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PAGE NO.
DATE

EM-III Assignment I

Roll no - 21409

Q1) If $v = 3x^2y - y^3$, find its harmonic conjugate u. Find $f(z) = u + iv$ in terms of z.

A) Given: $v = 3x^2y - y^3$

Differentiating v partially w.r.t x and y, we get

$$V_x = 6xy \quad ; \quad V_{xx} = 6y$$
$$V_y = 3x^2 - 3y^2 \quad ; \quad V_{yy} = -6y$$

then $V_{xx} + V_{yy} = 0$

$\therefore$ v is harmonic.

To find conjugate harmonic u of v:

From cauchy - riemann conditions:

$$u_x = V_y \quad \text{and} \quad u_y = -V_x$$

so $u_x = V_y = 3x^2 - 3y^2 \rightarrow (1)$

Integrating (1) partially w.r.t x, we get

$$u(x,y) = x^3 - 3xy^2 + c(y) \rightarrow (2)$$

Differentiating (2) partially w.r.t y and using second cauchy-riemann ($u_y = -V_x$), we have

$$-6xy + \frac{dc}{dy} = \frac{\partial u}{\partial y} = -V_x = -6xy$$

So $\frac{dc}{dy} = 0$ or $c =$ constant

Hence the conjugate harmonic u of v is

$$\boxed{u(x,y) = x^3 - 3xy^2 + c}$$

Roll no - 21409

PAGE NO :
DATE :

$\therefore f(z) = u + iv$

$= (x^3 - 3xy^2 + c) + i(3x^2y - y^3)$

$\boxed{f(z) = z^3 + c}$

Q2) If $f(z) = u + iv$ is analytic find $f(z)$, if $u - v = (x-y)(x^2 + 4xy + y^2)$

A) Given :

$u - v = (x-y)(x^2 + 4xy + y^2) \rightarrow (1)$

Differentiate w.r.t x

$u_x - v_x = (x^2 + 4xy + y^2) + (x-y)(2x + 4y) \rightarrow (2)$

Differentiate (1) w.r.t y:

$u_y - v_y = -(x^2 + 4xy + y^2) + (x-y)(4x + 2y) \rightarrow (3)$

using C-R equations in eqn (3)

$-v_x - u_x = -(x^2 + 4xy + y^2) + (x-y)(4x + 2y) \rightarrow (4)$

Adding (2) and (4)

$-2v_x = (x-y)(6x + 6y)$

$\therefore -v_x = 3(x-y)(x+y)$

$\therefore v_x = 3(y^2 - x^2) \rightarrow (5)$

subtract (4) from (2)

$2u_x = 2(x^2 + 4xy + y^2) + (x-y)(2y - 2x)$

$\therefore u_x = x^2 + 4xy + y^2 + 2xy - x^2 - y^2$

$u_x = 6xy \rightarrow (6)$

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Roll no - 21409

PAGE NO.
DATE

Now $f'(z) = u_x + iv_x$

$= 6xy + i[3(y^2-x^2)]$ from (5) & (6)

By milne - Thompson method

Put $x = z$, $y = 0$

$\therefore f'(z) = -3iz^2$

Integration gives:

$f(z) = -\frac{\cancel{3}iz^3}{\cancel{3}} + c$

$\boxed{f(z) = -iz^3 + c}$

Q3) Show that analytic function f(z) with constant modulus is constant

A) Let $f(z) = u+iv$ and $|f(z)| = c$, then $u^2+v^2=c^2$

Differentiating partially w.r.t x & y respectively

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \rightarrow (1)$

$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \Rightarrow u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \rightarrow (2)$

using cauchy Riemann equations (1) & (2)

$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \rightarrow (3)$

$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0 \rightarrow (4)$

Hence, $\left[u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}\right]^2 + \left[u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x}\right]^2 = 0$

$u^2\left[\frac{\partial u}{\partial x}\right]^2 + v^2\left[\frac{\partial u}{\partial x}\right]^2 + v^2\left[\frac{\partial u}{\partial y}\right]^2 + u^2\left[\frac{\partial u}{\partial y}\right]^2 = 0$

Roll no - 21409

PAGE NO.:
DATE:

$$(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=0$$

$$u^2+v^2=c \quad \text{(given)} \quad \rightarrow (5)$$

$$c^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]=0 \quad \rightarrow (6)$$

If $c=0$, then from (5), $u=0$, $v=0$

If $c\neq 0$ then by (6) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial y}{\partial y}\right)^2=0$

& by cauchy Riemann equation $\frac{\partial v}{\partial y}=0$, $\frac{\partial v}{\partial x}=0$

Integrating $\frac{\partial u}{\partial x}=0$ w.r.t x treating y as constant

$u=f(y)$. Differentiating $\frac{\partial u}{\partial y}=f'(y)=0$

$\therefore$ $f(y)=c$ or $u=$ constant

Similarly $v=$ constant. Hence $f(z)=$ constant.

