1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock
3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.
4. Difference equation for discrete-time system $\leftrightarrow$ Differential equation for continuous-time system
5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$
   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7
6. The analog output of the sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)
   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

     $$y(t) \quad \rightarrow \quad y(k) = y(kT) \quad \text{with} \quad t = kT$$

     where k is an integer and T is a fixed value (sample period, or sampling time).
   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

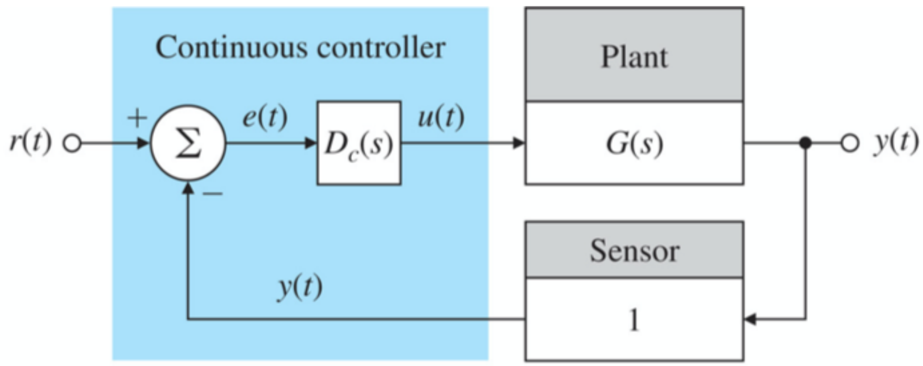
1 Digitization

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)

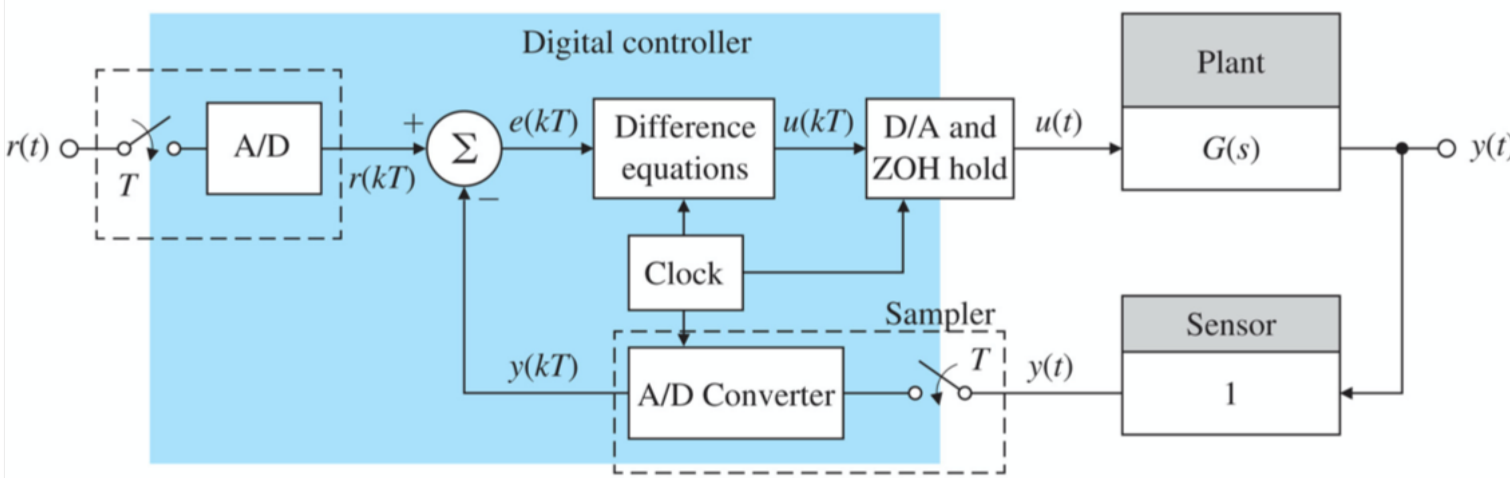
   - Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
   - Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
   - If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

8. A system having both discrete and continuous signals is called a 'sampled data system'.

1 Digitization



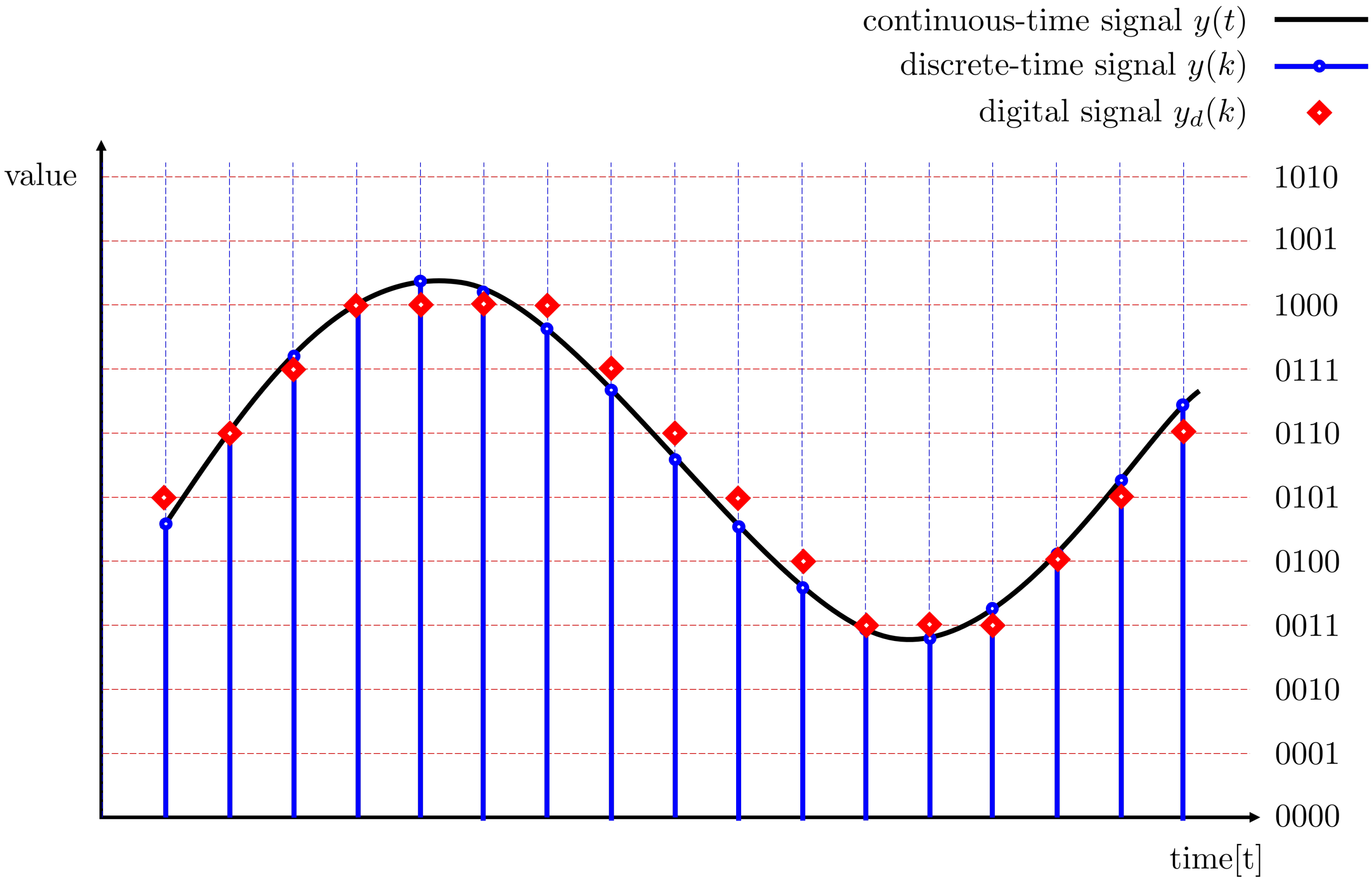
(a)



