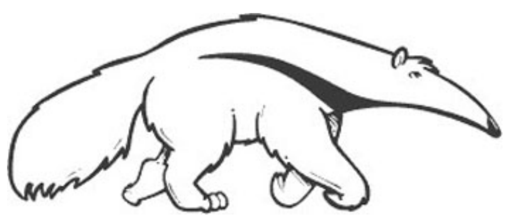
# Markov Models

Machine Learning & Data Mining

Prof. Alexander Ihler



Some slides adapted from Andrew Moore's lectures

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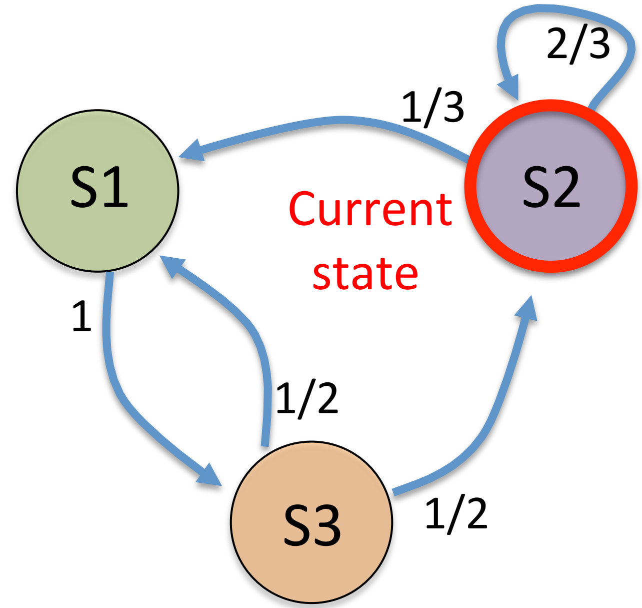
# Markov system

- System has d states, $s_1 \dots s_d$
- Discrete time intervals, t=0,1,…,T
- At time t, system is in state $x_t$

$$x_0 \sim p(x_0)$$

"State transition diagram"




d = 3
t = 0
$x_0$ = 2

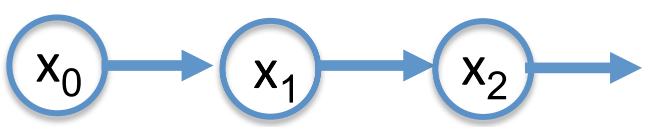
- At each t, system transitions to another state according to

$$p(x_{t+1} \,|\, x_t) =$$

$$p(x_{t+1} = s_1 \,|\, x_t = s_1) = 0 \qquad p(x_{t+1} = s_1 \,|\, x_t = s_2) = 0.33 \qquad p(x_{t+1} = s_1 \,|\, x_t = s_3) = 0.5$$
$$p(x_{t+1} = s_2 \,|\, x_t = s_1) = 0 \qquad p(x_{t+1} = s_2 \,|\, x_t = s_2) = 0.66 \qquad p(x_{t+1} = s_2 \,|\, x_t = s_3) = 0.5$$
$$p(x_{t+1} = s_3 \,|\, x_t = s_1) = 1 \qquad p(x_{t+1} = s_3 \,|\, x_t = s_2) = 0 \qquad p(x_{t+1} = s_3 \,|\, x_t = s_3) = 0$$

(think of this as a d$\times$d matrix "P")

- Bayes Net on states x over time: a "Markov chain"



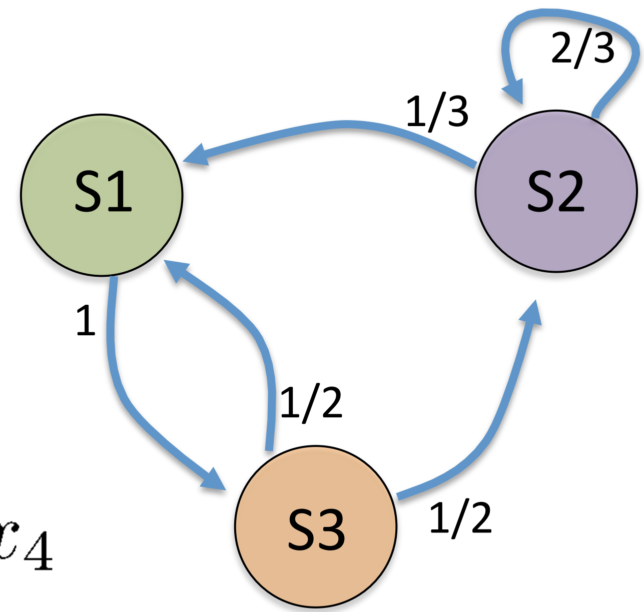

  – Each conditional probability distribution is identical ("homogeneous")

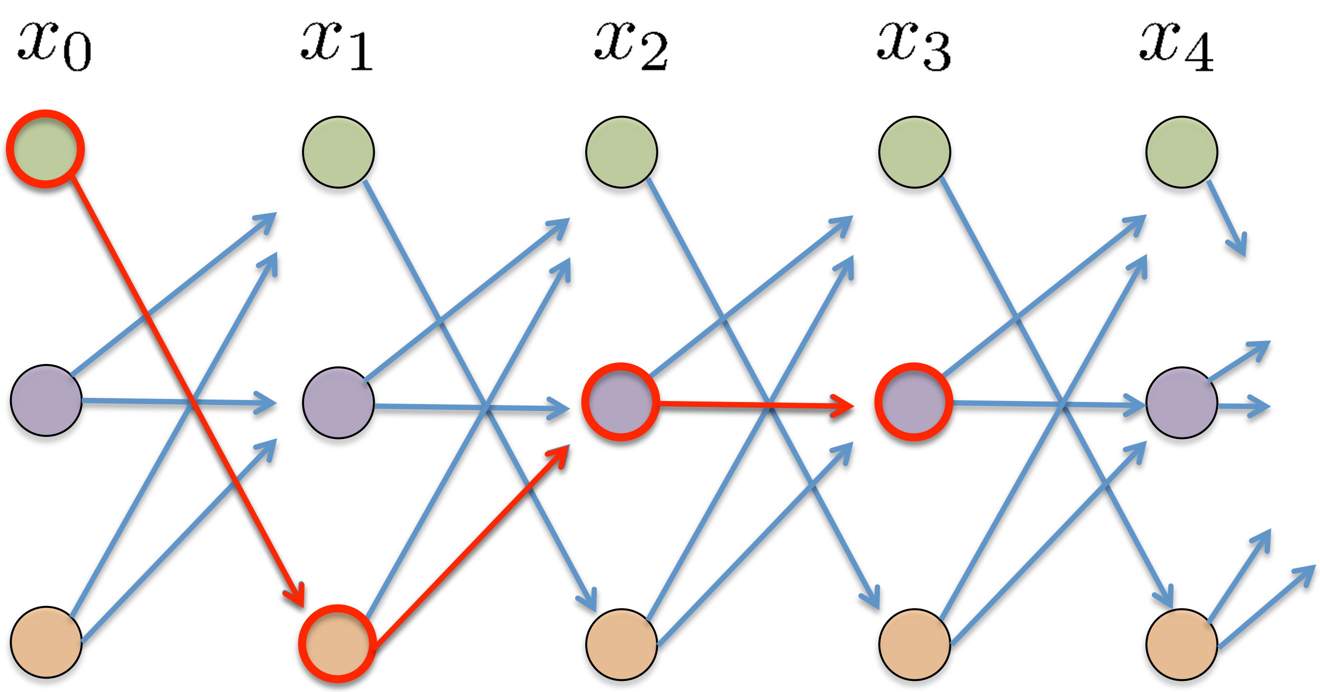
# Markov system

- Another view: "lattice of states"
- State sequence = path in lattice

$$[x_0, x_1, x_2, x_3, \ldots] = [s_1, s_3, s_2, s_2, \ldots]$$

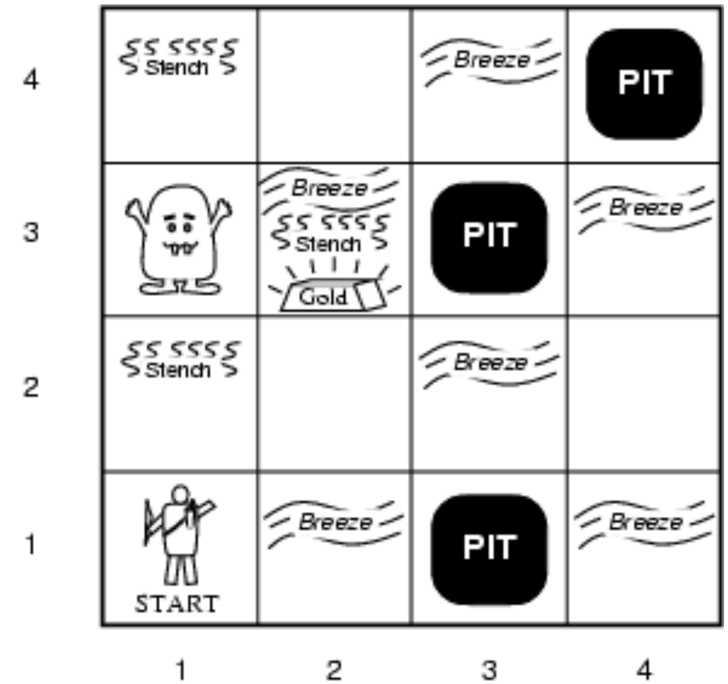
"State transition diagram"



