J Stat Phys (2016) 163:411–439
DOI 10.1007/s10955-016-1481-4

On the Emergence and Orbital Stability of Phase-Locked States for the Lohe Model

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Received: 13 November 2015 / Accepted: 9 February 2016 / Published online: 23 February 2016

Abstract We study the emergence and orbital stability of phase-locked states of the Lohe model, which was proposed as a non-abelian generalization of the Kuramoto phase model for synchronization. Lohe introduced a first-order system of matrix-valued ordinary differential equations for quantum synchronization and numerically observed the asymptotic formation and orbital stability of phase-locked states of the Lohe model. In this paper, we provide an analytical framework to confirm Lohe's observations of emergent phase-locked states. This extends earlier special results on lower dimensions to any finite dimension. For the construction and orbital stability of phase-locked states, we introduce Lyapunov functions to measure the ensemble diameter and dissimilarity between two Lohe flows, and using the time-evolution estimates of these Lyapunov functions, we present an admissible set of initial states, and show that an admissible initial state leads to a unique phase-locked asymptotic state.

Keywords Emergence · Kuramoto model · Lohe model · Quantum network · Quantum synchronization

Mathematics Subject Classification 82C10 · 82C22 · 35B37

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1 Introduction

Emergent phenomena are ubiquitous in complex systems, e.g., cardiac pacemaker cells [29], biological clocks in the brain [35], Josephson junction arrays [1], [30], and flashing of fireflies [4]. These complex systems often exhibit synchronous and anti-synchronous behaviors in their dynamics [1]. Recently, the collective dynamics of complex systems have received much attention in diverse scientific and engineering disciplines [1,4,14,29–31,34,35] because of their possible applications. Among many synchronization models, the Kuramoto model [21], [22] is the most widely used for the synchronous dynamics of weakly coupled limit-cycle oscillators. In this paper, we are interested in Lohe's matrix valued ordinary differential equation (ODE) model. This model corresponds to a non-abelian generalization of the Kuramoto model and a mathematical model for quantum synchronization [23]. More precisely, we consider a quantum network [20,36,37] consisting of N quantum nodes and quantum channels connecting all possible pairs of quantum nodes. Each quantum node can be viewed as a component of a physical system interacting via quantum channels. For instance, atoms at nodes can have effect spin-spin interactions generated by a single photon pulse traveling along the channels (see [20] for a detailed explanation). In [23,24], Lohe proposed a first-order system of matrix valued ODEs for synchronization over quantum networks and studied the existence of phase-locked states of the proposed model based on numerical simulations. Let U_i and $U_i^\dagger$ be a $d \times d$ unitary matrix and its Hermitian conjugate, and H_i a $d \times d$ Hermitian matrix whose eigenvalues correspond to the natural frequencies of the i-th Lohe oscillator. Then the Lohe model reads as follows:

$$\mathrm{i}\dot{U}_i U_i^\dagger = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(U_i U_j^\dagger - U_j U_i^\dagger\right), \quad i = 1, \ldots, N, \tag{1}$$

where K is a nonnegative coupling strength. In low dimensions, i.e., $d = 1, 2$, (1) can be reduced to the Kuramoto model [21,22] and a consensus model [25] on the three-sphere $\mathbb{S}^3$, respectively (see Sect. 2.1.2 for details). Because of the self-adjoint property of the right-hand side of (1), $U_i U_i^\dagger$ is preserved along the Lohe flow (see Lemma 1); thus, the ℓ^2-norm of U_i is bounded a priori. Therefore, the local flow of (1), whose existence is guaranteed by standard Cauchy–Lipschitz theory, can be extended to the global flow using the continuation arguments. Our main interest in the Lohe model lies in the formation of phase-locked states that arise from the asymptotic dynamics of the Lohe flow. As discussed in [1,2,11,12,14], the existence and stability of phase-locked states are important topics in the nonlinear dynamics of the Kuramoto model. Other interesting issues, such as dependence of emergent behavior on initial data and critical coupling strength for the emergent property will be discussed in Remark 6.

In this paper, we are interested in an analytical framework that guarantees the emergence and stability of phase-locked states from the initial states via the Lohe flow (1). In nonlinear dynamical systems theory, one of the central problems other than well-posedness is to understand the large-time dynamics such as existence of equilibria and limit-cycles, and their stability. Thus, the first step in this direction is to identify possible limiting dynamics of a given dynamical system. In our situation, this is equivalent to solving the corresponding nonlinear algebraic system, and then study some kind of structural similarity (orbital stability). As discussed in [23,24], phase-locked states $\{U_i\}$ for (1) correspond to special limit-cycles with the form of $U_i = U_i^\infty e^{-\mathrm{i}\Lambda t}$, where $U_i^\infty \in \mathcal{U}(d)$ is the unitary matrix, and Λ is a constant $d \times d$ Hermitian matrix satisfying the algebraic constraints:

$$U_i^\infty \Lambda U_i^{\infty\dagger} = H_i - \frac{\mathrm{i}K}{2N}\sum_{k=1}^N \left(U_i^\infty U_k^{\infty\dagger} - U_k^\infty U_i^{\infty\dagger}\right), \quad 1 \le i \le N. \tag{2}$$

Thus, to resolve the existence problem of phase-locked states, we need to determine the set of pairs $\{(U_i^\infty, \Lambda)\}$ satisfying algebraic system (2). However, this is often very difficult to solve directly or even impossible. Another interesting issue is to study the stability of phase-locked states. Here we adopt the concept of orbital stability which roughly says the preservation of nonlinear structures (i.e., shapes) under small perturbations (see Sect. 4.2.2). Note that the stability issue can be studied in a priori setting without knowing the existence of phase-locked states. The orbital stability implies the uniqueness of phase-locked states.

In this paper, we will adopt a time-asymptotic (or flow-based) approach to construct desired phase-locked states and their orbital stability by deriving differential inequalities for some Lyapunov functionals measuring closeness to phase-locked states and structural similarity between phase-locked states. This time-asymptotic approach turns out to be very useful for the Kuramoto flow, as noted in [11]. In this flow-based approach, we do not solve the nonlinear "*algebraic system* (2)" directly, but we instead construct phase-locked states $\{(U_i^\infty, \Lambda)\}$ as asymptotic states for the time-dependent system (1) starting from some admissible class of initial data under suitable conditions on the parameters H_i and K. The existence of phase-locked states, its structural information and stability follows naturally from this time-asymptotic approach simultaneously. In the sequel, we briefly explain our main ideas in the time-asymptotic approach. We first introduce scalar quantities measuring diameters of the sets $\{U_i\}$ and $\{H_i\}$:

$$D(U) := \max_{1\le i,j\le N} \|U_i - U_j\|, \qquad D(H) := \max_{1\le i,j\le N} \|H_i - H_j\|.$$

where $\|A\| = \sqrt{\mathrm{tr}(AA^\dagger)}$ is the Frobenius norm of the matrix A. By straightforward calculations (see Lemma 2), we derive key differential inequalities for $D(U)$:

$$\begin{aligned}
&-D(H) - K D(U)\left(1 + \frac{D(U)^2}{2}\right) \\
&\quad \le \frac{dD(U)}{dt} \le D(H) - K D(U)\left(1 - \frac{D(U)^2}{2}\right), \quad \text{a.e., } t \in (0, \infty).
\end{aligned} \tag{3}$$

These differential inequalities yield the exponential decay e^{-cKt} of $D(U)$ for the ensemble of identical oscillators and the existence of a trapping set for the ensemble of nonidentical oscillators, respectively (see Theorem 1 and Lemma 3). On the other hand, for the orbital stability of phase-locked states, we introduce another scalar-valued functional $d(U(t), \tilde{U}(t))$ for two solutions U and $\tilde{U}$:

$$d(U(t), \tilde{U}(t)) := \max_{1\le i,j\le N} \|U_i(t)U_j^\dagger(t) - \tilde{U}_i(t)\tilde{U}_j^\dagger(t)\|, \quad t > 0.$$

For one-dimensional setting $d = 1$, it follows from the relation $U_i = e^{-\mathrm{i}\theta_i}$ in (9) that we have

$$d(U(t), \tilde{U}(t)) = \max_{1\le i,j\le N} \left|1 - e^{-i(\tilde{\theta}_{ij}(t) - \theta_{ij}(t))}\right|, \qquad \theta_{ij} := \theta_i - \theta_j.$$

This yields

$$\begin{aligned}
&\lim_{t\to\infty} d(U(t), \tilde{U}(t)) = 0 \\
&\quad \Longleftrightarrow \quad \lim_{t\to\infty} (\tilde{\theta}_{ij}(t) - \theta_{ij}(t)) = 2n\pi, \quad n \in \mathbb{Z}, \quad 1 \le i, j \le N \\
&\quad \Longleftrightarrow \quad \Theta(t) \text{ and } \tilde{\Theta}(t) \text{ have the same asymptotic structure,}
\end{aligned}$$

where $\Theta = (\theta_1, \cdots, \theta_N)$ and $\tilde{\Theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_N)$. Hence the orbital stability of phase-locked states is reduced to the zero convergence of the functional $d(U(t), \tilde{U}(t))$. In Appendix B, we derive the differential inequalities: for some positive constants c_1 and c_2,

$$-c_1 K d(U, \tilde{U}) \le \frac{d}{dt} d(U, \tilde{U}) \le -c_2 K d(U, \tilde{U}), \quad \text{a.e. } t \in (0, \infty).$$

under suitable conditions on K and initial data (see Theorem 2). This clearly implies the exponential convergence of $d(U(t), \tilde{U}(t))$.

$$d(U(t), \tilde{U}(t)) \le C_0 e^{-c_2 K t} \quad \text{as } t \to \infty. \tag{4}$$

For the ensemble of nonidentical oscillators, we combine the estimates (3) and (4) to derive the existence of $\{(U_i^\infty, \Lambda)\}$ satisfying (2) in the asymptotic process (see Theorem 3).

The rest of this paper is organized as follows. In Sect. 2, we briefly introduce basic properties, special cases, and phase-locked states of the Lohe model. In Sect. 3, we define the ensemble diameter and derive Gronwall's differential inequality for the ensemble diameter, which is essential for our analysis in the following sections. In Sect. 4, we present our main results on the existence of phase-locked states under the two frameworks stated in terms of the coupling strength and the size of the ensemble diameter for identical and nonidentical oscillators. In our frameworks, the phase-locked state emerges asymptotically and is unique up to right multiplication by a unitary matrix. In Sect. 5, we summarize our main results and discuss future directions of research. In Appendix A, we provide a detailed derivation of an identity that helps describe the dynamics of the ensemble diameter. Finally, Appendix B is devoted to the derivation of Gronwall-type inequalities for the functional $d(U, \tilde{U})$ measuring the mismatch between two solutions to the Lohe flow.

2 Preliminaries

In this section, we study basic properties and a sufficient framework leading to the uniform boundedness of the solution to the Lohe matrix model.

2.1 The Lohe Model

Consider a complete quantum network where all nodes are connected with equal constant weight, and assume that quantum oscillators are located at each node with all-to-all couplings. Let U_i and $U_i^\dagger$ be the complex unitary $d \times d$ matrix and the corresponding Hermitian conjugate of the i-th Lohe oscillator, respectively. Let H_i be a given $d \times d$ Hermitian matrix whose eigenvalues correspond to the natural frequencies of the Lohe oscillator at node i. In two series of papers [23,24], the Australian physicist Max Lohe proposed a non-abelian matrix evolution model for the dynamics of U_i, which is given by the following first-order matrix-valued ODE system:

$$
\begin{aligned}
&\mathrm{i}\dot{U}_i U_i^\dagger = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^N \left(U_i U_j^\dagger - U_j U_i^\dagger\right), \quad i = 1, \ldots, N, \quad t > 0, \\
&U_i(0) = U_i^0,
\end{aligned} \tag{5}
$$

where the initial matrix U_i^0 is in the unitary group $\mathcal{U}(d)$, H_i is a $d \times d$ Hermitian matrix whose real eigenvalues correspond to the natural frequencies of the oscillator at node i, and K is the uniform, nonnegative coupling strength. In the sequel, we will see that the flow generated by (1) preserves the unitarity of U_i along the dynamics; hence, all components of U_i are bounded a priori by the unity, and the matrix valued ODE in (1) can be regarded as a d^2-coupled system of first-order ODEs. Thus, the standard Cauchy–Lipschitz theory for local solutions and the a priori uniform bound of the components u_{ij} yield a unique global solution to (5).

