A New Era of Prosthetic Arms: The Mathematical Model

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Abstract:

The InMoov robotic arm was built using the Flashforge Creator Pro 2. The arm is controlled by 6 MG996R DIGI HI TORQUE servo motors and an Arduino UNO. The purpose of 3D-printing a prosthetic arm is in hopes to help solve the ever-growing problem of amputees without prosthetics. Modeling the motors that provide functionality to this arm will hopefully advance the search for creating affordable prosthetics. The mathematical model defines the motor parameters with ordinary differential equations. These equations represent the motor dynamics function that describes the motor's behavior using the given parameters. The Python code will solve the ODEs using solve_ivp and plot the angular velocity and current over time. These graphs will describe the motors converting current into rotational motion. Finally, this paper presents a graphical representation of the relationship between the voltage applied to the DC motor and the corresponding terminal angular velocity at t=1s. This plot allows us to better understand the influence of voltage on the motor's performance.

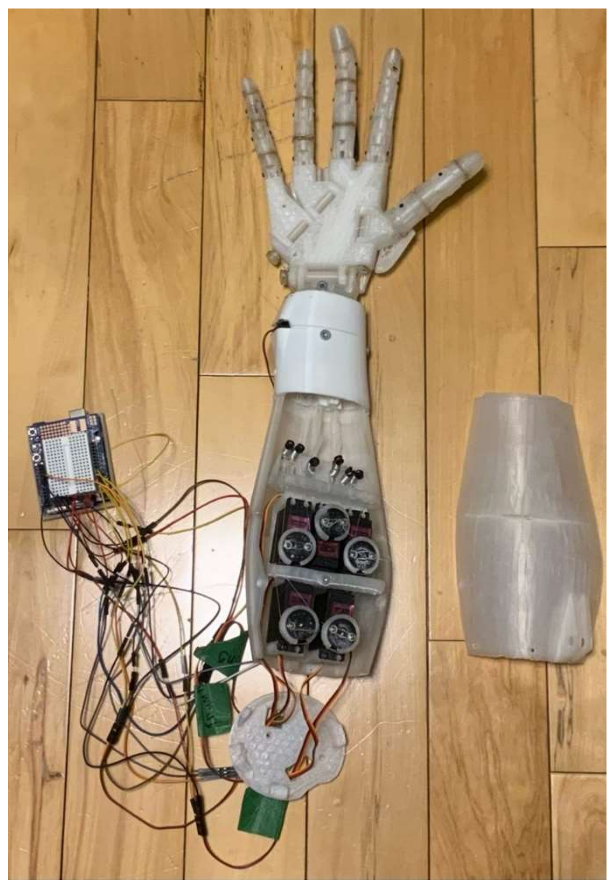
Introduction:

Over 100 million people are in need of prosthetic limbs or support devices for damaged limbs in the world (1). This data is even more staggering when on a global scale there is an amputation done on average every 30 seconds (3). Over 80% of these individuals in need of prosthetics have no access to this technology (2). This is due to 9.2% of the world's population or 719 million people living in extreme poverty, making medical aid itself difficult to come by (2). The cost of prosthetic limbs ranges from 5,000-70,000 $ depending on the type of prosthetic needed (4). This unfortunately makes prosthetics a luxury for many people, thus forcing millions of individuals to live without the complete capabilities of an average human being. All prosthetics can be divided into 3 separate categories, body-powered, myoelectric, and hybrid (6). Body-powered arms operate due to the movement of the wearer's body (6). Myoelectric prosthetics use electronics and sensors to operate the movement of the arm due to sensor readings (6). Hybrid prosthetics combine both body-powered and myoelectric prosthetics components (6). With the advancements in technology, a possible future of custom prosthetics being able to be printed for a fraction of the price due to 3d printers looks like a possibility. A few common types of plastics/materials used for prosthetics are polyethylene, polypropylene, polyurethane, and acrylics (7). These materials are very similar to the PLA/ABS plastics and nylon used by 3d printers (8). These materials are chosen specifically for their flexibility, strength, and overall durability. On average a kg of PLA or ABS filament costs anywhere from 15-50$ (5). Nylon costs anywhere from 40-100$, and other composite materials can cost anywhere from 40-500$ (5). 3D printers can provide a solution to making complex parts much more efficiently. This is done due to minimal material loss since no machining is required to shave away material from a larger block of that material. Rather the 3d printer prints all the components directly onto the build plate with the only loss in the material being support structures for the overall build. This ability will help drastically decrease the price of prosthetic arms and provide much larger flexibility regarding design and functionality. The 3d printers can only provide aid in developing the husk and components required for the prosthetic arms. For body-powered prosthetics, most of the limbs can be constructed with a 3D printer, filament, and any other required components. For both myoelectric and hybrid prosthetics more complex components are required to provide the functionality of the prosthesis. The main technological component used for these types of prosthetics is motors. This allows prosthetics to be much more versatile and someday be viewed as an improvement rather than an impairment. This paper will focus on a 3d-printed prosthetic arm that will be built and the development of a different model of the motors used to provide its motion.

1. Prosthetic Arm:

1.1 General Overview:

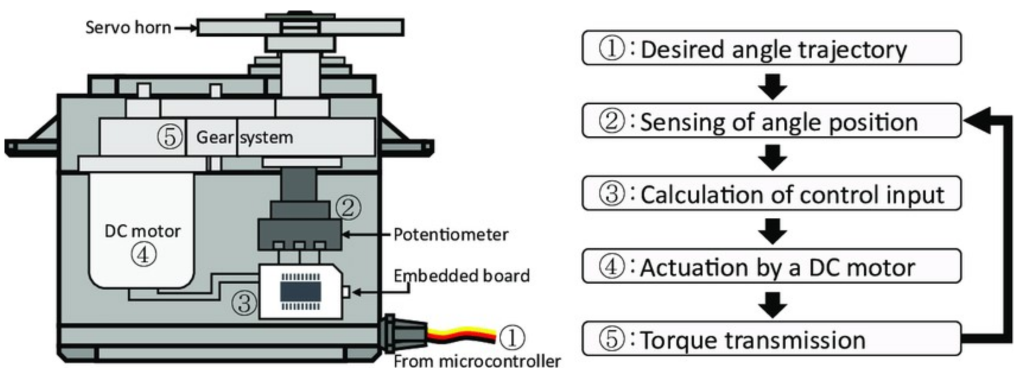
The prosthetic arm that was built for this student project belongs to the InMoov open-source community. A French sculptor and designer Gael Langevin created the 3D files for each component of the forearm, wrist, and hand that were used to build this project. The prosthetic hand works by using servo motors to rotate 2 strings per finger. The rotation of the servo from the resting or 0 position would be 90° on each side meaning the servo has a total range of 180°. The string on the right of each servo motor controls the closing motion of the finger. The rotation of the servo clockwise from the 0 position would cause tension on that string and cause the joints in the finger to collapse. The string on the left side of each servo motor controls the finger's closing motion. The rotation of the servo counterclockwise from the 0 position would cause tension on the left string and cause the joints in the finger to open. In total 6 servo motors have been used for this project, 5 for each finger and the 6th for the wrist. The specific servo motors used for this hand are the MG 996R DIGI HI TORQUE servo motor. It took roughly 1.3 kg or (2.87 lbs) of PLA plastic to print all the different components of the forearm. The filament used for the arm was the Mech Solutions Ltd transparent filament costing roughly 30$ for a 1kg spool. With all this under consideration, it would cost roughly 11.11 $/ lb with a total cost 31.89$ to print the entire forearm and hand. In addition, with the fishing line costing 19.56 $ and a 4 pack of servo motors costing 39.29$ thus needing 2 sets to have the required 6 servos the total price of the arm was roughly 130.03$. Even without labor or shipping, the cost of this prosthetic arm is a far cry from the 5000-70,000 $ required to get a prosthesis today.