(b)

1 Digitization

- Continuous-time signal: Both domain and range are continuous, $y(t)$
- Discrete-time signal: Domain is discrete and range is continuous, $y(k)$ or $y(kT)$
- Digital signal: Both domain and range are discrete, $y_d(k)$



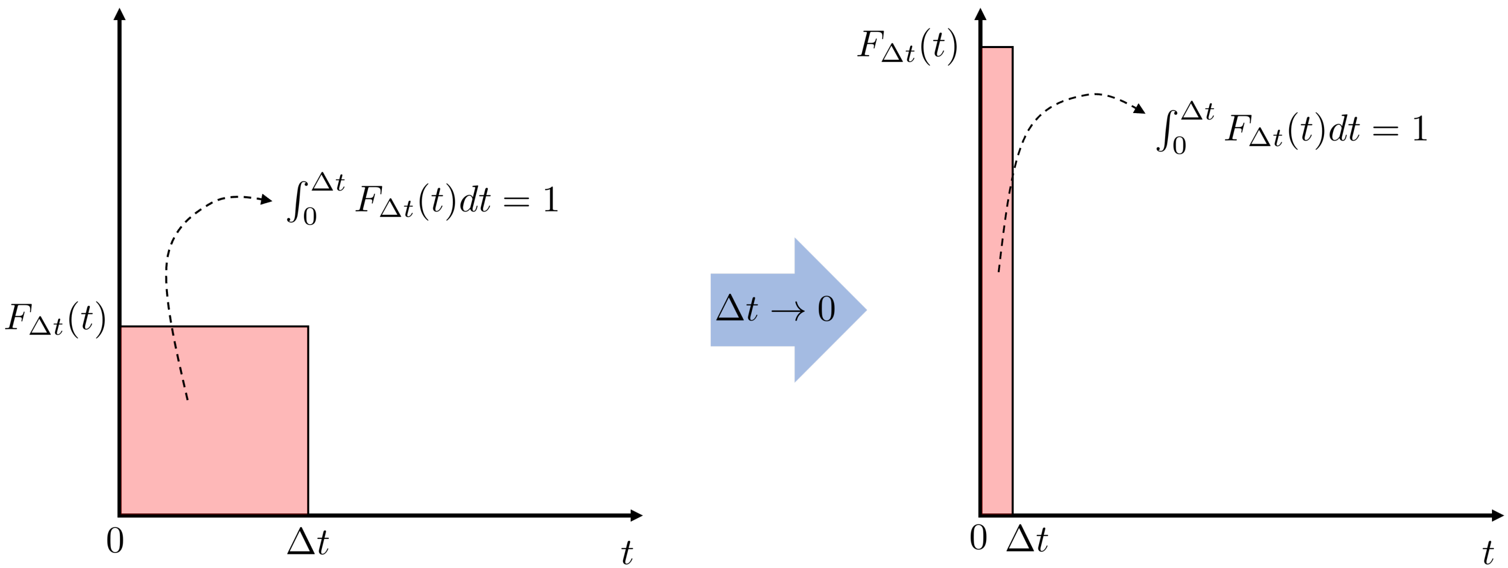
1 Digitization

★ Dirac delta function is mathematically defined as:

1. Approximation

$$F_{\Delta t}(t) = \begin{cases} 1/\Delta t & 0 < t \leq \Delta t \\ 0 & otherwise \end{cases}$$

$$\delta(t) = \lim_{\Delta t \to 0} F_{\Delta t}(t)$$



1 Digitization

2. Generalized function

- $\delta(t) = 0$ **for** $t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t)dt = 1$

★ Unit step function is mathematically defined as:

$$1(t) = \begin{cases} 1 & t > 0 \\ undefined & t = 0 \\ 0 & t < 0 \end{cases}$$

In contol theory

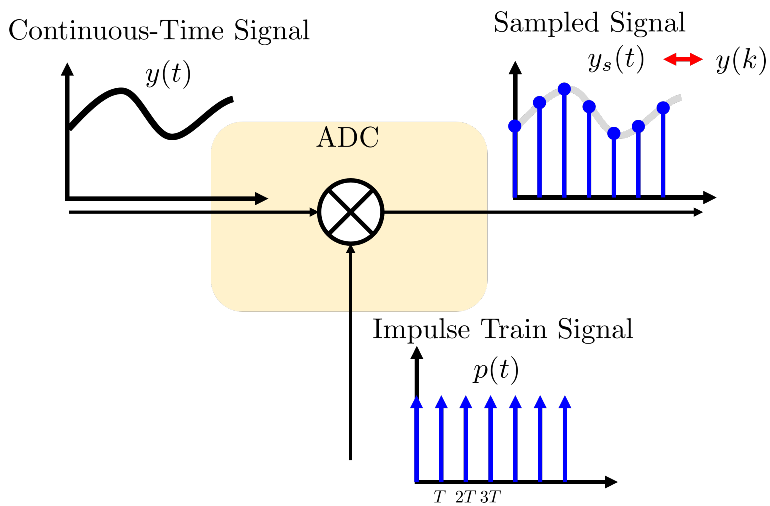
$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

★ Useful Properties

1. $\frac{d1(t)}{dt} = \delta(t)$ (수학적으로는 틀림, 개념적으로 사용)
2. $x(t)\delta(t - kT) = x(kT)\delta(t - kT)$
3. $\int_{-\infty}^{\infty} x(t)\delta(t - kT)dt = x(kT)$
   $\because \int_{-\infty}^{\infty} x(t)\delta(t - kT)dt = \int_{-\infty}^{\infty} x(kT)\delta(t - kT)dt = x(kT) \int_{-\infty}^{\infty} \delta(t - kT)dt = x(kT)$

1 Digitization

★ Sampling Process



1. Periodic Impulse Train: p(t) is periodic with period $T = 1/F_s$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

2. Sampled Signal: we can consider $y_s(t)$ te be the analog equivalent to discrete-time signal $y(k)$ or $y(kT)$

$$y_s(t) = y(t) \cdot p(t) = \sum_{k=-\infty}^{\infty} y(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} y(kT)\delta(t - kT)$$
$$\leftrightarrow y(k) = y(kT)$$

1 Digitization

★ Nyquist–Shannon Sampling Theorem

1. 간단하게, 아날로그 신호가 갖는 최대 주파수의 2배이상의 샘플링 주파수를 사용해야만 손실되는 정보없이 디지털 신호를 아날로그 신호로 복원할 수 있다.