2.1.1 Basic Properties

In this subsection, we list some invariant properties of (1).

Lemma 1 [23,24] *The following statements hold:*

1. *Let $\{U_i\}$ be a solution to (1) with initial data $\{U_i^0\}$. Then, $U_i U_i^\dagger$ is conserved along the Lohe flow:*

$$U_i(t)U_i^\dagger(t) = U_i^0 U_i^{\dagger 0}, \quad t \geq 0, \ 1 \leq i \leq N.$$

2. *The Lohe system (1) is invariant under right-translation by a unitary matrix in the sense that if $L \in U(d)$ and $V_i = U_i L$, then V_i satisfies*

$$
\begin{aligned}
&\mathrm{i}\dot{V}_i V_i^\dagger = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^N \left(V_i V_j^\dagger - V_j V_i^\dagger\right), \quad i = 1, \ldots, N, \quad t > 0, \\
&V_i(0) = U_i^0 L.
\end{aligned} \tag{6}
$$

Proof Let U_i be a solution to (5) with initial data $U_i^0 \in \mathcal{U}(d)$.

(i) We take the Hermitian conjugate of (5) using $H_i^\dagger = H_i$ to obtain

$$
\begin{aligned}
&-\mathrm{i}U_i\dot{U}_i^\dagger = H_i^\dagger + \frac{\mathrm{i}K}{2N}\sum_{j=1}^N \left(U_j U_i^\dagger - U_i U_j^\dagger\right), \quad \text{or equivalently} \\
&\mathrm{i}U_i\dot{U}_i^\dagger = -H_i + \frac{\mathrm{i}K}{2N}\sum_{j=1}^N \left(U_i U_j^\dagger - U_j U_i^\dagger\right).
\end{aligned} \tag{7}
$$

Adding (5) and (7) and dividing the resulting relation by i yields

$$\frac{d}{dt}(U_i U_i^\dagger) = 0, \qquad \text{i.e.,} \qquad U_i(t)U_i^\dagger(t) = U_i(0)U_i^\dagger(0) = I_d, \quad t \geq 0.$$

(ii) For $L \in U(d)$, we set $V_i := U_i L$. Then V_i satisfies

$$
\begin{aligned}
&\dot{V}_i V_i^\dagger = (\dot{U}_i L)(U_i L)^\dagger = \dot{U}_i L L^\dagger U_i^\dagger = \dot{U}_i U_i^\dagger, \\
&V_j V_i^\dagger = (U_j L)(U_i L)^\dagger = U_j L L^\dagger U_i^\dagger = U_j U_i^\dagger, \quad \text{similarly} \quad V_i V_j^\dagger = U_i U_j^\dagger.
\end{aligned} \tag{8}
$$

Since U_i satisfies (5), (8) implies that V_i satisfies (6). □

Remark 1 Note that for a solution U_i to (5), if we set $W_i := LU_i,\ L \in \mathcal{U}(d)$, then W_i satisfies

$$\mathrm{i}\dot{W}_i W_i^\dagger = \tilde{H}_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(W_i W_j^\dagger - W_j W_i^\dagger\right), \quad \tilde{H}_i = LH_iL^\dagger.$$

Since $\tilde{H}_i$ and H_i have the same eigenvalues, left multiplication by a unitary matrix describes the same physics as the original system.

2.1.2 Special Cases

In this subsection, following [23,24], we briefly discuss the special cases of (5) when $d = 1, 2$. For the one-dimensional case when $d = 1$, (5) reduces to the Kuramoto phase model [1] for classical synchronization:

$$U_i := e^{-\mathrm{i}\theta_i}, \qquad H_i := \Omega_i \in \mathbb{R}. \tag{9}$$

By direct calculation, we have

$$\mathrm{i}\dot{U}_i U_i^\dagger = \dot{\theta}_i, \quad U_iU_j^\dagger - U_jU_i^\dagger = e^{\mathrm{i}(\theta_j-\theta_i)} - e^{-\mathrm{i}(\theta_j-\theta_i)} = 2\mathrm{i}\sin(\theta_j-\theta_i).$$

Thus, system (5) becomes the Kuramoto phase model [1,21,22]:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i). \tag{10}$$

On the other hand, when we turn off coupling so that $K = 0$, U_i satisfies the time-dependent Schrödinger equation in finite-dimensional form:

$$\mathrm{i}\dot{U}U^\dagger = H, \quad \text{or} \quad \mathrm{i}\dot{U} = HU.$$

In this situation, U and H are regarded as the unitary time-evolution operator and Hamiltonian, respectively. Thus, (5) can be viewed as a generalization of the Kuramoto model and the free Schrödinger equation. For further motivation, consider the case when $d = 2$ in (5). In this case, we use the parametrization of the unitary matrix U_i in terms of Pauli's matrices $\{\sigma_k\}_{k=1}^3$:

$$U_i := e^{-\mathrm{i}\theta_i}\left(\mathrm{i}\sum_{k=1}^{3}x_i^k\sigma_k + x_i^4 I_2\right) = e^{-\mathrm{i}\theta_i}\begin{pmatrix} x_i^4+\mathrm{i}x_i^1 & x_i^2+\mathrm{i}x_i^3 \\ -x_i^2+\mathrm{i}x_i^3 & x_i^4-\mathrm{i}x_i^1\end{pmatrix},$$

where I_2 and σ_i are the identity matrix and Pauli matrices, respectively, defined by

$$I_2 := \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma_1 := \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \sigma_2 := \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad \sigma_3 := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

We also expand H_i:

$$H_i = \sum_{k=1}^{3}\omega_i^k\sigma_k + \nu_i I_2,$$

where $\omega_i = (\omega_i^1, \omega_i^2, \omega_i^3)$ is a real three-vector, and the natural frequency ν_i is associated with the $U(1)$ component of U_i. After some algebraic manipulations, we obtain $5N$ equations for the angles θ_i and the four-vectors x_i:

$$||x_i||^2\dot{\theta}_i = \nu_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i)\langle x_i, x_k\rangle, \quad 1 \le i \le N, \ t \in \mathbb{R},$$
$$||x_i||^2\dot{x}_i = \Omega_i x_i + \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k - \theta_i)(||x_i||^2 x_k - \langle x_i, x_k\rangle x_i), \tag{11}$$

where Ω_i is a real 4×4 antisymmetric matrix:

$$\Omega_i := \begin{pmatrix} 0 & -\omega_i^3 & \omega_i^2 & -\omega_i^1 \\ \omega_i^3 & 0 & -\omega_i^1 & -\omega_i^2 \\ -\omega_i^2 & \omega_i^1 & 0 & -\omega_i^3 \\ \omega_i^1 & \omega_i^2 & \omega_i^3 & 0 \end{pmatrix}.$$

By taking $\theta_i = 0$ and $\nu_i = 0$ in (11), we formally obtain the consensus model in (12) with $x_i \in \mathbb{S}^3$:

$$||x_i||^2\dot{x}_i = \Omega_i x_i + \frac{K}{N}\sum_{k=1}^{N}(||x_i||^2 x_k - \langle x_i, x_k\rangle x_i), \quad i = 1, \ldots, N. \tag{12}$$

The emergence of phase-locked states for systems (11) and (12) has already been studied [5–10].

2.2 Phase-Locked States

In this subsection, we discuss the phase-locked states of the Lohe flow. As discussed in Sect. 2.1, the Lohe model can be viewed as a non-abelian generalization of the Kuramoto model in (10); thus, the concept of phase-locked states of the Lohe model should generalize the corresponding concept of the Kuramoto model. For the Kuramoto model in (10), the phase-locked state $\Theta = (\theta_1, \ldots, \theta_N)$ is defined by the solution to (10) that satisfies the invariance property [1,11,14,15]:

$$|\theta_i(t) - \theta_j(t)| = \text{constant}, \quad t \ge 0. \tag{13}$$

Note that for $d = 1$, using relation (9), $U_i U_j^\dagger = e^{-\mathrm{i}(\theta_i - \theta_j)}$. Thus, relation (13) is equivalent to

$$U_i(t)U_j^\dagger(t) = \text{constant}, \quad t \ge 0.$$

Based on this special observation, we introduce a concept of phase-locked states for the Lohe model.

Definition 1 Let $\{U_i(t)\}$ be a solution to (5).

1. $\{U_i(t)\}$ is a *phase-locked state* if and only if $U_i(t)U_j(t)^\dagger$ is constant for all i, j, and $t \ge 0$.
2. The Lohe flow $\{U_i(t)\}$ achieves *asymptotic phase-locking* if and only if the limit of $U_i U_j^\dagger$ as $t \to \infty$ exists.

Remark 2 It follows from (8) that right multiplication of a phase-locked state by a unitary matrix is also a phase-locked state; likewise, a right multiplication of a Lohe flow achieving asymptotic phase-locking also achieves asymptotic phase-locking.

We now return to characterizing phase-locked states. For the Kuramoto model in (10), physics and control communities assume without any rigorous justification that for a large coupling strength K relative to the size of the natural frequencies $\{H_i\}$, phase-locked states Θ^∞ emerge from generic initial configurations. Recently, rigorous proofs have been obtained in [3,13] for identical oscillators and [17] for nonidentical oscillators (see [12,18,19,26–28] for the related complete synchronization problem). In the presence of complete synchronization, the asymptotic phase θ_i^∞ for the phase-locked states takes the following form:

$$\theta_i(t) = \theta_i^\infty + \Omega_c t, \qquad \Omega_c := \frac{1}{N}\sum_{j=1}^{N} \Omega_i.$$

In this case, the phase-locked states of the Kuramoto model correspond to the traveling solution on the unit circle with an average natural frequency Ω_c. This can also be generalized to the Lohe model.

Proposition 1 ([23,24]) The phase-locked states $\{U_i\}$ of (5) are of the form

$$U_i = U_i^\infty e^{-\mathrm{i}\Lambda t},$$

where $U_i^\infty \in \mathcal{U}(d)$ and Λ is the constant $d \times d$ Hermitian matrix satisfying

$$U_i^\infty \Lambda U_i^{\infty\dagger} = H_i - \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(U_i^\infty U_k^{\infty\dagger} - U_k^\infty U_i^{\infty\dagger}\right). \tag{14}$$

Remark 3 Note that since the phase-locked states are still phase-locked states after right multiplication, it follows that for any unitary matrix $L \in \mathcal{U}(d)$,

$$V_i = U_i L = U_i^\infty L e^{-\mathrm{i}L^\dagger \Lambda L t}$$

is also a phase-locked state for (5). This, in particular, shows that any uniqueness result for Λ must consider the conjugation of Λ by a unitary matrix to be equivalent to Λ. Meanwhile, Lohe [23,24] observed by numerical simulations that a Lohe flow with a generic initial condition achieves asymptotic phase-locking in a large coupling strength regime and that this phase-locked state is unique up to right multiplication by a unitary matrix.

3 Evolution of the Ensemble Diameter

In this section, we introduce the ensemble diameter and derive Gronwall inequalities for this ensemble diameter.

