1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock
3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.
4. Difference equation for discrete-time system $\leftrightarrow$ Differential equation for continuous-time system
5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$
   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7
6. The analog output of the sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)
   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

     $$y(t) \quad \rightarrow \quad y(k) = y(kT) \quad \text{with} \quad t = kT$$

     where k is an integer and T is a fixed value (sample period, or sampling time).
   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

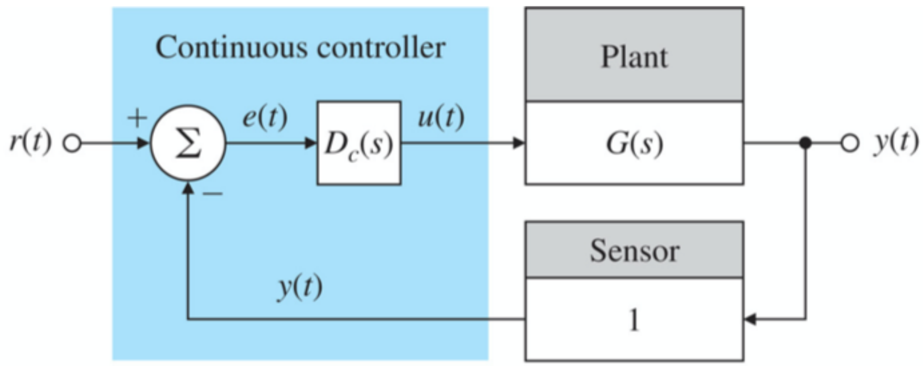
1 Digitization

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)

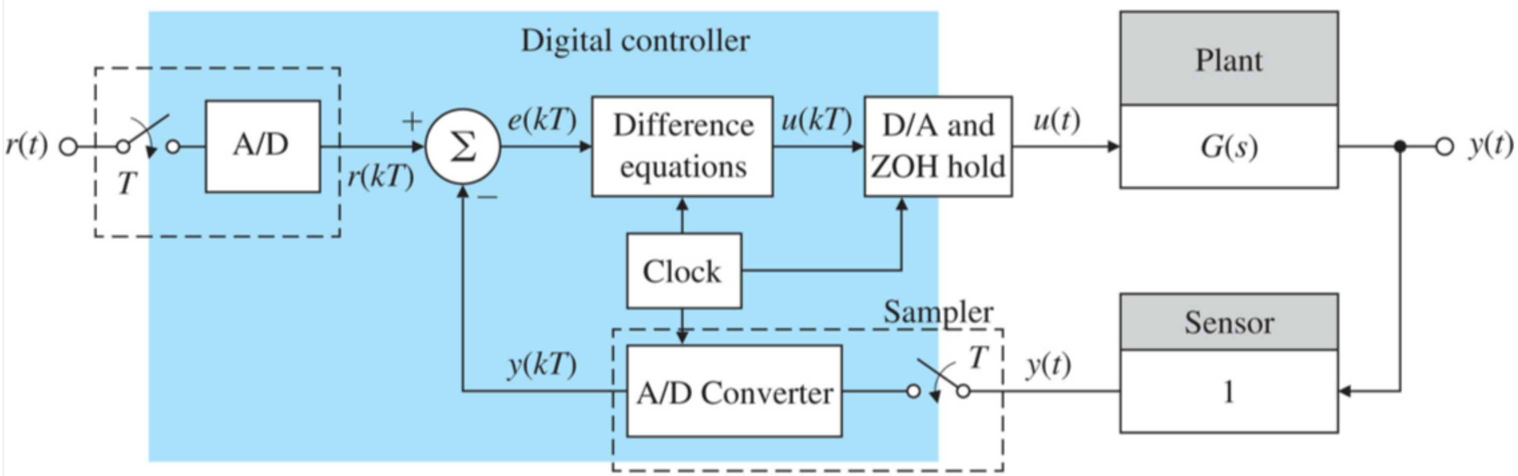
   - Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
   - Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
   - If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