2. Servo Motor:

2.1 General Overview:

A servo motor is a specific type of motor defined as a “closed-loop servomechanism that uses position feedback in order to control its rotational speed and position” (9). A servo motor can be broken down into 4 separate parts, a DC motor, a potentiometer, a gearbox, and a control unit.



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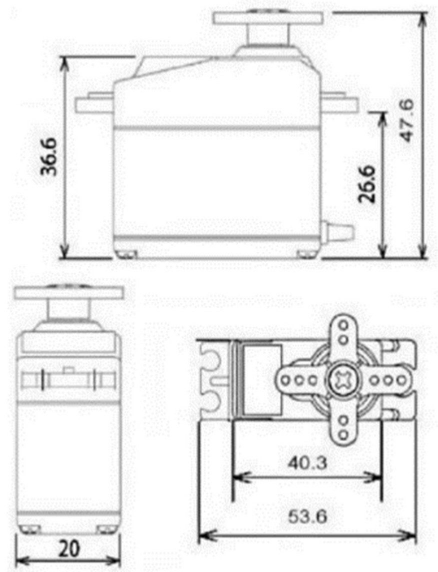
The DC motor within the servo motor will provide high speeds but low torque, the gearbox will then convert the high speed into high torque. The potentiometer and DC motor are connected to the control unit within the servo motor. Due to this, the potentiometer can calculate the angle of the servo motor and provide this information to the control unit. The control unit with this information and commands from the Arduino will inform the dc motor how much to rotate (12). The Dimensions of the MG966R servo motor:



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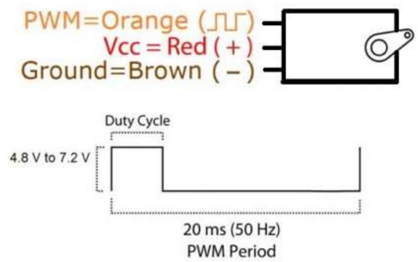
Arduino Servo motors work due to 3 connectors, a 5V connector, a ground connector, and the signal pin. The ground and 5V connectors provide power to the servo motor. The signal pin informs the Arduino Uno where the servo is located on the chip. The respective code for these servos must correlate to these specific pins and libraries required for servos. For the prosthetic arm the servo for the thumb is on pin 1, index finger on pin 2, middle finger on pin 3, ring finger on pin 4, pinky on pin 5, and wrist on pin 6. The MG 996R DIGI HI TORQUE servo motor can rotate from 0° - 180°, depending on the width of the PWM pulses (Pulse-width modulation) sent from the control system to the servos control unit (10).

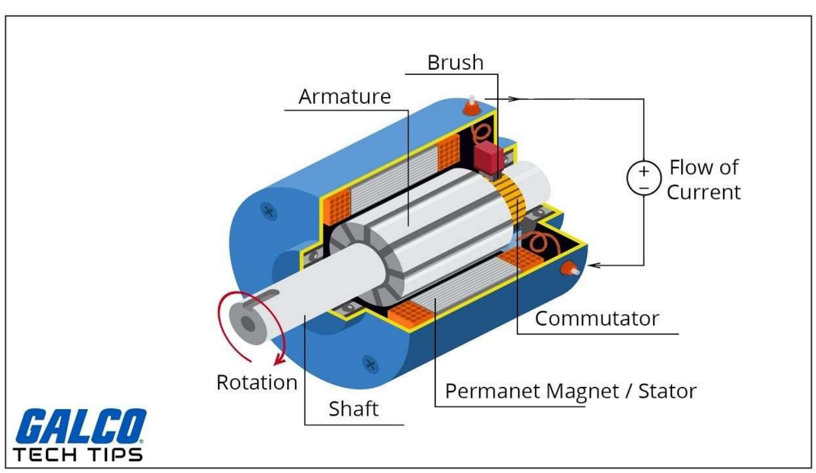
PWM pulses can be viewed as the method of transferring the code or the desired action to the servo motor. PWMs work by transmitting periodic pulses that correspond to the voltage required for that specific motor to function (11). PWMs for DC motors will control the speed of the motor depending on the width of the pulses. Pulses with larger widths will make the motor rotate faster, due to the motor receiving a consistent voltage for a longer period of time (11). Since pulses with smaller widths will only provide short bursts of voltage to the motor, the motor will rotate slowly. For Servo motors, the width of the PWM pulses will control the position of the servo. The widths of the PWMs range from 1ms to 2ms. 1ms pulses will rotate the servo to the 0° position and the 2ms pulses to the 180° position (10). Any width between 1-2 ms will rotate the servo anywhere between 0°-180°.



(20)

2.2 Dc Motor Overview:

To understand the mathematical model of the servo motor it is crucial to understand the mathematical functionality of a DC motor. This is due to the DC motor being the main component providing functionality to the servo motor. A DC motor is a motor that works due to direct current. The DC motor present within the MG 996R DIGI HI TORQUE servo motor is a brushed motor. A brushed DC motor is composed of 4 main components the stator, armature, commutator, and brushes (14).



(15)

The stator is composed of two permanent magnets fixed in position on opposite ends of the dc motor. The reason for the 2 magnets within the stator is that each magnet provides a constant magnetic pole both negative and positive. The armature lies within the two halves of the stator due to an axel and bearings thus allowing it to rotate. The armature itself is composed of copper wires wound around in specific patterns to allow magnetic induction. The armature is directly connected to the commutator. The commutator is composed of 2 rings that will receive and transmit current to the armature whenever they come into contact with the brushes. There are 2 brushes that are directly connected to the power source, one on the positive end and the other on the negative end.

The motor works by the commutator rings getting current from the brushes and providing that current to the armature. The armature with this current will induce a magnetic field. The importance of the stator being stationary and composed of both poles will be important now due to the induced magnetic field from the armature will react to one of the poles and rotate away from it. Just as the armature is about to reach the other stationary magnet of the stator, the rings of the commutator will be positioned in a certain way that it will come into contact with the opposite brush. Due to the opposite brush, the current flowing through the commutator will be the opposite thus causing a switch in polarity within the armature. This will cause the armature to not stop at that magnet and continue back to the other side where the cycle will repeat itself. This entire process can be explained and modeled by Differential equations.

3. Mathematical Model

3.1 General Overview:

To begin this process of modeling 2 constants must be found, K_T= the torque constant and K_E= the back electromagnetic force constant. Finding these constants requires some preliminary data that the manufacturers provide in data sheets. Most of the data is based on the different voltages that the servo will be experiencing. Due to the operating voltage of the MG996R servo ranging from 4.8-7.2 V with the average being 6V all the data and calculations will be based on this Voltage (12). With the operating voltage = 6V, the stall torque = 11 Kg-cm, no-load speed = 0.17 s/60°, stall current = 2.5 A, and the running current = 0.5-0.9 A with the average being = 0.7 A (12).