# Ex: "Wumpus World"

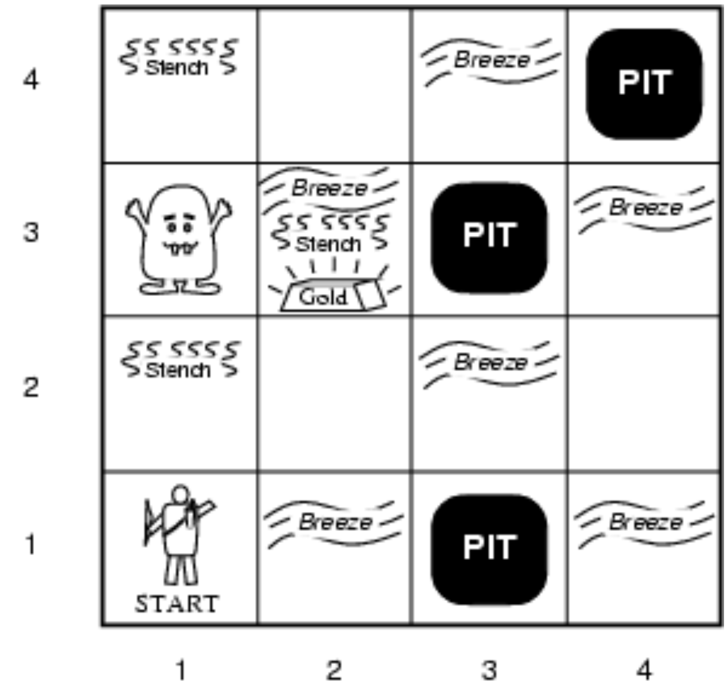
- Person & Wumpus in cave
- Wander randomly
- Cave is dark; assume layout known




- State = (location of person, location of wumpus)
  - \# states? (16) x (16) = 256; Initial state distribution, $p(x_0)$?
- Dynamics:
  - Person & Wumpus wander randomly to adjacent square at each time
- Some possible questions:
  - What's the expected time until the Wumpus eats us?
  - What's the probability we find the gold first?
  - What's the probability the Wumpus will eat us at the next step?

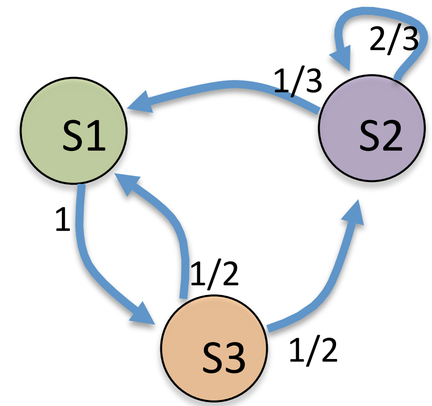
# Ex: “Wumpus World”

- Person & Wumpus in cave
- Wander randomly
- Cave is dark; assume layout known




- Example: “Given that it’s time t and we’re OK, what’s the probability the wumpus eats us at time t+1?”
- If
  - We are omnipotent (see entire cave) – easy to compute from dynamics
  - If we’re blind (no information at all) – Markov model
  - If we have some indirect information – hidden Markov model

# Computing probabilities

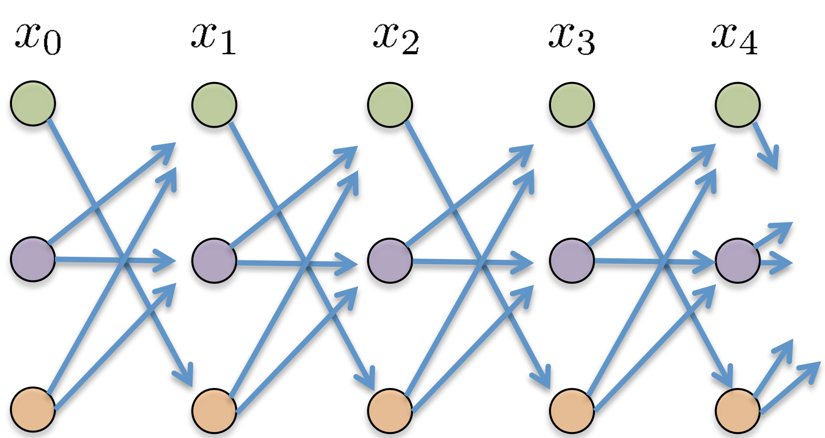



- How to compute the state distribution at time t?
- Simple answer: enumerate over all paths. Ex t=2, $x_0$=2:

  p([2,1,1]) = .33*0 p([2,1,2]) = .33*0 …

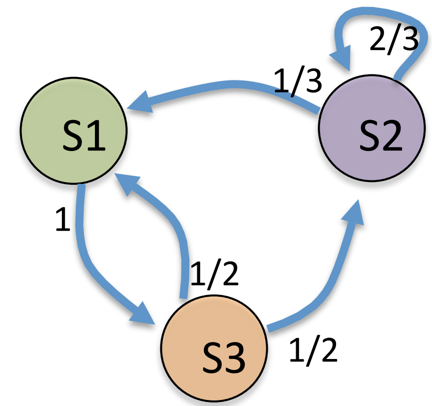
  p([2,1,3]) = .33*1 p([2,2,1]) = .66*.33 p([2,2,2]) = .66*.66

- Problem: number of paths of length t? $O(d^t)$
- How can we use the structure of the problem? (e.g., lattice)




Use induction
("dynamic programming")

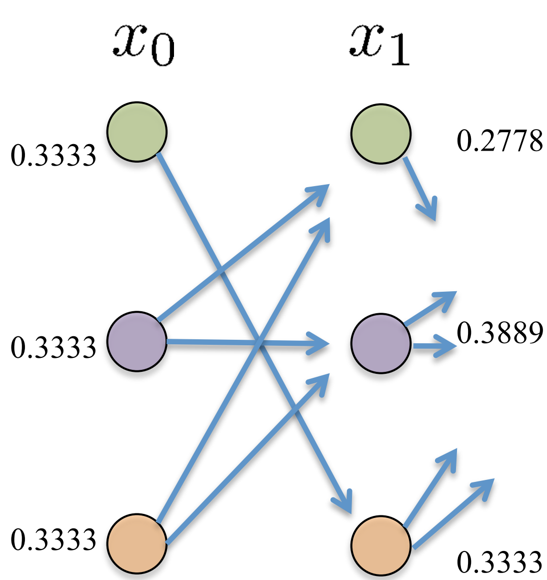
# Computing probabilities



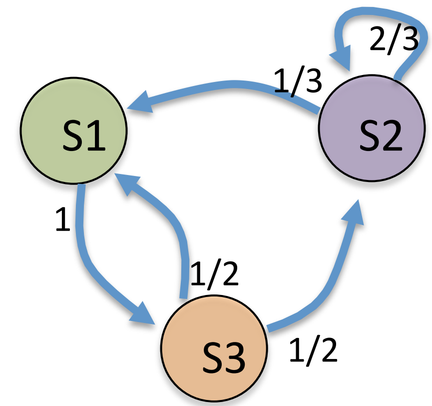

- We can compute the state distribution at time t:

$$p(x_1) = \sum_{x_0} p(x_0) \cdot p(x_1 \,|\, x_0)$$

$$\begin{pmatrix} 0.2778 \\ 0.3889 \\ 0.3333 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 0.333 \\ 0.333 \\ 0.333 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} 0.0000 & 0.0000 & 1.0000 \\ 0.3333 & 0.6667 & 0.0000 \\ 0.5000 & 0.5000 & 0.0000 \end{pmatrix}$$



