LINEAR PROGRAMMING PROBLEM

LINEAR PROGRAMMING

- Linear programming deals with optimization (maximization or minimization) of a function of variables known as objective functions.
- It is subjected to a set of linear equalities and /or inequalities known as constraints.
- Linear programming is a mathematical technique which involves the allocation of limited resources in an optimal manner, the basis of a given criterion of optimally.

FORMULATION OF LP PROBLEMS

The procedure for mathematical formulation of a LPP consist of the following steps

Step 1: To write down the decision variables of the problem

Step 2: To formulate the objective function to be optimized (maximized or minimized) as a linear function of the decision variables

Step 3: To formulate the other conditions of the problem such as resource limitation, market constraints, interrelation between variable etc, as linear inequations or equations in terms of the decision variables

Step 4: To add the non-negativity constraint from the considerations so that the negative values of the decision variables do not have any valid physical interpretation

The objective functions, the set of constraints and non-negative constraints form a Linear Programming Problem

GENERAL FORM OF LP PROBLEM

The general LP problem with $'n'$ decision variables and $'m'$ constraints

can be stated in the following form

Find the values of decision variables $x_1, x_2, \dots \dots, x_n$ so as to

Optimize (max or min) $Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$ (Objective function)

subject to the linear constraints

$a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n (\leq, =, \geq) b_1$

$a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n (\leq, =, \geq) b_2$ (constraints)

....

$a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mm}x_n (\leq, =, \geq) b_m$

And $x_1, x_2, \dots \dots, x_n \geq 0$ (non-negativity constraints)

EX. A dealer sells two articles A and B. He earns Rs 2 and 3 when he sells one piece of A and B respectively. He buys these articles at the rate of Rs 1 and Rs 2 respectively in the morning and sells them out in the evening. He invests only 100 Rs. Write down the equations of his profit and other conditions

ANS:

