Complex Analysis I

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Contents

Chapter 1

Not sure yet

1.1 Basics of analytic functions

1.1.1 Cauchy-Riemann equation

Let Ω be a connect open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$.

Definition 1.1.1. For $a \in \Omega$,

- $\lim\limits_{z \in a} f(z) = A \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall z \in \Omega$ and $0 < |z - a| < \delta \rightsquigarrow |f(z) - A| < \varepsilon$.
- $f(z)$ is **continuous** at a if $\lim\limits_{z \to a} f(z) = f(a)$.
- $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ provide the limit exists.

Observation : $f(z) = u(z) + iv(z)$ can be regard as $f(x, y) = u(x, y) + iv(x, y)$ where $z = x + iy$, then $f : (x, y) \mapsto (u(x, y), v(x, y))$. Recall that

- f is conti. at $z_0 = (x_0, y_0) \iff u, v$ are conti. at z_0
- f is differentiable at $z_0 \implies f$ is conti. at z_0.

Also, $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2 = d((x, y), (x_0, y_0))^2$, so we have same result what we learn in calculus in $\mathbb{R}^2$.

Now we see some different between $\mathbb{C}$ and $\mathbb{R}^2$:

Theorem 1.1.1 (Cauchy Riemann equation)**.** Let $u, v \in C^1(\Omega)$. Then

$$f \text{ is differentiable} \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof: Let $z = x + iy \in \Omega$. Since Ω is open, we can that the line segment $\overline{z(z + \Delta z)} \subseteq \Omega$, where $\Delta z = \Delta x + i\Delta y$.

- $(\Rightarrow)$: Since $f'(z)$ exists, we have

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - (u(x, y) + iv(x, y))}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

and

$$f'(z) = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - (u(x, y) + iv(x, y))}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$$

Hence we have $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$ or $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$.

- $(\Leftarrow)$: Since u, v are differentiable,

$$\begin{cases} \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + |\Delta z|\psi_1(\Delta z) \\ \Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + |\Delta z|\psi_2(\Delta z) \end{cases}$$

where $\psi_1(\Delta), \psi_2(\Delta z) \to 0$ as $\Delta z \to 0$. Combine with assumption we have

$$\begin{aligned} \frac{\Delta f}{\Delta z} = \frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z} &= \frac{1}{\Delta z}\left(\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(\Delta x + i\Delta y) + |\Delta z|(\psi_1(\Delta z) + \psi_2(\Delta z))\right) \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + \underbrace{\frac{|\Delta z|}{\Delta z}(\psi_1(\Delta z) + \psi_2(\Delta z))}_{\to 0 \text{ as } \Delta z \to 0} \end{aligned}$$

Hence $\lim\limits_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ exists.

□

Definition 1.1.2.

- $f(z)$ is said to be **analytic** in open set Ω if it has derivative at each point of Ω.
- $f(z)$ is called an **entire function** if it is analytic in $\mathbb{C}$.

Example 1.1.1. $f(z) = \mathrm{Re} z$ is continuous but nowhere analytic since $\frac{\partial u}{\partial x} = 1 \neq 0 = \frac{\partial v}{\partial y}$.

Corollary 1.1.1. If $f'(z) = 0$ in open connected subset Ω, then f is constant in Ω.

Proof: By $f'(z) = 0$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and thus $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Given point $z, z' \in \Omega$, use line segment $\overline{z_k z_{k+1}}$ which parallel x-axis or y-axis to connect $z := z_0, z' := z_n$. By partial derivative of u, v with respect to x, y are all zero we have

$$f(z) = f(z_1) = \cdots = f(z_{n-1}) = f(z')$$

□

1.1.2 Change of coordinate

1. complex conjugate

If $z = x + iy$, $\overline{z} = x - iy$, then $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2} \rightsquigarrow f(x, y) = f(z, \overline{z})$.
By chain rule,

$$\begin{cases} \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \end{cases} \implies \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \\ \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \end{cases}$$

Hence, f is differentiable $\iff \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \iff \frac{\partial f}{\partial \overline{z}} = 0$. So $f'(z) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}$.

Example 1.1.2. $f(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z] \rightsquigarrow f'(z) = n a_n z^{n-1} + \cdots + a_1$

Theorem 1.1.2 (Lucas's theorem). The half plane that contains the zeros of $f(z)$ also contains the zeros of $f'(z)$.

Proof: Recall that give two point a, b on line ℓ with $b \neq 0$, then we can write ℓ by $z = a + tb$ with $t \in \mathbb{R} \iff \operatorname{Im}\left(\frac{z-a}{b}\right) = 0$. So two half plane cut by ℓ are

$$H^+ := \operatorname{Im}\left(\frac{z-a}{b}\right) > 0 \text{ and } H^- := \operatorname{Im}\left(\frac{z-a}{b}\right) < 0$$

Let $f(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$ with $\alpha_i \in H^-$ $\forall i = 1, ..., n$. Notice that

$$\frac{f'(z)}{f(z)} = \frac{1}{z-\alpha_1} + \cdots + \frac{1}{z-\alpha_n}$$

Assume $z_0 \in H^+ \cup \ell$ i.e. $\operatorname{Im}(\frac{z_0-a}{b}) \geq 0$, then

$$\operatorname{Im}\left(\frac{z_0-\alpha_i}{b}\right) = \operatorname{Im}\left(\frac{z_0-a}{b}\right) - \operatorname{Im}\left(\frac{\alpha_i-a}{b}\right) > 0 \implies \operatorname{Im}\frac{b}{z_0-\alpha_i} < 0$$

Hence $\operatorname{Im}\frac{bf'(z_0)}{f(z_0)} = \sum_{i=1}^{n} \operatorname{Im}\frac{b}{z_0-\alpha_i} < 0$ i.e. $bf'(z_0) \neq 0 \rightsquigarrow f'(z_0) \neq 0$. $\square$

2. polar coordinate

Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases} \rightsquigarrow f(x,y) = f(r,\theta).$

$$\begin{cases} \frac{\partial r}{x} = \frac{x}{r} = \cos\theta \\ \frac{r}{y} = \frac{y}{r} = \sin\theta \end{cases} \text{ and } \begin{cases} \frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r} \\ \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r} \end{cases}$$

$$\implies \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \text{ and } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Hence

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x} \end{cases} \implies \begin{cases} \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta} & (1) \\ \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta} & (2) \end{cases}$$

$$\begin{cases} (1)\times\cos\theta + (2)\times\sin\theta & \implies r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial\theta} \\ -(1)\times\sin\theta + (2)\times\cos\theta & \implies r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial\theta} \end{cases}$$

and

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \left(\frac{\cos\theta - i\sin\theta}{r}\right)\left(\frac{\partial v}{\partial\theta} - i\frac{\partial v}{\partial\theta}\right)$$

1.1.3 Power series

Recall :

- $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, let $a_n = \max\{\alpha_1, ..., \alpha_n\} \rightsquigarrow a_n \nearrow$ (non-decrasing), so $\exists A_1$ s.t. $\lim_{n\to\infty} a_n = A_1$ (**least upper bound** or **supremum**). Let $A_k = \sup\{\alpha_n\}_{n=k}^{\infty} \rightsquigarrow A_k \searrow$ (non-increasing), so we can define **limit superior**

$$\overline{\lim}\alpha_n = \lim_{k\to\infty} A_k = A \in \mathbb{R} \text{ or } \pm\infty$$

If $A \in \mathbb{R}$, then by definition, $\forall \varepsilon > 0$, $\exists n_0$ s.t. $n \geq n_0$, $A_n < A + \varepsilon \rightsquigarrow \alpha_n < A + \varepsilon$. Similarly, we can define **limit inferior** by

$$\underline{\lim}\alpha_n = \lim_{k\to\infty} \inf_{n\geq k} \alpha_n = B$$

If $B \in \mathbb{R}$, then $\forall \varepsilon > 0$, $\exists n_0$ s.t. $n \geq n_0$, $\alpha_n > B - \varepsilon$.

- $\{\alpha_n\}$ converge ($\overline{\lim}\alpha = \underline{\lim}\alpha$) $\iff$ $\{\alpha_n\}$ is a Cauchy sequence :

  Proof:

  - $(\Rightarrow)$: $\forall \varepsilon > 0$, $\exists n_0$ s.t. $\forall n \geq n_0, |\alpha_n - A| < \varepsilon/2 \rightsquigarrow \forall m, n \geq n_0, |\alpha_n - \alpha_m| < \varepsilon$.
  - $(\Leftarrow)$: Assume $A = \overline{\lim}\alpha_n > \underline{\lim}\alpha = B$. Let $\varepsilon = \frac{A-B}{3}$, then $\exists n_0$ s.t.

  $$\begin{cases} \forall n \geq n_0,\ B - \varepsilon < \alpha_n < A + \varepsilon \\ \forall n, m \geq n_0,\ |\alpha_n - \alpha_m| < \varepsilon \end{cases}$$

  Then $\forall n, m \geq n_0$

  $$3\varepsilon = |A - B| \leq |A - \alpha_n| + |\alpha_n - \alpha_m| + |\alpha_m - B| < 3\varepsilon \ (\rightarrow\!\leftarrow)$$

  $\square$

- Let $S_n = \sum\limits_{k=1}^{n} \alpha_k$. $\sum\limits_{n=1}^{\infty} \alpha_n$ converges $\iff$ $\{S_n\}$ converges $\iff$ $\{S_n\}$: Cauchy. Especially $|\alpha_n| < \varepsilon$ i.e. $\lim\limits_{n\to\infty} \alpha_n = 0$.

- Since $|\alpha_n + \cdots + \alpha_{n+p}| \leq |\alpha_n| + \cdots + |\alpha_{n+p}|$, $\sum\limits_{n=1}^{\infty} |\alpha_n|$ converges $\Longrightarrow$ $\sum\limits_{n=1}^{\infty} \alpha_n$ converges, which is call **absolutely convergent**.

- **Uniformly converge** : $f_n(x) \xrightarrow{\text{unif.}} f(x)$ on Ω if $\forall \varepsilon > 0, \exists n_0$ s.t. $\forall n \geq n_0$

  $$|f(x) - f_n(x)| < \varepsilon \ \forall x \in \Omega$$

- **Weierstrass M-test** : If $\forall n >> 0$, $|f_n(x)| \leq M_n \ \forall x \in \Omega$. Then

  $$\sum_{n=1}^{\infty} M_n \text{ conv.} \Longrightarrow \sum_{n=1}^{\infty} f_n(x) \text{ unif. conv.}$$

Definition 1.1.3. $\sum\limits_{n=0}^{\infty} a_n z^n$ ($a_n \in \mathbb{C}$) is called a **power series**.

Theorem 1.1.3 (Abel's 1st theorem)**.** Given $\sum\limits_{n=0}^{\infty} a_n z^n$, $\exists\ 0 \leq R \leq \infty$ s.t.

(1) if $|z| < R$, then $\sum\limits_{n=0}^{\infty} a_n z^n$ absolutely. converge and for $0 \leq \rho < R$, the converge is uniform for $|z| < \rho$

(2) if $|z| > R$, then $\sum\limits_{n=0}^{\infty} a_n z^n$ is diverge

(3) if $|z| < R$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ converge for $|z| < R$.

Hadamard's formula : $R^{-1} = \overline{\lim} \sqrt[n]{|a_n|}$

Proof:

(1) For z with $|z| < R$, $\exists \rho$ s.t. $|z| < \rho < R \rightsquigarrow 1/\rho > 1/R$. By def of $\overline{\lim}$, $\exists n_0$ s.t. $\forall n \geq n_0$, $\sqrt[n]{|a_n|} < 1/\rho \rightsquigarrow |a_n| < \rho^{-n}$. So $|a_n z^n| < (|z|/\rho)^n$ $\forall n \geq n_0$. Since $\sum(\underbrace{|z|/\rho}_{<1})^n$ converge, $\sum_{n=0}^{\infty} |a_n z^n|$ converge.

For $0 \leq \rho < R$, pick ρ' s.t. $\rho < \rho' < R \rightsquigarrow \exists n_0$ s.t. $n \geq n_0$, $|a_n| < (1/\rho')^n$. So $|a_n z^n| < (\rho/\rho')^n$ $\forall |z| \leq \rho$ and thus $\sum a_n z^n$ conv. unif. by Weierstrass M-test.

(2) For z with $|z| > R$, $\exists \rho$ s.t. $|z| > \rho > R \rightsquigarrow 1/\rho < 1/R$. Then exists infinitely n s.t. $\sqrt[n]{|a_n|} > \rho^{-1} \rightsquigarrow |a_n z^n| > (\underbrace{|z|/\rho}_{>1})^n$ which is unbound i.e. $\lim_{n\to\infty} a_n z^n \neq 0$.

(3) For $|z| < R$, write $f(z) = S_n(z) + R_n(z)$ with $S_n(z) = \sum_{k=0}^{n-1} a_k z^k$. Let $f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} S_n'(z)$.

- Let $|z| < \rho < R$. $\exists n_0$ s.t. $n \geq n_0$, $|a_n| < \rho^{-n}$, then

$$|n a_n z^{n-1}| < \frac{n}{\rho} \left(\frac{|z|}{\rho} \right)^{n-1}$$

  Let $r = |z|/\rho < 1$, then by ratio test, $\sum n r^{n-1}/\rho$ converges and thus $f_1(z)$ converges in $|z| < R$.

- **Claim** : $f'(z_0) = f_1(z_0)$ for $|z_0| < R$
  subproof : For $n \geq n_0$,

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \left(\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right) + (S_n'(z_0) - f_1(z)) + \underbrace{\frac{R_n(z) - R_n(z_0)}{z - z_0}}_{(3)}$$

where $z \neq z_0$, $|z|, |z_0| < \rho < R$. Also

$$|(3)| = \left| \sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + \cdots + z_0^{k-1}) \right| \leq \sum_{k=n}^{\infty} \frac{k}{\rho} r^{k-1} : \text{converge, where } r = \max\left\{ \frac{|z|}{\rho}, \frac{|z_0|}{\rho} \right\}$$

$\forall\, \varepsilon > 0$,

$$\begin{cases} \exists n_1 \text{ s.t. } \forall n \geq n_1,\ |(3)| < \varepsilon/3 \\ \exists n_2 \text{ s.t. } \forall n \geq n_2,\ |S_n'(z_0) - f_1(z_0)| < \varepsilon/3 \end{cases}$$

Choose a fixed $n \geq n_0, n_1, n_2$, $\exists \delta$ s.t. $0 < |z - z_0| < \delta$,

$$\left| \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right| < \frac{\varepsilon}{3}$$

Hence, $f'(z_0)$ exists and equal to $f_1(z_0)$.

□

Theorem 1.1.4 (Abel's 2nd theorem)**.** If the convergence radius R of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is 1 and the series converges at $z = 1$, then $f(z) \to f(1)$ as $z \to 1$ in a such way that $|1-z|\Big/1-|z|$ is bounded.

Proof: Let $|1-z|/(1-|z|) < M$ If $\sum_{n=0}^{\infty} a_n = C$, then we consider $(a_0 - C) + \sum_{n=1}^{\infty} a_n z^n$. So we may assume "$f(1) = 0$". Write $S_n = \sum_{k=0}^{n} a_k$.

$$\begin{aligned} S_n(z) &= \sum_{k=0}^{n} a_k z^k = S_0 + \sum_{k=1}^{n} (S_k - S_{k-1}) z^k = \sum_{k=0}^{n-1} S_k (z^k - z^{k+1}) + S_n z^n \\ &= (1-z) \sum_{k=0}^{n-1} S_k z^k + S_n z^n \end{aligned}$$

$S_n z^n \to 0$ as $n \to \infty$ since $|z| < 1$ and $S_n \to 0$. For $|z| < 1$, $f(z) = \lim_{n\to\infty} S_n(z) = (1-z) \sum_{n=0}^{\infty} S_n z^n$. Let $n \geq n_0$, $|S_n| < \varepsilon$. Then

$$\begin{aligned} |f(z)| &\leq |1-z| \left| \sum_{k=0}^{n_0-1} S_k z^k \right| + \varepsilon |1-z| \sum_{k=n_0}^{\infty} |z^k| \\ &\leq |1-z| \left| \sum_{k=0}^{n_0-1} S_k z^k \right| + \underbrace{\frac{\varepsilon |1-z| |z|^{n_0}}{1-|z|}}_{<\varepsilon M} \end{aligned}$$

As $z \to 1$ subject to $|1-z|/(1-|z|) < M$, $f(z) \to 0 = f(1)$. $\square$

1.1.4 Basic example

Problem : Solve $f'(z) = f(z)$ with $f(0) = 1$.
Ans : Write $f(z) = \sum_{n=0}^{\infty} a_n z^n \rightsquigarrow f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. By assumption, $a_{n-1} = n a_n$ and thus $a_n = 1/n! \rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(0! = 1)$.

Definition 1.1.4. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

- $R = \overline{\lim} \sqrt[n]{n!} = \infty$:

$$(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \geq n^n \implies \sqrt[n]{n!} \geq \sqrt{n} \ \forall n$$

  So $f(z)$ is entire.

- $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$: $(e^z \cdot e^{c-z})' = e^z e^{c-z} - e^z e^{c-z} = 0 \rightsquigarrow e^z e^{c-z}$ is a constant. Substitute $z = 0 \rightsquigarrow e^z e^{c-z} = e^c$.

- $e^z e^{-z} = e^0 = 1 \rightsquigarrow e^z \neq 0 \ \forall z$.

For $z = iy \in$ imaginary axis,

$$\begin{cases} e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ \overline{e^{iy}} = 1 - iy + \frac{(iy)^2}{2!} - \frac{(iy)^3}{3!} + \cdots = e^{-iy} \end{cases}$$

$|e^{iy}|^2 = e^{iy}e^{-iy} = 1 \implies |e^{iy}| = 1 \implies |e^{x+iy}| = e^x$.

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right) = \cos y + i \sin y$$

$$\implies e^z = e^x(\cos y + i \sin y) \text{ and } \begin{cases} \cos y = \frac{e^{iy}+e^{-iy}}{2} \\ \sin y = \frac{e^{iy}-e^{-iy}}{2i} \end{cases}$$

Definition 1.1.5. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
($\rightsquigarrow \cos^2 z + \sin^2 z = 1$, $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$)

Definition 1.1.6. $f(z)$ has the period ω if $f(z + \omega) = f(z)$ $\forall z \in \Omega$ and $z + \omega \in \Omega$.

Proposition 1.1.1. The smallest positive period of e^{iz} is 2π. (then for $\cos z$, $\sin z$)

Proof:

- $e^{i(z+\omega)} = e^{iz} \implies e^{i\omega} = 1 \rightsquigarrow \omega \in \mathbb{R}$.
- Let $\varphi : (\mathbb{R}, +, 0) \to$ unit circle in $\mathbb{C}$ define by $y \mapsto e^{iy} = \cos y + i \sin y$, then $\ker \varphi = \langle 2\pi \rangle_{\mathbb{Z}}$.

$\square$

Now we consider the inverse function of e^z, denoted by $\log z$. $z = e^\omega$, where $z = re^{i\theta}$ and $\omega = u + iv$, then $r = e^u$ and $v = \arg z + 2k\pi$, so $\log z$ is a multiple-valued function. Note that $\arg z$ is discontinuous on the negative real axis. Let $\log z = \ln |z| + i \arg z$, $-\pi < \arg z < \pi$ which is called **principal branch**.

- $\log z$ is analytic on $\mathbb{C} \setminus \mathbb{R}^-$: $z = re^{i\theta}$, $-\pi < \theta < \pi \rightsquigarrow \log z = \ln r + i\theta$.

$$\begin{cases} r\frac{\partial \ln r}{\partial r} = \frac{\partial \theta}{\partial \theta} \\ r\frac{\partial \theta}{\partial r} = -\frac{\partial \ln r}{\partial \theta} \end{cases} \quad \text{and } \frac{1}{r}, 1, 0, 0 \text{ are conti.}$$

$$\implies (\log z)' = (\cos\theta - i \sin\theta)\left(\frac{\partial \ln r}{\partial r} + i\frac{\partial \theta}{\partial r}\right) = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{z}$$

1.2 Cauchy theorem

1.2.1 Line integral

Definition 1.2.1. Let $\begin{aligned} f : \ (a,b) &\longrightarrow \mathbb{C} \\ t &\longmapsto u(t) + iv(t) \end{aligned}$, then define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Property 1.2.1. $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Proof: Let $\theta = \arg\left(\int_a^b f(t)dt\right)$, then

$$\begin{aligned}\left|\int_a^b f(t)dt\right| &= \operatorname{Re}\left(e^{i\theta}\int_a^b f(t)dt\right) = \int_a^b \operatorname{Re}\left(e^{i\theta}f(t)\right)dt \\ &\leq \int_a^b |e^{i\theta}f(t)|dt = \int_a^b |f(t)|dt\end{aligned}$$

□

Definition 1.2.2.

- γ is a **piecewise smooth curve (arc)** in $\mathbb{C}$ if γ is parameterized by $z(t) = x(t) + iy(t)$, $t \in [\alpha, \beta]$ and exists a partition $\{[\alpha_i, \beta_i]\}$ of $[\alpha, \beta]$ s.t. $z|_{[\alpha_i,\beta_i]} \in C^1$.
- Let f be continuous on Ω and $\gamma \subset \Omega$, define

  $$\int_\gamma f(z)dz := \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} f(z(t))z'(t)dt$$

  By chain rule, the definition is independent of the choice of parameters of γ.

  - $dz = z'(t)dt = (x'(t) + iy'(t))dt$, $\overline{dz} = (x'(t) - iy'(t))dt$
  - $|dz| = \sqrt{x'(t)^2 + y'(t)^2}dt = ds$

Property 1.2.2.

$$\left|\int_\gamma fdz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \leq \int_a^b |f(z(t))||z'(t)|dt = \int_\gamma |f||dz|$$

Observation : If $f = u + iv$, then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(x'+iy')dt = \int_\gamma (ux'-vy')dt + i\int_\gamma (vx'+uy')dt \\ &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)\end{aligned}$$

Recall : Ω : open connected in $\mathbb{R}^2$, $A, B \in C^1(\Omega)$. Then $\int Adx+Bdy$ is only determined by P, Q in Ω and is independent of arcs connecting P and Q $\iff$ $\exists U \in C^1(\Omega)$ s.t. $dU = Adx + Bdy$ i.e. $\dfrac{\partial U}{\partial x} = A$, $\dfrac{\partial U}{\partial y} = B$. Actually, $U(x,y) = \int_\gamma Adx + Bdy$, where $\gamma \subset \Omega$ is any curve connected P, Q.

Proposition 1.2.1. Let f be continuous on Ω. Then $\int_\gamma fdz$ depends only on the end points of γ $\iff$ f is the derivative of an analytic function F on Ω.

Proof:

- $(\Leftarrow)$: Say $F = U + iV$, then $f = F' = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial y} - i\dfrac{\partial U}{\partial y} = u + iv$. Then

  $$udx - vdy = dU \;\; vdx + udy = dV$$

  By observation, $\int_\gamma fdz$ is independent on arcs.

- $(\Rightarrow)$: By recall, $\exists U, V \in C^1(\Omega)$ s.t.

$$dU = udx - vdy \ dV = vdx + udy$$

Let $F = U + iV \rightsquigarrow F$ is analytic and $F' = f$.

□

Example 1.2.1.

- $\forall n \in \mathbb{N},\ \int_\gamma (z-a)^n dz = 0\ \forall \gamma$: closed arc in $\mathbb{C}$, since $\left(\frac{(z-a)^{n+1}}{n+1}\right)' = (z-a)^n$.
- Let $C_r(a) := \{z \in \mathbb{C} : |z-a| = r\}$, then

$$\begin{aligned}\int_{C_r(a)} \frac{dz}{z-a} &= \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}} d\theta \ (z = a + re^{i\theta}) \\ &= 2\pi i\end{aligned}$$

We can't apply proposition 1.2.1 since we can't define a single-valued branch of $\log(z-a)$ in $B_r(a) := \{z : |z-a| < r\}$.

Theorem 1.2.1 (Cauchy theorem for a rectangle)**.** Let f be analytic in a rectangle R (i.e analytic in an open set containing R). Then

$$\int_{\partial R} f(z) dz = 0$$

where the preset orientation of ∂R is counterclockwise.

Proof: Divide R to four small rectangle $R_1^{(1)}, ..., R_1^{(4)}$. Define $\Gamma(R) := \int_{\partial R} f(z)dz$, then

$$\Gamma(R) = \Gamma(R_1^{(1)}) + \cdots + \Gamma(R_4^{(1)})$$

Then exists $R_1^{(k)}$ for some $k \in \{1, ..., 4\}$ s.t. $|\Gamma(R_1^{(k)})| \geq \frac{1}{4}|\Gamma(R)|$. Say $R_1 = R_1^{(k)}$ and define $R_2, ...$ by same method. Let d_i, L_i be the diameter, perimeter of R_i respectively. We obtain a nested rectangles $R \supset R_1 \supset R_2 \supset \cdots$ with $d_i = d_{i-1}/2$, then $\exists!\ z^* \in R_i\ \forall i$ i.e. $\forall \delta > 0$, $\exists n_0 \geq 0$ s.t. $\forall n \geq n_0$, $R_n \subset B_\delta(z^*)$. Since f is analytic at z^*, $\forall \varepsilon > 0$, $\exists \delta > 0 (\rightsquigarrow \exists n_0)$ s.t.

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \varepsilon \ \forall z \in R_n,\ n \geq n_0$$

$$\implies |f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon |z - z^*| \ \forall z \in R_n,\ n \geq n_0$$

By example 1.2.1, $\int_{\partial R_n} dz = 0$ and $\int_{\partial R_n} z dz = 0$, then

$$\begin{aligned}\Gamma(R_n) &= \int_{\partial R_n} (f(z) - f(z') - (z - z^*)f'(z^*))dz \\ \implies \frac{1}{4^n}|\Gamma(R)| &\leq |\Gamma(R_n)| \leq \varepsilon \int_{\partial R_n} |z - z^*||dz| \leq \varepsilon L_n d_n \quad \forall n \geq n_0 \\ \implies |\Gamma(R)| &\leq \varepsilon 4^n L_n d_n = \varepsilon L d \ \forall \varepsilon \implies \Gamma(R) = 0\end{aligned}$$

□

Theorem 1.2.2 (Stronger form)**.** Let $R' = R \setminus \{\xi_1, ..., \xi_n\}$ with $\xi_i \in (R \setminus \partial R) =: R^o$ and f be analytic in R'. If $\lim\limits_{z \to \xi_i} (z - \xi_i) f(z) = 0$, then

$$\int_{\partial R} f(z)dz = 0$$

Proof:

- $n = 1 : \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - \xi| < \delta \implies |z - \xi||f(z)| < \varepsilon$. Choose a square R_0 with center ξ s.t. $R \subset B_\delta(\xi)$. Extend the side length of R_0 and cut R into nine rectangle $R_0, R_1, ..., R_8$. We already know $\Gamma(R_i) = 0$, so $\Gamma(R) = \Gamma(R_0)$.

$$|\Gamma(R)| = |\Gamma(R_0)| = \left| \int_{\partial R_0} f(z)dz \right| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z - \xi|} \leq \varepsilon \frac{8}{L_0} L_0 = 8\varepsilon$$

  where L_0 is the perimeter of R_0. Hence, $\Gamma(R) = 0$.

- In general n, we just cut R into several rectangle and apply Cauchy theorem for a rectangle and the case of $n = 1$.

$\square$

Theorem 1.2.3 (local existence of primitives)**.** Any analytic function f in $B_\rho(a)$ has a primitive (antiderivatives) in $B_\rho(a)$.

Proof: For $z \in B_r(a)$, let γ_z connected a and z by one horizontal line first and one vertical line. Define $F(z) = \int_{\gamma_z} f(u)du$.
Claim : F is analytic in $B_\rho(a)$ and $F'(z) = f(z)$.
subproof : Apply Cauchy theorem for a rectangle we have

$$F(z + \Delta z) - F(z) = \int_{\overline{z(z+\Delta x)}} f(u)du + \int_{\overline{(z+\Delta x)(z+\Delta z)}} f(u)du$$

Since f is continuous at z, we can write $f(u) = f(z) + \delta(u)$, where $\delta(u) \to 0$ as $u \to z$.

$$\int_{\overline{z(z+\Delta x)}} f(z)du + \int_{\overline{(z+\Delta x)(z+\Delta z)}} f(z)du = f(z)\Delta z$$

$$\int_{\overline{z(z+\Delta x)}} |\delta(u)||du| + \int_{\overline{(z+\Delta x)(z+\Delta z)}} |\delta(u)||du| \leq (\sup |\delta(u)|)(\underbrace{|\Delta x| + |\Delta y|}_{\leq 2|\Delta z|})$$

where $\sup |\delta(u)|$ is consider $u \in \overline{z(z + \Delta x)} \cup \overline{(z + \Delta x)(z + \Delta z)}$.

$$\implies \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left(\quad \bigstar \quad \right)$$

where $\frac{1}{\Delta z}|\bigstar| \leq \frac{2|\Delta z| \sup |\delta(u)|}{|\Delta z|} \to 0$ as $\Delta z \to 0$. Hence, $F'(z) = f(z)$. $\square$

Theorem 1.2.4 (Cauchy theorem for a disk)**.** If f is a analytic in $B_\rho(a)$, then

$$\int_\gamma f(z)dz = 0$$

for all closed arc $\gamma \subset B_\rho(a)$.

Proof: Since f has a primitive in $B_\rho(a)$, by proposition 1.2.1 the statement will holds. $\square$

Corollary 1.2.1. If f is analytic in Ω and $B_\rho(a) \subsetneq \Omega$, then

$$\int_{C_\rho(a)} f(z)dz = 0$$

Proof: Choose a larger $B_{\rho'}(a')$ s.t. $B_\rho(a) \subsetneq B_{\rho'}(a') \subseteq \Omega$. Then $\gamma = C_\rho(a) \subseteq B_{\rho'}(a')$ and apply Theorem 1.2.4 $\square$

Theorem 1.2.5 (Stronger form)**.** Let $B = B_\rho(a) \setminus \{\xi_1, ..., \xi_n\}$ and f be analytic in B. If $\lim\limits_{z \to \xi_i} (z - \xi_i) f(z) = 0$, then f has a primitive in B. Moreover,

$$\int_\gamma f(z)dz = 0$$

for all closed arc in B.

Proof: For $z \in B$, define

$$F(z) = \int_{\gamma_z} f(u)du = \int_{\gamma'_z} f(u)du$$

where γ_z, γ'_z connected a and z and composed by finite horizontal line and vertical line not pass $\{\xi_1, ..., \xi_n\}$. The red equation will holds since $\gamma'_z - \gamma_z = \sum\limits_{\text{finite}} \pm \partial R_i$ and by stronger form of Cauchy theorem for a rectangle $\Gamma(R_i) = 0$. By the similar argument, $F'(z) = f(z)$. $\square$

1.2.2 Winding number

Theorem 1.2.6 (winding number)**.** Let γ be a closed arc and $a \notin \gamma$. Then

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n$$

for some nonnegative integer n.

Proof: Let $z : [\alpha, \beta] \to \gamma$ with $t \mapsto z(t)$ and $z|_{[\alpha_i, \beta_i]}$: smooth. Consider

$$p(x) = \int_\alpha^x \frac{z'(t)}{z(t) - a} dt$$

Then we have

$$\begin{cases} p(x) \text{ is continuous on } [\alpha, \beta] \\ p'(x) = \dfrac{z'(x)}{z(x) - a} \text{ on } (\alpha, \beta) \setminus \{t_1, ..., t_{n-1}\} \\ p(\beta) = \displaystyle\int_\gamma \frac{dz}{z-a} \end{cases}$$

Notice that $2\pi i$ is the period of e^x, so hope that $e^{p(\beta)} = 1$. Now

$$\left(e^{-p(x)}\right)' = -p'(x) e^{-p(x)} = \frac{-z'(x)}{z(x) - a} e^{-p(x)} \implies (z(x) - a)(e^{-p(x)})' + z'(x) e^{-p(x)} = 0$$

$$\implies \left(e^{-p(x)}(z(x) - a)\right)' = 0 \implies e^{-p(x)}(z(x) - a) = \text{constant} = e^{-p(\alpha)}(z(\alpha) - a) = z(\alpha) - a$$

Hence, $e^{p(x)} = \dfrac{z(x) - a}{z(\alpha) - a} \rightsquigarrow e^{p(\beta)} = 1 \implies p(\beta) = (2\pi i)n$ for some $n \in \mathbb{N}$. $\square$

Observation : Define
$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$
be the **winding number** of γ around a. It will be the number of turns of γ around a.

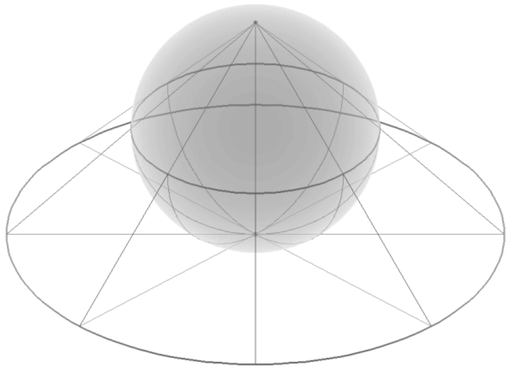
- $n(-\gamma, a) = -n(\gamma, a)$
- If $\gamma \subseteq B_\rho(a) \subseteq \Omega$, then $\forall a \in \Omega \setminus B_\rho(z)$, $n(\gamma, a) = 0$:
  Since $(z-a)^{-1}$ is analytic in $B_\rho(z)$.
- $n(\gamma, a)$ is constant for each region cut by γ and $n(\gamma, a) = 0$ $\forall a$ in unbounded region.
  Claim : If $\gamma \cap \overline{aa'} = \varnothing$, then $n(\gamma, a) = n(\gamma, a')$.
  subproof : for $z \in \overline{aa'}$, $\dfrac{z-a}{z-a'} \in \mathbb{R}_{\leq 0}$ and $\dfrac{z-a}{z-a'} \notin \mathbb{R}_{\leq 0}$ for all $z \notin \overline{aa'}$. Then $\log\left(\dfrac{z-a}{z-a'}\right)$ is analytic on $\mathbb{C} \setminus \overline{aa'}$. Hence,
  $$0 = \int_\gamma \left(\log\left(\frac{z-a}{z-a'}\right)\right)' dz = \int_\gamma (\log(z-a) - \log(z-a'))' dz = \int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-a'}\right) dz$$
  Hence, $n(\gamma, a) = n(\gamma, a')$. □
  In the same region, we can connected by polyline, hence $n(\gamma, z)$ is constant on same region. Choose a open ball $B_\rho(a)$ that cover γ, and choose a point $b \in \mathbb{C} \setminus B_\rho(a)$, then $n(\gamma, b) = 0$. Hence, $n(\gamma, z) = 0$ on unbound region.
- Let γ be the simple curve around 0, then $n(\gamma, 0) = 1$:
  Let $C = C_\rho(0)$ for some ρ s.t. $C \cap \gamma = \varnothing$. Choose $a_1, a_2 \in \gamma$, $b_1, b_2 \in C$ s.t. $a_2, b_2, 0, b_1, a_1$ collinear as this order. Let γ, C be cut by this line into $\gamma_1 \cup \gamma_2, C_1 \cup C_2$ respectively and C_1, γ_1 are in same side w.r.t. this line. Let $\sigma_1 = \gamma_1 + \overline{a_1b_1} - c_1 - \overline{a_2b_2}$, $\sigma_2 = \gamma_2 + \overline{a_2b_2} - C_1 - \overline{a_1b_1}$. By definition of winding number,
  $$n(\gamma, 0) = n(C, 0) + n(\sigma_1, 0) + n(\sigma_2, 0) = 1$$
  where $n(\sigma_1, 0) = n(\sigma_2, 0) = 0$ by 0 is in the unbounded region w.r.t. to σ_1, σ_2.

Let $f(z)$ be analytic in $B_\rho(b)$, $\gamma \subset B_\rho(b)$ and $a \in B_\rho(b) \setminus \gamma$. Then $F(z) = \dfrac{f(z) - f(a)}{z-a}$ is analytic for $z \neq a$ and $\lim\limits_{z \to a}(z-a)F(a) = \lim\limits_{z \to a}(f(z) - f(a)) = 0$. By Theorem 1.2.5,
$$\int_\gamma \frac{f(z) - f(a)}{z-a} dz = 0 \implies \int_\gamma \frac{f(z)}{z-a} = f(a)\int_\gamma \frac{dz}{z-a}$$
Then we have Cauchy integral formula :
$$f(a) = \frac{1}{2\pi i \cdot n(r, a)}\int_\gamma \frac{f(z)}{z-a} dz$$
In particular, if $n(\gamma, z) = 1$, then
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi$$

1.2.3 Simply connected

Set up

- extended complex plane (**Riemann sphere**) : $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with $a + \infty = \infty + 0 = \infty$, $b \cdot \infty = \infty \cdot b = \infty$ for $b \neq 0$. $a/0 = \infty$ for $a \neq 0$, $b/\infty = 0$ for $b \neq \infty$.



- γ : piecewise-smooth curve ($z|_{[\alpha_i,\beta_i]}$: smooth i.e. $z'(t)$ continuous and $z'(t) \neq 0$)
  - **simple** : $z(t_1) = z(t_2) \iff t_1 = t_2$
  - **closed** : $z(\alpha) = z(\beta)$
  - **Jordan curve** : simple closed curve
  - **opposite arc** : $-\gamma$ define by $z(-t)$.
- Let $\gamma_1, ..., \gamma_n$ be arcs in Ω. A **form sum** $\gamma_1 + \cdots + \gamma_n$ is called a **chain** in Ω.
- Define $\gamma_1 + \cdots + \gamma_n \sim \gamma_1' + \cdots + \gamma_m' \iff \int_{\gamma_1+\cdots+\gamma_n} f dz = \int_{\gamma_1'+\cdots+\gamma_m'} f dz \ \forall f$ on Ω, where

$$\int_{\gamma_1+\cdots+\gamma_n} f dz = \sum_{i=1}^{n} \int_{\gamma_i} f dz$$

  In general, we can write a chain $\gamma = b_1\gamma_1 + \cdots + b_n\gamma_n$, where γ_i are distinct arcs and $b_i \in \mathbb{Z}$.

- γ is a **cycle** if $\forall \gamma_i$ is closed

$$\implies \int_\gamma dF = 0 \text{ and define } n(\gamma, a) := \sum b_i n(r_i, a)$$

Definition 1.2.3. A region $\Omega \subseteq \mathbb{C}$ is simply connected if $\widetilde{\mathbb{C}} \setminus \Omega$ is connected.

Property 1.2.3. Ω is simply connected $\iff n(\gamma, a) = 0 \ \forall \gamma$: cycle in Ω and $a \in \widetilde{\mathbb{C}} \setminus \Omega$.

Proof:

- $(\Rightarrow)$ Since $\widetilde{\mathbb{C}} \setminus \Omega$ is connected and $\gamma \subseteq \Omega$, $\widetilde{\mathbb{C}} \setminus \Omega$ is contained in the unbounded region determined by $\gamma \rightsquigarrow n(\gamma, a) = 0 \ \forall a \in \widetilde{\mathbb{C}} \setminus \Omega$.

