^{\int_0^s \lambda(l)\mathrm{d}l} \mathrm{d}s},$$

wherein

$$\lambda(x) = \frac{1 + \frac{\chi}{\sigma^2} P(x)}{c - \chi P'(x)},$$

and

$$\kappa(x) = \frac{1 + \frac{\chi}{\sigma^2}}{c - \chi P'(x)},$$

and $P(x)$ is the unique solution of the elliptic equation

$$P(x) - \sigma^2 P''(x) = U(x), \forall x \in \mathbb{R}. \tag{4.1}$$

Namely, we have that

$$U'(x) = \frac{1}{c - \chi P'(x)} U(x) \left(\left(1 + \frac{\chi}{\sigma^2} P(x)\right) - \left(1 + \frac{\chi}{\sigma^2}\right) U(x) \right), \forall x \in \mathbb{R}. \tag{4.2}$$

Therefore, we have that

$$cU'(x) - \chi P'(x) U'(x) - \frac{\chi}{\sigma^2} U(x)(P(x) - U(x)) = U(x)(1 - U(x)), \forall x \in \mathbb{R}.$$

By using (4.1), we have that

$$cU'(x) - \chi (P'(x) U(x))' = U(x)(1 - U(x)), \forall x \in \mathbb{R}. \tag{4.3}$$

We prove that

$$U(\infty) := \lim_{x \to +\infty} U(x) = 1 \text{ and } U(-\infty) := \lim_{x \to -\infty} U(x) = 0.$$

Indeed, since $U'(x) \geq 0$ and $0 < U(x) \leq 1$ for any $x \in \mathbb{R}$, then $U(\infty)$ exists. By using P-equation (1.8), the function $x \to P(x)$ is increasing and bounded, and by Lebesgue's dominated convergence theorem, we have

$$P(\pm\infty) = U(\pm\infty).$$

Therefore,

$$\lim_{x \to \pm\infty} U'(x) = 0, \text{ and } \lim_{x \to \pm\infty} P'(x) = 0. \tag{4.4}$$

It follows from (4.3) and (4.4), we have that

$$\lim_{x \to \pm\infty} U(x)(1 - U(x)) = 0,$$

and since $x \to U(x)$ increasing and $U(0) = u_0 > 0$, this implies that

$$U(-\infty) = 0, \text{ and } U(+\infty) = 1.$$

This completes the proof of the Theorem 1.4.

5 Numerical simulations

We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows

$$u_0(x) = \frac{2\,\mathrm{e}^{-\beta(x+K)}}{1+\mathrm{e}^{-\beta(x+K)}}. \tag{5.1}$$

In the following numerical simulations, we solve the PDE numerically using the upwind scheme, and we refer to Leveque [15] and Toro [18] for more results on this subject. The numerical method used for the simulations is presented in Section A of the Appendix.

In this section, we set the parameters of the system (A.1) all equal to one. That is

$$\sigma = \chi = \lambda = \kappa = 1.$$

In Figure 4 we plot $x \to u_0(x)$ with the parameter values $\beta = 1$, and $K = 20$, and the corresponding traveling wave profile which coincides with $x \to u(20, x)$ the solution of system (A.1) at $t = 20$ days.

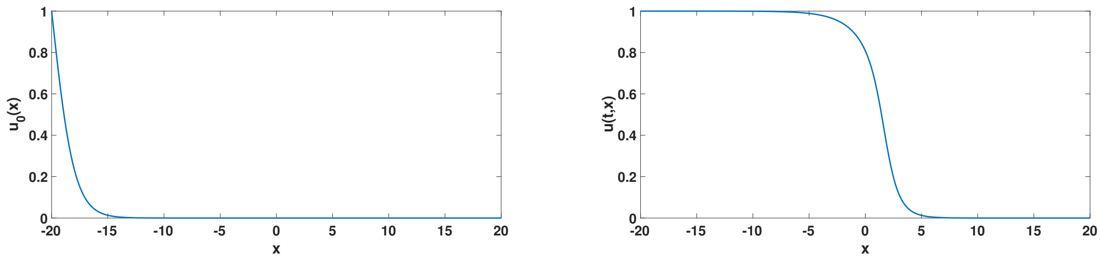

Figure 4: *On the left-hand side, we plot $x \to u_0(x)$ the initial distribution of system* (A.1)*, obtained by using formula* (5.1) *with $\beta = 1$ and $K = 20$. On the right-hand side, we plot the traveling wave profile which coincides with $x \to u(t, x)$ the solution of system* (A.1) *at $t = 20$ days.*

In Figure 5, we run a simulation from $t = 0$ until $t = 20$ of the model (A.1). We observe that the traveling wave appears almost immediately after the starting time $t = 0$.