Maximize $z = 2x_1 + 3x_2$

Subject to $x_1 + 2x_2 \leq 100$

and $x_1 \geq 0, x_2 \geq 0$

EX 1. Solve the following LPP by graphical method

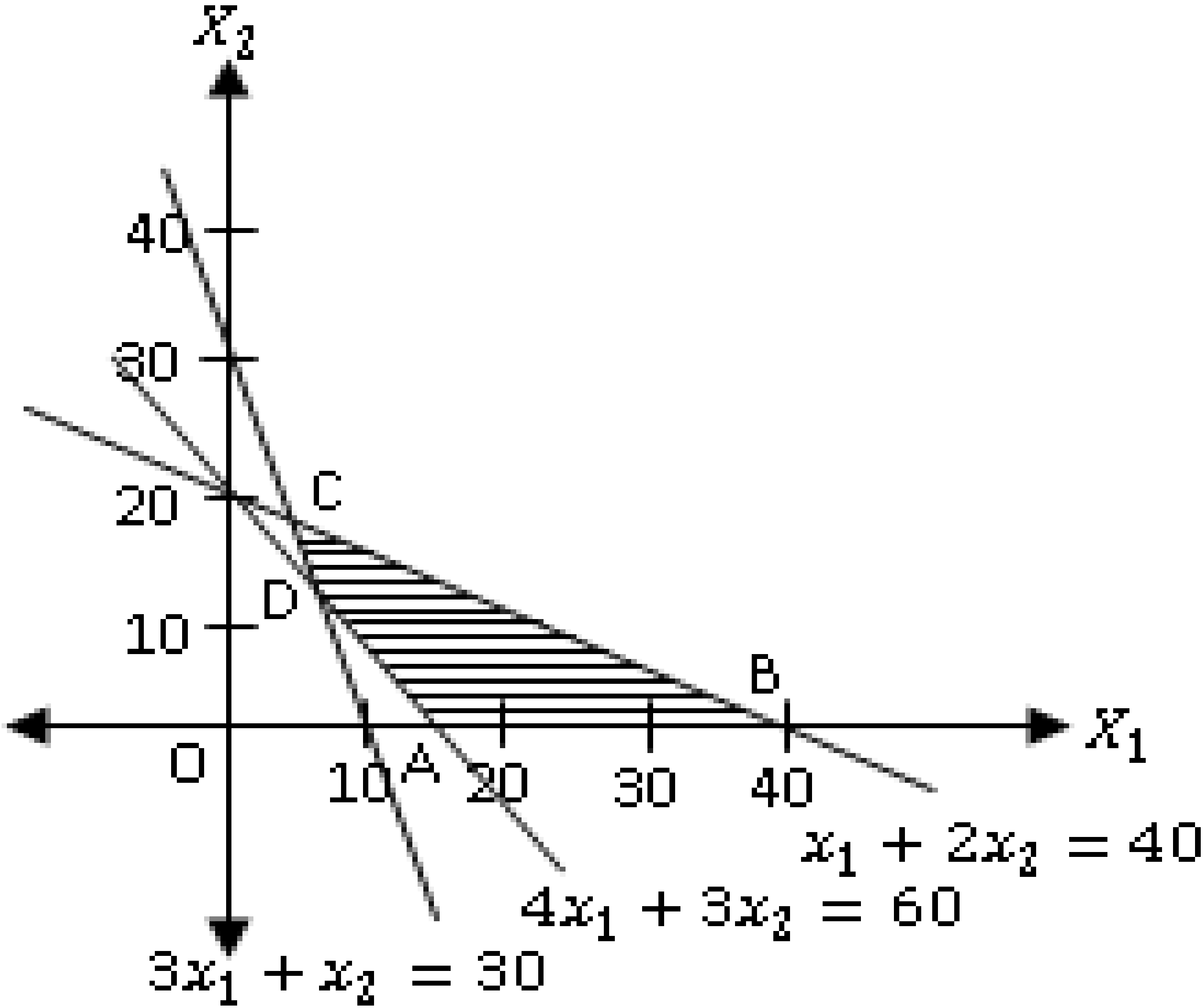
Minimize $Z = 20x_1 + 10x_2$

Subject to $x_1 + 2x_2 \leq 40,$

$$3x_1 + x_2 \geq 30,$$

$$4x_1 + 3x_2 \geq 60,$$

$$x_1, x_2 \geq 0$$



X_2
40
30
20
10
C
D
B
O
10
A
20
30
40
X_1
$x_1 + 2x_2 = 40$
$4x_1 + 3x_2 = 60$
$3x_1 + x_2 = 30$

Corners Points	Value of $Z = 20x_1 + 10x_2$
$A(15, 0)$	300
$B(40, 0)$	800
$C(4, 18)$	260
$D(6, 12)$	240 (Minimum value)

$\therefore$ The minimum value of Z occurs at $D(6, 12)$

Hence the optimal solution is

$$x_1 = 6, x_2 = 12$$

GENERAL FORM OF LP PROBLEM

The general LP problem with $'n'$ decision variables and $'m'$ constraints

can be stated in the following form

Find the values of decision variables $x_1, x_2, \dots \dots, x_n$ so as to

Optimize (max or min) $Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$
(Objective function)

subject to the linear constraints

$a_{11}x_1 + a_{12}x_2 + \ \dots \dots + a_{1n}x_n (\leq, =, \geq) b_1$

$a_{21}x_1 + a_{22}x_2 + \ \dots \dots + a_{2n}x_n (\leq, =, \geq) b_2$ (constraints)

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$a_{m1}x_1 + a_{m2}x_2 + \ \dots \dots + a_{mm}x_n (\leq, =, \geq) b_m$

And $\quad x_1, x_2, \dots \dots, x_n \geq 0$ (non-negativity constraints)

MATRIX FORM OF LPP

The linear programming problem can be expressed in the matrix form

as follows.

Maximize or Minimize $Z = CX$

Subject to $AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0$

Where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \dots \\ x_n \end{bmatrix}_{n\times 1}, b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \dots \\ b_m \end{bmatrix}_{m\times 1}, C = [C_1 \quad C_2 \quad \dots \dots \quad C_n]_{1\times n}$$

and $A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ \dots \dots & \dots \dots & \dots \dots & \dots \dots \\ a_{ml} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix}_{m\times n}$

IMPORTANT DEFINITION

1. **Solution**: Solution values of decision variables $x_j (j = 1, 2, \dots \dots, n)$ which satisfy the constraints of a general LP model is called a solution to that LP model.

IMPORTANT DEFINITION

2. **Feasible Solution**: Solution values of decision variables $x_j (j = 1, 2, \dots \dots, n)$ which satisfy the constraints and non negativity condition of a general LP model are said to constitute the feasible solution to that LP model

3. **Basic Solution:** For a set of $\boldsymbol{m}$ **equations in** $\boldsymbol{n}$ **variable** $(\boldsymbol{n} > m)$, a solution obtained by setting $(n - m)$ variables equal to zero and solving for remaining m equations in m variables is called a basic solution.