The torque constant can be calculated by formula 1:

$$\mathbf{1.}\ \boldsymbol{K_T} = \frac{\textbf{\textit{stall torque}}}{\textbf{\textit{rated current}}}$$

(18)

To use the stall torque in the equation it must be converted into SI units meaning the kg-cm must be converted into Nm. This is done by remembering that 1kg-cm is 0.098 Nm thus making the new stall torque = 1.078 Nm (16). Then the rated current must be estimated, the rated current is defined as the maximum current a motor is able to draw normally. The stall current is the maximum current the motor can handle, and the running current is the current the motor needs to function. Due to manufacturers not providing the rated current it can be estimated that it will fall between the stall and running current. The average of these two values will be estimated as the rated current.

$Rated\ Current = \frac{running\ current+stall\ current}{2} = \frac{0.7+2.5}{2} = 1.6\ A$

$$kt = \frac{\boldsymbol{stall\ torque}}{\boldsymbol{rated\ current}} = \frac{1.078Nm}{1.6A} = 0.67375\ \frac{Nm}{A}$$

The back EMF constant can be calculated by formula 2:

$$\mathbf{2.}\ \boldsymbol{K_E} = \frac{(\boldsymbol{Operating\ voltage})-(\boldsymbol{Running\ Current})\boldsymbol{X}(\boldsymbol{Resistance})}{(\boldsymbol{no-load\ Speed})} \tag{18}$$

To use this formula the resistance must be calculated, and the no load speed must be converted into $\frac{rad}{s}$.

$Resistance = \frac{V}{I} = \frac{6V}{0.7} = 8.57\Omega$

$no - load\ speed\ (W) = 0.17\frac{s}{60°} * \frac{360°}{1rev} = 1.02\frac{s}{rev} = \frac{0.98rev}{s} * \frac{2\pi\ rad}{rev} = 6.1575\frac{rad}{s}$

$$\boldsymbol{K_E} = \frac{6V-0.7A*8.57}{6.1575\frac{rad}{s}} = \frac{0.001}{6.1575} = 1.624 * 10^{-4}\frac{V}{\frac{rad}{S}}$$

While exploring the brushed servo motor's modeling, grasping the essentials of inductance/inductors and resistors/resistance is crucial, as these electrical properties significantly impact the motor's functionality and performance.

Inductance, symbolized by 'L' and measured in henrys (H), represents the capacity of an electrical circuit or component to store electrical energy as a magnetic field when current passes through it. Predominantly, inductance is connected to coils or inductors – passive electronic components made of a wire wound into a spiral or helical configuration. The generated magnetic field around the coil is directly proportional to the current flowing through it when electricity is applied (21). In the case of a brushed servo motor, the motor's windings function as inductors, accumulating and releasing energy during operation. The motor's inductance affects its speed and torque response by contributing to its transient behavior when the current in the windings shifts.

Conversely, resistors are electrical components that resist the flow of electric current, transforming electrical energy into heat. Resistance, denoted by the symbol 'R' and measured in ohms (Ω), quantifies this opposition (22). The intrinsic property of the conducting material in a brushed servo motor's windings creates a certain amount of resistance, leading to power losses, heat generation, and reduced motor efficiency.

To accurately model a brushed servo motor, it's essential to consider both inductance and resistance, as they shape the motor's electrical and mechanical traits. The motor's electrical behavior can be depicted by a series of differential equations incorporating its inductance (L), resistance (R), and back EMF (K_E) – which is related to the motor's rotational speed (24).

3.2 armature inductance:

Calculating the armature inductance from the available data is not straightforward, as the necessary information like the rate of change of current (di/dt) is not available. However, we can attempt an indirect approach by estimating the electrical time constant (τ) of the motor and then use it to find the armature inductance (L) using the formula:

$$\mathbf{3.}\ \boldsymbol{\tau = \frac{L}{R}}$$

(23)

where τ is the electrical time constant, L is the armature inductance, and R is the armature resistance. The electrical time constant is a measure of the speed at which the current or voltage in an electrical circuit rises or falls in response to a change.

When a step voltage is applied to such a circuit, the current doesn't instantly reach its maximum value but rather increases gradually due to the inductor's property of opposing changes in current, a property known as inductance. The time constant for an RL circuit is defined as the time it takes for the current to reach approximately 63.2% of its final steady-state value after the voltage is applied, or conversely, the time it takes to fall to 36.8% of its initial value after the voltage is switched off.

To estimate τ we can use the equation used to calculate the electrical time constant for the SM and NeoMetric motors:

$$\textbf{3. } \boldsymbol{\tau} = \frac{(\boldsymbol{R})\times(\boldsymbol{J})}{(\boldsymbol{K_E})\times(\boldsymbol{K_T})} \tag{25}$$

Where τ is the electrical time constant, J is the rotor inertia (refer to the next section), K_E is the back EMF constant, K_T is the torque constant and R is the winding resistance. Due to the lack of data, the average resistance of 0.8Ω for a small servo motor will be used in its place (29).

$$Eelectrical\ time\ constant = \frac{(R)\times(J)}{(K_E)\times(K_T)} = \frac{(0.8\ \boldsymbol{\Omega})\times(2.7*10^{-6}\ \boldsymbol{kg*m^2})}{(1.624*10^{-4}\frac{\boldsymbol{V}}{\boldsymbol{rad}/\boldsymbol{s}})\times(0.67375\frac{\boldsymbol{N*m}}{\boldsymbol{A}})} = \mathbf{0.019741}\ \boldsymbol{s}$$

Substituting back into the original equation and isolating for the armature inductance(L):

$$L = \tau \times R = (0.019741\ s)\times(8.57\ \Omega) = \mathbf{0.1692}\ \boldsymbol{H}$$

The estimated armature inductance is approximately 0.1692 H. However, please note that this is only an estimation and could be different from the actual value.

3.3 Moment of Inertia:

To calculate the moment of inertia of a solid cylinder, we can use the following formula:

$$\mathbf{4.}\ \boldsymbol{J = \left(\frac{1}{2}\right)(M \times r^2)}$$

(26)

where J is the moment of inertia (Kg m^2), M is the mass of the cylinder, and R is the radius of the cylinder.

From the specifications of the motor, we know that the total mass is 55 grams, the dimensions of the servo motor, and the diameter is 19.7mm. With the 19.7 mm dimension as the diameter of the cylinder and 40.7 mm as the height, we can calculate the moment of inertia as follows:

$$J = \frac{1}{2} \times M \times\ r^2 = \frac{1}{2}(0.055\ kg) \times \left(\frac{0.0197}{2}m\right)^2 = \mathbf{2.7 \times 10^{-6}}\ \boldsymbol{kg * m^2}$$

The moment of inertia of the solid cylinder rotor is approximately 0.0000027 kg * m^2.

3.4 Differential Equations:

The motor model consists of the following parameters:

- Armature resistance (R)
- Armature inductance (L)
- Back-emf constant (K_E)
- Torque constant (K_T)
- Rotor inertia (J)
- Rotor damping (set to zero in this case)
- DC supply voltage (V)

Using these parameters, we define a set of coupled differential equations to represent the motor dynamics:

$$5. \frac{d\omega}{dt} = \frac{K_T(i) - damping(\omega)}{J}$$

(27)

$$6. \frac{di}{dt} = \frac{V - R(i) - K_E(\omega)}{L}$$

(27)

1. Motor Torque Constant (K_T)

Torque is a measure of how much a force can cause an object to rotate about an axis. In an electric motor, this force is generated by the interaction between the magnetic field produced by the current flowing through the motor's coils and the magnetic field of the motor's permanent magnets. The torque constant \`kt\` quantifies this relationship and indicates how much torque the motor can produce for a given current. The torque \`τ\` is thus given by \`τ = kt * i\`, which reflects the physical principle that the torque is directly proportional to the current.

