Complex Numbers

Complex number – Def^n → A number of the form $x+iy$, where x, y are real numbers is defined as Complex no.

If $z = x+iy$ then x = Real part of z
y = Imaginary part of z
i.e. $Re(z) = x$ & $Im(z) = y$

If $x=0$ & $y \neq 0$ then $z = iy$ is purely Imaginary number.
If $x \neq 0$ & $y = 0$ then $z = x$ is ~~purely~~ real number.

1.

Algebra of Complex numbers –

If $z_1 = x_1+iy_1$ & $z_2 = x_2+iy_2$ are any two complex numbers.

① Equality → $z_1 = z_2$ iff $x_1 = x_2$ & $y_1 = y_2$

2) Addition → $z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$

3) Subtraction → $z_1 - z_2 = (x_1-x_2)+i(y_1-y_2)$

4) Multiplication → $z_1 \cdot z_2 = (x_1x_2 - y_1y_2)+i(x_1y_2+y_1x_2)$

5) Division → $\dfrac{z_1}{z_2} = \dfrac{\cancel{x_1 x_2}\; x_1x_2+y_1y_2}{(x_2^2+y_2^2)} + i\,\dfrac{y_1x_2 - x_1y_2}{x_2^2+y_2^2}$

6) Commutative law of Addition & multiplication –

$z_1+z_2 = z_2+z_1$ & $z_1z_2 = z_2z_1$

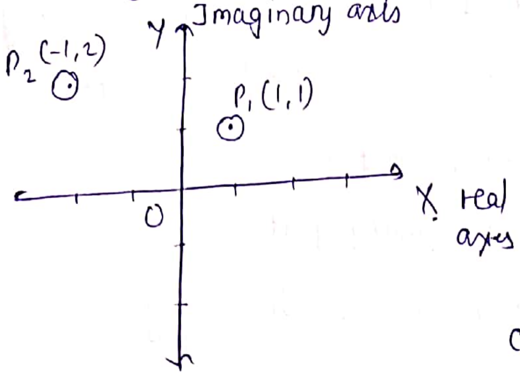
7) Associative law of Addition & multiplication:

$z_1+(z_2+z_3) = (z_1+z_2)+z_3$ where $z_3 = x_3+iy_3$

$z_1(z_2z_3) = (z_1z_2)z_3$

8) Distributive law → $z_1(z_2+z_3) = z_1z_2+z_1z_3$

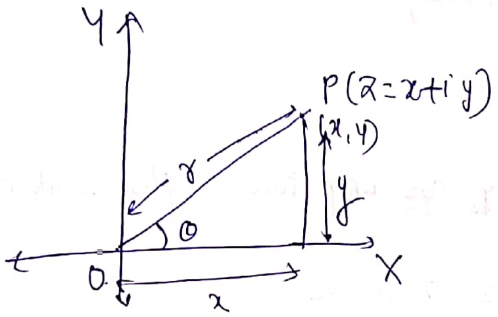
Graphical or Geometrical representation of complex numbers (Argand Diagram)



$z_1 = 1+i$, $z_2 = -1+2i$

$\Downarrow$ $P_1(1,1)$ $\Downarrow$ $P_2(-1,2)$

The method of representing complex numbers by points in a plane is called as Argand diagram.



A complex number $z = x+iy$ can also be represented by vector OP whose initial point is origin & whose terminal point P is the point (x,y)

The complex number $z = x+iy$ as

① The point z whose co-ordinates are (x,y)

② the vector OP from O to $P(x,y)$.

Complex Conjugate numbers –

If $z = x+iy$ then complex conjugate of z is $\bar{z} = x-iy$

Ex If $z = 5+2i$ then $\bar{z} = 5-2i$

It is just reflection of the point z in the x-axis.

Note → $Re(z) = \frac{1}{2}(z+\bar{z})$, $Im(z) = \frac{1}{2i}(z-\bar{z})$

$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Absolute value – (Modulus of complex number)

The absolute or modulus of complex number $z = x+iy$ is defined by non-negative real number $\sqrt{x^2+y^2}$ of denoted by $|z|$ i.e. $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} \geq 0$

Ex. $|-4+2i| = \sqrt{(-4)^2+(2)^2} = 2\sqrt{5}$.

Geometrically →

1. $|z|$ is the distance of the point $P(x,y)$ from the origin.
2. $|z_1 - z_2|$ is the distance betn the points z_1 & z_2

   i.e. $|z_1 - z_2| = |(x_1-x_2)+i(y_1-y_2)| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

   ~~Ex.~~

3. $|z_1| > |z_2|$ means that the point z_1 is farther from the origin than the point z_2 ∵ absolute values are real no's.

4. $z_1 > z_2$ or $z_1 < z_2$ are meaningless, unless z_1 & z_2 are both real.

Ex. $|z-i| = 5$ ⇒ represents the point 'z' on the circle of radius 5 with centre at $(0,1)$.

