Solved Examples

Ex. 1) Find the conditions under which $u = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic.

Sol^n - We have

$$u = ax^3 + bx^2y + cxy^2 + dy^3$$

$$\therefore \quad \frac{\partial u}{\partial x} = 3ax^2 + 2bxy + cy^2$$

and $$\frac{\partial u}{\partial y} = bx^2 + 2cxy + 3dy^2$$

Now $$\frac{\partial^2 u}{\partial x^2} = 6ax + 2by$$

and $$\frac{\partial^2 u}{\partial y^2} = 2cx + 6dy$$

Since u is harmonic

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \quad 6ax + 2by + 2cx + 6dy = 0$$

$$(6a + 2c)x + (2b + 6d)y = 0$$

$$\therefore \quad 6a + 2c = 0 \quad \text{and} \quad 2b + 6d = 0$$

$$\therefore \quad 3a + c = 0 \quad \text{and} \quad b + 3d = 0$$ are required conditions.

Ex. 2) If $f(z) = u+iv$ is analytic, find $f(z)$ if

$$u-v = (x-y)(x^2+4xy+y^2)$$

Sol^n: Given

$$u-v = (x-y)(x^2+4xy+y^2) \quad \text{——— (1)}$$

Diff. w.r. to x

$$u_x - v_x = (x^2+4xy+y^2)+(x-y)(2x+4y) \quad \text{—— (2)}$$

Diff(1) w.r. to y

$$u_y - v_y = -(x^2+4xy+y^2)+(x-y)(4x+2y) \quad \text{—— (3)}$$

Using C-R equations in eq^n (3)

$$-v_x - u_x = -(x^2+4xy+y^2)+(x-y)(4x+2y) \quad \text{—— (4)}$$

Adding (2) & (4)

$$-2v_x = (x-y)(6x+6y)$$

$$\therefore -v_x = 3(x-y)(x+y)$$

$$\therefore v_x = 3(y^2-x^2) \quad \text{——— (5)}$$

Substract (4) from (2)

$$2u_x = 2(x^2+4xy+y^2)+(x-y)(2y-2x)$$

$$\therefore u_x = x^2+4xy+y^2+2xy-x^2-y^2$$

$$u_x = 6xy \quad \text{——— (6)}$$

Now $f'(z) = u_x + iv_x$

$$= 6xy + i\left[3(y^2-x^2)\right] \quad \text{— from (5) and (6)}$$

By Milne-Thompson method Put $x=z$, $y=0$

$$\therefore f'(z) = -3iz^2$$

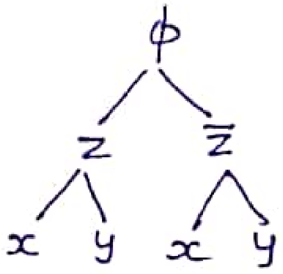
Integration gives

$$f(z) = -3i\frac{z^3}{3}+c = -iz^3+c$$

Ex. 3) Express Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$ in terms of variables z and $\bar{z}$.

Soln- We know $z=x+iy$ $\bar{z}=x-iy$



$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial x}+\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{\partial\bar{z}}{\partial x}$$

$$\phi_x=\phi_z\cdot(1)+\phi_{\bar{z}}(1)$$

$$\therefore\ \frac{\partial}{\partial x}\equiv\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}$$

Now $$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial x}\right]=\left[\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right]\left[\frac{\partial\phi}{\partial z}+\frac{\partial\phi}{\partial\bar{z}}\right]$$

$$=\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial z\partial\bar{z}}+\frac{\partial^2\phi}{\partial\bar{z}\partial z}+\frac{\partial^2\phi}{\partial\bar{z}^2}$$

$$=\frac{\partial^2\phi}{\partial z^2}+2\frac{\partial^2\phi}{\partial z\partial\bar{z}}+\frac{\partial^2\phi}{\partial\bar{z}^2}\quad\text{———(1)}$$

$\because\ \phi_{z\bar{z}}=\phi_{\bar{z}z}$

Now $$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{\partial\bar{z}}{\partial y}$$

$$=\frac{\partial\phi}{\partial z}\cdot(i)+\frac{\partial\phi}{\partial\bar{z}}\cdot(-i)$$

$$\therefore\ \frac{\partial}{\partial y}=i\frac{\partial}{\partial z}-i\frac{\partial}{\partial\bar{z}}$$

$$\frac{\partial^2\phi}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial y}\right]=\left[i\frac{\partial}{\partial z}-i\frac{\partial}{\partial\bar{z}}\right]\left[i\frac{\partial\phi}{\partial z}-i\frac{\partial\phi}{\partial\bar{z}}\right]$$

$$=-\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial z\partial\bar{z}}+\frac{\partial^2\phi}{\partial\bar{z}\partial z}-\frac{\partial^2\phi}{\partial\bar{z}^2}\quad\text{———(2)}$$

Add (1) & (2)

Note $$\frac{\partial^2\phi}{\partial z\partial\bar{z}}=\frac{\partial^2\phi}{\partial\bar{z}\partial z}$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=4\frac{\partial^2\phi}{\partial z\partial\bar{z}}$$

$$\therefore\ \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0\Rightarrow\frac{\partial^2\phi}{\partial z\partial\bar{z}}=0$$

This example suggest that $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ and $4\frac{\partial^2}{\partial z\partial\bar{z}}$ are equivalent operators.

Ex. 4) If $f(z)$ is analytic. Show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Soln: We know that

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}$$

$$\text{L.H.S.} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2$$

$$|f(z)|^2 = f(z)\cdot\overline{f(z)}$$
$$= f(z)f(\bar{z})$$

$$= 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}|f(z)|^2$$

$$= 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}\left[f(z)\cdot f(\bar{z})\right]$$

$$= 4\cdot\frac{\partial}{\partial z}\left[f(z)f'(\bar{z})\right]$$

$$= 4\cdot\left[f'(z)\cdot f'(\bar{z})\right]$$

$$= 4.\left[f'(z).\overline{f'(z)}\right]$$

$$= 4|f'(z)|^2$$

$$= \text{R.H.S.}$$

Assignment No. 3 Date 20/03/2020.

Ex. 1) Find the analytic function, whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Hint: $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

Ex. 2) Show that analytic function $f(z)$ with constant amplitude is constant.

Ex. 3) If $f(z)$ is analytic, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^4 = 16\,|f(z)|^2\,|f'(z)|^2$$

Hint: $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \equiv 4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$

Ex. 4) If $f(z) = u + iv$ is analytic function, find $f(z)$ if

$$u + v = e^{-x}(\cos y - \sin y)$$

Ex. 5) Prove that an analytic function with constant real part is constant.