GRAPHS

Outline

- Graph- Concept
- Graph terminology: vertex, edge, adjacent, incident, degree, cycle, path, connected component, spanning tree
- Types of graphs: undirected, directed, weighted
- Graph representations: adjacency matrix, array adjacency lists, linked adjacency lists
- Graph search methods: breath-first, depth-first search

What is a graph?

- A data structure that consists of a set of nodes (*vertices*) and a set of edges that relate the nodes to each other
- The set of edges describes relationships among the vertices

Formal definition of graphs

- A graph *G* is defined as follows:

  $$G=(V,E)$$

  V(G): a finite, nonempty set of vertices

  E(G): a set of edges (pairs of vertices)

- Vertices are also called nodes and points.
- Each edge connects two vertices.
- Edges are also called arcs and lines.
- Vertices *i* and *j* are adjacent vertices iff (*i*, *j*) is an edge in the graph
- The edge (*i*, *j*) is incident on the vertices *i* and *j*

Graphs

- Undirected edge has no orientation (no arrow head)
- Directed edge has an orientation (has an arrow head)
- Undirected graph – all edges are undirected
- Directed graph – all edges are directed

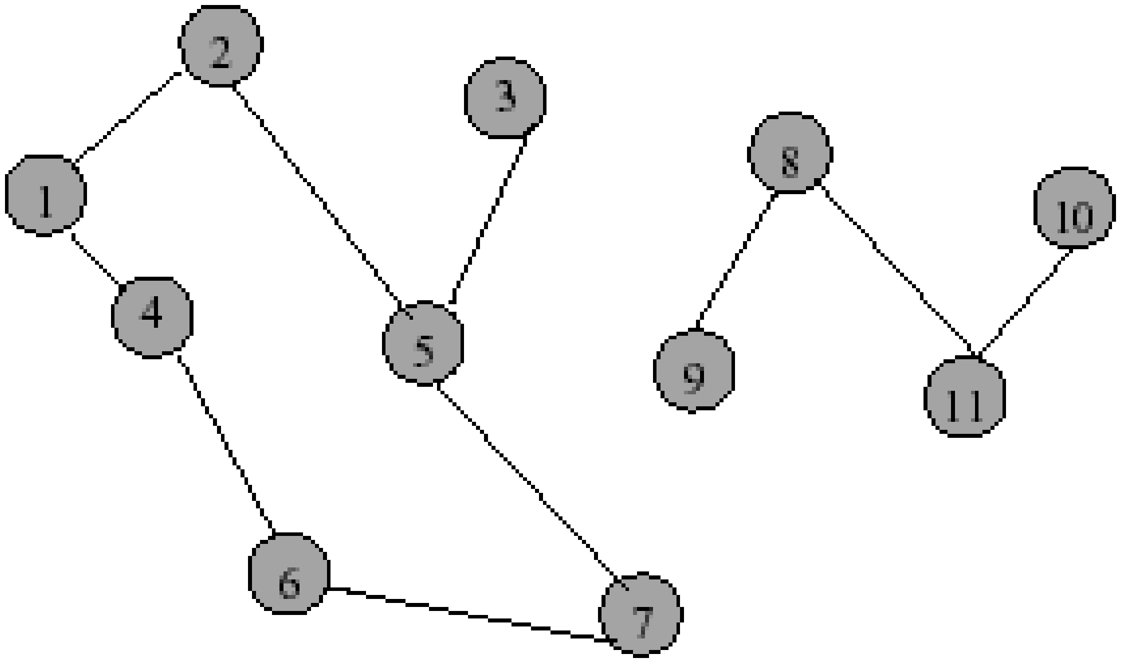
$$\begin{tikzcd}u \arrow[r, dash] & v\end{tikzcd}$$

undirected edge

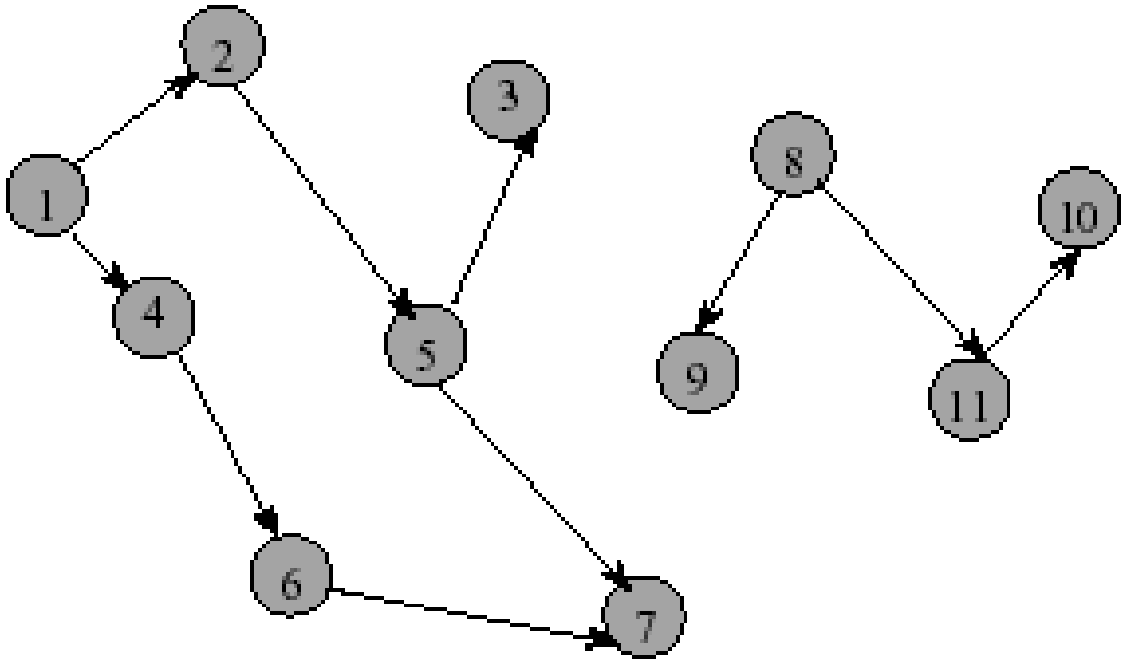
$$\begin{tikzcd}u \arrow[r] & v\end{tikzcd}$$

directed edge

Undirected Graph

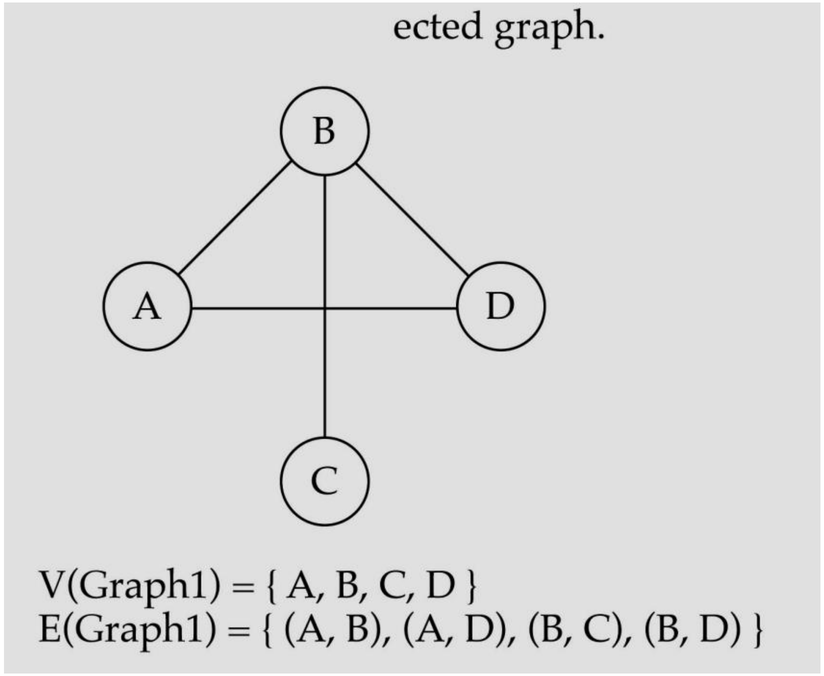


Directed Graph (Digraph)



Directed vs. undirected graphs

- When the edges in a graph have no direction, the graph is called *undirected*

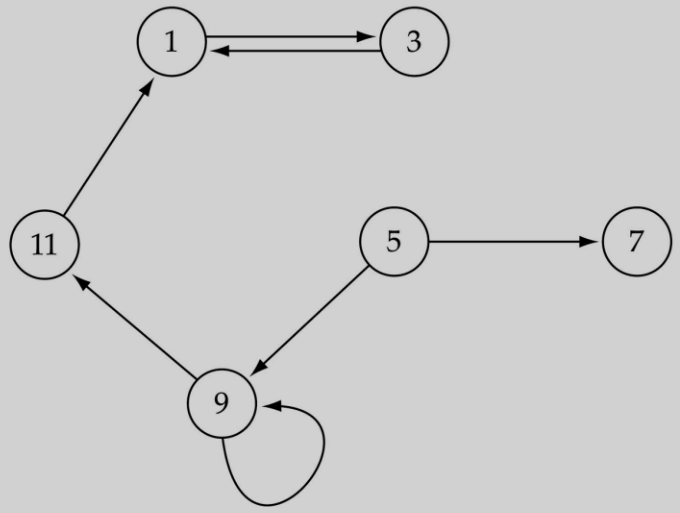


V(Graph1) = { A, B, C, D }
E(Graph1) = { (A, B), (A, D), (B, C), (B, D) }

Directed vs. undirected graphs (cont.)

- When the edges in a graph have a direction, the graph is called *directed* (or *digraph*)

(b) Graph2 is a directed graph.

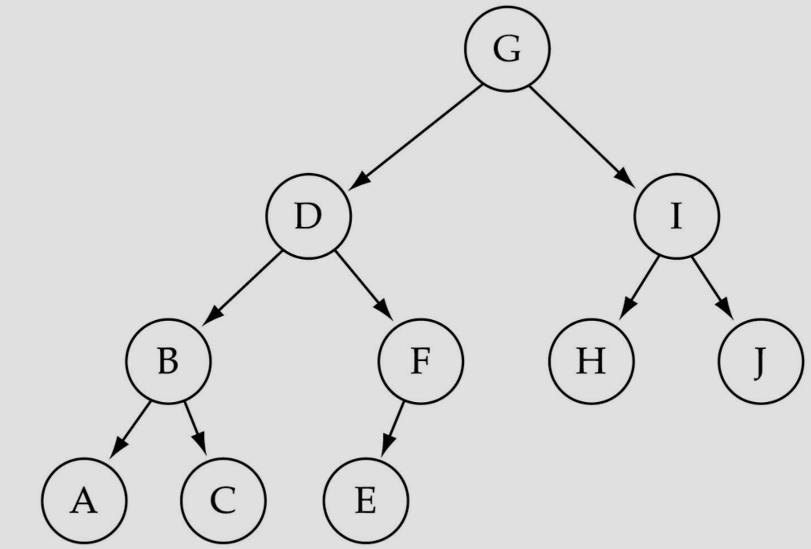


V(Graph2) = { 1, 3, 5, 7, 9, 11 }
E(Graph2) = {(1,3) (3,1) (5,9) (9,11) (5,7) 1), (9, 9), (11, 1) }

Trees vs graphs

- Trees are special cases of graphs!!

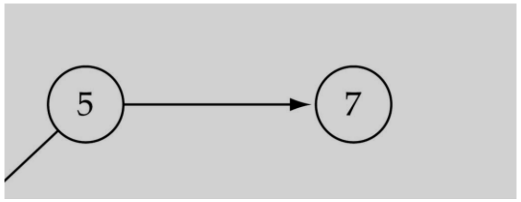
(c) Graph3 is a directed graph.



V(Graph3) = { A, B, C, D, E, F, G, H, I, J }
E(Graph3) = { (G, D), (G, J), (D, B), (D, F) (I, H), (I, J), (B, A), (B, C), (F, E) }

Graph terminology

- Adjacent nodes: two nodes are adjacent if they are connected by an edge



5 is adjacent to 7

7 is adjacent from 5

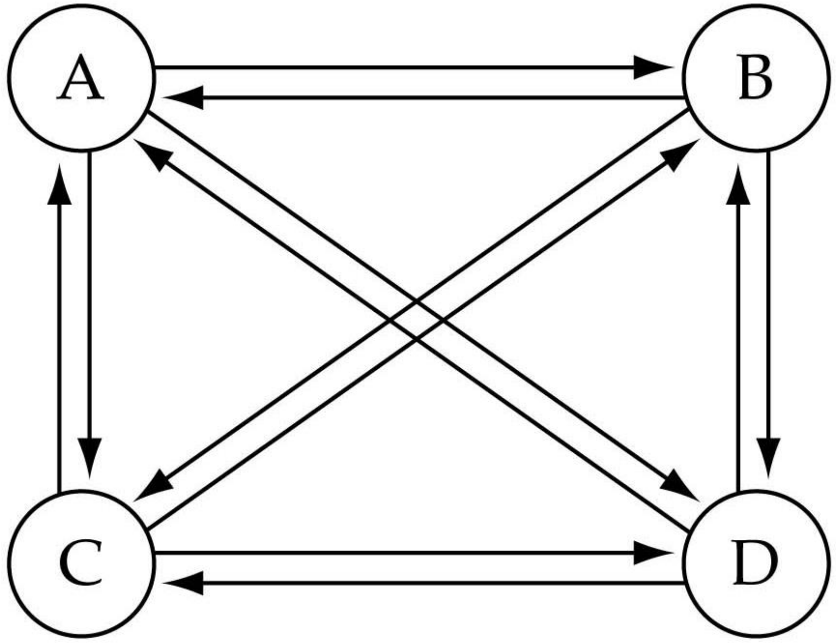
- Path: a sequence of vertices that connect two nodes in a graph
- Complete graph: a graph in which every vertex is directly connected to every other vertex

Graph terminology (cont.)

- What is the number of edges in a complete directed graph with N vertices?

