Full proofs for the postulates of a rational consequence relation for datalog

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Postulate 1 (Reflexivity)

$$\mathcal{K} \mathrel{|\!\!\approx} \beta \rightsquigarrow \beta$$

Reflexivity seems to be satisfied universally by any kind of reasoning that is based on some notion of consequence [1]. Our defeasible entailment check for the given defeasible rule is eventually reduced to a classical entailment check for a strict version of the rule, which will always be reflexive.

Proof:

1. In order for RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \beta$) to terminate, we have to break out of the while loop on line 2 of Algorithm 3, so we will either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

2. (a) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \rightarrow \bot$, we must have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \beta$ by classical inference.
   (b) Line 4 of Algorithm 3 will then return **True** for RationalClosure($\mathcal{K}$, β), meaning $\mathcal{K} \mathrel{|\!\!\approx} \beta \rightsquigarrow \beta$ for this case.

Case 2:

3. (a) Since $i_\beta > n$, we will only be dealing with the classical portion of the datalog program. $\mathcal{SR} \models \beta \rightarrow \beta$ will hold by classical inference.
   (b) Line 4 of Algorithm 3 will then return **True** for RationalClosure($\mathcal{K}$, β), meaning $\mathcal{K} \mathrel{|\!\!\approx} \beta \rightsquigarrow \beta$ for this case.

□

Postulate 2 (Left Logical Equivalence)

$$\frac{\beta = \gamma,\ \mathcal{K} \mathrel{|\!\!\approx} \beta \rightsquigarrow \eta}{\mathcal{K} \mathrel{|\!\!\approx} \gamma \rightsquigarrow \eta}$$

Left Logical Equivalence expresses the requirement that logically equivalent formulas have exactly the same consequences [1]. Since $\beta \equiv \gamma$, Algorithm 3 will consider the same portion of the knowledge base for both β and γ where they are not exceptional. Since $\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \eta$, we know that $\beta \rightarrow \eta$ holds for this portion of the knowledge base. Since $\beta \equiv \gamma$ and we are considering the same portion of the knowledge base, $\gamma \rightarrow \eta$ will hold. Algorithm 3 will, therefore, return **True** when checking $\mathcal{K} \approx\!\!\!\!| \; \gamma \rightsquigarrow \eta$.

Proof:

1. Given $\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \eta$, RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) returns **True**.
2. In order for RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) to terminate, we have to break out of the while loop on line 2 of Algorithm 3, so we either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

3. (a) Since $\beta \equiv \gamma$, RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) must give $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \gamma \rightarrow \bot$ where $i_\gamma = i_\beta$ for which $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \rightarrow \bot$ in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (b) Given $\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \eta$, $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (c) $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \equiv \bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR}$ and $\beta \equiv \gamma$, therefore $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \rightarrow \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \approx\!\!\!\!| \; \gamma \rightsquigarrow \eta$ for this case.

Case 2:

4. (a) If $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \equiv \emptyset$.
   (b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \rightarrow \eta$ must hold on line 4 of Algorithm 3.
   (c) Since $\beta \equiv \gamma$, RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) must not have any number i_γ for which $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \gamma \rightarrow \bot$ and $i_\gamma \leq n$, therefore $i_\gamma > n$.
   (d) If $i_\gamma > n$, then $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} = \emptyset$.
   (e) If $\mathcal{SR} \models \beta \rightarrow \eta$ and $\beta \equiv \gamma$, then $\mathcal{SR} \models \gamma \rightarrow \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \approx\!\!\!\!| \; \gamma \rightsquigarrow \eta$ for this case.

□

Postulate 3 (Right Weakening)

$$\frac{\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \eta, \ \models \eta \rightarrow \gamma}{\mathcal{K} \approx\!\!\!\!| \; \beta \rightsquigarrow \gamma}$$

Right Weakening implies that we may replace logically equivalent formulas in the head of the rule [1]. The portion of the knowledge base that Algorithm 3 considers for both $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$ and $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$ is the same due to exceptionality being determined by the body of a rule and these two rules have the same body. We know that $\beta \rightarrow \eta$ holds for this portion of the knowledge base and we know that $\models \eta \rightarrow \gamma$. Due to transitivity of strict (classical) implication, we know that $\beta \rightarrow \gamma$ will also hold for this portion of the knowledge base, so Algorithm 3 will return **True** when checking $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$.

