§4. Two important morphisms

~~§5. Separated~~ .

⊙ Separated morphisms

Recall that if X is a topological space, then

X is Hausdorff $\Leftrightarrow$ $\Delta = \{(a,a) \in X \times X \mid a \in X\}$ is closed w.r.t. the product topology.

(The $\underset{\text{Zariski}}{\text{topology}}$ of a scheme $X \times X$ is generally stronger than the prod. top.)

eg. $\underset{\mathbb{C}^2}{\overset{\shortparallel}{\mathbb{C}\times\mathbb{C}}} \supset \{(a,a) \mid a \in \mathbb{C}\} = V(x-y)$ $x=y$ closed w.r.t. Zariski top, not closed w.r.t. product top.

Def: $f \in \mathrm{Hom}_{Sch}(X, Y)$

- $$\begin{tikzcd} X \arrow[r, "\Delta", "\exists!"'] \arrow[rr, bend left, "Id"] \arrow[dr, bend right, "Id"'] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$, Δ is called the diagonal morphism.

Here, it is a relative version.
($Y = \mathrm{spec}\, k$, it is)

* U, V : affine $\Rightarrow$ $U \cap V \cong \Delta(X) \cap (U \times_R V)$ (affine) closed in $U \times_R V$ $\Rightarrow$ affine.

- f is separated if Δ is a closed immersion.
  Here, we say that X is separated over Y.
- X is separated if it is separated over spec $\mathbb{Z}$. ($\mathbb{Z} \longrightarrow \mathcal{O}_X(U)$ $\forall U$)

Prop 1: $f : X \to Y$ is separated $\Leftrightarrow$ $\Delta(X)$ is a closed subset of $X \times_Y X$.

(pf): "$\Rightarrow$" : o.k.

"$\Leftarrow$": • $\because$ $p_1 \circ \Delta = Id$

$\therefore$ $X \to \Delta(X)$ is a homeomorphism.

• Any $p \in X$, $f(p) \in \mathrm{spec}\, A \subset Y$ and take $\underset{p\in}{\mathrm{spec}}\, B \subset f^{-1}(\mathrm{spec} A)$.

So $f : \underset{U}{\overset{\shortparallel}{\mathrm{Spec}\, B}} \longrightarrow \underset{V}{\overset{\shortparallel}{\mathrm{spec} A}}$

Lemma 1.1: If $f : X \to Y$ with X, Y affine, then f is separated.

(pf): Let $X = \mathrm{spec}\, A$ and $Y = \mathrm{spec}\, B$.

$$\begin{tikzcd} A & A \otimes_B A \arrow[l, "\exists!"', "a\otimes 1"] & A \arrow[l, "1\otimes b"'] \arrow[ll, bend right, "Id"'] \\ & A \arrow[u] \arrow[ul, "Id"] & B \arrow[l] \arrow[u] \end{tikzcd}$$

$A \otimes_B A \longrightarrow\!\!\!\!\rightarrow A$

$a \otimes b \longmapsto\!\!\!\!\rightarrow ab$

$\Rightarrow$ $\underset{X}{\overset{\shortparallel}{\mathrm{Spec}\, A}} \xrightarrow[\Delta]{} \underset{X \times_Y X}{\overset{\shortparallel}{\mathrm{Spec}\, A \otimes_B A}}$ is a closed immersion.

"$\Leftarrow$": For $x, y \in X$ with $x \neq y$, $p = (x,y) \in X \times X$. $\because x \neq y$ $\therefore p \notin \Delta \to p \in \underset{U \times V}{\overset{\parallel}{B}} \subseteq (X \times X) \setminus \Delta \rightsquigarrow x \in U, y \in V$ & $U \cap V = \phi$

"$\Rightarrow$": Take $p \in X \times X \setminus \Delta$, say $p = (x,y)$ with $x \neq y$ and pick $\underset{\text{open}}{U, V \subseteq X}$ with $x \in U$, $y \in V$ & $U \cap V = \phi$.
Then $p \in U \times V$ & $\underset{\text{open}}{U \times V} \subseteq X \times X \setminus \Delta$. Hence $X \times X \setminus \Delta$ is open $\rightsquigarrow \Delta$ is closed

Hence $\Delta_U : U \longrightarrow U \times_V U$ is a closed immersion

$\Rightarrow \Delta_U^{\#} : \mathcal{O}_{U \times_V U} \twoheadrightarrow \Delta_{U*} \mathcal{O}_U$

$\Rightarrow \mathcal{O}_{X \times_Y X, \Delta(p)} \twoheadrightarrow \mathcal{O}_{X,p}$

Therefore, $\mathcal{O}_{X \times_Y X} \twoheadrightarrow \Delta_* \mathcal{O}_X$. ✡

Eg 1: Any immersion is separated.