- $(\Leftarrow)$: Assume $\widetilde{\mathbb{C}} \setminus \Omega = A \sqcup B$ with A, B closed and assume A is bouneded. Let δ be the shortest distance between A and B. Let

$$\{Q : Q \text{ is a square of side } = \frac{\delta}{2\sqrt{2}}\}$$

covers A and a be a center of some Q in A. Let $\gamma = \sum\limits_{Q_j \cap A \neq \varnothing} \partial Q_j$, then $\gamma \in \Omega$ and

$$n(\gamma, a) = \sum_{Q_j \cap A \neq \varnothing} n(\partial Q_j, a) = n(\partial Q, a) = 1 \; (\to\!\times\!\leftarrow)$$

since ∂Q is a simple curve around a and a is in the unbounded region w.r.t. others ∂Q_j.

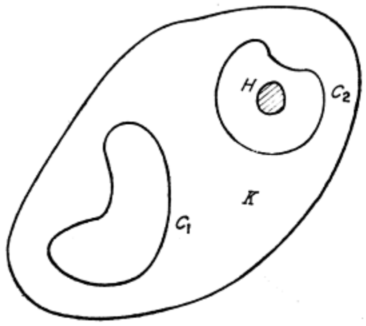
□

Remark 1.2.1. If Ω is not simply connected, then $\exists \gamma$ in Ω s.t. $n(\gamma, a) \neq 0$ for some $a \in \widetilde{\mathbb{C}} \setminus \Omega$. Now $(z-a)^{-1}$ is analytic in Ω, but

$$\int_\gamma \frac{1}{z-a} dz = n(\gamma, a) \neq 0$$

i.e. Cauchy theorem doesn't hold in this case.

Definition 1.2.4. γ in Ω is said to be **homologous to** 0 w.r.t. Ω if $n(\gamma, a) = 0 \; \forall a \in \widetilde{\mathbb{C}} \setminus \Omega$, and denoted by $\gamma \sim 0 \pmod{\Omega}$.



For example, $C_1 \sim 0 \pmod{K \setminus H}$, but $C_2 \not\sim 0 \pmod{K \setminus H}$.

Theorem 1.2.7 (Cauchy theorem)**.** If $f(z)$ is analytic in Ω, then $\displaystyle\int_\gamma f(z) dz = 0$ for all cycle $\gamma \sim 0$ in Ω.

Corollary 1.2.2. Let Ω be simply connected and f be analytic in Ω. Then

- $\displaystyle\int_\gamma f dz = 0$ for all cycle γ in Ω, since $\gamma \sim 0$.
- $\displaystyle\int f dz$ is independent of the path connecting P and Q. Which means $f dz = dF$ for some F i.e. f has a primitive.

Proof: (Cauchy theorem)

- Ω is bounded : For $\delta > 0$, let $\{S_i : i \in I\}$ be a subset of closed squares of side δ which are contained in Ω (Ω : bounded $\rightsquigarrow |I| < \infty$). Let $\Gamma_\delta = \sum\limits_{i \in I} \partial S_i$, $\Omega_\delta = \left(\bigcup\limits_{i \in I} S_i \right)^o$. Choose δ s.t. $\gamma \subset \Omega_\delta$. Let $\xi \in \Gamma_\delta \subseteq \overline{\Omega} \setminus \Omega_\delta$, then exists a square $S \notin \{S_i : i \in I\}$ s.t. $\xi \in S$. Let $\xi_0 \in S \setminus \Omega \rightsquigarrow \overline{\xi\xi_0} \subset S \rightsquigarrow \overline{\xi_0\xi} \cap \Omega_\delta = \varnothing$. Since $\gamma \sim 0 \pmod{\Omega}$, $n(\gamma, \xi_0) = 0$ and thus $n(\gamma, \xi) = n(\gamma, \xi_0) = 0$ since they are in same region w.r.t. Ω_δ.

- If $z \in S_j^o$, then

$$\frac{1}{2\pi i}\int_{\partial S_i}\frac{f(\xi)}{\xi - z}d\xi = \begin{cases} f(z) & \text{,if } i = j \\ 0 & \text{, if } i \neq j \end{cases}$$

since $\dfrac{f(\xi)}{\xi - z}$ is analytic on S_i when $i \neq j$ and by Cauchy integral formula when $i = j$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi \quad \forall z \in \bigcup_{j\in I} S_j^o \underset{\text{by conti.}}{\Longrightarrow} f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi \quad \forall z \in \Omega_\delta$$

Hence,

$$\begin{aligned}\int_\gamma f(z)dz &= \frac{1}{2\pi i}\int_\gamma\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi dz = \frac{1}{2\pi i}\int_{\Gamma_\delta}\int_\gamma\frac{f(\xi)}{\xi - z}dzd\xi \text{ (since it conti. on } \Gamma_\delta, \gamma) \\ &= \frac{1}{2\pi i}\int_{\Gamma_\delta}\left(f(\xi)\int_\gamma\frac{-1}{z-\xi}dz\right)d\xi = \frac{-1}{2\pi i}\int_{\Gamma_\delta}f(\xi)\underbrace{n(\gamma,\xi)}_{=0}d\xi = 0\end{aligned}$$

- If Ω is unbound : We replace Ω by $\Omega' := \Omega \cap B_R(0)$ for R large enough to get $\gamma \subset \Omega'$. Then $\forall a \in \widetilde{\mathbb{C}} \setminus \Omega'$,

$$\begin{cases} a \in \widetilde{\mathbb{C}} \setminus \Omega \implies n(\gamma, a) = 0 & \text{, since } \gamma \sim 0 \pmod{\Omega} \\ a \in \widetilde{\mathbb{C}} \setminus B_R(0) \implies n(\gamma, a) = 0 & \text{, since } \frac{1}{z-a} \text{ is analytic on } B_R(0) \end{cases}$$

Hence, $\gamma \sim 0 \pmod{\Omega'}$, which is follow from the bounded case.

□

1.3 Cauchy's integral theorem

Recall : Let f : analytic in $B_\rho(a)$, γ : closed arc in $B_\rho(a)$. For $z \neq \gamma$,

$$f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

In fact we can have more relax assumption : f : analytic in $B_\rho(a)\setminus\{a\}$ with $\lim\limits_{z\to a}(z-a)f(z) = 0$. Then

$$f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

Proof: For $z \in B_\rho(a)$, let $F(\xi) = \dfrac{f(\xi) - f(z)}{\xi - z}$, then F analytic on $B_\rho(a) \setminus \{a, z\}$. Also

$$\begin{cases} \lim\limits_{\xi\to z}(\xi - z)F(\xi) = 0 \\ \lim\limits_{\xi\to a}(\xi - a)F(\xi) = \lim\limits_{\xi\to a}\frac{(\xi-a)(f(\xi)-f(z))}{\xi-z} = 0 \end{cases}$$

By stronger Cauchy theorem,

$$\int_\gamma F(\xi)d\xi = 0 \implies f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

□

Lemma 1.3.1 (key lemma). If $\varphi(\xi)$ is continuous on γ, then

$$F_n(z) := \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n} d\xi$$

is analytic in each regions determined by γ and $F_n'(z) = nF_{n+1}(z)$.

Proof:

- $n = 1$:
  - continuous : $\forall z_0 \notin \gamma$, pick $\delta > 0$ s.t. $B_\delta(z_0) \cap \gamma = \varnothing$. If $|z - z_0| < \delta/2 \rightsquigarrow |\xi - z| > \delta/2 \ \forall \xi \in \gamma$. So
  $$|F_1(z) - F_1(z_0)| = \left| \int_\gamma \varphi(\xi) \frac{z - z_0}{(\xi - z)(\xi - z_0)} dz \right| \leq |z - z_0| M \frac{2}{\delta^2} L$$
  where $M = \max_{\xi \in \gamma} \varphi(\xi)$ and L be the length of γ. Hence, $F_1(z)$ is continuous.
  - differentiable :
  $$\frac{F(z) - F(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\xi)}{(\xi - z)(\xi - z_0)} d\xi \xrightarrow{z \to z_0} F'(z_0) = \int_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^2} d\xi = F_2(z_0)$$
  where red approaching is by
  $$\left| \int_\gamma \frac{\varphi(\xi)}{(\xi - z)(\xi - z_0)} d\xi - \int_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^2} d\xi \right| \leq \int_\gamma \left| \frac{\varphi(\xi)(z - z_0)}{(\xi - z)(\xi - z_0)^2} \right| |d\xi| \leq \frac{4ML}{\delta^3} |z - z_0|$$
- By induction on $n > 1$:
  - continuous :
  $$\begin{aligned} |F_n(z) - F_n(z_0)| &= \left| \int_\gamma \varphi(\xi) \left(\frac{1}{(\xi - z)^n} - \frac{1}{(\xi - z_0)^n} \right) dz \right| \\ &= \left| \int_\gamma \varphi(\xi) \left(\frac{\xi - z + z - z_0}{(\xi - z)^n (\xi - z_0)} - \frac{1}{(\xi - z_0)^n} \right) dz \right| \\ &= \left| \int_\gamma \frac{\varphi(\xi)}{\xi - z_0} \left(\frac{1}{(\xi - z)^{n-1}} - \frac{1}{(\xi - z_0)^{n-1}} \right) d\xi + (z - z_0) \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \right| \end{aligned}$$
  By induction hypothesis on $\varphi(\xi)/(\xi - z_0)$, we have
  $$\left| \int_\gamma \frac{\varphi(\xi)}{\xi - z_0} \left(\frac{1}{(\xi - z)^{n-1}} - \frac{1}{(\xi - z_0)^{n-1}} \right) d\xi \right| \to 0$$
  and similar method,
  $$\left| (z - z_0) \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \right| \leq \frac{2^n ML}{\delta^{n+1}} |z - z_0|$$
  Hence, $F_n(z)$ is continuous.
  - differentiable :
  $$\begin{aligned} \frac{F_n(z) - F_n(z_0)}{z - z_0} &= \frac{1}{z - z_0} \int_\gamma \left(\frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z)^{n-1}} - \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^{n-1}} \right) d\xi + \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \\ &\longrightarrow (n-1) \int_\gamma \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^n} d\xi + \int_\gamma \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^n} d\xi = nF_{n+1}(z) \end{aligned}$$
  the former is by induction hypothesis. □

Corollary 1.3.1. Let f be analytic in Ω. For $a \in \Omega$, $\exists B_\rho(a) \subset \Omega$, if $\gamma = C_\rho(a)$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi \text{ and } f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(z-\xi)^{n+1}}d\xi$$

Hence, if f is analytic in Ω, then $f \in C^\infty(\Omega)$.

Theorem 1.3.1 (Removable singularities). If f is analytic in $\Omega' = \Omega \setminus \{\xi_1, ..., \xi_n\}$ and $\lim\limits_{z\to\xi_i}(z-\xi_i)f(z) = 0$ $\forall i$, then $\exists!$ analytic function $\widetilde{f}$ in Ω s.t. $\widetilde{f}|_\Omega = f$.

Proof: Let $a = \xi_i \rightsquigarrow \exists B_\rho(a) \setminus \{a\} \subseteq \Omega'$. If $\widetilde{f}$ exists, then

$$\begin{cases} \widetilde{f}(a) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \dfrac{\widetilde{f}(\xi)}{\xi - a}d\xi = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \dfrac{f(\xi)}{\xi - a}d\xi \\ \widetilde{f}(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \dfrac{\widetilde{f}(\xi)}{\xi - z}d\xi = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \dfrac{f(\xi)}{\xi - z}d\xi = f(z)\ \forall z \neq a \end{cases}$$

So we should define $\widetilde{f}(a) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \dfrac{f(\xi)}{\xi - a}d\xi$ □

Observation : Let $F(z) = \dfrac{f(z) - f(a)}{z - a}$. $\because \lim\limits_{z\to a}(z-a)F(z) = 0 \therefore \exists!$ analytic function s.t.

$$f_1(z) = \begin{cases} F(z) & \text{, for } z \neq a \\ f'(z) & \text{, for } z = a \end{cases}$$

$\because \lim\limits_{z\to a}(z-a)\dfrac{f_1(z) - f_1(a)}{z-a} = 0 \therefore \exists!$ analytic function s.t.

$$f_2(z) = \begin{cases} \frac{f_1(z) - f_1(a)}{z-a} & \text{, for } z \neq a \\ f_1'(z) & \text{, for } z = a \end{cases}$$

That is $f_{k-1}(z) = f_{k-1}(a) + (z-a)f_k(z)$ $\forall k = 1, ..., n$, where $f_0(z) = f(z)$.

$$\implies f(z) = f(a) + (z-a)f_1(a) + \cdots + (z-a)^{n-1}f_{n-1}(a) + (z-a)^n f_n(z)$$

Differentiate n times and evaluation $z = a$, we can get $f^{(n)}(a) = n!f_n(a)$. Then we have

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z)$$

Here for $z \in B_\rho(a)$,

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f_n(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{1}{\xi - z}\left(\frac{f(\xi)}{(\xi-a)^n} - \sum_{k=0}^{n-1}\frac{f^k(a)}{k!(\xi-a)^{n-k}}\right)d\xi$$

Let $G_k(u) = \displaystyle\int_{C_\rho(a)} \frac{1/(\xi - z)}{(\xi-u)^k}d\xi$ for $u \in B_\rho(a)$. By key lemma, $G_{k+1}(u) = G_1^k(u)/k!$.

$$\begin{aligned} G_1(u) = \int_{C_\rho(a)} \frac{d\xi}{(\xi-u)(\xi-z)} &= \frac{1}{u-z}\int_{C_\rho(a)}\left(\frac{1}{\xi-u} - \frac{1}{\xi-z}\right)d\xi \\ &= \frac{2\pi i(n(C_\rho(a), u) - n(C_\rho(a), z))}{u-z} = 0 \end{aligned}$$

Hence, $G_k(u) = 0\ \forall k$ and thus

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - z)(\xi - a)^n} d\xi$$

Theorem 1.3.2 (Cauchy's estimate)**.** Let $M = \max\limits_{\xi \in C_\rho(a)} |f(\xi)|$, then

$$|f^n(a)| = \left| \frac{n!}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \right| \leq \frac{n!}{2\pi}\frac{M \cdot 2\pi\rho}{\rho^{n+1}} \leq n! M \rho^{-n}$$

Theorem 1.3.3 (Liouville's theorem)**.** If f is bounded entire function, then f is a constant.

Proof: Say $|f(z)| \leq M\ \forall z \in \mathbb{C}$. For all $a \in \mathbb{C}$, $|f'(a)| \leq M\rho^{-1} \to 0$ as $\rho \to \infty$. Hence, $f'(a) = 0\ \forall a \in \mathbb{C}$ which means f is constant. □

Theorem 1.3.4 (Fundamental theorem of algebra)**.** Given $p(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$, $n \geq 1$, then $\exists \alpha \in \mathbb{C}$ s.t. $p(\alpha) = 0$.

Proof: If $\forall z \in \mathbb{C}$, $p(z) \neq 0$, then $1/p(z)$ is entire. Also

$$\left|\frac{1}{p(z)}\right| \leq \frac{1}{|a_n||z|^n - |a_{n-1}||z|^{n-1} - \cdots - |a_1||z| - |a_0|} \to 0 \text{ as } |z| \to \infty$$

Then $\exists R \in \mathbb{R}$ s.t. $|z| > R$, $|1/p(z)| \leq 1$. Since $|z| \leq R$ is compact set, $M := \max\limits_{|z| \leq R} |1/p(z)|$ exists. Then $|1/p(z)| \leq \max\{1, M\}\ \forall z \in \mathbb{C}$. By Liouville's theorem, $1/p(z) = c$, which contradict to $n \geq 1$. □

Theorem 1.3.5 (Morera's theorem)**.** If $\displaystyle\int_\gamma f(z)dz = 0\ \forall \gamma$: closed arc in Ω, then $f(z)$ is analytic in Ω.

Proof: Since the line integral is independent on path, there exists F : analytic s.t. $F'(z) = f(z)$. Then $f'(z) = F''(z)$ i.e. f is analytic. □

Theorem 1.3.6 (zero order)**.** If $\exists a$ s.t. $f(a) = 0$ and $f^{(k)}(a) = 0\ \forall k \in \mathbb{N}$, then $f \equiv 0$.

Proof: $\forall n \in \mathbb{N}$, $f(z) = f_n(z)(z-a)^n$, for some analytic function $f_n(z)$ in Ω. For $z \in B_\rho(a)$

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - a)^n(\xi - z)} d\xi$$

$$\implies |f(z)| = |z-a|^n |f_n(z)| \leq \frac{|z-a|^n}{2\pi}\frac{M \cdot 2\pi\rho}{\rho^n(\rho - |a-z|)} = \left(\frac{|z-a|}{\rho}\right)^n \frac{M\rho}{\rho - |a-z|} \to 0 \text{ as } n \to \infty$$

Define

$$A_1 = \{z \in \Omega | f(z) = 0, f^{(k)}(z) = 0\ \forall k \geq 1\} \text{ which is open}$$

$$A_2 = \{z \in \Omega | f(z) \neq 0 \text{ or } f^k(z) \neq 0 \text{ for some } k \geq 1\} \text{ which is also open}$$

Since $\Omega = A_1 \sqcup A_2$ is open connected and $A_1 \neq 0$, $A_2 = \varnothing$ and thus $f \equiv 0$ in Ω. □

Definition 1.3.1. If $f \neq 0$ and $f(a) = 0$, then $\exists$ the smallest $m \in \mathbb{N}$ s.t. $f^m(a) \neq 0$. This m is called the **zero order** of a. Since we can write $f(z) = (z-a)^m f_m(z)$ with $f_m(z)$ is analytic and $f_m(a) = \frac{1}{m!} f^m(a) \neq 0$.

1.4 Singularity

Recall : If $f \not\equiv 0$ in Ω, m : the zero order of $a \rightsquigarrow f(z) = (z-a)^m f_m(z)$ and $f_m(a) = \frac{1}{m!} f^m(a) \neq 0 \rightsquigarrow \exists$ a neighborhood of a s.t. $f_m(a) \neq 0$ in this neighborhood $\rightsquigarrow f(z) \neq 0$ in this neighborhood except a. Then $z = a$ is an isolated zero.

Proposition 1.4.1. If f, g are analytic in Ω and $U \subset \Omega$ with an accumulation point $a \in U$, then $f = g$ on $U \implies f = g$ on Ω.

Proof: Assume $f \neq g$ on Ω and $(f-g)(a) = 0 \implies a$ is not isolated zero ($\rightarrow\leftarrow$). $\square$

Corollary 1.4.1. $f \equiv 0$ in a subregion of $\Omega \rightsquigarrow f \equiv 0$ in Ω.

Corollary 1.4.2. $f \equiv 0$ on an arc $\rightsquigarrow f \equiv 0$ in Ω.

Corollary 1.4.3. Let f be analytic in Ω and $f(z) = f(a) + \frac{f'(a)}{1!})(z-a) + \cdots$ in $B_\rho(a)$. Let R be the radius of convergence, then $f(z) = f(a) + \frac{f'(a)}{1!} + \cdots$ in $\Omega \cap B_R(a)$.

Definition 1.4.1. Let f be analytic in $0 < |z-a| < \delta$ except perhaps at a itself. We call a an isolated singularity.

- removable : $\lim\limits_{z \to a} (z-a) f(z) = 0 \rightsquigarrow f(a)$ can be define s.t. f is analytic in $|z-a| < \delta$.
- pole : $\lim\limits_{z \to a} f(z) = \infty \rightsquigarrow \exists \delta' \leq \delta$ s.t. $f(z) \neq 0$ for $0 < |z-a| < \delta' \rightsquigarrow g(z) = f(z)^{-1}$ is analytic for $0 < |z-a| < \delta' \rightsquigarrow \lim\limits_{z \to a} g(z) = \frac{1}{\infty} = 0 \rightsquigarrow g(z)$ has removable singularity. $\because g(z) \not\equiv 0$ in $B_{\delta'}(a) \setminus \{a\} \therefore g(z) = (z-a)^m g_m(z)$ with $g_m(a) \neq 0 \rightsquigarrow f(z) = g(z)^{-1} = (z-a)^{-m} \dfrac{1}{g_m(z)}$ and define m be the **order of pole** a.
- $f(z)$ is analytic in Ω except for removable singularity or poles $\rightsquigarrow f$: **meromorphic** in Ω.

Property 1.4.1. f, g : analytic in Ω with $g \neq 0 \rightsquigarrow \frac{f}{g}$ is meromorphic, since the only possible pole are the zero of g and a common zero of f, g is pole or removable.

Definition 1.4.2.

- An isolated singularity is called an **essential singularity** if is not a removable singularity or a pole.
- $f(\infty)$ is always not defined so ∞ is regard as an isolated singularity

$$\infty \text{ is a } \begin{cases} \text{removable} \\ \text{pole} \\ \text{essential} \end{cases} \quad \text{if } g(z) = f(z^{-1}) \text{ is a } \begin{cases} \text{removable} \\ \text{pole} \\ \text{essential} \end{cases} \quad \text{at } 0$$

Example 1.4.1. Classify singularity for

(a) $\dfrac{\sin z}{z}$:

$z = 0$ is singularity and $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \rightsquigarrow \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \rightsquigarrow$ removable

(b) $e^{1/z}$:

$z = 0$ is singularity : It is clear that is not removable. If it is a pole, then $z^m e^{1/z}$ is analytic at $z = 0$ ($\rightarrow\leftarrow$). Hence $z = 0$ is essential and thus $z = \infty$ is essential singularity for e^z.

(c) $\dfrac{1}{z^3(z-2)^2}$:

$z = 0$ is pole of order 3 and $z = 2$ is pole of order 2

(d) $\dfrac{\sin z}{z^4}$

$z = 0$ is pole of order 4

(e) $\dfrac{z+1}{z^{1/2}-1}$

$z = 1$ is pole of order 1, since $\dfrac{z+1}{z^{1/2}-1} = \dfrac{(z+1)(z^{1/2}+1)}{z-1}$.

Theorem 1.4.1 (Weierstrass-Casorati)**.** If $z = a$ is an essential singularity of f, then $\forall B_\rho(a)$, f comes arbitrary close to any complex value in $B_\rho(a)$.

Proof: If not, $\exists B_\rho(a)$, $A \in \mathbb{C}$, $\exists \delta > 0$ s.t. $|f(z) - A| > \delta$ for $0 < |z - a| < \rho$, then

$$\lim_{z\to a} \frac{f(z)-A}{z-a} = \infty$$

We can write $\dfrac{f(z)-A}{z-a} = (z-a)^{-m}\dfrac{1}{g_m(z)}$ with $g_m(z) \neq 0$.

- If $m = 1 \rightsquigarrow f(z) - A = \dfrac{1}{g_m(z)} \rightsquigarrow f$ is analytic at $z = a$
- If $m \geq 2 \rightsquigarrow f(z) = A + (z-a)^{-(m-1)}\dfrac{1}{g_m(z)} \rightsquigarrow \lim\limits_{z\to a} f(z) = \infty \rightsquigarrow z = a$ is a pole of $f(z)$.

$\square$

1.5 Analytic function as mappings

f : analytic in Ω, $f : \Omega \to \mathbb{C}$. We say that $\Omega \subseteq \mathbb{C}$ is z**-plane** and $\mathbb{C}$ is w**-plane**. Given a curve γ in Ω and a parameterize $z : [\alpha, \beta] \to \gamma$ with $t \mapsto z(t)$, Γ be the image of γ via f can be parameterized by $t \mapsto f(z(t)) =: w(t)$.

- $w'(t) = z'(t)f'(z(t))$: If $z'(t_0) \neq 0$, then $f'(z_0) \neq 0 \implies w'(t_0) \neq 0$. Then we have

$$\arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0)$$

So if $f'(z_0) \neq 0$, then f is conformal in the neighborhood of z_0.

- Now we still not know the image of analytic function, so we may ask that
  - whether $w_0 \in \operatorname{Im} f$ or not.
  - and how many such z_0? i.e. find the zero order of $f(z) - w_0$ at $z = z_0$
- Now we consider $f \not\equiv 0$:

- f has zero $z_1, ..., z_m$ (count multiplicity) in $B_\rho(a)$ and $\gamma \subset B_\rho(a)$ with $f \neq 0$ on γ.

  Write $f(z) = (z - z_1) \cdots (z - z_m) g(z)$ with $g(z) : \begin{cases} \text{analytic in } B_\rho(a) \\ \text{no zero in } B_\rho(a) \end{cases}$, then

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_m} + \frac{g'(z)}{g(z)}$$

$$\implies \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_k) + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz$$

  Since $g'(z)/g(z)$ is analytic in $B_\rho(a)$, by Cauchy theorem, $\displaystyle\int_\gamma \frac{g'(z)}{g(z)} dz = 0$ and thus

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_k)$$

- f has infinite zero in $B_\rho(a)$: Let $\gamma \subset B_{\rho'}(a) \subsetneq B_\rho(a)$

  Claim : exists only finite many $z_{i_1}, ..., z_{i_m}$ in $B_{\rho'}(a)$

  subproof : If $\exists$ infinitely many $z_j's$ in $B\rho'(a)$, then by Bolzano-Weiestrass theorem, exists an accumulation point of $z_j's$ in $\overline{B_{\rho'}(a)} \subseteq B_\rho(a) \rightsquigarrow f \equiv 0$ on $\overline{B_{\rho'}(a)} \rightsquigarrow f \equiv 0$ in Ω ($\to\!\!\leftarrow$).

  □

  For $z_k's$ outside $B_{\rho'}(a)$, $n(\gamma, z_k) = 0$. Hence,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_{i_k}) = \sum_i n(\gamma, z_i) \text{ (finite sum)}$$

  This formula is called **argument principal**.

- If $\gamma = C_\rho(b)$, then $n(\gamma, z_i) = 0$ or 1, then $\displaystyle\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#$of zeros inside γ. Moreover we have it equal to

$$\frac{1}{2\pi i} \int_\alpha^\beta \frac{f'(z(t))}{f(z(t))} z'(t) dt = \frac{1}{2\pi i} \int_\alpha^\beta \frac{w'(t)}{w(t)} dt = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = n(\Gamma, 0)$$

- Let $f(z) \neq w_0$ on γ. If $\{z_j(w_0) : j = 1, ...\}$ is the set of zeros of $f(z) = w_0$, then

$$\sum_j n(\gamma, z_j(w_0)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_\Gamma \frac{w}{w - w_0} dw = n(\Gamma, w_0)$$

  In particular, choose γ be the circle $C_\rho(b)$, then # of zeros of $f(z) - w_0$ inside $\gamma = n(\Gamma, w_0)$. Hence, if w_1, w_2 lie in the same region determined by Γ, then $\#f^{-1}(w_1) = \#f^{-1}(w_2)$ inside γ.

Property 1.5.1 (key result). If $f(z) - w_0$ has a zero with order being n, then for small $\varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < |w - w_0| < \delta$, $f(z) = w$ has exactly n roots in $|z - z_0| < \varepsilon$.

Proof: Pick $\varepsilon > 0$ s.t.

$$\begin{cases} f \text{ is analytic for } |z - z_0| < \varepsilon \\ z_0 \text{ is the only zero of } f(z) = w_0 \quad \text{(since } z_0 \text{ is isolated)} \\ f'(z) \neq 0 \text{ for } 0 < |z - z_0| < \varepsilon \end{cases}$$

We can get the third since

$$\begin{cases}\text{if } f'(z_0) = 0 : \text{since zero point is isolated or } f' \equiv 0 \text{ i.e. } f \text{ is constant} \rightsquigarrow n = \infty \\ \text{if } f'(z_0) \neq 0 : \text{we can choose it by continuous}\end{cases}$$

Choose ε suitable s.t. $B_\delta(w_0) \cap \Gamma = \varnothing$. Then $n(\Gamma, w_0) = \sum_{n \text{ times}} n(\gamma, z_0) = n$. Now $\forall w$ with $0 < |w - w_0| < \delta$, $\sum n(\gamma, z_i(w)) = n(\Gamma, w) = n(\Gamma, w_0) = n$ and $\because f'(z_i(w)) \neq 0 \therefore z_i(w)$ is simple zero of $f(z) = w$. □

Corollary 1.5.1.

- f is an open mapping : for U open in Ω, $\forall z_0 \in U$, $\exists B_\rho(z_0) \subseteq \Omega$, choose smaller ε with $0 < \varepsilon < \rho$ s.t. if $w_0 = f(z_0)$, $\exists \delta > 0$ s.t. $B_\delta(w_0) \subseteq f(B_\varepsilon(z_0)) \subseteq f(U) \implies f(U)$ is open in $\mathbb{C}$.
- f is analytic at z_0 and $f'(z_0) \neq 0$. For small ε (in above), there exists δ satisfy key result. We will prove that $B_\varepsilon(z_0)$ is homeomorphic to it image. Given a open ball $B_\rho(w)$ in $f(B_\varepsilon(z_0))$, say $w = f(z)$, then $f^{-1}(B_\delta(w)) \overset{f}{\twoheadrightarrow} B_\delta(w)$. Since $f'(z) \neq 0$, f will be $1-1$ and thus f will be homeomorphism (topologically).

  Conversely, if $f'(z_0) = 0$, then f is not topologically. Thus $n = 1 \rightsquigarrow f'(z_0) \neq 0$ $(\rightarrow\!\leftarrow)$.

Theorem 1.5.1 (maximal principal)**.** If $f(z)$ is analytic and non-constant, then $|f(z)|$ has no max in Ω.

Proof: For $z \in \Omega$, if $f(z) = w_0$, then $\exists \delta > 0$ s.t. $B_\delta(w_0) \subset f(\Omega)$, and there exists w s.t. $|w| > |w_0|$, so $|w_0|$ is not max. □

We have another form of maximal principal.

Theorem 1.5.2. If f is analytic in a bounded region Ω and continuous on $\partial\Omega$, then $|f(z)|$ attains its max on $\partial\Omega$.

Proof: Since $\Omega \cup \partial\Omega$ is a compact set, $M = \max_{z \in \Omega \cup \partial\Omega} |f(z)|$ exists. If $f \neq$ constant, then $M \notin \{|f(z)| : z \in \Omega\} \implies M \in \{|f(z)| : z \in \partial\Omega\}$ □

Proof: (Another proof for maximal principal) If not, $\exists z_0 \in \Omega$ s.t. $|f(z)| \leq |f(z_0)|$ $\forall \Omega$. Let $\gamma = C_\rho(z_0) \subseteq \Omega \rightsquigarrow z = z_0 + \rho e^{i\theta}$, $\theta \in (-\pi, \pi)$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z_0 + \rho e^{i\theta}) \cdot i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + \rho e^{i\theta}) d\theta$$

- If $\exists z \in \gamma$ s.t. $|f(z)| < |f(z_0)| \rightsquigarrow \exists [\theta_1, \theta_2] \subseteq [-\pi, \pi]$ s.t. $|f(z_0 + \rho e^{i\theta})| < |f(z_0)|$ for all $\theta \in [\theta_1, \theta_2]$. So

$$|f(z_0)| = \left| \frac{1}{2\pi} \left(\int_{-\pi}^{\theta_1} + \int_{\theta_1}^{\theta_2} + \int_{\gamma_2}^{\pi} \right) f(z_0 + \rho e^{i\theta}) d\theta \right| < |f(z_0)| \ (\rightarrow\!\leftarrow)$$

- $\forall z \in \gamma$, $|f(z)| = |f(z_0)|$. Since ρ is arbitrary (only need $C_\rho(z_0) \subseteq \Omega$), $|f(z)| = |f(z_0)|$ $\forall z \in \overline{B_\rho(z_0)}$. Set

$$S = \{z \in \Omega : |f(z)| = |f(z_0)|\}$$

which is open, since for all $z' \in S$, we can replace z_0 in above argument to get $|f(z)| = |f(z')| = |f(z_0)|\ \forall z \in B_\rho(z')$. Also $\Omega \setminus S = \{z \in \Omega : |f(z)| < |f(z_0)|\}$ is open. Since Ω is simply connect and $z_0 \in S$, $\Omega = S$ i.e. $|f(z)|$ is constant. Since

$$|f(z)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z+\rho e^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0+\rho e^{i\theta})|d\theta = |f(z)|$$

the equality holds, which means $f(z)$ is constant on $C_\rho(z)$ for all $z \in \Omega$ and suitable ρ i.e. $f(z)$ is constant in Ω.

□

Theorem 1.5.3 (Schwarz lemma)**.**

- If f is analytic in $B_1(0)$ and $\begin{cases} f(B_1(0)) \subseteq \overline{B_1(0)} \\ f(0) = 0 \end{cases}$, then $|f(z)| \leq |z|$ and $|f'(0)| = 1$
- If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ with $|c| = 1$.

Proof:

- Define $g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}$. For $0 < \rho < 1$, $\forall z \in B_\rho(0)$, by maximal principal,
$$|g(z)| \leq \max_{z \in C_\rho(0)} |g(z)| = \frac{|f(z)|}{\rho} \leq \frac{1}{\rho}$$
As $\rho \to 1$, $|g(z)| \leq 1$ on $B_1(0)$ i.e. $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
- If $|f(z)| = |z|$ for some $z \neq 0$ in $B_1(0)$, then $|g(z)| = 1 \rightsquigarrow |g(z)|$ attains a max in $B_1(0) \rightsquigarrow g(z) = c$ is a constant function. Since $|f(z)| = |z|$, $|c| = 1$. □

1.6 Automorphism of unit disk and half plane

1.6.1 Automorphism of unit disk

- $f(z) = e^{i\theta}z : B_1(0) \to B_1(0)$ is a rotation $\rightsquigarrow f \in \mathrm{Aut}(B_1(0))$, the group of **bianalytic map** from $B_1(0)$ to $B_1(0)$.
- For $0 \neq a \in B_1(0)$, $T_a(z) = \dfrac{a-z}{1-\overline{a}z} \rightsquigarrow T_a \in \mathrm{Aut}(B_1(0))$:
  - $|a| < 1 \rightsquigarrow |1/\overline{a}| > 1 \rightsquigarrow T_a$ is analytic in $B_1(0)$
  - $\forall |z| = 1$,
$$|T_a(z)| = \frac{|a-z|}{|1-\overline{a}z|}\frac{1}{|\overline{z}|} = \frac{|a-z|}{|\overline{z}-\overline{a}|} = 1$$
By maximal principal, $|T_a(z)| < 1\ \forall z \in B_1(0)$.
  - $T_a(0) = a, T_a(a) = 0 \rightsquigarrow T_a \circ T_a(0) = 0, T_a \circ T_a(a) = a \rightsquigarrow T_a \circ T_a(0) = 0, T_a \circ T_a(a) = a$. By Schwarz lemma, $T_a \circ T_a(z) = cz$. Evaluate a, then $c = 1 \rightsquigarrow T_a \circ T_a = \mathrm{id}$ on $B_1(0)$
  - $\mathrm{Aut}(B_1(0)) = \{e^{i\theta} \circ T_a : \theta \in \mathbb{R}, a \in B_1(0)\}$:
$\forall f \in \mathrm{Aut}(B_1(0))$, let a s.t. $f(a) = 0$. Define $g := f \circ T_a \in \mathrm{Aut}(B_1(0)) \rightsquigarrow g(0) = 0, g^{-1}(0) = 0$. By Schwarz lemma, $\begin{cases} |g'(0)| \leq 1 \\ |(g^{-1})'(0)| \leq 1 \end{cases} \implies |g'(0)| = 1 \rightsquigarrow g(z) = e^{i\theta}z$ i.e. $f = e^{i\theta} \circ T_a$.

1.6.2 Automorphism of half plane

Since we already knew all element in $\mathrm{Aut}(B_1(0))$, our idea is construct the bianalytic between $B_1(0)$ and half plane $\mathbb{H}$. Construct

$$\begin{aligned} S: \ \mathbb{H} &\longrightarrow B_1(0) \\ z &\longmapsto \frac{i-z}{i+z} \end{aligned}$$

which is analytic in $\mathbb{H}$. $|i+z| < |i-z| \ \forall z \in \mathbb{H}$, since z in the half plane divide by perpendicular bisector of $i, -i$ which contain i. Also $S^{-1}(z) = i\left(\dfrac{1-z}{1+z}\right)$ will sent $B_1(0)$ to $\mathbb{H}$, since

$$\operatorname{Im} S^{-1}(z) = \frac{(1-r^2)\cos^2\theta + (1+r^2)\sin^2\theta}{(1+r\cos\theta)^2 + (r\sin\theta)^2} > 0 \text{ if } z = re^{i\theta} \text{ with } r < 1$$

Hence, S is bianalytic from $B_1(0)$ to $\mathbb{H}$, which will induce the group homomorphism

$$\begin{aligned} \varphi: \ \mathrm{Aut}(B_1(0)) &\longrightarrow \mathrm{Aut}(\mathbb{H}) \\ f &\longmapsto s^{-1}\circ f \circ s \end{aligned}$$

and it clear that the inverse is

$$\begin{aligned} \varphi^{-1}: \ \mathrm{Aut}(\mathbb{H}) &\longrightarrow \mathrm{Aut}(B_1(0)) \\ g &\longmapsto s\circ f \circ s^{-1} \end{aligned}$$

Hence, $\mathrm{Aut}(B_1(0)) \simeq \mathrm{Aut}(\mathbb{H})$.

Definition 1.6.1. A **linear functional transformation** is $F_A(z) = \dfrac{az+b}{cz+d}$ with $ad-bc \neq 0$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. It is clear that $F_B \circ F_A = F_{AB}$.

Theorem 1.6.1. $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\} \simeq \{A \in \mathrm{SL}_2(\mathbb{R}) : \det A > 0\} =: \overline{\mathrm{SL}_2(\mathbb{R})}$

Proof:

- $A \in \overline{\mathrm{SL}_2(\mathbb{R})} \rightsquigarrow F_A \in \mathrm{Aut}(\mathbb{H})$:
  - pole is $z = -d/c \in \mathbb{R}$ which is not in $\mathbb{H} \rightsquigarrow F_A$ is analytic in $\mathbb{H}$
  - $\operatorname{Im}\dfrac{az+b}{cz+d} = \operatorname{Im}\dfrac{(az+b)(c\overline{z}+d)}{|cz+d|^2} = \operatorname{Im}\dfrac{adz+bc\overline{z}}{|cz+d|^2} = \dfrac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} > 0$
  - $F_A^{-1} = F_{A^{-1}} \rightsquigarrow F_A : \mathbb{H} \xrightarrow{\sim} \mathbb{H}$
- If $g \in \mathrm{Aut}(\mathbb{H})$ with $g(i) = i \rightsquigarrow \varphi^{-1}(g)(0) = s\circ g\circ s^{-1}(0) = 0$. By Schwarz lemma, $|(\varphi^{-1}(g))'(0)| \leq 1$. Similar we have $|\varphi(g)'(0)| \leq 1$. By Schwarz lemma, $\varphi(g) = e^{i\theta}$. Let

$$B = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in \overline{\mathrm{SL}_2(\mathbb{R})} \implies \begin{cases} F_B(i) = i \\ F_B'(i) = e^{i\theta} \end{cases} \implies \varphi^{-1}(F_B)(z) = cz$$

  Differentiate in both side and substitute $z = 0$ we have $e^{i\theta} = c$. Hence, $\varphi^{-1}(F_B) = \varphi^{-1}(g) \rightsquigarrow g = F_B$.