Proof:

1. Given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) returns **True**.
2. In order for RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) to terminate, we have to break out of the while loop on line 2 of Algorithm 3, so we either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

3. (a) Given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (b) With $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ and given $\models \eta \rightarrow \gamma$, we will get $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \gamma$ due to the transitivity of strict datalog implication.
   (c) Thus, $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \gamma$ will hold on line 4 of algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and cause RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) to return **True**, meaning $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$ for this case.

Case 2:

4. (a) If $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \equiv \emptyset$.
   (b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \equiv \emptyset$ and given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \rightarrow \eta$.
   (c) With $\mathcal{SR} \models \beta \rightarrow \eta$ and given $\models \eta \rightarrow \gamma$, we will get $\mathcal{SR} \models \beta \rightarrow \gamma$ due to the transitivity of strict datalog implication.
   (d) Thus, $\mathcal{SR} \models \beta \rightarrow \gamma$ will hold on line 4 of algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and cause RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) to return **True**, meaning $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$ for this case.

□

Postulate 4 (And)

$$\frac{\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma,\ \mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta}{\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma \wedge \eta}$$

And expresses the fact that the conjunction of two plausible consequences is also a plausible consequence [1]. The portion of the knowledge base that Algorithm 3 considers for $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$, $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, and $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma \wedge \eta$ is the same due to exceptionality being determined by the body of a rule and all of these rules having the same body. For this portion of the knowledge base, we know that both $\beta \to \gamma$ and $\beta \to \eta$. Due to classical conjunction introduction, $\beta \to \gamma \wedge \eta$ must also hold for this portion of the knowledge base. Algorithm 3 will, therefore, return **True** when checking $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma \wedge \eta$.

Proof:

1. Given $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma$ and $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$, RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) both return **True**.
2. In order for RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) to terminate, we have to break out of the while loop on line 2 of Algorithm 3, so we either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

3. (a) Given $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma$, $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \gamma$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$).
   (b) Given $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$, $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (c) If we have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \gamma$ and $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \gamma \wedge \eta$ must hold due to classical conjunction introduction.
   (d) If we have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \gamma \wedge \eta$ on line 4 of algorithm 3, then RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma \wedge \eta$) will return **True**, meaning $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma \wedge \eta$ for this case.

Case 2:

4. (a) If $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$.
   (b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma$, then $\mathcal{SR} \models \beta \to \gamma$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma$).
   (c) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \to \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (d) If we have $\mathcal{SR} \models \beta \to \gamma$ and $\mathcal{SR} \models \beta \to \eta$, then $\mathcal{SR} \models \beta \to \gamma \wedge \eta$ must hold due to classical conjunction introduction..
   (e) If we have $\mathcal{SR} \models \beta \to \gamma \wedge \eta$ on line 4 of algorithm 3, then RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \gamma \wedge \eta$) will return **True**, meaning $\mathcal{K} \approx\models \beta \rightsquigarrow \gamma \wedge \eta$ for this case.

□

Postulate 5 (Or)

$$\frac{\mathcal{K} \approx\models \beta \rightsquigarrow \eta,\ \mathcal{K} \approx\models \gamma \rightsquigarrow \eta}{\mathcal{K} \approx\models \beta \vee \gamma \rightsquigarrow \eta}$$

Or states that any formula that is, separately, a plausible consequence of two different formulas, should also be a plausible consequence of their disjunction [1]. For $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$, Algorithm 3 considers the portion of the knowledge base where β is not exceptional. η classically follows for this portion of the knowledge base. For $\mathcal{K} \approx\models \gamma \rightsquigarrow \eta$, Algorithm 3 considers the portion of the knowledge base where γ is not exceptional. η classically follows for this portion of the knowledge base. When Algorithm 3 checks if $\mathcal{K} \approx\models \beta \vee \gamma \rightsquigarrow \eta$, it will consider the largest portion of the knowledge base where at least one of β or γ is no longer exceptional. We know that at the point where at least one of β or γ is no longer exceptional, η classically follows for this portion of the knowledge base. So we know that $\beta \vee \gamma \rightarrow \eta$ for this portion of the knowledge base. Algorithm 3 will, therefore, return **True** when checking $\mathcal{K} \approx\models \beta \vee \gamma \rightsquigarrow \eta$.

