COMPLEX INTEGRATION

* Line Integral :-

Line integral of $f(z)$ taken along the curve 'C' is denoted as

$$I = \int_C f(z)\,dz$$

$\because$ $f(z) = u + iv$ & as $z = x + iy \Rightarrow$
$dz = dx + i\,dy$

$$\therefore \quad I = \int_C (u+iv)(dx + i\,dy)$$

$$I = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$ //

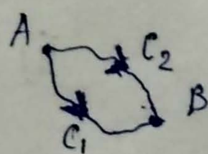
* Note:-

i) Line integral of $f(z)$ taken along the closed curve 'C' is denoted as.

$$I = \oint_C f(z)\,dz$$

ii) The value of I will depends upon the path of integration or the equation of the curve joining the points.

i.e. $I_1 = \int_{C_1} f(z)\,dz$ & $I_2 = \int_{C_2} f(z)\,dz$



$\Rightarrow I_1 \neq I_2$

iii) If $f(z)$ is analytic, then value of I will be independent of the paths joining the same points.

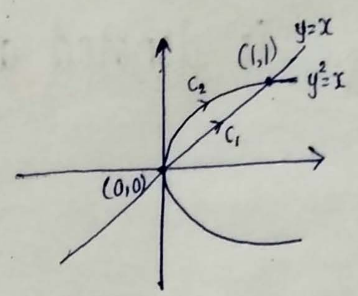
i.e. $I_1 = I_2$

* Examples:-

1) Evaluate $\int_C f(z)\,dz$ where $f(z)=\bar{z}$ and C is the curve i) C: straight line joining $y=x$ joining $(0,0)$ to $(1,1)$
ii) C: parabola $y^2=x$ joining $(0,0)$ & $(1,1)$

→ Let,

$$I = \int_C f(z)\,dz$$



Here $f(z) = \bar{z} = x - iy$

$dz = dx + i\,dy$

$$\therefore\ I = \int_C \bar{z}\,dz = \int_C (x-iy)(dx+idy)$$

$$I = \int_C (x\,dx + y\,dy) + i\int_C (x\,dy - y\,dx) \quad \text{—— (1)}$$

i) Along curve C: st. line $y=x$ joining $(0,0)$ & $(1,1)$

Consider $y = x \Rightarrow dy = dx$

along C, $x: 0$ to 1

$$\therefore\ I = \int_0^1 (x\,dx + x\,dx) + i\int_0^1 (x\,dx - x\,dx)$$

$$= \int_0^1 2x\,dx + 0 = [x^2]_0^1$$

$I = 1$ //

ii) Along curve C: parabola $y^2=x$ joining $(0,0)$ & $(1,1)$

Consider $y^2 = x$

$\Rightarrow dx = 2y\,dy$

along C, $y: 0$ to 1

$$\therefore\ I = \int_0^1 [y^2\,2y\,dy + y\,dy] + i\int_0^1 [y^2 dy - y\,2y\,dy]$$

$$= \int_0^1 (2y^3+y)\,dy + i\int_0^1 (-y^2)\,dy$$

$$= \left[\frac{y^4}{2} + \frac{y^2}{2}\right]_0^1 + i\left[-\frac{y^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2}+\frac{1}{2}\right) - i\,\frac{1}{3}$$

$$I = 1 - i\,\frac{1}{3}$$

Here Note (2) is verified

2) Evaluate $\int_C f(z)dz$ where $f(z)=z^2$ and C is the path joining the points $A(z=0)$ & $B(z=1+i)$, where

i) C: parabola $y=x^2$ joining A & B

ii) C: parabola $x=y^2$ joining A & B

→ Let,

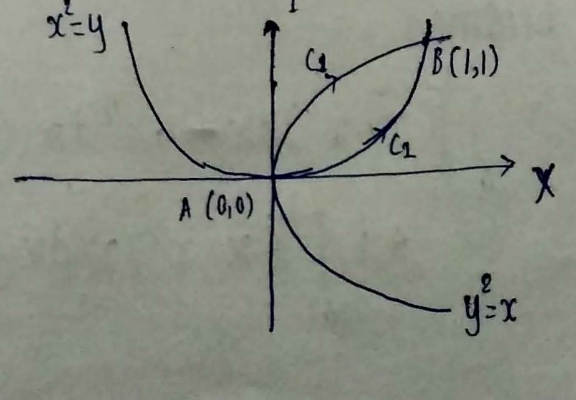
$$I = \int_C f(z)\,dz$$

Here $f(z) = z^2 = (x+iy)^2 = (x^2-y^2)+i(2xy)$

$dz = dx + i\,dy$

$$\therefore I = \int_C z^2\,dz$$

$$I = \int_C [(x^2-y^2)dx - 2xy\,dy] + i\int_C [(x^2-y^2)dy + 2xy\,dx] \quad \text{— (1)}$$



i) Along curve C: parabola $y^2 = x$ joining $A(z=0)$ i.e. $A(0,0)$ & $B(z=1+i)$ i.e. $B(1,1)$