3.1 Ensemble Diameter

In this subsection, we introduce the ensemble diameter measuring the size of ensemble $\{U_i\}$. First, we recall some basic properties of matrices that are used throughout this paper. Let $A = (a_{ij})$ be a $d \times d$ complex matrix. The Frobenius norm of A is defined as follows:

$$\|A\| := \left[\sum_{1 \le i,j \le d} |a_{ij}|^2 \right]^{\frac{1}{2}} = \left[\mathrm{tr}(AA^\dagger) \right]^{\frac{1}{2}}. \tag{15}$$

The Frobenius norm has the following properties:

$$\|AU\| = \|UA\| = \|A\|, \qquad A \in \mathbb{C}^{d\times d}, \;\; U \in \mathcal{U}(d), \tag{16}$$

and

$$\begin{aligned} &|\mathrm{tr}(A_1 A_2 \cdots A_k)| \le \|A_1\| \|A_2\| \cdots \|A_k\|, \\ &\|A_1 A_2 \cdots A_k\| \le \|A_1\| \|A_2\| \cdots \|A_k\|. \end{aligned} \tag{17}$$

We now introduce the ensemble diameter. For a finite collection of $\mathcal{C}^1$ square matrices $\{U_i(t)\}_{i=1}^N$ and $\{H_i\}_{i=1}^N$, set

$$D(U) := \max_{1\le i,j \le N} \|U_i - U_j\|, \qquad D(H) := \max_{1 \le i,j\le N} \|H_i - H_j\|. \tag{18}$$

Note that $D(U(t))$ is Lipschitz continuous in t, so it is differentiable almost everywhere for $t \in (0, \infty)$. In particular, for a given Lohe flow $\{U_i\}_{i=1}^N$, it follows from (16) and Lemma 1 (1) that

$$\|U_i - U_j\| = \|(U_i - U_j)U_j^\dagger\| = \|U_i U_j^\dagger - I\|, \quad \text{similarly} \quad \|U_i - U_j\| = \|U_i^\dagger U_j - I\|.$$

Thus, we have

$$D(U) = \max_{1\le i,j\le N} \|U_i U_j^\dagger - I\| = \max_{1\le i,j\le N} \|U_i^\dagger U_j - I\|, \quad t \ge 0.$$

3.2 Evolution of the Ensemble Diameter

In this subsection, we derive Gronwall type differential inequalities for $D(U)$ (see Lemma 2):

$$\begin{aligned} &-D(H) - K D(U)\Big(1 + \frac{D(U)^2}{2}\Big) \\ &\qquad \le \frac{dD(U)}{dt} \le D(H) - K D(U)\Big(1 - \frac{D(U)^2}{2}\Big), \quad \text{a.e., } t \in (0,\infty). \end{aligned} \tag{19}$$

For the ensemble of some restricted identical oscillators with $D(H) = 0$ and $D(U) \ll 1$, the differential inequalities (19) yield

$$\frac{dD(U)}{dt} \approx -cK D(U), \quad \text{a.e.,} \;\; t \in (0, \infty),$$

where c is some positive constant. Then, this yields the desired exponential decay of $D(U)$, which leads to the complete synchronization (see Theorem 1):

$$\lim_{t\to\infty} \max_{1\le i,j\le N} ||U_i(t) - U_j(t)|| = 0.$$

On the other hand, for an ensemble of non-identical oscillators, the inequalities (19) yield the existence of an invariant set in Lemma 3 which will be crucially used in the emergence of complete synchronization in the sense of Definition 1 (see Theorem 3).

We next return to the derivation of (19). Let U_i be a solution to the Lohe model in (5). Then U_i satisfies

$$\begin{aligned}\dot{U}_i U_i^\dagger &= -\mathrm{i}H_i + \frac{K}{2N}\sum_{k=1}^{N}\Big(U_k U_i^\dagger - U_i U_k^\dagger\Big),\\ U_i \dot{U}_i^\dagger &= \mathrm{i}H_i - \frac{K}{2N}\sum_{k=1}^{N}\Big(U_k U_i^\dagger - U_i U_k^\dagger\Big).\end{aligned} \tag{20}$$

Since the functional $D(U)$ is Lipschitz continuous, it is differentiable almost everywhere for $t \in (0, \infty)$. Thus, for a time t when $D(U)$ is differentiable, choose indices i and j depending on t such that

$$D(U(t)) = \|U_i(t) - U_j(t)\|.$$

Then, (15) is used to obtain

$$\begin{aligned}\frac{d}{dt}D(U)^2 = \frac{d}{dt}\|U_i - U_j\|^2 &= \frac{d}{dt}\mathrm{tr}\Big((U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big)\\ &= \mathrm{tr}\Big[\frac{d}{dt}\Big((U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big)\Big].\end{aligned} \tag{21}$$

On the other hand, note that

$$\begin{aligned}&\frac{d}{dt}\Big((U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big)\\ &\quad= \frac{d}{dt}\Big(U_iU_i^\dagger + U_jU_j^\dagger - U_iU_j^\dagger - U_jU_i^\dagger\Big)\\ &\quad= \frac{d}{dt}\Big(2I_d - U_iU_j^\dagger - U_jU_i^\dagger\Big) \qquad \text{since } U_i, U_j \in \mathcal{U}(d)\\ &\quad= \underbrace{-\Big(\dot{U}_iU_j^\dagger + \dot{U}_jU_i^\dagger\Big)}_{=:\mathcal{I}_1}\underbrace{-\Big(U_i\dot{U}_j^\dagger + U_j\dot{U}_i^\dagger\Big)}_{=:\mathcal{I}_2}.\end{aligned} \tag{22}$$

Below, we rewrite the $\mathcal{I}_i$ terms as follows:
• (Estimate of $\mathcal{I}_1$): We use (20) and the relations

$$\dot{U}_iU_j^\dagger = \dot{U}_iU_i^\dagger U_iU_j^\dagger, \qquad \dot{U}_jU_i^\dagger = \dot{U}_jU_j^\dagger U_jU_i^\dagger$$

to find

$$\begin{aligned}\dot{U}_iU_j^\dagger &= -\mathrm{i}H_iU_iU_j^\dagger + \frac{K}{2N}\sum_{k=1}^{N}\Big(U_kU_i^\dagger U_iU_j^\dagger - U_iU_k^\dagger U_iU_j^\dagger\Big),\\ \dot{U}_jU_i^\dagger &= -\mathrm{i}H_jU_jU_i^\dagger + \frac{K}{2N}\sum_{k=1}^{N}\Big(U_kU_j^\dagger U_jU_i^\dagger - U_jU_k^\dagger U_jU_i^\dagger\Big).\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\mathcal{I}_1 &= \mathrm{i}H_iU_iU_j^\dagger + \mathrm{i}H_jU_jU_i^\dagger\\ &\qquad - \frac{K}{2N}\sum_{k=1}^{N}\Big(U_kU_j^\dagger + U_kU_i^\dagger - U_iU_k^\dagger U_iU_j^\dagger - U_jU_k^\dagger U_jU_i^\dagger\Big).\end{aligned} \tag{23}$$

• (Estimate of $\mathcal{I}_2$): We use (20) and the relations

$$U_i\dot{U}_j^\dagger = U_iU_j^\dagger U_j\dot{U}_j^\dagger, \qquad U_j\dot{U}_i^\dagger = U_jU_i^\dagger U_i\dot{U}_i^\dagger$$

to find

$$\begin{aligned} U_i\dot{U}_j^\dagger &= \mathrm{i}U_iU_j^\dagger H_j + \frac{K}{2N}\sum_{k=1}^N \left[U_iU_j^\dagger U_jU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger\right], \\ U_j\dot{U}_i^\dagger &= \mathrm{i}U_jU_i^\dagger H_i + \frac{K}{2N}\sum_{k=1}^N \left[U_jU_i^\dagger U_iU_k^\dagger - U_jU_i^\dagger U_kU_i^\dagger\right]. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \mathcal{I}_2 \quad &= \quad -\mathrm{i}U_iU_j^\dagger H_j - \mathrm{i}U_jU_i^\dagger H_i \\ &\quad - \frac{K}{2N}\sum_{k=1}^N \Big(U_iU_k^\dagger + U_jU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger - U_jU_i^\dagger U_kU_i^\dagger\Big). \end{aligned} \tag{24}$$

In (21), we combine estimates (22), (23), and (24) to obtain

$$\frac{d}{dt}D(U)^2 = \mathcal{L} - \frac{K}{2N}\sum_{k=1}^N \mathcal{M}_k, \tag{25}$$

where the terms $\mathcal{L}$ and $\mathcal{M}_k$ are given by the following relations:

$$\begin{aligned} \mathcal{L} &:= \mathrm{i}\times \mathrm{tr}\left(H_iU_iU_j^\dagger - U_iU_j^\dagger H_j + H_jU_jU_i^\dagger - U_jU_i^\dagger H_i\right), \\ \mathcal{M}_k &:= \mathrm{tr}\Big(U_kU_j^\dagger - U_iU_k^\dagger U_iU_j^\dagger + U_iU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger \\ &\qquad + U_kU_i^\dagger - U_jU_k^\dagger U_jU_i^\dagger + U_jU_k^\dagger - U_jU_i^\dagger U_kU_i^\dagger\Big). \end{aligned}$$

In the following lemma, we estimate the terms $\mathcal{L}$ and $\mathcal{M}_k$ and derive the desired Gronwall inequalities.

Lemma 2 *Let $\{U_i\}$ be a solution to (5). Then the ensemble diameter $D(U)$ in (18) satisfies Gronwall's differential inequalities:*

$$\left|\frac{d}{dt}D(U)^2 + 2K\,D(U)^2\right| \le 2D(H)D(U) + K\,D(U)^4 \qquad a.e.\ t \in (0, \infty), \tag{26}$$

and

$$\left|\frac{d}{dt}D(U) + K\,D(U)\right| \le D(H) + \frac{K}{2}D(U)^3 \qquad a.e.\ t \in (0, \infty). \tag{27}$$

Proof Below, we estimate the terms $\mathcal{L}$ and $\mathcal{M}_k$ in terms of $D(H)$ and $D(U)$.

• (Estimate of $\mathcal{L}$): We use $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ to find

$$\begin{aligned} \mathcal{L} &= \mathrm{i}\times\mathrm{tr}\left(H_iU_iU_j^\dagger - H_jU_iU_j^\dagger + H_jU_jU_i^\dagger - H_iU_jU_i^\dagger\right) \\ &= \mathrm{i}\times\mathrm{tr}\left[(H_i - H_j)(U_iU_j^\dagger - U_jU_i^\dagger)\right]. \end{aligned}$$

Together with (17), this implies

$$|\mathcal{L}| \le \|H_i - H_j\|\cdot\|U_iU_j^\dagger - U_jU_i^\dagger\| \le 2D(H)D(U). \tag{28}$$

• (Estimate of $\mathcal{M}_k$): We claim that

$$\begin{aligned}\mathcal{M}_k &= 4\|U_i - U_j\|^2 - \mathrm{tr}\Big[(U_k - U_j)(U_k^\dagger - U_j^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &\quad + (U_k - U_i)(U_k^\dagger - U_i^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big]. \end{aligned} \tag{29}$$

Since the proof of (29) is long and tedious, we give its proof in Appendix A. Now, with estimate (29), we use (17) to obtain

$$\begin{aligned}|\mathcal{M}_k - 4D(U)^2| &= \Big|\mathrm{tr}\Big[(U_k - U_j)(U_k^\dagger - U_j^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &\quad + (U_k - U_i)(U_k^\dagger - U_i^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big]\Big| \\ &\le 2D(U)^4.\end{aligned}$$

It follows that

$$-2KD(U)^2 - KD(U)^4 \le -\frac{K}{2N}\sum_{k=1}^{N}\mathcal{M}_k \le -2KD(U)^2 + KD(U)^4. \tag{30}$$

Finally in (25), we combine (28) and (30) to obtain the estimate in (26). The second estimate in (27) follows directly from (26). □

4 Emergence and Stability of Phase-Locked States

In this section, we present the emergence and stability of phase-locked states of the Lohe matrix model.

4.1 Identical Oscillators

In this subsection, we provide a sufficient framework that guarantees the formation of phase-locked states for identical oscillators. We assume that the Hermitian matrices H_i satisfy

$$H_i = H_c, \quad \forall\, i = 1, \ldots, N, \qquad \text{i.e.,} \quad D(H) = 0.$$

In this situation, the matrices U_i satisfy

$$\mathrm{i}\dot{U}_i U_i^\dagger = H_c - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\Big(U_i U_j^\dagger - U_j U_i^\dagger\Big), \quad i = 1, \ldots, N. \tag{31}$$

To characterize the admissible class of initial data, for any positive constant α, we introduce the set $\mathcal{S}(\alpha)$:

$$\mathcal{S}(\alpha) := \{U \in \mathcal{U}(d)^N : D(U) < \alpha\}.$$

We now present the first result on exponential synchronization of identical oscillators.

