* Cauchy's Theorem :-

If $f(z)$ is analytic on and within closed curve C then

$$\boxed{\oint_C f(z)\,dz = 0}$$

The direction of description of C is taken as anticlockwise.

* Cauchy's Integral Formula :-

If $f(z)$ is analytic on and within closed curve 'C' and if 'a' is any point within 'C' then

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz}$$

* Remark :- Cauchy's Integral Formula for Derivative :-

$$\boxed{f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

where $f^n(a)$ = value of n^{th} order derivative of $f(z)$ at $z=a$.

* Note :-

i) Closed curve is also called as contour

ii) To solve the problems on complex integration we will make use of Cauchy's Integral Formula as.

1) $$\oint \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

2) $$\oint \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

* Examples:-

1) Evaluate : $\oint_C \frac{z^2+1}{z-2}\, dz$

where i) C is the circle $|z-2|=1$

ii) C is the circle $|z|=1$

→ Let

$$I = \oint_C \frac{z^2+1}{z-2}\, dz$$

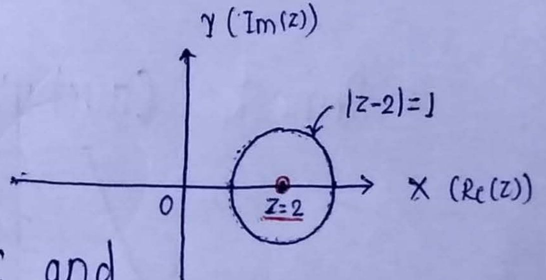
i) To evaluate I when $C: |z-2|=1$

Equate denominator of $\frac{z^2+1}{z-2}$ i.e. $z-2$ to zero

$\therefore\ z-2=0 \Rightarrow z=2\ [=(2,0)=2+0i]$

which lies within the closed curve 'C' : $|z-2|=1$

which is a circle $(x-2)^2+y^2=1$



$\therefore$ Consider,

$f(z) = z^2+1$

which is analytic on and within C and

$z=2$ is any point within 'C'

$\therefore$ By Cauchy's integral formula,

$$\oint_{C:|z-2|=1} \frac{z^2+1}{z-2}\, dz = 2\pi i\, f(2)$$

$$= 2\pi i\, [2^2+1]$$

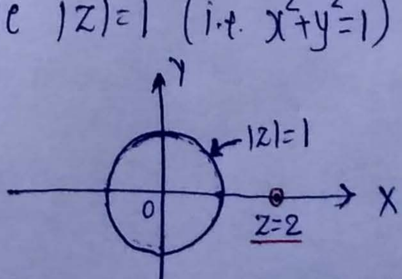
$$I = 10\pi i$$ //

ii) To evaluate I when $C: |z|=1$

As $z=2$ lies outside of the circle $|z|=1$ (i.e. $x^2+y^2=1$)

$\therefore$ Consider $f(z) = \frac{z^2+1}{z-2}$ which is

analytic on & within curve 'C'



$\therefore$ By Cauchy's theorem

$$I = 0$$ //

2) Evaluate : $\oint_C \frac{z^3-5}{(z+1)^2(z-2)}\,dz$, where C is the circle $|z|=1.5$

→ Let ,

$$I = \oint_C \frac{z^3-5}{(z+1)^2(z-2)}\,dz$$

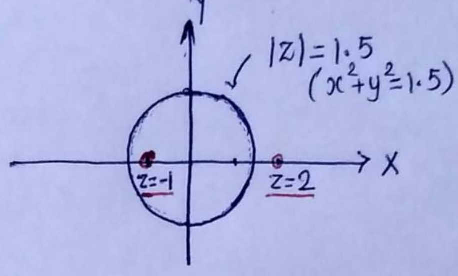
Equating denominator to zero, we get

$$z = -1, 2$$

Clearly, out of which

$z = -1$ lies inside c &

$z = 2$ lies outside c



∴ Consider,

$$f(z) = \frac{z^3-5}{z-2}$$

which is analytic on and within curve $C: |z|=1.5$

and $z=-1$, is any point in c with power >1

∴ By Cauchy's integral formula for derivative, we get

$$\oint_C \frac{z^3-5/z-2}{(z+1)^2}\,dz = \oint \frac{z^3-5/z-2}{(z+1)^{1+1}}$$

$$= \frac{2\pi i}{1!} f'(z=-1) \qquad \because n=1$$

$$= 2\pi i\left[\frac{d}{dz}\left(\frac{z^3-5}{z-2}\right)\right]_{z=-1}$$

$$= 2\pi i\left[\frac{2z^3-6z^2+5}{(z-2)^2}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{-2-6+5}{9}\right]$$

$$\oint_C \frac{z^3-5}{(z+1)^2(z-2)}\,dz = -\frac{2\pi i}{3} \quad //$$