*N * (N-1)*

$O(N^2)$



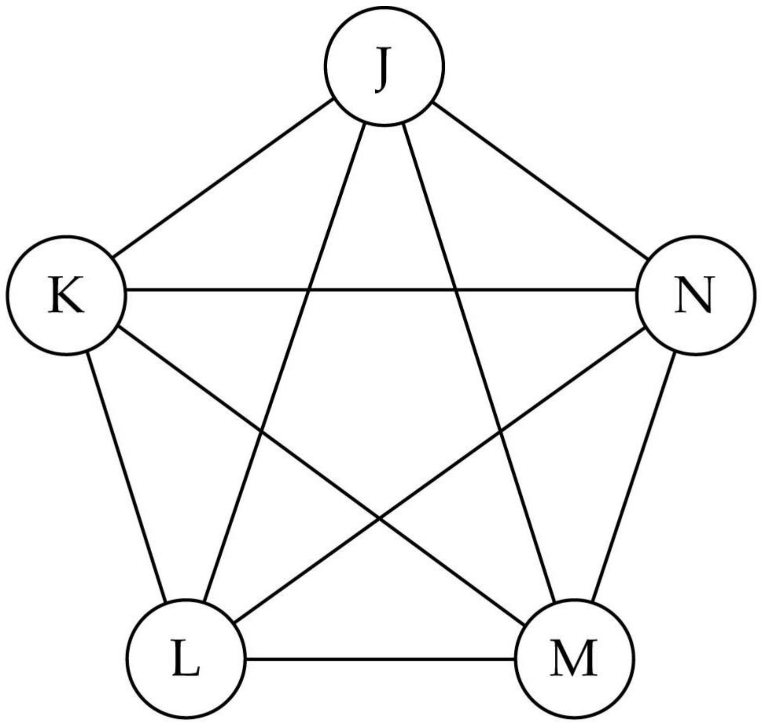
(a) Complete directed graph.

Graph terminology (cont.)

- What is the number of edges in a complete undirected graph with N vertices?

*N * (N-1) / 2*

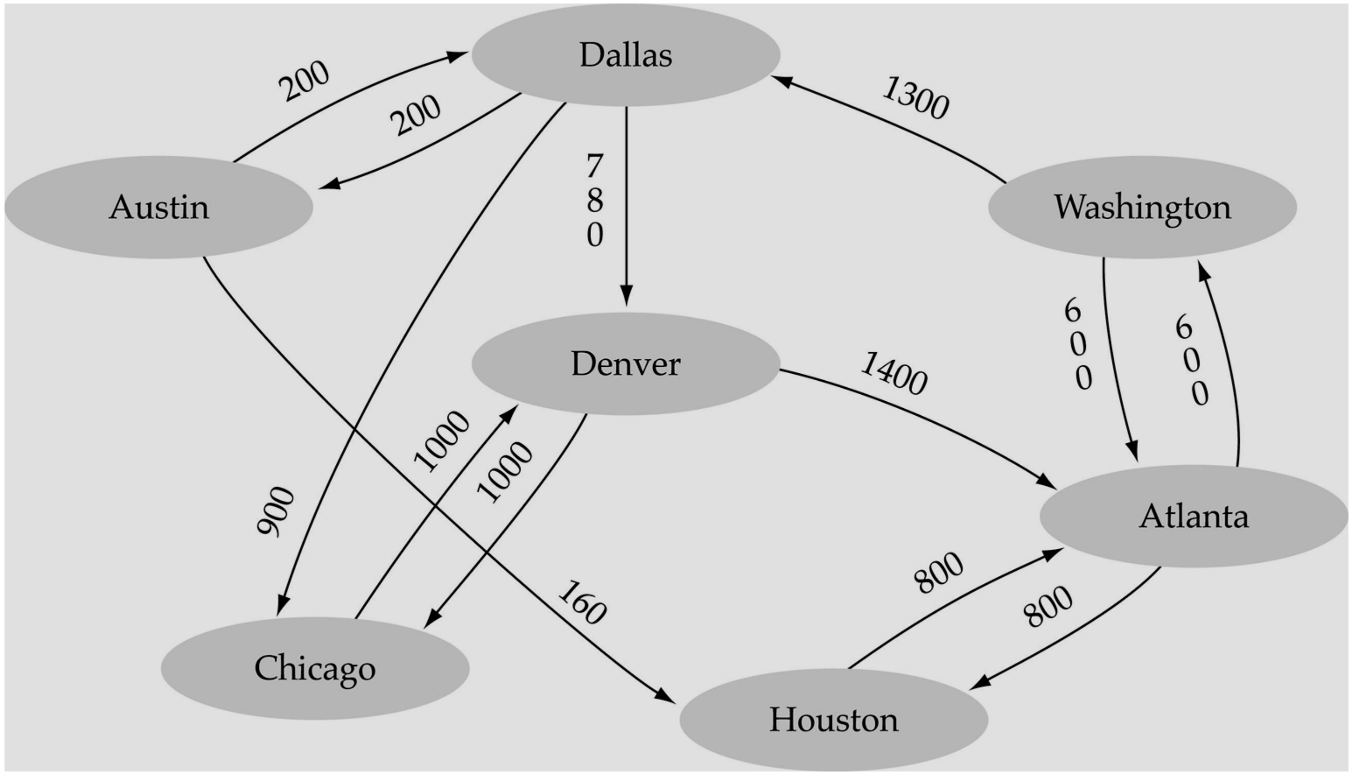
$O(N^2)$



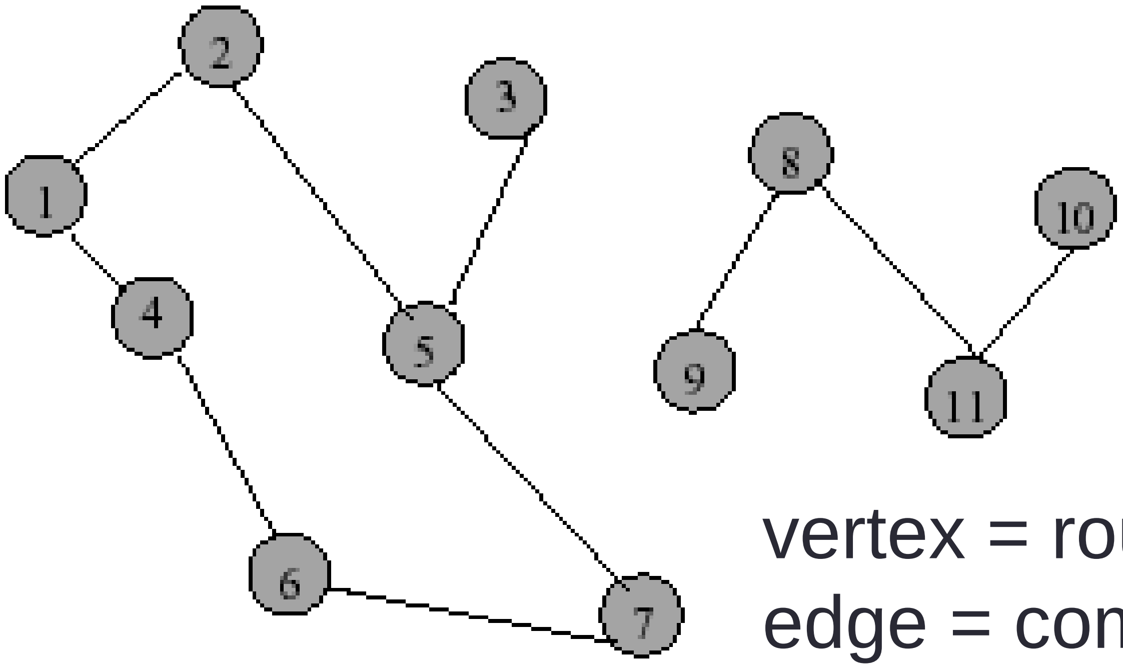
(b) Complete undirected graph.

Graph terminology (cont.)

- Weighted graph: a graph in which each edge carries a value



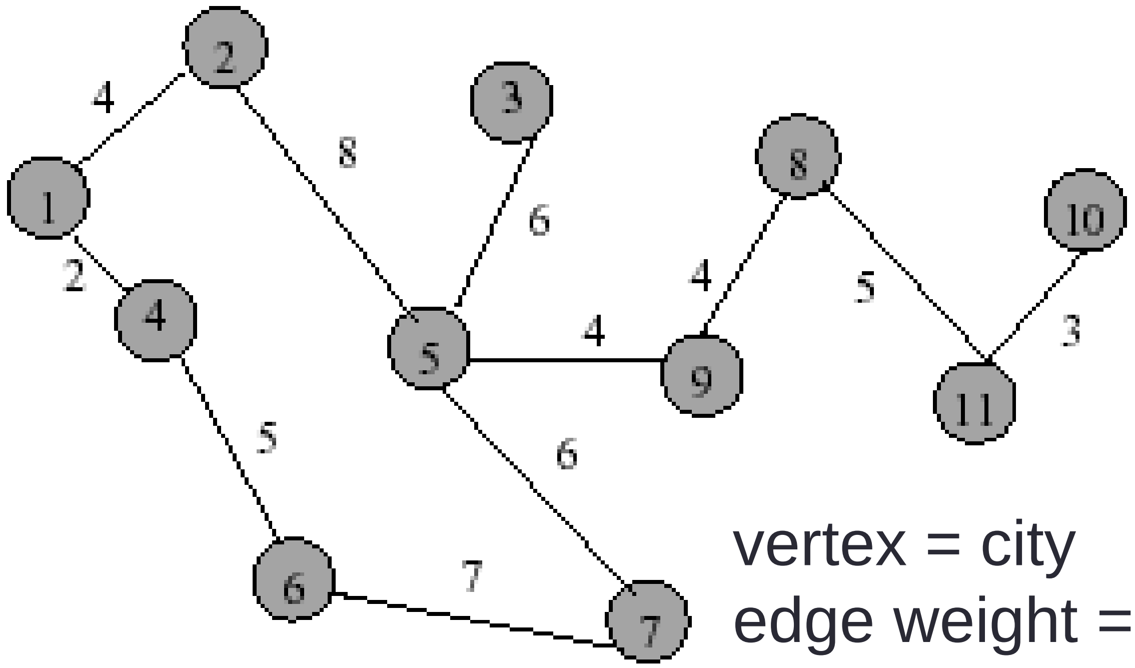
Applications – Communication Network



vertex = router

edge = communication link

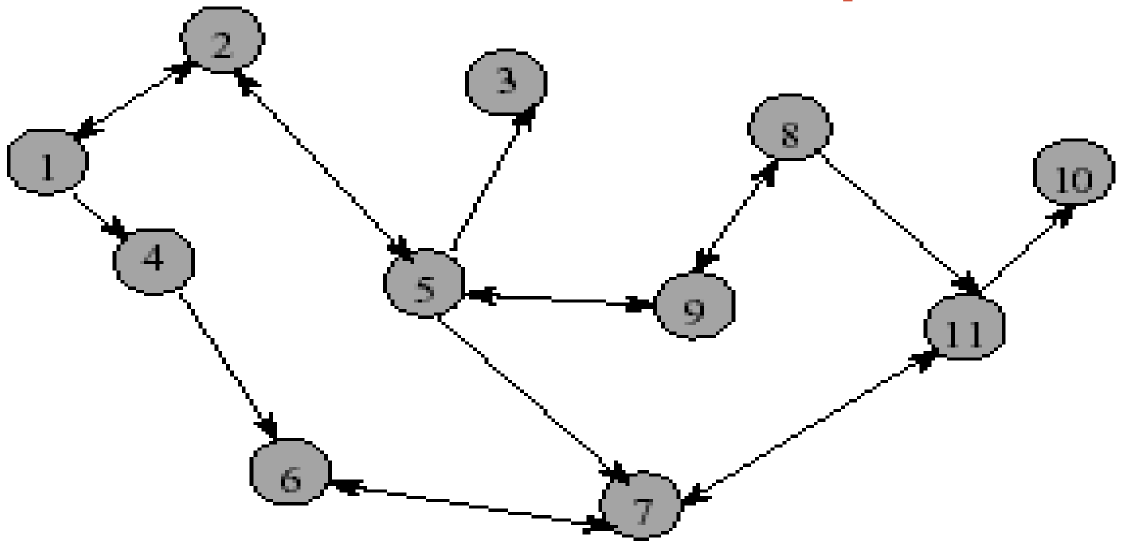
Applications - Driving Distance/Time Map



vertex = city

edge weight = driving distance/time

Applications - Street Map



- Streets are one- or two-way.
- A single directed edge denotes a one-way street
- A two directed edge denotes a two-way street
- Read Example 16.1 and see Figure 16.2

Path

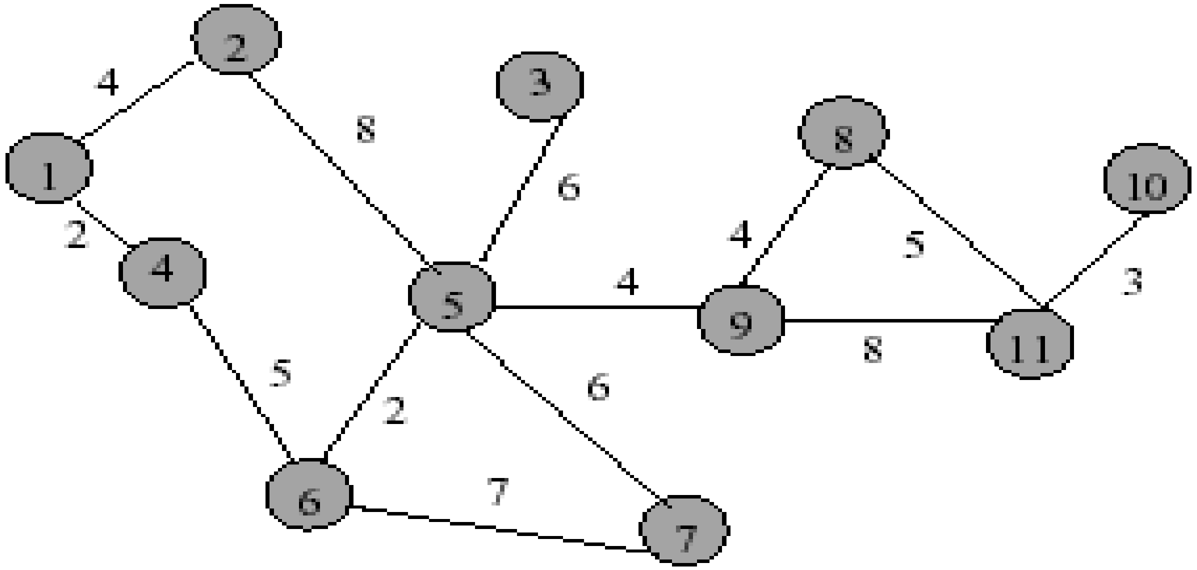
- A sequence of vertices $P = i_1, i_2, \ldots, i_k$ is an i_1 to i_k **path** in the graph $G=(V, E)$ iff the edge (i_j, i_{j+1}) is in E for every j, $1 \leq j < k$
- What are possible paths in Figure 16.2(b)?

Simple Path

- A **simple path** is a path in which all vertices, except possibly in the first and last, are different

Length (Cost) of a Path

- Each edge in a graph may have an associated **length (or cost).** The length of a path is the sum of the lengths of the edges on the path
- What is the length of the path 5, 9, 11, 10?



