1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock
3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.
4. Difference equation for discrete-time system $\leftrightarrow$ Differential equation for continuous-time system
5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$
   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7
6. The analog output of the sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)
   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

     $$y(t) \qquad \rightarrow \qquad y(k) = y(kT) \qquad \text{with} \quad t = kT$$

     where k is an integer and T is a fixed value (sample period, or sampling time).
   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

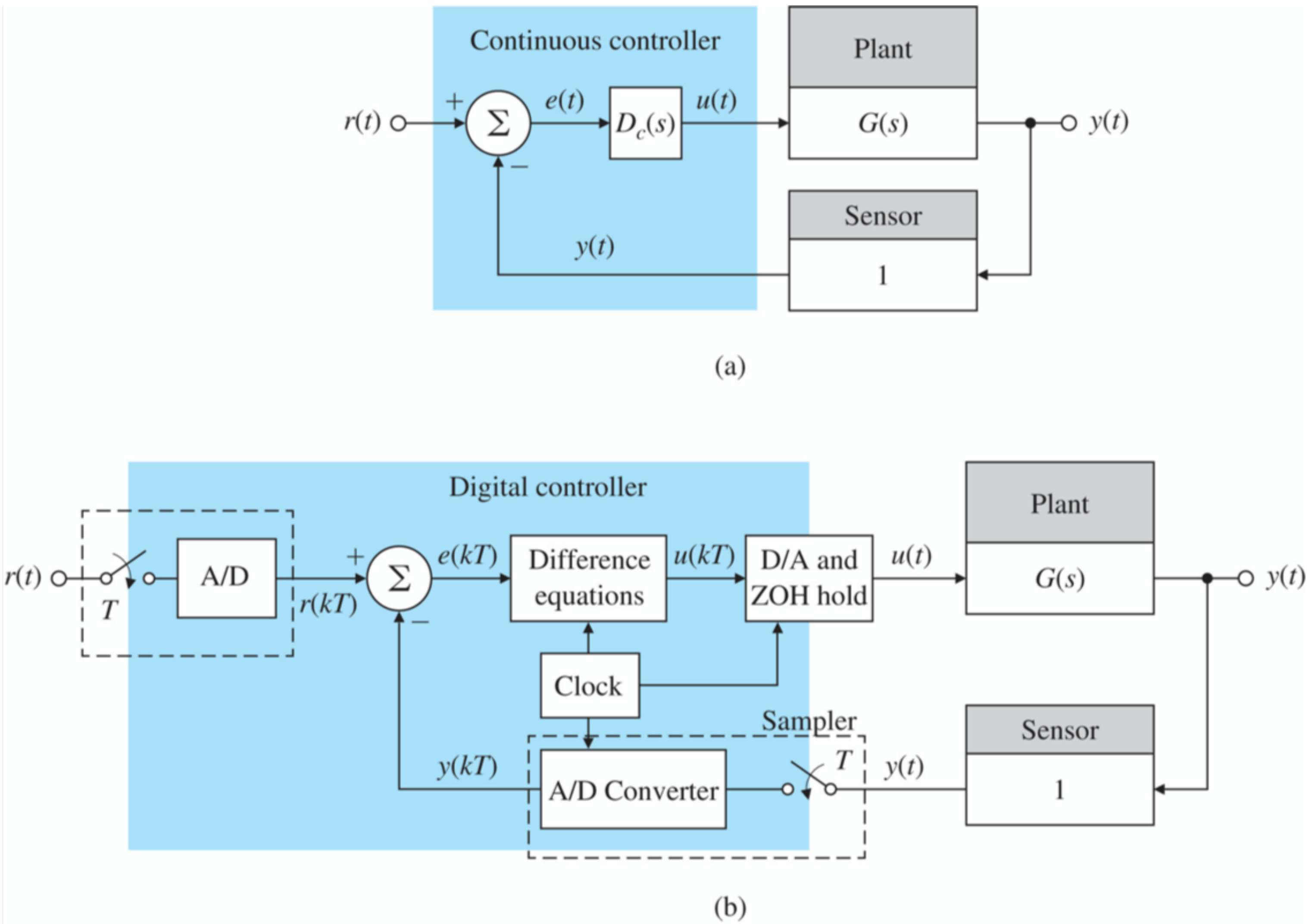
1 Digitization

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)

   - Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
   - Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
   - If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