2. Back EMF Constant (K_E)

When the motor's rotor spins, it generates a back electromotive force (back EMF) due to the changing magnetic field experienced by the coils (this is Faraday's law of electromagnetic induction). This back EMF opposes the applied voltage, reducing the effective voltage across the motor and thus the current flowing through it. The back EMF constant \`$K_E$\` quantifies this relationship and indicates how much back EMF is produced for a given angular velocity. The back EMF \`E\` is thus given by \`E = $K_E$* ω\`, which reflects the physical principle that the back EMF is directly proportional to the angular velocity.

3. Moment of Inertia (J)

The moment of inertia \`J\` is a measure of an object's resistance to changes in its rotational motion. For a cylindrical rotor of mass \`M\` and radius \`r\`, \`J\` is given by \`J = (1/2) * M * r^2\`. This is a direct consequence of the definition of moment of inertia, which, for a collection of particles, is the sum of the products of each particle's mass and the square of its distance from the axis of rotation. For a continuous body like a cylinder, this sum becomes an integral.

From these principles, we derive the differential equations:

$$d\omega/dt = (kt * i - damping * \omega) / J$$

$$di/dt = (V - R * i - ke * \omega) / L$$

These equations represent the dynamic behavior of a DC motor, linking the electrical input (current) and the mechanical output (angular velocity), considering various factors such as resistance, inductance, damping, and back EMF. The derivation of these equations is fundamentally grounded in the physical laws of electromagnetism and mechanics.

5. $d\omega/dt = (kt * i - damping * \omega) / J$

This equation is derived from Newton's second law, which states that the acceleration of an object is proportional to the net force acting on it and inversely proportional to its mass. In the context of rotational motion, this law is often written as:

Torque = Moment of Inertia * Angular Acceleration

or

$$\tau = J * d\omega/dt$$

The torque on the motor's rotor (τ) is the difference between the motor torque ($kt * i$) and the damping torque ($damping * \omega$). Therefore, we can write:

$$kt * i - damping * \omega = J * d\omega/dt$$

Rearranging this gives the fifth equation:

$$d\omega/dt = (kt * i - damping * \omega) / J$$

6. $di/dt = (V - R * i - ke * \omega) / L$

This equation is derived from Kirchhoff's voltage law, which states that the sum of the electrical potential differences (voltages) around any closed loop or mesh in a network is always equal to zero. In the case of a DC motor, the applied voltage (V) is distributed across the resistance (R * i), the back emf (ke * ω), and the change in magnetic flux through the motor's inductance (L * di/dt). Therefore, we can write:

$$V = R * i + ke * \omega + L * di/dt$$

Rearranging this gives the sixth equation:

$$di/dt = (V - R * i - ke * \omega) / L$$

In summary, these equations are derived from fundamental laws of physics (Newton's second law and Kirchhoff's voltage law) and describe the dynamic behavior of a DC motor. The fifth equation relates the motor's angular acceleration to the current flowing through it and the damping due to friction, while the sixth equation relates the rate of change of current to the applied voltage and the motor's angular velocity. We solve these differential equations numerically and plot the motor's angular velocity (ω) and current (i) as functions of time. The resulting graphs, Figure 1.1 (angular velocity vs. time) and Figure 1.2 (current vs. time), provide valuable insights into the motor's performance.

3.5 Python Code:

Now that the required constants are found, the python model can be made. In the following code, a DC motor is modeled using differential equations representing the motor dynamics. The system is simulated, and its angular velocity and current are plotted over time.

1. Import the required libraries:

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
```

NumPy is used for numerical computations, scipy.integrate.solve_ivp is used to solve ordinary differential equations, and Matplotlib is used for plotting the results.

2. Define motor parameters:

```
# Motor parameters
R = 8.57  # Ohm, armature resistance
L = 0.1692  # H, armature inductance
ke = 1.624e-4  # V/rad/s, back-emf constant
kt = 0.67375  # N*m/A, torque constant
J = 2.7 * 1e-6  # kg*m^2, rotor inertia
damping = 0  # N*m/(rad/s), rotor damping
V = 6  # V, DC supply voltage
```

3. Define the differential equation representing the motor dynamics:

```python
# ODE function representing the motor dynamics
def motor_dynamics(t, y):
    omega = y[0]
    i = y[1]

    d_omega_dt = (kt * i - damping * omega) / J
    d_i_dt = (V - R * i - ke * omega) / L

    return [d_omega_dt, d_i_dt]
```

The motor_dynamics function represents the motor's behavior as a system of ordinary differential equations , called by the solver (solve_ivp) to compute the motor's state (angular velocity and current) at each time step.

It takes two arguments:

1. t: The current time point, required by solve_ivp function's syntax.

2. y: A state vector with the angular velocity (omega) and current (i) at the current time point. A state vector is a representation of the current state of a system, which consists of multiple variables. In this case, the state vector y contains two variables, the angular velocity omega and the current i. This state vector is used to describe the current state of the motor at a specific time point.

The function calculates the derivatives of omega and i using motor dynamics equations derived from the electrical and mechanical characteristics of a DC motor: d_omega_dt and d_i_dt. It then returns a list [d_omega_dt, d_i_dt] containing these derivatives, In essence, the motor_dynamics function computes the rate of change of the motor's state (angular velocity and current) based on its current state and time, allowing the ODE solver to determine the motor's behavior under given parameters and initial conditions.

4. Set initial conditions through a list containing initial values for angular velocity and current:

```
# Initial conditions
initial_conditions = [0, 0]  # [omega_0, i_0]
```

5. Define the time span for the simulation:

```
# Time span
t_span = [0, 0.2]
```

6. Call solve_ivp to solve the ODE with the motor dynamics function, time span, initial conditions, and specified tolerances rtol and atoll:

```
# Solve the ODE
sol = solve_ivp(motor_dynamics, t_span, initial_conditions, rtol=1e-6, atol=1e-8)
```

solve_ivp: The solve_ivp function from the SciPy library is used to numerically solve initial value problems for systems of ODEs. It requires parameters such as the ODE function, time span, and initial conditions as arguments. Rtol and Atol are tolerance parameters in the solve_ivp function that control the error tolerances for the numerical solution of the system of ordinary differential equations. They help to determine the accuracy of the solution obtained by the numerical integration method used by solve_ivp.

7. Plot the results:

```
# Plot the results
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(10, 6))
ax1.plot(sol.t, sol.y[0], label='Angular velocity (rad/s)')
ax1.set_xlabel('Time (s)')
ax1.set_ylabel('Angular velocity (rad/s)')
ax1.grid()
                                                                    #
ax2.plot(sol.t, sol.y[1], label='Current (A)', color='r')
ax2.set_xlabel('Time (s)')
ax2.set_ylabel('Current (A)')
ax2.grid()

plt.show()
```

8. Visualize the operation of a DC motor under varying voltages:

```
                                                                    #
voltages = np.linspace(1,10, 100)
voltages_at_1s = []
angular_velocities_at_1s = []

for v in voltages:
    V = v
    sol = solve_ivp(motor_dynamics, t_span, initial_conditions, rtol=1e-6, atol=1e-8)

    # find the index in the time array closest to 1s
    idx_1s = np.abs(sol.t - 1).argmin()

    # store the voltage and corresponding angular velocity at t=1s
    voltages_at_1s.append(v)
    angular_velocities_at_1s.append(sol.y[0][idx_1s])