Subgraph & Cycle

- Let *G* = *(V, E)* be an undirected graph
- A graph *H* is a **subgraph** of graph *G* iff its vertex and edge sets are subsets of those of *G*
- A **cycle** is a simple path with the same start and end vertex
- List all cycles of the graph of Figure 16.1(a)?
  - 1, 2, 3, 1
  - 1, 4, 3, 1
  - 1, 2, 3, 4, 1

Graph Properties

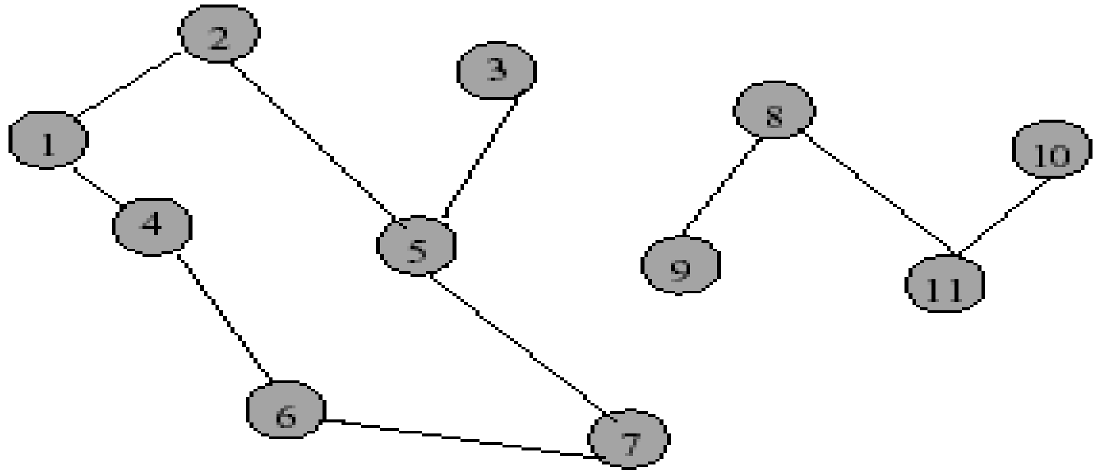
Number of Edges – Undirected Graph

- Each edge is of the form *(u,v)*, u != v.
- The no. of possible pairs in an n vertex graph is n*(n-1)
- Since edge *(u,v)* is **the same** as edge *(v,u)*, the number of edges in an undirected graph is n*(n-1)/2
- Thus, the number of edges in an undirected graph is ≤ **n*(n-1)/2**

Number of Edges - Directed Graph

- Each edge is of the form *(u,v)*, u != v.
- The no. of possible pairs in an n vertex graph is n*(n-1)
- Since edge *(u,v)* is **not the same** as edge *(v,u)*, the number of edges in a directed graph is n*(n-1)
- Thus, the number of edges in a directed graph is **≤ n*(n-1)**

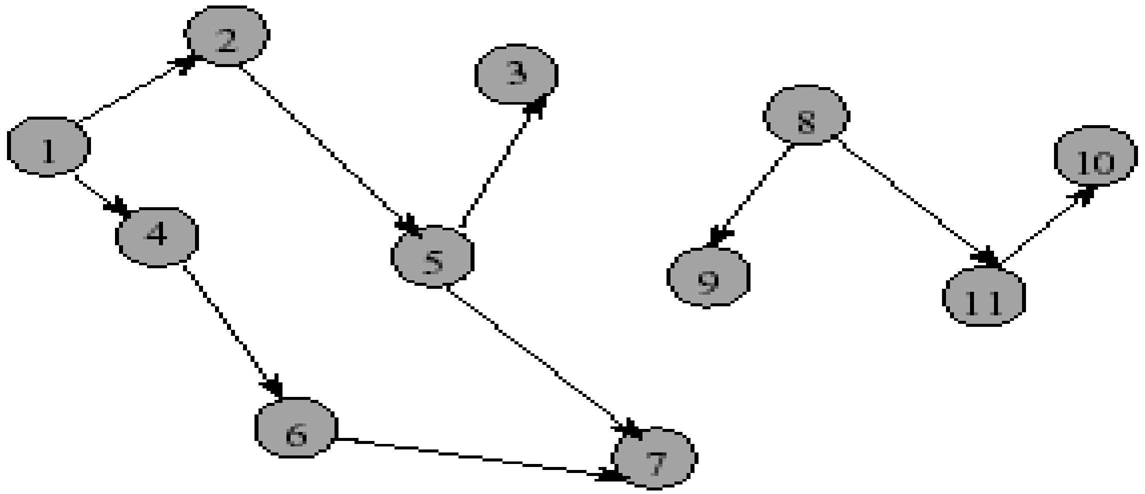
Vertex Degree



- The **degree** of vertex *i* is the no. of edges incident on vertex *i*.

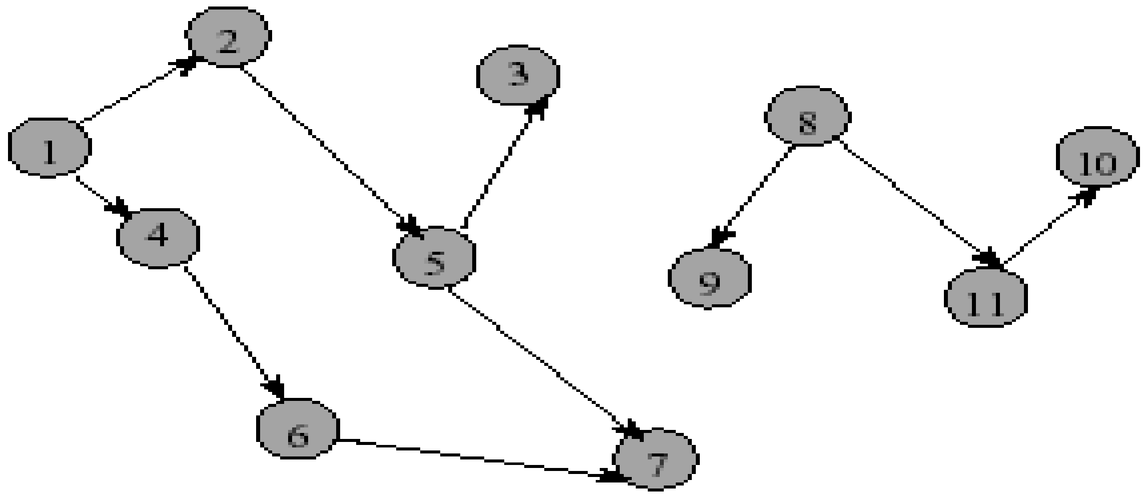
e.g., degree(2) = 2, degree(5) = 3, degree(3) = 1

In-Degree of a Vertex



- **In-degree** of vertex *i* is the number of edges incident to *i* (i.e., the number of incoming edges).
e.g., indegree(2) = 1, indegree(8) = 0

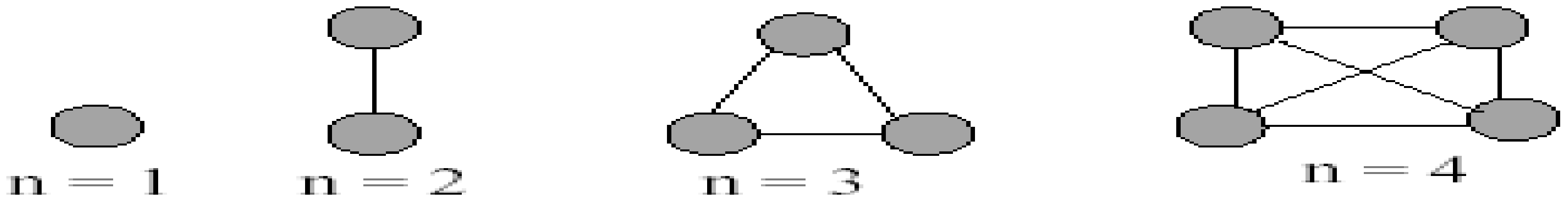
Out-Degree of a Vertex



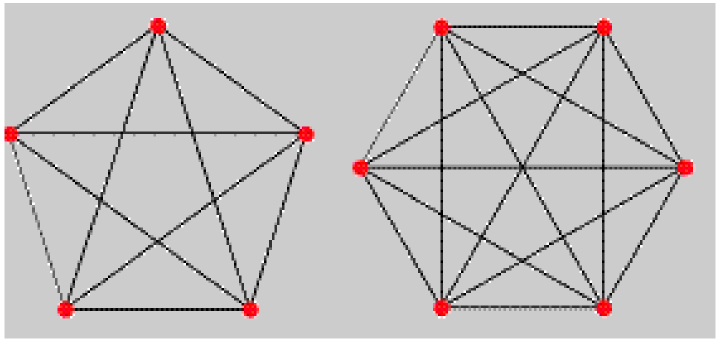
- **Out-degree** of vertex *i* is the number of edges incident from *i* (i.e., the number of outgoing edges).
- e.g., outdegree(2) = 1, outdegree(8) = 2

Complete Undirected Graphs

- A **complete undirected graph** has n(n-1)/2 edges (i.e., all possible edges) and is denoted by K_n



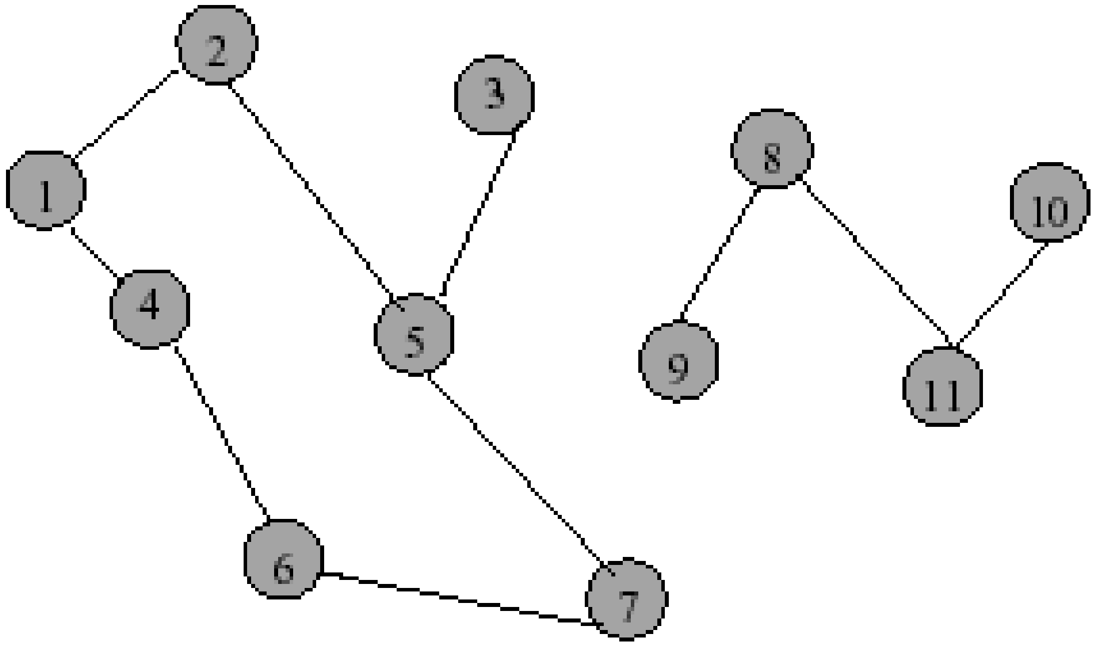
- What would a complete undirected graph look like when n=5? When n=6?



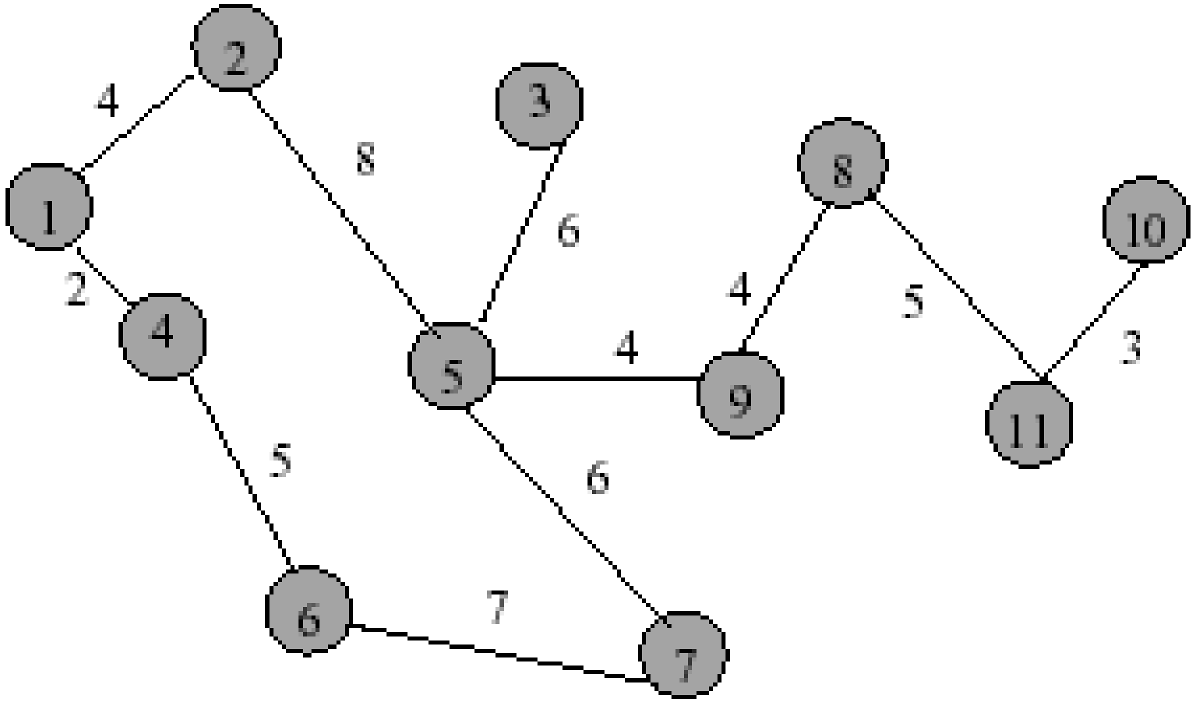
Connected Graph

- Let $G = (V, E)$ be an undirected graph
- G is **connected** iff there is a path between every pair of vertices in G

Example of Not Connected



Example of Connected Graph

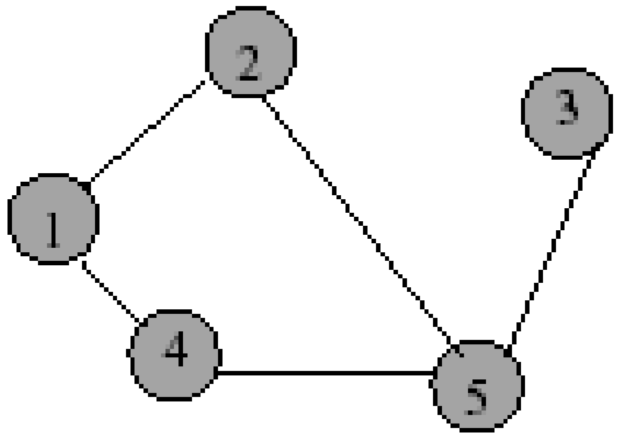


Representation of Unweighted Graphs

- The most frequently used representations for unweighted graphs are
  - Adjacency Matrix
  - Linked adjacency lists
  - Array adjacency lists

Adjacency Matrix

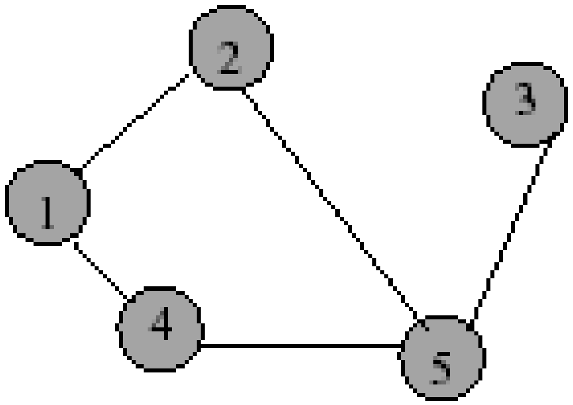
- 0/1 n x n matrix, where n = # of vertices
- A(i, j) = 1 iff (i, j) is an edge.

