Computational Physics Final Exam.

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Abstract

When you hand in your answer, you should gather all your program files including the Makefile into a single tar file using the command: tar -czf <file name>.tar.gz <file 1> <file 2> $\cdots$, and email it to `snucp2019@gmail.com`.

In the email, please write down your name and student ID, and describe the contents of attached files such that the TA can identify who submits the answer to the final exam. You must use the GNU make to compile your code. The Makefile should be included in the tar file. You can submit the answer only once. We will **not** receive any re-submission of the answer file. **Any re-submission will be discarded**. Since you have only one chance to submit the file, please be very careful when you submit the answer.

Numerical Algorithms

1. ⟨ `Finding Roots` ⟩ [30 points]

   Let us consider a function $f(x)$ defined as

$$
\begin{aligned}
f(x) &= \sin[\, g \ \sin(h)] && (1)\\
g(x) &= \frac{10x}{x^2+1} && (2)\\
h(x) &= \frac{10x^2}{x^4+1} e^{-x^2} && (3)
\end{aligned}
$$

   (a) Make a plot of $f(x)$ as a function of x in the range of $x \in [0, 10]$.

   `[HINT]` You may use the `gnuplot` program in the linux system.

   (b) Find all the roots in the range of $x \in [0, +\infty]$, using the Newton method. The values of the roots must have a numerical precision better than 1.0×10^{-10}.

2. ⟨ `Calculating 2-dimensional Integrals` ⟩ [30 points]

   Let us consider a function $f(x, y)$ defined as follows,

$$
\begin{aligned}
f(x, y) &= g \cdot \exp[-p] && (4)\\
g(x, y) &= \cos[j \cos(j)] \ \frac{1}{j^2+1} \ \frac{\sin[k \sin(k)]}{k^{7/2}} && (5)\\
p(x, y) &= x^2 + xy + y^2 && (6)\\
j(x, y) &= x^4 + x^2y^2 + y^4 && (7)\\
k(x, y) &= \sqrt{x^2 + y^2} && (8)
\end{aligned}
$$

   We want to calculate the following integral:

$$
I_2 = \int_0^{+\infty} dx \int_0^{+\infty} dy \ f(x, y) \tag{9}
$$

   Here, note that there is a removable singularity at $x = y = 0$.

   (a) In the class you learned the Trapezoidal rule. Extend it to the 2-dimensional case and obtain the integral I_2 with a numerical precision better than 1.0×10^{-10}.

   `[HINT]` You may try the lattice spacing $h = 0.1, 0.075, 0.05, 0.025$ and then fit the results to a simple fitting function and obtain the results in the limit of $h \to 0$.

(b) In the class you learned the Simpson rule. Repeat the previous problem (a) with the Simpson rule.

(c) Compare the results of the Trapezoidal rule with those of the Simpson rule.

(d) In the class, you learned the Monte Carlo method. Using the Metropolis algorithm, obtain the integral I_2.

`[HINT]` You may choose the weight function w as

$$w(x, y) = \frac{1}{j^2 + 1} \exp[-p] \tag{10}$$

(e) Compare the results of the Monte Carlo method with those of the quadrature method. Which one is more efficient? Explain the reason why it is more efficient.

3. ⟨ `Monte Carlo Methods for 4-dimensional Integral` ⟩ [40 points]

The boson propagator on the 4-dimensional Euclidean lattice is

$$B(k) = \frac{1}{\hat{k}^2} \tag{11}$$

$$\hat{k}^2 = \sum_{\mu=1}^{4} \hat{k}_\mu \cdot \hat{k}_\mu \tag{12}$$

$$\hat{k}_\mu = 2 \sin\left(\frac{k_\mu}{2}\right) \tag{13}$$

where k is a 4-dimensional vector: $k = (k_1, k_2, k_3, k_4)$. The tadpole diagram Z_n can be expressed as

$$Z_n \equiv \int_\Omega d\Omega \ \exp(ik \cdot n) \ B(k) \tag{14}$$

$$\int_\Omega d\Omega = \prod_{\mu=1}^{4} \int_{-\pi}^{+\pi} \frac{dk_\mu}{(2\pi)} \tag{15}$$

where n is a 4-dimensional vector: $n = (n_1, n_2, n_3, n_4)$ with $n_\mu \in Z$. Note that there exists a removable singularity at $k = 0$ in Z_n.

(a) Obtain the following integrals: Z_{0000}, Z_{1000}, Z_{1100}, Z_{1110}, and Z_{1111}.

`[HINT]` You may use the Monte Carlo method with $w(k) = 1$.

(b) [Advanced: extra points] Using the generating functional for the Bessel functions, prove the following identity.

$$Z_n = \int_0^{+\infty} dx \ e^{-8x} \prod_{\mu=1}^{4} I_{n_\mu}(2x) \tag{16}$$

where $I_{n_\mu}(2x)$ is a modified Bessel function. Write down all the details of the proof on a sheet of paper, and scan it to make a pdf file out of it, and submit the pdf file by email. You may not solve this problem if it is too difficult to prove.

`[HINT]` You may refer to Chapter 14 of the book: Mathematical Methods for Physicists (7th Edition) by Arfken, Weber, and Harris.

(c) Using the identity of Eq. (16), it is possible to convert the 4-dimensional intergal into a 1-dimensional integral. Using this identity, calculate the integrals in Problem 3a using the quadrature methods.

`[HINT]` You may use the numerical recipe subroutines for the Bessel functions which are given to you separately.

(d) Compare the results of Problem 3a with those for Problem 3c. Which method is more efficient? Explain the reason why it is more efficient.