UNIT II

SOUND WAVE AND ITS APPLICATIONS

2. Oscillation and Waves
3. Ultrasonic Waves
4. Non-destructive Testing
5. Architectural Acoustics

2
CHAPTER

Oscillation and Waves

2.1 INTRODUCTION

To and fro motion of a particle about the equilibrium position is known as *oscillation*. Any motion that repeats itself continuously on a definite path in definite interval of time is known as *periodic motion*, and interval of time is called the *time period* (T). T-Harmonic oscillations are the periodic motions which are very common in nature. Examples of this motion are motion of the Earth around the Sun, motion of the Moon around the Earth, oscillation of the pendulum of a clock, a mass attached to a spring, and motion of atoms within molecules or in a solid lattice. These oscillations are due to the changes in mechanical quantities such as displacement, velocity, acceleration, and density. Thus, these oscillations belong to the mechanical system. The oscillations belonging to electrical system are oscillations of electric and magnetic field vectors in visible light, radio waves, microwaves, etc. A tuned circuit and closed metal cavity oscillate electromagnetically; thus, these are the examples of electrical system. Although these oscillations belong to different systems, these are described by the same fundamental principle and mathematical equations.

Essential condition for a particle to execute a simple harmonic motion is given as $F \propto -y$, where F is the restoring force and y is the displacement. The model of harmonic oscillator is helpful in understanding important phenomenon of physics, e.g., elasticity, acoustics, optics, solid state physics, vibration of crystal lattices, and atomic and molecular physics. Many complex vibrations occurring in physical problems such as vibrations of string or membrane and the vibration of atoms in solids can also be described by simple harmonic vibrations. In fact it is difficult to realise exact simple harmonic motions in nature. It is because the linear dependence of restoring force and displacement do not hold good in actual systems. Thus, the systems can be treated as simple harmonic oscillators only within certain limits.

In this chapter, we will discuss basic features of different oscillatory systems such as simple harmonic motion, damped harmonic oscillation, forced vibration, coupled oscillation and attenuation coefficient of vibrating system. We will also focus on qualitative discussion on waves having periodic variation in space and time, wave equation, and reflection and transmission of waves at the boundary of two media.

2.2 SIMPLE HARMONIC MOTION

An oscillatory motion in which the amplitudes are equal on both sides of equilibrium position is known as *simple harmonic motion.* In order to understand it qualitatively, let us consider a particle P moving

around a circle with a uniform speed υ and N be the foot of perpendicular drawn from its instantaneous position to the diameter YY' of the circle. A complete oscillation can be described according to the motion of particle on the circumference of the circle along with the movement of N on the diameter YY', as shown in Fig. 2.1. During anti-clockwise motion of particle P, the foot of perpendicular changes its position at YY'. Let the motion of particle start from X, i.e., initially the particle is at X, and N is at O. When the particle reaches at Y, the foot of perpendicular coincides with its position at Y. As P moves along the circle and reaches from Y to X', N moves from Y and reaches at O. Further when P travels from X' to Y', N travels from O to Y'. Finally, when P moves from Y' to X, N reaches at O to complete one revolution. It is clear from the above discussion that during the time in which P completes one revolution, N travels through the diameter YY' twice. This oscillatory motion of N about centre O is called simple harmonic motion, which is defined as the projection of a uniform circular motion on the diameter of a circle.

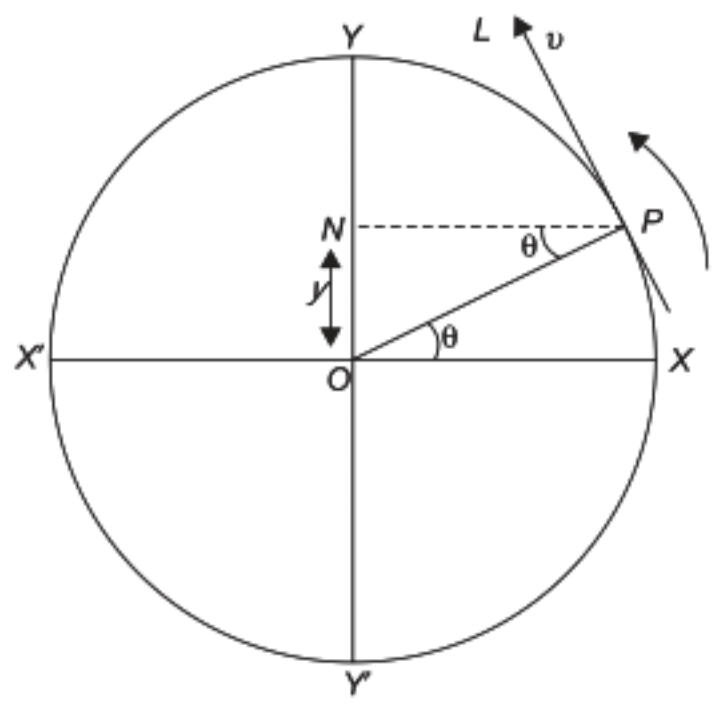


Fig. 2.1 Simple harmonic motion of particle *P*

If a is the radius of the circle, then θ is the angle made by radius vector by joining instantaneous position of particle and OX. From Fig. 2.1, it is clear that ON is the displacement of N during which particle moves from X to P. This displacement can be given as

$$y = a \sin \theta \tag{2.1}$$

If t is the time during which OP travels through an angle θ, then $\theta = \omega t$, and Eq. (2.1) can be written as

$$y = a \sin \omega t \tag{2.2}$$

Radius a is the maximum displacement on either side of O and is called the amplitude of motion.

2.2.1 Velocity of Motion

The velocity of N can be given by the component of the velocity of P parallel to YY'. From Fig. 2.1, it is clear that the velocity of P is υ along PL, i.e., along the tangent of the circle at point P. Its component parallel to YY' can be given as $\upsilon \cos \theta$. Now the velocity of N can be given as

$$\upsilon_N = \upsilon \cos \theta$$

[Because υ_N will be perpendicular to PN and is given as $\upsilon \sin \angle LPN = \upsilon \sin (90 - \theta) = \upsilon \cos \theta$]

$$= \upsilon \cos \omega t$$

$$= \upsilon (1 - \sin^2 \omega t)^{1/2}$$

From Eq. (2.1), $\sin \omega t = \dfrac{y}{a}$

Thus, $$\upsilon_N = \upsilon\left(1 - \frac{y^2}{a^2}\right)^{1/2}$$

$$= \frac{\upsilon}{a}(a^2 - y^2)^{1/2}$$

Since $\upsilon = a\omega$,

$$\upsilon_N = \omega\sqrt{(a^2 - y^2)} \tag{2.3}$$

From Eq. (2.3), it is clear that the velocity of N varies according to the displacement, i.e., the distance of N from O. When the displacement is zero, υ_N is maximum, while it is minimum, when y is maximum. This indicates that the velocity of a particle executing simple harmonic motion will be maximum at the mean position and will be zero at the extreme positions.

2.2.2 Acceleration

We know that in circular motion, the centripetal accleration is υ^2/a and is directed towards the centre along PO (see Fig. 2.1). Acceleration of N is given as the component of υ^2/a along NO. Now the acceleration of N, i.e., a_N can be given as

$$a_N = -a\omega^2 \sin\theta$$

$$= -\omega^2 a \sin\theta$$

or $$= -\omega^2 . y \tag{2.4}$$

where $y = a \sin\theta$.

Negative sign in Eq. (2.4) shows that the acceleration acts opposite to the direction in which y increases. According to Eq. (2.4), a body is in simple harmonic motion, if its acceleration of motion ($\omega^2 y$) is always directed towards a fixed point at the centre and is directly proportional to its distance from it.

2.2.3 Time Period

The time taken by the moving particle P to complete one oscillation is called *time period* or *periodic time*. If ω is the angular velocity of motion, then by definition

$$\omega = \frac{\text{Angle described in one revolution}}{\text{Time of one complete revolution}}$$

$$= \frac{2\pi}{T}$$

or $$T = \frac{2\pi}{\omega} \tag{2.5}$$

From Eq. (2.4),

$$\omega = \sqrt{\frac{a_N}{y}} = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

By putting the value of ω in Eq. (2.5), we get

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} \tag{2.6}$$

2.2.4 Frequency

Frequency is defined as the number of oscillations made by a moving body in one second. It is expressed as the reciprocal of periodic time, i.e.,

$$n = \frac{1}{T} = \frac{\omega}{2\pi} \tag{2.7}$$

where n is frequency.

2.2.5 Phase

The phase of an oscillating particle determines its position and direction of motion at a particular instant.

If at any instant, two vibrating particles are passing simultaneously through their equilibrium positions in the same direction, then at that instant they are in *same phase*, and if they are passing in opposite directions, then they are in *opposite phase*.

Suppose the time is measured from the instant, when the particle P is at P_0 moving on the reference circle (Fig. 2.2), then $P_0OX = \phi$, and the displacement y (ON) of N, from the equilibrium position O after a time t, will be given as

$$y = ON = OP \sin(\omega t + \phi)$$

$$y = a \sin(\omega t + \phi) \tag{2.8}$$

where a is the radius of the circle and ωt is the angle through which the particle P has rotated in moving from P_0 to P in time t. Hence, at the instant t, the phase of N will be measured by the total angle ($\omega t + \phi$), where ϕ is known as the initial phase or epoch of N. Equation (2.8) is known as the *general equation of a simple harmonic motion*.

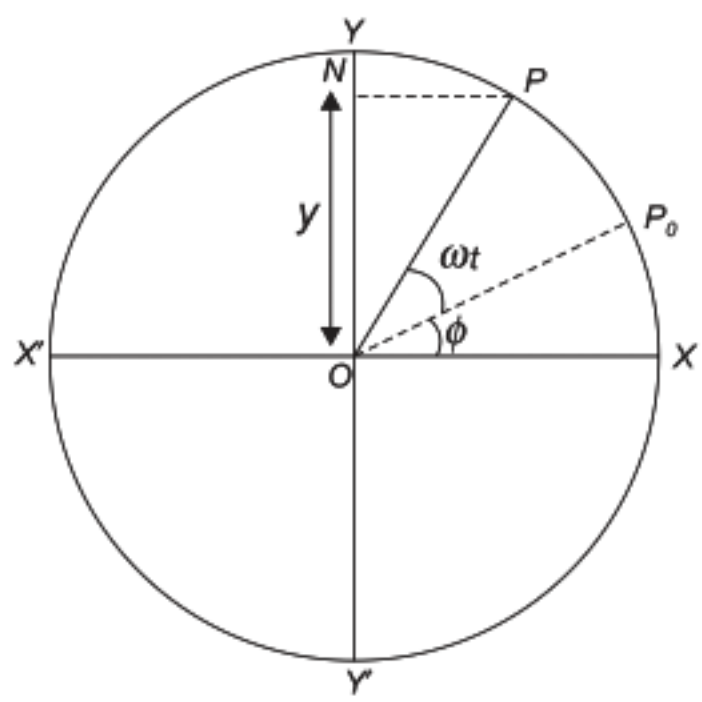


Fig. 2.2 P_0 and P are two different positions of a particle depicted at different instants

2.3 DAMPED HARMONIC OSCILLATIONS

A body executing single harmonic oscillations has constant amplitude only under ideal conditions. It is possible only when there is no dissipative or damping force acting on the body, thrown to execute simple harmonic motion. In all practical cases, it is observed that the vibrating system experiences a resisting force which causes decay of its amplitude.

If the bob of a pendulum is displaced slightly from its mean position, it starts oscillating. The oscillation continues with its natural frequency but the amplitude gradually dies away and finally comes to rest. Decay in the amplitude of oscillation is due to the various resisting forces opposing the motion of the body, which diminishes the amplitude or damps the oscillation. Such vibrations with decreasing amplitude are called damped oscillations. The energy given to the body initially to begin the oscillations is used up in doing work against damping forces. Thus, the energy given to the system decreases with time, and finally the system comes to rest.

2.3.1 Wave Equation of Damped Vibration

If y is the instantaneous displacement from the equilibrium position of a body of mass m, performing damped simple harmonic oscillation, then the forces acting on the body are as follows:

(i) ***Restoring force***: The tendency of this force is that it always tries to bring the body (executing oscillation) back to its mean position and is proportional to the displacement. Thus, it may be written as

$$F_R \propto -y$$

or

$$F_R = -ky \tag{2.9}$$

(ii) ***Damping force***: It is the retarding force proportional to the velocity acting in the opposite direction of motion. Thus, it may be written as

$$F_D \propto -\frac{dy}{dt}$$

or

$$F_D = -b\frac{dy}{dt} \tag{2.10}$$

(iii) ***Force of inertia***: Each body ready to set itself for oscillatory motion has inertia, which can be given as

$$F_I \propto \frac{d^2y}{dt^2}$$

or

$$F_I = m\frac{d^2y}{dt^2} \tag{2.11}$$

At equilibrium, the force of inertia is balanced by restoring force and damping force. Hence, the equation of motion of a body executing damped harmonic vibrations is given as

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

or $$\frac{d^2y}{dt^2}+\frac{bdy}{mdt}+\frac{k}{m}y=0$$

Putting b/m=2r and k/m = ω^2 , we obtain

$$\frac{d^2y}{dt^2}+2r\frac{dy}{dt}+\omega^2 y=0 \tag{2.12}$$

This is a homogenous linear differential equation of second order for damped harmonic oscillator. The general solution of eq. (2.12) is given as

$$y=a_0e^{-rt}\sin(\omega t+\phi) \tag{2.13}$$

where $$\omega=\sqrt{\frac{k}{m}-\left(\frac{b}{2m}\right)^2} \tag{2.14}$$

or $$\omega=\sqrt{\omega_0{}^2-\left(\frac{b}{2m}\right)^2} \tag{2.15}$$

where $\omega_0=\sqrt{\frac{k}{m}}$ is the natural frequency and ω is the frequency of damped oscillation.

2.3.2 Analysis of Damped Harmonic Oscillator

In order to analyse damped harmonic oscillations, let us start with equation (2.12) i.e.,

$$\frac{d^2y}{dt^2}+2r\frac{dy}{dt}+\omega_0{}^2y=0$$

where $2r=\frac{b}{m}=\frac{1}{\tau}$ and $\omega_0=\sqrt{\frac{k}{m}}$ is the natural frequency in absence of damping forces. τ is the relaxation time and r is damping constant.

To solve equation (2.12) let its possible solution be

$$y=Ae^{\alpha t} \tag{2.16}$$

using the value of y in equation (2.12) we get

$$A\left(\alpha^2+2r\alpha+\omega_0{}^2\right)e^{\alpha t}=0$$

or $$\alpha^2+2r\alpha+\omega_0{}^2=0$$

Now the roots of α can be given as

$$\alpha=-r\pm\sqrt{r^2-\omega_0{}^2}$$

Thus the solution of equation (12.2) are

$$y = A_1 e^{\left(-r+\sqrt{r^2-\omega_0{}^2}\right)t} \quad \text{and} \quad y = A_2 e^{\left(-r-\sqrt{r^2-\omega_0{}^2}\right)t}$$

Hence the general solution of equation (12.2) can be given as

$$y = A_1 e^{\left(-r+\sqrt{r^2-\omega_0{}^2}\right)t} + A_2 e^{\left(-r-\sqrt{r^2-\omega_0{}^2}\right)t}$$

or

$$y = e^{-rt}\left(A_1 e^{\left(\sqrt{r^2-\omega_0{}^2}\right)t} + A_2 e^{\left(-\sqrt{r^2-\omega_0{}^2}\right)t}\right) \tag{2.17}$$

where A_1 and A_2 are constants, can be determined with initial conditions.

Depending on the values of quantity $\sqrt{r^2-\omega_0{}^2}$ we may reach at different conditions of damping oscillations. According to the values of r and ω_0 this quantity may have imaginary, real and zero values. Corresponding to the different values of $\sqrt{r^2-\omega_0{}^2}$, we may have following three different situations.

1. **Underdamping (oscillatory):** $\omega_0^2 > \beta^2$
2. **Overdamping (non-oscillatory):** $\omega_0^2 < \beta^2$
3. **Critical damping (non-oscillatory):** $\omega_0^2 = \beta^2$

1. **Under damping case:** This condition occurred when $r < \omega_0$

   it means $\sqrt{r^2-\omega_0{}^2} = \sqrt{-1\left(\omega_0{}^2-r^2\right)} = i\sqrt{\omega_0{}^2-r^2} = i\omega(say)$

   where $i = \sqrt{-1}$ and $\omega = \sqrt{\left(\omega_0{}^2-r^2\right)}$ which is a real quantity. Now equation (2.17) can be given as

$$\begin{aligned} y &= e^{-rt}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right] \\ &= e^{-rt}\left[A_1\left(\cos\omega t + i\sin\omega t\right) + A_2\left(\cos\omega t - i\sin\omega t\right)\right] \\ &= e^{-rt}\left[\left(A_1+A_2\right)\cos\omega t + i\left(A_1-A_2\right)\sin\omega t\right] \end{aligned}$$

   since y is a real quantity, $\left(A_1+A_2\right)$ and $i\left(A_1-A_2\right)$ must be real quantities. Clearly, A_1 and A_2 are complex quantities. If $A_1 + A_2 = a_0 \sin\phi$ and $i\left(A_1-A_2\right) = a_0\cos\phi$ then

$$y = a_0 e^{-rt}\sin\left(\omega t + \phi\right) \tag{2.18}$$

   This equation is same as equation (2.13) represents a damped harmonic motion, whose periodic time is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0{}^2 - r^2}} \tag{2.19}$$

From the equation (2.19) is clear that the time period of damped harmonic oscillation (T) is greater than the time period $T_0 = \dfrac{2\pi}{\omega_0}$ of undamped oscillator.

The amplitude of the oscillatory motion equation (2.18) is given by $a = a_0 e^{-rt} = a_0 e^{-t/2\tau}$, where a_0 is the amplitude in the absence of damping. In case of damping the amplitude decreases exponentially with time till the oscillation dies out. Displacement current for damped harmonic motion is shown in Fig. 2.3. The total mechanical energy of the oscillator also decreases exponentially with time. Since energy is proportional to square of the amplitude (e^{-rt}). Thus energy also decreases exponentially with time, i.e., as e^{-2rt}. Displacement curve for damped harmonic motion is shown in Fig. 2.3.

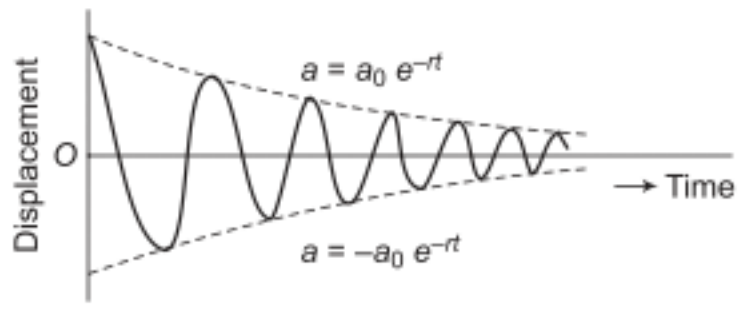


Fig. 2.3 Damped harmonic motion

Decrement: The ratio between amplitudes of two successive maxima i.e., $\dfrac{a_0 e^{-rt}}{a_0 e^{-r(t+T_1)}} = e^{rT_1}$ is known as the decrement of the oscillator. The natural logarithm of the decrement $\delta = \log e^{rT_1} = rT_1 = \dfrac{2\pi r}{\sqrt{\omega_0^2 - r^2}}$ is known as the logarithmic decrement of the oscillator.

2. **Overdamping case:** If damping is high i.e. $r > \omega_0$ then $\sqrt{r^2 - \omega_0^2} = \beta$ (say), is a real quantity and then from equation (2.17) we have

$$y = \left(A_1 e^{-(r-\beta)t} + A_2 e^{-(r+\beta)t}\right) \tag{2.20}$$

As $r > \beta$, both quantities of right hand side decreases exponentially with time and motion is non-oscillatory. Such motion is called dead beats or *aperiodic* and its main application is in dead beat galvanometers. Highly damped oscillation is shown in Fig. 2.4.

