Problem 1: Eigenvectors of 3x3 diagonal and non-diagonal matrix (work by hand)
Use pencil and paper to complete this. Submit a picture (or pictures) of your work.

1) 3x3 diagonal matrix
Show that the following vectors

$$\hat{\imath} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\jmath} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the eigenvectors of the following 3x3 diagonal matrix D:

$$\mathbf{D} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

That is, for each of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, show that the equation $\mathbf{D}\hat{\imath} = \lambda\hat{\imath}$ is true when λ is a scalar. Also, report the eigenvalue corresponding to each of these three eigenvectors.

$$D = \begin{array}{c} \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \end{array} \\ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \end{array} \qquad D\hat{\imath} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

$$|D - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix} = 0$$

$$(c-\lambda)(ab - a\lambda - b\lambda + \lambda^2) = 0$$

$$(\lambda - a)(\lambda - b)(\lambda - c) = 0$$

$$\lambda_1 = a \Rightarrow \hat{\imath}$$

$$\lambda_2 = b \Rightarrow \hat{\jmath}$$

$$\lambda_3 = c \Rightarrow \hat{k}$$

$$D\hat{\imath} = \lambda\hat{\imath}$$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$a \times 1 + 0 + 0 = a \times 1$$
$$a = a$$

$$0 \times 1 + b \times 0 + 0^2 = 0$$
$$0 = 0$$

$$0 \times 1 + 0^2 + c \times 0 = 0$$
$$0 = 0$$

QED

$$\therefore D\hat{\imath} = \lambda_1 \hat{\imath}$$

② $v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ $M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

$$|M - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

$$Mv = \lambda v$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$e_1 + e_2 + 2e_3 = \lambda e_1$

$e_1 + e_2 + 2e_3 = 3e_1$

$e_1 + e_2 + 4e_1 = 3e_1$

$e_2 + 2e_1 = 0$

$e_2 = -2e_1$

$2e_2 - e_3 = 3e_2$

$-e_3 = e_2$

$-e_3 = 2e_1 - 2e_3$

$e_3 = 2e_1$

$3e_3 = \lambda e_3$

$\lambda = 3$

in unit vector $\Rightarrow$ $v = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ for $\lambda_3 = 3$

Problem 2: Calculate Eigenvectors and Eigenvalues (work by hand)

For each matrix below, 1) write down the characteristic equation, 2) find the eigenvalues by solving the characteristic equation. Then find the eigenvector that corresponds to each eigenvalue by using $\mathbf{Mv} = \lambda\mathbf{v}$. Use pencil and paper to complete this. Submit a picture (or pictures) of your work.

a) $A = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$

$|A - \lambda I| = 0$

b) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

a

1. $\left| \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| \Rightarrow \left| \begin{bmatrix} 7-\lambda & 8 \\ 0 & 9-\lambda \end{bmatrix} \right| = 0$

2. $(7-\lambda)(9-\lambda) = 0$

$\lambda_1 = 7$ $\quad$ $\lambda_2 = 9$

$Av = \lambda v$

$\begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \lambda e_1 \\ \lambda e_2 \end{bmatrix}$ $\quad$ $7e_1 + 8e_2 = 9e$

$e_1 = 4e_2$

$7e_1 + 8e_2 = 7e_1$

$8e_2 = 0$

$e_2 = 0$

$0e_1 + 9e_2 = 7e_2$

$7e_1 = 7e_1$

$e_1 = 1$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = eigenvec. for $\lambda_1 = 7$

$7e_1 + 8e_2 = 9e_1$

$e_1 = 4e_2$

$e_1 = 4$

$9e_2 = 9e_2$

$e_2 = 1$

so $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ = eigenvec. for $\lambda_2 = 9$

(b) 1. $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = A$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - 2 = 0$$
$$\lambda^2 - 3\lambda = 0$$
$$\lambda(\lambda - 3) = 0$$

$\lambda_1 = 0$ $\quad$ $\lambda_2 = 3$

$$Av = \lambda v$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \lambda e_1 \\ \lambda e_2 \end{bmatrix}$$

$e_1 + e_2 = 0 \qquad 2e_1 + 2e_2 = 0$

$e_1 = -e_2 \qquad -2e_2 + 2e_2 = 0$

$e_2 = 1$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 0$

$e_1 + e_2 = 3e_1 \qquad 2e_1 + 2e_2 = 3e_2$

$e_2 = 2e_1 \qquad 2e_1 + 4e_1 = 6e_1$

$e_1 = 1$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda_2 = 3$

Problem 4: Computing $M^{\frac{1}{2}}$ (work by hand)

Use pencil and paper to complete this. Submit a picture (or pictures) of your work.