    # Plot the results
    fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(7, 4))
    fig.suptitle(f'V = {v:.3f} V', fontsize=16)
    ax1.plot(sol.t, sol.y[0], label='Angular velocity (rad/s)')
    ax1.set_xlabel('Time (s)')
    ax1.set_ylabel('Angular velocity (rad/s)')
    ax1.set_ylim(0, 15000)  # to ensure the y-axis goes from 0 to 15000
    ax1.grid()

    ax2.plot(sol.t, sol.y[1], label='Current (A)', color='r')
    ax2.set_xlabel('Time (s)')
    ax2.set_ylabel('Current (A)')
    ax2.set_ylim(0, 1)  # to ensure the y-axis goes from 0 to 1
    ax2.grid()

    plt.show()
```

The previous code simulates a DC motor operation under a range of voltage settings from 1 to 10 volts, while capturing the angular velocity at 1 second for each setting. The main loop applies each voltage to the motor, solves the differential equations describing the motor's behavior, and generates a plot of the motor's angular velocity and current over time. The angular velocity of the motor at each voltage setting at the one second mark is recorded and used for further analysis.

9. Perform Linear Regression to Visualize the Relationship of Voltage and Angular Velocity:

```
                                                                                #
from scipy import stats


# Fit a degree 1 polynomial to the data
slope, intercept, r_value, p_value, std_err = stats.linregress(voltages_at_1s, angular_velocities_at_1s)

# Generate values for the line of best fit
best_fit_line = slope * np.array(voltages_at_1s) + intercept

# Plot the data points
plt.scatter(voltages_at_1s, angular_velocities_at_1s, label='Data points')

# Plot the line of best fit
plt.plot(voltages_at_1s, best_fit_line, 'r', label=f'Best fit line: ω = {slope:.3f}V + {intercept:.3f}')

