Elin_Ahlstrand_Exercises_8

Problem 3: 1D nonlinear dynamics

Suppose we have the following differential equation:

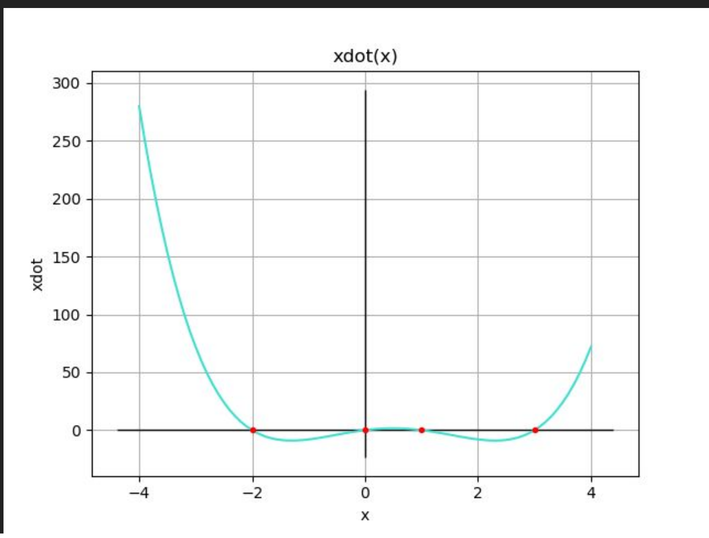
$$\dot{x} = x^4 - 2x^3 - 5x^2 + 6x$$

a) Plot in `Julia` the above differential equation with the x-axis being x and the y-axis being $\dot{x}$. In the plot, you can limit the range of x values to be between -4 and 4.

```
function dxdt(x)
      return (x.^4) .- (2*x.^3) .- (5*x.^2) .+ 6*x
end

figure(1); clf()

      x=-4:0.1:4
      plot(x, dxdt(x), color="turquoise")
      hlines(0, xlim()[1], xlim()[2], color="black", linewidth=1)
      vlines(0, ylim()[1], ylim()[2], color="black", linewidth=1)
      grid("on")
      xlabel("x"); ylabel("xdot"); title("xdot(x)")
      #
      dxdt_0 = find_zeros(dxdt, -4, 4)
      plot(dxdt_0, dxdt(dxdt_0), ".", color="red")
```

b) Using the plot in **a)**, identify the fixed points of the dynamics described by the differential equation above. How many fixed points do you find?

4 fixed points (red)



c) Note that when $x = 1$, $\dot{x} = 0$. Therefore, $x = 1$ is a fixed point (you can also verify this from the plot above). Find the first-order Taylor approximation of the above function around $x = 1$ *(Hint: Problem 3 in Exercises 7)* and plot the approximation on top of the plot you created in **a)**.

EX8

3

$$\dot{x}(x) = x^4 - 2x^3 - 5x^2 + 6x$$

1st App. $\dot{x}(x) = \dot{x}(x_0) + \frac{d\dot{x}}{dx}\Big|_{x_0} (x - x_0)$

@ $x_0 = 1 \Rightarrow \dot{x}(x_0) = 0$ (is a FP)

$x_0 = 1 \Rightarrow \dot{x}(x) \simeq \dot{x}(1) + \frac{d\dot{x}}{dx}\Big|_1 (x-1)$