1) Suppose that a 2x2 matrix D is defined as below:

$$\mathbf{D} = \begin{bmatrix} a^2 & 0 \\ 0 & c^2 \end{bmatrix}$$

Compute $\mathbf{D}^{\frac{1}{2}}$. That is, find a matrix X that will satisfy $\mathbf{X}^2 = \mathbf{D}$.

2) Suppose that a 2x2 matrix M is defined as below:

$M = \mathbf{V}\Lambda\mathbf{V}^{-1}$

where

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

a) Compute $\mathbf{V}^{-1}$.

b) Compute M.

1

$$D = X^2 = X \cdot X$$

$$\begin{bmatrix} a^2 & 0 \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \cdot \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

$x \times x = a^2$ $\quad x = 7$

$y \times y = c^2$ $\quad y = 7c$

$$\therefore X = \underline{\underline{\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}}}$$

2

a

$$V^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3\times 1 - 2\times 1} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}}}$$

$$\lambda V^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 4 & -4 \\ -18 & 27 \end{bmatrix}}}$$

b

$$V\lambda V^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -4 \\ -18 & 27 \end{bmatrix} = \begin{bmatrix} (12-18) & (-12+27) \\ (8-18) & (-8+27) \end{bmatrix} = \underline{\underline{\begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix}}} = M$$

c

$$V\lambda V^{-1} = M$$

$$V\lambda = MV = \begin{bmatrix} -6 & 15 \\ -10 & 19 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\lambda = V^{-1}MV = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 12 & 9 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\lambda$$

$$\mapsto \quad X = V \begin{bmatrix} \lambda_1^{1/2} & 0 \\ 0 & \lambda_2^{1/2} \end{bmatrix} V^{-1}$$

$$= V \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{9} \end{bmatrix} V^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 \\ -6 & 9 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}}}$$

```
using LinearAlgebra;

v1 = [2;1]; #v1 = v1/norm(v1)
v2 = [0.5;2.5]; #v2 = v2/norm(v2)

V = [v1 v2]

julia> V
2×2 Array{Float64,2}:
 2.0  0.5
 1.0  2.5

x1 = 2*v1
x = [x1 v2]

julia> x
2×2 Array{Float64,2}:
 4.0  0.5
 2.0  2.5

x_new = inv(V)*x

julia> x_new
2×2 Array{Float64,2}:
 2.0  0.0
 0.0  1.0


Lambda = x_new
M = V*Lambda*inv(V)

julia> M
2×2 Array{Float64,2}:
 2.11111   -0.222222
 0.555556   0.888889
```

```
# STRETCH & SQUEEZE
stretch = 1.3
squeeze = 0.8

ss = collect(Diagonal([stretch,squeeze]))
M_ss = M*ss
```

```
M_orig = M_ss*Lambda*inv(M_ss);

julia> M_orig
2×2 Array{Float64,2}:
 1.93827   0.234568
 0.246914  1.06173

eig = eigen(M_orig)

julia> eig
Eigen{Float64,Float64,Array{Float64,2},Array{Float64,1}}
eigenvalues:
2-element Array{Float64,1}:
 2.0
 1.0
eigenvectors:
2×2 Array{Float64,2}:
 0.967075  -0.242536
 0.254493   0.970143

#Same eigenvalues as in (1)

# Lambda = 2 refers to 1st eigenvector set
M_orig*[0.967075; 0.254493]

2-element Array{Float64,1}:
1.934149901234568
 0.5089863950617285

2*[0.967075; 0.254493]

2-element Array{Float64,1}:
 1.93415
 0.508986

# Lambda = 1 refers to 2nd eigenvector set
M_orig*[-0.242536;0.970143]

2-element Array{Float64,1}:
 -0.24253623456790135
  0.9701429382716049

1*[-0.242536;0.970143]

2-element Array{Float64,1}:
```

```
-0.24253623456790135
 0.9701429382716049
```