Proof:

1. Given $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$ and $\mathcal{K} \approx\models \gamma \rightsquigarrow \eta$, RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) both return **True**.
2. In order for both RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) to terminate, both calls to Algorithm 3 need to break out of the while loop on line 2, so we can have 4 different cases. For **Case 1** RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) reaches a point where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) reaches a point where $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \rightarrow \bot$ no longer holds. For **Case 2** RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) reaches a point where $i_\beta \leq n$ is no longer true and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) reaches a point where $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \rightarrow \bot$ no longer holds. For **Case 3** RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) reaches a point where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) reaches a point where $i_\gamma \leq n$ is no longer true. For **Case 4** RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$) reaches a point where $i_\beta \leq n$ is no longer true and RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$) reaches a point where $i_\gamma \leq n$ is no longer true.

Case 1:

3. (a) Given $\mathcal{K} \approx\models \beta \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ holds on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (b) Given $\mathcal{K} \approx\models \gamma \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \rightarrow \eta$ holds on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$).
   (c) We now have 3 subcases. We have **Case 1a** where $i_\beta < i_\gamma$. We have **Case 1b** where $i_\beta > i_\gamma$. We have **Case 1c** where $i_\beta = i_\gamma$.
   Case 1a:
   (d) i. Since $i_\beta < i_\gamma$, RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$) will break out of the while loop on line 2 of Algorithm 3 at a point when $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \rightarrow \bot$ no longer holds.

ii. Since we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$ holds, we then know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta \vee \gamma \rightsquigarrow \eta$ for this case.

Case 1b:

(e) i. Since $i_\beta > i_\gamma$, RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$) will break out of the while loop on line 2 of Algorithm 3 at the point when $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \bot$ no longer holds.

ii. Since we know that $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \to \eta$ holds, we then know that $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta \vee \gamma \rightsquigarrow \eta$ for this case.

Case 1c:

(f) i. Since $i_\beta = i_\gamma$, we will have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \equiv \bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j}$, so RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$) will break out of the while loop on line 2 of Algorithm 3 at a point when both $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \bot$ and $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \bot$ no longer hold.

ii. Since we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$ and $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \to \eta$ both hold, we know that both $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \eta$ and $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta \vee \gamma \rightsquigarrow \eta$ for this case.

Case 2:

4. (a) Since $i_\beta > n$ and $i_\gamma \leq n$, it must be that $i_\gamma < i_\beta$, so RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$) will break out of the while loop on line 2 of Algorithm 3 at the point when $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \bot$ no longer holds.
   (b) Given $\mathcal{K} \mathrel{|\!\approx} \gamma \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \to \eta$ holds on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\gamma \rightsquigarrow \eta$).
   (c) Since $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \gamma \to \eta$, then $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta \vee \gamma \rightsquigarrow \eta$ for this case.

Case 3:

5. (a) Since $i_\gamma > n$ and $i_\beta \leq n$, it must be that $i_\beta < i_\gamma$, so RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$) will break out of the while loop on line 2 of Algorithm 3 at the point when $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \to \bot$ no longer holds.
   (b) Given $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$ holds on line 4 of Algorithm 3 in RationalClosure($\mathcal{K}$, $\beta \rightsquigarrow \eta$).

(c) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \vee \gamma \rightarrow \eta$ must hold on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta \vee \gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta \vee \gamma \rightsquigarrow \eta$ for this case.

Case 4:

6. (a) Since $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$.
   (b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \rightarrow \eta$ holds on line 4 of Algorithm 3.
   (c) Since $i_\gamma > n$, then $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} = \emptyset$.
   (d) If $\bigcup_{j=i_\gamma}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \mathrel{|\!\approx} \gamma \rightsquigarrow \eta$, then $\mathcal{SR} \models \gamma \rightarrow \eta$ holds on line 4 of Algorithm 3.
   (e) Therefore $\mathcal{SR} \models \beta \vee \gamma \rightarrow \eta$ also holds on line 4 of Algorithm 3, thus returning **True** for RationalClosure($\mathcal{K}$, $\beta\vee\gamma \rightsquigarrow \eta$), meaning $\mathcal{K} \mathrel{|\!\approx} \beta\vee\gamma \rightsquigarrow \eta$ for this case.