# Computing probabilities



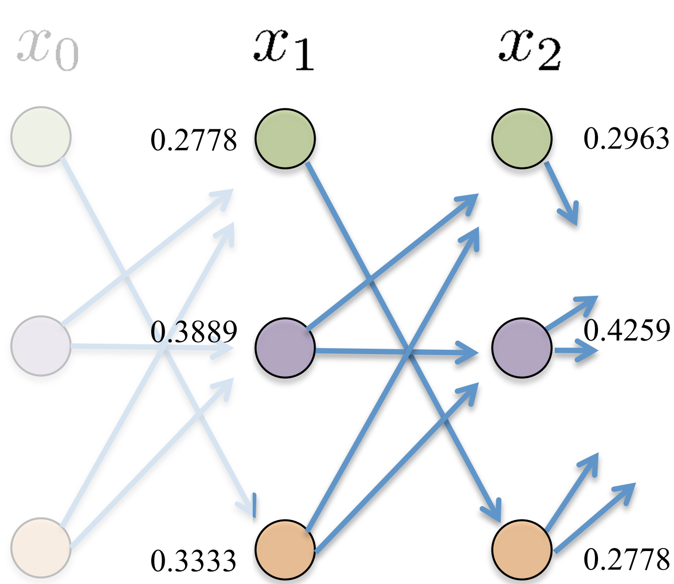

- We can compute the state distribution at time t:

$$p(x_1) = \sum_{x_0} p(x_0) \cdot p(x_1 \mid x_0)$$

$$p(x_2) = \sum_{x_1} p(x_1) \cdot p(x_2 \mid x_1)$$

$$\begin{pmatrix} 0.2963 \\ 0.4259 \\ 0.2778 \end{pmatrix}^T = \begin{pmatrix} 0.2778 \\ 0.3889 \\ 0.3333 \end{pmatrix}^T \begin{pmatrix} 0.0000 & 0.0000 & 1.0000 \\ 0.3333 & 0.6667 & 0.0000 \\ 0.5000 & 0.5000 & 0.0000 \end{pmatrix}$$

Computation? O( t d$^2$ )




What's the state occupancy distribution in the far future?

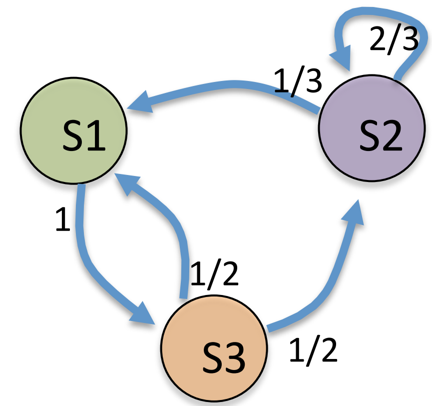
$$\lim_{t \to \infty} p(x_t) = ?$$

Does it depend on $x_0$?

In general:

$p(x_t) = p_0 \, P \, P \, \ldots = p_0 \, (P)^t$

# Computing probabilities

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$$p(x_2) = \sum_{x_1} p(x_1) \cdot p(x_2 \,|\, x_1)$$

**Notes:**

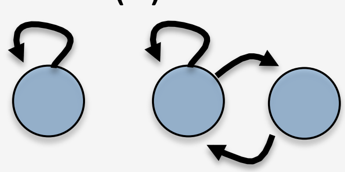
Stationary distribution: $s(x) \quad : \quad s(x_{t+1}) = \sum_{x_t} p(x_{t+1} \,|\, x_t) s(x_t)$

s(x) exists & is unique, so that $p(x_t)$ becomes independent of $p(x_0)$, if:

(a) p(.|.) is irreducible: $\forall i,j \; \exists t \; : \; \Pr[x_t = s_i \,|\, x_0 = s_j] > 0$

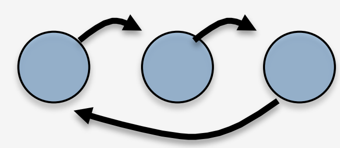
(b) p(.|.) is acyclic: $\gcd\{t \; : \; \Pr[x_t = s_i \,|\, x_0 = s_i] > 0\} = 1$

Ex: if not (a):

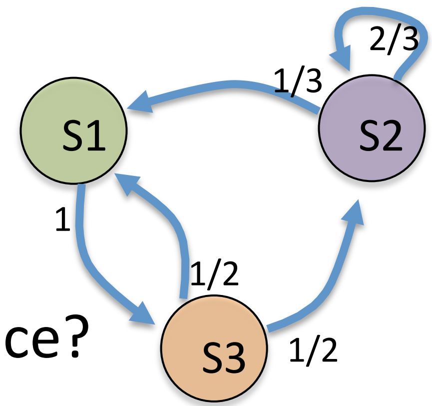


(Long-term prob will depend on initial state dist)

Ex: if not (b):



# Dynamic programming

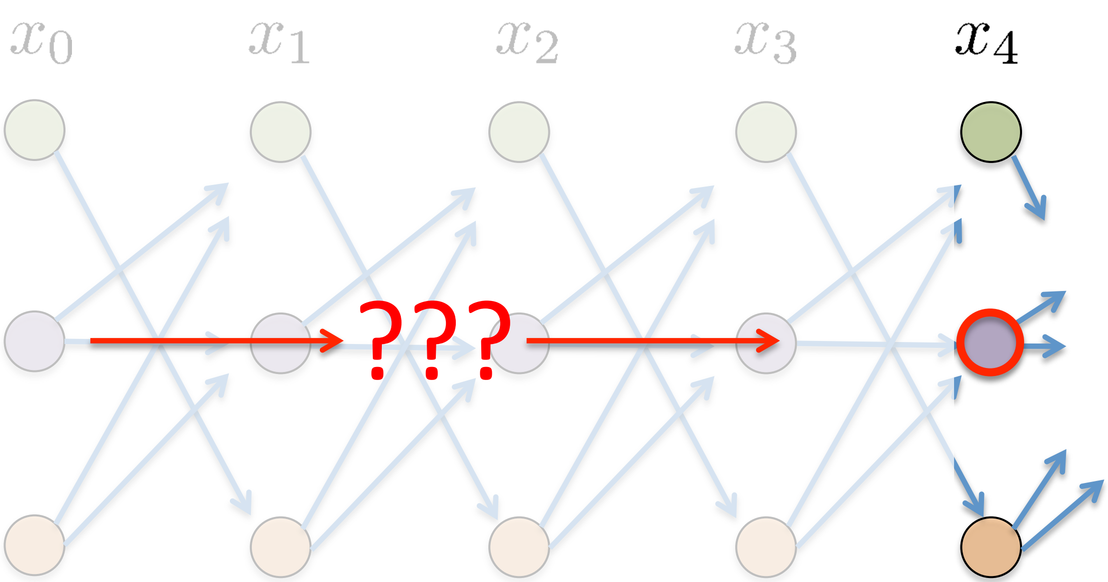



- Observe, say, $x_4 = 2$
- What's the (value of the) most likely state sequence?

$$r^* = \max_{x_0,\ldots,x_4} p(x_0, x_1, x_2, x_3, x_4)\delta(x_4 = s_2)$$

$$r(x_4) = \delta(x_4 = s_2)$$

$$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \end{pmatrix}$$



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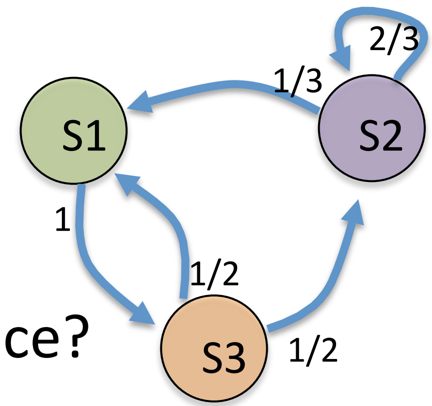