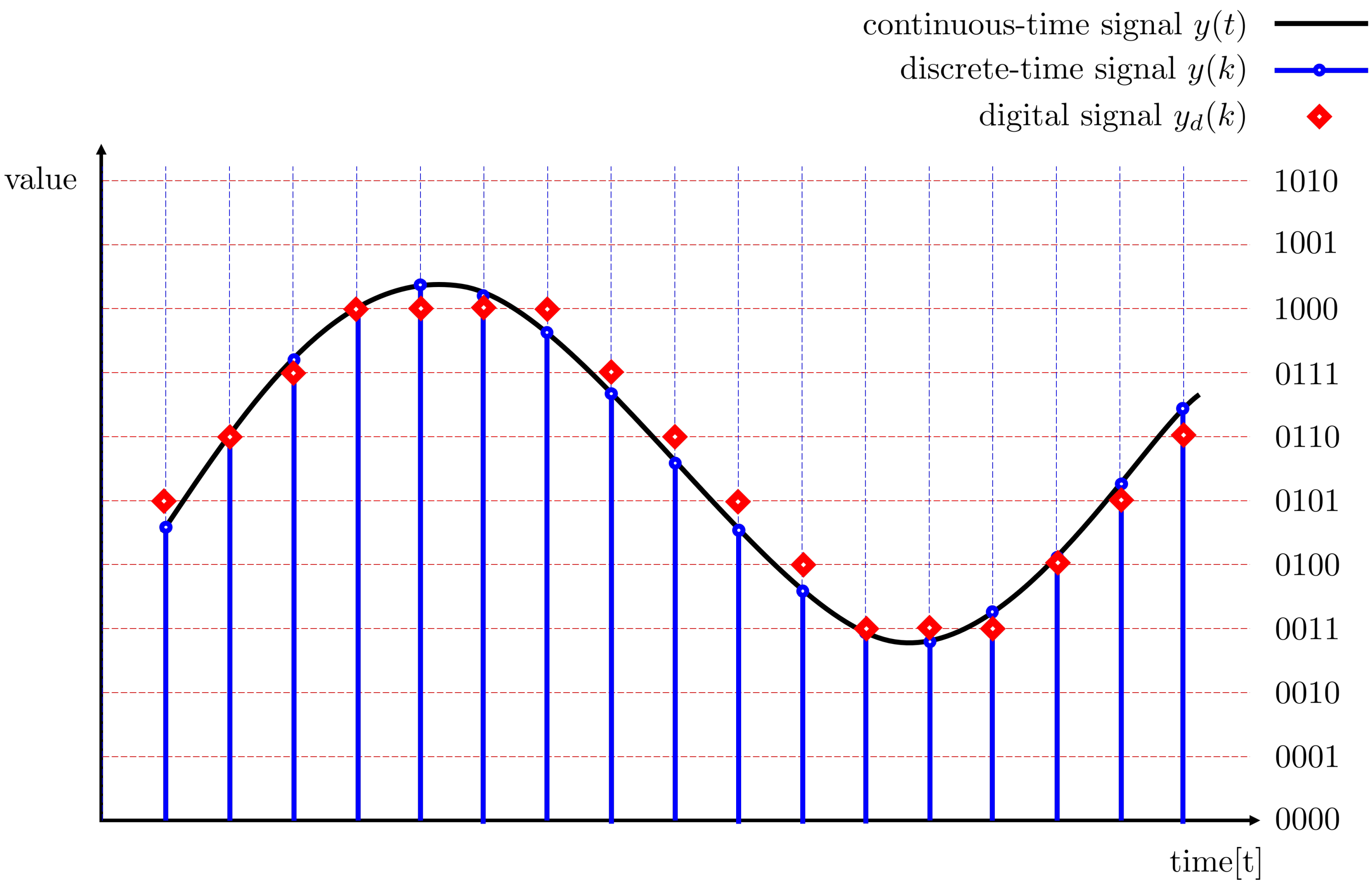
8. A system having both discrete and continuous signals is called a 'sampled data system'.

1 Digitization



1 Digitization

- Continuous-time signal: Both domain and range are continuous, $y(t)$
- Discrete-time signal: Domain is discrete and range is continuous, $y(k)$ or $y(kT)$
- Digital signal: Both domain and range are discrete, $y_d(k)$