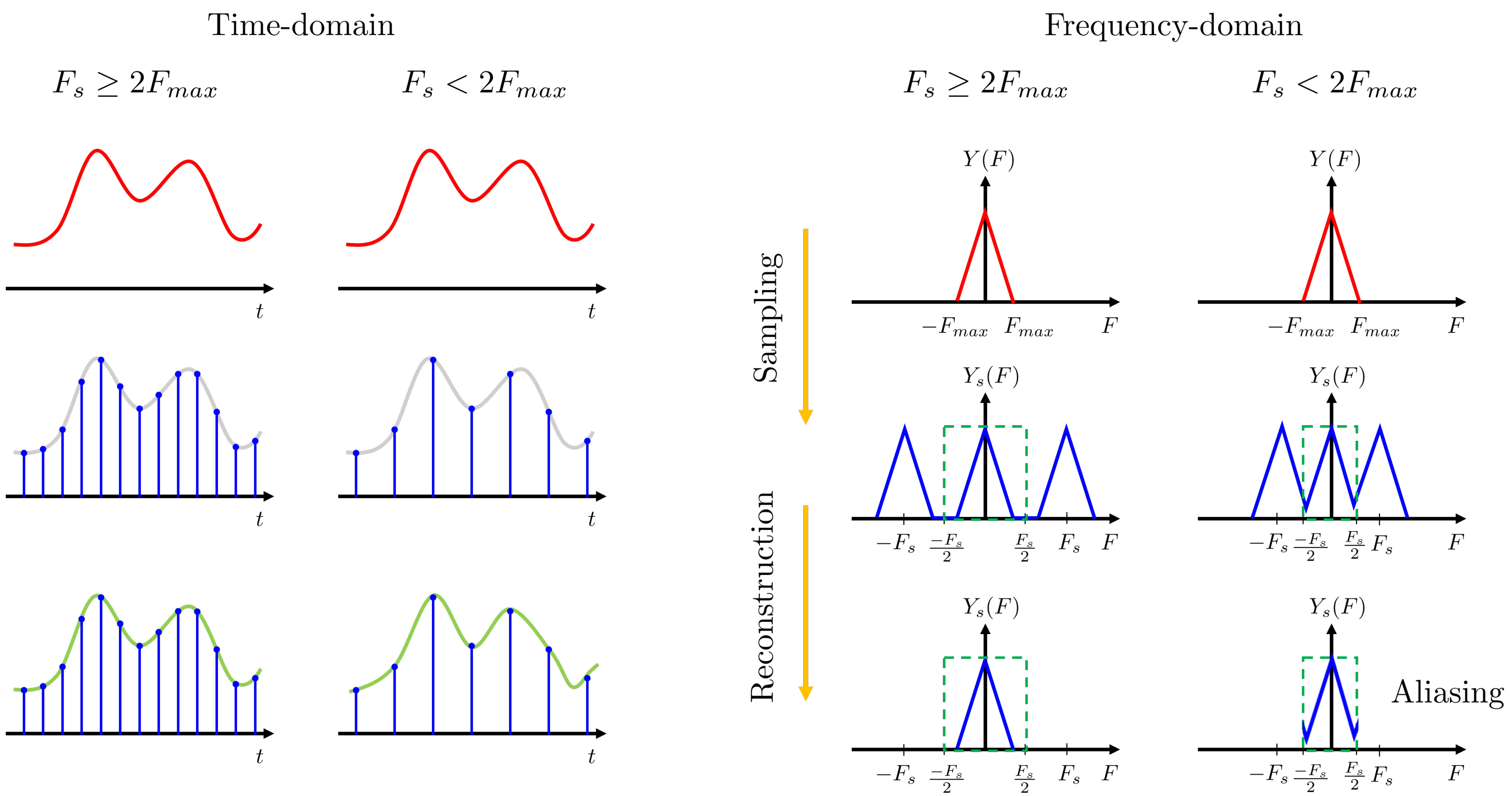
2. Nyquist freqeuncy(Folding Frequency): Sampling Frequency F_s의 절반 $F_{nyquist} = F_s/2$이며, 이는 신호의 최대 주파수 $F_{nyquist} \geq F_{max}$ 이어야 신호를 복원 할 수 있다.

3. Frequency Domain Analysis:

time-domain		frequency-domain
$y_s(t) = y(t) \cdot p(t)$	$\leftrightarrow$	$Y_s(F) = Y(F) * P(F)$

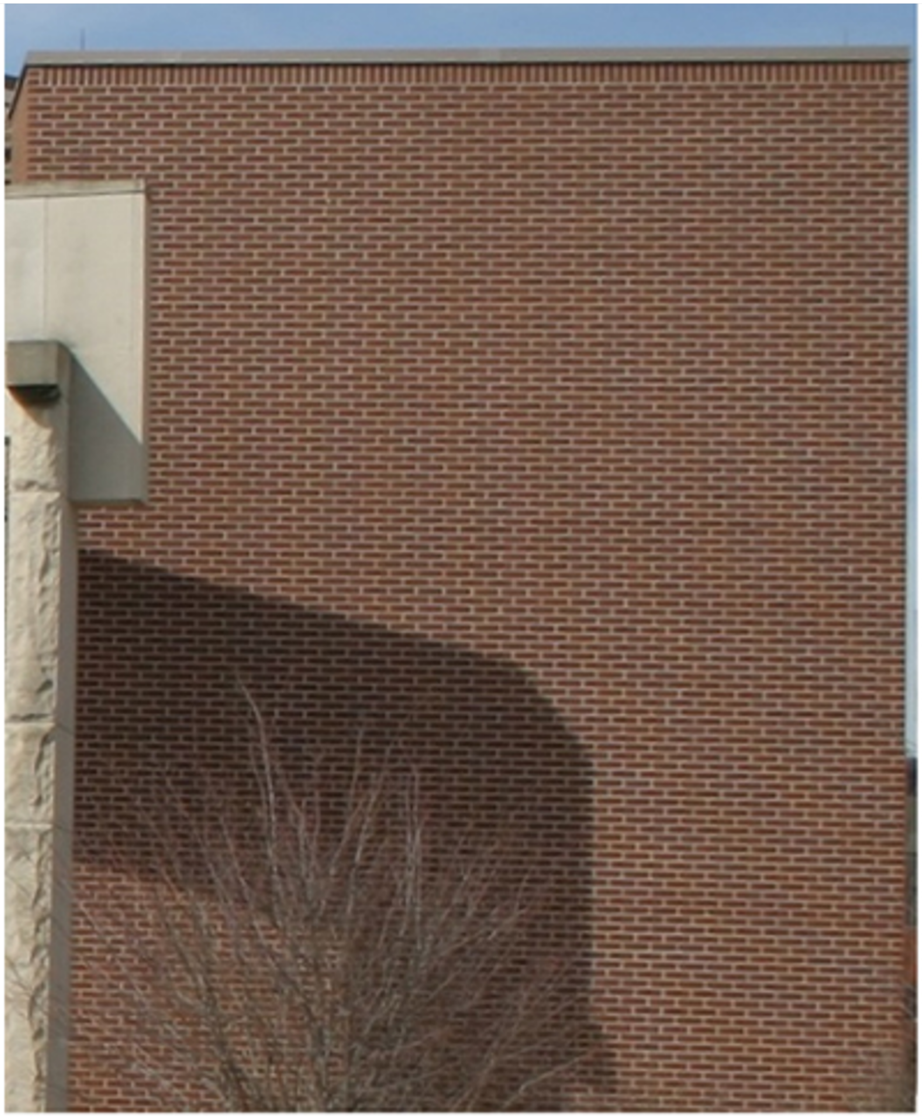
$$
\begin{aligned}
Y_s(F) &= Y(F) * P(F) \\
&= Y(F) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(F - kF_s)
\end{aligned}
$$

