Numerical Solution of the 1D Heat Equation using the Dufort-Frankel Method

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Abstract

This report presents a numerical solution to the one-dimensional heat equation, which models heat diffusion through a uniform rod over time. The problem is solved using the Dufort-Frankel method, an explicit, unconditionally stable finite difference scheme. The implementation is carried out in Python, and the numerical results are compared against the known analytical solution for a specific set of initial and boundary conditions. An error analysis is performed by calculating the difference between the numerical and analytical results, and the Mean Absolute Error (MAE) is reported. The findings demonstrate the accuracy and stability of the Dufort-Frankel method for this parabolic partial differential equation.

1 Introduction

The heat equation is a fundamental parabolic partial differential equation (PDE) that describes the distribution of heat (or variation in temperature) in a given region over time. In its one-dimensional form, it models the flow of heat in a thin, uniform rod. The equation is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature at position x and time t, and k is the thermal diffusivity of the material, a constant that governs the rate of heat conduction.

This project focuses on solving the heat equation over a spatial domain $0 \leq x \leq L$ and for a time duration $t \in [0, T]$. The specific problem considers a rod with the following conditions:

- **Initial Condition:** The initial temperature distribution along the rod is given by a sine wave:

$$u(x,0) = \sin\left(\frac{\pi x}{L}\right)$$

- **Boundary Conditions:** The ends of the rod are held at a constant temperature of zero for all time (Dirichlet boundary conditions):

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

To solve this problem numerically, we employ the Dufort-Frankel finite difference method. The results are then visualized and compared with the exact analytical solution to assess the accuracy of the numerical scheme.

2 Analytical Solution

For the given initial and boundary conditions, the heat equation can be solved analytically using the method of separation of variables. The exact solution provides a benchmark against which our numerical approximation can be validated. The analytical solution is:

$$u(x,t) = e^{-k\left(\frac{\pi}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right)$$

This function describes the temperature at any point x along the rod at any given time t, showing an exponential decay of the initial sinusoidal temperature profile.

3 Numerical Method: The Dufort-Frankel Scheme

To solve the PDE numerically, we discretize the continuous domain into a grid. The spatial domain $[0, L]$ is divided into $N_x - 1$ intervals of width $\Delta x = L/(N_x - 1)$, and the time domain $[0, T]$ is divided into N_t intervals of duration $\Delta t = T/N_t$. The temperature at a grid point $(x_i, t_j) = (i\Delta x, j\Delta t)$ is denoted by u_i^j.

The Dufort-Frankel method is an explicit, three-level finite difference scheme. It is derived by approximating the time derivative with a central difference and the spatial derivative with a modified central difference. The time derivative is:

$$\left.\frac{\partial u}{\partial t}\right|_{(i,j)} \approx \frac{u_i^{j+1} - u_i^{j-1}}{2\Delta t}$$

The key innovation of the Dufort-Frankel scheme is in the approximation of the second spatial derivative, where the central term u_i^j is replaced by a time average $\frac{u_i^{j+1}+u_i^{j-1}}{2}$:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(i,j)} \approx \frac{u_{i+1}^j - (u_i^{j+1} + u_i^{j-1}) + u_{i-1}^j}{(\Delta x)^2}$$

Substituting these approximations into the heat equation gives:

$$\frac{u_i^{j+1} - u_i^{j-1}}{2\Delta t} = k\frac{u_{i+1}^j - u_i^{j+1} - u_i^{j-1} + u_{i-1}^j}{(\Delta x)^2}$$

Rearranging to solve for the temperature at the next time step, u_i^{j+1}, yields the explicit formula:

$$u_i^{j+1} = \frac{(1-2r)u_i^{j-1} + 2r(u_{i+1}^j + u_{i-1}^j)}{1+2r}$$

where $r = \frac{k\Delta t}{(\Delta x)^2}$ is the mesh ratio.

3.1 Stability and Startup

A significant advantage of the Dufort-Frankel method is that it is **unconditionally stable** for all positive values of r. This removes the restrictive stability constraint of the standard Forward-Time Central-Space (FTCS) method.

However, as a three-level scheme, it requires temperature values from two previous time steps (j and $j-1$) to compute the next ($j+1$). To start the simulation, we only have the initial condition at $j=0$. Therefore, a different method must be used to find the values at the first time step, $j=1$. In the provided Python code, a simple and common approach is used: a first-order forward Euler (FTCS) step is performed to approximate u_i^1:

$$u_i^1 = u_i^0 + r(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0)$$

4 Implementation and Results

The numerical simulation was implemented in Python using the `numpy` and `matplotlib` libraries. The simulation parameters were set as follows:

- Length of the rod, $L = 1.0$
- Thermal diffusivity, $k = 0.01$
- Total simulation time, $T = 10.0$
- Number of spatial points, $N_x = 51$
- Number of time steps, $N_t = 1001$