- $\forall z_0 \in \mathbb{H}$, $\exists D \in \overline{\mathrm{SL}_2(\mathbb{R})}$ s.t. $F_D(i) = z_0$:
  Let $D_1 = \begin{pmatrix} \sqrt{\operatorname{Im} z_0} & 0 \\ 0 & \sqrt{\operatorname{Im} z_0}^{-1} \end{pmatrix} \rightsquigarrow F_{D_1}(i) = i \operatorname{Im} z_0$. Let $D_2 = \begin{pmatrix} 1 & \mathrm{Re} z_0 \\ 0 & 1 \end{pmatrix}$, then $F_{D_2} \circ F_{D_1}(i) = z_0$. Let $D = D_2 D_1$, then $F_D(i) = z_0$.
- $\forall f \in \mathrm{Aut}(\mathbb{H})$, $\exists z_0$ s.t. $f(z_0) = i \rightsquigarrow \exists D \in \overline{\mathrm{SL}_2(\mathbb{R})}$ s.t $F_D(i) = z_0 \rightsquigarrow g = f \circ F_D$ and $g(i) = i \rightsquigarrow g = F_B$ for some $B \in \overline{\mathrm{SL}_2(\mathbb{R})} \rightsquigarrow f = F_D^{-1} \circ F_B = F_{D^{-1}B}$

□

1.7 Residue

1.7.1 Laurent series

Recall : $f(z)$ is analytic in Ω and $z_0 \in \Omega$.

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^n(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

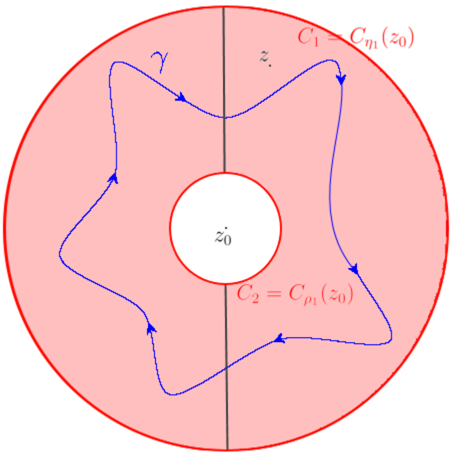
where

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}(\xi - z)} d\xi$$

Then we can general the Taylor expansion.

Theorem 1.7.1 (Laurent series). f : analytic in $\Omega : \rho < |z - z_0| < \eta$. Then $f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$ converge in Ω with $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}}$, where γ : simple closed arc in Ω, z_0 lies inside γ. The part of $\sum_{-\infty}^{-1} a_n (z - z_0)^n$ is called **singular part**.

Proof: Given $z \in \Omega$, choose $\rho < \rho_1 < \eta_1 < \eta$ such that $\rho_1 < |z - z_0| < \eta$, C_1 contain γ and C_2 inside γ. Choose two (black) segments connected C_1 and C_2 do not pass z and let L_1, L_2 be the right and left half curve in below, then $L_1 + L_2 = C_1 - C_2$. WLOG z inside L_1.



$$f(z) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{L_2} \frac{f(\xi)}{\xi - z} d\xi = \underbrace{\frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi}_{(1)} - \underbrace{\frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{\xi - z} d\xi}_{(2)}$$

$$(1) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)d\xi}{(\xi - z_0)\left(1 - \dfrac{z-z_0}{\xi - z_0}\right)} = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z_0}\left(1 + \frac{z-z_0}{\xi-z_0} + \cdots + \frac{\left(\dfrac{z-z_0}{\xi-z_0}\right)^n}{1 - \dfrac{z-z_0}{\xi-z_0}}\right)d\xi$$

$$\implies a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi$$

and the error is

$$R_n = \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\xi)d\xi}{(\xi-z_0)^n(z-z_0)} \implies |R_n| \le \frac{|z-z_0|^n}{2\pi}\frac{M\cdot 2\pi\eta_1}{\eta_1^n(\eta_1 - |z-z_0|)} \to 0$$

Similarly, we have

$$-(2) = \int_{C_2}\frac{f(\xi)}{z-z_0}\left(\frac{1}{1 - \dfrac{\xi - z_0}{z-z_0}}\right)d\xi \implies a_{-m} = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{(\xi-z_0)^{-m+1}}d\xi$$

$\square$

Remark 1.7.1. z_0 : isolated singularity $\rightsquigarrow$ f analytic in $\Omega : 0 < |z-z_0| < \eta \implies f(z) = \sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$

- $a_n = 0\ \forall n < 0 \rightsquigarrow z_0$: removable
- $a_n = 0\ \forall n < -m$ but $a_m \neq 0 \rightsquigarrow z_0$: pole of order m
- $a_n \neq 0\ \forall n \to \infty \rightsquigarrow z_0$: essential singularity

If z_0 is a pole of order m, then

$$\frac{1}{2\pi i}\int_{C_\rho(z_0)} f(z)dz = \underbrace{\frac{1}{2\pi i}\int_{C_0(z_0)}\frac{a_{-m}}{(z-z_0)^m}dz}_{\text{have primitive}} + \cdots + \frac{1}{2\pi i}\int_{C_0(z_0)}\frac{a_{-1}}{(z-z_0)}dz + \underbrace{\frac{1}{2\pi i}\int_{C_0(z_0)} p(z)dz}_{\text{analytic}}$$
$$= a_{-1}n(C_\rho(z_0), z_0) = a_{-1} =: \operatorname{Res}_{z=z_0} f(z)$$

Definition 1.7.1.

- A region Ω is called **multiply connected** if it is not simply connected
- Ω has the finite **connectivity** n if $\widetilde{C} \setminus \Omega$ has exactly n connected components $A_1, A_2, ..., A_n$ and usually let $\infty \in A_n$.

Recall $\forall\gamma$: cycle in Ω, $n(\gamma, a)$ is constant in $A_i\ \forall i$ and $n(\gamma,a) = 0$ on A_n. For $i = 1, ..., n-1$, since A_i is bounded, as in the proof of fact about simply connectivity, $\exists\gamma_i \subset \Omega$ s.t. $n(\gamma_i, a) = 1\ \forall a \in A_i$ and $n(\gamma_i, b) = 0\ \forall b \in A_j \neq A_i$. $\forall\gamma$: cycle in Ω, let $c_i = n(\gamma, a)\ \forall a \in A_i$. Since $\forall a \in \widetilde{\mathbb{C}} \setminus \Omega$, say $a \in A_i$, then

$$n(\gamma - c_1\gamma_1 - \cdots - c_{n-1}\gamma_{n-1}, a) = n(\gamma, a) - c_i n(\gamma_i, a) = 0$$

i.e. $\gamma \sim c_1\gamma_1 + \cdots + c_{n-1}\gamma_{n-1}$ w.r.t. Ω. Hence, if f is analytic in Ω, then

$$\int_\gamma f dz = c_1\int_{\gamma_1} f dz + \cdots + c_{n-1}\int_{\gamma_{n-1}} f dz$$

$f(z)$ is analytic in Ω except for isolated singularities $a_1, ..., a_n$. Let $\Omega' = \Omega \setminus \{a_1, ..., a_n\}$ and $\gamma_i = C_{\rho_i}(a_i)$ with $\begin{cases} 0 < |z-a| < \rho_i \subset \Omega' \\ \gamma_i \sim 0 \text{ w.r.t. } \Omega \end{cases}$. Let γ be a cycle in Ω' with $\gamma \sim 0$ w.r.t. Ω. Since

$$n(\sum_{i=1}^{n} n(\gamma, a_i)\gamma_i, a_j) = \sum_{i=1}^{n} n(\gamma, a_i) n(\gamma_i, a_j) = n(\gamma, a_j) \ \forall j$$

and $\gamma \sim \sum_{i=1}^{n} n(\gamma, a_i)\gamma_i$ w.r.t. Ω, $\gamma \sim \sum_{i=1}^{n} n(\gamma, a_i)\gamma_i$ w.r.t. Ω'. Hence, we have

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_{i=1}^{n} n(\gamma, a_i)\left(\frac{1}{2\pi i}\int_{\gamma_i} f(z)dz\right)$$

If a_i is pole, then $\frac{1}{2\pi i}\int_{\gamma_i} f(z)dz = \operatorname{Res}_{z=a_i} f(z)$. If all a_i are pole, we can rewrite it as

$$\sum_{i=1}^{n} n(\gamma, a_i) \operatorname{Res}_{z=a_i} f(z)$$

Property 1.7.1 (key fact). If z_0 is a pole of f of order m, then

$$a_{-1} = \operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} (z-z_0)^m f(z)$$

Proof: Since $(z-z_0)^{m-1} f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + p(z)(z-z_0)^m$. □

1.7.2 Evaluation definite integrals

(1) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta : z = e^{i\theta} \rightsquigarrow dz = ie^{i\theta} = izd\theta$, $\sin\theta = \frac{z-z^{-1}}{2}$, $\cos\theta = \frac{z+z^{-1}}{2}$.

$$L = \frac{-1}{4i}\int_{|z|=1} \underbrace{\frac{(z^2-1)^2}{z^2(2z^2+5z+2)}}_{=f} dz = \frac{-2\pi i}{4i}(\operatorname{Res}_{z=-1/2} f + \operatorname{Res}_{z=0} f) = \frac{-5}{4}$$

(2) $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2+1}dx$: Consider the curve Γ consists the segment from $-R$ to R and counterclockwise circular arc γ from R to $-R$ with radius R and center in 0.

$$\int_{\Gamma} \frac{1-e^{2\pi iz}}{z^2+1}dz = \int_{-R}^{R} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1}dx + \int_{\gamma} \frac{1-e^{2\pi iz}}{z^2+1}dz$$

Calculate residue, it will be $2\pi i \operatorname{Res}_{z=i} f = \pi(1-e^{-2})$. Also,

$$\left|\int_{\gamma} f dz\right| \le \int_{\gamma} \frac{1+|e^{2zi}|}{|z|^2-1}|dz| \le \int_{\gamma} \frac{1+|e^{-2\operatorname{Im} z}|}{|z|^2-1}|dz| \le \frac{2\pi R}{R^2-1} \to 0 \text{ as } R\to\infty$$

Hence,

$$\begin{aligned}\pi(1-e^{-2}) &= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1}dx + \int_{\gamma} \frac{1-e^{2\pi iz}}{z^2+1}dz\right) \\ &= \int_{-\infty}^{\infty} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1}dx\end{aligned}$$

Consider the real part and thus $L = \pi(1-e^{-2})$.

(3) $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}dx,\ a>0$: First we check that will converge.

- $\displaystyle\int_1^\infty \frac{\ln x}{x^2+a^2}$ will converge since $\dfrac{\ln x}{x^2+a^2} < \dfrac{1}{x^{1.5}}$ for x sufficiently large.
- $\displaystyle\int_0^1 \frac{\ln x}{x^2+a^2}dx$ will converge since in

$$\int_0^1 \frac{-\ln x dx}{x^2+a^2}dx \le \int_0^1 \frac{-\ln x dx}{a^2} = \frac{-1}{a^2}\left(x\ln x - x\right)\Big|_0^1 = \frac{1}{a^2}$$

Let C_ρ be the curve $\{\rho e^{i\theta} : 0 \le \theta \le \pi\}$ and $\gamma = \overline{(-R)(-r)} - C_r + \overline{rR} + C_R$. We define $\ln x$ by branch $-\pi/2$. By residue,

$$\int_\gamma \frac{\log z}{z^2+a^2}dz = 2\pi i\,\mathrm{Res}_{z=ai}\frac{\log z}{z^2+a^2} = 2\pi i \lim_{z\to ai}\frac{\log z}{z+ai} = \frac{\pi}{a}\left(\ln a + \frac{\pi}{2}\right)$$

On C_R, $z = Re^{i\theta}$, $0 \le \theta \le \pi$

$$\left|\frac{\log z}{z^2+a^2}\right| = \left|\frac{\ln R + \theta i}{R^2-a^2}\right| \le \frac{\ln R + \pi}{R^2-a^2} \implies \left|\int_{C_R}\frac{\log z}{z^2+a^2}\right| \le \frac{\pi R(\ln R+\pi)}{R^2-a^2} \to 0 \text{ as } R\to 0$$

On $-C_r$,

$$\left|\int_{-C_r}\frac{\log z}{z^2+a^2}dz\right| \le \left(\frac{-\ln r + \pi}{a^2-r^2}\right)\pi r \to 0 \text{ as } r\to 0$$

Hence,

$$\frac{\pi}{a}\left(\ln a + \frac{\pi}{2}i\right) = \lim_{\substack{r\to 0\\ R\to\infty}}\left(\int_{-R}^{-r}\frac{\ln z}{z^2+a^2}dz + \int_r^R \frac{\ln z}{z^2+a^2}dz + \int_{C_R} f dz + \int_{-C_r} f dz\right)$$
$$= \int_0^\infty \frac{\ln x}{x^2+a^2}dx + \int_{-\infty}^0 \frac{\ln(-x)+\pi i}{x^2+a^2}dx = 2\int_0^\infty \frac{\ln x}{x^2+a^2}dx + \pi i\int_0^\infty \frac{dx}{x^2+a^2}$$

Hence, $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}$

1.7.3 Rouche's theorem

Recall : Argument principle : Let $f \not\equiv 0$ be analytic in $B_\rho(a)$ and $\gamma \subseteq B_\rho(a)$ with $f \ne 0$ on γ. Let z_i be the roots of $f(z)=0$, then

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

Theorem 1.7.2 (General form)**.** Let $f(z)$ be meromorphic in Ω with zeros a_i's and the poles b_k's. Then $\forall \gamma \sim 0$ w.r.t. Ω and $a_j, b_k \notin \gamma$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)} = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

Proof: Rearrange a_i's and b_k's s.t. there is h_i $\widetilde{a}_i$'s in $\{a_1, a_2, ...\}$ and ℓ_j $\widetilde{b}_j$'s in $\{b_1, b_2, ...\}$, where $\widetilde{a}_i \neq \widetilde{a}_j, \widetilde{b}_i \neq \widetilde{b}_j$ $\forall i \neq j$. Since zero and pole are isolated, $\exists \gamma_i$ and γ_i' s.t.

$$\begin{cases} n(\gamma_i, \widetilde{a}_j) = n(\gamma_i', \widetilde{b}_j) = \delta_{ij} \\ n(\gamma_i, \widetilde{b}_k) = n(\gamma_i', \widetilde{a}_k) = 0 \\ \gamma_i \sim 0, \gamma_i' \sim 0 \text{ w.r.t. } \Omega \end{cases}$$

We have known that $\gamma \sim \sum_i n(\gamma, \widetilde{a}_i)\gamma_i + \sum_j n(\gamma, \widetilde{b}_j)\gamma_j'$ w.r.t. $\Omega' = \Omega \setminus \{a_i, b_j \ \forall i, j\}$. Observe that $\dfrac{f'(z)}{f(z)}$ is meromorphic. By residue formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, \widetilde{a}_i) \operatorname{Res}_{z=\widetilde{a}_i} f + \sum_j n(\gamma, \widetilde{b}_j) \operatorname{Res}_{z=\widetilde{b}_j} f$$

Now, for $a = \widetilde{a}_i$, $h = h_i$, write $f(z) = (z-a)^h f_h(z)$, $\dfrac{f'(z)}{f(z)} = \dfrac{h}{z-a} + \dfrac{f_h'(z)}{f_h(z)} \rightsquigarrow \operatorname{Res}_{z=a}\dfrac{f'(z)}{f(z)} = h$. For $b = \widetilde{b}_j$, $\ell = \ell_j$, write $f(z) = (z-b)^{-\ell} g_\ell(z)$, $\dfrac{f'(z)}{f(z)} = \dfrac{-\ell}{z-b} + \dfrac{g_\ell'(z)}{g_\ell(z)} \rightsquigarrow \operatorname{Res}_{z=b}\dfrac{f'(z)}{f(z)} = -\ell$. Hence,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, \widetilde{a}_i)h_i - \sum_j n(\gamma, \widetilde{b}_j)\ell_j = \sum_i n(\gamma, a_i) - \sum_k n(\gamma, b_k)$$

□

Remark 1.7.2. More general, if $g(z)$ is analytic in Ω, then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, a_i)g(a_i) - \sum_j n(\gamma, b_j)g(b_j)$$

which can prove by same method in above.

Theorem 1.7.3 (Rouche's theorem)**.** Let $\gamma \sim 0$ w.r.t. Ω and $n(\gamma, z) = 0$ or 1 $\forall z \notin \gamma$. Let f and g be analytic in Ω. If $|f(z) - g(z)| < |f(z)|$ on γ, then f and g have the same number of zeros inside γ.

Proof: By assumption, $f \neq 0$ and $g \neq 0$ on γ, $\left|\dfrac{g(z)}{f(z)} - 1\right| < 1$ on γ. Let $\omega = F(z) = \dfrac{g(z)}{f(z)} \rightsquigarrow$ $\Gamma := \operatorname{Im} F|_\gamma \subset B_1(1) \rightsquigarrow n(\Gamma, 0) = 0$. Then

$$\begin{aligned} 0 = n(\Gamma, 0) &= \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega} = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)dz}{F(z)} \\ &= \#\text{ zero of } F - \#\text{ pole of } F \text{ inside } \gamma \\ &= \#\text{ zero of } g - \#\text{ zero of } f \text{ inside } \gamma \end{aligned}$$

□

Example 1.7.1. Show that $e^z = az^n$ has exactly n solution in $|z| < 1$, where $a > e$.

Proof: $|e^z| = e^x \leq e \ \forall z \in C_1(0)$, $|-az^n| = a > e \ \forall z \in C_1(0) \implies |-az^n| > |e^z| \ \forall z \in C_1(0) \rightsquigarrow e^z = az^n$ and $-az^n$ have the same number of zeros in $|z| < 1 \rightsquigarrow e^z = az^n$ has exactly n solution in $|z| < 1$. □

Question 1 : Let $w = f(z)$ be analytic in Ω. For $w_0 \in f(\Omega)$, find $z_j(w_0) \in \Omega$ s.t. $f(z_j(w_0)) = w_0$.
Assume that for $|w - w_0| < \delta$, $f - w$ has exactly n roots $z_j(w)$ in $|z - z_0| < \varepsilon$.

- Set $g(z) = z \rightsquigarrow \displaystyle\sum_{j=1}^{n} z_j(w) = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)-w} z dz$. In particular, $n = 1$ we have
$$f^{-1}(w) = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)-w} z dz$$
- Set $g(z) = z^m \rightsquigarrow \displaystyle\sum_{j=1}^{n} z_j(w)^m = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z)-w} z^m dz$ which is analytic w.r.t. w. Since the elementary symmetric polynomial of $z_1(w), ..., z_n(w)$ is in the $\mathbb{C}$-algebra generated by k-power sum $z_k(w)$ of $z_1(w), ..., z_n(w)$ which is also analytic in Ω. Hence, we can calculate the roots of $z^n - s_1(w)z^{n-1} + \cdots + (-1)^n s_n(w)$ which is $z_1(w), ..., z_n(w)$.

Question 2 : Find the number of zero of f in $|z| < R$.
Write $f(z) = P_{n-1} + z^n f_n(z)$. If we can choose n s.t. $R^n |f_n(z)| < |P_{n-1}(z)|$ on $|z| = R$, then # zero of $f(z) =$ # zero of $P_{n-1}(z)$ in $|z| < R$.

1.8 Sum and product

Definition 1.8.1. We write $f_n \xrightarrow{\overline{\text{unif}}} f$ in Ω if f_n converge uniform on each compact subset in Ω.

Theorem 1.8.1 (Weierstrass theorem). f_n : analytic in Ω_n with $(\Omega_n \subset \Omega_{n+1})$ and $f_n \xrightarrow{\overline{\text{unif}}} f$ in $\Omega = \bigcup_n \Omega_n$. Then f is analytic and $f_n' \xrightarrow{\overline{\text{unfi}}} f'$.

Proof:

- For a fixed $\overline{B_\rho}(a) \subset \Omega$, $\exists n_0$ s.t. $\overline{B_\rho(a)} \subseteq \Omega_n \ \forall n \geq n_0$. By Cauchy integral formula,
$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f_n(\xi)}{\xi - z} d\xi \text{ in } B_\rho(a)$$
By assumption, since $f_n \to f$ uniformly converge in $B_\rho(a)$
$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{\lim\limits_{n\to\infty} f_n(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{\xi - z} d\xi$$
which is analytic.
- $\forall n \geq n_0$, $f_n'(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \frac{f_n(\xi)}{(\xi - z)^2} d\xi$ in $B_\rho(a)$. For all $\delta < \rho$, choose $\delta' \in (\delta, \rho)$. Since f_n uniformly converge to f in $\overline{B_{\delta'}(a)}$, for sufficiently large n, we have
$$|f'(z) - f_n'(z)| \leq \frac{1}{2\pi}\int_{C_{\delta'}(a)} \frac{|f(\xi) - f_n(\xi)|}{|\xi - z|^2} |d\xi| \leq \frac{\varepsilon\delta\pi}{2\pi|\delta' - \delta|^2} \ \forall z \in \overline{B_\delta(a)}$$

and thus f_n' is uniformly converge to f in $\overline{B_\delta(a)}$. Since any compact subset of Ω can be covered by $\{\overline{B_(\delta_1)(a_1)}, ..., \overline{B_(\delta_k)(a_k)}\}$, the result follow.

$\square$

Theorem 1.8.2 (Mittag-Leffler theorem)**.** Let $\{b_n\} \subset \mathbb{C} \setminus \{0\}$ with $\lim_{n\to\infty} b_n = \infty$ and let $P_m(z) \in \mathbb{C}[z]$ with $P_m(0) \neq 0$. Then $\exists$ a meromorphic function $f(z)$ in $\mathbb{C}$ with pole at $b_m{}'s$ and singularity part is $P_m(1/(z-b_m))$.

Proof:

- Since $P_m\left(\frac{1}{z-b_m}\right)$ is analytic for $|z| < |b_m|$, consider the Taylor series at $z = 0$

$$F_m(z) := P_m\left(\frac{1}{z-b_m}\right) = a_0^m + a_1^m + \cdots + \left(\frac{1}{2\pi i}\int_{C_{|b_m|/2}(0)} \frac{F_m(\xi)}{\xi^{n+1}(\xi - z)} d\xi\right) z^{m+1}$$

Let $M_m := \max_{|z|=|b_m|/2} |F_m(z)|$, $q_m(z) = a_0^m + \cdots + a_{n_m}^m z^{n_m}$ and choose n_m s.t. $2^{n_m} \geq M^m \cdot 2^m$. Then

$$|F_m(z) - q_m(z)| \leq \frac{M_m}{2\pi} \frac{|z|^{n_m+1}}{(|b_m|/2)^{n_m+1}} \frac{2\pi|b_m|/2}{|b_m|/4} = \frac{M_m}{2^{n_m}} \leq \frac{1}{2^m}$$

$\forall N \in \mathbb{N}$, $\exists n_N > 0$ s.t. $n \geq n_N$, $|b_n| > N$. So for $|z| \leq N/4 < |b_n|/4$ $\forall n \geq n_N$,

$$|F_n(z) - q_n(z)| \leq \left(\frac{1}{2}\right)^n \ \forall n \geq n_N, \ |z| \leq \frac{|N|}{4}$$

By Weierstrass M-test, $g_N(z) := \sum_{n=n_N}^{\infty} (F_n(z) - q_n(z))$ converge uniformly on $|z| \leq N/4$ and thus is analytic in $|z| < N/4$. Define

$$f_N(z) = \sum_{n=1}^{N_n - 1} (F_n(z) - q_n(z)) + g_N(z) : \text{meromorphic } |z| < N/4 \text{ with poles } b_1, ..., b_{n_N - 1}$$

Notice that

$$|g_{N+1} - g_N| = \left|\sum_{n=n_N}^{n_{N+1}-1} (F_n(z) - q_n(z))\right| \leq \sum_{n=n_N}^{n_{N+1}-1} \frac{1}{2^n} \leq \frac{1}{2^{n_N}} \to 0 \text{ as } N \to \infty$$

Then g_N is Cauchy sequence and thus $g_N \to g$ uniformly in $\mathbb{C}$.

$$|f_{N+1} - f_N| \leq \left|\sum_{n=n_N}^{n=n_{N+1}} (F_n(z) - q_n(z))\right| + |g_{N+1} - g_N| \leq \frac{2}{2^{n_N - 1}} \text{ for } |z| \leq N/4$$

Then f_N is Cauchy sequence and thus

$$f_n \to f = \sum_{n=1}^{\infty} (F_n(z) - q_n(z)) + g(z) \text{ as } N \to \infty$$

where $g(z)$ is entire function.

$\square$

Remark 1.8.1. If we consider $\{b_m\}_{m\in\mathbb{N}} \cup \{0, ..., 0\}$, then $\widetilde{f}(z) = f(z) + \sum_{i=1}^{\ell} \overline{P}_i(1/z)$.

Example 1.8.1. $f(z) = \dfrac{\pi^2}{\sin^2 \pi z}$ has pole when $z \in \mathbb{Z}$.

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\left(\sum\limits_{n=1}^{\infty} \frac{(-\pi z)^{2n-1}}{(2n-1)!}\right)^2} = \frac{1}{z^2}\left(1 - \frac{(\pi z)^2}{3!} + \cdots\right)^{-2} = \frac{1}{z^2}\left(1 - \left(\frac{(\pi z)^2}{3!} + \cdots\right) + \cdots\right)^2$$

The singularity part at 0 is $\frac{1}{z^2}$. Since $\sin^2 \pi(z-n) = \sin^2 \pi z$ $\therefore$ the singularity part at n is $\dfrac{1}{(z-n)^2}$. Then

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)$$

where $g(z)$ analytic in $\mathbb{C}$. Claim $g(z) = 0$:
subproof : g has period $\omega = 1$.

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

and thus

$$\frac{\pi^2}{|\sin^2 \pi z|} \le \frac{\pi^2}{|\cosh^2 \pi y| - \cos^2 \pi x} \le \frac{\pi^2}{|\cosh^2 \pi y| - 1} \xrightarrow[0\le x\le 1]{\text{unif.}} 0 \text{ as } |y| \to \infty$$

and $\sum\limits_{n=-\infty}^{\infty} \dfrac{1}{(z-n)^2} \xrightarrow[0\le x\le 1]{\text{unif.}} 0$ as $|y| \to \infty \implies g(z) \xrightarrow[0\le x\le 1]{\text{unif.}} 0$ as $|y| \to \infty$. Then $|g(z)|$ is bounded in $0 \le x \le 1$ and thus bounded in $\mathbb{C}$. By Lioville's theorem, $g = c$ is constant. Since $\lim\limits_{|y|\to\infty} g(z) = 0 \rightsquigarrow g = 0$.

Definition 1.8.2.

- $p_n \neq 0\ \forall n$, $q_n = p_1 p_2 \cdots p_n$, $\prod\limits_{n=0}^{\infty} p_n = \lim\limits_{n\to\infty} q_n$.
- In general, $\prod\limits_{n=1}^{\infty} p_n$ converge $\iff \#\{p_i | p_i = 0\} < \infty$ and $\prod\limits_{p_n \neq 0} p_n$ exists.

Fact 1.8.1.

- $\prod\limits_{n=1}^{\infty} p_n$ converge $\implies \lim\limits_{n\to\infty} p_n = 1 \rightsquigarrow \prod\limits_{n=1}^{\infty}(1 + a_n)$ with $\lim\limits_{n\to\infty} a_n = 0$.
- $\prod(1 + a_n)$ with $(1 + a_n) \neq 0 \iff \sum \log(1 + a_n)$ converge (principal branch).
- $\prod(1 + a_n)$ absolutely converge $\iff \sum |a_n|$ converge :

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1 \rightsquigarrow \forall \varepsilon > 0,\ (1-\varepsilon)|a_n| < |\log(1 + a_n)| < (1+\varepsilon)|\varepsilon|$$

- $g(z)$: entire $\rightsquigarrow f(z) = e^{g(z)}$: entire and $\neq 0$.

- $f(z)$: entire and never zero, for a fixed z_0

$$g(z) := \int_{\gamma_z} \frac{f'(\xi)}{f(\xi)} d\xi + c_0,\ e^{c_0} = f(z_0)$$

where γ_z. Then $g'(z) = \dfrac{f'(z)}{f(z)} \rightsquigarrow \dfrac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0 \rightsquigarrow f(z)e^{-g(z)} = c \xrightarrow{z=z_0} c = 1 \rightsquigarrow f(z) = e^{g(z)}$.

Theorem 1.8.3 (Weierstrass). Given $\{a_n\} \subseteq \mathbb{C} \setminus \{0\}$ with $\lim_{n\to\infty} a_n = \infty$, there exists entire functions with zeros $= \{a_n\}$. possibly including 0.

Proof:

- # of $\{a_n\} < \infty$: $f(z) = z^m e^{g(z)} \prod_{i=1}^{n}\left(1 - \frac{z}{a_i}\right)$, $g(z)$: entire function.
- # of $\{a_n\} = \infty$:
  - $\sum |a_n|^{-1}$ converge $\iff \sum \dfrac{|z|}{|a_n|}$ converge $\forall |z| \le R \iff \prod \left(1 - \dfrac{z}{a_n}\right)$ converge uniform $\forall |z| < R$. So $f(z) = z^m \prod_{n=1}^{\infty}\left(1 - \dfrac{z}{a_n}\right)$ analytic in $\mathbb{C}$.
  - In general, $\exists$ polynomial $p_n(z)$ s.t. $\prod\left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$ converge to entire function.
    subproof : For $R > 0$, say $|a_n| > R\ \forall n > N$. For $|z| \le R$, $\forall n \in \mathbb{N}$

$$\log\left(1 - \frac{z}{a_n}\right) = \frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots$$

Let $p_n(z) = \sum_{k=1}^{m_n} \dfrac{1}{k}\left(\dfrac{z}{a_n}\right)^k$ for some $m_n \in \mathbb{Z}_{\ge 0}$. If $R_n(z) = \log\left(1 - \dfrac{z}{a_n}\right) + P_n(z)$, then

$$|R_n(z)| \le \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}$$

Choose $m_n = n$, then by root test, $\sum_{n=N}^{\infty} \dfrac{1}{m_n + 1}\left(\dfrac{R}{|a_n|}\right)^{m_n+1}$ converge. Then

- $R_n(z) \to 0 \rightsquigarrow -\pi < \operatorname{Im} R_n(z) < \pi\ \forall n \ge N_0 > N$.
- $\sum_{n=N_0}^{\infty} R_n$ converge absolutely convergent and uniformly for $|z| \le R \rightsquigarrow \prod_{n=N_0}^{\infty}\left(1 - \dfrac{z}{a_n}\right) e^{p_n(z)}$ is analytic for $|z| \le R$ and thus

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$$

is analytic for $|z| \le R$, where $p_n(z) \in \mathbb{C}[z]$. By a similar argument for Mittag-Leffler theorem, $\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$ is analytic in $\mathbb{C}$.

Hence, in general,

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{m_n}\right)^{m_n}}$$

$\square$

Definition 1.8.3.

- $m_n = n$, $E_n(z/m) = \left(1 - \frac{z}{a_m}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right)$ is called **canonical factor**
- h is called the **genus of canonical product** of f if h is the smallest integer s.t.

$$\sum \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1} \text{ converge i.e. } \sum \frac{1}{|a_n|^{h+1}} \text{ converge}$$

Example 1.8.2. $f(z) = \sin \pi z = z e^{g(z)} \prod_{n\neq 0}\left(1 - \frac{z}{n}\right) e^{z/n}$, since $\sum n^{-1}$ diverge and $\sum n^{-2}$ converge $\rightsquigarrow m_n = 1\ \forall n$. Consider $f'(z)/f(z)$, we have

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \underbrace{\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)}_{(1)}$$

Recall that $\dfrac{\pi^2}{\sin^2 \pi z} = \sum\limits_{n=-\infty}^{\infty} \dfrac{1}{(z-n)^2}$ and integrate both side :

$$\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = \int_0^z \left(\frac{\pi^2}{\sin^2 \pi x} - \frac{1}{x^2}\right) dx = \pi \cot \pi z - \frac{1}{z}$$

Hence, $g'(z) = 0$ and thus $g(z)$ is constant. $\because \lim\limits_{z\to 0} \dfrac{\sin \pi z}{z} = \pi \therefore e^{g(z)} = \pi$. Now we check that (1) is converge for all $z \in \mathbb{C}$.

$$\sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right) = z \sum_{n\neq 0} \frac{1}{n(z-n)}$$

which will converge by comparison test with $\sum n^{-2}$.

Proposition 1.8.1. If $f(z)$ is meromorphic in $\mathbb{C}$, then $F(z) = f(z)/g(z)$, where $f(z), g(z)$: entire.

Proof: Let $g(z)$ be an entire function with zero = poles of $F(z) \rightsquigarrow g(z)F(z)$ is an entire function $f(z) \rightsquigarrow F(z) = f(z)/g(z)$. □

1.9 Gamma function

Recall that

$$\frac{\sin \pi z}{\pi} = z \prod_{n\neq 0}\left(1 - \frac{z}{n}\right) e^{z/n} \underbrace{\prod_{n\neq 0}\left(1 + \frac{z}{n}\right) e^{z/n}}_{:=G(z)}$$

Observation : zero of $G(z-1) = \{0, -1, -2, ...\} \rightsquigarrow G(z-1) = zG(z)e^{g(z)}$ for some $g(z)$: entire. Consider $\dfrac{(\cdot)'}{(\cdot)}$, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n} + \frac{1}{n}\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{n}\right) + g'(z) \implies g'(z) = 0 \text{ i.e. } g(z) = c$$

Let $z=1$, $1=G(0)=e^c\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n} \rightsquigarrow e^{-c}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}=\lim_{n\to\infty}(n+1)e^{-(1+\cdots+1/n)}$, then $c=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log(n+1)\right)=\gamma$ which is Euler's constant. Then $G(z-1)=zG(z)e^c$.

- $H(z):=e^{cz}G(z)\rightsquigarrow H(z-1)=e^{c(z-1)}G(z-1)=ze^{cz}G(z)=zH(z)$.
- $\Gamma(z):=1/zH(z)$, then $\Gamma(z-1)=\frac{1}{(z-1)H(z-1)}=\frac{1}{z(z-1)H(z)}=\frac{1}{z-1}\Gamma(z)$

$$\implies \Gamma(z)=(z-1)\Gamma(z-1)$$

In particular, $\Gamma(1)=1/H(1)=1/e^cG(1)=1$, $\Gamma(2)=1$, $\cdots\Gamma(n)=(n-1)!$.

•• $\Gamma(z)=1/(ze^{cz}G(z))=z^{-1}e^{-cz}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{-z/n}$ which is meromorphic function with poles $0,-1,-2,...$ and no zero.

•• $\Gamma(1-z)=(1-z)^{-1}e^{cz-z}\prod_{n=1}^{\infty}\left(1+\frac{1-z}{n}\right)^{-1}e^{(1-z)/n}$. Then

$$\begin{aligned}\Gamma(z)\Gamma(1-z)&=z^{-1}(1-z)^{-1}e^{-c}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1-z}{n}\right)^{-1}e^{1/n}\\&=\frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)\left(1+\frac{1-z}{n}\right)\left(\frac{n}{n+1}\right)\right)^{-1}\\&=\frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n+1}\right)\right)^{-1}\\&=\frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}(e^{z/n})^{-1}\prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)^{-1}(e^{-z/(n+1)})\\&=\frac{1}{z(1-z)}\frac{1}{G(z)}\cdot\frac{1-z}{G(-z)}=\frac{\pi}{\sin\pi z}\end{aligned}$$

In particular, $\Gamma(1/2)^2=\frac{\pi}{\sin(\pi/2)}\implies\Gamma(1/2)=\sqrt{\pi}$.

•• **Legendre's duplication formula** : $\sqrt{\pi}\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma(z+1/2)$:

••• $\frac{\Gamma'(z)}{\Gamma(z)}=\frac{-1}{z}-c+\sum_{n=1}^{\infty}\left(\frac{-1}{z+n}+\frac{1}{n}\right)$ and $\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)=\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}=\sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$ and thus

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right)=4\left(\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2}+\sum_{n=0}^{\infty}\frac{1}{(2z+1+n)^2}\right)=2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)$$

Integral in both side we have

$$\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}=\frac{\Gamma'(2z)}{\Gamma(2z)}+a$$

Integral in both side we have

$$\Gamma(z)\Gamma(z+1/2)=\Gamma(2z)e^{az+b}$$

Substitute $z=1,1/2$, we have

$$\begin{cases}\Gamma(1)\Gamma(3/2)=\Gamma(2)e^{a+b}\\\Gamma(1/2)\Gamma(1)=\Gamma(1)e^{a/2+b}\end{cases}\implies\begin{cases}e^{a+b}=\sqrt{\pi}/2\\e^{a/2+b}=\sqrt{\pi}\end{cases}\implies\begin{cases}e^a=1/4\\e^b=2\sqrt{\pi}\end{cases}$$

Hence, $2^{2z-1}\Gamma(z)\Gamma(z+1/2)=\Gamma(2z)\sqrt{\pi}$.

1.10 Entire function

Definition 1.10.1. $u : \mathbb{C} \to \mathbb{R}$ is **harmonic** if u_{xx}, u_{yy} continuous and $\Delta u = u_{xx} + u_{yy} = 0$.

Fact 1.10.1.

(1) $f = u + iv$: analytic $\rightsquigarrow u, v \in \mathcal{H}$:

By Cauchy Riemann equation, $u_x = v_y, u_y = -v_x \rightsquigarrow u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

(2) $u \in \mathcal{H}(\Omega)$ with Ω : simply connected $\rightsquigarrow \exists v \in \mathcal{H}(\Omega)$ s.t. $f = u + iv$ is analytic in Ω :

- $g = u_x - iu_y$ is analytic, since $\begin{cases} u_{xx} = (-u_y)_y & \text{, since } u \in \mathcal{H} \\ u_{xy} = -(-u_y)_x \end{cases}$
- Since Ω is simply connected, g has a primitive $f(z) = \displaystyle\int_{z_0}^{z} g(z)dz + u(x_0, y_0)$
- $f = U + iV \rightsquigarrow f' = U_x - iU_y$ and equal to $g = u_x - iu_y \implies U_x = u_x, U_y = u_y$ and thus $U = u + c$
- $f(z_0) = u(x_0, y_0) = U(x_0, y_0) + iV(x_0, y_0) \rightsquigarrow c = 0$.

(3) **Mean-value property** : $u \in \mathcal{H}$, let $v \in \mathcal{H}$ s.t. $f = u + iv$ is analytic.

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \implies u(x_0, y_0) = \frac{1}{2\pi i}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

(4) **Poisson's formula** : $u \in \mathcal{H}(\overline{B_R(0)})$, $\forall z \in B_R(0)$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} u(Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right) u(Re^{i\theta})d\theta$$

proof : For $a \in B_R(0)$, i.e. $|a| < R$. Consider $w = T(\xi) = \dfrac{R(R\xi + a)}{R + \overline{a}\xi}$, then

$$\begin{aligned} T: \quad |\xi| \le 1 \quad &\longrightarrow \quad |w| \le R \\ \xi = 0 \quad &\longmapsto \quad w = a \end{aligned}$$

Then $u(T(\xi)) \in \mathcal{H}(|\xi| < 1)$, then

$$u(a) = u(T(0)) = \frac{1}{2\pi}\int_{|\xi|=1} u(T(\xi))d\arg\xi$$

Notice that $\xi = |\xi|e^{i\arg\xi} \rightsquigarrow d\xi = i|\xi|e^{i\arg\xi}d\arg\xi \rightsquigarrow d\arg\xi = \dfrac{d\xi}{i\xi}$. Also

$$\xi = \frac{R(w-a)}{R^2 - \overline{a}w} \rightsquigarrow \frac{1}{\xi} = \frac{R^2 - \overline{a}w}{R(w-a)} \text{ and } d\xi = \frac{R(R^2 - |a|^2)}{(R^2 - \overline{a}w)^2}dw$$

Also, for $|\xi| = 1$, $|T(\xi)|^2 = \dfrac{R^2(R\xi + a)(R\overline{\xi} + \overline{a})}{(R + \overline{a}\xi)(R + a\overline{\xi})} = R^2$. Let $w = Re^{i\theta}$, then $-idw = wd\theta$.

$$\begin{aligned} \frac{d\xi}{i\xi} &= \frac{(R^2 - |a|^2)dw}{i(w-a)(R^2 - \overline{a}w)} = -i\left(\frac{1}{w-a} + \frac{\overline{a}}{R^2 - \overline{a}w}\right)dw = \left(\frac{w}{w-a} + \frac{\overline{a}w}{w\overline{w} - \overline{a}w}\right)d\theta \\ &= \left(\frac{a}{w-a} + \frac{\overline{a}}{\overline{w} - \overline{a}}\right) = \frac{R^2 - |a|^2}{|w-a|^2}d\theta \end{aligned}$$

Also,

$$\operatorname{Re}\left(\frac{w+a}{w-a}\right)=\frac{1}{2}\left(\frac{w+a}{w-a}+\frac{\overline{w}+\overline{a}}{\overline{w}-\overline{a}}\right)=\frac{R^2-|a|^2}{|w-a|^2}$$

which proved the equation.

