Vector Differentiation

06 October 2023
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$\phi(x,y,z) = x^2y^2z^2$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i(2xy^2z^2) + j(2x^2yz^2) + k(2x^2y^2z)$

GRADIENT:

If $\Phi(x,y,z)$ is a scalar point function then the gradient of Φ written as $\nabla\Phi$ or $grad\Phi$ is defined by

$$grad\ \Phi = \nabla\Phi = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$$

Remark: We also denote grad Φ as, $\text{grad}\ \Phi = \nabla\Phi = \sum i\frac{\partial\Phi}{\partial x} = i\frac{\partial\Phi}{\partial x} + j\frac{\partial\Phi}{\partial y} + k\frac{\partial\Phi}{\partial z}$

Note: **(1)** grad Φ is a vector point function

(2) If Φ is a constant then $\frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial z} = 0$ $\therefore$ grad $\Phi = 0$.

Results: **(1)** $\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$

(2) $\nabla(\Phi\Psi) = \Phi(\nabla\Psi) + (\nabla\Phi)\Psi$

(3) $\nabla f(u) = i\frac{\partial f(u)}{\partial x} + j\frac{\partial f(u)}{\partial y} + k\frac{\partial f(u)}{\partial z} = f'(u)\nabla u$

$\phi = \boxed{xyz = 3}$

GEOMETRICAL MEANING OF grad Φ :

$\nabla\Phi$ is a normal vector to the surface $\Phi(x,y,z) = c$ in the outward direction.

$\nabla\phi \to$ normal to surface

ANGLE BETWEEN TWO SURFACES:

We know that $\nabla\Phi$ is perpendicular to the tangent plane to the surface $\Phi(x,y,z) = c$. Hence, if $\Phi(x,y,z) = c_1$ and $\Psi(x,y,z) = c_2$ are two surfaces the angle between the two surface is equal to the angle between the normal i.e. the angle between $\nabla\Phi$ and $\nabla\Psi$.

If θ is the angle between them then $\theta = cos^{-1}\left|\frac{\nabla\Phi\cdot\nabla\Psi}{|\nabla\Phi||\nabla\Psi|}\right|$

If the surfaces are orthogonal then $\nabla\Phi\cdot\nabla\Psi = 0$

ϕ $\quad$ ψ

$\nabla\phi$ $\quad$ $\nabla\psi$

$\cos\theta = \frac{\nabla\phi\cdot\nabla\psi}{|\nabla\phi||\nabla\psi|}$

DIRECTIONAL DERIVATIVE:

$\nabla\Phi$ is a vector quantity its component (or resolved part) in the direction of a vector a is $\frac{\nabla\Phi\cdot a}{|a|}$

This component is called the directional derivative of Φ in the direction of a.

Thus, the directional derivative of Φ in the direction of $a = \frac{\nabla\Phi\cdot a}{|a|}$

Physically the directional derivative is the rate of change of Φ at (x, y, z) in the given direction.

$\frac{\nabla\phi\cdot\bar{a}}{|\bar{a}|}$

MAXIMUM DIRECTIONAL DERIVATIVE:

Since the resolved part of a vector is maximum in its own direction, the directional derivative is maximum in the direction $\nabla\Phi$. Since $\nabla\Phi$ is normal to the surface, we can also say that $\nabla\Phi$ is maximum in the direction of the normal to the surface and the maximum directional derivative is $|\nabla\Phi|$.

when $\bar{a} = \nabla\phi$

max d.d $= \frac{\nabla\phi\cdot\nabla\phi}{|\nabla\phi|} = |\nabla\phi|$

Examples:

Q.1 Find the angle between the normals to the surface $xy = z^2$ at the points $(1,4,2)$ and $(-3,-3,3)$

let $\phi = xy - z^2$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = i(y) + j(x) + k(-2z)$

let $\bar{a} = \nabla\phi\Big|_{(1,4,2)} = 4i + j - 4k$

let $\bar{a} = \nabla\phi|_{(1,4,2)}$

$\bar{b} = \nabla\phi|_{(-3,-3,3)} = -3i - 3j - 6k$

angle betn normals $= \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{4(-3)+(1)(-3)+(-4)(-6)}{\sqrt{4^2+1^2+(-4)^2}\ \sqrt{(-3)^2+(-3)^2+(-6)^2}}$

$= \dfrac{-12-3+24}{\sqrt{33}\ \sqrt{54}} = \dfrac{9}{3\sqrt{198}} = \dfrac{3}{\sqrt{198}}$

