Assignment 4

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Question 5

The following rules will be applied for finding relevant variables.

- The query node and the evidence nodes are relevant to the query.
- All ancestors of relevant nodes are relevant to the query.
- All other nodes are irrelevant to the query.

Part (a)

Show that $P(Z_0|X_0) = \frac{P(X_0|Z_0)P(Z_0)}{P(X_0)}$

$$\textit{Conditional probability: } P(Z_0|X_0) = \frac{P(Z_0, X_0)}{P(X_0)}$$

The relevant nodes to the query are Z_0 and X_0.

$$P(Z_0, X_0) = \sum P(Z_0, X_0) = P(X_0|Z_0)P(Z_0)$$

$$\Rightarrow P(Z_0|X_0) = \frac{P(Z_0, X_0)}{P(X_0)} = \frac{P(X_0|Z_0)P(Z_0)}{P(X_0)}$$

Part (b)

Show that $P(X_0) = \sum_{Z_0} P(X_0|Z_0)P(Z_0)$

The relevant nodes to the query are Z_0 and X_0. Then we sum over nodes to omit them from query.

$$\xrightarrow{part(a)} P(X_0) = \sum_{Z_0} P(X_0, Z_0) = \sum_{Z_0} P(X_0|Z_0)P(Z_0)$$

Part (c)

Show that $P(Z_1|X_1, X_0) = \frac{P(X_1|Z_1)P(X_0)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)}{P(X_1,X_0)}$

$$\textit{Conditional probability: } P(Z_1|X_1, X_0) = \frac{P(Z_1, X_1, X_0)}{P(X_1, X_0)}$$

The relevant nodes to the query are [Z_0, X_0, Z_1, X_1].

$$\begin{aligned}
P(Z_1, X_1, X_0) &= \sum_{Z_0} P(Z_1, Z_0, X_1, X_0) \\
&= \sum_{Z_0} P(X_1|Z_1)P(Z_1|Z_0)P(Z_0)P(X_0|Z_0) \\
&= P(X_1|Z_1) \sum_{Z_0} P(Z_1|Z_0)P(Z_0)P(X_0|Z_0) \\
\xrightarrow{part(a)} &= P(X_1|Z_1) \sum_{Z_0} P(Z_1|Z_0)[P(Z_0)P(X_0|Z_0)] \\
\xrightarrow{P(Z_0)P(X_0|Z_0)=P(X_0)P(Z_0|X_0)} &= P(X_1|Z_1) \sum_{Z_0} P(Z_1|Z_0)P(X_0)P(Z_0|X_0) \\
&= P(X_1|Z_1)P(X_0) \sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow P(Z_1|X_1, X_0) &= \frac{P(Z_1, X_1, X_0)}{P(X_1, X_0)} \\
&= \frac{P(X_1|Z_1)P(X_0)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)}{P(X_1, X_0)}
\end{aligned}$$

Part (d)

Show that $P(X_1, X_0) = P(X_0)\sum_{Z_1} P(X_1|Z_1)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)$

$$\xrightarrow{part(c)} P(Z_1, X_1, X_0) = P(X_1|Z_1)P(X_0) \sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)$$

$$\begin{aligned}
P(X_1, X_0) &= \sum_{Z_1} P(Z_1, X_1, X_0) \\
&= \sum_{Z_1} P(X_1|Z_1)P(X_0) \sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0) \\
&= P(X_0) \sum_{Z_1} P(X_1|Z_1) \sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)
\end{aligned}$$

Part (e)

Show that $P(Z_2|X_2, X_1, X_0) = \frac{P(X_2|Z_2)P(X_1,X_0)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|X_1,X_0)}{P(X_2,X_1,X_0)}$

$$\textit{Conditional probability: } P(Z_2|X_2, X_1, X_0) = \frac{P(Z_2, X_2, X_1, X_0)}{P(X_2, X_1, X_0)}$$

The relevant nodes to the query are $[Z_0,\ X_0,\ Z_1,\ X_1,\ Z_2,\ X_2]$.

$$\begin{aligned}
P(Z_2, X_2, X_1, X_0) &= \sum_{Z_1, Z_0} P(Z_2, Z_1, Z_0, X_2, X_1, X_0) \\
&= \sum_{Z_1, Z_0} P(X_2|Z_2)P(Z_2|Z_1)P(Z_1|Z_0)P(X_1|Z_1)P(X_0|Z_0)P(Z_0) \\
&= \sum_{Z_1}\sum_{Z_0} P(X_2|Z_2)P(Z_2|Z_1)P(Z_1|Z_0)P(X_1|Z_1)P(X_0|Z_0)P(Z_0) \\
&= P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|Z_0)P(X_1|Z_1)\sum_{Z_0} P(Z_0)P(X_0|Z_0) \\
&\xrightarrow{part(a)} = P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|Z_0)P(X_1|Z_1)\sum_{Z_0}[P(Z_0)P(X_0|Z_0)] \\
&\xrightarrow{P(Z_0)P(X_0|Z_0)=P(X_0)P(Z_0|X_0)} = P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|Z_0)P(X_1|Z_1)\sum_{Z_0} P(X_0)P(Z_0|X_0) \\
&= P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)P(X_1|Z_1)P(X_0)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0) \\
&\xrightarrow{part(c)} = P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)[P(X_1|Z_1)P(X_0)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)] \\
&= P(X_2|Z_2)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|X_1, X_0)P(X_1, X_0) \\
&= P(X_2|Z_2)P(X_1, X_0)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|X_1, X_0)
\end{aligned}$$

$$\begin{aligned}
P(Z_2|X_2, X_1, X_0) &= \frac{P(Z_2, X_2, X_1, X_0)}{P(X_2, X_1, X_0)} \\
&= \frac{P(X_2|Z_2)P(X_1, X_0)\sum_{Z_1} P(Z_2|Z_1)P(Z_1|X_1, X_0)}{P(X_2, X_1, X_0)}
\end{aligned}$$

Question 6

The rules mentioned in question 5 will be used for finding relevant variables.

