Machine Learning

Lecture 01-1: Basics of Probability Theory

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Outline

Random Experiments

- Probability associated with a **random experiment** — a process with uncertain outcomes
- Often kept implicit

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In machine learning, we often assume that data are generated by a hypothetical process (or a model), and task is to determine the structure and parameters of the model from data.

Sample Space

- **Sample space (aka population)** Ω: Set of possible outcomes and a random experiment.
- Example: Rolling two dice.

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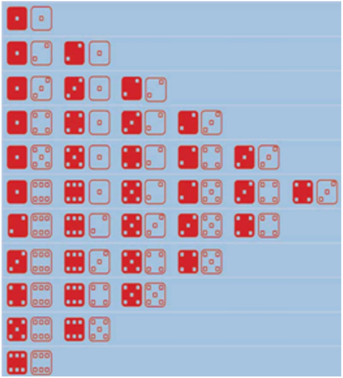

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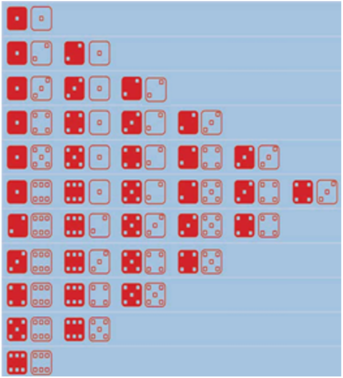

- Elements in a sample space are outcomes.

Events

- **Event**: A subset of the sample space.

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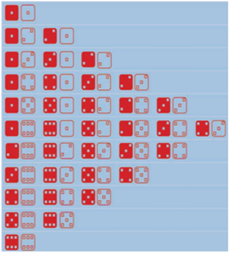

- Example: The two results add to 4.

Probability Weight Function

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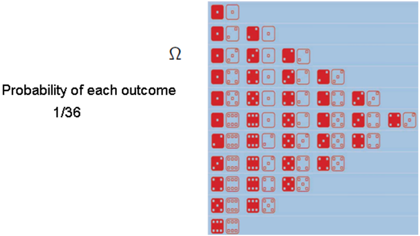

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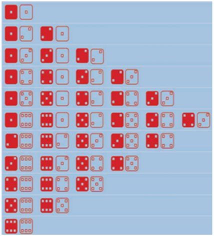
1/36



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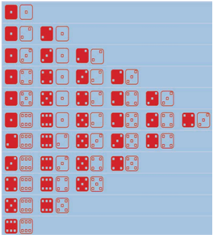

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In a more advanced treatment of Probability Theory, we would start with the concept of probability measure, instead of probability weights.

Random Variables

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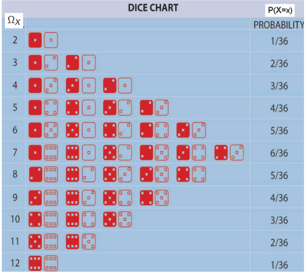
  - Example: X = sum of the two results. $X((2,5)) = 7; X((3,1)) = 4)$



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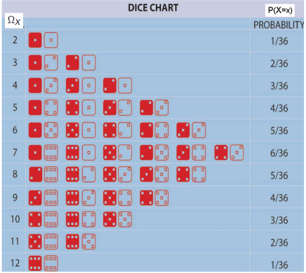

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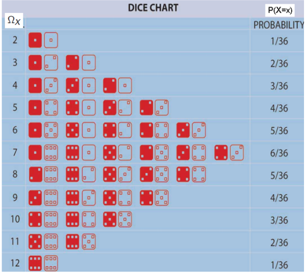

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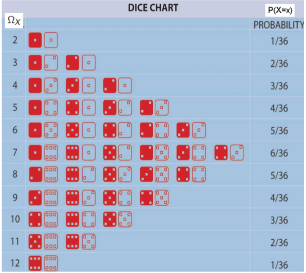
- **Domain** of a random variable: Set of all its possible values.

$$\Omega_X = \{2, 3, \ldots, 12\}$$

Random Variables and Event

- A random variable X taking a specific value x is an event:

$$\Omega_{X=x} = \{\omega \in \Omega | X(\omega) = x\}$$



- $\Omega_{X=4} = \{(1, 3), (2, 2,)(3, 1)\}$.

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- If X is continuous, we have a **density function** $p(X)$.

