Conformal Transformation

Conformal Transformation:

Consider the function of complex variable $w = f(z)$,

which transforms set of points $z_0, z_1, z_2, \dots, z_n$ from z-plane to the set of points $w_0, w_1, w_2, \dots, w_n$ in the w-plane, by using relation $w = f(z)$.

i.e. $w_0 = f(z_0),\ w_1 = f(z_1),\ w_2 = f(z_2), \dots \dots \dots, w_n = f(z_n)$.

First, we examine the linkage between the analyticity of a complex function $w = f(z)$ and the conformality of a mapping.

Definition:

A mapping is said to be conformal at a point if it preserves the angle of intersection between a pair of smooth arcs through that point.

i.e. The angle between any two intersecting arcs in the z-plane is equal to the angle between the images of the arcs in the w-plane under a linear mapping.

#Refer: https://www.slideshare.net/mariolabestia/afirstcourseincomplexanalysis [Page No. 390, Chapter 7].

Theorem on Conformal Mapping:

If $f(z)$ is an analytic function in a domain D containing z_0, and if $f'(z_0) \neq 0$, then $w = f(z)$ is a conformal mapping at z_0.

- i.e. If $w = f(z)$ is analytic function, then the translation $w = f(z)$ is conformal at all points of the z-plane where $f'(z) \neq 0$.
- If $f'(z_0) = 0$ then the mapping $w = f(z)$ will not be a Conformal Transformation, since the angle preservation property will not be satisfied for the curves intersecting at $z = z_0$.

Exercise:

Find all points where the mapping f(z) = sin z is conformal.

Theorem on Conformal Mapping:

Ex. Find all points where the mapping f(z) = sin z is conformal.

Solution: The function $f(z) = \sin z$ is analytic function, & we have $f'(z) = \cos z$.

we found that $\cos z = 0$ if and only if $z = \frac{(2n+1)\pi}{2}, \quad n = 0, \pm 1, \pm 2, \dots,$ and so each of these points is a critical point of $f(z)$.

Therefore, by Theorem on Conformal mapping,

$w = \sin z$ is a conformal mapping at z for all $z = \frac{(2n+1)\pi}{2}, n = 0, \pm 1, \pm 2, \dots$

Furthermore, $w = \sin z$ is not a conformal mapping at z if $z = \frac{(2n+1)\pi}{2}, n = 0, \pm 1, \pm 2, \dots$.

Some Important Transformations:

1. Translation:

Consider the transformation $w = z + h$

where $w = u + iv,\ z = x + iy,\ h = h_1 + ih_2$

$$w = z + h \qquad \Longrightarrow u + iv = (x + iy) + (h_1 + ih_2)$$

$$\Longrightarrow u + iv = (x + h_1) + i(y + h_2)$$

$$\Longrightarrow u = x + h_1 \quad \& \quad v = y + h_2$$

Geometrically: For all points (x, y) in the z-plane will be translated in w-plane by preserving its original shape, size and orientation.

Under the transformation $w = z + h$ rectangles or circles in z-plane will be mapped onto rectangles or circles in w-plane respectively.

Some Important Transformations:

2. Rotation & Magnification:

Consider the transformation $w = cz$

where $z = x + iy = re^{i\theta}$, $c = c_1 + ic_2 = ae^{i\alpha}$

where $|z| = r = \sqrt{x^2 + y^2}$ & $Amp(z) = \tan\theta = \frac{y}{x}$,

$|c| = a = \sqrt{c_1^2 + c_2^2}$ & $Amp(c) = \tan\alpha = \frac{c_2}{c_1}$

$$
\begin{aligned}
w = cz \qquad & \Longrightarrow |w| = |cz| = |c||z| = ar \\
& \Longrightarrow Amp(w) = Amp(cz) = \alpha + \theta \\
& \Longrightarrow Amp(w) = Amp(c) + Amp(z)
\end{aligned}
$$

Geometrically: For all points (x, y) in the z-plane will be translated in w-plane by magnifying its shape by amount a and rotation of z-plane through an angle α.