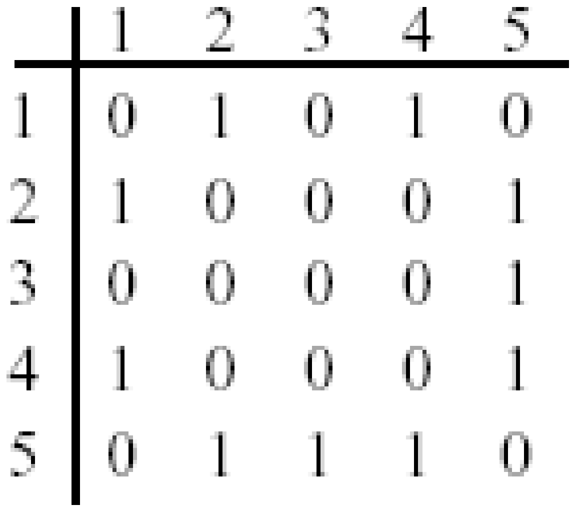


	1	2	3	4	5
1	0	1	0	1	0
2	1	0	0	0	1
3	0	0	0	0	1
4	1	0	0	0	1
5	0	1	1	1	0

Adjacency Matrix Properties

- Diagonal entries are zero.
- Adjacency matrix of an undirected graph is symmetric (A(i,j) = A(j,i) for all i and j).

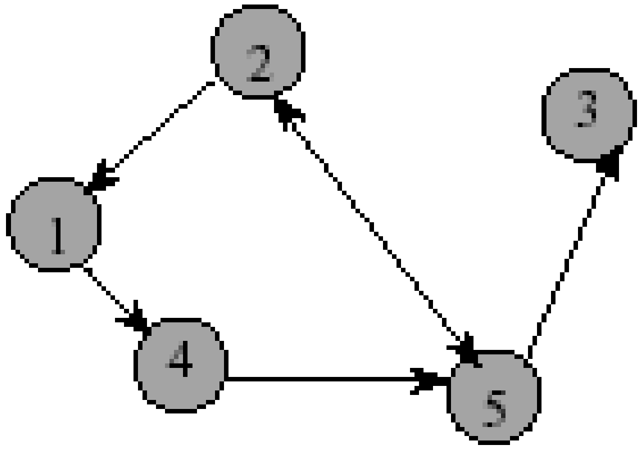


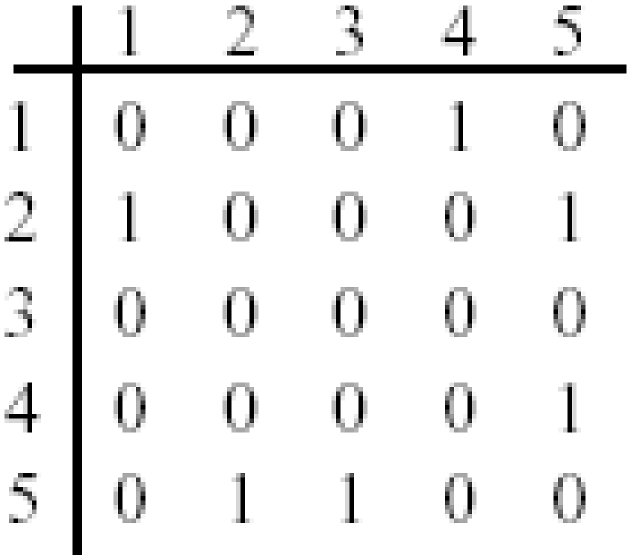


	1	2	3	4	5
1	0	1	0	1	0
2	1	0	0	0	1
3	0	0	0	0	1
4	1	0	0	0	1
5	0	1	1	1	0

Adjacency Matrix for Digraph

- Diagonal entries are zero(only if there is no self loop)
- Adjacency matrix of a digraph need not be symmetric.

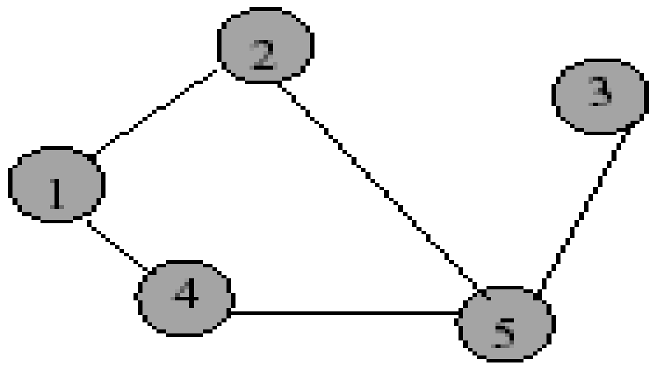




	1	2	3	4	5
1	0	0	0	1	0
2	1	0	0	0	1
3	0	0	0	0	0
4	0	0	0	0	1
5	0	1	1	0	0

Adjacency Lists

- Adjacency list for vertex *i* is a linear list of vertices adjacent from vertex *i*.
- An array of n adjacency lists for each vertex of the graph.



aList[1] = (2,4)

aList[2] = (1,5)

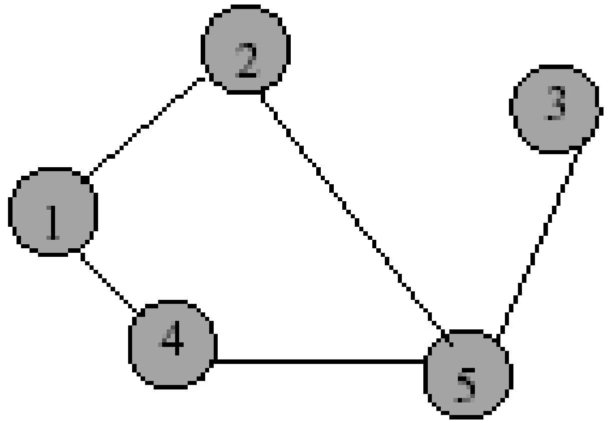
aList[3] = (5)

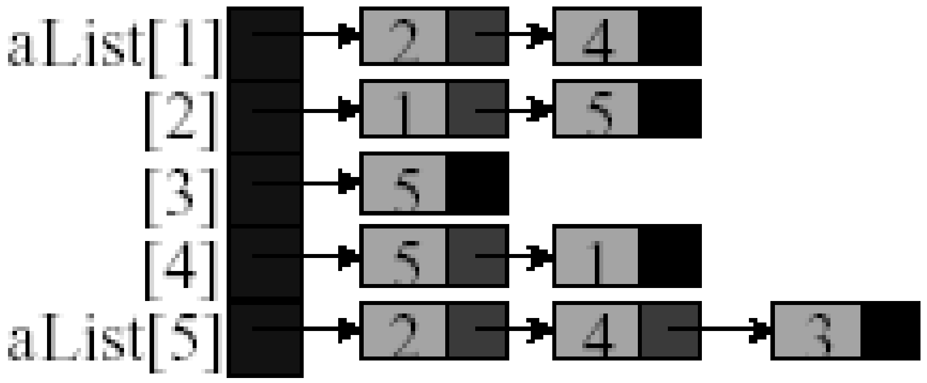
aList[4] = (5,1)

aList[5] = (2,4,3)

Linked Adjacency Lists

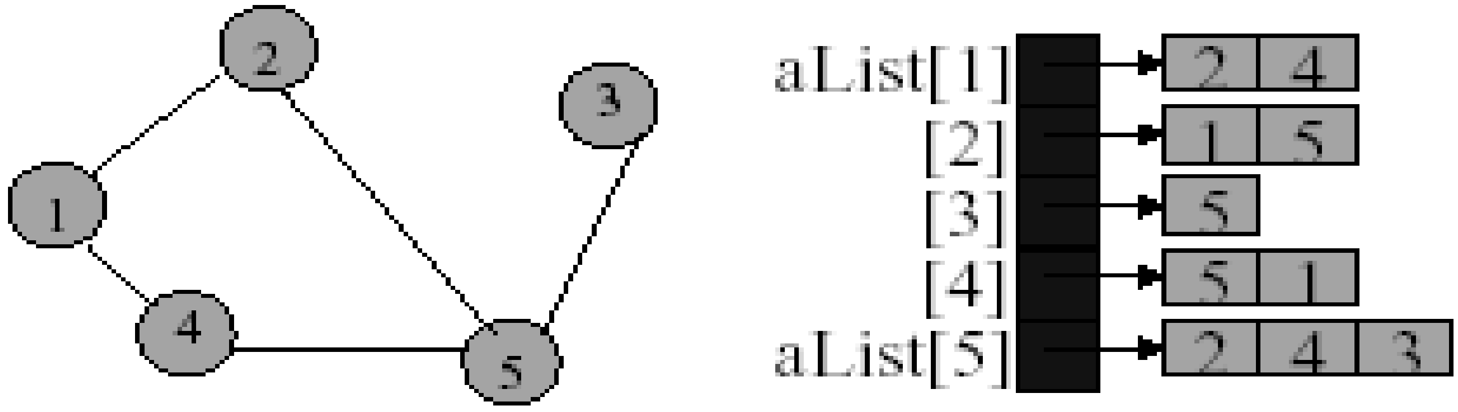
- Each adjacency list is a chain.
  Array length = n.
  # of chain nodes = 2e (undirected graph)
  # of chain nodes = e (digraph)





Array Adjacency Lists

- Each adjacency list is an array list.
  Array length = n.
  # of chain nodes = 2e (undirected graph)
  # of chain nodes = e (digraph)



Representation of Weighted Graphs

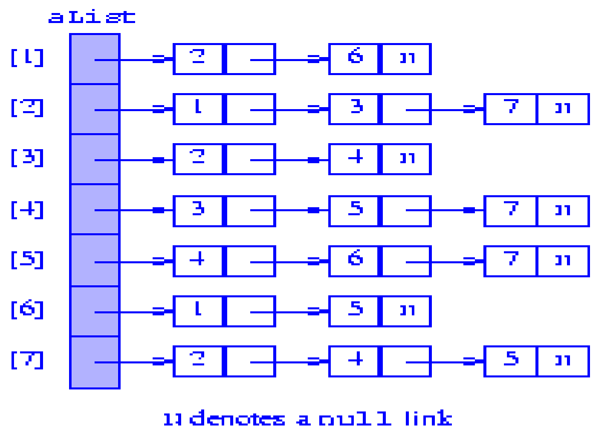
- **Weighted graphs** are represented with simple extensions of those used for unweighted graphs
- The **cost-adjacency-matrix** representation uses a matrix C just like the adjacency-matrix representation does
- Cost-adjacency matrix: $C(i, j)$ = cost of edge (i, j)
- Adjacency lists: each list element is a pair (adjacent vertex, edge weight)

For the digraph Figure 16.2(b)

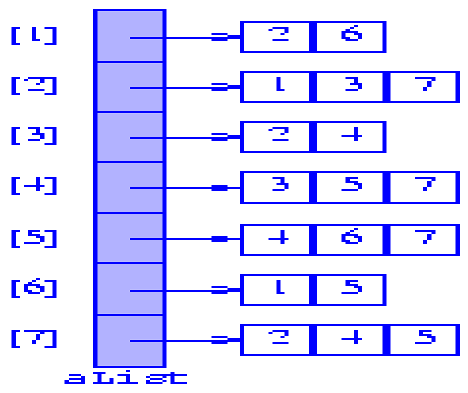
(a) adjacency matrix

$$\begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
5 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
6 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
7 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\hline
\end{array}$$

(b) Linked adjacency list



(c) Array adjacency list



Graph Traversals (Search)

- We have covered some of these with binary trees
  - Breadth-first search (BFS)
  - Depth-first search (DFS)
- A traversal (search):
  - An algorithm for systematically exploring a graph
  - Visiting (all) vertices
  - Until finding a goal vertex or until no more vertices

Only for connected graphs

Breadth-first search

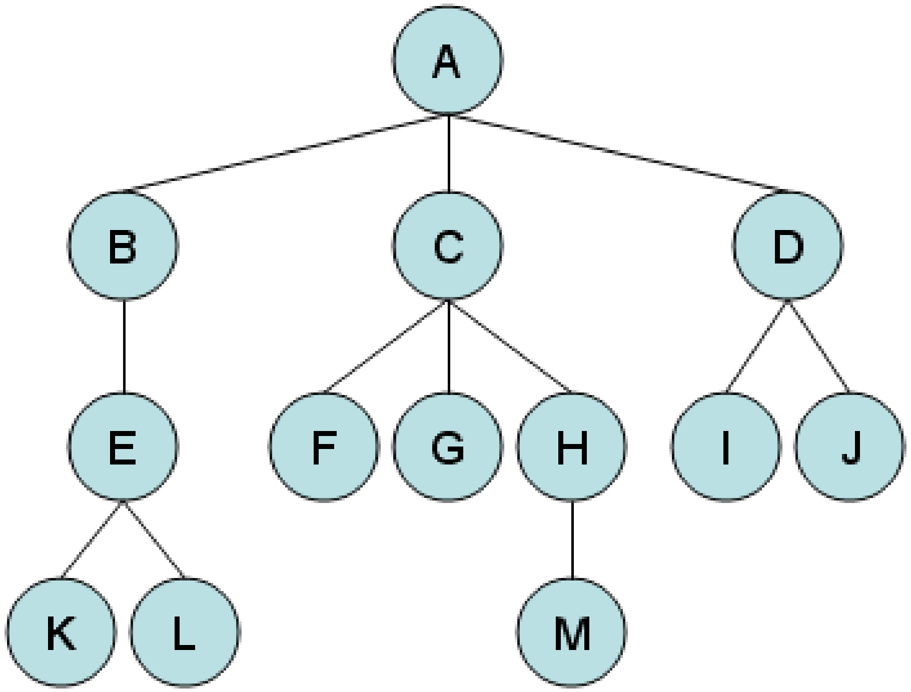
- One of the simplest algorithms
- Also one of the most important
  - It forms the basis for MANY graph algorithms

BFS: Level-by-level traversal

- Given a starting vertex s
- Visit all vertices at increasing distance from s
  - Visit all vertices at distance k from s
  - Then visit all vertices at distance k+1 from s
  - Then ….