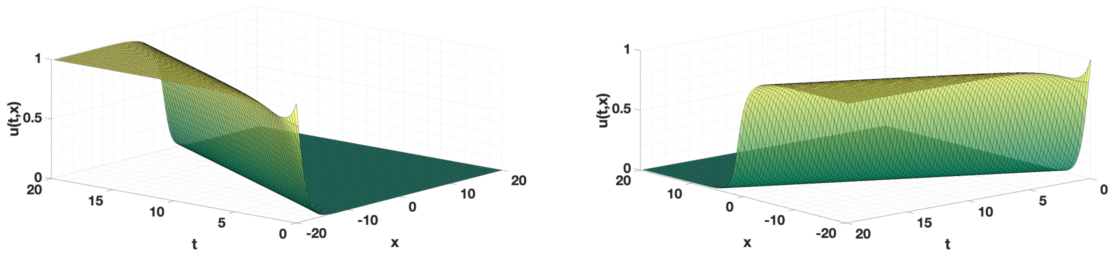

Figure 5: *In this figure, we plot the solution of the model* (A.1) *starting from the initial distribution* (5.1) *(with $\beta = 1$ and $K = 20$).*

Next we use the following initial value

$$u_0(x) = \max\left(1 - \beta\left(x + K\right), 0\right). \tag{5.2}$$

In Figure 6, we plot $x \to u_0(x)$ the initial distribution of system (A.1) (on the left-hand side), and the corresponding traveling wave profile which coincides with $x \to u(20, x)$ the solution of system (A.1) at $t = 20$ days.

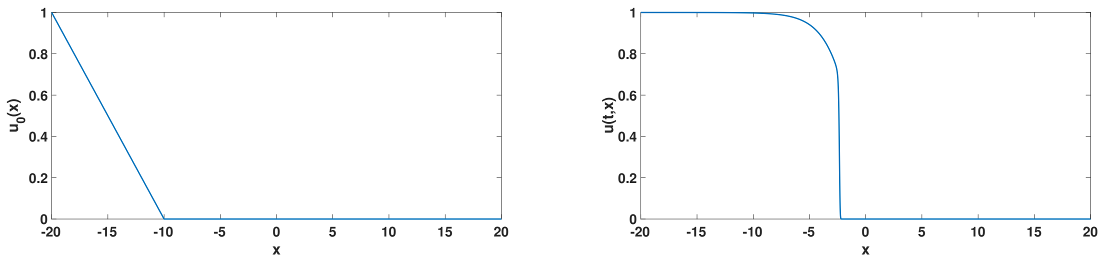

Figure 6: *On the left-hand side, we plot $x \to u_0(x)$ the initial distribution of system* (A.1)*, obtained by using formula* (5.2) *with $\beta = 0.1$ and $K = 20$. On the right-hand side, we plot the traveling wave profile which coincide with $x \to u(t,x)$ the solution of system* (A.1) *at $t = 20$ days.*

In Figure 7, we run a simulation from $t = 0$ until $t = 20$ of the model (A.1). We observe that the traveling wave appears almost immediately after the starting time $t = 0$.

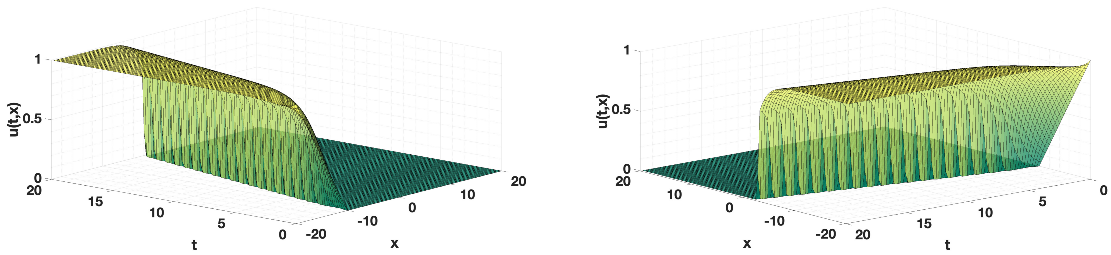

Figure 7: *In this figure, we plot the solution of the model* (A.1) *starting from the initial distribution* (5.2) *(with $\beta = 0.1$ and $K = 20$).*

On the one hand, our numerical simulations show that continuous traveling waves can be observed from an initial distribution decaying exponentially (slowly enough). On the other hand, sharp traveling waves can also be observed when starting the PDE with initial distributions equal to zero on the half-plane. So in practice, both types of traveling waves can be observed numerically.

