Machine Learning

Lecture 01-2: Basics of Information Theory

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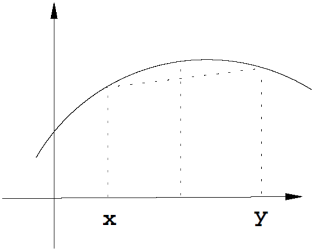
Outline

Concave functions



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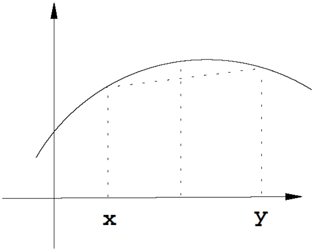

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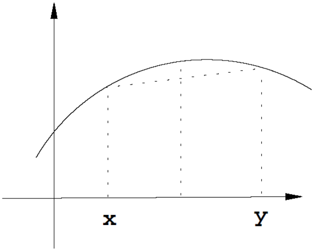

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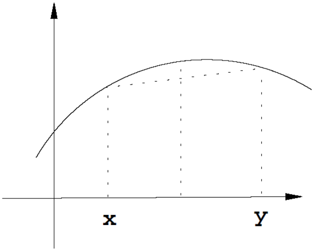

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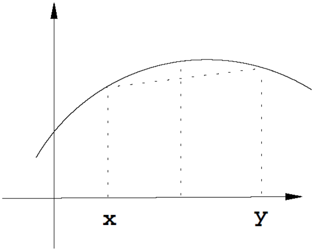

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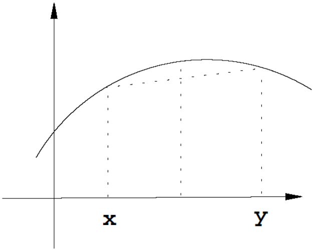

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Jensen's Inequality

Theorem (1.1)

Suppose function f is concave on interval I. Then

- *For any $p_i \in [0, 1], \sum_{i=1}^{n} p_i = 1$ and $x_i \in I$.*

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Exercise: Prove this (using induction).

Logarithmic function

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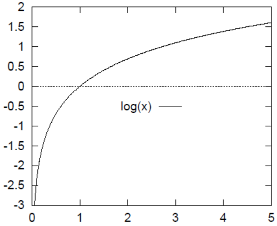

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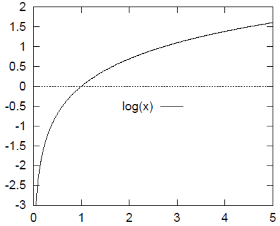

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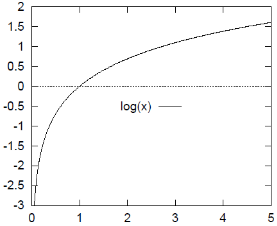

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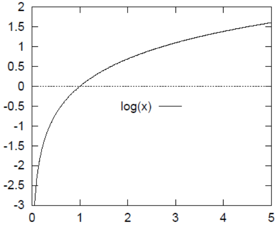

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- In words, exchanging $\sum_i p_i$ with *log* increases quantity. Or, swapping expectation and logarithm increases quantity:

$$E[\log x] \leq \log E[x].$$

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For real-valued variable, replace $\sum_X \ldots$ with $\int \ldots dx$.

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$$\begin{aligned} H(X) &= \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1(\log 2) \\ H(Y) &= \frac{1}{6}\log 6 + \ldots + \frac{1}{6}\log 6 = \log 6 \\ H(Z) &= \frac{1}{54}\log 54 + \ldots + \frac{1}{54}\log 54 = \log 54 \end{aligned}$$

Indeed we have:

$$H(X) < H(Y) < H(Z).$$

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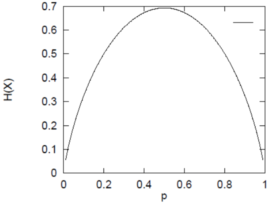

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  - Exercise: Give example.

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KL divergence between P and Q is larger than 0 unless P and Q are identical.

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$$H(P,Q) = \sum_X P(X) log \frac{1}{Q(X)} = -E_P[logQ(X)]$$

- Relationship with KL:

$$\begin{aligned} KL(P||Q) &= \sum_X P(X) log \frac{P(X)}{Q(X)} = E_P[logP(X)] - E_P[logQ(X)] \\ &= H(P,Q) - H(P) \end{aligned}$$

Or,

$$H(P,Q) = KL(P||Q) + H(P)$$

A corollary

Corollary (1.1)

(Gibbs Inequality)

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In general, let $f(X)$ be a non-negative function. Then

$$\sum_X f(X)\log Q(X) \leq \sum_X f(X)\log P^*(X)$$

where $P^*(X) = f(X)/\sum_X f(X)$.