Theorem 1 *Suppose that the Hermitian matrices* $H = \{H_i\}_{i=1}^N$ *and initial data* $U^0 = \{U_i^0\}_{i=1}^N$ *satisfy*

$$K > 0, \quad D(H) = 0, \quad U^0 \in \mathcal{S}(\sqrt{2}).$$

Then for any solution $\{U_i\}$ to (31), the ensemble diameter $D(U)$ approaches zero exponentially fast: for $t \geq 0$,

$$\sqrt{\frac{2D^2(U^0)}{(D^2(U^0)+2)e^{2Kt} - D^2(U^0)}} \leq D(U(t)) \leq \sqrt{\frac{2D^2(U^0)}{D^2(U^0)+(2-D^2(U^0))e^{2Kt}}}.$$

Proof It follows from (26) of Lemma 2 that

$$-2K D(U)^2 - K D(U)^4 \leq \frac{d}{dt} D(U)^2 \leq -2K D(U)^2 + K D(U)^4, \quad \text{a.e. } t > 0.$$

Set $Y = D(U)^2$. Then the nonnegative function Y satisfies

$$-2KY - KY^2 \leq \frac{dY}{dt} \leq -2KY + KY^2, \quad \text{a.e. } t > 0.$$

• Case A (Upper bound estimate): By the comparison principle with the explicit solution to the Riccati equation $y' = -2Ky + Ky^2$, we have

$$D(U(t)) \leq \sqrt{\frac{2D^2(U^0)}{D^2(U^0)+(2-D^2(U^0))e^{2Kt}}}, \quad t \geq 0. \tag{32}$$

• Case B (Lower bound estimate): Similar to Case A, we have

$$D(U(t)) \geq \sqrt{\frac{2D^2(U^0)}{(D^2(U^0)+2)e^{2Kt} - D^2(U^0)}}, \quad t \geq 0. \tag{33}$$

By combining estimates (32) and (33), we obtain the desired result. □

Remark 4 1. Theorem 1 implies exponential synchronization:

$$D(U(t)) = \mathcal{O}(1)e^{-Kt}, \quad \text{as } t \to \infty.$$

2. When exponential synchronization occurs, i.e., $U_i \approx U_j$, the asymptotic dynamics of U_i is governed by the free Schrödinger equation:

$$\mathrm{i}\dot{U}U^\dagger = H_c.$$

4.2 Nonidentical Oscillators

In this subsection, we discuss a sufficient framework leading to the formation of phase-locked states for nonidentical oscillators.

4.2.1 Existence of a Positively Invariant Set

In this subsection, we provide a positively invariant set for the Lohe flow in (5). The Gronwall inequality in (27) of Lemma 2 motivates a cubic polynomial f:

$$f(x) := x - \frac{1}{2}x^3, \quad x \geq 0,$$

so that the upper bound in (27) becomes

$$\frac{d}{dt} D(U) \leq D(H) - Kf(D(U)), \quad \text{a.e. } t > 0.$$

It is easy to verify that f satisfies the following:

(i) f has only two roots, 0 and $\sqrt{2}$, in the interval $[0, \infty)$.

(ii) $f \geq 0$ for $x \in [0, \sqrt{2}]$, $\quad f \nearrow$ for $x \in [0, \sqrt{\frac{2}{3}}]$, $\quad f \searrow$ for $x \in [\sqrt{\frac{2}{3}}, \sqrt{2}]$.

(iii) $\text{argmax}_{x\geq 0} f(x) = \sqrt{\frac{2}{3}}$, $\quad \max_{x\geq 0} f(x) = \frac{2}{3}\sqrt{\frac{2}{3}}$.

Henceforth, K_e will denote a given sufficient positive coupling strength so that $\frac{D(H)}{K_e} \leq \frac{2}{3}\sqrt{\frac{2}{3}}$. With this K_e, the equation

$$f(x) = \frac{D(H)}{K_e}, \quad x \in [0, \sqrt{2}],$$

has a solution in $[0, \sqrt{\frac{2}{3}}]$, which we denote by α_1, and a solution in $[\sqrt{\frac{2}{3}}, \sqrt{2}]$, which we denote by α_2.

Lemma 3 (Existence of a positively invariant set) *Suppose that the coupling strength K and initial data U^0 satisfy*

$$K > K_e > \frac{3}{2}\sqrt{\frac{3}{2}}D(H) \approx 1.8371 D(H), \quad U^0 \in \overline{\mathcal{S}(\alpha_2)},$$

where $\overline{\mathcal{S}(\alpha)}$ denotes the closure of $\mathcal{S}(\alpha)$:

$$\overline{\mathcal{S}(\alpha)} := \{U \in \mathcal{U}(d)^N : D(U) \leq \alpha\}.$$

Then the following assertions hold:

1. *As long as the diameter $D(U(t))$ remains in the interval $[\alpha_1, \alpha_2]$, it is strictly decreasing:*

$$\frac{d}{dt}D(U(t)) \leq c, \quad \text{a.e. } t > 0 \text{ for which } D(U(t)) \in [\alpha_1, \alpha_2],$$

   where $c < 0$ is a negative constant determined by $D(H)$, K, and K_e.
2. *The set $\mathcal{S}(\alpha_2)$ is a positively invariant set for the flow in (5):*

$$U(t) \in \mathcal{S}(\alpha_2), \quad t > 0.$$

3. *There exists a positive time $t_e \geq 0$ such that*

$$U(t) \in \mathcal{S}(\alpha_1), \quad t \geq t_e.$$

Proof (i) If $D(U(t_0)) \in [\alpha_1, \alpha_2]$ for time t_0, then

$$\frac{dD(U)}{dt}\bigg|_{t=t_0} \leq D(H) - Kf(D(U(t_0))) \leq D(H) - Kf(\alpha_1).$$

But by definition of α_1,

$$D(H) - Kf(\alpha_1) = D(H)\left(1 - \frac{K}{K_e}\right) < 0.$$

Thus, $c = D(H) - Kf(\alpha_1)$ satisfies the given conditions.

(ii) By (i), if $D(U^0) = \alpha_2$, then $D(U)$ is in a strictly decreasing mode at $t = 0+$. Moreover, since the equations in (5) are autonomous, we can apply the same arguments for the hitting times for $D(U(t)) = \alpha_2$. Thus, the region $S(\alpha_2)$ is a positively invariant set for (5).

(iii) By (i), $D(U)$ is strictly decreasing as long as it remains in the interval $[\alpha_1, \alpha_2]$, and thus, cannot remain in the interval $[\alpha_1, \alpha_2]$ (see the arguments in [11]). This proves finite-time entrance into $S(\alpha_1)$.

□

4.2.2 *Orbital Stability of the Lohe Flow*

In this subsection, we provide a stability estimate for the Lohe flow. Consider two Lohe flows $\{U_i\}$ and $\{\tilde{U}_i\}$ with common natural frequencies $H_i = \tilde{H}_i$ and initial data $\{U_i^0\}$ and $\{\tilde{U}_i^0\}$, respectively:

$$
\begin{aligned}
&\mathrm{i}\dot{U}_i U_i^\dagger = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(U_i U_j^\dagger - U_j U_i^\dagger\right), \quad i = 1, \cdots, N, \ t > 0,\\
&U_i(0) = U_i^0,
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
&\mathrm{i}\dot{\tilde{U}}_i \tilde{U}_i^\dagger = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(\tilde{U}_i \tilde{U}_j^\dagger - \tilde{U}_j \tilde{U}_i^\dagger\right), \quad i = 1, \cdots, N, \ t > 0,\\
&\tilde{U}_i(0) = \tilde{U}_i^0.
\end{aligned} \tag{35}
$$

We now introduce the functional $d(U(t), \tilde{U}(t))$. For any two Lohe flows $U(t)$ and $\tilde{U}(t)$, set

$$
\begin{aligned}
d(U(t), \tilde{U}(t)) &:= \max_{i,j} \|U_i(t)U_j^\dagger(t) - \tilde{U}_i(t)\tilde{U}_j^\dagger(t)\|\\
&= \max_{i,j} \|I_d - \tilde{U}_i^\dagger(t)U_i(t)U_j^\dagger(t)\tilde{U}_j(t)\|.
\end{aligned} \tag{36}
$$

To motivate the meaning of the functional $d(U, \tilde{U})$, we consider a familiar case when $d = 1$. In this case, (9) implies that for $\theta_{ij} := \theta_i - \theta_j$,

$$
d(U(t), \tilde{U}(t)) = \max_{i,j} \left|1 - e^{-i(\tilde{\theta}_{ij} - \theta_{ij})}\right|.
$$

Thus, it is easy to verify that

$$
d(U(t), \tilde{U}(t)) = 0 \quad \Longleftrightarrow \quad \Theta \text{ and } \tilde{\Theta} \text{ are congruent up to constant shift.}
$$

Hence, $d(U(t), \tilde{U}(t))$ measures the degree of maximal mismatch in configurations U and $\tilde{U}$. It is easy to verify that $d(\cdot, \cdot)$ can be controlled by the sum of ensemble diameters:

$$
\begin{aligned}
d(U(t), \tilde{U}(t)) &\le \max_{i,j} \|U_i(t)U_j^\dagger(t) - I\| + \max_{i,j} \|I - \tilde{U}_i(t)\tilde{U}_j^\dagger(t)\|\\
&= D(U(t)) + D(\tilde{U}(t)).
\end{aligned} \tag{37}
$$

We now present the orbital stability of the Lohe flow in (5).

Theorem 2 (Stability estimates) *Suppose that the coupling strength K and initial data U^0 and $\tilde{U}^0$ satisfy*

$$K > K_e > \frac{54}{17}D(H) \approx 3.1765D(H), \qquad U^0,\ \tilde{U}^0 \in \mathcal{S}(\alpha_1).$$

Then for the two Lohe flows $\{U_i\}$ and $\{\tilde{U}_i\}$, the following assertions hold:

1. *The relative positions synchronize exponentially fast:*

$$d(U^0, \tilde{U}^0)e^{-K(1+3\alpha_1)t} \le d(U(t), \tilde{U}(t)) \le d(U^0, \tilde{U}^0)e^{-K(1-3\alpha_1)t}, \quad t \ge 0.$$

2. *The normalized velocities $\dot{U}_i U_i^\dagger$ and $\dot{\tilde{U}}_i \tilde{U}_i^\dagger$ synchronize:*

$$\left\| \dot{\tilde{U}}_i \tilde{U}_i^\dagger - \dot{U}_i U_i^\dagger \right\| \le K d(U^0, \tilde{U}^0)e^{-K(1-3\alpha_1)t}.$$

3. *There exists a unitary matrix $L_\infty \in \mathcal{U}(d)$ independent of i such that*

$$\lim_{t\to\infty} U_i(t)^\dagger \tilde{U}_i(t) = L_\infty \quad \textit{and} \quad \lim_{t\to\infty} \|\tilde{U}_i(t) - U_i(t)L_\infty\| = 0, \quad i = 1, \dots, N.$$

The convergence is exponential with rate bounded above by $-K(1-3\alpha_1)$.

Proof Note that if we choose $K_e > \frac{54}{17}D(H)$, then

$$\alpha_1 < \frac{1}{3}, \qquad \frac{\sqrt{69}-1}{6} \approx 1.2178 < \alpha_2 < \sqrt{2}.$$

Thus, the results of Lemma 3 can be applied.

(i) It suffices to show that $d(U, \tilde{U})$ satisfies

$$-K(1+3\alpha_1)d(U, \tilde{U}) \le \frac{d}{dt}d(U, \tilde{U}) \le -K(1-3\alpha_1)d(U, \tilde{U}), \quad \text{a.e. } t \in (0, \infty). \tag{38}$$

Since the proof of (38) is long, we provide its proof in Appendix B.

