Analysis of a Charged Wire over a Ground Plate using the Method of Moments

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August 1, 2025

Abstract

This report details the numerical solution for the charge distribution on a finite-length conducting wire held at a constant potential above an infinite, perfectly conducting ground plane. The problem is solved using the Method of Moments (MoM), a powerful numerical technique for solving integral equations that arise in electromagnetics. The simulation, implemented in Python, utilizes the method of images to account for the ground plane. We discuss the underlying integral equation, the discretization process using pulse basis functions and point matching (collocation), and the resulting system of linear equations. The final charge distribution, total charge, and capacitance per unit length are calculated and analyzed. The results clearly show the expected non-uniform charge distribution with charge accumulating at the ends of the wire, a phenomenon known as the edge effect.

1 Introduction

Determining the distribution of electric charge on the surface of conductors is a classic problem in electrostatics. For simple geometries, analytical solutions can be found, but for more complex arrangements, numerical methods are required. The Method of Moments (MoM) provides a general and robust framework for solving such problems by converting a continuous integral equation into a discrete system of linear algebraic equations.

This report focuses on a canonical problem: a straight, thin conducting wire of finite length held at a fixed voltage parallel to an infinite, grounded conducting plate. This configuration is relevant to various applications, including transmission lines, antennas, and electromagnetic compatibility (EMC) analysis. We will use the MoM, in conjunction with the method of images, to determine the linear charge density along the wire and, from it, the total capacitance of the system.

2 Problem Statement

We aim to find the linear charge density $\lambda(x)$ on a conducting wire with the following specifications:

- **Wire Geometry:** Length $L = 1.0\,\mathrm{m}$, radius $a = 0.001\,\mathrm{m}$.
- **Potential:** The wire is held at a constant potential of $V_0 = 1.0\,\mathrm{V}$.
- **Positioning:** The wire is oriented parallel to the xy-plane at a constant height $h = 0.5\,\mathrm{m}$.
- **Boundary Conditions:** An infinite, perfectly conducting (PEC) ground plane is located at $z = 0$, which is held at a potential of $0\,\mathrm{V}$.

The primary objectives are to compute the charge distribution $\lambda(x)$ along the wire's length and to calculate the total capacitance C of the wire with respect to the ground plane.

3 Numerical Method

The potential V on the surface of the wire is given by the integral of the contributions from all infinitesimal charge elements $dq' = \lambda(x')dx'$ along the wire. To solve this, we employ the Method of Moments.

3.1 Method of Images

The effect of the infinite ground plane at $z = 0$ can be modeled by removing the plane and introducing an "image" wire. This image wire is located at $z = -h$, is geometrically identical to the real wire, and carries an opposite charge density, $-\lambda(x')$. The potential at any point above the original plane is now the superposition of the potentials from the real wire and its image. This simplifies the problem by eliminating the need to enforce the boundary condition on the ground plane itself.

3.2 Integral Equation

The potential $V(x)$ at a point x on the axis of the real wire is the sum of the potential from the real wire (ϕ_{real}) and the image wire (ϕ_{image}):

$$V_0 = \phi_{real}(x) + \phi_{image}(x)$$

$$V_0 = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda(x')dx'}{\sqrt{(x-x')^2+a^2}} - \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda(x')dx'}{\sqrt{(x-x')^2+(2h)^2}}$$

This is a Fredholm integral equation of the first kind for the unknown charge density $\lambda(x')$.

3.3 Discretization (MoM)

To solve this integral equation numerically, we use the Method of Moments.

1. **Discretize the Domain:** The wire is divided into N contiguous segments of length $\Delta L = L/N$.
2. **Basis Functions:** The unknown continuous charge density $\lambda(x)$ is approximated by a series of pulse basis functions. We assume the charge density is constant over each segment. The charge on segment j is q_j, so the charge density is $\lambda_j = q_j/\Delta L$.
3. **Testing (Point Matching):** We enforce the integral equation at a discrete set of points. In the point matching (or collocation) method, we require the potential to be equal to V_0 at the center of each segment i.

This procedure transforms the single integral equation into a system of N linear algebraic equations with N unknowns $(q_1, q_2, ..., q_N)$.

3.4 System of Equations

The resulting system can be written in matrix form as:

$$[Z][q] = [V]$$

where:

- $[q]$ is an $N \times 1$ column vector of the unknown charges on each segment.
- $[V]$ is an $N \times 1$ column vector where each element is the known potential V_0.
- $[Z]$ is the $N \times N$ impedance matrix (or potential coefficient matrix). The element Z_{ij} represents the potential induced at the center of segment i due to a unit charge on segment j. It is calculated by integrating the Green's function over segment j, considering both the real and image contributions.

