1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock
3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.
4. Difference equation for discrete-time system $\leftrightarrow$ Differential equation for continuous-time system
5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$
   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7
6. The analog output of the sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)
   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

     $$y(t) \qquad \rightarrow \qquad y(k) = y(kT) \qquad \text{with} \quad t = kT$$

     where k is an integer and T is a fixed value (sample period, or sampling time).
   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

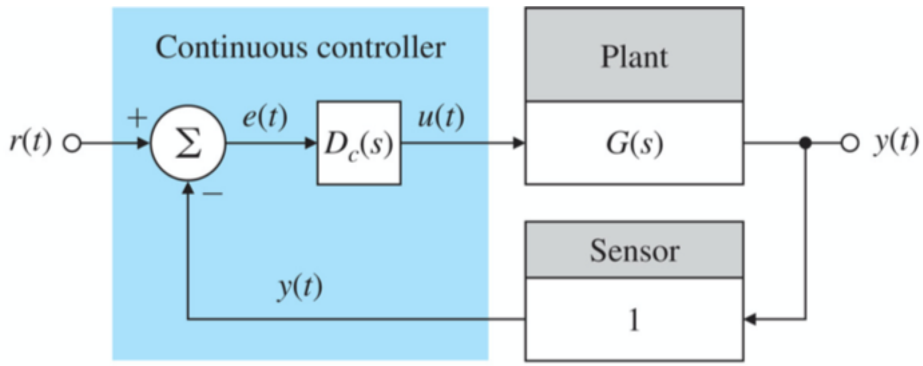
1 Digitization

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)

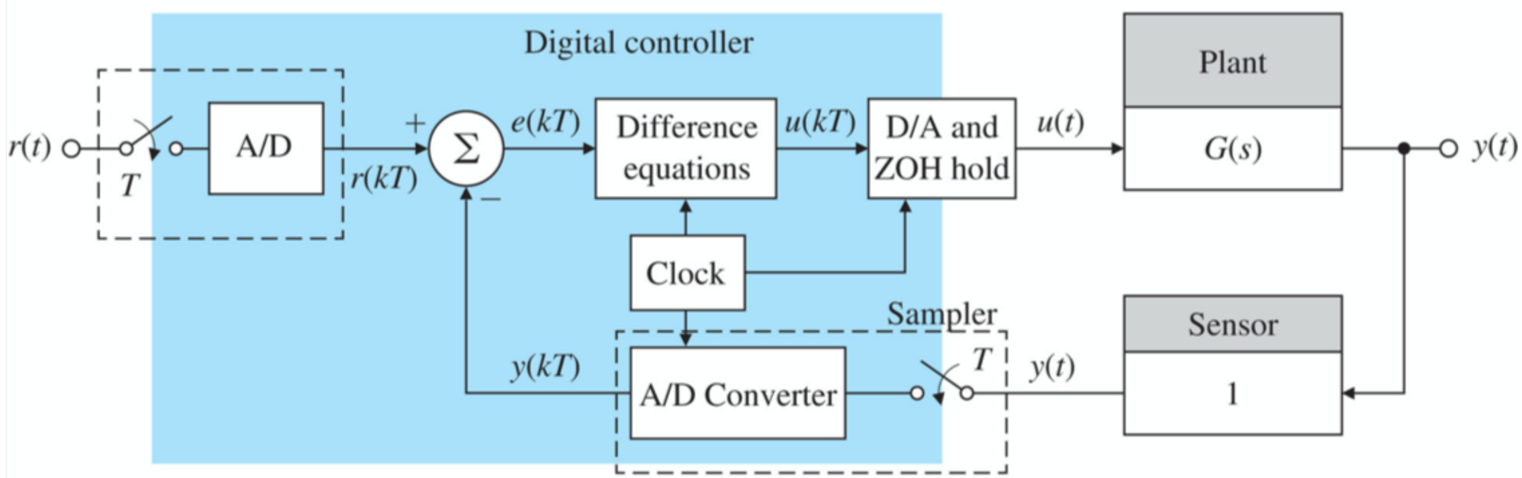
   - Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
   - Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
   - If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

8. A system having both discrete and continuous signals is called a 'sampled data system'.

1 Digitization



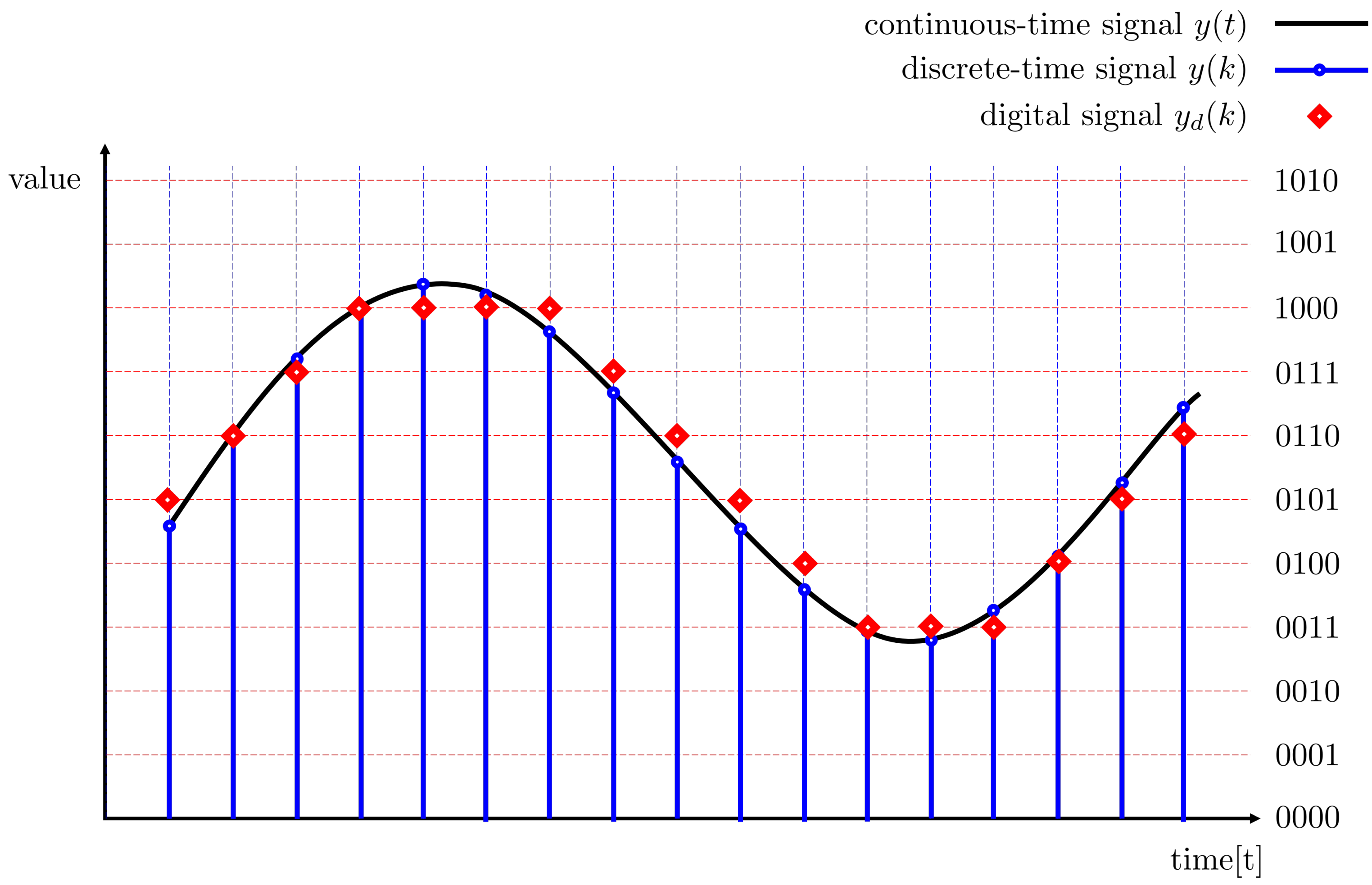
(a)



(b)

1 Digitization

- Continuous-time signal: Both domain and range are continuous, $y(t)$
- Discrete-time signal: Domain is discrete and range is continuous, $y(k)$ or $y(kT)$
- Digital signal: Both domain and range are discrete, $y_d(k)$



1 Digitization

★ Dirac delta function

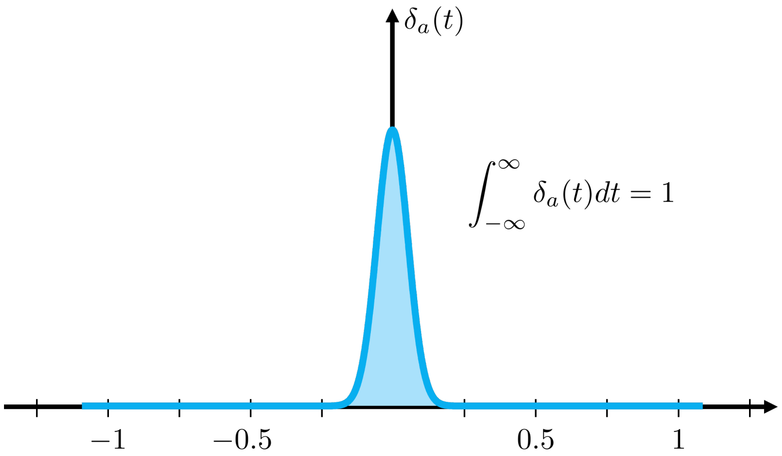
- Approximation

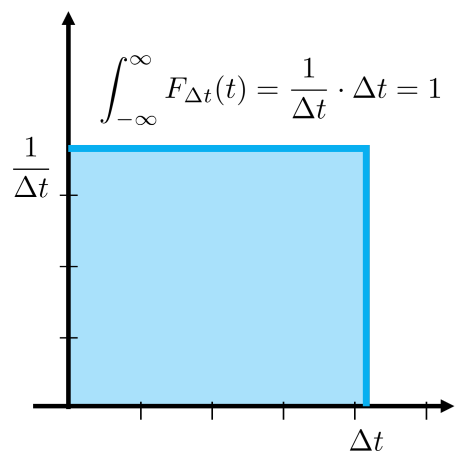
$$\delta_a(t) \doteq \frac{1}{|a|\sqrt{\pi}} e^{-\left(\frac{t}{a}\right)^2}$$

$$\delta(t) \doteq \lim_{a \to 0} \delta_a(t)$$

$$F_{\Delta t}(t) \doteq \begin{cases} 1/\Delta t & 0 < t \leq \Delta t \\ 0 & otherwise \end{cases}$$