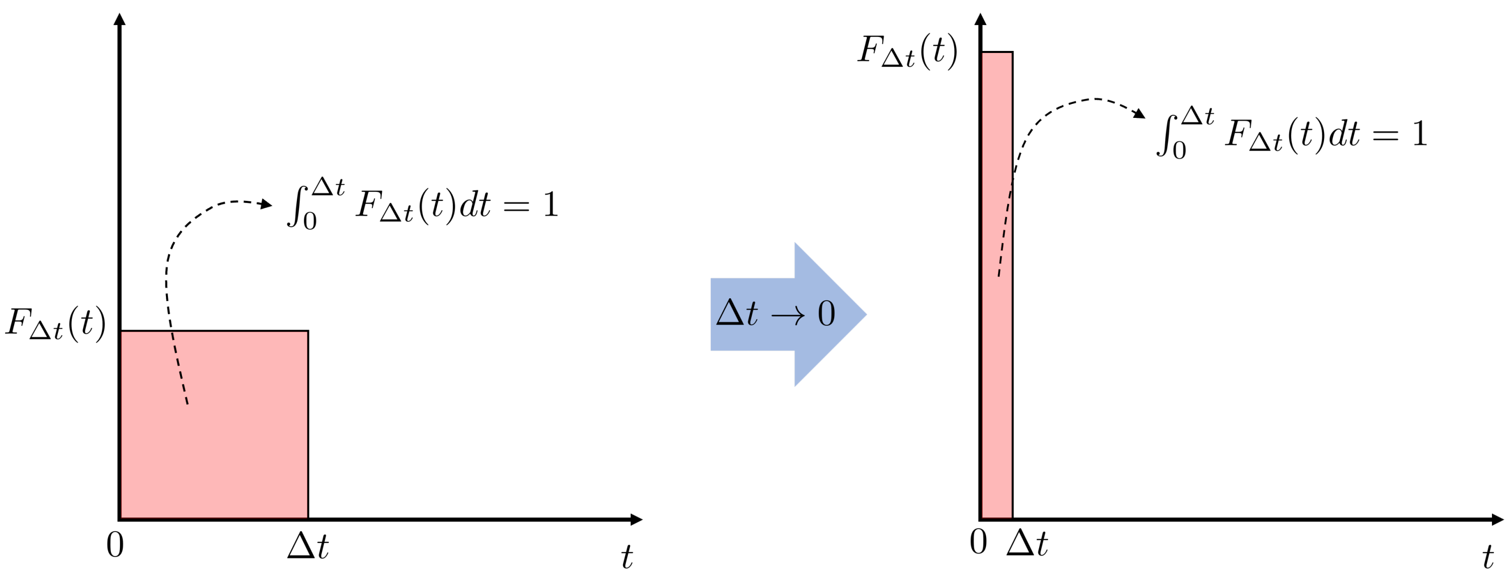
1 Digitization

★ Dirac delta function is mathematically defined as:

1. Approximation

$$F_{\Delta t}(t) = \begin{cases} 1/\Delta t & 0 < t \leq \Delta t \\ 0 & otherwise \end{cases}$$

$$\delta(t) = \lim_{\Delta t \to 0} F_{\Delta t}(t)$$



1 Digitization

2. Generalized function

- $\delta(t) = 0$ **for** $t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t)dt = 1$

★ Unit step function is mathematically defined as:

$$1(t) = \begin{cases} 1 & t > 0 \\ undefined & t = 0 \\ 0 & t < 0 \end{cases}$$

In contol theory

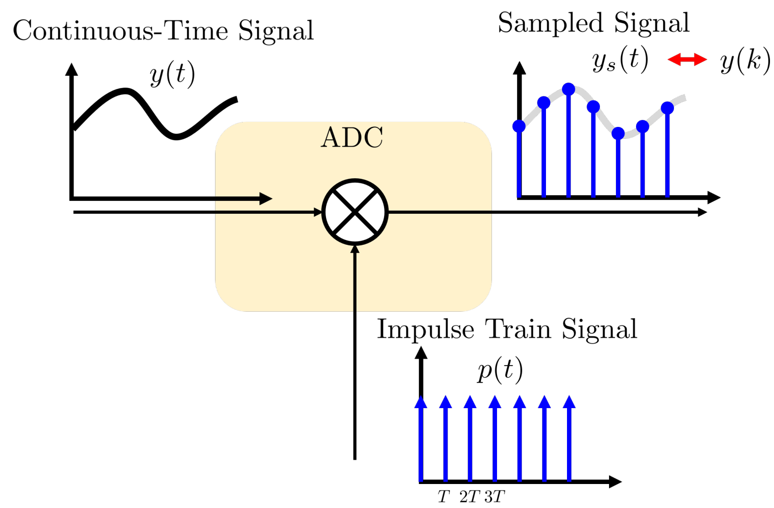
$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