Q4) If $f(z)$ is analytic show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4=16|f(z)|^2|f'(z)|^2$$

A) We know that

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=4\cdot\frac{\partial^2}{\partial z\cdot\partial\bar{z}} \quad \rightarrow (1)$$

$$L.H.S=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4$$

$$=\frac{4\partial^2}{\partial z\cdot\partial\bar{z}}|f(z)|^4 \quad \text{from (1)}$$

Roll no - 21409

PAGE NO.
DATE

$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left(f(z) f(\bar{z})\right)^2$ $\qquad |f(z)|^2 = f(z).\overline{f(z)}$

$\qquad\qquad\qquad\qquad = f(z).f(\bar{z})$

$= 4\frac{\partial^2}{\partial z.\partial \bar{z}}(f(z))^2 (f(\bar{z}))^2$

$= 4\frac{\partial}{\partial z}\left[(f(z))^2 . 2. f(\bar{z}).f'(\bar{z})\right]$

$= 8\left[2.f(z).f'(z).f(\bar{z}).f'(\bar{z})\right]$

$= 16\left[f(z).f(\bar{z})\, f'(z).f'(\bar{z})\right]$

$= 16\left[f(z).\overline{f(z)}\, f'(z).\overline{f'(z)}\right]$

$= 16|f(z)|^2|f'(z)|^2$

$=$ R.H.S

Hence proved.

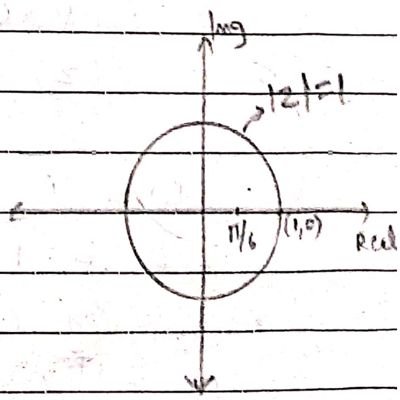
Q5) Evaluate $\oint_c \frac{\sin^2 z}{(z-\pi/6)^3}dz$, where c is $|z|=1$

A) Let $I = \oint_c \frac{\sin^2 z}{(z-\pi/6)^3}$

To evaluate I when $c = |z|=1$

equate denominator of $\frac{\sin^2 z}{(z-\pi/6)^3}$ i.e $(z-\pi/6)^3$ to 0

$\therefore z = \pi/6$ which lies within the closed curve 'c' : $|z|=1$



Consider

$f(z) = \sin^2 z$ which is analytic and within c and $z=\pi/6$ is any point within 'c'

$\therefore$ by cauchy's integral formula

$$\oint_{c:|z|=1} \frac{\sin^2 z}{(z-\pi/6)^3}dz = 2\pi i\, f(\pi/6)$$

Roll no-21409

PAGE NO :
DATE :

$$= 2\pi i \left(\sin^2\left(\frac{\pi}{6}\right) \right)$$

$$= 2\pi i \left(\frac{1}{4} \right)$$

$$\boxed{I = \frac{\pi i}{2}}$$

$$\therefore \oint_c \frac{\sin^2 z}{(z - \pi/6)^3} = \frac{\pi}{2} i \qquad \text{where } c \Rightarrow |z| = 1$$

Q6) Use cauchy's Integral formula to evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

A) let

$$I = \oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

To evaluate, equate denominator to zero

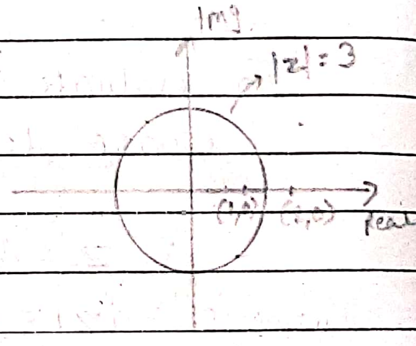
i.e $(z-1)(z-2) = 0$

$z = 1, \; z = 2$

clearly,

$z = 1$ lies inside c &

$z = 2$ lies outside c



$\therefore$ Consider

$$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2 \; \cancel{dz}}{\cancel{(z-1)}}$$

which is analytic on and within curve $C: |z| = 3$ and $z = 1$ is any point in c.

$\therefore$ By cauchy's integral formula for

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz = 2\pi i f(1)$$

$$= 2\pi i \left(\sin \pi + \cos \pi \right)$$

Roll no-21409

$$\boxed{\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}\, dz = -2\pi i}$$