$$\delta(t) \doteq \lim_{\Delta t \to 0} F_{\Delta t}(t)$$



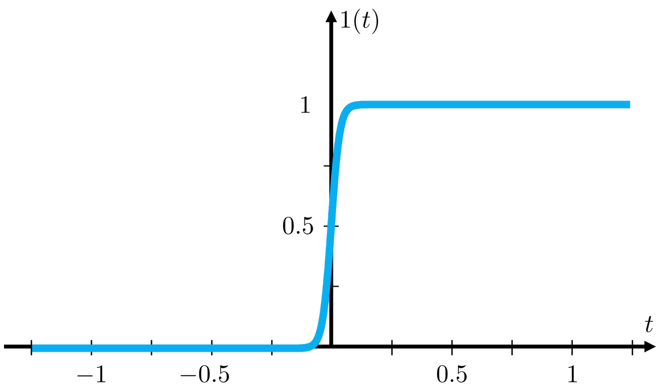


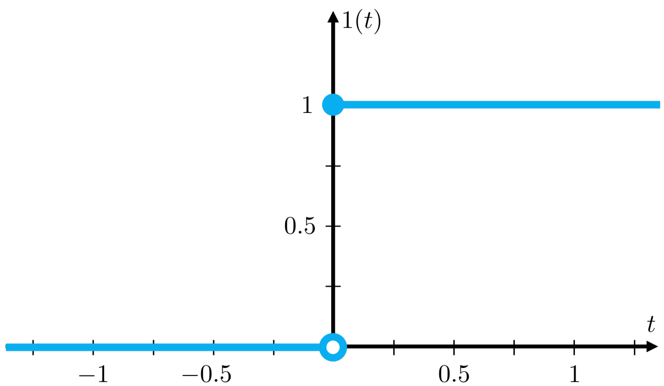
1 Digitization

★ Unit step function

$$1_k(t) = \frac{1}{1+e^{-2kt}} \qquad\qquad 1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$1(t) = \lim_{k\to\infty} 1_k(t)$$





★ Useful Properties

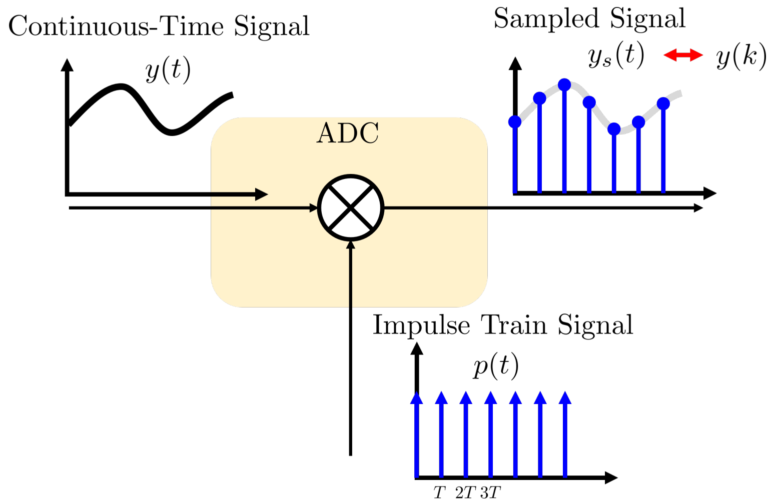
1. $\frac{d1(t)}{dt} = \delta(t)$ (수학적으로는 틀림, 개념적으로 사용)

2. $x(t)\delta(t-kT) = x(kT)\delta(t-kT)$

3. $\int_{-\infty}^{\infty} x(t)\delta(t-kT)dt = x(kT)$
$\because \int_{-\infty}^{\infty} x(t)\delta(t-kT)dt = \int_{-\infty}^{\infty} x(kT)\delta(t-kT)dt = x(kT)\int_{-\infty}^{\infty}\delta(t-kT)dt = x(kT)$

1 Digitization

★ Sampling Process



1. Periodic Impulse Train: p(t) is periodic with period $T = 1/F_s$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

2. Sampled Signal: we can consider $y_s(t)$ te be the analog equivalent to discrete-time signal $y(k)$ or $y(kT)$

$$y_s(t) = y(t) \cdot p(t) = \sum_{k=-\infty}^{\infty} y(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} y(kT)\delta(t - kT)$$
$$\leftrightarrow y(k) = y(kT)$$

1 Digitization

★ Nyquist–Shannon Sampling Theorem

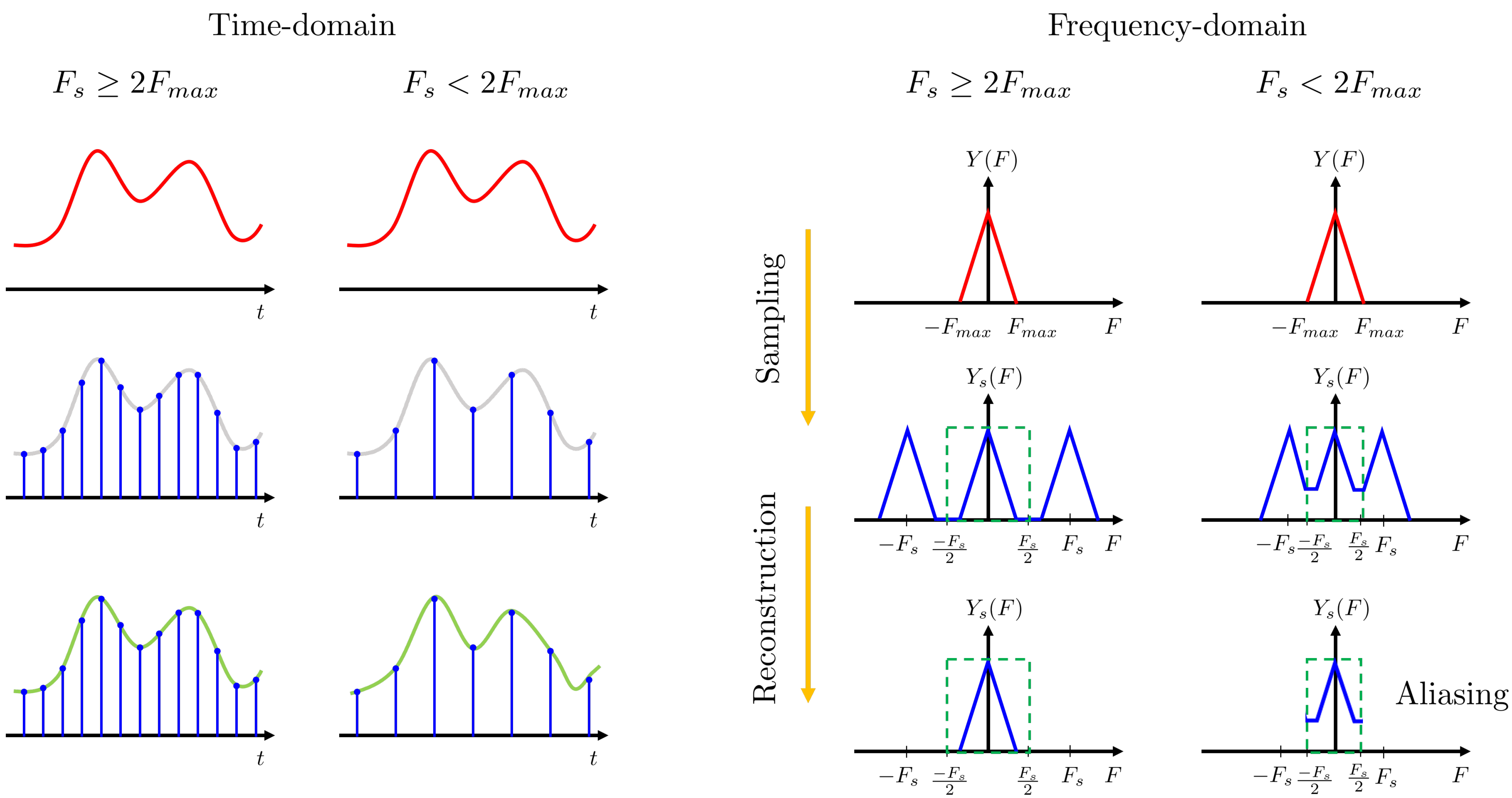
1. 간단하게, 아날로그 신호가 갖는 최대 주파수의 2배이상의 샘플링 주파수를 사용해야만 손실되는 정보없이 디지털 신호를 아날로그 신호로 복원할 수 있다.

2. Nyquist freqeuncy(Folding Frequency): Sampling Frequency F_s의 절반 $F_{nyquist} = F_s/2$이며, 이는 신호의 최대 주파수 $F_{nyquist} \geq F_{max}$ 이어야 신호를 복원 할 수 있다.