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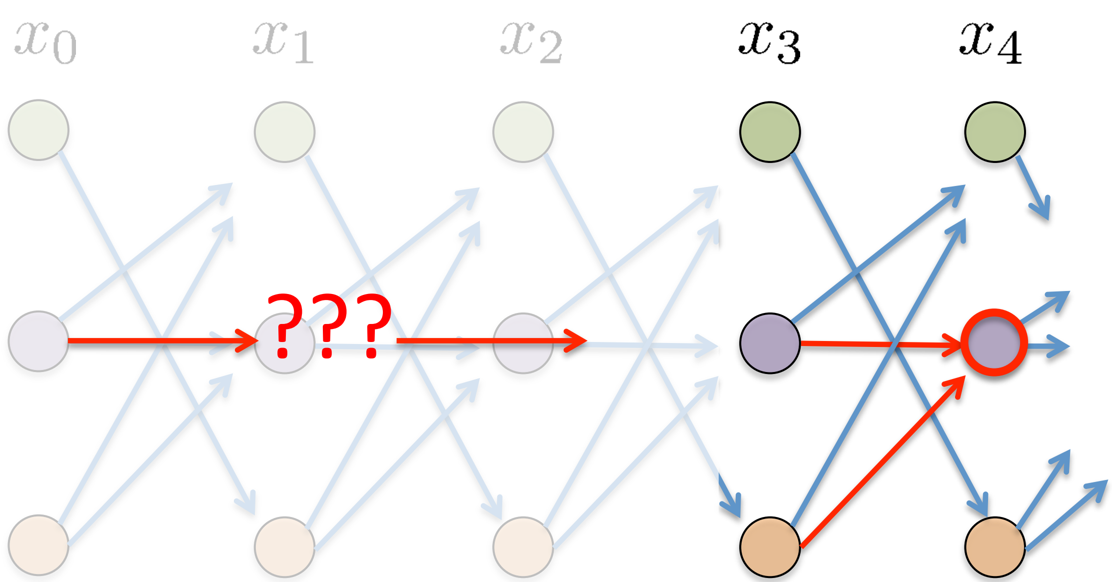
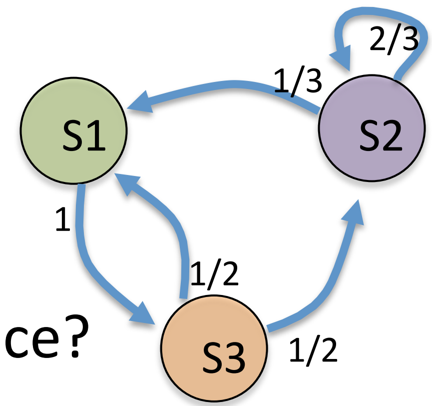



$$r(x_4) = \delta(x_4 = s_2)$$

$$r(x_3) = \max_{x_4} p(x_4 \,|\, x_3) r(x_4)$$

$$\begin{pmatrix} 0.0000 \\ 0.6667 \\ 0.5000 \end{pmatrix}$$

# Dynamic programming

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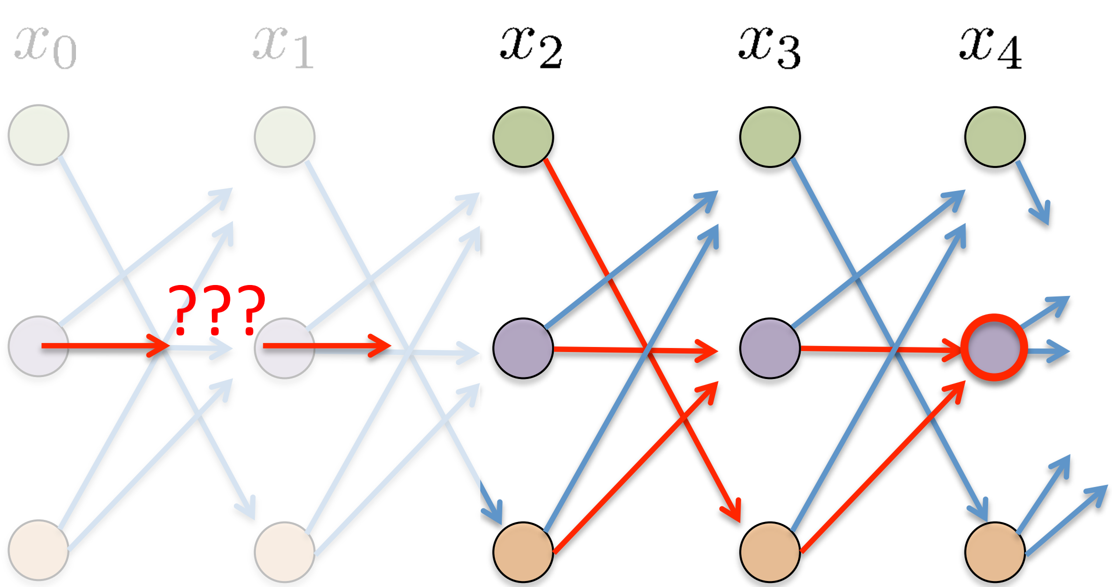


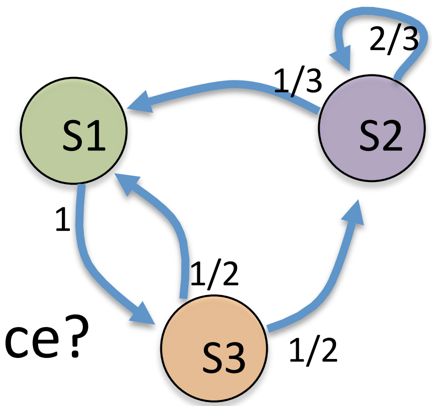

$$r(x_4) = \delta(x_4 = s_2)$$

$$r(x_3) = \max_{x_4} p(x_4 \mid x_3) r(x_4)$$

$$r(x_2) = \max_{x_3} p(x_3 \mid x_2) r(x_3)$$

$$\begin{pmatrix} 0.5000 \\ 0.4444 \\ 0.3333 \end{pmatrix}$$

# Dynamic programming

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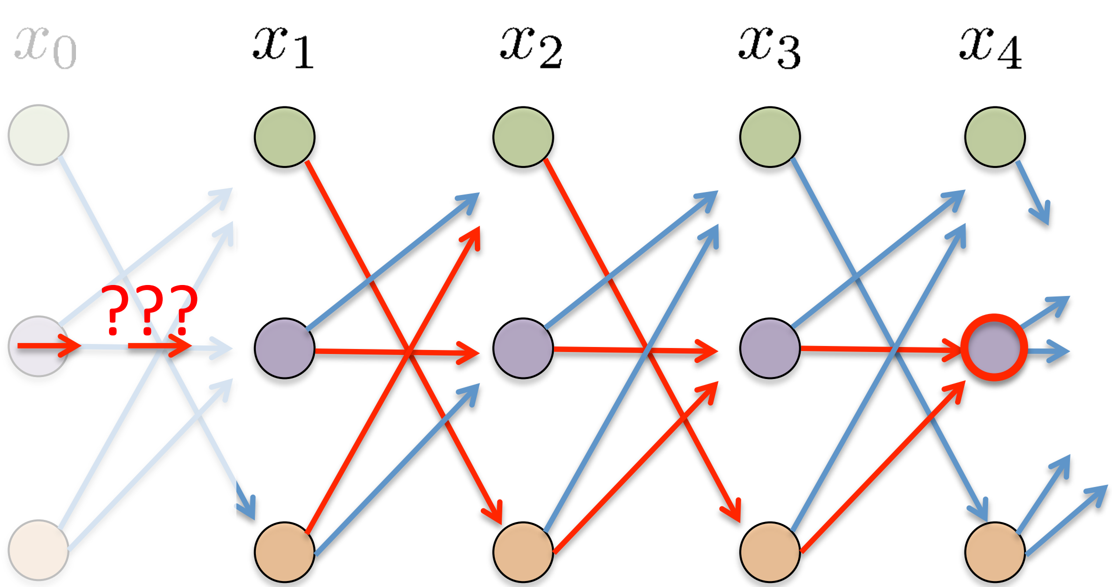