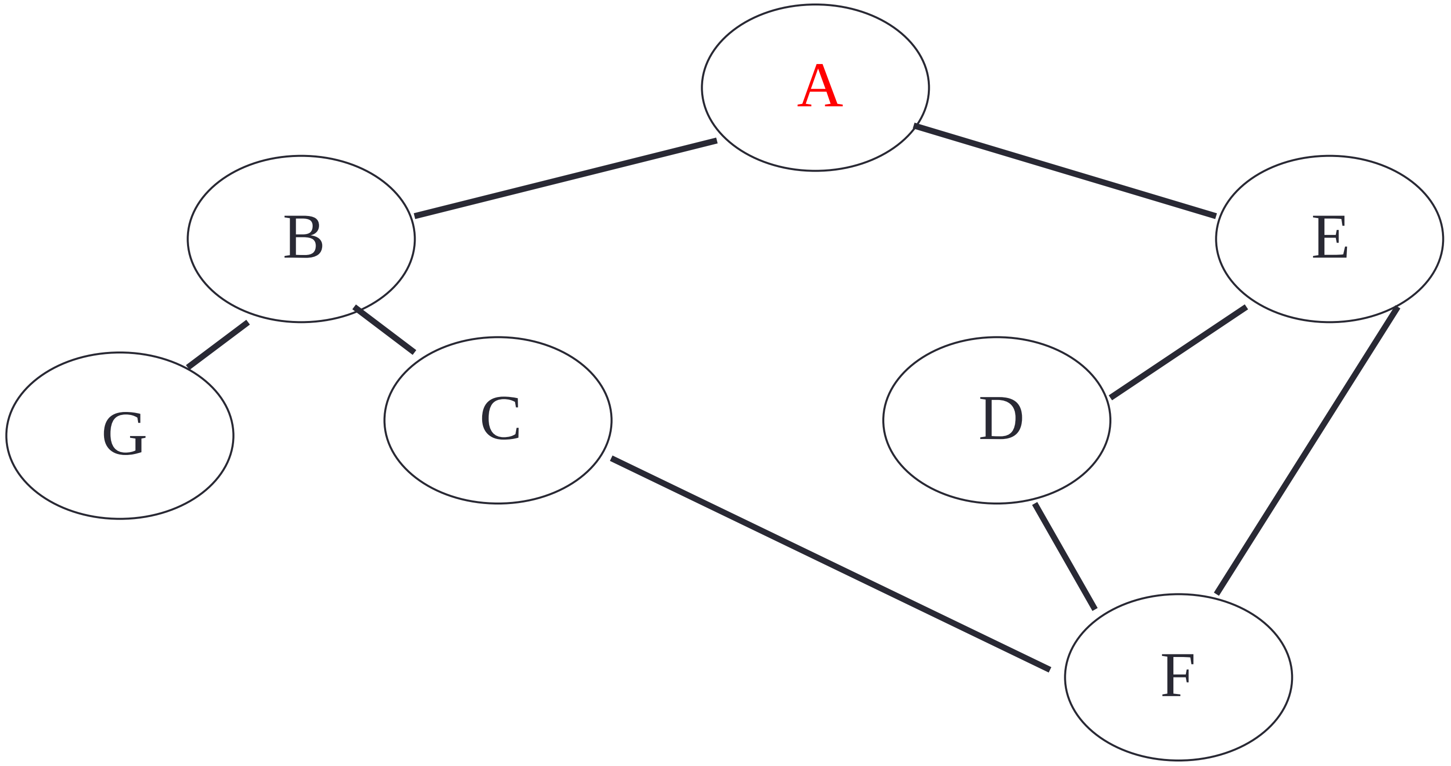
BFS in a tree

BFS: visit all siblings before their descendants



A B C D E F G H I J K L M

BFS: Graph

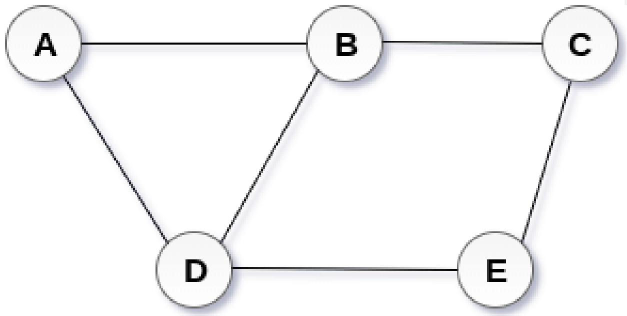


A B E G C D F

BFS(graph g, vertex s)

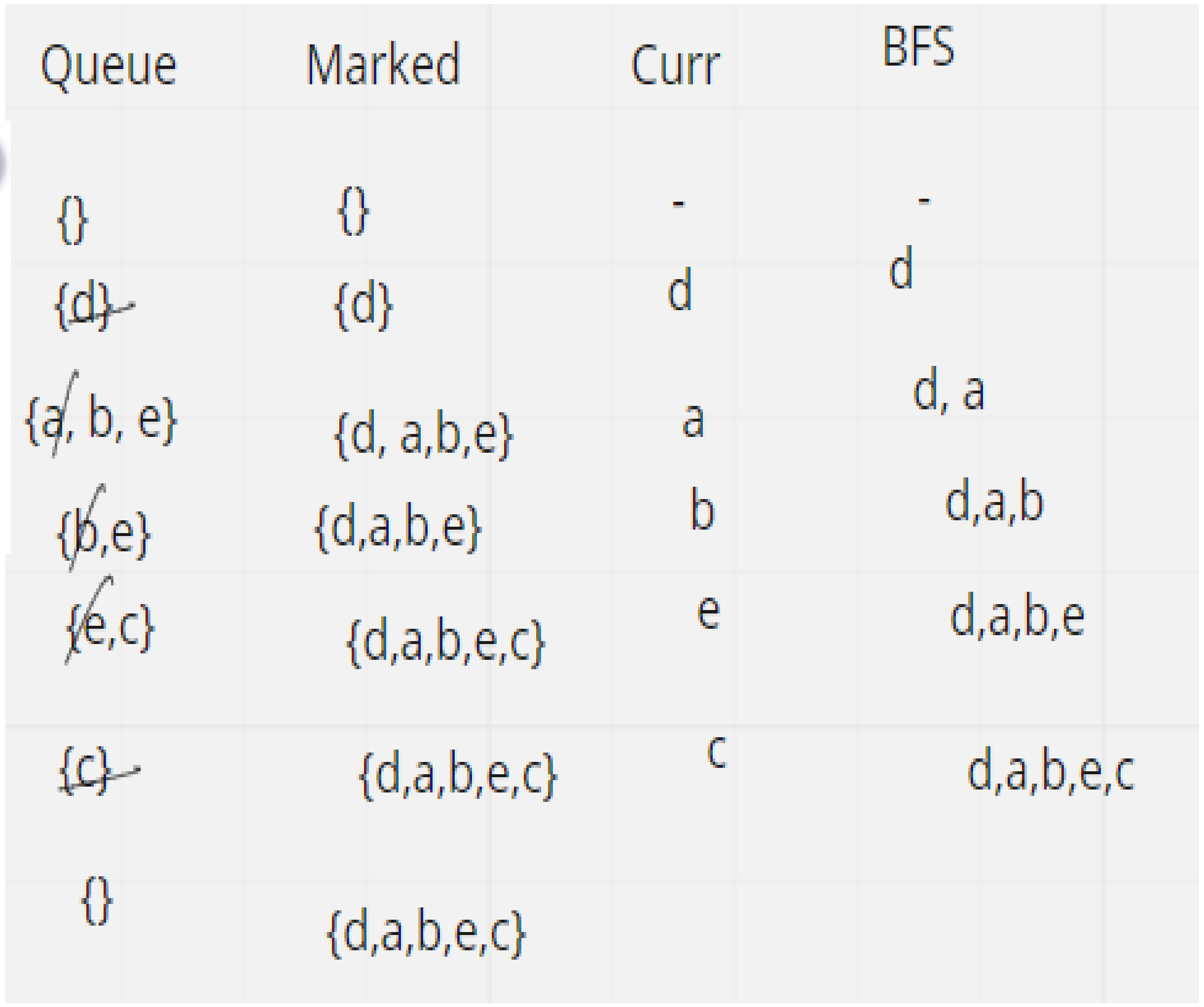
```
1.  unmark all vertices in G
2.  q ← new queue
3.  mark s // s is starting vertex
4.  enqueue(q, s)
5.  while (not empty(q))
6.     curr ← dequeue(q)
7.     visit curr // e.g., print its data
8.     for each edge <curr, V>
9.        if V is unmarked
10.          mark V
11.          enqueue(q, V)
```

BFS algorithm

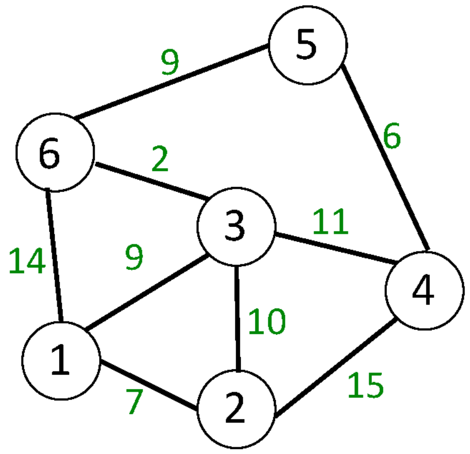


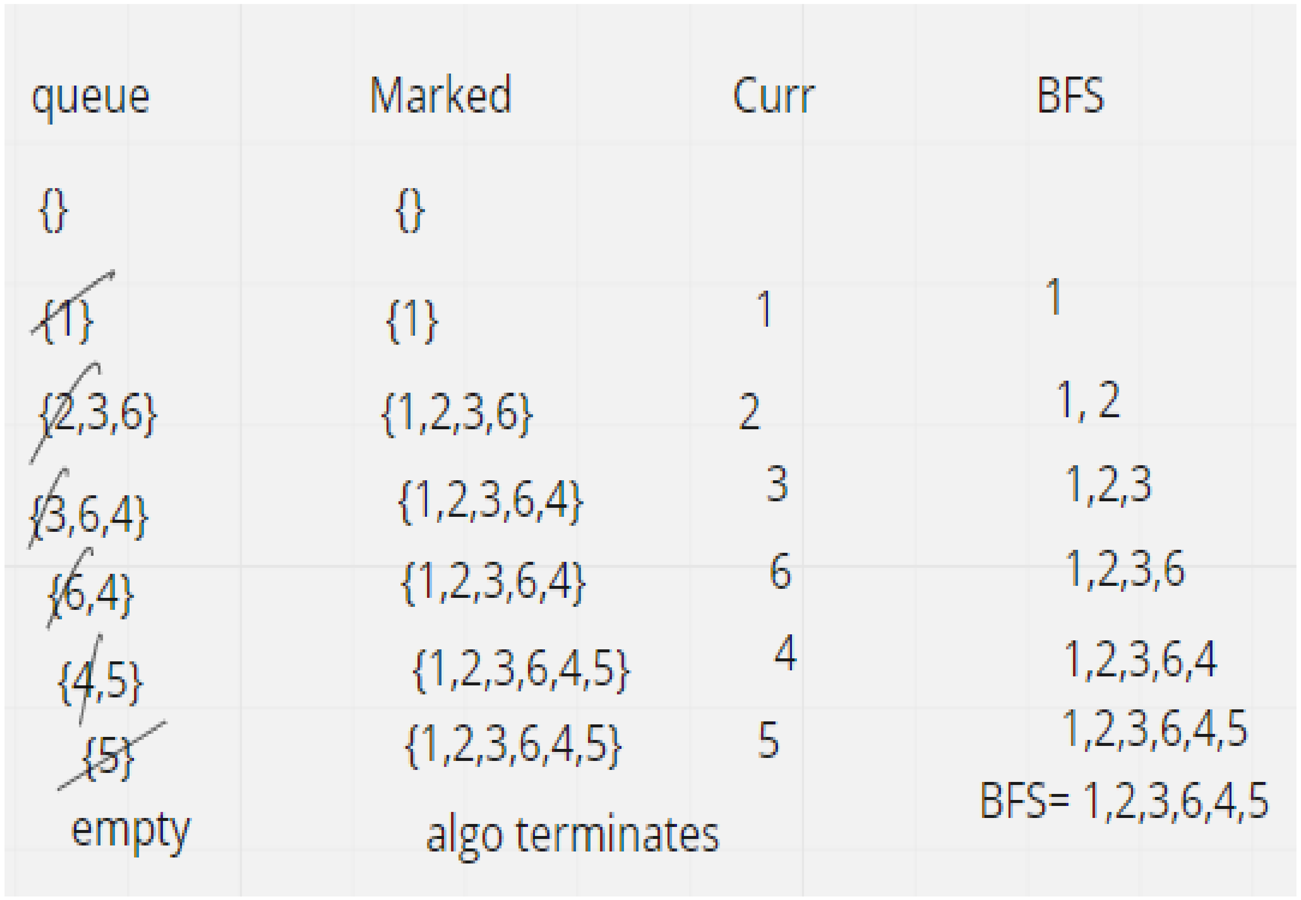
Undirected Graph

Starting vertex = d



Queue	Marked	Curr	BFS
{}	{}	-	-
~~{d}~~	{d}	d	d
{~~a~~, b, e}	{d, a,b,e}	a	d, a
{~~b~~,e}	{d,a,b,e}	b	d,a,b
{~~e~~,c}	{d,a,b,e,c}	e	d,a,b,e
~~{c}~~	{d,a,b,e,c}	c	d,a,b,e,c
{}	{d,a,b,e,c}		