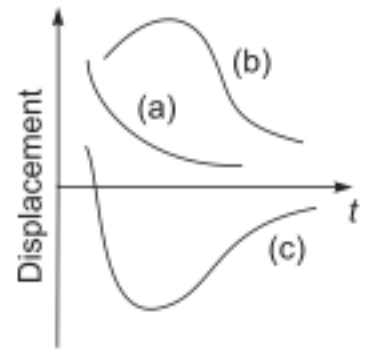


Fig. 2.4 Oscillations under overdamping with different initial conditions

Under the relative values of γ, β, A_1 and A_2 with initial position and velocity, the oscillator comes back to the equilibrium position in one of the following ways:

(a) The displacement decreases to zero exponentially in a continuous manner.
(b) The displacement first increases, attains a maximum, and the decreases to zero.
(c) The displacement changes sign, i.e., the oscillator overshoots to the other side of the equilibrium position, attains a maximum and then continuously comes back to equilibrium position.

Motion of simple pendulum in highly viscous liquid medium in an example of overdamped motion.

3. **Critical damping case:** This case occurs at $r = \omega_0$, at this condition equation (2.17) gives us $y = (A_1 + A_2)e^{-rt} = ce^{-rt}$ where $c = A_1 + A_2$

Since there is only one constant, thus it does not provide us the solution of equation (2.12) of second order.

Now let us consider

$\sqrt{r^2 - \omega_0^{\,2}} = h$ which is very small

Thus, equation (2.17) gives us

$$y = e^{-rt}\left[A_1 e^{ht} + A_2 e^{-ht}\right]$$

$$= e^{-rt}\left[A_1(1 + ht + ...) + A_2(1 - ht + ...)\right]$$

Due to small value of h, higher term will be negligible.

Hence $$y = e^{-rt}\left[(A_1 + A_2) + h(A_1 - A_2)t\right]$$

or $$y = e^{-rt}(P + Qt) \tag{2.21}$$

where $P = A_1 + A_2$ and $Q = (A_1 - A_2)$

Also from the expression $\frac{dy}{dt}$ we can write

$$\frac{dy}{dt} = e^{-rt}Q + e^{-rt}(-r)(P + Qt)$$

Initially at $t = 0$ the displacement of particle is $y = y_0$ and then velocity is υ_0, using these values in equation (2.21) we get

$$y_0 = P \text{ and } \upsilon_0 = Q - rP = Q - ry_0 \text{ or } Q = \upsilon_0 + ry_0$$

Thus $$y = \left[y_0 + (\upsilon_0 + ry_0)t\right]e^{-rt} \tag{2.22}$$

From equation (2.22) it is clear that initially displacement increases due to the factor $[y_0 + (\upsilon_0 + ry_0)t]$. But as time elapses, exponential term becomes dominating and displacement returns to zero. Such motion is known as critically damped or just *aperiodic*. Figure 2.5 shows the critically damped oscillation.

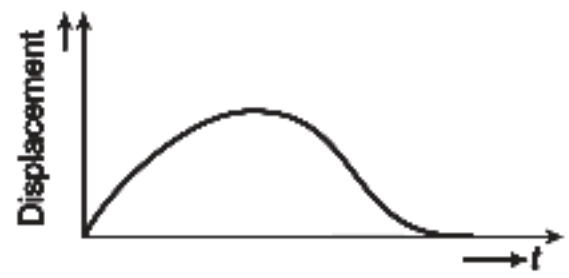


Fig. 2.5 Critically damped oscillation

2.3.3 Important Parameters under Damped Harmonic Oscillations

The secrets of damped harmonic oscillations can be observed in its different parameters as discussed below:

1. **Logarithmic Decrement (λ):** As it is mentioned in equation (2.18) the solution of damped harmonic oscillations can be given as

$$y = e^{-rt}\sin(\omega t + \phi)$$

The maximum value of $\sin(\omega t + \phi)$ is alternately +1 and –1, obviously the time displacement curve of vibrating body lies entirely between the curves $a = a_0 e^{-rt}$ and $a = -a_0 e^{-rt}$ as shown in Fig. 2.3.

If a_n and a_{n+1} are the successive amplitudes then

$$a_n = a_0 e^{-rt} \text{ and } a_{n+1} = a_0 e^{-r(t+T/2)}$$

So
$$\frac{a_n}{a_{n+1}} = \frac{e^{-rt}}{e^{-r(t+T/2)}} = e^{rT/2} = d\,(say) \tag{2.23}$$

where d is a constant for motion, called as decrement, indicating for the reduction in amplitude.

Now, $$\log_e\left(\frac{a_n}{a_{n+1}}\right) = \log_e d = \frac{rT}{2} = \lambda \tag{2.24}$$

The quantity λ is known as logarithmic decrement and is equal to the ratio of the natural logarithm of the ratio of two successive amplitudes of vibration.

2. **Damping Coefficient (r) :** The value of damping coefficient may be obtained directly from equation (2.24), as $\lambda = \frac{rT}{2} = \frac{r}{2\upsilon}$ (where ν is the frequency)

or
$$r = 2\lambda\upsilon$$

again from equation (2.24) above expression can be given as

$$r = 2.303 \quad 2\upsilon\log_{10}\left(\frac{a_n}{a_{n+1}}\right) \tag{2.25}$$

Thus by knowing ν and ratio $\left(\frac{a_n}{a_{n+1}}\right)$, r can be calculated

3. **Relaxation Time (τ):** Relaxation time is defined as a time in which the amplitude, velocity or total mechanical energy of oscillation falls to $\frac{1}{e}$ (times) of its original value. It is denoted by τ. Sometimes it is also termed as Time constant, Mean life time, Modulus of decay or Mean decay time.

According to the definition we may write

amplitude after time $(t+\tau) = \frac{1}{e}$ amplitude after time t

or $$\frac{1}{e} = \frac{\text{amplitude after time}(t+\tau)}{\text{amplitude after time } t}$$

or $$\frac{1}{e} = \frac{a_0 e^{-r(t+\tau)}}{a_0 e^{-rt}}$$

or $$\frac{1}{e} = e^{-r\tau} = \frac{1}{e^{r\tau}}$$

or $$r\tau = 1$$

or $$\tau = \frac{1}{r} = \frac{2m}{b} \tag{2.26}$$

Thus in terms of relaxation time, the amplitude is written as $a_0 e^{-t/\tau}$

2.3.4 Energy in Damped Harmonic Oscillation

According to the equation (2.16) the displacement of a damped harmonic oscillation can be given as

$$y = a_0 e^{-rt} \sin(\beta t + \phi)$$

where a_0 is maximum amplitude without damping and ϕ is phase constant.
r = damping constant

and $$\beta = \sqrt{\omega^2 - r^2}$$

Initially let us consider $y = 0$ and $\phi = 0$ at $t = 0$, thus the above equation becomes

$$y = a_0 e^{-rt} \sin \beta t$$

$$\frac{dy}{dt} = \beta a_0 e^{-rt} \cos \beta t + \sin \beta t \left(-r a_0 e^{-rt}\right)$$

or $$\frac{dy}{dt} = a_0 e^{-rt} (\beta \cos \beta t - r \sin \beta t)$$

For very small damping $r << \omega$, thus above equation become

$$\frac{dy}{dt} = \beta a_0 e^{-rt} \cos \beta t$$

$$\text{kinetic energy} = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 = \frac{1}{2}m\beta^2 a_0{}^2 e^{-2rt}\cos^2\beta t$$

$$\text{thus, kinetic energy} = \frac{1}{2}a_0^2 e^{-2rt}\cos^2\beta t \qquad \begin{cases} for\ r << \omega \\ \beta^2 = \omega^2 \\ \beta^2 = k/m \end{cases}$$

$$\text{potential energy} = \frac{1}{2}ky^2 = \frac{1}{2}ka_0{}^2 e^{-2rt}\sin^2\beta t$$

Now $$\text{total energy} = \frac{1}{2}ka_0{}^2 e^{-2rt}\cos^2\beta t + \frac{1}{2}ka_0{}^2 e^{-2rt}\sin^2\beta t$$

or $$E = \frac{1}{2}ka_0{}^2 e^{-2rt}$$

or $$E = E_0 e^{-2rt} \tag{2.27}$$

where $E_0 = \frac{1}{2}ka_0{}^2$ is the energy at $t = 0$.

Thus energy also decreases exponentially with time.
Equation (2.27) can also be expressed in terms of relaxation time as

$$E = E_0 e^{-t/\tau}$$

We can find the value of τ in case of energy reduction as

$$E = \frac{1}{e}\ (E_0 \text{ in time } \tau)$$

or $$E_0 e^{-2r\tau} = E_0 e^{-1}$$

or $$2r\tau = 1$$

or $$\tau = \frac{1}{2r} \tag{2.28}$$

2.3.5 Power Dissipation (P)

It is defined as the rate of decrease of total energy due to the damping and denoted by P.

$$P = -\frac{dE}{dt} = -\frac{d}{dt}\left(\frac{1}{2}ka_o^2 e^{-t/\tau}\right)$$

$$= \frac{1}{\tau}\cdot\frac{1}{2}\cdot ka_o^2 e^{-t/\tau}$$

or $$P = \frac{E}{\tau} \tag{2.29}$$

2.3.6 Quality Factor (Q)

Quality factor is a dimensionless factor that describes the damp behaviour of oscillator. It measures the rate at which the energy of an oscillator decays and defined as the reciprocal of fractional decrease of energy per cycle.

i.e. $$Q \propto \frac{1}{\dfrac{\text{energy dissipated per cycle}}{\text{energy stored per cycle}}}$$

or $$Q = 2\pi \frac{\text{average energy stored per cycle}}{\text{average energy dissipated per cycle}}$$

For the total energy E and power dissipated (energy decay) rate of an oscillator, quality factor can be given as.

$$Q = 2\pi \frac{E}{PT} = \frac{2\pi}{T} \cdot \frac{E}{E/\tau} = \omega\tau$$

or $$Q = \omega\tau \tag{2.30}$$

$$Q = \omega . \frac{1}{2r} = \frac{m\omega}{b}$$

or $$Q = \frac{m\omega}{b} \tag{2.31}$$

According to equation (2.31) quality factor is directly proportional to mass and inversely proportional to the damping acting on the system. Q-factor can also be expressed in terms of logarithmic decrement λ as $Q = \omega \frac{1}{r} = \frac{2\pi}{T} \cdot \frac{1}{r} = \frac{\pi}{\lambda}$.

or $$Q = \frac{\pi}{\lambda} \tag{2.32}$$

2.4 FORCED VIBRATION

When a body capable of oscillation is displaced from its equilibrium position and is left free, it starts to oscillate with a definite frequency. This frequency depends upon the intrinsic properties of the body and is known as the *natural frequency* of the body. Such oscillations of the body which are not affected by any external force are called *free oscillations*.

When an external periodic force is applied to an oscillating body, then in the beginning the body tends to oscillate with its natural frequency; however, the external periodic force tries to impose its own frequency upon the oscillating body. Thus, there is a tussle between the oscillating body and the external force due to which the oscillation of the body becomes irregular and dies out in a short time and finally the body oscillates with the frequency of the external periodic force with constant amplitude and regular oscillations. These oscillations are known as *forced oscillations*. Thus, when a body being acted by an external periodic force starts to oscillate with the frequency of the external periodic force, then the oscillations of the body are called *forced oscillations*.

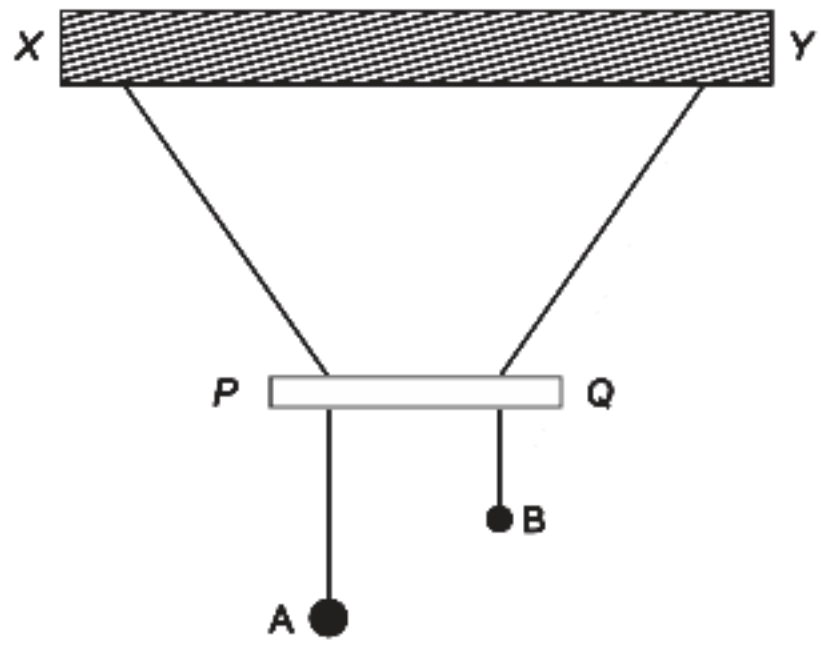


Fig. 2.6 Pendulum A is imposing its frequency on pendulum B to execute forced vibration

For example, consider Fig. 2.6 in which *A* and *B* are two pendulum having different lengths suspending from *PQ*. Due to their individual properties, those have different natural frequencies. When the pendulum *A* is displaced and left free, it oscillates with its natural frequency. These oscillations are applied in terms of periodic force on pendulum *B* through rod *PQ*. Thus pendulum *B* to acted upon by a periodic force of frequency equal to the frequency of pendulum *A*. Therefore, pendulum *B* starts to oscillate with the frequency of pendulum *A*. Now the oscillations of *B* are the forced oscillations. Analytical analyses of forced oscillations are discussed in following subsections.

2.4.1 Analytical Treatment of Forced Vibration

Let us consider a mechanical oscillator of mass m derived by an external force $F_0 \sin pt$ where p is the angular frequency of driving force having maximum value F_0. Let y is the instant displacement from the position of equilibrium during the oscillation, and dy/dt is its instantaneous velocity. The instant force acting on the system can be given as:

(i) **Restoring Force:** This force tries to bring the system back to its equilibrium position. It is proportional to the displacement with negative sign i.e.

$$F_{rs} \propto -y \text{ or } F_{rs} = -ky \text{ where } k \text{ for constant or stiffness constant.}$$

(ii) **Resisting (or Frictional) Force:** It results damping in the system and is proportional to the velocity but acting in opposite direction i.e.

$$F_{fs} = -b\frac{dy}{dt} \text{ where } b \text{ is positive constant}$$

(iii) **Force of Inertia:** The system possesses this force by the virtue its motion given as

$$F = m\frac{d^2y}{dt^2}$$

(iv) **External Periodic Force:** It is the driving force of frequency $p/2\pi$ and given as

$$F_{Ef} = F_0 \sin pt$$

2.4.2 Equation of Motion of Forced Vibration and Its Solution

The equation of motion of a particle under external deriving force can be written as

$$m\frac{d^2y}{dt^2} = F_0 \sin pt - ky - b\frac{dy}{dt}$$

or
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_0 \sin pt$$

Deflecting force + damping force + restoring force = deriving force (externally applied)

or
$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_0}{m}\sin pt$$

or
$$\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega^2 y = f_0 \sin pt \tag{2.33}$$

Where
$$2r = \frac{b}{m}, \omega^2 = \frac{k}{m}, f_0 = \frac{F_0}{m}$$

Equation (2.33) is an inhomogeneous linear differential equation of second order representing forced vibration.

The behaviour of mechanical system executing oscillations under external deriving force can be analysed from the solution of equation (2.33). The complete solutions of equation (2.33) consist of two parts.

(i) First part is the 'transient part' which dies away with time. It is in fact the solution of equation

$$\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega^2 y = 0$$

Which is in fact a differential equation of damped harmonic vibration, the solution can be given as

$$y = a_0 e^{-rt} \sin(\beta t + \phi) \tag{2.34}$$

Where a_0 is a constant and β is damped frequency.

(ii) The second part is the 'steady state part', which is obtained by trial method. Both the parts contribute to the solution initially but we shall concentrate on the steady state part as it is the ultimate behaviour of oscillator.

It is important to mention that when transient effect dies away system will oscillate with the frequency of impressed force. The solution of steady state part of equation (2.33) can be given as

$$y = A\sin(pt - \theta) \tag{2.35}$$

Where A is the maintained amplitude under external force and θ is the phase difference between deriving force and resulting displacement of the system.

Now from equation (2.35)

$$\frac{dy}{dt} = Ap\cos(pt-\theta), \frac{d^2y}{dt^2} = -p^2 A\sin(pt-\theta)$$

using these values in equation (2.33) we get

$$-p^2 A\sin(pt-\theta) + 2rpA\cos(pt-\theta) + \omega^2 A\sin(pt-\theta) = f_0 \sin\left[(pt-\theta)+\theta\right]$$

or

$$A\left(\omega^2 - p^2\right)\sin(pt-\theta) + 2r\,pA\cos(pt-\theta) = f_0 \sin(pt-\theta)\cos\theta + f_0\cos(pt-\theta)\sin\theta$$

If above equation holds good for all values of t, we may equate the coefficients of sine and cosine of $(pt-\theta)$. Thus we get

$$A\left(\omega^2 - p^2\right) = f_0\cos\theta \tag{2.36}$$

and

$$2rpA = f_0 \sin\theta \tag{2.37}$$

squaring and adding equations (2.36) and (2.37) we get

$$A^2\left\{\left(\omega^2 - p^2\right)^2 + 4r^2p^2\right\} = f_0^2$$

or

$$A = \frac{f_0}{\sqrt{\left\{\left(\omega^2 - p^2\right)^2 + 4r^2p^2\right\}}} \tag{2.38}$$

Now dividing equation (2.37) by (2.36) we get

$$\tan\theta = \frac{2rp}{\omega^2 - p^2}, \tag{2.39}$$

which gives the phase difference between the action of deriving force and response of the oscillator. For a given oscillator the phase difference θ depends on the frequency of the driving force.

Hence the solution given in equation (2.35) can be given as

$$y = \frac{f_0}{\sqrt{\left\{\left(\omega^2 - p^2\right)^2 + 4r^2p^2\right]}}\sin(pt-\theta) \tag{2.40}$$

Thus the complete solution of equation (2.33) can be given as

$$y = \frac{f_0}{\sqrt{\left\{\left(\omega^2 - p^2\right)^2 + 4r^2p^2\right\}}}\sin(pt-\theta) + a_0e^{-rt}\sin(pt+\theta) \tag{2.41}$$

Since the second part of above equation exists for a very short time, so the first part becomes more important.

2.4.3 Amplitude of Forced Vibration

From the equation (2.38) it is clear that the amplitude of forced vibration depends on the difference between driving frequency and natural frequency of oscillation (i.e. $\omega^2 - p^2$) variation of amplitude at different conditions are discusses below:

Case-I: when $p << \omega$, i.e. the frequency of driving force is low we have

$$A = \frac{f_0}{\omega^2} = \frac{F_0 / m}{k / m} = \frac{F_0}{k} \tag{2.42}$$

In this case amplitude depends only on the force constant k.

Case-II: ***(Amplitude Resonance)***

When $p = \omega$, i.e. the frequency of driving force is equal to the natural frequency of oscillator. In this case amplitude will be maximum

$$A = \frac{f_0}{\sqrt{\left\{\left(\omega^2 - p^2\right)^2 + 4r^2 p^2\right\}}} = \frac{f_0}{2r\omega}$$

or

$$A_{\max} = \frac{f_0}{2r\omega} \tag{2.43}$$

From equation (2.43) it is clear for low damping (i.e., small r) amplitude is high. If $r = 0$ $A_{\max} = \infty$ which is not possible because r is never zero.

