CSIT5900
Game Theory and Auction - A Short Introduction

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Multi-Agent Systems (MASs)

- An MAS is one where more than one agent co-exist and interact.
- In addition to issues that need to be dealt with in single agent case (such as checking if a newly observed/told fact is consistent with his current belief set), new issues in MASs include the following:
  - How do agents communicate with each other.
  - How do they cooperate with each other.
  - How do they act in face of adversity.

Examples

- Two person adversary games such as chess, go, and tic-tac-toe:
  - What's my opponent like?
  - How much time do I have for each move?
  - Why am I playing the game?
- Two teams compete against each other including bridge and most sport games.
- Resource allocations: a number of agents need to share certain resources, what's a fair way of dividing the resources?
- Market mechanisms in a society.

Game Theory

- Game theory studies how self-interested agents interact.
- It has been applied in economics, political science, social science, biology, computer science, and many other areas.
- Games in normal form are the simplest games studied in game theory and also most fundamental ones.

Game Theory (Definitions)

Wikipedia:

> *Game theory is a branch of applied mathematics that is used in the social sciences, most notably in economics, as well as in biology, engineering, political science, international relations, computer science, and philosophy. Game theory attempts to mathematically capture behavior in strategic situations, in which an individual's success in making choices depends on the choices of others. While initially developed to analyze competitions in which one individual does better at another's expense (zero sum games), it has been expanded to treat a wide class of interactions, which are classified according to several criteria. Today, "game theory is a sort of umbrella or 'unified field' theory for the rational side of social science, where 'social' is interpreted broadly, to include human as well as non-human players (computers, animals, plants)" (Aumann 1987).*

Stanford Encyclopedia of Philosophy:

> *Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences (or utilities) of those players, none of which might have been intended by any of them. The meaning of this statement will not be clear to the non-expert until each of the italicized words and phrases has been explained and featured in some examples.*

An Example

You and your friend share a network and both of you want to download a movie:

- If both of you do that, the network is jammed, and none of you is happy: say both of you value it 2.
- If only one of you do that, the network works perfectly: the one who did that is very happy (score of 5), the other is very unhappy (0).
- If none of you do that, then none of you is very happy, but then you can do something together. So let's assign it a score of 3.

What should you do? Does it depend on what you think your friend will do? Does it depend on if you two can talk about it? Will most rational people choose to do the same thing? What if the same situations need to be repeated a number of times?

These are some of the problems studied in game theory.

Modeling Agents' Interests

- Preferences: prefer A over B.
- Utility functions: A has value 100 and B 60 to me.

A utility function is a function from outcomes (situations) to real numbers between $[0, 1]$.

Theorem (von Neumann and Morgenstern, 1944) If a preference relation $\succeq$ satisfies the axioms completeness, transitivity, substitutability, decomposability, monotonicity, and continuity, then there exists a function u: $O \rightarrow [0, 1]$ with the properties:

1. $u(o_1) \geq u(o_2)$ iff $o_1 \succeq o_2$, and
2. $u([p_1 : o_1, ...p_k : o_k]) = \sum_{i=1}^{k} p_i u(o_i)$, where $[p_1 : o_1, ..., p_k : o_k]$ is a lottery (assigning probability p_i to outcome o_i) on O.

Games in Normal Form

A (finite, n-person) game is a tuple (N, A, u), where

- N is a set of players, indexed by i;
- $A = (A_1, ..., A_n)$, and each A_i is a set of actions (pure strategies) for agent i;
- $u = (u_1, ..., u_n)$, and each u_i is a utility function for agent i, which is a function from A to numbers.

Prisoner's Dilemma

- Two prisoners are interrogated separately.
- If both denies (cooperate), then each gets 1 year;
- If both admit (defect), then each serves 2 years;
- If one admits and the other denies, then the one who denies gets 3 years and the one who admits is free.

	C	D
C	3, 3	0,5
D	5, 0	1, 1

Coordination Games

- We can choose to drive either on the left or the right side of the road.
- Safe if we choose the same.
- Collision if we choose differently.