Consider $y^2 = x$

$\Rightarrow dx = 2y\,dy$

And along C, y : 0 to 1

Hence ① becomes.

$$I = \int_0^1 [(y^4 - y^2)2y\,dy - 2y^3 dy]$$

$$+ i\int_0^1 [(y^4 - y^2)\,dy + 4y^4 dy]$$

$$= \int_0^1 (2y^5 - 4y^3)\,dy + i\int_0^1 (5y^4 - y^2)\,dy$$

$$= \left[\frac{y^6}{3} - y^4\right]_0^1 + i\left[y^5 - \frac{y^3}{3}\right]_0^1$$

$$= \left[\frac{1}{3} - 1\right] + i\left[1 - \frac{1}{3}\right]$$

$$I = -\frac{2}{3} + i\frac{2}{3} = \frac{2}{3}(-1+i) \,//$$

ii) Along curve C: parabola $x^2 = y$ joining $A(0,0)$ & $B(1,1)$

Consider $x^2 = y$

$\Rightarrow dy = 2x\,dx$

Along C, x : 0 to 1

Hence ① becomes

$$I = \int_0^1 [(x^2 - x^4)\,dx + 4x^4\,dx$$

$$+ i\int_0^1 [(x^2 - x^4)2x\,dx + 2x^3 dx$$

$$I = \int_0^1 (x^2 - 5x^4)\,dx + i\int_0^1 (4x^3 - 2x^5)\,dx$$

$$= \left[\frac{x^3}{3} - x^5\right]_0^1 + i\left[x^4 - \frac{x^6}{3}\right]_0^1$$

$$= \left[\frac{1}{3} - 1\right] + i\left[1 - \frac{1}{3}\right]$$

$$I = -\frac{2}{3} + i\frac{2}{3} = \frac{2}{3}(-1+i)$$

Here note (3) is verified
(As $f(z)$ is analytic)

OR

As $f(z) = z^2 = (x^2 - y^2) + i\,2xy$

$\therefore$ $u = x^2 - y^2$ & $v = 2xy$

$u_x = 2x,\ u_y = -2y,\ v_x = 2y,\ v_y = 2x$

$\Rightarrow$ $u_x = v_y$ & $u_y = -v_x \quad \forall x, y$

$\Rightarrow$ CR equations are satisfied

$\Rightarrow$ $f(z)$ is analytic

Hence $I = \int_C f(z)\,dz$ is independent of path.

$$\therefore I = \int_{z=0}^{z=1+i} z^2\,dz$$

$$= \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{1}{3}\left[(1+i)^3 - 0\right]$$

$$= \frac{1}{3}\left[1 + 3i - 3 - i\right]$$

$$I = \frac{2}{3}[-1+i]$$

3) Evaluate $\int_{1-i}^{2+i} (2z+4)\,dz$ along the path $x=t+1,\ y=2t^2-1$.

→ Let,

$$I = \int_{1-i}^{2+i} (2z+4)\,dz$$

Consider, $f(z) = 2z+4$

$f(z)$ is not analytic function (?)

Hence I is dependent of path

$$\therefore\ I = \int_{1-i}^{2+i} [2(x+iy)+4]\,(dx+i\,dy)$$

$$= \int_{1-i}^{2+i} [(2x+4)dx-2y\,dy] + i[2y\,dx+(2x+4)dy]$$

Eq^n of curve is.

$$x = t+1 \quad \& \quad y = 2t^2-1$$

$$\therefore\ dx = dt \quad \& \quad dy = 4t\,dt$$

Along C, $x : 1$ to 2 & $y: -1$ to 1

$\Rightarrow$ $t : 0$ to 1

$$\therefore\ I = \int_0^1 [(2t+6)\,dt - (16t^3-8t)\,dt$$

$$+i\,[(4t^2-2)\,dt + (8t^2+24t)\,dt]$$

$$= \int_0^1 [-16t^3+10t+6]\,dt + i[12t^2+24t-2]\,dt$$

$$= \{[-4t^4+5t^2+6t] + i\,[4t^3+12t^2-2t]\}_0^1$$

$$= [-4+5+6] + i[4+12-2]$$

$$I = 7+14i \;//$$

Assignment

Q.1) Evaluate $\int_C f(z)\,dz$ where $f(z) = z^2 + 2z$ along the curve 'c' which is a straight line $y = x$ joining $z = 0$ & $z = 2+2i$

Q.2) Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$, along

i) the line $y = \frac{x}{2}$

ii) the real axis to 2 & then vertically to $2+i$

Q.3) Evaluate $\int_0^{1+i} (x^2 + iy)\,dz$ along the paths

i) $y = x$, ii) $y = x^2$

Q.4) Evaluate $\int_C \frac{2z-1}{z}$, where c is lower half of the circle $|z| = 3$ described in anticlockwise direction

Q.5) Evaluate $\int_{z=0}^{z=1+4i} (2z + \bar{z})\,dz$ along the path $x = t, y = 4t$