1 Digitization



1 Digitization

$F_s \geq 2F_{max}$



$F_s < 2F_{max}$

2 Dynamic Analysis of Discrete Systems

Continuous-Time		Discrete-Time	
periodic	Discrete Fourier Series	periodic	Fourier Series
absolutely integrable	Discrete-Time Fourier Transform	absolutely summable	Fourier Transform
causal	z-Transform	causal	Laplace Transform

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform
  1. Laplace transform and its important property

$$\mathcal{L}(f(t)) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) - f(0^-)$$

$$\Downarrow$$

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) \quad \textbf{where } f(0^+) = 0$$

0^-부터인 이유는 $f(t)$가 $\delta(t)$나 $\frac{d\delta(t)}{dt}$일때 Laplace Transform에 반영하기 위함, 이해를 돕기위해 정확한 정의는 아니지만 아래와 같은 정의 사용

2 Dynamic Analysis of Discrete Systems

2. z-transform is defined by

$$\begin{aligned}\mathcal{Z}(f(k)) = F(z) &= \sum_{k=0}^{\infty} f(k) z^{-k} \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\end{aligned}$$

$$\begin{aligned}\mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1) z^{-k} \\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots \\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right] \\ &= z^{-1}F(z)\end{aligned}$$

where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.

Example) $x(0) = 0, x(1) = 1, x(2) = 2, x(3) = 3, x(4) = 4$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \tag{1}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \tag{2}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \tag{3}$$

3. Important property between LT and z-transform

$$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$\begin{aligned} Y(z) &= -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) \\ &= (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \end{aligned}$$

now we have a discrete transfer function:

$$\begin{aligned} Y(z) - (-a_1 z^{-1} - a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ (1 + a_1 z^{-1} + a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

2 Dynamic Analysis of Discrete Systems

- (8.2.1) z-Transform

1. See the Table 8.1 for understanding between z-transform and LT

$F(s)$	$f(kT)$	$F(z)$	
-	$\delta(kT)$	1	1
-	$\delta(kT-k_0T)$	z^{-k_0}	z^{-k_0}
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$	$\frac{z^{-1}\sin aT}{1-(2\cos aT)z^{-1}+z^{-2}}$
$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$	$\frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}$

2. For parts of Table, we have

$$\mathcal{Z}(\delta(kT)) = \sum_{k=0}^{k=\infty} \delta(kT)z^{-k} = \delta(0\cdot T) + \delta(1\cdot T)z^{-1} + \cdots = 1 + 0z^{-1} + 0z^{-2} + \cdots = 1$$

$$\mathcal{Z}(\delta(k-k_0T)) = 0 + 0z^{-1} + \cdots + 1z^{-k_0} + \cdots + = z^{-k_0}$$

$$\mathcal{Z}(1(kT)) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1-z^{-1}} = (1-z^{-1})^{-1}$$

$$\mathcal{Z}(e^{-akT}) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots = \frac{1}{1-e^{-aT}z^{-1}}$$

3. z-Transform Property

- Linearity :

$$\alpha x_1(k) + \beta x_2(k) \qquad \leftrightarrow \qquad \alpha X_1(z) + \beta X_2(z)$$

- Time-shift :

$$x(k - k_0) \qquad \leftrightarrow \qquad z^{-k_0} X(z)$$

- Multiplication by k :

$$kx(k) \qquad \leftrightarrow \qquad -z\frac{dX(z)}{dz}$$

- Multiplication by a^k:

$$a^k x(k) \qquad \leftrightarrow \qquad X(\frac{z}{a})$$

- Multiplication by e^{-akT}:

$$e^{-akT} x(k) \qquad \leftrightarrow \qquad X(e^{aT} z)$$

$$
\begin{aligned}
\mathcal{Z}(1-e^{-akT}) &= \mathcal{Z}(1)-\mathcal{Z}(e^{-akT}) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}} \\
&= \frac{z^{-1}-e^{-aT}z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})} = \frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}(\cos(-akT)) &= \mathcal{Z}(\frac{e^{jakT}+e^{-jakT}}{2}) = \frac{\mathcal{Z}(e^{jakT})}{2} + \frac{\mathcal{Z}(e^{-jakT})}{2} \\
&= \frac{1}{2}\cdot\frac{1}{1-e^{jaT}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1-e^{-jaT}z^{-1}} \\
&= \frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}
\end{aligned}
$$

$$
\mathcal{Z}(\sin(-akT)) =?, \mathcal{Z}(\sinh(-akT)) =?, \mathcal{Z}(\cosh(-akT)) =?
$$

2 Dynamic Analysis of Discrete Systems

- (8.2.2) z-Transform Inversion

  1. A z-transform inversion technique that has no continuous counterpart is called 'long division'. For example, consider a first-order discrete system

  $$y(k) = \alpha y(k-1) + u(k) \quad \rightarrow \quad Y(z) = \alpha z^{-1} Y(z) + U(z) \quad \rightarrow \quad \frac{Y(z)}{U(z)} = \frac{1}{1-\alpha z^{-1}}$$

  For a unit-pulse input, its z-transform is

  $$U(z) = \mathcal{Z}(\delta(kT)) = 1$$

  so the long division becomes

  $$\begin{aligned} Y(z) &= \frac{1}{1-\alpha z^{-1}} \\ &= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \cdots \end{aligned}$$

  We see that the sampled time history of y is

  $$y(0) = 1 \qquad y(1) = \alpha \qquad y(2) = \alpha^2 \qquad y(3) = \alpha^3 \qquad \cdots$$

$$
\begin{array}{r l}
 & 1+\alpha z^{-1}+\alpha^{2}z^{-2}+\alpha^{3}z^{-3} \\ \cline{2-2}
1-\alpha z^{-1}) & 1 \\ \cline{2-2}
 & 1-\alpha z^{-1} \\ \cline{2-2}
 & \quad \alpha z^{-1}+0 \\
 & \quad \alpha z^{-1}-\alpha^{2}z^{-2} \\ \cline{2-2}
 & \qquad \alpha^{2}z^{-2}+0 \\
 & \qquad \alpha^{2}z^{-2}-\alpha^{3}z^{-3} \\ \cline{2-2}
 & \qquad\quad \alpha^{3}z^{-3}
\end{array}
$$

2 Dynamic Analysis of Discrete Systems

- (8.2.3) Relationship between s and z

1. Consider the continuous signal of

$$f(t) = e^{-at} \qquad t > 0 \qquad \rightarrow \qquad F(s) = \frac{1}{s+a}$$

and it corresponds to a pole $s = -a$.

2. Consider the discrete signal of

$$f(kT) = e^{-akT} \qquad \rightarrow \qquad F(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

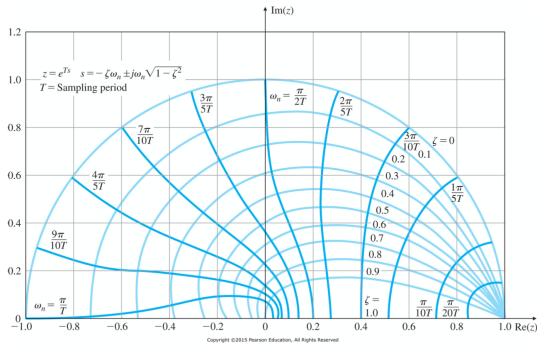
and it corresponds to a pole $z = e^{-aT}$.