8. A system having both discrete and continuous signals is called a 'sampled data system'.

1 Digitization



(a)



(b)

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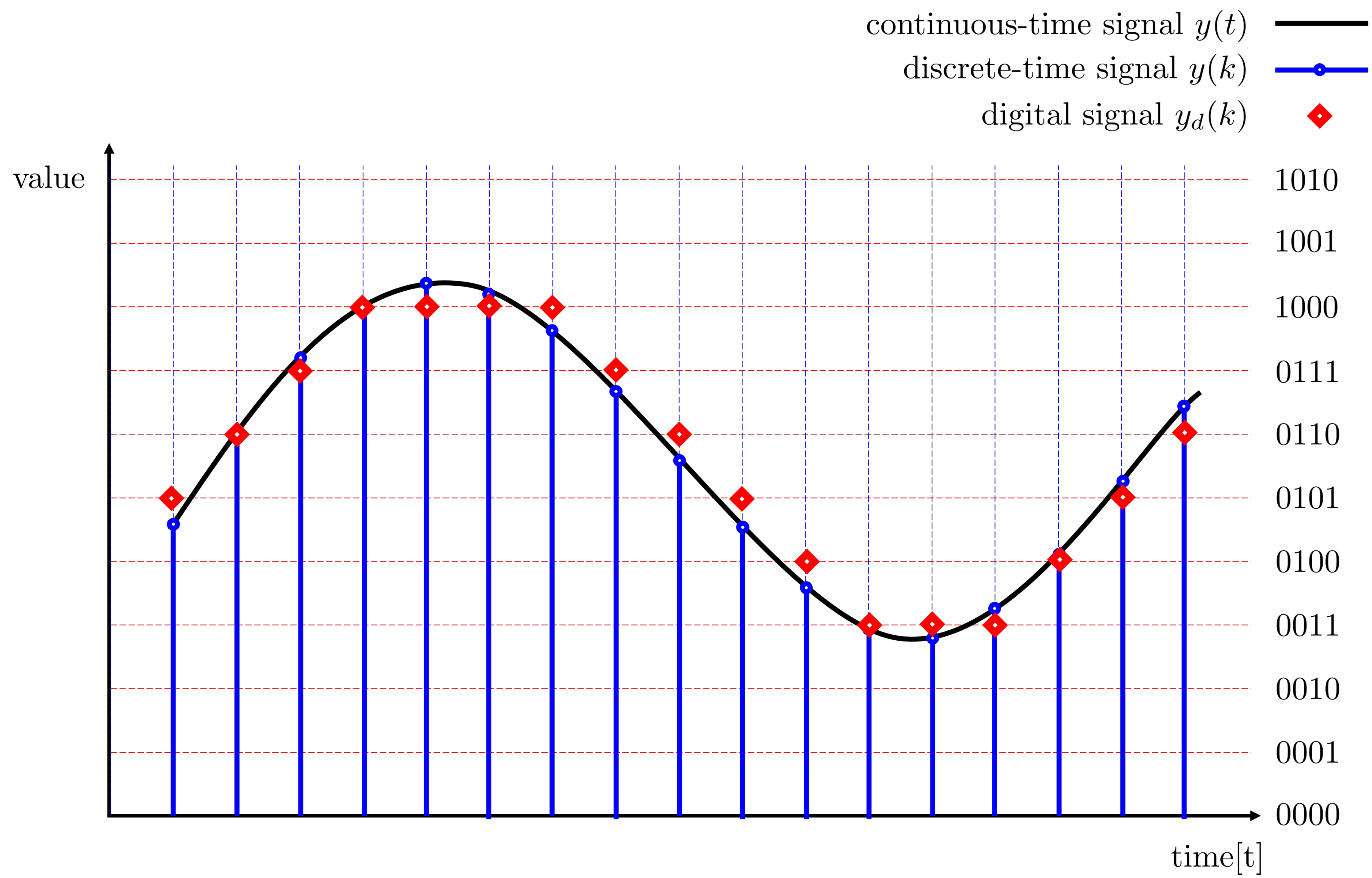
1 Digitization

