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On the Relaxation Dynamics of Lohe Oscillators on Some Riemannian Manifolds

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Abstract
We study the collective relaxation dynamics appearing in weakly coupled Lohe oscillators in a large coupling regime. The Lohe models on the unit sphere and unitary group were proposed as a nonabelian generalization of the Kuramoto model on the unit circle and their emergent dynamics has been extensively studied in previous literature for some restricted class of initial data based on the Lyapunov functional approach and order parameter approach. In this paper, we extend the previous partial results to cover a generic initial configuration via the detailed analysis on the order parameter measuring the modulus of the centroid. In particular, we present a detailed relaxation dynamics and structure of the resulting asymptotic states for the Lohe sphere model. We also present new gradient flow formulations for the Lohe matrix models with the same one-body Hamiltonians on some group manifolds. As a direct application of this new formulation, we show that every bounded Lohe flow which originated from any initial configuration converges asymptotically.

Keywords Complete synchronization · The Kuramoto model · Order parameter · Phase-locked state

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1 Introduction

Collective emergent dynamics often appears in many-body systems in our nature [1,4,31–33,39]; to name a few, synchronous firing of fireflies, applause in concert halls, Josephson junction arrays in solid mechanics, etc. However, systematic research on such collective phenomena was made only in the past five decades by the two pioneers A. Winfree and Y. Kuramoto in [22,38]. To model synchronization of weakly coupled limit-cycle oscillators, they introduced first-order continuous systems for the oscillators' phases, and showed that synchronous dynamics can emerge in the competition between the randomness due to distributed natural frequency and sinusoidal couplings between oscillators in a strong coupling regime. Although there were extensive research on the Kuramoto model in last four decades, there are still lots of mysterious phenomena waiting for scientific treatment. In this paper, we are interested in the relaxation dynamics of Lohe oscillators. In two recent papers [24,25], Lohe introduced a first-order consensus model for matrix oscillators and discussed several dynamic features of the proposed model based on analogy with the Kuramoto model and some numerical simulations. Let $U_i, U_i^\dagger$ and H_i be a unitary time-evolution operator, its Hermitian conjugate (transpose conjugate) and Hamiltonian associated with the i-th Lohe matrix oscillator, respectively. Then, the first-order evolution law for U_i is governed by the following system:

$$\begin{cases} \mathrm{i}\dot{U}_i U_i^\dagger = H_i + \frac{\mathrm{i}\kappa}{2N} \sum\limits_{j=1}^{N} \Big(U_j U_i^\dagger - U_i U_j^\dagger \Big), \quad t > 0, \quad i = 1, \ldots, N, \\ U_i(0) = U_i^0, \quad U_i^0 (U_i^0)^\dagger = I_d, \end{cases} \tag{1.1}$$

where κ is a nonnegative coupling strength. Note that the unitarity of U_i is propagated along the Lohe flow (1.1). In two dimensions $d = 2$, we can expand U_i and H_i using Pauli's matrices as a basis and rewrite the system (1.1) in terms of coefficients $x_i \in \mathbb{S}^3 \subset \mathbb{R}^4$ as follows.

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N} \sum_{k=1}^{N} (x_k - \langle x_i, x_k \rangle x_i), \quad i = 1, \ldots, N. \tag{1.2}$$

Throughout the paper, we call oscillators whose dynamics are governed by (1.1) and (1.2) as Lohe matrix oscillators and Lohe sphere oscillators, respectively. In a series of papers [6–10,17], the first author and his collaborators studied extensively the emergent dynamics of both models (1.1) and (1.2) using the Lyapunov functional approach and the order parameter approach, and established the existence of phase-locked states and their stability emerging from some restricted class of initial configurations. However, in analogy with the Kuramoto model, the emergent dynamics can be generic in a large coupling regime. This generality of emergent dynamics has not yet been fully established in previous literature. Numerical simulations support the emergence of phase-locking from generic initial configurations as long as the coupling strength is large enough. This issue motivates the current study of this paper. In this paper, we are interested in the relaxation dynamics of the ensemble of Lohe oscillators toward a position-locked state via the extension of the order parameter approach and gradient flow approach in [6,19]. Before we present our main results, we introduce two functionals measuring the degree of position concentration: for a given Lohe sphere configuration $X(t) = (x_1(t), \ldots, x_N(t))$, we set

$$D(X(t)) := \max_{1 \le i, j \le N} \|x_i(t) - x_j(t)\|, \quad \rho(t) := \Big\| \frac{1}{N} \sum_{i=1}^{N} x_i(t) \Big\|, \quad t \ge 0.$$

Note that

$$\lim_{t\to\infty} D(X) = 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \rho(t) = 1.$$

Next, we briefly discuss our main results. The first four results deal with the Lohe sphere oscillators on $\mathbb{S}^d$ and the last fifth result deals with the Lohe matrix oscillators on three group manifolds. First, we present a detailed relaxation dynamics of the identical Lohe sphere flow with $H_i = H,\ i = 1, \ldots, N$ from generic initial configurations with a positive order parameter. Our first result is about the a priori exponential synchronization (see Theorem 3.1):

$$\lim_{t\to\infty} D(X) = 0 \quad \Longrightarrow \quad D(X) \approx e^{-\kappa t} \quad \text{as } t \to \infty.$$

Of course the zero convergence of $D(X)$ is not completely resolved yet. In fact, our second result deals with the asymptotic behavior of the order parameter ρ. Second, we show that the asymptotic order parameter converges to some value ρ^∞, for the identical Lohe flow; more precisely, in any positive coupling regime, an initial configuration with non-overlapping positions tends either to a completely synchronized state, where every oscillator tends to a single common position, or to a $(N-1, 1)$-type bi-polar state, where all oscillators except one approach the same position. In terms of the order parameter, we have a dichotomy expressed by $\rho^\infty = 1, \frac{N-2}{N}$. Like the Kuramoto flow in [19], we believe that for generic initial configuration, ρ^∞ will be unity (we refer to Theorem 3.2 for a detailed discussion). Our third result asserts that for a generic initial configuration, there exists a positive coupling strength κ_∞ such that if the coupling strength κ is above the value κ_∞, then the Lohe sphere flow of nonidentical oscillators tends to the practical $\frac{N-1}{N}$-entrainment state asymptotically (see Theorem 4.1). The fourth result deals with practical partial entrainment. More precisely, we show that a generic flow tends to a practical $\left(\frac{1}{2} + \frac{\rho^0}{6}\right)$-entrainment state asymptotically. By reducing the required order level to $\left(\frac{1}{2} + \frac{\rho^0}{6}\right)$, the dependence of N vanishes in the sufficient condition of the coupling strength κ. Our last result shows that Lohe matrix models on three distinct group manifolds $\mathbb{U}(d)$, $S\mathbb{U}(d)$, $\mathbb{R} \times S\mathbb{U}(d)$ can be rewritten as gradient flows with analytical potentials. As noted in Sect. 5, the candidates for analytic potentials coincides with the potential for the Kuramoto model in the uni-dimensional case. Then, thanks to a new gradient flow formulation, we can see that all bounded flows are convergent. Of course, our general convergence results are restricted to the identical Hamiltonians $H_i = H$ in (1.1) which generalizes the analogous results for the Kuramoto flow in [19].

The rest of the paper is organized as follows. In Sect. 2, we briefly review two Lohe models, namely, the Lohe matrix model and Lohe sphere model, and we study their basic properties which will be crucially used in later sections. We also provide a brief summary for the previous results. In Sect. 3, we deal with Lohe sphere oscillators on $\mathbb{S}^d$ which is a straightforward generalization of (1.2), and provide two estimates on the exponential synchronization and asymptotic values of the order parameter in the relaxation dynamics. In Sect. 4, we introduce two refined concepts, namely "*asymptotic p-entrainment*" and "*practical p-entrainment*", and present two results on the formation of asymptotic entrainment from a generic initial configuration. In Sect. 5, we briefly discuss the convergence of bounded flows whose dynamics is governed by a gradient flow with analytical potential in a Riemannian setting, and we present new gradient flow formulations of the Lohe sphere model and the Lohe matrix models on group manifolds such as $\mathbb{U}(d)$, $S\mathbb{U}(d)$, and $\mathbb{R} \times S\mathbb{U}(d)$. Finally, Sect. 6 is devoted to the brief summary of our main results. In Appendix A, we provide supplementary calculations in relation with the proof of Theorem 4.2.

2 Preliminaries

In this section, we briefly introduce two Lohe models, namely the Lohe matrix and sphere models which exhibit emergent behaviors, and study their basic properties to be used in later sections.

2.1 The Lohe Matrix Model

Let U_i and $U_i^\dagger$ be the $d \times d$ unitary matrix and the corresponding Hermitian conjugate corresponding to the ith Lohe oscillator, respectively, and let H_i be a given $d \times d$ Hermitian matrix whose eigenvalues correspond to the natural frequencies of the ith Lohe oscillator. In [24,25], Max Lohe proposed a non-abelian matrix evolution model for U_i whose dynamics is given by the first-order matrix-valued ODE system:

$$\begin{cases} \mathrm{i}\dot{U}_i U_i^\dagger = H_i + \frac{\mathrm{i}\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger \right), \quad t > 0, \quad i = 1, \ldots, N, \\ U_i(0) = U_i^0, \end{cases} \tag{2.1}$$

where the initial matrix U_i^0 is in the unitary group $\mathbb{U}(d)$ and κ is the nonnegative uniform coupling strength. In the next lemma, we show that system (2.1) exhibits one conservation law and a kind of rotational symmetry.

Lemma 2.1 [24,25] (Conservation law and symmetry)

(1) *Let $\{U_i\}$ be a solution to (2.1) with initial data $\{U_i^0\}$. Then the quadratic quantity $U_i U_i^\dagger$ is conserved along the Lohe flow (2.1):*

$$U_i(t) U_i^\dagger(t) = U_i^0 U_i^{0\dagger}, \quad t \geq 0, \quad 1 \leq i \leq N.$$

(2) *The Lohe system (2.1) is invariant under right-translation by a unitary matrix in the sense that if $L \in \mathbb{U}(d)$ and $V_i = U_i L$, then V_i satisfies*

$$\begin{cases} \mathrm{i}\dot{V}_i V_i^\dagger = H_i + \frac{\mathrm{i}\kappa}{2N} \sum_{j=1}^N \left(V_j V_i^\dagger - V_i V_j^\dagger \right), \quad t > 0, \quad i = 1, \ldots, N, \\ V_i(0) = U_i^0 L. \end{cases}$$

Next, we study the connection between the Lohe matrix model and the Kuramoto model. As a matter of fact, the Lohe matrix model has been introduced as a nonabelian counterpart of the Kuramoto model (2.3) on $\mathbb{U}(d)$. To see this connection, we consider the uni-dimensional case and ansatzs for U_i and H_i:

$$U_i := e^{-\mathrm{i}\theta_i}, \quad H_i := \nu_i \in \mathbb{R}. \tag{2.2}$$

Then, we substitute (2.2) into (2.1) to recover the Kuramoto model [1,22,23]:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad t > 0, \tag{2.3}$$

System (2.3) has been extensively studied in [2,3,11–16,18,21,27–29,35–37]. On the other hand, if we set $\kappa = 0$ and multiply the unitary matrix U_i to both sides of (2.1), we obtain the Schrödinger equation in finite-dimensional form:

$$\mathrm{i}\dot{U}_i U_i^\dagger = H_i, \quad \text{or} \quad \mathrm{i}\dot{U}_i = H_i U_i.$$

In this case, U_i and H_i can be regarded as the unitary time-evolution operator and Hamiltonian, respectively. Thus, we can regard the Lohe matrix model (2.1) as a generalized model for the Kuramoto model and the Schrödinger equation simultaneously. In general, like the Kuramoto model, the Lohe matrix model does not admit the equilibrium solution. Thus, we need to study a weaker concept of equilibria, namely, *relative equilibria* or *phase-locked states*. Now, we recall the definition of phase-locked states to the Lohe matrix model as follows.

Definition 2.1 [17] Let $\{U_i(t)\}$ be a solution to (2.1).

(1) $\{U_i(t)\}$ is a *phase-locked state* if and only if $U_i(t)U_j(t)^\dagger$ is a constant of motion for all $t \geq 0$ and each pair i, j.
(2) The Lohe flow $\{U_i(t)\}$ achieves *asymptotic phase-locking* if and only if the limit of $U_i U_j^\dagger$ exists, as t tends to ∞, for each pair i, j.

Remark 2.1 For $d = 1$, via the ansatz (2.2), it is easy to see that

$$U_i(t)U_j(t)^\dagger = e^{-\mathrm{i}(\theta_i - \theta_j)} \ : \ \text{constant}, \quad \text{i.e.,} \quad \theta_i - \theta_j \ : \ \text{constant}$$

which coincides with the usual definition of phase-locked state for the Kuramoto model.

Next, we recall the structure of the phase-locked state for (2.1) as follows.

Proposition 2.1 [24,25] *The phase-locked states $\{U_i\}$ of (2.1) are of the form*

$$U_i = U_i^\infty e^{-\mathrm{i}\Lambda t},$$

where $U_i^\infty \in \mathbb{U}(d)$ and Λ is the constant $d \times d$ Hermitian matrix satisfying

$$U_i^\infty \Lambda U_i^{\infty\dagger} = H_i + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N}\left(U_k^\infty U_i^{\infty\dagger} - U_i^\infty U_k^{\infty\dagger}\right).$$

2.2 The Lohe Sphere Model

In this subsection, we discuss the Lohe sphere model (or the swarm model [30] on the d-sphere $\mathbb{S}^d$ in the control theory community for identical particles) that can be derived from the Lohe matrix model for $d = 2$. Next, we provide a heuristic argument for the derivation of the Lohe sphere model following [6,24,25].

Consider a two-dimensional case with $d = 2$ in (2.1). In this case, we can use Pauli's matrices $\{I_2, \sigma_1, \sigma_2, \sigma_3\}$ as a basis of the unitary group $\mathbb{U}(2)$:

$$\begin{cases} U_i := e^{-\mathrm{i}\theta_i}\left(\mathrm{i}\sum_{k=1}^3 x_i^k \sigma_k + x_i^4 I_2\right) = e^{-\mathrm{i}\theta_i}\begin{pmatrix} x_i^4 + \mathrm{i}x_i^1 & x_i^2 + \mathrm{i}x_i^3 \\ -x_i^2 + \mathrm{i}x_i^3 & x_i^4 - \mathrm{i}x_i^1 \end{pmatrix}, \\ H_i = \sum_{k=1}^3 \omega_i^k \sigma_k + \nu_i I_2, \end{cases} \tag{2.4}$$

where I_2 and σ_i are the identity matrix and Pauli matrices, respectively, defined by

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\omega_i = (\omega_i^1, \omega_i^2, \omega_i^3)$ is a real three-vector, and the natural frequency ν_i is associated with the $\mathbb{U}(1)$ component of U_i. After some algebraic manipulations, we obtain $5N$ equations for the angles θ_i and the four-vectors $x_i \in \mathbb{S}^3 \subset \mathbb{R}^4$:

$$\begin{aligned} \dot{\theta}_i &= \nu_i + \frac{\kappa}{N}\sum_{k=1}^{N} \sin(\theta_k - \theta_i)\langle x_i, x_k\rangle, \quad t \in \mathbb{R}, \quad 1 \le i \le N, \\ \dot{x}_i &= \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N} \cos(\theta_k - \theta_i)(x_k - \langle x_i, x_k\rangle x_i), \end{aligned} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^4$ and Ω_i is a real 4×4 skew-symmetric matrix:

$$\Omega_i := \begin{pmatrix} 0 & -\omega_i^3 & \omega_i^2 & -\omega_i^1 \\ \omega_i^3 & 0 & -\omega_i^1 & -\omega_i^2 \\ -\omega_i^2 & \omega_i^1 & 0 & -\omega_i^3 \\ \omega_i^1 & \omega_i^2 & \omega_i^3 & 0 \end{pmatrix}.$$

Here, we concern the equation on $\mathbb{S}^3$. We take $\theta_i = 0$ and $\nu_i = 0$ in (2.4) to obtain the swarming sphere model in (2.5) with $x_i \in \mathbb{S}^3$:

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}(x_k - \langle x_i, x_k\rangle x_i), \quad t > 0, \quad i = 1, \ldots, N. \tag{2.6}$$

It is easy to see that the unit modulus of x_i is invariant along the dynamics (2.6). Now, we formally extend system (2.6) as a dynamical system on the unit sphere $\mathbb{S}^d$ to obtain *the Lohe sphere model* on $\mathbb{S}^d$:

$$\dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{k=1}^{N}(x_k - \langle x_i, x_k\rangle x_i), \quad i = 1, \ldots, N, \tag{2.7}$$

where Ω_i is a skew-symmetric matrix in $\mathbb{R}^{d+1}$, called the natural frequency.
Below we introduce the emergent dynamics of (2.7). For this, we first introduce several functionals: For $t \ge 0$ and $X = (x_1, \ldots, x_N)$,

$$\begin{aligned} & x_c := \frac{1}{N}\sum_{i=1}^{N} x_i, \quad \rho := \|x_c\|, \quad \mathcal{A}(t) := \min_{1\le i\le N}\langle x_i(t), x_c(t)\rangle, \quad \mathcal{B}(t) := \min_{1\le i,j\le N}\langle x_i, x_j\rangle, \\ & D(\Omega) := \max_{1\le i,j\le N}\|\Omega_i - \Omega_j\|_\infty, \quad D_2(\Omega) := \max_{1\le i,j\le N}\|\Omega_i - \Omega_j\|_{op}, \\ & D(X(t)) := \max_{1\le i,j\le N}\|x_i(t) - x_j(t)\|. \end{aligned} \tag{2.8}$$

Here the norm $\|\cdot\|_{op}$ is the operator norm as a linear operator of $\mathbb{R}^{d+1}$. Then, it is easy to see that

$$0 \le \rho \le 1, \quad -1 \le \mathcal{A} \le 1.$$

and we use the definition of $\mathcal{A}$ to see

$$\mathcal{A} = \min_{1\le i\le N}\langle x_i(t), x_c(t)\rangle \le \frac{1}{N}\sum_{i=1}^{N}\langle x_i, x_c\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N} x_i, x_c \right\rangle = \|x_c\|^2 = \rho^2. \tag{2.9}$$

Lemma 2.2 [6] *Let* $X = (x_1, \ldots, x_N)$ *be the solution to system* (2.7) *satisfying the unit modulus conditions:*

$$\|x_i\| = 1, \quad i = 1, \ldots, N.$$

Then, we have the following assertions:

(1) *The order parameter ρ satisfies the following differential equation:*

$$\frac{d\rho^2}{dt} = \frac{2}{N}\sum_{i=1}^{N}\langle x_c, \Omega_i x_i\rangle + 2\kappa\left(\rho^2 - \frac{1}{N}\sum_{i=1}^{N}\langle x_i, x_c\rangle^2\right), \quad t > 0. \tag{2.10}$$

Then, we also have

$$\frac{d\rho^2}{dt} \geq -2D_2(\Omega)\rho + 2\kappa\left(\rho^2 - \frac{1}{N}\sum_{i=1}^{N}\langle x_i, x_c\rangle^2\right), \quad t > 0. \tag{2.11}$$

(2) *As long as $\mathcal{A}(t) > 0$, the functional $\mathcal{A}$ satisfies*

$$\frac{d\mathcal{A}}{dt} \geq -D_2(\Omega) + \kappa\mathcal{A}(1-\mathcal{A}). \tag{2.12}$$

Proof The detailed derivations for (2.10) and (2.12) can be found in Lemmas 3.4 and 3.6 of [6], respectively. □

Lemma 2.3 [6] *Suppose that the coupling strength and initial position satisfy*

$$\kappa > 0, \quad \|x_i^0\| = 1, \quad x_i^0 \neq x_j^0 \quad \text{if } i \neq j, \quad \rho^0 := \rho(0) > 0,$$

and let $X = (x_1, \ldots, x_N)$ be a solution to (2.7). Then, we have

$$\frac{d}{dt}\langle x_i, x_j\rangle = \langle(\Omega_i - \Omega_j)x_i, x_j\rangle + \kappa(1 - \langle x_i, x_j\rangle)\langle x_c, x_i + x_j\rangle, \quad t > 0,$$

or, equivalently,

$$\frac{d}{dt}(1 - \langle x_i, x_j\rangle) = \langle(\Omega_j - \Omega_i)x_i, x_j\rangle - \kappa(1 - \langle x_i, x_j\rangle)\langle x_c, x_i + x_j\rangle \quad t > 0.$$

Proof For the ensemble of identical oscillators with $\Omega_i = 0$, a detailed proof can be found in Lemma 4.5 in [6]. For the readers' convenience, we briefly discuss its proof. We take the inner product of (2.7) and x_j to obtain

$$\langle \dot{x}_i, x_j\rangle = \langle \Omega_i x_i, x_j\rangle + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle x_k, x_j\rangle - \langle x_i, x_k\rangle\langle x_i, x_j\rangle\Big). \tag{2.13}$$

Similarly, we have

$$\langle \dot{x}_j, x_i\rangle = -\langle \Omega_j x_i, x_j\rangle + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle x_k, x_i\rangle - \langle x_j, x_k\rangle\langle x_i, x_j\rangle\Big), \tag{2.14}$$

where we used the skew-symmetry of Ω_i:

$$\langle \Omega_j x_j, x_i\rangle = -\langle x_j, \Omega_j x_i\rangle = -\langle \Omega_j x_i, x_j\rangle.$$

Finally, we add (2.13) and (2.14) to obtain

$$\begin{aligned}\frac{d}{dt}\langle x_i, x_j\rangle &= \langle(\Omega_i - \Omega_j)x_i, x_j\rangle + \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle x_k, x_i + x_j\rangle - \langle x_i + x_j, x_k\rangle\langle x_i, x_j\rangle\Big)\\ &= \langle(\Omega_i - \Omega_j)x_i, x_j\rangle + \kappa\Big(\langle x_c, x_i + x_j\rangle - \langle x_i + x_j, x_c\rangle\langle x_i, x_j\rangle\Big)\\ &= \langle(\Omega_i - \Omega_j)x_i, x_j\rangle + \kappa(1 - \langle x_i, x_j\rangle)\langle x_c, x_i + x_j\rangle.\end{aligned}$$

□

Theorem 2.1 [5,6] (Identical oscillators) *Suppose that the coupling strength and initial data satisfy*

$$\kappa > 0, \quad \Omega_i = 0, \quad \|x_i^0\| = 1, \quad 1 \le i \le N,$$

and let $X = (x_1, \dots, x_N)$ be a solution to (2.7). Then, the following assertions hold.