- Number of basic solution is ${}^{n}C_{m} = {}^{n}C_{n-m}$
- The $(n-m)$ variables whose values did not appear in this solution are called **non-basic variables** and the
- Remaining m variables are called **basic variables**

4. **Basic Feasible Solution**: A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. i.e. All basic variables assume non-negative values

Basic Feasible Solutions are of two types

(1) Degenerate basic feasible solution: A basic feasible solution is called degenerate if at least one basic variable possesses zero value

(2) Non degenerate basic feasible solution: A basic feasible solution is called non degenerate if all m basic variables are non zero and positive

5. **Optimum basic feasible solution:** A basic feasible solution which optimizes (maximizes or minimizes) the objective function of the given LP model is called on optimum basic feasible solution

6. **Unbounded Solution**: A solution which can increase or decrease the value of objective function of LP problem indefinitely, is called unbounded solution

7. **Slack variables**: If the constraints of a general LPP be $\sum_{j=1}^{n} a_{ij}x_i \leq b_i \ (i = 1, 2, \ldots., m)$ then non negative variables S_i which are introduces to convert the inequalities $(\leq)$ to be equalities $\sum_{j=1}^{n} a_{ij}x_i + S_i = b_i \ (i = 1, 2, \ldots., m)$ are called slack variables.

8. **Surplus variables**: If the constraints of a general LPP be $\sum_{j=1}^{n} a_{ij} x_i \geq b_i \;\; (i = 1, 2, \ldots., m)$ then the non negative variables S_i which are introduced to convert the inequalities($\geq$) to be equalities $\sum_{j=1}^{n} a_{ij} x_i - S_i = b_i \; (i = 1, 2, \ldots., m)$ are called surplus variables

EX 1. Find all the basic solutions to the following problem,

Maximize $Z = x_1 + 3x_2 + 3x_3$ Subject to

$x_1 + 2x_2 + 3x_3 = 4$ and $2x_1 + 3x_2 + 5x_3 = 7$.

Also find which of the basic solutions are

(i) Basic feasible

(ii) Non degenerate basic feasible and

(iii) Optimal basic feasible

Solution: No of equation $m = 2$

No of variables $n = 3$

A basic solution can be obtained by setting any of the $(n - m) = 1$ variable equal to zero and then solving the constraints equation.

The total no. of basic solution is ${}^{n}C_{m} = {}^{3}C_{2} = 3$

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible ? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	5			

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	5			
2	x_1, x_3	x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	4			

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	5			
2	x_1, x_3	x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	4			
3	x_2, x_3	x_1	$2x_1 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	3			

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $x_1 = 2, x_2 = 1$	5	Yes	Yes	Yes
2	x_1, x_3	x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $x_1 = 1, x_3 = 1$	4	Yes	Yes	No
3	x_2, x_3	x_1	$2x_1 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $x_2 = -1, x_3 = 2$	3	No	No	No

EX 1. Find all the basic solutions to the following problem,

Maximize $z = 2x_1 - 2x_2 + 4x_3 - 5x_4$

Subject to $x_1 + 4x_2 - 2x_3 + 8x_4 = 2,$

$-x_1 + 2x_2 + 3x_3 + 4x_4 = 1, x_1, x_2, x_3, x_4 \geq 0$

Also find which of the basic solutions are

(i) Basic feasible

(ii) Non degenerate basic feasible and

(iii) Optimal basic feasible

Solution: No of equation $m = 2$

No of variables $n = 4$

A basic solution can be obtained by setting any of the $(n - m) = 2$ variable equal to zero and then solving the constraints equation.

The total no. of basic solution is

${}^{n}C_{m} = {}^{4}C_{2} = 6$

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1	Yes	No	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1	Yes	No	
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	yes	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1.5	Yes	No	
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	Yes	
3	x_2, x_3	x_1, x_4	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	-1	Yes	No	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1	Yes	No	
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	yes	
3	x_2, x_3	x_1, x_4	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	-1	Yes	No	
4	x_1, x_4	x_2, x_3	$x_1 + 8x_4 = 2,$ $-x_1 + 4x_4 = 1$ $x_1 = 0, x_4 = 1/4$	-1.25	Yes	No	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1.5	Yes	No	
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	Yes	
3	x_2, x_3	x_1, x_4	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	-1	Yes	No	
4	x_1, x_4	x_2, x_3	$x_1 + 8x_4 = 2,$ $-x_1 + 4x_4 = 1$ $x_1 = 0, x_4 = 1/4$	-1.25	Yes	No	
5	x_2, x_4	x_1, x_3	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ Unbounded	—	—	—	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution non degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1	Yes	No	
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	Yes	
3	x_2, x_3	x_1, x_4	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	-1	Yes	No	
4	x_1, x_4	x_2, x_3	$x_1 + 8x_4 = 2,$ $-x_1 + 4x_4 = 1$ $x_1 = 0, x_4 = 1/4$	-1.25	Yes	No	
5	x_2, x_4	x_1, x_3	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ Unbounded	—	—	—	
6	x_3, x_4	x_1, x_2	$-2x_3 + 8x_4 = 2$ $3x_3 + x_4 = 12$ $x_3 = 0, x_4 = 1/4$	-12.5	Yes	No	