$$r(x_4) = \delta(x_4 = s_2)$$

$$r(x_3) = \max_{x_4} p(x_4 \mid x_3) r(x_4)$$

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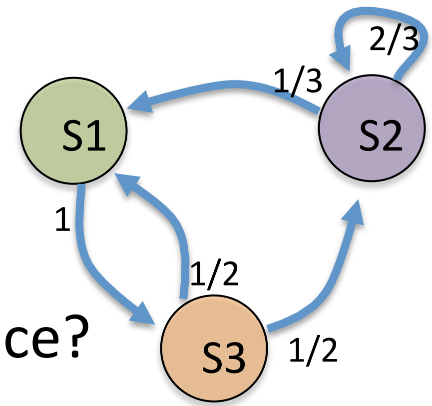
$$r(x_1) = \max_{x_2} p(x_2 \mid x_1) r(x_2)$$

$$\begin{pmatrix} 0.3333 \\ 0.2963 \\ 0.2500 \end{pmatrix}$$

# Dynamic programming

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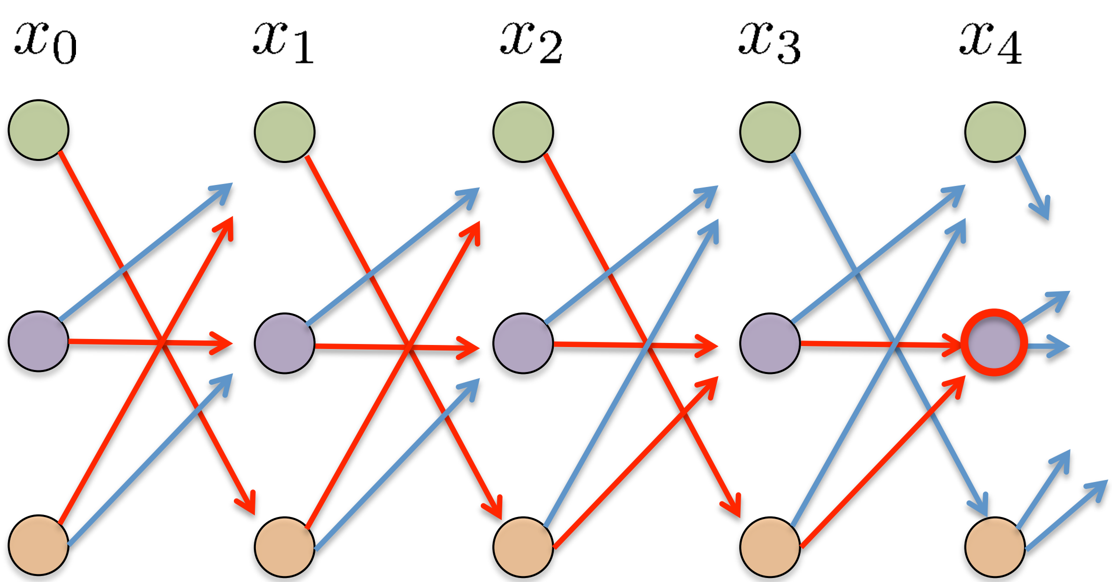
- What's the (value of the) most likely state sequence?

$$r^* = \max_{x_0,\ldots,x_4} p(x_0, x_1, x_2, x_3, x_4)\delta(x_4 = s_2)$$
$$= 0.0833$$

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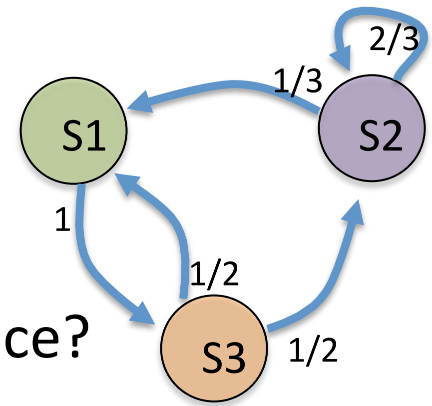
$$r(x_0) = \max_{x_1} p(x_1 \,|\, x_0) r(x_1)$$
$$r^* = \max_{x_0} p(x_0) r(x_0)$$

$$\begin{pmatrix} 0.0833 \\ 0.0658 \\ 0.0556 \end{pmatrix} = \begin{pmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{pmatrix} \begin{pmatrix} 0.2500 \\ 0.1975 \\ 0.1667 \end{pmatrix}$$

# Dynamic programming

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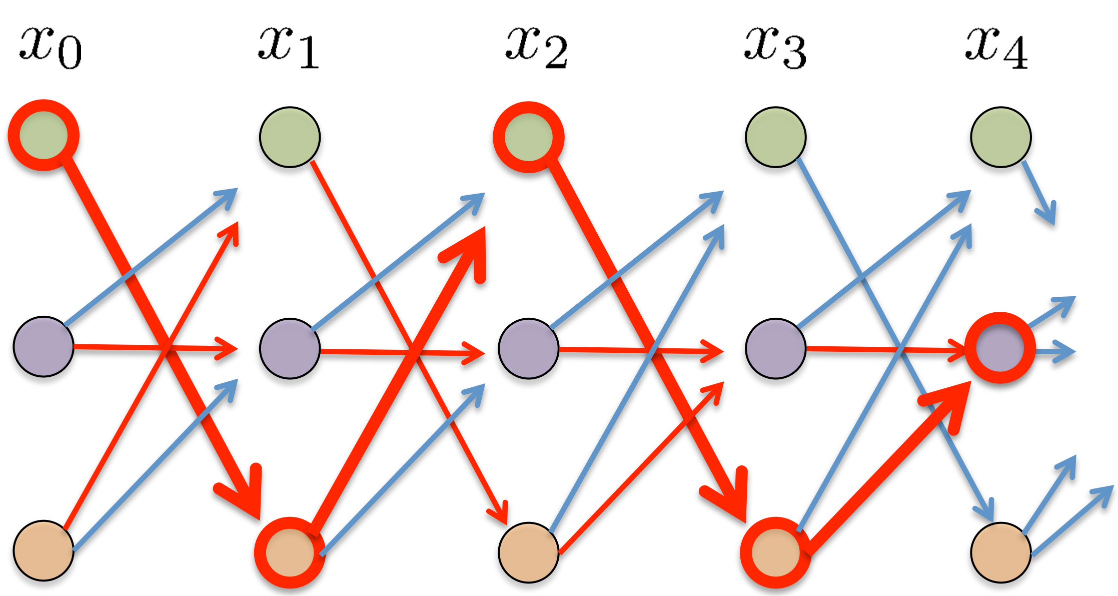
$$= 0.0833$$

$$\mathbf{x}^* = [s_1,\ s_3,\ s_1,\ s_3,\ s_2\,]$$

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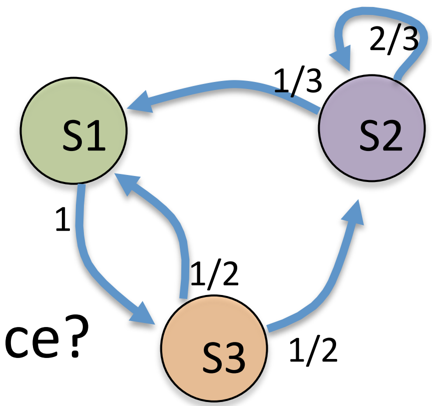
# Dynamic programming

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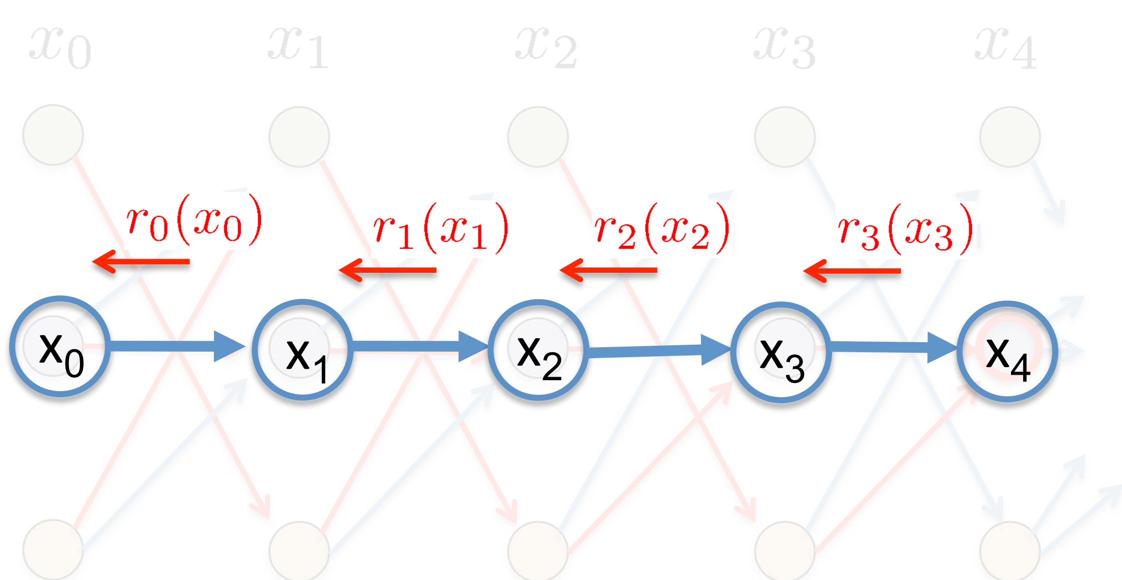
$$= 0.0833$$

$$\mathbf{x}^* = [s_1,\ s_3,\ s_1,\ s_3,\ s_2]$$

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$$r^* = \max_{x_0} p(x_0) r(x_0)$$