$\therefore\ \theta = \cos^{-1}\left(\dfrac{3}{\sqrt{198}}\right)$

Q.2 Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

Soln :- let $\phi = ax^2 - byz - (a+2)x$

$\psi = 4x^2y + z^3 - 4$

$\nabla\phi = i\dfrac{\partial\phi}{\partial x} + j\dfrac{\partial\phi}{\partial y} + k\dfrac{\partial\phi}{\partial z}$

$= i\left(2ax - (a+2)\right) + j(-bz) + k(-by)$

let $\bar{p} = \nabla\phi|_{(1,-1,2)} = (a-2)i - 2bj + bk$

Now $\nabla\psi = i\dfrac{\partial\psi}{\partial x} + j\dfrac{\partial\psi}{\partial y} + k\dfrac{\partial\psi}{\partial z}$

$= i(8xy) + j(4x^2) + k(3z^2)$

Let $\bar{q} = \nabla\psi|_{(1,-1,2)} = -8i + 4j + 12k$

The surfaces are orthogonal

$\therefore\ \bar{p}\cdot\bar{q} = 0 \Rightarrow -8(a-2) + 4(-2b) + 12(b) = 0$

$\Rightarrow -8a + 16 - 8b + 12b = 0$

$\Rightarrow -8a + 4b + 16 = 0$ ——— (1)

Now the point $(1,-1,2)$ lies on the surface

$ax^2 - byz = (a+2)x$

$a + 2b = (a+2)$ ———

$$\Rightarrow 2b = 2 \quad \Rightarrow \boxed{b=1}$$

Sub this in ① $\longrightarrow$ $-8a+4b+16=0$

$-8a+4+16=0 \Rightarrow a = \frac{20}{8} = \frac{5}{2}$

Q.3 Find the directional derivative of $\Phi(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the vector $i + 2j + 2k$.

Solⁿ :- $\phi = xy^2 + yz^3$ $\qquad \bar{a} = i+2j+2k$

$$d \cdot d = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(y^2) + j(2xy + z^3) + k(3yz^2)$$

$$\nabla\phi\Big|_{(2,-1,1)} = i - 3j - 3k \qquad (\bar{a} = i+2j+2k)$$

$$d \cdot d = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|} = \frac{(1)(1) + (-3)(2) + (-3)(2)}{\sqrt{1^2+2^2+2^2}} = \frac{-11}{3}$$

Q.4 Find the directional derivative of $\Phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at $(1,2,3)$

Solⁿ :- $\phi = x^2+y^2+z^2$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i(2x) + j(2y) + k(2z)$$

$$\nabla\phi\Big|_{(1,2,3)} = 2i + 4j + 6k$$

Now the direction of given line $\bar{a} = 3i+4j+5k$

$$d \cdot d = \frac{\nabla\phi \cdot \bar{a}}{|\bar{a}|} = \frac{(2)(3)+(4)(4)+6(5)}{\sqrt{3^2+4^2+5^2}} = \frac{6+16+30}{\sqrt{50}} = \frac{52}{\sqrt{50}}$$

Q.5 Find the directional derivative of $\Phi = \frac{y}{x^2+y^2}$ at $(0,1)$ in the direction making an angle of $30°$ with the positive x – axis .

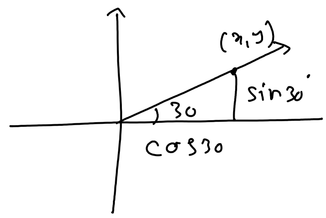
Soln :- $\phi = \dfrac{y}{x^2+y^2}$

$$\frac{\partial\phi}{\partial x} = y\left[\frac{-1}{(x^2+y^2)^2}\cdot 2x\right] = \frac{-2xy}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{(x^2+y^2)\cdot(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z} = \frac{-2xy}{(x^2+y^2)^2}\,i + \frac{(x^2-y^2)}{(x^2+y^2)^2}\,j$$

$$\nabla\phi\big|_{(0,1)} = 0i - j$$



Given direction $\rightarrow$ angle of 30° with positive x axis

Unit vector along given direction. $\cos 30\, i + \sin 30\, j$

$$\bar{a} = \frac{\sqrt{3}}{2}i + \frac{1}{2}j \rightarrow |\bar{a}| = 1$$

$$d.d = \frac{\nabla\phi\cdot\bar{a}}{|\bar{a}|} = (0i - j)\cdot\left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = \frac{-1}{2}$$

Q.6 In what direction from the point $(2,1,-1)$ is the directional derivative of $\Phi = x^2yz^3$ maximum? what is its magnitude ?