(ii) It follows from Lemma 3 (3) that the set $\mathcal{S}(\alpha_1)$ is a positively invariant set; thus, we have

$$U(t), \tilde{U}(t) \in \mathcal{S}(\alpha_1) \quad \text{for } t \ge 0.$$

On the other hand, (34) and (35) imply

$$\begin{aligned} \mathrm{i}(\dot{\tilde{U}}_i \tilde{U}_i^\dagger - \dot{U}_i U_i^\dagger) &= \left(H_i - \frac{\mathrm{i}K}{2N}\sum_{k=1}^N (\tilde{U}_i \tilde{U}_k^\dagger - \tilde{U}_k \tilde{U}_i^\dagger) \right) - \left(H_i - \frac{\mathrm{i}K}{2N}\sum_{k=1}^N (U_i U_k^\dagger - U_k U_i^\dagger) \right) \\ &= \frac{\mathrm{i}K}{2N}\sum_{k=1}^N \left[\tilde{U}_k \tilde{U}_i^\dagger - U_k U_i^\dagger - \tilde{U}_i \tilde{U}_k^\dagger + U_i U_k^\dagger \right]. \end{aligned}$$

This yields

$$\left\| \dot{\tilde{U}}_i \tilde{U}_i^\dagger - \dot{U}_i U_i^\dagger \right\| \le \frac{K}{2N}\sum_{k=1}^N \left(\|\tilde{U}_k \tilde{U}_i^\dagger - U_k U_i^\dagger\| + \|\tilde{U}_i \tilde{U}_k^\dagger - U_i U_k^\dagger\| \right) \le K d(U, \tilde{U}).$$

This result, along with (i), gives the desired estimate.

(iii) The proof of the third statement is divided into two steps. First, we show that the limit $U_i(t)^\dagger \tilde{U}_i(t)$ as $t \to \infty$ exists, and then, we show that it is independent of i.

• Step A: Note that $U_i^\dagger \tilde{U}_i$ satisfies

$$U_i(t)^\dagger \tilde{U}_i(t) = (U_i^0)^\dagger \tilde{U}_i^0 + \int_0^t \frac{d}{ds}\Big(U_i(s)^\dagger \tilde{U}_i(s)\Big)ds. \tag{39}$$

On the other hand, the integrand can be estimated as follows:

$$\begin{aligned}
\left\|\frac{d}{ds}U_i^\dagger \tilde{U}_i\right\| &= \|\dot{U}_i^\dagger \tilde{U}_i + U_i^\dagger \dot{\tilde{U}}_i\| \\
&= \|U_i\Big(\dot{U}_i^\dagger \tilde{U}_i + U_i^\dagger \dot{\tilde{U}}_i\Big)\tilde{U}_i^\dagger\| \qquad \text{by (16)} \\
&= \|U_i\dot{U}_i^\dagger + \dot{\tilde{U}}_i\tilde{U}_i^\dagger\| \qquad \text{by } \tilde{U}_i\tilde{U}_i^\dagger = I \text{ and } U_iU_i^\dagger = I \\
&= \|-\dot{U}_iU_i^\dagger + \dot{\tilde{U}}_i\tilde{U}_i^\dagger\| \qquad \text{by } U_i\dot{U}_i^\dagger = -\dot{U}_iU_i^\dagger \\
&\le Kd(U,\tilde{U}) \le Kd(U^0,\tilde{U}^0)e^{-K(1-3\alpha_1)t} \quad \text{by the result (ii).}
\end{aligned}$$

Hence, since $\left|\frac{d}{ds}\Big(U_i(s)^\dagger \tilde{U}_i(s)\Big)\right|$ decays to zero exponentially fast, it follows from (39) that the limit of $U_i(t)^\dagger \tilde{U}_i(t)$ exists as $t \to \infty$:

$$\begin{aligned}
\exists \lim_{t\to\infty} U_i(t)^\dagger \tilde{U}_i(t) &= (U_i^0)^\dagger \tilde{U}_i^0 + \int_0^\infty \frac{d}{ds}\Big(U_i(s)^\dagger \tilde{U}_i(s)\Big)ds \\
&=: L_i, \quad i = 1, \cdots, N.
\end{aligned} \tag{40}$$

• Step B: Since $d(U,\tilde{U}) \to 0$ as $t \to \infty$, it follows from (36) that

$$0 = \lim_{t\to\infty}\Big(I - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j\Big) = I - L_i^\dagger L_j, \quad \text{i.e.,} \quad L_i^\dagger L_j = I.$$

This implies

$$L_i = L_j =: L_\infty, \quad 1 \le i, j \le N. \tag{41}$$

Thus, by combining (40) and (41), we obtain the desired result. □

Remark 5 For the one-dimensional case, L_∞ corresponds to the phase shift between Θ and $\tilde{\Theta}$.

4.2.3 Emergence of Phase-Locked States

In this part, we provide the last result on the emergence of phase-locked states for the Lohe flow in (5). Before we state our main result, we briefly summarize how the phase-locked states can emerge from some restricted class of initial data, in the sense of Proposition 1.

• Step A: We first derive exponential convergences of $U_iU_1^\dagger$, $i = 1, \cdots, N$ to obtain

$$\exists\, V_i^\infty := \lim_{t\to\infty}(U_iU_1^\dagger)(t), \quad i = 1, \ldots, N.$$

• Step B: We next derive a useful identity. Using (1),

$$\begin{aligned}
\frac{d}{dt}U_iU_j^\dagger &= -\mathrm{i}H_iU_iU_j^\dagger + \mathrm{i}U_iU_j^\dagger H_j \\
&\quad + \frac{K}{2N}\sum_{k=1}^N\Big[U_kU_j^\dagger - U_iU_k^\dagger U_iU_j^\dagger + U_iU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger\Big].
\end{aligned}$$

Letting $t \to \infty$ and using the result of Step A, we derive the identity:

$$
\begin{aligned}
&-\mathrm{i}V_i^{\infty\dagger} H_i V_i^{\infty} + \frac{K}{2N}\sum_{k=1}^{N}\left[V_i^{\infty\dagger}V_k^{\infty} - V_k^{\infty\dagger}V_i^{\infty}\right] \\
&\quad = -\mathrm{i}V_j^{\infty\dagger} H_j V_j^{\infty} + \frac{K}{2N}\sum_{k=1}^{N}\left[V_j^{\infty\dagger}V_k^{\infty} - V_k^{\infty\dagger}V_j^{\infty}\right], \quad 1 \le i, j \le N.
\end{aligned}
$$

• Step C: Finally, we define a phase matrix Λ_i appearing in Proposition 1 by setting:

$$
-\mathrm{i}\Lambda_i := -\mathrm{i}V_i^{\infty\dagger} H_i V_i^{\infty} + \frac{K}{2N}\sum_{k=1}^{N}\left[V_i^{\infty\dagger}V_k^{\infty} - V_k^{\infty\dagger}V_i^{\infty}\right], \quad 1 \le i, j \le N.
$$

We further show using Step B that Λ_i is independent on the index i, i.e., $\Lambda =: \Lambda_i$ to see

$$
\Lambda = V_j^{\infty\dagger} H_j V_j^{\infty} + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left[V_j^{\infty\dagger}V_k^{\infty} - V_k^{\infty\dagger}V_j^{\infty}\right], \quad 1 \le j \le N.
$$

This is equivalent to the desired relation:

$$
V_j^{\infty}\Lambda V_j^{\infty\dagger} = H_j - \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left[V_j^{\infty}V_k^{\infty\dagger} - V_k^{\infty}V_j^{\infty\dagger}\right].
$$

In this way, we can obtain Λ and V_j^{∞} appearing in Proposition 1.

We are now ready to state our final result.

Theorem 3 *Suppose that the coupling strength K and initial data U^0 satisfy*

$$
K > \frac{54}{17}D(H), \quad U^0 \in \overline{\mathcal{S}(\alpha_2)}.
$$

Then for the Lohe flow $\{U_i\}$, the following assertions hold:

1. *$\{U_i\}$ achieves asymptotic phase-locking:*

$$
\lim_{t\to\infty} U_i U_j^{\dagger}
$$

converges exponentially fast, with exponential rate bounded above by $-K(1-3\alpha_1)$.

2. *There exists a phase-locked state $\{V_i\}$ and a unitary matrix $L \in \mathcal{U}(d)$ such that*

$$
D(V) < \alpha_1 \quad \text{and} \quad \lim_{t\to\infty} \|U_i - V_i L\| = 0.
$$

The convergence rate is exponentially fast, with rate bounded above by $-K(1-3\alpha_1)$.

3. *Phase locked states in $\overline{S(\alpha_2)}$ are unique up to right-multiplication. That is, if $\{V_i\}$ and $\{W_i\}$ are two phase-locked states in $\overline{S(\alpha_2)}$, then there exists a unitary matrix $L \in \mathcal{U}(d)$ such that*

$$
W_i = V_i L, \quad i = 1, \cdots, N.
$$

Proof Since $D(U^0) \le \alpha_2$ implies the existence of a finite-time $t_e \ge 0$ such that $D(U^0) < \alpha_1$ for $t \ge t_e$, and since the Lohe flow is autonomous, without loss of generality, we assume

$$
t_e = 0, \quad D(U^0) < \alpha_1.
$$

This allows us to apply the results of Theorem 2.

(i) For any $T \geq 0$, $\{U_i(t+T)\}$ is a Lohe flow with time-shifted initial data $\{U_i(T)\}$. Thus, it follows from the orbital stability estimate in Theorem 2 (1) that for $t \geq 0$,

$$\|(U_i U_j^\dagger)(t+T) - (U_i U_j^\dagger)(t)\| \leq \max_{i,j} \|(U_i U_j^\dagger)(T) - (U_i U_j^\dagger)(0)\| e^{-K(1-3\alpha_1)t}. \tag{42}$$

In particular, for $T = 1$ and $t = n$, estimate (42) yields

$$\|(U_i U_j^\dagger)(n+1) - (U_i U_j^\dagger)(n)\| \leq \max_{i,j} \|(U_i U_j^\dagger)(1) - (U_i U_j^\dagger)(0)\| e^{-K(1-3\alpha_1)n}, \quad n \in \mathbb{Z}_+.$$

This again yields that for $m \in \mathbb{Z}_+$,

$$\|(U_i U_j^\dagger)(n+m) - (U_i U_j^\dagger)(n)\| \leq \max_{i,j} \|(U_i U_j^\dagger)(1) - (U_i U_j^\dagger)(0)\| \frac{e^{-K(1-3\alpha_1)n}}{1 - e^{-K(1-3\alpha_1)}}, \quad n \in \mathbb{Z}_+.$$

Hence, the discrete sequence $\{U_i(n)U_j^\dagger(n)\}_{n\in\mathbb{N}}$ is Cauchy in the complete space $\mathcal{U}(d)$; hence, it converges to $L_{ij}^\infty \in \mathcal{U}(d)$ exponentially fast and satisfies

$$\|(U_i U_j^\dagger)(n) - L_{ij}^\infty\| \leq \max_{i,j} \|(U_i U_j^\dagger)(1) - (U_i U_j^\dagger)(0)\| \frac{e^{-K(1-3\alpha_1)n}}{1 - e^{-K(1-3\alpha_1)}}. \tag{43}$$

Furthermore, note that for $0 \leq s \leq 1$,

$$\begin{aligned} &\|U_i(n+s)U_j(n+s)^\dagger - U_i(n)U_j(n)^\dagger\| \\ &\quad \leq \left(\max_{\substack{1\leq i,j\leq N \\ 0\leq s\leq 1}} \|U_i(s)U_j(s)^\dagger - U_i(0)U_j(0)^\dagger\| \right) e^{-K(1-3\alpha_1)n}. \end{aligned} \tag{44}$$

Then (43) and (44) give the exponential convergence of $U_i U_j^\dagger$ with the aforementioned rate.
(ii) By the result of (i), the limit of $U_i U_1^\dagger$ as $t \to \infty$ exists, so we set

$$V_i^\infty := \lim_{t\to\infty} (U_i U_1^\dagger)(t), \quad i = 1, \cdots, N.$$