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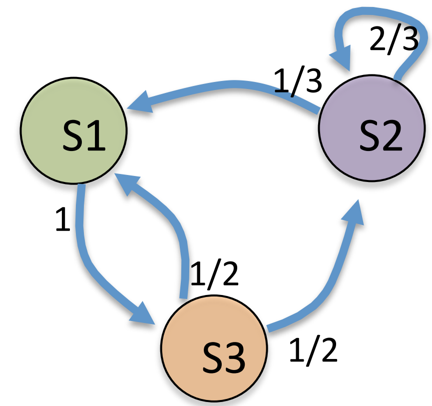
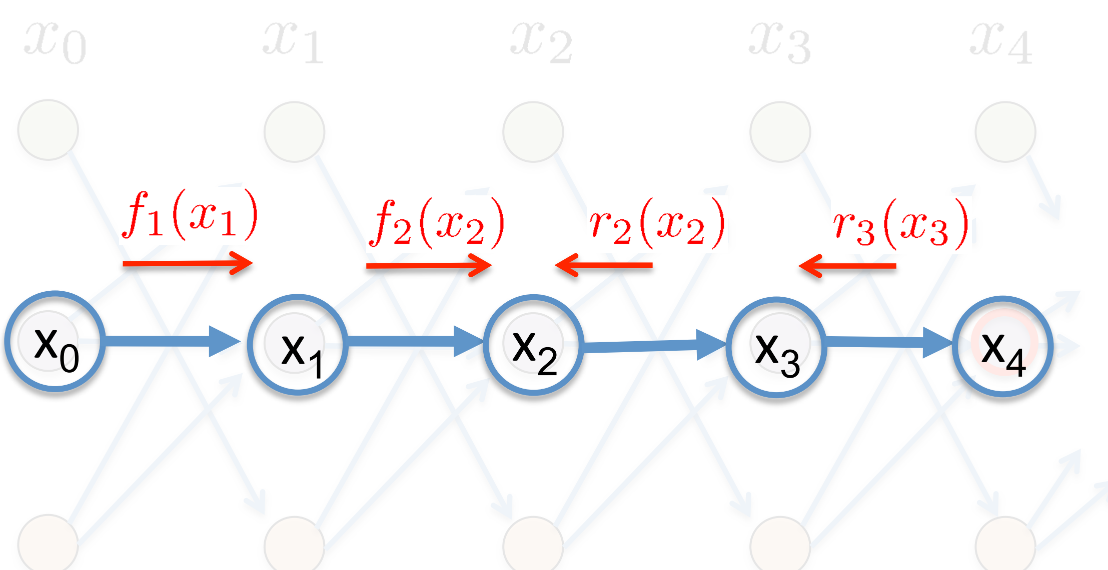
# Dynamic programming




- Observe, say, $x_4 = 2$
- Similar algorithm for computing marginals:

$$p(x_2 | x_4 = s_2) \propto \sum_{x_0,\ldots,x_4} p(x_0, x_1, x_2, x_3, x_4)\delta(x_4 = s_2)$$

$$\propto f_2(x_2) \cdot r_2(x_2)$$




$$r_4(x_4) = \delta(x_4 = s_2)$$

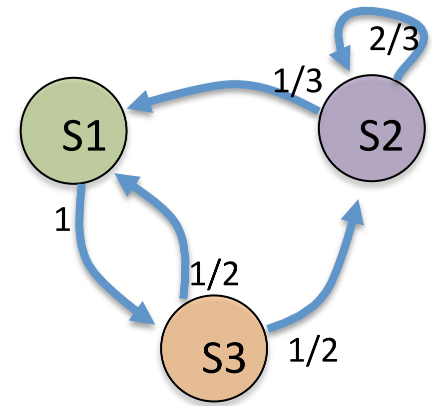
$$r_3(x_3) = \sum_{x_4} p(x_4 \,|\, x_3) r(x_4)$$

$$r_2(x_2) = \sum_{x_3} p(x_3 \,|\, x_2) r(x_3)$$

$$f_1(x_1) = \sum_{x_0} p(x_1 \,|\, x_0) p(x_0)$$

$$f_2(x_2) = \sum_{x_1} p(x_2 \,|\, x_1) f_1(x_1)$$

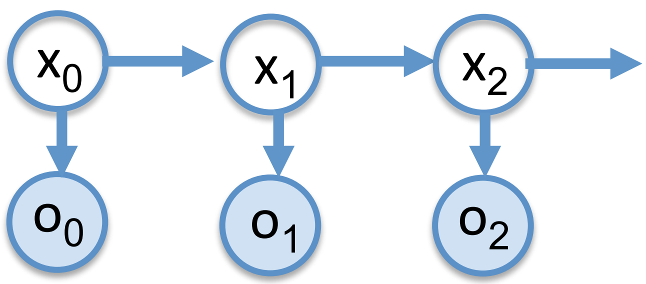
# Hidden Markov Model




- In addition to the Markov state variables $x_t$
- We also have "emission" variables, $o_t$
- Model is specified by

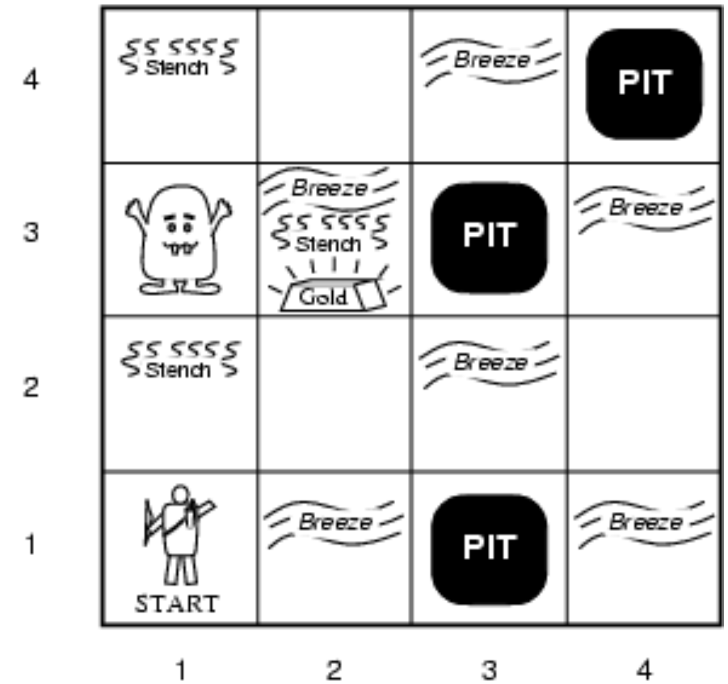
$$x_0 \sim p(x_0) \qquad x_{t+1} \sim p(x_{t+1} \mid x_t)$$
$$o_t \sim p(o_t \mid x_t)$$

- Bayes Net on states $x_t$ and observations $o_t$ over time t




- Typically, we'll observe the values of the o's (shaded)
  - Induces a model over the x's, and use this to answer queries about x's