Now concerning the traveling speed, we observe numerically that sharp traveling waves are slower than continuous traveling waves. The question of the minimal speed is left for future works.

6 Application to wound healing

The wound healing assay is used in a range of disciplines to study the coordinated movement of a cell population. We refer to the paper of Jonkman et al. [13] for a review on this topic. In this paper, we consider the cell-cell repulsion described by nonlinear diffusion, but cell-cell attraction also occurs and this problem was recently considered by Webb [19] (see also the references therein for more results).

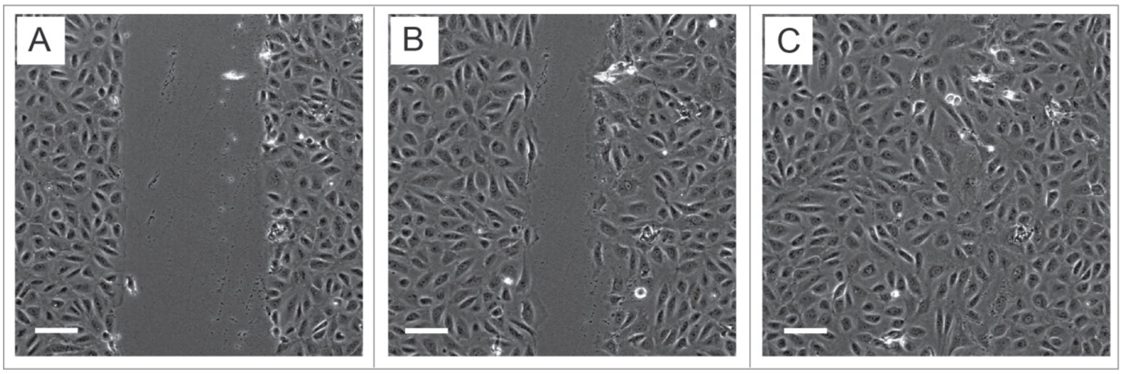

Figure 8: *Images from a scratch assay experiment at different time points. Human umbilical vein endothelial cells (HUVEC) were plated on gelatin-coated plastic dishes, wounded with a p20 pipette tip, and then imaged overnight using a microscope equipped with point visiting and live-cell apparatus. Scale bar* $= 120\ \mu m$*. This figure is taken from Jonkman et al. [13].*

In this section, we set the parameters of the system (A.1) as follows

$$\chi = \lambda = 4 \text{ and } \sigma = \kappa = 1.$$

Initial distribution for imperfect wound: We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows

$$u_0(x) = \frac{1}{2}\left(\frac{2\,\mathrm{e}^{-\beta(x+K)}}{1+\mathrm{e}^{-\beta(x+K)}}\right) + \frac{1}{2}\left(\frac{2\,\mathrm{e}^{-\beta(K-x)}}{1+\mathrm{e}^{-\beta(K-x)}}\right). \tag{6.1}$$

In Figure 9 we plot $x \to u_0(x)$ with the parameter values $\beta = 0.5$, and $K = 20$, and $x \to u(7, x)$ the solution of system (A.1) at $t = 7$ days.

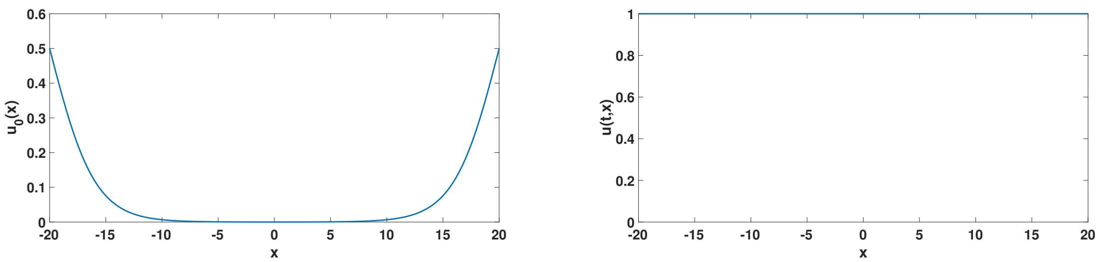

Figure 9: *On the left-hand side, we plot* $x \to u_0(x)$ *the initial distribution of system* (A.1)*, obtained by using formula* (6.1) *with* $\beta = 0.5$ *and* $K = 20$*. On the right-hand side, we plot* $x \to u(t, x)$ *the solution of system* (A.1) *at* $t = 7$ *days.*

In Figure 10, we run a simulation from $t = 0$ until $t = 7$ of the model (A.1). We observe that two traveling wave moving in opposite directions appears almost immediately after the starting time $t = 0$. They merge together to give a flat distribution approximately on day 2.