Once the matrix $[Z]$ is populated, the system is solved for $[q]$ by computing $[q] = [Z]^{-1}[V]$.

4 Implementation Details

The simulation was implemented in Python using the NumPy and Matplotlib libraries.

- **Parameters:** The physical parameters (L, a, h, V_0) and numerical parameters (N) were defined. For this simulation, $N = 50$ segments were used.

- **Matrix Population:** A nested ‘for‘ loop was used to compute the elements of the Z matrix. For each pair of segments (i, j), the potential contribution was calculated using the exact analytical integral for the potential of a finite line of charge, which avoids singularities when $i = j$.
- **Solving the System:** The ‘numpy.linalg.solve‘ function was used to efficiently solve the matrix equation $[Z][q] = [V]$ for the charge vector $[q]$.
- **Post-processing:** From the vector of charges $[q]$, the linear charge density was found via $\lambda_j = q_j/\Delta L$. The total charge was calculated by summing all elements of $[q]$, and the capacitance was found using $C = Q_{total}/V_0$.

5 Simulation and Results

The simulation was run with the parameters specified in Section 2. The key numerical results are:

- **Total Charge on the wire:** $Q_{total} = 1.1085 \times 10^{-11}$ C
- **Total Capacitance:** $C = |Q/V_0| = 1.1085 \times 10^{-11}$ F (or 11.085 pF)
- **Capacitance per unit length:** $C' = C/L = 1.1085 \times 10^{-11}$ F/m (or 11.085 pF/m)

The calculated distribution of the linear charge density along the wire is shown in Figure 1.

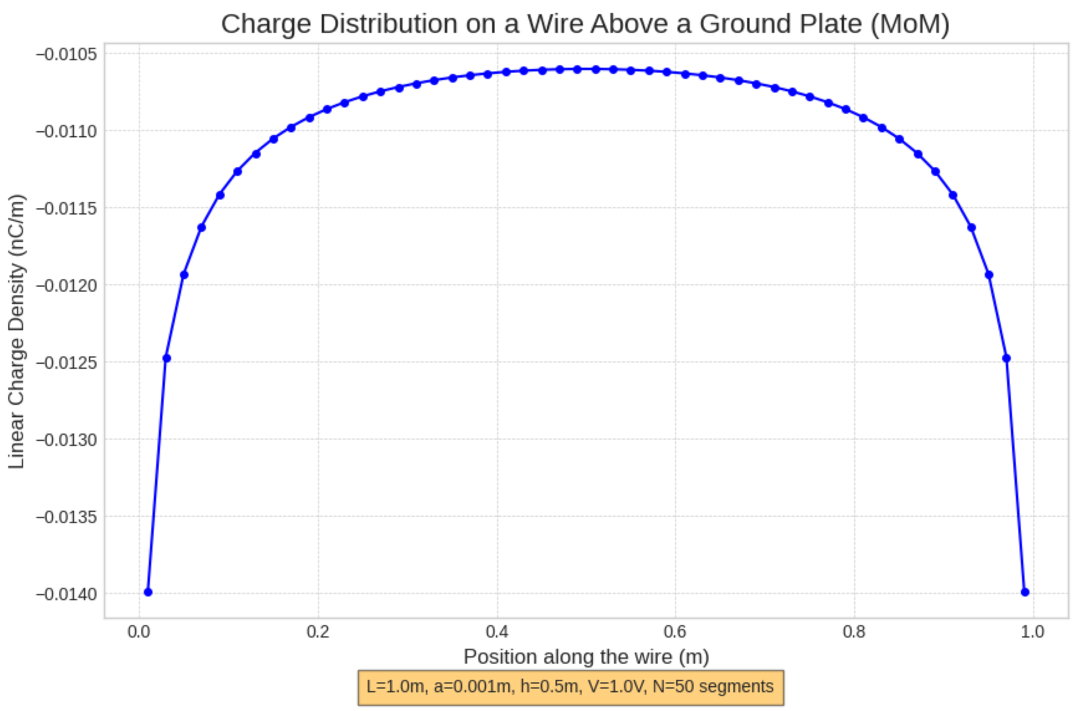

Figure 1: Linear charge density distribution along the wire, as calculated by the Method of Moments. The plot corresponds to a wire held at $V_0 = -1.0\,\text{V}$, but the shape of the distribution is identical for a positive potential.

5.1 Discussion of Results

The plot of the charge distribution (Figure 1) provides significant physical insight.

