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Computer Organization
and Architecture
6th Edition

Chapter 9

Computer Arithmetic

2.1 **Introduction to Arithmetic and Logical unit, Computer Arithmetic: Fixed and Floating point numbers, Signed numbers, Integer Arithmetic, 2's Complement arithmetic**

2.2 **Booth's Recoding and Booth's algorithm for signed multiplication, Restoring division and non-restoring division algorithms**

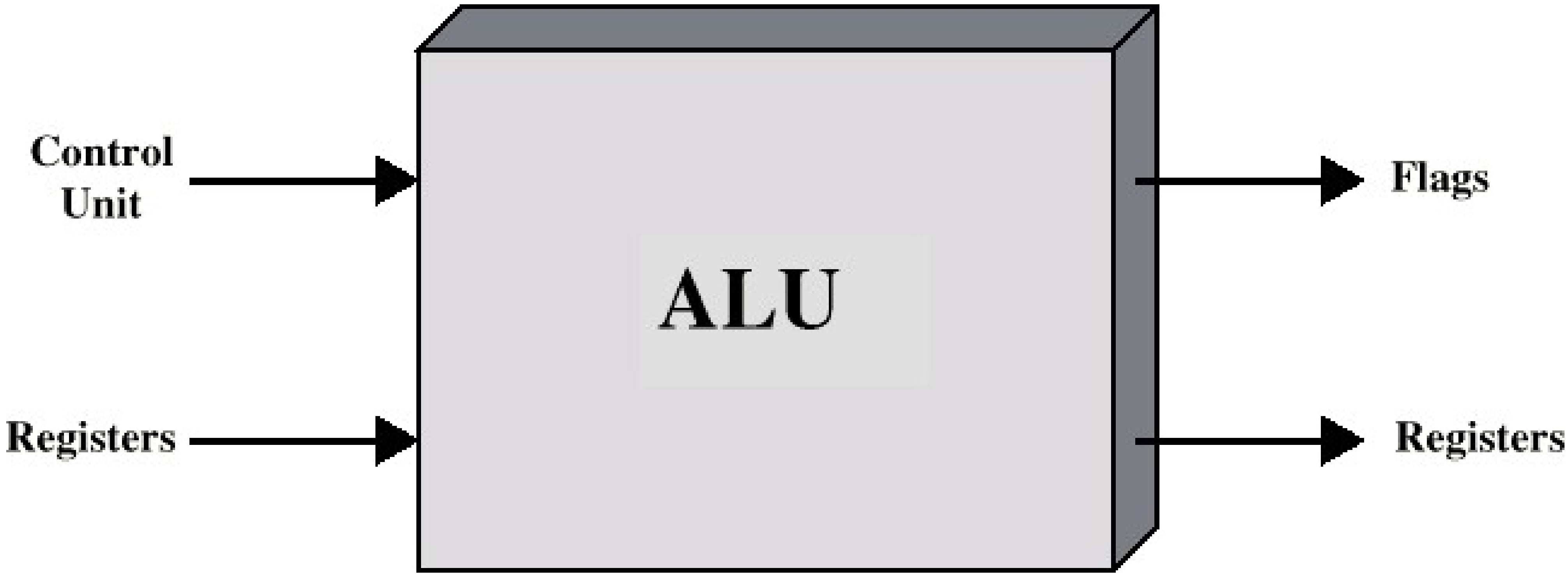
2.3 **IEEE floating point number representation and operations: Addition. Subtraction, Multiplication and Division**. **IEEE standards for Floating point representations :Single Precision and Double precision Format**

Arithmetic & Logic Unit

- Does the calculations
- Everything else in the computer is there to

  service this unit
- Handles integers
- May handle floating point (real) numbers
- May be separate FPU-floating-point unit (FPU), which operates on floating point numbers. (maths co-processor)

ALU Inputs and Outputs



- Operands for arithmetic and logic operations are presented to the ALU in registers, and the results of an operation are stored in registers.
- These registers are temporary storage locations within the processor that are connected by signal paths to the ALU.
- The ALU may also set flags as the result of an operation.
- The flag values are also stored in registers within the processor.
- The processor provides signals that control the operation of the ALU and the movement of the data into and out of the ALU.

Addition and Subtraction

- Normal binary addition
- Monitor sign bit for overflow
- Take twos compliment of substahend and add to minuend
  - i.e. a - b = a + (-b)