Polar form of complex number →

let $z = x+iy = \{(x,y) \mid x, y \in \mathbb{R}\}$, $\mathbb{R}$ ⇒ Set of Real numbers

(x,y) ⇒ cartesian co-ordinates of point P

(r,θ) ⇒ Polar —"— —"—

Here $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow z = x+iy = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

'θ' is amplitude or argument of z

& r is modulus of z.

Here $\arg(z) = \theta + 2n\pi$, $n = 0, 1, 2, \cdots$

where $\theta + 2n\pi$ is General value of amp. of z.

& if 'θ' $\in$ lies betn $-\pi$ to π is principal value of amplitude.

$\arg(z) \rightarrow$

1. If $x>0, y>0$ then $\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$
2. If $x<0, y>0$ then $\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$
3. If $x<0, y<0$ then $\theta = \tan^{-1}\left|\frac{y}{x}\right| \pm \pi$
4. If $x>0, y<0$ then $\theta = -\tan^{-1}\left|\frac{y}{x}\right|$

The Complex number $z = x+iy$ has following three forms-

$z = x+iy \Rightarrow$ Cartesian form

$= r(\cos\theta + i\sin\theta) \Rightarrow$ Polar form

$= re^{i\theta} \Rightarrow$ Exponential form.

Graphical or Geometrical representation-

1. $z_1 + z_2 \Rightarrow$ Parallelogram law of vectors
2. The product of complex numbers is a complex

number whose modulus is the product of their moduli & whose amplitude is the sum of their amplitudes

$|z_1 \cdot z_2| = r_1 r_2$

$Amp(z_1 \cdot z_2) = Amp(z_1) + Amp(z_2)$

③ $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ & $amp\left(\frac{z_1}{z_2}\right) = amp(z_1) - amp(z_2)$

DeMoivre's Theorem-

for any real number 'n',

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

Application of DeMoivre's Theorem to solve Algebraic Equations.

Here $z = x+iy = r(\cos\theta + i\sin\theta)$

$= r(\cos(2n\pi+\theta) + i\sin(2n\pi+\theta))$

$\therefore z^{1/q} = (x+iy)^{1/q} = r^{1/q}\left[\cos\left(\frac{2n\pi+\theta}{q}\right) + i\sin\left(\frac{2n\pi+\theta}{q}\right)\right]$

where $n = 0, 1, 2, \ldots q-1$

n is positive integer

Hyperbolic functions –

As $e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$\Rightarrow \cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

& $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Rel^n between circular & Exponential function (for x-real)

If $z = x + iy$

then $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ & $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

lly $\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$, $\cot z = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

$\sec z = \dfrac{2}{e^{iz} + e^{-iz}}$, $\operatorname{cosec} z = \dfrac{2i}{e^{iz} - e^{-iz}}$

Here, period of $\sin z$ & $\cos z$ are 2π

& period of $\tan z$ is π.

Hyperbolic function –

Hyperbolic Sine of 'x' denoted by Sinhx is defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic Cosine of x denoted by coshx is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

My $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$, $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$

As $\sinh(-x) = -\sinh x$ $\Rightarrow$ Odd function

$\cosh(-x) = \cosh x$ $\Rightarrow$ Even function.

Relation betn Circular & Hyperbolic functions –

1] $\sin(ix) = i\sinh x$

2] $\cos(ix) = \cosh x$

3] $\tan(ix) = i\tanh x$

4] $\cot(ix) = -i\coth x$

5) $\sec(ix) = \operatorname{sech} x$

6) $\operatorname{cosec}(ix) = -i\operatorname{cosech} x$

7] $\sinh(ix) = i\sin x$

8] $\cosh(ix) = \cos x$

9) $\coth(ix) = -i\cot x$

10) $\tanh(ix) = i\tan x$

11] $\operatorname{sech}(ix) = \sec x$

12] $\operatorname{cosech}(ix) = -i\operatorname{cosec} x$

Period of $\sinh z$ & $\cosh z$ are $2\pi i$.

Formulae of Hyperbolic functions -

1) $\cosh^2 x - \sinh^2 x = 1$

2) $\operatorname{sech}^2 x + \tanh^2 x = 1$

3) $\coth^2 x - \operatorname{cosech}^2 x = 1$

4) $\sinh(x \pm y) = \sinh x . \cosh y \pm \cosh x . \sinh y$

5) $\cosh(x \pm y) = \cosh x . \cosh y \pm \sinh x . \sinh y$

6) $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2(\cosh^2 x - 1)$
$= 1 + 2\sinh^2 x.$

7) $\sinh 2x = 2\sinh x . \cosh x.$

Differentiation -

If $y = \sinh x$ then $\frac{d}{dx}\sinh x = \cosh x$

If $y = \cosh x$ then $\frac{d}{dx}(\cosh x) = \sinh x$

If $y = \tanh x$ then $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Integration -

$$\int \cosh x \, dx = \sinh x$$

$$\int \sinh x \, dx = \cosh x$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x$$

Logarithm of Complex numbers -

If $z = x + iy$

$\operatorname{Log} z = \log r + i(2n\pi + \theta)$ => General value of Logarithm

$n = 0, 1, 2, \cdots,\ r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\frac{y}{x}$

& $\log z = \log r + i\theta$ => Principal value of Logarithm.