Z- Transform

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Z-TRANSFORM:

Definition: Let $\{f(k)\}$ be a given sequence, where k varies from $-\infty$ to ∞ , then Z – transform of $\{f(k)\}$ is defined as $Z\{f(k)\} = \sum_{k=-\infty}^{k=\infty} f(k)z^{-k}$ and is denoted by $F(z)$, where z is a complex number, Z is an operator of Z-transform.

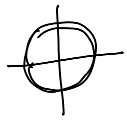
Thus Z- transform of the sequence $\{f(k)\}$ is $F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{k=\infty} f(k)z^{-k}$

Note: We shall require the following results in finding Z – transforms.

1. $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots\ldots = \frac{a}{1-r}$ if $|r| < 1$
2. $1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots = (1 + x)^{-1}$ if $|x| < 1$
3. $1 + x + x^2 + x^3 + \cdots . = \sum_{k=0}^{\infty} x^k = (1 - x)^{-1}$ if $|x| < 1$
4. $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\ldots = (1 + x)^n$
5. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots = e^x$
6. $\boldsymbol{log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots}$
7. The set $|z| < a$ means $\sqrt{x^2 + y^2} < a$ i.e $x^2 + y^2 < a^2$ This means $|z| < a$ is the set of points inside the circle of radius a and centre at the origin.

   By the same reasoning $|z| > a$ is the set of points outside the circle with radius a and centre at the origin.

$|z| = a \to$ circle



LINERITY PROPERTY OF Z – TRANSFORMATION:

If $\{f(k)\}$ and $\{g(k)\}$ are two sequence such that they can be added and a and b are constants,

then $Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + bZ\{g(k)\}$

CHANGE OF SCALE PROPERTY:

If $Z\{f(k)\} = F(z)$ then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$ also $Z\{a^{-k} f(k)\} = F(az)$

SHIFTING PROPERTY:

If $Z\{f(k)\} = F(z)$, then $Z\{f(k \pm n)\} = z^{\pm n}F(z)$.

$Z\{f(k+n)\} = z^{n} F(z)$

$Z\{f(k-n)\} = z^{-n} F(z)$

MULTIPLICATION BY k THEOREM:

If $F(z) = Z\{f(k)\}, then\ Z\{k\ f(k)\} = -z\frac{d}{dz}F(z)$.

Generalizing, we have $Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

Note: $\left(-z\frac{d}{dz}\right)^2 \neq z^2\frac{d^2}{dz^2}$, but it is repeated operation of $\left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)$

CONVOLUTION:

Definition: Let $\{f(k)\}$ and $\{g(k)\}$ be given two sequences. The convolution $\{f(k)\}$ and $\{g(k)\}$ is $\{h(k)\}$ which is defined by $\{h(k)\} = \{f(k)\} * \{g(k)\}$

Where $\{h(k)\} = \{g(k)\} * \{f(k)\}$

$\sum_{n=-\infty}^{\infty} f(n)\,g(k-n) = \sum_{n=-\infty}^{\infty} g(n)\,f(k-n)$

$h(k) = \sum f(n)\,g(k-n) = \sum_{n=-\infty}^{\infty} g(n)\,f(k-n)$

CONVOLUTION THEOREM:

If $\{h(k)\}$ is the convolution of $\{f(k)\}$ and $\{g(k)\}$ then $Z\{h(k)\} = Z\{f(k)\}.Z\{g(k)\}$.

Examples:

1. If $\{f(k)\} = \begin{cases} 4^k, \text{for } k < 0 \\ 3^k, \text{for } k \geq 0 \end{cases}$, find $Z\{f(k)\}$.

Sol^n :- $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$

$$k=-\infty$$

$$= \sum_{k=-\infty}^{-1} 4^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$= \left[\frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \cdots\right] + \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \cdots\right]$$