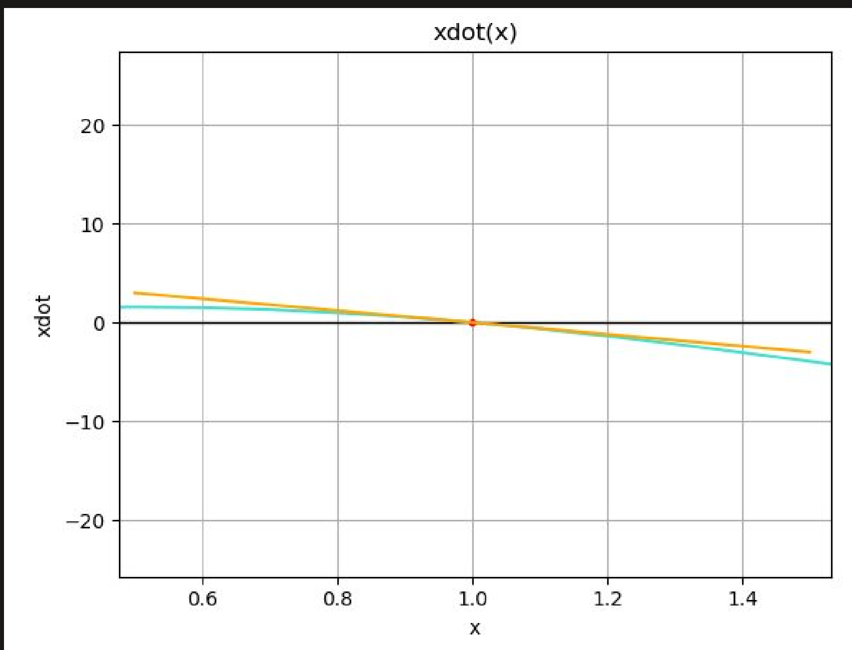
$= 0 + (-6)(x-1) = -6x + 6$

$\frac{d\dot{x}}{dx} = 4x^3 - 6x^2 - 10x + 6$

@ $x=1$

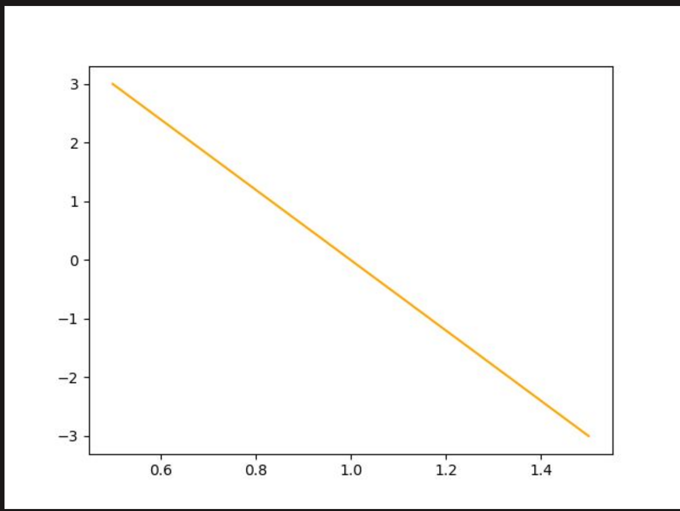
$4 - 6 - 10 + 6 = -6$

```
function x_dot(x)
      return -6 .* x .+ 6
end
      x=.5:0.1:1.5
      plot(x, x_dot(x), color="orange")
#
```