In presence of damping, maximum amplitude does not occur exactly at $p = \omega$, which may be clear from following analysis

A is maximum if denominator of equation (2.38) is minimum

i.e.

$$\frac{d}{dp}\left\{\left(\omega^2 - p^2\right)^2 + 4r^2 p^2\right] = 0$$

or

$$2\left(\omega^2 - p^2\right)(-2p) + 4r^2(2p) = 0$$

or

$$\omega^2 - p^2 = 2r^2$$

or

$$p = \sqrt{\omega^2 - 2r^2} \tag{2.44}$$

In the above equation when r is small p is nearly equal to ω. In ideal case only when $r = 0$, $p = \omega$. Actually the frequence at this condition is known is resonant frequency.

$$\nu_{\text{resonant}} = \sqrt{\left(\frac{\omega^2 - 2r^2}{2\pi}\right)} \tag{2.45}$$

Case-III When $p >> \omega$, i.e. deriving frequency is high we get

$$A = \frac{f_0}{p^2} = \frac{F_0}{mp^2} \tag{2.46}$$

above equation (2.46) indicates that the amplitude decreases as mass and deriving frequency increases.

2.4.4 Resonance

As we have read under forced oscillations that when an external periodic force is applied on an oscillating body having different frequency than the natural frequency of the body, then the body executes forced oscillation with the frequency of the applied force having a small amplitude. If in case the frequency of the external force is equal to the natural frequency of the body, then the amplitude of the forced oscillations of the body becomes very large. This phenomena is known as *resonance.* We can say that the resonance is an extreme case of forced oscillations. The value of maximum amplitude and resonant frequency in given Eqs. (2.43 and 2.45) respectively.

In order to explain the phenomena of resonance, let us consider four simple pendulums A, B, C, and D suspended from a thin cord, as shown in Fig. 2.7. The lengths of pendulums A and C are equal and their frequencies are also equal. The length of B is slightly greater than A, while the length of D is slightly less. When pendulum A is set into oscillations, the pendulums B, C, and D experience a periodic force of frequency equal to that of A, and are set in forced oscillations. It is observed that the pendulums B and D (whose natural frequency is different from A), execute forced vibrations with small amplitude, while the amplitude of forced oscillations of C goes on increasing and becomes equal to the amplitude of A. Now the oscillations of pendulum C are *resonant oscillations*.

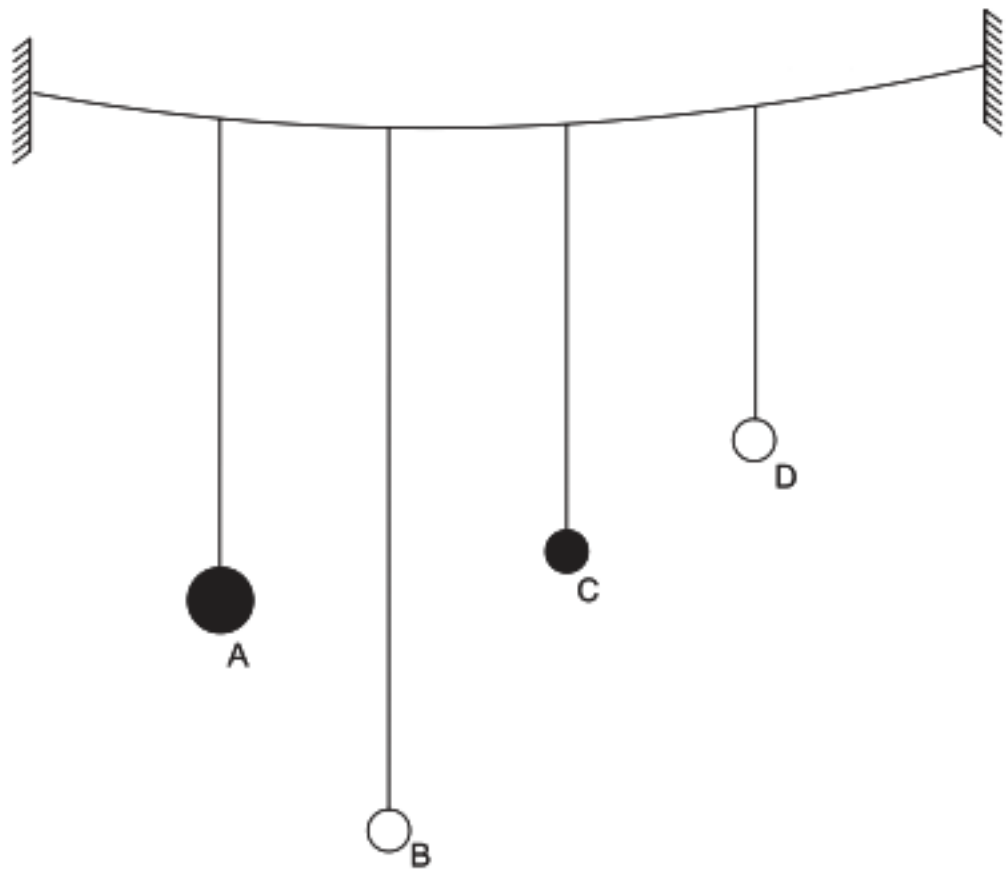


Fig. 2.7 Different pendulums showing resonance

2.4.5 Sharpness of Resonance

At resonance, the amplitude of the oscillating system becomes maximum. It decreases from this maximum value with the change (decrease or increase) of the frequency of the driving force.

The term sharpness of resonance refers to the rate of fall in amplitude with the change of forcing frequency on each side of the resonant frequency.

The analysis of sharpness and flatness of resonance can be done on the basis of graph plotted between A and p corresponding to different conditions given in Case-I, Case-II and Case-III of the section 2.4.3 the expressions of A for different values of p are given in equation (2.42) equation (2.43) and equation (2.46). According to these equations the variation of A with p has been shown in Fig. 2.8.

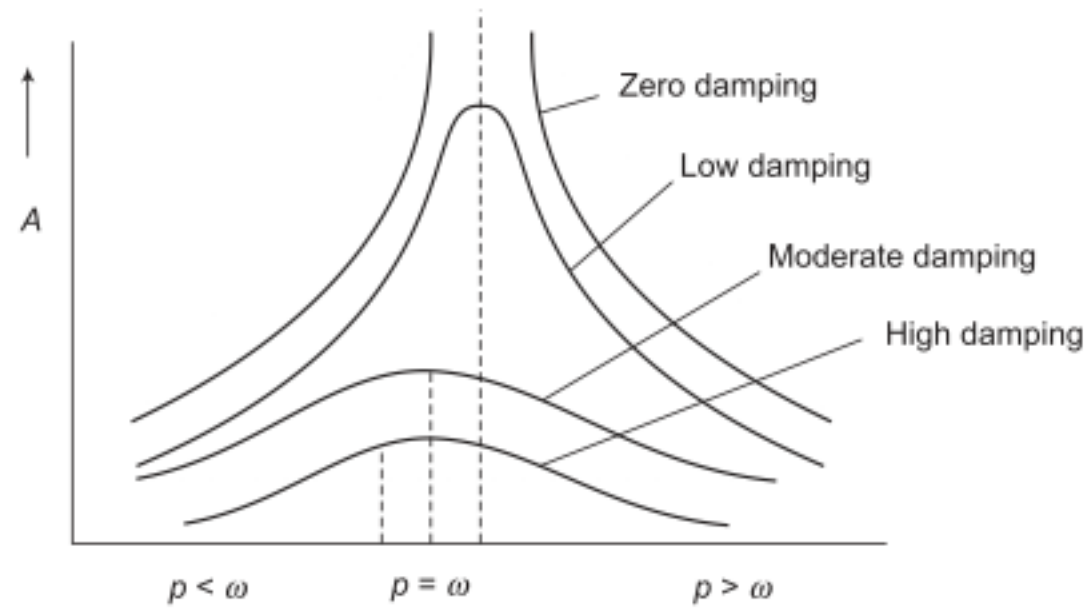


Fig. 2.8 Variation of *A* with *P* at different damping conditions

As it is clear from the above graph, when damping is low, the response (amplitude) fall is very rapid on other side of resonant frequency and we say that the resonance is sharp. On the other hand, for high damping the resonance falls off very slowly on either side of resonant frequency and the resonance is said to be flat. Thus the resonance is flat or sharp according to the damping when the oscillating system is large or small.

Familiar example of a flat or sharp resonance are the resonance of an air column and a sonometer wire with a turning fork respectively. Due to its large damping the air column the air column responds with tuning fork respectively. Due to its large damping the air column responds with tuning fork over a fairly wide range in the vicinity of resonance. Thus in this case it is very difficult to point out exact position of resonance and it is said to be flat. But in case of sonometer damping is small so the resonance is sharp and hence to have the resonant point the length of the wire is adjusted very precisely.

2.4.6 Average Power in Forced Vibration

The steady state oscillations in forced vibration is maintained on the cost of absorbed energy from the driving force which is spent in doing work against the damping force in each cycle.

In order to calculate the power supplied by external force, suppose that the force $F_0 \sin pt$ at any instant. Work done for the force $F_0 \sin pt$ with the displacement dy in time dt will be $F_0 \sin pt\, dy$. Thus the average power supplied in each cycle can be given as

$$P_{av} = \frac{1}{T}\int_0^T F_0 \sin pt \frac{dy}{dt} dt \tag{2.47}$$

we know $y = A\sin(pt - \theta)$

or $$\text{velocity} = \frac{dy}{dt} = Ap\cos(pt - \theta)$$

$$\therefore \qquad P_{av} = \frac{1}{T}\int_0^T F_0 \sin pt \cdot Ap\cos(pt-\theta)dt$$

$$= \frac{F_0 Ap}{T}\int_0^T (\sin pt\cos pt\cos\theta + \sin pt\sin pt\sin\theta)dt$$

$$= \frac{F_0 Ap}{T}\int_0^T (\sin pt\cos pt\cos\theta + \sin^2 pt\sin\theta)dt$$

$$= \frac{F_0 Ap}{2}\sin\theta$$

Because $\frac{1}{T}\int_0^T \sin pt\cos pt\, dt = 0$ and $\frac{1}{T}\int_0^T \sin^2 pt\, dt = 1/2$

using equation (2.39) $\sin\theta = \frac{2rp\,A}{f_0}$ where $r = \frac{b}{2m}$ and $f_0 = \frac{F_0}{m}$

Now $$P_{av} = F_0 A^2 p^2 \frac{b}{2m}\frac{m}{F_0}$$

or $$P_{av} = \frac{1}{2}bA^2p^2 \tag{2.48}$$

2.4.7 Power Dissipated

The power of deriving force is not stored in the forced oscillation but it is dissipated in doing work against damping forces $b\frac{dy}{dt}$. The work done against the damping force for displacement dy in time dt is $b\frac{dy}{dt}dy$.

Now the dissipated power can be given as

$$dP_{\text{dissipated}}\frac{dw}{dt} = b\frac{dy}{dt}.\frac{dy}{dt}$$

Thus the average power dissipated in each cycle

$$\left(P_{\text{dissipated}}\right)_{av} = \frac{1}{T}\int_0^T b\left(\frac{dy}{dt}\right)^2 dt$$

or $$\left(P_{\text{dissipated}}\right)_{av} = \frac{1}{T}\int_0^T bA^2p^2\cos^2(pt-\theta)dt$$

or $$\left(p_{\text{dissipated}}\right)_{av} = \frac{1}{2} bA^2 p^2 \tag{2.49}$$

$$\left\{\text{because } \frac{1}{T}\int_0^T \cos^2(pt-\theta) = \frac{1}{2}\right.$$

From equation (2.48) and equation (2.49) it is clear that

$$P_{av} = \left(P_{\text{dissipated}}\right)_{av}$$

which prove the initial statement that the power of deriving force is used against the damping forces. Maximum value of absorbed power can be given by the
Maximum value of A. From equation (2.30) and equation (2.43) we know

$$A = \frac{f_0}{\sqrt{\left\{\left(\omega^2 - p^2\right)^2 + 4r^2p^2\right\}}}$$

and $$A_{\max} = \frac{f_0}{2rp}$$

Thus $$\left(P_{av}\right)_{\max} = \frac{1}{2} b \frac{p^2 f_0^2}{4r^2p^2} = \frac{1}{2}\frac{2rmf_0^2}{4r^2} = \frac{mf_0^2}{4r}$$

or $$\left(P_{av}\right)_{\max} = \frac{mf_0^2}{4r} \tag{2.50}$$

2.4.8 Quality Factor in Terms of Bandwidth

The graph between average power (P_{av}) and the frequency of driving force p is known as absorption curve of oscillator; which is shown in Fig. (2.9). The plot represents the response of the oscillator. The peak occurs at the frequency $p = \omega$ when the power absorbed by the system from the driving force is maximum. The sharpness of its peak at resonance is also determined by the value of damping constant (b or r).

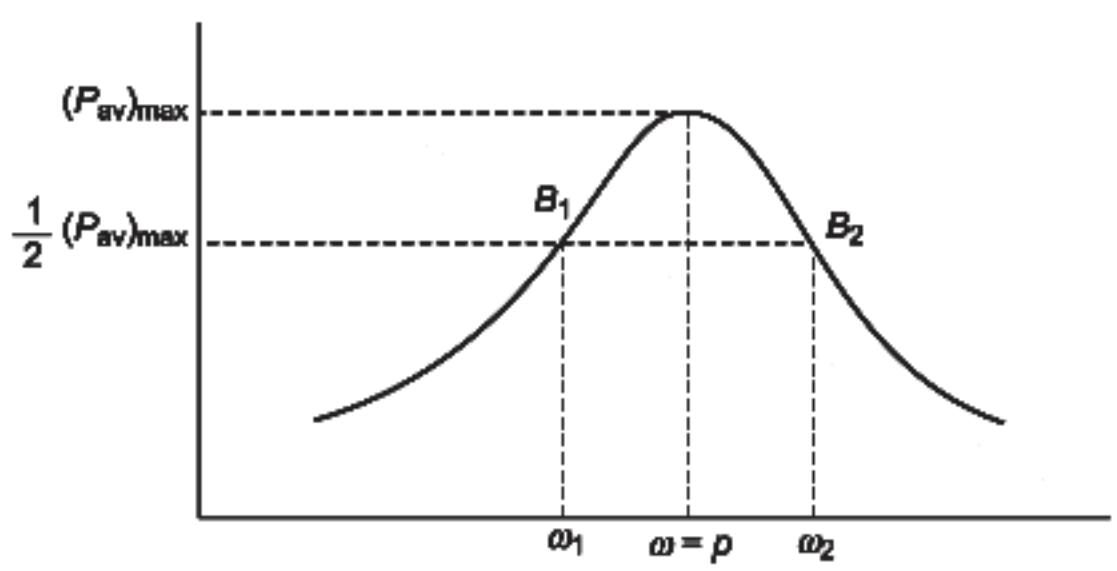


Fig. 2.9 Absorption curve of the oscillator

In the above absorption curve at B_1 and B_2 are two points on the graph where average power falls to half of the maximum power, which are known as half power points.

If ω_1 and ω_2 are the corresponding frequencies then $(\omega_1 - \omega_2)$ is called 'bandwidth' of resonance. Bandwidth can also be defined as the difference of frequencies on the absorption curve at which average power drops to half of its maximum power. The sharpness of resonance can also be measured by this quantity. If bandwidth is small, the resonance is said to be sharp and if it is large the resonance is said to be flat.

Calculating of Bandwidth $(\omega_1 - \omega_2)$: At ω_1 and ω_2 the average power $(P_{av}) = \frac{1}{2}(P_{av})_{\max}$

or $$\frac{mf_0^2 rp^2}{\left(\omega^2 - p^2\right)^2 + 4r^2p^2} = \frac{1}{2}\frac{mf_0^2}{4r}$$

or $$\left(\omega^2 - p^2\right)^2 = 4r^2p^2$$

or $$\left(\omega^2 - p^2\right)^2 = \pm 2rp$$

or $$(\omega - p) = \pm\frac{2rp}{(\omega + p)}$$

If $\omega_2 > \omega_1$ then

$$\omega_2 - p = +\frac{2rp}{(\omega + p)}$$

$$\omega_2 = p + \frac{2rp}{(\omega + p)}$$

and $$\omega_1 - p = -\frac{2rp}{(\omega + p)}$$

$$\omega_1 = p - \frac{2rp}{(\omega + p)}$$

Now $$\omega_2 - \omega_1 = \frac{4rp}{(\omega + p)}$$

If $\omega \simeq p$, $\omega_2 - \omega_1 = 2r$

Then $$(\omega_2 - \omega_1) = \frac{1}{\tau} \tag{2.51}$$

where $\tau = \frac{1}{2r}$ is the relaxation time.

2.4.9 Quality Factor

In terms of bandwidth quality factor of force oscillator is defined as

$$Q = \frac{\omega}{\omega_2 - \omega_1} \tag{2.52}$$

where ω is frequency at which $(P_{av})_{max}$ have peak value while ω_1 and ω_2 are the frequencies at which $(P_{av})_{max}$ drops to half of its maximum value on both the sides of ω as shown in Fig. 2.7.

In terms of Q-value the sharpness of resonance can be stated as:

For low values of $(\omega_2 - \omega_1)$, Q is high, the resonance in the system is very sharp and if bandwidth is large i.e. Q is low. The resonance is flat.

Since the bandwidth is defined as

$\omega_2 - \omega_1 = \frac{1}{\tau}$ so the equation (2.52) can be given as

$$Q = \frac{\omega}{1/\tau} = \omega\tau$$

or $$Q = \frac{\omega}{2r}$$

or $$Q = \frac{m\omega}{b} \tag{2.53}$$

From equation (2.53) it is clear that sharpness of resonance is directly proportional to the natural frequency of the system and inversely proportional to the damping constant.

SOLVED EXAMPLES

Example 2.1

A 1 gm particle is subjected to an elastic force of 20 dynes/cm and a frictional force of 3 dynes/cm s^{-1}. If it is displaced through 2 cm and then left. Find whether the resulting motion is oscillatory or not. If it is oscillatory then, find its period.

Solution

The equation of motion of a particle subjected to elastic force and friction force is.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

or $$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = 0 \text{ where } 2r = \frac{b}{m}, \omega^2 = \frac{k}{m}$$

Where $r < \omega$ above equation represents oscillatory motion which time period can be given as

$$T = \frac{2\pi}{\sqrt{\left(\omega^2 - r^2\right)}}$$

Given that $m = 1$ gm, $k = 20$ dynes/cm, $b = 3$ dynes/ cm s^{-1}

$$r = \frac{b}{2m} = \frac{3}{2\times 1} = 1.5,\ \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{1}} = \sqrt{20} = 4.47$$

From above calculation $r < \omega$ (1.5 < 4.47). Therefore the motion is oscillatory.

Now Time period T is given as

$$T = \frac{2\pi}{\sqrt{\left(\omega^2 - r^2\right)}} = \frac{6.283}{\sqrt{20 - 2.25}} = \frac{6.283}{4.213} = 1.495$$

Example 2.2

A damped harmonic oscillator starts from rest has an initial amplitude of 10 cm, reduces to 1 cm after 50 complete oscillations each of period 1 sec. Find the logarithmic decrement and quality factor of the system.

Solution

In the damped harmonic oscillator logarithmic decrement can be given as

$$\lambda = \frac{2.303}{p}\log_{10}\left(\frac{a_1}{a_p}\right)$$

where a_1 = initial amplitude

p = number of oscillator

a_p = amplitude after p vibrations

Now $$\lambda = \frac{2.303}{50}\log_{10}\left(\frac{10}{1}\right) = 0.046$$

Quality factor $Q = \frac{\pi}{\lambda} = \frac{3.14}{0.046} = 68.26$

Example 2.3

The amplitude of a damped oscillator of frequency 100 c/s is reduced to $\frac{1}{10}$ of its initial value after 1000 cycles. Calculate

(i) damping constant (ii) relaxation time (iii) its quality factor (iv) time in which its energy reduced to $\frac{1}{100}$ of its initial value.