	L	R
L	1, 1	-1,-1
R	-1, -1	1, 1

Battle of Sexes

- A man and a woman are sharing a TV.
- There are two channels, sport and movie.
- The man prefers to watch the sport channel.
- The woman prefers to watch the movie channel.

	S	M
S	1, 0	-1,-1
M	-1, -1	0, 1

The Matching Pennies Game

- We each flip a coin.
- I win if they match, and lose if they don't.

	H	T
H	1, -1	-1,1
T	-1, 1	1, -1

Solution Concepts: Nash Equilibria

- Given a profile $s = (a_1, ..., a_n)$, and a player i, we write $s_i = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$, and $s = (s_i, a_i)$.
- A strategy $a^* \in A_i$ is player i's best response to s_i if for any $a \in A_i$, $u_i(s_i, a) \leq u_i(s_i, a^*)$.
- A profile $s = (a_1, ..., a_n)$ is a Nash equilibrium if for each player i, a_i is player i's best response to s_i.
- Given a game, there may be one, more than one, or no Nash equilibria.

Zero-Sum games (Strictly Competitive Games)

A two-person game (A, B, u_1, u_2) is a zero-sum (or strictly competitive game) if for all $a \in A$, $b \in B$, $u_1(a, b) + u_2(a, b) = 0$.
In a zero-sum game, what is good for one player is bad for the other. A zero-sum game has a unique Nash equilibria payoff: if (a_1, b_1) and (a_2, b_2) are both Nash equilibria, then $u_i(a_1, b_1) = u_i(a_2, b_2)$, $i = 1, 2$.
In fact, if (a, b) is a Nash equilibrium, then

$$min_{y \in B} u_1(a, y) \geq min_{y \in B} u_1(x, y)$$

for any $x \in A$, and similarly for any $y \in B$,

$$min_{x \in A} u_2(x, b) \geq min_{x \in A} u_2(x, y)$$

Mixed Strategies

- A mixed strategy of an agent is a lottery of her actions $\pi : A \to [0,1]$ and $\sum_a \pi(a) = 1$.
- The utility function can be extended to mixed strategies:

$$u_i(\pi_1, ..., \pi_n) = \sum_{(a_1,...,a_n)} u_i(a_1, ..., a_n) \prod_{k=1}^{n} \pi_k(a_k)$$

- A mixed strategy profile is a (mixed) NE if no one can deviate for better by another mixed strategy.

Nash Theorem Every game has a mixed NE.

Auctions

The auction setting is important for two reasons.

- Auctions are widely used in consumer, corporate, and computer science settings: Trade goods; sell important public resources; run financial markets; allocate broadband bandwidth and processing powers.
- The second and more fundamental reason to care about auctions is that they provide a general theoretical framework for understanding resource allocation problems among self-interested agents: an auction is any protocol that allows agents to indicate their interest in one or more resources, and that uses these indications of interest to determine both an allocation of resources and a set of payments by the agents.

Single-Item Auctions

Examples:

1. English auctions: the auctioneer sets a starting price, and agents bid successively.
2. Japanese auctions: the auctioneer sets a starting price, and increases the price successively; agents decide whether to be "in" at each price point.
3. Dutch auctions: the auctioneer sets a high starting price, and decreases the price successively; the first agent signals a buy signal wins the item at that price.
4. Sealed-bid auctions: agents send in bids in secret and the auctioneer uses a protocal to decide the winner. The protocal must be known to the agents before they send in their bids.

Auctions as Structured Negotiations

An auction is a negotiation with the following rules:

- Bidding rules: How are offers made (by whom, when, what can their content be)?
- Clearing rules: When do trades occur, or what are those trades (who gets which goods, and what money changes hands) as a function of the bidding?
- Information rules: Who knows what when about the state of negotiation?

Sealed Bid Auctions

First Price Auction:

- Bidders send in their bids sealed.
- The highest bidder wins the item at the price of her bid.
- When there is a tie, break it randomly.

Second Price Auction:

- Bidders send in their bids sealed.
- The highest bidder wins the item at the price of the next highest bid.
- When there is a tie, break it randomly.

How should agents bid?