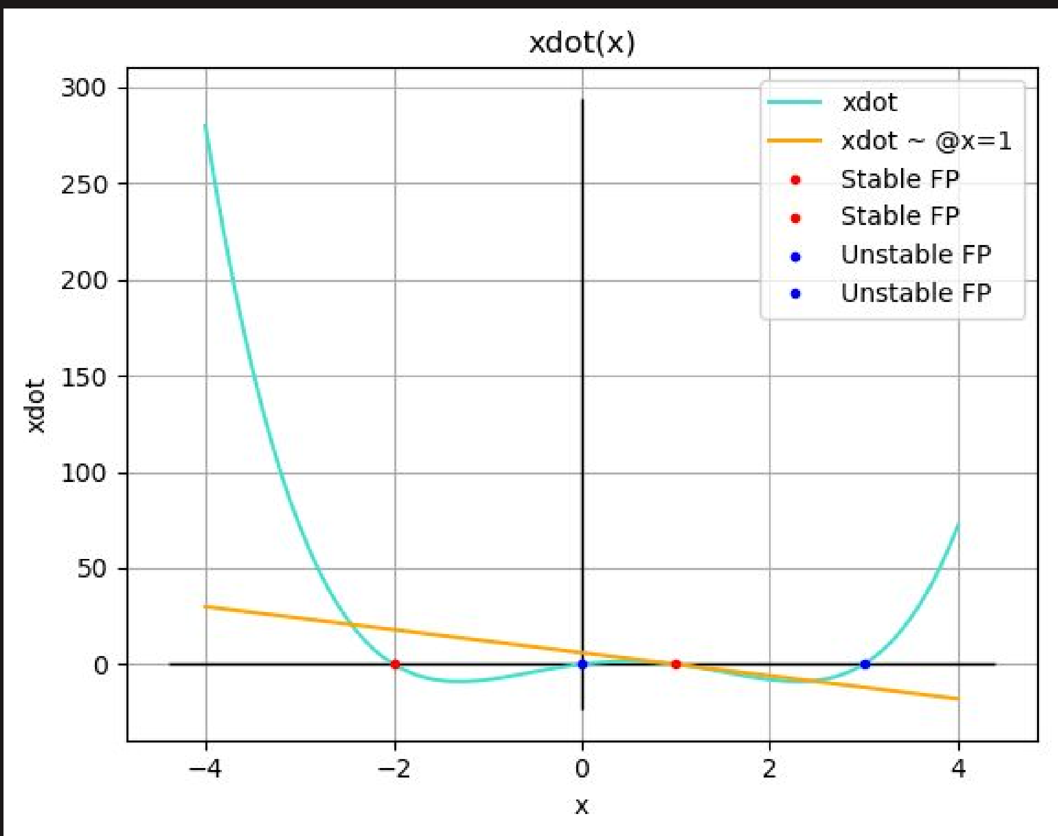


d) What is the slope of the approximation line you plotted in **c)**? Based on the slope of the line, can you tell whether $x = 1$ is a stable or an unstable fixed point?

```
Stable fixed point because the gradient of the slope < 0
```



e) In the plot you created in **c)**, mark the stable fixed points as red dots and unstable fixed points as blue dots. Make sure to label the axes and add a legend.



Problem 4: 2D linear dynamics - feedback loop of excitatory and inhibitory neurons
The firing rates of excitatory neuron (r_E) and inhibitory neuron (r_I) are following the 2D dynamics shown below:

$\dot{r_E} = 0.5 r_E - r_I$

$\dot{r_I} = r_E - 0.5 r_I$.

As you can see from the equations above, the excitatory neuron activates both the inhibitory neuron and the excitatory neuron itself. In contrast, the inhibitory neuron suppresses both neurons.

a) Using `Julia`'s function `eigen()`, derive the eigenvalues and eigenvectors of the matrix M where

$$M = \begin{bmatrix} 0.5 & -1 \\ 1 & -0.5 \end{bmatrix}$$

By looking at the eigenvalues, predict the behavior of this dynamical system. Does it oscillate?
* Note that

$$\begin{bmatrix} \dot{r_E} \\ \dot{r_I} \end{bmatrix} = M \begin{bmatrix} r_E \\ r_I \end{bmatrix}$$