(pf): • $f : Y \longrightarrow X$ is a closed immersion :
$y \longmapsto f(y) \in U$

Let $U = \operatorname{spec} A \subset X$ and thus $V = f^{-1}(U) \xrightarrow[\sim]{f} \operatorname{spec} A/I \hookrightarrow \operatorname{spec} A$.
$\phi^{\#}$

$\because \underset{\bar a}{A/I} \otimes_A \underset{\bar b}{A/I} \cong \dfrac{A/I}{I \cdot A/I = 0} \cong \underset{\overline{ab}}{A/I}$ $\therefore \Delta|_V : V \xrightarrow{\sim} V \times_U V$

$\left(\begin{array}{l} M \otimes_A A/I \cong M/IM \\ m \otimes \bar a \longmapsto \overline{am} \end{array} \right)$ $\Rightarrow \Delta : Y \xrightarrow{\sim} Y \times_X Y$

An isomorphism is a closed immersion.

• $g : Z \longrightarrow X$ is an open immersion :
$z \longmapsto g(z) \in U$

Let $U = \operatorname{spec} A \subset f(Z)$ and thus $V = f^{-1}(U) \xrightarrow{\sim} \operatorname{spec} A$.

Trivially, $\Delta|_V : V \xrightarrow{\sim} V \times_U V \Rightarrow \Delta : Z \xrightarrow{\sim} Z \times_X Z$

$\Rightarrow \Delta$ is a closed immersion ✡.

Eg 2: Let $X = \operatorname{spec} \mathbb{C}[X]$, $Y = \operatorname{spec} \mathbb{C}[Y]$ and
$U = X \setminus \{0\} = \operatorname{spec} \mathbb{C}[X, \frac{1}{X}]$, $V = Y \setminus \{0\} = \operatorname{spec} \mathbb{C}[Y, \frac{1}{Y}]$.

We have $\varphi : U \xrightarrow{\sim} V$ via $\mathbb{C}[Y, \frac{1}{Y}] \xrightarrow{\sim} \mathbb{C}[X, \frac{1}{X}]$.
$f(Y, \frac{1}{Y}) \longmapsto f(X, \frac{1}{X})$.

Let Z be the scheme obtained by gluing X and Y through φ.
Then Z is not separated over $\mathbb{C}$.
Indeed, $X \times_{\mathbb{C}} X$, $X \times_{\mathbb{C}} Y$, $Y \times_{\mathbb{C}} X$, $Y \times_{\mathbb{C}} Y$ cover $Z \times_{\mathbb{C}} Z$

and the closed points of $\left\{ \begin{array}{l} Z \times_{\mathbb{C}} Z = \{ (z_1, z_2) : z_1, z_2 : \text{closed points in } Z \} \\ \Delta(Z) = \{ (z, z) : z : \text{closed point} \in Z \} \end{array} \right.$

and $\Delta(Z) \cap (X \times_{\mathbb{C}} Y) = \{ (x, x) \mid x \in X \cap Y = \underset{\substack{\uparrow \\ \text{open}}}{U} \}$ is not closed in $(X \times_{\mathbb{C}} Y)$ since $(0_X, 0_Y)$ is missing.

✻.

$\forall p \in Y$, $\exists V_p$ s.t. $f^{-1}(V_p)$ is quasi-compact.

Prop 2: Let $f : X \to Y$ be a quasi-compact morphisms of schemes. Then $f(X)$ is a closed subset of Y

$\Leftrightarrow$ f is stable under specialization.

$(\underset{\substack{\uparrow \\ \text{a specialization of } x_1}}{x_0} \in \overline{\{x_1\}}$, if $x_1 \in f(X)$, then $x_0 \in f(X)$.)