On the other hand, note that

$$\begin{aligned} \frac{d}{dt} U_i U_j^\dagger &= \dot{U}_i U_j^\dagger + U_i \dot{U}_j^\dagger \\ &= \dot{U}_i U_i^\dagger U_i U_j^\dagger + U_i U_j^\dagger U_j \dot{U}_j^\dagger \\ &= \Big(-\mathrm{i}H_i - \frac{K}{2N} \sum_{k=1}^N \Big[U_i U_k^\dagger - U_k U_i^\dagger \Big] \Big) U_i U_j^\dagger \\ &\quad + U_i U_j^\dagger \Big(\mathrm{i}H_j + \frac{K}{2N} \sum_{k=1}^N \Big[U_j U_k^\dagger - U_k U_j^\dagger \Big] \Big) \\ &= -\mathrm{i}H_i U_i U_j^\dagger + \mathrm{i}U_i U_j^\dagger H_j \\ &\quad + \frac{K}{2N} \sum_{k=1}^N \Big[U_k U_j^\dagger - U_i U_k^\dagger U_i U_j^\dagger + U_i U_k^\dagger - U_i U_j^\dagger U_k U_j^\dagger \Big]. \end{aligned} \tag{45}$$

Again, by the result of (i), the right-hand side of (45) attains a limit value, and thus the left-hand side $\frac{d}{dt} U_i U_j^\dagger$ must also attain a limit value. But $U_i U_j^\dagger = U_i U_1^\dagger (U_j U_1^\dagger)^\dagger$ converges to the fixed value $V_i^\infty V_j^{\infty\dagger}$. Thus, the limit value of $\frac{d}{dt} U_i U_j^\dagger$ must be 0. This implies

$$\begin{aligned} 0 &= -\mathrm{i} H_i V_i^\infty V_j^{\infty\dagger} + \mathrm{i} V_i^\infty V_j^{\infty\dagger} H_j \\ &\quad + \frac{K}{2N} \sum_{k=1}^N \Big[V_k^\infty V_j^{\infty\dagger} - V_i^\infty V_k^{\infty\dagger} V_i^\infty V_j^{\infty\dagger} + V_i^\infty V_k^{\infty\dagger} - V_i^\infty V_j^{\infty\dagger} V_k^\infty V_j^{\infty\dagger} \Big]. \end{aligned} \tag{46}$$

Left-multiplying by $V_i^{\infty\dagger}$, right-multiplying by V_j^∞, and rearranging terms in (46) yields

$$\begin{aligned} &-\mathrm{i} V_i^{\infty\dagger} H_i V_i^\infty + \frac{K}{2N} \sum_{k=1}^N \Big[V_i^{\infty\dagger} V_k^\infty - V_k^{\infty\dagger} V_i^\infty \Big] \\ &\quad = -\mathrm{i} V_j^{\infty\dagger} H_j V_j^\infty + \frac{K}{2N} \sum_{k=1}^N \Big[V_j^{\infty\dagger} V_k^\infty - V_k^{\infty\dagger} V_j^\infty \Big], \quad 1 \le i, j \le N. \end{aligned}$$

Note that the left-hand side depends only on the index i, and that the right-hand side depends only on the index j. Hence, we introduce an index-independent matrix Λ that satisfies

$$-\mathrm{i}\Lambda := -\mathrm{i} V_i^{\infty\dagger} H_i V_i^\infty + \frac{K}{2N} \sum_{k=1}^N \Big[V_i^{\infty\dagger} V_k^\infty - V_k^{\infty\dagger} V_i^\infty \Big], \quad 1 \le i, j \le N.$$

Then, we have

$$V_i^\infty \Lambda V_i^{\infty\dagger} = H_i - \frac{\mathrm{i}K}{2N} \sum_{k=1}^N \Big[V_i^\infty V_k^{\infty\dagger} - V_k^\infty V_i^{\infty\dagger} \Big].$$

But this implies that $\{V_i^\infty\}$ and Λ satisfy (14) in Proposition 1. Therefore, $\{V_i\} := \{V_i^\infty e^{-\mathrm{i}\Lambda t}\}$ is a phase-locked state for (5). Moreover, V_i satisfies

$$V_i V_j^\dagger = V_i^\infty V_j^{\infty\dagger} = \lim_{t\to\infty} (U_i U_j^\dagger)(t),$$

and thus,

$$D(V) = \max_{i,j} \|V_i V_j^\dagger - I_d\| = \lim_{t\to\infty} \max_{i,j} \|U_i U_j^\dagger - I_d\| = \lim_{t\to\infty} D(U) < \alpha_1.$$

The existence of a unitary matrix $L \in \mathcal{U}(d)$ satisfying the given condition easily follows from part (3) of Theorem 2.

(iii) The comments in the beginning of this proof prove that any phase-locked states in $\overline{\mathcal{S}(\alpha_2)}$ must be in $\mathcal{S}(\alpha_1)$. It follows directly from part (1) of Theorem 2 that if $\{V_i\}$ and $\{W_i\}$ are two phase-locked states with $D(V^0) < \alpha_1$ and $D(W^0) < \alpha_1$, respectively, then

$$\|V_i V_j^\dagger - W_i W_j^\dagger\| \to 0, \quad \text{as } t \to \infty.$$

However, by definition of phase-locked states, $V_i V_j^\dagger$ and $W_i W_j^\dagger$ are constant. Hence,

$$V_i V_j^\dagger = W_i W_j^\dagger, \quad \text{or equivalently,} \quad W_i^\dagger V_i = W_j^\dagger V_j, \quad t \ge 0.$$

Thus, we can right-translate $\{V_i\}$ to $\{V_i'\}$ so that $\{V_i'\}$ agrees with $\{W_i\}$ at $t = 0$. This right-translation $\{V_i'\}$ will agree with $\{W_i\}$ for all times t. □

Remark 6 The main results of Theorem 3 are the existence of emergent phase-locked states from some admissible class of initial data and their orbital stability. As we have discussed in Sect. 2.1.2, for $d = 1$, the Lohe model is equivalent to the Kuramoto model, where the phase-locked states emerge from generic initial data in a large coupling regime (see [17]). In this recent work [17], the gradient flow structure of the Kuramoto model is crucially used. Thus, one natural question is whether we can extend our results in Theorem 3 for a generic initial data or not. Of course, we even do not know whether the Lohe model can be written as a gradient flow or not. Another interesting problem is to look for a critical coupling strength which guarantees the existence of phase-locked states (see [16,32,33] for the Kuramoto model).

5 Conclusion

In this paper, we provided sufficient frameworks for the existence and orbital stability of phase-locked states of the Lohe matrix model for quantum synchronization. The emergence of phase-locked states is the emergent property of synchronization models appearing in the study of complex systems. In classical oscillatory systems, the Kuramoto phase model plays a prototype role of synchronization and has been extensively studied by control theory and statistical physics communities in the last four decades. In particular, these communities have conjectured what conditions under which the Kuramoto model exhibits complete synchronization phenomena and what the emerging phase-locked states look like. Such naive questions have been addressed by the relevant communities in the last forty years. Despite recent progress in answering such questions, there are still several difficult questions related to the relaxation process toward phase-locked states from initial data. The Lohe model can be viewed as a multi-dimensional non-abelian generalization of the Kuramoto model and was proposed by the physics community as a possible quantum synchronization. In Lohe's original papers [23,24], he discussed the existence of phase-locked states via numerical simulations and posed several questions analogous to the Kuramoto model. In the special case when $d = 2$, the Lohe matrix model can be rewritten as a dynamical system on the three-sphere $\mathbb{S}^3$. In this special situation, analytical treatment is possible and emergence of phase-locked states has already been studied. However, such a rigorous analysis of the original Lohe model has remained an open problem to date. In this paper, we explicitly provided the existence of emergent phase-locked states and its orbital stability in terms of the coupling strength, size of Hamiltonian, and size of initial data. There are still many other interesting issues open at this moment such as the emergence of phase-locked states for generic initial data, detailed relaxation process and existence of critical coupling strength (see [32,33] for the Kuramoto model). These questions will be addressed in future works.

Acknowledgments The work of S.-Y. Ha is supported by the Samsung Science and Technology Foundation under Project Number SSTF-BA1401-03.

Appendix 1: Derivation of Relation (29)

In this appendix, we discuss the relation (29) that appears in the proof of Lemma 2. Recall the term $\mathcal{M}_k$:

$$\begin{aligned}\mathcal{M}_k = \mathrm{tr}\Big[&U_kU_j^\dagger - U_iU_k^\dagger U_iU_j^\dagger + U_iU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger + U_kU_i^\dagger \\ &-U_jU_k^\dagger U_jU_i^\dagger + U_jU_k^\dagger - U_jU_i^\dagger U_kU_i^\dagger\Big].\end{aligned}$$

Our goal is to establish the following relation:

$$\begin{aligned}\mathcal{M}_k = 4\|U_i - U_j\|^2 - \mathrm{tr}\Big[&(U_k - U_j)(U_k^\dagger - U_j^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &+ (U_k - U_i)(U_k^\dagger - U_i^\dagger)(U_i - U_j)(U_i^\dagger - U_j^\dagger)\Big].\end{aligned}$$

Rearranging the terms inside the bracket in $\mathcal{M}_k$ and using the properties $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $U_j, U_i \in \mathcal{U}_d$ yields

$$\begin{aligned}\mathcal{M}_k &= \mathrm{tr}\Big[(I_d - U_iU_j^\dagger)U_kU_j^\dagger + U_iU_k^\dagger(I_d - U_iU_j^\dagger) + (I_d - U_jU_i^\dagger)U_kU_i^\dagger + U_jU_k^\dagger(I_d - U_jU_i^\dagger)\Big] \\ &= \mathrm{tr}\Big[(U_kU_j^\dagger + U_iU_k^\dagger)(I_d - U_iU_j^\dagger) + (U_kU_i^\dagger + U_jU_k^\dagger)(I_d - U_jU_i^\dagger)\Big] \\ &= \mathrm{tr}\Big[(U_kU_j^\dagger + U_iU_k^\dagger)U_iU_i^\dagger(I_d - U_iU_j^\dagger) + (U_kU_i^\dagger + U_jU_k^\dagger)U_jU_j^\dagger(I_d - U_jU_i^\dagger)\Big] \\ &= \mathrm{tr}\Big[(U_kU_j^\dagger + U_iU_k^\dagger)U_i(U_i^\dagger - U_j^\dagger) - (U_kU_i^\dagger + U_jU_k^\dagger)U_j(U_i^\dagger - U_j^\dagger)\Big] \\ &= \mathrm{tr}\Big[\underbrace{\Big((U_kU_j^\dagger U_i + U_iU_k^\dagger U_i - U_kU_i^\dagger U_j - U_jU_k^\dagger U_j\Big)(U_i^\dagger - U_j^\dagger)}_{\mathcal{J}_1}\Big].\end{aligned} \tag{47}$$

Note that the terms $\mathcal{J}_1$ can be simplified as follows:

$$\begin{aligned}\mathcal{J}_1 &= U_kU_j^\dagger(U_i - U_j)(U_i^\dagger - U_j^\dagger) + U_iU_k^\dagger(U_i - U_j)(U_i^\dagger - U_j^\dagger) + U_iU_k^\dagger U_j(U_i^\dagger - U_j^\dagger) \\ &\quad - U_kU_i^\dagger(U_j - U_i)(U_i^\dagger - U_j^\dagger) - U_jU_k^\dagger(U_j - U_i)(U_i^\dagger - U_j^\dagger) - U_jU_k^\dagger U_i(U_i^\dagger - U_j^\dagger) \\ &= \Big(U_kU_j^\dagger + U_iU_k^\dagger + U_kU_i^\dagger + U_jU_k^\dagger\Big)(U_i - U_j)(U_i^\dagger - U_j^\dagger) + \Big(U_iU_k^\dagger U_j - U_jU_k^\dagger U_i\Big)(U_i^\dagger - U_j^\dagger) \\ &=: \mathcal{J}_{11} + \mathcal{J}_{12},\end{aligned} \tag{48}$$

where we used the cancellation:

$$U_kU_j^\dagger U_j(U_i^\dagger - U_j^\dagger) - U_kU_i^\dagger U_i(U_i^\dagger - U_j^\dagger) = 0.$$