(1) *If the initial data X^0 satisfies an extra condition*

$$0 < D(X^0) := D(X(0)) < \frac{1}{2}, \tag{2.15}$$

then the diameter $D(X)$ decays exponentially:

$$D(X(t)) \lesssim D(X^0)e^{-\kappa t}, \quad t \ge 0.$$

(2) *If the initial data satisfies an extra condition*

$$\mathcal{A}^0 := \mathcal{A}(0) > 0, \tag{2.16}$$

then order parameter ρ approaches the unity:

$$\lim_{t\to\infty} \rho(t) = 1.$$

Proof (1) The exponential decay of $D(X)$ follows from a Gronwall type differential inequality for $D(X)$(in Lemma 3.1 of [5]):

$$\frac{d}{dt}D(X) \le -\kappa D(X)(1 - 2D(X)).$$

(2) For a detailed proof, we refer to Theorem 4.4 in [6]. In the course of the proof, we used the fact that ρ is bounded and non-decreasing so that it has a limit as $t \to \infty$ and a contradiction argument. □

Next, we state the emergent dynamics of the ensemble of nonidentical Lohe sphere oscillators as follows.

Theorem 2.2 [6] (Nonidentical oscillators) *Suppose that the coupling strength and initial data satisfy*

$$\begin{aligned}&\kappa > 4ND_2(\Omega), \quad \|x_i^0\| = 1,\\ &\min\Big\{\rho^0, \min_{1\le i\le N}\langle x_i^0, x_c^0\rangle\Big\} > \frac{\kappa - \sqrt{\kappa^2 - 4ND_2(\Omega)\kappa}}{2\kappa}.\end{aligned} \tag{2.17}$$

Then we have practical synchronization:

$$\lim_{\kappa\to\infty} \liminf_{t\to\infty} \rho(t) = 1.$$

Remark 2.2 In the proof of Theorem 2.2, we use a differential inequality:

$$\frac{d\rho}{dt} \geq \frac{\kappa\rho(1-\rho)}{N},$$

which depends on the number of oscillators N. This is why N appears in the conditions (2.17).

In the following two sections, we will generalize or strengthen the results in Theorems 2.1 and 2.2 to cover generic initial configurations.

3 Identical Lohe Sphere Oscillators

In this section, we study the asymptotic dynamics of identical Lohe sphere oscillators with $\Omega_i = \Omega$ under generic initial configurations via the order parameter ρ.

Note that for identical oscillators with $\Omega_i = \Omega$, system (2.7) becomes

$$\dot{x}_i = \Omega x_i + \frac{\kappa}{N}\sum_{k=1}^{N}(x_k - \langle x_i, x_k\rangle x_i), \quad i = 1, \ldots, N.$$

Then, it follows from the operator splitting argument in [6] that we can assume that $\Omega = 0$ without loss of generality:

$$\dot{x}_i = \frac{\kappa}{N}\sum_{k=1}^{N}(x_k - \langle x_i, x_k\rangle x_i), \quad i = 1, \ldots, N. \tag{3.1}$$

In the following two subsections, we are interested in whether the restrictive conditions (2.15) and (2.16) in Theorem 2.1 can be replaced by generic conditions or not.

3.1 An Exponential Decay of $D(X)$

In this subsection, we provide an exponential synchronization estimate under the a priori assumption “complete positional synchronization". More precisely, our first result can be summarized as follows.

Theorem 3.1 (Exponential synchronization) *Suppose that the coupling strength, Ω_i and initial data satisfy*

$$\kappa > 0, \quad \Omega_i = 0, \quad \|x_i^0\| = 1, \quad i = 1, \ldots, N.$$

If X is a solution to (3.1) with complete positional synchronization:

$$\lim_{t\to\infty} D(X(t)) = 0,$$

then, there exist constants $C_1, C_2 > 0$ which depend on initial data such that

$$C_1 e^{-\kappa t} \leq D(X(t)) \leq C_2 e^{-\kappa t}, \quad \forall\, t > 0.$$

Proof We briefly outline the proof as follows.

- Step A For a lower bound estimate, we will prove the following estimates sequentially.

$$\rho \leq 1 + \mathcal{O}(e^{-2\kappa t}) \quad \Longrightarrow \quad D(X) \geq C_1 e^{-\kappa t}.$$

- Step B For an upper bound estimate, we will prove the following estimates sequentially.

$$\lim_{t\to\infty} D(X) = 0 \Longrightarrow \lim_{t\to\infty} \mathcal{B}(t) = 1$$
$$\Longrightarrow \mathcal{B}(t) \ge 1 - |\mathcal{O}(1)|e^{-2\kappa t} \quad \Longrightarrow \quad D(X) \le C_2 e^{-\kappa t}.$$

The detailed proof will be provided after a series of lemmas in the sequel. □

3.1.1 Basic Lemmas

We assume that complete synchronization occurs asymptotically:

$$\lim_{t\to\infty} D(X(t)) = 0. \tag{3.2}$$

Under this a priori estimate (3.2), we will provide estimates for the order parameter ρ, the angle functional $\mathcal{A}$ defined in (2.8) and a supplementary angle functional $\mathcal{B}$ in three lemmas.

Lemma 3.1 *Let $X = (x_1, \dots, x_N)$ be a solution to (3.1). Then, we have the following assertions.*

(1) *As long as $\mathcal{A} > 0$, we have*

$$\mathcal{A}(t) \ge 1 - \left(\frac{1}{\mathcal{A}^0} - 1\right) e^{-\kappa t} + \mathcal{O}(e^{-2\kappa t}), \quad \text{as } t \to \infty.$$

(2) *As long as $\mathcal{A} > 0$, the order parameter ρ satisfies differential inequalities:*

$$\kappa\rho(1-\rho) \le \frac{d\rho}{dt} \le \kappa\rho(1-\rho^2), \quad t > 0.$$

Proof (1) It follows from (2.12) with $D(\Omega) = 0$ that as long as $\mathcal{A} > 0$, we have

$$\frac{d\mathcal{A}}{dt} \ge \kappa\mathcal{A}(1-\mathcal{A}), \quad t > 0.$$

Hence, we have

$$\mathcal{A}(t) \ge \frac{1}{1 + \left(\frac{1}{\mathcal{A}^0} - 1\right) e^{-\kappa t}}, \quad t \ge 0,$$

which yields the desired estimate.

(2) We use (2.10) in Lemma 2.2 to find

$$\frac{d\rho^2}{dt} = 2\kappa\left(\rho^2 - \frac{1}{N}\sum_{i=1}^N \langle x_i, x_c\rangle^2\right). \tag{3.3}$$

To relate the term $\frac{1}{N}\sum_{i=1}^N \langle x_i, x_c\rangle^2$ with ρ, we take two steps. Note that there exists an orthogonal matrix $M(t) \in O(d+1)$ that varies smoothly in time such that

$$M(t)x_c(t) = (0, \dots, 0, \rho(t))^T.$$

We set

$$\bar{x}_i(t) := M(t)x_i(t), \quad \beta_i(t) := (\bar{x}_i(t))^{d+1}, \quad i = 1, \dots, N.$$

Then, since

$$\frac{1}{N}\sum_{i=1}^{N}\bar{x}_i(t) = M(t)x_c(t) = (0,\dots,0,\rho(t))^T,$$

we have

$$\frac{1}{N}\sum_{i=1}^{N}\beta_i(t) = \frac{1}{N}\sum_{i=1}^{N}(\bar{x}_i(t))^{d+1} = \frac{1}{N}\sum_{i=1}^{N}\Big(M(t)x_i(t)\Big)^{d+1} = (M(t)x_c(t))^{d+1} = \rho(t). \tag{3.4}$$

On the other hand, note that

$$\langle x_i, x_c\rangle = \langle M(t)x_i, M(t)x_c\rangle = \rho\beta_i. \tag{3.5}$$

Thus, we use (3.3) and (3.5) to obtain

$$\rho\beta_i = \frac{1}{N}\sum_{j=1}^{N}\langle x_j, x_i\rangle \geq \mathcal{A} > 0 \quad\Rightarrow\quad \beta_i > 0,$$

and

$$\frac{d\rho^2}{dt} = 2\kappa\Big(\rho^2 - \frac{1}{N}\sum_{i=1}^{N}\rho^2\beta_i^2\Big).$$

This again yields

$$\frac{d\rho}{dt} = \kappa\rho\Big(1 - \frac{1}{N}\sum_{i=1}^{N}\beta_i^2\Big). \tag{3.6}$$

Now, we claim:

$$\rho^2 \leq \frac{1}{N}\sum_{i=1}^{N}\beta_i^2 \leq \rho. \tag{3.7}$$

Clearly, (3.6) and (3.7) yield the desired differential inequalities.

- (Left-hand inequality in (3.7)) We use the Cauchy–Schwarz inequality and (3.4) to obtain

$$\frac{1}{N}\sum_i \beta_i^2 \geq \left(\frac{1}{N}\sum_{i=1}^{N}\beta_i\right)^2 = \rho^2.$$

- (Right-hand inequality in (3.7)) Consider the following optimization problem:

$$\begin{cases} \text{maximize } \frac{1}{N}\sum_i \hat{\beta}_i^2 \\ \text{subject to } \frac{1}{N}\sum_i \hat{\beta}_i = \rho, \quad \hat{\beta}_i \in [0,1]. \end{cases} \tag{3.8}$$

The object function in (3.8) is a continuous function defined on a compact domain and thus attains a maximum. Since the map $x \mapsto x^2$ is convex, a maximum point must have the form

$$(\hat{\beta}_1,\dots,\hat{\beta}_N) = (0,\dots,0,\lambda,\underbrace{1,\dots,1}_{m \text{ times}}), \quad \lambda \in [0,1), \quad \text{up to a permutation.} \tag{3.9}$$

Here, $m \in \mathbb{Z}$ is determined by the following relationship with $N\rho$,

$$m \le N\rho = \sum_{i=1}^{N} \hat{\beta}_i = m + \lambda < m + 1, \quad 0 \le m \le N.$$

Then, $\lambda = N\rho - m \in [0, 1)$. Hence

$$\frac{1}{N}\sum_i \hat{\beta}_i^2 \le \frac{1}{N}(\underbrace{1^2 + \cdots + 1^2}_{m \text{ times}} + \lambda^2) \le \frac{1}{N}(m + \lambda) = \rho.$$

This proves the claim (3.7), which along with (3.6), completes the proof. □

Remark 3.1 Below, we provide several implications of the results in Lemma 3.1.

1. The first estimate $\lim_{t\to\infty} \mathcal{A}(t) = 1$ together with the relation $\mathcal{A} \le \rho^2$ in (2.9) yield

$$\lim_{t\to\infty} \rho(t) = 1.$$

2. As a direct application of differential inequalities (2) in Lemma 3.1, for $\mathcal{A} > 0$,

$$\frac{1}{1 + \left(\frac{1}{\rho_0} - 1\right) e^{-\kappa t}} \le \rho(t) \le \frac{1}{\left[1 + \left(\frac{1}{\rho_0^2} - 1\right) e^{-2\kappa t}\right]^{1/2}}, \quad t \ge 0.$$

Lemma 3.2 *Let $X = (x_1, \ldots, x_N)$ be a given vector in $(\mathbb{S}^d)^N$. Then, we have*

$$D(X) \ge \sqrt{2 - 2\rho^2} \quad \text{and} \quad D(X) = \sqrt{2 - 2\mathcal{B}}.$$

Proof (i) By definition of $D(X)$ and $\|x_i - x_j\|^2 = 2 - 2\langle x_i, x_j\rangle$, we have

$$\begin{aligned} D(X) &\ge \left(\frac{1}{N^2}\sum_{i,j=1}^{N} \|x_i - x_j\|^2\right)^{1/2} = \left(\frac{1}{N^2}\sum_{i,j=1}^{N} (2 - 2\langle x_i, x_j\rangle)\right)^{1/2} \\ &= \left(\frac{1}{N^2}(2N^2 - N\sum_{i=1}^{N} \rho\beta_i)\right)^{1/2} = \sqrt{2 - 2\rho^2}. \end{aligned}$$

(ii) By definition of $D(X)$ and $\mathcal{B}$, we have

$$\begin{aligned} D(X) &= \max_{1\le i,j\le N} \|x_i - x_j\| = \max_{1\le i,j\le N} \sqrt{2 - 2\langle x_i, x_j\rangle} = \sqrt{2 - 2\min_{1\le i,j\le N} \langle x_i, x_j\rangle} \\ &= \sqrt{2 - 2\mathcal{B}}. \end{aligned}$$

□

Lemma 3.3 *Let $X = (x_1, \ldots, x_N)$ be a solution to (3.1). Then, the functional $\mathcal{B}$ defined in (2.8) satisfies*

$$\frac{d\mathcal{B}}{dt} \ge 2\kappa\mathcal{B}(1 - \mathcal{B}), \quad t > 0. \tag{3.10}$$

Proof (i) For a given $t > 0$, let i_t, j_t be indices such that

$$\mathcal{B}(t) = \langle x_{i_t}, x_{j_t}\rangle.$$

Then we have, by Lemma 2.3,

$$\frac{d\mathcal{B}}{dt} = \frac{d}{dt}\langle x_{i_t}, x_{j_t}\rangle = \kappa\langle x_c, x_{i_t} + x_{j_t}\rangle(1 - \langle x_{i_t}, x_{j_t}\rangle) = \kappa\langle x_c, x_{i_t} + x_{j_t}\rangle(1 - \mathcal{B}). \tag{3.11}$$

On the other hand, we use the definition of $\mathcal{B}$ to see

$$\langle x_c, x_{i_t}\rangle = \frac{1}{N}\sum_{k=1}^{N}\langle x_k, x_{i_t}\rangle \geq \frac{1}{N}\sum_{k=1}^{N}\mathcal{B} = \mathcal{B}, \tag{3.12}$$

and similarly for x_{j_t}. We combine (3.11) and (3.12), together with $\mathcal{B} \leq 1$, to obtain the desired estimate. □

Remark 3.2 It follows from the differential inequality in (3.10) that if $\mathcal{B}^0 := \mathcal{B}(0) > 0$, then we have

$$\mathcal{B}(t) \geq \frac{1}{1 + \left(\frac{1}{\mathcal{B}^0} - 1\right)e^{-2\kappa t}}, \quad t \geq 0.$$

We are now ready to provide a proof of Theorem 3.1.

3.1.2 Proof of Theorem 3.1

Below, we provide its proof into two steps.

- Step A (LHS inequality): It follows from the estimates in Remark 3.1 that we have

$$\rho(t) \leq 1 - \frac{1}{2}\left(\frac{1}{|\rho^0|^2} - 1\right)e^{-2\kappa t} + \mathcal{O}(e^{-4\kappa t}),$$

We apply the above estimate and Lemma 3.2 to find

$$\begin{aligned} D(X) &\geq \sqrt{2 - 2\rho^2} \geq \left[2\left(\frac{1}{|\rho^0|^2} - 1\right)e^{-2\kappa t} + \mathcal{O}(e^{-4\kappa t})\right]^{1/2} \\ &= \sqrt{2\left(\frac{1}{|\rho^0|^2} - 1\right)} \cdot e^{-\kappa t} + \mathcal{O}(e^{-3\kappa t}). \end{aligned} \tag{3.13}$$

- Step B (RHS inequality): By the assumption $\lim_{t\to\infty} D(X(t)) = 0$ and Lemma 3.2 we have

$$\lim_{t\to\infty}\mathcal{B}(t) = 1.$$

Thus we may change the starting time so that $\mathcal{B}^0 > 0$. This is possible since (3.1) is an autonomous system. Thus, we can use the result in Remark 3.2 to derive

$$\mathcal{B}(t) \geq 1 - \left(\frac{1}{\mathcal{B}^0} - 1\right)e^{-2\kappa t} + O(e^{-4\kappa t}).$$

Then, we substitute the above estimate into Lemma 3.2 to obtain

$$\begin{aligned} D(X) &= \sqrt{2 - 2\mathcal{B}} \leq \sqrt{2\left(\frac{1}{\mathcal{B}^0} - 1\right)e^{-2\kappa t} + \mathcal{O}(e^{-4\kappa t})} \\ &= \sqrt{2\left(\frac{1}{\mathcal{B}^0} - 1\right)} \cdot e^{-\kappa t} + \mathcal{O}(e^{-3\kappa t}). \end{aligned} \tag{3.14}$$

Finally, we combine (3.13) and (3.14) to derive the desired estimates. This completes the proof.

3.2 Asymptotics of the Order Parameter

In this subsection, we study possible asymptotic values of ρ as $t \to \infty$. From the definition of ρ and the relation (2.11) in Lemma 2.2, we can see that ρ is bounded above by 1 and non-decreasing. Thus, we have

$$\exists \lim_{t\to\infty} \rho(t) = \rho^\infty. \tag{3.15}$$

In this sequel, we study possible values of ρ^∞. In Theorem 2.1, we have shown that if $\mathcal{A}^0 > 0$, then we have $\rho^\infty = 1$. Thus, what matters is what will happen to ρ^∞ if $\mathcal{A}^0 < 0$. To motivate the asymptotics of ρ, we first consider the simplest system.

Example 3.1 Consider a two-particle system:

$$\dot{x}_1 = \frac{\kappa}{2}(x_2 - \langle x_2, x_1\rangle x_1), \quad \dot{x}_2 = \frac{\kappa}{2}(x_1 - \langle x_2, x_1\rangle x_2). \tag{3.16}$$

Note that

$$\begin{aligned} \rho^2 &= \|x_c\|^2 = \left\| \frac{x_1 + x_2}{2} \right\|^2 = \frac{1 + \langle x_1, x_2\rangle}{2}, \\ \langle x_c, x_1\rangle &= \langle x_c, x_2\rangle = \frac{1}{2}\Big(1 + \langle x_1, x_2\rangle\Big) = \rho^2. \end{aligned} \tag{3.17}$$

By Lemma 2.3, we have

$$\frac{d}{dt}\langle x_1, x_2\rangle = \kappa(1 - \langle x_1, x_2\rangle^2).$$

Then, by direct calculation, we have

$$\langle x_1, x_2\rangle(t) = \frac{(1 + \langle x_1^0, x_2^0\rangle)e^{2\kappa t} - (1 - \langle x_1^0, x_2^0\rangle)}{(1 + \langle x_1^0, x_2^0\rangle)e^{2\kappa t} + (1 - \langle x_1^0, x_2^0\rangle)}, \quad t \geq 0.$$

This yields

$$\lim_{t\to\infty} \langle x_1, x_2\rangle(t) = \begin{cases} 1, & \langle x_1^0, x_2^0\rangle \neq -1, \\ -1, & \langle x_1^0, x_2^0\rangle = -1. \end{cases} \tag{3.18}$$

Finally, we combine (3.17), (3.18) and the increasing property of ρ to find

$$\rho^\infty = \begin{cases} 1, & \rho^0 \neq 0, \\ 0, & \rho^0 = 0. \end{cases} \quad \text{and} \quad \lim_{t\to\infty} \langle x_c, x_1\rangle = \lim_{t\to\infty} \langle x_c, x_2\rangle = \begin{cases} 1, & \rho^0 \neq 0, \\ 0, & \rho^0 = 0. \end{cases} \tag{3.19}$$

Note that for generic initial data X^0, we have $\rho^\infty = 1$.

