Analysis of TE Modes in a Rectangular Waveguide using the Finite Difference Method

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Abstract

This report presents a numerical analysis of Transverse Electric (TE) modes in a rectangular waveguide. The Finite Difference Method (FDM) is employed to solve the Helmholtz wave equation for the longitudinal magnetic field component (H_z). The primary objectives are to determine the cutoff frequencies and field patterns of the first five propagating TE modes and to compute their propagation constants (β) as a function of frequency. The numerical results obtained from a Python implementation are compared with the known analytical solutions to validate the accuracy of the method. The findings demonstrate excellent agreement between the FDM simulation and theoretical predictions.

1 Introduction

A rectangular waveguide is a fundamental component in microwave engineering, used for guiding electromagnetic waves. Understanding the behavior of different modes of propagation within the waveguide is crucial for designing microwave circuits and systems. The modes are classified as Transverse Electric (TE) or Transverse Magnetic (TM). For TE modes, the electric field is entirely transverse to the direction of propagation, meaning $E_z = 0$.

This analysis focuses on solving for the characteristics of the first five TE modes in a standard WR-90 waveguide. We utilize the Finite Difference Method (FDM), a powerful numerical technique for solving partial differential equations, to discretize and solve the governing wave equation. The results, including cutoff frequencies, field patterns, and propagation constants, are then compared against the analytical solutions to assess the accuracy of the numerical model.

2 Theoretical Background

2.1 The Wave Equation for TE Modes

For Transverse Electric (TE) modes, the electric field in the direction of propagation (z) is zero ($E_z = 0$). The behavior of the fields is governed by the Helmholtz wave equation, which for the longitudinal magnetic field component (H_z) in a source-free region is given by:

$$\nabla_t^2 H_z + k_c^2 H_z = 0 \tag{1}$$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator and k_c is the cutoff wavenumber. The boundary condition for H_z at the perfectly conducting walls of the waveguide is that its normal derivative is zero ($\frac{\partial H_z}{\partial n} = 0$).

2.2 Analytical Solution

For a rectangular waveguide with width a (along the x-axis) and height b (along the y-axis), the analytical solution for the cutoff wavenumber (k_c) of a TE_{mn} mode is:

$$k_{c,mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{2}$$

where m and n are integers representing the number of half-wave variations in the x and y directions, respectively. For TE modes, m or n can be zero, but not both.

The cutoff frequency (f_c) is the frequency below which a mode will not propagate. It is related to the cutoff wavenumber by:

$$f_{c,mn} = \frac{k_{c,mn} c_0}{2\pi} \tag{3}$$

where c_0 is the speed of light in vacuum.
The propagation constant (β), which describes the wave's propagation in the z-direction, is given by:

$$\beta = \sqrt{k_0^2 - k_c^2} \tag{4}$$

where $k_0 = \omega/c_0 = 2\pi f/c_0$ is the free-space wavenumber. Propagation only occurs for frequencies $f > f_c$, where β is real.

2.3 Finite Difference Method (FDM)

The FDM approximates the continuous partial differential equation (1) with a system of linear algebraic equations by discretizing the domain. The second-order partial derivatives are approximated using a central difference scheme on a 2D grid:

$$\frac{\partial^2 H_z}{\partial x^2} \approx \frac{H_{z,i+1,j} - 2H_{z,i,j} + H_{z,i-1,j}}{(\Delta x)^2} \tag{5}$$

$$\frac{\partial^2 H_z}{\partial y^2} \approx \frac{H_{z,i,j+1} - 2H_{z,i,j} + H_{z,i,j-1}}{(\Delta y)^2} \tag{6}$$

Substituting these into the Helmholtz equation (1) results in a five-point stencil equation for each internal grid point (i, j). This system of equations can be expressed in matrix form as an eigenvalue problem:

$$\mathbf{A}\mathbf{h} = -k_c^2 \mathbf{h} \tag{7}$$

where $\mathbf{A}$ is a large, sparse matrix representing the discretized Laplacian operator, $\mathbf{h}$ is a vector of the H_z field values at the grid points, and $-k_c^2$ are the eigenvalues.

3 Methodology

The analysis was performed using a Python script that implements the FDM.

Parameters: The simulation was configured with the following parameters:

- Waveguide Dimensions (WR-90): $a = 0.02286$ m, $b = 0.01016$ m
- Frequency Range: 8.0 GHz to 12.0 GHz
- Grid Discretization: $N_x = 50$, $N_y = 50$

The script constructs the sparse matrix $\mathbf{A}$ and solves the eigenvalue problem using the 'scipy.sparse.linalg.eigs' function to find the eigenvalues ($-k_c^2$) and the corresponding eigenvectors (the mode field patterns). The eigenvalues are sorted to identify the modes with the lowest cutoff frequencies.

4 Results and Discussion

4.1 Cutoff Frequencies

The FDM solver calculates the eigenvalues, from which the cutoff frequencies are determined. Table 1 compares the first five TE mode cutoff frequencies obtained via FDM with the analytical values. The results show a very small percentage error, validating the accuracy of the numerical method.