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	x_3, x_4	$x_1 + 4x_2 = 2$ $-x_1 + 2x_2 = 1$ $x_1 = 0, x_2 = 1/2$	-1	Yes	No	No
2	x_1, x_3	x_2, x_4	$x_1 - 2x_3 = 2$ $-x_1 + 3x_3 = 1$ $x_1 = 8, x_3 = 3$	28	Yes	Yes	Yes
3	x_2, x_3	x_1, x_4	$4x_2 - 2x_3 = 2$ $2x_2 + 3x_3 = 1$ $x_3 = 0, x_2 = 1/2$	-1	Yes	No	No
4	x_1, x_4	x_2, x_3	$x_1 + 8x_4 = 2,$ $-x_1 + 4x_4 = 1$ $x_1 = 0, x_4 = 1/4$	-1.25	Yes	No	No
5	x_2, x_4	x_1, x_3	$4x_2 + 8x_4 = 2$ $2x_2 + 4x_4 = 1$ Unbounded	—	—	—	—
6	x_3, x_4	x_1, x_2	$-2x_3 + 8x_4 = 2$ $3x_3 + x_4 = 12$ $x_3 = 0, x_4 = 1/4$	-1.25	Yes	No	No

Find all basic solutions to the following problem. Which of them are basic feasible, non-degenerate, infeasible basic and optimal feasible solutions?

Maximize $z = 2x_1 + 3x_2$

Subject to $x_1 + 3x_2 \leq 6,$

$$3x_1 + 2x_2 \leq 6, x_1, x_2 \geq 0$$

Solution: First we convert the given problem to standard form

Maximize $z = 2x_1 + 3x_2 + 0S_1 + 0S_2$

Subject to $x_1 + 3x_2 + S_1 + 0S_2 = 6,$

$$3x_1 + 2x_2 + 0S_1 + S_2 = 6,$$

$$x_1, x_2, S_1, S_2 \geq 0$$

No of equation $m = 2$

No of variables $n = 4$

A basic solution can be obtained by setting any of the $(n - m) = 2$ variable equal to zero and then solving the constraints equation.

The total no. of basic solution is

$^{n}C_{m} = {}^{4}C_{2} = 6$

Sr. No. of Basic Solution	Basic Variables	Non basic Variables	Values of the basic variables given by the constraints equation	Values of the objective function	Is the solution feasible? (are all $x_j \geq 0$)	Is the solution degenerate (are all basic variables > 0)?	Is the solution feasible & Optimal
1	x_1, x_2	S_1, S_2	$x_1 + 3x_2 = 6,$ $3x_1 + 2x_2 = 6$ $x_1 = \frac{6}{7}, \quad x_2 = \frac{12}{7},$	$\frac{48}{7}$	Yes	Yes	No
2	x_1, S_1	x_2, S_2	$x_1 + S_1 = 6,$ $3x_1 + 0S_1 = 6$ $x_1 = 2, \quad S_1 = 4$	4	Yes	Yes	No
3	x_1, S_2	x_2, S_1	$x_1 + 0S_2 = 6,$ $3x_1 + S_2 = 6$ $x_1 = 6, \quad S_2 = -12,$	12	Yes	No	Yes
4	x_2, S_1	x_1, S_2	$3x_2 + S_1 = 6,$ $2x_2 + 0S_1 = 6$ $x_2 = 3, \quad S_1 = -3,$	9	Yes	No	No
5	x_2, S_2	x_1, S_1	$3x_2 + 0S_2 = 6,$ $2x_2 + S_2 = 6$ $x_2 = 2, \quad S_2 = 2$	6	Yes	Yes	No
6	S_1, S_2	x_1, x_2	$S_1 + 0S_2 = 6,$ $0S_1 + S_2 = 6$ $S_1 = 6, \quad S_2 = 6,$	0	Yes	Yes	No

Is the solution $x_1 = 1, x_2 = 1/2\,, x_3 = 0, x_4 = 0, x_5 = 0$ a basic solution of the following problem?

$$x_1 + 2x_2 + x_3 + x_4 = 2\ ,$$
$$x_1 + 2x_2 + (1/2)x_3 + x_5 = 2$$