- Continuous-time signal: Both domain and range are continuous, $y(t)$
- Discrete-time signal: Domain is discrete and range is continuous, $y(k)$ or $y(kT)$
- Digital signal: Both domain and range are discrete, $y_d(k)$



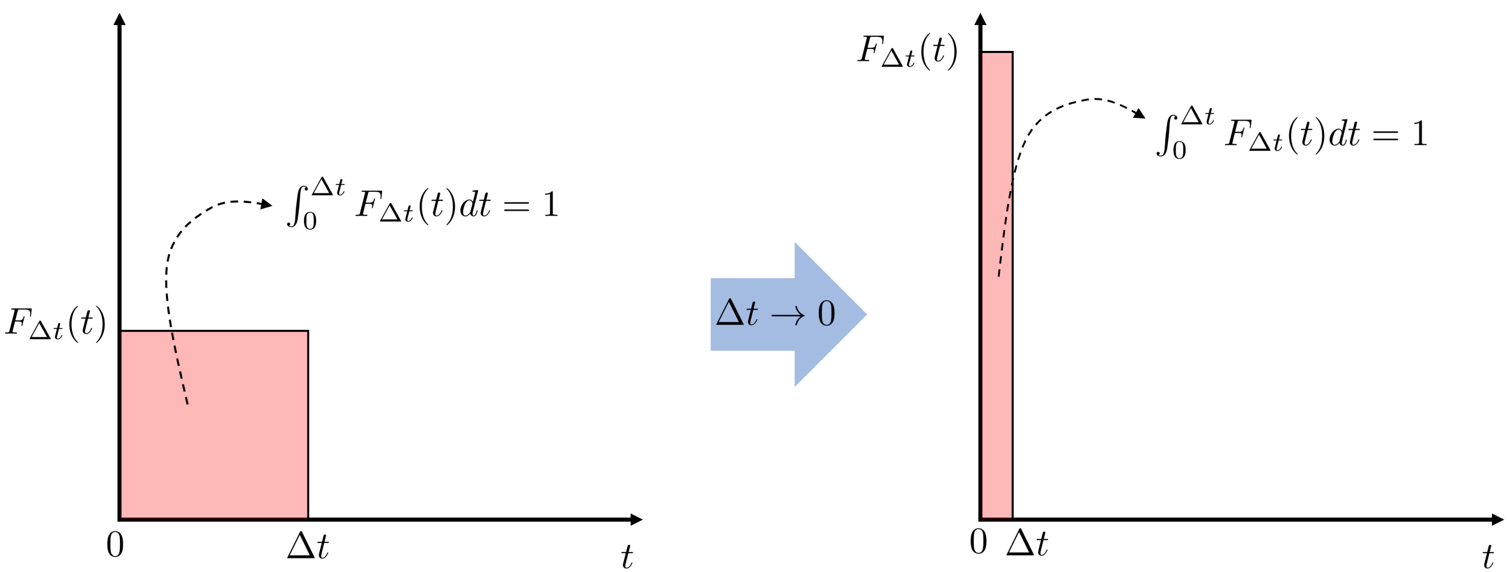
1 Digitization

★ Dirac delta function is mathematically defined as:

1. Approximation

$$F_{\Delta t}(t) = \begin{cases} 1/\Delta t & 0 < t \leq \Delta t \\ 0 & otherwise \end{cases}$$

$$\delta(t) = \lim_{\Delta t \to 0} F_{\Delta t}(t)$$



1 Digitization

2. Generalized function

- $\delta(t) = 0$ **for** $t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t)dt = 1$