(pf): "$\Rightarrow$" $x_1 \in f(X)$: closed $\Rightarrow$ $x_0 \in \overline{\{x_1\}} \subset f(X)$.

"$\Leftarrow$": • key case: $X = \operatorname{spec} A$, $Y = \operatorname{spec} B$ with (X) Y reduced and $\underset{\overline{f(X)}}{} \ Y = \overline{f(X)}$: $\quad \nearrow Y = f(X)$ $\quad \overset{\cap}{}$

•• $B \hookrightarrow A$:

Assume $\underset{f^{\#}: B \to A}{} \ q \longmapsto \langle 0 \rangle$. $\Rightarrow$

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ B/q \arrow[rr] & & A \end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd} \operatorname{spec} A \arrow[r] \arrow[dr] & \operatorname{Spec} B/q = V(q) \arrow[d] \\ & \operatorname{Spec} B = \overline{f(X)} \end{tikzcd}$$

Spec A → Spec B
$\nexists\, q \to \overset{\psi}{p} \in \overline{\{p'\}} = V(\overset{\vee}{p'})$ minimal prime.

$\Rightarrow \operatorname{Spec} B/q = \operatorname{Spec} B \Rightarrow q \subset \sqrt{0} = \langle 0 \rangle \Rightarrow q = \langle 0 \rangle$.

•• If p' is a minimal prime, then $B_{p'}$ is a field:

First, we claim that $\forall b \in p'$, $\exists x \in B \setminus p'$ $k \in \mathbb{N}$ s.t. $b^k x = 0$.

Indeed, consider $S = \{ b^k x \mid k \in \mathbb{N}, x \in B \setminus p' \}$ which is multiplicatively closed. If $0 \notin S$, then $\exists P_0 \in \operatorname{Spec} B$ and $P_0 \cap S = \phi$.

$\Rightarrow P_0 \subset p'$ and $P_0 \neq p'$ $\quad \underset{b}{\not\ni} \in$

since if $\exists y \in P_0$ but $y \notin p'$ then $yb \in S$, however $\underset{\times}{yb \in P_0}$.

Then it implies that $\forall b \in p'$, $s \notin p' \Rightarrow \frac{b}{s} \in \operatorname{nil}(B_{p'}) = \operatorname{nil}(B)_{p'} = \{0\}$.

•• $\forall p \in \operatorname{Spec} B$, $\exists q \in \operatorname{Spec} A$ s.t. $f(q) = p$:

Let $\underset{\subset p}{p'}$ be a minimal prime in B, i.e. $p \in \overline{\{p'\}} = V(p')$.

$S = \{ R \setminus q \mid \overset{\operatorname{spec} B}{\overset{\psi}{q}} \subset p \}$

$R \setminus q_1 \subseteq R \setminus q_2 \subseteq \cdots$
$\Rightarrow q_1 \supseteq q_2 \supseteq \cdots$

$\Rightarrow \bigcap_{i=1}^{\infty} q_i \in \operatorname{Spec} B \Rightarrow$ Zorn's lemma $\exists$ max. element $R \setminus p'$.

Then $B \hookrightarrow A \Rightarrow$

$$\begin{tikzcd} B_{p'} \arrow[r, hook] & A_{p'} \cong A \otimes B_{p'} & A \arrow[l] \\ \{0\} & \mathfrak{q}_0 \arrow[l] \arrow[r] \arrow[u, phantom, "\cup"] & \mathfrak{q}' \arrow[u, phantom, "\cup"] \\ & \mathfrak{q}'_{p'} \arrow[u, phantom, "\parallel"] & \text{Max} A_{p'} \arrow[ul] \end{tikzcd}$$

So $\underset{(\mathfrak{q}' \cap B)_{p'}}{\overset{\parallel}{\mathfrak{q}'_{p'} \cap B_{p'}}} = \underset{p'_{p'}}{\overset{\parallel}{\{0\}}} \Rightarrow \mathfrak{q}' \cap B = p' \Rightarrow p' \in f(X)$

By assumption, $p \in f(X)$.