As we can see in Example 3.1, it is important to see the asymptotic phase distribution of oscillators x_i if we want to know ρ^∞. In fact, we can show that all the x_i tend toward the direction of x_c or $-x_c$, in the following sense:

Lemma 3.4 *Let $X = (x_1, \ldots, x_N)$ be a solution to (3.1) with $\kappa > 0$ and $\Omega_i = 0$ for all i. Then,*

$$\lim_{t\to\infty} \langle x_c(t), x_j(t)\rangle = \pm\rho^\infty, \textit{ for all } j.$$

Proof We will use Barbalat's lemma to prove the existence of the limit value. We already know that ρ has a limit value ρ^∞. From Lemma 2.2,

$$\frac{d\rho^2}{dt} = 2\kappa\left(\rho^2 - \frac{1}{N}\sum_{i=1}^{N}\langle x_i, x_c\rangle^2\right). \tag{3.20}$$

From Barbalat's lemma, if we already know $\frac{d^2\rho}{dt^2}$ is bounded, then $\frac{d\rho^2}{dt}$ goes to zero, and then

$$(\rho^\infty)^2 - \frac{1}{N}\sum_{i=1}^{N}\lim_{t\to\infty}\langle x_i, x_c\rangle^2 = 0,$$

which proves our claim. It is quite straightforward to show that $\frac{d^2\rho}{dt^2}$ is bounded, which looks reasonable from the analyticity of the dynamics (2.7). First, from (3.20),

$$\left|\frac{d\rho}{dt}\right| \le \rho\kappa.$$

On the other hand, from Lemma 2.3,

$$\frac{d}{dt}\langle x_i, x_j\rangle = \kappa(1 - \langle x_i, x_j\rangle)\langle x_c, x_i + x_j\rangle,$$

hence we have

$$\left|\frac{d}{dt}\langle x_i, x_c\rangle\right| \le \left|\frac{1}{N}\sum_{j=1}^{N}\frac{d}{dt}\langle x_i, x_j\rangle\right| \le 4\rho\kappa.$$

Finally, using (3.20) again,

$$\frac{d}{dt}\left(\frac{d\rho^2}{dt}\right) = 2\kappa\left(\frac{d\rho^2}{dt} - \frac{1}{N}\sum_{i=1}^{N}\frac{d}{dt}\langle x_i, x_c\rangle^2\right),$$

so that we have

$$2\rho\left|\frac{d^2\rho}{dt^2}\right| + 2\left|\frac{d\rho}{dt}\right|^2 \le 2\kappa\left(2\rho\left|\frac{d\rho}{dt}\right| + \frac{2}{N}\sum_{i=1}^{N}|\langle x_i, x_c\rangle|\left|\frac{d}{dt}\langle x_i, x_c\rangle\right|\right) \le 20\rho^2\kappa^2,$$

hence the second derivative $\frac{d^2\rho}{dt^2}$ is bounded, and the conclusion holds. □

Remark 3.3 An argument similar to that of Lemma 3.4 will be employed in Remark 5.2, to obtain an analogous result for the Lohe model on $\mathbb{U}(d)$.

Below, we will show that for a many-body system with $N \ge 3$, there is a dichotomy: either all particles aggregate to the same position asymptotically, or to a bi-polar state with the property that $N-1$ particles aggregate to the same position $x_* \in \mathbb{S}^d$ and the remaining one oscillator approaches the antipodal position $-x_*$.

Theorem 3.2 *Suppose that the coupling strength,* Ω_i *and initial velocities satisfy*

$$\kappa > 0, \quad \Omega_i = 0, \quad \|x_i^0\| = 1, \quad x_i^0 \ne x_j^0 \;\; \textit{if}\; i \ne j, \quad \rho^0 > 0,$$

and let $X = (x_1, \ldots, x_N)$ *be a solution to* (3.1)*. Then, we have the following assertions:*

(1) *There exists at most one i such that*

$$\lim_{t\to\infty}\langle x_c(t), x_i(t)\rangle = -\rho^\infty,$$

and then others $j \neq i$ satisfy

$$\lim_{t\to\infty}\langle x_c(t), x_j(t)\rangle = \rho^\infty.$$

(2) *ρ^∞ takes at most two values:*

$$\rho^\infty \in \left\{\frac{N-2}{N}, 1\right\}.$$

Proof (1) First, from Lemma 3.4,

$$\lim_{t\to\infty}\langle x_c(t), x_j(t)\rangle = \pm\rho^\infty, \text{ for any } j.$$

Suppose that there are two distinct indices $i_0 \neq j_0$ such that

$$\lim_{t\to\infty}\langle x_c(t), x_{i_0}(t)\rangle = \lim_{t\to\infty}\langle x_c(t), x_{j_0}(t)\rangle = -\rho^\infty. \tag{3.21}$$

Then, for such indices i_0 and j_0, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\|x_{i_0}(t) - x_{j_0}(t)\| \\
&\leq \limsup_{t\to\infty}\left\|x_{i_0}(t) + \frac{1}{\rho(t)}x_c(t)\right\| + \limsup_{t\to\infty}\left\| -\frac{1}{\rho(t)}x_c(t) - x_{j_0}(t)\right\| \\
&= \limsup_{t\to\infty}\left[2 + \frac{2}{\rho(t)}\langle x_{i_0}(t), x_c(t)\rangle\right]^{1/2} + \limsup_{t\to\infty}\left[2 + \frac{2}{\rho(t)}\langle x_{j_0}(t), x_c(t)\rangle\right]^{1/2} \\
&= 0.
\end{aligned}$$

Hence, we have

$$\lim_{t\to\infty}\langle x_{i_0}(t), x_{j_0}(t)\rangle = 1. \tag{3.22}$$

On the other hand, it follows from (3.21) and non-decreasing ρ that we have

$$\lim_{t\to\infty}\langle x_c(t), x_{i_0}(t)\rangle = \lim_{t\to\infty}\langle x_c(t), x_{j_0}(t)\rangle = -\rho^\infty \leq -\rho^0.$$

Thus, there exists a positive constant $t_0 \geq 0$ such that

$$\langle x_c(t), x_{i_0}(t)\rangle, \quad \langle x_c(t), x_{j_0}(t)\rangle \leq -\frac{\rho^0}{2}, \quad \forall\, t \geq t_0.$$

This and Lemma 2.3 give

$$\begin{aligned}
\frac{d}{dt}\langle x_{i_0}, x_{j_0}\rangle &= \kappa(\langle x_c, x_{i_0}\rangle + \langle x_c, x_{j_0}\rangle)(1 - \langle x_{i_0}, x_{j_0}\rangle) \\
&\leq -\kappa\rho^0(1 - \langle x_{i_0}, x_{j_0}\rangle) \quad \text{for } t > t_0.
\end{aligned}$$

Again this yields

$$\frac{d}{dt}(1 - \langle x_i, x_j\rangle) \geq \kappa\rho^0(1 - \langle x_i, x_j\rangle) \quad \text{for } t > t_0. \tag{3.23}$$

Since $x_i^0 \neq x_j^0$, by the uniqueness of solutions to (3.1), we must have

$$x_i(t) \neq x_j(t) \quad \text{for all } t,$$

i.e., we have

$$1 - \langle x_i(t), x_j(t) \rangle > 0 \quad \text{for all } t. \tag{3.24}$$

We combine (3.23) and (3.24) to get

$$1 - \langle x_i, x_j \rangle(t) \ge e^{\kappa \rho^0 t}(1 - \langle x_i^0, x_j^0 \rangle), \quad t \ge t_0.$$

This contradicts (3.22). Thus we have the desired result.

(2) From the first estimate (1), the following dichotomy holds.

- Case A (complete positional synchronization): Suppose that

$$\lim_{t\to\infty} \langle x_c(t), x_i(t) \rangle = \rho^\infty, \quad \forall\ i = 1, \dots, N,$$

i.e., x_i will be aligned in the direction of x_c asymptotically. Since this is for every i, we have

$$\rho^\infty = \lim_{t\to\infty} \langle x_c(t), x_i(t) \rangle = \frac{1}{N} \sum_{i=1}^{N} \lim_{t\to\infty} \langle x_c(t), x_i(t) \rangle = \lim_{t\to\infty} \langle x_c(t), x_c(t) \rangle = (\rho^\infty)^2. \tag{3.25}$$

Therefore, we have

$$\rho^\infty = 1.$$

- Case B (bi-polar positional synchronization): It follows from the result (i) of Lemma 3.2 that

$$\lim_{t\to\infty} \langle x_c(t), x_i(t) \rangle = \rho^\infty, \quad i = 1, \dots, N-1, \quad \lim_{t\to\infty} \langle x_c(t), x_N(t) \rangle = -\rho^\infty,$$

up to a permutation of the indices. We use the argument in (3.25) to derive

$$\rho^\infty = \frac{N-2}{N}.$$

On the other hand, it is easy to see that

$$\begin{aligned} &\lim_{t\to\infty} \langle x_i(t), x_j(t) \rangle = 1, \quad 1 \le i, j \le N-1, \\ &\lim_{t\to\infty} \langle x_i(t), x_N(t) \rangle = -1, \quad i = 1, \dots, N-1. \end{aligned} \tag{3.26}$$

Finally, we combine the results in Case A and Case B to obtain the desired result. □

Remark 3.4 It follows from the differential inequalities of ρ that if $\rho^0 = 0$, then

$$\rho(t) = 0, \quad t \ge 0, \quad \text{i.e.,} \quad \rho^\infty = 0.$$

In the course of the proof of Theorem 3.2, we have seen that a dichotomy occurs. Either all particles aggregate to their center of mass x_c, or $N-1$ points aggregate to one position and the remaining one particle approaches the opposite position. In Theorem 3.1, we have shown that the first complete positional synchronization occurs exponentially fast. Next, we show that for the latter case, the $N-1$ particles also aggregate to the same position exponentially fast. To prove this, we introduce functionals with the first $N-1$ partial configuration $\{x_1, \dots, x_{N-1}\}$:

$$\widetilde{\mathcal{B}} := \min_{1 \le i,j \le N-1} \langle x_i, x_j \rangle, \quad \widetilde{D}(X) := \max_{1 \le i,j \le N-1} \|x_i - x_j\|. \tag{3.27}$$

Lemma 3.5 *Suppose that the coupling strength, Ω_i and initial configuration satisfy*

$$N \geq 3, \quad \kappa > 0, \quad \Omega_i = 0, \quad \|x_i^0\| = 1,$$
$$x_i^0 \neq x_j^0 \quad \textit{if } i \neq j, \quad \rho^0 > 0, \quad \widetilde{\mathcal{B}}^0 > \frac{1}{N-1},$$

and let $X = (x_1, \dots, x_N)$ *be a solution to* (3.1) *satisfying the a priori estimate:*

$$\lim_{t\to\infty} \langle x_c(t), x_N(t)\rangle = -\rho^\infty.$$

Then, we have

$$\frac{d\widetilde{\mathcal{B}}}{dt} \geq \frac{2\kappa(N-1)}{N}\left(\widetilde{\mathcal{B}} - \frac{1}{N-1}\right)(1-\widetilde{\mathcal{B}}), \quad t > 0.$$

Proof Let i_t and j_t be indices depending on time t such that

$$\langle x_{i_t}, x_{j_t}\rangle = \widetilde{\mathcal{B}}(t).$$

Then, we have

$$\langle x_c, x_{i_t}\rangle = \frac{1}{N}\left(\sum_{k=1}^{N-1}\langle x_k, x_{i_t}\rangle + \langle x_N, x_{i_t}\rangle\right) \geq \frac{1}{N}\left((N-1)\widetilde{\mathcal{B}} - 1\right). \tag{3.28}$$

Similarly, we have

$$\langle x_c, x_{j_t}\rangle = \frac{1}{N}\left(\sum_{k=1}^{N-1}\langle x_k, x_{j_t}\rangle + \langle x_N, x_{j_t}\rangle\right) \geq \frac{1}{N}\left((N-1)\widetilde{\mathcal{B}} - 1\right). \tag{3.29}$$

Thus, we use (3.28), (3.29) and Lemma 2.3 to obtain

$$\frac{d\widetilde{\mathcal{B}}}{dt} = \frac{d}{dt}\langle x_{i_t}, x_{j_t}\rangle = \kappa\langle x_c, x_{i_t} + x_{j_t}\rangle(1-\widetilde{\mathcal{B}}) \geq \frac{2\kappa(N-1)}{N}\left(\widetilde{\mathcal{B}} - \frac{1}{N-1}\right)(1-\widetilde{\mathcal{B}}),$$

which yields the desired differential inequality. □

We are now ready to provide the exponential decay of $\tilde{D}(X)$ as follows.

Proposition 3.1 *Suppose that the coupling strength,* Ω_i *and initial velocities satisfy*

$$N \geq 3, \quad \kappa > 0, \quad \Omega_i = 0, \quad \|x_i^0\| = 1, \quad x_i^0 \neq x_j^0 \quad \textit{if } i \neq j, \quad \rho^0 > 0,$$

and let $X = (x_1, \dots, x_N)$ *be a solution to* (3.1) *satisfying the a priori estimate:*

$$\lim_{t\to\infty} \langle x_c(t), x_N(t)\rangle = -\rho^\infty.$$

Then, there exists a nonnegative constant $t_0 \geq 0$ *such that*

$$\widetilde{D}(X(t)) \leq \exp\left[-\frac{\kappa(N-1)}{N}\left(\widetilde{\mathcal{B}}(t_0) - \frac{1}{N-1}\right)(t-t_0)\right]\widetilde{D}(X(t_0)), \quad t > t_0.$$

Proof It follows from Theorem 3.1 that there exists $t_0 \geq 0$ such that

$$\widetilde{\mathcal{B}}(t_0) > \frac{1}{N-1}.$$

By Lemma 3.5, we see that

$$\widetilde{\mathcal{B}}(t) \geq \widetilde{\mathcal{B}}(t_0), \quad t \geq t_0.$$

On the other hand, an argument similar to that given in Lemma 3.3 gives

$$\widetilde{D}(X) = \sqrt{2 - 2\widetilde{\mathcal{B}}}$$

and

$$\begin{aligned}\frac{d}{dt}\widetilde{D}(X) &= \frac{-1}{\sqrt{2-2\widetilde{\mathcal{B}}}}\frac{d\widetilde{\mathcal{B}}}{dt} \le \frac{-1}{\sqrt{2-2\widetilde{\mathcal{B}}}} \cdot \frac{\kappa(N-1)}{N}\left(\widetilde{\mathcal{B}} - \frac{1}{N-1}\right)(2-2\widetilde{\mathcal{B}}) \\ &\le -\frac{\kappa(N-1)}{N}\left(\widetilde{\mathcal{B}}(t_0) - \frac{1}{N-1}\right)\widetilde{D}(X), \quad t \ge t_0.\end{aligned}$$

Then this Gronwall type inequality completes the proof. □

Remark 3.5 As noted in the two-particle system, we conjecture that for a generic initial configuration X^0 with $\kappa > 0$, we have

$$\rho^\infty = 1.$$

This is because the bi-polar states are linearly unstable, as demonstrated in Appendix in Theorem B.1.

4 Nonidentical Lohe Sphere Oscillators

In this section, we study emergent dynamics of the ensemble of the Lohe sphere oscillators for a generic initial configuration by improving the result of Theorem 4.9 of [6] (see Theorem 2.2). First, we introduce new concepts such as asymptotic p-entrainment and practical p-entrainment as follows.

Definition 4.1 Let $X^0 = \{x_i^0\}_{i=1}^N$ be an ensemble of initial data on the Lohe sphere model (2.7).

(1) The ensemble X^0 exhibits asymptotic p-entrainment for some $p \in (0, 1)$ if there exist $S \subset \{1, \dots, N\}$ and $\kappa_0 > 0$ such that

$$\frac{|S|}{N} \ge p \quad \text{and} \quad \limsup_{t\to\infty} \max_{i,j\in S} \|x_i(t) - x_j(t)\| = 0, \quad \forall\, \kappa > \kappa_0.$$

(2) The ensemble X^0 exhibits practical p-entrainment for some $p \in (0, 1)$ if there exists $S \subset \{1, \dots, N\}$ such that for any $\epsilon > 0$, there exist $\kappa(\epsilon)$ and $T(\epsilon)$ satisfying

$$\frac{|S|}{N} \ge p \quad \text{and} \quad \sup_{t>T(\epsilon)} \max_{i,j\in S} \|x_i(t) - x_j(t)\| \le \epsilon, \quad \forall\, \kappa > \kappa(\epsilon). \tag{4.1}$$

Remark 4.1 Note that Theorem 3.2 states that identical Lohe sphere oscillators with a generic initial configuration achieves asymptotic $\frac{N-1}{N}$-entrainment under any positive coupling $\kappa > 0$.

4.1 Emergence of Practical Entrainment

In this subsection, we present practical entrainment for (2.7) introduced in Definition 4.1. For this, we first present several basic lemmas.

Consider a first-order differential inequality:

$$y' \ge f(y), \quad t > 0, \quad y(0) = y^0, \tag{4.2}$$

where f has two distinct positive real roots $y_- < y_+$ satisfying

$$f(y_\pm) = 0 \quad \text{and} \quad f(y) \begin{cases} < 0, \ y \in [0, y_-), \\ > 0, \ y \in (y_-, y_+), \\ < 0, \ y \in (y_+, \infty). \end{cases}$$

Lemma 4.1 *[6] Suppose that the differentiable function $y = y(t)$ satisfies (4.2) with $y^0 \in (y_-, \infty)$. Then, we have*

$$\liminf_{t\to\infty} y(t) \ge y_+, \quad \textit{and} \quad y(t) \ge y_-, \ \forall\, t > 0,$$

Proof The proof can be found in Lemma 3.7 [6]. □

Lemma 4.2 *Suppose that the coupling strength, natural frequencies, and initial data satisfy*

$$\kappa > \frac{2pD_2(\Omega)}{(2p-1)^2}, \quad D(\Omega) > 0, \quad \|x_i^0\| = 1,$$

and let $X(t) = (x_1, \ldots, x_N)$ be a solution to (2.7) and let $S \subset \{1, \ldots, N\}$ be a set with the following a priori estimate:

$$p := \frac{|S|}{N} > \frac{1}{2} \quad \textit{and} \quad \mathcal{B}_S(0) > \frac{1}{2p}\left[1 - \sqrt{1 - 4p\left(1 - p + \frac{D_2(\Omega)}{2\kappa}\right)}\right], \tag{4.3}$$

where

$$\mathcal{B}_S(t) := \min_{i,j\in S} \langle x_i(t), x_j(t)\rangle.$$

Then, we have

$$\liminf_{t\to\infty} \mathcal{B}_S(t) \ge \frac{1}{2p}\left[1 + \sqrt{1 - 4p\left(1 - p + \frac{D_2(\Omega)}{2\kappa}\right)}\right].$$

Thus, the ensembles satisfying (4.3) exhibit the practical p-entrainment since

$$\frac{1}{2p}\left[1 + \sqrt{1 - 4p\left(1 - p + \frac{D_2(\Omega)}{2\kappa}\right)}\right] \to 1 \ \textit{as}\ \kappa \to \infty.$$

Proof We may find $i_t, j_t \in S$ such that

$$\mathcal{B}_S(t) = \langle x_{i_t}, x_{j_t}\rangle.$$

Then, we use Lemma 2.3 to obtain

$$\begin{aligned} \frac{d\mathcal{B}_S}{dt} = \frac{d}{dt}\langle x_{i_t}, x_{j_t}\rangle &= \langle(\Omega_{i_t} - \Omega_{j_t})x_{i_t}, x_{j_t}\rangle + \kappa\Big(1 - \langle x_{i_t}, x_{j_t}\rangle\Big)\frac{1}{N}\sum_{k=1}^N \langle x_k, x_{i_t} + x_{j_t}\rangle \\ &= \langle(\Omega_{i_t} - \Omega_{j_t})x_{i_t}, x_{j_t}\rangle + \kappa(1 - \mathcal{B}_S)\frac{1}{N}\left(\sum_{k\in S}\langle x_k, x_{i_t} + x_{j_t}\rangle + \sum_{k\notin S}\langle x_k, x_{i_t} + x_{j_t}\rangle\right) \\ &\ge -\|(\Omega_{i_t} - \Omega_{j_t})x_{i_t}\| + \kappa(1 - \mathcal{B}_S)\frac{1}{N}\Big(|S|\cdot 2\mathcal{B}_S + (N - |S|)\cdot(-2)\Big) \\ &\ge -D_2(\Omega) + \kappa(1 - \mathcal{B}_S)\left(p\cdot 2\mathcal{B}_S + (1-p)\cdot(-2)\right) \\ &= 2\kappa\Big[-p\mathcal{B}_S^2 + \mathcal{B}_S - \left(1 - p + \frac{D_2(\Omega)}{2\kappa}\right)\Big], \end{aligned} \tag{4.4}$$

where we have used the fact

$$\langle x_i, x_j\rangle \geq \mathcal{B}_S \quad \text{for } i, j \in S, \quad \langle x_i, x_j\rangle \geq -1, \quad \text{for } i, j \in S^c.$$

Finally, we apply Lemma 4.1 with

$$y = \mathcal{B}_S, \quad f(y) = 2\kappa\Big[-py^2 + y - \Big(1 - p + \frac{D_2(\Omega)}{2\kappa}\Big)\Big]$$

to find the desired estimate. □

Next, we study the deviation of the Lohe flow for identical and nonidentical particles. For this, consider the Lohe sphere systems: for $i = 1, \dots, N$,

$$\begin{cases} \dot{y}_i = \frac{\kappa}{N}\sum_{j=1}^N \big(y_j - \langle y_i, y_j\rangle y_i\big), \quad t > 0, \\ \dot{x}_i = \Omega_i x_i + \frac{\kappa}{N}\sum_{j=1}^N \big(x_j - \langle x_i, x_j\rangle x_i\big), \\ y_i(0) = x_i(0) = y_i^0, \quad \|y_i^0\| = 1. \end{cases} \tag{4.5}$$

We define

$$\|\Omega\|_\infty := \max_{1\leq i\leq N} \|\Omega_i\|_\infty, \quad \|X - Y\|_\infty := \max_{1\leq i\leq N} \|x_i - y_i\|.$$

Lemma 4.3 *Let X and Y be solutions to (5.14) with the same initial data, respectively. Then we have*

$$\|X(t) - Y(t)\|_\infty \leq \frac{\|\Omega\|_{op}}{4\kappa}\big(e^{4\kappa t} - 1\big), \quad t \geq 0.$$

Proof Note that

$$\|\Omega_i x_i\| \leq \|\Omega\|_{op}$$

and

$$\begin{aligned} &\big\|\big(x_j - \langle x_i, x_j\rangle x_i\big) - \big(y_j - \langle y_i, y_j\rangle y_i\big)\big\| \\ &\quad\leq \big\|x_j - y_j\big\| + \big\|\langle y_i, y_j\rangle(y_i - x_i)\big\| + \big\|\langle y_i - x_i, y_j\rangle x_i\big\| + \big\|\langle x_i, y_j - x_j\rangle x_i\big\| \\ &\quad\leq 2\,\big\|x_j - y_j\big\| + 2\,\|x_i - y_i\| \leq 4\|X - Y\|_\infty. \end{aligned}$$

We choose i_t so that

$$\|X - Y\|_\infty = \|x_{i_t} - y_{i_t}\|.$$

Then, we have

$$\frac{d}{dt}\|X - Y\|_\infty = \frac{d}{dt}\|x_{i_t} - y_{i_t}\| \leq \|\Omega\|_{op} + 4\kappa\|X - Y\|_\infty.$$

This yields the desired estimate. □

Below, we present an improved result on the emergence of practical synchronization in Theorem 2.2 where the N-dependent differential inequality was employed:

$$\frac{d\rho}{dt} \geq \frac{\kappa\rho(1-\rho)}{N}.$$

This is why N appears in (2.17). Now we replace this inequality by the L.H.S. of Lemma 3.1, which does not depend on N:

$$\frac{d\rho}{dt} \geq \kappa\rho(1-\rho),$$

and emulate the proof of Theorem 2.2 to remove N-dependence in the following corollary.

