Computational Physics Midterm Exam.

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When you hand in the answer, you should gather all your program files including the Makefile into a single tar file using the command: tar -czf <file name>.tar.gz <file 1> <file 2> $\cdots$, and email it to `snucp2019@gmail.com`.

In the email, please write down your name and student ID, and describe the contents of attached files such that the TA can identify who submits the answer to the midterm exam. You must use GNU make to compile your code. The Makefile should be included in the tar file. You can submit the answer only once. We will not receive any resubmission of the answer file. Any resubmission will be discarded. Since you have only one chance to submit the file, please be very careful when you submit the answer.

C Programming Language

1. $\langle$ `File I/O` $\rangle$ [30 points]

   Find the data file "`data.20191105`". This file has the data format of two columns: the first column gives values for the x variable and the second column gives values for the function $f(x)$. Please note that $0 \leq x \leq 5$. Each line of the data has a x value and the corresponding $f(x)$ value. The data set is composed of all the data between $x = 0$ and $x = 5$.

   (a) $\langle$ `File I/O` $\rangle$ Make a code to read in the data of $(x, f(x, i))$ and determine the number of the whole data sets. You must use the `feof()` function to check the end of file.

(b) $\langle$ `Statistical Analysis` $\rangle$ Make a code to calculate the average and statistical error of $f(x)$ for each x value. Let us say that $f(x,i)$ represents the data at coordinate x for the ith data set. Then the statistical average ($\langle f(x)\rangle$) and error ($\sigma(x)$) is defined as

$$\langle f(x)\rangle = \frac{1}{N}\sum_i f(x,i) \tag{1}$$

$$\sigma^2(x) = \frac{1}{N(N-1)}\sum_i \Big[f(x,i) - \langle f(x)\rangle\Big]^2 \tag{2}$$

(c) $\langle$ `Print results` $\rangle$ Print the results of the average and error of $f(x)$ as a function of x.

(d) $\langle$ `Interpretation` $\rangle$ From the statistical analysis on the results for $f(x)$, let us guess that the functional form is $f(x) = a \cdot x + b$, and then obtain the statistical average of a and b.

(e) $\langle$ `Advanced question` $\rangle$ Obtain the statistical error of a and b. [HINT] If this question is too difficult for you, then you may skip it.

2. $\langle$ `Factorial` $\rangle$ [30 points]

When you calculate factorial of a number (ex: 100!), you will be caught in a problem of precision. The number is too big to be represented by the integer (int) type in C/C++ language. Note that the integer type (int) number should be smaller than $2^{32} = 4294967296$. Therefore if the number is bigger than 2^{32}, it is not possible to represent it using the integer type. In the case of gamma functions or factorial of a number (ex: 100!), it is usually much bigger than 2^{32}. Hence, in order to calculate this big number, one needs a code which can handle an arbitrarily high precision in integer arithmetics. The main goal of this problem is to write a program which can calculate factorial (gamma function) of an arbitrary integer to a full precision. Note that one should not use a double precision floating point number, which can not represent the full precision. One should use an array of integers to represent the gamma function of an arbitrary positive integer. The code should be general enough to calculate $x!$ for an arbitrary integer x and print it in its full precision.

[HINT] For the mathematical definition of the gamma function and factorial, refer to Chap. 13 of the book: Mathematical Methods for Physicists (7th Edition) by Arfken, Weber, and Harris.

(a) In the class, you learned how to define arrays on the data memory using the `malloc()` function. You must use these to represent a very big number, which can not be represented by the interger type.

(b) In the class, you learned self-referential structures. You must use these to represent a very big number, which can not be represented by the interger type.

(c) [HINT] Think about an array of integers and each element of the array represent a part of a big integer. For example, let each integer in the array represent two digits of the big integer. Then all you need to know is how to multiply a number to this array.

(d) The code must also convert the full precision number of $x!$ into a double precision number and print it.

3. $\langle$ `Bernoulli Numbers` $\rangle$ [40 points]

The Bernoulli numbers satisfies the following recursion relation:

$$B_0 = 1, \qquad B_1 = -\frac{1}{2} \tag{3}$$

$$N - \frac{1}{2} = \sum_{n=1}^{N} \binom{2N+1}{2n} B_{2n} \qquad \text{with} \quad N \geq 1 \tag{4}$$

$$N - 1 = \sum_{n=1}^{N-1} \binom{2N}{2n} B_{2n} \qquad \text{with} \quad N \geq 2 \tag{5}$$

where the binomial coefficient is defined as

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!} \tag{6}$$

[HINT] For the mathematical definition of the Bernoulli numbers, refer to Chap. 12 of the book: Mathematical Methods for Physicists (7th Edition) by Arfken, Weber, and Harris.

(a) Using the code in the previous problem to calculate $n!$, calculate the binomial coefficient in double precision, and obtain B_{2n} with $1 \leq n \leq 10$ by solving Eq. (4).

(b) Using the code in the previous problem to calculate $n!$, calculate the binomial coefficient in double precision, and obtain B_{2n} with $1 \leq n \leq 10$ by solving Eq. (5).
[HINT] You may compare the answer with that of Problem 3a in order to cross-check it.

(c) Using the code in the previous problem to calculate $n!$, calculate the binomial coefficient exactly, and obtain B_{2n} with $1 \leq n \leq 10$ exactly by solving Eq. (4).
[HINT] Express the answer B_{2n} as a fractional number:

$$B_{2n} = \frac{x}{y}, \qquad \text{with} \quad x, y \in Z\,. \tag{7}$$

where x and y must be integers.

(d) $\langle$ Advanced question $\rangle$ Express the answer in Eq. (7) in such a form that x and y do not share a common integer factor $n \geq 2$ with each other. In other words, obtain the answer such that the numerator and denominator are not divisible by $n \geq 2$ and $n \in Z$.
[HINT] If this question is too difficult for you, then you may skip it.