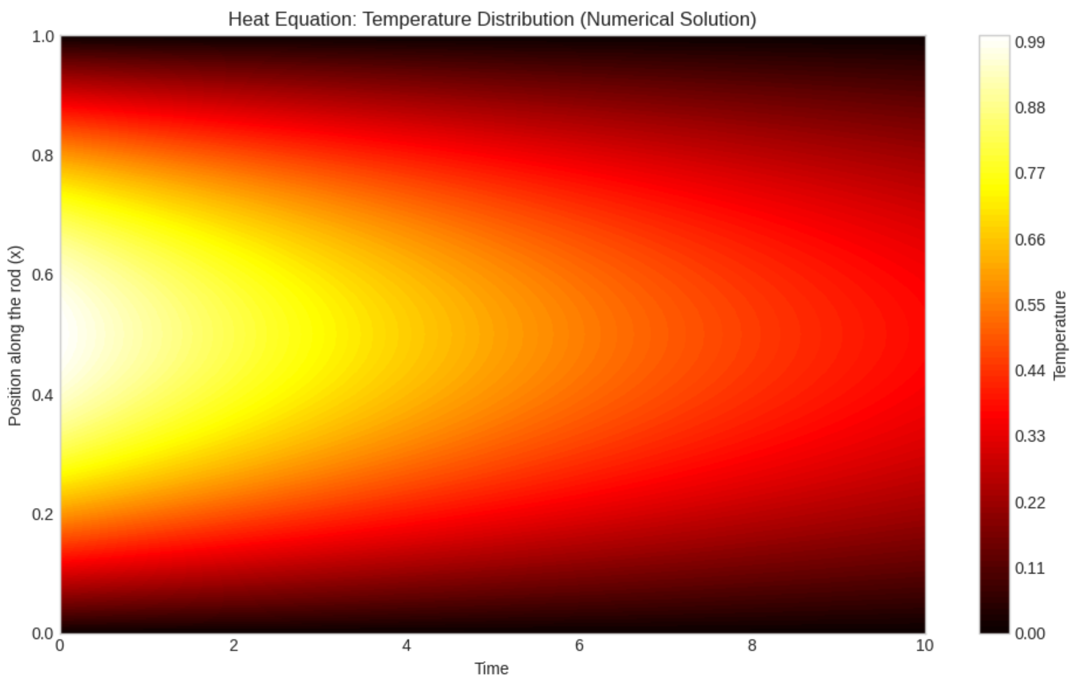

Figure 1: Contour plot of the numerical solution for temperature distribution $u(x,t)$ over time and position.

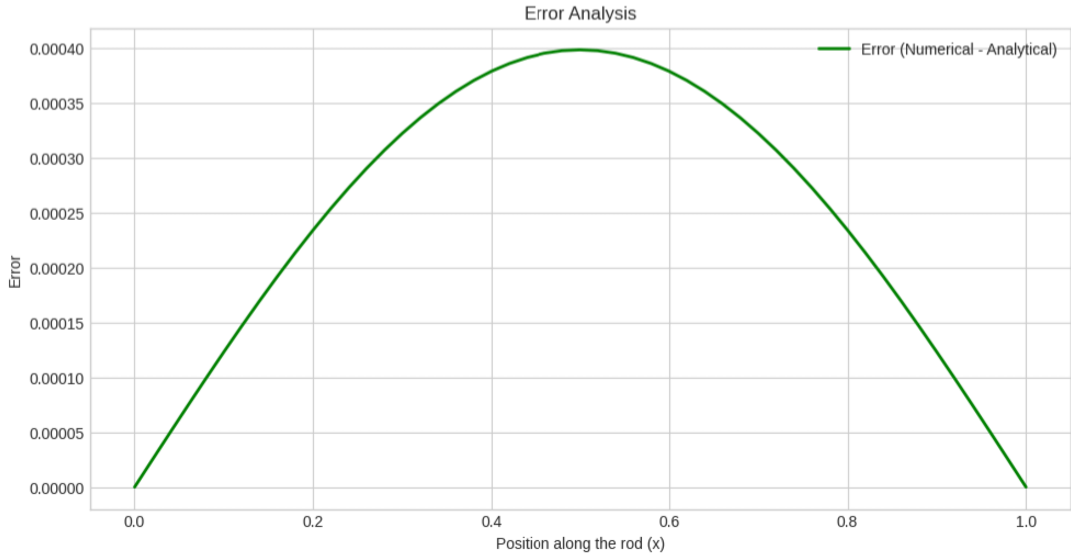

Figure 2: Error between the numerical and analytical solutions.

The results of the simulation are presented graphically. Figure 1 shows the temperature distribution over time and space as a contour plot. It clearly illustrates the diffusion of heat from the center of the rod towards the boundaries, which are held at zero temperature.

Figure 2 plots the absolute error, calculated as the difference between the numerical and analytical solutions ($u_{\text{numerical}} - u_{\text{analytical}}$) at the final time $T = 10.0$. The error is very small, on the order of 10^{-4}, which confirms the accuracy of the method.

To quantify the overall error, the Mean Absolute Error (MAE) was calculated:

$$\text{MAE} = \frac{1}{N_x} \sum_{i=0}^{N_x-1} |u_{\text{numerical},i} - u_{\text{analytical},i}|$$

The computed MAE for this simulation was approximately **0.000248**.

5 Conclusion

This project successfully implemented the Dufort-Frankel method to solve the one-dimensional heat equation. The numerical results show excellent agreement with the analytical solution, as evidenced by the graphical comparison and the very low Mean Absolute Error. The use of an unconditionally stable scheme like Dufort-Frankel is advantageous as it allows for flexibility in the choice of time and space steps without compromising the stability of the solution. The simulation effectively models the diffusion of heat in the rod, demonstrating the power of finite difference methods in solving partial differential equations.