Corollary 4.1 *Suppose that the coupling strength and initial data satisfy*

$$\kappa > 4D_2(\Omega), \quad \|x_i^0\| = 1, \quad \min\{\rho^0, \min_{1\le i\le N}\langle x_i^0, x_c^0\rangle\} > \frac{\kappa - \sqrt{\kappa^2 - 4D_2(\Omega)\kappa}}{2\kappa},$$

and let X be a solution to (2.7). Then, we have

$$\lim_{\kappa\to\infty}\liminf_{t\to\infty}\rho(t) = 1.$$

Proof It follows from the same argument in Lemma 3.1 that we have

$$\frac{d\rho}{dt} \ge -D_2(\Omega) + \kappa\rho(1-\rho).$$

The rest of the proof follows from Lemma 4.1. □

In the following theorem, we show that in a large coupling regime, the Lohe sphere flow evolves toward the $\frac{N-1}{N}$-entrainment state asymptotically.

Theorem 4.1 *Suppose that $D(\Omega)$ and the initial data satisfy*

$$D(\Omega) > 0, \quad \|x_i^0\| = 1, \quad x_i^0 \ne x_j^0 \ \text{if}\ i \ne j, \quad \rho^0 > 0.$$

Then, there exists a positive coupling strength κ_∞ depending on the initial data such that if $\kappa > \kappa_\infty$, then for the solution X to (2.7), X evolves toward a practical $\frac{N-1}{N}$-entrainment state.

Proof For a two-particle system it is easy to see the emergence of practical entrainment, as demonstrated in Example 3.1. Thus, we consider $N \ge 3$. As noted in Theorem 3.2 and Proposition 3.1, the dynamics of identical Lohe sphere oscillators are well understood in any positive coupling regime. Note that x_i satisfies

$$\frac{1}{\kappa}\frac{dx_i}{dt} = \frac{\Omega_i}{\kappa}x_i + \frac{1}{N}\sum_{j=1}^N\left(x_j - \langle x_i, x_j\rangle x_i\right). \tag{4.6}$$

Then, in terms of a rescaled time-variable $\tau = \kappa t$, system (4.6) becomes

$$\frac{dx_i}{d\tau} = \frac{1}{\kappa}\Omega_i x_i + \frac{1}{N}\sum_{j=1}^N\left(x_j - \langle x_i, x_j\rangle x_i\right). \tag{4.7}$$

Thus as $\kappa \to \infty$, the dynamics (4.7) becomes close to the identical Lohe sphere flow with $\Omega_i = 0$. Hence, we might be able to use the results for identical Lohe sphere flow in a large coupling regime to study the asymptotics of nonidentical Lohe sphere oscillators. Consider the identical Lohe sphere flow Y defined in Lemma 4.3, and introduce a slow time scale $\tau = \kappa t$ along with a flow $\tilde{Y}$:

$$\tilde{y}_i(\tau) = y_i\left(\frac{\tau}{\kappa}\right).$$

Then $\tilde{y}_i$ satisfies

$$\begin{cases} \dfrac{d\tilde{y}_i}{d\tau} = \dfrac{1}{N}\displaystyle\sum_{j=1}^N\left(\tilde{y}_j - \langle\tilde{y}_i, \tilde{y}_j\rangle\tilde{y}_i\right), \quad \tau > 0, \\ \tilde{y}_i(0) = x_i^0. \end{cases}$$

Then, it follows from Theorem 3.2 that $\tilde{y}$ exhibits $\frac{N-1}{N}$-entrainment and thus there exist $\tau_0 > 0$ and $S \subset \{1, \ldots, N\}$ with $|S| \geq N-1$ so that

$$\widetilde{\mathcal{B}}_S(\tau_0) := \min_{i,j\in S}\langle \tilde{y}_i(\tau_0), \tilde{y}_j(\tau_0)\rangle > \frac{1}{N-1}.$$

We use Lemma 4.3 to see that

$$\begin{aligned}\left|\mathcal{B}_S\left(\frac{\tau_0}{\kappa}\right) - \tilde{\mathcal{B}}_S(\tau_0)\right| &= \left|\min_{i,j\in S}\left\langle x_i(\frac{\tau_0}{\kappa}), x_j(\frac{\tau_0}{\kappa})\right\rangle - \min_{i,j\in S}\left\langle y_i(\frac{\tau_0}{\kappa}), y_j(\frac{\tau_0}{\kappa})\right\rangle\right| \\ &\leq 2\left\|X\left(\frac{\tau_0}{\kappa}\right) - Y\left(\frac{\tau_0}{\kappa}\right)\right\|_\infty \\ &\leq \frac{\|\Omega\|_{op}}{2\kappa}(e^{4\tau_0}-1).\end{aligned}$$

Let $p = \frac{N-1}{N}$. Then $p > \frac{1}{2}$, and since

$$\lim_{\kappa\to\infty}\frac{1-[1-4p(1-p+D_2(\Omega)/\kappa)]^{1/2}}{2p} = \frac{1}{N-1},$$

we may find κ_∞ such that $\kappa > \kappa_\infty$ implies

$$\widetilde{\mathcal{B}}_S(\tau_0) - \frac{\|\Omega\|_{op}}{2\kappa}(e^{4\tau_0}-1) > \frac{1-[1-4p(1-p+D_2(\Omega)/\kappa)]^{1/2}}{2p}$$

and

$$\kappa > \frac{2p}{(2p-1)^2}D_2(\Omega).$$

The assumptions of Lemma 4.3 are satisfied for $\kappa > \kappa_\infty$ at $T_0 = \frac{\tau_0}{\kappa}$. □

4.2 Emergence of Practical Partial Entrainment

Recall that in Theorem 4.1, we have shown that $\frac{N-1}{N}$-entrainment is possible from any generic initial configuration. However, we had no upper bound for the sufficient coupling strength $\kappa(\varepsilon)$ - in particular, $\kappa(\varepsilon)$ may be dependent on N - and similarly for the time $T(\varepsilon)$. Thus, in this part, we describe a scheme for p-entrainment, where p may be less than $\frac{N-2}{N}$ but $\kappa(\varepsilon)$ and $T(\varepsilon)$ are completely determined by ε, ρ^0, $D(\Omega)$ and d. Next, we state our main results in this subsection as follows.

Theorem 4.2 *Suppose that the coupling strength, $D(\Omega)$ and initial data X^0 satisfy*

$$\kappa > 0, \quad D(\Omega) > 0, \quad \|x_i^0\| = 1, \quad \rho^0 > 0, \tag{4.8}$$

and let X be a solution to (2.7). Then, this ensemble exhibits practical $\left(\frac{1}{2}+\frac{\rho^0}{6}\right)$-entrainment, with $\kappa(\varepsilon)$ and $T(\varepsilon)$ depending only on ε, ρ^0, $D(\Omega)$, and d.

Proof The proof needs several technical lemmas, which include lemmas in Appendix A. We will come back to its detailed proof at the end of this subsection. □

Next, we study several technical lemmas for the proof of Theorem 4.2.

4.2.1 Technical Lemmas

We begin with some estimates for the behavior of the order parameter ρ. Recall that the order parameter ρ satisfies the differential inequality (2.11):

$$\frac{d\rho^2}{dt} \ge \kappa\rho^2\left\{-\frac{2D_2(\Omega)\rho}{\kappa\rho^2}+2\left[1-\frac{1}{N}\sum_{i=1}^N\left(\frac{\langle x_i, x_c\rangle}{\rho}\right)^2\right]\right\}, \quad t>0. \tag{4.9}$$

In the following lemma, we show that there exists a time when the R.H.S. of (4.9) becomes less than a positive value, $2D_2(\Omega)\rho$.

Lemma 4.4 *Suppose that the coupling strength, $D(\Omega)$ and initial data X^0 satisfy*

$$N\ge 3, \quad \kappa>0, \quad D(\Omega)>0, \quad \|x_i^0\|=1, \quad \rho^0>0, \tag{4.10}$$

and let X be a solution to (2.7). Then, there exists a time $t_0 \ge 0$ for which

$$\rho(t_0)\ge\rho^0 \quad \text{and} \quad 1-\frac{1}{N}\sum_{i=1}^N\left(\frac{\langle x_i, x_c\rangle}{\rho(t)}\right)^2\Big|_{t=t_0} \le \frac{2D_2(\Omega)}{\kappa\rho(t)}\Big|_{t=t_0}. \tag{4.11}$$

Proof If the second inequality in (4.11) holds for $t_0=0$, we are done. Thus, we assume that the second inequality in (4.11) does not hold at $t=0$. By continuity, there exists $\delta>0$ such that

$$1-\frac{1}{N}\sum_{i=1}^N\left(\frac{\langle x_i, x_c\rangle}{\rho(t)}\right)^2 > \frac{2D_2(\Omega)}{\kappa\rho(t)}, \quad t\in[0,\delta). \tag{4.12}$$

Then, from the inequality (4.9), we have

$$\frac{d\rho^2}{dt}\ge 2D_2(\Omega)\rho, \quad \text{i.e.,} \quad \frac{d\rho}{dt}\ge D_2(\Omega), \quad \forall\, t\in[0,\delta).$$

Note that as long as the condition (4.12) holds, ρ is strictly increasing and hence the value of ρ is larger than ρ^0. Next, we argue that the relation (4.12) cannot hold for all time, i.e., the relation $(4.11)_2$ should hold for some finite time. For this, we consider the set:

$$\mathcal{T} := \left\{t_1>0 \,\middle|\, (4.11)_2 \text{ does not hold for } t\in(0,t_1)\right\}.$$

Then, the set $\mathcal{T}$ is nonempty by assumption. Note that if the second inequality of (4.11) does not hold, then we have

$$\frac{d\rho^2}{dt}\ge 2D_2(\Omega)\rho.$$

We denote $T^* := \sup\mathcal{T}$, and we will show that $T^*<\infty$. Suppose not, i.e., $T^*=\infty$, then we have

$$\frac{d\rho^2}{dt}\ge D_2(\Omega)\rho, \quad t\in(0,\infty),$$

or equivalently

$$\frac{d\rho}{dt}\ge D_2(\Omega), \quad t\in(0,\infty).$$

This yields that for $t\ge 0$, we have

$$\rho(t) \geq \rho^0 \quad \text{and} \quad \rho(t) - (\rho^0) \geq D_2(\Omega)t. \tag{4.13}$$

Note that the R.H.S. of the second inequality tends to ∞ as $t \to \infty$, there exists $t_* > 0$ such that

$$\rho(t_*) > \rho^0 + 1,$$

which is contradictory to the boundedness of $\rho \leq 1$. Therefore, $T^* < \infty$ and there exists $t_0 > T^*$ satisfying (4.11). □

Remark 4.2 The upper bound for t_0 can be estimated as follows. It follows from $(4.13)_2$ that we have

$$1 \geq \rho(t_0) \geq \rho^0 + D_2(\Omega)t_0, \quad \text{i.e.,} \quad t_0 \leq \frac{(1-\rho^0)}{D_2(\Omega)}.$$

Lemma 4.5 *Suppose that in addition to* (4.10)*, there exist constants* $\alpha_0 \in (0, 1)$, $n_0 \in \mathbb{N}$ *and* $\kappa_0 > 0$ *satisfying the relations:*

$$\begin{aligned}
&\frac{1}{2} < p = \frac{n_0}{N} \leq 1, \quad \kappa_0 > \frac{2p}{(2p-1)^2} D_2(\Omega), \quad 1 - 2\alpha_0 > \frac{1}{2p}, \\
&p > \frac{1}{2} + \frac{\rho^0}{6}, \quad \frac{\rho^0 + (1-\alpha_0)}{1+(1-\alpha_0)} - \frac{1}{N}\left\lfloor \frac{N}{\alpha_0} \cdot \frac{2D_2(\Omega)}{\kappa_0 \rho^0} \right\rfloor > \frac{n_0 - 1}{N}.
\end{aligned} \tag{4.14}$$

Then, for $\kappa > \kappa_0$, p *and* t_0 *a time for which* (4.11) *holds , there exists a set* $S \subset \{1, \ldots, N\}$ *such that*

$$|S| \geq n_0 \quad \text{and} \quad \min_{i,j \in S} \langle x_i, x_j \rangle > \frac{1 - [1 - 4p(1 - p + D_2(\Omega)/\kappa)]^{1/2}}{2p}, \quad t \geq t_0. \tag{4.15}$$

Proof For notational simplicity, we set

$$\beta_i(t_0) = \frac{\langle x_i(t_0), x_c(t_0) \rangle}{\rho(t_0)}, \quad i = 1, \ldots, N \quad \text{and} \quad \beta_0 := \sqrt{1 - \alpha_0},$$

and suppress t_0 dependence in $\beta_i(t_0)$, i.e.,

$$\beta_i = \beta_i(t_0), \quad i = 1, \ldots, N,$$

and define index sets:

$$S_+ := \{i \mid \beta_i > \beta_0\}, \quad S_0 := \{i \mid -\beta_0 \leq \beta_i \leq \beta_0\}, \quad S_- := \{i \mid \beta_i < -\beta_0\}.$$

Then, clearly we have

$$S_+ \cup S_0 \cup S_- = \{1, \ldots, N\}.$$

Next, we claim that the set S_+ plays the role of S in the statement of Lemma 4.5, i.e., S_+ satisfies the conditions (4.15). This will be verified in two steps.

- Step A (S_+ satisfies $(4.15)_1$): By condition $(4.11)_1$ and the relation $\rho = \|x_c\|$, we have

$$\begin{aligned}
\rho^0 \leq \rho(t_0) &= \frac{1}{N}\sum_i \beta_i < \frac{|S_+|}{N} + (-\beta_0)\frac{|S_-|}{N} + \beta_0 \frac{|S_0|}{N} \\
&= (1+\beta_0)\frac{|S_+|}{N} + 2\beta_0 \frac{|S_0|}{N} - \beta_0,
\end{aligned} \tag{4.16}$$

  where we used the relations:

$$|\beta_i| \le 1, \quad \frac{|S_-|}{N} = 1 - \frac{|S_+|}{N} - \frac{|S_0|}{N}. \tag{4.17}$$

On the other hand, we use the relation (4.11)$_2$ and (4.17) to see

$$\begin{aligned} 1 - \frac{2D_2(\Omega)}{\kappa\rho_0} &\le 1 - \frac{2D_2(\Omega)}{\kappa\rho(t_0)} \le \frac{1}{N}\sum_{i=1}^N \beta_i^2 \le \frac{|S_+|}{N} + \frac{|S_-|}{N} + \beta_0^2\frac{|S_0|}{N} \\ &= 1 - (1-\beta_0^2)\frac{|S_0|}{N} = 1 - \alpha_0\frac{|S_0|}{N}. \end{aligned}$$

This yields

$$|S_0| \le \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho_0}, \quad \text{i.e.,} \quad |S_0| \le \left\lfloor \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho_0} \right\rfloor. \tag{4.18}$$

Then, it follows from (4.16) and (4.18) that we have

$$\begin{aligned} \frac{|S_+|}{N} &> \frac{\rho^0+\beta_0}{1+\beta_0} - \frac{2\beta_0}{1+\beta_0}\cdot\frac{|S_0|}{N} \ge \frac{\rho^0+\beta_0^2}{1+\beta_0^2} - 1\cdot\frac{1}{N}\left\lfloor \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho_0} \right\rfloor \\ &> \frac{n_0-1}{N}. \end{aligned} \tag{4.19}$$

Here in the first inequality, we used

$$\frac{\rho^0+\beta_0}{1+\beta_0} \ge \frac{\rho^0+\beta_0^2}{1+\beta_0^2} \iff 1-\beta_0 \ge \rho^0(1-\beta_0); \quad \frac{2\beta_0}{1+\beta_0} < 1$$

and in the last inequality of (4.19), we used the last inequality of (4.14). Hence $|S_+| \ge n_0$.

- Step B (S_+ satisfies (4.15)$_2$): Let $i, j \in S_+$. Then, we use

$$\langle x_i(t_0), x_c(t_0)\rangle = \rho(t_0)\beta_i > \rho(t_0)\beta_0,$$

to get

$$\left\| x_i(t_0) - \frac{\beta_0}{\rho(t_0)}x_c(t_0) \right\|^2 | = 1+\beta_0^2 - 2\frac{\beta_0}{\rho(t_0)}\langle x_i(t_0), x_c(t_0)\rangle < 1-\beta_0^2 = \alpha_0. \tag{4.20}$$

Similarly, we have

$$\left\| x_j(t_0) - \frac{\beta_0}{\rho(t_0)}x_c(t_0) \right\| < \sqrt{\alpha_0}, \tag{4.21}$$

Then, we use (4.20) and (4.21) to get

$$\|x_i(t_0) - x_j(t_0)\| < 2\sqrt{\alpha_0}, \quad \text{i.e.,} \quad \langle x_i(t_0), x_j(t_0)\rangle = 1 - \frac{\|x_i(t_0)-x_j(t_0)\|^2}{2} > 1-2\alpha_0.$$

Hence, we have

$$\min_{i,j\in S}\langle x_i(t_0), x_j(t_0)\rangle > 1-2\alpha_0 > \frac{1}{2\frac{n_0}{N}} = \frac{1}{2p} > \frac{1-[1-4p(1-p+D_2(\Omega)/\kappa)]^{1/2}}{2p},$$

where we used the third relation in (4.14). □

4.2.2 Proof of Theorem 4.2

Suppose that

$$N \geq 3, \quad \kappa > 0, \quad D(\Omega) > 0, \quad \|x_i^0\| = 1, \quad \rho^0 > 0.$$

We split its proof into two steps as follows.

