PROBLEM SET #5

For the problems below, you need to write programs using Python. Turn in your own source programs written independently together with a report in pdf format by email (both to the instructor at wkim@astro.snu.ac.kr and the TA at moon@astro.snu.ac.kr.)

1. *Leap-frog Integrator*: Consider a binary system consisting of two stars with masses $m_1 = 1$ and $m_2 = 0.5$ placed in the x-y plane. Initially ($t = 0$), the two stars are located at $\mathbf{r}_1 = (x_1, y_1) = (-0.5, 0)$ and $\mathbf{r}_2 = (1, 0)$, and have velocities of $\mathbf{v}_1 = (0.01, 0.05)$ and $\mathbf{v}_2 = (0.02, 0.2)$. The two stars orbit with each other due to the mutual gravity. The relevant equation of motion is

$$\ddot{\mathbf{r}}_i = -m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \quad (i \neq j), \tag{1}$$

for $i, j = 1$ or 2.

(a) Integrate Equation (1) from $t = 0$ to $t = 50$ using the Leap-frog scheme, and plot the orbits of the two stars in the x-y plane. You need to choose a small enough dt for accurate orbit calculations.

(b) Indicate the motion of the center of mass of the binary in the figure you draw in Part (a).

(c) Plot the angular momentum $\mathbf{L}$ and the total energy E defined by

$$\mathbf{L} = m_1 \mathbf{r}_1 \times \mathbf{v}_1 + m_2 \mathbf{r}_2 \times \mathbf{v}_2 \tag{2}$$

and

$$E = \frac{1}{2} m_1 \mathbf{v}_1^2 + \frac{1}{2} m_2 \mathbf{v}_2^2 - \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{3}$$

over $t = 0 - 1000$. Comment on the accuracy of the calculated orbits in terms of the energy conservation.

2. Consider the following differential equation

$$y'' + [\lambda - 2Q\cos(2x)]y = 0, \quad \text{over} \quad 0 \leq x \leq \pi, \tag{4}$$

subject to the boudnary conditions $y(0) = 1$ and $y'(0) = y'(\pi) = 0$. Here, λ and Q are constants.

(a) Solve Equation (4) for $Q = 5$ and find the corresponding eigenvalue λ. Plot the resulting y.

(b) Repeat the calculations for $0 \leq Q \leq 10$, and plot the resulting λ as a function of Q.

3. Solve the following tridiagonal system

$$u_{i-1} - 2u_i + u_{i+1} = -\left(\frac{2\pi}{n}\right)^2 \cos\left(\frac{2\pi i}{n}\right), \quad \text{for} \quad i = 1, 2, \cdots, n, \tag{5}$$

for $n = 256$, using

(a) the Gauss-Jordan elimination method, and

(b) the forward and backward substitutions method.

Compare your results with the solution $u(x) = \cos(x) + constant$ over $0 \le x \le 2\pi$.

4. Consider a 6-level model of the N^+ (which produces the [N II] lines). The statistical equilibrium between transitions into and out of any level i requires

$$n_i \left[n_e \sum_{k \neq i} q_{ik} + \sum_{k<i} A_{ik} \right] = n_e \sum_{k \neq i} n_k q_{ki} + \sum_{k>i} n_k A_{ki}. \tag{6}$$

This leads to six equations in the six unknowns $n_1, \cdots, n_6$, but it can be shown that this system is *degenerate*, i.e., any one of the equations can be constructed as a linear combination of the other five. We thus introduce another equation, which is the equation of normalization: $\sum_{i=1}^{6} n_i = 1$. This plus any five of the others may then be solved for the relative populations n_i.

In this problem, you are asked to develop a computer routine to solve for n_i of the N^+ ions. The required values of the atomic parameters are statistical weights ω_i, differences between the energy levels E_{ij}, Einstein A-values A_{ji}, and collisional parameters Ω_{ji}, which are given in Figure 1 and Table 1. The temperature dependence of the collision parameter Ω_{ji} can be fit to a power law: $\Omega_{ji}(t) = t^{\beta_{ji}} \Omega_{ji}(1.0)$, where the temperature is $t \equiv T/(10^4 \text{ K})$. The rate of downward collisions $j \to i$ is given by

$$q_{ji} = \frac{8.629 \times 10^{-8}}{t^{1/2}} \frac{\Omega_{ji}(t)}{\omega_j} \text{ cm}^3 \text{ s}^{-1}, \quad (j > i), \tag{7}$$

while the rate of upward collisions can be found from

$$q_{ij} = \frac{\omega_j}{\omega_i} q_{ji} e^{-1.1605 E_{ij}/t}, \tag{8}$$

where the energy separation E_{ij} is in eV.