★ Useful Properties

1. $\frac{d1(t)}{dt} = \delta(t)$ (수학적으로는 틀림, 개념적으로 사용)
2. $x(t)\delta(t-kT) = x(kT)\delta(t-kT)$
3. $\int_{-\infty}^{\infty} x(t)\delta(t-kT)dt = x(kT)$
   $\because \int_{-\infty}^{\infty} x(t)\delta(t-kT)dt = \int_{-\infty}^{\infty} x(kT)\delta(t-kT)dt = x(kT)\int_{-\infty}^{\infty} \delta(t-kT)dt = x(kT)$

1 Digitization

★ Sampling Process



1. Periodic Impulse Train: p(t) is periodic with period $T = 1/F_s$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

2. Sampled Signal: we can consider $y_s(t)$ te be the analog equivalent to discrete-time signal $y(k)$ or $y(kT)$

$$y_s(t) = y(t) \cdot p(t) = \sum_{k=-\infty}^{\infty} y(t)\delta(t - kT) = \sum_{k=-\infty}^{\infty} y(kT)\delta(t - kT)$$
$$\leftrightarrow y(k) = y(kT)$$

1 Digitization

★ Nyquist–Shannon Sampling Theorem

1. 간단하게, 아날로그 신호가 갖는 최대 주파수의 2배이상의 샘플링 주파수를 사용해야만 손실되는 정보없이 디지털 신호를 아날로그 신호로 복원할 수 있다.

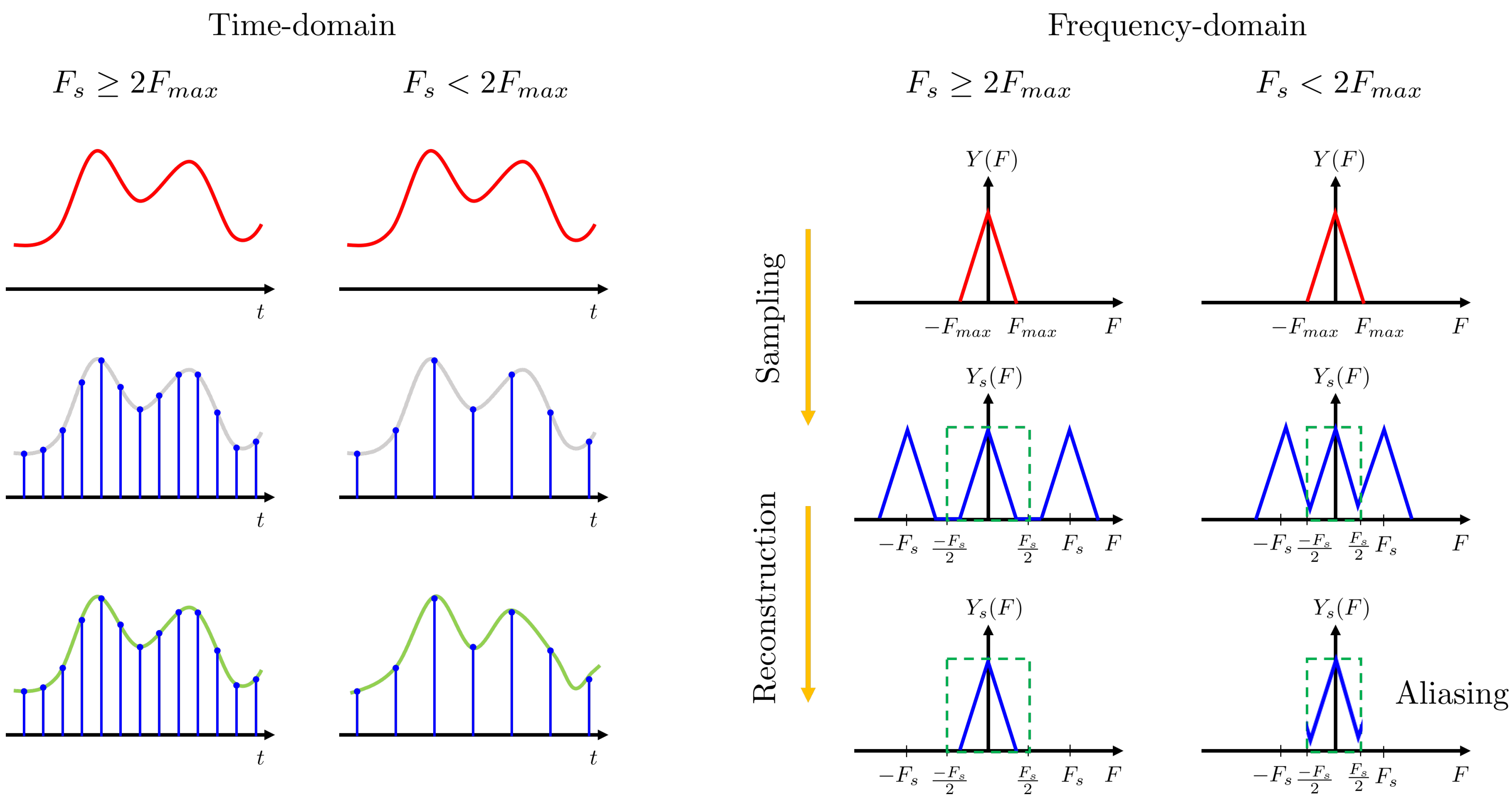
2. Nyquist freqeuncy(Folding Frequency): Sampling Frequency F_s의 절반 $F_{nyquist} = F_s/2$이며, 이는 신호의 최대 주파수 $F_{nyquist} \geq F_{max}$ 이어야 신호를 복원 할 수 있다.