3. Frequency Domain Analysis:

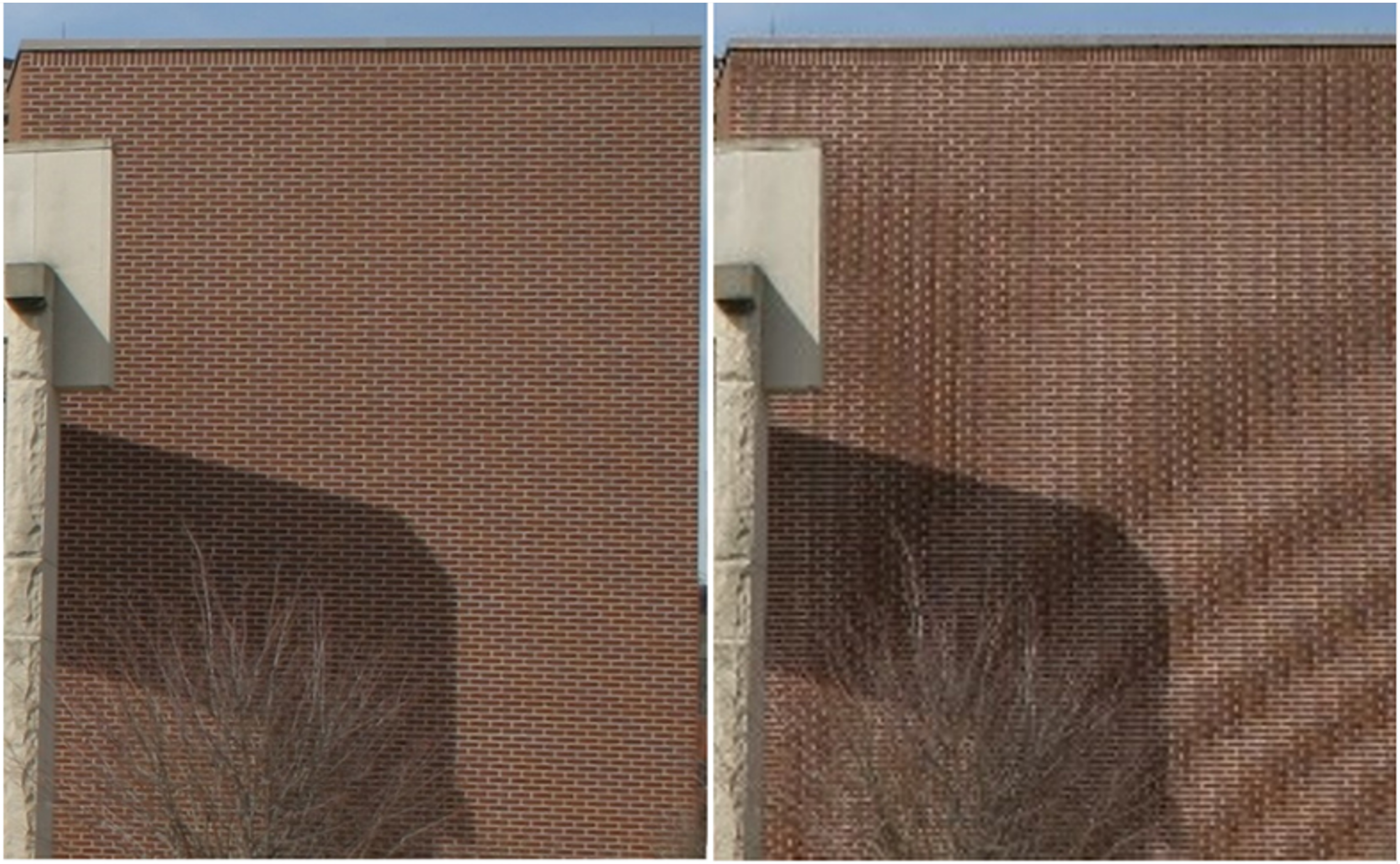
$$
\begin{array}{ccc}
\text{time-domain} & & \text{frequency-domain} \\
y_s(t) = y(t) \cdot p(t) & \leftrightarrow & \begin{aligned} Y_s(F) =& Y(F) * P(F) \\ =& Y(F) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s) \\ =& \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(F - kF_s) \end{aligned}
\end{array}
$$

1 Digitization



1 Digitization

$F_s \geq 2F_{max}$ $\qquad$ $F_s < 2F_{max}$



2 Dynamic Analysis of Discrete Systems

Discrete-Time		Continuous-Time	
periodic	Discrete Fourier Series	periodic	Fourier Series
absolutely summable	Discrete-Time Fourier Transform	absolutely integrable	Fourier Transform
causal	z-Transform	causal	Laplace Transform

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform
  1. Laplace transform and its important property

$$\mathcal{L}(f(t)) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) - f(0^-)$$

$$\Downarrow$$

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) \quad \text{where } f(0^+) = 0$$

0^-부터인 이유는 $f(t)$가 $\delta(t)$나 $\frac{d\delta(t)}{dt}$일때 Laplace Transform에 반영하기 위함, 이해를 돕기위해 정확한 정의는 아니지만 아래와 같은 정의 사용

2. z-transform is defined by

$$\begin{aligned} \mathcal{Z}(f(k)) = F(z) &= \sum_{k=0}^{\infty} f(k) z^{-k} \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots \end{aligned} \qquad \begin{aligned} \mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1) z^{-k} \\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots \\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right] \\ &= z^{-1}F(z) \end{aligned}$$

where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.

Example) $x(0) = 0, x(1) = 1, x(2) = 2, x(3) = 3, x(4) = 4$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \tag{1}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \tag{2}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \tag{3}$$

3. Important property between LT and z-transform

$$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$\begin{aligned} Y(z) &= -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) \\ &= (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \end{aligned}$$

now we have a discrete transfer function:

$$\begin{aligned} Y(z) - (-a_1 z^{-1} - a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ (1 + a_1 z^{-1} + a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

2 Dynamic Analysis of Discrete Systems

- (8.2.1) z-Transform

1. See the Table 8.1 for understanding between z-transform and LT

$F(s)$	$f(kT)$	$F(z)$	
-	$\delta(kT)$	1	1
-	$\delta(kT-k_0T)$	z^{-k_0}	z^{-k_0}
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$	$\frac{z^{-1}\sin aT}{1-(2\cos aT)z^{-1}+z^{-2}}$
$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$	$\frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}$

2. For parts of Table, we have

$$\mathcal{Z}(\delta(kT)) = \sum_{k=0}^{k=\infty} \delta(kT)z^{-k} = \delta(0\cdot T) + \delta(1\cdot T)z^{-1} + \cdots = 1 + 0z^{-1} + 0z^{-2} + \cdots = 1$$

$$\mathcal{Z}(\delta(k-k_0T)) = 0 + 0z^{-1} + \cdots + 1z^{-k_0} + \cdots + = z^{-k_0}$$

$$\mathcal{Z}(1(kT)) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1-z^{-1}} = (1-z^{-1})^{-1}$$

$$\mathcal{Z}(e^{-akT}) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots = \frac{1}{1-e^{-aT}z^{-1}}$$

3. z-Transform Property

- Linearity :

$$\alpha x_1(k) + \beta x_2(k) \quad \leftrightarrow \quad \alpha X_1(z) + \beta X_2(z)$$

- Time-shift :

$$x(k - k_0) \quad \leftrightarrow \quad z^{-k_0} X(z)$$

- Multiplication by k :

$$kx(k) \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$$

- Multiplication by a^k:

$$a^k x(k) \quad \leftrightarrow \quad X(\frac{z}{a})$$

- Multiplication by e^{-akT}:

$$e^{-akT} x(k) \quad \leftrightarrow \quad X(e^{aT} z)$$

$$
\begin{aligned}
\mathcal{Z}(1-e^{-akT}) &= \mathcal{Z}(1) - \mathcal{Z}(e^{-akT}) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}} \\
&= \frac{z^{-1}-e^{-aT}z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})} = \frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}(\cos(-akT)) &= \mathcal{Z}(\frac{e^{jakT}+e^{-jakT}}{2}) = \frac{\mathcal{Z}(e^{jakT})}{2} + \frac{\mathcal{Z}(e^{-jakT})}{2} \\
&= \frac{1}{2}\cdot\frac{1}{1-e^{jaT}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1-e^{-jaT}z^{-1}} \\
&= \frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}
\end{aligned}
$$

(8장 숙제-1) z-transform 특성들을 이용하여 다음을 풀어서 제출하라.

- $\mathcal{Z}(\sin(-akT))$

- $\mathcal{Z}(\cos(-akT+3akT))$

- $\mathcal{Z}(e^{-bkT}\cos(-akT+2\pi k))$

2 Dynamic Analysis of Discrete Systems

- (8.2.2) z-Transform Inversion

  1. A z-transform inversion technique that has no continuous counterpart is called 'long division'. For example, consider a first-order discrete system

     $$y(k) = \alpha y(k-1) + u(k) \quad \rightarrow \quad Y(z) = \alpha z^{-1} Y(z) + U(z) \quad \rightarrow \quad \frac{Y(z)}{U(z)} = \frac{1}{1-\alpha z^{-1}}$$

     For a unit-pulse input, its z-transform is

     $$U(z) = \mathcal{Z}(\delta(kT)) = 1$$

     so the long division becomes

     $$\begin{aligned} Y(z) &= \frac{1}{1-\alpha z^{-1}} \\ &= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \cdots \end{aligned}$$

     We see that the sampled time history of y is

     $$y(0) = 1 \qquad y(1) = \alpha \qquad y(2) = \alpha^2 \qquad y(3) = \alpha^3 \qquad \cdots$$

$$
\begin{array}{r|l}
 & 1+\alpha z^{-1}+\alpha^2 z^{-2}+\alpha^3 z^{-3} \\
\hline
1-\alpha z^{-1}) & 1 \\
\hline
 & 1-\alpha z^{-1} \\
\hline
 & \quad \alpha z^{-1}+0 \\
 & \quad \alpha z^{-1}-\alpha^2 z^{-2} \\
\hline
 & \qquad \alpha^2 z^{-2}+0 \\
 & \qquad \alpha^2 z^{-2}-\alpha^3 z^{-3} \\
\hline
 & \qquad\quad \alpha^3 z^{-3}
\end{array}
$$

2 Dynamic Analysis of Discrete Systems

- (8.2.3) Relationship between s and z

1. Consider the continuous signal of

$$f(t) = e^{-at} \qquad t > 0 \qquad \rightarrow \qquad F(s) = \frac{1}{s+a}$$

and it corresponds to a pole $s = -a$.

2. Consider the discrete signal of

$$f(kT) = e^{-akT} \qquad \rightarrow \qquad F(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

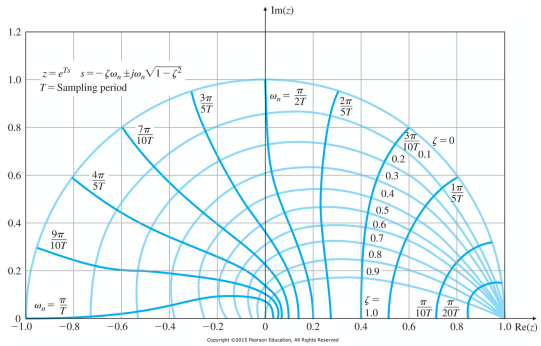
and it corresponds to a pole $z = e^{-aT}$.