Solution

At any instant amplitude of damped harmonic oscillator is given as $a_0 e^{-rt}$, where r is damping constant

(i) If a is initial amplitude and $\frac{a}{10}$ be the amplitude after 10 second then

$$\frac{a}{10} = ae^{-10r} \quad or \log e^{10} = 10r$$

or $$r = \frac{2.3\log_{10} 10}{10} = 0.23$$

(ii) Relaxation time : $= \tau = \frac{1}{2r} = \frac{1}{2\times 0.23} = 2.175$

(iii) Quality factor $= Q = \omega\tau = 2\pi n\tau = 2\times 3.14\times 100\times 2.17$

or $Q = 1362.76$

(iv) The energy of the oscillator is $E_0 e^{-2rt}$

If t is the time after which the energy reduced to $\frac{E_0}{100}$.Then

$$\frac{E_0}{100} = E_0 e^{-2rt} \quad or 10^{-2} = e^{-2rt}$$

$$2\log 10 = 2rt$$

$$2.3\times 2\log_{10} 10 = 2rt$$

$$4.6 = 2rt$$

or $$t = \frac{4.6}{2r} = \frac{4.6}{2\times 0.23} = 10 \text{ second}$$

Example 2.4

Calculate the average energy stored in a 10 gm mass attached to a spring and vibrating with an amplitude 2 cm in resonances with a periodic force whose frequency is 10 Hz. If the quality factor of the oscillator be 100, how much energy is being dissipated per second?

Solution

During resonance average energy stored

$$= \text{maximum potential energy}$$

$$= \frac{1}{2} m\omega_0{}^2 x_0{}^2$$

$$= \frac{1}{2}\times 0.01\times (2\times 3.14\times 10)^2 \times (0.02)^2$$

$$= 0.0079 \text{ Joule}$$

Now $$Q = \frac{2\pi.(\text{average energy stored})}{\text{energy dissipated per cycle}}$$

∴ Energy dissipated in 1/10 second (i.e. time for one vibration)

$$= \frac{2\pi}{Q} \times \text{average energy stored}$$

$$= \frac{2 \times 3.14}{100} \times 0.0079$$

∴ Energy dissipated per second

$$= \frac{10 \times 2 \times 3.14}{100} \times 0.0079 = 0.0050 \text{ Joule}$$

Example 2.5

In an oscillatory circuit $L = 0.1$ H, $C = 0.0010$ μF. Find the maximum value of the resistance for the circuit to be oscillatory.

Solution

The maximum value of R for the circuit to be oscillatory can be obtained by

$$\frac{R^2}{4L^2} = \frac{1}{LC} \quad or \quad R^2 = \frac{4L}{C}$$

$$R = \sqrt{\left(\frac{4L}{C}\right)} = \sqrt{\left(\frac{4 \times 0.1}{10 \times 10^{-10}}\right)} = 2.0 \times 10^4 \text{ ohm}$$

Thus for the discharge to be oscillatory, R should be less than 2.0×10^4 ohm

Example 2.6

In an experiment the frequency of the driving force is changing at constant amplitude. If the amplitude of forced vibration increases from 0.01 mm at very low frequency to maximum 5 mm when frequency is 100 Hz Calculate:

(a) Relaxation time

(b) Q value of the system

(c) The half width of the resonance curve at half maximum power

Solution

(a) when the driving frequency is low $A = f_0/\omega^2$
the amplitude is maximum i.e., at resonance

$$A_{max} = \frac{f_0}{2r\omega} = \frac{f_0\tau}{\omega}$$

therefore $\frac{A_{max}}{A} = \omega\tau \; or \; \tau = \frac{A_{max}}{A\omega}$

$$A_{max} = 5 \text{ mm} = 0.5 \text{ cm}$$

$$A = 0.01mm = 0.001cm$$

$$\tau = \frac{0.5}{0.001 \times 2 \times \pi \times 100} = 0.796\text{s}$$

(b) $Q = \omega.\tau = 2\pi \times 100 \times 0.796 = 499.88$

$= 500$

(c) Half width of resonance curve

$$= \frac{1}{2}(\omega_2 - \omega_1) = \frac{1}{2\tau}$$

$$= \frac{1}{2 \times 0.796} = 0.628 \text{ rad/s}$$

or $= 0.1$ cycle/s

Example 2.7

Calculate the quality factor and bandwidth of a harmonic oscillator of mass 4 gm driven by a periodic force for which force constant $k = 4 \times 10^4$ dynes/cm and relaxation time $\tau = \frac{1}{4}s$.

Solution

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4 \times 10^4}{4}} = 10^2 = 100/\text{cycle}$$

$$Q = \omega\tau = 100 \times \frac{1}{4} = 25$$

$$\text{bandwidth} = (\omega_2 - \omega_1) = \frac{\omega}{Q} = \frac{100}{25} = 4/\text{second}$$

Example 2.8

A series LCR circuit has L =4 mH, C=0.4 μF and R=20 ohm. Calculate the resonant frequency of the circuit. What is the separation between the two half power points and hence the sharpness.

Solution

Resonant frequency $(\omega) = \frac{1}{\sqrt{LC}}$

$$L = 4 \times 10^{-3}\text{H},\ c = 4 \times 10^{-7}\text{F}$$

Now $\omega = \sqrt{\frac{1}{4 \times 10^{-3} \times 4 \times 10^{-7}}} = \frac{1}{4} \times 10^5$ rad/s

Band width $(\omega_2 - \omega_1) = \frac{R}{L} = \frac{20 \text{ ohm}}{4 \times 10^{-3}\text{H}} = 5 \times 10^3$ rad/s

The sharpness is $\dfrac{\omega}{(\omega_2 - \omega_1)} = \dfrac{0.25 \times 10^5}{5 \times 10^3} = 0.05 \times 100 = 5$

Example 2.9

A circuit contains a resistance of 2 ohm and an induction of 0.50 H, and alternating effective emf of 500 volts at a frequency of 100 Hz applied to it. Find the value of the effective current in the circuit and power factor.

Solution

$$\text{Impedance}(z) = \sqrt{R^2 + p^2L^2} = \sqrt{R^2 + (2\pi n)^2 L^2}$$

$$= \sqrt{(2)^2 + 4 \times (3.14)^2 (100)^2 (0.5)^2}$$

$$= \sqrt{98600}$$

$$= 314 \text{ ohm}$$

$$\text{effective current} = \frac{\text{Effective emf}}{z} = \frac{500}{314} = 1.54 \text{ Amp}$$

$$\text{power factor} = \frac{R}{z} = \frac{R}{\sqrt{R^2 + p^2L^2}} = \frac{2}{314} = \frac{1}{157}$$

2.5 COUPLED VIBRATIONS

A system whose motion and position are characterized by two or more independent quantities is known as system having two or more degrees of freedom. The vibrations of such system which involve more than one vibration system are known as *coupled vibrations*. The nature has many fascinating examples of such systems, such as molecules and elementary particles.

A simple illustration of coupled vibration is provided by the oscillations of a pair of pendulum *CD* and *EF* which have nearly equal mass and length hanging from a thick cord *AB*, as shown in Fig. 2.10. When pendulum *CD* is displaced from its initial position, vibration starts in it. Since *EF* is attached with *CD* through cord *AB*, so the reaction of *EF* on *CD* decreases the energy and hence its amplitude. Thus, the energy of amplitude of *EF* gradually increases at the cost of that of *CD* and finally *CD* comes to momentary rest. Now in this situation *EF* is vibrating and *CD* is experiencing the periodic force to be in vibration at the cost of the energy and amplitude of *EF*. In this way *EF* now becomes the driving system and *CD* is the driven system. This process of exchange of energy continues while the maximum amplitude decreases in successive cycles until they come to rest when the initial excitation energy is dissipated in doing work against the resisting forces.

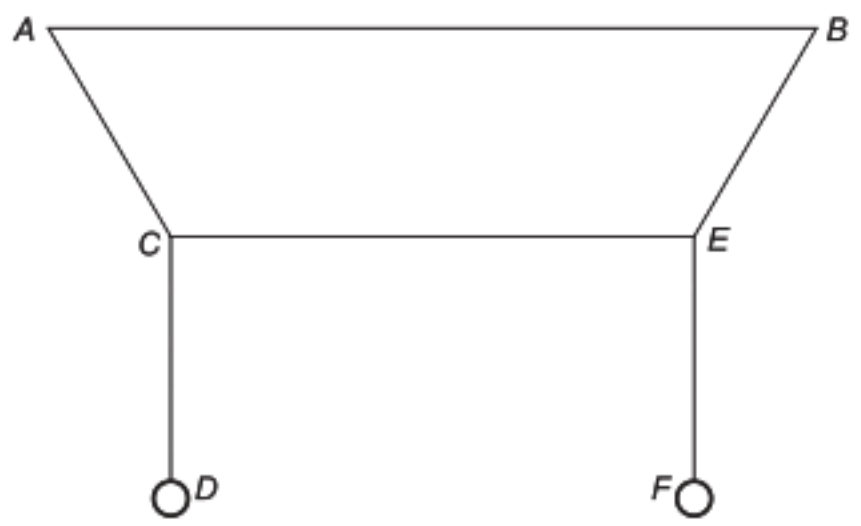


Fig. 2.10 Pendulum *CD* and *EF* showing coupled vibrations

The rate of exchange of energy can be determined according to the degree of coupling between the pendulums. It is observed that when coupling is loose, the rate of transfer of energy is small, but when coupling is tight, the rate of transfer of energy is high.

If the mutual interaction is controlled by the masses or inertia of the system and connecting link does not play any role, then the coupling is known as *mass-controlled* or *inertia coupling*. Examples are two pendulums connected by a cord or two electrical LC circuits coupled by mutual induction. There is another category of coupling which is controlled by the tension of the connecting link between the two systems and the inertial effect is negligible. Such coupling is known as *stiffness coupling*. The pendulums coupled by spring is an example of stiffness coupling. In order to derive the differential equation for coupled oscillations for its analytical treatment, let us consider the example of coupled pendulum. The detail analysis of this system is discussed in the following section.

2.5.1 Coupled Pendulum

This system consists of two identical simple pendulums whose bobs are connected by an elastic spring of stiffness constant K. Both pendulums having equal mass are suspended on a light rod of length l as shown in Fig. 2.11.

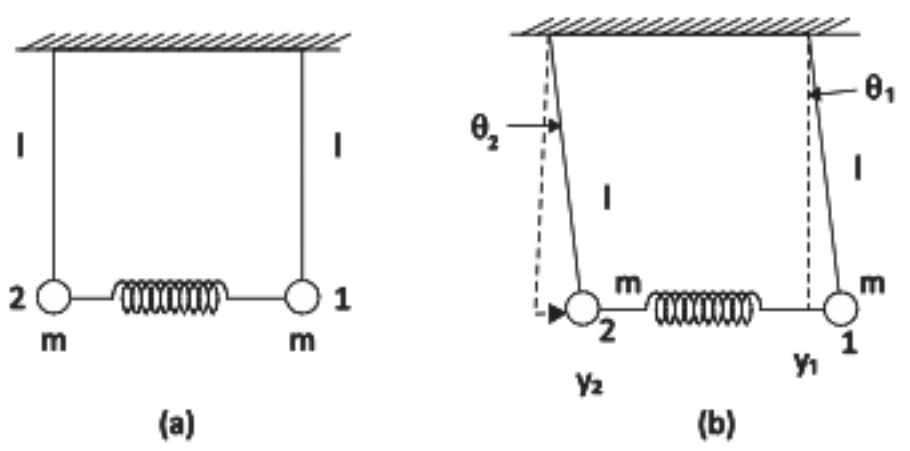


Fig. 2.11 Coupled pendulum.

Suppose we draw 1 aside, holding 2 fixed and then release both from equilibrium position Fig 2.11(b). Pendulum 1 oscillates but its amplitude continuously decreases. Pendulum 2, initially undisplaced, also begins to oscillate with increasing amplitude very soon the amplitude of oscillations of pendulum 1 and 2 becomes equal. The amplitude of 1 continues to decrease becomes zero, while that of 2 continues to increase and becomes equal to the amplitude initially given to 1. Thus, the starting condition is reversed.

Again the motion of 2 is transferred back to 1 and so on. The energy initially given to 1 thus shuttles back and forth between 1 and 2 through the coupling spring.

Let the displacement of the bobs 1 and 2 at any instant be y_1 and y_2, respectively, such that $y_1 > y_2$. The spring is stretched by an amount $y_1 - y_2$ and pulls 1 and 2 with an elastic force. Thus, the net restoring force (gravity + elasticity) on 1 is

$$-mg \sin \theta_1 - k(y_1 - y_2)$$

$$= -mg \frac{y_1}{l_1}(y_1 - y_2)$$

Similarly, the net restoring force on 2 is

$$-mg \sin \theta_2 + k(y_1 - y_2)$$

$$= -mg \frac{y_2}{l} + k(y_1 - y_2)$$

The elastic forces are negative and positive for pendulum 1 and 2 as it opposes the motion of 1 and helps that of 2.

Therefore, the equations of motion of the two bobs are

$$m\frac{d^2 y_1}{dt^2} = -mg\frac{y_1}{l} - k(y_1 - y_2)$$

and

$$m\frac{d^2 y_2}{dt^2} = -\frac{mg}{l}y_2 + k(y_1 - y_2)$$

We can write the above equations as

$$\frac{d^2 y_1}{dt^2} = -\omega_0^2 y_1 - \omega_0^2 (y_1 - y_2) \tag{2.54}$$

$$\frac{d^2 y_2}{dt^2} = -\omega_0^2 y_2 + \omega_0^2 (y_1 - y_2) \tag{2.55}$$

where $\omega_0 = g/l$ is the natural frequency of each pendulum and $\omega_0^2 = k/m$ is coupling frequency of the spring.

2.5.2 Normal Frequency

In order to calculate the normal frequencies, let us suppose that the system is allowed to vibrate in one of its normal modes with frequency w. The solution of equations (2.54) and (2.55) may be assumed to be $y_1 = A_1 e^{iwt}$ and $y_2 = A_2 e^{iwt}$, where A_1 and A_2 being the amplitudes of oscillations.

Thus,

$$\frac{d^2 y_1}{dt^2} = -\omega^2 A_1 e^{iwt} = -\omega^2 y_1$$

and

$$\frac{d^2 y_2}{dt^2} = -\omega^2 A_2 e^{iwt} = -\omega^2 y_2$$

Substituting these values in equations (2.54) and (2.55), we get

$$-\omega^2 y_1 = -\omega_0^2 y_1 - \omega_0^2 (y_1 - y_2)$$

or
$$(-\omega^2 + \omega_0^2 + \omega_0^2) y_1 = \omega_0^2 y_2 \tag{2.56}$$

and
$$-\omega^2 y^2 = -\omega_0^2 y_2 + \omega_0^2 (y_1 - y_2)$$

or
$$(-\omega^2 + \omega_0^2 + \omega_0^2) y_2 = \omega_0^2 y_1 \tag{2.57}$$

From equation (2.56), we can write

$$\frac{y_1}{y_2} = \frac{\omega_0^2}{(-\omega^2 + \omega_0^2 + \omega_0^2)} \tag{2.58}$$

and from equation (2.57), we get

$$\frac{y_1}{y_2} = \frac{(-\omega^2 + \omega_0^2 + \omega_0^2)}{\omega_0^2} \tag{2.59}$$

From equations (2.58) and (2.59), we get

$$\frac{\omega_0^2}{-\omega^2 + \omega_0^2 + \omega_0^2} = \frac{(-\omega^2 + \omega_0^2 + \omega_0^2)}{\omega_0^2}$$

or

$$(-\omega^2 + \omega_0^2 + \omega_0^2)^2 = (\omega_0^2)^2$$

or

$$-\omega^2 + \omega_0^2 + \omega_0^2 = \pm\omega_0^2 \tag{2.60}$$

from equation (2.60), we can write

$$\omega^2 = \omega_0^2 \tag{2.61}$$

or

$$\omega^2 = \omega_0^2 + 2\omega_0^2 \tag{2.62}$$

or

$$\omega^2 = \omega_0^2 + \frac{2k}{m} \tag{2.63}$$

Thus, the normal frequencies would be either

$$\omega_0 \quad \text{or} \quad \left(\omega_0^2 + \frac{2k}{m}\right)^{\frac{1}{2}}$$

ω_0 is called lower mode frequency and $\left(\omega_0^2 + \frac{2k}{m}\right)^{\frac{1}{2}}$ is higher mode frequency.

The ratio of the displacement of two bobs in different modes determine the configuration of the system.

From equation (2.58), we have $\frac{y_1}{y_2} = \frac{\omega_0^2}{(-\omega^2 + \omega_0^2 + \omega_0^2)}$

In the lower mode $\frac{y_1}{y_2} = 1$, i.e., $y_1 = y_2$

In this mode, both the bobs have equal displacements in the same direction. It means the spring remains unstretched, i.e., stiffness of the coupling has no effect.

In the higher mode, when $\omega^2 = \omega^2 + 2k/m$

$$\frac{y_1}{y_2} = -1, \quad \text{i.e.,} \quad y_1 = -y_2$$

In this mode, the bobs have equal but opposite displacements. The spring is stretched and compressed alternately, which causes the greater restoring force.

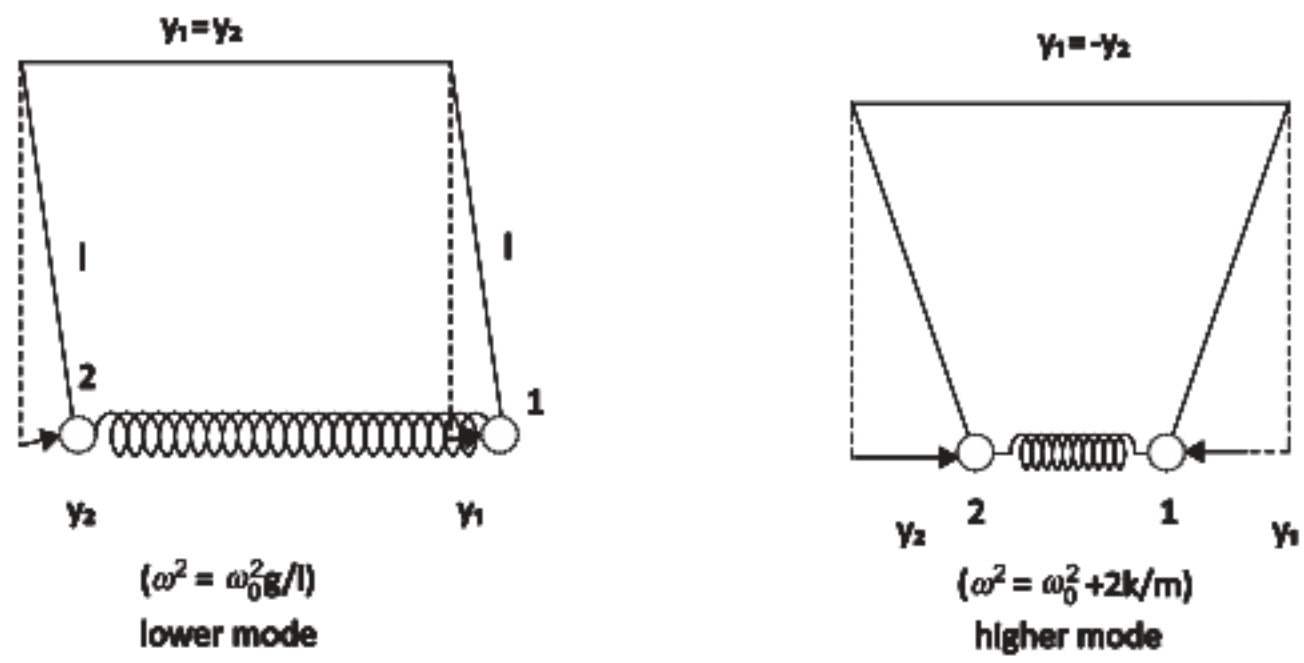


Fig. 2.12 Lower and higher modes of vibration of coupled pendulum

NORMAL COORDINATES

The oscillation of coupled system can be described in terms of normal coordinates which have been discussed below.

