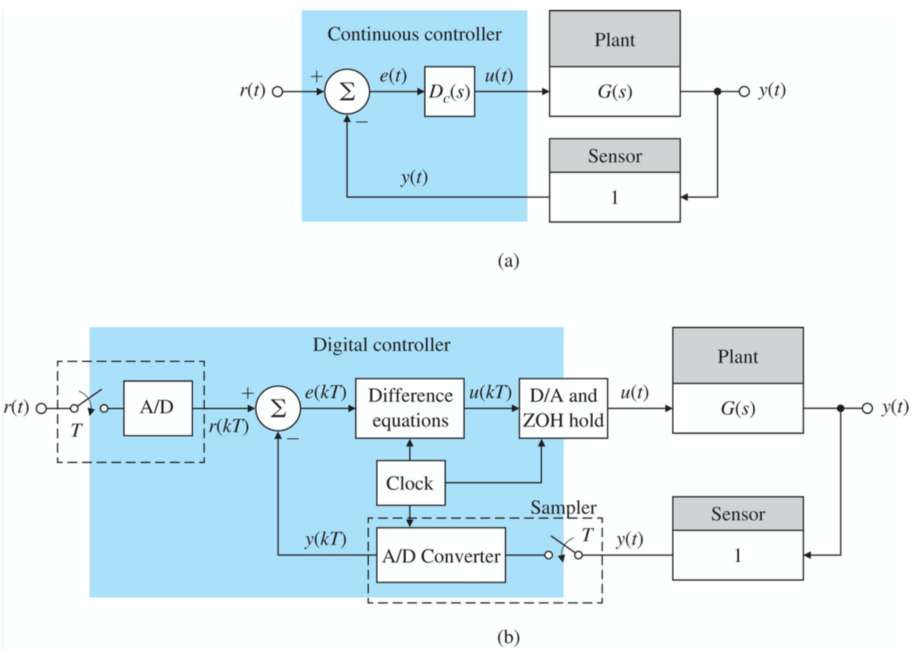
제 8 장

Digital Control

1 Digitization

1. Most control systems use digital computers (usually microprocessors) to implement the controller.
2. Sampler and A/D Converter, D/A Converter and ZOH (Zeroth-Order Holding), and Clock



3. The computation of error signal $e(t)$ and the dynamic compensation $D_c(s)$ can all be accomplished in a digital computer.

4. Difference equation for discrete-time system $\leftrightarrow$ Differential equation for continuous-time system

5. Two basic techniques for finding the difference equations for the digital controller, from $D_c(s)$ to $D_d(z)$

   - Discrete equivalent - section 8.3
   - Discrete design - section 8.7

6. The analog output of the sensor is sampled and converted to a digital number in the analog-to-digital (A/D) converter. (Sampler and ADC)

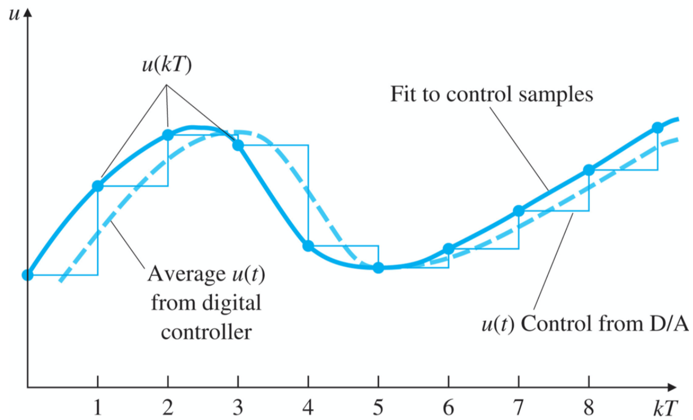
   - Conversion from the continuous analog signal $y(t)$ to the discrete digital samples $y(kT)$ occurs repeatedly at instants of time T apart where T is the sample period $[s]$ and $1/T$ is the sample rate $[Hz]$.

   $$y(t) \quad \rightarrow \quad y(k) = y(kT) \quad \text{with} \quad t = kT$$

   where k is an integer and T is a fixed value (sample period, or sampling time).

   - The sampled signal is $y(kT)$, where k can take on any integer value.
   - It is often written simply as $y(k)$. We call this type of variable a discrete signal.

7. The D/A converter changes the digital binary number to an analog voltage, and a zeroth-order hold maintains the same voltage throughout the sample period T. (DAC and ZOH)



- Because each value of $u(kT)$ in Fig. 8.1(b) is held constant until the next value is available from the computer, the continuous value of $u(t)$ consists of steps (see Fig. 8.2) that, on average, are delayed from a fit to $u(kT)$ by $T/2$ as shown in the figure.
- Sample rates should be at least 20 times the bandwidth in order to assure that the digital controller will match the performance of the continuous controller.
- If we simply incorporate this $T/2$ delay into a continuous analysis of the system, an excellent prediction results in, especially, for sample rates much slower than 20 times bandwidth.

8. A system having both discrete and continuous signals is called a ‘sampled data system’.

2 Dynamic Analysis of Discrete Systems

- z-transform for discrete time systems $\leftrightarrow$ Laplace transform for continuous time systems.
- (8.2.1) z-Transform
  1. Laplace transform and its important property

     $$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt \qquad \mathcal{L}(\dot{f}(t)) = sF(s)$$

     where $f(0^+) = 0$
  2. z-transform is defined by

     $$\begin{aligned}\mathcal{Z}(f(k)) = F(z) &= \sum_{k=0}^{\infty} f(k)z^{-k} \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\end{aligned}$$

     $$\begin{aligned}\mathcal{Z}(f(k-1)) &= \sum_{k=0}^{\infty} f(k-1)z^{-k} \\ &= f(-1) + f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \cdots \\ &= z^{-1}\left[f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots\right] \\ &= z^{-1}F(z)\end{aligned}$$

     where $f(k)$ is the sampled version of $f(t)$ and z^{-1} represents one sample delay, and $f(-1) = 0$.
  3. Important property between LT and z-transform

     $$z = e^{sT} \qquad \leftrightarrow \qquad s = \frac{1}{T}\ln z$$

4. For example, the general second-order difference equation

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted from this form to the z-transform of the variables $y(k)$ and $u(k)$ by invoking above relations,

$$Y(z) = (-a_1 z^{-1} - a_2 z^{-2}) Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2}) U(z)$$

now we have a discrete transfer function:

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

- (8.2.2) z-Transform Inversion

1. See the Table 8.1 for understanding between z-transform and LT

$F(s)$	$f(kT)$	$F(z)$	
-	**1**, $k=0$ **and 0**, $k \neq 0$	**1**	
-	**1**, $k=k_0$ **and 0**, $k \neq k_0$	z^{-k_0}	
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$
$\frac{1}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$	$\frac{z^{-1}(1-e^{-aT})}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z \sin aT}{z^2-(2\cos aT)z+1}$	$\frac{z^{-1}\sin aT}{1-(2\cos aT)z^{-1}+z^{-2}}$
$\frac{s}{s^2+a^2}$	$\cos akT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$	$\frac{(1-z^{-1}\cos aT)}{1-(2\cos aT)z^{-1}+z^{-2}}$

2. For parts of Table, we have

$$\mathcal{Z}(\delta(t)) = 1 + 0z^{-1} + 0z^{-2} + \cdots = 1$$

$$\mathcal{Z}(\delta(t = k_0T)) = 0 + 0z^{-1} + \cdots + 1z^{-k_0} + \cdots + = z^{-k_0}$$

$$\mathcal{Z}(1(t)) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = (1 - z^{-1})^{-1}$$

$$\mathcal{Z}(e^{-at}) = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + \cdots = \frac{1}{1 - e^{-aT}z^{-1}}$$

3. The differentiator s is transformed into z-domain

$$\frac{1}{s} \quad \leftrightarrow \quad \frac{1}{1-z^{-1}} \qquad\qquad s \quad \leftrightarrow \quad (1-z^{-1})$$

4. z-transform of ramp signal $t = kT$ becomes

$$\begin{aligned}
\mathcal{Z}(t) &= 0 + Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + \cdots \\
&= T[z^{-1} + 2z^{-2} + 3z^{-3} + \cdots] \\
z^{-1}\mathcal{Z}(t) &= T[z^{-2} + 2z^{-3} + 3z^{-4} + \cdots] \\
(1-z^{-1})\mathcal{Z}(t) &= T[z^{-1} + z^{-2} + z^{-3} + \cdots] = T\frac{z^{-1}}{1-z^{-1}} \\
\mathcal{Z}(t) &= \frac{Tz^{-1}}{(1-z^{-1})^2}
\end{aligned}$$

5. A z-transform inversion technique that has no continuous counterpart is called 'long division'. For example, consider a first-order discrete system

$$y(k) = \alpha y(k-1) + u(k) \qquad \rightarrow \qquad \frac{Y(z)}{U(z)} = \frac{1}{1-\alpha z^{-1}}$$

For a unit-pulse input, its z-transform is

$$U(z) = 1$$

so the long division becomes

$$\begin{aligned} Y(z) &= \frac{1}{1-\alpha z^{-1}} \\ &= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} \cdots \end{aligned}$$

We see that the sampled time history of y is

$$y(0) = 1 \qquad y(1) = \alpha \qquad y(2) = \alpha^2 \qquad y(3) = \alpha^3 \qquad \cdots$$

- (8.2.3) Relationship between s and z

1. Consider the continuous signal of

$$
\begin{aligned}
f(t) &= e^{-at} \qquad t > 0 \\
F(s) &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = \frac{1}{s+a}
\end{aligned}
$$

   and it corresponds to a pole $s = -a$.

2. Consider the discrete signal of

$$
\begin{aligned}
f(kT) &= e^{-akT} \\
F(z) &= \sum_{k=0}^{\infty} f(kT)z^{-k} = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \cdots \qquad \text{무한등비급수} \\
&= \frac{\text{초기치}}{1 - \text{공비}} = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}
\end{aligned}
$$

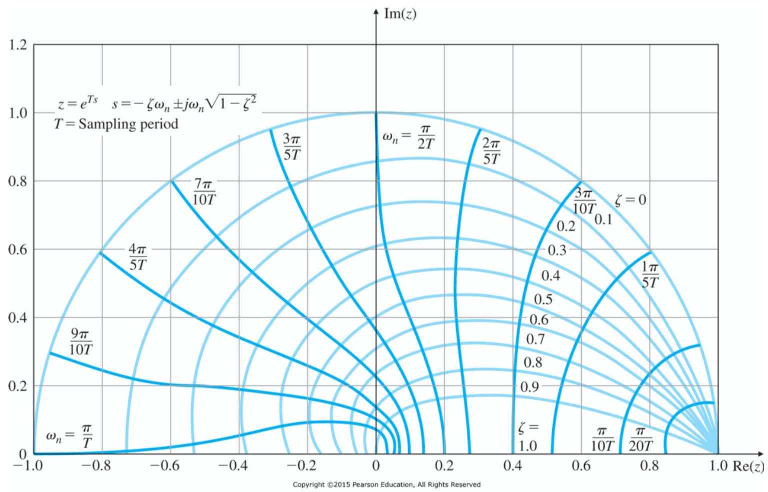
   and it corresponds to a pole $z = e^{-aT}$.

3. The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

$$
\begin{aligned}
z &= e^{sT} = e^{-aT+jbT} = e^{-aT}(\cos bT + j\sin b) \\
&= e^{-\sigma T}(\cos\omega_d T + j\sin\omega_d T) \\
&= e^{-\zeta\omega_n T}(\cos\omega_n\sqrt{1-\zeta^2}T + j\sin\omega_n\sqrt{1-\zeta^2}T)
\end{aligned}
$$

   where T is the sample period, and $s = -\sigma + j\omega_d = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$

4. See Fig. 8.4, and it shows the mapping of lines of constant damping ζ and natural frequency ω_n from s-plane to the upper half of the z-plane, using $z = e^{sT}$.



a) The stability boundary $s = 0 \pm j\omega$ becomes the unit circle $|z| = 1$ in the z-plane; inside the unit circle is stable, outside is unstable

b) The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around the origin $s = 0$, in the s-plane.

c) The z-plane locations give response information normalized to the sample rate rather than to time as in the s-plane.

d) The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T =$ circular sample rate in radians per second.