3. The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

$$\begin{aligned} z &= e^{sT} \\ &= e^{-aT+jbT} = e^{-aT}(\cos bT + j\sin b) \\ &= e^{-\sigma T}(\cos \omega_d T + j \sin \omega_d T) \\ &= e^{-\zeta\omega_n T}(\cos \omega_n\sqrt{1-\zeta^2}T + j\sin \omega_n\sqrt{1-\zeta^2}T) \end{aligned}$$

where T is the sample period, and $s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

4. See Fig. 8.4, and it shows the mapping of lines of constant damping ζ and natural frequency ω_n from s-plane to the upper half of the z-plane, using $z = e^{sT}$.



http://controlsystemsacademy.com/0003/0003.html

a) The stability boundary $s = 0 \pm j\omega$ becomes the unit circle $|z| = 1$ in the z-plane; inside the unit circle is stable, outside is unstable
s-plane에서의 stability boundary는 imaginary축 $s = \pm j\omega$ 인데 z-plane에서의 stability boundary는 unit circle $|z| = e^{sT}|_{s=\pm j\omega} = 1$이 된다.

b) The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around the origin $s = 0$, in the s-plane.
z-plane 에서의 $z = 1$근방은 s-plane에서의 $s = 0$ 근방과 같다.

c) The z-plane locations give response information normalized to the sample rate rather than to time as in the s-plane.
z-plane 에서의 response information은 s-plane에서와 같이 시간에 대한 정보가 아닌, sample rate로 normalized된 정보를 제공한다.

d) The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T =$ circular sample rate in radians per second.
ω_s가 $2\pi/T$ 일때 음의 z축은 $\omega_s/2$로 표현된다.

e) Vertical lines in the left half of the s-plane (the constant real part of s) map into *circles* within the unit circle of the z-plane
s-plane에서의 좌반면은 z-plane에서의 unit circle 내부로 매핑된다.

f) Horizontal lines in the s-plane (the constant imaginary part of s) map into *radial lines* in the z-plane.
s-plane에서의 수평선은 z-plane에서의 radial line들로 매핑된다.

g) Frequencies greater than $\omega_s/2$, called the Nyquist frequency, appear in the z-plane on the top of corresponding lower frequencies because of the circular characteristics of e^{sT}. This overlap is called *aliasing* or folding.

5. As a result, it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples.

- (8.2.4) Final Value Theorem

1. Discrete final value theorem is

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} sX(s) \qquad\qquad \lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z)$$

   if all the poles of $(1-z^{-1})X(z)$ are inside the unit circle.

2. For example, to find the DC gain of the TF

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{z+0.16}$$

   we let $u(k) = 1$ for $k \geq 0$, so that

$$U(z) = \frac{1}{1-z^{-1}} \quad \text{and} \quad X(z) = \frac{0.58(1+z)}{(1-z^{-1})(z+0.16)}$$

   Applying the final value theorem yields

$$x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z) = \frac{0.58\cdot 2}{1+0.16} = 1$$

   so the DC gain of $G(z)$ is unity.

3 Design using Discrete Equivalents

- 1장부터 7장까지 continuous compensation $D_c(s)$를 설계하는 법에 대해서 배웠으며, 이번 절에서는 $D_c(s)$를 discrete compensation $D_d(z)$로 변환하는 방법들과 각 방법들의 성능에 대해서 배운다.
- 다만, 이 방법들은 모두 근사법이며 완벽하게 $D_c(s)$와 동일한 성능을 갖는 $D_d(z)$를 설계할 수는 없다.
- 일반적인 $D_c(s)$는 Differential equation으로 표현되며 이에 대응하는 $D_d(z)$는 Difference equation으로 표현하게 된다.
- (8.3.1 - 8.3.4), (Tustin,ZOH,MPZ,MMPZ)

3 Design using Discrete Equivalents

- (8.3.1) Tustin's Method

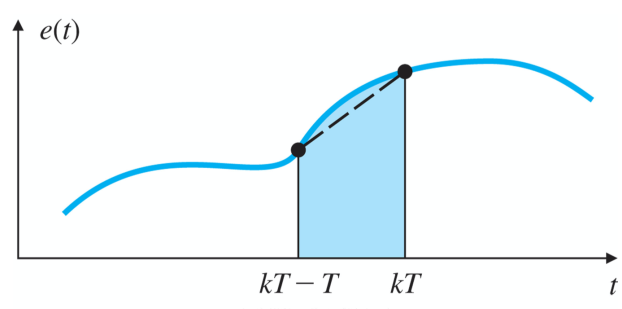
1. Tustin's method is a digitization technique that approaches the problem as one of numerical integration. Suppose

$$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}$$

which is integration. Therefore, it is corresponding to the *trapezoidal integration* as follows:

$$\begin{aligned} u(kT) &= \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt \\ &= u(kT-T) + \text{area under } e(t) \text{ over last period, } T, \\ &= u(kT-T) + T\frac{[e(kT-T)+e(kT)]}{2} \\ u(k) &= u(k-1) + T\frac{[e(k-1)+e(k)]}{2} \end{aligned}$$

where T is the sample period.



2. Taking z-transform,

$$\frac{U(z)}{E(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

3. In fact, the Tustin's method approximates $z = e^{sT}$ as follows:

$$s \approx \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

   where it can be derived from the Taylor's series expansions as follows:

$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} = \frac{1+\frac{sT}{2}+\frac{s^2T^2}{2^2}+\cdots}{1-\frac{sT}{2}+\frac{s^2T^2}{2^2}-\cdots} \approx \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} = \frac{2+sT}{2-sT} \qquad \rightarrow \qquad s \approx \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

4. For $D_c(s) = \frac{a}{s+a}$ as an example, we have

$$\begin{aligned}
D_c(s) &= \frac{a}{s+a} \\
D_d(z) &= \frac{a}{s+a}\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a} \\
&= \frac{aT(1+z^{-1})}{2(1-z^{-1}) + aT(1+z^{-1})} \\
&= \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}}
\end{aligned}$$

$$\begin{aligned}
D_d(z) = \frac{U(z)}{E(z)} &= \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}} \\
U(z) &= \frac{aT(1+z^{-1})}{(2+aT) - (2-aT)z^{-1}} E(z) \\
((2+aT) - (2-aT)z^{-1})U(z) &= (aT(1+z^{-1})E(z) \\
(2+aT)u(k) - (2-aT)u(k-1) &= aT[e(k) + e(k-1)] \\
\therefore \qquad u(k) &= \frac{(2-aT)}{(2+aT)}u(k-1) + \frac{aT}{(2+aT)}[e(k) + e(k-1)]
\end{aligned}$$

3 Design using Discrete Equivalents

5. (Example 8.1) Determine the difference equation with a sample rate of 25 times bandwidth using Tustin's approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} \qquad \text{Lead compensator}$$

Since the bandwidth is approximately $\omega_{bd} = 10[rad/s]$, the sampling rate should be

$$\omega_s = 25 \times \omega_{bd} = 250[rad/s] \quad \rightarrow \quad f_s = \frac{\omega_s}{2\pi} \approx 40[Hz] \quad \rightarrow \quad T = \frac{1}{f_s} = \frac{1}{40} = 0.025[s]$$

The discrete TF can be obtained as

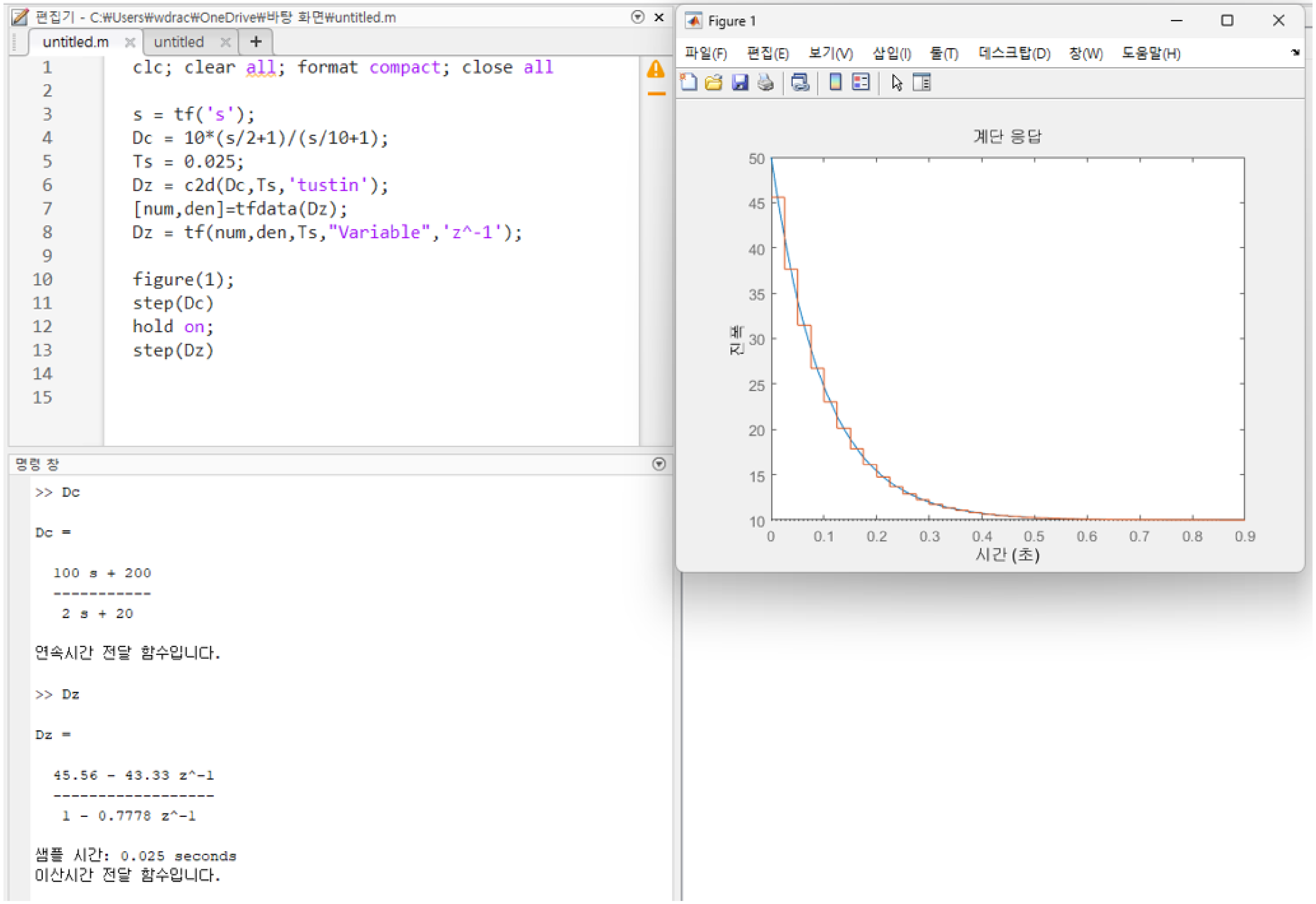
$$\begin{aligned} D_d(z) &= 10\frac{\frac{1}{T}\frac{1-z^{-1}}{1+z^{-1}}+1}{\frac{1}{5T}\frac{1-z^{-1}}{1+z^{-1}}+1} = 10\frac{5(1-z^{-1})+5T(1+z^{-1})}{(1-z^{-1})+5T(1+z^{-1})} \\ &= 50\frac{(1+T)-(1-T)z^{-1}}{(1+5T)-(1-5T)z^{-1}} = 50\frac{1.025-0.975z^{-1}}{1.125-0.875z^{-1}} = \frac{45.556-43.333z^{-1}}{1-0.778z^{-1}} \end{aligned}$$