The equation of motion of two identical pendulum equation (2.54) and (2.55) are coupled equations so they cannot be solved independently. They are solved simultaneously.

Adding equations (2.54) and (2.55), we get

$$\frac{d^2}{dt^2}(y_1 + y_2) + \omega_0^2(y_1 + y_2) = 0 \tag{2.64}$$

On subtracting (2.54) and (2.55), we get

$$\frac{d^2}{dt^2}(y_1 + y_2) + (\omega_0^2 + 2\omega_0^2)(y_1 + y_2) = 0 \tag{2.65}$$

Let us now choose two new coordinates, Y_1 and Y_2, such that

$$Y_1 = y_1 + y_2 \tag{2.66}$$

$$Y_2 = y_1 + y_2 \tag{2.67}$$

Now using (2.66) and (2.67) in (2.64) and (2.65), we get

$$\frac{d^2}{dt^2} Y_1 + \omega_0^2 (Y_1) = 0 \tag{2.68}$$

and

$$\frac{d^2}{dt^2} Y_1 + (\omega_0^2 + 2\omega_0^2) Y_2 = 0 \tag{2.69}$$

The above equations are independent of each other and involve only one variable. By solving them independently, we get

$$Y_1 = C \cos(\omega_o t + \phi_1) \tag{2.70}$$

$$Y_2 = D \cos(\omega' t + \phi_2) \tag{2.71}$$

where

$$\omega' = (\omega_0^2 + 2\omega_0^2)^{1/2}$$

Here C and D are constants depending upon the initial conditions. These equations represent simple harmonic oscillations of frequency ω_o and ω' respectively. The coordinates Y_1 and Y_2 are called the normal coordinates. The normal coordinate Y_1 oscillates with frequency $\omega_0 \left(= \sqrt{(g/l)}\right)$ and Y_2 oscillates with frequency

$$\omega' = (\omega_0 + 2\,\text{k/m})^{1/2} \tag{2.72}$$

Thus, the general motion of coupled oscillations may be considered as the superposition of two normal mode of vibrations. Hence the concept of normal mode is of fundamental importance in the study of coupled system.

2.6 ATTENUATION COEFFICIENT OF A VIBRATING SYSTEM

We know that the energy of an oscillator is proportional to the square of its amplitude. In case of damped oscillator, the amplitude decays exponentially with time as e^{-rt}, which means that energy will decay as $(e^{-rt})^2 = e^{-2rt}$. So, the decay rate of energy depends on $2r$ which is the damping force per unit mass at an instant when the vibrating body is moving with a unit velocity. The following three characteristics may give the attenuation of a vibrating system completely.

2.6.1 Logarithmic Decrement

It measures the rate at the amplitude of vibration decays as shown in Fig. 3. Let A and B are two successive maxima corresponding to displacements a_n and a_{n+1} and separated by a time period T/2. The amplitude of oscillatory motion shown in Fig. 2.13 is given by $a = a_0 e^{-kt} = a_0 e^{-t/2\tau}$ where a_0 is the amplitude in absence of damping. In the present case of damping, the amplitude decreases exponentially with time.

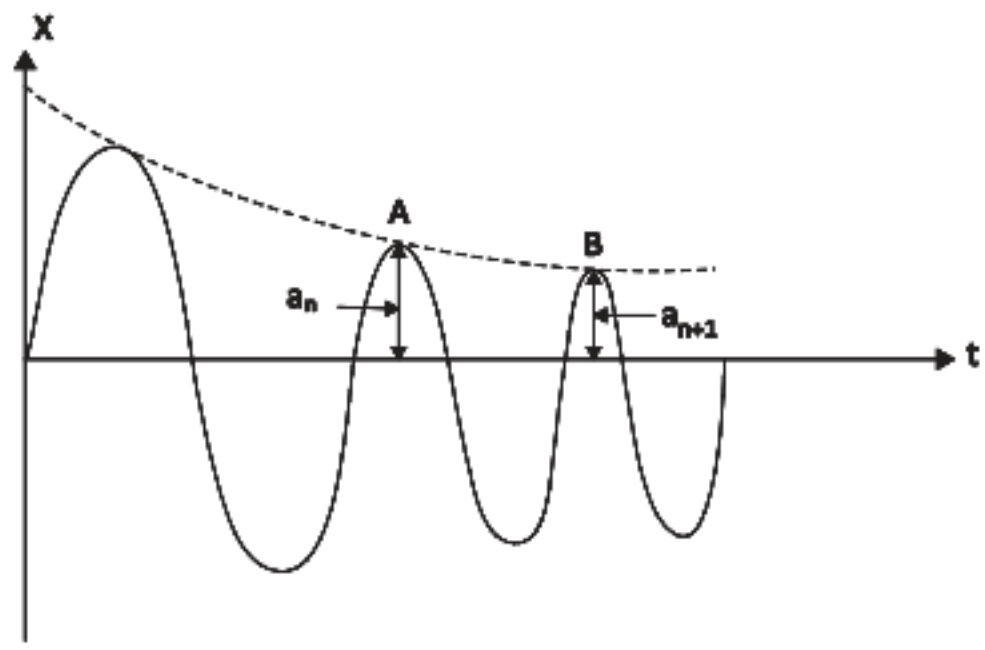


Fig. 2.13 Damping in a vibration.

Now

$$a_n = a_0 e^{-kt} \quad \text{and} \quad a_{n+1} = a_0 e^{-k(t+T/2)}$$

$$\frac{a_n}{a_{n+1}} = \frac{e^{-kt}}{e^{-k(t+T/2)}} = e^{kT/2} = d \tag{2.73}$$

where, a is constant of motion

d is called the decrement, the indicating for the reduction in amplitude.

Now, $\log_e d = kT/2 = \lambda$ (2.74)

This quantity λ is called the logarithmic decrement and is equal to the ratio of natural logarithm of the ratio of two successive amplitudes of vibration.

2.6.2 Relaxation Time (Modulus of Decay)

An alternate method of describing the attenuation of damped vibration is by means of time required to elapse for the amplitude to decay to $\frac{1}{e}$ of its original value a_0

At time t, the amplitude is given by

$$a_t = a_0 e^{-rt}$$

for $t = 2\tau$

$$a_t = a_0 e^{-1}$$

$\tau = \frac{1}{2r}$ is called relaxation time or modulus of decay.

2.6.3 Quality Factor Q

To represent the efficiency of an oscillating system, a term known as quality factor Q, is widely used. The quality factor of an oscillator is defined as 2π times the ratio of an energy stored to the average energy loss per period i.e.

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy loss per period}}$$

$$= \frac{2\pi E}{PT} = \frac{E\omega}{P} \left[\because \frac{2\pi}{T} = \omega\right]$$

or

$$Q = \omega\tau \left[\because P = \frac{E}{\tau}\right] \tag{2.75}$$

Evidently, Q is a dimensionless quantity.

In case of low damping $Q = \omega_c \tau$

But

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \tau = \frac{1}{r}$$

so

$$Q = \frac{1}{r}\sqrt{\frac{k}{m}}$$

Thus, high value of Q means that the damping of the oscillating system is low. This Q is a measure of the extent to which oscillator is free from damping. For an undamped oscillator $r = 0$ so that Q is infinite.

The energy of the oscillation ($E = E_0 e^{-t/\tau}$) decreases to e^{-1} of its initial value in the time τ and in this time oscillator performs $\frac{\omega_0 \tau}{2\pi} = \frac{Q}{2\pi}$ oscillations.

SOLVED EXAMPLES

Example 2.10

If the quality factor of an undamped tuning fork of frequency 256 is 10^3, calculate the time in which its energy is reduced to $\frac{1}{e}$ of its energy in the absence of damping. How many oscillations the tuning fork will make in this time?

Solution

In case of damped harmonic oscillator, the energy at any time t is given by

$$E = E_0 e^{-2rt} = E_0 e^{-t/\tau}$$

where E_o is the initial energy in absence of damping and τ is the relaxation time

when $t = \tau$

$$E = \frac{E_0}{e}$$

Also, quality factor Q is given by

$$Q = \omega_0 t$$

$$t = \frac{Q}{\omega_0} = \frac{10^3}{2\pi \times 256} = \frac{10^3}{1528.2} = 0.63 \text{ sec}$$

So, the energy is reduced $\frac{1}{e}$ times to its initial value in 0.63 second. The number of oscillations made by tuning fork in time t will be $\left(\frac{\omega_0}{2\pi}\right)\tau$.

Example 2.11

In an experiment the frequency of a sinusoidal driving force is changed while its amplitude is kept constant. If the amplitude of forced vibration increases from 0.010 mm at very low frequencies to a maximum value of 5 mm when the frequency is 200 Hz, calculate:

(a) Relaxation time
(b) Q-value of the system
(c) The half width of the resonance curve at half max power

Solution

(a) When the driving frequency is low

$$A = \frac{f_0}{\omega^2}$$

when the amplitude is maximum i.e. at resonance

$$A_{max} = \frac{f_0}{2r\omega} = \frac{f_0\tau}{\omega}$$

Therefore

$$\frac{A_{max}}{A} = \omega\tau \text{ or } \tau = \frac{A_{max}}{A}$$

Here

$$A_{max} = 5 \text{ mm} = 0.5 \text{ cm}$$
$$A = 0.01 \text{ mm} = 0.001 \text{ cm}$$
$$\omega = 2\pi \times 200$$
$$\tau = \frac{0.5}{0.001 \times 2\pi \times 200} = 0.40 \text{ second}$$

(a) $Q = 2\omega\tau = \pi \times 200 \times 0.40 = 502.4$

(b) Half width of resonance curve

$$= \frac{1}{2}(\omega_2 - \omega_1) = \frac{1}{2\pi}$$

$$= \frac{1}{2 \times 0.4} = 1.25 \text{ radian/second}$$

$$= 0.2 \text{ cycle/second}$$

Example 2.12

Calculate the quality factor and bandwidth of a harmonic oscillator of mass 1 gm driven by period force. (Given force constant $k = 10^4$ dyne/cm and relaxation time $\tau = \frac{1}{2}$ second)

Solution

We know that

$$\omega = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{10^4}{1}\right)^{1/2} = 10^2 = 100/\text{cycle}$$

$$Q = \omega\tau = 100 \times \frac{1}{2} = 50$$

Example 2.13

In case of forced harmonic oscillator, the amplitude of vibration increases from 0.02 mm at very low frequencies to a value 5 mm at the frequency 100 Hz. Find: (a) Q-factor of the system (b) damping constant k and relaxation time, and (c) half width of resonance curve.

Solution

(a) $Q = \omega_0\tau = \frac{\omega_o}{2k} = \frac{A_{\max}}{A \text{ at zero or very low frequency}}$

$$= \frac{5}{0.02} = 250$$

(b) Resonant frequency = 100 Hz, $\omega_0 = 2\pi \times 100$

$$\tau = \frac{Q}{\omega_0} = \frac{250}{2\pi \times 100} = 0.4 \text{ second}$$

and

$$k = \frac{1}{2\tau} = \frac{1}{2 \times 0.4} = 1.25 \text{ per seond}$$

(c) Half width $\Delta p = \sqrt{3k} = \sqrt{3} \times 1.25 = 2.2$ rad/s

Example 2.14

A damped vibrating system starting from rest reaches a first amplitude of 500 mm, which reduces to 50 mm after 100 oscillations each of period 2.3 seconds. Find the damping constant and correction for the first displacement for damping.

Solution

The amplitude of damped vibrations, as we know, is given by

$$y = y_0 e^{-kT} \quad \ldots(i)$$

Here, time period for 1 oscillation, i.e., T = 2.3 sec.

First amplitude of the system will be at T/4 and the amplitude after 100 oscillation, i.e., 201th amplitude will occur at (100 T + T/4) time. Hence,

$$a_1 = a_0 e^{-kT/4} \quad \text{and} \quad a_0 e^{-k(100\,T + T/4)}$$

$$\therefore \quad \frac{a_{201}}{a_1} = e^{-100kT} \quad \text{or} \quad \frac{50}{100} = e^{-100 \times 2.3k}$$

$$\text{or} \quad 10^{-1} = e^{-230k} \quad \text{or} \quad \log_e 10 = 230k$$

$$\therefore \quad k = \frac{2.3}{230} = 0.01$$

$$\therefore \quad a_1 = a_0 e^{-kT/4} \quad \therefore\ 500 = a_0 e^{-0.01(23/4)}$$

$$\text{or} \quad a_0 = 500\, e^{0.023} \text{ or } \log_{10} a_0 = \log_{10} 500 + \frac{0.023}{4} \cdot \frac{1}{2.3}$$

$$\text{or} \quad \log_{10} = 2.669 + 0.0025 = 2.7015$$

$$\text{when,} \quad a_0 = 502.9 \text{ mm}$$

Thus, in absence of the damping the amplitude of motion would have been 502.9 mm.

Example 2.15

The time period of pendulum is 2 sec. If the angular retardation due to air is 0.04 times the angular velocity of the pendulum and the original amplitude is 1^0, calculate the amplitude after 10 oscillations.

Solution

The amplitude of damped oscillations at any time is given by

$$y = y_0 e^{-kT} \quad \ldots(i)$$

where y_0 is the initial amplitude and k, the damping constant $2k$, the frictional retardation per unit mass per unit velocity.

Here $y_0 = 1°$, $2k = 0.04$ and $t = 2$ sec

Time required for 10 oscillations = 10 × 2 = 20 sec.

Substituting these values in relation (i), we get

$$y = 1 \times e^{-0.02 \times 20}$$

$$\log_{10} y = 0 - \frac{0.4}{0.3} = 00.1739 = \bar{1}.8261$$

$$y = 0.67° = 0.67 \times 60' = 40.2'$$

Example 2.16

In the suspension of a galvanometer coil exerts a restoring couple 5×10^{-5} n-m/rad the period is 6.28 seconds and the amplitude decreases to 1/10 of its original value in 92 seconds. Find the value of I, γ, and Q factor.

Solution

Period T = $\dfrac{2\pi}{\sqrt{w_0^2 - k^2}} = 6.28$ or $\omega_0^2 = k^2 = 1$

Now $\dfrac{a}{a_0} = e^{-kt}$, $\therefore$ here $\dfrac{1}{10} = e^{-92k}$ or $\log_e 10 = 92\,k$

$$\therefore \qquad k = \frac{2.3}{92} = \frac{1}{40}$$

Substituting this value of k in relation (i), we have

$$\omega_0^2 - \left(\frac{1}{40}\right)^2 = 1 \text{ or } \omega_0 = 1 \text{ nearly}$$

Now $\qquad \omega_0^2 = \dfrac{C}{I} \quad \therefore I = 5 \times 10^{-5}/I$

Thus $\qquad I = 5 \times 10^{-5}$ kg $\times$ m^2

$$\frac{\gamma}{I} = 2k \text{ or } \frac{\gamma}{5 \times 10^{-5}} = 2 \times \frac{1}{40}$$

$\therefore \qquad \gamma = 2.5 \times 10^{-6}$ n-m sec

$$Q = \frac{\omega_0}{2k} = \frac{1}{2 \times \frac{1}{40}} = 20$$

2.7 WAVE MOTION

Wave motion is a basic and most important method by which energy can be transferred from one place to another. There are different forms of energies such as light, heat, sound, X-ray radiation, radio waves, and wireless which are transmitted by wave motion. It is a form of disturbance which travels through the medium. Due to the repeated periodic motion of the particles of the medium from their mean position, the disturbance is handed over from one particle to the next. For example, when a stone is dropped in a pond of water, waves are produced at the striking point of the water surface (Fig. 2.14). In this wave, water particles vibrate up and down from its mean position and waves travel outward from its point of origin.

Fig. 2.14 Ripple at the surface of water due to the striking

In order to transfer the energy through wave motion, the medium must possess negligible frictional resistance, elastic property, and property of inertia.

2.7.1 Characteristics of Wave Motion

There are some important characteristics of wave motion which are as follows:

(i) It is a disturbance produced in the medium due to the repeated periodic motion of the particles of the medium.

(ii) In the wave motion, wave travels in the forward direction while particles of the medium vibrate about their mean positions.

(iii) There is a regular phase change between the particles of the medium.

(iv) The velocity of wave is different from the velocity of the particle. The velocity of wave is uniform, while the velocity of particle is different at different positions.

From the characteristics of wave motion, it is clear that a wave is the periodic variation in space and time.

2.8 WAVE AS PERIODIC VARIATION IN SPACE AND TIME

The speed of a wave in a medium depends only upon the properties of the medium, and it can be described in terms of the periodicity of the vibrations of medium particles in the space. A one-dimensional sinusoidal wave travelling in the +ve x direction is shown in Fig. 2.15. In this figure, the abcissa represents the time and the ordinate represents the corresponding displacement of vibrating particles of the medium. It is clear from the figure that

when $t = 0, y = 0$ (minimum)

at $t = T/4, y = a$ (maximum)

when $t = T/2, y = 0$ (minimum)

at $t = \frac{3T}{4}, y = -a$ (maximum in –ve direction)

when $t = T, y = 0$ (minimum)

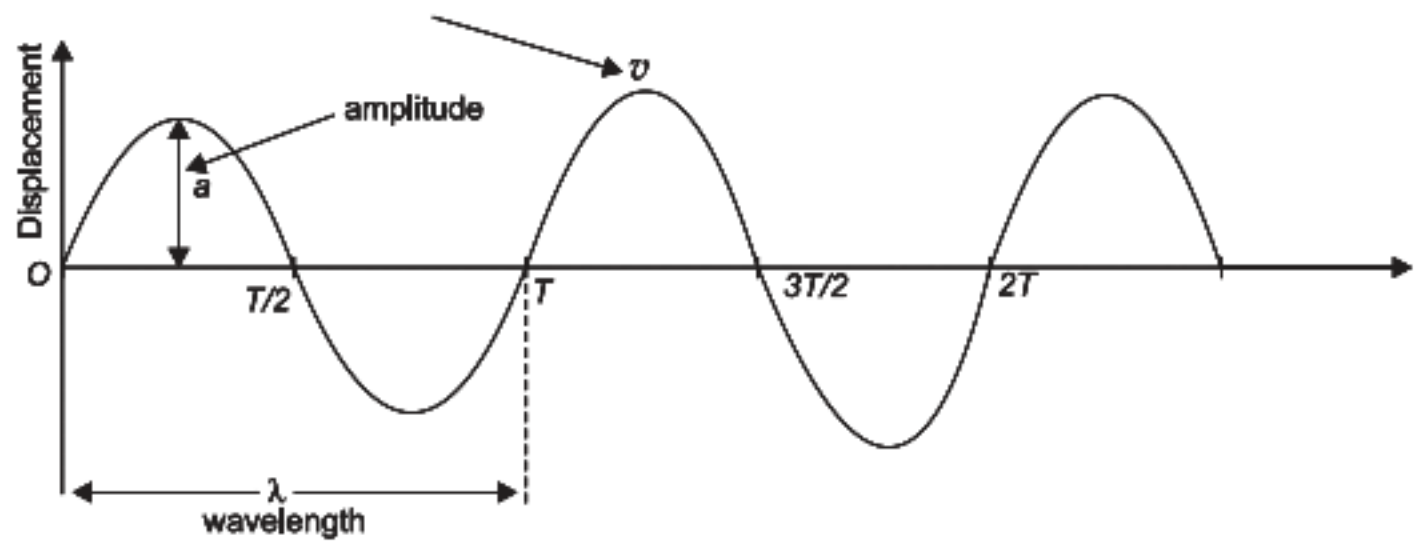


Fig. 2.15 A one-dimensional sinusoidal wave travelling to the right with a speed υ

The slope of the curve at any point represents the velocity of vibrating particle at that particular instant. The rate of change of slope represents the acceleration of the point. Some important terms related to a wave motion are as follows:

(i) *Time period (T)*: The time taken by the oscillating body to complete one oscillation is known as time period or periodic time.

(ii) *Frequency (n)*: It is the number of vibrations performed by the particle in one second. It is given by

$$n = \frac{1}{T}$$

(iii) *Wavelength (λ)*: The distance travelled by the wave in one time period, i.e., in T, is known as wavelength.