# Ex: “Wumpus World”

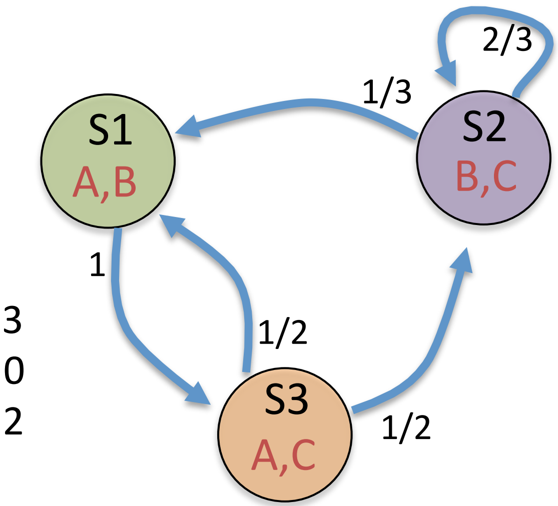



- Person & Wumpus in cave
- Wander randomly
- Cave is dark; assume layout known

- Observe “a bit about” state
  - Walls (tell us something about our location)
  - Breeze (tell us we are next to one of the pits)
  - Smell (tell us something about Wumpus location)

- Still can’t observe the complete state, but more information now

# Ex: Hidden Markov model




d = 3
t = 0
$x_0$ = 2

- Initial state distribution

$$x_0 \sim p(x_0)$$

- State transition probabilities

$$p(x_{t+1} \mid x_t) =$$

$$p(x_{t+1} = s_1 \mid x_t = s_1) = 0 \quad p(x_{t+1} = s_1 \mid x_t = s_2) = 0.33 \quad p(x_{t+1} = s_1 \mid x_t = s_3) = 0.5$$
$$p(x_{t+1} = s_2 \mid x_t = s_1) = 0 \quad p(x_{t+1} = s_2 \mid x_t = s_2) = 0.66 \quad p(x_{t+1} = s_2 \mid x_t = s_3) = 0.5$$
$$p(x_{t+1} = s_3 \mid x_t = s_1) = 1 \quad p(x_{t+1} = s_3 \mid x_t = s_2) = 0 \quad p(x_{t+1} = s_3 \mid x_t = s_3) = 0$$

(think of this as a d$\times$d matrix “P”)

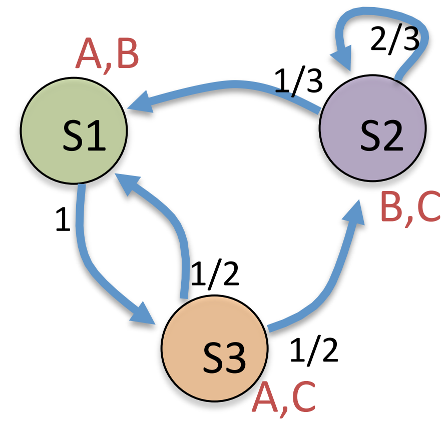
- Observation probabiliites

$$p(o_t \mid x_t) =$$

$$p(o_t = A \mid x_t = s_1) = 0.5 \quad p(o_t = A \mid x_t = s_2) = 0 \quad p(o_t = A \mid x_t = s_3) = 0.5$$
$$p(o_t = B \mid x_t = s_1) = 0.5 \quad p(o_t = B \mid x_t = s_2) = 0.5 \quad p(o_t = B \mid x_t = s_3) = 0$$
$$p(o_t = C \mid x_t = s_1) = 0 \quad p(o_t = C \mid x_t = s_2) = 0.5 \quad p(o_t = C \mid x_t = s_3) = 0.5$$

(think of this as a k$\times$d matrix “Q”)

# Ex: state estimation




Initial distribution:

$$p(x_0) = \begin{pmatrix} 0.333 \\ 0.333 \\ 0.333 \end{pmatrix}$$

Observe: [A,A,B,A,...]
What state are we in?
Depends on both P, Q

Observe A:

$$p(x_0, o_0) = \begin{pmatrix} 0.333 * 0.5 \\ 0.333 * 0 \\ 0.333 * 0.5 \end{pmatrix} = \begin{pmatrix} 0.166 \\ 0.0 \\ 0.166 \end{pmatrix} \Rightarrow$$

"posterior"

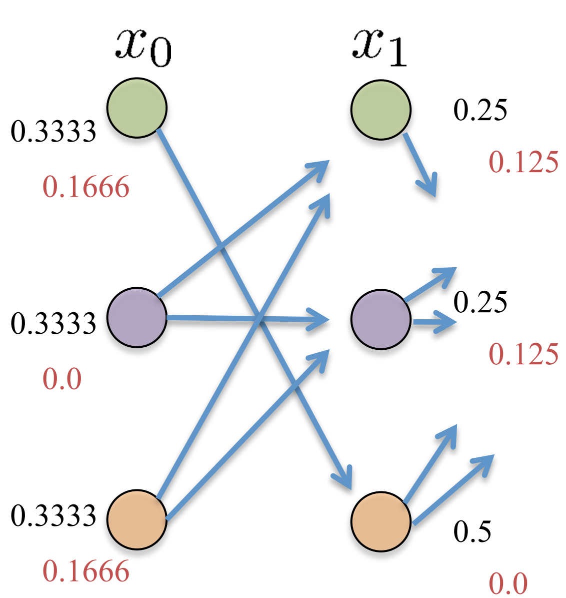
$$p(x_0|o_0) = \begin{pmatrix} 0.5 \\ 0.0 \\ 0.5 \end{pmatrix}$$

$$p(o_0) = 0.1666 + 0.1666 = 0.3333$$

$$\Downarrow$$

$$p(x_1|o_0) = \sum_{x_0} p(x_0|o_0) \cdot p(x_1 \mid x_0)$$

$$\begin{pmatrix} 0.25 \\ 0.25 \\ 0.5 \end{pmatrix}^T = \begin{pmatrix} 0.5 \\ 0.0 \\ 0.5 \end{pmatrix}^T \begin{pmatrix} 0.0000 & 0.0000 & 1.0000 \\ 0.3333 & 0.6667 & 0.0000 \\ 0.5000 & 0.5000 & 0.0000 \end{pmatrix}$$




Observe B:

$$p(x_1, o_1|o_0) = p(x_1|o_0)\, p(o_1|x_1)$$

$$\begin{pmatrix} 0.25 * 0.5 \\ 0.25 * 0.5 \\ 0.5 \;\; * 0 \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0.125 \\ 0.0 \end{pmatrix} \Rightarrow$$

$$p(x_1|o_0, o_1) \quad \begin{pmatrix} 0.5 \\ 0.5 \\ 0.0 \end{pmatrix}$$

$$p(o_1|o_0) = 0.125 + 0.125 = 0.25$$

# State estimation: filtering

- Estimate state distribution at time t given observations up to t

Forward messages:

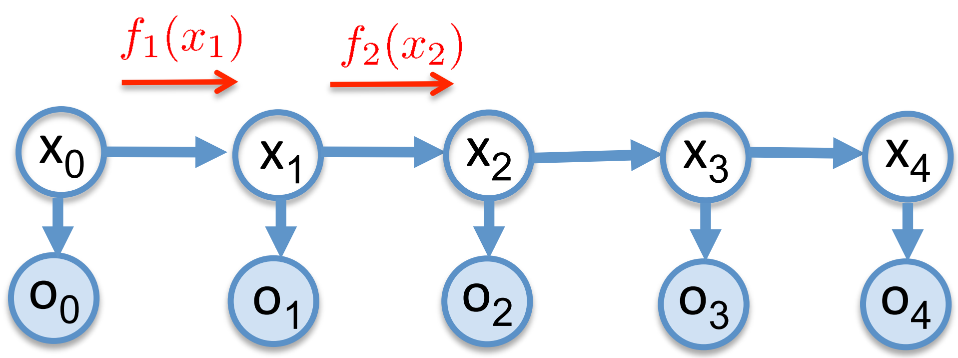
$$\begin{aligned} f_t(x_t) &= p(x_t \mid o_0, \ldots, o_t) \\ &= \frac{1}{Z_t} \sum_{x_{t-1}} p(x_t, x_{t-1}, o_t | o_0, \ldots, o_{t-1}) \\ &= \frac{1}{Z_t} p(o_t|x_t) \sum_{x_{t-1}} p(x_t|x_{t-1}) f_{t-1}(x_{t-1}) \end{aligned}$$

$Z_t$ is the scalar that normalizes $f_t(x_t)$:

$$Z_t = p(o_t|o_0, \ldots, o_{t-1})$$

Observation likelihood:

$$\begin{aligned} p(O = o) &= p(o_0)\, p(o_1|o_0) \ \ldots \\ &= \prod_t Z_t \end{aligned}$$