- So we only need addition and complement circuits

A	B	Sum
0	0	0
0	1	1
1	0	1
1	1	0, Carry 1
1	1	1,Carry 1

Example of 2's Compliment

5 = 00000101

Complement Digits

11111010

+1

Add 1

-5 = 11111011

0 1 1 0 1 1 1 0 — Original binary value

1 0 0 1 0 0 0 1 — 1's complement

	1 0 0 1 0 0 0 1
+	1
	1 0 0 1 0 0 1 0

1 0 0 1 0 0 1 0 — 2's complement

-13 = 11110011

Complement Digits

00001100

+1

Add 1

13 = 00001101

Find 2's compliment

1000

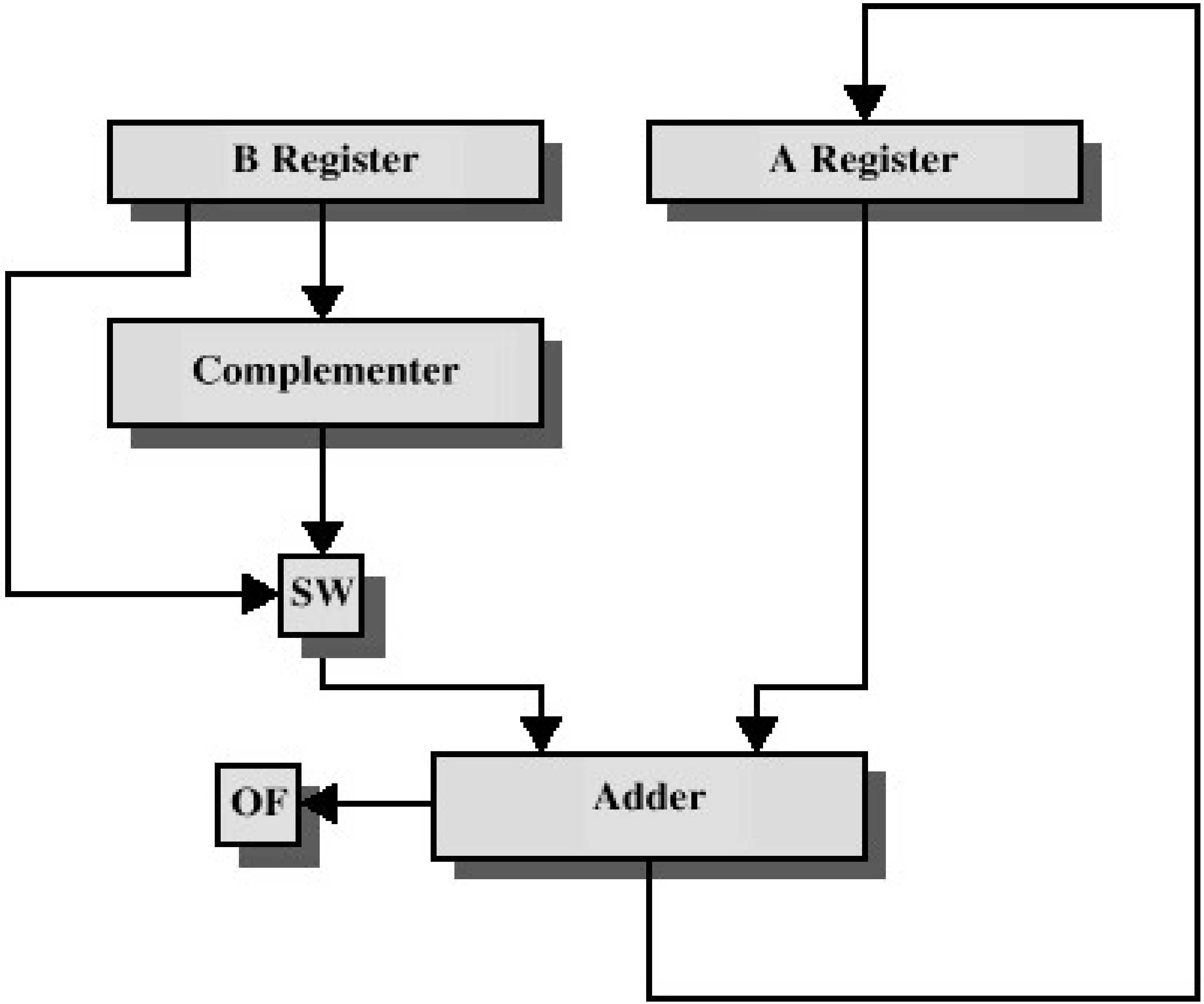
0100

111001

101010

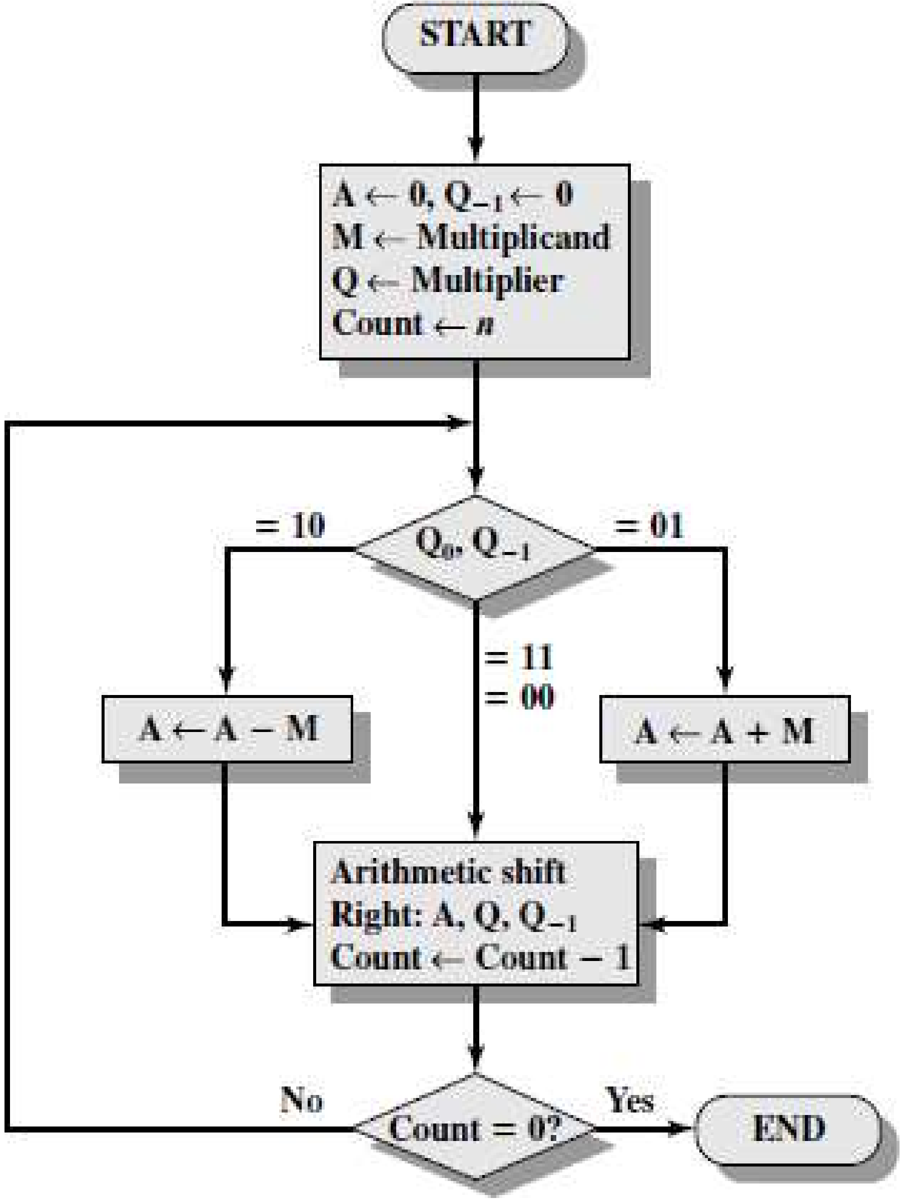
100000

Hardware for Addition and Subtraction



OF = overflow bit
SW = Switch (select addition or subtraction)

Booth's Algorithm



Q0	Q-1	Result
0	0	Only shift
1	1	
0	1	A=A + M ,then shift
1	0	A= A – M , then shift

M =7

Q =3

M = 0 1 1 1

Q = 0 0 1 1

- M = 1 0 0 1

Example of Booth's Algorithm:7(M)*3(Q)

A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial Values	
1001	0011	0	0111	A = A - M	First Cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second Cycle
0101	0100	1	0111	A = A + M	Third Cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth Cycle

Answer is in A and Q→0001 0101 =21

A	Q	Q_{-1}	M		
0000	0011	0	0111	Initial values	
1001	0011	0	0111	A ← A − M	First cycle
1100	1001	1	0111	Shift	
1110	0100	1	0111	Shift	Second cycle
0101	0100	1	0111	A ← A + M	Third cycle
0010	1010	0	0111	Shift	
0001	0101	0	0111	Shift	Fourth cycle

Figure 9.13 Example of Booth's Algorithm (7 × 3)

