Final-Exam

This exam is taken home and 48 hours long in duration. You may use books, class notes, or even internet. However, please work independently; you should not discuss with other people except the instructor. Write programs using Python. Make sure to turn in your own source programs together with a report in pdf format by email to wkim@astro.snu.ac.kr before the due date. There will be 30% grade deduction per hour for late submission.

1. (10 Points) Find three smallest roots of

$$\int_0^x \frac{\sin t}{t} dt = \frac{3}{2}. \tag{1}$$

Give your answer correct to five significant digits.

2. (20 Points) Consider the following differential equation:

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x - A\sin(t) = 0, \tag{2}$$

for constants μ and A.

(a) By fixing $A = 0$, solve Equation (2) from $t = 0$ to 102.3 subject to $x(0) = 2$ and $dx/dt(0) = 0$ for $\mu = 0.1$, 1, and 4. Make plots for x *vs.* t and dx/dt *vs.* x.

(b) Repeat Part (a) for $A = 1.2$.

(c) Let $\{x_i\}$ (with $i = 0, \cdots, 1023$) denote the values of x in Part (a) at $t = i\Delta t$ with $\Delta t = 0.1$. Perform the Fourier transform of $\{x_i\}$ and plot the resulting power spectra. Find the frequencies of the modes with the first to third largest power.

(d) Repeat Part (c) for $A = 1.2$.

3. (30 Points) This is an extension of Problem 2 in HW Set #4 to calculate the angular dependence of emergent radiation intensity (i.e., limb darkening). In Step 4 and Step 7 of Problem 2 in HW Set #4, include the following substeps:

Step 4b: For each emergent photon with $z > z_{\max}$, record its emergent direction, $\mu = \cos\theta$. (Due to symmetry, you do not have to consider angular dependence in the ϕ-direction.)

Step 7b: Place all emergent photons in 10 uniform bins in μ with a bin width $\Delta\mu = 0.1$ in the range of [0,1]. Let n_i denote the number of photons in the i-th bin. Then the fractional emergent *energy* of photons in the angle between μ_i and $\mu_i + \Delta\mu$ is given by $E_i = n_i/n_{\rm tot}$, where $n_{\rm tot} = \sum_{i=0}^{9} n_i$ is the total number of emergent photons. The normalized emergent intensity is given by $I_i = E_i/(2\mu_i\Delta\mu)$.

After Step 7b, plot I_i as a function of θ_i and compare it with the prediction $I(\theta) = (2 + 3\cos\theta)/5$ of the gray, LTE atmosphere under the Eddington approximation. Discuss physically why I is a decreasing function of θ.

4. **(30 Points)** Write your *own* code for optimization based on the Powell's Method that should *not* rely on the `scipy` library. (Only partial points will be given if you use the `scipy` library.)

   (a) Use your code to minimize Beale's function

$$f(x, y) = (1.5 - x + xy)^2 + (2.25 - x + xy^2)^2 + (2.625 - x + xy^3)^2 \tag{3}$$

   stating from $(x, y) = (2, -4)$, $(0, 0)$, $(3, 4)$, and $(-3, -2)$, and plot the trajectories reaching to the respective minimum on top of the contours of Beale's function.

   (b) The `fakedata1.dat` file in the class web page contains three-column data: the first two columns give (x_i, y_i), while the last column is σ_i representing the error in y. These data can be fitted by a Gaussian function $y = a_1 \exp[-(x - a_2)^2/a_3^2]$ with three parameters a_1, a_2, and a_3. Use `scipy.odr` to find the best-fit parameters a_1, a_2, and a_3.

   (c) For the data given in Part (b) above, use your own optimization code with Powell's method to find a_1, a_2, and a_3 that minimize

$$\chi^2 = \sum_i^N \left[\frac{y_i - y(x_i; a_1, a_2, a_3)}{\sigma_i}\right]^2, \tag{4}$$

   and compare your results with those of Part (b).

5. **(40 Points)** Consider a sphere with the density profile

$$\rho(r) = \begin{cases} \rho_0, & \text{for} \quad r < r_0, \\ \rho_0 (r_0/r)^2, & \text{for } r_0 < r < r_e, \\ 0, & \text{for} \quad r > r_e, \end{cases} \tag{5}$$

   where ρ_0 is the core density, r_0 is the core radius, and r_e is the edge radius. Find ρ_0 that makes the total mass $M(r_e)$ within r_e equal to unity. Take $M(r_e) = 1$, $r_0 = 1$, and $r_e = 4$ for the problems below.

   (a) Realize the sphere described by Equation (5) using $N = 10^3$ random particles. Calculate the (dimensionless) gravitational potential energy V, moment of inertia I, and center of mass $\mathbf{r}_{\rm CM}$. From your particle distribution, calculate $\rho(r)$ and compare your result with Equation (5).

   (b) Repeat Part (a) with $N = 5 \times 10^3$ particles.

   (c) After setting the initial velocities of each particle to zero, evolve the sphere constructed with $N = 300$ to $t = 5$ by taking $dt = 0.01$. Plot the temporal changes of V, K, and the total energy $E = V + K$.