(iv) *Amplitude (a)*: The maximum displacement of a vibrating particle from the position of equilibrium is known as amplitude.

(v) *Velocity (υ)*: According to the definition of velocity, we know that

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}}$$

From the definition of wavelength, it is clear that the wavelength is a distance travelled in T second, thus velocity can be expressed as

$$\upsilon = \frac{\lambda}{T}$$

Since $\frac{1}{T} = n$ (frequency),

$$\upsilon = n \cdot \lambda \tag{2.76}$$

2.9 TYPES OF WAVES

The two types of wave motion are as follows:

(i) *Transverse waves*: Transverse waves are those waves in which the particles of the medium vibrate about their mean position perpendicular to the direction of propagation of wave. Light waves are examples of transverse waves. A transverse wave is shown in Fig. 2.12(a).

(ii) *Longitudinal waves*: These are the disturbances of the medium in which medium particles vibrate about their mean position in the direction of the wave propagation. The example of such waves are the sound waves. A transverse and longitudinal wave is shown in Fig. 2.16(a) and Fig. 2.16(b) respectively.

Fig. 2.16 (a) Transverse wave and (b) longitudinal wave

2.10 WAVE EQUATION

From the reference circle showing simple harmonic oscillation of a particle P, the displacement can be given as (see Section 2.2)

$$y = a \sin \omega t$$

If we consider another particle Q at a distance X from P and wave is travelling with a velocity υ from P to Q, the displacement of Q can be given as

$$y = a \sin (\omega t - \phi) \tag{2.77}$$

where ϕ is the phase difference between particles P and Q. Since particle is moving with angle 2π for a path difference λ, we can write

$$\text{Phase difference } \phi = \frac{2\pi}{\lambda} \text{ (Path difference)}$$

Now for path difference x, phase difference can be given as

$$\phi = \frac{2\pi}{\lambda}(x) \tag{2.78}$$

From the definition of angular velocity, we know that

$$\omega = \frac{2\pi}{T} = 2\pi n$$

$$\omega = \frac{2\pi\upsilon}{\lambda} \tag{2.79}$$

[Because from Eq. (2.76), $\upsilon = n\lambda$]

Using the values of ϕ and ω in Eq. (2.77), we get

$$y = a \sin\left(\frac{2\pi\upsilon t}{\lambda} - \frac{2\pi x}{\lambda}\right)$$

or

$$y = a \sin \frac{2\pi}{\lambda}(\upsilon t - x) \tag{2.80}$$

Equation (2.80) represents general equation for a simple harmonic wave. From this wave equation, it is clear that a wave is the periodic variation in space and time.

If a particle is at a distance x in the negative direction, then the equation for displacement can be given similar to Eq. (2.80) as follows:

$$y = a \sin \frac{2\pi}{\lambda}(\upsilon t + x) \tag{2.81}$$

2.11 DIFFERENTIAL EQUATION OF WAVE MOTION

From the general equation of wave motion as given in Eq. (2.80), we know that

$$y = a \sin \frac{2\pi}{\lambda}(\upsilon t - x)$$

Differentiating above equation with respect to time, we get

$$\frac{dy}{dt} = \frac{2\pi a\upsilon}{\lambda} \cos \frac{2\pi}{\lambda}(\upsilon t - x) \tag{2.82}$$

Differentiating Eq. (2.82) with respect to time again, we get

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 a\upsilon^2}{\lambda^2} \sin \frac{2\pi}{\lambda}(\upsilon t - x) \tag{2.83}$$

In order to get the value of compression, differentiating Eq. (2.80) with respect to x, we get

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(\upsilon t - x) \tag{2.84}$$

The value of compression in terms of distance can be obtained by differentiating Eq. (2.84) with respect to x, i.e.,

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(\upsilon t - x) \tag{2.85}$$

From Eqs. (2.82) and (2.84), we can write

$$\frac{dy}{dt} = -\upsilon \frac{dy}{dx} \tag{2.86}$$

and from Eqs. (2.83) and (2.85), we get

$$\frac{d^2y}{dt^2} = \upsilon^2 \frac{d^2y}{dx^2} \tag{2.87}$$

Equation (2.87) represents the differential equation of wave motion.

Now the general equation of wave motion can be given as

$$\frac{d^2y}{dt^2} = k \frac{d^2y}{dx^2} \tag{2.88}$$

where $k = \upsilon^2$ or $\upsilon = \sqrt{k}$

It is obvious from Eq. (2.88) that the value of wave velocity can be calculated if the value of k is known.

2.12 REFLECTION AND TRANSMISSION OF TRANSVERSE WAVES AT A BOUNDARY

When a wave reaches at a point where there is sudden change of impedance in its path, a part of its amplitude is reflected and transmitted depending upon the impedance of the medium. Such behaviour is observed in all types of waves such as acoustic waves, voltage and current waves, and electromagnetic waves. Let us consider two strings of linear mass densities m_1 and m_2 joined together at point $x = 0$ and stretched under the same tension T along the x-axis. Let the wave velocities and impedances concern to the strings at left and right to $x = 0$ be υ_1, υ_2 and Z_1, Z_2, respectively. Here

$$\upsilon_1 = \sqrt{\frac{T}{m_1}},\ \upsilon_2 = \sqrt{\frac{T}{m_2}},\ Z_1 = m_1\upsilon_1, \text{ and } Z_2 = m_2\upsilon_2$$

Let a wave move along with the string and be incident at the junction $x = 0$ where there is discontinuity in the impedance (Fig. 2.17).

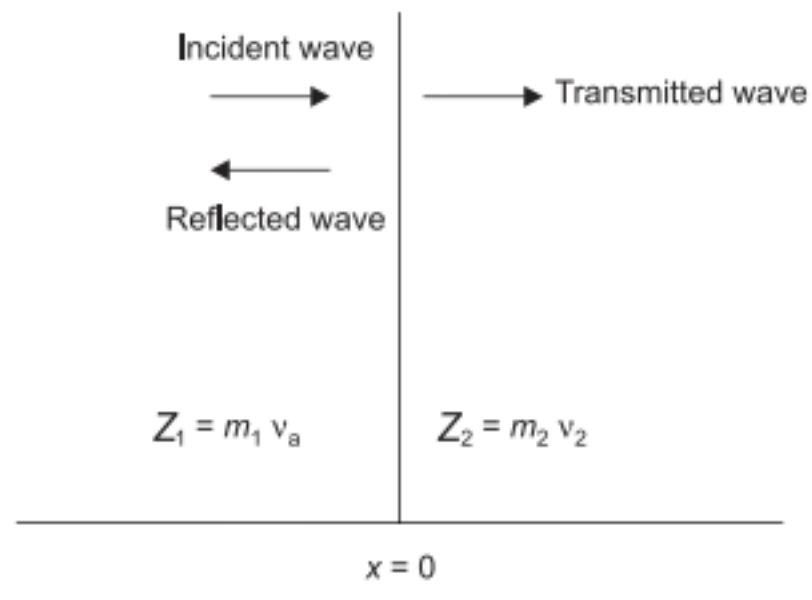


Fig. 2.17 Reflection and transmission of wave at the boundary of two media

When a wave is incident at the boundary (at $x = 0$) of two mediums, a part of it is reflected and a part of it is transmitted into the region of impedance Z_2.

Now the wave equations for incident, reflected, and transmitted waves are as follows:

$$y_i = a_i \sin\left(\omega t - \frac{x}{\upsilon_1}\right) \tag{2.89}$$

$$y_r = a_r \sin\left(\omega t + \frac{x}{\upsilon_1}\right) \tag{2.90}$$

$$y_t = a_t \sin\left(\omega t - \frac{x}{\upsilon_2}\right) \tag{2.91}$$

where all the symbols have their usual meaning and subscripts, i, r, and t stand for incident, reflected, and transmitted waves, respectively. The frequency of wave does not change with the change in medium.

We can calculate reflection and transmission amplitude coefficients considering the following two boundary conditions which must be satisfied at the boundary:

(i) There should be no discontinuity of displacement at $x = 0$, i.e., the displacement should be same immediately to the left and right of $x = 0$, for all times. From this boundary condition, we write

$$y_i + y_r = y_t \tag{2.92}$$

(ii) Continuity of transverse force must be maintained at the boundary, i.e., the forces just on the two sides of boundary must be equal. Transverse force at the boundary is given as

$$T\left[\frac{dy}{dx}\right] \text{ at } x = 0 \tag{2.93}$$

Applying the second boundary condition on the problem, we write

$$T\left[\frac{d}{dx}(y_i + y_r)\right] = T\frac{dy_t}{dx} \text{ at } x = 0 \tag{2.94}$$

Now by putting the values of y_i, y_r, and y_t at $x = 0$ from Eqs. (2.89), (2.90), and (2.91) in Eq. (2.92), we get

$$a_i \sin \omega t + a_r \sin \omega t = a_t \sin \omega t$$

or

$$a_i + a_r = a_t \tag{2.95}$$

Differentiating Eqs. (2.89), (2.90), and (2.91) with respect to x, we get

$$\frac{dy_i}{dx} = -\frac{a_i}{\upsilon_1}\cos\omega\left(t - \frac{x}{\upsilon_1}\right)$$

$$\frac{dy_r}{dx} = \frac{a_r}{\upsilon_1}\cos\omega\left(t + \frac{x}{\upsilon_1}\right)$$

$$\frac{dy_t}{dx} = \frac{-a_t}{\upsilon_2}\cos\omega\left(t - \frac{x}{\upsilon_2}\right)$$

Using the values of $dy_i/dx, dy_r/dx$, and dy_t/dx in Eq. (2.94), we get

$$-T\frac{a_i}{\upsilon_1}\cos\omega t + T\frac{a_r}{\upsilon_1}\cos\omega t = -T\frac{a_t}{\upsilon_2}\cos\omega t$$

or

$$T\left(\frac{a_i - a_r}{\upsilon_1}\right) = T\frac{a_t}{\upsilon_2} \tag{2.96}$$

We know that $\upsilon_1 = \sqrt{\frac{T}{m_1}}$ or $\upsilon_1^2 = \frac{T}{m_1}$

or $$\frac{T}{\upsilon_1} = m_1 \upsilon_1 = Z_1 \tag{2.97}$$

Similarly $$\frac{T}{\upsilon_2} = m_2 \upsilon_2 = Z_2 \tag{2.98}$$

Using the value of T/υ_1 and T/υ_2 in Eq. (2.90), we get

$$Z_1(a_i - a_r) = Z_2\, a_t$$

or $$a_i - a_r = \frac{Z_2}{Z_1} a_t \tag{2.99}$$

Adding Eqs. (2.95) and (2.99), we get

$$2a_i = \left(1 + \frac{Z_2}{Z_1}\right) a_t$$

or $$\frac{a_t}{a_i} = \frac{2}{1 + \frac{Z_2}{Z_1}}$$

or $$\frac{a_t}{a_i} = \frac{2Z_1}{Z_1 + Z_2} \tag{2.100}$$

Subtracting Z_1/Z_2 × {Eq. (2.99)} from Eq. (2.95), we get

$$a_i\left(1 - \frac{Z_1}{Z_2}\right) + a_r\left(1 + \frac{Z_1}{Z_2}\right) = 0$$

or $$\frac{a_r}{a_i} = -\frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} = -\frac{Z_2 - Z_1}{Z_2 + Z_1}$$

or $$\frac{a_r}{a_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{2.101}$$

The term a_t/a_i is known as *transmission coefficient* and is represented by T and a_r/a_i is known as *reflection coefficient* and is represented by R.

Now

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{2.102}$$

and $$T = \frac{2Z_1}{Z_1 + Z_2} \tag{2.103}$$

From the expressions of reflection and transmission coefficients, it is clear that it is independent to the frequency of wave. It depends only on the ratio of impedance of two mediums.

2.12.1 Special Cases

(i) *Impedance matching*: At $Z_1 = Z_2$, $R = 0$ and $T = 1$. It means that in this case there is no reflection, and the entire wave will be transmitted as such.

(ii) *When* $Z_2 > Z_1$: At this condition, R is negative and T is positive. Negative R means, the displacement is changed by π on reflection. Positive T means that there is no change in the phase of transmitted wave.

(iii) *When* $Z_1 > Z_2$: At this condition, R is positive and $R < 1$. It means that reflected wave suffers no change of phase, and reflected amplitude a_r is smaller than a_i.

(iv) *When* $Z_2 = \infty$: At this condition $R = -1$ and $T = 0$. It means nothing will be transmitted but the entire wave will be reflected with a phase change of π.

(v) *When* $Z_2 = 0$: At this condition, $R = 1$ and $T = 2$. This explains the flick at the end of a whip or free-ended string when a wave reaches it.

2.13 REFLECTION AND TRANSMISSION OF LONGITUDINAL SOUND WAVES AT BOUNDARIES

When a sound wave meets at the junction of two mediums having different acoustic impedances, then a part of this wave will be reflected and a part of it will be transmitted. Let us consider a sound wave incident normally at the boundary $x = 0$. Let us also consider that the impedance of the medium at the left of $x = 0$ is $Z_1 = \rho_1 \upsilon_1$, while it is $Z_2 = \rho_2 \upsilon_2$ at its right side, as is shown in Fig. 2.18.

In this figure, ρ_1 and ρ_2 are the densities and υ_1 and υ_2 are the velocities of the wave in the media I and II, respectively.

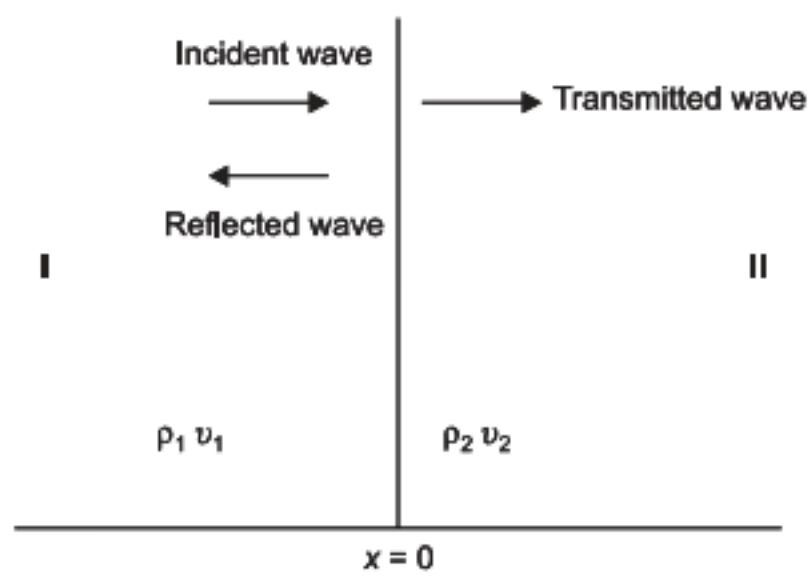


Fig. 2.18 Reflection and transmission of sound wave at the boundaries

The equations of incident, reflected, and transmitted waves are as follows:

$$y_i = a_i \sin \omega\left(t - \frac{x}{\upsilon_1}\right) \tag{2.104}$$

$$y_r = a_r \sin \omega\left(t + \frac{x}{\upsilon_1}\right) \tag{2.105}$$

$$y_t = a_t \sin \omega\left(t - \frac{x}{\upsilon_2}\right) \tag{2.106}$$

In above equations all the symbols have their usual meaning.

In order to calculate the reflection and transmission coefficients, the following two conditions must be satisfied at the boundary:

(i) The particle velocity must be continuous at the junction, i.e.,

$$\left(\frac{dy_i}{dt}\right) + \left(\frac{dy_r}{dt}\right) = \left(\frac{dy_t}{dt}\right) \tag{2.107}$$

(ii) The acoustic excess pressure must be continuous, i.e.,

$$P_i + P_r = P_t \tag{2.108}$$

According to the physical meaning of the above expression, the two media are in complete contact everywhere across the boundary.

If K is the bulk modulus, then by Hooke's law

$$K = \frac{\text{Stress}}{\text{Strain}} = \frac{-P}{(dy/dx)} \tag{2.109}$$

The negative sign indicates that an increase in pressure produces decrease in the volume.

Using Eq. (2.109), we can write

$$P_i = -K_1\left(\frac{dy_i}{dx}\right) = P_1 v_2\left(\frac{dy_i}{dx}\right)$$

$$= P_1 \upsilon_1\left(\frac{dy_i}{dt}\right) \qquad \left[\because\ \upsilon_1 = \sqrt{\left(\frac{K_1}{\rho_1}\right)} \text{ and } \frac{dy_i}{dt} = -\upsilon_1 \frac{dy_i}{dx}\right]$$

Now $$P_i = Z_1\left(\frac{dy_i}{dt}\right) \tag{2.110}$$

Similarly,

$$P_r = -Z_1\left(\frac{dy_r}{dt}\right) \qquad \text{[Because for reflected wave, } Z_1 = -\rho_1 \upsilon_1\text{]} \tag{2.111}$$

and $$P_t = Z_2\left(\frac{dy_t}{dt}\right) \tag{2.112}$$

Using Eqs. (2.110), (2.111), and (2.112) in Eq. (2.108) we get

$$Z_1\left(\frac{dy_i}{dt}\right) - Z_1\left(\frac{dy_r}{dt}\right) = Z_2\left(\frac{dy_t}{dt}\right) \tag{2.113}$$

Putting the value of dy_t/dt from Eq. (2.107) in Eq. (2.113), we get

$$Z_1\left(\frac{dy_i}{dt}\right) - Z_1\left(\frac{dy_r}{dt}\right) = Z_2\left[\left(\frac{dy_i}{dt}\right) + \left(\frac{dy_r}{dt}\right)\right]$$

or

$$(Z_1 - Z_2)\frac{dy_i}{dt} = (Z_1 + Z_2)\frac{dy_r}{dt}$$

$$\frac{dy_r / dt}{dy_i / dt} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right) \tag{2.114}$$

Now by putting the value of $\frac{dy_r}{dt}$ from Eq. (2.107) in Eq. (2.113), we get

$$Z_1\left(\frac{dy_i}{dt}\right) - Z_1\left(\frac{dy_t}{dt} - \frac{dy_i}{dt}\right) = \left(\frac{dy}{dt}\right)$$

$$2Z_1\frac{dy_i}{dt} = (Z_1 + Z_2)\frac{dy_t}{dt}$$

or

$$\frac{dy_t / dt}{dy_i / dt} = \frac{2Z_1}{(Z_1 + Z_2)} \tag{2.115}$$

Equations (2.114) and (2.115) represent the expressions for reflected and transmitted particle velocities, respectively. With the help of Eqs. (2.110), (2.111), and (2.112), we can write

$$\frac{P_r}{P_i} = \frac{Z_1(dy_r / dt)}{Z_1(dy_i / dt)} = -\frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \tag{2.116}$$

$$\frac{P_t}{P_i} = \frac{Z_2(dy_t / dt)}{Z_1(dy_i / dt)} = \frac{Z_2}{Z_1}\frac{(dy_t / dt)}{(dy_i / dt)} = \frac{Z_2}{Z_1}.\frac{2Z_1}{(Z_1 + Z_2)} = \frac{2Z_2}{Z_1 + Z_2} \tag{2.117}$$

Equations (2.116) and (2.117) represent the expressions of reflected and transmitted acoustic pressures, respectively.

2.14 REFLECTION AND TRANSMISSION OF SOUND INTENSITY

The expression for the intensity of sound wave is given as

$$I = \frac{P_m^2}{2\rho v} \qquad \text{(where } P_m \text{ is the pressure amplitude)}$$

$$= \frac{k^2 (dy/dx)^2}{2p\upsilon}$$

or
$$I = \frac{\rho^2 \upsilon^4 (dy/dx)^2}{2\rho\upsilon} = \frac{1}{2}\rho\upsilon\left(\frac{dy}{dt}\right)^2 \qquad \left(\text{Because } P = -k\frac{dy}{dx} \text{ and } \upsilon = \sqrt{\frac{k}{\rho}}\right) \tag{2.118}$$

and we also know that $\frac{dy}{dx} = -\frac{1}{\upsilon}\frac{dy}{dt}$.