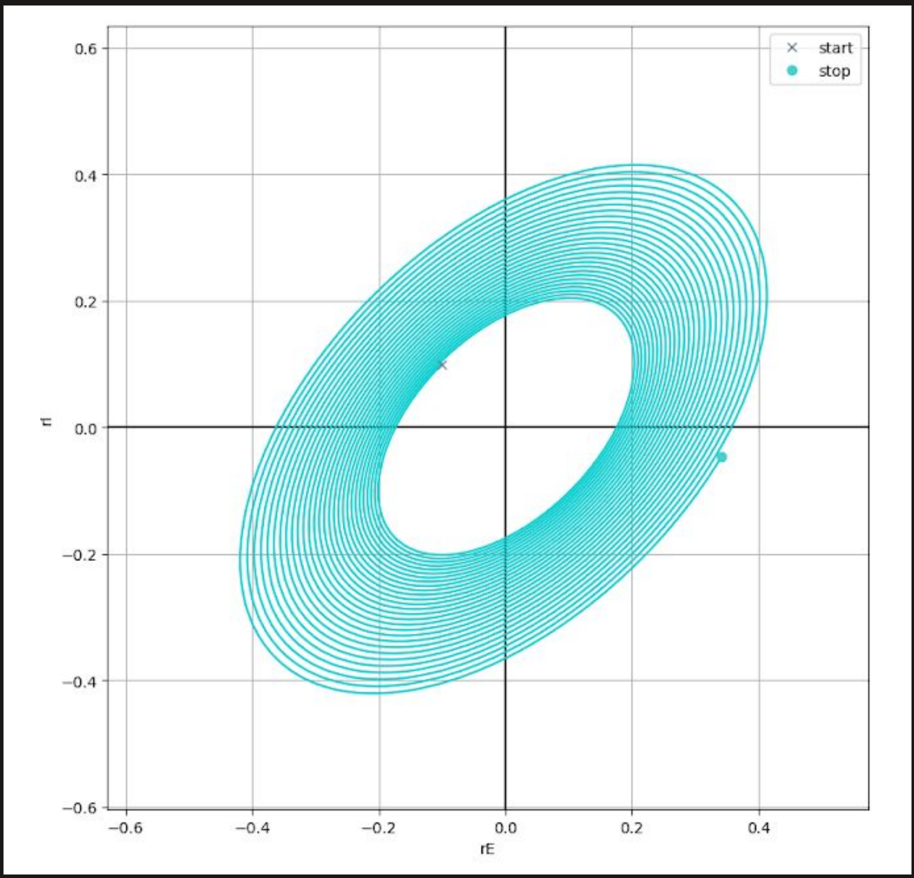
```
Eigenvalues are a complex-conjugate, suggesting a spiral point plot. The
real part of the eigenvalues is negative, suggesting that the system has
a stable node. Since the eigenvalues have an imaginary component, the
system will oscillate around this node.

M = [0.5 -1; 1 -.5]

M_eig = eigen(M)

Eigen{Complex{Float64},Complex{Float64},Array{Complex{Float64},2},Array{
Complex{Float64},1}}
eigenvalues:
2-element Array{Complex{Float64},1}:
 -5.551115123125783e-17 + 0.8660254037844386im
 -5.551115123125783e-17 - 0.8660254037844386im
eigenvectors:
2×2 Array{Complex{Float64},2}:
 0.707107+0.0im          0.707107-0.0im
 0.353553-0.612372im  0.353553+0.612372im
```

c) We can tell that $\begin{bmatrix} r_E \\ r_I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point because $\begin{bmatrix} \dot{r_E} \\ \dot{r_I} \end{bmatrix} = M \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Is it stable or unstable? You can answer this question by either (a) describing what it is that you see in the eigenvalues of *M* that leads you to your conclusion; or (b) using Euler integration to plot, and include in your answer, plots of dynamical trajectories in this system. If you want to, you can adapt the code used in lecture and posted on the calendar.

-5.551115123125783e-17 (real part of M eigenvalues) ~ 0
→ the eigenvalues are essentially the same as the stable point (0,0)

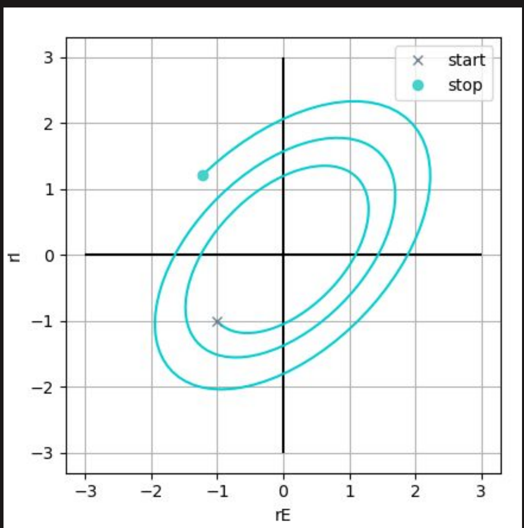
Since the real component is technically = 0, we have a system that is approximately a center-equilibrium. This system will have *almost* concentric periodic orbits around the central node.