- General case : $\overset{}{(X, (\mathcal{O}_X)_{red})}$ the associated sheaf to $U \to \mathcal{O}_X(U)_{red} = \frac{\mathcal{O}_X(U)}{\sqrt{0}}$; $\mathcal{O}_X(U) \to \frac{\mathcal{O}_X(U)}{\sqrt{0}}$

Consider $(X_{red} \longrightarrow) X \longrightarrow \overline{f(X)} \longrightarrow Y$

with reduced induced structure.

so we can assume that $(X,)\ Y$ ~~(are)~~ is reduced and $Y = \overline{f(X)}$.

Hope " $\forall y \in Y \Rightarrow y \in f(X)$."

For $\overset{y \in U}{p} \in \operatorname{Spec} B \subset Y$, $f : \underbrace{f^{-1}(U)}_{\text{quasi-compact}} \longrightarrow U$, so we can assume that Y is affine.

By assumption of quasi-compactness, $X = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$, so $y \in \overline{f(X)} = \bigcup_{i=1}^{n} \overline{f(\operatorname{Spec} A_i)}$

$\Rightarrow y \in f(\operatorname{Spec} A_t)$ for some t.

Hence we can reduce it to the key case. ※

補 : U, V : affine in X which is separated over an affine scheme $S \Rightarrow U \cap V = \Delta(X) \cap (U \times_S V)$: closed in $\overset{\text{affine}}{U \times_S V} \rightsquigarrow$ affine

◉ Valuative Criterion

Recall that an integral domain R is called a valuation ring if $\exists$ a valuation $v : \underset{\text{the q.f. of } R}{K} \longrightarrow \underset{\text{totally ordered abelian group}}{G}$ s.t. $R = \{x \in K \mid v(x) \geq 0\}$

- $v(xy) = v(x) + v(y)$
- $v(x+y) \geq \min\{v(x), v(y)\}$

- R is a local ring with $m = \{x \in K \mid v(x) > 0\}$
- R is a valuation ring $\Leftrightarrow \forall x \in K \setminus \{0\}$, $x \in R$ or $x^{-1} \in R$.

Def: R, S : local rings in a field k.
S is said to dominate R if $R \subset S$ and $m_S \cap R = m_R$.

Prop 3: A local ring R with q.f. K is a valuation ring
$\Leftrightarrow$ R is max with respect to the domination order in $\{S \mid S$: local ring in $K\}$

(pf) "$\Rightarrow$": If a local ring $S \subset K$ dominates R, then (R=S, otherwise) for $\exists\, a \in S \setminus R$, $a^{-1} \in R$.
and thus $a^{-1} \in S$. (Note : $m_S \cap R = m_R$)

We claim that $\underset{= m_S \cap R}{a^{-1} \in m_R}$ and thus $a^{-1} \in m_S \Rightarrow 1 = \underset{S}{\hat{a}} \cdot a^{-1} \in m_S$. ※

Indeed, if $a^{-1} \notin m_R$, then a^{-1} is a unit in R, i.e. $a = (a^{-1})^{-1} \in R$ ※.

since a is not a unit in R, $a^{-1} \in m_R = m_S \cap R$. However, $1 = a a^{-1} \in m_S$ ※

"$\Leftarrow$": **Claim**: $x \in K \setminus \{0\} \Rightarrow$ either $m[x] \neq R[x]$ or $m[x^{-1}] \neq R[x]$. where $m = m_R$

(pf): If not, $\exists$

$$u_0 + u_1 x + \cdots + u_m x^m = 1 \quad \text{—— (1)} \qquad , u_i \in m$$
$$v_0 + v_1 x^{-1} + \cdots + v_n x^{-n} = 1 \quad \text{—— (2)} \qquad v_j \in m$$

where m, n are chosen as small as possible.

Assume that $m \geq n$. (similar for $m \leq n$)

$$(2) \times x^m : \quad (\underbrace{1 - v_0}_{\text{unit in } R}) x^m = v_1 x^{m-1} + \cdots + v_n x^{m-n} \Rightarrow x^m = v_1' x^{m-1} + \cdots + v_n' x^{m-n} \text{ —— (2')}$$

We plug (2)′ into (1) to get a relation with smaller degree of x ※.

Let L be the algebraic closure of R/m and $g : R \to R/m \to L$.
If $(\underset{\text{a subring in } K}{R'}, g') \geq (R, g)$, i.e. $R' \supseteq R$, $g'|_R = g$, $g' : R' \to L$
then $p' := \ker g'$
$\Rightarrow p' \cap R = m$
$\Rightarrow R'_{p'}$ dominates R
$(R \hookrightarrow R' \Rightarrow R_p \hookrightarrow R'_{p'})$
$\Rightarrow R'_{p'} = R$
$\Rightarrow R' = R$.

no need !