Unsupervised Learning

- Unknown true distribution $P(\mathbf{x})$.

$$P(\mathbf{x}) \xrightarrow{\text{sampling}} \mathcal{D} = \{\mathbf{x}_i\}_{i=1}^{N} \xrightarrow{\text{learning}} Q(\mathbf{x})$$

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  - Same as **maximizing likelihood**: $\log Q(\mathcal{D})$.

Supervised Learning

- Unknown true distribution $P(\mathbf{x}, y)$, where y is **label** of input $\mathbf{x}$.

$$P(\mathbf{x}, y) \xrightarrow{\text{sampling}} \mathcal{D} = \{\mathbf{x}_i, y_i\}_{i=1}^{N} \xrightarrow{\text{learning}} Q(y|\mathbf{x})$$

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  - Same as **maximizing loglikelihood**: $\sum_{i=1}^N \log Q(y_i|\mathbf{x}_i)$,
  - Or **minimizing the negative loglikelihood (NLL)**: $-\sum_{i=1}^N \log Q(y_i|\mathbf{x}_i)$

Jensen-Shannon divergence

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- **Jensen-Shannon divergence** is one symmetrized version of KL:

$$JS(P||Q) = \frac{1}{2}KL(P||M) + \frac{1}{2}KL(Q||M)$$

where $M = \frac{P+Q}{2}$

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  where $M = \frac{P+Q}{2}$

- Properties:
  - $0 \leq JS(P||Q) \leq \log 2$
  - $JS(P||Q) = 0$ if $P = Q$
  - $JS(P||Q) = \log 2$ if P and Q has disjoint support.

Outline

Mutual information

- The **mutual information** of X and Y:

Mutual information

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$$I(X;Y) = H(X) - H(X|Y)$$

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$$I(X;Y) = H(X) - H(X|Y)$$

- Average reduction in uncertainty about X from learning the value of Y, or
- Average amount of information Y conveys about X.

Mutual information and KL Divergence

- Note that:

$$I(X;Y) = \sum_{X} P(X) log \frac{1}{P(X)} - \sum_{X,Y} P(X,Y) log \frac{1}{P(X|Y)}$$

Mutual information and KL Divergence

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- Due to equivalent definition:

$$I(X;Y) = H(X) - H(X|Y) = I(Y;X) = H(Y) - H(Y|X)$$

Property of Mutual information

Theorem (1.3)

$$I(X;Y) \geq 0$$

with equality holds iff $X \perp Y$.

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Proof: Follows from previous slide and Theorem 1.2.

Conditional Entropy Revisited

Theorem (1.4)

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Mutual information and Entropy

- From definition of mutual information

$$I(X;Y) = H(X) - H(X|Y)$$

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we get

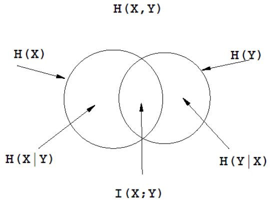
$$H(X) + H(Y) = H(X,Y) + I(X;Y)$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

- Consequently
  - $H(X,Y) \leq H(X) + H(Y)$ with equality holds iff $X \perp Y$.

Mutual information and entropy

Venn Diagram: Relationships among joint entropy, conditional entropy, and mutual information



$$H(X) + H(Y) = H(X, Y) + I(X; Y)$$

$$I(X; Y) = H(X) - H(X|Y)$$

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Conditional Mutual information

- The **conditional mutual information** of X and Y given Z:

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- Average amount of information Y conveys about X given Z.

Conditional mutual information and KL Divergence

Note:

$$I(X;Y|Z) = \sum_{X,Z} P(X,Z) log \frac{1}{P(X|Z)} - \sum_{X,Y,Z} P(X,Y,Z) log \frac{1}{P(X|Y,Z)}$$

Conditional mutual information and KL Divergence

Note:

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&= \sum_{Z} P(Z) KL(P(X,Y|Z), P(X|Z)P(Y|Z)) \geq 0.
\end{aligned}
$$

Property of conditional mutual information

Theorem (1.5)

$$I(X;Y|Z) \geq 0$$

$$H(X|Z) \geq H(X|Y,Z)$$

with equality hold iff $X \perp Y|Z$.

Interpretation:

- More observations reduce uncertainty on average except for the case of conditional independence.

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- More observations reduce uncertainty on average except for the case of conditional independence.
- X and Y are independently given Z iff X contain no information about Y given Z and vice versa:

$$X \perp Y|Z \equiv I(X;Y|Z) = 0.$$

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- More observations reduce uncertainty on average except for the case of conditional independence.
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Another characterization of conditional independence.