- Step A: We claim that for $N \geq 3$ and $\rho^0 > 0$, there exists $0 < \alpha_0 < 1$, $n_0 \in \mathbb{N}$, $\kappa_0 > 0$ satisfying (4.14).
- Case A (N even): We choose $m_0 \in \mathbb{Z}$ such that

$$\frac{2m_0}{N} < \rho^0 \leq \frac{2(m_0+1)}{N},$$

and for such m_0, we set

$$n_0 = \left\lfloor \frac{m_0}{2} \right\rfloor + \frac{N}{2} + 1, \quad \kappa_0 = \frac{1152}{7} \frac{D_2(\Omega)}{(\rho^0)^3}, \quad \alpha_0 = \begin{cases} \rho^0/8, & m_0 \leq 1, \\ \frac{m_0}{4N}, & m_0 \geq 2. \end{cases} \tag{4.22}$$

- Case B (N odd): We choose $m_0 \in \mathbb{Z}$ such that

$$\frac{2m_0 - 1}{N} \leq \rho^0 < \frac{2m_0+1}{N},$$

and for such m_0, we set

$$n_0 = \left\lfloor \frac{m_0}{2} \right\rfloor + \frac{N+1}{2}, \quad \kappa_0 = \frac{640}{3} \frac{D_2(\Omega)}{(\rho^0)^3}, \quad \alpha_0 = \begin{cases} \frac{1}{4N}, & m_0 \leq 1, \\ \frac{m_0}{8N}, & m_0 \geq 2. \end{cases} \tag{4.23}$$

Then, for both cases, the relation $\frac{N}{2} < n_0 \leq N$ is easily verified. The other conditions will be checked in Appendix A.
- Step B: We use Lemma 4.2 to conclude the p-entrainment. We set

$$\begin{aligned} \kappa(\varepsilon) &:= \max\left\{ \frac{1152}{7} \frac{D_2(\Omega)}{(\rho^0)^3}, \kappa_1(\varepsilon, \rho^0) \right\}, \\ T(\varepsilon) &:= t_0 + T_1(\varepsilon) \leq \frac{1}{2D_2(\Omega)\rho^0} + T_1(\varepsilon). \end{aligned} \tag{4.24}$$

Here κ_1 and t_0 are the coupling strength and time appearing in Lemma 4.5, and T_1 is the time we should wait for

$$\sup_{t > T(\varepsilon)} \max_{i,j \in S} \|x_i(t) - x_j(t)\| \leq \varepsilon.$$

Note that $\kappa(\varepsilon)$ and $T_1(\varepsilon)$ are functions of ε, ρ^0, $D_2(\Omega)$, and d. This is because the function

$$f(y) = 2\kappa\left[-py^2 + y - \left(1 - p + \frac{D_2(\Omega)}{2\kappa}\right)\right]$$

used in Lemma 4.2 via Lemma 4.1 depends only on p, d, ρ^0 and κ. In detail, for a proper value of $\kappa(\varepsilon)$, we can set

$$\frac{1 + [1 - 4p(1 - p + D_2(\Omega)/\kappa)]^{1/2}}{2p} > 1 - \frac{1}{2}\cos\varepsilon.$$

On the other hand, from Step A,

$$\min_{i,j\in S}\langle x_i(t_0), x_j(t_0)\rangle > \frac{1-[1-4p(1-p+D_2(\Omega)/\kappa)]^{1/2}}{2p}.$$

Hence for $\kappa(\varepsilon)$,

$$\min_{i,j\in S}\langle x_i(t_0+T_1(\varepsilon)), x_j(t_0+T_1(\varepsilon))\rangle > 1-\cos\varepsilon,$$

for a finite time $T_1(\varepsilon)$ which may depend on $\kappa(\varepsilon)$, ε, ρ^0, $D_2(\Omega)$, and d. Therefore,

$$\sup_{t>T(\varepsilon)}\max_{i,j\in S}\|x_i(t)-x_j(t)\| \le \varepsilon.$$

5 Gradient Flow Formulation of the Lohe Models

In this section, we present gradient flow formulations of the Lohe matrix models for $\Omega_i = \Omega$, and using this new gradient flow formulation, we show that all initial configurations tend to the complete entrainment state asymptotically.

5.1 A Gradient Flow on Riemannian Manifolds

Recall that for a finite-dimensional gradient flow with an analytical potential on the Euclidean space $\mathbb{R}^d$, it is well known that a uniformly bounded flow converges. Below, we extend this result to a Riemannian setting. First, we recall the Łojasiewicz inequality in a Riemannian framework where the proof of Łojasiewicz inequality in $\mathbb{R}^d$ can be generalized in a straightforward manner.

Theorem 5.1 *Let $(\mathcal{M}, g)$ be a Riemannian $C^\omega-$manifold with Riemannian metric g. Then, for an open subset $U \subset \mathcal{M}$, a point $p \in U$, and a real analytic function $f : U \to \mathbb{R}$, there exist constants $\gamma \in [\frac{1}{2}, 1)$, C_L and an open neighborhood V satisfying $p \in V \subset U$ such that*

$$|f(z)-f(p)|^\gamma \le C_L\|\nabla f(z)\|, \quad \forall z \in V. \tag{5.1}$$

Proof For the Euclidean space $\mathcal{M} = \mathbb{R}^d$, the Łojasiewicz inequality (5.1) has been proved in [26]. Since the inequality (5.1) has a local nature, the proof for the Euclidean space can be trivially extended to the Riemannian manifold $\mathcal{M}$. □

We now apply Theorem 5.1 to study the asymptotic behavior of gradient flows with analytical potentials on Riemannian manifolds. The following theorem is an extended version of the arguments in [13], which was used for the Kuramoto model in $\mathbb{R}^N$.

Theorem 5.2 *Let $f : U \to \mathbb{R}$ be a real analytic function defined on an open subset U of a Riemannian manifold $(\mathcal{M}, g)$, and suppose the smooth path $\theta : [0,\infty) \to U$ is an integral curve of $-\nabla f$ with starting point $\theta^0 \in U$, i.e.,*

$$\theta(0) = \theta^0, \quad \theta_*\left(\frac{d}{dt}\right) = \frac{d\theta(t)}{dt} = -\nabla f(\theta(t)).$$

(Here, the gradient vector $-\nabla f$ is obtained from the covariant vector $-df$ under the natural identification induced by the Riemannian metric g on $\mathcal{M}$. The pushforward θ_* is the

corresponding derivative as well.) If the image of θ *is contained in a compact subset* K *of* U*, then*

$$\exists \lim_{t\to\infty} \theta(t) = \theta_\infty \in K \quad and \quad \lim_{t\to\infty} \left\| \theta_* \left(\frac{d}{dt} \right) \right\| = 0.$$

Proof Without loss of generality, we may assume the initial condition θ^0 is not an equilibrium. Then, $\theta(t)$ can never be an equilibrium point in finite time since f is analytic. Since the image of θ is contained in the compact metric space K, we may find a sequence $\{t_n\}_{n=1}^{\infty}$ and a point $\theta_\infty \in M$ such that

$$t_1 \le t_2 \le \cdots \le t_n \to \infty, \quad \lim_{n\to\infty} \theta(t_n) = \theta_\infty.$$

On the other hand, $f(\theta(t))$ satisfies

$$\frac{d}{dt} f(\theta(t)) = -\|\nabla f(\theta(t))\|^2,$$

and since f is bounded on K, there exists a limit $\lim_{t\to\infty} f(\theta(t)) = f_\infty$ and we must have $f(\theta_\infty) = f_\infty$ by continuity of f. Then, we apply Theorem 5.1 at θ_∞ to obtain constants $\gamma \in [\frac{1}{2}, 1)$, C_L and an open neighborhood $\theta_\infty \in V \subset U$ such that

$$|f(z) - f_\infty|^\gamma \le C_L \|\nabla f(z)\|, \quad \forall z \in V.$$

Now consider the metric d induced on M through the Riemannian metric, and choose $r > 0$ such that $B_r(\theta_\infty) \subset V$. Define the function $h : [0, \infty) \to [0, \infty)$ by

$$h(t) = (f(\theta(t)) - f_\infty)^{1-\gamma}.$$

Clearly, as $t \to \infty$, $h \downarrow 0$. So for any $\varepsilon \in (0, \varepsilon_0)$ there exists some $T_0 \in \{t_n\}$ such that

$$|h(t) - h(T_0)| \le \frac{\varepsilon(1-\gamma)}{3C_L}, \quad \forall t \ge T_0 \quad \text{and} \quad \|\theta(T_0) - \theta_\infty\| < \frac{\varepsilon}{3}.$$

We now intend to show that $\theta(t) \in B_\varepsilon(\theta_\infty)$ for $t \ge T_0$. The following set

$$\mathcal{T} = \{t_1 > T_0 : \theta(t) \in B_\epsilon(\theta_\infty) \text{ for all } t \in (T_0, t_1)\}$$

is nonempty by continuity of $\theta(t)$. Also, for any $t_1 \in \mathcal{T}$,

$$\begin{aligned} \frac{dh(t)}{dt} &= (1-\gamma)(f(\theta(t)) - f_\infty)^{-\gamma} \frac{d}{dt} f(\theta(t)) = -(1-\gamma)(f(\theta(t)) - f_\infty)^{-\gamma} \|\nabla f(\theta(t))\|^2 \\ &\le -\frac{1-\gamma}{C_L} \|\nabla f(\theta(t))\|, \quad t \in (T_0, t_1). \end{aligned}$$

Hence

$$\int_{T_0}^{t_1} \left\| \theta_* \left(\frac{d}{d\tau} \right) \right\| d\tau = \int_{T_0}^{t_1} \|\nabla f(\theta(\tau))\| d\tau \le -\frac{C_L(h(t_1) - h(T_0))}{1-\gamma} \le \frac{\varepsilon}{3}. \tag{5.2}$$

Now, we set

$$T_1 = \sup \mathcal{T}.$$

If $T_1 < \infty$ then clearly $T_1 \in \mathcal{T}$. Then by inequality (5.2),

$$\begin{aligned} d(\theta(T_1), \theta_\infty) &\le d(\theta(T_1), \theta(T_0)) + d(\theta(T_0), \theta_\infty) \le \int_{T_0}^{T_1} \|\theta_*(\frac{d}{d\tau})\| d\tau + \|\theta(T_0) - \theta_\infty\| \\ &\le \frac{2\epsilon}{3}. \end{aligned}$$

This contradicts the definition of T_1 since we may choose an element of $\mathcal{T}$ greater than T_1 by continuity of $\theta(t)$. Hence $T_1 = \infty$, i.e.,

$$\theta(t) \in B_\epsilon(\theta_\infty), \quad t \geq T_0.$$

By taking the limit $t_1 \to \infty$ in (5.2), we obtain

$$\int_{T_0}^{\infty} \|\dot{\theta}(\tau)\| d\tau \leq \frac{\varepsilon}{3},$$

which implies the existance of the limit

$$\lim_{t\to\infty} \theta(t) = \theta_\infty.$$

On the other hand, by the continuity of ∇f, the limit

$$\lim_{t\to\infty} \left\| \theta_* \left(\frac{d}{dt} \right) \right\| = \lim_{t\to\infty} \| - \nabla f(\theta(t))\| = \| - \nabla f(\theta_\infty)\|,$$

must be 0 since the limit $\lim_{t\to\infty} \theta(t)$ exists. □

Corollary 5.1 *Let $f : \mathcal{M} \to \mathbb{R}$ be a real analytic function defined on a compact Riemannian manifold M, and suppose that the smooth path $\theta : [0, \infty) \to \mathcal{M}$ is an integral curve of $-\nabla f$ with starting point $\theta^0 \in \mathcal{M}$, i.e.*

$$\theta(0) = \theta^0, \quad \theta_* \left(\frac{d}{dt} \right) = -\nabla f(\theta(t)).$$

Then the limit

$$\lim_{t\to\infty} \theta(t) = \theta_\infty \in \mathcal{M}$$

exists and

$$\lim_{t\to\infty} \left\| \theta_* \left(\frac{d}{dt} \right) \right\| = 0.$$

Proof We set $\mathcal{M} = U = K$ in Theorem 5.2 to get the proof. □

5.2 The Kuramoto Model is a Gradient Flow

We briefly discuss the gradient flow formulation of the Kuramoto model as a motivation to the Riemannian manifolds:

$$\dot{\theta}_i = \nu_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \tag{5.3}$$

Recall that the Lohe matrix model in one-dimension $d = 1$ is reduced to the Kuramoto model via the correspondence relation

$$U_i = e^{-\mathrm{i}\theta_i}, \quad H_i = \nu_i, \quad i = 1, \ldots, N. \tag{5.4}$$

Then, we define an order parameter (R_k, ϕ_k) and potential V_k as follows:

$$
\begin{aligned}
R_k e^{\mathrm{i}\phi_k} &:= \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}, \quad \Theta := (\theta_1, \ldots, \theta_N), \\
\nu := (\nu_1, \ldots, \nu_N), &\quad V_k(\Theta) := -\nu \cdot \Theta - \frac{\kappa}{N}\sum_{i,j}\cos(\theta_j - \theta_i).
\end{aligned}
\tag{5.5}
$$

We use (5.4) to rewrite the potential V_k in terms of R:

$$
V_k(\Theta) = -\nu \cdot \Theta - \kappa N R_k^2. \tag{5.6}
$$

Then it is easy to see that the Kuramoto flow (5.3) is a gradient flow:

$$
\dot{\Theta} = -\nabla_\Theta V_k(\Theta).
$$

This fact is used in [13] to prove the emergence of synchronization in the Kuramoto model whose domain is $\mathbb{R}^N$. Hence it is natural to ask whether the Lohe matrix and sphere models can be written as a gradient flow or not. This will be the main focus of the next subsections.

5.3 Gradient Flow Formulations of the Lohe Models

In this subsection, we present gradient flow formulations for generalized Lohe models on four Riemannian manifolds:

$$
\mathcal{M}: \ \mathbb{S}^d, \quad \mathbb{U}(d), \quad S\mathbb{U}(d), \quad \mathbb{R} \times S\mathbb{U}(d).
$$

In the following, we consider Lohe flows on the above Riemannian manifolds and present a gradient flow formulation.

5.3.1 A Lohe Flow on $\mathbb{S}^d$

Consider the Lohe sphere model for identical oscillators:

$$
\dot{x}_i = \Omega + \frac{\kappa}{N}\sum_{k=1}^{N}(x_k - \langle x_i, x_k\rangle x_i) = \Omega + \kappa \mathbb{P}^{\perp}_{x_i}\left(\frac{1}{N}\sum_{k=1}^{N} x_k\right), \quad i = 1, \ldots, N. \tag{5.7}
$$

where Ω is a skew-symmtric matrix and $\mathbb{P}^{\perp}_{x_i}$ is the orthogonal projection onto the tangent plane perpendicular to x_i:

$$
\mathbb{P}^{\perp}_{x_i} y = y - \langle x_i, y\rangle x_i.
$$

Without loss of generality, we assume $\Omega = 0$. Now, we introduce a potential function $\mathcal{V}_0$:

$$
\mathcal{V}_0(X) := -\frac{\kappa}{2N}\sum_{i,k=1}^{N}\langle x_i, x_k\rangle. \tag{5.8}
$$

Note that the potential function $\mathcal{V}_0$ is analytic, and recall that the surface gradient is defined as the projection of the gradient onto the tangent plane.

Proposition 5.1 *The Lohe sphere model (5.7) with $\Omega = 0$ is a gradient flow:*

$$
\dot{x}_i = -\nabla_{x_i}\mathcal{V}_0\Big|_{T_{x_i}\mathbb{S}^d},
$$

where $\nabla_{x_i}\mathcal{V}_0\Big|_{T_{x_i}\mathbb{S}^d}$ *denotes the orthogonal projection of the gradient vector* $\nabla_{x_i}\mathcal{V}_0 \in \mathbb{R}^{d+1}$ *onto the tangent plane* $T_{x_i}\mathbb{S}^d$ *at* x_i, *which is the induced gradient vector on the manifold* $\mathbb{S}^d$.

Proof It follows from (5.8) that for each $i = 1, \dots, N$,

$$\nabla_{x_i}\mathcal{V}_0 = -\frac{\kappa}{N}\sum_{k=1}^{N} x_k.$$

This yields

$$\nabla_{x_i}\mathcal{V}_0\Big|_{T_{x_i}\mathbb{S}^d} = -\frac{\kappa}{N}\sum_{k=1}^{N} x_k\Bigg|_{T_{x_i}\mathbb{S}^d} = -\frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k - \langle x_k, x_i\rangle x_i\right).$$

□

As a direct application of Theorem 5.2 and Proposition 5.1, we obtain the following convergence result for the Lohe flow from any initial data.

Corollary 5.2 *Let* $X = X(t)$ *be a Lohe flow whose dynamics is governed by the system* (5.7) *with* $\Omega = 0$. *Then, there exists an equilibrium* X^∞ *such that*

$$X^\infty = \lim_{t\to\infty} X(t).$$

Proof Since $(\mathbb{S}^d)^N$ is a compact manifold, the Lohe flow X is always bounded. Thus it follows from Theorem 5.2 and Proposition 5.1 that any Lohe flow converges. □

Next, we briefly argue that why we need to restrict $\Omega_i = 0$. One natural question is the existence of potential functions when the natural frequencies are not identical. For simplicity, let $\kappa = 0$ so that the interaction terms disappear:

$$\dot{x}_i = \Omega_i x_i, \quad i = 1, \dots, N,$$

where the Ω_i's are skew-symmetric matrices. In this case, a possible ansatz on the nontrivial potential function might be

$$\mathcal{V}(x) = -\sum_{j=1}^{N} x_j^t \Omega_j x_j,$$

but then the derivatives of $\mathcal{V}$ are zero due to skew-symmetry of Ω_i:

$$-\frac{\partial \mathcal{V}}{\partial x_i} = (\Omega_i^t + \Omega_i)x_i = 0, \quad \text{i.e.,} \quad \mathcal{V} = \text{constant}.$$

On the other hand, for the Kuramoto model, the potential for nonidentical oscillators still exists on the covering space $(\mathbb{R}^1)^N$ of $(\mathbb{S}^1)^N$ as we can see in (5.5), which is not well-defined on $(\mathbb{S}^1)^N$.

For the Lohe sphere model, let us consider $d > 1$, since $d = 1$ is exactly the same as the Kuramoto model. The sphere $\mathbb{S}^d$ for $d > 1$ is a simply connected manifold, which has no proper covering space. Therefore, the potential function should be on $(\mathbb{S}^d)^N$ if it exists. However, it cannot exist on $(\mathbb{S}^d)^N$ since this model has a periodic solution. We can easily see this from the following example.

Example 5.1 Let $\kappa = 0$ and

$$\Omega_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \Omega_k = 0 \ \text{ for any } \ k \neq 1.$$

Then the Ω_i's are skew-symmetric matrices. With initial data $x_1^0 = (1, 0, \ldots, 0)^t$, it has a periodic solution

$$x_1(t) = \exp(\Omega_1 t) x_1^0 = (\cos(t), 0, \ldots, 0, \sin(t))^t,$$

with $x_k(t) = x_k^0$ for $k \neq 1$.

Remark 5.1 The Lohe sphere model on $\mathbb{S}^d$ for nonidentical oscillators does not have a potential function on $\mathbb{S}^d$. However, for $d = 1$, it has a potential on its covering space $(\mathbb{R}^1)^N$, as same as the Kuramoto model.

5.3.2 A Lohe Flow on $\mathbb{U}(d)$

In this part, we consider the Lohe matrix model on the unitary group $\mathbb{U}(d)$ with identical Hamiltonians $H_i = H$:

$$\begin{cases} \dot{U}_i U_i^\dagger = -\mathrm{i}H + \frac{\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger \right), & \text{or equivalently} \\ \dot{U}_i = -\mathrm{i}HU_i + \frac{\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger U_i - U_i U_j^\dagger U_i \right), & t > 0, \\ U_i(0) = U_i^0, & i = 1, \ldots, N. \end{cases} \tag{5.9}$$

Note that we may assume $H = 0$ in (5.9) without loss of generality, if necessary by considering a change of variable $U_i \to e^{\mathrm{i}Ht} U_i$, so that

$$\dot{U}_i = \frac{\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger U_i - U_i U_j^\dagger U_i \right), \quad t > 0. \tag{5.10}$$

On the other hand, motivated by the Kuramoto model in Sect. 5.2, we introduce an order parameter R and a potential $\mathcal{V}_1$ for (5.9) with $H_i = 0$:

$$R^2 := \frac{1}{N^2} \sum_{i,j=1}^N \mathrm{tr}\left(U_i U_j^\dagger \right) \quad \text{and} \quad \mathcal{V}_1\left(\{U_i\}_{i=1}^N \right) := -\frac{\kappa}{2} N R^2. \tag{5.11}$$

Here, the Riemannian metric of $\mathbb{U}(d)$ is induced by the natural inclusion $\mathbb{U}(d) \hookrightarrow M_{d,d}(\mathbb{C})$. For the one-dimensional case, the relations (5.11) are exactly the same as (5.5) and (5.6) via the relation (5.4).