For $x \in K \setminus \{0\}$, by the above claim, we assume that $m[x] \neq R[x]$.
$\exists$ max ideal m' in $R[x]$, s.t. $m[x] \leq m'$.
Since $m' \cap R \geq m$, $m' \cap R = m$. Now we have $\left(\begin{matrix} R_m \hookrightarrow R[x]_{m'} \\ \shortparallel \\ R \end{matrix} \right)$ $R[x]_{m'}$ dominates $R_m = R$.
By the max property of R, $R[x]_{m'} = R_m \Rightarrow R[x] = R$, i.e. $x \in R$ (local property)
Similarly, if $m[x^{-1}] \neq R[x^{-1}]$, then $x^{-1} \in R$.

※.

<u>Prop 4</u> : Let R be a valuation ring with q.f. K.

Then $\mathrm{Hom}_{Sch}(\mathrm{Spec} R, X)$

$\mathrm{Spec} K \longrightarrow X \Leftrightarrow \exists x \in X \ \& \ \kappa(x) \hookrightarrow K$

$\updownarrow$

$\{ \{x_0, x_1\} \subset X \mid x_0 \in \overline{\{x_1\}},\ k(x_1) \le K,\ R \text{ dominates } \mathcal{O}_{\overline{\{x_1\}}, x_0} \text{ on } \overline{\{x_1\}} \text{ with its reduced induced structure} \}$

(pf): " $\downarrow$ " : Given $f : T = \mathrm{Spec} R \longrightarrow X$

closed point : $t_0 = m \longmapsto x_0$; $\mathcal{O}_{T,t_0} = R_m = R$ $\Rightarrow f^{\#}_{x_1} : \mathcal{O}_{X,x_1} \to K$ a local homo. i.e $m_{X,x_1} \mapsto \langle 0 \rangle$

generic point : $t_1 = \langle 0 \rangle \longmapsto x_1$; $\underbrace{\mathcal{O}_{T,t_1} = R_{\langle 0 \rangle} = K.}$ $\Rightarrow k(x_1) \hookrightarrow K$

Observe that $f^{-1}(\overline{\{x_1\}})$ is closed and $t_1 \in f^{-1}(\overline{\{x_1\}})$

$\Rightarrow T = \overline{\{t_1\}} \le f^{-1}(\overline{\{x_1\}}) \Rightarrow T = f^{-1}(\overline{\{x_1\}})$ and $x_0 \in f(T) = \overline{\{x_1\}}$.

and T is reduced $\Rightarrow f : \underset{\overset{\shortparallel}{T}}{T_{red}} \to Z = \overline{\{x_1\}}$ with reduced induced structure

It induces the local homomorphisms :

$$\begin{tikzcd}
k(x_1) = \mathcal{O}_{Z,x_1} \arrow[r, hook, "Z = \overline{\{x_1\}} = V(p)"] & \mathcal{O}_{T,t_1} = K \\
\mathcal{O}_{Z,x_0} \arrow[u, hook] \arrow[r, hook] & \mathcal{O}_{T,t_0} = R \arrow[u, hook]
\end{tikzcd}$$

the function field of $Z \rightsquigarrow K(Z) = \mathcal{O}_{Z,x_1}$

the associated sheaf to $U \to \mathcal{O}_X(U)_{red} = \dfrac{\mathcal{O}_X(U)}{\sqrt{0}}$

$(X, (\mathcal{O}_X)_{red}) = X_{red}$

$\mathcal{O}_X(U) \to \dfrac{\mathcal{O}_X(U)}{\sqrt{0}}$

" $\uparrow$ " : $\mathcal{O}_Z(Z) \nearrow \mathcal{O}_{Z,x_0} \hookrightarrow R \Rightarrow \mathrm{Spec} R \longrightarrow Z \overset{\text{closed immersion}}{\hookrightarrow} X_{red} \longrightarrow X$

☆.

<u>Main theorem</u> (Valuative criterion of separatedness)

Let $f \in \mathrm{Hom}_{Sch}(X, Y)$ and X be noeth.