□

Postulate 6 (Cautious Monotonicity)

$$\frac{\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \gamma,\ \mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \eta}{\mathcal{K} \mathrel{|\!\approx} \beta \wedge \gamma \rightsquigarrow \eta}$$

Cautious Monotonicity expresses that learning a new fact that could have been plausibly concluded should not invalidate previous conclusions [1]. For $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \gamma$ and $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \eta$, Algorithm 3 considers the portion of the knowledge base where β is not exceptional. We know that $\beta \rightarrow \gamma$ and $\beta \rightarrow \eta$ for this portion of the knowledge base. Since $\beta \rightarrow \gamma$, we know that $\beta \wedge \gamma$ will not be exceptional for this same portion of the knowledge base. Algorithm 3 will, therefore, consider this same portion of the knowledge base when checking $\mathcal{K} \mathrel{|\!\approx} \beta \wedge \gamma \rightsquigarrow \eta$. Since $\beta \rightarrow \eta$ for this portion of the knowledge base, $\beta \wedge \gamma \rightarrow \eta$ will also hold for this portion of the knowledge base due to classical monotonicity. Algorithm 3 will, therefore, return **True** when checking $\mathcal{K} \mathrel{|\!\approx} \beta \wedge \gamma \rightsquigarrow \eta$.

Proof:

1. Given $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \gamma$ and $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \eta$, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$) both return **True**.
2. In order for RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \gamma$) and RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$) to terminate, both calls to Algorithm 3 have to break out of the while loop on line 2, so we either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

3. (a) Given $\mathcal{K} \mathrel{|\!\approx} \beta \rightsquigarrow \gamma$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \gamma$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \gamma$).

(b) Given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$).

(c) Since we have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \gamma$, we equivalently have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \neg(\beta \wedge \neg\gamma)$. Therefore $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \neg\gamma$ cannot hold, so the equivalent $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \gamma \to \bot$ will not hold on line 2 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$).

(d) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \to \eta$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \gamma \to \eta$ must hold by classical monotonicity on line 4 of Algorithm 3.

(e) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \gamma \to \eta$, then RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) must return **True**, meaning $\mathcal{K} \approx\!\!\!| \beta \wedge \gamma \rightsquigarrow \eta$ for this case.

Case 2:

4. (a) Since $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$.

(b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \gamma$, then $\mathcal{SR} \models \beta \to \gamma$ holds on line 4 of Algorithm 3.

(c) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \to \eta$ holds on line 4 of Algorithm 3.

(d) Since $\mathcal{SR} \models \beta \to \gamma$, we equivalently have $\mathcal{SR} \models \neg(\beta \wedge \neg\gamma)$. Therefore, $\mathcal{SR} \models \beta \wedge \neg\gamma$ cannot hold, so the equivalent $\mathcal{SR} \models \beta \wedge \gamma \to \bot$ does not hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$).

(e) Since $\mathcal{SR} \models \beta \to \eta$, then $\mathcal{SR} \models \beta \wedge \gamma \to \eta$ must hold by classical monotonicity on line 4 of Algorithm 3.

(f) Since $\mathcal{SR} \models \beta \wedge \gamma \to \eta$, then RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) must return **True**, meaning $\mathcal{K} \approx\!\!\!| \beta \wedge \gamma \rightsquigarrow \eta$ for this case.

□

Postulate 7 (Rational Monotonicity)

$$\frac{\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta,\ \mathcal{K} \not\approx\!\!\!| \beta \rightsquigarrow \neg\gamma}{\mathcal{K} \approx\!\!\!| \beta \wedge \gamma \rightsquigarrow \eta}$$