3. Frequency Domain Analysis:

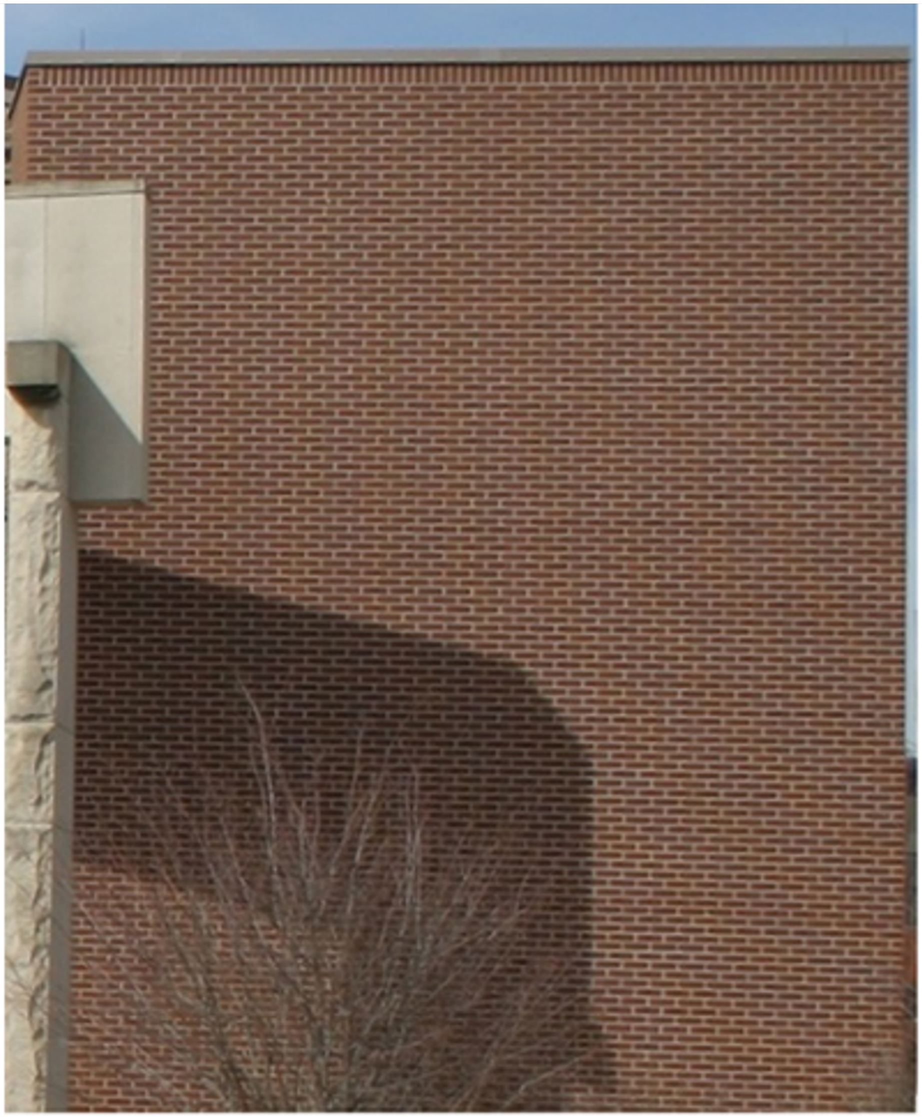
time-domain		frequency-domain
$y_s(t) = y(t) \cdot p(t)$	$\leftrightarrow$	$Y_s(F) = Y(F) * P(F)$

$$
\begin{aligned}
Y_s(F) &= Y(F) * P(F) \\
&= Y(F) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(F - kF_s) \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} Y(F - kF_s)
\end{aligned}
$$