• Case A (Estimate of $\mathcal{J}_{11}$): For further simplification, we use the trick

$$U_kU_j^\dagger = (U_kU_j^\dagger - I_d)U_jU_j^\dagger + I_d = (U_k - U_j)U_j^\dagger + I_d$$

to find

$$\begin{aligned}\mathcal{J}_{11} &= 4(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &\quad + \Big((U_k - U_j)U_j^\dagger + (U_i - U_k)U_k^\dagger + (U_k - U_i)U_i^\dagger + (U_j - U_k)U_k^\dagger\Big)(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &= 4(U_i - U_j)(U_i^\dagger - U_j^\dagger) \\ &\quad - \Big((U_k - U_j)(U_k^\dagger - U_j^\dagger) + (U_k - U_i)(U_k^\dagger - U_i^\dagger)\Big)(U_i - U_j)(U_i^\dagger - U_j^\dagger).\end{aligned} \tag{49}$$

• Case B (Estimate of $\mathcal{J}_{12}$): Next, we show that

$$\mathrm{tr}\mathcal{J}_{12} = 0.$$

By expanding $\mathcal{J}_{12}$ and using the linearity of the trace, the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and the unitarity of U_i, U_j, we obtain

$$
\begin{aligned}
\mathrm{tr}\mathcal{J}_{12} &= \mathrm{tr}(U_iU_k^\dagger U_jU_i^\dagger - U_jU_k^\dagger - U_iU_k^\dagger + U_jU_k^\dagger U_iU_j^\dagger) \\
&= \mathrm{tr}\Big[(U_iU_k^\dagger)(U_jU_i^\dagger)\Big] - \mathrm{tr}(U_jU_k^\dagger) - \mathrm{tr}(U_iU_k^\dagger) + \mathrm{tr}\Big[(U_jU_k^\dagger)(U_iU_j^\dagger)\Big] \\
&= \mathrm{tr}\Big[(U_jU_i^\dagger)(U_iU_k^\dagger)\Big] - \mathrm{tr}(U_jU_k^\dagger) - \mathrm{tr}(U_iU_k^\dagger) + \mathrm{tr}\Big[(U_iU_j^\dagger)(U_jU_k^\dagger)\Big] \\
&= \mathrm{tr}(U_jU_k^\dagger) - \mathrm{tr}(U_jU_k^\dagger) - \mathrm{tr}(U_iU_k^\dagger) + \mathrm{tr}(U_iU_k^\dagger) \\
&= 0.
\end{aligned} \tag{50}
$$

Finally, in (47), we combine estimates (48), (49), and (50) to obtain the desired estimate. This completes the proof of (29).

Appendix 2: Derivation of Gronwall's Inequalities in (38)

In this appendix, we provide the proof of claim (38) in the proof of Theorem 2.

For a given time t, choose indices i and j depending on t such that

$$
d(U(t), \tilde{U}(t))^2 = \|U_i(t)U_j^\dagger(t) - \tilde{U}_i(t)\tilde{U}_j^\dagger(t)\|^2.
$$

Then, we have

$$
\begin{aligned}
\frac{d}{dt}d(U, \tilde{U})^2 &= \frac{d}{dt}\mathrm{tr}\Big[(U_iU_j^\dagger - \tilde{U}_i\tilde{U}_j^\dagger)(U_jU_i^\dagger - \tilde{U}_j\tilde{U}_i^\dagger)\Big] \\
&= \frac{d}{dt}\mathrm{tr}\Big(2I_d - U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger\Big) \\
&= -\mathrm{tr}\Big[\frac{d}{dt}\Big((U_iU_j^\dagger)(\tilde{U}_j\tilde{U}_i^\dagger) + (\tilde{U}_i\tilde{U}_j^\dagger)(U_jU_i^\dagger)\Big)\Big].
\end{aligned} \tag{51}
$$

On the other hand, it follows from (45) that

$$
\begin{aligned}
\frac{d}{dt}U_iU_j^\dagger &= -\mathrm{i}(H_iU_iU_j^\dagger - U_iU_j^\dagger H_j) + \frac{K}{2N}\sum_{k=1}^N \Big[U_kU_j^\dagger - U_iU_k^\dagger U_iU_j^\dagger + U_iU_k^\dagger - U_iU_j^\dagger U_kU_j^\dagger\Big], \\
\frac{d}{dt}\tilde{U}_j\tilde{U}_i^\dagger &= -\mathrm{i}(H_j\tilde{U}_j\tilde{U}_i^\dagger - \tilde{U}_j\tilde{U}_i^\dagger H_i) + \frac{K}{2N}\sum_{k=1}^N \Big[\tilde{U}_k\tilde{U}_i^\dagger - \tilde{U}_j\tilde{U}_k^\dagger\tilde{U}_j\tilde{U}_i^\dagger + \tilde{U}_j\tilde{U}_k^\dagger - \tilde{U}_j\tilde{U}_i^\dagger\tilde{U}_k\tilde{U}_i^\dagger\Big].
\end{aligned} \tag{52}
$$

In (51), we use relation (52) to obtain

$$
\frac{d}{dt}d((U(t), \tilde{U}(t))^2 = \bar{\mathcal{L}} - \frac{K}{2N}\sum_{k=1}^N \bar{\mathcal{M}}_k, \tag{53}
$$

where the terms $\bar{\mathcal{L}}$ and $\bar{\mathcal{M}}_k$ are given by the following relations:

$$
\bar{\mathcal{L}} = \mathrm{i} \times \mathrm{tr}\Big[H_iU_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger - U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger H_i + H_i\tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger H_i\Big] \tag{54}
$$

and

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&U_kU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger - U_iU_k^\dagger U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger + U_iU_k^\dagger\tilde{U}_j\tilde{U}_i^\dagger - U_iU_j^\dagger U_kU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger\\ &+ U_iU_j^\dagger\tilde{U}_k\tilde{U}_i^\dagger - U_iU_j^\dagger\tilde{U}_j\tilde{U}_k^\dagger\tilde{U}_j\tilde{U}_i^\dagger + U_iU_j^\dagger\tilde{U}_j\tilde{U}_k^\dagger - U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger\tilde{U}_k\tilde{U}_i^\dagger\\ &+ \tilde{U}_k\tilde{U}_j^\dagger U_jU_i^\dagger - \tilde{U}_i\tilde{U}_k^\dagger\tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger + \tilde{U}_i\tilde{U}_k^\dagger U_jU_i^\dagger - \tilde{U}_i\tilde{U}_j^\dagger\tilde{U}_k\tilde{U}_j^\dagger U_jU_i^\dagger\\ &+ \tilde{U}_i\tilde{U}_j^\dagger U_kU_i^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_k^\dagger U_jU_i^\dagger + \tilde{U}_i\tilde{U}_j^\dagger U_jU_k^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger U_kU_i^\dagger\Big].\end{aligned}$$

Lemma 4 *The terms $\bar{\mathcal{M}}_k$ can be decomposed into three parts:*

$$\bar{\mathcal{M}}_k = \bar{\mathcal{A}}_k + \bar{\mathcal{B}}_k + \bar{\mathcal{C}}_k, \tag{55}$$

where

$$\begin{aligned}\bar{\mathcal{A}}_k := tr\Big[&\tilde{U}_i^\dagger U_iU_k^\dagger\tilde{U}_k(I_d - \tilde{U}_k^\dagger U_kU_i^\dagger\tilde{U}_i)(\tilde{U}_k^\dagger\tilde{U}_i - I_d)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (\tilde{U}_i^\dagger\tilde{U}_k - I_d)(I_d - \tilde{U}_k^\dagger U_kU_i^\dagger\tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (\tilde{U}_j^\dagger U_kU_j^\dagger\tilde{U}_j - I_d)(I_d - \tilde{U}_j^\dagger U_jU_k^\dagger\tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (I_d - \tilde{U}_j^\dagger U_jU_k^\dagger\tilde{U}_k)(\tilde{U}_k^\dagger\tilde{U}_j - I_d)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (I_d - \tilde{U}_i^\dagger U_iU_k^\dagger\tilde{U}_k)(\tilde{U}_k^\dagger\tilde{U}_i - I_d)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\\ &+ (\tilde{U}_i^\dagger U_kU_i^\dagger\tilde{U}_i - I_d)(I_d - \tilde{U}_i^\dagger U_iU_k^\dagger\tilde{U}_k)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\\ &+ (\tilde{U}_j^\dagger\tilde{U}_k - I_d)(I_d - \tilde{U}_k^\dagger U_kU_j^\dagger\tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\\ &+ \tilde{U}_j^\dagger U_jU_k^\dagger\tilde{U}_k(I_d - \tilde{U}_k^\dagger U_kU_j^\dagger\tilde{U}_j)(\tilde{U}_k^\dagger\tilde{U}_j - I_d)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\Big],\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{B}}_k := tr\Big[&(\tilde{U}_i^\dagger U_iU_k^\dagger\tilde{U}_k - I_d)(I_d - \tilde{U}_k^\dagger U_kU_i^\dagger\tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (\tilde{U}_j^\dagger U_jU_k^\dagger\tilde{U}_k - I_d)(I_d - \tilde{U}_k^\dagger U_kU_j^\dagger\tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\Big],\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{C}}_k := 2tr\Big[&(I_d - \tilde{U}_k^\dagger U_kU_i^\dagger\tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j) + (I_d - \tilde{U}_j^\dagger U_jU_k^\dagger\tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_iU_j^\dagger\tilde{U}_j)\\ &+ (I_d - \tilde{U}_i^\dagger U_iU_k^\dagger\tilde{U}_k)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i) + (I_d - \tilde{U}_k^\dagger U_kU_j^\dagger\tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_jU_i^\dagger\tilde{U}_i)\Big].\end{aligned}$$

Proof We verify relation (55) in several steps.

• Step A: The terms in $\bar{\mathcal{M}}_k$ can be rearranged as follows:

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&- U_iU_k^\dagger U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger + U_iU_j^\dagger\tilde{U}_j\tilde{U}_k^\dagger - U_iU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger\tilde{U}_k\tilde{U}_i^\dagger + U_kU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger\\ &- U_iU_j^\dagger U_kU_j^\dagger\tilde{U}_j\tilde{U}_i^\dagger + U_iU_j^\dagger\tilde{U}_k\tilde{U}_i^\dagger - U_iU_j^\dagger\tilde{U}_j\tilde{U}_k^\dagger\tilde{U}_j\tilde{U}_i^\dagger + U_iU_k^\dagger\tilde{U}_j\tilde{U}_i^\dagger\\ &- \tilde{U}_i\tilde{U}_k^\dagger\tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger + \tilde{U}_i\tilde{U}_j^\dagger U_jU_k^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_i^\dagger U_kU_i^\dagger + \tilde{U}_k\tilde{U}_j^\dagger U_jU_i^\dagger\\ &- \tilde{U}_i\tilde{U}_j^\dagger\tilde{U}_k\tilde{U}_j^\dagger U_jU_i^\dagger + \tilde{U}_i\tilde{U}_j^\dagger U_kU_i^\dagger - \tilde{U}_i\tilde{U}_j^\dagger U_jU_k^\dagger U_jU_i^\dagger + \tilde{U}_i\tilde{U}_k^\dagger U_jU_i^\dagger\Big].\end{aligned}$$

• Step B: Using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, it follows that

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&-\tilde{U}_i^\dagger U_i U_k^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_k^\dagger U_i U_j^\dagger \tilde{U}_j - \tilde{U}_i^\dagger \tilde{U}_k \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_i^\dagger U_k U_j^\dagger \tilde{U}_j \\ &- U_k U_j^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i U_j^\dagger + \tilde{U}_k \tilde{U}_i^\dagger U_i U_j^\dagger - \tilde{U}_k^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + U_k^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i \\ &- \tilde{U}_k^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + U_k^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j - U_k U_i^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger + \tilde{U}_k \tilde{U}_j^\dagger U_j U_i^\dagger \\ &- \tilde{U}_j^\dagger \tilde{U}_k \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_j^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_j^\dagger U_j U_k^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_k^\dagger U_j U_i^\dagger \tilde{U}_i \Big].\end{aligned}$$

