Simulation of a Radioactive Decay Problem

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1 Problem Statement

Consider a radioactive decay problem involving two types of nuclei, A and B, with populations $N_A(t)$ and $N_B(t)$, respectively. Assume that type A nuclei decay into type B nuclei, and similarly, type B nuclei decay into type A nuclei. The equations for the rate of change of the populations of these two nuclei are as follows:

$$\frac{dN_A}{dt} = \frac{N_B}{\tau} - \frac{N_A}{\tau} \tag{1}$$

$$\frac{dN_B}{dt} = \frac{N_A}{\tau} - \frac{N_B}{\tau} \tag{2}$$

For simplicity, we assume that both types of decay occur with the same time constant τ, and we take $\tau = 1s$. Consider the initial conditions $N_A(t=0) = 100$ and $N_B(t=0) = 0$. Solve this system of equations for the number of nuclei A and B as a function of time. Show that your numerical results are consistent with the idea that the system reaches a steady state where $N_A(t)$ and $N_B(t)$ are constant.

2 Analytical Solution

To solve the coupled equations analytically, we use the method of matrices and eigenvalues. For this purpose, we define the matrix N:

$$N = \begin{pmatrix} N_A \\ N_B \end{pmatrix}$$

Equations 1 and 2 can be simplified as:

$$\frac{dN}{dt} = AN$$

where:

$$A = \frac{1}{\tau} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Now, we find the eigenvalues of matrix A:

$$\det(A - \lambda I) = 0$$

$$\det\left(\frac{1}{\tau}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\left(-\frac{1}{\tau}-\lambda\right)^2-\left(\frac{1}{\tau}\right)^2=0\rightarrow\begin{cases}\lambda_1=0\\\lambda_2=-\frac{2}{\tau}\end{cases}$$

Now we find the eigenvectors for each of the obtained eigenvalues. For $\lambda_1=0$:

$$(A-0\cdot I)a_1=0\rightarrow\frac{1}{\tau}\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}=0\rightarrow a_{11}=a_{12}\rightarrow a_1=\begin{pmatrix}1\\1\end{pmatrix}$$

For $\lambda_2=-\frac{2}{\tau}$:

$$\left(A+\frac{2}{\tau}I\right)a_2=0\rightarrow\frac{1}{\tau}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}a_{21}\\a_{22}\end{pmatrix}=0\rightarrow a_{21}=-a_{22}\rightarrow a_2=\begin{pmatrix}1\\-1\end{pmatrix}$$

The general solution will be as follows:

$$N(t)=c_1a_1e^{\lambda_1t}+c_2a_2e^{\lambda_2t}$$

$$N(t)=c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-\frac{2t}{\tau}}$$

Applying the initial conditions, we have:

$$N(0)=\begin{pmatrix}N_A(0)\\N_B(0)\end{pmatrix}=\begin{pmatrix}100\\0\end{pmatrix}=c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}\rightarrow\begin{cases}c_1+c_2=100\\c_1-c_2=0\end{cases}\rightarrow c_1=c_2=50$$

So, assuming $\tau=1$, we will have:

$$N(t)=50\begin{pmatrix}1\\1\end{pmatrix}+50\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t}$$

$$\rightarrow\begin{cases}N_A(t)=50(1+e^{-2t})\\N_B(t)=50(1-e^{-2t})\end{cases}$$

Thus, the analytical solution of the equations is obtained.

3 Numerical Solution

For the numerical solution, we use the Euler method:

$$\frac{d}{dt}N(t)=\lim_{\Delta t\to0}\frac{N(t+\Delta t)-N(t)}{\Delta t}\approx\frac{N(t+\Delta t)-N(t)}{\Delta t}$$

To solve these two equations, we write the attached MATLAB code. The results are as follows: As you can see, after a while, the two graphs converge and will be equal as $t\rightarrow\infty$.

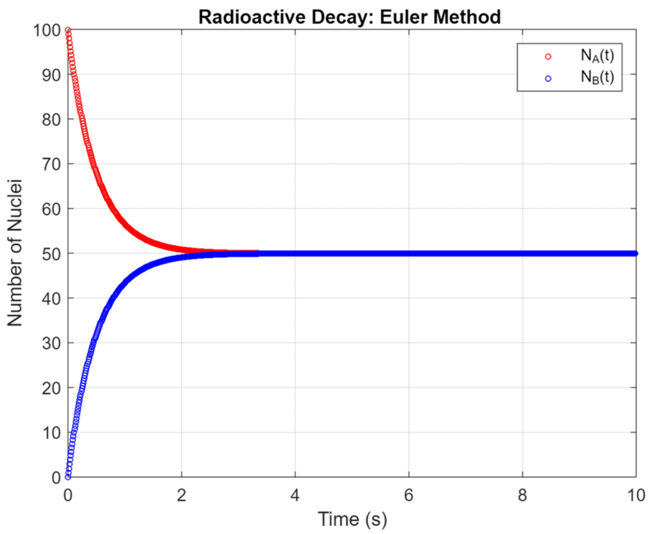

Figure 1: Plot of the number of particles A and B versus time ($\Delta t = 0.01$)

4 Comparison of Analytical and Numerical Results

For each of particles A and B, we plot the analytical and numerical data.