Examples-size of n determines answer

Solve using Booths Algorithm

A. M = 5 , Q = 5

B. M = 12, Q = 11

C. M = 9, Q = - 3

D. **M = -13 (0011)** , Q = 6

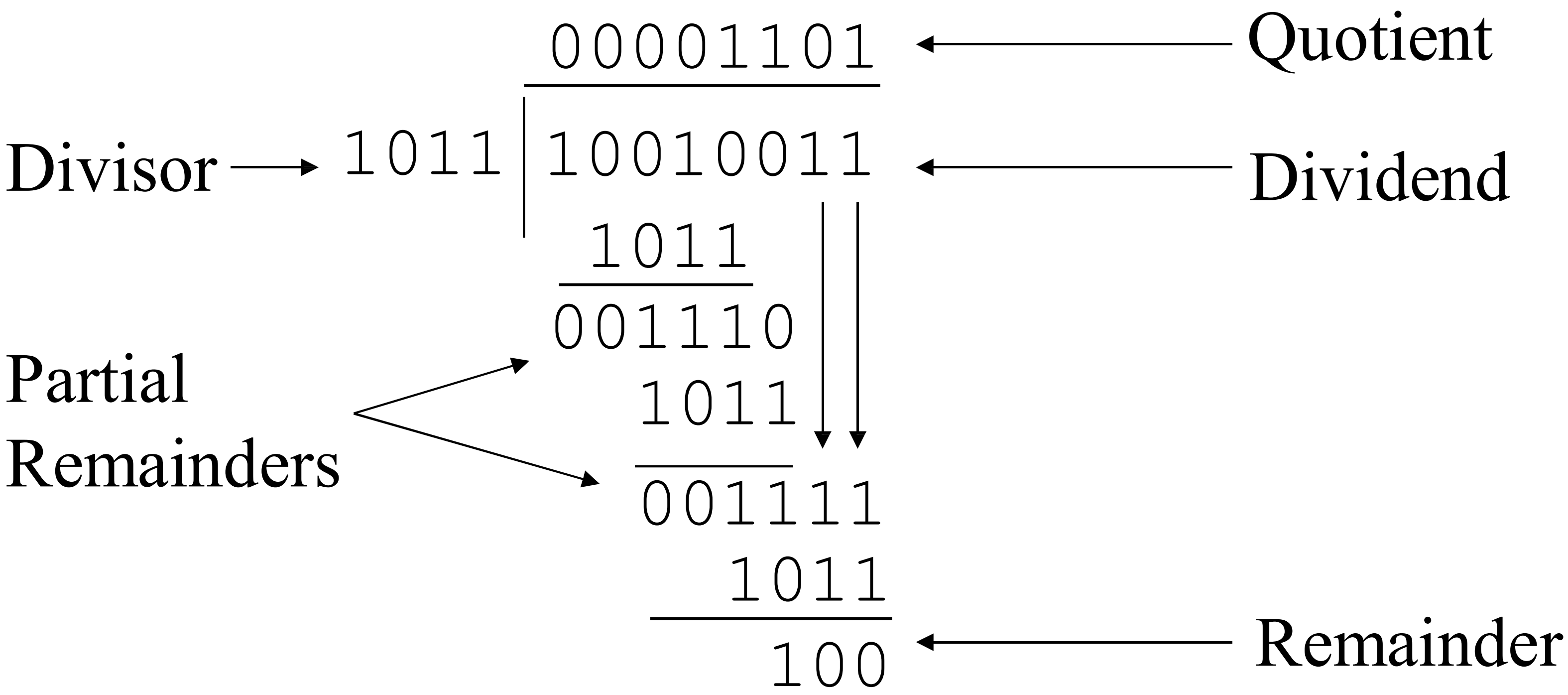
-M=13 (1101)

A. M = -19 , Q = -20

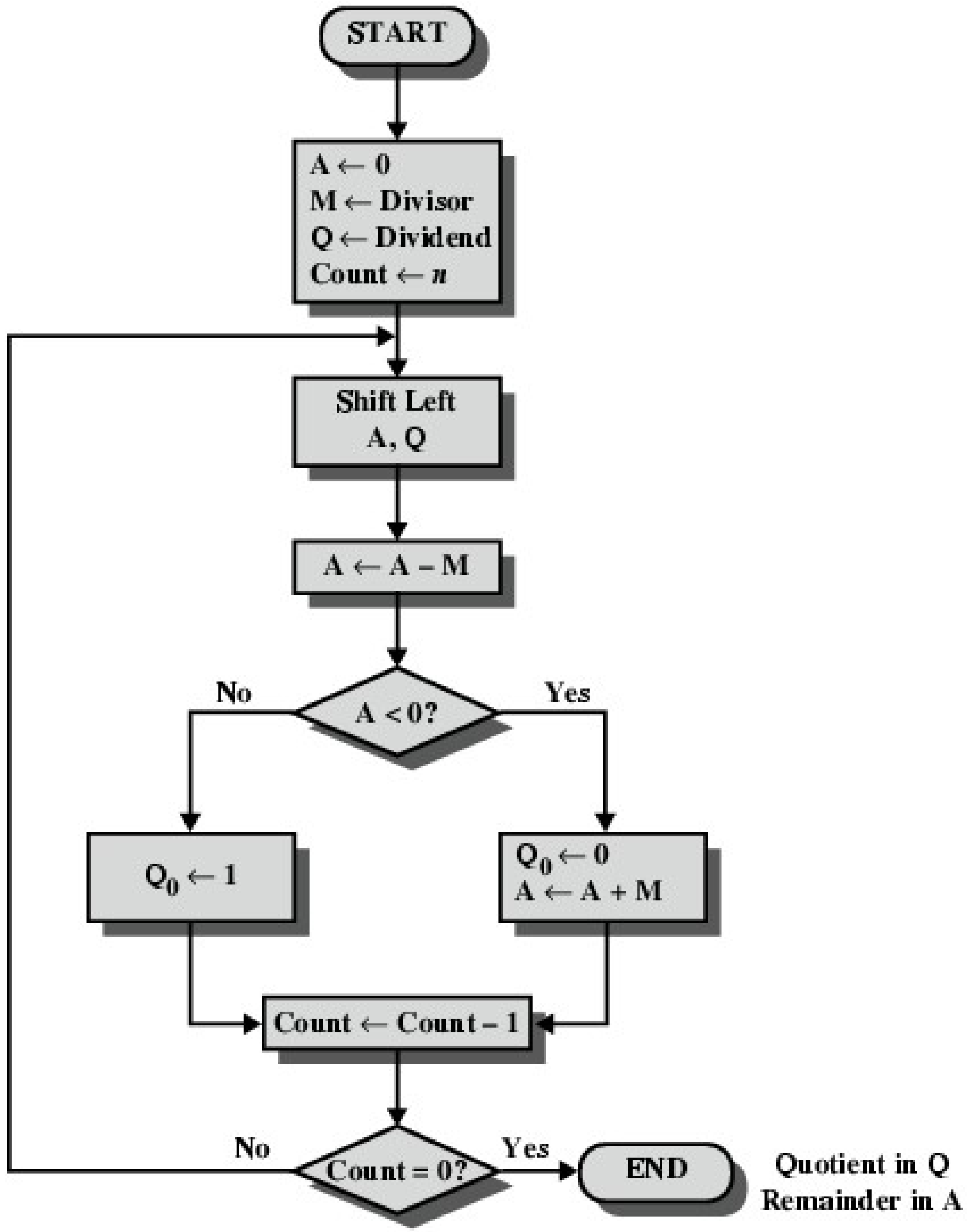
Division

- More complex than multiplication
- Negative numbers are really bad!
- Based on long division

Division of Unsigned Binary Integers



Flowchart for Restoring Division



$M \leftarrow 3\overline{)7} \rightarrow Q$

$M = 0011$

$-M = 1101$

$Q = 0111$

$$\begin{array}{r|l|l} 2 & 7 & \\ 2 & 3 & 1 \\ & 1 & 1 \end{array} \qquad \begin{array}{r|l|l} 2 & 3 & \\ & 1 & 1 \end{array}$$

1 1 0 0

1

A: 1101
M = 0011
0000

cycle 1

A	Q	
0 0 0 0	0 1 1 1	
0 0 0 0	1 1 1 $\square$	Shift left A, Q
1 1 0 1	1 1 1 $\square$	A = A − M
0 0 0 0	1 1 1 $\boxed{0}$	set q_0 = 0; A = A + M → Restore

	A	Q	
0001	0001	110 $\boxed{\ }$	shift left
1101			$A = A - M$
1110	1110	110 $\boxed{0}$	
			set $Q_0 = 0$
	Restore		$A = A + M$
1110			
0011	0001	1100	
0001			

0001

0 0 1 1   1 0 0 $\boxed{}$ shift left

1111
0011
1101
0000

0 0 0 0   1 0 0 $\boxed{1}$ A = A − M

set q_0 = 1

0 0 0 0   1 0 0 1

0001
1101
1110

0 0 0 1   0 0 1 $\boxed{}$ shift left

1 1 1 0   0 0 1 $\boxed{0}$ A = A − M

set q_0 = 0

A = A + M

$\underbrace{0\ 0\ 0\ 1}_{\text{Remainder} = 1}$ $\underbrace{0\ 0\ 1\ 0}_{\text{Quotient} = 2}$