★ Unit step function is mathematically defined as:

$$1(t) = \begin{cases} 1 & t > 0 \\ undefined & t = 0 \\ 0 & t < 0 \end{cases}$$

In contol theory

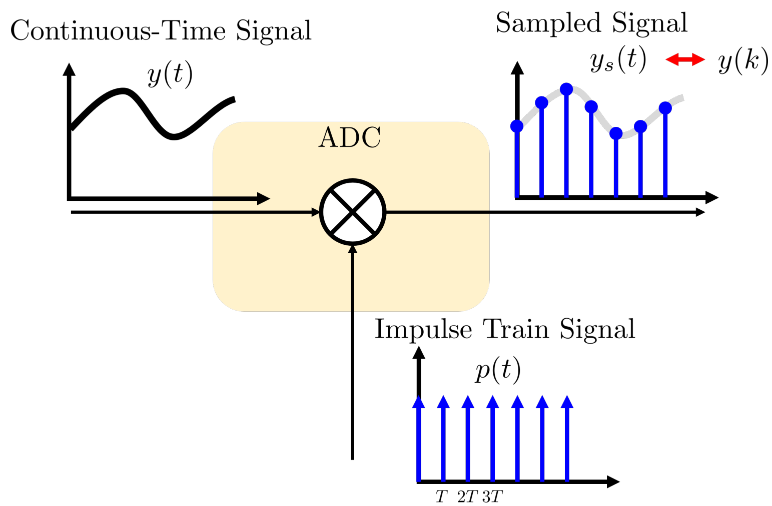
$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

★ Useful Properties

1. $\frac{d1(t)}{dt} = \delta(t)$ (수학적으로는 틀림, 개념적으로 사용)
2. $x(t)\delta(t - kT) = x(kT)\delta(t - kT)$
3. $\int_{-\infty}^{\infty} x(t)\delta(t - kT)dt = x(kT)$
   $\because \int_{-\infty}^{\infty} x(t)\delta(t - kT)dt = \int_{-\infty}^{\infty} x(kT)\delta(t - kT)dt = x(kT)\int_{-\infty}^{\infty} \delta(t - kT)dt = x(kT)$

1 Digitization

★ Sampling Process



1. Periodic Impulse Train: p(t) is periodic with period $T = 1/F_s$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

2. Sampled Signal: we can consider $y_s(t)$ te be the analog equivalent to discrete-time signal $y(k)$ or $y(kT)$

$$y_s(t) = y(t) \cdot p(t) = \sum_{k=-\infty}^{\infty} y(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} y(kT)\delta(t - kT)$$
$$\leftrightarrow y(k) = y(kT)$$

1 Digitization

★ Nyquist–Shannon Sampling Theorem

1. 간단하게, 아날로그 신호가 갖는 최대 주파수의 2배이상의 샘플링 주파수를 사용해야만 손실되는 정보없이 디지털 신호를 아날로그 신호로 복원할 수 있다.