1 Digitization



$F_s \geq 2F_{max}$



$F_s < 2F_{max}$

2 Dynamic Analysis of Discrete Systems

Continuous-Time		Discrete-Time	
periodic	Discrete Fourier Series	periodic	Fourier Series
absolutely integrable	Discrete-Time Fourier Transform	absolutely summable	Fourier Transform
causal	z-Transform	causal	Laplace Transform

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform
  1. Laplace transform and its important property

$$\mathcal{L}(f(t)) = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad\qquad \mathcal{L}(\dot{f}(t)) = sF(s) - f(0^-)$$

$$\Downarrow$$

$$\mathcal{L}(f(t)) = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \qquad\qquad \mathcal{L}(\dot{f}(t)) = sF(s) \quad \text{where } f(0^+) = 0$$

0^-부터인 이유는 $f(t)$가 $\delta(t)$나 $\frac{d\delta(t)}{dt}$일때 Laplace Transform에 반영하기 위함, 이해를 돕기위해 정확한 정의는 아니지만 아래와 같은 정의 사용

2. z-transform is defined by

$$\begin{aligned}\mathcal{Z}(f(k)) = F(z) &= \sum_{k=0}^{\infty} f(k) z^{-k} \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\end{aligned}$$

$$\begin{aligned}\mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1) z^{-k} \\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots \\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right] \\ &= z^{-1}F(z)\end{aligned}$$

where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.

Example) $x(0) = 0, x(1) = 1, x(2) = 2, x(3) = 3, x(4) = 4$

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} \tag{1}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} \tag{2}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} \tag{3}$$

3. Important property between LT and z-transform

$$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$\begin{aligned} Y(z) &= -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) \\ &= (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \end{aligned}$$

now we have a discrete transfer function:

$$\begin{aligned} Y(z) - (-a_1 z^{-1} - a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ (1 + a_1 z^{-1} + a_2 z^{-2}) Y(z) &= (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z) \\ \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

2 Dynamic Analysis of Discrete Systems

- (8.2.1) z-Transform

1. See the Table 8.1 for understanding between z-transform and LT

$F(s)$	$f(kT)$	$F(z)$	
-	$\delta(kT)$	1	1
-	$\delta(kT-k_0T)$	z^{-k_0}	z^{-k_0}
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$	$\frac{z^{-1}\sin aT}{1-(2\cos aT)z^{-1}+z^{-2}}$
$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$	$\frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}$

2. For parts of Table, we have

$$\mathcal{Z}(\delta(kT)) = \sum_{k=0}^{k=\infty} \delta(kT)z^{-k} = \delta(0\cdot T) + \delta(1\cdot T)z^{-1} + \cdots = 1 + 0z^{-1} + 0z^{-2} + \cdots = 1$$

$$\mathcal{Z}(\delta(k-k_0T)) = 0 + 0z^{-1} + \cdots + 1z^{-k_0} + \cdots + = z^{-k_0}$$

$$\mathcal{Z}(1(kT)) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1-z^{-1}} = (1-z^{-1})^{-1}$$

$$\mathcal{Z}(e^{-akT}) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots = \frac{1}{1-e^{-aT}z^{-1}}$$

3. z-Transform Property

- Linearity :

$$\alpha x_1(k) + \beta x_2(k) \quad \leftrightarrow \quad \alpha X_1(z) + \beta X_2(z)$$

- Time-shift :

$$x(k - k_0) \quad \leftrightarrow \quad z^{-k_0} X(z)$$

- Multiplication by k :

$$kx(k) \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$$

- Multiplication by a^k:

$$a^k x(k) \quad \leftrightarrow \quad X(\frac{z}{a})$$

- Multiplication by e^{-akT}:

$$e^{-akT} x(k) \quad \leftrightarrow \quad X(e^{aT} z)$$

$$\begin{aligned}\mathcal{Z}(1-e^{-akT}) &= \mathcal{Z}(1)-\mathcal{Z}(e^{-akT}) = \frac{1}{1-z^{-1}}-\frac{1}{1-e^{-aT}z^{-1}}\\ &= \frac{z^{-1}-e^{-aT}z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})} = \frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}\end{aligned}$$

$$\begin{aligned}\mathcal{Z}(\cos(-akT)) &= \mathcal{Z}(\frac{e^{jakT}+e^{-jakT}}{2}) = \frac{\mathcal{Z}(e^{jakT})}{2}+\frac{\mathcal{Z}(e^{-jakT})}{2}\\ &= \frac{1}{2}\cdot\frac{1}{1-e^{jaT}z^{-1}}+\frac{1}{2}\cdot\frac{1}{1-e^{-jaT}z^{-1}}\\ &= \frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}\end{aligned}$$

$$\mathcal{Z}(\sin(-akT)) =?, \mathcal{Z}(\sinh(-akT)) =?, \mathcal{Z}(\cosh(-akT)) =?$$