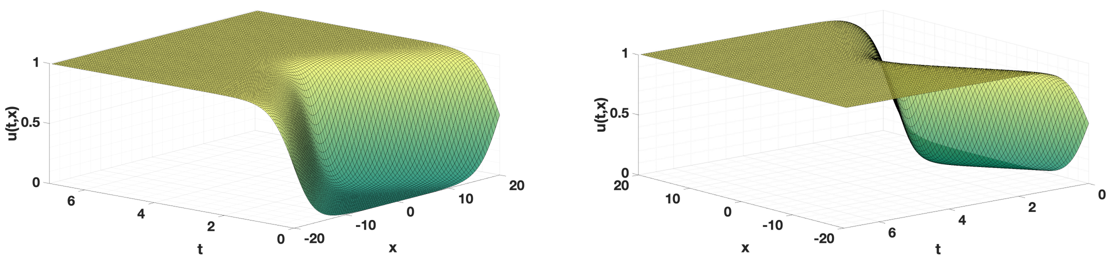

Figure 10: *In this figure, we plot the solution of the model* (A.1) *starting from the initial distribution* (6.1) *(with* $\beta = 0.5$ *and* $K = 20$*).*

Initial distribution for perfect wound: We choose a bounded interval $[-K, K]$ and an initial distribution $u_0 \in C([-K, K])$ as follows

$$u_0(x) = \frac{1}{2} \left(\max\left(1 - \beta\left(x + K\right), 0\right)\right) + \frac{1}{2} \left(\max\left(1 - \beta\left(K - x\right), 0\right)\right). \tag{6.2}$$

In Figure 11 we plot $x \to u_0(x)$ with the parameter values $\beta = 0.07$, and $K = 20$, and $x \to u(7, x)$ the solution of system (A.1) at $t = 7$ days.

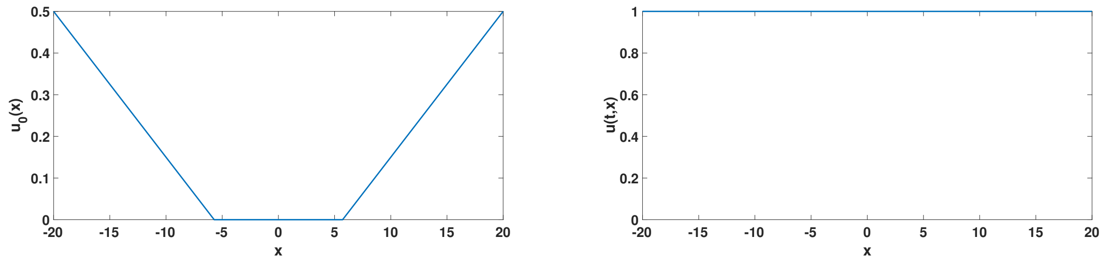

Figure 11: *On the left-hand side, we plot $x \to u_0(x)$ the initial distribution of system (A.1), obtained by using formula (6.2) with $\beta = 0.07$ and $K = 20$. On the right-hand side, we plot $x \to u(t, x)$ the solution of system (A.1) at $t = 7$ days.*

In Figure 12, we run a simulation from $t = 0$ until $t = 7$ of the model (A.1) for the parameter values $\sigma = 1$, and $\chi = 1$. We observe that two traveling wave moving in opposite directions appears almost immediately after the starting time $t = 0$. They merge together to give a flat distribution approximately on day 5.

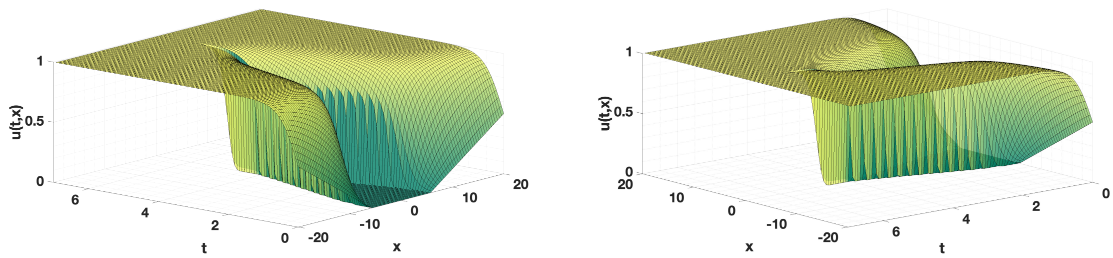

Figure 12: *In this figure, we plot the solution of the model (A.1) starting from the initial distribution (6.2) (with $\beta = 0.07$ and $K = 20$).*

It is observed that the speed of healing depends strongly on the imperfection of the wound. If we compare the two simulations, we see that the wound seems much larger in Figure 9 than in Figure 11. But the time required for healing is about 2 days in Figure 10 whereas it is about 5 days in Figure 12. Therefore, the imperfection of the wound has a strong influence on the healing time.