Some Important Transformations:

3. Inversion:

Consider the transformation $w = \frac{1}{z}$

where $w = u + iv = \rho e^{i\phi}$, $z = x + iy = re^{i\theta}$

where$|z| = r = \sqrt{x^2 + y^2}$

$\& \ Amp(z) = \tan\theta = \frac{y}{x}$

$$w = \frac{1}{z} \qquad \Longrightarrow \rho e^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}\, e^{-i\theta}$$

$$\Longrightarrow \rho = \frac{1}{r} \quad \& \quad \phi = -\theta$$

Geometrically: Let point P(r, θ) in the z-plane and P will be translated to point Q$(\frac{1}{r}, -\theta)$ in w-plane.

This transformation $w = \frac{1}{z}$ is an inversion of z with respect to the unit circle $|z| = 1$ and then the reflection of this inverse along X-axis (Real axis).

Some Important Transformations:

4. Bilinear Transformation:

Consider the linear transformation

$$w = \frac{az+b}{cz+d} \quad \text{..........} \ (1)$$

where a, b, c and d are complex numbers and $ad - bc \neq 0$

is called a bilinear transformation (or linear fractional transformation). It is so named because it takes the form of the ratio of two linear functions.

❖ Bilinear transformation is a combination of three basictrasformations:

i) Translation
ii) Magnification & Rotation
iii) Inversion

❖ Bilinear transformation maps Circle onto Circle.

❖ The cross ratio is constant.

Refer: https://www.slideshare.net/mariolabestia/afirstcourseincomplexanalysis [Page No. 400, Chapter 7].

Bilinear Transformation:

Suppose we have four coefficients in the bilinear transformation, but only three of them are independent. There exists a unique bilinear transformation that maps three distinct points z_1, z_2, z_3 in the z-plane onto three distinct points w_1, w_2, w_3 in the w-plane, we assume that the six points are all finite.

Since z_j is mapped to w_j, where $j = 1, 2, 3$, it follows that

$$w_j = \frac{az_j + b}{cz_j + d}, \qquad j = 1, 2, 3$$

From eq. (1),

$$w - w_j = \frac{(ad - bc)(z - z_j)}{(cz + d)(cz_j + d)}, \qquad j = 1, 2$$

and $$w_3 - w_j = \frac{(ad - bc)(z_3 - z_j)}{(cz_3 + d)(cz_j + d)}, \qquad j = 1, 2$$

Bilinear Transformation:

we obtain the following formula for the required bilinear transformation:

$$\left.\frac{w-w_1}{w-w_2}\middle/\frac{w_3-w_1}{w_3-w_2}\right. = \left.\frac{z-z_1}{z-z_2}\middle/\frac{z_3-z_1}{z_3-z_2}\right.$$

$$\Longrightarrow \frac{(w-w_1)(w_3-w_2)}{(w-w_2)(w_3-w_1)} = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} \quad \dots\dots(2)$$

Here we substitute the values of $z_1, z_2,\ z_3$ and w_1, w_2, w_3 in eq.(2)

To find the relation between w and z or vice-versa.

#Refer https://www.slideshare.net/JITENDRASUWASIYA/complex-analysis-by-cambridge (Page No. 377).

Bilinear Transformation:

What happens when some of these points are not finite?

For example, when $z_1 \to \infty$, the right-hand side of eq. (2) is then replaced by

$$\lim_{z_1 \to \infty} \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{z_3 - z_2}{z - z_2}$$

This technique can be applied to other limiting cases, like $z_2 \to \infty$, $w_1 \to \infty$, etc., to find the corresponding reduced form of the bilinear transformation formula.

#Refer https://www.slideshare.net/JITENDRASUWASIYA/complex-analysis-by-cambridge (Page No. 377).

Example:

Find the bilinear transformation that carries the points $-1, \infty, i$ onto the points (a) $i, 1, 1+i$; (b) $\infty, i, 1$.