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- The frequentist interpretation is meaningful only when experiment can be repeated under the same condition.

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    - Primary school student: 0.5
    - Me: 0.8
    - Geographer: 1 or 0

- Arguments such as **Dutch book** are used to explain why one's probability beliefs must satisfy Kolmogorov's axioms.

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- The subjectivist interpretation makes concepts such as conditional independence easy to understand.

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  - If the doctor finds that the eyes of the patient are yellow, his belief about patient suffering from Hepatitis B would be > 0.1.

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That is: posterior $\propto$ prior $\times$ likelihood

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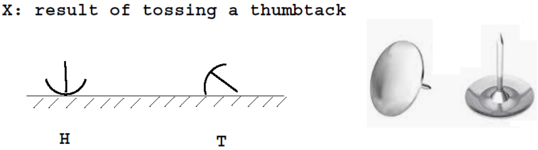
- Task: To estimate parameter $\theta = P(X{=}H)$.



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- $\theta = 0.01$ almost contradicts with the data. It does not explain the data well.

  However, it is more consistent with the data than $\theta = 0$ because $P(\mathcal{D}|\theta = 0.01) > P(\mathcal{D}|\theta = 0)$.

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- So $\theta = 0.5$ is more consistent with the data than $\theta = 0.01$ because $P(\mathcal{D}|\theta = 0.5) > P(\mathcal{D}|\theta = 0.01)$

  It explains the data the best, and is hence the most likely.

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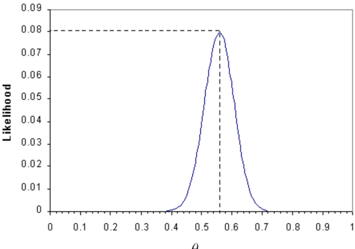
MLE best explains data or best fits data.



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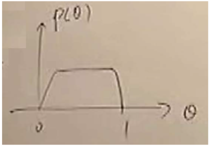
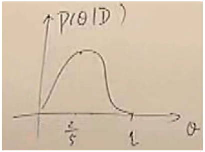
	Before Seeing Data	After Seeing Data: {2H, 3T}
MLE	?	2/5
Bayesian Estimation		

Two Views on Parameter Estimation

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$$P(D_{m+1} = H|\mathcal{D}) = \int P(D_{m+1} = H, \theta|\mathcal{D}) d\theta$$

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Full Bayesian: Take expectation over θ.

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Two Views on Parameter Estimation

Bayesian Estimation:

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- **Bayesian MAP**:

$$P(D_{m+1} = H|\mathcal{D}) = \theta^* = \arg\max p(\theta|\mathcal{D})$$

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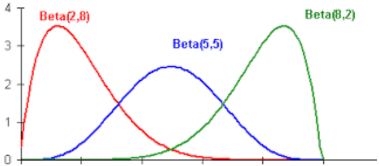
Beta Distribution



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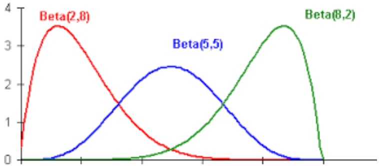

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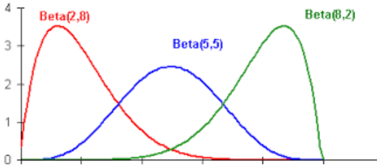

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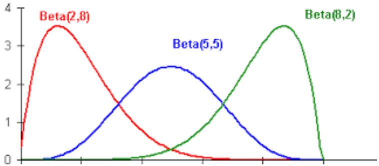

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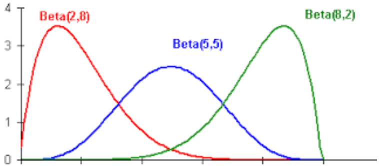

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- The larger the equivalent sample size, the more confident we are in our prior.

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- Conjugate families allow closed-form for posterior distribution of parameters and closed-form solution for prediction.

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- After taking data $\mathcal{D}$ into consideration, now our **updated belief** on $X{=}T$ is $\frac{m_t+\alpha_t}{m+\alpha}$.

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  - In case 2,
  $$P(D_{m+1} = H|\mathcal{D}) = \frac{30,000+100}{100,0000+100+100} \approx 0.3$$
  Data prevail.

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- In this course, we will focus on MLE.