3-(b) Proof

$$Z_n = \int_{\Omega} d\Omega\, e^{i\vec{k}\cdot\vec{n}} B(k) = \frac{1}{(2\pi)^4}\int_{[-\pi,\pi]^4} \frac{1}{4}\frac{e^{ik_\nu n_\nu}}{\sin(k_\mu/2)\sin(k_\mu/2)}\, dk_1 dk_2 dk_3 dk_4$$

$$= \frac{1}{(2\pi)^4}\int_{[-\pi,\pi]^4} e^{ik_\nu n_\nu} \frac{1}{2\,(\sum_\mu 1-\cos k_\mu)}\, dk_1 dk_2 dk_3 dk_4 \ .$$

since $\displaystyle \frac{1}{2\,(\sum_\mu 1-\cos k_\mu)} = \int_0^\infty e^{-2x\,[\sum_\mu (1-\cos k_\mu)]}\, dx,$

$$= \frac{1}{(2\pi)^4}\int_{[-\pi,\pi]^4} e^{ik_\nu n_\nu}\int_0^\infty e^{-2x\,[\sum_\mu (1-\cos k_\mu)]}\, dx\ dk_1 dk_2 dk_3 dk_4$$

$$= \frac{1}{(2\pi)^4}\int_{[-\pi,\pi]^4}\int_0^\infty e^{ik_\nu n_\nu} e^{-2x\,[\sum_\mu (1-\cos k_\mu)]}\, dx\ dk_1 dk_2 dk_3 dk_4$$

$$= \frac{1}{(2\pi)^4}\int_0^\infty e^{-8x}\int_{[-\pi,\pi]^4} e^{ik_\mu n_\mu + 2x\cos k_\mu}\, dk_1 dk_2 dk_3 dk_4 dx \quad (*)$$

Arfken Exercise 14.1.15 (b) gives $\displaystyle J_{n_\mu}(2i\,x) = \frac{i^{-n_\mu}}{2\pi}\int_0^{2\pi} e^{i(2ixcos\theta + n_\mu\theta)}\, d\theta$

Arfken Equaton 14.99 gives $I_{n_\mu}(2x) = i^{-n_\mu} J_{n_\mu}(2i\,x)$

By comvining these, we get $\displaystyle I_{n_\mu}(2\,x) = \frac{(-1)^{n_\mu}}{2\pi}\int_0^{2\pi} e^{i(2ixcos\theta + n_\mu\theta)}\, d\theta$

By set $\theta = \pi - k_\mu$, we get

$$I_{n_\mu}(2\,x) = \frac{(-1)^{n_\mu}}{2\pi}\int_{-\pi}^{\pi} e^{i(-2ixcosk_\mu + n_\mu(\pi - k_\mu))}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2xcosk_\mu - in_\mu k_\mu}\, dk_\mu$$

Finally $k_\mu \to -k_\mu$ gives $\displaystyle I_{n_\mu}(2\,x) = \int_{-\pi}^{\pi} e^{2xcosk_\mu + in_\mu k_\mu}\, dk_\mu$

from (*),

$$Z_n = \frac{1}{(2\pi)^4}\int_0^\infty e^{-8x}\int_{[-\pi,\pi]^4} e^{ik_\mu n_\mu + 2x\cos k_\mu}\, dk_1 dk_2 dk_3 dk_4 dx = \int_0^\infty e^{-8x}\prod_\mu I_{n_\mu}(2x)\, dx.$$