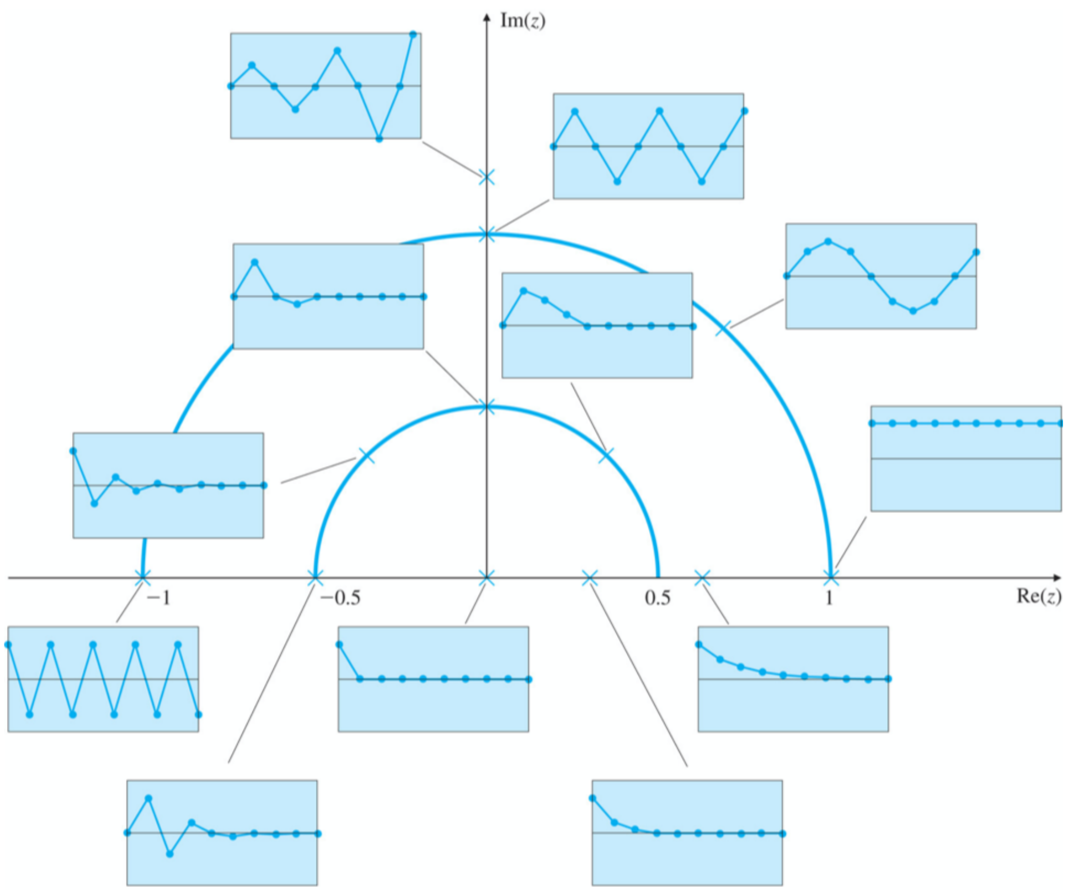
e) Vertical lines in the left half of the s-plane (the constant real part of s) map into *circles* within the unit circle of the z-plane

f) Horizontal lines in the s-plane (the constant imaginary part of s) map into *radial lines* in the z-plane.

g) Frequencies greater than $\omega_s/2$, called the Nyquist frequency, appear in the z-plane on the top of corresponding lower frequencies because of the circular characteristics of e^{sT}. This overlap is called *aliasing* or folding.

5. As a result, it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples.

6. The figure sketches time responses that would result from poles at the indicated locations.



- (8.2.4) Final Value Theorem

1. Discrete final value theorem is

$$\lim_{t\to\infty} x(t) = x_{ss} = \lim_{s\to 0} sX(s) \qquad\qquad \lim_{k\to\infty} x(k) = x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z)$$

   if all the poles of $(1-z^{-1})X(z)$ are inside the unit circle.

2. For example, to find the DC gain of the TF

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{z+0.16}$$

   we let $u(k)=1$ for $k \geq 0$, so that

$$U(z) = \frac{1}{1-z^{-1}}$$

   and

$$X(z) = \frac{0.58(1+z)}{(1-z^{-1})(z+0.16)}$$

   Applying the final value theorem yields

$$x_{ss} = \lim_{z\to 1}(1-z^{-1})X(z) = \frac{0.58\cdot 2}{1+0.16} = 1$$

   so the DC gain of $G(z)$ is unity.

3 Design using Discrete Equivalents

- It is important to remember that how to convert $D_c(s)$ into $D_d(z)$ is approximation; there is no exact solution for all possible inputs because $D_c(s)$ responds to the complete time history of $e(t)$, whereas $D_d(z)$ has access to only the samples $e(kT)$.

- (8.3.1) Tustin's Method

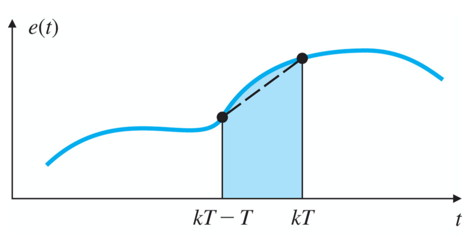
  1. Tustin's method is a digitization technique that approaches the problem as one of numerical integration. Suppose

$$\frac{U(s)}{E(s)} = D_c(s) = \frac{1}{s}$$

which is integration. Therefore, it is corresponding to the *trapezoidal integration* as follows:

$$\begin{aligned} u(kT) &= \int_0^{kT-T} e(t)dt + \int_{kT-T}^{kT} e(t)dt \\ &= u(kT-T) + \text{area under } e(t) \text{ over last period, } T, \\ u(k) &= u(k-1) + T\frac{[e(k-1)+e(k)]}{2} \end{aligned}$$

where T is the sample period.



