PROBLEM SET #4

For the problems below, you need to write programs using Python. Turn in your own source programs written independently together with a report in pdf format by email (both to the instructor at wkim@astro.snu.ac.kr and the TA at moon@astro.snu.ac.kr.)

1. Use a Monte-Carlo method to estimate the volume of a d-dimensional sphere with radius of unity for $d = 3$, 4, and 5, by taking 10^7 particles. Repeat the calculations 100 times and give the mean values and standard deviations. Discuss your results compared to the true value of $V_d = \pi^{d/2}/\Gamma(d/2+1)$.

2. Suppose a photon incident on the bottom of a plane-parallel slab with optical depth $\tau_{\rm max}$. We assume that the slab is infinite in the x and y directions: z-axis is normal to the slab. The photon can be scattered (with no absorption) at any point within the slab. We begin with $z = 0$ (bottom of the slab) and follow the photon's trajectories up to $z_{\rm max} = 1$ (top of the slab) by taking following steps:

Step 1: We inject a photon from below whose flux is isotropic in any direction. The probability for a certain injection angle at $z = 0$ (with respect to the z-axis), is given by $p(\mu)d\mu = 2\mu d\mu$ with $0 \le \mu \le 1$, where $\mu \equiv \cos\theta$. Use a uniform deviate ξ_1 to sample $\mu = \sqrt{\xi_1}$. Use another uniform deviate ξ_2 to obtain $\phi = 2\pi\xi_2$. Calculate the initial direction of the photon $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Note that the initial position of the photon is $(x, y, z) = (0, 0, 0)$.

Step 2: When the optical depth of a photon is τ, the probability P that a photon interacts with the medium is given by $P = 1 - e^{-\tau}$. Noting that P is random over [0, 1), τ can be sampled from a uniform deviate ξ_3 as $\tau = -\ln(\xi_3)$. Pick up a value for τ.

Step 3: The distance traveled by a photon along a ray is given by $L = \tau z_{\rm max}/\tau_{\rm max}$. Update the photon's new position as $x = x + L\sin\theta\cos\phi$, $y = y + L\sin\theta\sin\phi$, and $z = z + L\cos\theta$.

Step 4: If $z < 0$, add one to the number of reflected photons. If $z > z_{\rm max}$, add one to the number of transmitted photons. In either case skip to Step 7 below.

Step 5: Assume that the photons are scattered uniformly into 4π steradians. Generate the new direction by sampling uniformly for ϕ in the range 0 to 2π and μ in the range -1 to 1: $\phi = 2\pi\xi_4$ and $\mu = 2\xi_5 - 1$, where ξ_4 and ξ_5 are two independent uniform deviates in [0, 1).

Step 6: Repeat Steps 2–5 until the fate of the photon has been determined.

Step 7: Repeat Steps 1–6 with additional incident photons until sufficient data has been obtained.

(a) Calculate the transmission and reflection probabilities for $\tau_{\rm max} = 0.01$, 0.1, 1, and 10. Begin with 10^3 incident photons and increase this number until satisfactory statistics are obtained. Give a qualitative explanation of your results.

(b) Draw some typical paths of photons in the x–z plane.

3. *The Bondi or Parker Wind Problem*: The Bondi (stellar wind) problem involves the spherical accretion (outflow) of gas by (from) a gravitating point mass M. In spherical symmetry, the steady flow of an isothermal gas with density ρ and velocity v obeys the following equations

$$\left(u - \frac{1}{u}\right)\frac{du}{dx} = \frac{2}{x} - \frac{1}{x^2}, \tag{1}$$

where u and x are the dimensionless velocity and distance from the point mass.

(a) Solve Equation (1) inward starting from $x = 5$ to $x = 0.1$ for three values of $u = 3$, 0.1, and 0.01 at $x = 5$, and plot the results on the u–x plane. You will see that the case with $u(x = 5) = 0.1$ does not give the solution you want. Why does this happen? How can you circumvent this problem?

(b) Now you want to obtain the solutions with a sonic transition (that is, solutions with $u = 1$ at some point). In view of Equation (1), $u = 1$ should occur at $x = 1/2$ for regular solutions. Show that the transonic solutions should satisfy

$$\Delta u = \pm 2\Delta x, \tag{2}$$

where Δu and Δx denote the small changes in u and x around 1 and 1/2, respectively. (That is, $x = 0.5 + \Delta x$ and $u = 1 + \Delta u$ for $|\Delta x|, |\Delta u| \ll 1$.) The minus sign in Equation (2) corresponds to the Bondi accretion, while the plus sign is for the isothermal Parker winds.

(c) Draw the transonic solutions for winds and accretion on the u–x plane, with x in the range between 0.1 and 5.

4. *Lane-Emden equation*: A polytrope refers to a self-gravitating spherical object in hydrostatic equilibrium. Its density distribution is described by $\Theta(\xi)$ that satisfies the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n(\xi), \tag{3}$$

where ξ denotes the dimensionless radius and n is the polytropic index. The proper boundary conditions are $\Theta = 1$ and $d\Theta/d\xi = 0$ at $\xi = 0$.

(a) One can seek for a power series solution

$$\Theta(\xi) = \sum_{m=0}^{\infty} a_m \xi^m \tag{4}$$

of Equation (3) for arbitrary index n, with a_m's being coefficients. Find the values of a_0, a_1, a_2, a_3, and a_4 in terms of n.

(b) Write a program to solve Equation (3) for arbitrary n. Check your codes for $n = 0, 1, 5$ by comparing your results with the analytic solutions:

$$\Theta_0 = 1 - \frac{1}{6}\xi^2, \qquad \Theta_1 = \frac{\sin\xi}{\xi}, \qquad \Theta_5 = \left(1 + \frac{1}{3}\xi^2\right)^{-1/2}, \tag{5}$$

(c) Make a table that shows the values of the first zeros, ξ_1, and $d\Theta/d\xi|_{\xi_1}$ $n = 0$, 0.5, 1.0, 1.5, $\cdots$, 4.0, 4.5, 4.9. Your answers should be accurate to 6 digits at least.