Proposition 5.2 *The Lohe model* (5.9) *with* $H_i = 0$ *is a gradient flow with analytical potential* $\mathcal{V}_1$ *in* (5.11)*:*

$$\dot{U}_i = -\left. \frac{\partial \mathcal{V}_1}{\partial U_i} \right|_{T_{U_i} \mathbb{U}(d)}, \quad i = 1, \ldots, N, \quad t > 0.$$

Proof The function $\mathcal{V}_1$ has an obvious polynomial extension to all of $M_{d,d}(\mathbb{C})^N = \mathbb{R}^{2d^2 N}$ viewed as a real analytic manifold. Since each variable U_i is in $M_{d,d}(\mathbb{C}) = \mathbb{R}^{2d^2}$, the partial derivatives of a matrix can be calculated by the partial derivatives of each real and imaginary component of U_i on $\mathbb{R}^{2d^2}$. Let $u_i^{kl} = a_i^{kl} + \mathrm{i} b_i^{kl}$ be the (k,l)-element of matrix U_i. First, we use $\mathrm{tr}(U_i U_j^\dagger) = \sum_{k,l=1}^d u_i^{kl} \bar{u}_j^{kl} = \sum_{k,l=1}^d \left[(a_i^{kl} a_j^{kl} + b_i^{kl} b_j^{kl}) + \mathrm{i}(a_j^{kl} b_i^{kl} - a_i^{kl} b_j^{kl}) \right]$ to see

$$\begin{aligned}\mathcal{V}_1 &= -\frac{\kappa}{2N} \sum_{i,j=1}^N \mathrm{tr}(U_i U_j^\dagger) = -\frac{\kappa}{2N} \sum_{i,j=1}^N \sum_{k,l=1}^d \left[(a_i^{kl} a_j^{kl} + b_i^{kl} b_j^{kl}) + \mathrm{i}(a_j^{kl} b_i^{kl} - a_i^{kl} b_j^{kl}) \right] \\ &= -\frac{\kappa}{2N} \sum_{i,j=1}^N \sum_{k,l=1}^d \left[a_i^{kl} a_j^{kl} + b_i^{kl} b_j^{kl} \right],\end{aligned}$$

where we cancel the imaginary term by symmetry of the indices i, j. This yields

$$\frac{\partial \mathcal{V}_1}{\partial a_i^{kl}} = -\frac{\kappa}{N} \sum_{j=1}^N a_j^{kl}, \quad \frac{\partial \mathcal{V}_1}{\partial b_i^{kl}} = -\frac{\kappa}{N} \sum_{j=1}^N b_j^{kl},$$

and thus reverting back to the coordinates of $M_{d,d}(\mathbb{C})^N = \mathbb{R}^{2d^2 N}$, we have

$$\begin{aligned}\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i} M_{d,d}(\mathbb{C})} &= \sum_{k,l=1}^d \left(\frac{\partial \mathcal{V}_1}{\partial a_i^{kl}} + \mathrm{i} \frac{\partial \mathcal{V}_1}{\partial b_i^{kl}} \right) E^{kl} = \sum_{k,l=1}^d \left(-\frac{\kappa}{N} \sum_{j=1}^N a_j^{kl} - \mathrm{i} \frac{2\kappa}{N} \sum_{j=1}^N b_j^{kl} \right) E^{kl} \\ &= -\frac{\kappa}{N} \sum_{j=1}^N \sum_{k,l=1}^d \left(a_j^{kl} + \mathrm{i} b_j^{kl} \right) E^{kl} = -\frac{\kappa}{N} \sum_{j=1}^N \sum_{k,l=1}^d u_j^{kl} E^{kl} \\ &= -\frac{\kappa}{N} \sum_{j=1}^N U_j,\end{aligned}$$

where E^{kl} denotes the $d \times d$ matrix whose (k,l)-coordinate is 1 and the other coordinates are 0.

Now, we should project it on the tangent space $T_{U_i}\mathbb{U}(d)$ of $\mathbb{U}(d)$ at U_i. The orthogonal projection $\pi : T_{I_d} M_{d,d}(\mathbb{C}) \to \mathfrak{u}(d)$ is given by $A \mapsto \frac{1}{2}(A - A^\dagger)$ since $\mathfrak{u}(d) = T_{I_d}\mathbb{U}(d) = \{X \in M_{d,d}(\mathbb{C}) \mid X + X^\dagger = 0\}$ is the set of skew Hermitian matricies (See Chapter 4 in [34]). Since $T_{U_i}\mathbb{U}(d)$ is the right translate $\mathfrak{u}(d)U_i$ of $\mathfrak{u}(d)$, we can see that the orthogonal projection $\pi_{U_i} : T_{U_i} M_{d,d}(\mathbb{C}) \to T_{U_i}\mathbb{U}(d)$ is given by $AU_i \mapsto \pi(A)U_i = \frac{1}{2}(A - A^\dagger)U_i$ for an element $AU_i \in T_{U_i} M_{d,d}(\mathbb{C})$. Hence we may calculate

$$\begin{aligned}\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i}\mathbb{U}(d)} &= \pi_{U_i}\left(\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i} M_{d,d}(\mathbb{C})} \right) = \pi\left(\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i} M_{d,d}(\mathbb{C})} U_i^\dagger \right) U_i \\ &= \pi\left(-\frac{\kappa}{N} \sum_{j=1}^N U_j U_i^\dagger \right) U_i = -\frac{\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger \right) U_i.\end{aligned}$$

Therefore, we have

$$-\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i}\mathbb{U}(d)} U_i^\dagger = \frac{\kappa}{2N} \sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger \right),$$

which yields the desired estimate. □

As a direct application of Theorem 5.2 and Proposition 5.2, we have the convergence of the flow $e^{-Ht}U_i$ as $t \to \infty$.

Corollary 5.3 *Suppose that the Hamiltonians H_i are the same:*

$$H_i = H, \quad i = 1, \ldots, N,$$

and let $U_i = U_i(t)$ be a solution to the Cauchy problem (5.9). Then, the quantities $e^{-\mathrm{i}Ht}U_i$ converge for any initial configuration $\{U_i^0\}$.

Proof Upon change of variables $U_i \mapsto e^{\mathrm{i}Ht}U_i$, we may assume $H = 0$. Then we are done by Theorem 5.2 and Proposition 5.2. □

Remark 5.2 The above corollary implies that for the Lohe model (2.1) with positive coupling strength and zero natural frequencies,

$$\lim_{t\to\infty} \dot{U}_i(t) = 0 \text{ for all } i.$$

This can be proved by more elementary means similar to those of Lemma 3.4, without employing the profound result of Łojasiewicz's inequality and its implications on real analytic flows. However, this does not imply that the relative phases $U_iU_j^\dagger$ should converge, because the set of pairs of $U_iU_j^\dagger$, $(i, j = 1, \ldots, N)$ of equilibrium points is not in general discrete. This in turn highlights the significance of Łojasiewicz's inequality in proving the convergence of U_i.

Proof We first prove the existence of ρ^∞ where ρ is defined by the mean value,

$$\rho = \|U_c\|, \quad U_c = \frac{1}{N}\sum_{i=1}^N U_i,$$

where we use the trace norm

$$\|A\| = \sqrt{\mathrm{tr}(AA^\dagger)}, \quad A \in M_{d,d}(\mathbb{C}).$$

Note that

$$\dot{U}_iU_i^\dagger = \frac{\kappa}{2N}\sum_{j=1}^N \left(U_jU_i^\dagger - U_iU_j^\dagger\right) = \frac{\kappa}{2}(U_cU_i^\dagger - U_iU_c^\dagger),$$

so we may write

$$\dot{U}_i = \frac{\kappa}{2}(U_c - U_iU_c^\dagger U_i).$$

From the definition of the norm,

$$\begin{aligned}
\|\dot{U}_i\|^2 &= \frac{\kappa^2}{4}\mathrm{tr}(U_c - U_iU_c^\dagger U_i)(U_c^\dagger - U_i^\dagger U_cU_i^\dagger) \\
&= \frac{\kappa^2}{4}\mathrm{tr}(U_cU_c^\dagger - U_iU_c^\dagger U_iU_c^\dagger - U_cU_i^\dagger U_cU_i^\dagger + U_iU_c^\dagger U_iU_i^\dagger U_cU_i^\dagger) \\
&= \frac{\kappa^2}{4}\mathrm{tr}(U_cU_c^\dagger - U_iU_c^\dagger U_iU_c^\dagger - U_i^\dagger U_cU_i^\dagger U_c + U_c^\dagger U_c) \\
&= \frac{\kappa^2}{2}\mathrm{Re}\ \mathrm{tr}(U_cU_c^\dagger - U_iU_c^\dagger U_iU_c^\dagger).
\end{aligned}$$

On the other hand,

$$\dot{U}_c = \frac{\kappa}{2N}\sum_{i=1}^{N}(U_c - U_i U_c^\dagger U_i).$$

Thus,

$$\frac{d}{dt}\rho^2 = \frac{d}{dt}\|U_c\|^2 = 2\mathrm{Re}\ \mathrm{tr}(\dot{U}_c U_c^\dagger) = \frac{\kappa}{N}\mathrm{Re}\ \mathrm{tr}\sum_{i=1}^{N}(U_c U_c^\dagger - U_i U_c^\dagger U_i U_c^\dagger) = \frac{2}{\kappa N}\sum_{i=1}^{N}\|\dot{U}_i\|^2,$$

hence ρ is non-decreasing and bounded above by $\sqrt{d}$, which implies the existence of ρ^∞. Note that

$$\|\dot{U}_i\| = \left\|\frac{\kappa}{2}(U_i - U_c U_i^\dagger U_i)\right\| \le \frac{\kappa}{2}(\|U_i\| + \|U_c U_i^\dagger U_i\|) \le \frac{\kappa}{2}\sqrt{d}(1+d),$$

and thus that

$$\|\dot{U}_c\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\dot{U}_i\right\| \le \frac{1}{N}\sum_{i=1}^{N}\|\dot{U}_i\| \le \frac{\kappa}{2}\sqrt{d}(1+d).$$

Now, according to the previous calculations,

$$\begin{aligned}
\frac{d}{dt}\|\dot{U}_i\|^2 &= \frac{d}{dt}\frac{\kappa^2}{2}\mathrm{Re}\ \mathrm{tr}(U_c U_c^\dagger - U_i U_c^\dagger U_i U_c^\dagger)\\
&= \frac{\kappa^2}{2}\mathrm{Re}\ \mathrm{tr}(\dot{U}_c U_c^\dagger + U_c \dot{U}_c^\dagger - \dot{U}_i U_c^\dagger U_i U_c^\dagger - U_i \dot{U}_c^\dagger U_i U_c^\dagger - U_i U_c^\dagger \dot{U}_i U_c^\dagger - U_i U_c^\dagger U_i \dot{U}_c^\dagger)\\
&= \kappa^2\mathrm{Re}\ \mathrm{tr}(\dot{U}_c U_c^\dagger - \dot{U}_i U_c^\dagger U_i U_c^\dagger - U_i \dot{U}_c^\dagger U_i U_c^\dagger),
\end{aligned}$$

and hence we conclude that

$$\begin{aligned}
\left|\frac{d}{dt}\|\dot{U}_i\|^2\right| &\le \kappa^2(\|\dot{U}_c\|\|U_c^\dagger\| + \|\dot{U}_i U_c^\dagger\|\|U_i U_c^\dagger\| + \|U_i \dot{U}_c^\dagger\|\|U_i U_c^\dagger\|)\\
&\le \frac{\kappa^3}{2}d(1+d)(1+2d).
\end{aligned}$$

Therefore, $\frac{d^2\rho}{dt^2}$ is bounded, and by Barbalat's lemma we obtain the conclusion of the remark. □

As we discussed on $\mathbb{S}^d$, we can consider the existence of potential functions for nonidentical oscillators. In $\mathbb{U}(d)$, a possible potential function for $\kappa = 0$ is

$$\mathcal{V}(U) = \mathrm{i}\sum_{j=1}^{N}\mathrm{tr}(U_j^\dagger H_j U_j).$$

However, this is a constant function on $\mathbb{U}(d)$ since all U_j are unitary. We can rigorously prove the nonexistence of potential functions on $\mathbb{U}(d)$ by showing periodic solutions.

Example 5.2 For the case of $\kappa = 0$ and $H_1 = h_1 I_d$ with a nonzero real scalar h_1, the oscillator U_1 follows the equation $\dot{U}_1 = -\mathrm{i}h_1 U_1$. Therefore, we have a periodic solution $U_1(t) = \exp(-\mathrm{i}h_1 t)U_1^0$.

Since $\mathbb{U}(d)$ is not a simply connected manifold, we have the following potential function on the covering space $\mathbb{R} \times S\mathbb{U}(d)$ similar to the Kuramoto model.

Remark 5.3 The universal covering space of $\mathbb{U}(d)$ is $\mathbb{R} \times S\mathbb{U}(d)$. A Lohe flow on $\mathbb{U}(d)$ with nonidentical oscillators can have a potential function if and only if the natural frequencies H_i are all scalars. We will see this and analyze a Lohe flow on $\mathbb{R} \times S\mathbb{U}(d)$ in a later subsection.

5.3.3 A Lohe Flow on $S\mathbb{U}(d)$

Consider a Cauchy problem for the generalized Lohe model on $S\mathbb{U}(d)$, which was introduced in [20]:

$$\begin{cases} \dot{U}_i U_i^\dagger = -\mathrm{i}H_i + \frac{\kappa}{2N}\sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger - \frac{1}{d}\mathrm{tr}\left[U_j U_i^\dagger - U_i U_j^\dagger \right] I_d \right), & t > 0, \\ U_i(0) = U_i^0, \quad i = 1, \dots, N, \end{cases} \tag{5.12}$$

where $H_i \in \mathrm{i}\mathfrak{su}(d)$(recall that $\mathfrak{su}(d)$ consists of skew-hermitian matrices with zero trace). First, we show that system (5.12) preserves the structure of $S\mathbb{U}(d)$.

Lemma 5.1 *Special unitarity is preserved along the dynamics (5.12), i.e.,*

$$U_i^0 \in S\mathbb{U}(d) \quad \Longrightarrow \quad U_i(t) \in S\mathbb{U}(d), \quad t \ge 0.$$

Proof Since the R.H.S. of $(5.12)_1$ is anti-Hermitian, it is clear to see the unitarity of U_i. Thus, we only need to check whether $\det U_i = 1$ or not. Note that $\dot{U}_i U_i^\dagger$ has trace zero, and hence $U_i^{-1}\dot{U}_i = U_i^{-1}(\dot{U}_i U_i^\dagger)U_i$ has trace zero. We use the Jacobi's determinant formula (See Proposition 15.21 and Problem 15.7 in [34] for detail) to see that

$$\frac{d}{dt}\det U_i = \det U_i \cdot \mathrm{tr}(U_i^{-1}\dot{U}_i) = 0,$$

so that $\det U_i = 1$ for $t \ge 0$. □

Proposition 5.3 *Suppose that the constant hamiltonians H_i are the same, i.e., $H_i = H$, $i = 1, \dots, N$, and let $U_i = U_i(t)$ be a solution to the Cauchy problem* (5.12)*. Then, the Lohe model* (5.12) *can be rewritten as a gradient flow with analytical potential $\mathcal{V}_1$ in* (5.11)*:*

$$\dot{U}_i = -\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i}\mathbb{U}(d)}, \quad i = 1, \dots, N, \quad t > 0.$$

Here, the Riemannian metric on $S\mathbb{U}(d)$ is given by the inclusion $S\mathbb{U}(d) \hookrightarrow M_{d,d}(\mathbb{C})$.

Proof The proof is almost the same with that of Proposition (5.2). Again the function $\mathcal{V}_1$ has an obvious polynomial extension to all of $M_{d,d}(\mathbb{C})^N = \mathbb{R}^{2d^2N}$ viewed as a real analytic manifold, and has gradient

$$\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i}M_{d,d}(\mathbb{C})} = -\frac{\kappa}{N}\sum_{j=1}^N U_j. \tag{5.13}$$

Now we want to project it on the tangent space $T_{U_i}S\mathbb{U}(d)$. The orthogonal projection $\pi : T_{I_d}M_{d,d}(\mathbb{C}) \to \mathfrak{su}(d)$ is given by $A \mapsto \frac{1}{2}\left(A - A^\dagger - \frac{1}{d}\mathrm{tr}(A - A^\dagger)I_d\right)$ where $\mathfrak{su}(d) = \{X \in M_{d,d}(\mathbb{C}) \mid X + X^\dagger = 0,\ \mathrm{tr}(X) = 0\}$. Note that the derivative of ($\det X = 1$) at I_d is ($\mathrm{tr}(X) = 0$) because of Jacobi's determinant formula. (See Chapter 4 in [34] for details) The projection π comes from the composition of the projection from $M_{d,d}(\mathbb{C})$ onto $\mathfrak{u}(d)$ and the projection from $\mathfrak{u}(d)$ onto $\mathfrak{su}(d)$. Since the projection $\pi_1 : \mathfrak{u}(d) \to \mathfrak{su}(d)$ is given by the projection into the traceless matrices, $A \mapsto A - \frac{1}{d}\mathrm{tr}(A)I_d$, the projection π is obtained from the composition

$$\begin{aligned} &M_{d,d}(\mathbb{C}) \to \mathfrak{u}(d) \to \mathfrak{su}(d), \\ &\qquad A \mapsto \frac{1}{2}(A - A^\dagger) \mapsto \frac{1}{2}\left(A - A^\dagger - \frac{1}{d}\mathrm{tr}(A - A^\dagger)I_d\right). \end{aligned}$$

Therefore, the orthogonal projection of (5.13) on $T_{U_i}\mathbb{U}(d)$ is given by $AU_i \mapsto \pi(A)U_i$,

$$-\left.\frac{\partial \mathcal{V}_1}{\partial U_i}\right|_{T_{U_i}\mathbb{U}(d)} U_i^\dagger = \frac{\kappa}{2N}\sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger - \frac{1}{d}\mathrm{tr}\left[U_j U_i^\dagger - U_i U_j^\dagger\right] I_d \right).$$

□

Then, as a direct application of Proposition 5.3, we have the following corollary.

Corollary 5.4 *Suppose that Hamiltonians H_i are the same:*

$$H_i = H, \quad i = 1, \ldots, N,$$

and let $U_i = U_i(t)$ be a solution to the Cauchy problem (5.10) with $H = 0$. Then, the quantities $e^{-iHt}U_i$ converge for any initial configuration $\{U_i^0\}$.

Remark 5.4 A Lohe flow on $S\mathbb{U}(d)$ with nonidentical oscillators does not have a potential function since $S\mathbb{U}(d)$ is simply connected and there is a periodic solution as in Example 5.2.

5.3.4 A Lohe Flow on $\mathbb{R} \times S\mathbb{U}(d)$

Consider the universal covering space of $\mathbb{U}(d)$:

$$\mathbb{R} \times S\mathbb{U}(d) \to \mathbb{U}(d), \quad (\theta, V) \mapsto e^{\mathrm{i}\theta} V.$$

If we use this representation as a parametrization $U_i = e^{\mathrm{i}\theta_i} V_i$ of each particle described under the Lohe model, $\{\theta_i, V_i\}$ behaves as

$$\mathrm{i}\dot{\theta}_i I_d + \dot{V}_i V_i^\dagger = -\mathrm{i}H_i + \frac{\kappa}{2N}\sum_{j=1}^N \left[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \right].$$

Note that $V_i \in S\mathbb{U}(d)$, so

$$\dot{V}_i V_i^\dagger \in \mathfrak{su}(d), \quad \text{hence} \quad \mathrm{tr}(\dot{V}_i V_i^\dagger) = 0.$$

We may take the trace on both sides and use the above relation in order to split the derivatives for the $\mathbb{R} \times S\mathbb{U}(d)$ model:

$$\begin{aligned} \dot{\theta}_i &= -\frac{1}{d}\mathrm{tr}H_i - \frac{i\kappa}{2dN}\sum_{j=1}^N \mathrm{tr}\left[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \right], \\ \dot{V}_i V_i^\dagger &= -\mathrm{i}H_i + \frac{\mathrm{i}}{d}\mathrm{tr}H_i I_d + \frac{\kappa}{2N}\sum_{j=1}^N \Big[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \\ &\qquad - \frac{1}{d}\mathrm{tr}\left[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \right] I_d \Big]. \end{aligned} \tag{5.14}$$

As a special case, we choose $H_i = -\nu_i I_d$, $\nu_i \in \mathbb{R}$ to rewrite (5.14) as follows.

$$\begin{aligned} \dot{\theta}_i &= \nu_i - \frac{i\kappa}{2dN}\sum_{j=1}^N \mathrm{tr}\left[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \right], \\ \dot{V}_i &= \frac{\kappa}{2N}\sum_{j=1}^N \Big[V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \\ &\qquad - \frac{1}{d}\mathrm{tr}\Big(V_j V_i^\dagger e^{\mathrm{i}(\theta_j - \theta_i)} - V_i V_j^\dagger e^{\mathrm{i}(\theta_i - \theta_j)} \Big) I_d \Big] V_i. \end{aligned} \tag{5.15}$$

We set a real analytic potential on $(\mathbb{R} \times S\mathbb{U}(d))^N$:

$$\begin{aligned}
&\mathcal{V}_2\left(\{\theta_i, V_i\}_{i=1}^N\right)\\
&\quad = -d\sum_{i=1}^N \nu_i\theta_i - \frac{\kappa}{2}NR^2 = -\sum_{i=1}^N d\nu_i\theta_i - \frac{\kappa}{2N}\sum_{i,j=1}^N \mathrm{tr}\Big(e^{\mathrm{i}(\theta_i-\theta_j)}V_iV_j^\dagger\Big),
\end{aligned} \tag{5.16}$$

where R is the order parameter defined by the relation (5.11).