When dt is sufficiently small, we see that the system shows what are almost concentric orbits.

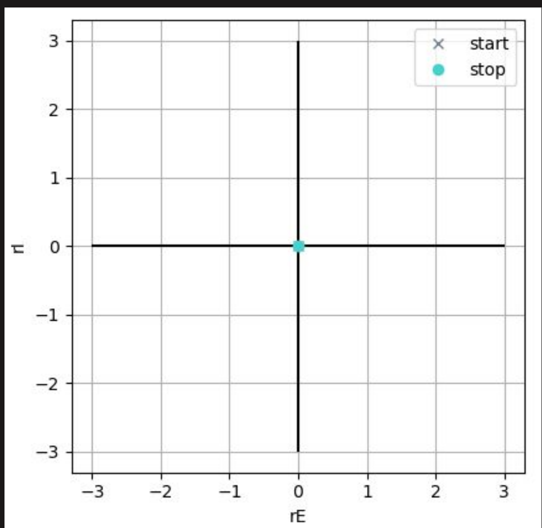


```
dt = 0.01; t = 0:dt:200
x = zeros(2,length(t))
x[:,1] = [-.1,.1]

for i=1:length(t)-1
      x[:,i+1] = x[:,i] + dt*M*x[:,i]  # Euler recipe
end

figure(1); clf(); axlim = 3
      xlim([-axlim, axlim]); ylim([-axlim, axlim])
      hlines(0, -axlim, axlim, color="black")
      vlines(0, -axlim, axlim, color="black")
      axis("scaled"); grid("on")

      plot(x[1,:], x[2,:], color="darkturquoise")
      # plot(x[1,:], x[2,:], "d")
      plot(x[1,1], x[2,1], "x", color="slategrey", label="start")
      plot(x[1,end], x[2,end], "o", color="mediumturquoise",
label="stop")
      xlabel("rE"); ylabel("rI")
      legend()
```

Start = [-1,-1]

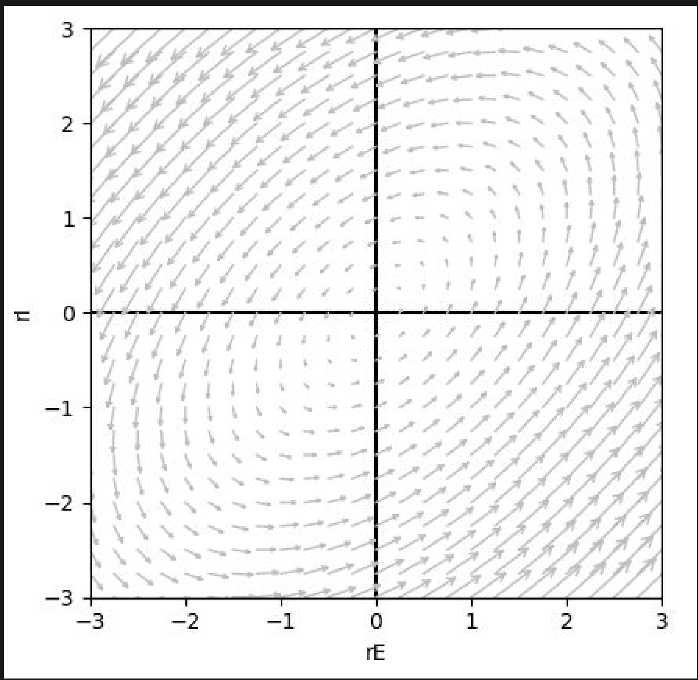


Start = [0,0]



The system looks to be unstable as the spiral is moving away from the fixed point [0,0], which is unexpected as the real component of the eigenvalues of M are negative (even though they are ~ 0). The fact that the system oscillates is congruent with the existence of imaginary parts of the eigenvalues.

Quiver Plot



Oscillates almost stably around a central node.