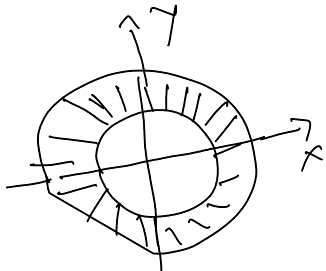
$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \quad \text{if } |r|<1$$

$$= \frac{z/4}{1-z/4} + \frac{1}{1-\frac{3}{z}} \quad \text{if} \quad \left|\frac{z}{4}\right|<1 \text{ and } \left|\frac{3}{z}\right|<1$$

$$= \frac{z}{4-z} + \frac{z}{z-3} \quad \text{if} \quad |z|<|4| \;\&\; |3|<|z|$$

$$= \frac{z^2-3z+4z-z^2}{(z-4)(z-3)} \quad \text{if} \quad 3<|z|<4$$

$$\therefore Z\{f(k)\} = \frac{z}{(z-4)(z-3)} \quad \text{if} \quad 3<|z|<4$$



ROC $3<|z|<4$

2. Find the $Z-$transform of $f(k) = k\alpha^k, k \geq 0$

Soln:- $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$= \sum_{k=0}^{\infty} k\alpha^k z^{-k}$$

$$= 0 + 1\cdot\alpha z^{-1} + 2\cdot\alpha^2 z^{-2} + 3\cdot\alpha^3 z^{-3} + \cdots$$

$$= \frac{\alpha}{z} + 2\left(\frac{\alpha}{z}\right)^2 + 3\cdot\left(\frac{\alpha}{z}\right)^3 + \cdots$$

$$= \frac{\alpha}{z}\left[1 + 2\left(\frac{\alpha}{z}\right) + 3\left(\frac{\alpha}{z}\right)^2 + \cdots\right]$$

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots = \cancel{(1+x)^{n}}$$

$$1 + 2x + 3x^2 + \cdots = (1-x)^{-2}$$

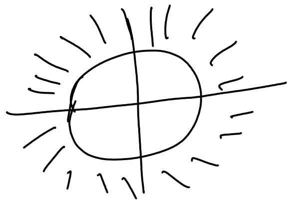
$-2 \quad \alpha z \quad \text{if } |\alpha| < 1$

$1 + 2x + 3x^2 + \cdots = (1-x)$

$$= \frac{\alpha}{z}\left[1-\frac{\alpha}{z}\right]^{-2} = \frac{\alpha}{z}\cdot\frac{1}{\left(1-\frac{\alpha}{z}\right)^2} = \frac{\alpha z}{(z-\alpha)^2} \quad \text{if } \left|\frac{\alpha}{z}\right| < 1$$

i.e. $|\alpha| < |z|$

$\therefore$ ROC is $|z| > |\alpha|$



3. Find the $Z-$transform of $f(k) = \frac{a^k}{k!}, k \geq 0$

Sol^n:

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{k!} z^{-k}$$

$$= 1 + \frac{a}{1!} z^{-1} + \frac{a^2}{2!} z^{-2} + \frac{a^3}{3!} z^{-3} + \cdots$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots = e^x$

$= e^{a/z}$ ROC is all of z-plane except $z=0$

4. Find the $Z-$transform of $f(k) = {}^nC_k, 0 \leq k \leq n$

Sol^n:

$$Z[f(k)] = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=0}^{n} {}^nC_k\, z^{-k}$$

$$= {}^nC_0 z^0 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} + {}^nC_3 z^{-3} + \cdots$$

$$= 1 + {}^nC_1\left(\frac{1}{z}\right) + {}^nC_2\left(\frac{1}{z^2}\right) + {}^nC_3\left(\frac{1}{z^3}\right) + \cdots + {}^nC_n\left(\frac{1}{z}\right)^n$$

$1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots + {}^nC_n x^n = (1+x)^n$

$= \left(1 + \frac{1}{z}\right)^n$ ROC is z-plane except z -

5. Find $Z\{2^k \cos(3k+2)\}, k \geq 0$

$$Z[\sin \alpha k] - \frac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

$$Z[\cos \alpha k] = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

Sol^n:

$$Z[\cos(3k+2)] = Z[\cos 3k \cdot \cos 2 - \sin 3k \sin 2]$$

$$= \cos 2 \; Z[\cos 3k] - \sin 2 \; Z[\sin 3k]$$

$$= \cos 2 \left[\frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1}\right] - \sin 2 \left[\frac{z \sin 3}{z^2 - 2z\cos 3 + 1}\right]$$

$$= \frac{z[z\cos 2 - \cos 2\cos 3 - \sin 2 \sin 3]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - (\cos 2\cos 3 + \sin 2 \sin 3)]}{z^2 - 2z\cos 3 + 1} = \frac{z[z\cos 2 - \cos 1]}{z^2 - 2z\cos 3 + 1}$$

By change of scale property

$$Z[a^k f(k)] = F\left(\frac{z}{a}\right)$$

$$\therefore Z[2^k \cos(3k+2)] = \frac{\frac{z}{2}\left[\frac{z}{2}\cos 2 - \cos 1\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1} = \frac{z[z\cos 2 - 2\cos 1]}{z^2 - 4z\cos 3 + 4}$$

