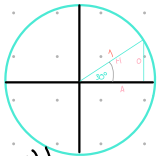
d) By hand and in Julia, compute d, d*d, d2*d. Using either Julia or by hand, plot each of these points on the real-imaginary plane, like in a). What do you notice about the effect of multiplying by d? (Hint: if you don't quite see the pattern by d2*d,keep multiplying by dand see what you get!)

e) By hand, rewrite d in the form rei, where r is a real scalar. By hand compute d*d, d2*d analytically, i.e., without solving for actual numbers on each step. How does your answer here relate to the previous answer? (Hint: in Julia, is atan()).

$0.966 + i\,0.259$

(d) $d = \cos(30°) + i\sin(30°)$



$\cos = \frac{A}{H}$ $\cos(30°) = A$

$$d \times d = (\cos(30°) + i\sin(30°)) \times (\cos(30°) + i\sin(30°))$$
$$= \cos^2(30°) + 2i\sin(30°)\cos(30°) - \sin^2(30°) = \cos(60°) + i\sin(60°)$$

$\cos(2\theta) = \cos^2(\theta) + \sin^2(\theta)$
$\sin(2\theta) = 2\sin\theta\cos\theta$

DOUBLE ANGLE

$$d^2 \times d = (\cos(30°) + i\sin(30°)) \times (\cos^2(30°) + 2i\sin(30°)\cos(30°) - \sin^2(30°))$$
$$= \cos^3(30°) + 3i\sin(30°)\cos^2(30°) - 3\sin^2(30°)\cos(30°) - i\sin^3(30°)$$
$$= \cos(90°) + i\sin(90°)$$

$\cos(3\theta) = \cos^3(\theta) - 3\sin^2\theta\cos\theta$
$\sin(3\theta) = -\sin^3\theta + 3\cos^2\theta\sin\theta$

TRIPPLE ANGLE

With every power, we are simply increasing the angle by a factor of itself => angle is trippled

(e) $d = \cos(30°) + i\sin(30°)$ in re^{i}
$\hookrightarrow 30° = \frac{\pi}{12}$ r

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{i\frac{\pi}{12}} = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$$
$$i\sin(x) = i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right.$$
$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} \ldots$$
$$e^{ix} = \cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} \ldots$$

$$d \times d = e^{ix} \times e^{ix} = e^{i2x}$$ where $x = \frac{\pi}{12}$
$$d^2 \times d = e^{i2x} \times e^{ix} = e^{i3x}$$

$$d^3 = e^{i\frac{\pi}{4}}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right) \qquad re^{i\phi} = a + ib$$

$$re^{i\frac{\pi}{4}} = d^3$$
$$= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}\right)$$

TRUE

$$\frac{\pi}{4} = \tan^{-1}(d^3)$$

We see the same trend here => angle is trippled!

b) By hand, use the diagonalized representation of M (i.e. $M = V\Lambda V^{-1}$, where Λ is a diagonal matrix of eigenvalues and V is a matrix whose columns are the corresponding eigenvectors) to compute $M^{1/2}$. Report the values of $\Lambda^{1/2}$. You can use Julia to compute V^{-1}. (Note: $1/\sqrt{2} \approx 0.7071$ and `1e-16` is effectively equal to 0 when using Julia (this is the residual result of rounding on a computer). Hint: you can use `Julia` to find the values of $\Lambda^{1/2}$ but you must show by hand that this is true).

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix}$$

$M^{1/2} = X$ such that $X * X = M$

$$V^{-1} = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

$$M = V \lambda V^{-1}$$

$$X = V \lambda^{1/2} V^{-1}$$

$$X * X = V \lambda^{1/2} V^{-1} V \lambda^{1/2} V^{-1}$$

$$M = V \lambda^{1/2} \lambda^{1/2} V^{-1}$$

$$M = V \lambda V^{-1}$$

$$\lambda^{1/2} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}^{1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1-i) \end{bmatrix}$$

$$V \lambda^{1/2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1-i) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1+i) & \frac{1}{2}(-1-i) \end{bmatrix} \Rightarrow \times V^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = M^{1/2}$$