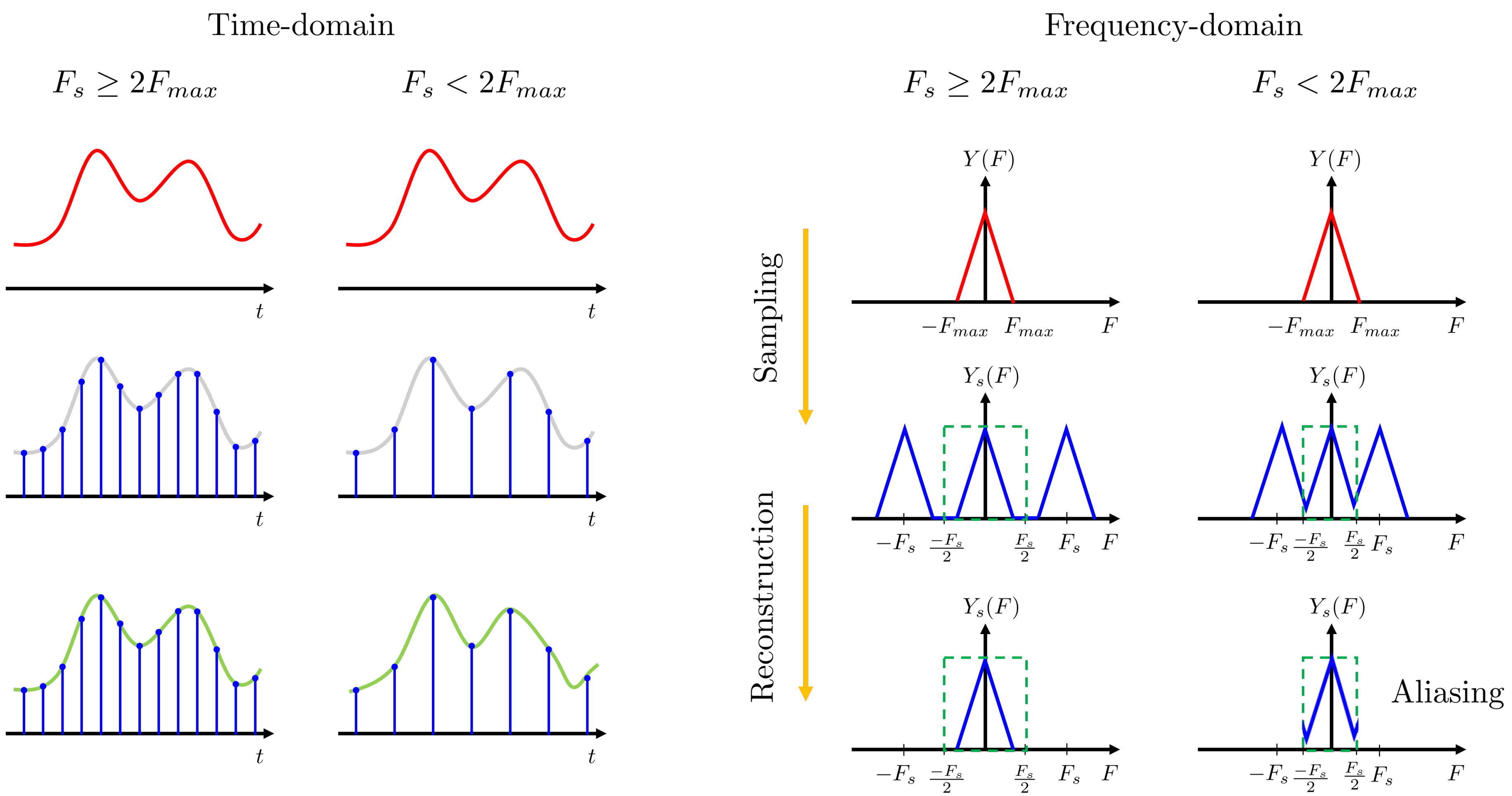
2. **Nyquist freqeuncy(Folding Frequency): Sampling Frequency** F_s의 절반 $F_{nyquist} = F_s/2$이며, 이는 신호의 최대 주파수 $F_{nyquist} \geq F_{max}$ 이어야 신호를 복원 할 수 있다.

3. Frequency Domain Analysis:

time-domain		frequency-domain
$y_s(t) = y(t) \cdot p(t)$	$\leftrightarrow$	$Y_s(F) = Y(F) * P(F)$