Appendix

A Upwind method applied to the numerical scheme

In Section 5, we use the following system of PDE to run the numerical simulations

$$\begin{cases} \partial_t u(t,x) = \chi \partial_x \left(u(t,x) \partial_x p(t,x)\right) + \lambda\, u(t,x)(1 - u(t,x)/\kappa), \quad t \in (0,T], x \in [-K, K], \\ p(t,x) - \sigma^2 \partial_{xx} p(t,x) = u(t,x), \quad t \in (0,T], x \in [-K, K], \\ \partial_x p(t, -K) = \partial_x p(t, +K) = 0, \quad t \in (0,T], \end{cases} \tag{A.1}$$

with

$$u(t,x) = u_0(x) \in L^\infty_+ \left([-K, K], \mathbb{R}\right).$$

Now we use the finite volume method to consider equation (A.1). Our numerical scheme reads as follows

$$u_i^{n+1} = u_i^n - \chi \frac{\Delta t}{\Delta x} \left(\phi(u_{i+1}^n, u_i^n) - \phi(u_i^n, u_{i-1}^n) \right) + \Delta t u_i^n (1 - u_i^n), \quad i = 1, 2, \dots, M, \tag{A.2}$$

where the flux $\phi(u_{i+1}^n, u_i^n)$ for $i = 0, \dots, M$ defined as

$$\phi(u_{i+1}^n, u_i^n) = \left(v_{i+\frac{1}{2}}^n \right)^+ u_i^n - \left(v_{i+\frac{1}{2}}^n \right)^- u_{i+1}^n = \begin{cases} v_{i+\frac{1}{2}}^n u_i^n, & v_{i+\frac{1}{2}}^n \geq 0, \\ v_{i+\frac{1}{2}}^n u_{i+1}^n, & v_{i+\frac{1}{2}}^n < 0, \end{cases} \tag{A.3}$$

where

$$x^+ = \max(0, x), \text{ and } x^- = \max(0, -x),$$

and

$$v_{i+\frac{1}{2}}^n = -\frac{p_{i+1}^n - p_i^n}{\Delta x}, i = 0, 1, \dots, M, \tag{A.4}$$

where

$$v_{0+\frac{1}{2}}^n = v_{M+\frac{1}{2}}^n = 0.$$

Moreover the vector P^n is defined by

$$P^n := \left(I - \frac{\sigma^2}{\Delta x^2} A \right)^{-1} U^n, \tag{A.5}$$

where

$$A = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}_{M \times M}. \tag{A.6}$$

Indeed, we have

$$p_i^n - \frac{\sigma^2}{\Delta x^2} \left(p_{i+1}^n - 2p_i^n + p_{i-1}^n \right) = u_i^n, \quad i = 1, 2, \dots, M, \tag{A.7}$$

and since we use the Neumann boundary condition, we must impose

$$p_0^n = p_1^n \text{ and } p_M^n = p_{M+1}^n.$$

Since the Neumann boundary condition corresponds to a no flux boundary condition, we have

$$\phi(u_1^n, u_0^n) = 0, \text{ and } \phi(u_{M+1}^n, u_M^n) = 0, \tag{A.8}$$

which corresponds to $p_0^n = p_1^n$ and $p_{M+1}^n = p_M^n$. Therefore, the numerical scheme at the boundary becomes

$$\begin{aligned} u_1^{n+1} &= u_1^n - \chi \frac{\Delta t}{\Delta x} \phi(u_2^n, u_1^n) + \Delta t \, u_1^n (1 - u_1^n), \\ u_M^{n+1} &= u_M^n + \chi \frac{\Delta t}{\Delta x} \phi(u_{M+1}^n, u_M^n) + \Delta t \, u_M^n (1 - u_M^n). \end{aligned} \tag{A.9}$$

Due to the boundary condition, we have the conservation of mass for equation (A.1) when the reaction term equals zero.

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