6. If $f(k) = 4^k\, U(k)$ and $g(k) = 5^k U(k)$, then find the $Z-$ transform of $f(k) * g(k)$

$U(k) \to$ unit step discrete function

$$U(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

$$f(k) = 4^k\, U(k) = 4^k \qquad k \geq 0$$

$$g(k) = 5^k\, U(k) = 5^k \qquad k \geq 0$$

$$\cancel{Z[f(k)] = \sum^{\infty} 4^k z^{-k} = 1 + \frac{4}{z} + \frac{4}{z} \quad \frac{4}{z^3} + \cdots}$$

$z[f(k)] = \sum_{k=0}^{\infty} 4^k z^{-k} = \cancel{1 + \frac{4}{z} + \frac{4}{z} \quad \frac{4}{z^3} + \cdots}$

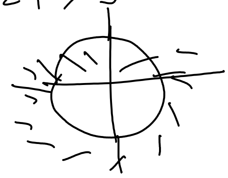
$= \frac{1}{1-\frac{4}{z}}$ for $\left|\frac{4}{z}\right|$

$z(f(k)) = \frac{z}{z-4}$ for $4<|z|$

Similarly $z[g(k)] = \frac{z}{z-5}$ for $5<|z|$

$z[f(k) * g(k)] = z[f(k)] \cdot z[g(k)]$

$= \left(\frac{z}{z-4}\right)\left(\frac{z}{z-5}\right) = \frac{z^2}{(z-4)(z-5)}$ for $|z|>5$

ROC 

7. Find $Z\{(k+1)a^k\}, k \geq 0$

$z[(k+1)a^k] = z[ka^k] + z[a^k]$

$z[a^k] = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots$

$= \frac{1}{1-\frac{a}{z}}$ for $\left|\frac{a}{z}\right| < 1$

$\therefore z[a^k] = \frac{z}{z-a}$ for $|a| < |z|$

$z[ka^k] = -z\frac{d}{dz}(z(a^k))$ (multiplication by k

$= -z\frac{d}{dz}\left(\frac{z}{z-a}\right) = -z\left[\frac{(z-a)(1) - z(1)}{(z-a)^2}\right]$

$= -z\left[\frac{-a}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$

$z[(k+1)a^k] = z[ka^k] + z[a^k] = \frac{az}{(z-a)^2} + \frac{z}{z-a}$

$= \frac{z^2}{(z-a)^2}$ for $|z| > |a|$

8. Find $Z\{k^2 a^{k-1} U(k-1)\}$

Soln: $z[U(k)] = \sum_{k=0}^{\infty} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$

Sol: $Z[U(k)] = \sum_{k=0} 1 \cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} \cdots$

$= \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$ for $\left|\frac{1}{z}\right| < 1$ i.e $|z| > 1$

By change of scale $Z[a^k U(k)] = F\left(\frac{z}{a}\right) = \frac{z/a}{z/a - 1} = \frac{z}{z-a}$

By shifting property

if $Z[f(k)] = F(z)$ then $Z[f(k-n)] = z^{-n} F(z)$

$\therefore Z[a^{k-1} U(k-1)] = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$

Now property of multiplication by k

$$Z[k^2 a^{(k-1)} U(k-1)] = \left(-z \frac{d}{dz}\right)^2 \left(\frac{1}{z-a}\right)$$

$$= \left(-z \frac{d}{dz}\right)\left(-z \frac{d}{dz}\right)\left(\frac{1}{z-a}\right)$$

$$= \left(-z \frac{d}{dz}\right)\left(-z\left(\frac{-1}{(z-a)^2}\right)\right)$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-a)^2}\right) = -z \left[\frac{(z-a)^{\cancel{2}} \cdot 1 - z \cdot 2\cancel{(z-a)}}{(z-a)^4}\right]$$

$$= \frac{-z[z-a-2z]}{(z-a)^3} = z(z+$$

INVERSE Z – TRANSFORM:

Definition: Let $\{f(k)\}$ be a given sequence. If $Z\{f(k)\} = F(z)$, then the inverse Z – transform is defined by , $Z^{-1}\{F(z)\} = \{f(k)\}$

Note: The inverse Z – transform can only be settled when the region of convergence (ROC) is given.

Consider $F(z) = \frac{P(z)}{Q(z)}$ is a rational function of z, Where $P(z)$ and $Q(z)$ are algebraic polynomials in z.