# State estimation: smoothing

- Estimate state distribution at time t given future observations

Forward messages:

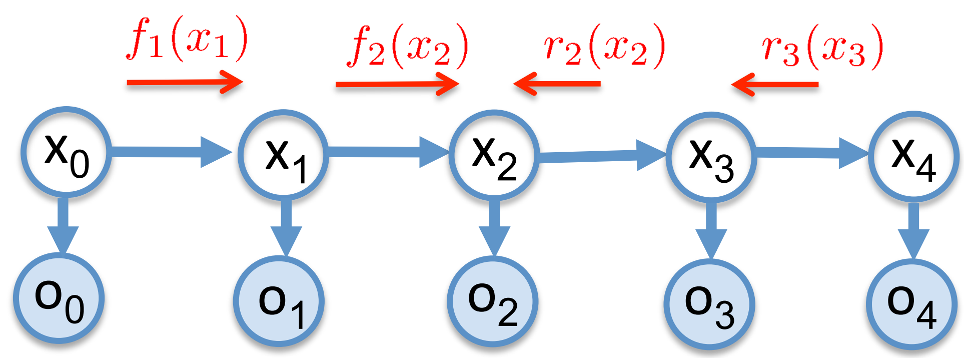
$$
\begin{aligned}
f_t(x_t) &= p(x_t \mid o_0, \dots, o_t) \\
&= \frac{1}{Z_t} \sum_{x_{t-1}} p(x_t, x_{t-1}, o_t | o_0, \dots, o_{t-1}) \\
&= \frac{1}{Z_t} p(o_t | x_t) \sum_{x_{t-1}} p(x_t | x_{t-1}) f_{t-1}(x_{t-1})
\end{aligned}
$$

Reverse messages:

$$
\begin{aligned}
r_t(x_t) &\propto p(o_{t+1}, \dots, o_T \mid x_t) \\
&\propto \sum_{x_{t+1}} p(x_{t+1} | x_t)\, p(o_{t+1} | x_{t+1})\, r_{t+1}(x_{t+1})
\end{aligned}
$$

$Z_t$ is the scalar that normalizes $f_t(x_t)$:

$$Z_t = p(o_t | o_0, \dots, o_{t-1})$$




Observation likelihood:

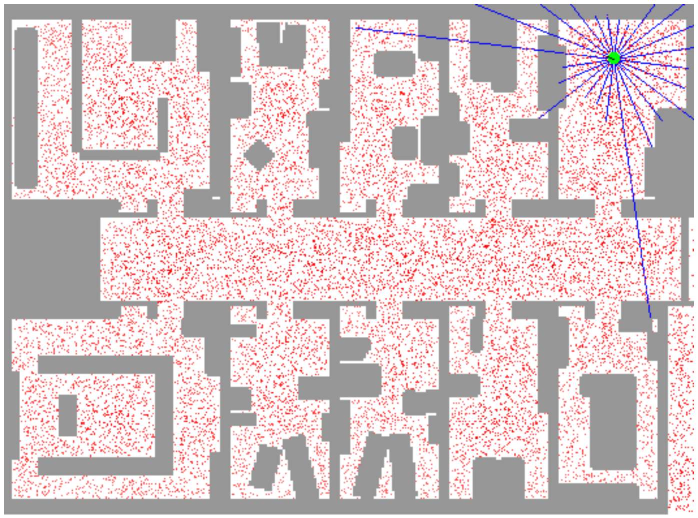
$$
\begin{aligned}
p(O = o) &= p(o_0)\, p(o_1 | o_0) \, \dots \\
&= \prod_t Z_t
\end{aligned}
$$

Marginal probabilities:

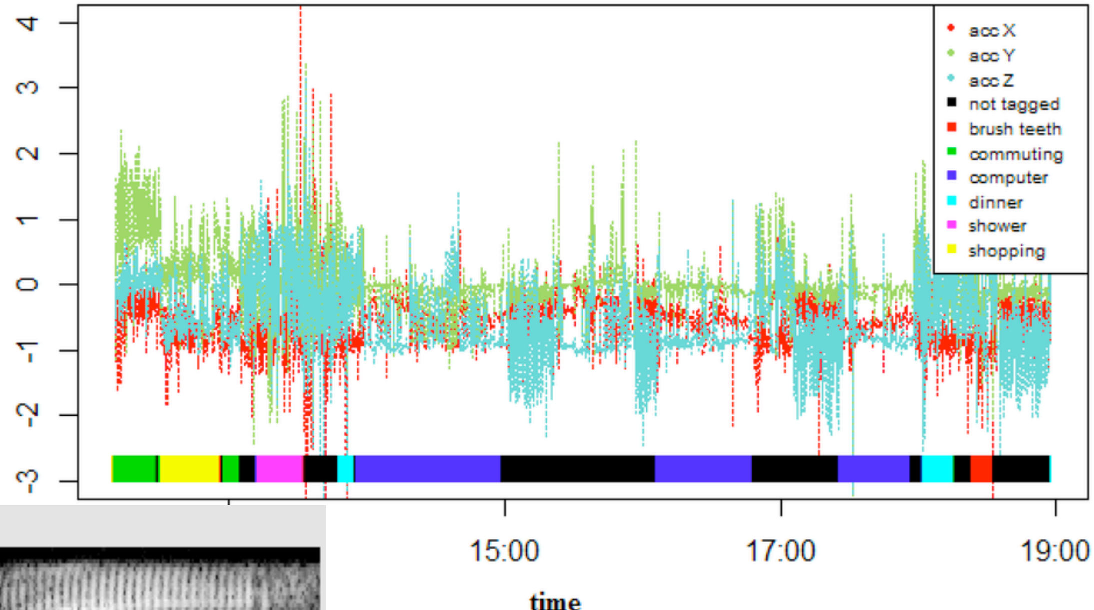
$$p(x_2 | O = o) = \frac{1}{p(O = o)} \sum_{\mathbf{x} \setminus x_2} p(\mathbf{x}, O = o) \propto f_2(x_2) \cdot r_2(x_2)$$

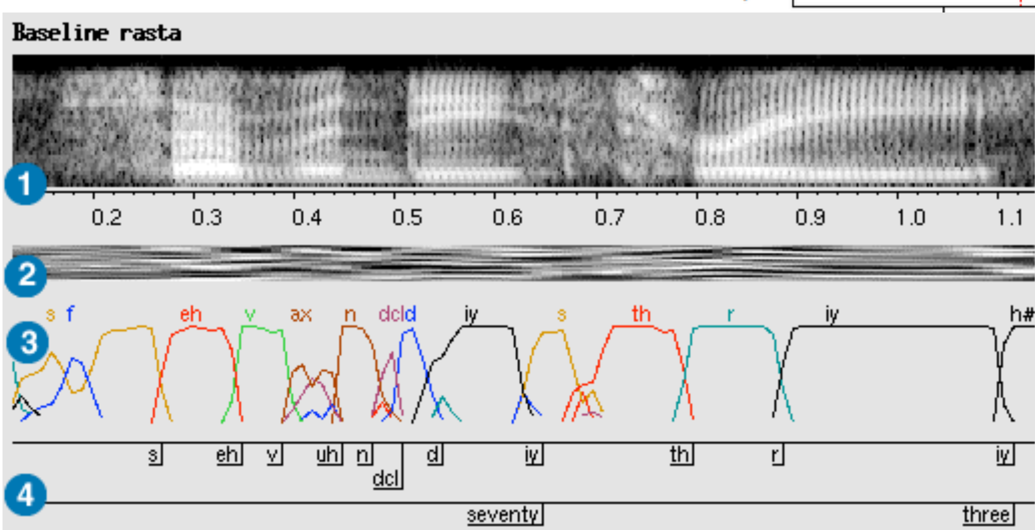
# Example HMM applications

Robot state estimation (animation: Deiter Fox, UW)



Activity recognition (from [Garcia-Ceja et al. 2014])




Speech recognition
(image from Dan Ellis' webpage)

# Summary

- Markov models, hidden Markov models
- Dynamic programming
  - For state distribution at time t (“forward-backward”)
  - For most probable sequence of states (“Viterbi”)
- Learning HMMs
  - Expectation-Maximization (also “Baum-Welch”)