(a) Write a computer program to calculate the populations for any given n_e, T pair.

(b) Find the populations for the following values of n_e and T: $T = 10^4$ K and $n_e = 10, 10^2, 10^3, 10^4, 10^5$ cm^{-3}; also $n_e = 10^4$ cm^{-3} and $T =$ 5,000, 7,000, 12,000, 15,000, 20,000 K. Make sure that all n_i's should be positive.

(c) The ratio of the $\lambda 6584$ line to the $\lambda 5755$ line of [N II]

$$R = \frac{4\pi j_\nu(4 \to 3)}{4\pi j_\nu(5 \to 4)} = \frac{N_4 A_{43}\, 5755}{N_5 A_{54}\, 6584} \tag{9}$$

can be an important indicator of the electron temperature in ionized nebulae. Plot this ratio over the temperature range 5,000 K $< T <$ 20,000 K for densities $n_e = 10, 10^2, 10^3, 10^4$, and 10^5 cm^{-3}.

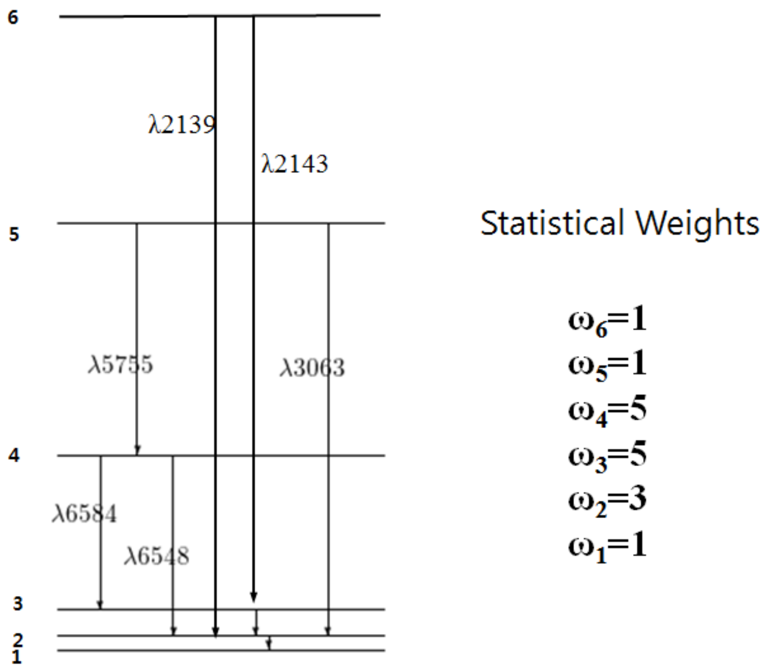

Figure 1: The first six energy levels and statistical weights of N II

i	j	λ_{ij}	E_{ij} (eV)	A_{ji}	$\Omega_{ji}(1.0)$	β_{ji}
1	2	203.53μm	0.00609	2.08×10^{-6}	0.408	0.125
1	3	76.136μm	0.01628	1.16×10^{-12}	0.272	0.21
1	4	6527.8Å	1.89879	5.35×10^{-7}	0.2934	0.048
1	5	$--$	4.05244	0.0	0.0326	0.050
1	6	$--$	5.80061	0.0	0.1323	0.025
2	3	121.64μm	0.01019	7.46×10^{-6}	1.120	0.17
2	4	6548.8Å	1.89270	1.01×10^{-3}	0.8803	0.048
2	5	3063.2Å	4.04635	3.38×10^{-2}	0.0977	0.050
2	6	2139.0Å	5.79452	$4.80\times10^{+1}$	0.3968	0.025
3	4	6584.3Å	1.88251	2.99×10^{-3}	1.4672	0.048
3	5	3070.9Å	4.03616	1.51×10^{-4}	0.1628	0.050
3	6	2142.8Å	5.78433	$1.07\times10^{+2}$	0.6613	0.025
4	5	5755.3Å	2.15365	1.12	0.8338	-0.18
4	6	$--$	3.90182	0.0	0.0	0.0
5	6	$--$	1.74817	0.0	0.0	0.0

Table 1: Data for all possible transitions between the levels