Now the intensity coefficients of reflection and transmission are given as

$$\frac{I_r}{I_i} = \frac{(1/2)Z_1(dy_r/dt)^2}{(1/2)Z_1(dy_i/dt)^2} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \tag{2.119}$$

and
$$\frac{I_t}{I_i} = \frac{(1/2)Z_2(dy_t/dt)^2}{(1/2)Z_1(dy_i/dt)^2} = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2} \tag{2.120}$$

A great disparity has been observed between the acoustic impedance of air and water. Usually sound wave energy suffers a total internal reflection at the air–water interface. Due to this reason, we do not hear the noise of sea animals inside water.

SOLVED EXAMPLES

Example 2.15

What is the difference between the oscillations given by

$$y_1 = A\cos\omega t \text{ and } y_2 = A\cos(\omega t + \pi)$$

Solution

The oscillations given by the above equations have same amplitude and frequency, but there is a phase difference of π between them. Both equations represent simple harmonic motion. From the above equations, it is also clear that one particle starts its motion from one extreme, $y = A$, while other starts from the opposite extreme, $y = -A$.

Example 2.16

A particle is executing SHM of amplitude 0.2 cm. If its velocity at the mean position is 5 m/s, then find its frequency.

Solution

We know that $\upsilon = \omega\sqrt{(a^2 - y^2)}$

At mean position, $y = 0$ and velocity is maximum.

$$\upsilon_{max} = a\omega$$

or
$$\omega = \frac{\upsilon_{max}}{a} = \frac{500}{0.2} = 2500$$

Frequency n is given by

$$n = \frac{\omega}{2\pi} = \frac{2500}{2 \times 3.14}$$

$$= 398.089 \approx 398 \text{ c/s}$$

Example 2.17

A point mass of 25 g is executing harmonic oscillations having an amplitude of 5 cm and energy 4×10^{-5} J. Write down equation of motion considering initial phase of 30°.

Solution

The energy of an oscillating particle is given as

$$E = \frac{1}{2}m\omega^2 a^2$$

or
$$\omega = \frac{1}{a}\sqrt{\left(\frac{2E}{m}\right)}$$

Given that $m = 25$ g

$a = 5$ cm

$E = 4 \times 10^{-5}$ J $= 400$ erg

Now
$$\omega = \frac{1}{5}\sqrt{\frac{2 \times 400}{25}} = \frac{1}{5}\sqrt{\frac{800}{25}}$$

$$= \frac{\sqrt{32}}{5} = 1.131 \text{ Hz}$$

Example 2.18

The equation of a progressive wave moving on a string is $y = 4 \sin \pi (0.01x - 2t)$. In this equation, y and x are in centimetres and t in seconds. Calculate amplitude, frequency, and velocity of the wave. If two particles at any instant are situated at 200 cm apart, what will be the phase difference between these particles?

Solution

Given wave equation is

$$y = 4 \sin \pi (0.01x - 2t)$$

This equation can be rewritten as

$$y = 4 \sin 2\pi (0.005x - t) \tag{1}$$

We know that the standard equation of wave is

$$y = a \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \tag{2}$$

Equation (1) can be given as

$$y = 4 \sin 2\pi\left(\frac{x}{200} - \frac{t}{T}\right) \tag{3}$$

Comparing Eqs. (2) and (3), we get

$$a = 4 \text{ cm}$$
$$\lambda = 200 \text{ cm}$$
$$T = 1 \text{ s}$$

and $$n = 1 \text{ c/s}$$

Now the wave velocity is given as

$$\begin{aligned} \upsilon &= n\lambda \\ &= 1 \times 200 \\ &= 200 \text{ cm/s} \end{aligned}$$

If the distance between two particles is 200 cm, then the corresponding phase difference is given by

$$\begin{aligned} \Delta\phi &= \frac{2\pi}{\lambda}(\Delta x) \\ &= \frac{2\pi}{200} \times 200 \\ &= 2\pi = 360° \end{aligned}$$

It means that both particles will be in same phase.

Example 2.19

A wave train is travelling in a gas in positive x direction with an amplitude of 4 cm, velocity 25 m/s, and frequency 100 c/s. Calculate the displacement, particle velocity, and acceleration at a distance of 150 cm from the origin after an interval of 3 s.

Solution

The displacement of a plane progressive wave is given as

$$y = a \sin \frac{2\pi}{\lambda}(\upsilon t - x)$$

Given that $a = 4$ cm, $\upsilon = 25$ m/s $= 2500$ cm/s, $x = 150$ cm, and $t = 3$ s.

We know that

$$\lambda = \frac{\upsilon}{n} = \frac{2500}{100} = 25 \text{ cm}$$

$$y = 4 \sin \frac{2\pi}{25}(2500 \times 3 - 150)$$

$$= 4 \sin 2\pi(294)$$
$$= 4 \sin 580\pi$$
$$= 0$$

The particle velocity is given as

$$\frac{dy}{dt} = \frac{2\pi a\upsilon}{\lambda}\cos\frac{2\pi}{\lambda}(\upsilon t - x)$$

As above $\sin\frac{2\pi}{\lambda}(\upsilon t - x) = 0$

so, $\cos\frac{2\pi}{\lambda}(\upsilon t - x) = 1$

So, $$\frac{dy}{dt} = \frac{2\pi\upsilon a}{\lambda} = \frac{2 \times 3.14 \times 2500 \times 4}{25}$$

$$= 2512 \text{ cm/s} = 25.12 \text{ m/s}$$

Particle acceleration is given as

$$\frac{d^2y}{dt^2} = \frac{-4\pi^2\upsilon^2}{\lambda^2} a \sin\frac{2\pi}{\lambda}(\upsilon t - x)$$

Since, $\sin\frac{2\pi}{\lambda}(\upsilon t - x) = 0$, $\frac{d^2y}{dt^2} = 0$.

Example 2.20

If a straight tunnel is bored through the centre of the Earth and a ball of mass m is dropped into it, then prove that the motion of ball is a simple harmonic motion. Find the periodic time of the motion of ball.

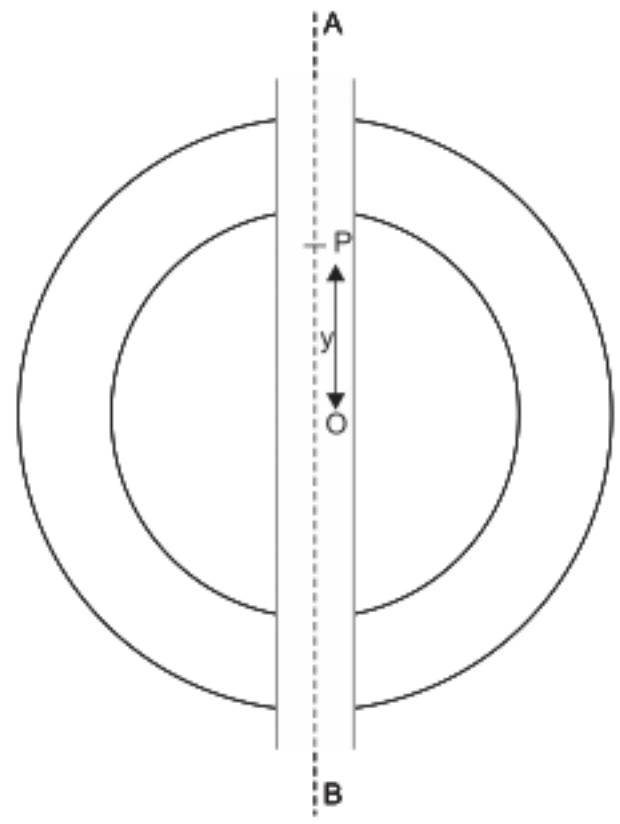
Solution

Let us consider O as the centre of the Earth, R its radius, and ρ its density. AB is the tunnel bored through the centre and a ball of mass m is dropped in this tunnel, as is shown in the figure. Mass of the Earth is given as

$$M = \frac{4}{3}\pi R^3\rho$$

If the ball is at a distance y from the centre, then the portion of the Earth mass of radius y exerting force on the ball is given as

$$M' = \frac{4}{3}\pi y^3 \rho$$



Now the force of attraction between the Earth of radius y and the particle is given as

$$F = -\frac{GMm}{y^2} = -\frac{G\left(\frac{4}{3}\pi y^3\right)\rho m}{y^2}$$

$$= -\left(\frac{4}{3}G\pi\rho m\right)y$$

$$= -ky \qquad \left[\text{where } k = \frac{4}{3}G\pi\rho m = \text{constant}\right]$$

or $\qquad F \propto -y$

which is the essential condition for simple harmonic motion. Thus, the motion of ball in the tunnel AB will be simple harmonic motion.

The periodic time can be calculated as

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{(4/3)\pi G\rho m}} = \sqrt{\left(\frac{3\pi}{G\rho}\right)}$$

$$= \sqrt{\frac{3\times 3.14}{(6.67\times 10^{11}\times 5.5\times 10^3)}} \qquad \left[\begin{array}{l}\text{Because } G = 6.67\times 10^{-11}\text{ MKS unit}\\ \text{and } \rho = 5.5\times 10^3\text{ kg/m}^3\end{array}\right]$$

$$= 5050 \text{ s}$$

Example 2.21

Show that the general relations for the period and frequency of any simple harmonic motion are

$$T = 2\pi\sqrt{\frac{y}{\alpha}} \text{ and } n = \frac{1}{2\pi}\sqrt{\frac{\alpha}{y}}$$

Solution

Equation of simple harmonic motion can be given as

$$y = a \sin(\omega t + \phi)$$

$$\therefore \quad \frac{dy}{dt} = -a\omega \cos(\omega t + \phi)$$

Now $$\alpha = \frac{d^2y}{dt^2} = -\omega^2 a \sin(\omega t + \phi) = -\omega^2 y$$

or $$\omega = \sqrt{\frac{\alpha}{y}}$$ [negative sign is only for the direction of acceleration]

or $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{y}{\alpha}}$$

or $$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\alpha}{y}}$$

Example 2.22

If a simple pendulum is set up in a lift which has a downward acceleration greater than 'g', then how does the pendulum behave?

Solution

If a pendulum of mass m is set up in a lift moving with acceleration a, then the tension T in the string is given as

$$mg - T = ma$$

or $$T = m(g - a)$$

If $a > g$, then the tension in the string will be negative, i.e., T and mg are in the same direction. This means that the pendulum bob floats with the string pulling it down. It can oscillate in this position with time period

$$T = 2\pi\sqrt{\left(\frac{L}{a-g}\right)}$$

Example 2.23

Velocity of radio waves is 4×10^5 km/s. Calculate the frequency in megahertz of the station broadcasting at (i) 515 m and (ii) 40 m.

Solution

(i) Given that $\upsilon = 4 \times 10^8$ m/s

$\lambda = 515$ m

We know that $\upsilon = n\,\lambda$

or $$n = \frac{\upsilon}{\lambda} = \frac{4 \times 10^8 \text{ m/s}}{515}$$

$= 0.7767 \times 10^6$ Hz

$= 0.7767$ MHz

(ii) Given that $\upsilon = 4 \times 10^8$ m/s

$x = 40$ m

We know that $\upsilon = n\,\lambda$

or $$n = \frac{\upsilon}{\lambda} = \frac{4 \times 10^8 \text{ m/s}}{40 \text{ m}}$$

$= 10 \times 10^6$ Hz

$= 10$ MHz

Example 2.24

Human ear can hear frequencies between 33 Hz to 20 kHz. What wavelength corresponds to these frequencies, if the speed of sound is 330 m/s?

Solution

Given that $\upsilon = 330$ m/s, $n_1 = 33$ Hz and $n_2 = 20 \times 10^3$ Hz

Corresponding to n_1, $\lambda_1 = \dfrac{\upsilon}{n_1} = \dfrac{330}{33} = 10$ m

and corresponding to n_2, $\lambda_2 = \dfrac{\upsilon}{n_2} = \dfrac{330}{20 \times 10^3} = 0.0165$ m $= 1.65$ cm

Example 2.25

A simple harmonic oscillator of mass 20 gram and force constant 80 N/m is displaced by 0.8 cm from the equilibrium position and released to execute simple harmonic oscillation. Find its amplitude, angular frequency, frequency, time period, maximum speed, total energy, initial phase and kinetic energy when it is half way between the equilibrium position and an extreme position.

Solution

Mass of oscillator $= m = 20$ g $= 0.02$ kg, force constant $k = 80$ N/m. Amplitude $= a = 0.8$ cm $= 0.008$ m.

Angular frequency = $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{80\,\text{N/m}}{0.02\,\text{kg}}} = 63.25$ rad/s

Frequency = $\nu = \frac{\omega}{2\pi} = \frac{63.25\text{ rad/s}}{6.28} = 10.07$ Hz

Time period = $T = \frac{1}{\nu} = 0.099\ \text{s}^{-1}$.

Maximum speed = $v_{max} = \omega_a = (63.25\text{ rad/s})(0.008\text{ m})$

$= 0.506$ m/s

Total energy = $\frac{1}{2} m\omega^2 a^2$

$= \left(\frac{1}{2}\right)(0.02\text{ kg})\,(63.25\text{ rad/s})^2(0.008\text{ m})^2 = 2.53 \times 10^{-3}$ J

If the displacement is represented by $x = a\cos(\omega t + \phi)$, then the initial displacement (at $t = 0$) is $x(t = 0) = a\cos\phi = (0.8\text{ cm})\cos\phi = 0.8$ cm

So, $\cos\phi = 1$, and hence the initial phase = $\phi = 0$

Kinetic energy = $\frac{1}{2} mv^2 = \frac{1}{2} m\omega^2(a^2 - x^2)$

When the oscillator is half way between the equilibrium position and the extreme point, $x = a/2$, so kinetic energy is

$$\text{K.E.} = \frac{1}{2} m\omega^2\left(a^2 - \frac{a^2}{4}\right) = \frac{3}{8} m\omega^2 a^2 = 1.92 \times 10^{-3}\text{ J}$$

Example 2.26

Show that displacement versus velocity graph for simple harmonic oscillator is an ellipse.

Solution

Let the displacement be $x = a\cos(\omega t + \phi)$,

$$\cos(\omega t + \phi) = \frac{x}{a}$$

So, velocity is $v = -\omega a\sin(\omega t + \phi)$, $\sin(\omega t + \phi) = -\frac{v}{\omega a}$

$$\frac{x^2}{a^2} + \frac{v^2}{\omega^2 a^2} = \cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1$$

This is the equation of an ellipse.

Example 2.27

The time period of a simple harmonic oscillator is 2 seconds. It is subjected to a damping force proportional to speed, with damping coefficient $\beta = 1\ s^{-1}$. Find the time period and log decrement. What should be the value of β for the motion to be critically damped?

Solution

The natural angular frequency is $\omega_0 = \frac{2\pi}{T} = \frac{2 \times 3.14}{2s} = 3.14\ s^{-1}$

The new time period is $T_1 = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2 \times 3.14}{\sqrt{(3.14 s^{-1})^2 - (1 s^{-1})^2}} = 2.11\ s$

Log decrement $= \delta = \beta T_1 = (1\ s^{-1})(2.11\ s) = 2.11$

For critical damping, the value of damping coefficient should be equal to $\beta = \omega_0 = \pi s^{-1} = 3.14\ s^{-1}$

Example 2.28

A simple harmonic oscillator of mass 10 g is subjected to a restoring force , of force constant 90 dyn/cm. Find the time period of the oscillator. If it is subjected to a damping force proportional to the velocity, what is the maximum value of damping constant for which the motion will remain oscillatory?

Solution

The angular frequency is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{90\ dyn/cm}{10 g}} = 3\ s^{-1}$

If b = damping constant, and β = damping coefficient $= \frac{b}{2m}$, the motion will remain oscillatory if $\omega_0^2 > \beta^2$. Hence, the maximum value of damping coefficient is $\beta_{max} = \omega_0 = 3\ s^{-1}$. So, the maximum value of damping constant is

$$(b)_{max} = 2m\beta = 2 \times 10\ g \times 3\ s^{-1} = 60\ gs^{-1}$$

Example 2.29

In a forced damped oscillator , the damping coefficient $\beta = 0.8\ s^{-1}$, and the resonant frequency is 800 Hz. Find the Q-factor of the oscillator.

Solution

Resonant angular frequency $= \omega_R = \omega_0 = 2\pi \times 800\ s^{-1}$

So $\quad Q = \frac{\omega_0}{2\beta} = \frac{2\pi \times 800 s^{-1}}{2 \times 0.8 s^{-1}} = 3140$

Example 2.30

Two pendulums of mass 50 g each are suspended by massless rigid rods of length 1 m each. The two masses are coupled by a massless spring of force constant $k = 250$ dyn/cm. Determine the normal mode frequencies of the coupled oscillator. (Take, $g = 10$ m/s^2)

Solution

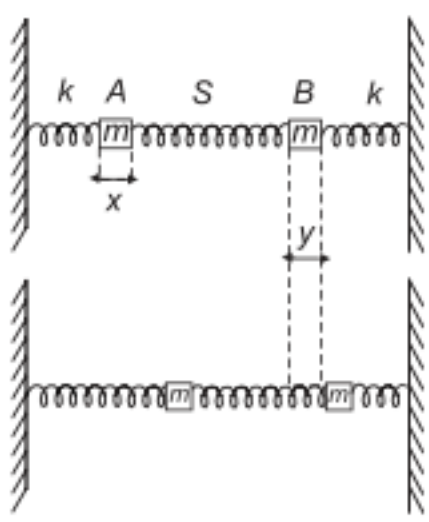
The two normal mode frequencies are:

$$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{3.14 \times 2}\sqrt{\frac{10\,\text{m/s}^2}{1\,\text{m}}} = 0.503 \text{ Hz}$$

and $$\nu_2 = \frac{1}{2\pi}\sqrt{\left(\frac{g}{l} + \frac{2k}{\text{m}}\right)} = \frac{1}{2 \times 3.14}\sqrt{\frac{10\,\text{m/s}^2}{1\,\text{m}} + \frac{2 \times 250\,\text{dyn/cm}}{50\,\text{g}}} = 0.712 \text{ Hz}$$

Example 2.31

Two identical rectangular blocks of mass 1 kg each are connected independently to two rigid supports by identical springs so as to oscillate along a line on a horizontal , frictionless table as shown in the following figure. The two blocks are connected by another spring and oscillate as a coupled system. The angular frequencies of the normal mode of oscillation are $\omega_1 = 16$ rad/s and $\omega_2 = 20$ rad/s. Evaluate the force constants of the springs.



Solution

Mass of each block = $m = 1$ kg

Let k = force constant of the springs connecting the two blocks

The angular frequencies of normal mode of oscillation are given by

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad w_2 = \sqrt{\frac{k+2s}{m}}$$

or $$w_1^2 = \frac{k}{m} \quad \text{and} \quad w_2^2 = \frac{k+2s}{m}$$

so, $$k = mw_1^2 = (1\text{ kg})\,(256\text{ s}^{-2}) = \frac{256\text{ kg}\cdot\text{s}^{-2}}{1\text{ m}} = 256\text{ N/m}$$

i.e. the force constant of each spring connecting the block to the rigid support is 256 N/m

Again $$w_2^2 - w_1^2 = \frac{k+2s}{m} - \frac{k}{m} = \frac{2s}{m}$$

or, $$s = \frac{m}{2}(w_2^2 - w_1^2) = \frac{1\text{ kg}}{2}(400\text{ s}^{-2} - 256\text{ s}^{-2}) = 72\text{ N/m}$$

So the force constant of the spring connecting the blocks is 72 N/m.

Example 2.32

A harmonic wave is represented by the wave equation

$$\psi(x, t) = 3\text{ cm}\sin(0.6x - 2.2t + \pi)$$

where x, is in cm and t is in s. Determine the amplitude, frequency, wavelength, phase velocity and phase constant of the wave.