$M = 27$; $Q = 55$

$M = 011011$

$-M = 100101$

$Q = 110111$

	A	Q	
	000000	110111	
	000001	10111	shift left A,
cle 1	100110	10111 $\boxed{0}$	A=A−M
	000001	101110	Set $Q_0=0$; A=A+

cycle2	000011	01110□	shift left A,Q
	101000	01110$\boxed{0}$	A=A-M
			set Q_0=0
	000011	011100	A=A+M
cycle3	000110	11100□	shift left A,Q
	101011	11100$\boxed{0}$	A=A-M
	000110	111000	Set Q_0=0; A=A+M
cycle4	001101	11000□	shift left A,Q
	110010	11000$\boxed{0}$	A=A-M

cycle 4	0011101	11000 $\boxed{\ }$	shift left A, Q
	1110010	11000 $\boxed{0}$	A=A−M
	0011101	110000	set Q_0=0., A=A+M
cycle 5	011011	10000 $\boxed{\ }$	shift left A, Q,
	000000	10000 $\boxed{1}$	set Q_0 = 1
cycle 6	000001	00001 $\boxed{\ }$	shift left A, Q
	100110	00001 $\boxed{0}$	A=A−M
			set Q_0 = 0,
	000001	000010	A= A+M

R = 1 Q = 2

M = 27 ; Q = 55

M = 011011

−M = 100101

Q = 110111

	A	Q	
cycle 1	000000	110111	
	000001	10111□	shift left A,Q
	100110	10111$\boxed{0}$	A = A − M
	000001	101110	Set Q_0=0, A=A+M
cycle 2	000011	01110□	shift left A,Q
	101000	01110$\boxed{0}$	A = A − M
			set Q_0=0
	000011	011100	A = A + M
cycle 3	000110	11100□	shift left A,Q
	101011	11100$\boxed{0}$	A = A − M
	000110	111000	Set Q_0=0 ; A=A+M
cycle 4	001101	11000□	shift left A,Q
	110010	11000$\boxed{0}$	A = A − M
	001101	110000	set Q_0=0, A=A+M
cycle 5	011011	10000□	shift left A,Q
	000000	10000$\boxed{1}$	set Q_0 = 1
cycle 6	000001	00001□	shift left A,Q
	100110	00001$\boxed{0}$	A = A − M
			Set Q_0 = 0
	000001	000010	A = A + M

R = 1 Q = 2

Solve using Restoring Division

A. M = 5 , Q = 5 , A=0000 , Q=0010

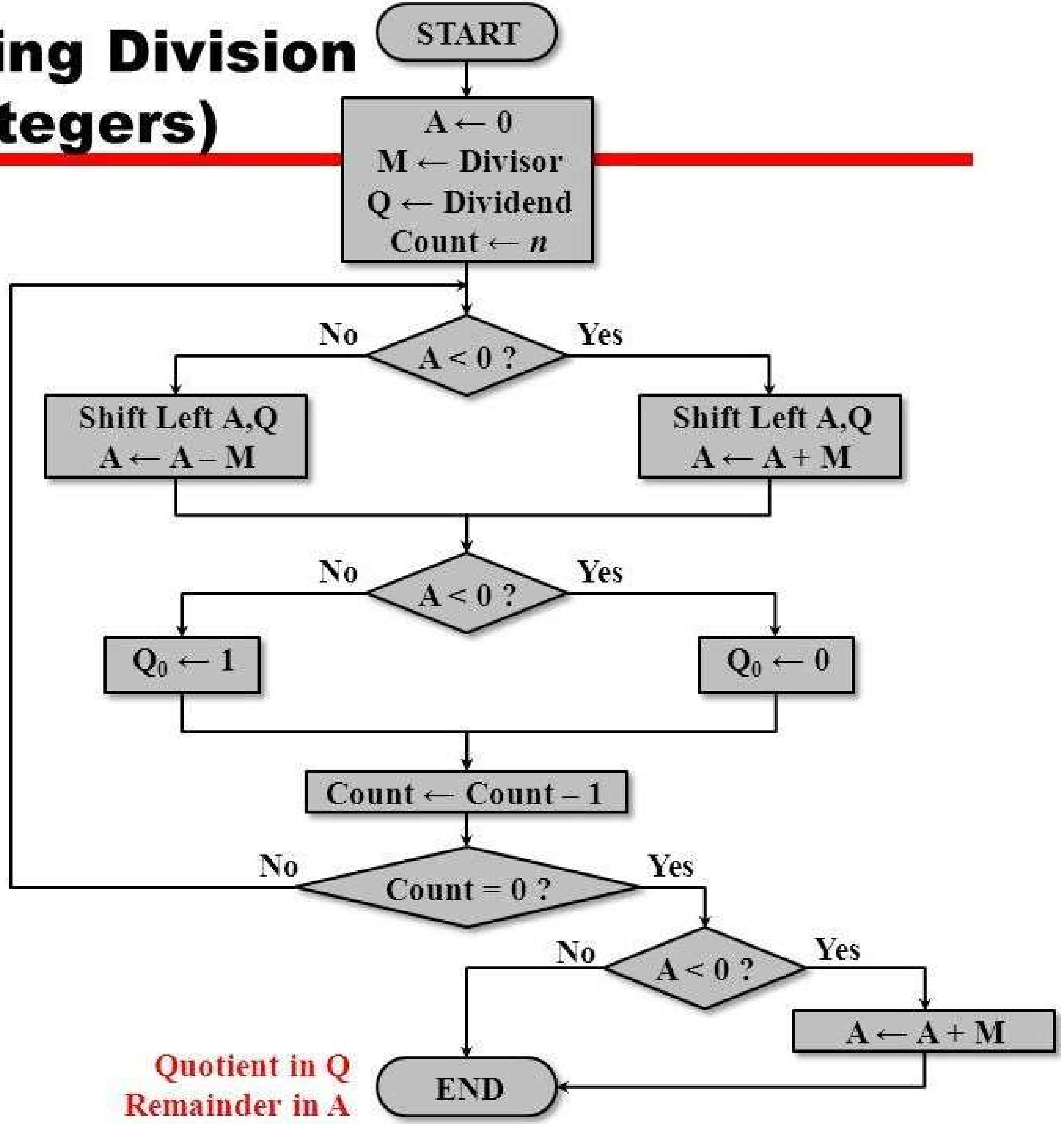
B. M = 12, Q = 26, A=00010 , Q= 00010

C. M = 9, Q = 19 , A=00001 , Q =00010

D. M = 32 , Q = 59, A=011011, Q=000001

E. M = 17 , Q = 42, A=001000 , Q=000010

Non-Restoring Division (Positive Integers)



$M = 2$; $Q = 4$

$M = 0010$

$-M = 1110$

$Q = 0100$

	A	Q	
	0 0 0 0	0 1 0 0	
Shift left A, Q	0 0 0 0	1 0 0 $\boxed{}$	①
A = A − M, set q_0 = 0	1 1 1 0	1 0 0 $\boxed{0}$	①
Shift left A, Q	1 1 0 1	0 0 0 $\boxed{}$	②
A = A + M, set q_0 = 0	1 1 1 1	0 0 0 $\boxed{0}$	②

shift left A,Q	1 1 1 0	0 0 0 $\square$	(3)
A = A + M	0 0 0 0	0 0 0 $\boxed{1}$	
set q_0 = 1			
shift left A,Q	0 0 0 0	0 0 1 $\square$	(4)
A = A − M	1 1 1 0	0 0 1 $\boxed{0}$	
set q_0 = 0			

count = 0

A = A + M

$$\begin{array}{rl} & 1110 \ (A) \\ + & 0010 \ (M) \\ \hline \cancel{1} & 0000 \end{array}$$