Theorem 1.10.1 (Jensen's formula)**.** f : analytic for $|z|\leq\rho$ with $f(0)\neq 0$. Then

$$\log|f(0)|=-\sum_{i=1}^{n}\log\left(\frac{\rho}{|a_i|}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|d\theta$$

where $a_1,...,a_n$ are zero of f in $\overline{B_\rho(0)}$.

Proof:

- If $f\neq 0$ in $|z|\leq\rho$: OK!
- $\forall i,\ |a_i|=\rho$: By induction on n :

  $n=1$: Let $g=\dfrac{f}{z-a_1}\rightsquigarrow g\neq 0$ in $|z|\leq\rho$ and thus

  $$\log|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|g(\rho e^{i\theta})|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left(\log|f(\rho e^{i\theta})|-\log\rho|e^{i\theta}-e^{i\theta_0}|\right)d\theta$$

  where $a_1=\rho e^{i\theta_0}$. We can calculate that

  $$\frac{1}{2\pi}\int_0^{2\pi}\log|1-e^{i\theta}|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log 2\left|\sin\frac{\theta}{2}\right|d\theta=\log 2+\frac{2}{\pi}\underbrace{\int_0^{\pi/2}\log\sin x dx}_{:=I}\tag{1}$$

  Notice that $x^{1/2}\log\sin x|=x^{1/2}\log x+x^{1/2}\log(\sin x/x)\to 0$ as $x\to 0$, so I converge. Consider $x=\pi/2-\theta$, then

  $$I=\int_{\pi/2}^{0}\log\cos x(-dx)=\int_0^{\pi}\log\cos x dx$$

  and thus

  $$2I=\int_0^{\pi}\log\sin x dx=2\int_0^{\pi/2}\log\sin 2\theta d\theta=\frac{1}{2}\int_0^{\pi/2}\log(2\sin\theta\cos\theta)d\theta=\pi\log 2+4I$$

  So we have $I=\dfrac{-\pi}{2}\log 2$ and thus $(1)=0$. So

  $$\log|f(0)|-\log|\rho|=\log|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta-\log|\rho|\implies\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta$$

  For $n>1$, we can do same argument.

- In general, let $F(z)=f(z)\displaystyle\prod_{i=1}^{n}\frac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\neq 0$ in $|z|<\rho$, since $|\rho^2/\overline{a_i}|\geq\rho$. Also, $|F(z)|=|f(z)|$ on $z=\rho$, so

  $$\log|f(0)|+\sum_{i=1}^{n}\log\frac{\rho}{|a_i|}=\log|F(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|F(\rho e^{i\theta})|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|d\theta$$

□

Notice that $\log(z)$ is analytic in $\mathbb{C}$ except one line, and $\log|z|$ be the real part of $\log(z)$ which is harmonic. Then we have below formula.

Theorem 1.10.2 (Poisson-Jensen's formula)**.** For $z \in B_\rho(0)$ with $f(z) \neq 0$, by Possion formula for $\log|F(z)|$,

$$\log|f(z)| + \sum_{i=1}^{m} \log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| = \log|F(z)| = \int_0^{2\pi} \operatorname{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right) \log|f(\rho e^{i\theta})| d\theta$$

Definition 1.10.2. Let f be an entire function. The **order** of f is defined by

$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(\rho)}{\log \rho} \text{ where } M(\rho) = \max_{|z|=\rho} |f(z)|$$

Fact 1.10.2. λ is the smallest number s.t. $M(\rho) \leq e^{\rho^{\lambda+\varepsilon}}$ for any $\varepsilon > 0$ as soon as large enough.

Proof:

- $\lambda = \lim_{\delta\to\infty} \sup_{\rho\geq\delta} \dfrac{\log\log M(\rho)}{\log\rho} \rightsquigarrow \forall\varepsilon > 0$, $\exists\delta_0 > 0$ s.t. for all $\delta > \delta_0$

  $$\left|\sup_{\rho\geq\delta} \frac{\log\log M(\rho)}{\log\rho} - \lambda\right| < \varepsilon \implies \frac{\log\log M(\rho)}{\log\rho} < \lambda + \varepsilon \ \forall \rho \geq \delta_0$$

  and thus $M(\rho) \leq e^{\rho^{\lambda+\varepsilon}}$.

- For $\mu < \lambda$, let $\varepsilon = (\lambda-\mu)/3 \rightsquigarrow \exists\rho > 0$ s.t. $\dfrac{\log\log M(\rho)}{\log\rho} > \lambda - \varepsilon = \mu + 2\varepsilon$ i.e. $M(\rho) > e^{\mu+2\varepsilon}$.

□

Theorem 1.10.3 (Main theorem)**.** Let $f(z)$ be the entire function with order $\lambda < \infty$ and h be the largest integer $\leq \lambda$ i.e. $h \leq \lambda < h+1$. If $a_1, a_2, ...$ be the zero of $f(z)$ and $0 \neq a_i\ \forall i$, then

- $\sum |a_n|^{-(h+1)}$ converge
- $f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \dfrac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$ with $g(z)$ is a polynomial with deg $\leq h$.

Proof:

- Assume $\mu(\rho)$ be the number of a_i's with $|a_i| \leq \rho$, then $n \leq \mu(|a_n|)$. By Jensen's formula,

  $$\log|f(0)| = -\sum_{i=1}^{\mu(2\rho)} \log\left|\frac{2\rho}{a_i}\right| + \frac{1}{2\pi}\int_0^{2\pi} |f(2\rho e^{i\theta})| d\theta$$

  Observer that if $\rho \leq |a_i| \leq 2\rho \rightsquigarrow 0 \leq \log\left|\dfrac{2\rho}{a_i}\right| \leq \log 2$. Then

  $$\mu(\rho)\log 2 \leq \sum_{|a_i|\leq\rho} \log\frac{2\rho}{|a_i|} \leq \sum_{i=1}^{\mu(2\rho)} \log\frac{2\rho}{|a_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(2\rho e^{i\theta})| d\theta - \log|f(0)|$$

and $\log|f(2\rho e^{i\theta})| \le \log M(2\rho) < (2\rho)^{\lambda+\varepsilon}$.

$$\implies \mu(\rho) \le \frac{1}{\log 2}(2^{\lambda+\varepsilon}\rho^{\lambda+\varepsilon} - \log|f(0)|) < K(2\rho)^{\lambda+\varepsilon}$$

for some constant $K > 0$. So $n \le \mu(|a_n|) < K(2|a_n|)^{\lambda+\varepsilon}$. Choose $\varepsilon > 0$ s.t. $\lambda+\varepsilon < h+1$ and thus

$$|a_n|^{-(h+1)} = (|a_n|^{-(\lambda+\varepsilon)})^{\frac{h+1}{\lambda+\varepsilon}} \le \frac{2^{h+1}K^{\frac{h+1}{\lambda+\varepsilon}}}{n^{\frac{h+1}{\lambda+\varepsilon}}}$$

Since $\frac{h+1}{\lambda+\varepsilon} > 1$, $\sum |a_n|^{-(h+1)}$ converge.

- By Poisson-Jensen's formula,

$$\log|f(z)| = -\sum_{i=1}^{\mu(\rho)} \log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)\log|f(\rho e^{i\theta})|d\theta$$

Note $f = u+iv$, then $f' = 2\frac{\partial u}{\partial z}$, $\frac{f'(z)}{f(z)} = (\log f(z))' = 2\frac{\partial}{\partial z}\log|f(z)|$

  - $\frac{\partial}{\partial z}\left(\sum 2\log\left|\frac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\right|\right) = \sum \frac{\partial}{\partial z}\log\left(\frac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\right)\left(\frac{\rho^2-a_i\overline{z}}{\rho(\overline{z}-\overline{a_i})}\right) = -\sum\left(\frac{1}{z-a_i} + \frac{\overline{a_i}}{\rho^2-\overline{a_i}z}\right)$
  - $2\frac{\partial}{\partial z}\mathrm{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right) = \left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)' = \frac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^2}$

Hence,

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{\mu(\rho)}\frac{1}{z-a_i} + \sum_{i=1}^{\mu(\rho)}\frac{\overline{a_i}}{\rho^2-\overline{a_i}z} + \frac{1}{\pi}\int_0^{2\pi}\frac{\rho e^{i\theta}}{(\rho e^{i\theta}-z)^2}\log|f(\rho e^{i\theta})|d\theta$$

Differentiate h times, we have

$$\left(\frac{f'(z)}{f(z)}\right)^{(h)} = \sum_{i=1}^{\mu(\rho)}\frac{-h!}{(a_i-z)^{h+1}} + \underbrace{\sum_{i=1}^{\mu(\rho)}\frac{h!\cdot\overline{a_i}^{h+1}}{(\rho^2-\overline{a_i}z)^{h+1}}}_{(2)} + \underbrace{\frac{2}{\pi}\int_0^{2\pi}\frac{(h+1)!\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+2}}\log|f(\rho^{i\theta})|d\theta}_{(3)}$$

  - (3) : If $\rho > 2|z|$, then

$$|(3)| \le \frac{(h+1)!\cdot 2}{\pi}\int_0^{2\pi}\frac{\log M(\rho)}{(\rho-|z|)^{h+2}}d\theta = \frac{4(h+1)!\log M(\rho)}{\rho^{h+1}(1-|z|/\rho)^{h+2}}$$

    since $\log M(\rho) \le \rho^{\lambda+\varepsilon}\ \forall\varepsilon>0 \rightsquigarrow \rho^{-(h+1)}\log M(\rho) \le \rho^{\lambda-h-1+\varepsilon}$. Choose ε s.t. $\lambda-h-1+\varepsilon < 0$. Hence, $(3)\to 0$ as $\rho\to\infty$.
  - (2) : $\rho > 2|z|$, $|a_i| \le \rho$, then

$$|(2)| \le h!\sum_{i=1}^{\mu(\rho)}\frac{\rho^{h+1}}{(\rho^2/2)^{h+1}} = h!\cdot\mu(\rho)2^{h+1}\rho^{-(h+1)} < h!\cdot K2^{h+1+\lambda+\varepsilon}\rho^{\lambda+\varepsilon-h-1} \to 0$$

    as $\rho\to\infty$.

Therefore,

$$\left(\frac{f'(z)}{f(z)}\right)^{(h)} = -h!\sum_{i=1}^{\infty}\frac{1}{(a_i-z)^{h+1}}$$

Let $p(z) = \prod \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$ and $f(z) = e^{g(z)}p(z)$. Then

$$\left(\frac{p'(z)}{p(z)}\right)^{(h)} = \sum_{n=1}^{\infty} \frac{-h!}{(a_n - z)^{h+1}} = \left(\frac{f'(z)}{f(z)}\right)^{(h)}$$

and $g^{(h+1)}(z) = \left(\frac{f'(z)}{f(z)}\right)^{(h)} - \left(\frac{p'(z)}{p(z)}\right)^{(h)} = 0 \implies g(z) \in \mathbb{C}[z]$ and $\deg g \leq h$. □

Definition 1.10.3. f has genus h if h is the smallest integer s.t.

$$\begin{cases} \sum |a_n|^{-(h+1)} \text{ converge} \\ \deg g(x) \leq h \end{cases}$$

Theorem 1.10.4. Let h be the genus of f and λ be the order of f, then $h \leq \lambda \leq h+1$.

Proof:

- If h is finite, then $\lambda \leq h+1$:

  Claim : $\log |E_h(z)| \leq (2h+1)|z|^{h+1}$:

  subproof :

  - If $|z| \leq 1$, $\log(1-z) = -z - \frac{z^2}{2} - \cdots$, then $E_n(z) = e^{\log(1-z)+z+\cdots+\frac{1}{h}z^h} = e^{-\sum\limits_{n=h+1}^{\infty} \frac{z^n}{n}}$

  $$\implies |E_n(z)| \leq e^{\sum\limits_{n=h+1}^{\infty} \frac{|z|^n}{n}} \implies \log |E_n(z)| \leq \sum_{n=h+1}^{\infty} \frac{|z|^n}{n} \leq \frac{1}{h+1} \cdot \frac{|z|^{h+1}}{1-|z|}$$

    and thus $(1-|z|) \log |E_n(z)| \leq \dfrac{|z|^{h+1}}{h+1} \leq |z|^{h+1}$
  - $h = 0$: $\log |E_0(z)| = \log |1-z| \leq \log(1+|z|) \leq |z|$
  - For $h \geq 1$: We induction on h. $\log |E_h(z)| \leq \log |E_{h-1}(z)| + \dfrac{|z|^h}{h} \leq \log |E_{h-1}(z)| + |z|^h$
    - $|z| \geq 1$: $\log |E_h(z)| \leq (2h-1)|z|^h + |z|^h \leq (2h+1)|z|^{h+1}$
    - $|z| \leq 1$: $\log |E_h(z)| \leq |z| \log |E_h(z)| + |z|^{h+1} \leq |z|(2h|z|^h) = (2h+1)|z|^{h+1}$ □

  By Claim,

  $$\log |f(z)| \leq \log |e^{g(z)}| + \log |p(z)| \leq |g(z)| + \sum_n \log |E_h(z/a_n)|$$
  $$\leq |z|^{h+1} \left(\frac{|b_q|}{|z|^{h+1-q}} + \cdots + \frac{|b_0|}{|z|^{h+1}}\right) + (2h+1)|z|^{h+1} \sum_n |a_n|^{-(h+1)}$$

  Hence,

  $$\log \log |f|(z)| \leq (h+1) \log |z| + \log \left(\frac{|b_q|}{|z|^{h+1-q}} + \cdots + \frac{|b_0|}{|z|^{h+1}} + (2h+1) \sum_n |a_n|^{-(h+1)}\right)$$

  When z on $C_r(0)$:

  $$\frac{\log \log M(r)}{\log r} \leq h + 1 + \log \left(O(|r|^{q-h-1}) + (2h+1) \sum_n |a_n|^{-(h+1)}\right) \Big/ \log r$$

  As $r \to \infty$, we have $\lambda \leq h+1$ and hence λ is finite.

- If λ is finite, let h_0 be the smallest integer h_0 s.t. $h_0 \leq \lambda$. By Theorem 1.10.3, $\begin{cases} \sum |a_n|^{-(h_0+1)} \text{ converge} \\ \deg g \leq h_0 \end{cases}$

  By definition of genus, $h \leq h_0 \leq \lambda$.

Theorem 1.10.5. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ and $\alpha = \liminf \dfrac{\log(1/|c_n|)}{n \log n}$. Then

- $\alpha > 0 \implies f$ is entire of order α
- $\alpha = 0 \implies f$ has infinite order

Also, if $f(z)$ is entire of finite order λ, then $\lambda = 1/\alpha$.

Proof:

- $\alpha > 0 : \forall \varepsilon > 0, \exists n_0$ s.t. $\forall n > n_0$, $\log(1/|c_n|) > (\alpha - \varepsilon) n \log n$ i.e. $|c_n| \leq n^{-n(\alpha-\varepsilon)}$, then $\sum c_n z^n$ converge for all $z \in \mathbb{C} \rightsquigarrow f$ is entire.

  - α is finite : Notice that $|c_n|$ is bounded, say $|c_n| \leq A$ with $A > 1$. $\forall r > 1$, for $|z| \leq r$,

    $$|f(z)| \leq A r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n n^{-n(\alpha-\varepsilon)}$$

    where $n_0 = \lfloor (2r)^{1/(\alpha-\varepsilon)} \rfloor$. Then $\forall n > n_0$, $n \geq (2r)^{1/(\alpha-\varepsilon)} \rightsquigarrow r n^{-(\alpha-\varepsilon)} \leq 1/2$ and thus

    $$\sum_{n=n_0+1}^{\infty} r^n n^{-n(\alpha-\varepsilon)} \leq \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \leq 1$$

    Hence,

    $$\begin{aligned} |f(z)| \leq 2A r^{n_0} &\implies \log|f(z)| \leq \log 2 + n_0 \log r \leq (2r)^{1/\alpha-\varepsilon} \log r \\ &\implies \frac{\log\log M(r)}{\log r} \leq \frac{\frac{1}{\alpha-\varepsilon}\log r + \log\log r}{\log r} \end{aligned}$$

    as $r \to \infty$, we have $\lambda \leq \frac{1}{\alpha-\varepsilon}$ for all $\varepsilon > 0 \implies \lambda \leq \frac{1}{\alpha}$.

  - $\alpha = \infty$: By definition, $\forall N > 0, \varepsilon > 0$, $\exists n_0$ s.t. $\forall n > n_0$,

    $$\log(1/|c_n|) > (N - \varepsilon) n \log n \implies \cdots \implies \lambda \leq \frac{1}{N} \xrightarrow{N\to\infty} \lambda = 0 = \frac{1}{\infty}$$

  - If $0 < \alpha < \infty \rightsquigarrow \lambda \geq \frac{1}{\alpha}$: $\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $(\alpha + \varepsilon) n_\varepsilon \log n_\varepsilon > \log(1/|c_{n_\varepsilon}|)$ i.e.

    $$|c_{n_\varepsilon}| > n^{-n_\varepsilon(\alpha+\varepsilon)} \implies |c_{n_\varepsilon}| r^{n_\varepsilon} > (r n_\varepsilon^{-(\alpha+\varepsilon)})^{n_\varepsilon}$$

    Choose $r = (2n)^{(\alpha+\varepsilon)}$, then

    $$|c_{n_\varepsilon}| r^{n_\varepsilon} > 2^{n_\varepsilon(\alpha+\varepsilon)} \implies \log|c_{n_\varepsilon}| r^{n_\varepsilon} > \frac{r^{1/(\alpha+\varepsilon)}(\alpha+\varepsilon)}{2} \log 2$$

    By Cauchy estimate,

    $$|c_{n_\varepsilon}| = \frac{|f^{(n_\varepsilon)}(0)|}{n!} \leq M(r) r^{-n_\varepsilon} \implies \log\log M(r) \geq \frac{1}{\alpha+\varepsilon} \log r + \log \frac{(\alpha+\varepsilon)\log 2}{2}$$

    Hence, $\lambda \geq \frac{1}{\alpha+\varepsilon}$ $\forall \varepsilon > 0 \rightsquigarrow \lambda \geq \frac{1}{\alpha}$.

- If $\alpha = 0 : \forall \dfrac{1}{N}$, $\lambda \geq \dfrac{1}{1/N + \varepsilon} \xrightarrow{\varepsilon\to 0} \lambda \geq N \xrightarrow{N\to\infty} \lambda = \infty$.

$\square$

1.11 Gamma function

Recall that we had already define

$$\Gamma(z) = z^{-1}e^{-cz}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$$

where $c = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$. And our goal in this section is

Theorem 1.11.1 (Stirling's formula)**.**

$$\log\Gamma(z) = \frac{1}{2}\log 2\pi - z + \left(z-\frac{1}{2}\right)\log z + J(z) \text{ or } \Gamma(z) = \sqrt{2\pi}z^{z-1/2}e^{-z}e^{J(z)}$$

where error function is

$$J(z) = \frac{1}{\pi}\int_0^{\infty}\frac{z}{v^2+z^2}\log(1-e^{2\pi v})^{-1}dv$$

Proof:

- First we have
$$\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)' = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{(z+k)^2}$$
$$\pi\cot\pi z = \frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right)$$

- Let $\varphi(\xi) = \dfrac{\pi\cot\pi\xi}{(\xi+z)^2}$ with a fixed z with Re$z > 0$, then
$$\varphi(\xi) = \underbrace{\frac{1}{(\xi+z)^2}\cdot\frac{1}{\xi}}_{\varphi_1(\xi)} + \underbrace{\frac{1}{(\xi+z)^2}\sum_{n\neq 0}\left(\frac{1}{\xi-n}+\frac{1}{n}\right)}_{\varphi_2(\xi)}$$

  Let γ_n be consisted by $V_0^{(Y)}, V_{n+1/2}^{(Y)}$, H_Y and H_{-Y}, where
$$V_x^{(Y)} = \{\mathrm{Re}z = x, |\operatorname{Im} z| \le Y\} \text{ and } H_Y = \{0 \le \mathrm{Re}z \le n+1/2, \operatorname{Im} z = Y\}$$

  - $\displaystyle\frac{1}{2\pi i}\int_{\gamma_n}\varphi_2(\xi)d\xi = \sum_{k=1}^{n}\mathrm{Res}_{\xi=k}\,\varphi_2(\xi) = \sum_{k=1}^{n}\frac{1}{(z+k)^2}$
  - Let $c_\varepsilon = \{\varepsilon e^{-i\theta} : -\pi/2 \le \theta \le \pi/2\}$, then
$$\frac{1}{2\pi i}\int_{\gamma_n}\varphi_1(\xi)d\xi = \lim_{\varepsilon\to 0}\int_{c_\varepsilon}\varphi_1(\xi)d\xi = \frac{-1}{2}\lim_{\varepsilon\to}\int_{C_\varepsilon}\varphi_1(\xi)d\xi = \frac{-1}{2}\mathrm{Res}_{\xi=0}\,\varphi_1(\xi) = \frac{-1}{2z^2}$$

  Hence,
$$\frac{1}{2\pi i}\int_{\gamma_n}\varphi_1(\xi)d\xi = \frac{-1}{2z^2} + \sum_{k=1}^{n}\frac{1}{(z+k)^2}$$

- On $H_{\pm Y}$: Let $\xi = u + iv \in H_{\pm Y}$, then

$$\cot \pi\xi = i\frac{e^{2\pi i\xi}+1}{e^{2\pi i\xi}-1} = i\frac{e^{-2\pi v+i(2\pi u)}+1}{e^{-2\pi v+i(2\pi u)}-1} \xrightarrow{u\in[0,n+1/2]} \begin{cases} -i & \text{as } Y\to\infty \\ i & \text{as } Y\to-\infty \end{cases}$$

Since $\left|\frac{1}{(z+\xi)^2}\right| = \frac{1}{|z+u+iv|^2} \to 0$ as $v \to \pm\infty$, we have

$$\lim_{Y\to\infty}\int_{H_{\pm Y}} \varphi(\xi)d\xi = 0$$

- On $V_{n+1/2}^{(Y)}$: For $\xi = n + 1/2 + iy$, $e^{2\pi i\xi} = e^{-2\pi y+2n\pi i+\pi i} = -e^{-2\pi y}$ and thus

$$|\cot\pi\xi| = \left|\frac{-e^{-2\pi y}+1}{-e^{-2\pi y}-1}\right| \le \frac{e^{-2\pi y}+1}{e^{-2\pi y}+1} = 1$$

Hence,

$$\left|\int_{V_{n+1/2}^{(Y)}}\varphi(\xi)d\xi\right| \le \int_{V_{n+1/2}^{(Y)}}\frac{|d\xi|}{|\xi+z|^2} = \int_{V_{n+1/2}^{(Y)}}\frac{|d\xi|}{(\xi+z)(\overline{\xi+z})}$$

Since $\xi \in V_{n+1/2}^{(Y)}$, $\xi+\overline{\xi} = 2n+1$ and thus $\overline{\xi+z} = 2n+1+\overline{z}-z$. Then $\frac{1}{|\xi+z|^2}$ has pole at $-z, 2n+1+\overline{z}$, which doesn't contain in bounded region of γ_n.

□

1.12 Prime number theorem

Goal : Let $\pi(x)$ be the number of prime $\le x$, we will prove that $\pi(x) \sim \frac{x}{\ln x}$ as $x \to \infty$.

Definition 1.12.1 (Riemann ζ−function).

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

is analytic for $\mathrm{Re}(s) > 1$, since $\sum_{n=1}^{\infty} n^{-\mathrm{Re}(s)}$ uniformly converge for $\mathrm{Re}s \ge \rho > 1$.

Fact 1.12.1. $\zeta(s) = \prod_{p:\text{prime}}(1-p^{-s})^{-1}$ for $\mathrm{Re}s > 1$.

Proof: By UFD property of $\mathbb{N}$ and absolutely convergent of $\zeta(s)$,

$$\zeta(s) = \sum_{r_2,r_3,\dots\in\mathbb{Z}_{\ge0}}(2^{r_2}3^{r_3},\cdots)^s = \prod_{p:\text{prime}}\left(\sum_{r\in\mathbb{Z}_{\ge0}}p^{-rs}\right) = \prod_{p:\text{prime}}(1-p^{-s})^{-1}$$

□

Fact 1.12.2. $\zeta(s) - \frac{1}{s-1}$ extends analytically to $\mathrm{Re}(s) > 0$.

Proof: For $\mathrm{Re}(s) \geq 1$, $\zeta(s) - \dfrac{1}{s-1} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx$

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| = \left| \int_n^{n+1} \left(s \int_n^x \frac{dy}{y^{s+1}} \right) dx \right| = \left| \int_n^{n+1} s \cdot \frac{(x-n)}{y_x^{s+1}} dx \right| \text{ (MVT)}$$
$$\leq \int_n^{n+1} (x-n) \max_{n \leq y \leq n+1} \left| \frac{s}{y^{s+1}} \right| dx \leq \frac{|s|}{2n^{\mathrm{Re}(s)+1}}$$

□

Fact 1.12.3. If $\mathrm{Re}(s) > 1$, then $\log \zeta(s) = \sum_{p,m} \dfrac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$ for some $c_n \geq 0$.

Proof: Recall for $0 \leq x < 1$, $\log \left(\dfrac{1}{1-x} \right) = \sum_{m=1}^{\infty} \dfrac{x^m}{m}$. Now, for $s > 1$,

$$\log \zeta(s) = \sum_p \log \left(\frac{1}{1-p^{-s}} \right) = \sum_p \sum_m \frac{p^{-sm}}{m} = \sum_m \sum_p \frac{p^{-sm}}{m}$$

since it is absolutely converge. By Fact 1.12.1, $\zeta(s)$ has no zero for $\mathrm{Re}(s) > 1 \rightsquigarrow \log \zeta(s)$ is analytic for $\mathrm{Re}(s) > 1$ and RHS is also analytic for $\mathrm{Re}(s) > 1$. So for $\mathrm{Re}(s) > 1$,

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-sm}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

where $c_n = \begin{cases} m^{-1} & \text{if } n = p^m \\ 0 & \text{otherwise} \end{cases}$. □

Fact 1.12.4. $\zeta(s)$ has no zero on line $\mathrm{Re}(s) = 1$.

Proof: **Claim** : If $x > 1$ and $y \in \mathbb{R}$, then

$$\log |\zeta(x)^3 \zeta(x+iy)^4 \zeta(x+2iy)| \geq 0$$

subproof : Let $s = x + iy \rightsquigarrow \mathrm{Re}(n^{-s}) = e^{-x \log n} \cos(y \log n) =: n^x \cos \theta_n$. Then it will equal to

$$\begin{aligned}
&3 \log |\zeta(x)| + 4 \log |\zeta(x+iy)| + \log |\zeta(x+2iy)| \\
=&3 \mathrm{Re} \log \zeta(x) + 4 \mathrm{Re} \log \zeta(x+iy) + \mathrm{Re} \log \zeta(x+2iy) \\
=&3 \sum_{n=1}^{\infty} c_n n^{-x} + 4 \sum_{n=1}^{\infty} c_n n^{-x} \cos \theta_n + \sum_{n=1}^{\infty} c_n n^{-x} \cos(2\theta_n) \\
=&\sum_{n=1}^{\infty} c_n n^{-x} \cdot 2(\cos \theta_n + 1)^2 \geq 0
\end{aligned}$$

□

Now, suppose $\zeta(1 + iy_0) = 0$ for some $y_0 \neq 0$. By Fact 1.12.2, $\zeta(s)$ is analytic at $s = 1 + iy_0$, then order of s is ≥ 1. By factorization, $|\zeta(x+iy_0)|^4 \leq c_1(x-1)^4$ as $x \to 1$, where $c_1 > 0$. By fact 1.12.2, $\zeta(s)$ has simple pole at $z = 1$, then $|\zeta(x)|^3 \leq c_2 |x-1|^{-3}$ as $x \to 1$ for some $c_2 > 0$. Since $\zeta(s)$ is analytic at $s = 1 + 2iy_0$, $|\zeta(x+2iy_0)|$ is bounded for $x \to 1$. Then

$$|\zeta(x)^3 \zeta(x+iy_0)^4 \zeta(x+2iy_0)| \to 0 \text{ as } x \to 1$$

Then $\log |\zeta(x)^3 \zeta(x+iy_0)^4 \zeta(x+2iy_0)| < 0$ as $x \to 1$ ($\rightarrow\!\leftarrow$). □

Fact 1.12.5. $D(s)=\sum_p \frac{\log p}{p^s} \rightsquigarrow D(s)-\frac{1}{s-1}$ is analytic for $\mathrm{Re}(s)\geq 1$.

Proof: $\frac{\zeta'(s)}{\zeta(s)}=(\log\zeta(s))'=\sum_p\left(\log\frac{1}{1-p^{-s}}\right)'=\sum_p\frac{-\log p}{p^s-1}$,

$$D(s)+\frac{\zeta'(s)}{\zeta(s)}=\sum_p\frac{-\log p}{p^s(p^s-1)} \text{ converge for } \mathrm{Re}(s)>\frac{1}{2}$$

Note $\zeta(s)$ has a simple pole at $s=1$, say $\zeta(s)=(s-1)^{-1}g(s)$, then

$$\frac{\zeta'(s)}{\zeta(s)}=\frac{-1}{s-1}+\frac{g'(s)}{g(s)} \rightsquigarrow D(s)-\frac{1}{s-1}=-\frac{g'(s)}{g(s)}+\sum_p\frac{-\log p}{p^s(p^s-1)}$$

Since $g(s)$ has not zero for Re$s\geq 1$ (By Fact 1.12.1,1.12.4), $D(s)-(s-1)^{-1}$ is analytic for $\mathrm{Re}(s)\geq 1$. $\square$

Fact 1.12.6. For $x>0$, $\mathscr{I}(x):=\sum_{p\leq x}\log p=O(x)$ as $x\to\infty$.

Proof: For $n\in\mathbb{N}$, $(1+1)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}\geq\binom{2n}{n}>\prod_{n<p\leq 2n}p=e^{\mathscr{I}(2n)-\mathscr{I}(n)}$. Then

$$\mathscr{I}(2n)-\mathscr{I}(n)<(\log 2)(2n)$$

- $\forall x>0$, $\exists n$ s.t. $n\leq x<n+1 \rightsquigarrow \mathscr{I}(x)=\mathscr{I}(n)$
- $n=2m \rightsquigarrow \begin{cases} 2m\leq x\leq 2m+1 & \Longrightarrow \mathscr{I}(2m)=\mathscr{I}(x) \\ m\leq x/2\leq m+1/2 & \Longrightarrow \mathscr{I}(m)=\mathscr{I}(x/2)\end{cases}$. So

$$\mathscr{I}(x)-\mathscr{I}(x/2)=\mathscr{I}(2m)-\mathscr{I}(m)<(\log 2)(2m)\leq(\log 2)x$$

- $n=2m+1 \rightsquigarrow \begin{cases} 2m+1\leq x<2m+2 & \Longrightarrow \mathscr{I}(2m+1)=\mathscr{I}(x) \\ m+1/2\leq x/2<m+1 & \Longrightarrow \mathscr{I}(m)=\mathscr{I}(x/2)\end{cases}$. So

$$\mathscr{I}(x)-\mathscr{I}(x/2)=\mathscr{I}(2m+1)-\mathscr{I}(m)\leq\mathscr{I}(2m)-\mathscr{I}(m)+\log(2m+1)<(\log 2)(2m)+\log(2m+1)<cx$$

Let $\ell_x\in\mathbb{N}$ s.t. $x/2^{\ell_x}<2$. Then $\mathscr{I}(x/2^{\ell_x})=0$ and thus

$$\mathscr{I}(x)=\sum_{k=1}^{\ell_x}(\mathscr{I}(x/2^{k-1})-\mathscr{I}(x/2^k))<c\sum_{k=1}^{\ell_x}x/2^{k-1}<2cx$$

$\square$

Theorem 1.12.1 (Convergence theorem)**.** Let $s(t)$ be a bounded locally integrable complex-valued function for $t>0$ and suppose $f(s)=\int_0^\infty s(t)e^{-st}dt$ holds for $\mathrm{Re}(s)>0$ and $f(s)$ can extend to an analytic function on $\Omega\supset\{\mathrm{Re}(s)\geq 0\}$. Then $\int_0^\infty s(t)dt$ exists and equals $f(0)$. In general, $\int_0^\infty s(t)e^{-st}dt$ converge for $\mathrm{Re}(s)=0$ and equals to $f(s)$.

Remark 1.12.1. $f(t) = \int_0^\infty s(t)e^{-st}dt$ is called the **Laplace transformation** of $s(t)$.

Proof:

- Let $R > 1$, for all z on y-axis with $|z| \leq R$, $\exists B_{\rho_z}(z) \subseteq \Omega$. Then $\{B_{\rho_z/3}(z)\}$ is the open cover of $\{yi : |y| \leq R\}$. Say $\{B_{\rho_{z_i}/3}(z_i)\}$ is it finite subcover, then $\{B_{2\rho_{z_i}/3}(z_i)\}$ cover $\{yi : |y| \leq R\}$. Let $\delta_R = \min_i(\rho_{z_i}/3, 1/2)$, then f is analytic on

$$U_R = \{-\delta_R \leq \mathrm{Re}(s) \leq 0, |\mathrm{Im}(s)| \leq R\} \cup \{\mathrm{Re}(s) \geq 0, |s| \leq R\}$$

Let $M_R := \max_{s \in U_R} |f(s)|$ and $\gamma = \partial U_R$. Let $C_1 = \gamma \cap \{\mathrm{Re}s \geq 0\}$ and $C_2 = \gamma \cap \{\mathrm{Re}s \leq 0\}$. Define

$$S_N(s) = \int_0^N \underbrace{s(t)e^{-st}}_{g(s,t)}dt$$

- $S_N(s)$ is entire : $g(s,t) : \mathbb{C} \times [0, N] \to \mathbb{C}$, for fixed t, $g(s,t)$ is analytic in $\mathbb{C}$. Let $a_1, ..., a_n$ be the discontinuity points of $s(t)$ and let $a_0 = 0, a_{n+1} = N$. For any closed curve $\gamma \subset \mathbb{C}$,

$$\int_\gamma S_N(s)ds = \sum_{m=0}^{n} \int_\gamma \int_{a_m}^{a_{m+1}} g(s,t)dtds = \sum_{m=0}^{n} \int_{a_m}^{a_{m+1}} \left(\int_\gamma g(s,t)ds \right) dt = 0$$

The second equality is by Fubini theorem and last equality is by Morera's theorem. Now, by Morera's theorem, $S_N(s)$ is analytic in $\mathbb{C}$.

- Observe that

$$\frac{1}{2\pi i} \int_\gamma f(s)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s} = \mathrm{Res}_{s=0} \left(f(s)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{1}{s} \right) = f(0)$$

Similarly for $S_N(s)$, we have

$$S_N(0) = \frac{1}{2\pi i} \int_{C_R(0)} S_N(s)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s}$$

For $C_R(0) - C_1 : z(t) = Re^{i(\pi+t)} = -Re^{it}$ for $t \in [-\pi/2, \pi/2]$, so

$$\int_{C_R(0)-C_1} S_N(s)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s} = \int_{C_1} S_N(-s)e^{-sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s}$$

Hence,

$$\begin{aligned} f(0) - S_N(0) &= \frac{1}{2\pi i} \int_{C_1} (f - S_N)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s} && \text{(I)} \\ &- \frac{1}{2\pi i} \int_{C_1} S_N(-s)e^{-sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s} && \text{(II)} \\ &+ \frac{1}{2\pi i} \int_{C_2} f(s)e^{sN} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s} && \text{(III)} \end{aligned}$$

  - (I) : Let L be the bound of $s(t)$, then

$$|f(s) - S_N(s)| = \left| \int_N^\infty s(t)e^{-st}dt \right| \leq L \int_N^\infty e^{-xt}dt = \frac{Le^{-xN}}{x}$$

where $x = \mathrm{Re}(s) > 0$. On $|s| = R$,

$$\left| e^{sN}\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right| = e^{xN}\left|\frac{1}{s}+\frac{s}{R^2}\right| = e^{xN}\left|\frac{\overline{s}+s}{R^2}\right| = \frac{2xe^{xN}}{R^2}$$

Hence,

$$|(\mathrm{I})| \leq \frac{1}{2\pi}\int_{C_1}\frac{2L}{R^2}|ds| = \frac{1}{2\pi}\cdot\frac{2L\cdot\pi R}{R^2} = \frac{L}{R}$$

- (II) :

$$|S_N(-s)| = \left|\int_0^N s(t)e^{sN}dt\right| \leq L\int_0^N e^{xt}dt = \frac{L(e^{xN}-1)}{x}$$

$$\left|e^{-sN}\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right| = \frac{e^{-xN}\cdot 2x}{R^2}$$

Hence, $|(\mathrm{II})| \leq \dfrac{L}{R}$

- (III) :

$$\begin{aligned}
|(\mathrm{III})| &\leq \frac{M_R}{2\pi}\left(\int_{-R}^{R} e^{-\delta_R N}\left|1+\frac{(\delta_R+iy)^2}{R^2}\right|\frac{dy}{|-\delta_R+iy|} + 2\int_{-\delta_R}^{0} e^{xN}\left(1+\frac{|x\pm iR|^2}{R^2}\right)\frac{dx}{|x\pm iR|}\right)\\
&\leq \frac{M_R}{2\pi}\left(\int_{-R}^{R} e^{-\delta_R N}\underbrace{\left(1+\frac{\delta_R^2+R^2}{R^2}\right)}_{\leq 3}\frac{dy}{R} + 2\int_{-\delta_R}^{0} e^{xN}\frac{3dx}{R}\right)\\
&\leq \frac{M_R}{2\pi}\left(6e^{-\delta_R N} + \frac{6(1-e^{\delta_R N})}{RN}\right) \to 0 \text{ as } N\to\infty
\end{aligned}$$

Choose $R = 3L/\varepsilon$, then $|(\mathrm{I})+(\mathrm{II})| \leq 2\varepsilon/3$. Now choose N sufficiently large s.t. $|(\mathrm{III})| \leq \varepsilon/3$, then $|f(0) - S_N(0)| \leq \varepsilon$. Hence, $\lim\limits_{N\to\infty} S_N(0) = f(0)$.

□

Corollary 1.12.1. Let $\{c_n\} \subseteq \mathbb{R}_{\geq 0}$ and let $D(s) = \sum\limits_n \dfrac{c_n \log n}{n^s}$. Suppose $s(x) = \sum\limits_{n\leq x} c_n \log n$ is $O(x)$ and that $D(s) - \dfrac{1}{s-1}$ is analytic for $\mathrm{Re}(s) \geq 1$. Then $\sum\limits_{n\leq x} c_n \log n \sim x$.