There are three methods to find inverse Z – transform of $F(z)$

1. Direct division **2.** Binomial expansion **3.** Partial fractions

DIRECT DIVISION METHOD:

In this method, we divide the numerator by the denominator and obtain a power series. i.e

if $F(z) = \frac{P(z)}{Q(z)}$, we actually divide $P(z)$ by $Q(z)$.

BINOMIAL EXPANSION METHOD:

To apply Binomial Expansion method we take a suitable factor common depending upon ROC from the denominator so that the denominator is of the form $(1-r)$ where $|r| < 1$ and then use Binomial Theorem.

PARTIAL FRACTION METHOD:

If $F(z) = \frac{P(z)}{Q(z)}$ can be resolved into partial fractions, (linear – repeated or non – repeated and quadratic factors).
We express $F(z)$ as the sum of such factors and using Binomial expansion we find its inverse Z – transform.

Examples:

1. Find the inverse Z –transform of $\frac{1}{z-a}$ if **(i)** $|z| > |a|$ **(ii)** $|z| < |a|$

Sol^n:- (i) if $|z| > |a|$ ie $\left|\frac{a}{z}\right| < 1$

$$F(z) = \frac{1}{z-a} = \frac{1}{z\left(1-\frac{a}{z}\right)} = \frac{1}{z}\left(1-\frac{a}{z}\right)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$= \frac{1}{z}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\cdots+\left(\frac{a}{z}\right)^{-1}+\cdots\right]$$

$$F(z) = \frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\cdots+\frac{a^{k-1}}{z^k}+\cdots$$

$$\sum f(k) z^{-k}$$

$\therefore$ coefficient of z^{-k} is a^{k-1}, $k \geq 1$

$$\therefore z^{-1}[F(z)] = a^{k-1} \text{ for } {-1}$$

(ii) if $|z| < |a|$ ie $\left|\frac{z}{a}\right| < 1$

$$F(z) = \frac{}{z+a} = \frac{1}{a\left(\frac{z}{a}-1\right)} = \frac{-1}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$= \frac{-1}{a}\left[1+\frac{z}{a}+\left(\frac{z}{a}\right)^2+\cdots\cdots+\left(\frac{z}{a}\right)^{k-1}+\cdots\right]$$

$$= -\left[\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\cdots+\frac{z^{k-1}}{a^k}+\cdots\right]$$

$$= -a^{-1}z^0 - a^{-2}z^1 - a^{-3}z^2 \cdots\cdots - a^{k}z^{k-1} - a^{-k-}z^k$$

Coefficient — $z^k = -a^{-k-}$, $k \geq 0$

Coefficient of $z^{-k} = -a^{k-1}$, $k \leq 0$

$$\therefore z^{-1}[F(z)] = -a^{k-1}, \quad k \leq 0$$

2. Find the inverse Z –transform of $\frac{1}{(z-a)^2}$ (i) $|z| < a$, (ii) $|z| > a$

Soln:- (i) $|z| < a$ i.e $\left|\frac{z}{a}\right| < 1$

$F(z) = \frac{1}{(z-a)^2} = \frac{1}{a^2\left(\frac{z}{a}-1\right)^2} = \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2}$

$= \frac{1}{a^2}\left[1 + 2\cdot\frac{z}{a} + 3\cdot\frac{z^2}{a^2} + \cdots + (k+1)\frac{z^k}{a^k} + \cdots\right]$

$= \frac{1}{a^2} + 2\cdot\frac{z}{a^3} + 3\cdot\frac{z^2}{a^4} + \cdots + (k+1)\frac{z^k}{a^{k+2}} + \cdots$

$\therefore$ coefficient of $z^k = \frac{k+1}{a^{k+2}}$, $k \geq 0$

coefficient of $z^{-k} = \frac{-k+1}{a^{-k+2}}$, $k \leq 0$

$\therefore Z^{-1}[F(z)] = \frac{-k+1}{a^{-k+2}}$, $k \leq 0$

(ii) $|z| > a$ $\therefore \left|\frac{z}{a}\right| > 1$ $\left|\frac{a}{z}\right| < 1$

$F(z) = \frac{1}{(z-a)^2} = \frac{1}{z^2\left(1-\frac{a}{z}\right)^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2}$

$= \frac{1}{z^2}\left(1 + 2\cdot\frac{a}{z} + 3\cdot\frac{a^2}{z^2} + \cdots + (k-1)\frac{a^{k-2}}{z^{k-2}} + \cdots\right)$