plt.xlabel('Voltage (V)')
plt.ylabel('Angular velocity at t=1s (rad/s)')
plt.title('Relationship between Voltage and Angular Velocity at t=1s')
plt.legend()
plt.grid()
plt.show()
                                                                                #
```

This final section performs a linear regression on the recorded voltages and corresponding angular velocities at one second, fitting a straight line that best describes their relationship. The slope and intercept of this line are calculated, and the line of best fit is generated. This line, alongside the original data points, is then plotted in a scatter graph, showing the relationship between the applied voltage and the resulting angular velocity of the motor at one second. The plot includes labels, a legend, and a grid for clarity.

3.6 Data:

Figure 1.1 depicting the angular velocity of the servo's underlying dc motor (0 damping)

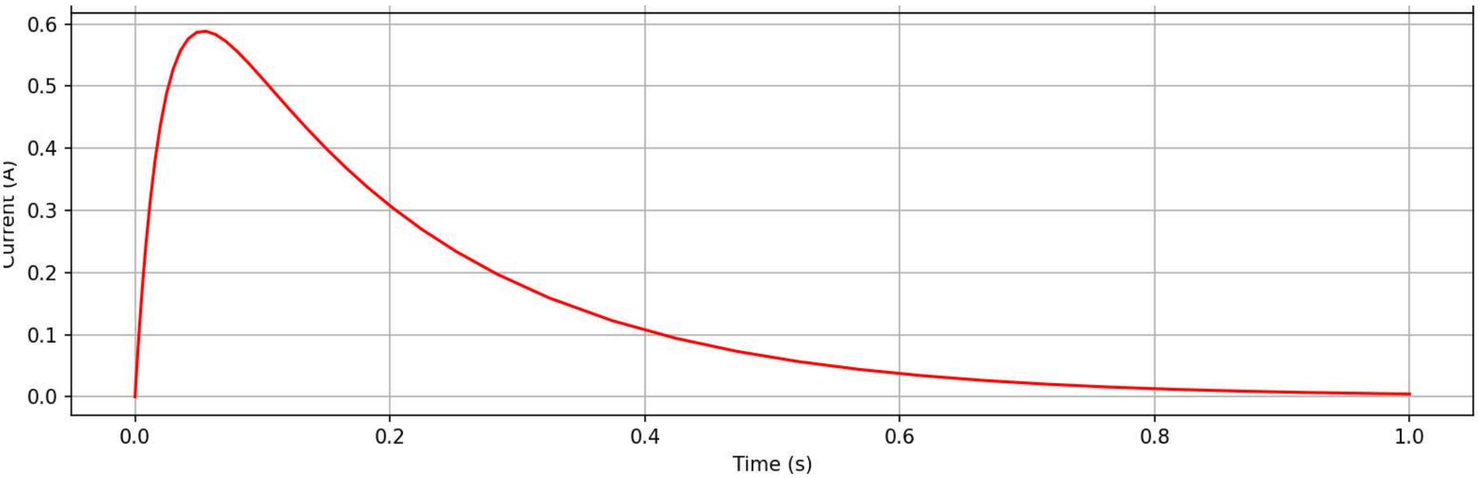


Figure 1.2 depicting the current of the servo's underlying dc motor (0 damping)

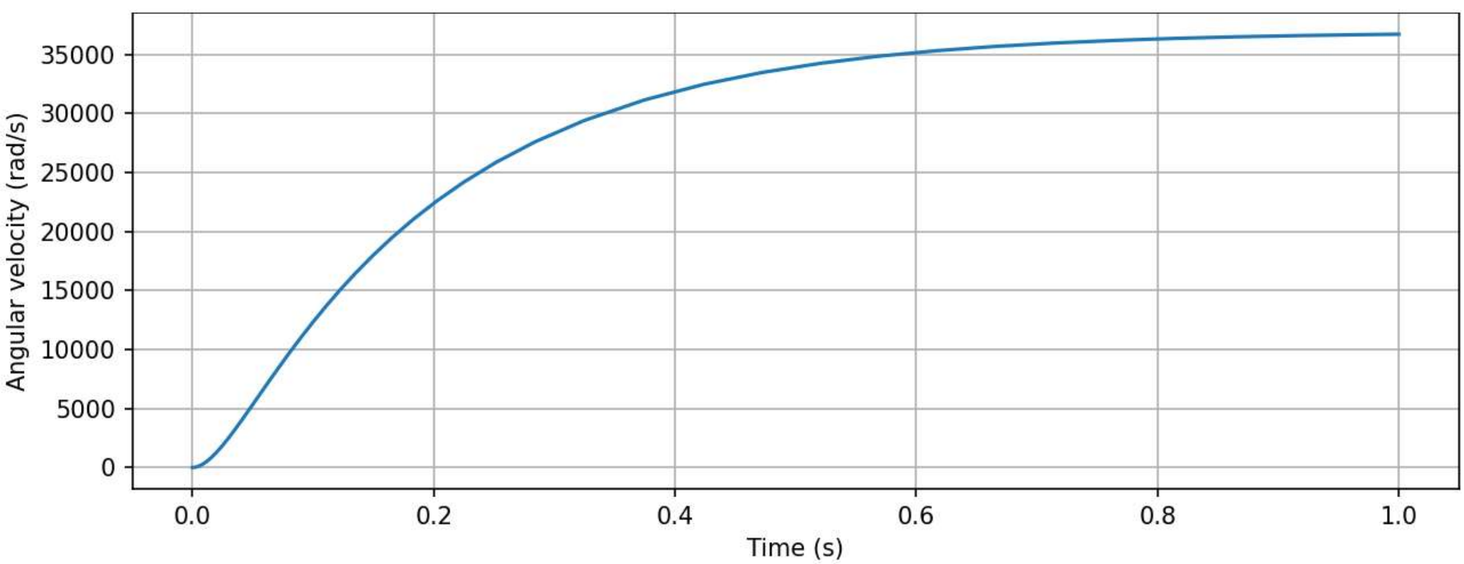


Figure 1.3

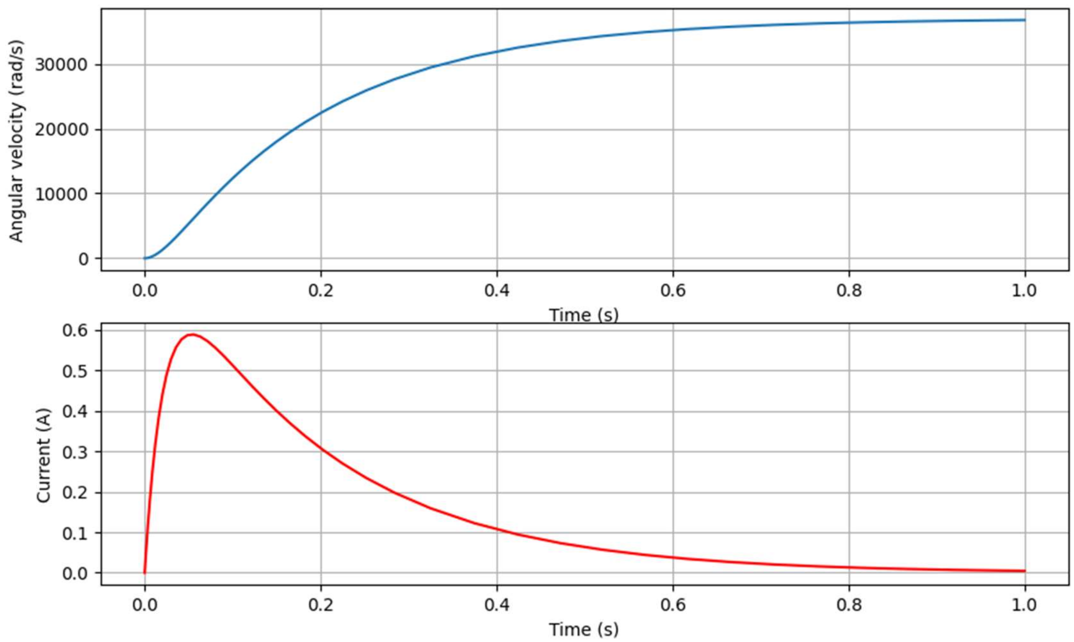
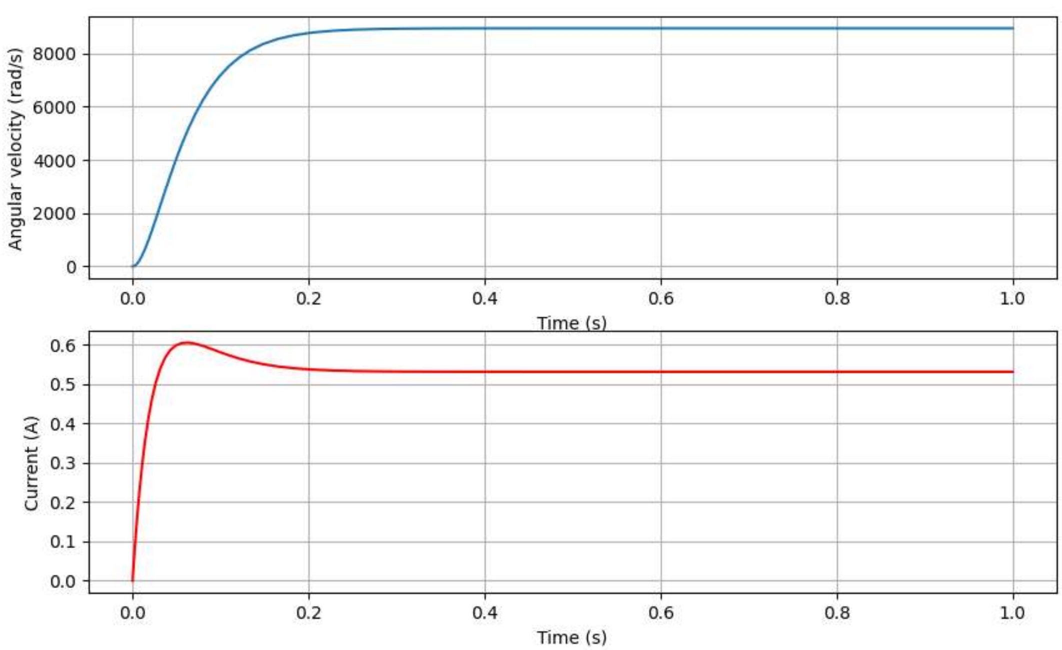


Figure 1.4 setting the damping to 0.00004 N*m/(rad/s)



Figures 2.00 – 2.99 (100 graphs of varied voltages) NOT direct output from code.

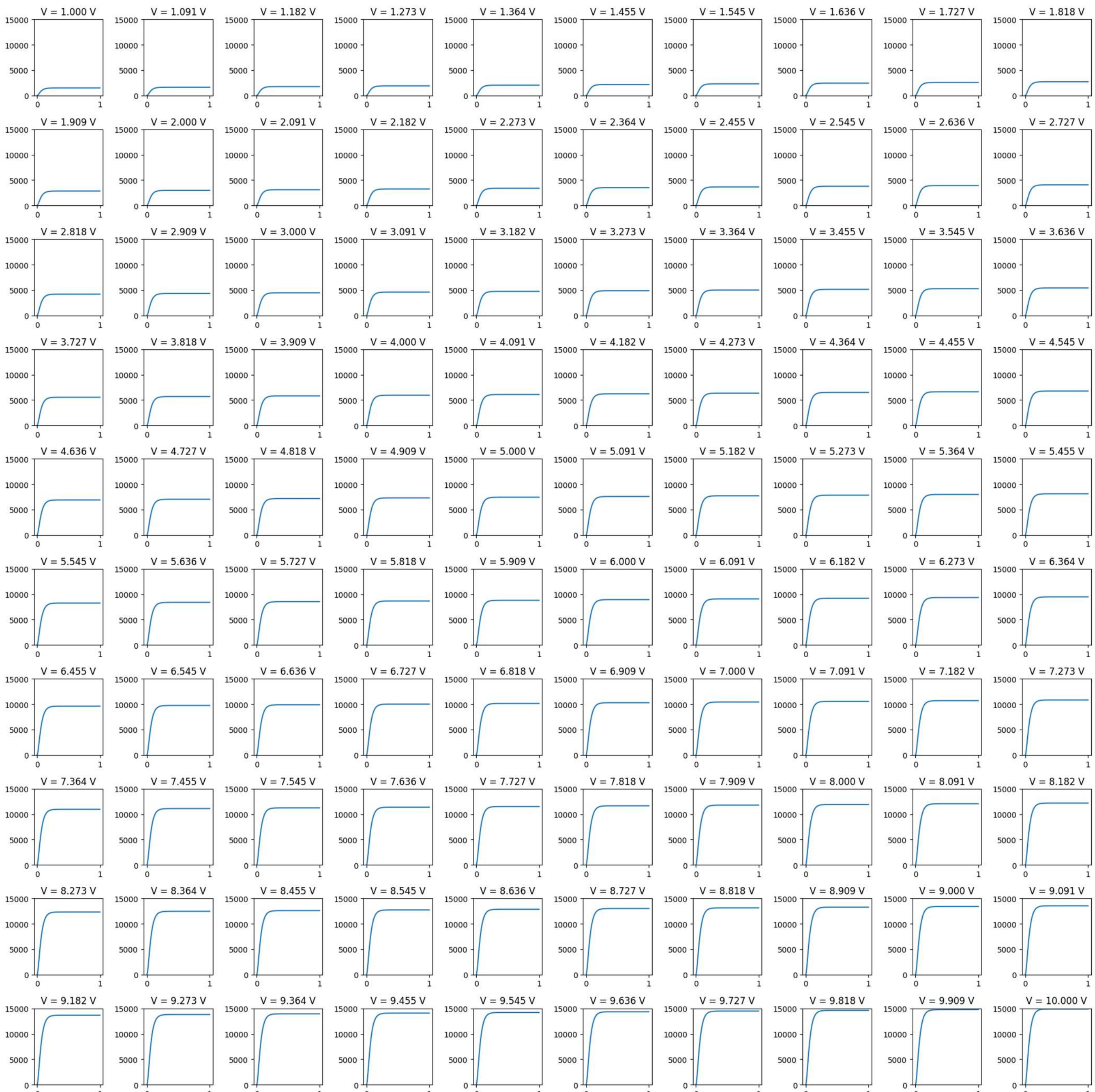
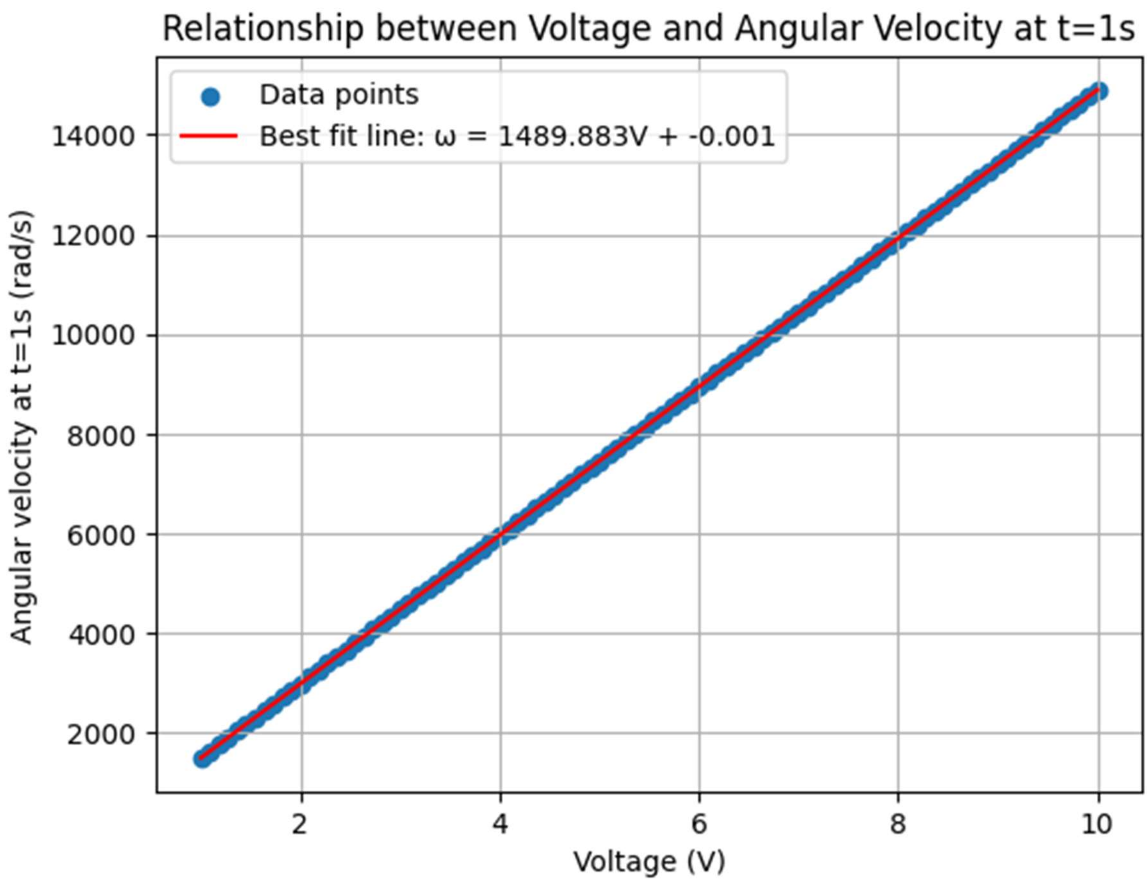


Figure 3.0 Relationship between Terminal Angular Velocity and Supplied Voltage



3.7 Analysis:

This research paper investigates the performance of a DC motor by simulating its behavior using a mathematical model and evaluating the results graphically. The following is an in-depth analysis of the motor's angular velocity and current over time, considering the impact of motor parameters and the absence of load torque and damping.

Figure 1.1 illustrates the motor's angular velocity rising almost linearly with time, due to the absence of load torque and damping, which lets the motor accelerate freely. The motor's speed eventually reaches a constant maximum value at steady-state.

Figure 1.2 displays the motor's current initially spiking, then gradually decreasing. This is due to the motor's back-emf, which starts small and increases with the motor's acceleration. As the back-emf grows, the voltage difference across the armature decreases, causing the current to drop (28).

At steady-state, the motor attains a constant maximum speed with minimal current draw, as the back-emf balances the supply voltage. In this condition, the motor maintains its maximum speed unaffected by load torque or damping, which are absent in this model.

In an ideal scenario, when a motor reaches its maximum speed or no-load speed, the back electromotive force (EMF) equals the supplied voltage, thus reducing the current to theoretically zero. However, in real-world scenarios, it's not common for the current to go all the way to zero while the motor is still running. This is due to several reasons:

1. Friction and load: In practice, motors always have some load, which includes friction in the bearings and air resistance. When a load is applied, it reduces the speed of the motor,

which in turn reduces the back EMF. Consequently, more current is drawn to maintain the speed of the motor.

2. Losses: Motors aren't 100% efficient. There are always losses such as winding resistance and magnetic losses, which will also draw some current.

3. Imperfect counteracting EMF: The back EMF is ideally equal to the applied voltage when the motor is running at maximum speed, but due to design imperfections and non-ideal magnetic field distribution, it might not perfectly counteract the applied voltage.

Thus, under normal operation, while the current drawn by the motor decreases significantly as it reaches its maximum speed, it won't usually reach absolute zero as long as the motor is running.

This analysis demonstrates how motor parameters and the absence of load torque and damping influence the motor's performance. Figures 1.1 and 1.2 show the motor reaching a constant maximum speed with minimal current at steady-state, reflecting the interplay between back-emf and acceleration.

As we turn our attention to Figure 1.4, we find a graphical representation that significantly enhances our understanding of the behavior of a DC motor. By presenting these parameters graphically, we can observe the dynamics and steady-state performance of the motor under different damping conditions. In the following discussion, we will delve deeper into interpreting these graphs, correlating them with the mathematical model, and elucidating the physical principles that dictate the motor's behavior. By exploring Figure 1.4, we will be able to better understand the role of damping in the system and its implications for motor performance and energy efficiency. In the exploration of the dynamics of a DC motor, we've observed some significant effects when introducing damping into the system. By setting the damping coefficient to a seemingly minuscule 0.00004 N*m/(rad/s), we have noticed a considerable shift in the system's behavior. This seemingly insignificant change has a profound impact on both the current and angular velocity of the motor, illustrating the importance of the damping effect in dynamic systems.