Proposition 5.4 *The Lohe model (5.15) is a gradient flow with analytical potential $\mathcal{V}_2$ in (5.16):*

$$\dot{\theta}_i = -\nabla_{\theta_i}\mathcal{V}_2\Big|_{T\mathbb{R}}, \quad \dot{V}_i = -\frac{\partial \mathcal{V}_2}{\partial V_i}\Big|_{T_{V_i}\mathbb{SU}(d)},$$

where the Riemannian metric of the $\mathbb{SU}(d)$ components of $(\mathbb{R}\times \mathbb{SU}(d))^N$ are given by the inclusion $SU(d) \hookrightarrow M_{d,d}(\mathbb{C})$, and those of the $\mathbb{R}$ components of $(\mathbb{R}\times SU(d))^N$ are given as d times of the standard Riemannian metric of $\mathbb{R}$.

Proof Note that the function $\mathcal{V}_2$ given in (5.16) is defined on $(\mathbb{R}\times M_{d,d}(\mathbb{C}))^N$ viewed as a real analytic manifold, and has gradient

$$\begin{aligned}
-\frac{\partial \mathcal{V}_2}{\partial\theta_i}\Big|_{T\mathbb{R}} &= \frac{1}{d}\left(\nu_i d - \frac{i\kappa}{2N}\sum_{j=1}^N \mathrm{tr}\left[V_jV_i^\dagger e^{\mathrm{i}(\theta_j-\theta_i)} - V_iV_j^\dagger e^{\mathrm{i}(\theta_i-\theta_j)}\right]\right),\\
&= \nu_i - \frac{i\kappa}{2dN}\sum_{j=1}^N \mathrm{tr}\left[V_jV_i^\dagger e^{\mathrm{i}(\theta_j-\theta_i)} - V_iV_j^\dagger e^{\mathrm{i}(\theta_i-\theta_j)}\right],\\
-\frac{\partial \mathcal{V}_2}{\partial V_i}\Big|_{TM_{d,d}(\mathbb{C})} &= \frac{\kappa}{N}\sum_{j=1}^N V_j e^{\mathrm{i}(\theta_j-\theta_i)},
\end{aligned}$$

where we consider $\mathbb{R}^N\times M_{d,d}(\mathbb{C})^N$ with its ordinary metric times d on its $\mathbb{R}^N$ component and with its ordinary metric on its $M_{d,d}(\mathbb{C})^N$ component. We perform an orthogonal projection onto $\mathfrak{su}(d)$ to obtain

$$\begin{aligned}
&-\frac{\partial \mathcal{V}_2}{\partial V_i}\Big|_{T_{V_i}SU(d)} V_i^\dagger\\
&= \frac{\kappa}{2N}\sum_{j=1}^N\left[V_jV_i^\dagger e^{\mathrm{i}(\theta_j-\theta_i)} - V_iV_j^\dagger e^{\mathrm{i}(\theta_i-\theta_j)} - \frac{1}{d}\mathrm{tr}\left[V_jV_i^\dagger e^{\mathrm{i}(\theta_j-\theta_i)} - V_iV_j^\dagger e^{\mathrm{i}(\theta_i-\theta_j)}\right]I_d\right].
\end{aligned}$$

Therefore, we can conclude that the Lohe flow (5.16) is a gradient flow with analytic potential $\mathcal{V}_2$. □

As a direct corollary of Theorem 5.2 and Proposition 5.4, we have the following a priori convergence result.

Corollary 5.5 *Let (θ_i, V_i) be a solution to system (5.15) such that the phase variables θ_i are uniformly bounded. Then, the quantities $U_i = e^{\mathrm{i}\theta_i}V_i$ converge.*

6 Conclusion

In this paper, we have studied the relaxation dynamics of the ensemble of Lohe sphere and matrix oscillators from generic initial configurations. For an ensemble of Lohe oscillators under the same Hamiltonians, we show that exponential position synchronization occurs asymptotically under the a priori assumption that complete position synchronization occurs. Moreover, we also showed that there are only two possible asymptotic states when the initial positions are all different and have a positive order parameter. This kind of conditions reminds us of the corresponding condition for the Kuramoto model in [19]. For distinct constant Hamiltonians, the relaxation dynamics are rather complicated. In this heterogeneous ensemble, we show that if the coupling strength is sufficiently large, then the heterogeneous ensemble tends to the practical $\frac{N-1}{N}$-entrained state asymptotically. However, in the intermediate coupling strength regime, we show that practical partial entrainment states will emerge from an initial configuration with positive order parameter. For the convergence result of the Lohe matrix oscillators, we employ a gradient flow approach. For the Kuramoto model, the gradient flow structure of the Kuramoto model plays a key role in the resolution of the complete synchronization in [19]. However, it is not known whether the Lohe matrix models admit gradient flow formulations with analytical potentials in a general setting. Fortunately, our finding in this paper says that at least for Lohe matrix models with the same Hamiltonian, the Lohe matrix flows can admit a gradient flow formulation so that the Lohe matrix flow on compact group manifolds such as $\mathbb{U}(d)$ and $S\mathbb{U}(d)$ always converge from any initial configuration. Of course, we cannot resolve the gradient flow formulation of the Lohe matrix model with heterogeneous Hamiltonians. We confirmed that it is no longer a gradient flow according to Remark 5.1, 5.3, and 5.4. Hence the asymptotic behaviors of nonidentical oscillators are still open problems for both sphere and matrix model. However, similar behaviors as in the Kuramoto model are observed through numerical simulations. We leave this issue for a future work.

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Appendix A: Verification of Conditions (4.14)

In this section, we show that the choices explicitly verify that the choices (4.22) and (4.23) satisfy the conditions (4.14), that is the conditions

$$\frac{2p}{(2p-1)^2} D_2(\Omega) < \kappa_0, \tag{A.1}$$

$$1 - 2\alpha_0 > \frac{1}{2p}, \tag{A.2}$$

$$p > \frac{1}{2} + \frac{\rho^0}{6}, \tag{A.3}$$

$$\frac{\rho^0 + (1-\alpha_0)}{1 + (1-\alpha_0)} - \frac{1}{N}\left\lfloor \frac{N}{\alpha_0} \cdot \frac{2D_2(\Omega)}{\kappa_0 \rho^0} \right\rfloor > \frac{n_0 - 1}{N}. \tag{A.4}$$

A.1 *N* is even

In this case, recall that we choose

$$\begin{aligned}
&\frac{2m_0}{N} < \rho^0 \le \frac{2(m_0+1)}{N}, \quad n_0 = \left\lfloor \frac{m_0}{2} \right\rfloor + \frac{N}{2} + 1, \\
&\kappa_0 = \frac{1152}{7}\frac{D_2(\Omega)}{(\rho^0)^3}, \quad \alpha_0 = \begin{cases} \rho^0/8, & m_0 \le 1, \\ \frac{m_0}{4N}, & m_0 \ge 2. \end{cases}
\end{aligned} \tag{A.5}$$

Here, n_0 was roughly chosen so that $p = \frac{n_0}{N} \approx \frac{1}{2} + \frac{m_0}{2N} \approx \frac{1}{2} + \frac{\rho^0}{4}$ and hence that (A.3) would hold in most cases. On the other hand, (A.2) suggests that

$$\alpha_0 < \frac{1}{2} - \frac{1}{4p} \approx \frac{1}{2} - \frac{1}{4}\frac{1}{1+(p-1)} \approx \frac{1}{2} - \frac{1}{4}(1-(p-1)) = \frac{p}{4} = \frac{m_0}{4N}.$$

When $m_0 \ge 2$, this happens to give $\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)} > \frac{n_0-1}{N}$, hence we can finally choose κ_0 large enough so as to make (A.1) and (A.4) true. On the other hand, when $m_0 \le 1$, we need to choose a smaller α_0, because then $n_0 = \frac{N}{2} + 1$ and so

$$\frac{\rho^0 + (1-\alpha_0)}{1+(1-\alpha_0)} > \frac{n_0-1}{N} = \frac{1}{2} \quad \Leftrightarrow \quad \alpha_0 < 2\rho^0,$$

while ρ_0 could be very small. The denominator 8 was chosen because $\frac{m_0}{4N} \le \frac{1}{4\cdot 2} = \frac{1}{8}$. The funny coefficient $\frac{1152}{7}$ in the choice for κ_0 will be explained in the demonstration of (A.4) when $m_0 \ge 2$; the condition (A.1) is weaker than (A.4).

- Case A ($m_0 \le 1$): We verify the relations (A.1)–(A.4) one by one. Note that by our choice, we have

$$n_0 = \frac{N}{2} + 1, \quad \rho^0 \le \frac{4}{N}, \quad \alpha_0 = \frac{\rho^0}{8}.$$

- (Verification of (A.1)) We have

$$p = \frac{n_0}{N} = \frac{1}{2} + \frac{1}{N},$$

so

$$\frac{2p}{(2p-1)^2} = \frac{1+(2/N)}{(2/N)^2} = \frac{N^2}{16}\left(4 + \frac{8}{N}\right) \le \frac{1}{(\rho^0)^2}(4+4) = \frac{8}{(\rho^0)^2} \le \frac{1152}{7(\rho^0)^3},$$

hence

$$\frac{2p}{(2p-1)^2} D_2(\Omega) < \frac{1152}{7}\frac{D_2(\Omega)}{(\rho^0)^3} = \kappa_0.$$

- (Verification of (A.2)) We have

$$(1-2\alpha_0)\frac{2n_0}{N} = (1 - \frac{\rho^0}{4})(1 + \frac{2}{N}) \ge (1 - \frac{1}{N})(1 + \frac{2}{N}) > 1, \quad \text{i.e.,} \quad 1 - 2\alpha_0 > \frac{1}{2p},$$

where we use the fact that $N \ge 3$.

- (Verification of (A.3)) We use (A.5) and $\left\lfloor \frac{m_0}{2} \right\rfloor = 0$ to find

$$p=\frac{n_0}{N}=\frac{1}{2}+\frac{1}{N}\geq\frac{1}{2}+\frac{\rho^0}{4}>\frac{1}{2}+\frac{\rho^0}{6}.$$

- (Verification of (A.4)) By the choice of α_0 and κ_0 in (A.5), we have

$$0<\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}=\frac{N}{\rho^0/8}\cdot\frac{7(\rho^0)^2}{576}=\frac{7N\rho^0}{72}\leq\frac{28}{72}<1,\quad\text{i.e.,}\quad\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor=0.$$

Hence, we have

$$\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{1}{N}\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor=\frac{1+7\rho^0/8}{2-\rho^0/8}-0>\frac{1}{2}=\frac{n_0-1}{N}.$$

- Case B ($m_0\geq 2$): Similar to Case A, we check the relations (A.1)–(A.4) one by one.
- (Verification of (A.1)): Since

$$n_0=\left\lfloor\frac{m_0}{2}\right\rfloor+\frac{N}{2}+1>\frac{m_0}{2}+\frac{N}{2},$$

we have

$$2p-1=\frac{2n_0}{N}-1>\frac{m_0}{N}.\tag{A.6}$$

This yields

$$\begin{aligned}\frac{2p}{(2p-1)^2}&=\frac{1}{2p-1}\left(\frac{1}{2p-1}+1\right)<\frac{N}{m_0}\left(\frac{N}{m_0}+1\right)\\&=\frac{N}{2(m_0+1)}\cdot\frac{2(m_0+1)}{m_0}\left(\frac{N}{2(m_0+1)}\cdot\frac{2(m_0+1)}{m_0}+1\right)\\&\leq\frac{1}{\rho^0}\cdot 4\left(\frac{1}{\rho^0}\cdot 4+1\right)\leq\frac{4}{\rho^0}\left(\frac{5}{\rho^0}\right)\leq\frac{1152}{7(\rho^0)^3},\end{aligned}$$

Hence

$$\frac{2p}{(2p-1)^2}D_2(\Omega)<\frac{1152}{7}\frac{D_2(\Omega)}{(\rho^0)^3}=\kappa_0.$$

- (Verification of (A.2)) We use $\alpha_0=\frac{m_0}{4N}$ and (A.6) to obtain

$$(1-2\alpha_0)p=(1-2\alpha_0)\frac{2n_0}{N}>\left(1-\frac{m_0}{2N}\right)\left(1+\frac{m_0}{N}\right)=1+\frac{m_0}{2N}\left(1-\frac{m_0}{N}\right)>1,$$

i.e., we have

$$1-2\alpha_0>\frac{1}{2p}.$$

- (Verification of (A.3)) We again use the relation (A.6) to get

$$p=\frac{n_0}{N}>\frac{1}{2}+\frac{m_0}{2N}=\frac{1}{2}+\frac{2(m_0+1)}{N}\cdot\frac{m_0}{4(m_0+1)}\geq\frac{1}{2}+\rho^0\cdot\frac{2}{4\cdot 3}=\frac{1}{2}+\frac{\rho^0}{6}.$$

- (Verification of (A.4)) By the choice of $\alpha_0=\frac{m_0}{4N}$ and κ_0, we have

$$\begin{aligned}\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}&=\frac{4N^2}{m_0}\cdot\frac{7(\rho^0)^2}{576}=\frac{7N^2(\rho^0)^2}{144m_0}\leq\frac{7\cdot 4(m_0+1)^2}{144m_0}\\&=\frac{7m_0}{36}\left(1+\frac{1}{m_0}\right)^2.\end{aligned}\tag{A.7}$$

On the other hand, we have

$$\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{n_0-1}{N}>\frac{\frac{2m_0}{N}+1-\frac{m_0}{4N}}{1+1-\frac{m_0}{4N}}-\left(\frac{m_0}{2N}+\frac{1}{2}\right)=\frac{4N+7m_0}{8N-m_0}-\frac{N+m_0}{2N}$$
$$=\frac{m_0(7N+m_0)}{2N(8N-m_0)}. \tag{A.8}$$

Thus, we use $m_0 \geq 2$, (A.7) and (A.8) to find

$$\begin{aligned}&\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{n_0-1}{N}-\frac{1}{N}\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{K_0\rho^0}\right\rfloor\\&\quad>\frac{m_0(7N+m_0)}{2N(8N-m_0)}-7\frac{(m_0+1)^2}{36Nm_0}\\&\quad=\frac{14N(5m_0^2-8m_0-4)+25m_0^3+14m_0^2+7m_0}{36Nm_0(8N-m_0)}\\&\quad=\frac{14N(m_0-2)(5m_0+2)+25m_0^3+14m_0^2+7m_0}{36Nm_0(8N-m_0)}>0.\end{aligned} \tag{A.9}$$

i.e. we have

$$\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{1}{N}\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{K_0\rho^0}\right\rfloor>\frac{n_0-1}{N}.$$

The role of the coefficient $\frac{1152}{7}$ appears in the last line of (A.9). For any coefficient ζ replacing $\frac{1152}{7}$, the last line of (A.9) would appear as

$$\frac{Na(m_0)+b(m_0)}{Nm_0(8N-m_0)},$$

where $a(m_0)$ is a quadratic in m_0 and $b(m_0)$ is a cubic in m_0, depending on ζ. If $a(2)<0$, then we may choose N sufficiently large so that this rational function takes a negative value. Hence we must have $a(2)\geq 0$, which forces us to choose $\zeta \geq \frac{1152}{7}$. This choice of ζ happens to make the rational function positive for all valid values of m_0 and N.

A.2 *N* is odd

Recall that we choose m_0 to satisfy

$$\frac{2m_0-1}{N}\leq\rho^0<\frac{2m_0+1}{N}. \tag{A.10}$$

Then, for such m_0, we set

$$n_0=\left\lfloor\frac{m_0}{2}\right\rfloor+\frac{N+1}{2},\quad \kappa_0=\frac{640}{3}\frac{D_2(\Omega)}{(\rho^0)^3},\quad \alpha_0=\begin{cases}\frac{1}{4N}, & m_0\leq 1,\\ \frac{m_0}{8N}, & m_0\geq 2.\end{cases} \tag{A.11}$$

We again claim that the choices (A.10)–(A.11) satisfy (A.1)–(A.4). The rationale for choosing n_0, and α_0 when $m_0 \geq 2$, is similar to A.1. However, when $m_0 \leq 1$, we have $n_0=\frac{N+1}{2}$ and $\frac{n_0-1}{N}=\frac{N-1}{2N}$, which is less than $\frac{1}{2}$, so we can be a little more lenient with our choice of α_0 so as to make $\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}>\frac{n_0-1}{N}$ even when $\rho^0\approx 0$. Again, the coefficient

$\frac{640}{3}$ in the choice of κ_0 is necessitated by (A.4) when $m_0 \geq 2$; the condition (A.1) is weaker than (A.4).

- Case C ($m_0 \leq 1$): Note that the relations (A.11) yield

$$\alpha_0 = \frac{1}{4N}, \quad n_0 = \frac{N+1}{2} \quad \text{and} \quad \rho^0 < \frac{3}{N}. \tag{A.12}$$

- (Verification of (A.1)) We use the relations (A.12) to obtain

$$\frac{2p}{(2p-1)^2} = \frac{2\frac{n_0}{N}}{\left(2\frac{n_0}{N}-1\right)^2} = N(N+1) = \frac{9(N+1)}{N}\cdot\frac{N^2}{9} < 12\cdot\frac{1}{(\rho^0)^2} < \frac{640}{3(\rho^0)^3}.$$

  Thus, we have

$$\frac{2p}{(2p-1)^2}D_2(\Omega) < \frac{640}{3}\frac{D_2(\Omega)}{(\rho^0)^3} = \kappa_0.$$

- (Verification of (A.2)) We again use (A.12) to get

$$(1-2\alpha_0)2p = (1-2\alpha_0)\frac{2n_0}{N} = (1-\frac{1}{2N})\cdot\frac{N+1}{N} = 1+\frac{1}{2N}\left(1-\frac{1}{N}\right) > 1.$$

- (Verification of (A.3)) We use (A.12) to get

$$p = \frac{n_0}{N} = \frac{1}{2}+\frac{1}{2N} > \frac{1}{2}+\frac{\rho^0}{6}.$$

- (Verification of (A.4)) Note that

$$0 < \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0} = 4N^2\cdot\frac{3(\rho^0)^2}{320} = \frac{3N^2(\rho^0)^2}{80} < \frac{27}{80} < 1, \quad \text{i.e.,} \quad \left\lfloor \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor = 0.$$

  This yields

$$\begin{aligned}&\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)} - \frac{1}{N}\left\lfloor \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor - \frac{n_0-1}{N}\\ &\geq \frac{1-\frac{1}{4N}}{2-\frac{1}{4N}} - 0 - \frac{N-1}{2N} = \frac{4N-1}{8N-1} - \frac{N-1}{2N} = \frac{7N-1}{2N(8N-1)} > 0,\end{aligned}$$

  i.e.,

$$\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)} - \frac{1}{N}\left\lfloor \frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor > \frac{n_0-1}{N} > 0.$$

- Case D ($m_0 \geq 2$): By definition of n_0 in (A.11), we have

$$\begin{aligned} n_0 &= \left\lfloor\frac{m_0}{2}\right\rfloor + \frac{N+1}{2} > \frac{m_0}{2} - 1 + \frac{N+1}{2} = \frac{m_0-1}{2} + \frac{N}{2},\\ n_0 &= \left\lfloor\frac{m_0}{2}\right\rfloor + \frac{N+1}{2} \leq \frac{m_0}{2} + \frac{N+1}{2}.\end{aligned} \tag{A.13}$$

  These yield

$$2p = \frac{2n_0}{N} > \frac{m_0-1}{N}+1 \quad \text{and} \quad 2p < \frac{m_0+1}{N}+1. \tag{A.14}$$

- (Verification of (A.1)) It can easily be seen that $x \mapsto \frac{2x}{(2x-1)^2} = \frac{1}{2x-1} \cdot (\frac{1}{2x-1} + 1)$ is decreasing for $x > \frac{1}{2}$. Thus we can use $(\mathrm{A.14})_1$ to see We use (A.10) and (A.14) to see

$$
\begin{aligned}
\frac{2p}{(2p-1)^2} &< \frac{\frac{m_0-1}{N}+1}{(\frac{m_0-1}{N})^2} < \frac{N(m_0-1+N)}{(m_0-1)^2} < \frac{N(m_0-1+N)}{(m_0-1)^2} \cdot \underbrace{\left(\frac{2m_0+1}{N} \cdot \frac{1}{\rho^0}\right)^2}_{>1} \\
&\le \frac{2N^2}{(m_0-1)^2} \cdot \left(\frac{2(m_0-1)+3}{N} \cdot \frac{1}{\rho^0}\right)^2 \\
&= 2 \cdot \left(\frac{2(m_0-1)+3}{m_0-1}\right)^2 \cdot \frac{1}{(\rho^0)^2} \\
&\le 2 \cdot 5^2 \cdot \frac{1}{(\rho^0)^2} < \frac{640}{3(\rho^0)^3},
\end{aligned}
$$

where in the third inequality, we used (A.10) to find the relation:

$$
\frac{2m_0-1}{N} \le \rho^0 \le 1 \quad \Rightarrow \quad m_0 - 1 < 2m_0 - 1 \le N.
$$

