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EM-III Assignment -1

Q.1. If $v = 3x^2y - y^3$, find its harmonic conjugate u. Find $f(z) = u + iv$ in terms of z.

given,

$$v = 3x^2y - y^3$$

Differentiating v partially w.r.t. x & y, we get

$$v_x = 6xy \quad ; \quad v_{xx} = 6y$$
$$v_y = 3x^2 - 3y^2 \quad ; \quad v_{yy} = -6y$$

then $v_{xx} + v_{yy} = 0$

$\therefore$ v is harmonic.

To find conjugate harmonic u of v from cauchy - riemann conditions:

$$u_x = v_y \quad \& \quad u_y = -v_x$$

so, $u_x = v_y = 3x^2 - 3y^2$ — (1)

Integrating (1) partially w.r.t x, we get

$$u(x,y) = x^3 - 3xy^2 + c(y) \quad — (2)$$

Differentiating (2) partially w.r.t. y & using second cauchy-riemann $(u_y = -v_x)$, we have

$$-6xy + \frac{\partial c}{dc} = \frac{\partial u}{\partial y} = -v_x = -6xy$$

$$\frac{dc}{dy} = 0 \quad \text{or} \quad c = \text{constant}$$

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hence the conjugate harmonic u of v is

$$\boxed{u(x,y) = x^3 - 3xy^2 + c}$$

$\therefore$ $f(z) = u + iv$

$= (x^3 - 3xy^2 + c) + i(3x^2y - y^3)$

$$\boxed{f(z) = z^3 + c}$$

Q.2. If $F(z) = u + iv$ is analytic find $f(z)$ if $u - v = (x-y)(x^2+4xy+y^2)$ -

given,

$u - v = (x-y)(x^2+4xy+y^2)$ — ①

Differentiate w.r.t. x

$u_x - v_x = (x^2+4xy+y^2) + (x-y)(2x+4y)$ - ②

Differentiate ① w.r.t. y

$u_y - v_y = -(x^2+4xy+y^2) + (x-y)(4x+2y)$ — ③

using C-R equations in eqⁿ ③

$-v_x - u_x = -(x^2+4xy+y^2) + (x-y)(4x+2y)$ ④

Adding ② & ④

$-2v_x = (x-y)(6x+6y)$

$\therefore -v_x = 3(x-y)(x+y)$

$\therefore v_x = 3(y^2 - x^2)$ — ⑤

subtract ④ from ②

$2u_x = 2(x^2+4xy+y^2) + (x-y)(2y-2x)$

$\therefore u_x = x^2 + 4xy + y^2 + 2xy - x^2 - y^2$

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$u_x = 6xy$ — (6)

Now $f'(z) = u_x + iv_x$

$= 6xy + i[3(y^2 - x^2)]$ from (5) & (6)

By milne - Thompson method by substituting $x = z$ & $y = 0$ we get,

$\therefore f'(z) = -3iz^2$

Integrating both sides we get,

$f(z) = \frac{-\cancel{3}iz^3}{\cancel{3}} + c$

$$\boxed{f(z) = -iz^3 + c}$$

Q.3. show that analytic function $f(z)$ with constant modulus is constant.

→ Let $f(z) = u + iv$ be analytic function with constant argument & let $\arg(f(z)) = K =$ constant.

$\therefore \arg(u + iv) = K$

$\tan^{-1}\left(\frac{v}{u}\right) = K$

$\therefore v = u \tan K$ — (1)

case i: suppose $K = 0$, then $v = 0$

then $v_x = v_y = 0$

By C-R eq^n we get $u_x = u_y = 0$

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$\therefore$ u is a constant function

$\therefore$ $f(z) = u + iv$ = constant fun^n.

case ii) : Differentiating eq^n ① w.r.t x, we get

$$V_x = u_x \tan k$$

$$\therefore \quad u_x \tan k - V_x = 0 \quad — ②$$

Diff eq^n ① w.r.t y, we get,

$$V_y = u_y \tan k$$

By C.R. eq^n, we get

$$u_x = -V_x \tan k$$

$$\therefore \quad -u_x + V_x \tan k = 0 \quad — ③$$

multiplying eq^n ② by tan k, we get

$$u_x \tan^2 k - V_x \tan k = 0 \quad — ④$$

Adding eq^n ③ & ④, we get

$$u_x (1 + \tan^2 k) = 0$$

$$u_x = 0$$

similarly, we get $V_x = 0$

$$\therefore \quad f'(z) = u_x + iV_x = 0$$

$\therefore$ f(z) is a constant function.