3. The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

$$\begin{aligned} z &= e^{sT} \\ &= e^{-aT+jbT} = e^{-aT}(\cos bT + j\sin b) \\ &= e^{-\sigma T}(\cos \omega_d T + j \sin \omega_d T) \\ &= e^{-\zeta\omega_n T}(\cos \omega_n \sqrt{1-\zeta^2}T + j \sin \omega_n \sqrt{1-\zeta^2}T) \end{aligned}$$

where T is the sample period, and $s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

4. See Fig. 8.4, and it shows the mapping of lines of constant damping ζ and natural frequency ω_n from s-plane to the upper half of the z-plane, using $z = e^{sT}$.



http://controlsystemsacademy.com/0003/0003.html

a) The stability boundary $s = 0 \pm j\omega$ becomes the unit circle $|z| = 1$ in the z-plane; inside the unit circle is stable, outside is unstable
s-plane에서의 stability boundary는 imaginary축 $s = \pm j\omega$ 인데 z-plane에서의 stability boundary는 unit circle $|z| = e^{sT}|_{s=\pm j\omega} = 1$이 된다.

b) The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around the origin $s = 0$, in the s-plane.
z-plane 에서의 $z = 1$근방은 s-plane에서의 $s = 0$ 근방과 같다.

c) The z-plane locations give response information normalized to the sample rate rather than to time as in the s-plane.
z-plane 에서의 response information은 s-plane에서와 같이 시간에 대한 정보가 아닌, sample rate로 normalized된 정보를 제공한다.

d) The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T =$ circular sample rate in radians per second.
ω_s가 $2\pi/T$ 일때 음의 z축은 $\omega_s/2$로 표현된다.

e) Vertical lines in the left half of the s-plane (the constant real part of s) map into *circles* within the unit circle of the z-plane
s-plane에서의 좌반면은 z-plane에서의 unit circle 내부로 매핑된다.

f) Horizontal lines in the s-plane (the constant imaginary part of s) map into *radial lines* in the z-plane.
s-plane에서의 수평선은 z-plane에서의 radial line들로 매핑된다.

g) Frequencies greater than $\omega_s/2$, called the Nyquist frequency, appear in the z-plane on the top of corresponding lower frequencies because of the circular characteristics of e^{sT}. This overlap is called *aliasing* or folding.

5. As a result, it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples.

2 Dynamic Analysis of Discrete Systems

- (8.2.4) Final Value Theorem

1. Discrete final value theorem is

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} sX(s) \qquad\qquad \lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1 - z^{-1})X(z)$$

   if all the poles of $(1 - z^{-1})X(z)$ are inside the unit circle.

2. For example, to find the DC gain of the TF

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{z+0.16}$$

   we let $u(k) = 1$ for $k \geq 0$, so that

$$U(z) = \frac{1}{1 - z^{-1}} \quad \text{and} \quad X(z) = \frac{0.58(1+z)}{(1 - z^{-1})(z + 0.16)}$$

   Applying the final value theorem yields

$$x_{ss} = \lim_{z\to 1}(1 - z^{-1})X(z) = \frac{0.58 \cdot 2}{1 + 0.16} = 1$$

   so the DC gain of $G(z)$ is unity.

3 Design using Discrete Equivalents

- 1장부터 7장까지 continuous compensation $D_c(s)$를 설계하는 법에 대해서 배웠으며, 이번 절에서는 $D_c(s)$를 discrete compensation $D_d(z)$로 변환하는 방법들과 각 방법들의 성능에 대해서 배운다.

- 다만, 이 방법들은 모두 근사법이며 완벽하게 $D_c(s)$와 동일한 성능을 갖는 $D_d(z)$를 설계할 수는 없다.

- 일반적인 $D_c(s)$는 Differential equation으로 표현되며 이에 대응하는 $D_d(z)$는 Difference equation으로 표현하게 된다.

- (8.3.1 - 8.3.4), (Tustin,ZOH,MPZ,MMPZ)

3 Design using Discrete Equivalents

- (8.3.1) Tustin's Method

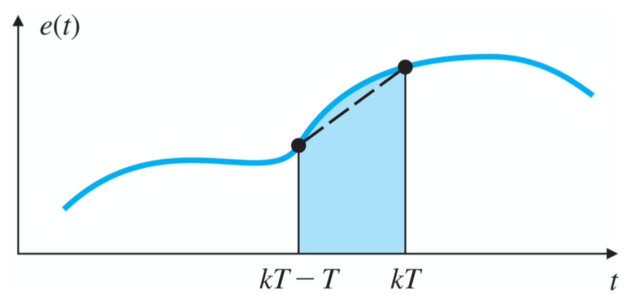
  1. Tustin's method is a digitization technique that approaches the problem as one of numerical integration. Suppose

     $$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}$$

     which is integration. Therefore, it is corresponding to the *trapezoidal integration* as follows:

     $$\begin{aligned} u(kT) &= \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt \\ &= u(kT-T) + \text{area under } e(t) \text{ over last period, } T, \\ &= u(kT-T) + T\frac{[e(kT-T)+e(kT)]}{2} \\ u(k) &= u(k-1) + T\frac{[e(k-1)+e(k)]}{2} \end{aligned}$$

     where T is the sample period.



2. Taking z-transform,

$$\frac{U(z)}{E(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

3. In fact, the Tustin's method approximates $z = e^{sT}$ as follows:

$$s \approx \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

where it can be derived from the Taylor's series expansions as follows:

$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} = \frac{1+\frac{sT}{2}+\frac{s^2T^2}{2^2}+\cdots}{1-\frac{sT}{2}+\frac{s^2T^2}{2^2}-\cdots} \approx \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} = \frac{2+sT}{2-sT} \qquad \rightarrow \qquad s \approx \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

4. For $D_c(s) = \frac{a}{s+a}$ as an example, we have

$$
\begin{aligned}
D_c(s) &= \frac{a}{s+a} \\
D_d(z) &= \frac{a}{s+a}\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a} \\
&= \frac{aT(1+z^{-1})}{2(1-z^{-1}) + aT(1+z^{-1})} \\
&= \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}}
\end{aligned}
$$

$$
\begin{aligned}
D_d(z) &= \frac{U(z)}{E(z)} = \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}} \\
U(z) &= \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}} E(z) \\
((2+aT) - (2-aT)z^{-1})U(z) &= (aT(1+z^{-1})E(z) \\
(2+aT)u(k) - (2-aT)u(k-1) &= aT[e(k) + e(k-1)] \\
\therefore \qquad u(k) &= \frac{(2-aT)}{(2+aT)}u(k-1) + \frac{aT}{(2+aT)}[e(k) + e(k-1)]
\end{aligned}
$$

3 Design using Discrete Equivalents

5. (Example 8.1) Determine the difference equation with a sample rate of 25 times bandwidth using Tustin's approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} \qquad \text{Lead compensator}$$

Since the bandwidth is approximately $\omega_{bd} = 10[rad/s]$, the sampling rate should be

$$\omega_s = 25 \times \omega_{bd} = 250[rad/s] \qquad \rightarrow \qquad f_s = \frac{\omega_s}{2\pi} \approx 40[Hz] \qquad \rightarrow \qquad T = \frac{1}{f_s} = \frac{1}{40} = 0.025[s]$$