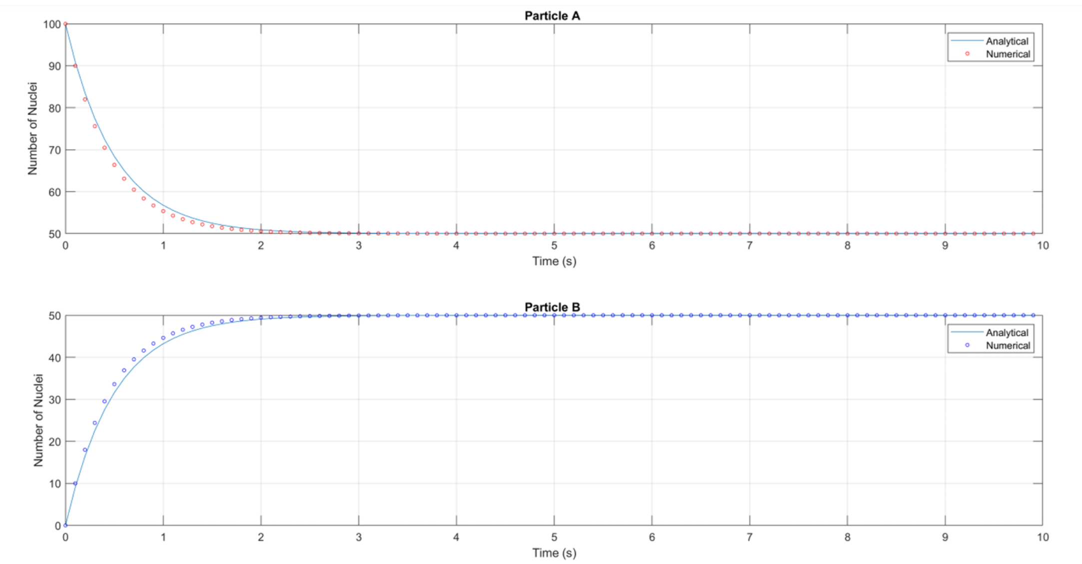

Figure 2: Comparison of numerical and analytical values for the solved problem with $\Delta t = 0.1$

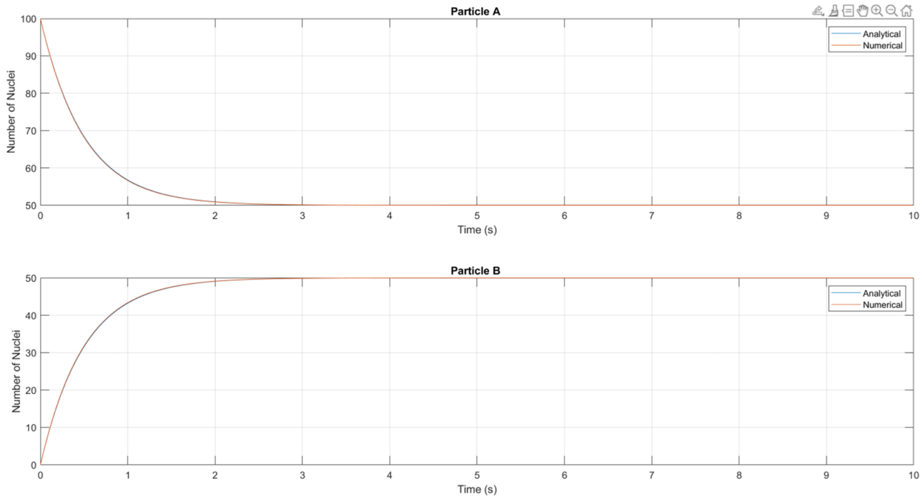

Figure 3: Comparison of numerical and analytical values for $\Delta t = 0.01$ using a line plot

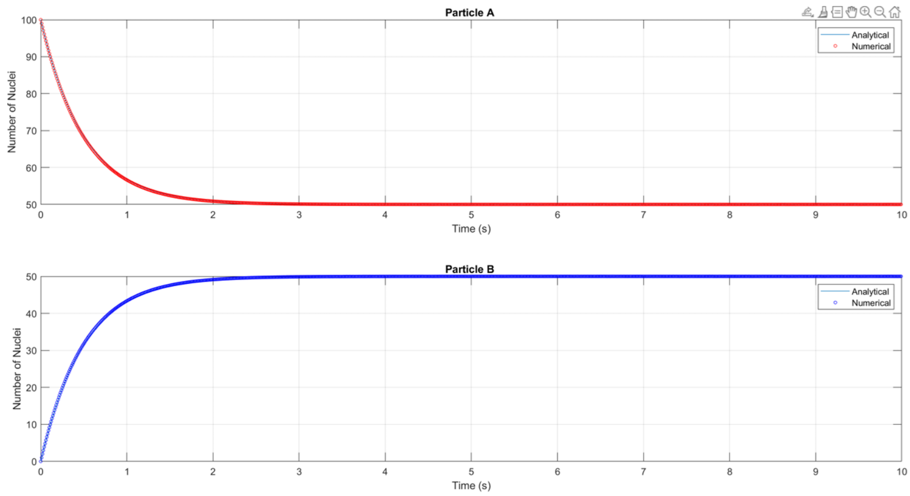

Figure 4: Comparison of numerical and analytical values for $\Delta t = 0.01$ using a scatter plot for the numerical solution

5 Margin and Stability Conditions

The stability of the Euler method depends on the time step Δt and the characteristics of the problem. The stability condition for the Euler method, especially for solving linear ordinary differential equations, is as follows:

$$|\lambda_{max}| \leq \frac{2}{\Delta t}$$

According to previous calculations, $\lambda_{max} = -\frac{2}{\tau}$, and for $\tau = 1$, we will have $\lambda_{max} = -2$. Using the above relation, we can write:

$$\Delta t \leq 1$$

Therefore, for the Euler method to be stable, the time step Δt must be chosen to be less than 1.

6 Conclusion

As is clear from the graphs, the number of particles A and B will become equal at infinite time. This result is confirmed by both numerical and analytical methods. On the other hand, the error of the numerical data compared to the analytical method is very negligible, which is why the Euler method is considered a good method for the approximate solution of this problem. To obtain higher accuracy, we must take smaller time steps to get more accurate answers.