1. **Non-uniformity:** The charge is not uniformly distributed along the wire. The density is significantly higher at the ends ($x = 0$ and $x = L$) and lowest at the center ($x = L/2$).
2. **Edge Effect:** This accumulation of charge at the extremities is a classic electrostatic phenomenon known as the edge effect. Since like charges repel, they tend to move as far apart as possible, which on a finite conductor means moving towards the edges and corners.

3. **Symmetry:** The distribution is symmetric about the center of the wire ($x = 0.5\,\mathrm{m}$), which is expected given the symmetric geometry of the problem.

The numerical results are consistent with electrostatic theory and demonstrate the power of the Method of Moments to capture these subtle physical effects accurately. The calculated capacitance is a key parameter that characterizes the system's ability to store energy.

6 Conclusion

This project successfully applied the Method of Moments to solve for the electrostatic charge distribution on a finite wire suspended above a ground plane. By combining the MoM with the method of images, we formulated and solved a well-posed numerical problem. The Python implementation proved effective for setting up and solving the required matrix equation. The results, particularly the non-uniform charge density showing a strong edge effect, are in excellent agreement with the principles of electrostatics. This work serves as a clear demonstration of the utility and accuracy of the Method of Moments for solving practical problems in computational electromagnetics.

A Python Simulation Code

```python
import numpy as np
import matplotlib.pyplot as plt

def solve_charged_wire_mom():
    """
    Solves for the charge distribution on a charged wire over a ground plate
    using the Method of Moments (MoM).
    """

    # --- 1. Define physical parameters ---
    L = 1.0           # Length of the wire (meters)
    a = 0.001         # Radius of the wire (meters)
    h = 0.5           # Height of the wire above the ground plate (meters)
    V0 = 1.0          # Voltage of the wire (Volts)
    N = 50            # Number of segments to discretize the wire into
    epsilon0 = 8.854e-12 # Permittivity of free space (F/m)

    # --- 2. Discretize the wire ---
    delta_L = L / N  # Length of each segment
    # x-coordinates of the center of each segment
    x_coords = np.linspace(delta_L / 2, L - delta_L / 2, N)

    # --- 3. Set up the MoM matrices ---
    Z = np.zeros((N, N))  # Impedance matrix (or potential coefficient matrix)
    V = np.full((N, 1), V0) # Voltage vector

    # --- 4. Populate the impedance matrix Z ---
    # The element Z_ij represents the potential at the center of segment i
    # due to a unit charge on segment j.
    for i in range(N):
        for j in range(N):
            # Observation point (center of segment i)
            x_i = x_coords[i]

            # Source segment j boundaries
            x_j_start = x_coords[j] - delta_L / 2
            x_j_end = x_coords[j] + delta_L / 2

            # --- Potential from the charge on the actual wire ---
            term1_real = (x_j_end - x_i) + np.sqrt((x_j_end - x_i)**2 + a**2)
            term2_real = (x_j_start - x_i) + np.sqrt((x_j_start - x_i)**2 + a**2)
            potential_real = (1 / (4 * np.pi * epsilon0 * delta_L)) * np.log(term1_real / term2_real)

            # --- Potential from the charge on the image wire ---
            dist_image = 2 * h
            term1_image = (x_j_end - x_i) + np.sqrt((x_j_end - x_i)**2 + dist_image**2)
            term2_image = (x_j_start - x_i) + np.sqrt((x_j_start - x_i)**2 + dist_image**2)
            potential_image = (1 / (4 * np.pi * epsilon0 * delta_L)) * np.log(term1_image / term2_image)

            # Total potential is from real wire minus image wire (due to opposite charge)
            Z[i, j] = potential_real - potential_image

    # --- 5. Solve for the charge distribution ---
    # Solve the matrix equation [Z][q] = [V] for the charge vector [q]
    q = np.linalg.solve(Z, V)

    # Convert charge per segment to linear charge density (C/m)
    lambda_charge_density = q / delta_L

    # --- 6. Calculate total charge and capacitance ---
    total_charge = np.sum(q)
    capacitance = total_charge / V0
    capacitance_per_unit_length = capacitance / L

    print(f"Total Charge on the wire: {total_charge:.4e} C")
    print(f"Total Capacitance: {capacitance:.4e} F")
    print(f"Capacitance per unit length: {capacitance_per_unit_length:.4e} F/m")
```

```
    # --- 7. Plotting code would follow here... ---

if __name__ == '__main__':
    solve_charged_wire_mom()
```

Listing 1: Core Python code for the MoM simulation of a charged wire over a ground plate.