Solution:

Let $z_1 = -1, z_2 = \infty, \ z_3 = i$

a) Let $w_1 = i, w_2 = 1, \ w_3 = 1 + i$

$\therefore$ from eq.(2) and as $z_2 \to \infty$, we get,

$$\frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \lim_{z_2 \to \infty} \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$$

$$\Longrightarrow \frac{(w - i)(1 + i - 1)}{(w - 1)(1 + i - i)} = \frac{(z - z_1)}{(z_3 - z_1)}$$

$$\Longrightarrow \frac{(w-i)(i)}{(w-1)(1)} = \frac{(z+1)}{(1+i)}$$

Rearranging the terms, we obtain $w = \frac{z+2+i}{z+2-i}$.

Example:

Find the bilinear transformation that carries the points $-1, \infty, i$ onto the points (a) $i, 1, 1+i$; (b) $\infty, i, 1$.

Solution:

Let $z_1 = -1, z_2 = \infty, \ z_3 = i$

b) Let $w_1 = \infty, w_2 = i \ , \ w_3 = 1$

$\therefore$ from eq.(2) and as $w_1 \to \infty$ & $z_2 \to \infty$, we get,

$$\lim_{w_1 \to \infty} \frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)} = \lim_{z_2 \to \infty} \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$$

$$\Longrightarrow \frac{(w_3 - w_2)}{(w - w_2)} = \frac{(z - z_1)}{(z_3 - z_1)}$$

$$\Longrightarrow \frac{(1 - i)}{(w - i)} = \frac{(z + 1)}{(1 + i)}$$

Rearranging the terms, we obtain $w = \frac{iz + 2 + i}{z + 1}$.

Example:

Find the map of the straight line $\boldsymbol{y = x}$ under the transformation

$$\boldsymbol{w = \frac{z - 1}{z + 1}}$$

Solution:

Let $w = \frac{z-1}{z+1}$ $\quad\Longrightarrow zw + w = z - 1$

$$\Longrightarrow z(w - 1) = -1 - w = -(1 + w)$$

$$\Longrightarrow z = -\frac{1+w}{w-1} = \frac{1+w}{1-w}$$

$$\Longrightarrow z = x + iy = \frac{1+w}{1-w} = \frac{(1+u)+iv}{(1-u)-iv}$$

$$\Longrightarrow z = x + iy = \frac{(1+u)+iv}{(1-u)-iv} \times \frac{(1-u)+iv}{(1-u)+iv}$$

$$\Longrightarrow x + iy = \frac{\left(1-u^2-v^2\right)+i[v(1-u)+v(1+u)]}{(1-u)^2+v^2}$$

Example:

Find the map of the straight line $y = x$ under the transformation

$$\boldsymbol{w = \frac{z-1}{z+1}}$$

Solution:

$$\Rightarrow x + iy = \frac{(1-u^2-v^2)+i[v(1-u)+v(1+u)]}{(1-u)^2+v^2}$$

$$\Rightarrow x = \frac{(1-u^2-v^2)}{(1-u)^2+v^2}, \qquad y = \frac{[v(1-u)+v(1+u)]}{(1-u)^2+v^2}$$

Hence map of $y = x$ is

$$\Rightarrow x = \frac{(1-u^2-v^2)}{(1-u)^2+v^2} = \frac{[v(1-u)+v(1+u)]}{(1-u)^2+v^2} = y$$

$$\Rightarrow 1 - u^2 - v^2 = 2v$$

$$\Rightarrow u^2 + v^2 + 2v = 1 \qquad \Rightarrow u^2 + (v+1)^2 = 2$$

Which is a circle in w-plane with centre $(0, -1)$ and radius $= \sqrt{2}$.

Exercise:

Ex.1) Find the bilinear transformation which sends the points $1, i, 1-i$ from z-plane into the points $i, 0, -i$ of the w-plane.

Ex.2) Find the bilinear transformation which sends the points $-i, 0, 2+i$ from z-plane into the points $0, -2i, 4$ of the w-plane.

Ex.3) Find the bilinear transformation which sends the points $1, 0, i$ from z-plane into the points $\infty, -2, -\frac{1}{2}(1+i)$ of the w-plane.

Ex.4) Show that under the transformation $w = z + \frac{4}{z}$, the circle $|z| = 2$ is mapped onto the straight line, but the circle $|z| = 3$ is mapped on to the ellipse.