$\underbrace{0000}_{A = \text{Remainder}}$ $\underbrace{0010}_{Q = \text{Quotient}}$

Solve using Non Restoring

A. M = 5 ,   Q = 5,A=0000,Q=0001.

B. M = 12,   Q = 26,A=000010, Q=000010.

C. M = 9,   Q = 19,A=00001,Q=00010.

D. M = 32 , Q = 59,A=011011,Q=000001.

E. M = 17 , Q = 42,A=001000,Q=000010

Booths Recoding / Bit pair recording

STEPS

Booth's Recoding algorithm

5 × 3

M Q

M 0101 Q = 0011

−M 1011

Step 1 : Table for M

Operation	Value
0	0000 0000
+1	0000 0101
−1	1111 1011
+2	0000 1010
−2	1111 0110

left shift +1

left shift −1

Step 2 : Value of Q

Q_{-1}

$0 \quad 0 \quad 1 \quad 1 \quad 0$

$0 \quad +1 \quad 0 \quad -1$

2(1st nos) + 2nd nos

$2(0) + 1 \qquad 2(0) + (-1)$

$1 \qquad -1$

Step 3: M * Q

0 1 0 1

1 1

1 1 1 1 1 0 1 1

0 0 0 1 0 1 + +

Discard carry 1

0 0 0 0 1 1 1 1

$2^3 \; 2^2 \; 2^1 \; 2^0$

$8 + 4 + 2 + 1 = 15$

Solve using Booths Recoding

1. M= 5 , Q = 4 (4 bits)= 00010100 (20)

2. M=9 , Q = -6 (5 bits)=11110 01010 (-54)

3. M=15 , Q=-10 (5 bits)=11011 01010(-150)

4. M= -13 ,Q = -20 (6 bits)=000100000100(260)

Sample mix problems-Kindly refrain referring to flowchart.

1. Booth's Algorithm = 000 100 000 100(260)

A= 110011 (Multiplicand)

B= 101100 (Multiplier)

2. Booth's Recoding = 0110 1010/11011 01010

M= (15)

Q= (-10)

3. Non Restoring Division

M=11 , Q= 21 , A= 01010 , Q= 00001

4. Restoring Division

M=14 , Q= 15, A=00001 , Q = 00001

Floating Point

Sign bit	Biased Exponent	Significand or Mantissa

- A floating point number, is a positive or negative whole number with a decimal point. For example, 5.5, 0.25, and -103.342 are all floating point numbers, while 91, and 0 are not

$$\textbf{+/- .significand x } \mathbf{2}^{\textbf{exponent}}$$

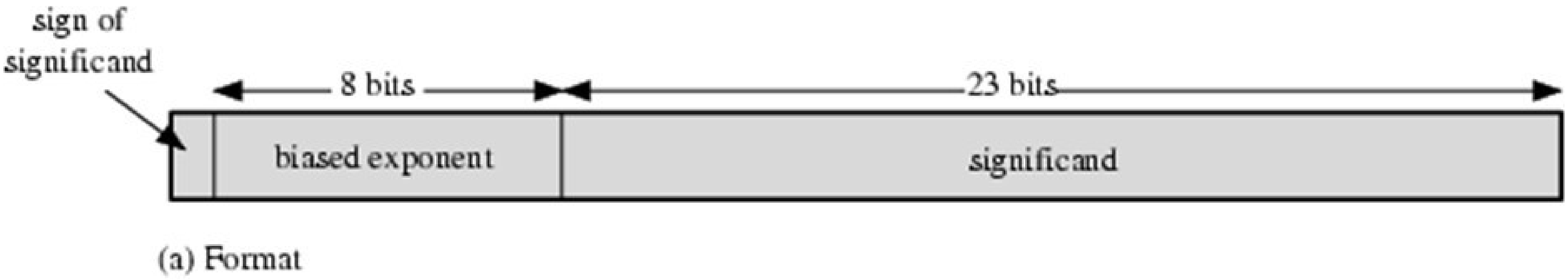
- Misnomer
- Point is actually fixed between sign bit and body of mantissa
- Exponent indicates place value (point position)

$$\pm S \times B^{\pm E}$$

This number can be stored in a binary word with three fields:

- Sign: plus or minus
- Significand S
- Exponent E

Floating Point Examples



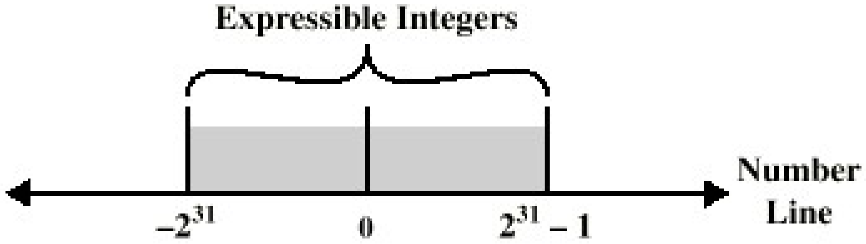
Typical 32-Bit Floating-Point Format

The leftmost bit stores the **sign** of the number
The **exponent** value is stored in the next 8 bits.
The representation used is known as a **biased representation**.
A fixed value, called the bias, is subtracted from the field to get the true exponent value.

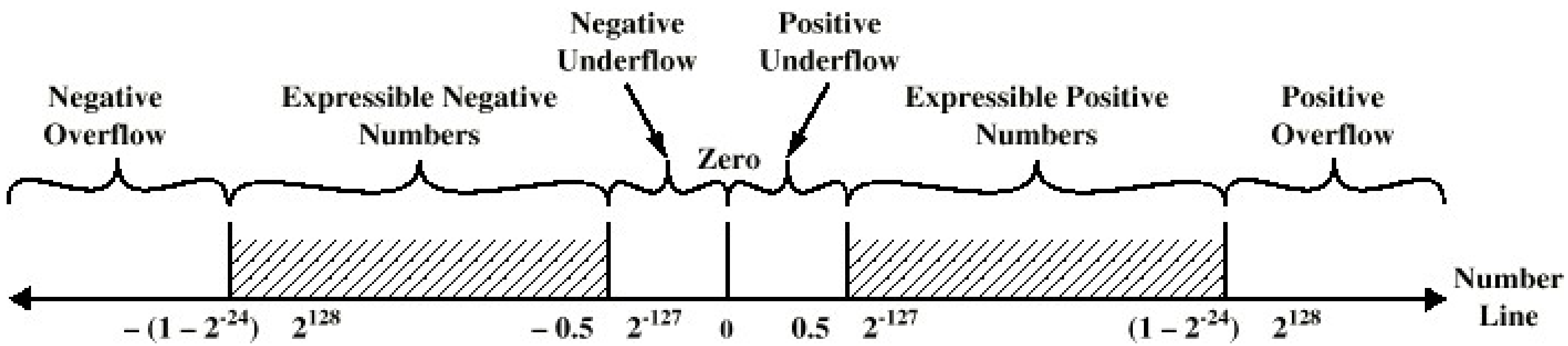
Signs for Floating Point

- Mantissa is stored in 2s compliment
- Exponent is in excess or biased notation
    - e.g. Excess (bias) 128 means
    - 8 bit exponent field
    - Pure value range 0-255
    - Subtract 128 to get correct value
    - Range -128 to +127