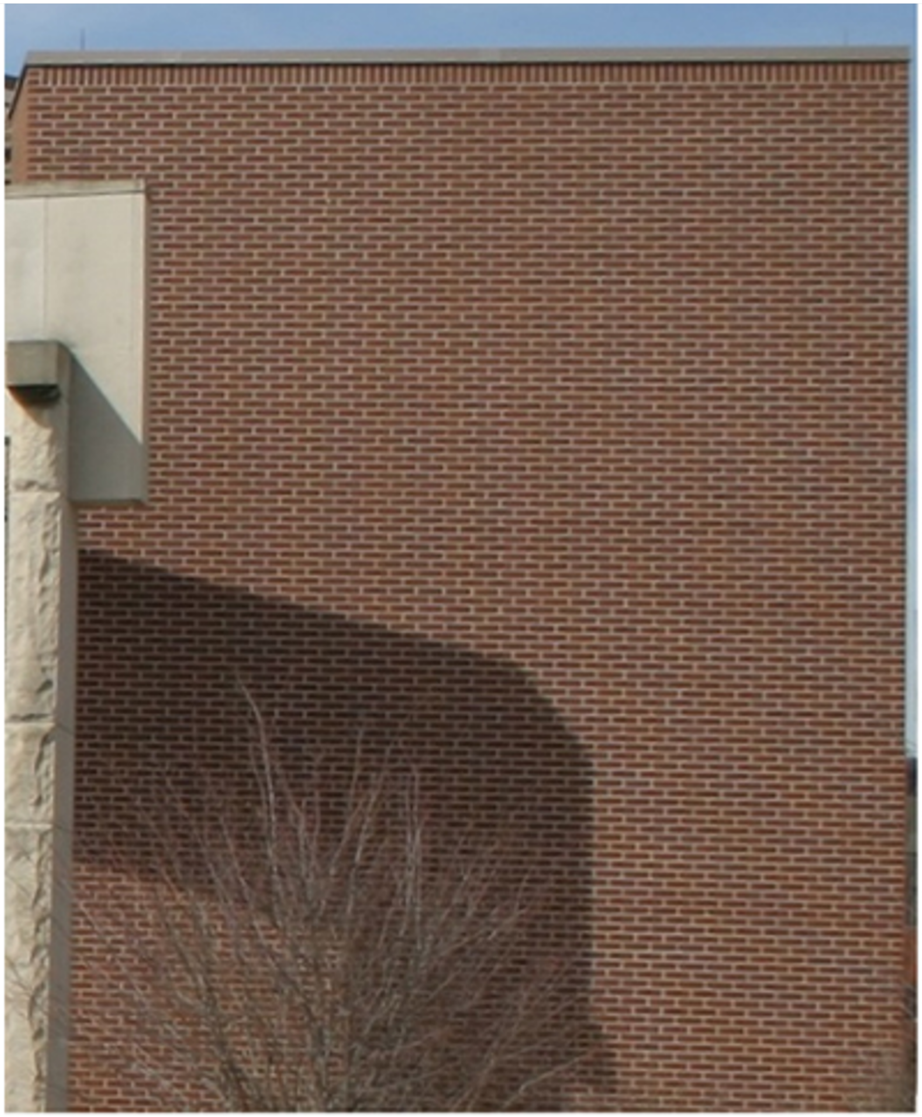
$$
\begin{aligned}
Y_s(F) &= Y(F) * P(F) \\
&= Y(F) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(F - kF_s)
\end{aligned}
$$

1 Digitization



1 Digitization

$F_s \geq 2F_{max}$



$F_s < 2F_{max}$

2 Dynamic Analysis of Discrete Systems

Condition	Discrete-Time	Continuous-Time
Periodic Signal	Discrete Fourier Series	Fourier Series
Absolute Summable/Integrable Signal	Discrete-Time Fourier Transform	Fourier Transform
Causal Signal	Unilateral z-Transfrom	Unilateral Laplace Transform

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform

  1. Laplace transform and its important property

$$\mathcal{L}(f(t)) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) - f(0^-)$$

$$\Downarrow$$

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s) \quad \text{where } f(0^+) = 0$$

0^-부터인 이유는 $f(t)$가 $\delta(t)$나 $\frac{d\delta(t)}{dt}$일때 Laplace Transform에 반영하기 위함, 이해를 돕기위해 정확한 정의는 아니지만 아래와 같은 정의 사용

1 Digitization

2. z-transform is defined by

$$\mathcal{Z}(f(k)) = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k} \\ = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

$$\begin{aligned} \mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1) z^{-k} \\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots \\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right] \\ &= z^{-1}F(z) \end{aligned}$$

where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.

Example) $x(0) = 0, x(1) = 1, x(2) = 2, x(3) = 3, x(4) = 4$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \tag{1}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \tag{2}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \tag{3}$$

3. Important property between LT and z-transform

$$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$Y(z) = (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z)$$

now we have a discrete transfer function:

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

1 Digitization

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