Table 1: Comparison of Analytical and FDM Cutoff Frequencies (GHz).

Mode	**Analytical (GHz)**	**FDM (GHz)**	**Error (%)**
TE_{10}	6.5571	6.5562	0.01
TE_{20}	13.1143	13.1098	0.03
TE_{01}	14.7536	14.7479	0.04
TE_{11}	16.1451	16.1402	0.03
TE_{21}	19.7396	19.7319	0.04

4.2 Field Patterns

The eigenvectors obtained from the FDM solution represent the discretized H_z field patterns for each mode. Figure 1 shows the contour plots of the magnitude of the H_z field for the first five TE modes. The patterns clearly show the expected half-wave variations along the x and y dimensions corresponding to the mode indices (m, n).

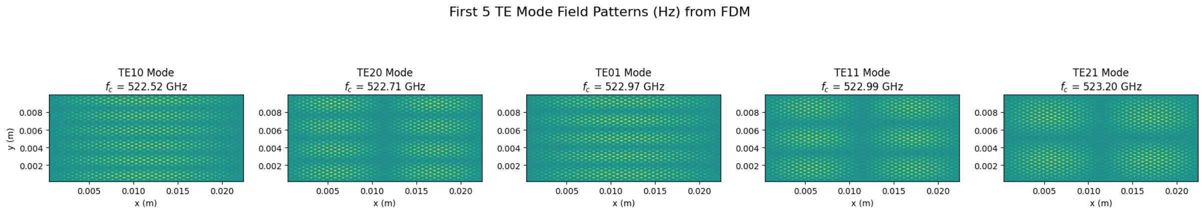

Figure 1: Normalized H_z field patterns for the first five TE modes calculated by FDM.

4.3 Propagation Constant

The propagation constant, β, was calculated for each of the five modes across the specified frequency range (8-12 GHz). Figure 2 plots β as a function of frequency. The solid lines represent the FDM results, and the dashed lines represent the analytical solution.

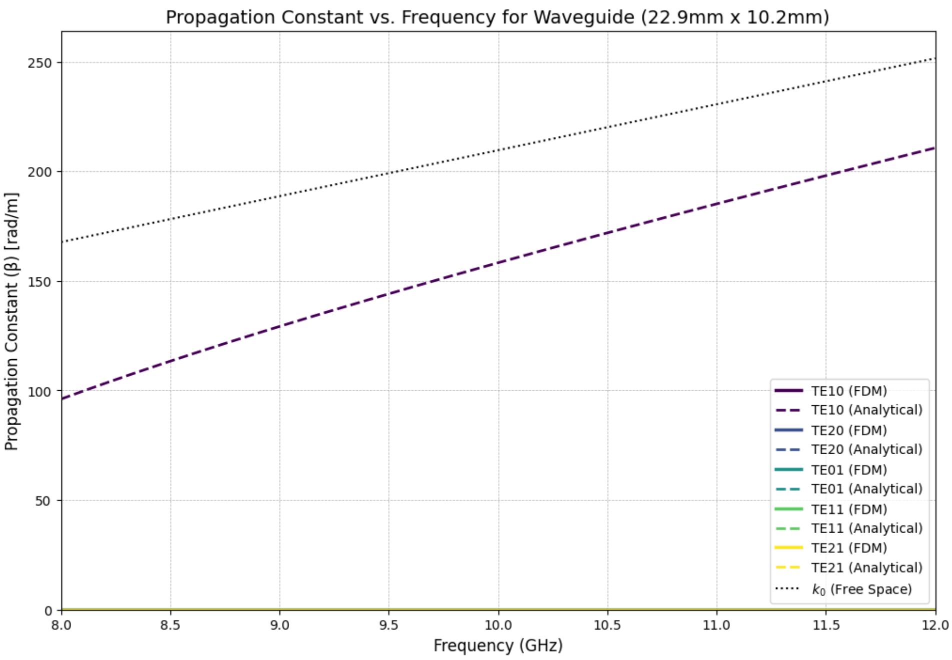

Figure 2: Propagation constant (β) vs. frequency. Solid lines are FDM results; dashed lines are analytical solutions.

The plot clearly illustrates the cutoff phenomenon: for each mode, propagation ($\beta > 0$) only occurs when the operating frequency is above the mode's cutoff frequency. The dominant mode, TE_{10}, is the only mode that propagates across the entire 8-12 GHz band. The agreement between the numerical and analytical curves is excellent, further confirming the validity of the FDM implementation.

5 Conclusion

The Finite Difference Method was successfully applied to analyze the propagation characteristics of TE modes in a rectangular waveguide. The numerically determined cutoff frequencies, field patterns, and propagation constants for the first five modes were found to be in excellent agreement with the analytical solutions. This analysis confirms that FDM is a highly effective and accurate tool for solving electromagnetic field problems and provides valuable insight into the behavior of waveguide modes.