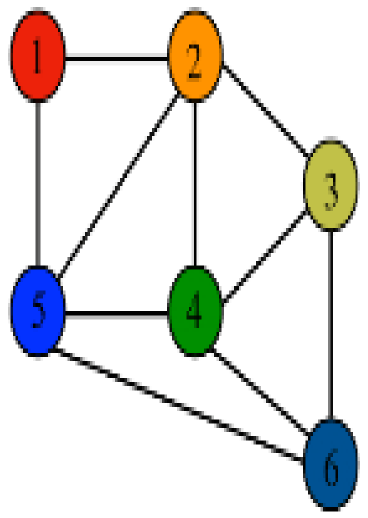


queue	Marked	Curr	BFS
{}	{}		
~~{1}~~	{1}	1	1
{~~2~~,3,6}	{1,2,3,6}	2	1, 2
{~~3~~,6,4}	{1,2,3,6,4}	3	1,2,3
{~~6~~,4}	{1,2,3,6,4}	6	1,2,3,6
{~~4~~,5}	{1,2,3,6,4,5}	4	1,2,3,6,4
~~{5}~~	{1,2,3,6,4,5}	5	1,2,3,6,4,5
empty	algo terminates		BFS= 1,2,3,6,4,5

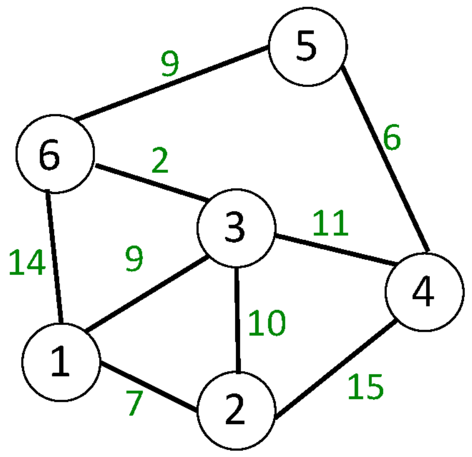
Interesting features of BFS

- Complexity: O(|V| + |E|)
  - All vertices put on queue exactly once
  - For each vertex on queue, we expand its edges
  - In other words, we traverse all edges once
- BFS finds shortest path from s to each vertex
  - Shortest in terms of number of edges
  - Why does this work?

  - Takes too much memory.
  - Runs out of memory before it runs out of time.



stack	Marked	curr	DFS
{}	{}	-	-
{1}	{1}	1	1
{2,5}	{1,2,5}	2	1,2
{3,4,5}	{1,2,5,3,4}	3	1,2,3
{6,4,5}	{1,2,5,3,4,6}	6	1,2,3,6
{4,5}	{1,2,5,3,4,6}	4	1,2,3,6,4
{5}	{1,2,5,3,4,6}	5	1,2,3,6,4,5
Empty	{1,2,5,3,4,6}	-	DFS: 1,2,3,6,4,5



Stack	Marked	Curr	DFS
{}	{}		
~~{1}~~	{1}	1	1
{~~2~~,3,6}	{1,2,3,6}	2	1,2
{~~4~~,3,6}	{1,2,3,6,4}	4	1,2,4
{~~5~~,3,6}	{1,2,3,6,4,5}	5	1,2,4,5
{~~3~~,6}	{1,2,3,6,4,5}	3	1,2,4,5,3
~~{6}~~	{1,2,3,6,4,5}	6	1,2,4,5,3,6
empty	algo terminates	DFS: 1,2,4,5,3,6	

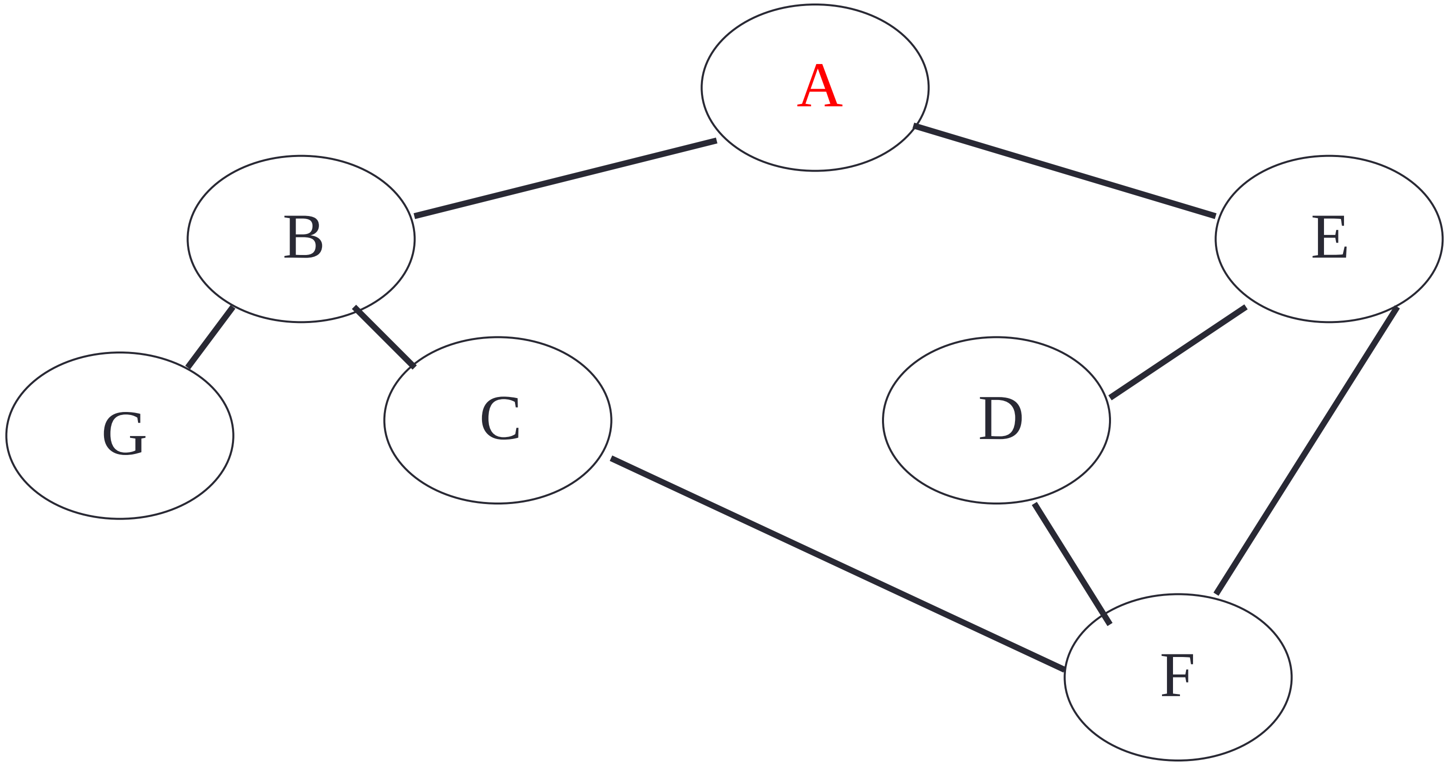
Depth-first search

- Again, a simple and powerful algorithm
- Given a starting vertex s
- Pick an adjacent vertex, visit it.
  - Then visit one of its adjacent vertices
  - …..
  - Until impossible, then backtrack, visit another

DFS(graph g, vertex s)

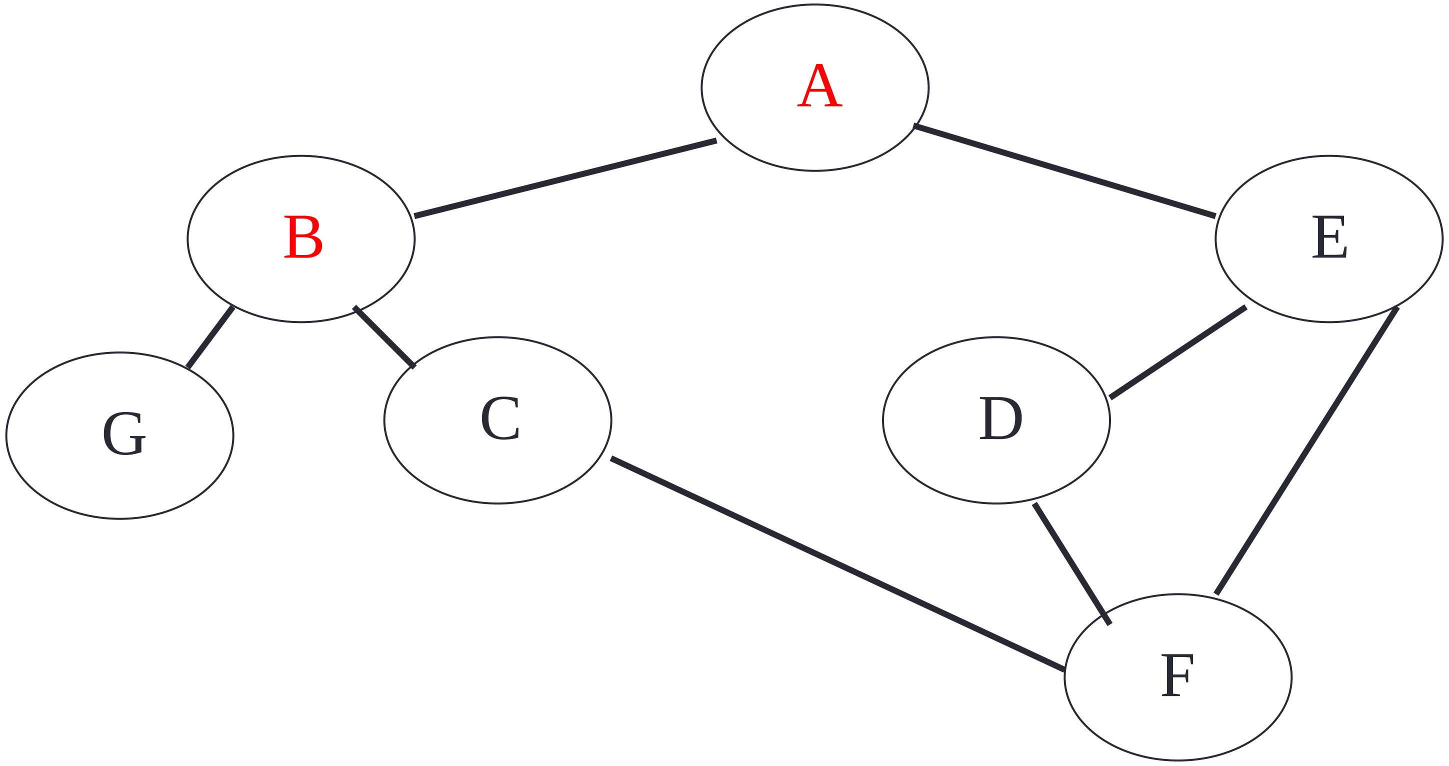
```
1.  unmark all vertices in G
2.  Stack ← new stack
3.  Push(stack, s)
4.  while (not empty(stack))
5.    curr ← pop(stack)
6.  If not marked curr
7.    visit curr // e.g., print its data
8.    Mark curr
9.    for each edge <curr, V>
10.     if V is unmarked
11.       push(stack, V)
```

Current vertex: A



Start with A. Mark it.

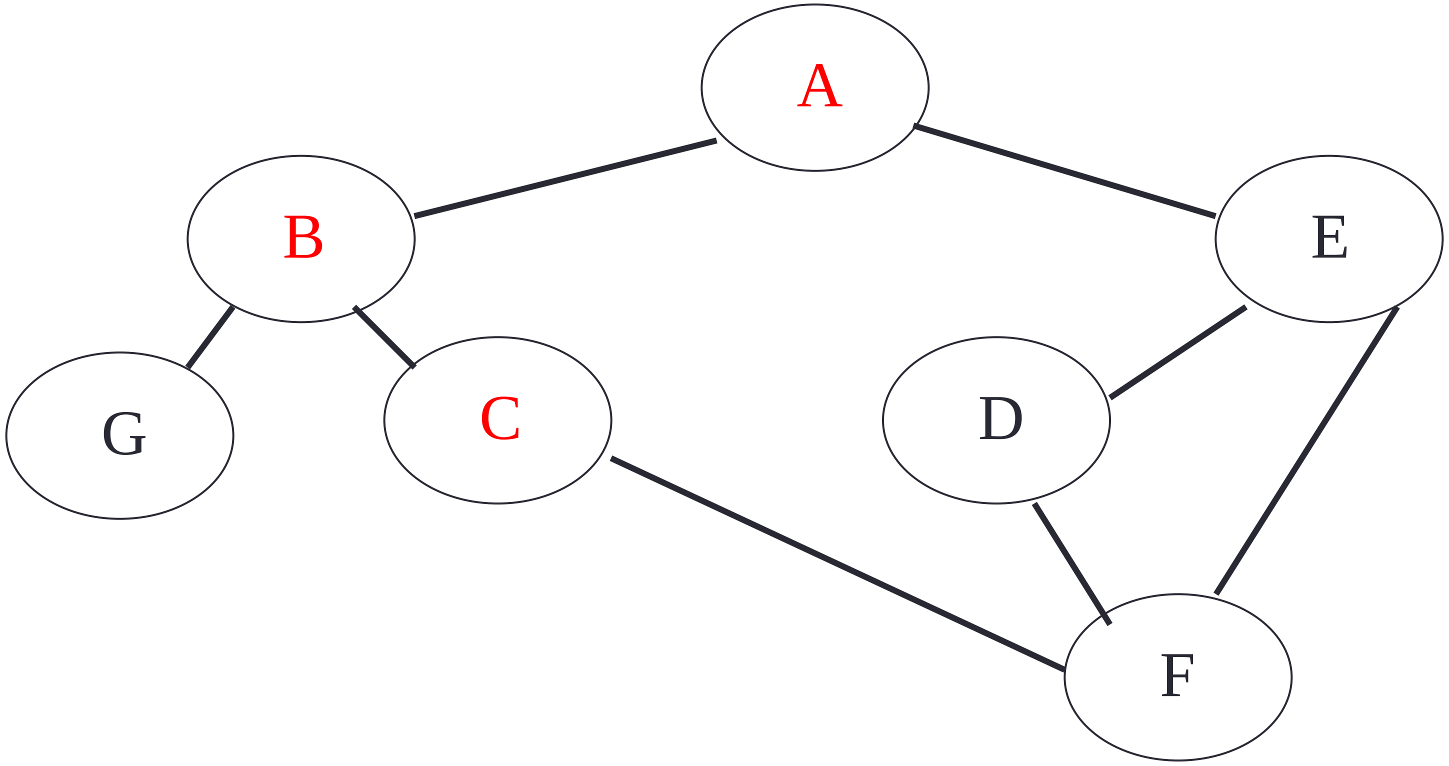
Current: B



Expand A’s adjacent vertices. Pick one (B).