Part (a)

Show that $P(X_2|Z_1) = \sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)$

The relevant nodes to the query are $[Z_0,\ Z_1,\ Z_2,\ X_2]$.

$$\begin{aligned}
P(X_2|Z_1) &= \frac{P(X_2, Z_1)}{P(Z_1)} \\
&= \frac{\sum_{Z_2,Z_0} P(X_2, Z_2, Z_1, Z_0)}{\sum_{Z_0} P(Z_1, Z_0)} \\
&= \frac{\sum_{Z_2}\sum_{Z_0} P(X_2|Z_2)P(Z_2|Z_1)P(Z_1|Z_0)P(Z_0)}{\sum_{Z_0} P(Z_1|Z_0)P(Z_0)} \\
&= \frac{\sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)\sum_{Z_0} P(Z_1|Z_0)P(Z_0)}{\sum_{Z_0} P(Z_1|Z_0)P(Z_0)} \\
&= \frac{[\sum_{Z_0} P(Z_1|Z_0)P(Z_0)]\sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)}{\sum_{Z_0} P(Z_1|Z_0)P(Z_0)} \\
&= \sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)
\end{aligned}$$

Part (b)

Show that $P(X_2, X_1|Z_0) = \sum_{Z_1} P(X_2|Z_1)P(X_1|Z_1)P(Z_1|Z_0)$

The relevant nodes to the query are $[Z_0,\ Z_1,\ Z_2,\ X_1,\ X_2]$.

$$\begin{aligned}
P(X_2, X_1|Z_0) &= \frac{P(X_2, X_1, Z_0)}{P(Z_0)} \\
&= \frac{\sum_{Z_2,Z_1,Z_0,X_2,X_1} P(X_2, X_1, Z_0)}{P(Z_0)} \\
&= \frac{\sum_{Z_2}\sum_{Z_1} P(X_2|Z_2)P(Z_2|Z_1)P(Z_1|Z_0)P(Z_0)P(X_1|Z_1)}{P(Z_0)} \\
&= \frac{P(Z_0)\sum_{Z_1}(Z_1|Z_0)P(X_1|Z_1)\sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)}{P(Z_0)} \\
&= \sum_{Z_1}(Z_1|Z_0)P(X_1|Z_1)[\sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)] \\
\xrightarrow{part(a)} &= \sum_{Z_1}(Z_1|Z_0)P(X_1|Z_1)P(X_2|Z_1) = \sum_{Z_1} P(X_2|Z_1)P(X_1|Z_1)P(Z_1|Z_0)
\end{aligned}$$

Part (c)

Show that $P(Z_1|X_0, X_1, X_2) = \frac{P(X_2|Z_1)P(X_1,X_0)P(Z_1|X_1,X_0)}{P(X_0,X_1,X_2)}$

The relevant nodes to the query are [Z_0, Z_1, Z_2, X_0,X_1, X_2].

$$\textit{Conditional probability: } P(Z_1|X_0, X_1, X_2) = \frac{P(Z_1, X_0, X_1, X_2)}{P(X_0, X_1, X_2)}$$

$$\begin{aligned}
P(Z_1, X_0, X_1, X_2) &= \sum_{Z_0,Z_2} P(Z_0, Z_1, Z_2, X_0, X_1, X_2) \\
&= \sum_{Z_0}\sum_{Z_2} P(X_2|Z_2)P(X_1|Z_1)P(X_0|Z_0)P(Z_2|Z_1)P(Z_1|Z_0)P(Z_0) \\
&\xrightarrow{part(a)} = P(X_1|Z_1)\sum_{Z_0} P(X_0|Z_0)P(Z_1|Z_0)P(Z_0)[\sum_{Z_2} P(X_2|Z_2)P(Z_2|Z_1)] \\
&= P(X_1|Z_1)\sum_{Z_0} P(X_0|Z_0)P(Z_1|Z_0)P(Z_0)P(X_2|Z_1) \\
&= P(X_2|Z_1)P(X_1|Z_1)\sum_{Z_0} P(Z_1|Z_0)P(Z_0)P(X_0|Z_0) \\
&\xrightarrow{Bayes\ rule} = P(X_2|Z_1)P(X_1|Z_1)\sum_{Z_0} P(Z_1|Z_0)P(Z_0)\frac{P(Z_0|X_0)P(X_0)}{P(Z_0)} \\
&= P(X_2|Z_1)P(X_1|Z_1)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)P(X_0) \\
&= P(X_2|Z_1)[P(X_1|Z_1)P(X_0)\sum_{Z_0} P(Z_1|Z_0)P(Z_0|X_0)] \\
&\xrightarrow{Q5\ part(c)} = P(X_2|Z_1)P(Z_1|X_1, X_0)P(X_1, X_0) \\
&= P(X_2|Z_1)P(X_1, X_0)P(Z_1|X_1, X_0)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow P(Z_1|X_0, X_1, X_2) &= \frac{P(Z_1, X_0, X_1, X_2)}{P(X_0, X_1, X_2)} \\
&= \frac{P(X_2|Z_1)P(X_1, X_0)P(Z_1|X_1, X_0)}{P(X_0, X_1, X_2)}
\end{aligned}$$