Solved examples on Cauchy's theorem and Cauchy's integral formula -

Ex. 1) Evaluate $\oint_C \frac{dz}{z-z_0}$ where

i) c is the closed contour containing the point z_0.

ii) point z_0 is outside the contour c.

Soln: i) Let $I = \oint_C \frac{dz}{z-z_0}$.

Choose $f(z) = 1$

$\therefore$ By Cauchys integral formula

$$\oint_C \frac{dz}{z-z_0} = 2\pi i f(z_0)$$

$$= 2\pi i \times (1) \quad \text{as} \quad f(z_0) = 1.$$

$$= 2\pi i$$

ii) Let $I = \oint_C \frac{dz}{z-z_0}$

Choose $f(z) = \frac{1}{z-z_0}$ It is analytic on and within a given region c. f(z) is not analytic at $z = z_0$, but c does not contain z_0.

$\therefore$ By Cauchy's theorem

$$\oint_C f(z)dz = 0 \quad \text{i.e.,} \quad \oint_C \frac{dz}{z-z_0} = 0$$

Ex. 2) Use Cauchy's integral formula to evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz \quad \text{where } c \text{ is } |z| = 3.$$

Soln - Let $$I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

$$I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz$$

$$\because \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= I_1 + I_2$$

Now $$I_1 = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$ Using Cauchy's integral formula.

$$= 2\pi i f(2)$$ Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

$f(2) = \sin 4\pi + \cos 4\pi = 1$

$$= 2\pi i (1)$$
$$= 2\pi i$$

$$I_2 = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz$$

$$= 2\pi i f(1)$$ By Cauchy's integral formula

$$= 2\pi i (-1)$$ As $f(1) = \sin \pi + \cos \pi$

$= -1$

$$= -2\pi i$$

$$\therefore I = I_1 + I_2 = 4\pi i$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i$$

Ex. 3) Evaluate $\oint_C \frac{e^z}{(z+1)^2(z+2)^2}\,dz$, where c is the contour $|z+1| = \frac{1}{2}$.

Soln - Let $I = \oint_C \frac{e^z}{(z+1)^2(z+2)^2}\,dz$

Choose $f(z) = \frac{e^z}{(z+2)^2}$ which is analytic on and within a circle $|z+1| = \frac{1}{2}$

$\therefore$ Using corollary to Cauchy's integral formula,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$ where a lies inside the region.

$$\therefore \quad I = \oint_C \frac{e^z}{(z+1)^2(z+2)^2}\,dz = \frac{2\pi i}{1!} f'(-1)$$ Here $a = -1$

$$= 2\pi i\, f'(-1)$$

We have $$f(z) = \frac{e^z}{(z+2)^2}$$

$$f'(z) = \frac{(z+2)^2 e^z - e^z 2(z+2)}{(z+2)^4}$$

$$f'(-1) = \frac{e^{-1} - 2e^{-1}}{1} = -e^{-1}$$

$$\therefore \quad I = \oint_C \frac{e^z}{(z+1)^2(z+2)^2}\,dz = \frac{-2\pi i}{e}.$$

Ex. 4) If $f(z_0) = \oint_C \frac{3z^3+5z+2}{(z-z_0)} dz$ where c is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$. Find (i) $f(1)$ (ii) $f''(1-i)$

Soln- (i) We have

$$f(z_0) = \oint_C \frac{3z^3+5z+2}{(z-z_0)} dz \quad \text{Take } z_0 = 1$$

$$\therefore f(1) = \oint_C \frac{3z^3+5z+2}{(z-1)} dz = 2\pi i\, g(1)$$ By Cauchy's integral formula

Here $g(z) = 3z^3+5z+2$

which is analytic on given region.

$\therefore g(1) = 10$

$\therefore \underline{f(1) = 20\pi i}$

ii) We have

$$f(z_0) = \oint_C \frac{3z^3+5z+2}{(z-z_0)} dz$$

Diff. w. r. to z_0

$$f'(z_0) = \oint_C \frac{3z^3+5z+2}{(z-z_0)^2} dz$$

Again diff. w. r. to z_0

$$f''(z_0) = 2\oint_C \frac{3z^3+5z+2}{(z-z_0)^3} dz$$

$$\therefore f''(1-i) = 2\oint_C \frac{3z^3+5z+2}{[z-(1-i)]^3} dz$$

$$= 2 \times \frac{2\pi i}{2!} g''(1-i) \quad \text{Here } g(z) = 3z^3+5z+2$$

$$g'(z) = 9z^2+5$$

$$g''(z) = 18z$$

$$\therefore f''(1-i) = 2\pi i\; 18(1-i)$$

$$= 36\pi(1+i)$$