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Q.4. If $f(z)$ is analytic : show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4 = 16|f(z)|^2|f'(z)|^2$$

we know that

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{4\,\partial^2}{\partial z\cdot\partial\bar{z}} \quad -①$$

$$L.H.S = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^4$$

$$= \frac{4\partial^2}{\partial z\cdot\partial\bar{z}}|f(z)|^4 \quad (\text{from } ①)$$

$$= \frac{4\partial^2}{\partial z\partial\bar{z}}\left(f(z)f(\bar{z})\right)^2$$

$$= \frac{4\,\partial^2}{\partial z\cdot\partial\bar{z}}(f(z))^2(f(\bar{z}))^2$$

$$= 4\frac{\partial}{\partial z}\left[(f(z))^2\cdot 2\,f(\bar{z})\cdot f'(z)\right]$$

$$= 8\left[2\cdot f(z)\cdot f'(z)\cdot f(\bar{z})\cdot f'(\bar{z})\right]$$

$$= 16\left[f(z)\cdot f(\bar{z})\,f'(z)\cdot f(\bar{z})\right]$$

$$= 16\left[f(z)\cdot f(\bar{z})\,f'(z)\cdot f'(z)\right]$$

$$= 16\,|f(z)|^2\,|f'(z)|^2$$

$$= R.H.S$$

Q.5 Evaluate $\oint \frac{\sin^2 z}{(z-\pi/6)^3}\,dz$, wheee c is

$|z| = 1$

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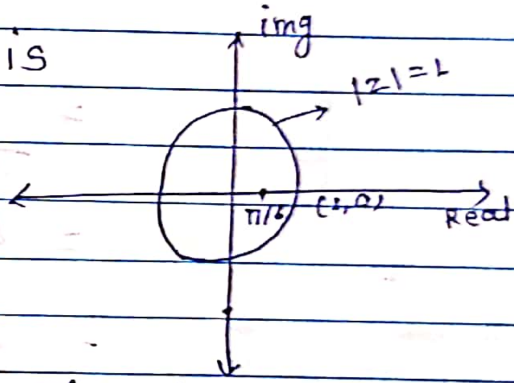
let $I = \oint \frac{\sin^2 z}{(z - \pi/6)^3}$

To evaluate I when $c = |z| = 1$.
equate denominator of $\frac{\sin^2 z}{(z-\pi/6)^3}$ i.e.
$(z - \pi/6)^3$ to 0

$\therefore$ $z = \pi/6$ which lies within the closed curve 'c' $|z| = 1$
consider
$f(z) = \sin^2 z$ which is analytic & within c &
$z = \pi/6$ is any point within 'c'.



$\therefore$ By Cauchy's Integral formula

$$\oint_{C: |z|=1} \frac{\sin^2 z}{(z - \pi/6)^3} dz = 2\pi i \, f(\pi/6)$$

$$= 2\pi i \left(\sin^2 \left(\frac{\pi}{6}\right)\right)$$

$$= 2\pi i \left(\frac{1}{4}\right)$$

$$\boxed{I = \frac{\pi}{2} i}$$

$\therefore \oint \frac{\sin^2 z}{(z - \pi/6)^3} = \frac{\pi}{2} i$ where $c \Rightarrow |z| = 1$

Q.6. Use cauchy's Integral formula to evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is circle $|z| = 3$

→ let

$$I = \oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

To evaluate, equate denominator to 0

i.e. $(z-1)(z-2) = 0$

$z=1$, $z=2$

clearly

$z=1$ lies inside c &

$z=2$ lies outside c

∴ consider

$f(z) = \sin \pi z^2 + \cos \pi z^2$

which is analytic on & within curve $c: |z| = 3$ & $z=1$ is any point in c

∴ by cauchy's Integral formula

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz = 2\pi i f(1)$$

$$= 2\pi i (\sin \pi + \cos \pi)$$

$$\boxed{\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz = -2\pi i}$$