2. Taking z-transform,

$$\frac{U(z)}{E(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

3. In fact, the Tustin's method approximates $z = e^{sT}$ as follows:

$$s \approx \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

where it can be derived from the Taylor's series expansions as follows:

$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} = \frac{1+\frac{sT}{2}+\frac{s^2T^2}{2^2}+\cdots}{1-\frac{sT}{2}+\frac{s^2T^2}{2^2}-\cdots} \approx \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} = \frac{2+sT}{2-sT} \qquad \rightarrow \qquad s \approx \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

4. For $D_c(s) = \frac{a}{s+a}$ as an example, we have

$$D_d(z) = \frac{U(z)}{E(z)} = \frac{a}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+a} = \frac{aT(1+z^{-1})}{2(1-z^{-1})+aT(1+z^{-1})} = \frac{aT(1+z^{-1})}{(2+aT)-(2-aT)z^{-1}}$$

$$(2+aT)u(k)-(2-aT)u(k-1) = aT[e(k)+e(k-1)]$$

$$\therefore \qquad u(k) = \frac{(2-aT)}{(2+aT)}u(k-1)+\frac{aT}{(2+aT)}[e(k)+e(k-1)]$$

5. (Example 8.1) Determine the difference equation with a sample rate of 25 times bandwidth using Tustin's approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1}$$

Since the bandwidth is approximately $\omega_{bd} = 10[rad/s]$, the sampling rate should be

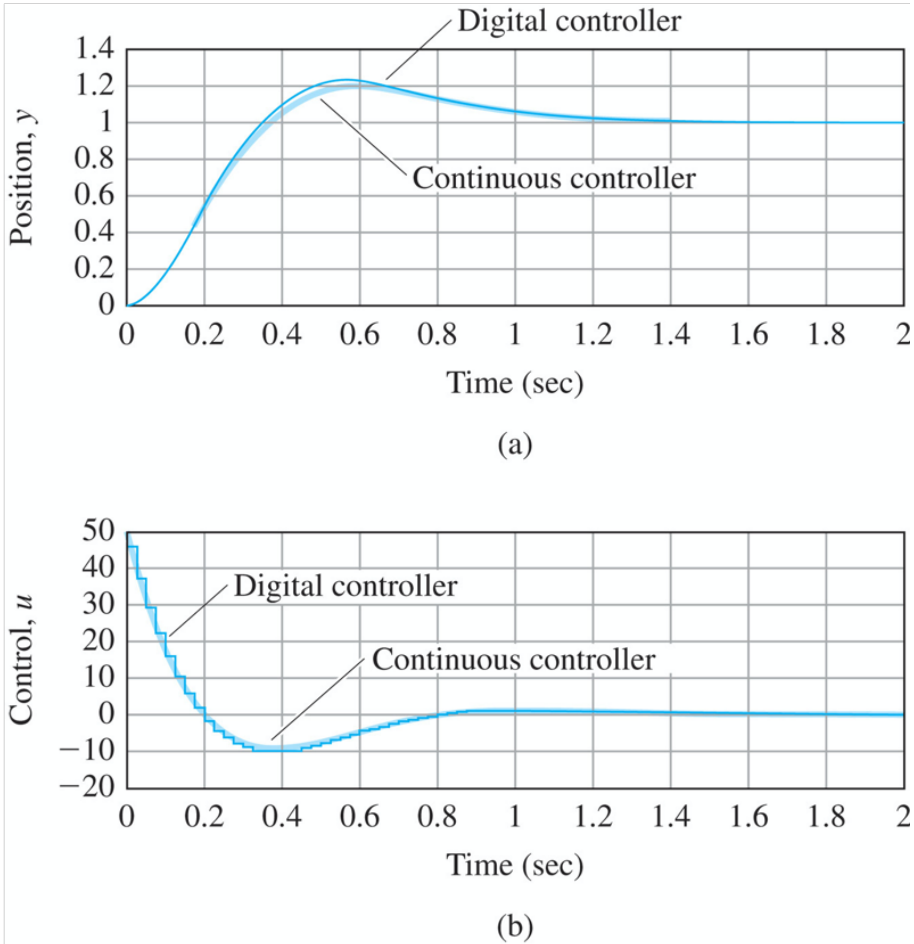
$$\omega_s = 25 \times \omega_{bd} = 250[rad/s] \qquad \rightarrow \qquad f_s = \frac{\omega_s}{2\pi} \approx 40[Hz] \qquad \rightarrow \qquad T = \frac{1}{f_s} = \frac{1}{40} = 0.025[s]$$

The discrete TF can be obtained as

$$\begin{aligned} D_d(z) &= 10\frac{\frac{1}{T}\frac{1-z^{-1}}{1+z^{-1}}+1}{\frac{1}{5T}\frac{1-z^{-1}}{1+z^{-1}}+1} = 10\frac{5(1-z^{-1})+5T(1+z^{-1})}{(1-z^{-1})+5T(1+z^{-1})} \\ &= 50\frac{(1+T)-(1-T)z^{-1}}{(1+5T)-(1-5T)z^{-1}} = 50\frac{1.025-0.975z^{-1}}{1.125-0.875z^{-1}} = \frac{45.556-43.333z^{-1}}{1-0.778z^{-1}} \end{aligned}$$

Finally, the difference equation is

$$u(k) = 0.778u(k-1) + 45.556[e(k) - 0.951e(k-1)]$$


Digital controller
Position, y
Continuous controller
Time (sec)
(a)
Control, u
Digital controller
Continuous controller
Time (sec)
(b)