Proof: We write the sum $D(s)$ as Riemann Stieltjes integral (note that $s(x)$ is positive and non-decreasing) and by integration by part,

$$\begin{aligned}
D(s) &= \int_1^\infty t^{-s}ds(t) = s\int_1^\infty s(t)t^{-s-1}dt\\
&= s\int_0^\infty s(e^u)e^{-us}du \qquad (t = e^u)
\end{aligned}$$

For $\mathrm{Re}(s) > 0$,

$$\frac{D(s+1)}{s+1} - \frac{1}{s} = \int_0^\infty (s(e^t)e^{-t}-1)e^{-ts}dt$$

Notice that $\dfrac{D(s+1)}{s+1} - \dfrac{1}{s}$ is analytic for $\mathrm{Re}(s) \geq 0$ and $s(e^t)e^t - 1$ is bounded since $s(x) = O(x)$. By convergence theorem,

$$\int_0^\infty (s(e^t)e^{-t}-1)dt = \int_1^\infty \frac{s(u)-u}{u^2}du \qquad (u = e^t)$$

converges.

- Suppose $\exists \varepsilon > 0$ s.t. $s(x) > (1+\varepsilon)x\ \forall x \gg 0$, then $\forall x \gg 0$

$$\int_x^{(1+\varepsilon)x} \frac{s(t)-t}{t^2}dt \geq \int_x^{(1+\varepsilon)x} \frac{s(x)-t}{t^2}dt \geq \int_x^{(1+\varepsilon)x} \frac{(1+\varepsilon)x-t}{t^2}dt \overset{t=ux}{=} \int_1^{1+\varepsilon} \frac{(1+\varepsilon)-u}{u^2}du > 0$$

contradict with $\displaystyle\int_1^{\infty} \frac{s(u)-u}{u^2}du$ converge.

- Suppose $\exists \varepsilon > 0$ s.t. $s(x) < (1-\varepsilon)x\ \forall x \gg 0$, then $\forall x \gg 0$

$$\int_{(1-\varepsilon)x}^{x} \frac{s(t)-t}{t^2}dt \geq \int_{(1-\varepsilon)x}^{x} \frac{s(x)-t}{t^2}dt \leq \int_{(1-\varepsilon)x}^{x} \frac{(1-\varepsilon)x-t}{t^2}dt \overset{t=ux}{=} \int_{1-\varepsilon}^{1} \frac{(1-\varepsilon)-u}{u^2}du < 0$$

contradict with $\displaystyle\int_1^{\infty} \frac{s(u)-u}{u^2}du$ converge.

Hence, $\forall \varepsilon > 0$, $(1-\varepsilon)x < s(s) < (1+\varepsilon)x\ \forall x \gg 0 \rightsquigarrow s(x) \sim x$. □

Lemma 1.12.1 (key lemma)**.** Let $f(x)$ satisfy $\sum_{p\leq x} f(p)\log p \sim rx$. Then $\sum_{p\leq x} f(p) \sim \dfrac{rx}{\log x}$

Proof: Let $\theta(x) = \sum_{p\leq x} f(p)\log p$ and $\varphi(x) = \sum_{p\leq x} f(p)$. By Riemann Stieltjes integral,

$$\varphi(x) \sim \int_a^x d\varphi(t) = \int_a^x \frac{1}{\log t}d\theta(t) = \frac{\theta(t)}{\log t}\bigg|_a^x + \int_a^x \frac{\theta(t)dt}{t(\log t)^2} \sim \frac{rx}{\log x} + \int_a^x \frac{rdt}{(\log t)^2}$$

Notice that

$$\begin{aligned}\int_a^x \frac{dt}{(\log t)^2} &= \left(\int_a^{\sqrt{x}} + \int_{\sqrt{x}}^x\right)\frac{dt}{(\log t)^2} \leq \int_a^{\sqrt{x}} \frac{\sqrt{x}dt}{t(\log t)^2} + \int_{\sqrt{x}}^x \frac{dt}{(\log\sqrt{x})^2} \\ &= \sqrt{x}\left(\frac{1}{\log a} - \frac{1}{\log\sqrt{x}}\right) + \frac{x-\sqrt{x}}{(\log\sqrt{x})^2} = o\left(\frac{x}{\log x}\right)\end{aligned}$$

Hence, $\varphi(x) \sim \dfrac{rx}{\log x}$. □

Proof: (prime number theorem)

- Let $\mathscr{I}(x) = \sum_{p\leq x}\log p = \sum_{n\leq x} c_n \log n$, where $c_n = \begin{cases} 1 & \text{if } n = p \\ 0 & \text{otherwise}\end{cases} \rightsquigarrow \mathscr{I}(x) = O(x)$ by Fact 1.12.6.
- Let $D(s) = \sum_n \dfrac{c_n \log n}{n^s} = \sum_p \dfrac{\log p}{p^s} \rightsquigarrow D(s) - \dfrac{1}{s-1}$ is analytic for $\mathrm{Re}(s) \geq 1$ by Fact 1.12.5.
- By Corollary 1.12.1, $\sum_{p\leq x}\log p \sim x$.
- Let $f(p) \equiv 1$. By key lemma, $\pi(x) = \sum_{p\leq x} f(p) \sim \dfrac{x}{\log x}$

□

1.13 Normal family

Definition 1.13.1.

- A family $\mathcal{F}$ of functions $: \Omega \to X$ with (X, d) is a metric space is **normal** if $\forall \{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converge uniformly on each compact subset of Ω.
- An **exhaustion** of Ω is nest compact set $E_1 \subseteq E_2 \subseteq \cdots \subseteq \Omega$ s.t. $E_i \subseteq E_{i+1}^\circ$ and $\forall$ compact subset E of Ω, $\exists k$ s.t. $E \subseteq E_k$ i.e. $\bigcup\limits_{k=1}^{\infty} E_k = \Omega$. e.g.

$$E_k := \{z \in \Omega : |z| \leq k \text{ and } |z - z_0| \geq k^{-1}\ \forall z_0 \in \mathbb{C} \setminus \Omega\}$$

Definition 1.13.2. Let $\mathfrak{X} = \{f : \Omega \to X\}$ be the function space equip the metric as follow :

- Replace d by δ :

$$\delta(a, b) = \frac{d(a, b)}{1 + d(a, b)}$$

  Then δ satisfies the triangle inequality and δ is bounded.

  - $d(a, b) < \varepsilon \rightsquigarrow \delta(a, b) < d(a, b) < \varepsilon$
  - $\delta(a, b) < \varepsilon \rightsquigarrow d(a, b) < \dfrac{\varepsilon}{1 - \varepsilon}$, if we take $\varepsilon = N^{-1}$, then $d(a, b) < \dfrac{1}{N - 1}$
- $\delta_k(f, g) := \sup\limits_{z \in E_k} \delta(f(z), g(z))$
- $\rho(f, g) := \sum\limits_{k=1}^{\infty} \delta_k(f, g) \cdot 2^{-k}$ be the metric define on $\mathfrak{X}$.

Fact 1.13.1. $f_n \xrightarrow{\overline{\text{unif}}} f \iff \lim\limits_{n \to \infty} \rho(f, f_n) = 0$

Proof:

- $(\Rightarrow)$: $\forall \varepsilon > 0$, $\exists k_0$ s.t. $\sum\limits_{k=k_0+1}^{\infty} 2^{-k} < \varepsilon/2$. Since $f \xrightarrow{\overline{\text{unif}}} f$ on E_{k_0}, $\exists n_0$ s.t. $\forall n \geq n_0, \forall z \in E_{k_0}$, $\delta(f_n(z), f(z)) < \varepsilon/2$. Then

$$\rho(f_n, f) = \sum_{k=1}^{k_0} 2^{-k} \delta_k(f_n, f) + \sum_{k=k_0+1}^{\infty} 2^{-k} \delta_k(f_n, f) \leq \sum_{k=1}^{k_0} 2^{-k} \frac{\varepsilon}{2} + \varepsilon/2 < \varepsilon$$

- $(\Leftarrow)$: Let $E \underset{\text{cpt.}}{\subset} \Omega$ and $E \subseteq E_k$. By assumption, $\forall \varepsilon > 0$, $\exists n_0 > 0$ s.t. $\forall n \geq n_0$, $\rho(f, f_n) < \varepsilon \cdot 2^{-k}$, then

$$\varepsilon > 2^k \rho(f_n, f) \geq \delta_k(f_n, f) > \delta(f_n(z), f(z))\ \forall z \in E_k, \forall n \geq n_0$$

□

Proposition 1.13.1. $\mathcal{F}$ is normal $\iff$ the closure of $\mathcal{F}$ w.r.t. ρ is compact.

Proof:

- $(\Rightarrow)$: $\forall \{g_n\} \subseteq \overline{\mathcal{F}}$, $\exists \{f_n\} \subseteq \mathcal{F}$ s.t. $\rho(g_n, f_n) < n^{-1}$. Since $\mathcal{F}$ is normal, $\exists \{f_{i_k}\}$ s.t. $f_{i_k} \xrightarrow{\text{unif}} f$. By Fact 1.13.1, $\lim_{n\to\infty} \rho(f_{i_k}, f) = 0$ and $f \in \overline{\mathcal{F}}$. Also, $g_{i_k} \to f \in \overline{\mathcal{F}}$.

- $(\Leftarrow)$: $\forall \{f_n\} \subseteq \mathcal{F} \subseteq \overline{\mathcal{F}}$, $\exists \{f_{i_k}\}$ and $f \in \overline{F}$ s.t. $\lim_{k\to\infty} \rho(f_{i_k}, f) = 0$. By Fact 1.13.1, $f_{i_k} \xrightarrow{\text{unif}} f$.

□

Definition 1.13.3. A metric space (X, d) is **totally bounded** if $\forall \varepsilon > 0$, $\exists x_1, ..., x_n \in X$ s.t.

$$\forall x \in X,\ \exists i \text{ s.t. } d(x, x_i) < \varepsilon$$

Fact 1.13.2. $Y \subseteq X$ is totally bounded $\iff$ $\overline{Y}$ is totally bounded.

Proof:

- $(\Rightarrow)$: $\forall \varepsilon > 0$, $\exists y_1, ..., y_n \in Y$ s.t. $\forall y \in Y$, $\exists i$ s.t. $d(y, y_i) < \varepsilon/2$. Now, $\forall \widetilde{y} \in \overline{Y}$, $\exists y \in Y$ s.t. $d(\widetilde{y}, y) < \varepsilon/2$ and $\exists i$ s.t. $d(y, y_i) < \varepsilon/2$. Then

$$d(\widetilde{y}, y_i) \le d(\widetilde{y}, y) + d(y, y_i) < \varepsilon$$

- $\forall \varepsilon > 0$, $\exists \widetilde{y_i}, ..., \widetilde{y_n} \in \overline{Y}$ s.t. $\forall \widetilde{y} \in \overline{Y}$, $\exists i$ s.t. $d(\widetilde{y}, \widetilde{y_i}) < \varepsilon/2$. Pick $y_i \in Y$ s.t. $d(\widetilde{y_i}, y_i) < \varepsilon/2$ $\forall i = 1, ..., n$. Then $\forall y \in Y \subseteq \overline{Y}$, $\exists i$ s.t. $d(y, \widetilde{y}) < \varepsilon/2$ and thus

$$d(y, y_i) \le d(y, \widetilde{y_i}) + d(\widetilde{y_i}, y_i) < \varepsilon$$

□

Fact 1.13.3. If X is total bounded, then $\{y_n\} \subseteq X$ has a Cauchy subsequence.

Proof: May assume $|\{y_n\}| = \infty$. $\forall k \in \mathbb{N}$, $\exists x_1^{(k)}, ..., x_{n_k}^{(k)}$ s.t. $\forall x \in X$, $\exists i_k$ s.t. $d(x, x_{i_k}^{(k)}) < (2k)^{-1}$. For $k = 1$, $\exists i_1$ s.t. $\{y_j | d(y_j, x_{i_1}^{(1)}) < 1/2\} = \{y_{j_1}^{(1)}, y_{j_2}^{(1)}, ...\}$ is infinite set. Then for all m, n,

$$d(y_{j_m}^{(1)}, d_{j_n}^{(1)}) \le d(y_{j_m}^{(1)}, x_{i_1}^{(1)}) + d(y_{j_n}^{(1)}, x_{i_1}^{(1)}) < 1$$

Construct by induction, $\exists i_k$ s.t. $\{y_{j_i}^{(k-1)} | d(y_{j_i}^{(k-1)}, x_{i_k}^{(k)}) < 1/(2k)\} = \{y_{j_1}^{(k)}, y_{j_1}^{(k)}, ...\}$ is infinite set. Then for all m, n,

$$d(y_{j_m}^{(k)}, d_{j_n}^{(k)}) \le d(y_{j_m}^{(k)}, x_{i_k}^{(k)}) + d(y_{j_n}^{(k)}, x_{i_k}^{(k)}) < \frac{1}{k}$$

Consider $\{y_{i_k}^{(k)} : k \in \mathbb{N}\}$, then $d(y_{i_m}^{(m)}, y_{i_n}^{(n)}) < \dfrac{1}{\min(m, n)}$ $\forall m, n$ i.e. $\{y_{i_k}^{(k)}\}$ is Cauchy sequence.

□

Fact 1.13.4. $\mathfrak{X} = \{f : \Omega \to X\}$ is complete $\iff$ X is complete.

Proof:

- $(\Rightarrow)$: Consider constant function.

- $(\Leftarrow)$: $\forall z \in \Omega$, $\exists k$ s.t. $z \in E_k$. $\forall N \in \mathbb{N}$, $\exists n_N$ s.t. $\forall m, n > n_N$, $\rho(f_n, f_m) < 2^{-k}/N$. Then $\delta_k(f_n, f_m) < 1/N$ and thus $\delta(f_n(z), f_m(z)) < 1/N$. Then $\{f_{n_1}(z), f_{n_2}(z), ...\}$ is a Cauchy sequence in X. Define $f(z) = \lim_{k\to\infty} f_{n_k}(z)$ and thus $f_{n_i} \xrightarrow{\text{unif}} f$.

□

Proposition 1.13.2. Let X be complete. Then $\mathcal{F}$ is normal $\iff$ $\mathcal{F}$ is total bounded.

Proof: By Fact 1.13.4, X : complete $\implies$ $\mathfrak{X}$: complete. By Fact 1.13.2, we may assume $\mathcal{F}$ is closed.

- $(\Rightarrow)$: $\mathcal{F}$ is normal $\rightsquigarrow$ $\mathcal{F}$ is compact $\rightsquigarrow$ $\mathcal{F}$ is total bounded.
- $(\Leftarrow)$: $\mathcal{F}$ is complete totally bounded metric space. By Fact 1.13.3, $\mathcal{F}$ is compact and thus is normal (By Proposition 1.13.1).

□

Proposition 1.13.3. $\mathcal{F}$ is totally bounded $\iff$ $\forall E \underset{\text{cpt.}}{\subset} \Omega$, $\forall \varepsilon > 0$, $\exists f_1, ..., f_n \in \mathcal{F}$ s.t.

$$\forall f \in \mathcal{F}\ \exists j \text{ s.t. } \delta(f(x), f_j(x)) < \varepsilon\ \forall x \in E$$

Proof:

- $(\Rightarrow)$: Let $E \underset{\text{cpt.}}{\subset} \Omega \rightsquigarrow \exists k$ s.t. $E \subset E_k$. $\forall \varepsilon > 0$, $\exists f_1, ..., f_n \in \mathcal{F}$, $\forall f \in \mathcal{F}$ s.t. $\rho(f.f_j) < \varepsilon/2^k$. Then $\delta_k(f, f_j) < \varepsilon$.
- $(\Leftarrow)$: $\forall \varepsilon > 0$, pick k_0 s.t. $\sum\limits_{k=k_0}^{\infty} 2^{-k} < \varepsilon/2$. Then $\forall f \in \mathcal{F}, \exists j$ s.t. $\delta(f(x), f_j(x)) < \varepsilon/2\ \forall x \in E_{k_0}$. Then
$$\rho(f, f_j) = \sum_{k<k_0} \delta_k(f, f_j) \cdot 2^{-k} + \sum_{k \geq k_0} \delta(f, f_j) \cdot 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem 1.13.1 (Arzela-Ascoli theorem)**.** $\forall f \in \mathcal{F}$, $f : \Omega \xrightarrow{\text{cont.}} X$ with X : complete. Then $\mathcal{F}$ is normal $\iff$

(1) $\mathcal{F}$ is equicontinuous on each compact subset $E \subset \Omega$

(2) $\forall z \in \Omega$, $\{f(z) | f \in \mathcal{F}\} \subset E \underset{\text{cpt.}}{\subset} X$ for some E.

Proof:

- $(\Rightarrow)$:
  (1) By Proposition 1.13.3, $\forall \varepsilon > 0$, $\exists f_1, ..., f_n$ s.t. $\forall f \in \mathcal{F}$, $\exists j$ s.t. $\delta(f(z), f_j(z)) < \varepsilon/3\ \forall z \in E$. Since E is compact, f_j is uniformly continuous on E. Then $\exists \delta > 0$ s.t. $\forall z, z_0 \in E$ with $d(z, z_0) < \delta \rightsquigarrow \delta(f_j(z), f_j(z_0)) < \varepsilon/3$. Hence,
  $$\delta(f(z), f(z_0)) \leq \delta(f(z), f_j(z)) + \delta(f_j(z), f_j(z_0)) + \delta(f_j(z_0), f(z_0)) < \varepsilon$$
  (2) **Claim** : $Z = \overline{\{f(z) : f \in \mathcal{F}\}}$ is compact in X.
  subproof : Let $\{x_n\} \subset Z$, $\forall n \in \mathbb{N}$, $\exists f_n(z) \in Z$ s.t. $\delta(f_n(z), x_n) < n^{-1}$. Since $\mathcal{F}$ is normal, $\exists \{f_{i_m}\} \subset \{f_n\}$ and f continuous in Ω s.t. $\lim\limits_{m\to\infty} f_{i_m} = f$ and thus $\lim\limits_{m\to\infty} x_{i_m} = f(z)$.
- $(\Leftarrow)$: Since $\mathbb{Q} \times \mathbb{Q}$ is countably dense in $\mathbb{R} \times \mathbb{R} = \mathbb{C}$, we can enumerate $(\mathbb{Q} \times \mathbb{Q}) \cap \Omega$ by $\{\xi_n : n \in \mathbb{N}\}$. Given $\{f_n\} \subseteq \mathcal{F}$.

- For $\{f_n(\xi_1) : n \in \mathbb{N}\}$, exists converge subsequence $\{f_{n_k^{(1)}}(\xi_1) : k \in \mathbb{N}\}$ by (2). For $\{f_{n_k^{(i)}}(\xi_{i+1}) : k \in \mathbb{N}\}$, exists converge subsequence $\{f_{n_k^{(i+1)}}(\xi_{i+1}) : k \in \mathbb{N}\}$ by (2). Pick $\{f_{n_k^{(k)}}\} \subset \{f_n\}$, then $\{f_{n_k^{(k)}}(\xi_i) : k \in \mathbb{N}\}$ converge for all i. So we may assume $\{f_i\}$ converge at ξ_i $\forall i$.
- Let $E \underset{\text{cpt.}}{\subset} \Omega$. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z, z' \in E$ with $d(z,z') < \delta \rightsquigarrow \delta(f(z), f(z')) < \varepsilon/3$ $\forall f \in \mathcal{F}$. Since E is compact, $\exists \{B_{\delta/2}(z_i)\}_{i=1}^m$ cover E. Pick $\xi_{\ell_i} \in B_{\delta/2}(z_i)$ $\forall i = 1, ..., m$. $\exists n_0$ s.t. $\forall n, m > n_0$, $\delta(f_n(\xi_{\ell_i}), f_m(\xi_{\ell_i})) < \varepsilon/3$ $\forall i = 1, ..., m$. $\forall z \in E$, say $z \in B_{\delta/2}(z_t)$ for some $t \in \{1, ..., m\}$. Then $d(z, \xi_{\ell_t}) \leq d(z, z_t) + d(z_t, \xi_{\ell_t}) < \delta$ and thus $\forall m, n > n_0$

$$\delta(f_n(z), f_m(z)) \leq \delta(f_n(z), f_n(\xi_{\ell_t})) + \delta(f_n(\xi_{\ell_t}), f_m(\xi_{\ell_t})) + \delta(f_m(\xi_{\ell_t}), f_m(z)) < \varepsilon$$

  Hence, $\{f_n\}$ converge uniformly on each compact subset.

$\square$

Definition 1.13.4 (metric on $\widetilde{\mathbb{C}}$). For $z, z' \in \widetilde{\mathbb{C}}$, the distance of z, z' is the distance of z, z' on Riemann Sphere with Euclidean norm

$$d(z,z') = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$

Notice that $z \to z_0 \iff d(z, z_0) \to 0$ and if $z \to \infty \iff d(z, \infty) = 0$.

Theorem 1.13.2 (Hurwitz's theorem (I)). If f_n is analytic and has no zero in Ω and $f \xrightarrow{\text{unif}} f$ in Ω, then $f = 0$ or f has no zero in Ω.

Proof: Suppose $f \not\equiv 0$ in Ω. Since the zero of f is isolated, $\forall z_0 \in \Omega$, $\exists \rho > 0$ s.t. f has not zero in $\{z | 0 < |z - z_0| \leq \rho\} \subset \Omega$. And

$$\frac{1}{f_n(z)} \to \frac{1}{f(z)} \text{ uniformly on } C_\rho(z) \text{ and } f_n'(z) \to f'(z) \text{ uniformly on } C_\rho(z)$$

By argument principle,

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_\rho(z)} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{C_\rho(z)} \lim_{n\to\infty} \frac{f_n'(z)}{f_n(z)} dz = \frac{1}{2\pi i} \int_{C_\rho(z)} \frac{f'(z)}{f(z)} dz$$

and thus $f(z_0) \neq 0$. $\square$

Proposition 1.13.4. Let $f_n : \Omega \to \widetilde{\mathbb{C}}$: meromorphic function $\forall n$. If $f_n \xrightarrow{\text{unif}} f$, then f is meromorphic or $f \equiv \infty$.

Proof:

- If $f(z_0) \neq 0$: f is bounded near $z_0 \rightsquigarrow$ for $n \gg 0$, f_n is bounded near $z_0 \rightsquigarrow n \gg 0$, $f_n \neq \infty$ near $z_0 \rightsquigarrow f_n$ analytic near z_0. By Weierstrass theorem, f is analytic near z_0.
- If $f(z_0) = \infty$: $f_n^{-1} \xrightarrow{\text{unif}} f^{-1}$ near $z_0 \rightsquigarrow f$ is analytic near $z_0 \rightsquigarrow z_0$ is pole of f.

$\square$

Remark 1.13.1. If f_n is analytic near z_0 and $f(z_0) = \infty \rightsquigarrow f_n^{-1}$ has no zero near z_0 and f^{-1} has at z_0, then by Hurwitz's theorem I, $f^{-1} = 0 \rightsquigarrow f = \infty$. Hence,

$$f_n : \Omega \to \widetilde{\mathbb{C}} : \text{analytic and } f_n \xrightarrow{\overline{\text{unif}}} f, \text{ then } f : \text{analytic or } f \equiv \infty$$

Theorem 1.13.3 (Montel theorem)**.** Let $\mathcal{F}$ is family of analytic functions. If the function in $\mathcal{F}$ are uniformly bounded on each compact subset $E \subset \Omega$, then $\mathcal{F}$ is normal.

Proof:

- **Claim** : Let $\gamma = C_\rho(z) \subset \Omega$ and $M = \max_{z\in\gamma} |f(z)|$. For $z_1, z_2 \in B_{\rho/2}(z)$,

$$|f(z_1) - f(z_2)| \le \frac{4M}{\rho}|z_1 - z_2|$$

  subproof : By Cauchy integral formula,

$$|f(z_1) - f(z_2)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(\xi)(z_1 - z_2)d\xi}{(\xi - z_1)(\xi - z_2)}\right| \le \frac{4M}{\rho}|z_1 - z_2|$$

- $\forall E \underset{\text{cpt.}}{\subset} \Omega$, $\forall z \in E \rightsquigarrow \exists\overline{B_\rho(z)} \subset \Omega$, then $\{B_{\rho/4}(z) : z \in E\}$ cover E. By compactness, $\{B_{\rho_i/4}(z_i) : i = 1, ..., n\}$ covers E. $\forall i = 1, ..., n$, let $|f(z)| \le M_i$ $\forall f \in \mathcal{F}$ $\forall z \in C_{\rho_i}(z_i)$. Let $M = \max_i M_i$ and $\rho = \min_i \rho_i$. $\forall \varepsilon > 0$, let $\delta = \min\left(\frac{\rho\varepsilon}{4M}, \frac{\rho}{4}\right)$. For $z, z' \in E$ with $|z - z'| < \delta$, say $z' \in B_{\rho_i/4}(z_i) \rightsquigarrow |z - z_i| \le |z - z'| + |z' - z_i| < \rho_i/2$. By Claim, $\forall f \in \mathcal{F}$,

$$|f(z) - f(z')| \le \frac{4M_i|z - z'|}{\rho_i} \le \varepsilon$$

  Also, $\forall z \in \Omega$, $\{f(z) : z \in \Omega\}$ is bounded, then $\{f(z) : z \in \Omega\}$ contained in a compact subset of $\mathbb{C}$. By Arzela-Ascoli theorem, $\mathcal{F}$ is normal. □

Theorem 1.13.4. If the analytic function in $\mathcal{F}$ are uniformly bounded on each compact subset, then so are the derivatives of $\mathcal{F}$.

Proof: By Cauchy integral formula,

$$|f'(z)| = \frac{1}{2\pi}\left|\int_{C_\rho(z_0)} \frac{f(\xi)d\xi}{(\xi - z)^2}\right| \le \frac{4M}{\rho} \ \forall z \in B_{\rho/2}(z_0) \ \forall f \in \mathcal{F}$$

□

Definition 1.13.5. A family $\mathcal{F}$ of meromorphic function $f : \Omega \to \widetilde{\mathbb{C}}$ is normal if $\forall\{f_n\} \subset \mathcal{F}$, $\exists\{f_{n_i}\}$ s.t. $f_{n_i} \xrightarrow{\overline{\text{unif}}} f$ or $f_{n_i} \xrightarrow{\overline{\text{unif}}} \infty$.

Remark 1.13.2. Consider $f_n(z) = nz^2 - n \xrightarrow{\overline{\text{unif}}} \infty$, but $f_n'(z) = 2nz \to \begin{cases} \infty & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$.

Definition 1.13.6. $f : \Omega \to \widetilde{\mathbb{C}}$: meromorphic. The **spherical derivative** of f is

$$f^\sharp(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

which is come from $d(f(z), f(z_0)) = \dfrac{2|f(z)-f(z_0)|}{\sqrt{(1+|f(z)|^2)(1+|f(z_0)|^2)}}$. Notice that if $f \neq 0$, then $f^\sharp = (f^{-1})^\sharp$, so we can define the spherical derivative at pole by $f^\sharp = (f^{-1})^\sharp$.

Fact 1.13.5. If f has a pole at a, then

$$\begin{cases} f^\sharp(a) = 0 \iff \text{the order of } a \geq 2 \\ a \text{ is simple pole } \rightsquigarrow f^\sharp(a) = \dfrac{2}{|\operatorname{Res}_{z=a} f|} \end{cases}$$

Theorem 1.13.5 (Marty theorem). A family $\mathcal{F}$ of meromorphic functions is normal $\iff$ $\{f^\sharp : f \in \mathcal{F}\}$ is uniformly bounded on each compact subset.

Proof:

- $(\Rightarrow)$: If $\{f^\# : f \in \mathcal{F}\}$ is not uniformly bounded on $E \underset{\text{cpt.}}{\subset} \Omega$. Choose $f_n \in \mathcal{F}$ s.t. $\max\limits_{z \in E} f_n^\#(z) > n$. Then exists $f_{n_k} \xrightarrow{\overline{\text{unif}}} f$ on Ω. For all $z \in E$, exists closed disk contain z and contained in Ω s.t. f is analytic or $1/f$ is analytic. If f is analytic on the closed disk, then f is bounded i.e. f_{n_k} has no pole for $k \gg 0$. By Weierstrass theorem, $f'_{n_k} \to f'$ uniformly on the smaller closed disk. Hence, $f_{n_k}^\# \to f^\#$ on that smaller closed disk and thus $f^\#$ continuous on it. If $1/f$ is analytic on the closed disk, by same argument combine $(1/f)^\# = f^\#$, we still have $f_{n_k}^\# \to f^\#$ converge uniformly on smaller closed disk and $f^\#$ continuous on it. Since E is compact, $f_{n_k}^\# \to f^\#$ converge uniformly on E. Since $f^\#$ continuous on compact set E, $\max\limits_{z \in E} f^\#(z)$ exists and thus $f_{n_k}^\#(z)$ is bounded for $k > N$ $(\rightarrow\!\leftarrow)$.
- $(\Leftarrow)$: Let $E \underset{\text{cpt.}}{\subset} \Omega$, there exists M s.t. $f^\#(z) < M$ $\forall z \in E$. Let γ be the segment from z_1 to z_2, then

$$d(f(z_1), f(z_2)) \leq \int_{f\circ\gamma} \frac{|dw|}{1+|w|^2} = \int_\gamma \frac{|f'(z)dz|}{1+|f(z)|^2} = \frac{1}{2}\int_\gamma f^\#(z)|dz| \leq \frac{M}{2}|z_2 - z_1|$$

and hence $\mathcal{F}$ is equicontinuous. $\{f(z) : f \in \mathcal{F}\} \subset \widetilde{\mathbb{C}}$: compact. By Arzela-Ascoli theorem, $\mathcal{F}$ is normal.

$\square$

1.14 Riemann mapping theorem

1.14.1 Riemann mapping theorem

Theorem 1.14.1 (Riemann mapping theorem). Let $\Omega \subsetneq \mathbb{C}$ be simply connected. Then for $z_0 \in \Omega$, $\exists!$ biholomorphism $F : \Omega \to B_1(0) =: \mathbb{D}$ s.t. $F(z_0) = 0$ and $F'(z_0) > 0$.

Proof: (uniqueness) Let $F_1, F_2 : \Omega \to \mathbb{D}$. Consider $f = F_2 \circ F_1^{-1} : \mathbb{D} \to \mathbb{D}$, then $f(0) = 0$ and $f'(0) = \dfrac{F_2'(z_0)}{F_1'(z_0)} > 0$. By Schwarz's lemma, $|f(z)| = |z| \rightsquigarrow f(z) = az$ with $|a| = 1$. Since $f'(0) > 0 \rightsquigarrow a = 1$ and thus $F_1 = F_2$. $\square$

Theorem 1.14.2 (Hurwitz's theorem (II)). Let $\{f_n\}$ be a sequence of holomorphic function in Ω. If $f_n \xrightarrow{\overline{\text{unif}}} f$ and $f_n : \Omega \to \mathbb{C}$ is injective $\forall n$, then f is injective or f is constant.

Proof: Assume f is not constant on Ω and f is not $1-1$, say $f(a) = w = f(b)$ with $a \neq b$. Let $w_n = f_n(a) \rightsquigarrow \lim_{n\to\infty} w_n = w$. Since $f \neq$ constant, $\exists \rho > 0$ s.t. $\forall z \in B_\rho(b) \setminus \{b\}$, $f(z) \neq f(b) \rightsquigarrow a \notin B_\rho(b)$. Since f_n is injective $\forall n$, $f_n - w_n$ has not solution in $B_\rho(b)$. But $f_n - w_n \xrightarrow{\overline{\text{unif}}} f - w$ and $f - w$ has a zero b. By Hurwitz's theorem (I), $f - w \equiv 0$ in $B_\rho(b)$ ($\rightarrow\!\leftarrow$). □

Now we fix $z_0 \in \Omega$ and let $\mathcal{F} = \{f : \Omega \to \mathbb{D} | f(z_0) = 0, f :$ holomorphic and injective$\}$.

Lemma 1.14.1. $\mathcal{F} \neq \varnothing$:

Proof: Since $\Omega \neq \mathbb{C}$, $\exists a \in \mathbb{C} \setminus \Omega \rightsquigarrow z - a \neq 0$ on Ω. Since Ω is simply connected, exists a holomorphic branch of $\log(z-a)$. Say it is $\ell : \Omega \underset{\text{inj.}}{\overset{\text{hol.}}{\to}} \mathbb{C}$ and $e^{\ell(z)} = z - a$. Since ℓ is injective, for $z_1, z_1 \in \Omega$ and $z_1 \neq z_2 \rightsquigarrow \ell(z_1) - \ell(z_2) \notin 2\pi i\mathbb{Z}$.
Claim : $\exists \varepsilon > 0$ s.t. $|\ell(z) - (\ell(z_0) + 2\pi i)| > \varepsilon \ \forall z \in \Omega$.
subproof : If not, $\exists \{z_n\} \subset \Omega$ s.t. $\ell(z_n) \to \ell(z_0) + 2\pi i \rightsquigarrow e^{\ell(z_n)} \to e^{\ell(z_0)+2\pi i} = e^{\ell(z_0)} \rightsquigarrow z_n \to z_0 \rightsquigarrow \ell(z_n) \to \ell(z_0)$ ($\rightarrow\!\leftarrow$) □
Consider $g(z) = \dfrac{1}{\ell(z) - \ell(z_0) - 2\pi i} \rightsquigarrow g$ is bounded by claim and holomorphic, injective on Ω.
Say $g : \Omega \to B_R(0)$ with $g(z_0) = \alpha$, where $R = \varepsilon^{-1}$. Then $f(z) := \dfrac{g(z) - \alpha}{R + \alpha} \in \mathcal{F}$. □

Lemma 1.14.2. $\lambda := \sup_{f\in\mathcal{F}} |f'(z_0)|$, then $\lambda > 0$

Proof: Recall a holomorphism is locally injective on $\Omega \iff f'(z) \neq 0 \ \forall z \in \Omega$. Since f is injective $\forall f \in \mathcal{F} \rightsquigarrow |f'(z)| > 0 \ \forall f \in \mathcal{F} \rightsquigarrow \lambda > 0$. □

Lemma 1.14.3. $\exists f \in \mathcal{F}$ s.t. $|f'(z_0)| = \lambda$ i.e. $\lambda < \infty$.

Proof: Let $\{f_n\} \subset \mathcal{F}$ s.t. $\lim_{n\to\infty} |f_n'(z_0)| = \lambda$. Since $|f_n(z)| < 1$, by Montel theorem, $\exists \{f_{n_i}\} \subset \{f_n\}$ s.t. $f_{n_i} \xrightarrow{\overline{\text{unif}}} f$. By Weierstrass theorem, f is holomorphic and $f_n' \xrightarrow{\overline{\text{unif}}} f' \rightsquigarrow |f(z)| \leq 1$, $f(z_0) = \lim_{i\to\infty} f_{n_i}(z_0) = 0$ and $|f'(z_0)| = \lim_{i\to\infty} |f_{n_i}'(z_0)| = \lambda$.

- $|f(z)| < 1$: Since $f \neq$ constant, by maximal principle, $|f(z)| < 1$ in Ω.
- f is injective : It follows from Hurwitz's theorem II.

Hence, $f \in \mathcal{F}$ □

Proof: (existence of Riemann mapping theorem) Let F be given in Lemma 1.14.3 : $F(z_0) = 0, F'(z_0) \neq 0$, say $\arg F'(z_0) = \theta$. Consider $e^{-i\theta}F \rightsquigarrow (e^{-i\theta}F)'(z_0) > 0$, so we may assume $F'(z_0) > 0 \rightsquigarrow F : \Omega \to \mathbb{D}$: injective.
Claim : F is surjective : If not, $\exists a \in \mathbb{D}$ s.t. $F(z) = a$ has no solution in Ω. Let $T_a(z) = \dfrac{a - z}{1 - \overline{a}z} \in \mathrm{Aut}(\mathbb{D}) \rightsquigarrow$ "$T_a(z) = 0 \iff z = a$". Consider $g(z) = \sqrt{T_a \circ F(z)} = \exp(T_a(F(z))/2)$ which has a holomorphic branch since Ω is simply connected. Define $G(z) = T_{\sqrt{a}} \circ g(z)$ which is injective and $G(z_0) = 0 \rightsquigarrow G \in \mathcal{F}$. We claim that $|G'(z_0)| > |F'(z_0)|$ and thus contradict to defintion of F. Let $S : \mathbb{D} \to \mathbb{D}$ define by $z \to z^2$, then $F = \underbrace{T_a^{-1} \circ S \circ T_{\sqrt{a}}^{-1}}_{:=\Phi} \circ G \rightsquigarrow \Phi : \mathbb{D} \to \mathbb{D}$ with $\Phi(0) = 0$. By Schwarz's lemma, $|\Psi'(0)| \leq 1$. If $|\Psi'(0)| = 1$, then $\Psi(z) = cz$ with $|c| = 1 \rightsquigarrow \Psi$ is $1-1 \rightsquigarrow S$ is $1-1$ in $\mathbb{D}$ ($\rightarrow\!\leftarrow$). So $|\Psi'(0)| < 1$ and thus

$$|F'(z_0)| = |\Psi'(G(z_0))| \cdot |G'(z_0)| < |G'(z_0)|$$

□

Example 1.14.1.

- $\mathrm{Aut}(\mathbb{D}) = \{e^{i\theta} \circ T_a : \theta \in \mathbb{R}, a \in \mathbb{D}\}$
- $\mathrm{Aut}(\mathbb{H}) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$

Corollary 1.14.1. If $\Omega \subset \widetilde{\mathbb{C}}$ is simply connected, then either $\Omega = \widetilde{\mathbb{C}}$ or $\Omega \xrightarrow{\sim} \mathbb{C}$ or $\Omega \xrightarrow{\sim} \mathbb{D}$.

Proof: Let $Z = \widetilde{\mathbb{C}} \setminus \Omega$.

- If $Z = \varnothing \rightsquigarrow \Omega = \widetilde{\mathbb{C}}$.
- If $Z \neq \varnothing$, say $a \in Z$. If $\infty \in Z$, $\Omega \xrightarrow{\sim} \mathbb{C}$ or $\mathbb{D}$ by Riemann mapping theorem. If $a \neq \infty$, consider $T(z) = \dfrac{1}{z-a}$, then

$$\begin{array}{rccc} T: & \widetilde{\mathbb{C}} & \xrightarrow{\sim} & \widetilde{\mathbb{C}} \\ & a & \longmapsto & \infty \\ & \Omega & \longrightarrow & T(\Omega) \subset \mathbb{C} \end{array}$$

  $\Omega \xrightarrow{\sim} T(\Omega) \xrightarrow{\sim} \mathbb{C}$ or $\mathbb{D}$.

□

Remark 1.14.1.