$= \frac{1}{z^2} + 2\cdot\frac{a}{z^3} + 3\cdot\frac{a^2}{z^4} + \cdots + \frac{(k-1)a^{k-2}}{z^k} + \cdots$

coefficient of $z^{-k} = (k-1)a^{k-2}$ $k \geq 2$

$\therefore Z^{-1}[F(z)] = (k-1)a^{k-2}$ $k \geq 2$

3. Find the inverse Z –trnasform of $F(z) = \frac{1}{(z-3)(z-2)}$ if ROC is (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$

Soln:- $F(z) = \frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$

$$1 = A(z-2) + B(z-3)$$

$$z = 2 \Rightarrow 1 = -B \Rightarrow B = -1$$

$$z = 3 \Rightarrow 1 = A$$

$z-2$

1

(i) $|z| < R(z) \Rightarrow \left|\cancel{\frac{z}{z_2}}3\right| < 1$

Obviously $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

$$F(z) = \frac{1}{z-3} - \frac{1}{z-2} = \frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots + \frac{z^k}{3^k} + \cdots\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \cdots + \frac{z^k}{2^k} + \cdots\right]$$

$$= \left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots - \frac{z^k}{3^{k+1}} \cdots\right] + \left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots + \frac{z^k}{2^{k+1}} \cdots\right]$$

coefficient of $z^k = \frac{-1}{3^{k+1}} + \frac{1}{2^{k+1}} = -3^{-k-1} + 2^{-k-1}, \quad k \geq 0$

Coefficient of $z^{-k} = -3^{k-1} + 2^{k-1}, \quad k \leq 0$

$$z^{-1}[F(z)] = -3^{k-1} + 2^{k-1}, \quad k \leq 0$$

(ii) $2 < |z| < 3$

$2 < |z| \quad \left|\frac{2}{z}\right| < 1$ and $|z| < 3$ ie $\left|\frac{z}{3}\right| < 1$

$$F(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{-1}{3}\left[1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots + \frac{z^k}{3^k} + \cdots\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots + \frac{2^{k-1}}{z^{k-1}} + \cdots\right]$$

$1 \quad 1 \quad z \quad z^2 \quad z^k \quad [\quad 2^{k-1} \quad]$

$$= \left[-\frac{1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} \cdots - \frac{z^k}{3^{k+1}} \cdots\right] + \left[-\frac{1}{z} - \frac{2}{z^2} - \frac{2^2}{z^3} \cdots - \frac{2^{k-1}}{z^k} \cdots\right]$$

From First series

$$\text{coefficient of } z^k = -\frac{1}{3^{k+1}} \qquad k \geq 0$$

$$= -3^{-k-1}$$

$$\text{coefficient of } z^{-k} = -3^{k-1} \qquad k \leq 0$$

From second series

$$\text{coefficient of } z^{-k} = -2^{k-1} \qquad k \geq 1$$

$$\therefore\ z^{-1}[F(z)] = \begin{cases} -3^{k-1}, & k \leq 0 \\ -2^{k-1} & k \geq 1 \end{cases}$$

(iii) $|z| > 3 \Rightarrow \left|\frac{z}{3}\right| > 1 \Rightarrow \left|\frac{3}{z}\right| < 1$

obviously $|z| > 3 > 2 \Rightarrow \left|\frac{z}{2}\right| > 1 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$F(z) = \frac{1}{z\left(1-\frac{3}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots+\left(\frac{3}{z}\right)^{k-1}\cdots\right] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots+\left(\frac{2}{z}\right)^{k-1}\cdots\right]$$

$$= \left[\frac{1}{z}+\frac{3}{z^2}+\frac{3^2}{z^3}+\cdots+\frac{3^{k-1}}{z^k}\cdots\right] + \left[-\frac{1}{z}-\frac{2}{z^2}-\frac{2^2}{z^3}\cdots-\frac{2^{k-1}}{z^k}\cdots\right]$$

$$\text{coefficient of } z^{-k} = 3^{k-1} - 2^{k-1}, \quad k \geq 1$$

$$z^{-1}[F(z)] = 3^{k-1} - 2^{k-1}, \quad k \geq 1$$

4. Find inverse Z –transform of $F(z) = \frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}, 3 < |z| < 4$

5. Find the inverse Z −transform of $\frac{2z^2-10z+13}{(z-3)^2(z-2)}, 2 < |z| < 3$