Auctions as Games

First price auction with common knowledge of agents' values:

- a set of agents $N = \{1, ..., n\}$;
- the same set of actions for each agent $\{x \mid 0 \leq x \leq 1\}$ (bid at x);
- utility functions $u_i(x_1, ..., x_n) = v_i - x_i$ if agent i wins the auction, and 0 otherwise, where v_i is the value that the item is worth to agent i.

Questions: Does it matter how ties are broken? What are the Nash equilibria?

Auctions as Games

Second price auction with common knowledge of agents' values:

- a set of agents $N = \{1, ..., n\}$;
- the same set of actions for each agent $\{x \mid 0 \leq x \leq 1\}$ (bid at x);
- utility functions $u_i(x_1, ..., x_n) = v_i - x_j$ if agent i wins the auction, and 0 otherwise, where v_i is the value that the item is worth for agent i, and x_j is the second highest value in $\{x_1, ..., x_n\}$.

Questions: Does it matter how ties are broken? What are the Nash equilibria?

First Price Auction with Common Same value

Assumptions:

- one single item up for auction using the first price mechanism;
- ties are broken randomly;
- N bidders, each can bid using prices from a fixed set P;
- each has value 1 for the item.

Case 1

Consider $N = 2$ (two bidders), and $P = \{0, 1\}$ (each can bid either 0 or 1). This yields the following game:

	0	1
0	0.5, 0.5	0, 0
1	0, 0	0, 0

When both bid 0, they all get expected payoff of 0.5 because the tie is broken randomly and the winner gets payoff 1. So two Nash equilibria: $(0, 0)$ and $(1, 1)$.

Case 1 - A Generalization

Suppose in general there are two bids $0 \leq a < b \leq 1$:

	a	b
a	(1-a)/2, (1-a)/2	0, 1-b
b	1-b, 0	(1-b)/2, (1-b)/2

- (b, b) will always be a Nash equilibrium.
- (a, a) is a Nash equilibrium iff $(1-a)/2 \geq 1-b$.
- (a, b) cannot be a Nash equilibrium: it is a Nash equilibrium iff $0 \geq (1-b)/2$ and $1-b \geq (1-a)/2$, i.e. $a = b = 1$, which is impossible.
- (b, b) will be the only Nash equilibrium if $(1-a)/2 < 1-b$, i.e. $a > 2b-1$.
- An example of last case: a=0.3, b=0.6. In this case, the game becomes the Prisoner's dilemma:

	0.3	0.6
0.3	0.35, 0.35	0, 0.4
0.6	0.4, 0	0.2, 0.2

General Case

In general, suppose there are $n \geq 2$ players, and each can bid $0 \leq a_1 < \cdots < a_m \leq 1$, $m \geq 2$.

- $(a_m, ..., a_m)$ is always a Nash equilibrium;
- $(a_i, ..., a_i)$, $1 \leq i < m$, is a Nash equilibrium iff $(1 - a_i)/n \geq 1 - a_{i+1}$;
- suppose $x = (x_1, ..., x_n)$ has two different elements, and the largest element is a_i which occurs k times.
  - if $k = 1$, then x is not a Nash equilibrium: if $a_i = 1$, then the player who bids a_i can increase her expected payoff by lowering her bid to a_{i-1}; if $a_i < 1$, then players whose bids are lower than a_i can increase their expected payoff by raising their bids to a_i.
  - If $k > 1$, then x is a Nash equilibrium iff $a_i = a_m = 1$.

If each player can bid arbitrarily in a continuous close interval $[l, m]$, then the unique Nash equilibrium is $(m, ..., m)$. If the upper bound is open: $[l, m)$, then there is no Nash equilibrium.

Auctions as Games

- But it is not reasonable to expect agents will know each other's valuations of the item.
- What the item is worth to an agent is her own private information.
- How to model uncertainties: probabilities.

Auctions as Bayesian Games

In general, a sealed single item auction consists of

- a set of agents $N = \{1, ..., n\}$;
- for each agent, a set of possible private values, and a set of possible bids - we assume they are both $[0, 1]$ here;
- a common joint prior on value profiles: $p : [0, 1]^n \to [0, 1]$;
- a payment function σ from bid profiles and agents to [0.1]:

$$\sigma : [0, 1]^n \times N \to [0, 1],$$

  meaning that $\sigma(b, i)$ is what i needs to pay when the bid profile is b;
- a winner selection function τ from bid profiles to agents.