- (8.3.2) Zeroth-Order Hold (ZOH) Method

  1. Tustin's method essentially assumed that the input to the controller varied linearly early between the past sample and the current sample.
  2. Another assumption is that the input to the controller remains constant throughout the sample period. $\rightarrow$ ZOH
  3. One input sample produces a square pulse of height $e(k)$ that lasts for one sample period T.
  4. For a constant positive step input, $e(k)$, at time k, $E(s) = e(k)/s$, so the result would be

     $$D_d(z) = \mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Furthermore, a constant negative step, one cycle delayed, would be

     $$D_d(z) = z^{-1}\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

     Therefore, the discrete TF for the square pulse is

     $$D_d(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right)$$

5. (Example 8.2) Determine the difference equation with a sample period $T = 0.025[s]$ using ZOH approximation.

$$D_c(s) = 10\frac{s/2+1}{s/10+1} = 10\frac{5s+10}{s+10}$$

The discrete TF using ZOH is

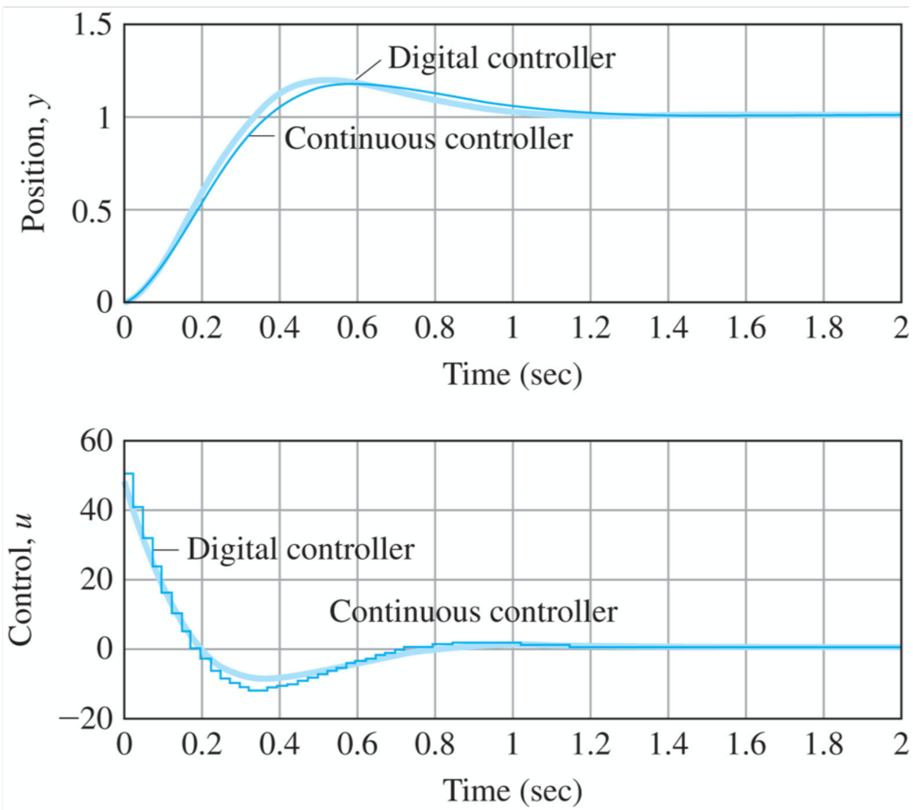
$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{5}{s+10}+\frac{10}{s(s+10)}\right) \\
&= 10(1-z^{-1})\left(\frac{5}{1-e^{-0.25}z^{-1}}+\frac{z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= 10(1-z^{-1})\left(\frac{5(1-z^{-1})+z^{-1}(1-e^{-0.25})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

where $\mathcal{Z}\left\{\frac{1}{s+10}\right\} = \frac{1}{1-e^{-10T}z^{-1}}$ with $e^{-10T} = e^{-0.25} = 0.779$. Or,

$$\begin{aligned}
D_d(z) = 10(1-z^{-1})\mathcal{Z}\left(\frac{5s+10}{s(s+10)}\right) &= 10(1-z^{-1})\mathcal{Z}\left(\frac{1}{s}+\frac{4}{s+10}\right) \\
&= 10(1-z^{-1})\left(\frac{1}{1-z^{-1}}+\frac{4}{1-e^{-0.25}z^{-1}}\right) \\
&= 10(1-z^{-1})\left(\frac{(1-e^{-0.25}z^{-1})+4(1-z^{-1})}{(1-z^{-1})(1-e^{-0.25}z^{-1})}\right) \\
&= \frac{50-47.79z^{-1}}{1-0.779z^{-1}}
\end{aligned}$$

Finally, the difference equation is

$$
\begin{aligned}
u(k) &= 0.779u(k-1) + 50e(k) - 47.79e(k-1) \\
&= 0.779u(k-1) + 50[e(k) - 0.956e(k-1)]
\end{aligned}
$$



- (8.3.3) Matched Pole-Zero (MPZ) Method

1. Another digitization method, called the matched pole-zero (MPZ) method, is suggested by matching the poles and zeros between s and z planes, using $z = e^{sT}$.
2. Because physical systems often have more poles than zeros, it is useful to arbitrarily add zeros at $z = -1$, resulting in a $(1 + z^{-1})$ term in $D_d(z)$.
   a) Map poles and zeros according to the relation $z = e^{sT}$
   b) If the numerator is of lower order than the denominator, add powers of $(1 + z^{-1})$ to the numerator until numerator and denominator are of equal order.
   c) Set the DC or low frequency gain of $D_d(z)$ equal to that of $D_c(s)$.
3. For example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s+b} \qquad\qquad D_d(z) = K_d \frac{1 - e^{-aT} z^{-1}}{1 - e^{-bT} z^{-1}}$$

where K_d is found by the DC-gain

$$\lim_{s \to 0} D_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z \to 1} D_d(z) = K_d \frac{1 - e^{-aT}}{1 - e^{-bT}}$$