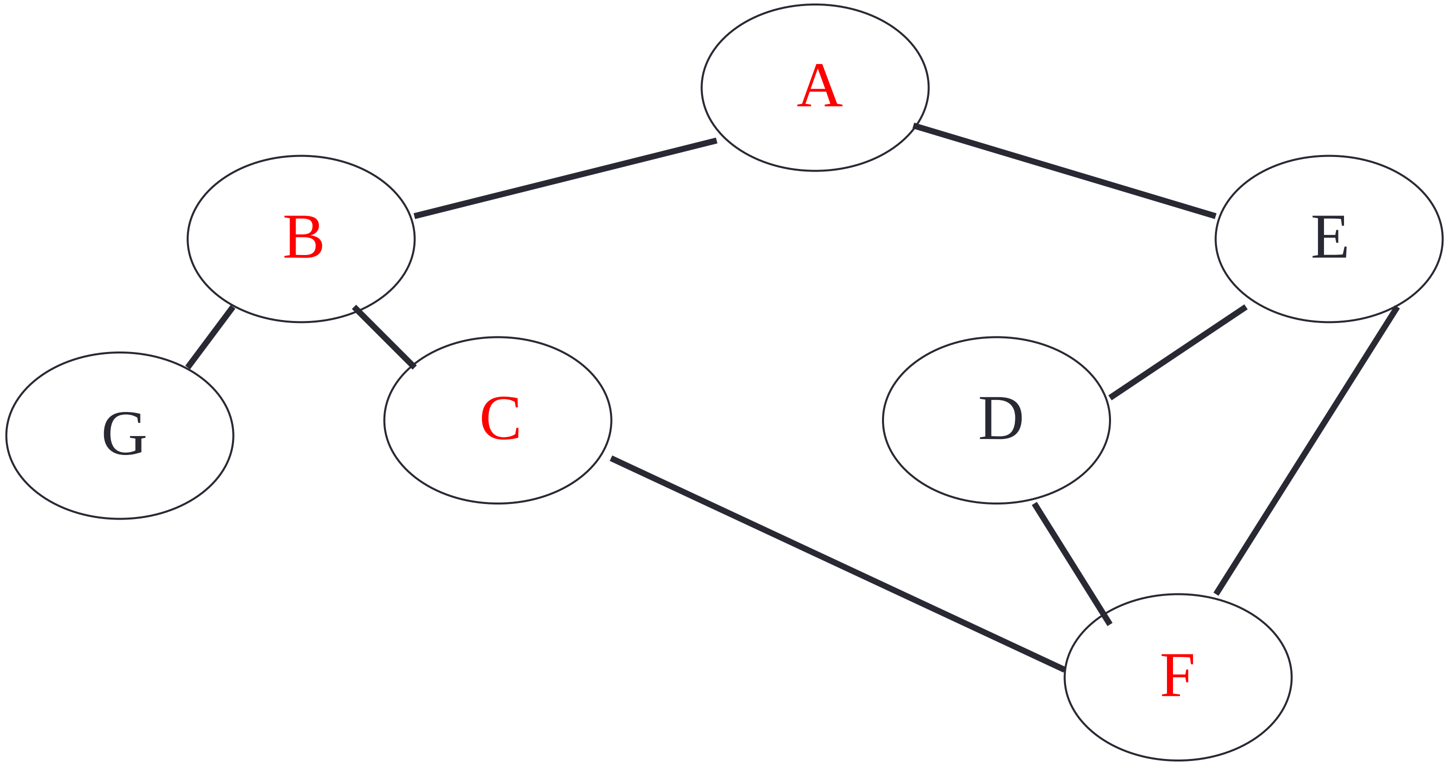
Mark it and re-visit.

Current: C



Now expand B, and visit its neighbor, C.

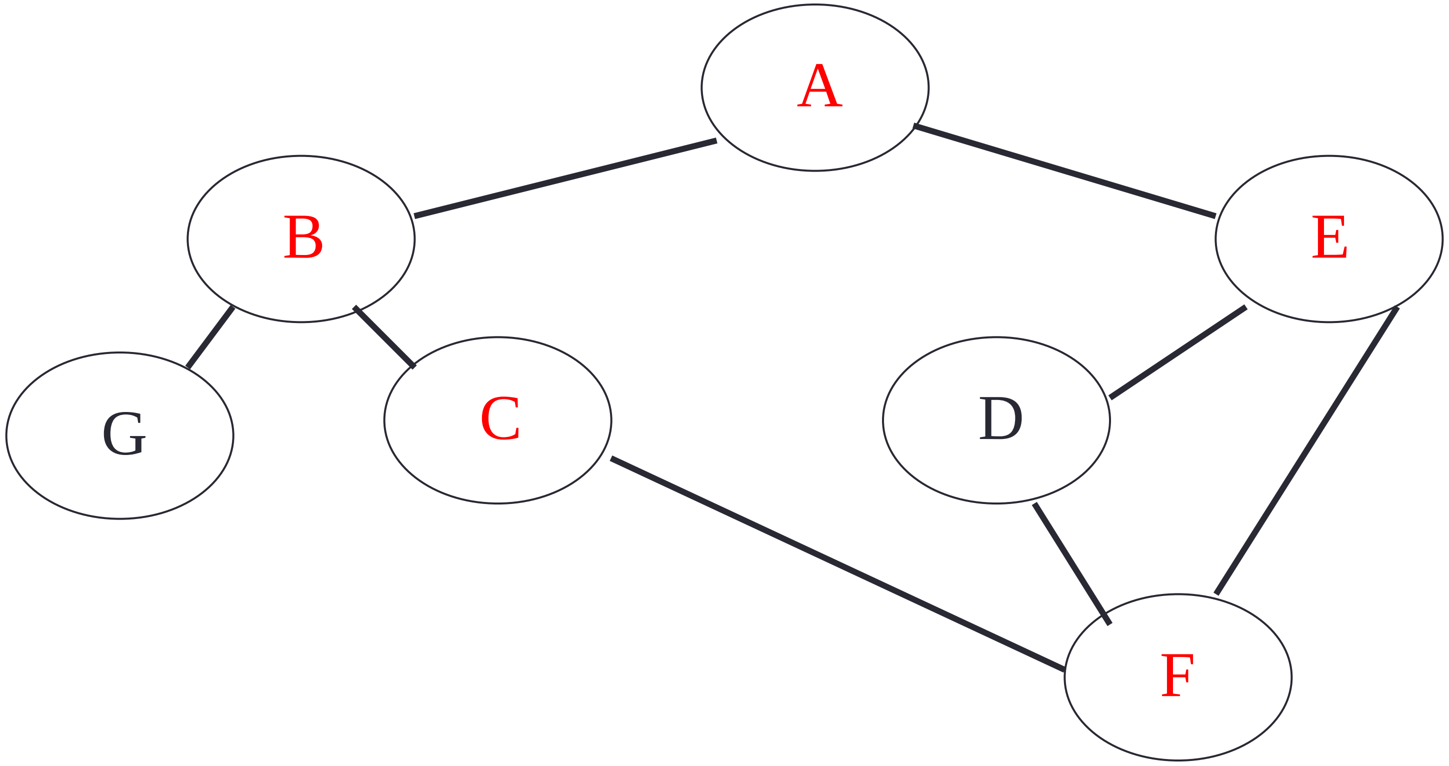
Current: F



Visit F.

Pick one of its neighbors, E.

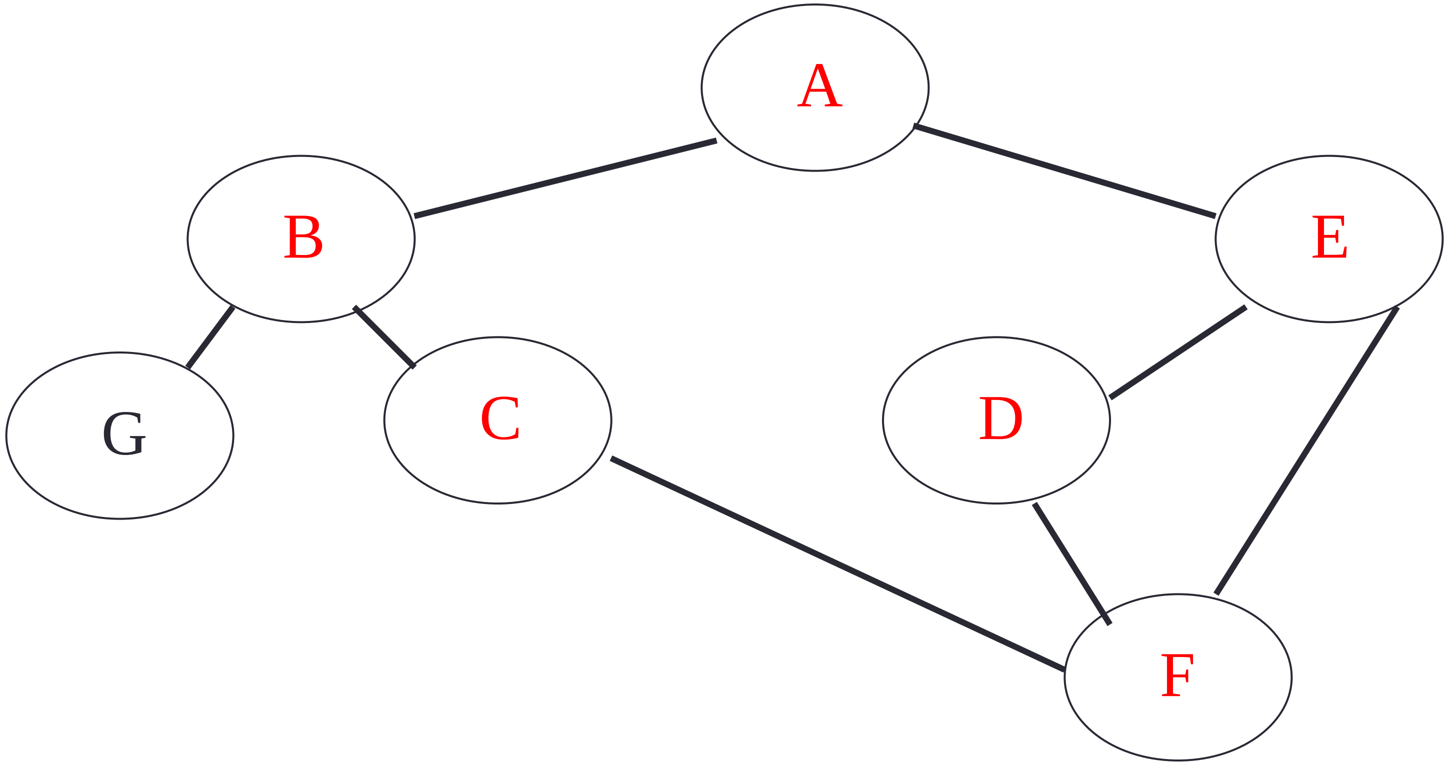
Current: E



E’s adjacent vertices are A, D and F.

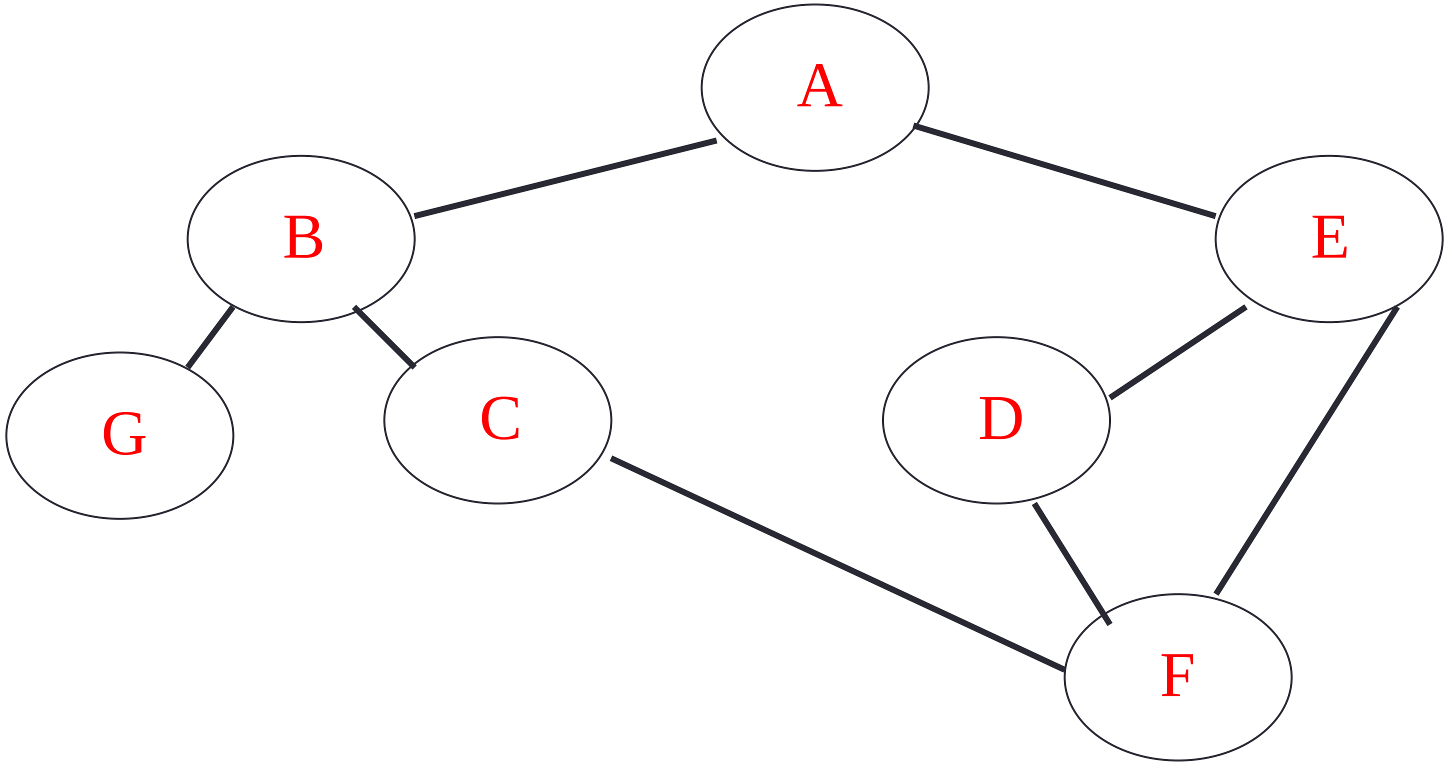
A and F are marked, so pick D.

