Introduction

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**Definition of a Periodic Function:**

When at equal intervals of abscissa 'x', the value of each ordinate $f(x)$ repeats itself, i.e $f(x) = f(x+T)$, for all $x$, then $y = f(x)$ is called a periodic function having period T.

**eg.** $\sin x$ and $\cos x$ are periodic of period $2\pi$, since $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \ldots..$

and $\cos x = \cos(x+2\pi) = \cos(x+4\pi) = \ldots..$

**Definition:**

Let $f(x)$ be a periodic function of period $2\pi$, defined in the interval $(c, c+2\pi)$, satisfying Dirichlet's Conditions as **(i)** $f(x)$ and its integrals are finite and single valued,

**(ii)** $f(x)$ has discontinuities finite in number ,

**(iii)** $f(x)$ has finite number of maxima and minima ,

then $f(x)$ can be expanded as an infinite trigonometric series as: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

which is called **Fourier Series**, where $a_0, a_n, b_n (n = 1,2,3, \ldots)$ are called Fourier coefficients or Fourier constants.

**EULER'S FORMULAE:**

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$ , $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\cos nx\, dx$ , $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\sin nx\, dx$ ......(I)

These values of $a_0, a_n, b_n$ represented by (I) are known as Euler's Formulae.

**TO ESTABLISH EULER'S FORMULAE:**

The following definite integrals will be required to establish Euler's formulae:

1. $\int_c^{c+2\pi} \cos mx\, dx = \left|\frac{\sin mx}{m}\right|_c^{c+2\pi} = 0, \{for\ all\ m \neq 0\}$
2. $\int_c^{c+2\pi} \sin mx\, dx = -\left|\frac{\cos mx}{m}\right|_c^{c+2\pi} = 0, \{for\ all\ m \neq 0\}$
3. $\int_c^{c+2\pi} \sin mx\, cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi}(\sin(m+n)x + \sin(m-n)x)\, dx = 0$
4. $\int_c^{c+2\pi} \cos mx . \cos nx\, dx$
$$= \begin{cases} \frac{1}{2}\int_c^{c+2\pi}(\cos(m+n)x + \cos(m-n)x)\, dx = 0, & \{for\ m \neq n, by\ virtue\ of\ (1)\} \\ \int_c^{c+2\pi} cos^2 mx\, dx = \frac{1}{2}\int_c^{c+2\pi}[1 + \cos 2mx]\, dx = \pi, & \{for\ m = n\} \end{cases}$$
5. $\int_c^{c+2\pi} \sin mx . \sin nx\, dx$
$$= \begin{cases} \frac{1}{2}\int_c^{c+2\pi}(\cos(m-n)x - \cos(m+n)x)\, dx = 0, & \{for\ m \neq n, by\ virtue\ of\ (1)\} \\ \int_c^{c+2\pi} sin^2 mx\, dx = \frac{1}{2}\int_c^{c+2\pi}[1 - \cos 2mx]\, dx = \pi, & \{for\ m = n\} \end{cases}$$

**Theorem:** Establish Euler's formulae represented by (I)

**Proof:** Let $f(x)$ be represented in the interval $(c, c+2\pi)$ by the Fourier series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx) + \sum_{n=1}^{\infty}(b_n \sin nx)$ ................(*)

**To find $a_0$**, integrating both sides of (*) w.r.t x from $c$ to $+2\pi$ , we have

$$\int_c^{c+2\pi} f(x)dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(a_n \cos nx)\, dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(b_n \sin nx)\, dx$$

$= \frac{a_0}{2}|c + 2\pi - c| + 0 + 0$ {by the integrals (1) and (2) in the above section }

$= a_0 . \pi$

Hence $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

**To find $a_n$**, multiplying both sides of (*) by cos nx and then integrating w.r.t x from $c$ to $c+2\pi$, we have

$$\int_c^{c+2\pi} f(x)\cos nx\,dx$$
$$= \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(a_n\cos nx)\cos nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(b_n\sin nx)\cos nx\,dx$$

$= 0 + a_n.\pi + 0$ { by the integrals (1), (3) and (4) in the above section }

$= a_n.\pi$

Hence $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\cos nx\,dx$

**To find $b_n$**, multiply both sides of (*) by sin nx and then integrating w.r.t x from $c$ to $c+2\pi$, we have

$$\int_c^{c+2\pi} f(x).\sin nx\,dx$$
$$= \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(a_n\cos nx)\sin nx\,dx + \int_c^{c+2\pi}\sum_{n=1}^{\infty}(b_n\sin nx)\sin nx\,dx$$