• Step C: Again, we use $\mathrm{tr}(UAU^\dagger) = \mathrm{tr}(A)$ for unitary U to find

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&-\tilde{U}_i^\dagger U_i U_k^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_k^\dagger U_i U_j^\dagger \tilde{U}_j - \tilde{U}_i^\dagger \tilde{U}_k \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_i^\dagger U_k U_j^\dagger \tilde{U}_j \\ &- (\tilde{U}_j^\dagger) U_k U_j^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i U_j^\dagger (\tilde{U}_j) + (\tilde{U}_j^\dagger) \tilde{U}_k \tilde{U}_i^\dagger U_i U_j^\dagger (\tilde{U}_j) \\ &- \tilde{U}_k^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + (\tilde{U}_j^\dagger U_j) U_k^\dagger \tilde{U}_j \tilde{U}_i^\dagger U_i (U_j^\dagger \tilde{U}_j) \\ &- \tilde{U}_k^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + (\tilde{U}_i^\dagger U_i) U_k^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j (U_i^\dagger \tilde{U}_i) \\ &- (\tilde{U}_i^\dagger) U_k U_i^\dagger \tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger (\tilde{U}_i) + (\tilde{U}_i^\dagger) \tilde{U}_k \tilde{U}_j^\dagger U_j U_i^\dagger (\tilde{U}_i) \\ &- \tilde{U}_j^\dagger \tilde{U}_k \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_j^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_j^\dagger U_j U_k^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_k^\dagger U_j U_i^\dagger \tilde{U}_i \Big],\end{aligned}$$

which factors into

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&(\tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_i - \tilde{U}_k^\dagger \tilde{U}_i)(-\tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) + (\tilde{U}_i^\dagger \tilde{U}_k - \tilde{U}_i^\dagger U_k U_i^\dagger \tilde{U}_i)(-\tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - \tilde{U}_j^\dagger \tilde{U}_k)(-\tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) + (\tilde{U}_k^\dagger \tilde{U}_j - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_j)(-\tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (\tilde{U}_k^\dagger \tilde{U}_i - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_i)(-\tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) + (\tilde{U}_i^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_i^\dagger \tilde{U}_k)(-\tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \\ &+ (\tilde{U}_j^\dagger \tilde{U}_k - \tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j)(-\tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) + (\tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_j - \tilde{U}_k^\dagger \tilde{U}_j)(-\tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \Big].\end{aligned}$$

• Step D: Each of these eight summands is the product of two terms, and adding the first multiplicative terms of all eight summands results in zero. Thus, we have

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&(\tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_i - \tilde{U}_k^\dagger \tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) + (\tilde{U}_i^\dagger \tilde{U}_k - \tilde{U}_i^\dagger U_k U_i^\dagger \tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - \tilde{U}_j^\dagger \tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) + (\tilde{U}_k^\dagger \tilde{U}_j - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_j)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (\tilde{U}_k^\dagger \tilde{U}_i - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_i)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) + (\tilde{U}_i^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_i^\dagger \tilde{U}_k)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \\ &+ (\tilde{U}_j^\dagger \tilde{U}_k - \tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) + (\tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_j - \tilde{U}_k^\dagger \tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \Big].\end{aligned}$$

• Step E: By the unitarity of U_i,

$$\begin{aligned}\bar{\mathcal{M}}_k = \mathrm{tr}\Big[&\tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k (I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i) \tilde{U}_k^\dagger \tilde{U}_i (I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ \tilde{U}_i^\dagger \tilde{U}_k (I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ \tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j (I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k) \tilde{U}_k^\dagger \tilde{U}_j (I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \\ &+ (I_d - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k) \tilde{U}_k^\dagger \tilde{U}_i (I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \\ &+ \tilde{U}_i^\dagger U_k U_i^\dagger \tilde{U}_i (I_d - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i)\end{aligned}$$

$$
\begin{aligned}
&+ \tilde{U}_j^\dagger \tilde{U}_k (I_d - \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \\
&+ \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k (I_d - \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j) \tilde{U}_k^\dagger \tilde{U}_j (I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) \Big]. \qquad (56)
\end{aligned}
$$

Note that there is a factor of two in $\bar{\mathcal{C}}_k$, and that matrix multiplication is distributive. In (56), the first line is the sum of the first term of $\bar{\mathcal{A}}_k$, the first term of $\bar{\mathcal{B}}_k$, and half of the first term of $\bar{\mathcal{C}}_k$. The second line is the sum of the second term of $\bar{\mathcal{A}}_k$ and half of the first term of $\bar{\mathcal{C}}_k$. The third and fourth lines are the sum of the third and fourth terms of $\bar{\mathcal{A}}_k$, respectively, with each half of the second term of $\bar{\mathcal{C}}_k$. The fifth and sixth lines are the sum of the fifth and sixth terms of $\bar{\mathcal{A}}_k$, respectively, with each half of the third term of $\bar{\mathcal{C}}_k$. The seventh line is the sum of the seventh term of $\bar{\mathcal{A}}_k$ and half of the fourth term of $\bar{\mathcal{C}}_k$. The eighth line is the sum of the eighth term of $\bar{\mathcal{A}}_k$, the second term of $\bar{\mathcal{B}}_k$, and half of the fourth term of $\bar{\mathcal{C}}_k$. □

Proposition 2 *Suppose that the coupling strength K and initial data U^0 and $\tilde{U}^0$ satisfy*

$$
K > K_e > \frac{54}{17} D(H) \approx 3.1765 D(H), \qquad U^0,\ \tilde{U}^0 \in \mathcal{S}(\alpha_1).
$$

Then for any two Lohe flows $\{U_i\}$ and $\{\tilde{U}_i\}$,

$$
-2K(1+3\alpha_1) d(U,\tilde{U})^2 \le \frac{d}{dt} d(U,\tilde{U})^2 \le -2K(1-3\alpha_1) d(U,\tilde{U})^2.
$$

Proof It follows from (53) that

$$
\frac{d}{dt} d((U(t),\tilde{U}(t))^2 = \bar{\mathcal{L}} - \frac{K}{2N} \sum_{k=1}^{N} \bar{\mathcal{M}}_k.
$$

Below, we estimate $\bar{\mathcal{L}}$ and $\bar{\mathcal{M}}_k$ separately.

• Case A (Estimate of $\bar{\mathcal{L}}$): We use the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and (54) to find

$$
\bar{\mathcal{L}} = \mathrm{i} \times \Big[\mathrm{tr}(H_i U_i U_j^\dagger \tilde{U}_j \tilde{U}_i^\dagger) - \mathrm{tr}(U_i U_j^\dagger \tilde{U}_j \tilde{U}_i^\dagger H_i) + \mathrm{tr}(H_i \tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger) - \mathrm{tr}(\tilde{U}_i \tilde{U}_j^\dagger U_j U_i^\dagger H_i) \Big] = 0.
$$

• Case B (Estimate of $\bar{\mathcal{M}}_k$): We use the decomposition of $\bar{\mathcal{M}}_k$ in Lemma 4 to derive the estimate

$$
\bar{\mathcal{M}}_k = 4 d(U,\tilde{U})^2 (1 + \text{small quantities}).
$$

• Case B.1 (Estimate of $\bar{\mathcal{A}}_k$): First, we estimate the four terms below; the remaining terms can be estimated similarly. It follows from (16) and (17) that

$$
\begin{aligned}
\diamond\ & \Big| \mathrm{tr}\Big(\tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k (I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)(\tilde{U}_k^\dagger \tilde{U}_i - I_d)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \Big) \Big| \\
& \le \|\tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k\| \cdot \|I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i\| \cdot \|\tilde{U}_k^\dagger \tilde{U}_i - I_d\| \cdot \|I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j\| \\
& \le D(\tilde{U}) d(U,\tilde{U})^2, \\
\diamond\ & \Big| \mathrm{tr}\Big((\tilde{U}_i^\dagger \tilde{U}_k - I_d)(I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \Big) \Big| \\
& \le \|\tilde{U}_i^\dagger \tilde{U}_k - I_d\| \cdot \|I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i\| \cdot \|I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j\| \\
& \le D(\tilde{U}) d(U,\tilde{U})^2, \\
\diamond\ & \Big| \mathrm{tr}\Big((\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - I_d)(I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) \Big) \Big| \\
& \le \|\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - I_d\| \cdot \|I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k\| \cdot \|I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j\|
\end{aligned}
$$

$$\leq D(U)d(U,\tilde{U})^2,$$

$$\diamond \quad \left|\mathrm{tr}\Big((I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k)(\tilde{U}_k^\dagger \tilde{U}_j - I_d)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j)\Big)\right| \leq D(\tilde{U})d(U,\tilde{U})^2. \tag{57}$$

In the third estimate of (57), we used the relation:

$$\|\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - I_d\| = \|\tilde{U}_j(\tilde{U}_j^\dagger U_k U_j^\dagger \tilde{U}_j - I_d)\tilde{U}_j^\dagger\| = \|U_k U_j^\dagger - I_d\| = \|U_k - U_j\| \leq D(U).$$

The other terms can be estimated similarly; thus, we obtain

$$|\bar{\mathcal{A}}_k| \leq (2D(U) + 6D(\tilde{U}))d(U,\tilde{U})^2 \leq 8\alpha_1 d(U,\tilde{U})^2.$$

$\diamond$ Case B.2 (Estimate of $\bar{\mathcal{B}}_k$): Similar to Case A, we obtain using (37) that

$$|\bar{\mathcal{B}}_k| \leq 2d(U,\tilde{U})^3 \leq 2(D(U) + D(\tilde{U}))d(U,\tilde{U})^2 \leq 4\alpha_1 d(U,\tilde{U})^2.$$

$\diamond$ Case B.3 (Estimate of $\bar{\mathcal{C}}_k$): Recall that

$$\begin{aligned}\bar{\mathcal{C}}_k := 2\mathrm{tr}\Big[&(I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j) + (I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j)\\ &+ (I_d - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i) + (I_d - \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j)(I_d - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i)\Big].\end{aligned}$$

Directly expanding the expression for $\bar{\mathcal{C}}_k$ yields

$$\begin{aligned}\bar{\mathcal{C}}_k = 2\mathrm{tr}\Big[&(I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j)\\ &+ (I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j)\\ &+ (I_d - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i)\\ &+ (I_d - \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)\Big].\end{aligned}$$

Using the relation $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\mathrm{tr}(UAU^\dagger) = \mathrm{tr}(A)$ for unitary U to cancel terms with index k yields

$$\begin{aligned}\bar{\mathcal{C}}_k &= 2\mathrm{tr}\Big[(I_d - \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j)\\ &\qquad + (I_d - \tilde{U}_j^\dagger U_j U_k^\dagger \tilde{U}_k - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j + U_k^\dagger \tilde{U}_k \tilde{U}_i^\dagger U_i)\\ &\qquad + (I_d - \tilde{U}_i^\dagger U_i U_k^\dagger \tilde{U}_k - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + U_k^\dagger \tilde{U}_k \tilde{U}_j^\dagger U_j)\\ &\qquad + (I_d - \tilde{U}_k^\dagger U_k U_j^\dagger \tilde{U}_j - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i + \tilde{U}_k^\dagger U_k U_i^\dagger \tilde{U}_i)\Big]\\ &= 4\mathrm{tr}\Big[2I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j - \tilde{U}_j^\dagger U_j U_i^\dagger \tilde{U}_i\Big]\\ &= 4\mathrm{tr}\Big[(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j)(I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j)^\dagger\Big]\\ &= 4\|I_d - \tilde{U}_i^\dagger U_i U_j^\dagger \tilde{U}_j\|^2\\ &= 4d(U(t),\tilde{U}(t))^2.\end{aligned}$$

Note that indices i and j are chosen so that the last equality holds. Finally, in (53), we combine all estimates in Case A and Case B to obtain

$$\left|\frac{d}{dt}d(U,\tilde{U})^2 + 2Kd(U,\tilde{U})^2\right| \leq 6\alpha_1 K d(U,\tilde{U})^2.$$

This completes the proof. □

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