- Let $U_0 = \widetilde{C} \setminus \{\infty\}$ and $U_\infty = \widetilde{C} \setminus \{0\}$. Then

$$\begin{array}{rccc} \varphi_0: & U_0 & \xrightarrow{\sim} & \mathbb{C} \\ & z & \longmapsto & z \end{array} \quad \text{and} \quad \begin{array}{rccc} \varphi_\infty: & U_\infty & \xrightarrow{\sim} & \mathbb{C} \\ & z & \longmapsto & z^{-1} \end{array}$$

  and $U_0 \cap U_\infty \xrightarrow{\sim} U_0 \cap U_\infty$ define by $z \mapsto z^{-1}$.
- $\Omega \subset \widetilde{\mathbb{C}}$ and $f : \Omega \to \mathbb{C}$: continuous
  - $a \neq \infty$, f is holomorphic at a iff $f \circ \varphi_0^{-1}$ is holomorphic at $\phi_0(a)$
  - $a = \infty$, f is holomorphic at a iff $f \circ \varphi_\infty^{-1}$ is holomorphic at $\phi_\infty(a)$ i.e. $f(z^{-1})$ is holomorphic at 0.
- Consider a line $(x(t), y(t))$ on $\mathbb{C}$ and it image on Riemann sphere $(\xi(t), \eta(t), \rho(t))$. The tangent slope at $(0,0,1)$ is $\displaystyle\lim_{t\infty\infty} \frac{\eta(t)}{\xi(t)} = \lim_{t\to\infty} \frac{y(t)}{x(t)} = \text{constant}$.
- $\widetilde{\mathbb{C}}$ is compact : Let $\{z_n\} \subset \widetilde{\mathbb{C}}$
  - $\{z_n\} \subset \mathbb{C}$ is bounded $\rightsquigarrow \exists \{z_{n_i}\}$ converge.
  - $\{z_n\} \subset \mathbb{C}$ is unbounded $\rightsquigarrow \exists \{z_{n_i}\}$ s.t. $|z_{n_i}| \to \infty$ as $i \to \infty$, $d(z_{n_i}, \infty) = \dfrac{2}{\sqrt{1 + |z_{n_i}|^2}} \to 0 \rightsquigarrow \displaystyle\lim_{i\to\infty} z_{n_i} = \infty$.
  - $\infty \in \{z_n\}$:
    - ∞ is an accumulation point, then $\exists \{z_{n_i}\}$ s.t. $\displaystyle\lim_{i\to\infty} z_{n_i} = \infty$
    - ∞ is not an accumulation point, then $\exists n_0$ st. $\{z_n\}_{n=n_0}^\infty$ is bounded, then exists converge subsequence.

Definition 1.14.1. $\mathbb{P}^1 := (\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\})/\sim$, where $(x_1,y_1)\sim(x_2,y_2) \iff \exists\lambda\in\mathbb{C}^\times$ s.t. $(x_1,y_1)=\lambda(x_2,y_2)$. Denote $[x,y]$ be the equivalent class for (x,y).

Remark 1.14.2.

$$\begin{array}{ccc} U_0=\{[x,y]\in\mathbb{P}^1 : x\neq 0\} & \longrightarrow & U_\infty=\{[x,y]\in\mathbb{P}^1 : y\neq 0\} \\ [x,y]=[1,y/x]\mapsto y/x\in\mathbb{C} & & [x,y]=[x/y,1]\mapsto x/y\in\mathbb{C} \\ z & \longmapsto & z^{-1} \end{array}$$

1.14.2 Automorphism of $\mathbb{C},\mathbb{C}^\times,\widetilde{\mathbb{C}}$

Lemma 1.14.4. If $g:\mathbb{C}^\times\to\mathbb{C}$ is holomorphic and injective, then g cannot have an essential singularity at 0.

Proof: If not, recall Weierstrass-Casorati theorem, $\forall B_\rho(a)$, g can arbitrary closed to any complex value on $B_\rho(a)\rightsquigarrow g(\mathbb{D})$ is dense in $\mathbb{C}$. By open mapping theorem, $U=g(B_1(2))$ is an open neighborhood of $g(2)\rightsquigarrow U\cap g(\mathbb{D})\neq\varnothing$. Say $w\in U\cap g(\mathbb{D})$ and $w=g(z_1)=g(z_2)$ with $z_1\in\mathbb{D}$, $z_2\in B_1(2)$, but $z_1\neq z_2$ since $\mathbb{D}\cap B_1(2)=\varnothing$, which contradict to injective. □

Lemma 1.14.5 (Generalization of Liouville's theorem)**.** $f:\mathbb{C}\to\mathbb{C}$ is entire s.t. $|f(z)|\leq M(1+|z|^n)$ $\forall z\in\mathbb{C}$ for some $M>0$. Then f is polynomial of degree $\leq n$.

Proof: By Cauchy estimate, for $R>1$ and $k>n$

$$|f^k(0)|\leq\frac{k!\cdot M(1+R^n)}{R^k}\leq 2M\cdot k!\cdot R^{n-k}\to\infty \text{ as } R\to\infty$$

Hence, $f^k(0)=0$ $\forall k>n$. By Taylor series, $a_k=\dfrac{f^{(k)}(0)}{k!}=0$ $\forall k>n\rightsquigarrow f$ is polynomial of degree $\leq n$. □

Theorem 1.14.3. $\mathrm{Aut}(\mathbb{C})=\{az+b : a,b\in\mathbb{C}, a\neq 0\}$

Proof: $(\supseteq)$: OK. $(\subseteq)$: For $f\in\mathrm{Aut}(\mathbb{C})$, $g(z)=f(z^{-1}):\mathbb{C}^\times\to\mathbb{C}$ is holomorphic and injective. By Lemma 1.14.4, g has a removable singularity or pole at $z=0\rightsquigarrow\exists M>0$ and n s.t. $|z|^n|g(z)|<M^{-1}$ on $|z|<1$ i.e. $|f(z)|\leq M|z|^n$ $\forall z>1$. On $|z|\leq 1$, $|f(z)|\leq M'$. Then

$$|f(z)|\leq\max(M,M')(1+|z|^n)\ \forall z\in\mathbb{C}$$

By Lemma 1.14.5, f is polynomial of degree $\leq n$. By fundamental theorem of algebra, f has at exactly n roots in $\mathbb{C}$. Since f is $1-1$, $f(z)$ must be $a(z-\alpha)^n$. If $n>1$, then near α, f is n to $1(\rightarrow\!\leftarrow)$. Hence, $n=1$. □

Theorem 1.14.4. $\mathrm{Aut}(\mathbb{C}^\times)=\{az|a\neq 0\}\cup\{az^{-1} : a\neq 0\}$

Proof: $(\supseteq)$: OK. $(\subseteq)$: For $f\in\mathrm{Aut}(\mathbb{C}^\times)$, by Lemma 1.14.4, $z=0$ is either removable singularity or pole.

- $z=0$ is removable singularity : We can define $\widetilde{f}:\mathbb{C}\to\mathbb{C}$. We claim that $\widetilde{f}$ is injective. If $\widetilde{f}(0)=\widetilde{f}(z_0)=w_0$ for some $z_0\neq 0$. By argument principle, exists a neighborhood V of w_0 and $B_\rho(z_0)$, $B_r(0)$ with $B_\rho(z_0)\cap B_r(0)=\varnothing$ s.t. $\forall w_0\neq w\in V$, $\exists z_0\neq z_1\in B_\rho(z_0)$ and $0\neq z_2\in B_r(0)$ s.t. $\widetilde{f}(z_1)=w=\widetilde{f}(z_2)\rightsquigarrow f(z_1)=f(z_2)$ $(\rightarrow\!\leftarrow)$. Hence, $f(0)=0$ and thus $\widetilde{f}\in\mathrm{Aut}(\mathbb{C})\rightsquigarrow\widetilde{f}(z)=az+b$. Since $\widetilde{f}(0)=0\rightsquigarrow b=0$.

- $z = 0$ is pole : $h(z) = f(z)^{-1} \rightsquigarrow h : \mathbb{C} \to \mathbb{C}$ s.t. $h(0) = 0 \rightsquigarrow h(z) = cz \rightsquigarrow f(z) = (1/c)/z$.

$\square$

Theorem 1.14.5. $\mathrm{Aut}(\widetilde{\mathbb{C}}) \simeq \mathrm{SL}_2(\mathbb{C})/\langle -I \rangle$.

Proof: We have know that $f(z) = \dfrac{az+b}{cz+d}$ with $ad - bc = 1 \rightsquigarrow f \in \mathrm{Aut}(\widetilde{\mathbb{C}})$. Conversely, let $T \in \mathrm{Aut}(\widetilde{\mathbb{C}})$, say $T(z_0) = \infty$ and $T(\infty) = w_0$.

- If $z_0 = \infty \rightsquigarrow w_0 = \infty \rightsquigarrow F := T|_{\mathbb{C}} \in \mathrm{Aut}(\mathbb{C}) \rightsquigarrow F(z) = az + b$ with $a \neq 0 \rightsquigarrow \widetilde{F} \in \mathrm{SL}_2(\mathbb{C})$, where $\widetilde{F}$ is normalization transformation.
- $z_0 \neq \infty \rightsquigarrow w_0 \neq \infty$. $F(z) := T(z_0+z) - w_0 \rightsquigarrow F(\infty) = 0$ and $F(0) = \infty \rightsquigarrow F|_{\mathbb{C}^\times} \in \mathrm{Aut}(\mathbb{C}^\times)$. Since $z = 0$ is a pole of $f \rightsquigarrow f(z) = \dfrac{a}{z} \rightsquigarrow T(z) = \dfrac{a}{z - z_0} + w \rightsquigarrow \widetilde{T} \in \mathrm{SL}_2(\mathbb{C})$.

$\square$

1.14.3 Cross ratio

We already know the automorphism of $\widetilde{\mathbb{C}}$, then three point can identify the unique automorphism of $\widetilde{\mathbb{C}}$.

- $S : z_1, z_2, z_3 \to 1, 0, \infty \rightsquigarrow S(z) = \dfrac{z - z_2}{z - z_3} \cdot \dfrac{z_1 - z_3}{z_1 - z_2}$
- $S : 1, 0, \infty \to 1, 0, \infty \rightsquigarrow S = \mathrm{id}$

Definition 1.14.2. The **cross ratio** of z_0, z_1, z_2, z_3 is defined by

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = S(z_0)$$

where $S : z_1, z_2, z_3 \to 1, 0, \infty$. Then we have some property for cross ratio :

- Given $T \in \mathrm{Aut}(\widetilde{\mathbb{C}})$, then $(Tz_0, Tz_1, Tz_2, Tz_3) = (z_0, z_1, z_2, z_3)$:
  Since $S \circ T^{-1} : Tz_1, Tz_2, Tz_3 \to 1, 0, \infty$, we have
  $$(Tz_0, Tz_1, Tz_2, Tz_3) = S \circ T^{-1}(T(z_0)) = S(z_0) = (z_0, z_1, z_2, z_3)$$
- $z_1, z_2, z_3 \xrightarrow{w(z)} w_1, w_2, w_3 \rightsquigarrow (z, z_1, z_2, z_3) = (w(z), w_1, w_2, w_3)$.

Definition 1.14.3. z and z^* **symmetric** w.r.t. C for if $\forall z_1, z_2, z_3 \in \mathbb{C}$,

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

- If C is a straight line : Consider $z_3 = \infty \rightsquigarrow \dfrac{z^* - z_2}{z - z_2} = \dfrac{\overline{z} - \overline{z_2}}{\overline{z_1} - \overline{z_2}} \rightsquigarrow \mathrm{Im}\,\dfrac{z^* - z_2}{z_1 - z_2} = -\,\mathrm{Im}\,\dfrac{z - z_2}{z_1 - z_2}$
- $C = C_R(a)$:
  $$\begin{aligned}\overline{(z, z_1, z_2, z_3)} = \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} &= \left(\overline{z} - \overline{a}, \frac{R^2}{z_1 - a}, \frac{R^2}{z_2 - a}, \frac{R^2}{z_3 - a}\right) \\ &= \left(\frac{R^2}{\overline{z} - \overline{a}}, z_1 - a, z_2 - a, z_3 - a\right) = \left(\frac{R^2}{\overline{z} - \overline{a}} + a, z_1, z_2, z_3\right)\end{aligned}$$
  Then $z^* = \dfrac{R^2}{\overline{z} - \overline{a}} + a$ and $\dfrac{z^* - a}{z - a} = \dfrac{R^2}{|z - a|^2}$.

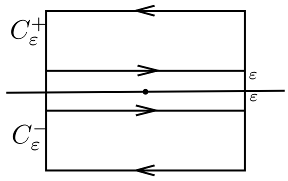
1.15 Conformal mapping

Theorem 1.15.1 (Schwarz reflection principle)**.** Let Ω be symmetric w.r.t. x-axis and $\Omega \cap x$-axis $= (a, b)$, $\Omega+ := \Omega \cap \mathbb{H}$. Assume $f : \Omega^+ \to \mathbb{C}$ is holomorphic and f is continuous on $\Omega^+ \cup (a, b)$ s.t. $f((a, b)) \subseteq \mathbb{R}$. Let $\Omega^- = \{\overline{z} : z \in \Omega^+\}$ and

$$\widetilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \Omega^+ \cup (a, b) \\ \overline{f(\overline{z})} & \text{if } z \in \Omega^- \end{cases}$$

Then $\widetilde{f}$ is holomorphic on Ω.

Proof: Define $g(z) = \overline{f(\overline{z})}$ for $z \in \Omega^-$. We claim that g is holomorphic in Ω^- and also continuous on (a, b).
subproof : $\forall z_0 \in \Omega^-$, $\overline{z_0} \in \Omega$. Say $f(z) = \sum a_n(z - \overline{z_0})^n$ near $\overline{z_0} \rightsquigarrow g(z) = \sum \overline{a_n}(z - z_0)^n$ near z_0 i.e. g is holomorphic near z_0. Also, $g|_{(a,b)} = f|_{(a,b)}$. $\square$
By claim, $\overline{f}$ holomorphic on Ω^+, Ω^- and is continuous on $\Omega^+ \cup (a, b) \cup \Omega^- = \Omega$. Now, for $z_0 \in (a, b)$, let $B_\rho(z_0) \subset \Omega$. For any rectangle $R \subseteq B_\rho(z_0)$. If $\partial R \cap (a, b) = \varnothing$, then $\partial R \subset \Omega^+$ or Ω^-. By Cauchy theorem, $\displaystyle\int_{\partial R} \widetilde{f} dz = 0$. If $\partial R \cap (a, b) \neq \varnothing$. Consider $C_\varepsilon^+, C_\varepsilon^-$ be defined in below,



Since $\widetilde{f}$ is continuous at (a, b) and $\widetilde{f}$ analytic in Ω^+, Ω^-, we have

$$\int_{\partial R} \widetilde{f}(z)dz = \lim_{\varepsilon \to 0} \left(\int_{C_\varepsilon^+} \widetilde{f}(z)dz + \int_{C_\varepsilon^-} \widetilde{f}(z)dz \right) = \lim_{\varepsilon}(0 + 0) = 0$$

and hence it will holds for all closed curve with same argument in Cauchy theorem. By Morera's theorem, $\widetilde{f}$ is holomorphic at z_0. $\square$

Remark 1.15.1.

- $f : B_1(0) \to \mathbb{C}$ holomorphic and is continuous on $\overline{B_1(0)}$ s.t. $f(C_1(0)) \subset \mathbb{R}$. Then $\exists \widetilde{f} : \mathbb{C} \to \widetilde{\mathbb{C}}$ s.t. $\widetilde{f}|_{\overline{B_1(0)}} = f$ and
$$\widetilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \overline{B_1(0)} \\ \overline{f(1/\overline{z})} & \text{if } |z| > 1 \end{cases}$$
- $f : B_1(0) \to \mathbb{C}$ holomorphic and is continuous on $\overline{B_1(0)}$ s.t. $f(C_1(0)) \subset C_1(0)$. Then $\exists \widetilde{f} : \mathbb{C} \to \widetilde{\mathbb{C}}$ s.t. $\widetilde{f}|_{\overline{B_1(0)}} = f$ and
$$\widetilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \overline{B_1(0)} \\ \dfrac{1}{\overline{f(1/\overline{z})}} & \text{if } |z| > 1 \end{cases}$$
- In general, Schwarz reflection principal follow from $\widetilde{f}(z) = f(z^*)^*$.

Goal : Construct a conformal map from $\mathbb{H}$ to a polygon P.
Theoretical existence : By Riemann mapping theorem, $\exists F_0 : P \xrightarrow{\sim} \mathbb{D} \rightsquigarrow F_0^{-1} : \mathbb{D} \to P$. Then we can extend it to $\overline{\mathbb{D}} \to \overline{P}$ s.t. $\partial\mathbb{D} \xrightarrow{\sim} \partial P$ by Lemma 1.15.1. Let $w : \mathbb{H} \to \mathbb{D}$ define $w(z) = \dfrac{i-z}{i+z}$ and it is continuous on $\mathbb{R} \rightsquigarrow f \circ w : \mathbb{H} \to P$ is conformal.
Construct explicitly : Want to construct $F : \mathbb{H} \to P$ and $a_i \in \mathbb{R}$ are the vertices of P s.t. $F : a_i \mapsto w_i$.

- If we try $f'(z) = (z-a)^\alpha = e^{\alpha(\log|z-a| + i\arg(z-a))}$ with $\alpha \in \mathbb{R}$
  - $z < a : \arg(z-a) = \pi \rightsquigarrow \arg(z-a)^\alpha = \alpha\pi$
  - $z > a : \arg(z-a) = 0 \rightsquigarrow \arg(z-a)^\alpha = 0$

  In fact, $f'(z) = \lambda(z-a)^\alpha \rightsquigarrow \arg f'(z) = \arg\lambda + \alpha\arg(z-a)$
- Now we try $f'(z) = \lambda(z-a)^{-\alpha}(z-b)^{-\beta}$ with $a < b$:
  - $z < a : \arg f'(z) = \arg\lambda - \alpha\pi - \beta\pi$
  - $a < z < b : \arg f'(z) = \arg\lambda - \beta\pi$
  - $z > b : \arg f'(z) = \arg\lambda$

  Notice that $\overline{a_i a_{i+1}} \mapsto \overline{w_i w_{i+1}}$ and thus $\arg f'(z)$ is constant in (a_i, a_{i+1}), which is the angle of $\overline{w_i w_{i+1}}$ and positive real axis. Let $f'(z) = \lambda(z-a_1)^{-\alpha_1}\cdots(z-a_n)^{-\alpha_n}$. For some argument, $\alpha_i\pi$ must be the exterior angle of vertex w_i of P. Then $\sum\limits_{i=1}^n \alpha_i\pi = 2\pi \rightsquigarrow \sum\limits_{i=1}^n \alpha_i = 2$. Notice that $f'(z)$ is holomorphic on the region Ω by deleting n cuts i.e.

  $$\Omega = \mathbb{C} \setminus \{\mathrm{Im}\, z \le 0 \text{ and } \mathrm{Re} z = a_i \text{ for some } i\}$$

  Fix $b \in \Omega$, $\forall z \in \Omega$,

  $$f(z) = \int_b^z (\xi - a_1)^{-\alpha_1}(\xi - a_2)^{-\alpha_2}\cdots(\xi - a_n)^{-\alpha_n} d\xi$$

  is holomorphic on Ω, since Ω is simply connected.

  Claim : $f(z)$ is continuous at a_i $\forall i = 1, ..., n$

  subproof : The only part of $f'(z)$ is not holomorphic at a_i is the term $(z-a_i)^{-\alpha_i}$. The other terms are holomorphic. Fix i and let

  $$\phi(z) = (z-a_1)^{-\alpha_1}\cdots\widehat{(z-a_i)^{-\alpha_i}}\cdots(z-a_n)^{-\alpha_n}$$

  Then $\phi(z)$ is holomorphic at $a_i \rightsquigarrow \phi(z) = \phi(a_i) + (z-a_i)\psi(z)$, where $\psi(z)$ is holomorphic near a_i. Then

  $$f'(z) = (z-a_i)^{-\alpha_i}\phi(z) = \phi(a_i)(z-a_i)^{-\alpha_i} + (z-a_i)^{1-\alpha_i}\psi(z)$$

  Notice that $0 < \alpha_i < 1$, then first term have primitive $\dfrac{\phi(a_i)(z-a_i)^{1-\alpha_i}}{1-\alpha_i}$ and the second term is holomorphic near a_i, so $f(a_i)$ exists and f continuous at a_i.

Definition 1.15.1. The **Schwarz-Christoffel integral** is given by

$$S(z) = \int_0^z \frac{d\xi}{(\xi - a_1)^{\alpha_1}\cdots(\xi - a_n)^{\alpha_n}}$$

where $a_1 < a_2 < \cdots < a_n$ and $\alpha_i < 1$ $\forall i$ and $1 < \sum\limits_{i=1}^n \alpha_i$

- $\Omega := \mathbb{C} \setminus \bigcup_{k=1}^{n} \{a_k + iy : y \leq 0\} \rightsquigarrow S(z)$ is holomorphic on Ω.

- For large $|\xi|$, $\exists c > 0$ s.t. $\left|\prod_{k=1}^{n}(\xi - a_k)^{-\alpha_k}\right| \leq c|\xi|^{-\sum_{k=1}^{n}\alpha_k}$. Since $\sum_{k=1}^{n}\alpha_k > 1 \rightsquigarrow |z| \gg 0$

$$\int_2^\infty \frac{1}{|\xi|^{\sum a_k}} d\xi \text{ conv.} \rightsquigarrow \int_{iy}^{i\infty} \frac{1}{|\xi|^{\sum a_k}} d\xi \text{ conv.}$$

By Cauchy theorem, $\lim_{r\to\infty} S(re^{i\theta}) = w_\infty$ converge independent on θ. When $\sum_{i=1}^{n}\alpha_i \in (1, 2)$, $S(\mathbb{H})$ is the polygon with verties $w_1, ..., w_n, w_\infty$. If $\sum_{i=1}^{n}\alpha_i = 2$, $w_\infty \in \overline{w_1 w_n}$.

Lemma 1.15.1. Let $f : \mathbb{D} \to P$ be the conformal mapping, then $\exists \widetilde{f} : \overline{\mathbb{D}} \xrightarrow{\sim} \overline{P}$ s.t. $\partial\mathbb{D} \xrightarrow{\widetilde{f}} \partial P$.

Proof:

- $\forall z_0 \in \partial\mathbb{D}$, $r \in (0, 1/2)$, choose arbitrary $z_r, z_r' \in \mathbb{D} \cap C_r(z_0)$ and let $\rho(r) = |f(z_r) - f(z_r')|$.

  Claim : $\exists r_n \to 0$ s.t. $\rho(r_n) \to 0$.

  subproof : If not, $\exists c > 0$ and $R \in (0, 1/2)$ s.t. $\rho(r) \geq c$ $\forall 0 < r \leq R$. Let γ_r be the arc on $C_r(z_0)$ connected z_r, z_r', then

$$\begin{aligned} c \leq \rho(r) = |f(z_r) - f(z_r')| &= \left|\int_{\gamma_r} f'(z)dz\right| = \left|\int_{\theta_r}^{\theta_r'} f'(z)rd\theta\right| \\ &\leq \int_{\theta_r}^{\theta_r'} |f'(z)r|d\theta \leq \left(\int_{\theta_r}^{\theta_r'} |f'(z)|^2 rd\theta\right)^{1/2} \left(\int_{\theta_r}^{\theta_r'} rd\theta\right)^{1/2} \end{aligned}$$

  The last inequality is from Cauchy Schwarz inequality. Hence,

$$\frac{c^2}{2\pi r} \leq 2\pi \int_{\theta_r}^{\theta_r'} |f'(z)|^2 rd\theta$$

  Integral r from 0 to R in both side, we have

$$\int_0^R \frac{c^2}{2\pi r}dr \leq \int_0^R \int_{\theta_r}^{\theta_r'} |f'(z)|^2 rd\theta dr \leq \int_{\mathbb{D}\cap B_R(z_0)} |f'(z)|^2 dxdy = \text{Area}(\mathbb{D} \cap B_R(z_0)) < \infty$$

  The last equality is follow from

$$\text{Area}(f(\Omega)) = \iint_{f(\Omega)} dxdy = \iint_\Omega \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dxdy = \iint_\Omega |f'(z)|^2 dxdy$$

- $\lim_{z\to z_0} f(z)$ exists : If not, $\exists \lim_{n\to\infty} z_n = \lim_{n\to\infty} z_n' = z_0$ but $q_1 = \lim_{n\to\infty} f(z_n) \neq \lim_{n\to\infty} f(z_n') = q_2 \rightsquigarrow$ $q_1, q_2 \in \partial P$. If not, $\exists B_r(q_1) \subset P \rightsquigarrow f^{-1}(B_r(q_1)) \subset \mathbb{D} \rightsquigarrow z_n \not\to q_1$ $(\rightarrow\!\leftarrow)$. Exists ρ_1, ρ_2 s.t. for $n \geq n_0$, $F(z_n) \in B_{\rho_1}(q_1), F(z_n') \in B_{\rho_2}(q_2)$ s.t. $\inf_{\substack{x\in B_{\rho_1}(q_1) \\ y\in B_{\rho_2}(q_2)}} |x - y| = d > 0$. Construct the curve $\Gamma_1 : [0, 1] \to B_{\rho_1}(q_1)$ by $\Gamma_1(0) = f(z_{n_0}), \Gamma_1(1 - 2^{-k}) = f(z_{n_0+k})$ and connect z_{n_0+i}, z_{n_0+i+1} by curve in $B_{\rho_1}(z_1) \cap P$ (since it is path connected), then the end point of Γ_1 is q_1. Similarly, construct Γ_2. Let $\gamma_1 = F^{-1}(\Gamma_1)$, $\gamma_2 = F^{-1}(\Gamma_2) \rightsquigarrow z_n \in \gamma_1, z_n' \in \gamma_2$ $\forall n \geq n_0$. For r sufficiently small, $\exists z_r \in C_1(z_0) \cap \gamma_1 \neq \varnothing$, $z_r' \in C_2(z_0) \cap \gamma_2 \neq \varnothing$, but $|f(z_r) - f(z_r')| > d$ $(\rightarrow\!\leftarrow)$.

- Define $f(z_0) := \lim_{z \to z_0} f(z)\ \forall z \in \partial\mathbb{D} \rightsquigarrow f$ is continuous on $\partial\mathbb{D}$:

  For $z_0 \in \partial\mathbb{D}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z \in \mathbb{D}$ with $|z - z_0| < \delta \rightsquigarrow |f(z) - f(z_0)| < \varepsilon$. $\forall \varepsilon > 0$, $\forall z \in \partial\mathbb{D} \cap B_\delta(z_0)$, choose $w \in \mathbb{D} \cap B_\delta(z_0)$ s.t. $|f(z) - f(w)| < \varepsilon$, then

$$|f(z) - f(z_0)| \le |f(z) - f(w)| + |f(w) - f(z_0)| < 2\varepsilon$$

- Let $g = f^{-1}$ on $P \rightsquigarrow \widetilde{g} : \overline{P} \to \overline{\mathbb{D}}$ and $\partial P \to \partial\mathbb{D}$ and thus $\widetilde{g} = \widetilde{f}^{-1}$.

$\square$

Theorem 1.15.2 (Main theorem)**.** Given $F : \mathbb{H} \to P$ (n-polygon with vertices $w_1, ..., w_n$) $F : \underbrace{a_i}_{\in\mathbb{R}} \mapsto w_i$ a conformal map, $\exists c_1, c_2 \in \mathbb{C}$ s.t.

$$F(z) = c_1 \int_0^z \frac{d\xi}{(\xi - a_1)^{\alpha_1} \cdots (\xi - a_n)^{\alpha_n}} + c_2$$

with $\alpha_1 + \cdots + \alpha_n = 2$.

Proof: Let α_i be defined above.

- Consider $h(z) = (F(z) - w_k)^{1/(1-\alpha_k)}\ \forall z \in H = \{z \in \overline{\mathbb{H}} : a_{k-1} \le \mathrm{Re} z \le a_{k+1}\}$. Notice that

$$\frac{1}{1-\alpha_k} \arg(w_{k+1} - w_k) = \frac{\arg(w_k - w_{k-1}) + \alpha_k \pi}{1 - \alpha_k} = \frac{\arg(w_k - w_{k-1}) + \pi}{1 - \alpha_k} - \pi$$

  and thus

$$\exp\left(\frac{\arg(w_{k+1} - w_k)}{1 - \alpha_k}\right) = -\exp\left(\frac{\arg(w_k - w_{k-1}) + \pi}{1 - \alpha_k}\right) = -\exp\left(\frac{\arg(w_k - w_{k-1})}{1 - \alpha}\right)$$

  which show that $h([a_{k-1}, a_k])$ is a segment. We may assume $h : [a_{k-1}, a_k] \to \mathbb{R}$ by scaling and $h(\mathbb{H}) \subset \mathbb{H}$. By Schwarz reflection principle, h extend to $\widetilde{h} : \widetilde{H} = \{a_{k-1} \le \mathrm{Re} z \le a_{k+1}\} \to \mathbb{C}$.

- For $z \in H$, $\dfrac{F'(z)}{F(z) - w_k} = (1 - \alpha_k)\dfrac{h'(z)}{h(z)} \rightsquigarrow \begin{cases} F'(z) \neq 0 \implies h'(z) \neq 0 \\ \forall z \in H^-,\ \widetilde{h}(z) = \overline{h(\overline{z})} \rightsquigarrow \widetilde{h}'(z) \neq 0 \end{cases}$

  $\forall z \in (a_{k-1}, a_{k+1})$, (待補)

- $F' = (1 - \alpha_k) h^{-\alpha_k} h' \implies F'' = (1 - \alpha_k)(-\alpha_k h^{-\alpha - 1}(h')^2 + h^{-\alpha_k} h'')$. Note that $h(a_k) = 0$, $h'(a_k) \neq 0 \rightsquigarrow a_k$: zero of order 1.

$$\frac{F''}{F'} = -\alpha_k \frac{h'}{h} + \frac{h''}{h'}$$

$\square$

1.16 Harmonic functions and subharmonic function

Recall that :

- $u : \Omega \to \mathbb{R}$ is harmonic if $u \in C^2(\Omega)$ and $\Delta u = u_{xx} + u_{yy} = 0$
- If Ω is simply connected and $u \in \mathcal{H}(\Omega) \rightsquigarrow \exists v \in \mathcal{H}(\Omega)$ s.t. $f = u + iv$ is holomorphic in Ω.

Definition 1.16.1. $h : U \to \mathbb{R}$ continuous and $\overline{B_\rho(a)} \subset U$. The **anverage** of h on $C_r(a) \subseteq U$ is

$$A(r) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})d\theta \text{ for } r \in (0, \rho)$$

Fact 1.16.1. $\lim_{r\to 0} A(r) = h(a)$

Proof:

$$|A(r) - h(a)| = \left|\frac{1}{2\pi}\int_0^{2\pi} (h(a + re^{i\theta}) - h(a))d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|h(a + re^{i\theta}) - h(a)\right| d\theta \le M(r) \to 0$$

where $M(r)$ is from uniformly continuous on compact set. □

Proposition 1.16.1 (Mean value property)**.** If u is harmonic, then

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta$$

Proof: By Green theorem,

$$\int_{C_\rho(a)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = \iint_{\overline{B_\rho(a)}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy = 0$$

Parameterize $C_\rho(a)$ into $x = a + r\cos\theta$, $y = a + r\sin\theta$. Then

$$0 = r\int_0^{2\pi} (u_x \cos\theta + u_y \sin\theta)d\theta = r\int_0^{2\pi} \frac{\partial}{\partial r}u(a + re^{i\theta})d\theta$$

$$\implies \frac{d}{dr}\left(\int_0^{2\pi} \frac{\partial}{\partial r}u(a + re^{i\theta})d\theta\right) = \int_0^{2\pi} \frac{\partial u}{\partial r}(a + re^{i\theta}) = 0$$

and thus $A(r)$ is constant on $(0, \rho)$. By fact 1.16.1, this constant is $u(a)$. □

Proposition 1.16.2. (Maximal principle) Let $u \in \mathcal{H}(\Omega)$ and $u(z) \le M$ on Ω. If $\exists u \in \Omega$ s.t. $u(a) = M$, then u is constant.

Proof: Let $V = \{z \in \Omega : u(z) = M\} \ne \varnothing$ by assumption, which is a closed set. For $b \in V$, let $B_\rho(b) \subset \Omega$, then

$$u(b) = \frac{1}{2\pi}\int_0^{2\pi} u(b + re^{i\theta})d\theta \text{ for } 0 < r < \rho$$

$$\implies 0 = \frac{1}{2\pi}\int_0^{2\pi} u(b + re^{i\theta}) - u(b)d\theta \le 0 \rightsquigarrow u(b + re^{i\theta}) = u(b) \in V \ \forall\theta$$

Then V is open. By Ω is connected, f is constant. □

Remark 1.16.1. Mean value property (M-V) $\implies$ Maximal principle (M-P).

Definition 1.16.2. u is called **C-harmonic** in Ω if $u \in \mathcal{H}(\Omega)$ and $u \in C^0(\overline{\Omega})$ and denoted by $u \in c\mathcal{H}(\Omega)$

Corollary 1.16.1. If $u_1, u_2 \in c\mathcal{H}(\Omega)$ with Ω : bounded and $u_1 \equiv u_2$ on $\partial\Omega$, then $u_1 \equiv u_2 \in \Omega$.

Proof: Let $u := u_1 - u_2$ attain max 0 on $\partial\Omega$, by M-P, $u_1 \leq u_2$ in Ω. Similarly, $u_2 \leq u_1$ in Ω and hence $u_1 \equiv u_2$ in $\overline{\Omega}$. $\square$

Theorem 1.16.1. If $u \in C^0(\overline{\mathbb{D}})$, define

$$P_u = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} u_0(e^{i\theta})d\theta$$

Then $P_u \in c\mathcal{H}(\mathbb{D})$ and $P_u = u$ on $P_u = u$ on $C_1(0)$.

Proof:

- Recall that $\dfrac{1-|z|^2}{|e^{i\theta}-z|^2} = \operatorname{Re}\left(\dfrac{e^{i\theta}+z}{e^{i\theta}-z}\right)$. Let

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})d\theta \text{ in } \mathbb{D} \rightsquigarrow g \text{ is continuous in } \mathbb{D}$$

  $\forall \gamma$: closed curve in $\mathbb{D}$,

$$\int_\gamma g(z)dz = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left(\int_\gamma \frac{e^{i\theta}+z}{e^{i\theta}-z}dz\right)d\theta = 0$$

  since $\dfrac{e^{i\theta}+z}{e^{i\theta}-z}dz$ analytic in $\mathbb{D}$. By Morera's theorem, g is holomorphic on $\mathbb{D} \rightsquigarrow P_u = \operatorname{Re} g$ is harmonic in $\mathbb{D}$.

- $\lim\limits_{z\to e^{i\theta}} P_u(z) = u(e^{i\theta})$: Substitute $z = re^{i\psi}$ in $\dfrac{1-|z|^2}{|e^{i\theta}-z|^2}$, it equal to $\dfrac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}$, which is called **Poisson kernel** and denoted by $k(\theta, z)$.

  - $k(\theta,z) > 0 : 1-2r\cos\theta + r^2 \geq (r-1)^2 > 0$
  - Since $w \equiv 1 \in \mathcal{H}(\mathbb{D})$, by Poisson formula,

$$1 = w(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} w(e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} k(\theta,z)d\theta$$

  - For $\delta > 0$, if $|\theta-\psi| > \delta$, then in $k(\theta,z)$, the denominator $\geq 1-2r\cos\delta + r^2$ and the numerator $\to 0$ as $z \to \partial\mathbb{D}$. Notice that

$$|P_u(z) - u(e^{i\psi})| = \left|\frac{1}{2\pi}\int_0^{2\pi} k(\theta,z)\left(u(e^{i\theta}) - u(e^{i\psi})\right)d\theta\right|$$

    Let $M = \max\limits_\theta |u(e^{i\theta})|$. Since u_0 is continuous on $\overline{\mathbb{D}}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|\theta-\psi| < \delta \implies |u(e^{i\theta}) - u(e^{i\psi})| < \varepsilon$. So

$$|P_u(z) - u(e^{i\psi})| \leq \underbrace{\frac{M}{\pi}\int_{|\theta-\psi|\geq\delta} k(\theta,z)d\theta}_{\to 0 \text{ as } z\to\partial\mathbb{D}} + \underbrace{\frac{1}{2\pi}\int_{\psi-\delta}^{\psi+\delta} \varepsilon k(\theta,z)d\theta}_{\leq\varepsilon}$$

    Hence, $\lim\limits_{z\to e^{i\psi}} P_u(z) = u(e^{i\psi})$. $\square$

Theorem 1.16.2. Let $u \in C^0(\Omega)$. Then u satisfy M-V $\iff u \in \mathcal{H}(\Omega)$.

Proof:

- $(\Leftarrow)$: OK!
- $(\Rightarrow)$: Pick $v \in \mathcal{H}(\Omega) \rightsquigarrow u - v$ satisfy M-V $\rightsquigarrow u - v$ satisfy M-P. Now, let $\overline{B_\rho(a)} \subseteq \Omega$ and $u_0 = u|_{\overline{B_\rho(a)}}$. For simplicity, we may assume $a = 0$, $\rho = 1$. Let $v = P_{u_0}$, then by lemma, $u_0 - v, v - u_0$ satisfy M-P and $u_0 \equiv v$ on $C_1(0) \rightsquigarrow u_0 \equiv v$ on $\overline{\mathbb{D}}$ and u_0 is harmonic and thus u is harmonic.

$\square$

Remark 1.16.2. The condition of M-V only require locally.

Theorem 1.16.3 (Harnack's inequality)**.** Let $u \in \mathcal{H}(\overline{B_\rho(0)})$ and $z = re^{i\theta}$, $r < \rho$. If $u(z) \geq 0$, then

$$\frac{\rho - r}{\rho + r} u(0) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(0)$$

Proof: Notice that $\dfrac{\rho - r}{\rho + r} \leq \dfrac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} \leq \dfrac{\rho + r}{\rho - r}$ if $|z| = r$. Since $u(z) \geq 0$,

$$\frac{\rho - r}{\rho + r} u(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta \leq \frac{\rho + r}{\rho - r} u(0)$$

as the result. $\square$

Theorem 1.16.4 (Harnack's principle)**.** Let $u_n \in \mathcal{H}(\Omega_n)$ $\forall n \in \mathbb{N}$. Let Ω be a region s.t. $\forall z_0 \in \Omega$, $z_0 \in \Omega_n$ for almost n, and $u_n(z) \leq u_{n+1}(z)$ for almost n near z_0. Then $u_n \xrightarrow{\text{unif}} \infty$ or $u_n \xrightarrow{\text{unif}} u \in \mathcal{H}(\Omega)$.

Proof:

- If $z_0 \in \Omega$ s.t. $u_n(z_0) \to \infty$ as $n \to \infty$: $\exists r > 0, n_0 > 0$ s.t. $\{u_n(z)\}_{n=n_0}^\infty$ is non-decreasing $\forall z \in B_r(z_0)$. $\forall n > n_0$, let $v_n := u_n - u_{n_0} > 0$. For $z \in \overline{B_{r/2}(z_0)}$,

$$v_n(z) \geq \frac{1}{3} v_n(z_0) \to \infty \rightsquigarrow u_n(z) \to \infty$$

- If $z_0 \in \Omega$ s.t. $\lim\limits_{n\to\infty} u_n(z_0) < \infty$: Similarly, $v_n(z) \leq 3v_n(z_0) < \infty$ $\forall z \in \overline{B_{r/2}(z_0)} \rightsquigarrow \lim\limits_{n\to\infty} u_n(z) < \infty$ $\forall z \in \overline{B_{r/2}(z_0)}$.
- Let $U = \{z \in \Omega : u_n(z) \to \infty \text{ as } n \to \infty\}$, which is open. Also $\Omega \setminus U = \{z \in \Omega : \lim\limits_{n\to\infty} u_n(z) < \infty\}$, which is also open. By Ω is simply connected, one of $U, \Omega \setminus U$ is empty set.
  - If $U \neq \varnothing$, then $U = \Omega$. On compact subset, the convergence is uniformly by usual argument.
  - If $U = \varnothing$: $\forall n > n_0$,

  $$u_{m+n}(z) - u_n(z) \leq 3(u_{m+n}(z_0) - u_n(z_0)) < M \ \forall z \in \overline{B_{r/2}(z_0)}$$

  where M is independent on m, so $u_n \to u$ on $\overline{B_{r/2}(z_0)}$ is uniformly. Consider

  $$u_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u_n(z + \rho e^{i\theta}) d\theta \xrightarrow{n\to\infty} u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + \rho e^{i\theta}) d\theta \ \forall \rho \leq \frac{r}{2}$$

By Theorem 1.16.2, $u \in \mathcal{H}(\Omega)$.