2 Dynamic Analysis of Discrete Systems

- (8.2.2) z-Transform Inversion

1. A z-transform inversion technique that has no continuous counterpart is called 'long division'. For example, consider a first-order discrete system

$$y(k) = \alpha y(k-1) + u(k) \quad \rightarrow \quad Y(z) = \alpha z^{-1} Y(z) + U(z) \quad \rightarrow \quad \frac{Y(z)}{U(z)} = \frac{1}{1 - \alpha z^{-1}}$$

For a unit-pulse input, its z-transform is

$$U(z) = \mathcal{Z}(\delta(kT)) = 1$$

so the long division becomes

$$\begin{aligned} Y(z) &= \frac{1}{1 - \alpha z^{-1}} \\ &= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \cdots \end{aligned}$$

We see that the sampled time history of y is

$$y(0) = 1 \qquad y(1) = \alpha \qquad y(2) = \alpha^2 \qquad y(3) = \alpha^3 \qquad \cdots$$

$$
\begin{array}{r l}
 & 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \\ \cline{2-2}
1 - \alpha z^{-1}) & 1 \\ \cline{2-2}
 & 1 - \alpha z^{-1} \\ \cline{2-2}
 & \quad \alpha z^{-1} + 0 \\
 & \quad \alpha z^{-1} - \alpha^2 z^{-2} \\ \cline{2-2}
 & \qquad \alpha^2 z^{-2} + 0 \\
 & \qquad \alpha^2 z^{-2} - \alpha^3 z^{-3} \\ \cline{2-2}
 & \qquad\quad \alpha^3 z^{-3}
\end{array}
$$

2 Dynamic Analysis of Discrete Systems

- (8.2.3) Relationship between s and z

1. Consider the continuous signal of

$$f(t) = e^{-at} \qquad t > 0 \qquad \rightarrow \qquad F(s) = \frac{1}{s+a}$$

and it corresponds to a pole $s = -a$.

2. Consider the discrete signal of

$$f(kT) = e^{-akT} \qquad \rightarrow \qquad F(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}$$

and it corresponds to a pole $z = e^{-aT}$.

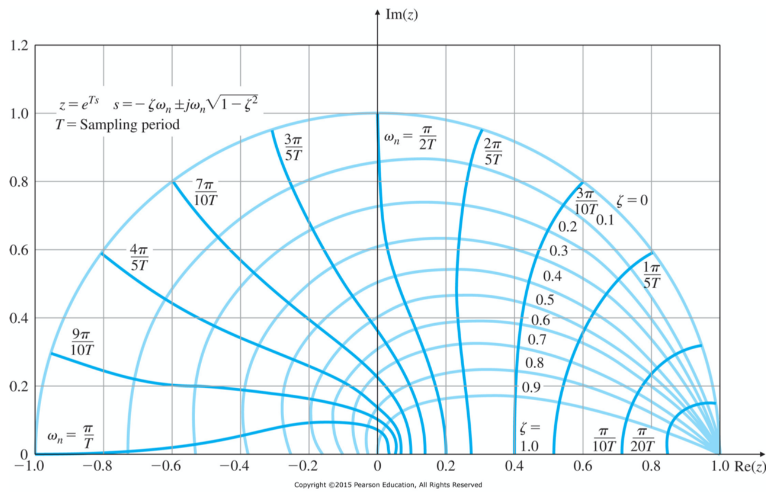
3. The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

$$\begin{aligned} z &= e^{sT} \\ &= e^{-aT+jbT} = e^{-aT}(\cos bT + j\sin b) \\ &= e^{-\sigma T}(\cos\omega_d T + j\sin\omega_d T) \\ &= e^{-\zeta\omega_n T}(\cos\omega_n\sqrt{1-\zeta^2}T + j\sin\omega_n\sqrt{1-\zeta^2}T) \end{aligned}$$

where T is the sample period, and $s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

2 Dynamic Analysis of Discrete Systems

4. See Fig. 8.4, and it shows the mapping of lines of constant damping ζ and natural frequency ω_n from s-plane to the upper half of the z-plane, using $z = e^{sT}$.



a) The stability boundary $s = 0 \pm j\omega$ becomes the unit circle $|z| = 1$ in the z-plane; inside the unit circle is stable, outside is unstable
s-plane에서의 stability boundary는 imaginary축 $s = \pm j\omega$ 인데 z-plane에서의 stability boundary는 unit circle $|z| = e^{sT}|_{s=\pm j\omega} = 1$이 된다.

b) The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around the origin $s = 0$, in the s-plane.
z-plane 에서의 $z = 1$근방은 s-plane에서의 $s = 0$ 근방과 같다.

c) The z-plane locations give response information normalized to the sample rate rather than to time as in the s-plane.
z-plane 에서의 response information은 s-plane에서와 같이 시간에 대한 정보가 아닌, sample rate로 normalized된 정보를 제공한다.

d) The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T =$ circular sample rate in radians per second.
ω_s가 $2\pi/T$ 일때 음의 z축은 $\omega_s/2$로 표현된다.