Then f is separated $\Leftrightarrow$ $\forall$ field K and $\forall$ valuation ring R with q.f. K

$$\begin{tikzcd}
\{t_1\} = \mathrm{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d] \\
\mathrm{Spec} R = T \arrow[r] & Y
\end{tikzcd}$$

$\exists$ at most one morphism $T \to X$ s.t. the whole diagram commutes.

eg: $\operatorname{spec}\mathbb{Z}/_{18\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{spec}\mathbb{Z}/_{12\mathbb{Z}} \cong \operatorname{spec}\mathbb{Z}/_{6\mathbb{Z}}$ $\quad$ $\mathbb{Z}\to\mathbb{Z}/_{18\mathbb{Z}}$, $\operatorname{spec}\mathbb{Z}/_{18\mathbb{Z}} \to \operatorname{spec}\mathbb{Z}$

$3\mathbb{Z}/_{18\mathbb{Z}} \longmapsto 3\mathbb{Z}$

$\operatorname{spec}\mathbb{Z}/_{12\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{spec}\mathbb{Z}/_{25\mathbb{Z}} = \operatorname{spec} 0 = \emptyset$

$$\begin{tikzcd}
\operatorname{spec}\mathbb{C}[X] \times_{\mathbb{C}} \operatorname{spec}\mathbb{C}[Y] \arrow[r, phantom, "="] & \operatorname{spec}\mathbb{C}[X,Y] \arrow[dl] \arrow[d, leftrightarrow] \arrow[dr] & \\
\operatorname{spec}\mathbb{C}[X] & \mathbb{C}^2 & \operatorname{spec}\mathbb{C}[Y]
\end{tikzcd}$$

underlying sets are not the same

$$\begin{tikzcd}
\operatorname{spec}\mathbb{C}[X] \arrow[d, leftrightarrow] & \operatorname{spec}\mathbb{C}[Y] \arrow[d, leftrightarrow] \\
\mathbb{C} \arrow[r, dashed, no head] & \mathbb{C}
\end{tikzcd}$$

(pf): "⇒":

$$\begin{tikzcd}
\{t_1\} = \mathcal{U} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, shift left, "h_1"] \arrow[ur, shift right, "h_2"'] & Y
\end{tikzcd}$$

$\overset{\sim}{\longrightarrow}$ $h|_{\mathcal{U}} = \Delta\circ h_1|_{\mathcal{U}}$

$$\begin{tikzcd}
X\times_Y X \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\
X \arrow[u, "\Delta"] & T \arrow[ul, "\exists! h"'] \arrow[r, "h_1"] \arrow[d, "h_2"] \arrow[l, "h|_{\mathcal{U}}"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

t_0, t_1, x_0, x_1, x_1

x_1'
$\shortparallel$

$h_1(t_1) = h_2(t_1) = x_1 \Rightarrow h(t_1) \in \Delta(X)$

$x_0' \in \overline{\{x_1'\}}$ $\quad$ $h(t_0) = \Delta(x_0') \rightsquigarrow p_1h(t_0) = p_2h(t_0)$ $\rightsquigarrow h_1(t_0) = h_2(t_0)$

$\because \Delta(X)$ is closed $\therefore h(t_0) \in \Delta(X)$ i.e. $h_1(t_0) = h_2(t_0) = x_0$

$$\begin{tikzcd}
\mathcal{O}_{Z,x_0} \arrow[r, hook] \arrow[d, hook] & R \arrow[d, hook] \\
\text{Also,} \quad \mathcal{O}_{Z,x_1} = k(x_1) \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & K \\
\mathcal{O}_{Z,x_0} \arrow[u, hook] \arrow[r, hook] & R \arrow[u, hook]
\end{tikzcd}$$

are the same, by prop 4, $h_1 = h_2$.

by $\mathcal{U} \to X$ the same

"⇐": By prop 1, it is sufficient to show "$\Delta(X)$ is a closed subset of $X\times_Y X$".

Since X is noeth, Δ is quasi-compact.

By prop 2, it is sufficient to show "Δ is stable under specialization".

Let $x_1 \in \Delta(X)$ and $x_0 \in \overline{\{x_1\}}$.