Expressible Numbers



(a) Twos Complement Integers



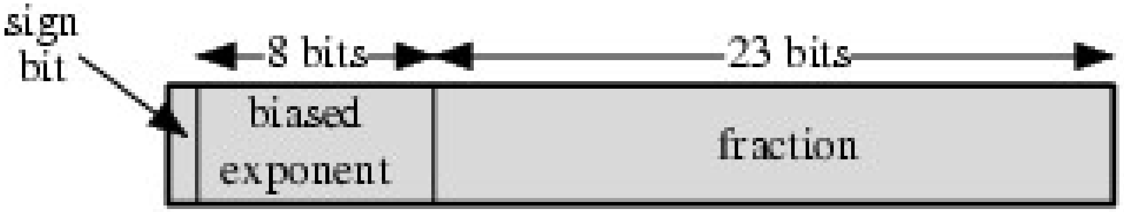
(b) Floating-Point Numbers

IEEE 754

- Standard for floating point storage
- developed to facilitate the portability of programs from one processor to another
- Defines 32 and 64 bit standards with 8 and 11 bit exponent respectively
- the standard defines two extended formats, single and double, whose exact format is implementation dependent.
- The extended formats include additional bits in the exponent (extended range) and in the significand (extended precision).
- The extended formats are to be used for intermediate calculations.
- Extended formats (both mantissa and exponent) for intermediate results

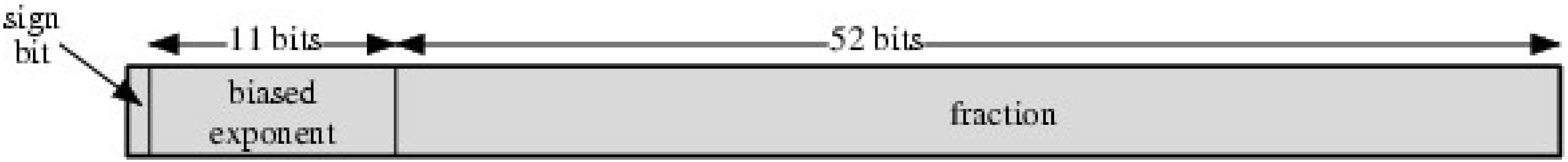
IEEE 754 Formats

$(1.N)2^{E-127}$



(a) Single format

32 BIT



(b) Double format

64 BIT

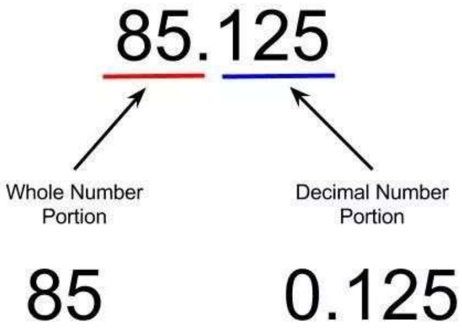
$(1.N)2^{E-1023}$

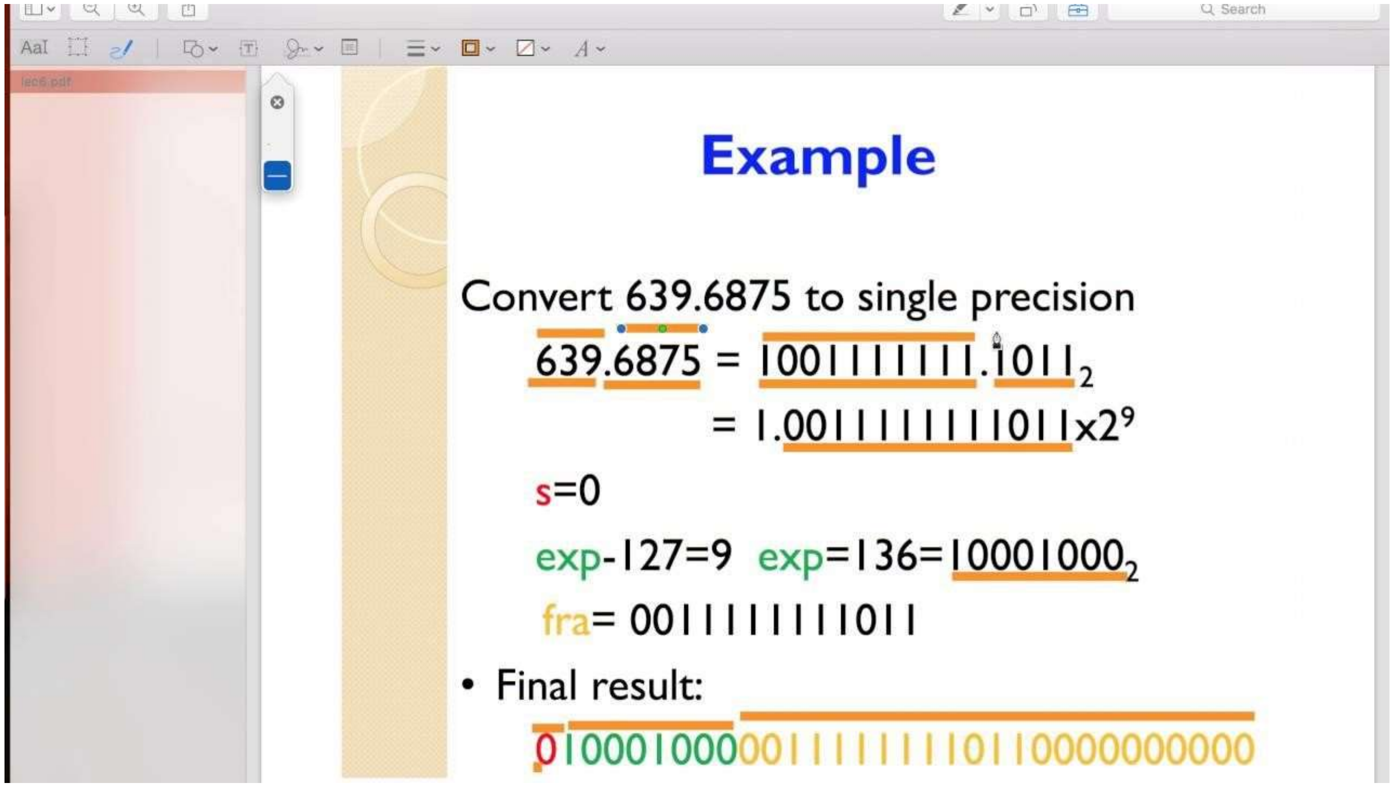
Steps

- 1. Convert Decimal to Binary
- 2. Normalization
  - Rewriting Step 1 into (1.N) form
  - Ex: 1 1 1 . 0 1 1 = **1 .** 1 1 0 1 1 x 2^{2}
  - Ex: 0 . 0 0 0 1 0 = 0 0 0 0 **1 .** 0 x 2^{-4}

Exponent

- 3.Biasing
  - Applying Single Precision (E – 1 2 7) & Double Precision (E – 1 0 2 3) on exponent from Step 2
- 4. Representation in Single (32 bit)and Double Precision (64 bit) Format


85.125
Whole Number Portion
Decimal Number Portion
85
0.125


Example
Convert 639.6875 to single precision
639.6875 = 1001111111.1011₂
= 1.0011111111011x2⁹
s=0
exp-127=9 exp=136=10001000₂
fra= 0011111111011
• Final result:
01000100000111111110110000000000