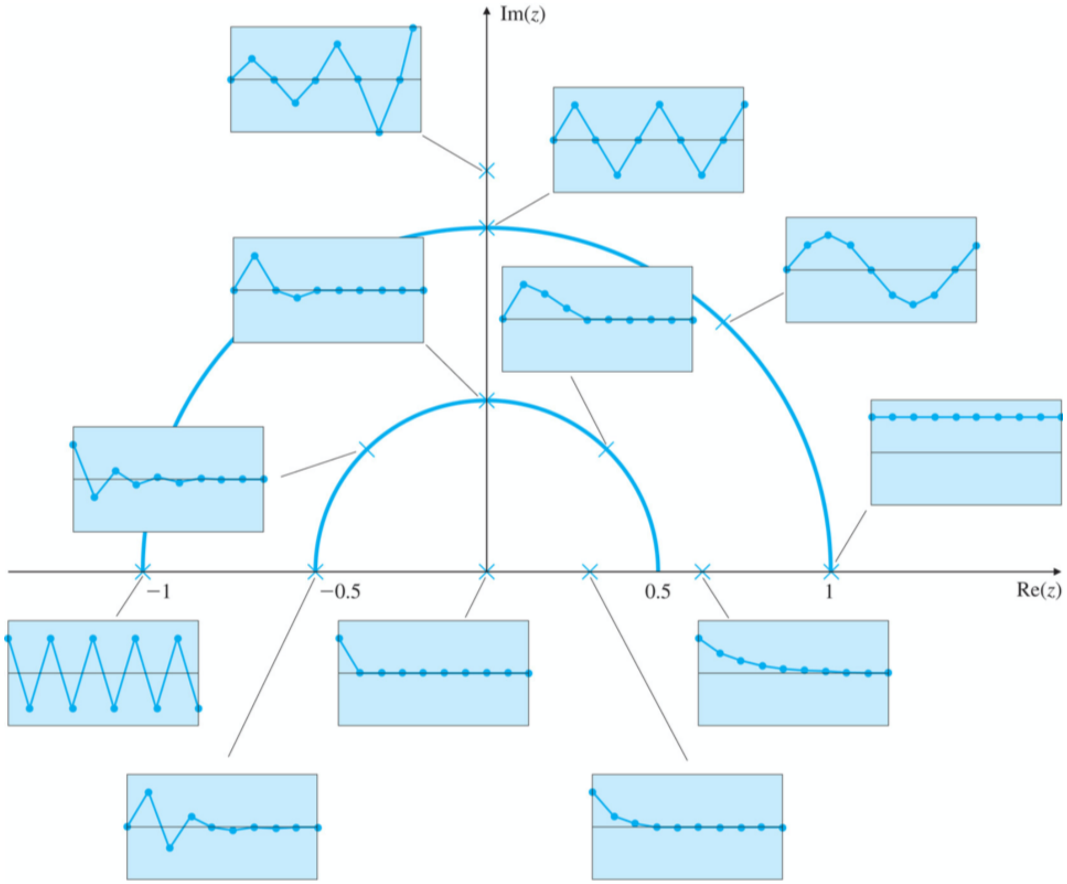
e) Vertical lines in the left half of the s-plane (the constant real part of s) map into *circles* within the unit circle of the z-plane
s-plane에서의 좌반면은 z-plane에서의 unit circle 내부로 매핑된다.

f) Horizontal lines in the s-plane (the constant imaginary part of s) map into *radial lines* in the z-plane.
s-plane에서의 수평선은 z-plane에서의 radial line들로 매핑된다.

g) Frequencies greater than $\omega_s/2$, called the Nyquist frequency, appear in the z-plane on the top of corresponding lower frequencies because of the circular characteristics of e^{sT}. This overlap is called *aliasing* or folding.

5. As a result, it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples.

6. The figure sketches time responses that would result from poles at the indicated locations.



2 Dynamic Analysis of Discrete Systems

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1 Digitization

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