Finally, the difference equation is

$$u(k) = 0.778u(k-1) + 45.556[e(k) - 0.951e(k-1)]$$

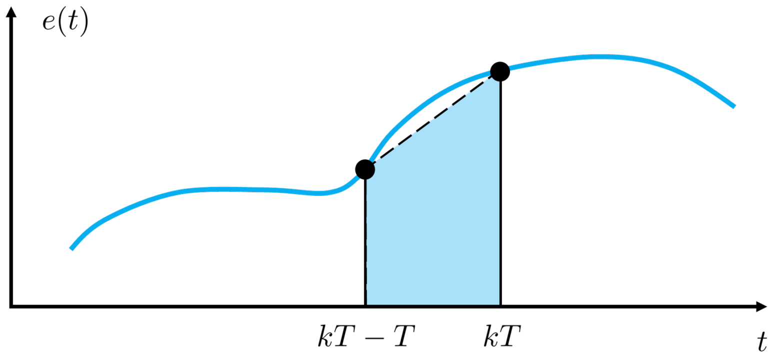
3 Design using Discrete Equivalents

Matlab Example



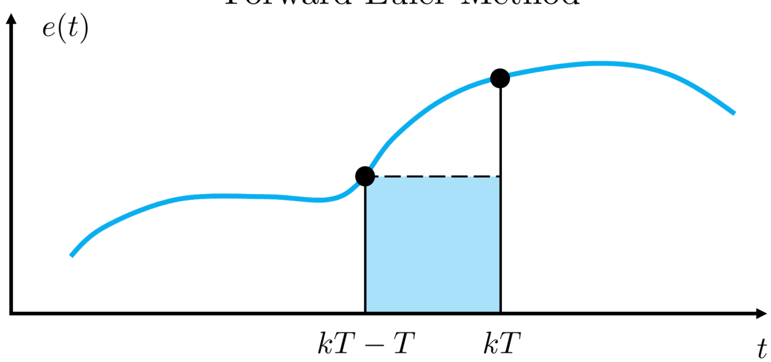
3 Design using Discrete Equivalents

Tustin's Method



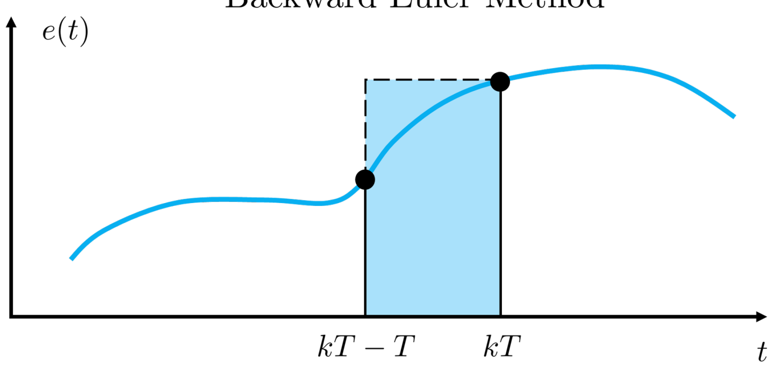
$$
\begin{aligned}
u(kT) &= u(kT-T) + T\frac{e(kT-T)+e(kT)}{2}\\
U(z) &= z^{-1}U(z) + T\frac{z^{-1}E(z)+E(z)}{2}\\
\frac{U(z)}{E(z)} &= \frac{T(1+z^{-1})}{2(1-z^{-1})} \leftrightarrow \frac{U(s)}{E(s)} = \frac{1}{s}\\
\therefore s &\rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}
\end{aligned}
$$

Forward Euler Method



$$
\begin{aligned}
u(kT) &= u(kT-T) + Te(kT-T)\\
U(z) &= z^{-1}U(z) + Tz^{-1}E(z)\\
\frac{U(z)}{E(z)} &= \frac{Tz^{-1}}{1-z^{-1}} \leftrightarrow \frac{U(s)}{E(s)} = \frac{1}{s}\\
\therefore s &\rightarrow \frac{1-z^{-1}}{Tz^{-1}}
\end{aligned}
$$

Backward Euler Method



$$
\begin{aligned}
u(kT) &= u(kT-T) + Te(kT)\\
U(z) &= z^{-1}U(z) + TE(z)\\
\frac{U(z)}{E(z)} &= \frac{T}{(1-z^{-1})} \leftrightarrow \frac{U(s)}{E(s)} = \frac{1}{s}\\
\therefore s &\rightarrow \frac{1-z^{-1}}{T}
\end{aligned}
$$

3 Design using Discrete Equivalents

- (8.3.2) Zeroth-Order Hold (ZOH) Method

  1. Tustin's method essentially assumed that the input to the controller varied linearly early between the past sample and the current sample.
  2. Another assumption is that the input to the controller remains constant throughout the sample period. $\rightarrow$ ZOH
  3. One input sample produces a square pulse of height $e(k)$ that lasts for one sample period T.
  4. For a constant positive step input, $e(k)$, at time k, $E(s) = e(k)/s$, so the result would be

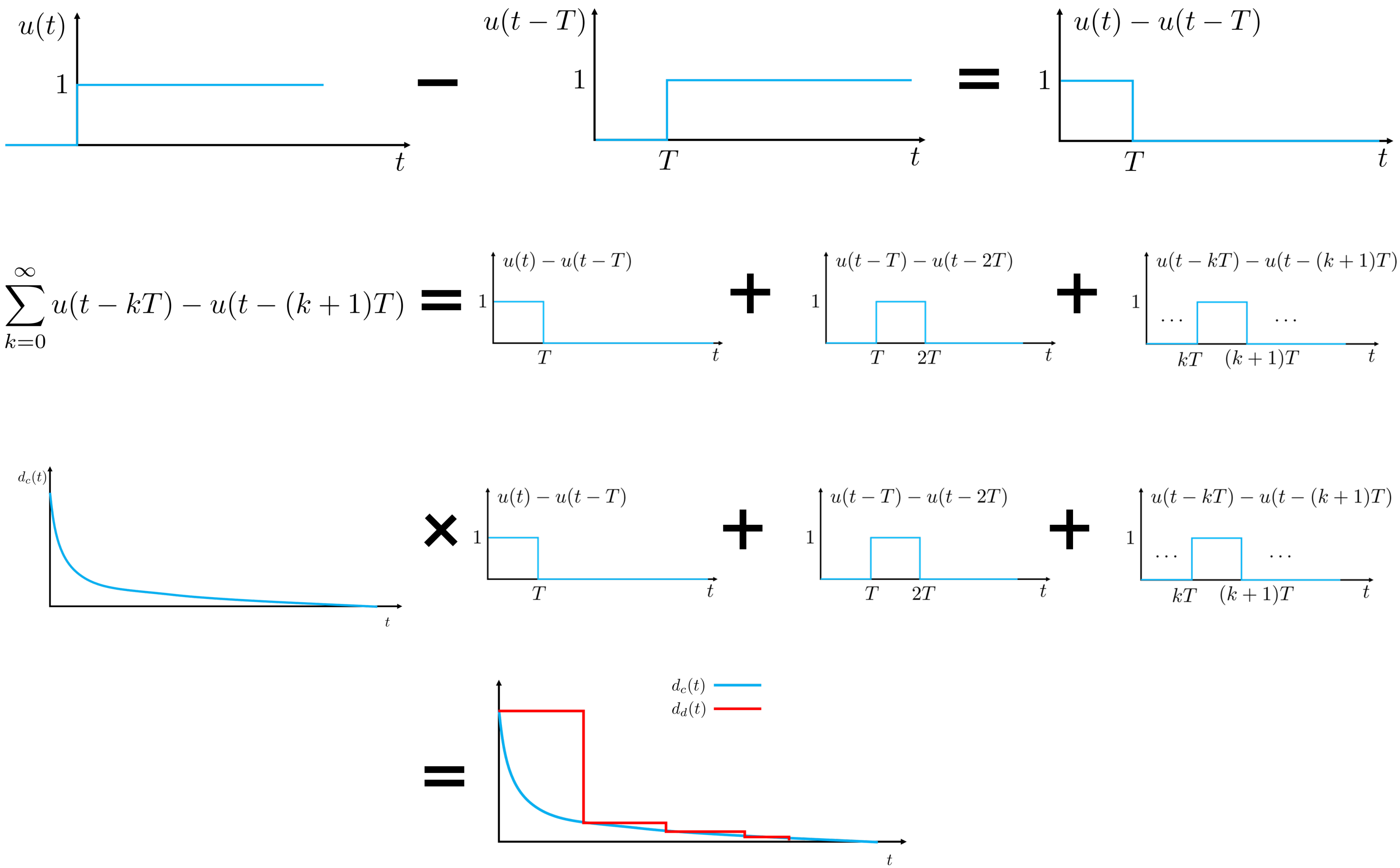
     $$D_d(z) = \mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Furthermore, a constant negative step, one cycle delayed, would be

     $$D_d(z) = z^{-1}\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Therefore, the discrete TF for the square pulse is

     $$D_d(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$



$u(t)$
1
t
$-$
$u(t-T)$
1
T
t
$=$
$u(t)-u(t-T)$
1
T
t
$\sum_{k=0}^{\infty} u(t-kT)-u(t-(k+1)T)$
$=$
$u(t)-u(t-T)$
1
T
t
$+$
$u(t-T)-u(t-2T)$
1
T
$2T$
t
$+$
$u(t-kT)-u(t-(k+1)T)$
1
$\dots$
$\dots$
kT
$(k+1)T$
t
$d_c(t)$
t
$\times$
$u(t)-u(t-T)$
1
T
t
$+$
$u(t-T)-u(t-2T)$
1
T
$2T$
t
$+$
$u(t-kT)-u(t-(k+1)T)$
1
$\dots$
$\dots$
kT
$(k+1)T$
t
$=$
$d_c(t)$
$d_d(t)$
t