Current: D



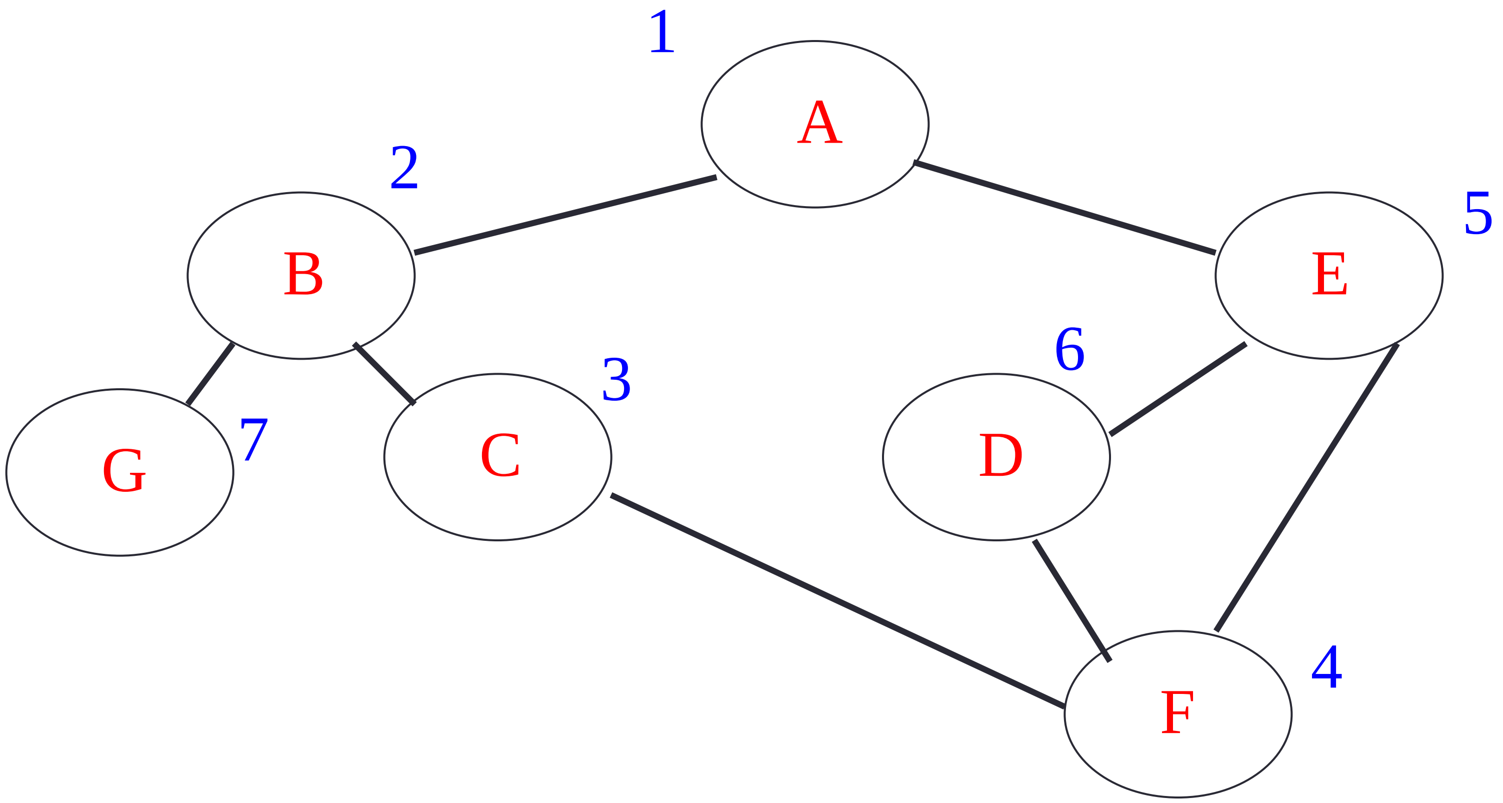
Visit D. No new vertices available. Backtrack to E. Backtrack to F. Backtrack to C. Backtrack to B

Current: G



Visit G. No new vertices from here. Backtrack to B. Backtrack to A. E already marked so no new.

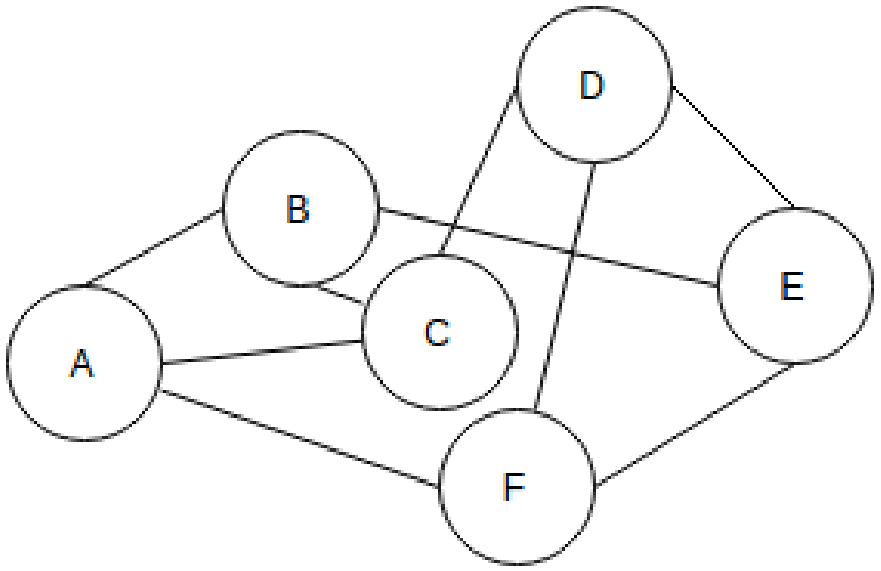
Current:

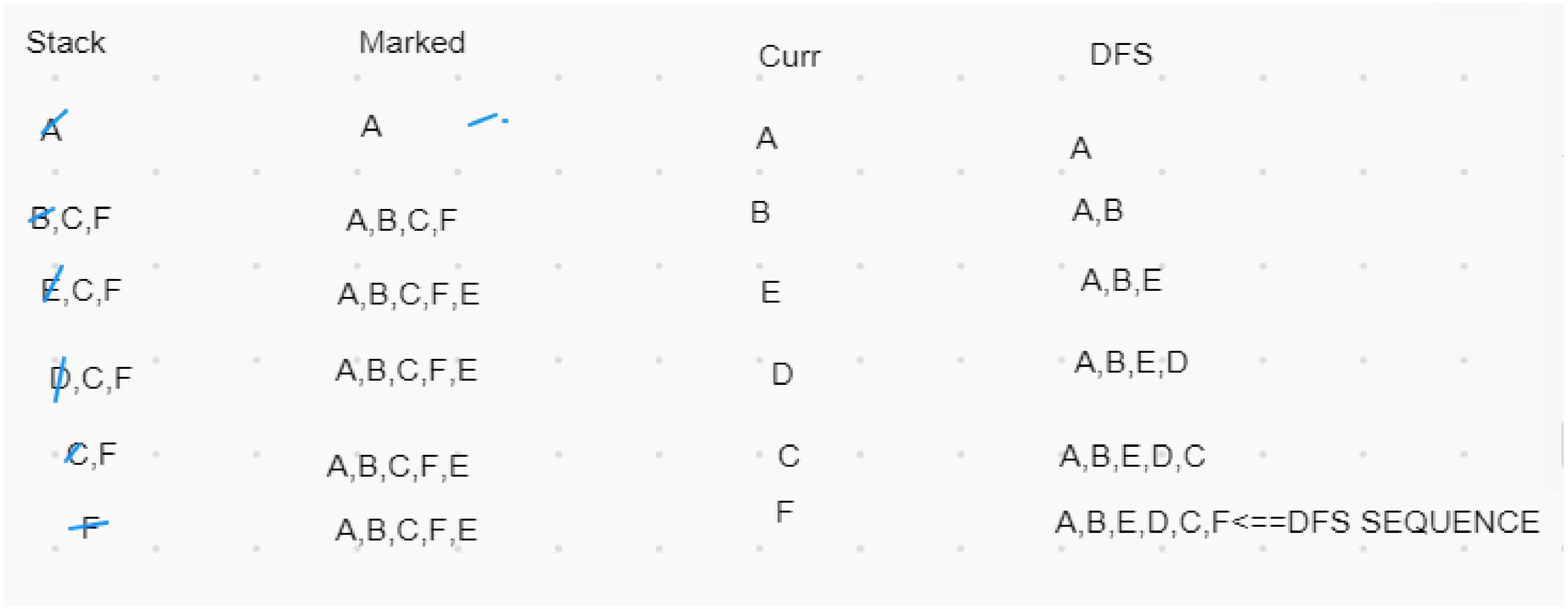


Done. We have explored the graph in order:

A B C F E D G

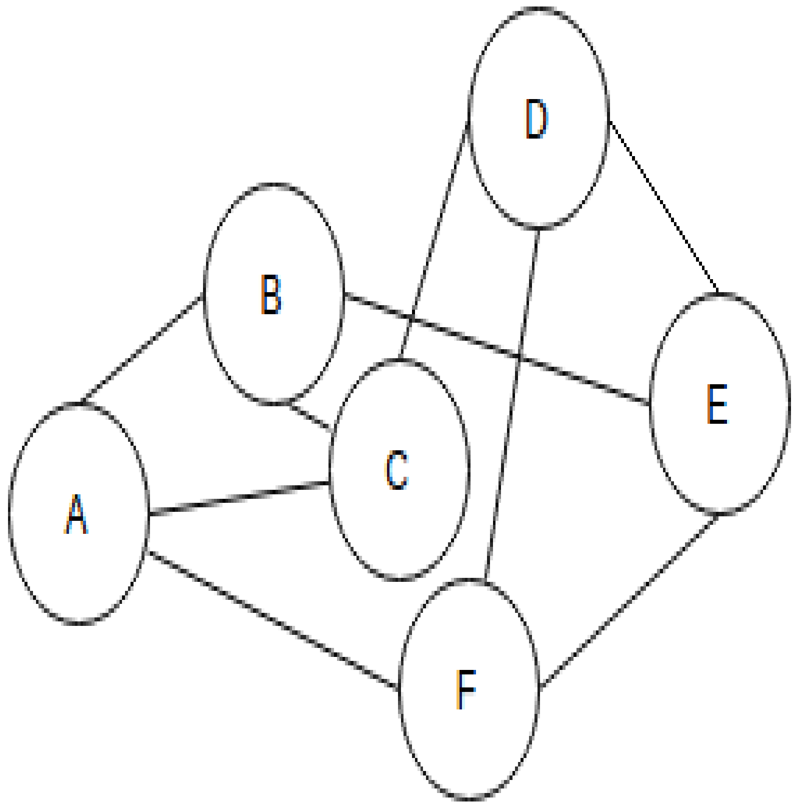
Method 1

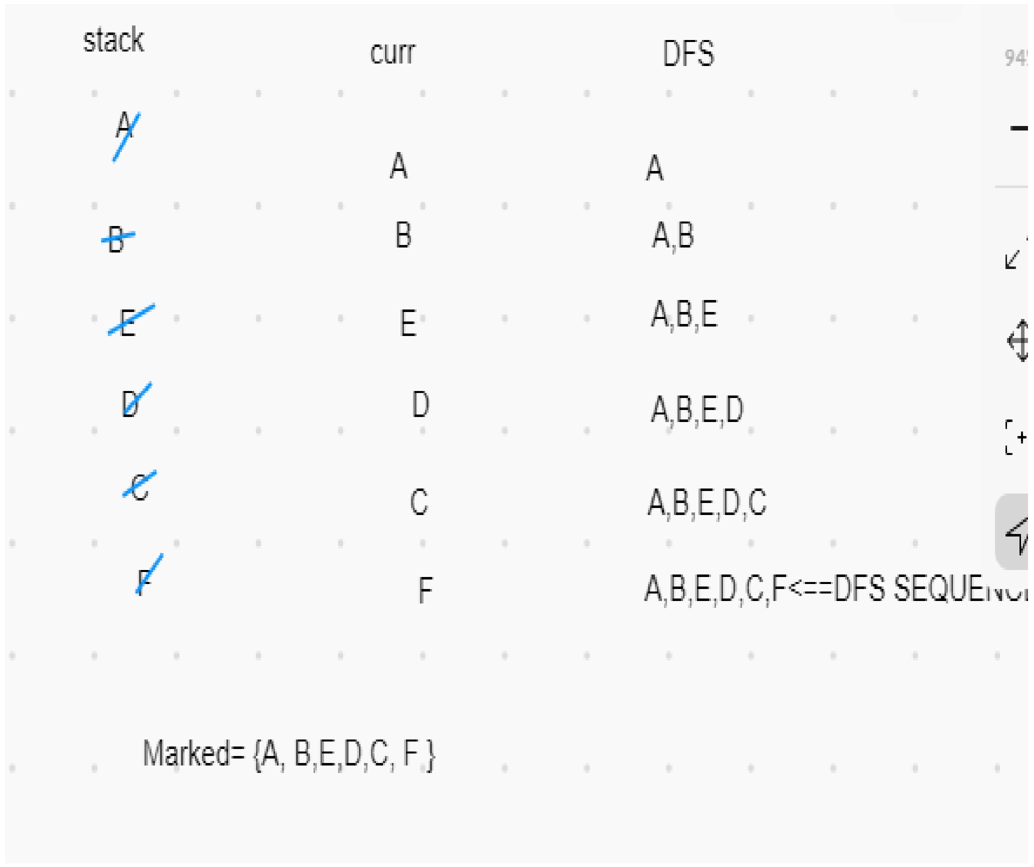




Stack	Marked	Curr	DFS
~~A~~	A	A	A
~~B~~,C,F	A,B,C,F	B	A,B
~~E~~,C,F	A,B,C,F,E	E	A,B,E
~~D~~,C,F	A,B,C,F,E	D	A,B,E,D
~~C~~,F	A,B,C,F,E	C	A,B,E,D,C
~~F~~	A,B,C,F,E	F	A,B,E,D,C,F<==DFS SEQUENCE

Method 2





stack	curr	DFS
~~A~~	A	A
~~B~~	B	A,B
~~E~~	E	A,B,E
~~D~~	D	A,B,E,D
~~C~~	C	A,B,E,D,C
~~F~~	F	A,B,E,D,C,F<==DFS SEQUE

Marked= {A, B,E,D,C, F }

Interesting features of DFS

- Complexity: O(|V| + |E|)
  - All vertices visited once, then marked
  - For each vertex on stack, we examine all edges
  - In other words, we traverse all edges once
- DFS does not necessarily find shortest path
  - Why?
- Not a good choice when the goal node is at shallow level on right side of the graph