Firstly, the maximum angular velocity of the motor decreased dramatically from approximately 35,000 rad/sec to around 8,500 rad/sec. This reduction indicates the impact of friction and other resistive forces represented by the damping term in the system of equations. These forces oppose the motion of the rotor and cause the angular velocity to reach a lower maximum value. This change in velocity reflects the transformation of some of the motor's mechanical energy into heat, reducing the overall kinetic energy available for rotational motion.

Moreover, we've noticed that the current within the system stabilized at a much higher value than without damping. Specifically, the steady-state current was approximately 0.5 amps, compared to essentially zero in the undamped system. This indicates that with damping, more current is continuously required to overcome the resistive forces acting on the rotor. Consequently, more power is needed to maintain the motor's operation under these conditions. This behavior also underscores an essential aspect of the electromechanical interplay in a DC motor. Namely, the angular velocity and current are not independent of one another but are intricately linked through the motor's electrical and mechanical characteristics. Variations in one (due to changes in parameters such as the damping coefficient) inevitably lead to adjustments in the other.

Hence, the damping effect serves as a vital factor to consider when modeling and predicting the behavior of a DC motor. Whether designing for energy efficiency, torque control, or overall performance, accounting for the damping factor ensures a more realistic representation of the motor's behavior under various conditions. It also underscores the need to consider all aspects of a system — electrical, mechanical, and thermal — to understand its overall performance accurately. The dampening effect in DC motors can be seen as an essential real-world phenomenon that must be accounted for in any accurate system model. While ideal motor models often ignore such factors for simplicity, real-world applications and analyses must consider the impacts of damping. Therefore, the accurate tuning and control of these motors necessitate a deep understanding of the complex interplay between electrical current, mechanical rotation, and the resistive forces at play.

Finally, Figure 3.0 presents a graphical representation of the relationship between the voltage applied to the DC motor and the corresponding terminal angular velocity at t=1s. This plot allows us to better understand the influence of voltage on the motor's performance. By examining this graph and interpreting its results, we can gain a more profound insight into the interaction between electrical input (voltage) and mechanical output (angular velocity) in a DC motor system. A key observation from Figure 3.0 is the almost linear relationship between the applied voltage and the terminal angular velocity at t=1s. This linear relationship can be expressed as $\omega = 1489.883V - 0.001$. This means that for every increase in the voltage by 1 unit, the angular velocity at 1s increases by approximately 1489.883 rad/s.

The mathematical and physical interpretation of this relationship is deeply rooted in the principles of electromagnetism and mechanics which govern the behavior of a DC motor. Here are some key points to consider:

1. Electrical Input: In our model, the voltage V is the primary electrical input to the system. This voltage drives the current through the motor's windings, creating a magnetic field. The strength of this magnetic field is directly proportional to the amount of current flowing through the motor, which in turn is dependent on the input voltage.

2. Magnetic Interaction: The magnetic field generated by the current interacts with the permanent magnet's field in the motor. This interaction results in a force which drives the rotation of the motor's rotor. The magnitude of this force, and thus the acceleration of the motor, is directly proportional to the magnetic field strength, which is influenced by the input voltage.

3. Mechanical Output: The angular velocity of the motor is the key mechanical output of the system. Based on Newton's second law, the acceleration (rate of change of velocity) of the motor is proportional to the force applied. Given that the force is influenced by the input voltage, it follows that the motor's angular velocity will also have a dependency on the voltage.