3 Design using Discrete Equivalents

$$
\begin{aligned}
d_d(t) &= d_c(t)\left(\sum_{k=0}^{\infty} u(t-kT) - u(t-(k+1)T)\right) \qquad = \sum_{k=0}^{\infty} d_c(t)u(t-kT) - \sum_{k=0}^{\infty} d_c(t)u(t-(k+1)T) \\
&= d_c(t) * u(t) - d_c(t) * u(t-T) \\
D_d(z) &= \mathcal{Z}\left(D_c(s)\cdot\left(\frac{1}{s} - e^{-sT}\frac{1}{s}\right)\right) \\
&= \mathcal{Z}\left((1-e^{-sT})\frac{D_c(s)}{s}\right) \\
&= (1-z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right) \qquad (\because z = e^{sT})
\end{aligned}
$$

5. (Example 8.2) Determine the difference equation with a sample period $T = 0.025[s]$ using ZOH approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} = 10\frac{5s+10}{s+10}$$

The discrete TF using ZOH is

$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{5}{s+10}+\frac{10}{s(s+10)}\right)\\
&= 10(1-z^{-1})\left(\frac{5}{1-e^{-0.25}z^{-1}}+\frac{z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right)\\
&= 10(1-z^{-1})\left(\frac{5(1-z^{-1})+z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right)\\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

where $\mathcal{Z}\left\{\frac{1}{s+10}\right\} = \frac{1}{1-e^{-10T}z^{-1}}$ with $e^{-10T} = e^{-0.25} = 0.779$. Or,

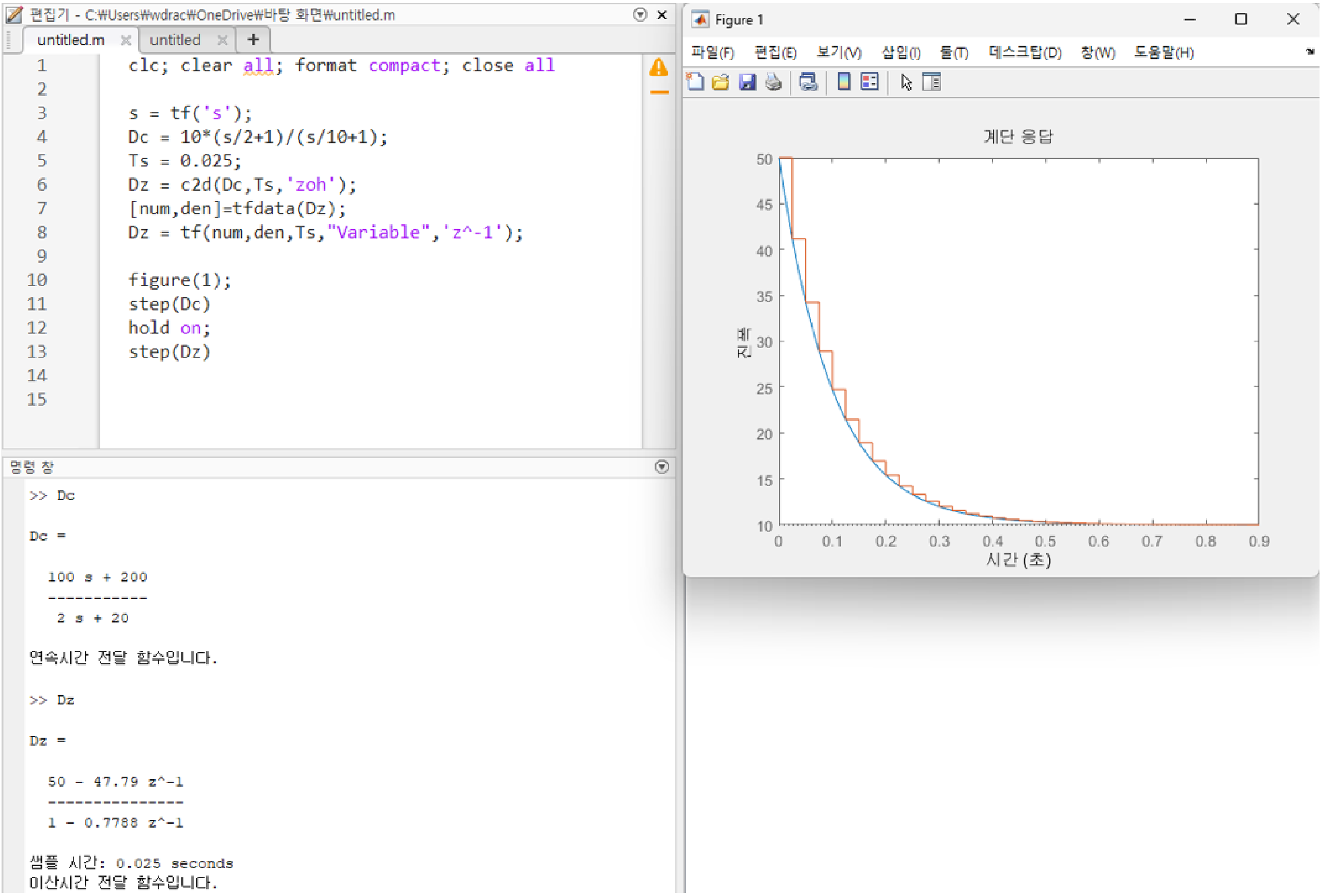
$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{1}{s} + \frac{4}{s+10}\right) \\
&= 10(1-z^{-1})\left(\frac{1}{1-z^{-1}} + \frac{4}{1-e^{-0.25}z^{-1}}\right) \\
&= 10(1-z^{-1})\left(\frac{(1-e^{-0.25}z^{-1}) + 4(1-z^{-1})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

Finally, the difference equation is

$$\begin{aligned}
u(k) &= 0.779u(k-1) + 50e(k) - 47.79e(k-1) \\
&= 0.779u(k-1) + 50[e(k) - 0.956e(k-1)]
\end{aligned}$$

3 Design using Discrete Equivalents

Matlab Example



3 Design using Discrete Equivalents

- (8.3.3) Matched Pole-Zero (MPZ) Method

1. Another digitization method, called the matched pole-zero (MPZ) method, is suggested by matching the poles and zeros between s and z planes, using $z = e^{sT}$.
2. Because physical systems often have more poles than zeros, it is useful to arbitrarily add zeros at $z = -1$, resulting in a $(1 + z^{-1})$ term in $D_d(z)$.
   a) Map poles and zeros according to the relation $z = e^{sT}$
   b) If the numerator is of lower order than the denominator, add powers of $(1 + z^{-1})$ to the numerator until numerator and denominator are of equal order.
   c) Set the DC or low frequency gain of $D_d(z)$ equal to that of $D_c(s)$.
3. For example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s+b} \qquad\qquad D_d(z) = K_d \frac{1 - e^{-aT} z^{-1}}{1 - e^{-bT} z^{-1}}$$

where K_d is found by the DC-gain

$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

Thus the result is

$$K_d = K_c \frac{a}{b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

4. As another example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s(s+b)} \qquad\qquad D_d(z) = K_d \frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}$$

where K_d is found by the DC-gain *by deleting the pure integration term* both sides

$$\lim_{s\to 0} sD_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z\to 1}(z-1)D_d(z) = K_d \frac{2(1-e^{-aT})}{1-e^{-bT}}$$

The result is

$$K_d = K_c \frac{a}{2b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

5. (Example 8.3) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$, another real pole at $s = -1.58$, using MPZ digitization

$$G(s) = \frac{1}{s^2}$$

Let us assume that the lead compensator is used

$$D_c(s) = K_c \frac{s+b}{s+a}$$

Then, we have the characteristic equation

$$\begin{aligned} 1 + G(s)D_c(s) = 1 + K_c \frac{s+b}{s^2(s+a)} &= s^3 + as^2 + K_c s + K_c b \\ \alpha_c(s) &= (s^2 + 2\zeta\omega_n s + {\omega_n}^2)(s-p) \\ &= (s^2 + 0.42s + 0.09)(s+1.58) = s^3 + 2s^2 + 0.7536s + 0.1422 \end{aligned}$$

with $a = 2$, $b = 0.19$, and $K_c = 0.7536$. Now we have the lead compensator:

$$D_c(s) = 0.7536 \frac{s+0.19}{s+2} \qquad \rightarrow \qquad D_c(s) = 0.81 \frac{s+0.2}{s+2}$$

3 Design using Discrete Equivalents

Let us determine the sampling rate and sampling period as follows:

$$\omega_s = 0.3 \times 20 = 6[rad/s] \qquad \rightarrow \qquad f_s = \frac{\omega_s}{2\pi} \approx 1[Hz] \qquad \rightarrow \qquad T = 1[s]$$

The MPZ digitization yields

$$D_d(z) = K_d \frac{1 - e^{-0.2} z^{-1}}{1 - e^{-2} z^{-1}} = K_d \frac{1 - 0.818 z^{-1}}{1 - 0.135 z^{-1}}$$

where the final value theorem gives

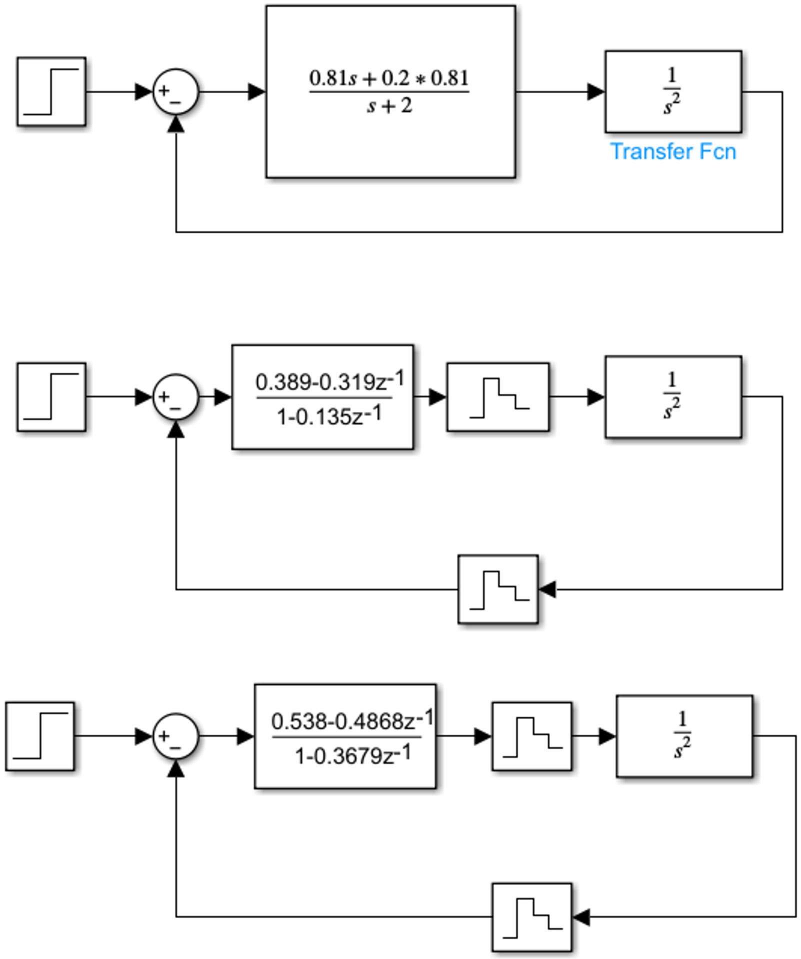
$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$
$$0.81 \frac{0.2}{2} = K_d \frac{1 - 0.818}{1 - 0.135} \qquad \rightarrow \qquad K_d = 0.385$$