Thus the result is

$$K_d = K_c \frac{a}{b} \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

4. As another example, the MPZ approximation

$$D_c(s) = K_c \frac{s+a}{s(s+b)} \qquad\qquad D_d(z) = K_d \frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}$$

where K_d is found by the DC-gain *by deleting the pure integration term* both sides

$$\lim_{s\to 0} sD_c(s) = K_c \frac{a}{b} \qquad \rightleftarrows \qquad \lim_{z\to 1}(z-1)D_d(z) = K_d \frac{2(1-e^{-aT})}{1-e^{-bT}}$$

The result is

$$K_d = K_c \frac{a}{2b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

5. (Example 8.3) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$, another real pole at $s = -1.58$, using MPZ digitization

$$G(s) = \frac{1}{s^2}$$

Let us assume that the lead compensator is used

$$D_c(s) = K_c \frac{s+b}{s+a}$$

Then, we have the characteristic equation

$$1 + G(s)D_c(s) = 1 + K_c \frac{s+b}{s^2(s+a)} = s^3 + as^2 + K_c s + K_c b$$

$$\alpha_c(s) = (s^2 + 0.42s + 0.09)(s + 1.58) = s^3 + 2s^2 + 0.7536s + 0.1422$$

with $a = 2$, $b = 0.19$, and $K_c = 0.7536$. Now we have the lead compensator:

$$D_c(s) = 0.7536 \frac{s+0.19}{s+2} \quad \rightarrow \quad D_c(s) = 0.81 \frac{s+0.2}{s+2}$$

Let us determine the sampling rate and sampling period as follows:

$$\omega_s = 0.3 \times 20 = 6[rad/s] \quad \rightarrow \quad f_s = \frac{\omega_s}{2\pi} \approx 1[Hz] \quad \rightarrow \quad T = 1[s]$$

The MPZ digitization yields

$$D_d(z) = K_d \frac{1 - e^{-0.2} z^{-1}}{1 - e^{-2} z^{-1}} = K_d \frac{1 - 0.818 z^{-1}}{1 - 0.135 z^{-1}}$$

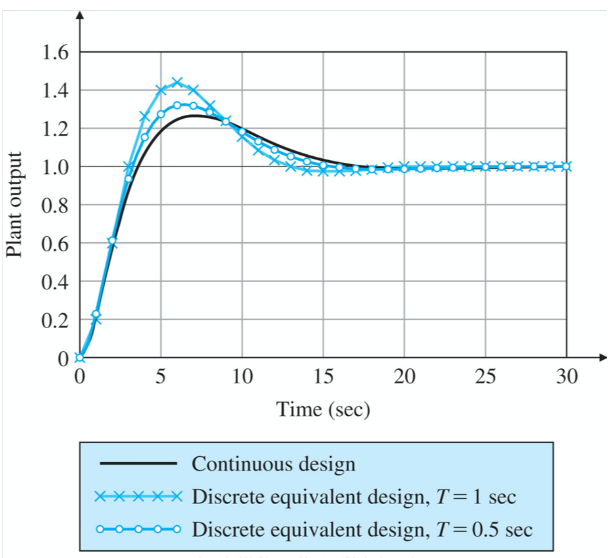
where the final value theorem gives

$$0.81 \frac{0.2}{2} = K_d \frac{1 - 0.818}{1 - 0.135} \qquad \rightarrow \qquad K_d = 0.385$$

The difference equation becomes

$$u(k) = 0.135 u(k-1) + 0.385[e(k) - 0.818 e(k-1)]$$

For the step responses,



- (8.3.4) Modified Matched Pole-Zero (MMPZ) Method

1. Modify Step 2 in the MPZ so that the numerator is of lower order than denominator by 1. For example, if

$$D_c(s) = K_c \frac{s+a}{s(s+b)}$$

we skip Step 2 to get

$$D_d(z) = K_d \frac{z^{-1}(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})} \qquad \text{where} \quad K_d = K_c \frac{a}{b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)$$

We can see the difference equation as follow:

$$u(k) = (1+e^{-bT})u(k-1) - e^{-bT}u(k-2) + K_d[e(k-1) - e^{-aT}e(k-2)]$$

where it makes use of $e(k-1)$ that are one cycle old, not $e(k)$.

- (8.3.5) Comparison of Digital Approximation Methods

1. Let us compare four approximation methods with the sampling rate

$$D_c(s) = \frac{5}{s+5}$$

2. Tustin's method

$$\begin{aligned} D_d(z) &= \frac{5}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+5} = \frac{5T(1+z^{-1})}{2(1-z^{-1})+5T(1+z^{-1})} = \frac{5T+5Tz^{-1}}{(2+5T)-(2-5T)z^{-1}} \\ &= \left(\frac{5T}{2+5T}\right)\frac{1+z^{-1}}{1-\left(\frac{2-5T}{2+5T}\right)z^{-1}} \end{aligned}$$

3. ZOH

$$\begin{aligned} D_d(z) &= (1-z^{-1})\mathcal{Z}\left(\frac{D_c(s)}{s}\right) = (1-z^{-1})\mathcal{Z}\left(\frac{5}{s(s+5)}\right) = (1-z^{-1})\frac{(1-e^{-5T})z^{-1}}{(1-z^{-1})(1-e^{-5T}z^{-1})} \\ &= (1-e^{-5T})\frac{z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

4. MPZ

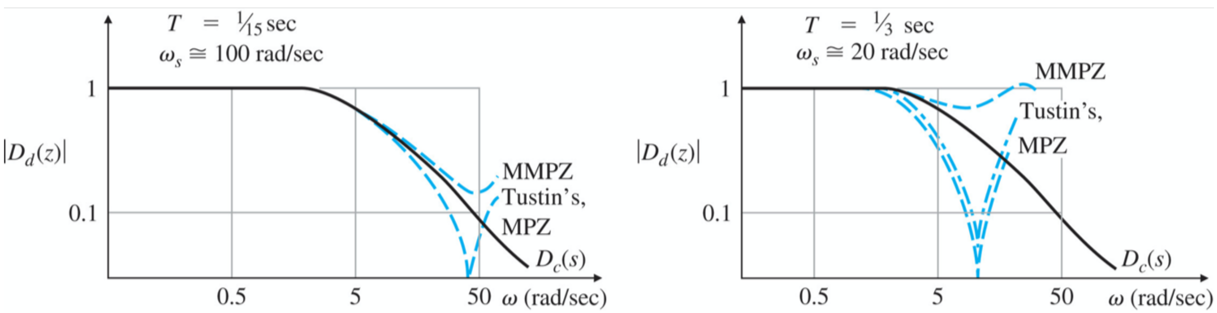
$$\begin{aligned} D_d(z) &= K_d\frac{(1+z^{-1})}{1-e^{-5T}z^{-1}} \quad \text{where} \quad K_d\frac{2}{1-e^{-5T}} = 1 \\ &= \left(\frac{1-e^{-5T}}{2}\right)\frac{1+z^{-1}}{1-e^{-5T}z^{-1}} \end{aligned}$$