The discrete TF can be obtained as

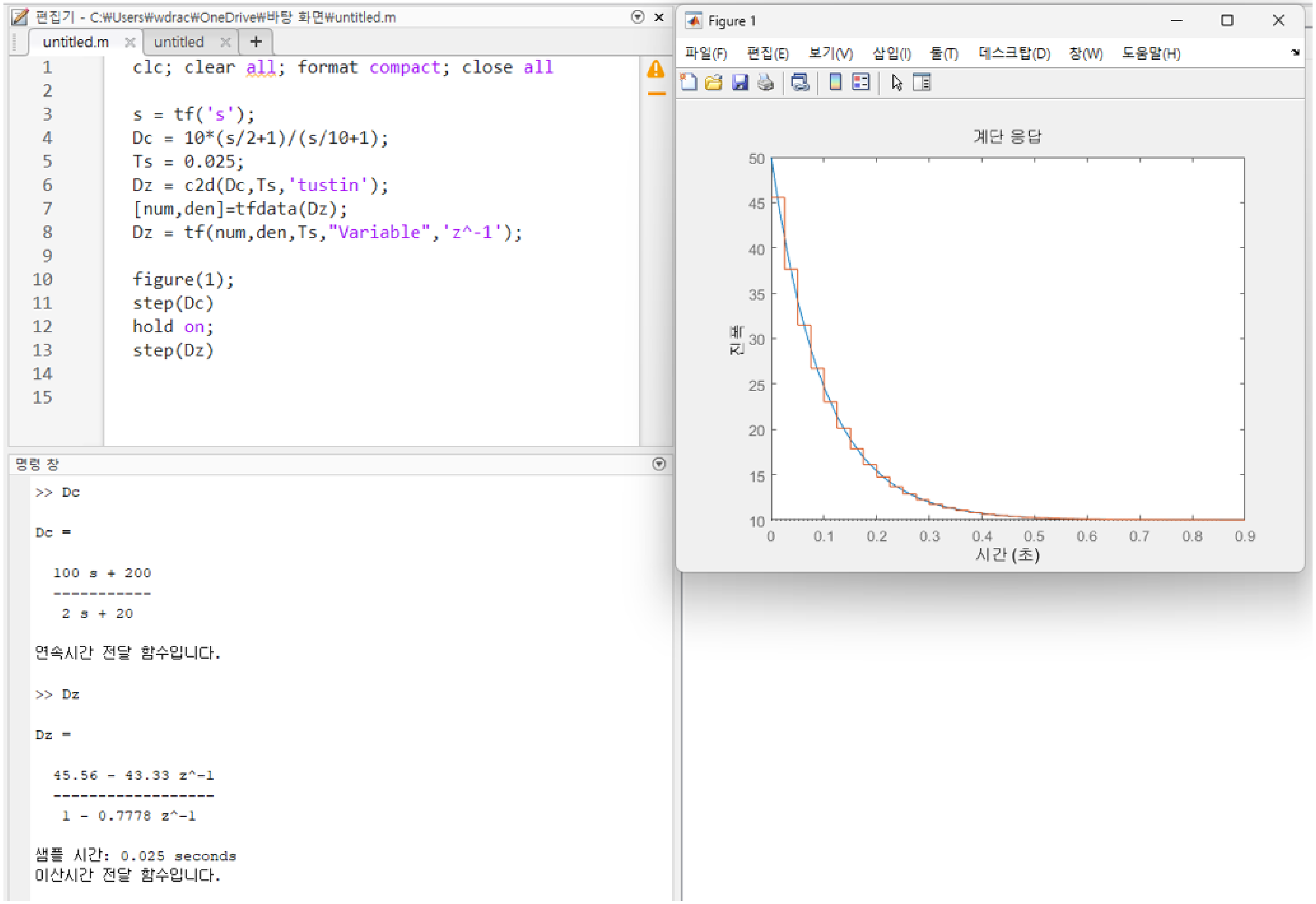
$$\begin{aligned} D_d(z) &= 10\frac{\frac{1}{T}\frac{1-z^{-1}}{1+z^{-1}}+1}{\frac{1}{5T}\frac{1-z^{-1}}{1+z^{-1}}+1} = 10\frac{5(1-z^{-1})+5T(1+z^{-1})}{(1-z^{-1})+5T(1+z^{-1})} \\ &= 50\frac{(1+T)-(1-T)z^{-1}}{(1+5T)-(1-5T)z^{-1}} = 50\frac{1.025-0.975z^{-1}}{1.125-0.875z^{-1}} = \frac{45.556-43.333z^{-1}}{1-0.778z^{-1}} \end{aligned}$$

Finally, the difference equation is

$$u(k) = 0.778u(k-1) + 45.556[e(k) - 0.951e(k-1)]$$

3 Design using Discrete Equivalents

Matlab Example



3 Design using Discrete Equivalents

- (8.3.2) Zeroth-Order Hold (ZOH) Method

  1. Tustin's method essentially assumed that the input to the controller varied linearly early between the past sample and the current sample.
  2. Another assumption is that the input to the controller remains constant throughout the sample period. $\rightarrow$ ZOH
  3. One input sample produces a square pulse of height $e(k)$ that lasts for one sample period T.
  4. For a constant positive step input, $e(k)$, at time k, $E(s) = e(k)/s$, so the result would be

     $$D_d(z) = \mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

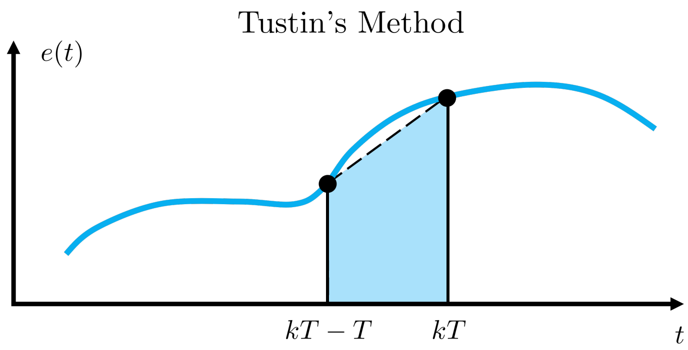
     Furthermore, a constant negative step, one cycle delayed, would be

     $$D_d(z) = z^{-1}\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

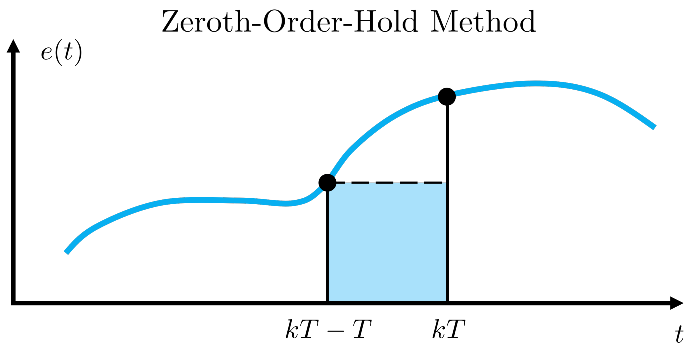
     Therefore, the discrete TF for the square pulse is