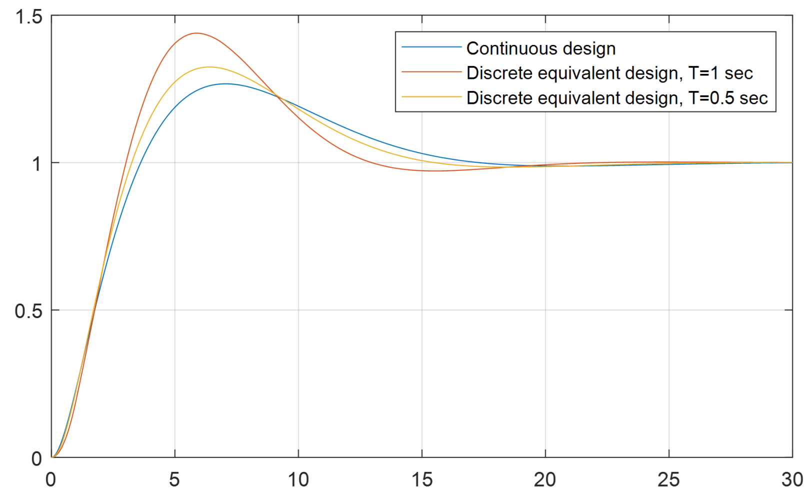
The difference equation becomes

$$u(k) = 0.135 u(k-1) + 0.385[e(k) - 0.818 e(k-1)]$$

For the step responses,

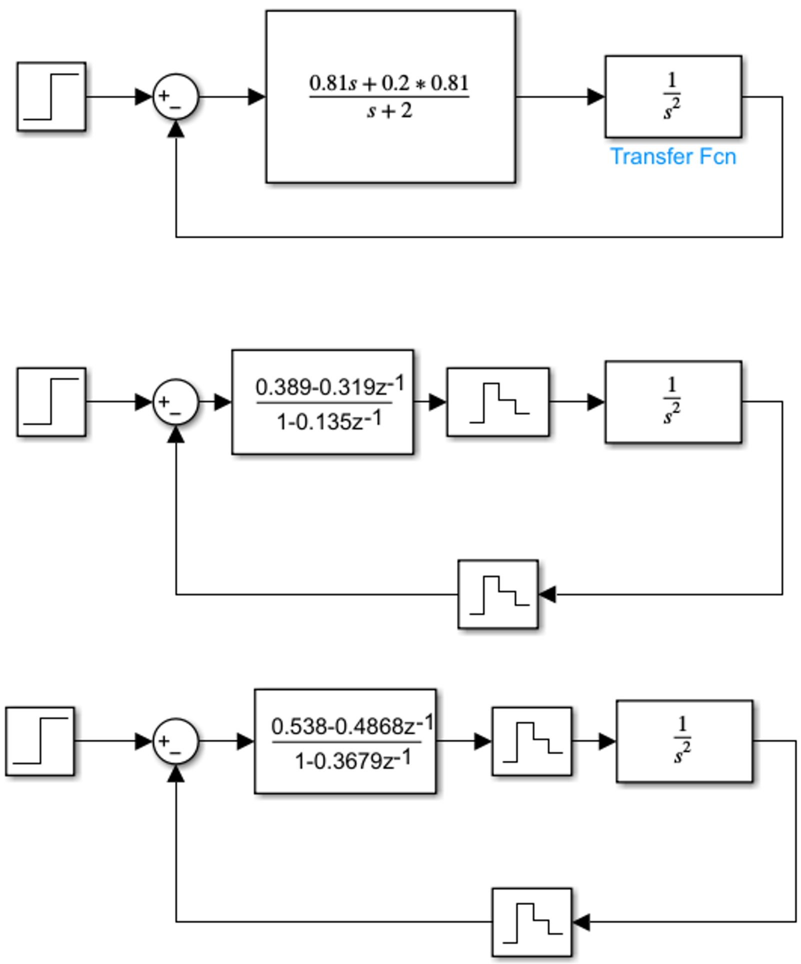
3 Design using Discrete Equivalents

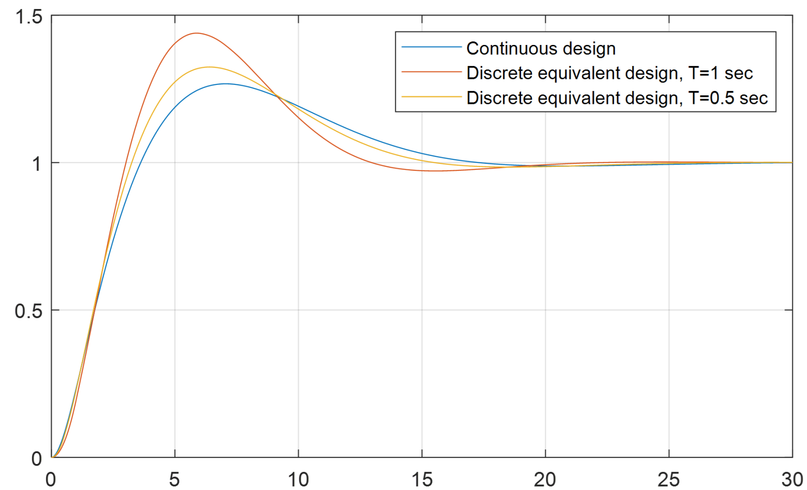




https://github.com/tjdalsckd/ControlSystem_Lecture8/blob/main/matlab/fig_8_14.slx

3 Design using Discrete Equivalents





https://github.com/tjdalsckd/ControlSystem_Lecture8/blob/main/matlab/fig_8_14.slx

3 Design using Discrete Equivalents

- (8.3.4) Modified Matched Pole-Zero (MMPZ) Method

  1. Modify Step 2 in the MPZ so that the numerator is of lower order than denominator by 1. For example, if

  $$D_c(s) = K_c \frac{s+a}{s(s+b)}$$

  we skip Step 2 to get

  $$D_d(z) = K_d \frac{z^{-1}(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})} \qquad \text{where} \quad K_d = K_c \frac{a}{b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

  We can see the difference equation as follow:

  $$u(k) = (1+e^{-bT})u(k-1) - e^{-bT}u(k-2) + K_d[e(k-1) - e^{-aT}e(k-2)]$$

  where it makes use of $e(k-1)$ that are one cycle old, not $e(k)$.

3 Design using Discrete Equivalents

- (8.3.5) Comparison of Digital Approximation Methods

1. Let us compare four approximation methods with the sampling rate

$$D_c(s) = \frac{5}{s+5}$$

2. Tustin's method

$$\begin{aligned} D_d(z) &= \frac{5}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + 5} = \frac{5T(1+z^{-1})}{2(1-z^{-1}) + 5T(1+z^{-1})} = \frac{5T + 5Tz^{-1}}{(2+5T) - (2-5T)z^{-1}} \\ &= \left(\frac{5T}{2+5T}\right)\frac{1+z^{-1}}{1 - \left(\frac{2-5T}{2+5T}\right)z^{-1}} \end{aligned}$$

3. ZOH

$$\begin{aligned} D_d(z) &= (1-z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right) = (1-z^{-1})\mathcal{Z}\left(\frac{5}{s(s+5)}\right) = (1-z^{-1})\frac{(1-e^{-5T})z^{-1}}{(1-z^{-1})(1-e^{-5T}z^{-1})} \\ &= (1-e^{-5T})\frac{z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

4. MPZ

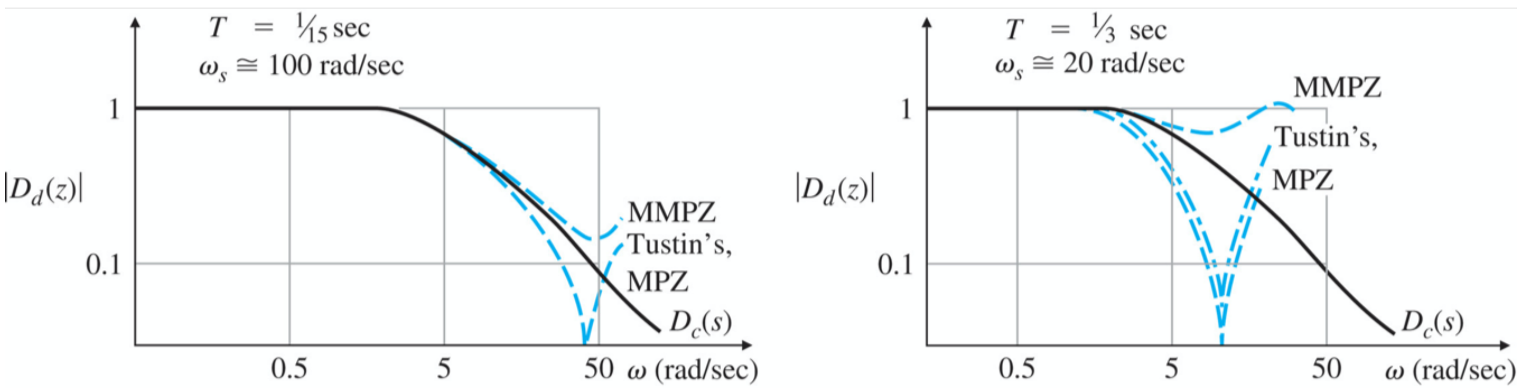
$$\begin{aligned} D_d(z) &= K_d\frac{(1+z^{-1})}{1-e^{-5T}z^{-1}} \qquad \text{where} \quad K_d\frac{2}{1-e^{-5T}} = 1 \\ &= \left(\frac{1-e^{-5T}}{2}\right)\frac{1+z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