$\square$

Definition 1.16.3.

- $u \in C^2(\Omega)$ is **subharmonic** if $\Delta u \geq 0$ in Ω.
- $u \in C^0(\overline{\Omega})$ is **subharmonic** if $\forall z \in \Omega$, $\exists \rho > 0$ s.t. $\overline{B_\rho(a)} \subset \Omega$ and $\forall r \leq \rho\ \forall h \in c\mathcal{H}(B_r(a))$ with $u \leq h$ on $C_r(a) \implies u \leq h$ in $B_r(a)$.

And denoted by $u \in \mathcal{S}h(\Omega)$.

Fact 1.16.2. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then definition above equivalent.

Proof:

- $(\Rightarrow)$: $\Delta(u) \geq 0 = \Delta(h) \rightsquigarrow \Delta(u-h) \geq 0$.

  Claim : $\max\limits_{\overline{B_\rho(a)}}(u-h) = \max\limits_{C_\rho(a)}(u-h)$:

  subproof : Let $v = u - h$. Define $\widetilde{v} = v(z) + \varepsilon e^x$ for $\varepsilon > 0$. Since $\Delta e^x = e^x > 0$, $\Delta\widetilde{v} > 0$. Assume $\exists z_0 \in B_\rho(a)$ s.t. $\widetilde{v}(z_0) = \max\limits_{\overline{B_\rho(a)}} \widetilde{v} \rightsquigarrow \nabla\widetilde{v}(z_0) = 0$ and the Hessian $\begin{pmatrix} \widetilde{v}_{xx} & \widetilde{v}_{xy} \\ \widetilde{v}_{yx} & \widetilde{v}_{yy} \end{pmatrix}(z_0)$ is negative semi-definite, then the trace $\Delta\widetilde{v}(z_0) = \widetilde{v}_{xx} + \widetilde{v}_{yy} \leq 0$ $(\rightarrow\!\leftarrow)$. So $\forall z \in \overline{B_\rho(a)}$,

$$v(z) < \widetilde{v}(z) \leq \max_{C_\rho(a)} \widetilde{v} \leq \max_{C_\rho(a)} v + \varepsilon \max_{C_\rho(a)} e^x \xrightarrow{\varepsilon\to 0} v(z) \leq \max_{C_\rho(a)} v$$

$\square$

- $(\Leftarrow)$: If not, say $\Delta u(z_0) < 0$ for some $z_0 \in \Omega \rightsquigarrow \Delta u(z) < 0$ for $z \in B_\rho(z_0) \subset \Omega$. Let $v = P_u$ in $B_\rho(z_0) \rightsquigarrow v \in \mathcal{H}(B_\rho(z_0))$ and $v = u$ on $C_\rho(z_0)$. Since $\Delta(v-u) = -\Delta u > 0$, by above, $v - u$ satisfy M-P on $B_\rho(z_0)$, $v < u$ in $B_\rho(z_0)$ or $v = u$ in $B_\rho(z_0)$, but $v \geq u$ by assumption or $\Delta u = \Delta v = 0$ $(\rightarrow\!\leftarrow)$.

$\square$

Theorem 1.16.5. $u \in C^0(\Omega)$. Then u is subharmonic $\iff \forall z \in \Omega$, $\exists \rho > 0$ s.t. $\overline{B_\rho(z)} \subseteq \Omega$ and

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

for all $r \leq \rho$ (mean value inequality).

Proof:

- $(\Rightarrow)$: For all $z_0 \in \Omega$, let ρ be given by definition of subharmonic. For all $r \leq \rho$, define $h = P_{u|_{C_r(z_0)}} \rightsquigarrow h \in c\mathcal{H}(B_r(z_0))$, $u = h$ on $C_r(z_0)$. Since u is subharmonic, $u \leq h$ for all $z \in \overline{B_r(z_0)}$ and thus

$$u(z_0) \leq h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

- ($\Leftarrow$) : For $a \in \Omega$ and ρ be corresponding radius in assumption. If $\exists r \leq \rho$ and $h \in c\mathcal{H}(B_r(a))$ with $u \leq h$ on $C_r(a)$, but $\exists z_0 \in B_r(a)$ s.t. $u(z_0) > h(z_0)$. Let $M = \max\limits_{B_r(a)}(u - h) \rightsquigarrow M \geq u(z_0) - h(z_0) > 0$ and let
$$V = \{z \in \overline{B_r(a)} : u(z) - h(z) = M\}$$
Then $V \cap C_\rho(a) = \varnothing$ i.e. $V \subset B_r(a)$. Since V is closed, $\partial V \subset V \subset B_r(a)$. Pick $z_1 \in \partial V$ and $s > 0$ small enough which can apply mean valued inequality s.t. $B_s(z_1) \subset B_r(a)$ but $C_s(z_1) \not\subset V$, which means there exists arc of $C_s(z_1)$ s.t. $u - h < M$ on that arc. Then
$$M = u(z_1) - h(z_1) \leq \frac{1}{2\pi}\int_0^{2\pi} \left(u(z_1 + se^{i\theta}) - h(z_1 + se^{i\theta})\right) d\theta < M \ (\rightarrow\leftarrow)$$

□

Theorem 1.16.6. If $u \in \mathcal{S}h(\Omega)$ and any open connect subset Ω' of Ω, then u is constant on Ω' or u doesn't attain maximum in Ω'.

Proof: If u attain maximum M in Ω', then $V = \{z \in \Omega' : u(z) = M\}$ is a closed set. For all $z_0 \in V$, by Theorem 1.16.5, $\exists\rho$ s.t. $\overline{B_\rho(z_0)} \subset \Omega'$ and u satisfy mean value inequality on $C_r(z_0) \forall r \leq \rho$ and thus
$$M = u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} M d\theta = M \ \forall r \leq \rho$$
i.e. $u(z_0 + re^{i\theta}) = u(z_0)\ \forall\theta \in [0, 2\pi], \forall r \leq \rho \rightsquigarrow B_\rho(z_0) \subset V \rightsquigarrow V$ is clopen. Since Ω' is connected and $V \neq 0 \rightsquigarrow V = \Omega'$ i.e. u is constant on Ω'. □

Corollary 1.16.2. If $u \in C^0(\overline{\Omega})$, then TFAE

(1) $\forall z \in \Omega$, $\exists\rho > 0$ s.t. $\overline{B_\rho(a)} \subset \Omega$ and $\forall r \leq \rho$, $\forall h \in c\mathcal{H}(B_r(a))$ with $u \leq h$ on $C_r(a) \implies u \leq h$ in $B_r(a)$.

(2) $\forall\overline{B_\rho(a)} \subset \Omega$ and $\forall h \in c\mathcal{H}(B_\rho(a))$ with $u \leq h$ on $C_\rho(a) \implies u \leq h$ in $B_\rho(a)$.

(3) $\forall z \in \Omega$, $\exists\rho > 0$ s.t. u satisfy mean value inequality on $C_r(z)\ \forall r \leq \rho$.

(4) $\forall\overline{B_\rho(a)} \subset \Omega$, u satisfy mean value inequality on $B_\rho(a)$.

Proof:

- (1) $\iff$ (3) : By Theorem 1.16.5
- (3) $\Rightarrow$ (2) : $\forall\overline{B_\rho(a)} \subset \Omega$ and $\forall h \in c\mathcal{H}(B_\rho(a))$ with $u \leq h$ on $C_\rho(a)$. If $u - h$ is constant on $\overline{B_\rho(a)}$, then $u - h = u(z) - h(z) \leq 0\ \forall z \in C_\rho(a)$. If not, since $\overline{B_\rho(a)}$ is compact and by Theorem 1.16.6, the maximal of $u - h$ in $\overline{B_\rho(a)}$ attain at point on $C_\rho(a)$ and thus $u - h \leq \max\limits_{C_\rho(a)}(u - h) \leq 0$ on $B_\rho(a)$.
- (2) $\iff$ (4) : Apply the proof in Theorem 1.16.5.
- (4) $\Rightarrow$ (3) : Trivial.

□

Property 1.16.1.

- $u_1, u_2 \in \mathcal{S}h(\Omega) \rightsquigarrow u_1 + ku_2 \in \mathcal{S}h(\Omega)$ for $k > 0$: Since $u_1 + ku_2$ satisfy mean value inequality.
- $u_1, u_2 \in \mathcal{S}h(\Omega) \rightsquigarrow u = \max(u_1, u_2) \in \mathcal{S}h(\Omega)$: Since u satisfy mean value inequality.
- If $v \in \mathcal{S}h(\Omega)$ and $u = \begin{cases} P_v & \text{in } B_\rho(a) \\ v & \text{in } \Omega \setminus B_\rho(a) \end{cases}$, then $u \in \mathcal{S}h(\Omega)$:

  $P_v \in c\mathcal{H}(B_\rho(a)) \rightsquigarrow u \in C^0(\Omega)$ and u satisfy mean value inequality, since $v \le P_v$ in $B_\rho(a)$.

1.17 Dirichlet problem

1.17.1 Uniqueness of Dirichlet problem

Dirichlet problem : Let Ω be a bounded region and $h \in C^0_{\mathbb{R}}(\partial\Omega)$ (continuous real function). Find $u \in c\mathcal{H}(\Omega)$ s.t. $u = h$ on $\partial\Omega$.

Theorem 1.17.1 (uniqueness of Dirichlet problem)**.** Exists at most one solution of u.

Proof:

- $h \equiv 0 \rightsquigarrow u \equiv 0$: Apply M-P on u and $-u$, we get $u \le 0$ and $-u \le 0$ in $\overline{\Omega} \rightsquigarrow u \equiv 0$ in $\overline{\Omega}$.
- $u_1, u_2 \in c\mathcal{H}(\Omega)$ with $u_1 = u_2 = h$ on $\partial\Omega \rightsquigarrow u_1 - u_2 \equiv 0$ on $\partial\Omega$ and thus $u_1 - u_2 \equiv 0$ in $\overline{\Omega}$.

$\square$

Remark 1.17.1. Existence for $\Omega = \overline{\mathbb{D}}$: $h \in C^0_{\mathbb{R}}(C_1(0)) \rightsquigarrow u = P_h \in c\mathcal{H}(\mathbb{D})$ and $u = h$ on $C_1(0)$.

Example 1.17.1. Let $h(e^{i\varphi}) = \begin{cases} 0 & \text{if } \phi \in (0, \pi) \\ 1 & \text{if } \phi \in (\pi, 2\pi) \end{cases}$, then for $z = re^{i\psi}$

$$\begin{aligned} u(z) &= \frac{1}{2\pi} \int_0^{2\pi} k(\theta, z) h(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_\pi^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \psi) + r^2} d\theta \\ &= \frac{1}{\pi} \left(\tan^{-1} \left(\frac{1+r}{1-r} \tan \left(\frac{\theta - \psi}{2} \right) \right) \right) \Bigg|_\pi^{2\pi} = \frac{1}{\pi} \tan^{-1} \left(\frac{1 - r^2}{2r \sin\psi} \right) \end{aligned}$$

1.17.2 Existence of Dirichlet problem

Perron method : f : bounded function on $\Gamma = \partial\Omega$, define the **Perron family**

$$\mathcal{F}_f = \{ v \in \mathcal{S}h(\Omega) : \overline{\lim_{z \to \xi}} v(z) \le f(\xi), \ \forall \xi \in \Gamma \}$$

is noempty, since if $|f| \le M \rightsquigarrow -M \in \mathcal{F}_f$.

Lemma 1.17.1. $u(z) := \sup_{v \in \mathcal{F}_f} v(z)$ is harmonic in Ω.

Proof:

- $v \in \mathcal{F}_f \rightsquigarrow v \le M \rightsquigarrow u \le M$: For $v \in \mathcal{F}_f$ and $\varepsilon > 0$, let

  $$E_\varepsilon = \{ z \in \overline{\Omega} : v(z) \ge M + \varepsilon \}$$

  is closed and bounded $\rightsquigarrow E_\varepsilon$ is compact. If $E_\varepsilon \ne \varnothing$, then $\exists$ max of v in E_ε and thus in Ω ($\longrightarrow\!\!\!\!\!\!\leftarrow$ by M-P). Hence, $E_\varepsilon = \varnothing$ $\forall \varepsilon > 0$, let $\varepsilon \to 0 \rightsquigarrow v \le M$.

- $\forall z_0 \in \Omega$, u is harmonic near z_0 : Let $\overline{B_\rho(z_0)} \subset \Omega$, by definition, $\exists \{v_n\} \subset \mathcal{F}_f$ s.t. $\lim_{n\to\infty} v_n(z_0) = u(z_0)$. Set $V_n = \max\{v_1, ..., v_n\} \rightsquigarrow V_n \in \mathcal{S}h(\Omega)$ and $\{V_n\}$: non-decreasing. Let $U_n = \begin{cases} P_{V_n} & \text{in } B_\rho(z_0) \\ V_n & \text{in } \Omega \setminus B_\rho(z_0) \end{cases} \rightsquigarrow U_n \in \mathcal{F}_f$ and $V_n \leq U_n$.
  - $v_n(z_0) \leq V_n(z_0) \leq U_n(z_0) \leq u(z_0) \xrightarrow{n\to\infty} \lim_{n\to\infty} U_n(z_0) = u(z_0)$. $\{U_n\} \subset c\mathcal{H}(B_\rho(z_0))$ non-decreasing, by Harnack principle, $U_n \xrightarrow{\overline{\text{unif}}} U \in c\mathcal{H}(B_\rho(z_0))$. $\forall z \in B_\rho(z_0)$, $U_n(z) - U_{n+1}(z) \leq 0$ and $U(z_0) = \lim_{n\to\infty} U_n(z_0) = u(z_0)$.
  - For $z_1 \in B_\rho(z_0)$, $\exists \{w_n\} \subset \mathcal{F}_f$ s.t. $\lim_{n\to\infty} w_n(z_1) = u(z_1)$. Let $\overline{w_n} = \max(v_n, w_n) \in \mathcal{F}_f$, then

$$\begin{cases} v_n(z_0) \leq \overline{w_n}(z_0) \leq u(z_0) \text{ and } v_n(z_0) \to u(z_0) & \implies \lim_{n\to\infty} v_n(z_0) = u(z_0) \\ w_n(z_1) \leq \overline{w_n}(z_1) \leq u(z_1) \text{ and } w_n(z_1) \to u(z_1) & \implies \lim_{n\to\infty} w_n(z_0) = u(z_1) \end{cases}$$

Let $W_n = \max\{\overline{w_1}, ..., \overline{w_n}\}$ and $Q_n = \begin{cases} P_{W_n} & \text{in } B_\rho(z_0) \\ W_n & \text{in } \Omega \setminus B_\rho(z_0) \end{cases}$. By Haenack principle, $Q_n \xrightarrow{\overline{\text{unif}}} Q \in c\mathcal{H}(B_\rho(z_0))$ and

$$v_n, w_n \leq \overline{w_n} \leq W_n \leq Q_n \leq u \implies \begin{cases} Q(z_0) = \lim_{n\to\infty} Q_(z_0) = u(z_0) \\ Q(z_1) = \lim_{n\to\infty} Q_n(z_1) = u(z_1) \end{cases}$$

Since $U_n \leq Q_n \leq u$, $U \leq Q \leq u$ in $B_\rho(z_0) \rightsquigarrow U - Q \leq 0$ in $B_\rho(z_0)$ and $U(z_0) = u(z_0) = Q(z_0)$. By M-P, $U \equiv Q$ in $B_\rho(z_0) \rightsquigarrow U(z_1) = Q(z_1) = u(z_1) \rightsquigarrow U = u$ in $B_\rho(z_0)$ i.e. $u \in \mathcal{H}(B_\rho(z_0))$. $\square$

Lemma 1.17.2 (barrior lemma). $z_0 \in \Gamma = \partial\Omega$. If $\exists w \in c\mathcal{H}(\Omega)$ s.t. $w(z_0) = 0$ and $w(z) > 0$ $\forall z \in \Gamma \setminus \{z_0\}$. If f is continuous at z_0, then $\lim_{z\to z_0} u(z) = f(z_0)$, where $u = \sup_{v\in\mathcal{F}_f} v(z)$.

Proof:

- $\overline{\lim_{z\to z_0}} u(z) \leq f(z_0)$: $\forall \varepsilon > 0$, $\exists B_\rho(z_0)$ s.t. $\forall z \in \Gamma \cap B_\rho(z_0)$, $|f(z) - f(z_0)| < \varepsilon$. Let $w_0 = \min_{\Gamma\setminus B_\rho(z_0)} w > 0$. Consider

$$V_+(z) = f(z_0) + \varepsilon + \frac{M - f(z_0)}{w_0} w(z)$$

  - $z \in \Gamma \cap B_\rho(z_0) : V_+(z) \geq f(z_0) + \varepsilon > f(z)$
  - $z \in \Gamma \setminus B_\rho(z_0) : V_+(z) \geq f(z_0) + \varepsilon + M - f(z_0) > f(z)$.

  By M-P, for $v \in \mathcal{F}_f$, $v(z) < V_+(z)$ $\forall z \in \Omega \rightsquigarrow u(z) \leq V_+(z)$ $\forall z \in \Omega$. Then

$$\overline{\lim_{z\to z_0}} u(z) \leq \overline{\lim_{z\to z_0}} V_+(z) = f(z_0) + \varepsilon$$

- $\underline{\lim_{z\to z_0}} u(z) \geq \varepsilon$: $\forall \varepsilon > 0$, consider

$$V_-(z) = f(z_0) - \varepsilon - \frac{M + f(z_0))}{w_0} w(z)$$

- •• $z \in \Gamma \cap B_\rho(z_0) : V_+(z) \le f(z_0) - \varepsilon < f(z)$
- •• $z \in \Gamma \setminus B_\rho(z_0) : V_+(z) \le f(z_0) - \varepsilon - M - f(z_0) < f(z)$

Then $V_- \in \mathcal{F}_f \rightsquigarrow V_- \le u \rightsquigarrow \underline{\lim}_{z\to z_0} u(z) \ge \underline{\lim}_{z\to z_0} V_-(z) = f(z_0) - \varepsilon$

□

Theorem 1.17.2 (Existence of Dirichlet problem)**.** If $\forall z_0 \in \Gamma$, $\exists$ line segment $\subseteq \mathbb{C} \setminus \overline{\Omega}$, but z_0 is an endpoint if this line segment, then the Dirichlet problem can be solved for Ω and h.

Proof: It is suffices to show that for all $z_0 \in \Gamma$, there exists $w \in c\mathcal{H}(\Omega)$ s.t. $w(z_0) = 0$ and $w(z) > 0\ \forall z \in \Gamma \setminus \{z_0\}$. Then the result follow from Lemma 1.17.2 and Lemma 1.17.1. Now, for $z_0 \in \Gamma$, say $\overline{z_1 z_0}$ is that line segment in assumption. By rotation and translation, we may assume $z_0 = 0$, $z_1 = -a < 0$. Notice that $\dfrac{z}{z+a}$ has a holomorphic branch $g(z)$ defined on $\mathbb{C} \setminus [-a, 0]$, since $\dfrac{z}{z+a} = -x \in \mathbb{R}_{\le 0} \iff z \in [-a, 0]$. Consider

$$\sqrt{\frac{z}{z+a}} = \left|\frac{z}{z+a}\right|^{1/2}\left(\cos\left(\frac{1}{2}\arg\frac{z}{z+a}\right) + i\sin\left(\frac{1}{2}\arg\frac{z}{z+a}\right)\right)$$

Then $w(z) = \mathrm{Re}\sqrt{\dfrac{z}{z+a}} \in c\mathcal{H}(\Omega)$ is the barrier, since

$$\frac{1}{2}\arg\frac{z}{z+a} \in (-\pi/2, \pi/2) \implies \mathrm{Re}\frac{z}{z+a} > 0 \text{ for } z \in \Gamma \setminus \{0\}$$

□

1.17.3 harmonic measure

Definition 1.17.1. $\Omega \subset \mathbb{C}$ is of connectivity $n > 1$ if $\widetilde{\mathbb{C}} \setminus \Omega = E_1 \sqcup \cdots \sqcup E_n$ with $\infty \in E_n$ with E_i : simply connected.

Let $C_i = \partial E_i$ with C_n is counterclockwise and C_i is clockwise $\forall i = 1, ..., n-1$. Let $\partial\Omega = C = C_1 + \cdots + C_n$. Since $\widetilde{\mathbb{C}} \setminus E_n$ is simply connected and contain in $\mathbb{C}$, by Riemann mapping theorem, $\varphi_0 : \widetilde{\mathbb{C}} \setminus E_n \to \mathbb{D}$ and send $C_n \to C_1(0)$. Apply Riemann mapping on $\widetilde{\mathbb{C}} \setminus \varphi(E_1)$, say $\varphi : \widetilde{\mathbb{C}} \setminus \varphi(E_1) \to \widetilde{\mathbb{C}} \setminus \mathbb{D}$, which send $C_1 \to C_1(0)$ and $C_1(0) \subset \widetilde{\mathbb{C}} \setminus \varphi(E_1)$ will send to a smooth curve. Continue this method, Ω will conformal to a region with smooth boundary.

Definition 1.17.2. Harmonic measure $w_k(z)$ of C_k w.r.t. Ω is

$$\begin{cases} \Delta w_k = 0 \text{ in } \Omega \rightsquigarrow w_k \in \mathcal{H}(\Omega) \\ w_k \equiv \delta_{ik} \text{ on } C_i \end{cases}$$

Then $0 < w_k < 1$ in Ω and $w_1 + \cdots + w_n \equiv$ on C.

Remark 1.17.2. The existence of harmonic measure is given by Dirichlet problem. Since we can use Riemann mapping theorem send C_i to circle. Apply Schwarz reflection principle, w_i can be extend cross $C_i \rightsquigarrow w_i \in \mathcal{H}(\Omega')$ with $\Omega' \supset \Omega$.

Proposition 1.17.1. $\lambda_1 w_1 + \cdots + \lambda_{n-1} w_{n-1}$ ($\lambda_i \in \mathbb{R}$) has single-valued conjugate, then $\lambda_i = 0\ \forall i$.

Proof: If $\lambda_1 w_1 + \cdots + \lambda_{n-1} w_{n-1}$ with f : analytic in Ω, then f extends to $\Omega' \subset \Omega$. Then $\mathrm{Re}(f)|_{C_i} = \lambda_i \ \forall i = 1, ..., n-1$ and $\mathrm{Re}(f)|_{C_n} = 0$ i.e. each C_i is mapped to a vertical line segment. Let a not on any vertical line segment, then $\arg(f-a)$ is defined (single value) on each C_i. Recall that

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_\gamma d\log(z-a) = \frac{1}{2\pi i}\left(\int_\gamma d\log|z-a| + \int_\gamma d\arg(z-a)\right)$$

Since $\arg(f-a)$ is single value on $f(C_i)$, we have

$$\int_{f(C_i)} \frac{dw}{w-a} = \int_{C_i} d\log|f-a| + \int_{C_i} d\arg(f-a) = 0 \implies \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)dz}{f(z)-a} = 0$$

Then $a \notin f(\Omega)$. Then $f(\Omega') \in \{\lambda_1, ..., \lambda_{n-1}, 0\}$, but f is continuous, it must be constant and thus $f \equiv 0$ on $\Omega' \rightsquigarrow \lambda = 0 \ \forall i$. $\square$

Observation : $u \in \mathcal{H}(\Omega)$, if exists $v \in \mathcal{H}(\Omega)$ s.t. $f = u + iv$ is holomorphic in Ω, then

$$\begin{cases} du = \dfrac{\partial u}{\partial x}dx + \dfrac{\partial u}{\partial y}dy \\ dv = \dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy = -\dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial x}dy \end{cases}$$

In general, u is no single-value conjugate function, and in these circumstances it is better not use the notation dv. Instead we write

$$^*du = -\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}dy$$

and call the **conjugate differential** of du.

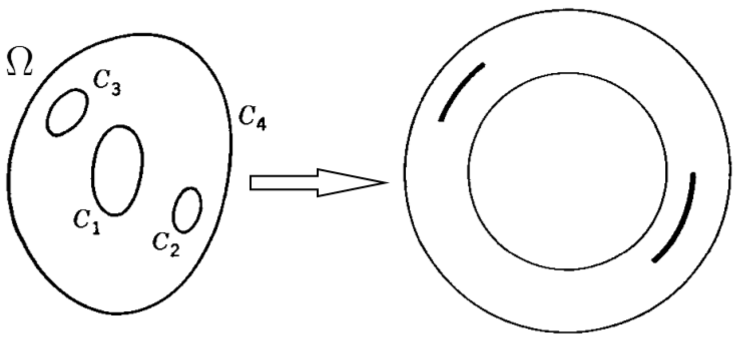
Example 1.17.2. If γ is smooth defined by $z = z(t)$ and let $\alpha = \arg z'(t)$, then

$$dx = |dz|\cos\alpha, \ dy = |dz|\sin\alpha$$

Let $\beta = \alpha - \dfrac{\pi}{2}$, then

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos\beta + \frac{\partial u}{\partial y}\sin\beta = \frac{\partial u}{\partial x}\frac{dy}{|dz|} + \frac{\partial u}{\partial y}\left(-\frac{dx}{|dz|}\right) \rightsquigarrow \frac{\partial u}{\partial n}|dz| = {}^*du$$

Goal : Construct F which is holomorphic in Ω' and



- We may assume $\partial\Omega$ is smooth. Define

$$\alpha_{kj} = \int_{C_j} {}^*dw_k = \int_{C_j}\left(-\frac{\partial w_k}{\partial y}dx + \frac{\partial w_k}{\partial y}dy\right) \in \mathbb{R}$$

- $A := (\alpha_{ij})_{(n-1)\times(n-1)}$ is invertible : If $Av = 0$ for $v = (\lambda_1, \cdots, \lambda_{n-1})^T$ i.e.

$$\lambda_1\alpha_{1,i} + \cdots + \lambda_{n-1}\alpha_{n-1,i} = 0\ \forall i \implies \int_{C_i} {}^*d(\underbrace{\lambda_1 w_1 + \cdots + \lambda_{n-1}w_{n-1}}_{:=u\in\mathcal{H}(\Omega')}) = 0\ \forall i = 1, \cdots, n-1$$

Claim : ${}^*du = dv$ for some v ($\rightsquigarrow u + iv$ is analytic and thus $\lambda_i = 0\ \forall i$)

subproof : $\forall$ closed curve γ in Ω, if the inside of γ not contain any C_i, then by Cauchy theorem, $\int_\gamma {}^*du = 0$. If the inside of γ contain $C_{i_1}, ..., C_{i_r}$. Since domain contain Ω, $\gamma \sim C_{i_1} + \cdots + C_{i_r} \implies \int_\gamma {}^*du = 0$. Hence, ${}^*du = dv$ for some v. $\square$

- $\exists!(\lambda_1, ..., \lambda_{n-1})$ s.t.

$$\begin{cases}\lambda_1\alpha_{1,1} + \cdots + \lambda_{n-1}\alpha_{n-1,1} = 2\pi \\ \lambda_1\alpha_{1,i} + \cdots + \lambda_{n-1}\alpha_{n-1,i} = 0 \text{ for } i = 2, ..., n-1\end{cases}$$

and let $u := \lambda_1 w_1 + \cdots + \lambda_{n-1}w_{n-1} \in \mathcal{H}(\Omega')$. By Green theorem

$$\alpha_{k,1} + \cdots + \alpha_{k,n} = \int_{\partial\Omega} {}^*dw_k = \iint_\Omega \left(\frac{\partial^2 w_k}{\partial x^2} + \frac{\partial^2 w_k}{\partial y^2}\right) dxdy = 0\ \forall k = 1, ..., n-1$$

Then $\lambda_1\alpha_{1,n} + \cdots + \lambda_{n-1}\alpha_{n-1,n} = -2\pi$. We have $\int_{C_1} {}^*du = 2\pi$, $\int_{C_n} {}^*du = -2\pi$, $\int_{C_i} {}^*du = 0\ \forall i = 2, ..., n-1$. If ${}^*du = dv$, then v is multiple valued with period 2π on C_1, -2π on C_n, and 0 on $C_j\ \forall j = 2, ..., n-1$. Then $f = u + iv$ is multiple valued with period $2\pi i$ on C_1, $-2\pi i$ on C_n, and 0 on $C_j\ \forall j = 2, ..., n-1$. $\mathrm{Re} f = u \rightsquigarrow \mathrm{Re} f|_{C_i} = \lambda_i\ \forall i = 1, ..., n-1, \mathrm{Re} f|_{C_n} = 0$. Let $F = e^f \rightsquigarrow F$ is single valued.

- Notice that $F(C_1) = C_{e^{\lambda_1}}(0)$, $F(C_n) = C_1(0)$ and $F(C_i) \subset C_{e^{\lambda_i}}(0)$. For $|w_0| \neq e^{\lambda_i}, 1$,

$$I_j(w_0) = \frac{1}{2\pi i}\int_{C_j} \frac{F'(z)}{F(z) - w_0} dz = \frac{1}{2\pi i}\int_{C_j} d\log(F(z) - w_0) = \frac{1}{2\pi}\int_{C_j} d\arg(F(z) - w_0)$$

First,

$$I_j(0) = \frac{1}{2\pi i}\int_{C_j} f' dz = \frac{1}{2\pi}\int_{C_j} (du + {}^*du) = \begin{cases}1 & \text{if } j = 1 \\ 0 & \text{if } j = 2, ..., n-1 \\ -1 & \text{if } j = n\end{cases}$$

Then $F(C_1)$ wind counterclockwise one time and $F(C_n)$ wind clockwise one time. Since the winding number is constant in each region w.r.t. the curve, we have

  - $I_j(w_0) = 0$ for $|w_0| \neq e^{\lambda_j}\ \forall j = 2, ..., n-1$
  - $I_1(w_0) = \begin{cases}0 & \text{if } |w_0| > e^{\lambda_1} \\ 1 & \text{if } |w_0| < e^{\lambda_1}\end{cases}$
  - $I_1(w_0) = \begin{cases}0 & \text{if } |w_0| > 1 \\ -1 & \text{if } |w_0| < 1\end{cases}$

By argument principle,

$$I(w_0) := I_1(w_0) + \cdots + I_n(w_0)$$

is the number of zero of $F(z) - w_0$ in Ω. Since F is an open mapping, $F : \Omega \to F(\Omega)$ open in $\mathbb{C}$. If $w_0 \in F(\Omega)$, $I(w_0) \geq 1$. The possible situation is $I_1(w_0) = 1$, $I_j(w_0) = 0\ \forall j = 2, ..., n-1$ and $I_n(w_0) = 0$ i.e. $1 < |w_0| < e^{\lambda_1}$.

(待補)

1.18 Elliptic functions

1.18.1 periodic function

Definition 1.18.1. Let f be a meromorphic function in $\mathbb{C}$. The period module M of f is

$$M = \{\omega \in \mathbb{C} : f(z+\omega) = f(z)\ \forall z \in \mathbb{C}\}$$

- $\omega_1, \omega_2 \in M \rightsquigarrow \omega_1 \pm \omega_2 \in M \rightsquigarrow M$ is $\mathbb{Z}$-module
- $f \neq$ constant $\rightsquigarrow M$ is discrete : If M has an accumulation point, then $f|_M = f(0) \rightsquigarrow f \equiv f(0)$ $(\rightarrow\!\leftarrow)$.

Proposition 1.18.1. $M = \{0\}$, $\langle \omega \rangle_{\mathbb{Z}}$ or $\langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ with $\omega_1/\omega_2 \notin \mathbb{R}$.

Proof: If $M \neq \{0\}$, for $r \gg 0$ with $\overline{B_r(0)} \cap M \neq \{0\}$. Since M is discrete, $\overline{B_r(0)} \cap M$ is finite. We may choose $\omega_1 \in M$ s.t. $|\omega_1|$ is min. If $M = \langle \omega_1 \rangle_{\mathbb{Z}}$, then done. Otherwise, $\exists \omega_2 \in M \setminus \langle \omega_1 \rangle_{\mathbb{Z}}$ s.t. $|\omega_2|$ is min. If $\omega_2/\omega_1 \in \mathbb{R} \setminus \mathbb{Z}$, then $\exists n \in \mathbb{Z}$ s.t. $n < \omega_2/\omega_1 < n+1 \rightsquigarrow |n\omega_1 - \omega_2| < |\omega_1|$ $(\rightarrow\!\leftarrow)$. Since $\omega_2/\omega_1 \notin \mathbb{R}$, $\mathbb{C} = \langle \omega_1, \omega_2 \rangle_{\mathbb{R}}$.
Claim : $M = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$:
subproof : $\forall \omega \in M$, $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}$ and choose $m_1, m_2 \in \mathbb{Z}$ s.t. $|\lambda_i - m_i| \leq \dfrac{1}{2}$. Let $\omega' = \omega - m_1\omega_1 - m_2\omega_2 \in M$. Then

$$|\omega'| \leq |\lambda_1 - m_1||\omega_1| + |\lambda_2 - m_2||\omega_2| \leq \frac{1}{2}\left(|\omega_1| + |\omega_2|\right) \leq |\omega_2|$$

The first equality of inequality will not holds since $\omega_2/\omega_1 \notin \mathbb{R}$. Hence, $\omega' \in \langle \omega_1 \rangle_{\mathbb{Z}} \rightsquigarrow \omega \in \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$. $\square$

Remark 1.18.1.

- $M = \langle \omega \rangle_{\mathbb{Z}} \rightsquigarrow \exists! F$: meromorphic in $\mathbb{C}^\times$ s.t. $f(z) = F(e^{2\pi i z/\omega})$: Consider

  $$U = \{z \in \mathbb{C} : 0 \leq \operatorname{Im} 2\pi z/\omega < 2\pi\}$$

  Then $\xi = \exp(2\pi i z/\omega)$ maps U to $\mathbb{C}^\times$. For $\xi \in \mathbb{C}^\times$, $\exists! z \in U$ s.t. $\xi = e^{2\pi i z/\omega} \rightsquigarrow F(\xi) = f(z)$ and f is meromorphic $\implies F$ is meromorphic.

- Assume $D = \{\xi : r_1 < |\xi| < r_2\} \subset \mathbb{C}^\times$ and F has no pole in D. By Laurent expansion, $F(\xi) = \sum\limits_{n=-\infty}^{\infty} c_n \xi^n$, where

  $$c_n = \frac{1}{2\pi i} \int_{|\xi|=r} F(\xi)\xi^{-n-1} d\xi,\ r_1 < r < r_2$$

  Then $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n e^{2\pi i n z/\omega}$ and

  $$c_n = \frac{1}{\omega} \int_a^{a+\omega} f(z) e^{2\pi i n z/\omega} dz$$

Definition 1.18.2. A meromorphic function in $\mathbb{C}$ is **elliptic** (double periodic) if its periods module $M = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ with $\omega_2/\omega_1 \notin \mathbb{R}$.

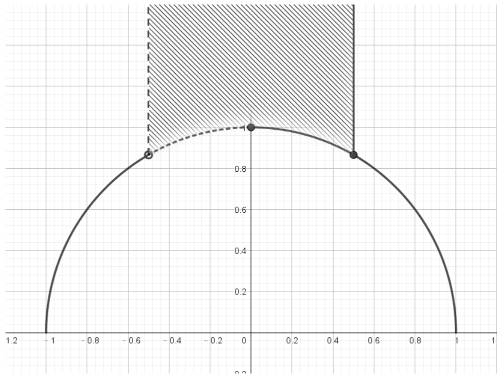
Proposition 1.18.2 (Canonical basis)**.** $\exists(\omega_1, \omega_2)$ s.t. $\tau = \dfrac{\omega_2}{\omega_1}$ satisfies

(1) $\operatorname{Im}\tau > 0$

(2) $\frac{-1}{2} < \mathrm{Re}\tau \leq \frac{1}{2}$

(3) $|\tau| \geq 1$

(4) $\mathrm{Re}\tau \geq 0$ if $|\tau| = 1$



The region defined in above is called **fundamental region** R.

Proof: Choose ω_1, ω_2 in Proposition 1.18.1, then $|\omega_1| \leq |\omega_2| \leq |\omega_1 \pm \omega_2| \rightsquigarrow |\tau| \geq 1$ and

$$|\omega_2|^2 \leq |\omega_1|^2 + |\omega_2|^2 \pm (\omega_2\overline{\omega_1} + \omega_1\overline{\omega_2}) \implies |\mathrm{Re}\tau| = \frac{1}{2}\left|\frac{\omega_2\overline{\omega_1} + \omega_1\overline{\omega_2}}{|\omega_1|^2}\right| \in \left[\frac{-1}{2}, \frac{1}{2}\right]$$

- If $\operatorname{Im}\tau < 0$, then replace (ω_1, ω_2) by $(-\omega_1, \omega_2)$
- If $\mathrm{Re}\tau = \frac{-1}{2}$, then replace (ω_1, ω_2) by $(\omega_1, \omega_1 + \omega_2)$
- If $|\tau| = 1$, $\mathrm{Re}\tau < 0$, then replace (ω_1, ω_2) by $(-\omega_2, \omega_1)$. Notice that

$$\mathrm{Re}\frac{\omega_2}{\omega_1} = \frac{a_1a_2 + b_1b_2}{a_1^2 + b_1^2} \text{ and } \mathrm{Re}\frac{-\omega_1}{\omega_2} = \frac{-(a_1a_2 + b_1b_2)}{a_2^2 + b_2^2}$$

□

Remark 1.18.2.

- If $M = \langle\omega_1, \omega_2\rangle_{\mathbb{Z}} = \langle\omega_1', \omega_2'\rangle_{\mathbb{Z}}$, then exists the basis transformation over $\mathbb{Z}$

$$\begin{pmatrix}\omega_2' \\ \omega_1'\end{pmatrix} = \underbrace{\begin{pmatrix}a & b \\ c & d\end{pmatrix}}_{:=A}\begin{pmatrix}\omega_2 \\ \omega_1\end{pmatrix}$$

  where $A \in M_{2\times2}(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{R})$. Then

$$\begin{pmatrix}\omega_2' & \overline{\omega_2}' \\ \omega_1' & \overline{\omega_1}'\end{pmatrix} = \begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_2 & \overline{\omega_2} \\ \omega_1 & \overline{\omega_1}\end{pmatrix}$$

- $\tau = \frac{\omega_2}{\omega_1}$, $\tau' = \frac{\omega_2'}{\omega_1'} \rightsquigarrow \tau' = \frac{a\tau + b}{c\tau + d}$ and $\operatorname{Im}\tau' = \mathrm{sign}(ad - bc)\frac{\operatorname{Im}\tau}{|c\tau + d|^2}$.
- If $\tau, \tau' \in R \rightsquigarrow \tau = \tau'$
  - (1) $\rightsquigarrow ad - bc = 1$
  - By symmetry, we may assume $\operatorname{Im}\tau' \geq \operatorname{Im}\tau \rightsquigarrow |c\tau + d| \leq 1$
    - $c = 0 \implies d = \pm1$, $ad = 1 \rightsquigarrow a = d = \pm1 \rightsquigarrow \tau' = \tau \pm b$.
      (2) $\implies |b| = |\mathrm{Re}\tau - \mathrm{Re}\tau'| < 1 \rightsquigarrow b = 0 \rightsquigarrow \tau = \tau'$
    - $c \neq 0 : |\tau + d/c| \leq |c|^{-1}$. If $|c| \geq 2 \rightsquigarrow d(\tau, x-\text{axis}) \leq 2^{-1}$, but $d(\tau, x-\text{axis}) \geq \frac{\sqrt{3}}{2}$.
      Hence, $|c| = 1 \rightsquigarrow |\tau \pm d| \leq 1 \rightsquigarrow d = 0$ or ±1.