     $$D_d(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

3 Design using Discrete Equivalents

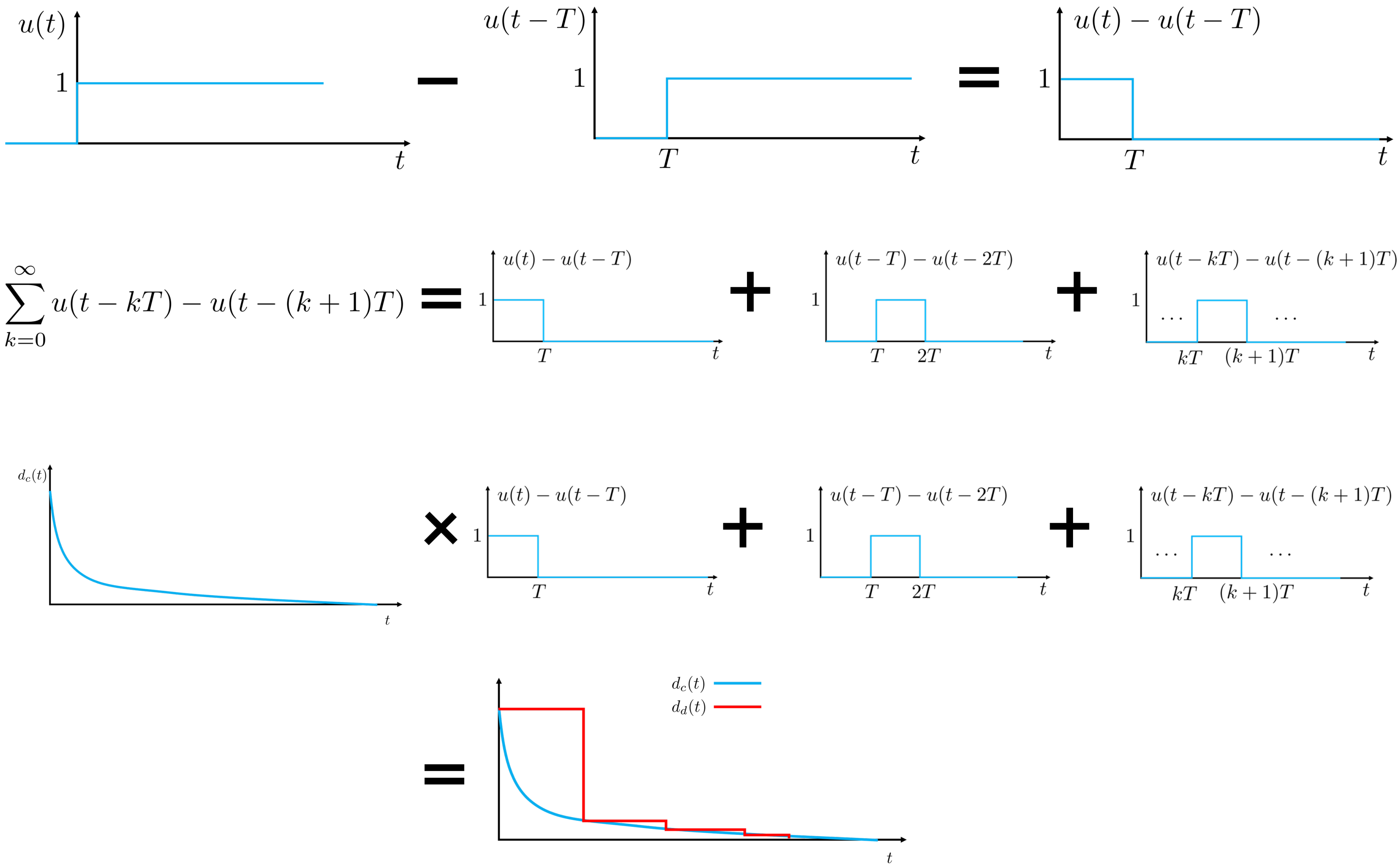


$$u(k) = u(k-1) + T\frac{[e(k-1) + e(k)]}{2}$$



$$u(k) = u(k-1) + Te(k-1)$$

3 Design using Discrete Equivalents



3 Design using Discrete Equivalents

$$\begin{aligned}
d_d(t) &= d_c(t)(\sum_{k=0}^{\infty} u(t-kT) - u(t-(k+1)T)) \qquad = \sum_{k=0}^{\infty} d_c(t)u(t-kT) - \sum_{k=0}^{\infty} d_c(t)u(t-(k+1)T) \\
&= d_c(t) * u(t) - d_c(t) * u(t-T) \\
D_d(z) &= \mathcal{Z}(D_c(s) \cdot (\frac{1}{s} - e^{-sT}\frac{1}{s})) \\
&= \mathcal{Z}((1-e^{-sT})\frac{D_c(s)}{s}) \\
&= (1-z^{-1})\mathcal{Z}(\frac{D_c(s)}{s}) \qquad (\because z = e^{sT})
\end{aligned}$$

5. (Example 8.2) Determine the difference equation with a sample period $T = 0.025[s]$ using ZOH approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} = 10\frac{5s+10}{s+10}$$

The discrete TF using ZOH is

$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{5}{s+10} + \frac{10}{s(s+10)}\right) \\
&= 10(1-z^{-1})\left(\frac{5}{1-e^{-0.25}z^{-1}} + \frac{z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= 10(1-z^{-1})\left(\frac{5(1-z^{-1}) + z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

where $\mathcal{Z}\left\{\frac{1}{s+10}\right\} = \frac{1}{1-e^{-10T}z^{-1}}$ with $e^{-10T} = e^{-0.25} = 0.779$. Or,

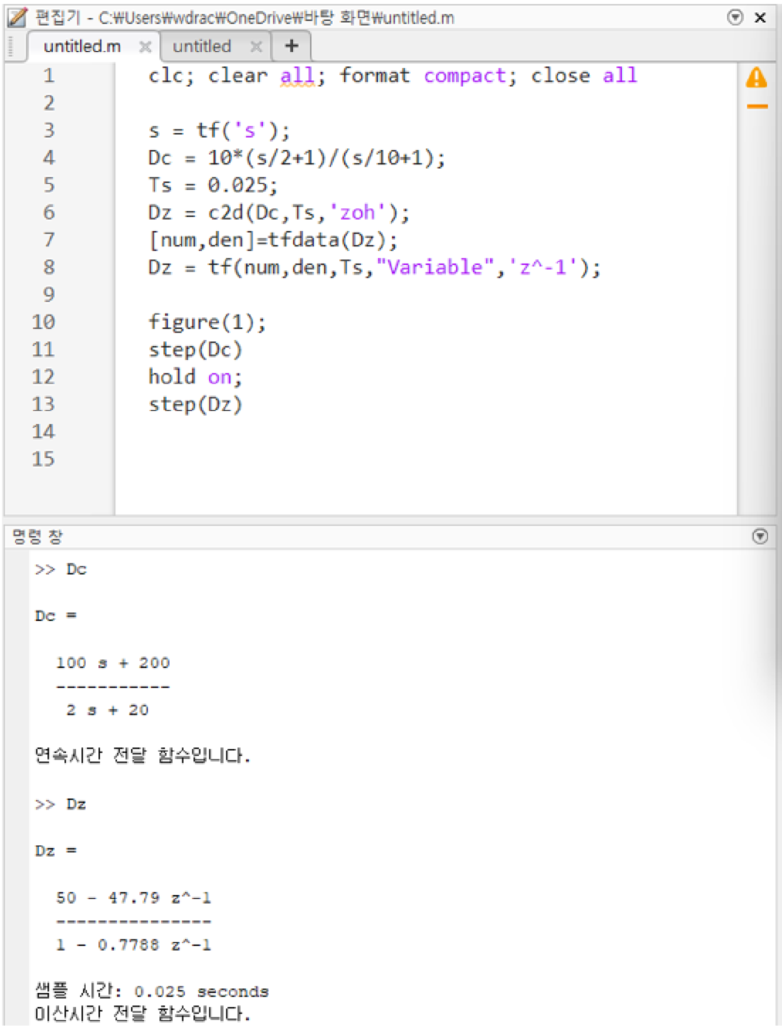
$$
\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{1}{s} + \frac{4}{s+10}\right) \\
&= 10(1-z^{-1})\left(\frac{1}{1-z^{-1}} + \frac{4}{1-e^{-0.25}z^{-1}}\right) \\
&= 10(1-z^{-1})\left(\frac{(1-e^{-0.25}z^{-1}) + 4(1-z^{-1})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50 - 47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}
$$