5. MMPZ

$$D_d(z) = K_d \frac{z^{-1}}{1 - e^{-5T} z^{-1}} \quad \text{where} \quad K_d \frac{1}{1 - e^{-5T}} = 1$$
$$= (1 - e^{-5T}) \frac{z^{-1}}{1 - e^{-5T} z^{-1}}$$

6. It is noted that Tustin and MPZ bring the similar structures each other, while ZOH and MMPZ show the similar structures, as shown in Table 8.2

7. Tustin and MPZ methods show a notch at $\omega_s/2$ because of their zero at $z = -1$ from $1 + z^{-1}$ term.



3 Design using Discrete Equivalents

- (8.3.6) Applicability Limits of the Discrete Equivalent Design Method

  1. The system can often be *unstable* for rates slower than approximately $5\omega_{bd}$, and
  2. the damping would be *degraded* significantly for rates slower than about $10\omega_{bd}$
  3. At sample rates $\geq 20\omega_{bd}$, design by discrete equivalent yields *reasonable* results, and
  4. at sample rates of 25 times the bandwidth or higher, discrete equivalents can be used *with confidence*.
  5. ZOH brings $T/2$ delay in the control system. A method to account for the $T/2$ delay is to include an approximation of the delay into the original plant model:

$$G_{ZOH}(s) = \frac{2/T}{s + 2/T}$$

4 Hardware Characteristics

- Analog-to-Digital (A/D) Converter는 센서로 획득한 전압값을 컴퓨터에서 연산할 수 있는 digital값으로 변환해주는 장치
  - Counting scheme
  - Successive-approximation technique
- Digital-to-Analog (D/A) Converter 컴퓨터로부터 만들어진 digital word를 전압값으로 변환해주는 장치, 일반적으로 A/D converter보다 싸다!
- 컴퓨터는 compensation $D_d(z)$를 프로그래밍하고, 계산하는 장치이다.

- Analog Anti-Alias Prefilter는 아날로그 센서와 A/D converter 사이에 위치한다.
  - Aliasing은 아날로그 신호의 Maximum Frequency의 두배 이상의 Sampling Frequency를 사용하지 않은 경우, 아날로그 신호 주파수의 변형을 일으켜 디지털 신호로 변환 후에 다시 원래의 아날로그 신호를 복원하지 못하게 되는 현상을 뜻함.

$$y(t) = \sin(2\pi \cdot 55t) + \sin(2\pi \cdot 120t) \qquad \textbf{Maximum Frequency is 120 [Hz]}$$

1. 만일 Maximum Frequency의 2배보다 작은 Sampling Frequency F_s = 200 [Hz], $kT = k/200$ 라면,

$$\begin{aligned}
y(kT) &= \sin(2\pi \cdot \frac{55}{200}k) + \sin(2\pi \cdot \frac{120}{200}k) \\
&= \sin(2\pi \cdot \frac{55}{200}k) + \sin(\frac{6\pi}{5}k) \\
&= \sin(2\pi \cdot \frac{55}{200}k) + \sin((\pi + \frac{1}{5}\pi)k) \\
&= \sin(2\pi \cdot \frac{55}{200}k) + \sin((2\pi - \frac{4}{5}\pi)k) \\
&= \sin(2\pi \cdot \frac{55}{200}k) + \sin((-\frac{4}{5}\pi)k) \qquad (\because \sin(\alpha) = \sin(2\pi k + \alpha)) \\
y_r(t) &= \sin(2\pi \cdot 55t) + \sin(-2\pi \cdot 80t)
\end{aligned}$$

기존 아날로그 신호의 120[Hz]는 Aliasing으로 인해 Nyquist Frequency 기준으로 대칭되어, 80[Hz]성분의 주파수로 변경됨.

- 결국 Aliasing이 발생하지 않도록 하려면 Sampling Frequency 보다 큰 주파수에 대해서는 모두 Low-pass Filter 처리해서 제거한 후에 샘플링을 진행해야 하며 이러한 방식의 LPF를 Anti-Alias Prefilter 라는 이름으로 사용한다.

$$y(t) = \sin(2\pi \cdot 55t) + \cancel{\sin(2\pi \cdot 120t)} \qquad \text{Anti-Aliasing over } 2F_s$$

$$y(kT) = \sin(2\pi \cdot \frac{55}{200}k)$$
$$y_r(t) = \sin(2\pi \cdot 55t)$$

- In a continuous system, noise components with a frequency much higher than the control-system bandwidth normally have a small effect because the system will not respond at the high frequency.
- However, in a *digital system*, the frequency of the noise can be *aliased down* to the vincinity of the system bandwidth so the closed-loop system would respond to the noise.
- The solution to prevent aliasing is to place an analog prefilter before the sampler. In many cases, a simple first-order low-pass filter will do - that is -

$$H_p(s) = \frac{a}{s+a}$$

  where the *breakpoint* a is selected to be lower than Nyquist rate $\omega_s/2$ so that any noise present with frequencies greater than Nyquist rate is attenuated by the prefilter.
- If ω_s is chosen to be $25 \times \omega_{bd}$, the anti-aliasing filter breakpoint a should be selected lower

than $\omega_s/2$, so that

$$a = 10 \times \omega_{bd} \qquad \leftarrow \qquad \omega_s = 25 \times \omega_{bd}$$

would be a reasonable choice.

5 Sample-Rate Selection

- The inherent approximation for the discrete TF may give rise to *decreased performance* or even *system instability* as the sample rate is lowered. This can lead the designer to conclude that a faster sample rate is required.
- The *sampling theorem* states that in order to reconstruct an unknown, band-limited, continuous signal from samples of that signal, we must *sample at least twice as fast as the highest frequency contained in the signal*. $\omega_s = 2\omega_{bd}$
- In the z-plane, the highest frequency that can be represented by a discrete system is $\omega_s/2$.
- For a very high frequency noise, it would be foolish to sample fast enough to attenuate the disturbance without the use of a prefilter.

6 Discrete Design

- This plant model can be used as part of a discrete model of the feedback system including the compensation $D_d(z)$.

- Analysis and design using this discrete model is called *discrete design* or alternatively, *direct digital design*.

- For a plant described by $G(s)$ and preceeded by a ZOH, the discrete TF was essentially given by

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

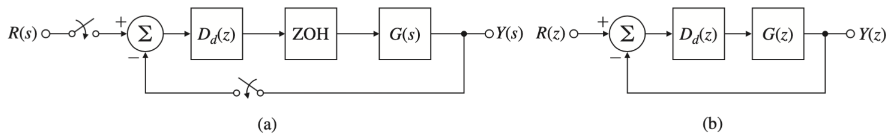


Figure 8.17
Comparison of: (a) a mixed system; and (b) its pure discrete equivalent

- The closed-loop poles or the roots of the discrete characteristic equation

$$1 + D_d(z)G(z) = 0$$

- The root-locus techniques used in continuous systems to find roots of a polynomial in s apply equally well and without modification to the polynoimal in z.

- The interpretation of the results is that the stability boundary is now the unit circle instead of the imaginary axis.

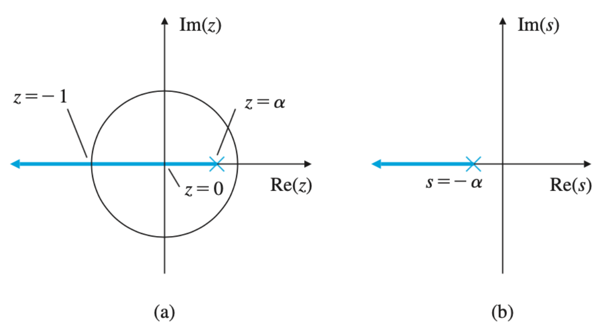
(Example 8.4) When $G(s) = \frac{a}{s+a}$ and $D_d(z) = K$, draw the root locus with respect to K?
(Answer)

$$
\begin{aligned}
G(z) &= (1 - z^{-1})\mathcal{Z}\left\{\frac{a}{s(s+a)}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+a}\right\} \\
&= (1 - z^{-1})\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}\right) \\
&= \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} \\
&= \frac{(1 - \alpha)z^{-1}}{1 - \alpha z^{-1}} \qquad \text{where} \quad \alpha = e^{-aT}
\end{aligned}
$$

The discrete characteristic equation becomes

$$
1 + D_d(z)G(z) = 1 + K\frac{(1 - \alpha)z^{-1}}{1 - \alpha z^{-1}} = 0
$$

Figure 8.18
Root loci for: (a) the z-plane; and (b) the s-plane



In the continuous case, the system remains stable for all values of K. In the discrete case, the system becomes oscillatory with decreasing damping ratio as z goes from 0 to -1 and eventually becomes unstable. This instability is due to the lagging effect of the ZOH.

Feedback properties

- Proportional

$$u(k) = Ke(k) \qquad \leftrightarrow \qquad D_d(z) = K$$

- Derivative

$$u(k) = KT_D[e(k) - e(k-1)] \qquad \leftrightarrow \qquad D_d(z) = KT_D(1 - z^{-1})$$

- Integral

$$u(k) = u(k-1) + \frac{K}{T_I}e(k) \qquad \leftrightarrow \qquad D_d(z) = \frac{K}{T_I}\left(\frac{1}{1 - z^{-1}}\right)$$

- Lead

$$u(k) = \beta u(k-1) + K[e(k) - \alpha e(k-1)] \qquad \leftrightarrow \qquad D_d(z) = K\frac{1 - \alpha z^{-1}}{1 - \beta z^{-1}}$$

(Example 8.5) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$ using discrete design

(Answer)

$$G(s) = \frac{1}{s^2} \qquad \rightarrow \qquad G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^3}\right\} = \frac{T^2}{2}\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}$$

which, with $T = 1$, becomes

$$G(z) = \frac{1}{2}\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}$$

Let us assume that the PD compensator is used

$$D_d(z) = K(1 - \alpha z^{-1})$$

The desired pole locations of $\omega_n = 0.3$ and $\zeta = 0.7$ become $z = 0.78 \pm 0.18j$

$$1 + D_d(z)G(z) = 1 + K\frac{1}{2}\frac{z^{-1}(1 + z^{-1})(1 - \alpha z^{-1})}{(1 - z^{-1})^2} = 0$$

Now we have

$$\alpha = 0.85 \qquad\qquad K = 0.374$$

and

$$D_d(z) = 0.374(1 - 0.85z^{-1})$$

The difference equation becomes

$$u(k) = 0.374[e(k) - 0.85e(k-1)]$$

(8장 숙제) 8장 연습문제에서 기말고사에 출제될 만한 문제 3개를 선택하여 풀어 제출하라? (마감 기말고사 전)