Soln :- $\phi = x^2yz^3$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i(2xyz^3)+j(x^2z^3)+k(3x^2yz^2)$$

$$\nabla\phi\big|_{(2,1,-1)} = -4i-4j+12k$$

Directional derivative is maximum in the direction of $\nabla\phi$

Hence d.d. is maximum in the direction of $-4i-4j+12k$

It's magnitude $= |\nabla\phi|$

Its max tude $= |\nabla\phi|$

$$= \sqrt{4^2+4^2+12^2}$$

$$= \sqrt{16+16+144} = \sqrt{176} = 4\sqrt{11}$$

Q.7 Find the values of a, b, c if the directional derivative of $\Phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has maximum magnitude 64 in the direction parallel to the $z-$axis.

Sol^n:- we have $\phi = axy^2 + byz + cz^2x^3$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = i(ay^2 + 3cz^2x^2) + j(2axy + bz) + k(by + 2czx^3)$$

$$\nabla\phi|_{(1,2,-1)} = i(4a+3c) + j(4a-b) + k(2b-2c)$$

The max d.d. is in the direction of $\nabla\phi$

ie in the direction of $(4a+3c)i + (4a-b)j + (2b-2c)k$

But by given data, the d.d. is max in the direction of

z-axis ie $0i + 0j + k$

$\Rightarrow$ $\underline{4a+3c=0}$, $\underline{4a-b=0}$

$\therefore$ $\nabla\phi = 2b-2c\,k$

max mag $= |\nabla\phi| = 64$

$\Rightarrow |2b-2c| = 64$ $\Rightarrow$ $\underline{2b-2c=64}$ $\Rightarrow$ $b-c=32$

$\therefore$ $a=6$, $b=24$, $c=$

DIVERGENCE:

Let $F(x,y,z) = f_1 i + f_2 j + f_3 k$ be a vector point function defined in a certain region of space, where the components f_1, f_2, f_3 are functions of x, y, z then the divergence of F written as $\nabla \cdot F$ or $divF$ is defined by

$divF = \nabla \cdot F = (\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k) \cdot (f_1 i + f_2 j + f_3 k) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$\bar{F} = x^2 i +$ [illegible] $j + x^3z^3 k$

Note: $\nabla \cdot F$ is a scalar point function

$\nabla \cdot \bar{F} = 2x + xz + 3x^3z^2$

CURL :

Let $F(x,y,z) = f_1 i + f_2 j + f_3 k$ be a vector point function defined in a certain region of space then the curl of F, written as $\nabla \times F$ or curl F is defined by

$$\text{Curl}F = \nabla \times F = \left(\frac{\partial}{\partial x}\imath + \frac{\partial}{\partial y}\jmath + \frac{\partial}{\partial z}k\right) \times (f_1\imath + f_2\jmath + f_3k) = \begin{vmatrix} \imath & \jmath & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note: $\nabla \times F$ is a vector point function

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & & x^3z^3 \end{vmatrix} = i\left(\frac{\partial}{\partial y}(x^3z^3) - \frac{\partial}{\partial z}(xyz)\right)$$
$$-j\left(\frac{\partial}{\partial x}(x^3z^3) - \frac{\partial}{\partial z}(x^2)\right)$$
$$+k\left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(x^2)\right)$$

$$\text{Curl } \bar{F} = i(-xy) - j(3x^2z^3) + k(yz)$$

1.	**Gradient of Scalar Point Function is Vector Point Function**
2.	**Divergence of Vector Point Function is Scalar Point**
3.	**Curl of Vector Point Function is Vector Point Function**

Examples:

Q.1 If $\Phi = x^3 + y^3 + z^3 - 3xyz$, find **(a)** $\bar{r}.\nabla\Phi$ **(b)** div $\bar{F}$ **(c)** curl $\bar{F}$ where $\bar{F} = \nabla\Phi$.

Soln:- $\bar{F} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$$\bar{F} = \nabla\phi = i(3x^2 - 3yz) + j(3y^2 - 3xz) + k(3z^2 - 3xy)$$

(a) $\bar{r}\cdot\nabla\phi = (xi + yj + zk)\cdot((3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k)$

$$= x(3x^2 - 3yz) + y(3y^2 - 3xz) + z(3z^2 - 3xy)$$
$$= 3x^3 - 3xyz + 3y^2 - 3xyz + 3z^2 - 3xyz$$
$$= 3(x^3 + y^3 + z^3 - 3xyz)$$