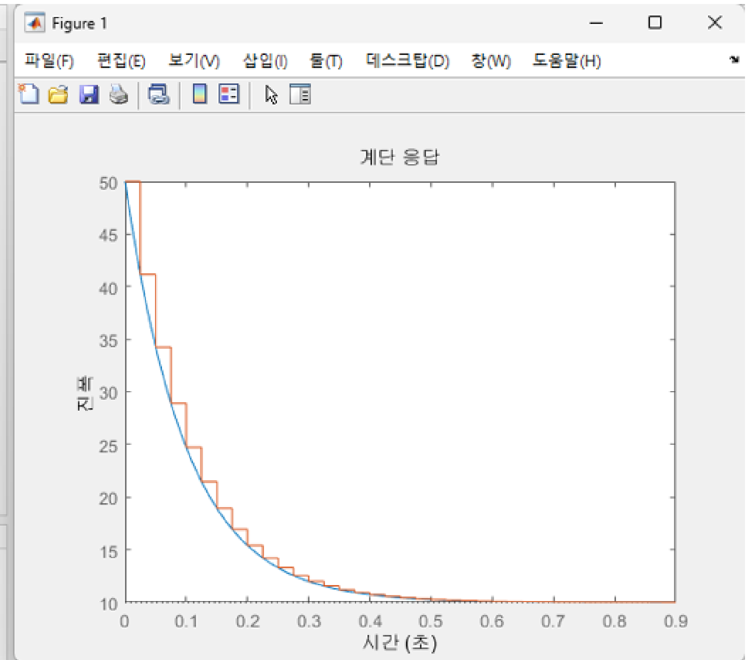
Finally, the difference equation is

$$
\begin{aligned}
u(k) &= 0.779u(k-1) + 50e(k) - 47.79e(k-1) \\
&= 0.779u(k-1) + 50[e(k) - 0.956e(k-1)]
\end{aligned}
$$

3 Design using Discrete Equivalents

Matlab Example





3 Design using Discrete Equivalents

- (8.3.3) Matched Pole-Zero (MPZ) Method

1. Another digitization method, called the matched pole-zero (MPZ) method, is suggested by matching the poles and zeros between s and z planes, using $z = e^{sT}$.
2. Because physical systems often have more poles than zeros, it is useful to arbitrarily add zeros at $z = -1$, resulting in a $(1 + z^{-1})$ term in $D_d(z)$.
   a) Map poles and zeros according to the relation $z = e^{sT}$
   b) If the numerator is of lower order than the denominator, add powers of $(1 + z^{-1})$ to the numerator until numerator and denominator are of equal order.
   c) Set the DC or low frequency gain of $D_d(z)$ equal to that of $D_c(s)$.
3. For example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s+b} \qquad\qquad D_d(z) = K_d \frac{1 - e^{-aT} z^{-1}}{1 - e^{-bT} z^{-1}}$$

where K_d is found by the DC-gain

$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

Thus the result is

$$K_d = K_c \frac{a}{b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

4. As another example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s(s+b)} \qquad\qquad D_d(z) = K_d \frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}$$

where K_d is found by the DC-gain *by deleting the pure integration term* both sides

$$\lim_{s\to 0} sD_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z\to 1}(z-1)D_d(z) = K_d \frac{2(1-e^{-aT})}{1-e^{-bT}}$$

The result is

$$K_d = K_c \frac{a}{2b} \left(\frac{1-e^{-bT}}{1-e^{-aT}} \right)$$

5. (Example 8.3) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$, another real pole at $s = -1.58$, using MPZ digitization

$$G(s) = \frac{1}{s^2}$$

Let us assume that the lead compensator is used

$$D_c(s) = K_c \frac{s+b}{s+a}$$

Then, we have the characteristic equation

$$\begin{aligned} 1 + G(s)D_c(s) = 1 + K_c \frac{s+b}{s^2(s+a)} &= s^3 + as^2 + K_c s + K_c b \\ \alpha_c(s) &= (s^2 + 2\zeta\omega_n s + {\omega_n}^2)(s-p) \\ &= (s^2 + 0.42s + 0.09)(s+1.58) = s^3 + 2s^2 + 0.7536s + 0.1422 \end{aligned}$$

with $a = 2$, $b = 0.19$, and $K_c = 0.7536$. Now we have the lead compensator:

$$D_c(s) = 0.7536 \frac{s+0.19}{s+2} \qquad \rightarrow \qquad D_c(s) = 0.81 \frac{s+0.2}{s+2}$$

3 Design using Discrete Equivalents

Let us determine the sampling rate and sampling period as follows:

$$\omega_s = 0.3 \times 20 = 6[rad/s] \qquad \rightarrow \qquad f_s = \frac{\omega_s}{2\pi} \approx 1[Hz] \qquad \rightarrow \qquad T = 1[s]$$

The MPZ digitization yields

$$D_d(z) = K_d \frac{1 - e^{-0.2} z^{-1}}{1 - e^{-2} z^{-1}} = K_d \frac{1 - 0.818z^{-1}}{1 - 0.135z^{-1}}$$

where the final value theorem gives

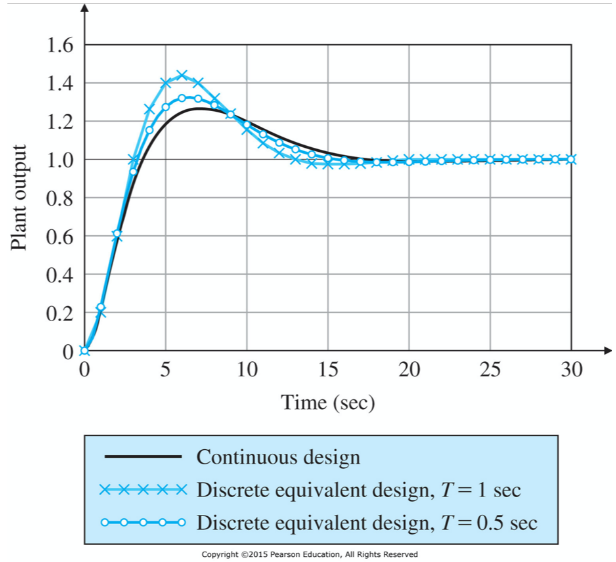
$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

$$0.81 \frac{0.2}{2} = K_d \frac{1 - 0.818}{1 - 0.135} \qquad \rightarrow \qquad K_d = 0.385$$

The difference equation becomes

$$u(k) = 0.135u(k-1) + 0.385[e(k) - 0.818e(k-1)]$$

For the step responses,



3 Design using Discrete Equivalents