5. MMPZ

$$D_d(z) = K_d \frac{z^{-1}}{1 - e^{-5T} z^{-1}} \quad \text{where} \quad K_d \frac{1}{1 - e^{-5T}} = 1$$
$$= (1 - e^{-5T}) \frac{z^{-1}}{1 - e^{-5T} z^{-1}}$$

6. It is noted that Tustin and MPZ bring the similar structures each other, while ZOH and MMPZ show the similar structures, as shown in Table 8.2

7. Tustin and MPZ methods show a notch at $\omega_s/2$ because of their zero at $z = -1$ from $1 + z^{-1}$ term.



- (8.3.6) Applicability Limits of the Discrete Equivalent Design Method

  1. The system can often be *unstable* for rates slower than approximately $5\omega_{bd}$, and
  2. the damping would be *degraded* significantly for rates slower than about $10\omega_{bd}$
  3. At sample rates $\geq 20\omega_{bd}$, design by discrete equivalent yields *reasonable* results, and
  4. at sample rates of 25 times the bandwidth or higher, discrete equivalents can be used *with confidence*.
  5. ZOH brings $T/2$ delay in the control system. A method to account for the $T/2$ delay is to include an approximation of the delay into the original plant model:

$$G_{ZOH}(s) = \frac{2/T}{s + 2/T}$$

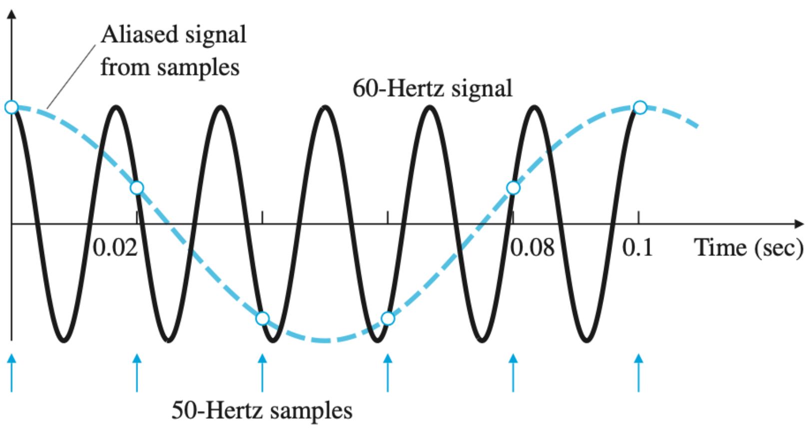
4 Hardware Characteristics

- Analog-to-Digital (A/D) Converters are devices that convert a voltage level from a sensor to a digital word usable by the computer
  - Counting scheme
    - The input voltage may be converted to a train or pulses whose frequency is proportional to the voltage level.
    - The pulses are then counted over a fixed period using a binary counter, thus resulting in binary representation of the voltage level.
    - Counter-based converter might require as many as 2^n cycles.
  - Successive-approximation technique
    - It is based on successively comparing the input voltage to reference levels representing the various bits in the digital word.
    - One clock cycle is required to set each bit, so an n-bit converter would require n cycles.
- If more than one channel of data needs to be sampled and converted to digital words, it is usually accomplished using a multiplexer rather than by multiple A/D converters.
- The multiplexer sequentially connects the converter into the channel being sampled.

- Digital-to-Analog (D/A) Converters are used to convert the digital words from the computer to a voltage level and are sometimes referred to as *sample and hold* devices.
  - Because no counting or iteration is required for such conveters, they tend to be much faster than A/D converters.
  - A/D converters that use the successive approximation method of converion include D/A converters as components.
  - The price of D/A converteris is comparable to A/D converters, but usually somewhat lower.
- Computer is the device where the compensation $D_d(z)$ is programmed and the calculations are carried out.

- Analog Anti-Alias Prefilters are often placed between the analog sensor and the A/D converter.
  - An example of aliasing is shown Fig. 8.16, where 60Hz oscillatory signal is being sampled at 50Hz. The figure shows the result from the samples as a 10Hz signal and also shows the mechanism by which the frequency of the signal is aliased from 60 to 10Hz.

Figure 8.16
An example of aliasing



  - Its function is to reduce the higher frequency noise components in the analog signal in order to prevent *aliasing*.
  - Aliasing will occur any time the sample rate is not at least twice as fast as any of the frequencies in the signal being sampled.
  - To prevent aliasing of a 60Hz signal, the sample rate would have to be faster than 120Hz.

- Aliasing can be explained from the *sampling theorem* of Nyquist and Shannon. For the signal to be reconstructed from the samples, it must have no frequency component greater than half the sample rate (*Nyquist rate* of $\omega_s/2$).
- In a continuous system, noise components with a frequency much higher than the control-system bandwidth normally have a small effect because the system will not respond at the high frequency.
- However, in a *digital system*, the frequency of the noise can be *aliased down* to the vincinity of the system bandwidth so the closed-loop system would respond to the noise.
- The solution to prevent aliasing is to place an analog prefilter before the sampler. In many cases, a simple first-order low-pass filter will do - that is -

  $$H_p(s) = \frac{a}{s+a}$$

  where the *breakpoint* a is selected to be lower than Nyquist rate $\omega_s/2$ so that any noise present with frequencies greater than Nyquist rate is attenuated by the prefilter.
- If ω_s is chosen to be $25 \times \omega_{bd}$, the anti-aliasing filter breakpoint a should be selected lower than $\omega_s/2$, so that

  $$a = 10 \times \omega_{bd} \qquad \leftarrow \qquad \omega_s = 25 \times \omega_{bd}$$

  would be a reasonable choice.