Solved Example

Eg 12.25

Step 1: Converting Dec to Bin

2	12	
2	6	0
2	3	0
	1	1
		1

$$
\begin{array}{r}
.25 \\
\times\ 2 \\
\hline
0.50 \\
\times\ 2 \\
\hline
1.00
\end{array}
\rightarrow \text{stop}
$$

$$\underline{12}\ .\ \underline{25}$$

$$1100\ .\ 01$$

Step 2 : Normalization (1. N)

1.100001×2^{3} → Exponent

Step 3 : Biasing

Single Precision	Double precision
$E - 127$	$E - 1023$
$3 = E - 127$	$3 = E - 1023$
$E = 127 + 3$	$E = 1023 + 3$
$= 130$	$= 1026$

2	130	
2	65	0
2	32	1
2	16	0
2	8	0
2	4	0
2	2	0
	1	0
		1

2	1026	
2	513	0
2	256	1
2	128	0
2	64	0
2	32	0
2	16	0
2	8	0
2	4	0
2	2	0
	1	0

Single Precision (32 bits)

Sign bit	Biased Exponent	Mantissa/Significand
0	10000010	10001
1 bit	8 bits	23 bits

Double Precision (64 bits)

Sign bit		
0	10000000010	10001
1 bit	11 bits	52 bits

Solve

25.44	SP- 0|100000|1001 0111 0000 1010 0011 110
	DP- 0|10000000011|1001 0111 0000 1010 0011 110
0.00635	SP- 0|1110111|00000001101000…
	DP- 0|1111110111|00000001101000…
-125.10	SP- 1 | 10000101| 1111 010001
	DP- 1 | 10000000101| 1111 010001
-13.54	SP- 1|10000010|10110001010
	DP- 1|10000000010|10110001010

Sample Problems to Solve

1) 178.1875

SP 0|10000110|01100100011

DP 0|10000000110|

1) 309.175

SP 0|10000111|01011101001011

DP 0|10000000111|

1) 1259.125

SP 0|10001001|0011101011001000...(9 zeroes)

DP 0|10000001001|**010100111100**

1) 0.0625

SP 0|01111011|0000000....

DP 0|01111111|00000.....

Division of signed numbers

1. Load the divisor into the M register and the dividend into the A, Q registers. The dividend must be expressed as a $2n$-bit twos complement number. Thus, for example, the 4-bit 0111 becomes 00000111, and 1001 becomes 11111001.
2. Shift A, Q left 1 bit position.
3. If M and A have the same signs, perform $A \leftarrow A - M$; otherwise, $A \leftarrow A + M$.
4. The preceding operation is successful if the sign of A is the same before and after the operation.
   a. If the operation is successful or $A = 0$, then set $Q_0 \leftarrow 1$.
   b. If the operation is unsuccessful and $A \neq 0$, then set $Q_0 \leftarrow 0$ and restore the previous value of A.
5. Repeat steps 2 through 4 as many times as there are bit positions in Q.
6. The remainder is in A. If the signs of the divisor and dividend were the same, then the quotient is in Q; otherwise, the correct quotient is the twos complement of Q.

The reader will note from Figure 9.17 that $(-7) \div (3)$ and $(7) \div (-3)$ produce different remainders. This is because the remainder is defined by

$$D = Q \times V + R$$

where

D = dividend
Q = quotient
V = divisor
R = remainder

The results of Figure 9.17 are consistent with this formula.

A	Q	M = 0011
0000	0111	Initial value
0000	1110	shift
1101		subtract
0000	1110	restore
0001	1100	shift
1110		subtract
0001	1100	restore
0011	1000	shift
0000		subtract
0000	1001	set $Q_0 = 1$
0001	0010	shift
1110		subtract
0001	0010	restore

(a) (7)/(3)

Solve

a) 7 / -3

b) -7 (Q) / 3 (M)

c) -7 (Q) / -3 (M)

A	Q	M = 1101
0000	0111	Initial value
0000	1110	shift
1101		add
0000	1110	restore
0001	1100	shift
1110		add
0001	1100	restore
0011	1000	shift
0000		add
0000	1001	set $Q_0 = 1$
0001	0010	shift
1110		add
0001	0010	restore

(b) (7)/(–3)

A	Q	M = 0011
1111	1001	Initial value
1111	0010	shift
0010		add
1111	0010	restore
1110	0100	shift
0001		add
1110	0100	restore
1100	1000	shift
1111		add
1111	1001	set $Q_0 = 1$
1111	0010	shift
0010		add
1111	0010	restor

(c) (–7)/(3)

A	Q	M = 1101
1111	1001	Initial value
1111	0010	shift
0010		subtract
1111	0010	restore
1110	0100	shift
0001		subtract
1110	0100	restore
1100	1000	shift
1111		subtract
1111	1001	set $Q_0 = 1$
1111	0010	shift
0010		subtract
1111	0010	restore

(d) (–7)/(–3)

- Dividend negative → Remainder –ve

Table 9.5 Floating-Point Numbers and Arithmetic Operations

Floating Point Numbers	Arithmetic Operations
$X = X_S \times B^{X_E}$ $Y = Y_S \times B^{Y_E}$	$\left.\begin{array}{l} X + Y = (X_S \times B^{X_E - Y_E} + Y_S) \times B^{Y_E} \\ X - Y = (X_S \times B^{X_E - Y_E} - Y_S) \times B^{Y_E} \end{array}\right\} X_E \leq Y_E$ $X \times Y = (X_S \times Y_S) \times B^{X_E + Y_E}$ $\dfrac{X}{Y} = \left(\dfrac{X_S}{Y_S}\right) \times B^{X_E - Y_E}$

Examples:

$$X = 0.3 \times 10^2 = 30$$
$$Y = 0.2 \times 10^3 = 200$$

$$X + Y = (0.3 \times 10^{2-3} + 0.2) \times 10^3 = 0.23 \times 10^3 = 230$$
$$X - Y = (0.3 \times 10^{2-3} - 0.2) \times 10^3 = (-0.17) \times 10^3 = -170$$
$$X \times Y = (0.3 \times 0.2) \times 10^{2+3} = 0.06 \times 10^5 = 6000$$
$$X \div Y = (0.3 \div 0.2) \times 10^{2-3} = 1.5 \times 10^{-1} = 0.15$$

4 phases of FP Arithmetic +/-

- Check for zeros
- Align significands (adjusting exponents)
- Add or subtract significands
- Normalize result

Floating Point Addition

Add the following two decimal numbers in scientific notation:

8.70×10^{-1} with 9.95×10^{1}

Rewrite the smaller number such that its exponent matches with the exponent of the larger number.

$8.70 \times 10^{-1} = 0.087$ (Note !) $\times 10^{1}$

Add the mantissas

$9.95 + 0.087 = 10.037$ and

write the sum 10.037×10^1

Put the result in Normalised Form

$10.037 \times 10^1 = 1.0037 \times 10^2$

(shift mantissa, adjust exponent)