- **••••** $|\tau+1| \le 1 : \tau = e^{2\pi i/3} \notin R$
- **••••** $|\tau-1| \le 1 : \tau = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \rightsquigarrow \operatorname{Im}\tau' = \operatorname{Im}\tau = \frac{\sqrt{3}}{2} \rightsquigarrow \tau = \tau'$
- **••••** $d = 0 : |\tau| \le 1$, by (4) $\rightsquigarrow \operatorname{Re}\tau \ge 0$. Since $bc = -1$, $b/c = -1$, then

$$\tau' = \pm a - \tau^{-1} = \pm a - \overline{\tau} \rightsquigarrow \operatorname{Re}\tau' + \operatorname{Re}\overline{\tau} = \pm a \in [-1, 1]$$

$$\begin{cases} a = 1 \rightsquigarrow \tau' + \overline{\tau} = 1 \implies \tau = \tau' = \frac{1}{2} + iy \\ a = 0 \rightsquigarrow \tau' = -\overline{\tau} \rightsquigarrow \tau = \tau' = i \end{cases}$$

- For a fixed $\tau \in R$, the number of pair (ω_1, ω_2) with $\tau = \omega_2/\omega_1$ is
  - $(\omega_1, \omega_2), (-\omega_1, -\omega_2) \rightsquigarrow \# = 2$
  - $\tau = i \rightsquigarrow \tau = -\frac{1}{\tau} \rightsquigarrow \# = 4$
  - $\tau = e^{\pi i/3} \rightsquigarrow \tau = \frac{-(\tau+1)}{\tau}$ or $\tau = \frac{-1}{\tau+1} \rightsquigarrow \# = 6.$

Definition 1.18.3.

- $z_1, z_2 \in \mathbb{C}$, $z_1 \equiv z_2 \pmod{M}$ if $z_1 - z_2 \in M$
- For $a \in \mathbb{C}$, define
$$P_a = \{a + r_1\omega_1 + r_2\omega_2 : 0 \le r_1, r_2 < 1\}$$
- $\mathcal{E}_M$ be the elliptic function with period module M.

Property 1.18.1.

(1) $f \in \mathcal{E}_M$ is analytic $\implies$ f is constant :
f is bounded on $\overline{P_a}$ and thus bounded on $\mathbb{C}$. By Liouville theorem, f is constant.

(2) $f \in \mathcal{E}_M$, the sum of residues of f is 0 :
Choose a s.t. no pole on ∂P_a, the sum of the residues of f in P_a° is

$$\frac{1}{2\pi i}\int_{\partial P_a} f(z)dz = \frac{1}{2\pi i}\left(\int_a^{a+\omega_1} + \int_{a+\omega_1+\omega_2}^{a+\omega_2} + \int_{a+\omega_1}^{a+\omega_1+\omega_2} + \int_{a+\omega_2}^{a}\right) f(z)dz = 0$$

(3) $f \in \mathcal{E}_M$, f has no single pole in P_a :
By (2).

(4) $f \in \mathcal{E}_M$ is not constant, then # of zeros = # of poles :
subproof : Note $f' \in \mathcal{E}_M$ and $\frac{f'}{f} \in \mathcal{E}_M$ has simple pole at the zeros and poles of f with corresponding residue is the multiplicity. By (2), $\frac{1}{2\pi i}\int_{\partial P_a} \frac{f'(z)dz}{f(z)} = 0$ and thus

$$\#\text{ of zeros} = \#\text{ of poles}$$

(5) For $c \in \mathbb{C}$, $f(x) - c$ and $f(x)$ have same poles $\rightsquigarrow$ # of zero of $f(z) = c$ equal to # of zero of $f(z) = 0$, and defined this number be the **order** of f.

(6) $a_1, ..., a_n$: incongruent zeros of f, $b_1, ..., b_n$: incongruent poles of f. Then

$$a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n \pmod{M}$$

subproof : Choose a s.t. f has no zeros and poles on ∂P_a. We have

$$\frac{1}{2\pi i}\int_{\partial P_a} \frac{zf'(z)}{f(z)}dz = a_1 + \cdots + a_n - b_1 - \cdots - b_n$$

$$\frac{1}{2\pi i}\left(\int_a^{a+\omega_1} - \int_{a+\omega_2}^{a+\omega_1+\omega_2}\right)\frac{zf'(z)}{f(z)}dz = \frac{-\omega_2}{2\pi i}\int_a^{a+\omega_1}\frac{f'(z)}{f(z)}dz \tag{*}$$

Notice that $\frac{1}{2\pi i}\int_a^{a+\omega_1}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\log f(z)\Big|_a^{a+\omega_1} = \frac{1}{2\pi}\arg f(z) \in \mathbb{Z}$ and thus $(*) \in \langle\omega_2\rangle_{\mathbb{Z}}$ and thus get the result.

1.18.2 Weierstrass elliptic function

If the singular part of f is $\frac{1}{z^2} \rightsquigarrow f(z)-f(-z)$ is analytic and thus $f(z)-f(-z) = c$. Substitute $z = \frac{\omega_1}{2} \rightsquigarrow c = f(\omega_1/2) - f(-\omega_1/2) = 0 \implies f(z) = f(-z)$ and thus

$$f(z) = z^{-2} + a_0 + a_1 z^2 + \cdots + a_2 z^4 + \cdots$$

Since $z = 0$ is pole then $z = \omega \in M$ is also pole. Then we may consider

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in M\setminus\{0\}}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

which is called **Weierstrass $\wp$ function** or **Weierstrass elliptic function**.

- $\wp$ will converge uniformly on each compact subset of $\mathbb{C}$: For a given z, $\forall\omega \in M$ with $|\omega| > 2|z|$

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \frac{|z|\cdot|z-2\omega|}{|\omega|^2\cdot|z-\omega|^2} \le \frac{|z|(|z|+2|\omega|)}{|\omega|^2(|\omega|-|z|)^2} \le \frac{10|z|}{|\omega|^3}$$

Claim : $\sum_{\omega\in M\setminus\{0\}} |\omega|^{-3} < \infty$ (then done)

subproof : Let $R \ni \tau = \omega_2/\omega_1 = s + it$, then

$$|n + m\tau| = \sqrt{|n+ms|^2 + |mt|^2} \ge \frac{1}{\sqrt{2}}(|n+ms| + |mt|) \ge \frac{1}{\sqrt{2}}(|n| + (t-|s|)|m|)$$

Since $\tau \in R$, $t \ge \frac{\sqrt{3}}{2}$, $|s| \le \frac{1}{2} \rightsquigarrow t - |s| > 0$. Let $k = \frac{|\omega_1|}{\sqrt{2}}\min(1, t-|s|)$ and thus

$$|n\omega_1 + m\omega_2| \ge \frac{|\omega_2|}{\sqrt{2}}(|n| + (t-|s|)|m|) \ge k(|n| + |m|)$$

Since there exists $4n$ pairs (n_1, n_2) s.t. $|n_1| + |n_2| = n$,

$$\sum_{\omega\in M\setminus\{0\}} \le \frac{1}{k^3}\sum_{n=1}^{\infty}\frac{n}{n^3} < \infty$$

- $\wp'(z) = -2\sum_{\omega\in M} \dfrac{1}{(z-\omega)^3} \in \mathcal{E}_M$. Then

$$\begin{cases} \wp(z+\omega_1) - \wp(z) = A \\ \wp(z+\omega_2) - \wp(z) = B \end{cases}$$

  Substitute $z = \dfrac{\omega_1}{2}, \dfrac{\omega_2}{2}$ respectively and by $\wp$ is even function, we have $A = B = 0$ i.e. $\wp \in \mathcal{E}_M$.

- Since $\left(\wp - \dfrac{1}{z^2}\right)(0) = 0$, we may assume $\wp(z) = \dfrac{1}{z^2} + \sum_{n=1}^{\infty} a_{2n} z^{2n} \rightsquigarrow f(z) := \wp(z) - \dfrac{1}{z^2}$ is analytic and thus $a_{2n} = \dfrac{f^{(2n)}(0)}{(2n)!}$. Since $f^{(n)}(z) = (-1)^n (n+1)! \sum_{\omega\in M} \dfrac{1}{(z-\omega)^{n+2}}$,

$$a_{2n} = (2n+1)\sum_{\omega\in M} \frac{1}{\omega^{2(n+1)}}$$

  which is exists since $\sum \omega^k$ converge for $k > 2$ by the subproof in above.

- Define **Eisenstein series** $G_n = \sum_{\omega\neq 0} \dfrac{1}{\omega^{2(n+1)}} \rightsquigarrow G_n$ absolutely converge. Then we have

$$\begin{cases} \wp(z) = z^{-2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \\ \wp'(z) = -2z^{-3} + 6G_2 z + 20 G_3 z^3 + \cdots \\ \wp(z)^3 = z^{-6} + 9G_2 z^{-2} + 15 G_3 + \cdots \\ \wp'(z)^2 = 4z^{-6} - 24 G_2 z^{-2} - 80 G_3 + \cdots \end{cases}$$

  and thus

$$\wp'(z)^2 - 4\wp(z)^3 + 60\wp(z) = -140 G_3 + \cdots$$

  is analytic elliptic function and thus is constant. So we have

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 0$$

  where $g_2 = 60G_2$ and $g_3 = 140 G_3$.

- Since $\wp'(z) = \sqrt{4\wp^3 - g_2\wp - g_3}$ and thus

$$z = \int_0^z \frac{d\wp(z)}{\sqrt{\wp(z)^3 - g_2\wp(z) - g_3}} = \int_0^{\wp(z)} \frac{dw}{\sqrt{w^3 - g_2 w - g_3}}$$

  Then $\wp(z)$ is the inverse function of elliptic integral.

Fact 1.18.1. $\mathcal{E}_M$ forms a field and $\begin{array}{ccc} \mathbb{C} & \hookrightarrow & \mathcal{E}_M \\ c & \mapsto & f \equiv c \end{array} \implies \mathcal{E}_M$ is a $\mathbb{C}$-algebra.

Proof:

- $f \in \mathcal{E}_M$ with $\begin{cases} f: \text{ even} \\ \{\text{pole of } f\} \subset M \end{cases} \implies f \in \mathbb{C}[\wp]$ say $f = a_0 + a_1\wp + \cdots + a_n\wp^n$, where $2n$ is order of f :

  Let $f \neq$ constant $\rightsquigarrow f$ has a least one pole in M, and hence a pole at $0 \rightsquigarrow f = a_{-2n} z^{-2n} + \cdots$, $n \geq 1$, $a_{-2n} \neq 0$. Also, $\wp(z)^n = z^{-2n} + \cdots$. Let $g = f - a_{-2n}\wp(z)^n \rightsquigarrow$ order of $g \leq 2(n-1)$. By induction, $g \in \mathbb{C}[\wp]_{n-1} \rightsquigarrow f \in \mathbb{C}[\wp]_n$.

- $f \in \mathcal{E}_M$ with f : even $\implies f \in \mathbb{C}(\wp)$:

  Let $f \neq$ constant. If has a pole $a \notin M$, then $(\wp(z)-\wp(a))^N f(z)$ has a removable singularity at $z = a$ for $N \gg 0$. Since f has only finitely many poles mod M, say $a_1, ...a_m$, and $\exists N_1, ..., N_m$ s.t.

  $$g(z) = f(z)\prod_{j=1}^{m}(\wp(z) - \wp(a_j))^{N_j}$$

  has no pole outside $M \implies g(z) \in \mathbb{C}[\wp] \implies f \in \mathbb{C}(\wp)$.

- For $f \in \mathcal{E}_M$: Notice that

  $$f(z) = \underbrace{\frac{1}{2}(f(z) + f(-z))}_{:=f_1(z)} + \underbrace{\frac{1}{2}(f(z) - f(-z))}_{:=f_2(z)}$$

  Then $f_1(z)$ is even and thus $f_1 \in \mathbb{C}(\wp)$. Since $f_2/\wp'$ is even, $f_2/\wp' \in \mathbb{C}(\wp)$ and thus $f_2 \in \wp'\mathbb{C}(\wp)$. Hence, $\mathcal{E}_M = \mathbb{C}(\wp) + \wp'\mathbb{C}(\wp)$. $\square$

- Let $\xi(z)$ satisfy $\begin{cases} \xi'(s) = -\wp(z) \\ \lim\limits_{z\to 0}\left(\xi(z) - \dfrac{1}{z}\right) = 0 \end{cases}$, then

  $$\xi(z) = \frac{1}{z} + \sum_{\omega\neq 0}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

  which will converge since $\left|\dfrac{1}{z-\omega} + \dfrac{1}{\omega} + \dfrac{z}{\omega^2}\right| = O\left(\dfrac{1}{|\omega|^3}\right)$ on each compact subset. Since $\wp(z+\omega) - \wp(z) = 0$ and ξ is odd function, we have

  $$\begin{cases} \xi(z+\omega_1) - \xi(z) = \eta_1 & \xrightarrow{z=\omega_1/2} \eta_1 = \xi(\omega_1/2) - \xi(-\omega_1/2) = 2\xi(\omega_1/2) \\ \xi(z+\omega_2) - \xi(z) = \eta_2 & \xrightarrow{z=\omega_2/2} \eta_2 = \xi(\omega_2/2) - \xi(-\omega_2/2) = 2\xi(\omega_2/2) \end{cases}$$

  Legendre's relation : $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$: Choose $a \notin M$, then

  $$\begin{aligned} 1 = \frac{1}{2\pi i}\int_{\partial P_a}\xi(z)dz &= \frac{1}{2\pi i}\left(\int_a^{a+\omega_1}(\xi(z) - \xi(z+\omega_2)dz) + \int_a^{a+\omega_2}(\xi(z+\omega_1) - \xi(z))dz\right) \\ &= \frac{1}{2\pi i}\left(-\eta_2\omega_1 + \eta_1\omega_2\right) \end{aligned}$$

Remark 1.18.3. Consider

$$\begin{array}{cccc} f: & P_0 & \longrightarrow & \mathbb{P}^2 \\ & z \neq 0 & \longmapsto & [1 : \wp(z) : \wp'(z)] \end{array}$$

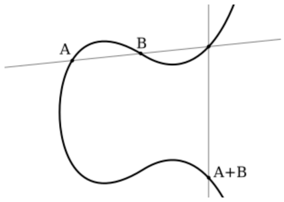
We can naturally extend f to $z = 0$ by

$$0 \longmapsto [1, z^{-2} + \cdots, -2z^{-3} + \cdots]_{z\to 0} = [z^3, z + \cdots, -2 + \cdots]_{z\to 0} = [0 : 0 : 1]$$

Recall that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, then $f(P_0)$ satisfy the homogeneous equation

$$z_0z_2^2 - 4z_1^3 - g_2z_0^2z_1 - g_2z_0^3 = 0$$

Consider the equation $y^2 = 4x^3 - g_2 x - g_3$ in $x - y$ plane, we can define the group structure by



i.e. the three intersection of line ℓ with elliptic curve A, B, C will satisfy

$$A + B + C = 0$$

where A, B, C may not be distinct and allow infinite point.

1.19 Weierstrass theory

Recall that $\wp(z) = \dfrac{1}{z^2} + \sum\limits_{\omega \neq 0} \left(\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2} \right)$, $\wp'(z) = -2 \sum\limits_{\omega} \dfrac{1}{(z-\omega)^3}$, then

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$$

Let e_1, e_2, e_3 be three roots of $4x^3 - g_2 x - g_3$, then

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

Observation : $\wp'(\omega_1 - z) = \wp'(-z) = -\wp'(z) \xrightarrow{z=\omega_1/2} \wp'(\omega_1/2) = 0$. Similarly, $\wp'(\omega_2/2) = \wp'((\omega_1 + \omega_2)/2) = 0$. So we may let $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, $e_3 = \wp((\omega_1 + \omega_2)/2)$. Notice that e_1, e_2, e_3 are homogeneous of order -2 in ω_1, ω_2 i.e. $e_k(t\omega_1, t\omega_2) = t^{-2} e_k(\omega_1, \omega_2)$. Then $\dfrac{e_3 - e_2}{e_1 - e_2}$ is a well-defined function on $\{[\omega_1 : \omega_2] = [1, \tau] : \omega_2/\omega_1 \notin \mathbb{R}\}$ and thus $\lambda(\tau) = \dfrac{e_3 - e_2}{e_1 - e_2}$ is meromorphic in $\mathbb{H}$.

$$\begin{cases} e_1 \neq e_2 & \Longrightarrow \lambda(\tau) \text{ is holomorphic in } \mathbb{H} \\ e_3 \neq e_2 & \Longrightarrow \lambda \neq 0 \text{ in } \mathbb{H} \\ e_1 \neq e_3 & \Longrightarrow \lambda \neq 1 \text{ in } \mathbb{H} \end{cases}$$

- $\mathrm{GL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) : ad - bc = \pm 1 \right\} \curvearrowright$ the $\mathbb{Z}-$basis of M.

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{:=A} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix}$$

Since M

Chapter 2

Homework

2.1

Problem 2.1.1. Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x).$$

Problem 2.1.2. Determine all values of $2^i, i^i, (-1)^{2i}$.

Problem 2.1.3. Express $\arctan w$ in terms of the logarithm.

Problem 2.1.4. Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when :

(a) $a_n = (\log n)^2$

(b) $a_n = n!$

(c) $a_n = \dfrac{n^2}{4^n + 3n}$

(d) $a_n = (n!)^3/(3n)!$

(e) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha, \beta, \gamma : z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, ...$

(f) Find the radius of convergence of the Bessel function of order r :

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}$$

Problem 2.1.5. Expand $(1-z)^{-m}$ in powers of z. Here m is a fixed positive integer. Also show that if

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,$$

the one obtains the following asymptotic relation for the coefficients :

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \text{ as } n \to \infty.$$

Problem 2.1.6. Show that for $|z| < 1$, one has

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^{n+1}}} = \frac{z}{1-z},$$

and

$$\sum_{n=0}^{\infty} \frac{2^n z^{2^n}}{1 + z^{2^n}} = \frac{z}{1-z}.$$

justify any change in the order of summation.

2.2

Problem 2.2.1. Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a, b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reserve orientation. Prove that for any continuous function f on γ

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

Problem 2.2.2. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

(a) Evaluate the integrals

$$\int_\gamma z^n dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Problem 2.2.3. It is possible to define $n(\gamma, a)$ for any continuous closed curve γ that does not pass through a, whether piecewise differentiable or not. For this purpose γ is divided into subarces $\gamma_1, ..., \gamma_n$, each contained in a disk that does not include a. Let σ_k be the directed line segment from the initial to the terminal point of γ_k, and set $\sigma = \sigma_1 + \cdots + \sigma_n$. We define $n(\gamma, a)$ to be the value of $n(\sigma, a)$. To justify the definition, prove the following :

(a) the result is independent of the subdivision;

(b) if γ is piecewise differentiable the new definition is equivalent to the old;

(c) If γ lies inside of a circle, then $n(\gamma, a) = 0$ for all points a outside of the same circle. As a function of a the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.

Problem 2.2.4. The **Jordan cure theorem** asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma, a) \neq 0$.

We may assume that $\mathrm{Re} z > 0$ on γ, and that there are points $z_1, z_2 \in \gamma$ with $\mathrm{Im}\, z_1 < 0$, $\mathrm{Im}\, z_2 > 0$. These point may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the arcs of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0, and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \gamma$ or γ.

The positive real axis intersects both γ_1 and γ_2. Choose the notation so that the intersection x_2 farthest to the right is with γ_2.

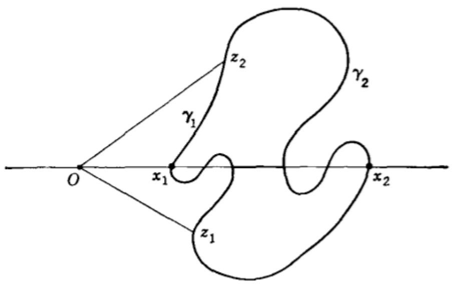


FIG. 4-6. Part of the Jordan curve theorem.

Prove the following :

(a) $n(\sigma_1, x_2) = 0$, hence $n(\sigma_1, z) = 0$ for $z \in \gamma_2$;

(b) $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$;

(c) the first intersection x_1 of the positive real axis with γ lies on γ_1;

(d) $n(\sigma_2, x_1) = 1$, hence $n(\sigma_2, z) = 1$ for $z \in \gamma_1$;

(e) there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma, x) = 1$ or -1.

2.3

Example 2.3.1. Compute

$$\int_{|z|=1} \frac{e^z}{z} dz.$$

Example 2.3.2. Compute
$$\int_{|z|=2} \frac{dz}{z^2+1}$$
by decomposition of the integrand in partial fractions.

Example 2.3.3. Compute
$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$
under the condition $|a| \neq \rho$.

2.4

Problem 2.4.1. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Problem 2.4.2. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

Problem 2.4.3. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

Problem 2.4.4. Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $\alpha \leq t \leq \beta$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then
$$F(z) = \int_\alpha^\beta \varphi(z,t)dt$$
is analytic in z and
$$F'(z) = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z} dt. \tag{1}$$
to prove this represent $\varphi(z,t)$ as a Cauchy integral
$$\varphi(z,t) = \frac{1}{2\pi i} \int_C \frac{\varphi(\xi,t)}{\xi - z} d\xi$$
Fill in the necessary details to obtain
$$F(z) = \int_C \left(\frac{1}{2\pi i} \int_\alpha^\beta \varphi(\xi,t)dt \right) \frac{d\xi}{\xi - z}$$
and use Lemma 3 to prove (1).

Problem 2.4.5. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion
$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$
is equal to 0. Prove that f is a polynomial.

2.5

Problem 2.5.1. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that fg has the order $h+k$, f/g the order $h-k$, and $f+g$ an order which does not exceed $\max(h,k)$.

Problem 2.5.2. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Problem 2.5.3. Show that any function which is meromorphic in the extended plane is rational.

Problem 2.5.4. Prove that an isolated singularity of $f(z)$ is removable as soon as either $\mathrm{Re} f(z)$ or $\mathrm{Im}\, f(z)$ is bounded above or below.

Problem 2.5.5. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.

2.6

Problem 2.6.1. Determine explicitly the largest disk about the origin whose image under the mapping $w = e^z$ is one to one.

Problem 2.6.2. If $f(z)$ is analytic at the origin and $f'(0) \neq 0$, prove the existence of an analytic $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of 0.

Problem 2.6.3. Let f be analytic in $B_1(0)$. Show that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \leq \frac{1}{1-|z|^2}$$

for all $z \in B_1(0)$ with $f(z) \neq 1$.

Problem 2.6.4. If γ is a piecewise differentiable arc contained in $|z| < 1$ the integral

$$\int_\gamma \frac{|dz|}{1-|z|^2}$$

is called the **noneuclidean length** (or **hyperbolic length**) of γ. Show that an analytic function $f(z)$ with $|f(z)| < 1$ for $|z| < 1$ maps every γ on an arc with smaller or equal noneuclidean length.

Problem 2.6.5. Prove that the arc of smallest noneuclidean length that joins two given points in the unit disk is a circular arc which is orthogonal to the unit circle. (Make use of a linear transformation that carries one end point to the origin, the other to a point on the positive real axis.)
The shortest noneuclidean length is called the **noneuclidean distance** between the end points. Derive a formula for the noneuclidean distance between z_1 and z_2 :

$$\frac{1}{2}\log\frac{1+\left|\dfrac{z_1-z_2}{1-\overline{z}_1 z_2}\right|}{1-\left|\dfrac{z_1-z_2}{1-\overline{z}_1 z_2}\right|}$$

2.7

Problem 2.7.1. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|}$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y} \quad (z = x + iy)$$

Problem 2.7.2. In Ex.1 and 2, prove that equality implies that $f(z)$ is a linear transformation.

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Problem 2.7.3. Derive corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

Problem 2.7.4. Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

Problem 2.7.5. How should noneuclidean length in the upper half plane be defined?

2.8

Problem 2.8.1. Find the poles and residues of the following functions :

$$(a)\ \frac{1}{z^2 + 5z + 6} \qquad (b)\ \frac{1}{(z^2 - 1)^2} \qquad (c)\ \frac{1}{\sin z}$$

$$(d)\ \cot z \qquad (e)\ \frac{1}{\sin^2 z} \qquad (f)\ \frac{1}{z^m (1 - z)^n}$$

Problem 2.8.2. Evaluate the following integrals by the method of residues :

(a) $\displaystyle\int_0^{\pi/2} \frac{dx}{a + \sin^2 x},\ |a| > 1$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

(c) $\displaystyle\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)^2} dx,\ a \in \mathbb{R}$

(d) $\displaystyle\int_0^{\infty} \frac{\cos x}{x^2 + a^2} dx,\ a \in \mathbb{R}$

Problem 2.8.3. Evaluate $\displaystyle\int_0^{\infty} \log(1 + x^2) \frac{dx}{x^{1+\alpha}},\ 0 < \alpha < 2.$

Problem 2.8.4. Complex integration can sometimes be used to evaluate area integrals. As an illustration, show that if $f(z)$ is analytic and bounded for $|z| < 1$ and if $|\xi| < 1$, then

$$f(\xi) = \frac{1}{\pi} \iint\limits_{|z|<1} \frac{f(z)dxdy}{(1-\overline{z}\xi)^2}$$

Remark. This is known as **Bergman's kernel formula**. To prove it, express the area integral in pole coordinates, then transform the inside integral to line integral which can be evaluated by residues.

Problem 2.8.5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

where are the poles of $(1+z^4)^{-1}$?

Problem 2.8.6. Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$$

for all $\xi \in \mathbb{R}$.

2.9

Problem 2.9.1. Suppose f and g are holomorphic in a region containing the disc $|z| \leq 1$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Let

$$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

Show that if ε is sufficiently small, then

(a) $f_\varepsilon(z)$ has a unique zero in $|z| \leq 1$, and

(b) if z_ε is this zero, the mapping $\varepsilon \mapsto z_\varepsilon$ is continuous.

Problem 2.9.2. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

Problem 2.9.3. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

2.10

Problem 2.10.1. Find the Hadamard products for :

(a) $e^z - 1$

(b) $\cos \pi z$

Problem 2.10.2. Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists entire function which satisfy $f(a_n) = A_n$.

Problem 2.10.3. Prove that

$$\sin \pi(z+\alpha) = \sin \pi\alpha e^{\pi z \cot \pi\alpha} \prod_{n=-\infty}^{\infty} \left(1 + \frac{z}{n+\alpha}\right) e^{-z/(n+\alpha)}$$

whenever α is not an integer.

Problem 2.10.4. What is the genus of $\cos \sqrt{z}$.

Problem 2.10.5. Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real.

2.11

Problem 2.11.1. Prove the formula of Gauss :

$$(2\pi)^{\frac{n-1}{2}}\Gamma(z) = n^{z-1/2}\Gamma\left(\frac{z}{n}\right)\Gamma\left(\frac{z+1}{n}\right)\cdots\Gamma\left(\frac{z+n-1}{n}\right)$$

Problem 2.11.2. Show that

$$\Gamma\left(\frac{1}{6}\right) = 2^{-1/3}\left(\frac{3}{\pi}\right)^{1/2}\Gamma\left(\frac{1}{3}\right)^2$$

Problem 2.11.3. What are the residues of $\Gamma(z)$ at the pole $z = -n$?

2.12

Problem 2.12.1. Assume that $f(z)$ has genus zero so that

$$f(z) = z^m \prod_n \left(1 - \frac{z}{a_n}\right)$$

Compare $f(z)$ with

$$g(z) = z^m \prod_n \left(1 - \frac{z}{|a_n|}\right)$$

and show that the maximum modulus $\max\limits_{|z|=r} |f(z)|$ is $\leq$ the maximum modules of g, and that the minimum modulus of f is $\geq$ the minimum modulus of g.

Problem 2.12.2. Find the order of growth of the following entire functions :

(a) $p(z)$ where p is polynomial.

(b) e^{bz^n} for $b \neq 0$.

(c) e^{e^z}.

Problem 2.12.3. Show that if τ is fixed with $\mathrm{Im}(\tau) > 0$, the Jacobi theta function

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is order 2 as a function of z.

Problem 2.12.4. Let $t > 0$ be given and fixed, and define $F(z)$ by

$$F(z) = \prod_{n=1}^{\infty} (1 - e^{-2\pi n t} e^{2\pi i z}).$$

Note that the product defines an entire function of z.

(a) Show that $|F(z)| \leq A e^{a|z|^2}$, hence F is order 2.

(b) F vanishes exactly when $z = -int + m$ for $n \geq 1$ and n, m integers. Thus, if z_n is an enumeration of these zeros we have

$$\sum \frac{1}{|z_n|^2} = \infty \text{ but } \sum \frac{1}{|z_n|^{2+\varepsilon}} < \infty.$$

2.13

Problem 2.13.1. Prove that the order of the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$$

is $1/\alpha$.

Problem 2.13.2. If $\lambda \neq 0$, and $p(z)$ is a nonzero polynomial, $e^{\lambda z} - p(z)$ has an infinitely of zeros.

Problem 2.13.3. If $f(z)$ is of order ρ, and $g(z)$ of order $\rho' \leq \rho$, and the zeros of $g(z)$ are all zeros of $f(z)$, then $f(z)/g(z)$ is of order ρ at most.

2.14

Problem 2.14.1. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the partial sums

$$A_n = a_1 + \cdots + a_n$$

are bounded. Prove that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in this half-plane.

Problem 2.14.2. Consider the following function

$$\widetilde{\zeta}(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

(a) Prove that the series defining $\widetilde{\zeta}(s)$ converges for $\mathrm{Re}(s) > 0$ and defines a holomorphic function in that half-plane.

(b) Show that for $s > 1$ one has $\widetilde{\zeta}(s) = (1 - 2^{1-s})\zeta(s)$.

(c) Conclude, since $\widetilde{\zeta}$ is given as an alternating series, that ζ has no zeros on the segment $0 < \sigma < 1$. Extend this last assertion to $\sigma = 0$ by using the function equation.

Problem 2.14.3. Show that for every $c > 0$

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{c-iN}^{c+iN} a^s \frac{ds}{s} = \begin{cases} 1 & \text{if } a > 1, \\ 1/2 & \text{if } a = 1, \\ 0 & \text{if } 0 \le a < 1. \end{cases}$$

The integral is taken over the vertical segment from $c - iN$ to $c + iN$.

2.15

Problem 2.15.1. In the theory of primes, a better approximation to $\pi(x)$ (instead of $x/\log x$) turns out to be $\mathrm{Li}(x)$ defined by

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

(a) Prove that

$$\mathrm{Li}(x) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad \text{as } x \to \infty,$$

and that as a consequence

$$\pi(x) \sim \mathrm{Li}(x) \text{ as } x \to \infty.$$

(b) Refine the previous analysis by showing that for every integer $N > 0$ one has the following asymptotic expansion

$$\mathrm{Li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \cdots + (N-1)!\frac{x}{(\log x)^N} + O\left(\frac{x}{(\log x)^{N+1}}\right)$$

as $x \to \infty$.

Problem 2.15.2. Let

$$\varphi(x) = \sum_{p \le x} \log p$$

where the sum is taken over all prime $\le$. Prove that the following are equivalent as $x \to \infty$:

(a) $\varphi(x) \sim x$,

(b) $\pi(x) \sim x/\log x$.

(c) $\psi(x) \sim x$,

(d) $\psi_1(x) \sim x^2/2$.

Problem 2.15.3. If p_n denotes the n^{th} prime, the prime number theorem implies that $p_n \sim n \log n$ as $n \to \infty$.

(a) Show that $\pi(x) \sim x/\log x$ implies that

$$\log \pi(x) + \log\log x \sim \log x.$$

(b) As a consequence, prove that $\log \pi(x) \sim \log x$, and take $x = p_n$ to conclude the proof.

2.16

Problem 2.16.1. Prove that in any region Ω the family of analytic functions with positive real part is normal. Under what added condition is it locally bounded?

Problem 2.16.2. Show that the functions z^n, n a nonnegative integer, form a normal family in $|z| < 1$, also in $|z| > 1$, but not in any region that contains a point on the unit circle.

Problem 2.16.3. If $f(z)$ is analytic in the whole plane, show that the family formed by all functions $f(kz)$ with constant k is normal in the annulus $r_1 < |z| < r_2$ if and only if f is a polynomial.

2.17

Problem 2.17.1 (Marty theorem)**.** A family $\mathcal{F}$ of meromorphic function is normal $\iff$ $\{f^{\#} : f \in \mathcal{F}\}$ is uniformly bounded on each compact subset.

Problem 2.17.2. If the family $\mathfrak{F}$ of analytic (or meromorphic) functions is not normal in Ω, show that there exists a point z_0 such that $\mathfrak{F}$ is not normal in any neighborhood of z_0.

2.18

Problem 2.18.1. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\overline{z'} = -1$.

Problem 2.18.2. Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}$$

Problem 2.18.3. Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

Problem 2.18.4. If z_0 is real and Ω is symmetric with respect to the real axis, prove by the uniqueness that f satisfies the symmetry relation $f(\overline{z}) = \overline{f(z)}$.

Problem 2.18.5. What is the corresponding conclusion if Ω is symmetric with respect to the point z_0?

2.19

All mappings below are to be conformal.

Problem 2.19.1. Map the common part of the disks $|z| < 1$ and $|z - 1| < 1$ on the inside of the unit circle. Choose the mapping so that the two symmetries are preserved.

Problem 2.19.2. Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle so that the point at ∞ correspond to each other.

Problem 2.19.3. Map the inside of the lemniscate $|z^2 - a^2| = \rho^2$ $(\rho > a)$ on the disk $|w| < 1$ so that symmetries are preserved.

Problem 2.19.4. Map the part of the z-plane to the left of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ on a half plane.

2.20

Problem 2.20.1. Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0, and so that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

Problem 2.20.2. Does there exist a holomorphic surjection from the unit disc to $\mathbb{C}$?

Problem 2.20.3. Other examples of eliptic integrals providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$\int_0^x \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}, \text{ with } \lambda \in \mathbb{R} \text{ and } \lambda \neq 1$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda = -1$, the image of

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are $\dfrac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

Problem 2.20.4. We consider conformal mappings to triangles.

(a) Show that

$$\int_0^z z^{-\beta_1}(1-z)^{-\beta_2}dz,$$

with $0 < \beta_1 < 1$, $0 < \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of 0, 1, and ∞, and with angles $\alpha_1\pi$, $\alpha_2\pi$ and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is

$$\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$$

2.21

Where all Schwarz-Christoffel is from disk to polygon.

Problem 2.21.1. If a vertex of the polygon is allowed to be at ∞, what modification does the formula undergo? If in this context $\beta_k = 1$, what is the polygon like?

Problem 2.21.2. Show that the mappings of a disk onto a parallel strip, or onto a half strip with two right angles, can be obtained as special cases of the Schwarz-Christoffel formula.

Problem 2.21.3. Show that

$$F(w) = \int_0^w (1 - w^n)^{-2/n} dw$$

maps $|w| < 1$ onto the interior of a regular polygon with n sides.

2.22

Problem 2.22.1. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that

$$M(r) \leq M(r_1)^\alpha M(r_2)^{1-\alpha}$$

where $\alpha = \dfrac{\log(r_2/r)}{\log(r_2/r_1)}$ (Hadamard's three-circle theorem). Discuss cases of equality.

Problem 2.22.2. Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity (Poisson's integral for the half plane).

Problem 2.22.3. In above, assume that U has a jump at 0, for instance $U(0+) = 0$, $U(0-) = 1$. Show that $P_U(z) - \dfrac{1}{\pi}\arg z$ tends to 0 as $z \to 0$. Generalize to arbitrary jumps and to the case of the circle.

Problem 2.22.4. If C_1 and C_2 are complementary arcs on the unit circle, set $U = 1$ on C_1, $U = 0$ on C_2. Find $P_U(z)$ explicitly and show that $2\pi P_U(z)$ equals the length of the arc, opposite to C_1, cut off by the straight lines through z and the end points of C_1.

Problem 2.22.5. Show that the mean-value formula remains valid for $u = \log|1 + z|$, z_0, $r = 1$, and use this fact to compute

$$\int_0^\pi \log\sin\theta d\theta$$

2.23

Problem 2.23.1. Show that the function $|x|$, $|z|^\alpha$ ($\alpha \geq 0$), $\log(1 + |z|^2)$ are subharmonic.

Problem 2.23.2. If $f(z)$ is analytic, prove that $|f(z)|^\alpha$ $(\alpha \geq 0)$ and $\log(1+|f(z)|^2)$ are subharmonic.

Problem 2.23.3. Formulate and prove a theorem to the effect that a uniform limit of subharmonic function is subharmonic.

2.24

Problem 2.24.1. Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1 \in \mathbb{R}$.

(a) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0 = \dfrac{1}{q}\omega_1$.

(b) If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m - n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m, n range over the integers.

Problem 2.24.2. Prove that the series

$$\sum_{n+m\tau \in \Lambda^*} \frac{1}{|n+m\tau|^2} \text{ where } \tau \in \mathbb{H}$$

does not converge. In fact, show that

$$\sum_{1 \leq n^2+m^2 \leq R^2} \frac{1}{n^2+m^2} = 2\pi \log R + O(1)$$

Problem 2.24.3. Let f and g be elliptic functions for the same lattice.

(1) If f and g have the same poles, and for each pole respectively the same principal parts, then f and g differ by an additive constant.

(2) If f and g have the same pole set and the same zero set, and if for any pole or zero the corresponding multiplicities coincide, then f and g differ by a multiplicative constant.

Problem 2.24.4. Let

$$\mathcal{F} := \{z \in \mathbb{C} : z = t_1\omega_1 + t_2\omega_2, 0 \leq t_1, t_2 \leq 1\}$$

be the fundamental region of the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with respect to a fixed basis ω_1, ω_2. Show that the Euclidian volume of the fundamental parallelogram is $|\operatorname{Im}(\overline{\omega_1}\omega_2)|$. Thus formula is independent of the choice of the basis.

2.25

Problem 2.25.1. For an odd elliptic function associated to the lattice L, the half-lattice points $\omega/2$, $\omega \in L$, are either zeros or poles.

Problem 2.25.2. Let f be an elliptic function of order m. Then its derivative f' is also an elliptic function of some order n, and the following inequality holds :

$$m + 1 \leq n \leq 2m$$

Construct examples for the extreme cases $n = m + 1$ and $n = 2m$.

Problem 2.25.3. Any elliptic function of order ≤ 2 with period lattice L, whose pole set is contained in L, is of the form $z \to a + b\wp(z)$.

Problem 2.25.4. For each of the elliptic functions $(\wp')^{-n}$, $1 \leq n \leq 3$, find the corresponding normal form $R(\wp) + S(\wp)\wp'$ with rational functions R and S.

Problem 2.25.5. Let us set $g_2 = g_2(L)$, $g_3 = g_3(L)$ for the g-invariants of a fixed lattice L. Let f be a meromorphic, non-constant function in some domain, which satisfies the same algebraic differential equation as $\wp$, i.e.

$$f'^2 = 4f^3 - g_2 f - g_3$$

Show that f is composition of $\wp$ with a translation, i.e. there exists an $a \in \mathbb{C}$ with $f(z) = \wp(z + a)$.