Rational Monotonicity expresses the fact that only the negation that only additional information that negates a previously drawn plausible conclusion should force us to withdraw that plausible conclusion [1]. For $\mathcal{K} \approx\!\!\!| \beta \rightsquigarrow \eta$ and $\mathcal{K} \not\approx\!\!\!| \beta \rightsquigarrow \neg\gamma$, Algorithm 3 considers the portion of the knowledge base where β is not exceptional. We know that $\beta \to \eta$ and that is is not the case that $\beta \to \neg\gamma$ for this portion of the knowledge base. We, therefore, know that $\beta \wedge \gamma$ will not be exceptional for this same portion of the knowledge base. Algorithm 3 will, therefore, consider this same portion of the knowledge base when checking $\mathcal{K} \approx\!\!\!| \beta \wedge \gamma \rightsquigarrow \eta$. Since $\beta \to \eta$ for this portion of the knowledge base, $\beta \wedge \gamma \to \eta$ will also hold for this portion of the knowledge base due to classical monotonicity. Algorithm 3 will, therefore, return **True** when checking $\mathcal{K} \approx\!\!\!| \beta \wedge \gamma \rightsquigarrow \eta$.

Proof:

1. Given $\mathcal{K} \approx\!\!\!\mid \beta \rightsquigarrow \eta$, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$) returns **True**.
2. Given $\mathcal{K} \not\approx\!\!\!\mid \beta \rightsquigarrow \neg\gamma$, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \neg\gamma$) returns **False**.
3. In order for RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$) and RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \neg\gamma$) to terminate, both calls to Algorithm 3 have to break out of the while loop on line 2, so we either have **Case 1** where $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \bot$ no longer holds or **Case 2** where $i_\beta \leq n$ is no longer true.

Case 1:

4. (a) Given $\mathcal{K} \approx\!\!\!\mid \beta \rightsquigarrow \eta$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ must hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (b) Given $\mathcal{K} \not\approx\!\!\!\mid \beta \rightsquigarrow \neg\gamma$, we know that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \neg\gamma$ does not hold on line 4 of Algorithm 3 in RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \rightsquigarrow \eta$).
   (c) In order to have $\mathcal{K} \approx\!\!\!\mid \beta \wedge \gamma \rightsquigarrow \eta$, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) must return **True**.
   (d) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \rightarrow \bot$, we will have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \wedge \gamma \rightarrow \bot$ by classical monotonicity.
   (e) Since we have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \wedge \gamma \rightarrow \bot$, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) will progress out of its while loop on line 2 of Algorithm 3.
   (f) Since $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \rightarrow \eta$ and $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \rightarrow \neg\gamma$, we can have $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \gamma \rightarrow \eta$ by classical monotonicity.
   (g) By having $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \models \beta \wedge \gamma \rightarrow \eta$ on line 4 of Algorithm 3, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) will return **True**, thereby giving $\mathcal{K} \approx\!\!\!\mid \beta \wedge \gamma \rightsquigarrow \eta$ for this case.

Case 2:

5. (a) Since $i_\beta > n$, then $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$.
   (b) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \approx\!\!\!\mid \beta \rightsquigarrow \eta$, then $\mathcal{SR} \models \beta \rightarrow \eta$.
   (c) If $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} = \emptyset$ and given $\mathcal{K} \not\approx\!\!\!\mid \beta \rightsquigarrow \neg\gamma$, then $\mathcal{SR} \not\models \beta \rightarrow \neg\gamma$.
   (d) Since there is no number $i_\beta \leq n$ such that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \rightarrow \bot$, there will be no number $i_\beta \leq n$ such that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \wedge \gamma \rightarrow \bot$ due to classical monotonicity.
   (e) Since there is no number $i_\beta \leq n$ such that $\bigcup_{j=i_\beta}^{j\leq n} \overrightarrow{R_j} \cup \mathcal{SR} \not\models \beta \wedge \gamma \rightarrow \bot$, we will have $i_\beta > n$, so RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) will progress out of its while loop on line 2 of Algorithm 3.
   (f) Since $\mathcal{SR} \models \beta \rightarrow \eta$ and $\mathcal{SR} \not\models \beta \rightarrow \neg\gamma$, we can have $\mathcal{SR} \models \beta \wedge \gamma \rightarrow \eta$ by classical monotonicity.
   (g) By having $\mathcal{SR} \models \beta \wedge \gamma \rightarrow \eta$ on line 4 of Algorithm 3, RationalClosure($\mathcal{R}$ of $\mathcal{K}$, $\beta \wedge \gamma \rightsquigarrow \eta$) will return **True**, thereby giving $\mathcal{K} \approx\!\!\!\mid \beta \wedge \gamma \rightsquigarrow \eta$ for this case.

$\square$

References

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