Solution

The general form of a wave function for a harmonic wave is

$$\psi(x, t) = A\sin(kx - \omega t + \alpha)$$

Comparing this with the given expression, we get

Amplitude = A = 3 cm, wave vector = k = 0.6 cm^{-1}

$$\omega = 2.2\text{ rad/s}$$

Frequency = $\nu = \dfrac{\omega}{2\pi} = \dfrac{2.2\text{ rad/s}}{2\times 3.14} = 0.35\text{ Hz}$

Wavelength = $\lambda = \dfrac{2\pi}{k} = \dfrac{2\times 3.14}{0.6\text{ cm}^{-1}} = 10.47\text{ cm}$

Phase velocity = $v = \nu\lambda$ = (0.35 Hz) (10.47 cm) = 3.66 cm/s

The phase constant = initial phase = $\alpha = \pi$

Example 2.33

Two harmonic waves $\psi_1(x, t) = 8\text{ cm}\sin(5x - 2\pi t)$ and $\psi_2(x, t) = (10\text{ cm})\sin(7x - 4\pi t)$ superpose. Find the group velocity of the resultant wave. (Given x is in cm and t is in s)

Solution

Angular frequencies of the individual waves are:

$$w_1 = 2\pi\text{ rad/s and } w_2 = 4\pi\text{ rad/s}$$

The wave vectors are: $k_1 = 5\ \text{cm}^{-1}$ and $k_2 = 7\ \text{cm}^{-1}$

So group velocity $= v_g = \dfrac{\Delta w}{\Delta k} = \dfrac{w_2 - w_1}{k_2 - k_1}$

$$= \frac{(4\pi - 2\pi)\ \text{rad/s}}{(7\ \text{cm}^{-1} - 5\ \text{cm}^{-1})} = 3.14\ \text{cm/s}$$

Example 2.34

Two harmonic waves of the same amplitude and the same frequency, but different phases superpose. The amplitude of the resultant wave is the same as that of any individual wave. Determine the phase difference between the two individual waves.

Solution

Let the two waves be represented by wave functions $\psi_1 = A \sin(\omega t + \phi_1)$ and $\psi_2 = A \sin(\omega t + \phi_2)$

The resultant wave is, from the principle of superposition

$$\psi = \psi_1 + \psi_2 = A \sin(\omega t + \phi_1) + A \sin(\omega t + \phi_2)$$

Since the amplitude of the resultant wave is the same as that of each individual waves

$$A = \sqrt{A^2 + A^2 + 2A \cdot A\cos(\phi_2 - \phi_1)} = A\sqrt{2(1 + \cos(\phi_2 + \phi_1)}$$

$$= A\sqrt{2.2\cos^2\left(\frac{\phi_2 - \phi_1}{2}\right)} = 2A\cos\frac{(\phi_2 - \phi_1)}{2}$$

So, $\cos\dfrac{\phi_2 - \phi_1}{2} = 1/2 \Rightarrow \phi_2 - \phi_1 = 120°$

Example 2.35

The bulk modulus and density of water are $2.1 \times 10^9\ \text{N/m}^2$ and $10^3\ \text{kg/m}^3$, respectively. Find the speed of longitudinal waves in water.

Solution

The speed of longitudinal wave in water is

$$v = \sqrt{\frac{k}{\rho}} = \sqrt{\frac{2.1 \times 10^9\ \text{N/m}^2}{10^3\ \text{kg/m}^3}} = 1.4 \times 10^3\ \text{m/s}$$

Example 2.36

The mass of uniform string of length 25 m is 100 g. Find the speed of transverse waves in the string when it is under tension of 16 N.

Solution

Mass per unit length of the string is

$$m = \frac{0.1\text{ kg}}{25\text{ m}} = 4 \times 10^{-2}\text{ kg/m}$$

Tension in the string = $T = 16$ N

So, the speed of propagation of transverse waves in the string is

$$v = \sqrt{\frac{T}{m}} = \sqrt{\frac{16\text{ N}}{4 \times 10^{-2}\text{ kg/m}}} = 20\text{ m/s}$$

Example 2.37

The oscillation of a tuning fork of frequency 200 c/s dies away to $\frac{1}{e}$ times their amplitude in 1 sec. Show that the reduction in frequency due to air damping is exceedingly small.

Solution

The amplitude at any instant = $a_0 e^{-rt}$

Where $t = 0$, initial amplitude = a_0

When $t = 1$ sec, amplitude = $\frac{1}{e} a_0$

$$\frac{a_0}{e} = a_0 e^{-r \times 1} \text{ or } e^{-1} = e^{-r}$$

$$\therefore \quad r = 1$$

The angular frequency in presence of damping

$$= \sqrt{(w^2 - r^2)}$$

$$= \sqrt{((2 \times 3.14 \times 200)^2 - 1)}$$

i.e. Damped frequency = $\sqrt{(1.6 \times 10^6 - 1)}$ ...(i)

Undamped frequency = $w^2 = 1.6 \times 10^6$ c/s ...(ii)

From eq. (i) and eq. (ii) we find the damped and undamped frequencies are almost same. Hence the effect of damping is exceedingly small.

2.17 FORMULAE AND HIGHLIGHTS

1. To and fro motion of a particle about the equilibrium position is known as *oscillation.*
2. Essential condition for SHM is

   $$F \propto -y$$

3. Displacement of a particle executing SHM is given as

   $$y = a \sin \omega t$$

4. Velocity, acceleration, and frequency of a particle executing SHM are given as

$$v = \omega\sqrt{a^2 - y^2},\ a = -\omega^2 y, \text{ and } T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

5. The phase of an oscillating particle determines its position and direction of motion at a particular instant.
6. Expression for the frequency of damped oscillations can be given as

$$\omega = \sqrt{\omega_0 - \left(\frac{b}{2m}\right)^2}$$

7. At the condition when natural frequency of oscillating particle becomes equal to the frequency of external force, the amplitude of oscillations becomes very large. This phenomena is known as *resonance*.
8. When coupling is loose, the rate of transfer of energy is small; however, when coupling is tight, the rate of transfer of energy is high.
9. Velocity of wave is expressed as $\upsilon = n\,\lambda$.
10. In transverse wave, medium particle vibrates perpendicular to its direction of propagation, but in case of longitudinal waves, medium particle vibrates parallel to its direction of propagation.
11. General equation of a simple harmonic wave is given as

$$y = a \sin \frac{2\pi}{\lambda}(\upsilon t - x)$$

12. Differential equation of wave motion is given as

$$\frac{d^2y}{dt^2} = \upsilon^2 \frac{d^2y}{dx^2}$$

13. Differential equation of damped harmonic equation is given as $\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega^2 y$.
14. Underdamping condition occurred when $r < \omega_0$, for overdamped $r > \omega_0$ and critical damped condition occurred at $r = \omega_0$.
15. Logarithmic decrement is given as $\log_e\left(\frac{a_n}{a_{n+1}}\right) = \frac{rT}{2} = \lambda$.
16. Relaxation time $\tau = \frac{1}{r} = \frac{2m}{b}$.
17. Energy in damped harmonic oscillation is given as $E = \frac{1}{2}ka_0^2 e^{-2rt}$.
18. Power dissipation is given as $P = E/\tau$.
19. Quality factor is given as $Q = \pi/\lambda$.
20. Differential equation of forced vibration is given as $\frac{d^2y}{dt^2} + 2r\frac{dy}{dt} + \omega^2 y = f_0 \sin pt$.
21. Maximum amplitude at resonance is given as $A_{max} = f_0/2r\omega$.

22. Average power in forced vibration is given as $P_{av} = \frac{1}{2} bA^2 p^2$.
23. Dissipated power in forced vibration is equal to the average power.
24. Bandwidth $\omega_2 - \omega_1 = \frac{1}{\tau}$.
25. Quality factor is given as $Q = \frac{m\omega}{b}$.
26. Normal frequencies of coupled vibrations is ω_0 or $\left(\omega_0^2 + \frac{2k}{m}\right)^{\frac{1}{2}}$.
27. Logarthmic decrement in coupled vibration is given as $\log_e d = \frac{kT}{2} = \lambda$.
28. Reflection coefficient of a transverse wave at the boundary of two mediums can be given as

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

and transmission coefficient is given as

$$T = \frac{2Z_1}{Z_1 + Z_2}$$

29. Reflection and transmitted acoustic pressure of longitudinal wave at the boundary of two mediums is given as

$$R = \frac{P_r}{P_i} = \frac{-Z_1(dy_r / dt)}{Z_1(dy_i / dt)} = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$

$$T = \frac{P_t}{P_i} = \frac{Z_2(dy_t / dt)}{Z_1(dy_i / dt)} = \frac{2Z_2}{Z_1 + Z_2}$$

30. Intensity coefficient of reflection and transmission for sound is given as

$$R_1 = \frac{I_r}{I_i} = \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2}$$

and $$R_T = \frac{I_t}{I_i} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}$$

EXERCISES

Section A
Theoretical Questions

1. Enumerate the characteristics of SHM. Derive an expression for the amplitude of a body executing SHM.
2. Define and explain wave motion, transverse waves, and longitudinal waves.

3. Derive the theoretical expression for plane progressive wave. Discuss its important properties.
4. Define and explain the following:
   (i) Velocity of motion (ii) Acceleration (iii) Time period (iv) Phase and frequency
5. Define damped harmonic oscillations. Explain the effect of damping on oscillatory motion.
6. Obtain the differential equation of damped harmonic oscillations.
7. Differentiate between free and forced vibrations. State the necessary conditions for forced vibration.
8. What is resonance? Explain how the condition of resonance is achieved.
9. What do you understand by coupled oscillations? Discuss stiffness coupling between two pendulums.
10. Discuss the phenomena of reflection and transmission of transverse wave at the boundary of two media.
12. Describe the phenomena of damped harmonic oscillations and obtain the differential equation for damped oscillations.
13. What are the different conditions in damped harmonic oscillations to obtain the expressions for displacement in each cases?
14. Mention the important parameter of harmonic oscillators. Obtain the expressions for the same.
15. Obtain the expression for total energy in damped harmonic oscillation.
16. Give the analytical treatment of forced vibration.
17. Define the phenomena of resonance and describe the sharpness of resonance.
18. Prove that the power dissipated in the forced vibration is equal to the average power.
19. Explain the quality factor is terms of bandwidth.
20. Describe coupled vibrations. Also obtain the expression for normal frequencies.
21. Discuss the attenuation in the vibrating system.
22. Discuss the phenomena of reflection and transmission of longitudinal wave at the boundary of two media.
23. Derive the expression for reflection and transmission coefficients of sound intensity at the boundary of two media.

Section B
Numerical Problems

1. A particle executes SHM of amplitude 0.1 cm, its velocity passing through the mean position is 3 m/s. What is its frequency? (*Ans*. 477 cycle/s)
2. A simple progressive wave is represented by the equation $y = 0.5 \sin (314t - 12.56x)$, where y and x are expressed in metres and t in seconds. Find (i) amplitude, (ii) wavelength, (iii) speed of wave, (iv) frequency, and (v) phase difference, for the points 7.5 m apart.
   (*Ans*. (i) 0.5 m, (ii) 0.5 m, (iii) 25 m/s, (iv) 0.50 c/s, (v) zero)
3. A particle executes simple harmonic motion. The velocity of the particle when it is passing through the points 3 cm and 4 cm from the centre are 16 cm/s and 12 cm/s, respectively. Calculate

(i) amplitude, (ii) period of motion, and (iii) maximum velocity of the particle.

(*Ans.* (i) $a = 5$ cm, (ii) $T = 1.57$ s, (iii) 20 cm/s)

[*Hint*: $v^2 = \omega^2 (a^2 - y^2)$]

4. Calculate the fundamental frequency of a sonometer wire of length 20 cm, tension 25 N, cross-sectional area 10^{-2} m^2, and density of the material of wire is 10^4 kg/m^3. (*Ans.* 125 Hz)

$$\left[Hint: x = \frac{1}{2l}\sqrt{\frac{T}{m}}\right]$$

5. Show that in simple harmonic wave, dy/dt is ahead of the phase by $\pi/2$ with respect to y and $\ddot{y}$ ahead by a phase difference of $\pi/2$ with respect to dy/dt.

$$\left[Hint: y = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right.$$

$$\frac{dy}{dt} = \dot{y} = \frac{2\pi}{T} a \cos 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$$

or

$$\dot{y} = \frac{2\pi}{T} a \sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \frac{\lambda}{2}\right]$$

Similarly, $\frac{d^2y}{dt^2} = \ddot{y} = \left(\frac{2\pi}{T}\right)^2 \sin 2\pi\left\{\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \lambda\right\}\Bigg]$

6. A simple harmonic wave is represented by

$$y = 10 \sin\left[\frac{2\pi t}{T} + \alpha\right]$$

The time period is 30. At time $t = 0$, the displacement is 5 cm. Calculate (i) phase angle at $t = 7.5$ s, and (ii) phase difference between two positions at a time interval of 6 s.

(*Ans.* (i) $2\pi/3$ rad, (ii) $2\pi/5$ rad)

7. The equation of a wave travelling along a spring is $y = 4 \sin \pi (0.10x - 2.0\, t)$, where y and x are in metres and t in second. Find (i) amplitude, (ii) wavelength, (iii) initial phase at the origin,(iv) speed, and (v) frequency of wave.

(*Ans.* (i) $a = 4$ cm, (ii) $\lambda = 200$ cm, (iii) $\phi = 0$, (iv) $v = 20$ m/s, (v) $n = 1.0$ Hz)

[*Hint*: $y = a \sin \frac{2\pi}{\lambda}(x - vt + \phi)$]

8. A body executing SHM describes 120 vibrations per minute and has a velocity of 5 m/s. What is the length of its path? What is the velocity, when it is halfway between its mean position and an extremity of its path? (*Ans.* (i) Length of path = 0.788 m, (ii) Velocity = 4.288 m/s)

[*Hint*: At mean position, $v = a\omega$

Total path = 2 × amplitude

Velocity, when the body is halfway between the mean position and an extremity, i.e., $y = a/2$ is given as

$$\upsilon = \omega\sqrt{a^2 - \frac{a^2}{4}}\Bigg]$$

9. A tuning fork of frequency 1000 Hz produces a wave of wavelength 20 cm in air. Calculate the velocity of sound in air. (*Ans*. 200 m/s)
   [*Hint*: $\upsilon = n\lambda$]
10. A radio station broadcasts at a frequency of 15×10^6 Hz. Calculate the length of the radio waves. (*Ans*. 20 m)
   [*Hint*: $\upsilon = n\lambda$; the velocity of radio waves is equal to the velocity of light, i.e., 3×10^8 m/s.]

Section C
Multiple Choice Questions

1. According to the essential condition of SHM
   (a) $F \propto y$ (b) $F \propto -y$
   (c) $F \propto x$ (d) $F \propto a$
2. Simple harmonic progressive waves are expressed as
   (a) $y = a \sin \omega t$ (b) $y = a \sin \omega t \cos kt$
   (c) $y = a \sin (\omega t - kx)$ (d) $y = a \cos kx$
3. Usually, sound waves are
   (a) transverse waves (b) electromagnetic waves
   (c) both transverse and longitudinal (d) longitudinal waves
4. Which of the following phenomena cannot take place with sound waves?
   (a) polarisation (b) reflection
   (c) interference (d) diffraction
5. The phase difference between two waves represented by $y = a \sin (\omega t - kx)$ and $y = b \cos (\omega t - kx)$ is
   (a) $\pi/4$ (b) $\pi/2$
   (c) π (d) 2π
6. If a straight tunnel is bored through the centre of the Earth and a body of mass m is dropped in it, then its motion will be
   (a) linear (b) unpredictable
   (c) simple harmonic motion (d) none of these
7. When a body vibrates without the influence of external force, then its vibration is known as
   (a) natural vibration (b) forced vibration
   (c) damped vibration (d) none of these

8. In damped vibration, amplitude of vibration
   (a) decreases (b) decreases then increases
   (c) increases (d) increases then decreases
9. When a vibrating system is subjected to an external periodic force, it is said to be in a state of
   (a) forced vibration (b) natural vibration
   (c) damped vibration (d) none of these
10. At the condition of resonance, the frequency of impressed force is same as the
   (a) natural frequency (b) damping frequency
   (c) both (a) and (b) (d) none of these
11. At resonance, the amplitude of vibration becomes
   (a) less (b) large
   (c) equal to initial value (d) average of natural frequency and forced vibration
12. Transmission coefficient at the junction of two media of impedance Z_1 and Z_2 is unity if
   (a) $Z_1 < Z_2$ (b) $Z_1 > Z_2$
   (c) $Z_1 = Z_2$ (d) none of these
13. Underdamping condition occurrs when
   (a) $r < \omega_0$ (b) $r > \omega_0$
   (c) $r = \omega_0$ (d) None of these
14. Critically damped condition occurrs when
   (a) $r < \omega_0$ (b) $r > \omega_0$
   (c) $r = \omega_0$ (d) None of these
15. Logarithmic decrement is given as
   (a) $\lambda = \frac{r}{2T}$ (b) $\lambda = \frac{rT}{2}$
   (c) $\lambda = rT$ (d) $\lambda = \frac{2r}{T}$
16. Relaxation time is expressed as
   (a) $\tau = \frac{b}{2m}$ (b) $\tau = \frac{2m}{b}$
   (c) $\tau = \frac{m}{b}$ (d) $\frac{2b}{m}$
17. Energy in damped harmonic oscillation is given as
   (a) $\frac{1}{2}ka_0^2e^{-2rt}$ (b) $\frac{1}{2}ka_0^2e^{T}$
   (c) $ka_0^2e^{-2r}$ (d) $k^2a_0^2e^{-2rt}$

18. Quality factor (Q) is expressed as

(a) $Q = \pi\lambda$ (b) $Q = \frac{\pi}{\lambda}$

(c) $Q = \frac{\lambda}{\pi}$ (d) $Q = \frac{2\pi}{\lambda}$

19. At resonance amplitude will be

(a) $\frac{f_0}{2r}$ (b) $\frac{f_0}{2\omega}$

(c) $\frac{f_0}{2r\omega}$ (d) $\frac{f_0}{4r\omega}$

20. Average power in force vibration is equal to

(a) zero (b) dissipated power

(c) maximum power (d) none of these

21. Quality factor of force oscillator is

(a) $\omega\tau$ (b) $\frac{\omega}{\tau}$

(c) $\frac{\tau}{\omega}$ (d) $\frac{1}{\omega\tau}$

22. In the coupled vibrations ω_0 and $\left(\omega_0^2 + \frac{2k}{m}\right)^{\frac{1}{2}}$ are known as

(a) Lower and higher mode frequency respectively

(b) Higher and lower mode frequency respectively

(c) Higher mode frequency

(d) Lower mode frequency

Answers

1. (b)	2. (c)	3. (d)	4. (a)	5. (b)	6. (c)
7. (a)	8. (a)	9. (a)	10. (a)	11. (b)	12. (c)
13. (a)	14. (c)	15. (b)	16. (b)	17. (a)	18. (b)
19. (c)	20. (b)	21. (a)	22. (a)		

Section D
Fill in the Blanks

1. To and fro motion of a particle about the equilibrium position is known as .....
2. The phase of an oscillating particle determines its ..... and ..... of motion at a particular instant.
3. The expression for the ..... of damped oscillation is given as $\omega = \sqrt{\omega_0 - (b/2m)^2}$

4. Maximum value of displacement about the mean position is known as ..... .
5. Number of oscillations per second is called as .....
6. In a ..... harmonic vibration, amplitude decreases continually.
7. Frequency depending on intrinsic properties of the body is known as ..... .
8. During resonance, the amplitude of vibration becomes ..... .
9. In the transverse waves, the particles of the medium vibrates ..... to the direction of wave propagation.
10. In the longitudinal waves, the particles of the medium vibrates ..... to the direction of wave propagation.

Answers

1. oscillation
2. position, direction
3. frequency
4. amplitude
5. frequency
6. damped
7. natural frequency
8. large
9. perpendicular
10. parallel