(b) $\text{div } \bar{F} = \nabla\cdot\bar{F}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((3x^2 - 3yz)i + (3y^2 - 3xz)j + (3z^2 - 3xy)k\right)$$
$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$\text{Div } \bar{F} = 6x$ ——— $6z = 6(x+y+z)$

(c) $\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= i\left[\frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz)\right] \ldots$

$$= i\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] - j\left[\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$= i\left[-3x+3x\right] - j\left[-3y+3y\right] + k\left[-3z+3z\right] = \bar{0}$$

Q.2 If $\bar{f} = (x+y+1)i + j - (x+y)k$, prove that $\bar{f}$. Curl $\bar{f} = 0$

Soln:- $\bar{f} = (x+y+1)i + j - (x+y)k$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= i\left[-1-0\right] - j\left[-1-0\right] + k\left[0-1\right]$$

$$\text{curl } \bar{F} = -i + j - k$$

$$\bar{f}\cdot \text{curl } \bar{f} = \left[(x+y+1)i + j - (x+y)k\right]\cdot(-i+j-k)$$

$$= -(x+y+1) + 1 + (x+y)$$

$$\bar{f}\cdot\text{curl} f = 0$$

Q.3 If $\bar{a}$ is a constant vector and $\bar{r} = xi + yj + zk$, prove that,

(i) $div\,\bar{a} = 0$ **(ii)** $curl\,\bar{a} = \bar{0}$ **(iii)** $grad\,r = \frac{1}{r}\bar{r}$ **(iv)** $div\,\bar{r} = 3$ **(v)** $curl\,\bar{r} = \bar{0}$

Soln:- $\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{r} = xi + yj + zk$

$$\text{div } \bar{a} = \nabla\cdot\bar{a} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(a_1 i + a_2 j + a_3 k)$$

$$= \frac{\partial}{\partial x}(a_1) + \frac{\partial}{\partial y}(a_2) + \frac{\partial}{\partial z}(a_3)$$

$$= \frac{\partial}{\partial x}(a_1) + \frac{\partial}{\partial y}(a_2) + \frac{\partial}{\partial z}(a_3)$$

$$= 0 + 0 + 0 = 0$$

(b) $\text{curl } \bar{a} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(a_3) - \frac{\partial}{\partial z}(a_2)\right] - j\left[\frac{\partial}{\partial x}(a_3) - \frac{\partial}{\partial z}(a_1)\right]$$
$$+ k\left[\frac{\partial}{\partial x}(a_2) - \frac{\partial}{\partial y}(a_1)\right]$$
$$= 0i + 0j + 0k = \bar{0}$$

(c) $\text{grad } r = \nabla r$

$$= i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}$$

$\left[\begin{array}{l} \bar{r} = xi + yj + zk \\ r = \sqrt{x^2+y^2+z^2} \end{array} \right]$

$\underline{\underline{\text{or}}}$ $\nabla f(r) = \dfrac{f'(r)\,\bar{r}}{r} \Rightarrow \nabla r = \dfrac{(1)\,\bar{r}}{r} = \dfrac{\bar{r}}{r}$

$f(r) = r \Rightarrow f'(r) = 1$

(d) $\text{div } \bar{r} = \nabla \cdot \bar{r} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(xi + yj + zk)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

(e) $\text{curl } \bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= i\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - j\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + k\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$
$$= 0i + 0j + 0k = \bar{0}$$

Q.4 If $\bar{a}$ is constant vector and $\bar{r} = xi + yj + zk$, prove that, $div\ (\bar{a} \times \bar{r}) = 0$.

$\bar{a} = a_1 i + a_2 j + a_3 k$

$\bar{r} = xi + yj + zk$

$r = xi + yj + zk$

$$\operatorname{div}(\bar{a} \times \bar{r}) = \nabla \cdot (\bar{a} \times \bar{r}) = [\nabla \ \bar{a} \ \bar{r}]$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}[a_2 z - a_3 y] - \frac{\partial}{\partial y}[a_1 z - a_3 x] + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 + 0 + 0 = 0$$

Solenoidal Vector field

A vector F whose $divF = 0$ is called **solenoid**

Irrotational Vector Field

A vector F whose $curl\ F = 0$ is called **Irrotational.**

In general, if $\nabla \times F = 0$ i.e curl $F = 0$ then we can find scalar field Φ so that $F = \nabla \Phi$

A ~~vector field F which~~ can be derived from a scalar field Φ so that $F = \nabla\Phi$ is called a **conservation vector field** and Φ is called the **scalar potential**

conversely also, if $F = \nabla\Phi$ then $\nabla \times F = 0$ i.e curl $F = 0$

SOME EXAMPLES:

1. If $\overline{F} = (x + 3y)i + (y - 2z)j + (az + x)k$ is solenoidal, find the value of a.

Since $\bar{F}$ is solenoidal $\Rightarrow \operatorname{div} \bar{F} = 0$

$$\therefore \nabla \cdot \bar{F} = 0 \Rightarrow \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left((x+3y)i + (y-2z)j + (az+x)k\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(az+x) = 0$$

$$\Rightarrow 1 + 1 + a = 0 \quad \Rightarrow \quad a = -2$$

2. Find a, b, c if $\overline{F} = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ is irrotational.

Since $\overline{F}$ is irrotational $\Rightarrow$ curl $\overline{F} = \overline{0}$

$$\text{curl } \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+bz^3 & 3x^2-cz & 3xz^2-y \end{vmatrix} = \overline{0}$$

$$\Rightarrow i\left[-1+c\right] - j\left[3z^2 - 3bz^2\right] + k\left(6x - ax\right) = \overline{0}$$

$$\Rightarrow -1+c = 0, \quad 3z^2 - 3bz^2 = 0, \quad 6x - ax = 0$$

$$\Rightarrow c = 1, \quad b = 1, \quad a = 6$$

3. A vector field is given by $\overline{F} = (x^2 + xy^2)i + (y^2 + x^2y)j$. Show that $\overline{F}$ is irrotational and find its scalar potential.

Solⁿ :-

$$\text{Curl } \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(y^2+x^2y\right)\right] - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(x^2+xy^2\right)\right]$$

$$+ k\left[\frac{\partial}{\partial x}\left(y^2+x^2y\right) - \frac{\partial}{\partial y}\left(x^2+xy^2\right)\right]$$

$$= i(0) + j(0) + k(0) = \overline{0}$$

$\therefore$ Curl $\overline{F} = \overline{0}$ $\Rightarrow$ $\overline{F}$ is irrotational

To find scalar potential

$$\bar{F} = \nabla \phi$$

$$(x^2+xy^2)i + (y^2+x^2y)j = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = x^2+xy^2 \quad \Rightarrow \quad \phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + f(y)$$

$$\frac{\partial \phi}{\partial y} = y^2+x^2y \quad \Rightarrow \quad \phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + g(x)$$

$$\therefore \quad \phi = \frac{x^2y^2}{2} + \frac{x^3}{3} + \frac{y^3}{3} + C$$

IDENTITIES INVOLVING GRAD, DIV, CURL:

1. $grad\,(\Phi \pm \Psi) = grad\,\Phi \pm grad\,\Psi$
$\nabla(\Phi \pm \Psi) = \nabla\Phi \pm \nabla\Psi$

2. $div\,(f \pm g) = div\,f \pm div\,g$
$\nabla \cdot (f \pm g) = \nabla \cdot f \pm \nabla \cdot g$

3. $Curl\,(f \pm g) = Curl\,f \pm Curl\,g$
$\nabla \times (f \pm g) = \nabla \times f \pm \nabla \times g$

4. $grad\,(\Phi\Psi) = \Phi\,grad\Psi + \Psi\,grad\Phi$
$\nabla(\Phi\Psi) = \Phi\,\nabla\Psi + \Psi\,\nabla\Phi$ where Φ and Ψ are scalar functions

5. $grad\,(f \cdot g) = f \times (curl\,g) + g \times (curl f) + (f \cdot \nabla)g + (g \cdot \nabla)f$
$\nabla(f \cdot g) = f \times (\nabla \times g) + g \times (\nabla \times f) + (f \cdot \nabla)g + (g \cdot \nabla)f$

6. $div\,(\Phi f) = \Phi div f + f \cdot grad\,\Phi$
$\nabla \cdot (\Phi f) = \Phi(\nabla \cdot f) + f \cdot (\nabla\Phi)$

7. $div\,(f \times g) = g \cdot curl f - f \cdot curl g$
$\nabla \cdot (f \times g) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g)$

8. $curl\,(f \times g) = f div\,g - g\,div\,f + (g \cdot \nabla)f - (f \cdot \nabla)g$
$\nabla \times (f \times g) = f(\nabla \cdot g) - g\,(\nabla \cdot f) + (g \cdot \nabla)f - (f \cdot \nabla)g$

9. $curl\,(\Phi f) = \Phi(curl f) + (grad\,\Phi) \times f$
$\nabla \times (\Phi f) = \Phi(\nabla \times f) + (\nabla\,\Phi) \times f$