The almost linear relationship is a direct consequence of these principles. If we recall Ohm's Law (**V = IR**), we know that the current I flowing through the motor is directly proportional to the voltage V (ignoring back-emf at this point for simplicity). In the context of a DC motor, when a voltage V is applied across the motor's windings, a current I is induced. According to Lorentz Law (**F = ILB**), the force exerted on the motor's windings is proportional to the current. In a DC motor, the current flowing through the windings creates a magnetic field. This field interacts with the magnetic field of the motor's permanent magnets, producing a force F on the windings. This force is directly proportional to the current I, the length L of the wire, and the magnetic field strength B. This force leads to an acceleration of the motor's rotor due to Newton's Second Law, and thus a change in angular velocity.

As a result, the linear relationship we observe is a translation of these proportional relationships from electrical input to mechanical output. The slope of the line, 1489.883 rad/s/V, can be seen as a transfer rate that represents how efficiently the motor converts electrical energy (input voltage) into mechanical energy (angular velocity). The minor deviation from a perfect linearity expressed by the -0.001 term in the equation could be attributed to other factors such as the presence of back-emf (which tends to reduce the effective voltage and hence the current) and other mechanical factors such as friction and air resistance, which have not been accounted for in the model.

In conclusion, Figure 3.0 and the equation $\boldsymbol{\omega = 1489.883V - 0.001}$ provide a meaningful interpretation of the DC motor's behavior, reflecting the transformation of electrical energy into mechanical energy.

Conclusion:

In conclusion, this paper provided a comprehensive analysis of a prosthetic arm using an MG 996R DIGI HI TORQUE servo motor, focusing on the servo motor's mathematical modeling and performance over time. The prosthetic arm, designed by French sculptor and designer Gael Langevin, was constructed using 3D printed components from the InMoov open-source community. The servo motor controls the prosthetic hand's movement through the rotation of two strings per finger, allowing for a range of 180° motion. The servo motor's functionality is contingent on its four key components: a DC motor, a potentiometer, a gearbox, and a control unit. The servo motor's dimensions, connections, and PWM pulses are crucial to its performance, as they impact the servo's control and range of motion.

Mathematically, the servo motor can be modeled by analyzing the underlying DC motor and accounting for factors such as torque constant, back EMF constant, and inductance. Using Python, the servo motor's behavior was simulated over time by solving a set of coupled differential equations representing the motor's dynamics. The resulting graphs, Figure 1.1 (angular velocity vs. time) and Figure 1.2 (current vs. time), illustrate the motor's performance in the absence of load torque and damping.

This study dove further into the servo motor's characteristics by exploring the relationship between the applied voltage and the motor's terminal angular velocity. This correlation was meticulously analyzed through the simulation of 100 different voltage conditions, generating distinct graphs (Figures 2.00-2.99) for each scenario. Subsequently, the angular velocity at t=1s was recorded for each voltage condition, leading to the generation of Figure 3.00, which depicts the relationship between the applied voltage and the terminal angular velocity. The linear relationship observed, defined by the equation $\omega=1489.883V - 0.001$, showcases the predictable behavior of the motor as voltage varies. This relationship offers insightful revelations into the behavior of the servo motor, its efficiency, and its predictability, all of which are crucial for the functionality of the prosthetic arm.

Ultimately, this research paper offers valuable insights into the prosthetic arm's functionality and performance, particularly regarding the critical role of the MG 996R DIGI HI TORQUE servo motor. By understanding and modeling the servo motor's dynamics, this research paper paves the way for further optimization and potential improvements in the prosthetic arm's design and capabilities. Additionally, the mathematical models and simulation techniques employed in this paper can be adapted and applied to other mechatronic systems, thereby extending the knowledge, and understanding of the engineering community at large.

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