A strategy is now a function from agent's private values to bids. Assuming agents are risk-neutral, the utility function for agent i is then

$$u_i(f_1, ..., f_n) = \sum_v p(v) u_i(f_1(v_1), ..., f_n(v_n), v_1, ..., v_n),$$

where $u_i(b, v) = v_i - \sigma(b, i)$ if $\tau(b) = i$, and $-\sigma(b, i)$ otherwise.

Nash Equilibria for Auctions

- For second-price auction, the Nash quilibrium is to bid your valuation: $(v_1, ..., v_n)$.
- For first-price auction, if each player's valuation is drawn independently and uniformly at random, then $(v_1(n-1)/n, ..., v_n(n-1)/n)$ is a Nash equilibria.

Revenue Equivalence

A landmark result of auction theory is Vickrey's celebrated revenue equivalence theorem: any auction in which

- the bidder with the highest valuation always wins
- the bidder with the lowest possible valuation expects zero payoff
- all bidders are risk neutral, and
- bidders' valuations are drawn from a strictly increasing and atomless distribution

will lead to the same expected revenue for the seller (auctioneer).

Iterated Prisoner's Dilemma

The general form of the prisoner's dilemma:

	Cooperate	Defect
Cooperate	R=3, R=3	S=0, T=5
Defect	T=5, S=0	P=1, P=1

Here R is rewards for mutual cooperation, P penalty for mutual defection, T the temptation to defect, and S sucker's payoff.

- In general, $T > R > P > S$, and for iterated prisoner's dilemma: $2R > T + S$ so that cooperate twice is better than taking turns to exploit each other.
- The game has a dominant Nash equilibrium (D, D), and this is the strategy to play if the players are to play the games for a known number of times.

Computer Prisoner's Dilemma Tournament (Axelrod 1984)

Rules:

- Each program is going to play another player an unknown number of times, determined by a probability (the probability of ending the match is set to be 0.00346).
- Each program has available to it the history of interactions so far.

Round one:

- 14 entries + RANDOM, round robin, each pair of entries was matched in 5 games of 200 moves each. An entry's score is the sum of its scores in all games it played.
- The winner was Tit For Tat (TFT) submitted by Anatol Rapoport, a psychologist from University of Toronto.
- TFT cooperates on the first move and there after plays whatever move the other player played last time.

Round two: conducted after the results of round one were announced:

- 62 entries + RANDOM, from 6 countries.
- TFT were again the winner.

References

Axelrod, Robert (1984), The Evolution of Cooperation, Basic Books, ISBN 0-465-02122-0

A List of Strategies:

`http://www.iterated-prisoners-dilemma.net/prisoners-dilemma-strategies.shtml`

Some Formal Properties (Axelrod)

Discount factor It is typical to discount future. In iterated prisoner's dilemma, future encounters can be similarly discounted, say by a weight w, $0 \leq w \leq 1$, so scores from the nth encounter contributes only $w^{(n-1)}s$, where s is the payoff received at the nth game. In the iterated game setting, the weight is the same as the probability of playing your oponent again the next time.

Proposition 1 If the discount weight w is sufficient large, then there is no strategy that is best against all possible strategies used by the other player. Think about which strategy is best against ALL D or Permanent Retaliation (C until the oponent plays D, then ALL D).

Properties (Cont'd)

TFT is also robust: performs well in a wide variety of environments. This is an informal statement supported by the tournaments and further experiments.

A strategy is colelctively stable if no strategy can invade it. A strategy S can invade strategy G if given a large group of individuals all using G, a new individual using S can get a better score than the individuals using G.

Proposition 2 TFT is collectively stable if and only if, w is large enough. This value of w is a function of the four payoff parameters T, R, P, S.

Proposition 3 ALL D is always collectively stable. However, ALL D can be invaded by a cluster of individuals, and this cluster needs only be a small proposition of the original population.

Proposition 4 A strategy is nice if it is not the one to defect the first. If a nice strategy cannot be invaded by an individual, then it cannot be invaded by any cluster of the original population.