$= 0 + 0 + b_n.\pi$ { by the integrals (2) , (3) and (5) in the above section }

$= b_n.\pi$

Hence $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x).\sin nx\,dx$

**Fourier Series for f(x) [ Even / Odd / Neither Even nor Odd ] in different interval**

| INTERVAL | FOURIER SERIES | $a_0$ | $a_n$ | $b_n$ |
|---|---|---|---|---|
| $(0,2\pi)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx) + (b_n\sin nx)$ | $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$ | $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx$ | $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx\,dx$ |
| $(0,2l)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l})) + (b_n\sin(\frac{n\pi x}{l}))$ | $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$ | $a_n = \frac{1}{l}\int_0^{2l} f(x).\cos(\frac{n\pi x}{l})\,dx$ | $b_n = \frac{1}{l}\int_0^{2l} f(x).\sin(\frac{n\pi x}{l})\,dx$ |
| $(-\pi,\pi)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx) + (b_n\sin nx)$ | $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$ | $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\cos nx\,dx$ | $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\,dx$ |
| $(-l,l)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l})) + (b_n\sin(\frac{n\pi x}{l}))$ | $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$ | $a_n = \frac{1}{l}\int_{-l}^{l} f(x).\cos(\frac{n\pi x}{l})\,dx$ | $b_n = \frac{1}{l}\int_{-l}^{l} f(x).\sin(\frac{n\pi x}{l})\,dx$ |
| Even function in $(-\pi,\pi)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx)$ | $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ | $a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx$ | $b_n = 0$ |
| Even function in $(-l,l)$ | $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi x}{l}))$ | $a_0 = \frac{2}{l}\int_0^{l} f(x)dx$ | $a_n = \frac{2}{l}\int_0^{l} f(x).\cos(\frac{n\pi x}{l})\,dx$ | $b_n = 0$ |
| Odd function in $(-\pi,\pi)$ | $f(x) = \sum_{n=1}^{\infty}(b_n\sin nx)$ | $a_0 = 0$ | $a_n = 0$ | $b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin nx\,dx$ |
| Odd function in $(-l,l)$ | $f(x) = \sum_{n=1}^{\infty}(b_n\sin(\frac{n\pi x}{l}))$ | $a_0 = 0$ | $a_n = 0$ | $b_n = \frac{2}{l}\int_0^{l} f(x).\sin(\frac{n\pi x}{l})\,dx$ |

## SOME IMPORTANT RESULTS:

**1.** Bernoulli's generalized formula of integration by parts:

$\int uv = uv_1 - u'v_2 + u''v_3 -$……………… until derivatives of $u$ vanish,

where $u', u'', \ldots\ldots\ldots\ldots$ are the successive derivatives of $u$ and $v_1, v_2, \ldots\ldots\ldots\ldots$ are the successive integrations of $v$.

**2.** $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

$ax$ $ax$

**3.** $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

**4. Property of definite integrals:**

$\int_{-a}^{a} f(x)dx = 0,$ when $f(x)$ is odd function

$= 2\int_0^a f(x)dx$ when $f(x)$ is even function

## PARSEVAL'S IDENTITY:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$ is the Fourier series in $(0, 2c)$ then prove that

$$\frac{1}{c}\int_0^{2c}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\} \quad \text{............(I)}$$

The formula represented by (I) is known **as Parseval's Identity.**

**Proof:** The Fourier for $f(x)$ in $(0, 2c)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$ ..............(II)

where
$$\left.\begin{aligned} a_0 &= \frac{1}{c}\int_0^{2c} f(x)dx \\ a_n &= \frac{1}{c}\int_0^{2c} f(x).\cos\frac{n\pi x}{c}dx \\ b_n &= \frac{1}{c}\int_0^{2c} f(x).\sin\frac{n\pi x}{c}dx \end{aligned}\right\} \quad \text{............(III)}$$

Multiplying both sides of (II) by $f(x)$ and integrating term by term w.r.t x from 0 to $2c$.

$$\int_0^{2c}|f(x)|^2\,dx = \frac{a_0}{2}\int_0^{2c} f(x)dx + \sum_{n=1}^{\infty}\left(a_n\int_0^{2c} f(x).\cos\frac{n\pi x}{c}dx + b_n\int_0^{2c} f(x).\sin\frac{n\pi x}{c}dx\right)$$

$\therefore \int_0^{2c}|f(x)|^2dx = c.\left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ {Using the result (III)}

$\frac{1}{c}\int_0^{2c}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ which is the required Parseval's identity.

| **Intervals** | **Parseval's identity** |
|---|---|
| $(0, 2c)$ | $\frac{1}{c}\int_0^{2c}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ |
| $(0, 2\pi)$ | $\frac{1}{\pi}\int_0^{2\pi}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ |
| $(-c, c)$ | $\frac{1}{c}\int_{-c}^{c}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ |
| $(-\pi, \pi)$ | $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$ |
| half – range cosine series in $(0, c)$ | $\frac{2}{c}\int_0^{c}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)\right\}$ |
| half – range cosine series in $(0, \pi)$ | $\frac{2}{\pi}\int_0^{\pi}|f(x)|^2dx = \left\{\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)\right\}$ |
| half – range sine series in $(0, c)$ | $\frac{2}{c}\int_0^{c}|f(x)|^2dx = \sum_{n=1}^{\infty}(b_n^2)$ |
| half – range sine series in $(0, \pi)$, | $\frac{2}{\pi}\int_0^{\pi}|f(x)|^2dx = \sum_{n=1}^{\infty}(b_n^2)$ |