Check for overflow/underflow of the exponent after normalisation

- **Overflow**

  The exponent is too large to be represented in the Exponent field

- **Underflow**

  The number is too small to be represented in the Exponent field

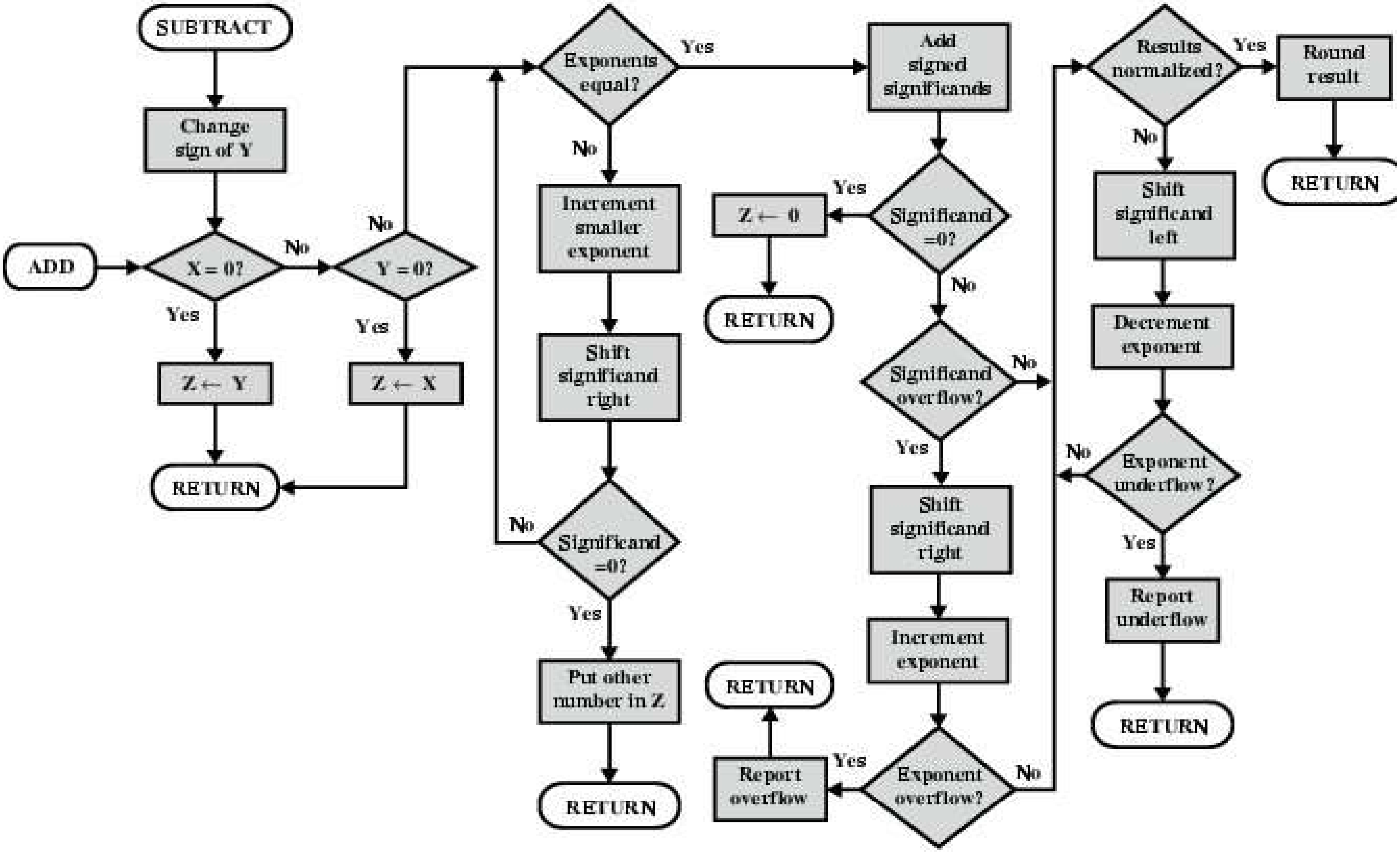
Round the result

If the mantissa does not fit in the space reserved for it, it has to be rounded off.

For Example: If only 4 digits are allowed for mantissa

1.0037×10^2 ===> 1.004×10^2

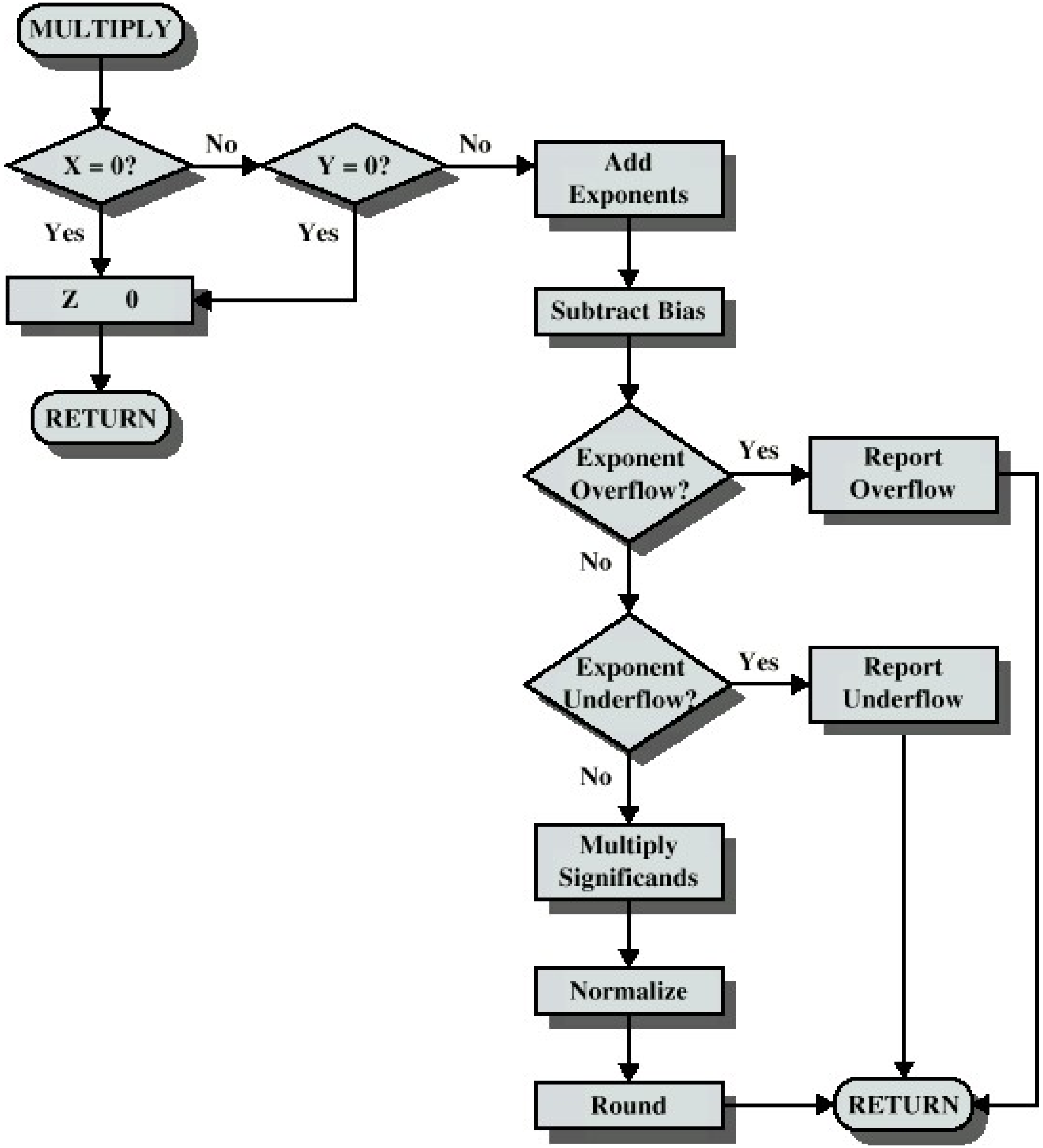
FP Addition & Subtraction Flowchart



FP Arithmetic x/÷

- Check for zero
- Add/subtract exponents
- Multiply/divide significands (watch sign)
- Normalize
- Round
- All intermediate results should be in double length storage

Floating Point Multiplication



Floating Point Division