5 Sample-Rate Selection

- The inherent approximation for the discrete TF may give rise to *decreased performance* or even *system instability* as the sample rate is lowered. This can lead the designer to conclude that a faster sample rate is required.

- The *sampling theorem* states that in order to reconstruct an unknown, band-limited, continuous signal from samples of that signal, we must *sample at least twice as fast as the highest frequency contained in the signal*. $\omega_s = 2\omega_{bd}$

- In the z-plane, the highest frequency that can be represented by a discrete system is $\omega_s/2$.

- For a very high frequency noise, it would be foolish to sample fast enough to attenuate the disturbance without the use of a prefilter.

6 Discrete Design

- This plant model can be used as part of a discrete model of the feedback system including the compensation $D_d(z)$.
- Analysis and design using this discrete model is called *discrete design* or alternatively, *direct digital design*.
- For a plant described by $G(s)$ and preceeded by a ZOH, the discrete TF was essentially given by

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

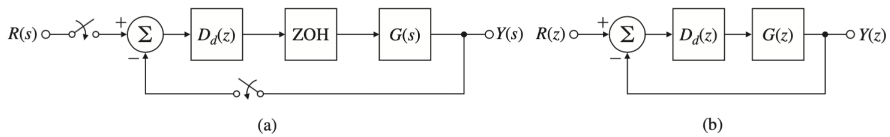


Figure 8.17
Comparison of: (a) a mixed system; and (b) its pure discrete equivalent

- The closed-loop poles or the roots of the discrete characteristic equation

$$1 + D_d(z)G(z) = 0$$

- The root-locus techniques used in continuous systems to find roots of a polynomial in s apply equally well and without modification to the polynoimal in z.
- The interpretation of the results is that the stability boundary is now the unit circle instead of the imaginary axis.

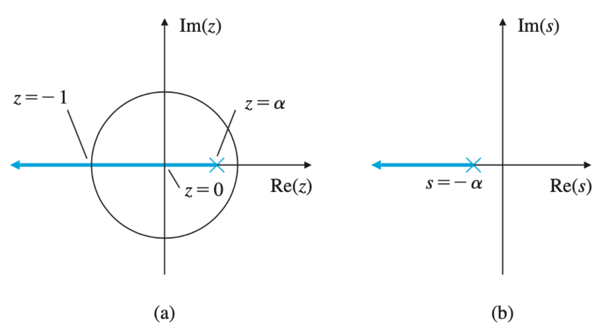
(Example 8.4) When $G(s) = \frac{a}{s+a}$ and $D_d(z) = K$, draw the root locus with respect to K?
(Answer)

$$
\begin{aligned}
G(z) &= (1 - z^{-1})\mathcal{Z}\left\{\frac{a}{s(s+a)}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s} - \frac{1}{s+a}\right\} \\
&= (1 - z^{-1})\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-aT}z^{-1}}\right) \\
&= \frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} \\
&= \frac{(1 - \alpha)z^{-1}}{1 - \alpha z^{-1}} \qquad \text{where} \quad \alpha = e^{-aT}
\end{aligned}
$$

The discrete characteristic equation becomes

$$
1 + D_d(z)G(z) = 1 + K\frac{(1 - \alpha)z^{-1}}{1 - \alpha z^{-1}} = 0
$$

Figure 8.18
Root loci for: (a) the *z*-plane; and (b) the *s*-plane



In the continuous case, the system remains stable for all values of K. In the discrete case, the system becomes oscillatory with decreasing damping ratio as z goes from 0 to -1 and eventually becomes unstable. This instability is due to the lagging effect of the ZOH.

Feedback properties

- Proportional

$$u(k) = Ke(k) \qquad \leftrightarrow \qquad D_d(z) = K$$

- Derivative

$$u(k) = KT_D[e(k) - e(k-1)] \qquad \leftrightarrow \qquad D_d(z) = KT_D(1 - z^{-1})$$

- Integral

$$u(k) = u(k-1) + \frac{K}{T_I}e(k) \qquad \leftrightarrow \qquad D_d(z) = \frac{K}{T_I}\left(\frac{1}{1 - z^{-1}}\right)$$

- Lead

$$u(k) = \beta u(k-1) + K[e(k) - \alpha e(k-1)] \qquad \leftrightarrow \qquad D_d(z) = K\frac{1 - \alpha z^{-1}}{1 - \beta z^{-1}}$$

(Example 8.5) Design a digital controller to have a closed-loop natural frequency $\omega_n = 0.3$ and a damping ratio $\zeta = 0.7$ using discrete design
(Answer)

$$G(s) = \frac{1}{s^2} \qquad \rightarrow \qquad G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^3}\right\} = \frac{T^2}{2}\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}$$

which, with $T = 1$, becomes

$$G(z) = \frac{1}{2}\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}$$

Let us assume that the PD compensator is used

$$D_d(z) = K(1 - \alpha z^{-1})$$

The desired pole locations of $\omega_n = 0.3$ and $\zeta = 0.7$ become $z = 0.78 \pm 0.18j$

$$1 + D_d(z)G(z) = 1 + K\frac{1}{2}\frac{z^{-1}(1 + z^{-1})(1 - \alpha z^{-1})}{(1 - z^{-1})^2} = 0$$

Now we have

$$\alpha = 0.85 \qquad\qquad K = 0.374$$

and

$$D_d(z) = 0.374(1 - 0.85z^{-1})$$

The difference equation becomes

$$u(k) = 0.374[e(k) - 0.85e(k-1)]$$

(8장 숙제) 8장 연습문제에서 기말고사에 출제될 만한 문제 3개를 선택하여 풀어 제출하라? (마감 기말고사 전)