Thus, we have

$$
\frac{2p}{(2p-1)^2} D_2(\Omega) < \frac{640}{3} \frac{D_2(\Omega)}{(\rho^0)^3} = \kappa_0.
$$

- (Verification of (A.2)) We use $\alpha_0 = \frac{m_0}{8N}$ and (A.14) to obtain

$$
\begin{aligned}
(1-2\alpha_0)2p = (1-2\alpha_0)\frac{2n_0}{N} &> (1-\frac{m_0}{4N})(1+\frac{m_0-1}{N}) = 1 + \frac{3m_0-1}{4N} - \frac{m_0(m_0-1)}{4N^2} \\
&\ge 1 + \frac{m_0}{2N} - \frac{m_0(m_0-1)}{4N^2} = 1 + \frac{m_0}{4N}\left(2 - \frac{m_0-1}{N}\right) > 1,
\end{aligned}
$$

where in the last inequality, we also used the following relation from (A.14):

$$
2 \ge 2p > \frac{m_0-1}{N} + 1 \quad \Longrightarrow \quad N > m_0 - 1.
$$

Hence, we have

$$
1 - 2\alpha_0 > \frac{1}{2p}.
$$

- (Verification of (A.3)) We use $\lfloor \frac{m_0}{2} \rfloor \ge \frac{m_0-1}{2}$ and (A.11) to find

$$
n_0 \ge \frac{m_0+N}{2}.
$$

This yields

$$
p = \frac{n_0}{N} \ge \frac{1}{2} + \frac{m_0}{2N} = \frac{1}{2} + \frac{2m_0+1}{N} \cdot \frac{m_0}{2(2m_0+1)} > \frac{1}{2} + \rho^0 \cdot \frac{2}{2 \cdot 5} > \frac{1}{2} + \frac{\rho^0}{6}.
$$

- (Verification of (A.4)) We use $\alpha_0 = \frac{m_0}{8N}$ and $\kappa_0 = \frac{640}{3}\frac{D_2(\Omega)}{(\rho^0)^3}$ to obtain

$$
\frac{N}{\alpha_0} \cdot \frac{2D_2(\Omega)}{\kappa_0 \rho^0} = \frac{8N^2}{m_0} \cdot \frac{3(\rho^0)^2}{320} = \frac{3N^2(\rho^0)^2}{40m_0} < \frac{3(2m_0+1)^2}{40m_0}. \tag{A.15}
$$

On the other hand, we have

$$\begin{aligned}\frac{\rho^0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{n_0-1}{N} &\geq \frac{\frac{2m_0-1}{N}+1-\frac{m_0}{8N}}{1+1-\frac{m_0}{8N}}-\frac{m_0-1+N}{2N}\\ &=\frac{8N+15m_0-8}{16N-m_0}-\frac{m_0-1+N}{2N}\\ &=\frac{m_0(15N+m_0-1)}{2N(16N-m_0)}.\end{aligned}\tag{A.16}$$

Thus, it follows from (A.15), (A.16) and $m_0 \geq 2$ that we have

$$\begin{aligned}&\frac{\rho_0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{n_0-1}{N}-\frac{1}{N}\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{\kappa_0\rho^0}\right\rfloor\\ &\quad>\frac{m_0(15N+m_0-1)}{2N(16N-m_0)}-\frac{3(2m_0+1)^2}{40Nm_0}\\ &\quad=\frac{12N(9m_0^2-16m_0-4)+32m_0^3-8m_0^2+3m_0}{40m_0N(16N-m_0)}\\ &\quad=\frac{12N(m_0-2)(9m_0+2)+8m_0^2(4m_0-1)+3m_0}{40m_0N(16N-m_0)}\\ &\quad>0.\end{aligned}\tag{A.17}$$

This yields

$$\frac{\rho_0+(1-\alpha_0)}{1+(1-\alpha_0)}-\frac{1}{N}\left\lfloor\frac{N}{\alpha_0}\cdot\frac{2D_2(\Omega)}{K_0\rho_0}\right\rfloor>\frac{n_0-1}{N}.$$

The role of the coefficient $\frac{640}{3}$ appears in the penultimate term in (A.17), and the reason for this choice is the same as Case B.

Appendix B: Linear Instability of the Bipolar State

In this section, we discuss the linear stability of the bipolar state for the Lohe model with identical oscillators:

$$\dot{x}_i=\Omega x_i+\frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k-\frac{\langle x_i,x_k\rangle}{\langle x_i,x_i\rangle}x_i\right),\quad i=1,\ldots,N.\tag{B.1}$$

By replacing the variables x_i by $e^{-\Omega t}x_i$, we may assume $\Omega=0$, so that

$$\dot{x}_i=\frac{\kappa}{N}\sum_{k=1}^{N}\left(x_k-\frac{\langle x_i,x_k\rangle}{\langle x_i,x_i\rangle}x_i\right),\quad i=1,\ldots,N.\tag{B.2}$$

By the $O(d)$-symmetry of the above model, i.e., by replacing y_i by Ay_i for some $A\in O(d)$, and by permuting the indices, we may assume the bipolar state $(y_1,\ldots,y_N)$ in question consists of the following:

$$y_i=\begin{cases}(0,\ldots,0,1), & 1\leq i\leq r,\\ (0,\ldots,0,-1), & r<i\leq N,\end{cases}$$

where $r = 0, \ldots, N$ describes the number of oscillators on the north pole. We consider a perturbation $\{x_i\}$ of $\{y_i\}$:

$$x_i = \begin{cases} (x_i^1, \ldots, x_i^d, \sqrt{1-(x_i^1)^2 - \ldots - (x_i^d)^2}), & 1 \le i \le r, \\ (x_i^1, \ldots, x_i^d, -\sqrt{1-(x_i^1)^2 \ldots - (x_i^d)^2}), & r < i \le N, \end{cases}$$

where $|x_i^a| \ll 1/\sqrt{d}$ for $i = 1, \ldots, N$, $a = 0, \ldots, d-1$. Discarding quadratic terms, we can see that

$$\langle x_i, x_j \rangle = \begin{cases} 1 + O(\|\mathbf{x}\|^2), & 1 \le i, j \le r \text{ or } r < i, j \le N, \\ -1 + O(\|\mathbf{x}\|^2), & 1 \le i \le r < j \le N \text{ or } 1 \le j \le r < i \le N, \end{cases}$$

where $\|\mathbf{x}\|^2 := \sum_{i=1}^N \sum_{a=0}^{d-1} (x_i^a)^2$. Hence, if we project (B.2) onto the first d variables, we obtain the system of equations

$$\dot{x}_i^a = \frac{\kappa}{N}\sum_{k=1}^N x_k^a - \frac{\kappa(2r-N)}{N} x_i^a + O(\|\mathbf{x}\|^3), \quad i = 1, \ldots, r, \quad a = 0, \ldots, d-1. \tag{B.3}$$

$$\dot{x}_i^a = \frac{\kappa}{N}\sum_{k=1}^N x_k^a - \frac{\kappa(N-2r)}{N} x_i^a + O(\|\mathbf{x}\|^3), \quad i = r+1, \ldots, N, \quad a = 0, \ldots, d-1. \tag{B.4}$$

Note that we do not need the $(d+1)$th component, since it is determined by the first d components. Linearizing (B.3), (B.4) yields

$$\dot{z}_i^a = \frac{\kappa}{N}\sum_{k=1}^N z_k^a - \frac{\kappa(2r-N)}{N} z_i^a, \quad i = 1, \ldots, r, \quad a = 0, \ldots, d-1. \tag{B.5}$$

$$\dot{z}_i^a = \frac{\kappa}{N}\sum_{k=1}^N z_k^a - \frac{\kappa(N-2r)}{N} z_i^a, \quad i = r+1, \ldots, N, \quad a = 0, \ldots, d-1. \tag{B.6}$$

Then our result on linear stability is as follows:

Theorem B.1 *Let* (B.5), (B.6) *be the linear approximation to* (2.7) *at the bipolar state* $(y_1, \ldots, y_N)$.

(1) *When* $r = 0$ *or* $r = N$, *i.e., near the completely synchronized state, the linear system* (B.5), (B.6) *is diagonalizable with eigenvalue* 0 *with multiplicity* d, *and* -1 *with multiplicity* $d(N-1)$. *Hence the linear stability is indeterminate.*
(2) *When* $1 \le r \le N-1$, *i.e., near a bipolar state that is not the completely synchronized state, the linear system* (B.5), (B.6) *is diagonalizable with eigenvalue* 0 *with multiplicity* d, 1 *with multiplicity* d, $\frac{2r-N}{N}$ *with multiplicity* $d(N-r-1)$, $\frac{N-2r}{N}$ *with multiplicity* $d(r-1)$; *hence it is linearly unstable.*

Remark B.1 In both cases, 0 is an eigenvalue with multiplicity at least d. This follows from the fact that (2.7) possesses an $O(d)$-symmetry, and from the fact that the orbit of the corresponding action of $O(d)$ on the bipolar states in $(S^d)^N$ has dimension d.

Proof This system is just the linear ODE

$$\mathbf{z} = \kappa \Lambda \mathbf{z},$$

where

$$\mathbf{z}^T = (z_1^0, \ldots, z_N^0, \ldots, z_1^{d-1}, \ldots, z_N^{d-1}),$$

and Λ is the block matrix

$$\Lambda = diag(\underbrace{\Lambda_0, \ldots, \Lambda_0}_{\text{d copies}}).$$

Here, Λ_0 is defined as

$$\Lambda_0 := \frac{1}{N}A - \frac{N-2r}{N}I_N - 2\frac{2r-N}{N}B$$

where A is the $N \times N$-matrix all of whose entries are 1, and B is the $N \times N$-matrix in block form

$$B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$ unit matrix, and the 0's represent zero matrices of the appropriate dimension. According to Corollary C.1 in the next section, Λ is diagonalizable, with eigenvalues

$$\begin{cases} 0 \text{ with multiplicity } d, \ -1 \text{ with multiplicity } d(N-1) & \text{if } r = 0, N, \\ 0 \text{ with multiplicity } d, \ 1 \text{ with multiplicity } d, \\ \frac{2r-N}{N} \text{ with multiplicity } d(N-r-1), \ \frac{N-2r}{N} \text{ with multiplicity } d(r-1) & \text{if } 1 \le r \le N-1. \end{cases}$$

$\square$

Appendix C: Calculation of Some Eigenvalues

Lemma C.1 *Let k be a field of characteristic zero. Let $N \in \mathbb{N}$, $m \in \mathbb{Z}$ such that $0 \le r \le N$. Let A be the $N \times N$-matrix all of whose entries are $1 \in k$, and let B be the $N \times N$-matrix in the block form*

$$B = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$ unit matrix over k, and the 0's represent zero matrices of the appropriate dimension. Then for any $\mu \in k$, the matrix $A - \mu B$ has characteristic polynomial

$$\phi(t) = \begin{cases} t^{N-1}(t-N) & \text{if } r = 0, \\ t^{N-r-1}(t+\mu)^{r-1}(t^2 + (\mu - N)t + \mu(r-N)) & \text{if } 1 \le r \le N-1, \\ (t+\mu)^{N-1}(t + (\mu - N)) & \text{if } r = N, \end{cases} \tag{C.1}$$

over k.

Proof • Case A ($r = 0$): Then $B = 0$, so we are searching for the characteristic polynomial of A. If $N = 1$, then it is immediate that the characteristic polynomial is $\phi(t) = t - 1$, as asserted above. So we assume $N \ge 2$. Then it is easy to verify that the relation

$$A^2 - NA = 0,$$

holds,as well as the relations

$$A \ne 0, \quad A - NI_N \ne 0.$$

Hence the minimal polynomial $m(t)$ of A is $m(t) = t^2 - Nt = t(t - N)$, and since the minimal polynomial and the characteristic polynomial must share the same irreducible factors, we conclude that $\phi(t)$ must be of the form

$$\phi(t) = t^{N-a}(t - N)^a,$$

where $a \in \mathbb{Z}$ is an integer $1 \le a \le N - 1$. This allows us to calculate, using the binomial theorem,

$$(\text{coefficient of } t^{N-1} \text{ in } \phi(t)) = -aN,$$

but we can also directly calculate

$$(\text{coefficient of } t^{N-1} \text{ in } \phi(t)) = -\text{tr}(A) = -N.$$

Hence we must have $a = 1$, i.e., $\phi(t) = t^{N-1}(t - N)$, as asserted.

- Case B ($r = N$): Then $B = I_N$, the $N \times N$ identity matrix, so

$$\phi(t) = det(tI_N - (A - \mu B)) = det((t + \mu)I_N - A) = (t + \mu)^{N-1}(t + \mu - N),$$

by using the result of Case A. This agrees with our assertion (C.1).

- Case C ($1 \le r \le N - 1$): Without loss of generality, we may consider μ to be a variable over k, i.e., that the field of rational functions $k(\mu)$ is a transcendental extension of k, and consider $A - \mu B$ as a matrix over $k(\mu)$. Then we must verify the relation (C.1) over the polynomial ring $k(\mu)[t]$. First, it is easy to verify the relations

$$A^2 = NA, \quad B^2 = B, \quad ABA = rA. \tag{C.2}$$

Using these relations, we proceed to verify the following by direct expansion:

$$\begin{aligned}
(A - \mu B)^2 &= NA + \mu^2 B - \mu AB - \mu BA,\\
(A - \mu B)^3 &= (N^2 - \mu r)A - \mu^3 B + (\mu^2 - \mu N)AB + (\mu^2 - \mu N)BA + \mu^2 BAB,\\
(A - \mu B)^4 &= (N^3 - 2N\mu r + r\mu^2)A + \mu^4 B + (-\mu^3 + \mu^2 N - \mu N^2 + \mu^2 r)AB\\
&\quad + (-\mu^3 + \mu^2 N - \mu N^2 + \mu^2 r)BA + (-2\mu^3 + \mu^2 N)BAB.
\end{aligned}$$

This permits us to verify the relation

$$\begin{aligned}
&(A - \mu B)^4 + (2\mu - N)(A - \mu B)^3\\
&\quad + (\mu^2 - 2\mu N + \mu r)(A - \mu B)^2 - \mu^2(N - r)(A - \mu B) = 0.
\end{aligned}$$

(The rationale for the preceding computation is that, the relations (C.2) force the powers of $A - \mu B$ to be linear combinations of the five terms A, B, AB, BA, BAB only, and hence that five powers of $A - \mu B$ suffice to give a linear relation. Luckily, we needed only four.) Hence the minimal polynomial $m(t)$ of $A - \mu B$ over $k(\mu)$ is a divisor of

$$\begin{aligned}
&t^4 + (2\mu - N)t^3 + (\mu^2 - 2\mu N + \mu r)t^2 - \mu^2(N - r)t\\
&= t(t + \mu)(t^2 + (\mu - N)t + \mu(r - N)).
\end{aligned}$$

But $t^2 + (\mu - N)t + \mu(r - N)$ is irreducible over $k(\mu)$ if and only if it factors into linear factors, that is to say, if its discriminant (as a quadratic polynomial in t)

$$(\mu - N)^2 - 4\mu(r - N) = \mu^2 - 2(2r - N)\mu + N^2$$

is a square in $k(\mu)$, which holds if and only if its discriminant (as a quadratic polynomial in μ)

$$4(2r-N)^2 - 4N^2 = 16r(r-N) = 0$$

is zero in k. Since k has characteristic zero, this is equivalent to

$$r = 0 \text{ or } r = N \quad \text{in } \mathbb{Z}.$$

These cases were already excluded. Hence $t^2 + (\mu - N)t + \mu(r-N)$ is irreducible over $k(\mu)$. Since $m(t)$ and $\phi(t)$ share the same irreducible factors, we conclude that

$$\phi(t) = t^{N-a-2b}(t+\mu)^a(t^2 + (\mu - N)t + \mu(r-N))^b,$$

where a, b are nonnegative integers such that $a + 2b \leq N$. As in Case A, we use the binomial theorem to calculate

$$(\text{coefficient of } t^{N-1} \text{ in } \phi(t)) = a\mu + b(\mu - N),$$

and compare this with

$$(\text{coefficient of } t^{N-1} \text{ in } \phi(t)) = -\mathrm{tr}(A) = r\mu - N.$$

Thus

$$a\mu + b(\mu - N) = r\mu - N$$

holds in $k(\mu)$, and since k has characteristic zero, we must have

$$a = r-1, \; b = 1 \quad \text{in } \mathbb{Z},$$

i.e.,

$$\phi(t) = t^{N-r-1}(t+\mu)^{r-1}(t^2 + (\mu - N)t + \mu(r-N)),$$

as asserted. □

Remark C.1 We will only be interested in the case $\mu = 4r - 2N$. The reason for introducing μ as a variable is that, in the case $1 \leq r \leq N-1$, the calculation of the characteristic polynomial becomes easier once we consider μ as a transcendental element over k.

Corollary C.1 *The matrix*

$$\Lambda_0 := \frac{1}{N}A - \frac{N-2r}{N}I_N - 2\frac{2r-N}{N}B$$

over $\mathbb{R}$ is diagonalizable, with eigenvalues

$$\begin{cases} 0 \textit{ with multiplicity } 1, \quad -1 \textit{ with multiplicity } N-1 & \textit{if } r = 0, N, \\ 0 \textit{ with multiplicity } 1, \; 1 \textit{ with multiplicity } 1, & \\ \frac{2r-N}{N} \textit{ with multiplicity } N-r-1, \; \frac{N-2r}{N} \textit{ with multiplicity } r-1 & \textit{if } 1 \leq r \leq N-1. \end{cases}$$

Proof Diagonalizability follows from the symmetry of Λ_0 and the spectral theorem for real symmetric matrices. Let $\mu = 2(2r-N)$. Then we can see that

$$\begin{aligned} t^2 + (\mu - N)t + \mu(r-N) &= t^2 + (4r-3N)t + 2(2r-N)(r-N) \\ &= (t + 2r - N)(t + 2r - 2N), \end{aligned}$$

so (C.1) becomes

$$\phi(t) = \begin{cases} t^{N-1}(t-N) & \text{if } r=0, \\ t^{N-r-1}(t+4r-2N)^{r-1}(t+2r-N)(t+2r-2N) & \text{if } 1 \le r \le N-1, \\ (t+2N)^{N-1}(t+N) & \text{if } r=N, \end{cases} \quad \text{(C.3)}$$

(we used $\mu = 2N$ when $r = N$). Thus, the desired characteristic polynomial is

$$\begin{aligned} & det\left[tI_N - \left(\frac{1}{N}A - \frac{N-2r}{N}I_N - 2\frac{2r-N}{N}B\right)\right] \\ &= \frac{1}{N^N} det\left[NtI_N - (A-(N-2r)I_N - 2(2r-N)B)\right] \\ &= \frac{1}{N^N} det\left[(Nt+N-2r)I_N - (A-\mu B)\right] \\ &= \frac{1}{N^N}\phi(Nt+N-2r) \\ &= \begin{cases} \frac{1}{N^N}(Nt+N-2\cdot 0)^{N-1}(Nt+N-2\cdot 0-N) & \text{if } r=0, \\ \frac{1}{N^N}(Nt+N-2r)^{N-r-1}((Nt+N-2r)+(4r-2N))^{r-1} & \\ \cdot((Nt+N-2r)+2r-N)((Nt+N-2r)+(2r-2N)) & \text{if } 1 \le r \le N-1, \\ \frac{1}{N^N}(Nt+N-2\cdot N+2N)^{N-1}((Nt+N-2\cdot N)+N) & \text{if } r=N, \end{cases} \\ &= \begin{cases} t(t+1)^{N-1} & \text{if } r=0, \\ t(t-1)(t+\frac{N-2r}{N})^{N-r-1}((t+\frac{2r-N}{N}))^{r-1} & \text{if } 1 \le r \le N-1, \\ t(t+1)^{N-1} & \text{if } r=N. \end{cases} \end{aligned}$$

This completes the proof.

(The diagonalizability of Λ_0, which is equivalent to the diagonalizability of $A - \mu B$, can also be shown as follows. If $r = 0$ or N, we use the diagonalizability of A. If $1 \le r \le N-1$, we use the fact that the minimal polynomial of $A - \mu B$ divides $t(t+\mu)(t^2 + (\mu - N)t + \mu(r-N)) = t(t+4r-2N)(t+2r-N)(t+2r-2N)$, which consists of distinct linear factors, and hence the minimal polynomial of $A - \mu B$ must factor into distinct linear factors.) □

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