Analytic function

A function $f(z)$ is said to be analytic at a point $z = z_0$ if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

It is said to be analytic in a region R if it is analytic at every point in region R.

Ex. 1) Every polynomial function of z is analytic everywhere.

2) e^z, $\cos z$, $\sin z$, $\log z$ are analytic functions.

3) $f(z) = \frac{P(z)}{Q(z)}$ is analytic everywhere except at zeroes of $Q(z)$.

4) $f(z) = \frac{1}{z}$ is analytic everywhere except at $z=0$.

5) $f(z) = \frac{z+1}{z^2-5z+6}$ is analytic everwhere except at $z=2$ and $z=3$.

Singular point - A point $z = z_0$ is said to be a singular point of a function $f(z)$ if $f'(z_0)$ does not exist.

In example (4) $z=0$ is the singular point.

* Necessary condition for $f(z)$ to be analytic –

If $f(z) = u+iv$ is analytic in a region R then u, v satisfies the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

provided the four partial derivatives u_x, u_y, v_x, v_y exist.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ these equations are known as Cauchy - Riemann equations. (CR equations)

* Polar form of Cauchy-Riemann equations –

If $f(z) = u+iv$ is an analytic function and $z = re^{i\theta}$ where u, v, r, θ are all real. Then Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\cdot\frac{\partial u}{\partial \theta}$$

* If $f(z) = u+iv$ is an analytic function in region R, then the curves $u =$ constant, $v =$ constant form two orthogonal families.

* Harmonic function- A function $f(x, y)$ is said to be harmonic if it is continuous and has continuous first and second order partial derivatives and satisfies Laplace equation.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

* If $f(z) = u+iv$ is analytic, then both u and v are harmonic.

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

* Expressions for $f'(z)$ –

Let $f(z) = u+iv$ be analytic.

$$f(x+iy) = u+iv \quad \text{———— (1)}$$

Diff. w.r.to x.

$$f'(x+iy) = u_x + i v_x$$

$$\therefore f'(z) = u_x + i v_x.$$

Now By CR equations

$$f'(z) = u_x - i u_y \quad \text{as} \quad u_y = -v_x$$

$$= v_y + i v_x \quad \text{as} \quad u_x = v_y$$

Diff (1) w.r.to y

$$i f'(x+iy) = u_y + i v_y$$

$$\therefore f'(z) = v_y - i u_y$$

At last,

$$f'(z) = u_x + i v_x$$

$$= v_y - i u_y$$

$$= u_x - i u_y$$

$$= v_y + i v_x$$

* Milne-Thomson method –

This method is used to find $f(z) = u(x,y)+iv(x,y)$ in terms of z when $u(x,y)$ and $v(x,y)$ are given.

Substitute $x = z$ and $y = 0$ to obtain $f(z)$ in terms of z.

* To find analytic function $f(z)$ whose real or imaginary part is given.

Ex. 1) If $u = \frac{1}{2}\log(x^2+y^2)$, find v such that $f(z) = u+iv$ is analytic. Determine $f(z)$ in terms of z.

1) $u = \frac{1}{2}\log(x^2+y^2)$

Diff. u w.r. to x

$$\frac{\partial u}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

Using CR equations

$u_x = v_y$ and $u_y = -v_x$

$$v_y = u_x = \frac{x}{x^2+y^2}$$

Integrate w.r. to y

$$v(x,y) = \int \frac{x}{x^2+y^2}\,dy + f(x)$$

$$v(x,y) = x\cdot\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

$$v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + f(x)$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) + f'(x)$$

$$\frac{-y}{x^2+y^2} = \frac{-y}{x^2+y^2} + f'(x) \Rightarrow f'(x) = 0 \quad \therefore f(x) = c$$

$$\therefore v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + c$$

Determine analytic function $f(z)$

$$\therefore f(z) = u+iv = \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(\frac{y}{x}\right)+c\right]$$

By Milne Thompson method

Put $x = z$ and $y = 0$.

$$\therefore f(z) = \frac{1}{2}\log(z^2) + ic$$

$$f(z) = \log z + ic$$

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Ex. 1) If $v = \frac{-y}{x^2+y^2}$, find u such that $f(z) = u+iv$ is analytic and determine $f(z)$ in terms of z.

Ex. 2) Show that following function is harmonic. and find their harmonic conjugate. Also find corrasponding analytic function $f(z)$ in terms of z.

$u = x^4 - 6x^2y^2 + y^4$

Ex. 3) Find the orthogonal trajectories of $x^2 - y^2 = c$.

Ex. 4) Find the analytic function $f(z)$ whose imaginary part is $r^n \sin(n\theta)$