Let $Z = \overline{\{x_1\}}$ with the reduced induced structure.

We know that $K := K(Z) = \mathcal{O}_{Z,x_1}$ is the function field of Z

and $\mathcal{O}_{Z,x_0} \hookrightarrow K$.

By prop 3, $R \overset{\exists}{=} \underbrace{\max}_{\text{Zorn's lemma}}\{\overset{\text{element in}}{S} \mid \underbrace{\mathcal{O}_{Z,x_0}}_{S \text{ dominates}}$ and $\underset{\text{local ring}}{S} \subset K\}$ is a valuation ring.

$\{$local rings in K containing $\mathcal{O}_{Z,x_0}\}$

i.e. $\mathcal{O}_{Z,x_0} \hookrightarrow R \hookrightarrow K$ with R a valuation ring.

By prop 4,

$$\begin{tikzcd}[row sep=small]
& & X \arrow[dr, "f"] & \\
T \arrow[r, "\exists g"] & X\times_Y X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & Y \\
& & X \arrow[ur, "f"'] &
\end{tikzcd}$$

$t_0 \longmapsto x_0$

$t_1 \longmapsto x_1$

$\because x_1 \in \Delta(X)$ $\quad$ $\therefore$ $p_1(x_1) = p_1\Delta(a) = a$
$\qquad\ \overset{\shortparallel}{\Delta(a)}$ $\qquad\qquad$ $p_2(x_1) = p_2\Delta(a) = a$ $\Rightarrow$

$$\begin{tikzcd}
\{t_1\} = \mathcal{U} \arrow[r] \arrow[d] \arrow[r, bend left, "x_1"] & X \arrow[d] \\
T \arrow[r] \arrow[ur, shift left, "p_1\circ g"] \arrow[ur, shift right, "p_2\circ g"'] & Y
\end{tikzcd}$$

$\Rightarrow p_1\circ g = p_2\circ g$!!

So $p_1\circ g(t_0) = p_2\circ g(t_0) \Rightarrow p_1(x_0) = p_2(x_0)$ ~~$\Rightarrow$~~ $x_0 \in \Delta(X)$.

$$\begin{tikzcd}
X\times_Y X \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& X \arrow[ul, "\Delta"'] \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

x_0 $\quad$ $p_1(x_0) = p_2(x_0)$

X: Noetherian
⇒ quasi-compact
$\mathcal{U}$:
$\forall\, \mathcal{U} \underset{\text{open}}{\subseteq} X$
Any collection $\mathcal{F}$ of open sets in X has a max element.
~~Otherwise~~ by ACC for open sets.
Now, for $X = \bigcup_\alpha \mathcal{U}_\alpha$, define $\mathcal{F} = \{\bigcup_{\text{finite}} \mathcal{U}_{\alpha_i} \mid$ for a finite union of $\mathcal{U}_\alpha$'s$\}$
Say $\mathcal{U}_{\alpha_1}\cup\cdots\cup\mathcal{U}_{\alpha_n}$ is the max one.
Claim: $X = \mathcal{U}_{\alpha_1}\cup\cdots\cup\mathcal{U}_{\alpha_n}$
If not, $\exists\, x \in X \setminus (\mathcal{U}_{\alpha_1}\cup\cdots\cup\mathcal{U}_{\alpha_n})$
$\overset{\mathcal{U}_\beta}{\ni}$
$\rightsquigarrow \mathcal{U}_{\alpha_1}\cup\cdots\cup\mathcal{U}_{\alpha_n}\cup\mathcal{U}_\beta \supsetneq \mathcal{U}_{\alpha_1}\cup\cdots\cup\mathcal{U}_{\alpha_n}$ ✗

$$\begin{tikzcd}
T \arrow[dr, "g"] \arrow[drr, bend left, "p_1\circ g"] \arrow[ddr, bend right, "p_2\circ g"'] & & \\
& X\times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[dr, "p_1\circ g"] \arrow[drr, bend left, "p_1\circ g"] \arrow[ddr, bend right, "p_2\circ g"'] & & \\
& X \arrow[d, "\Delta"] \arrow[r, "id"] \arrow[dd, bend right, "id"'] & X \arrow[dd, "f"] \\
& X\times_Y X \arrow[r] \arrow[d] & \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

$g = \Delta\circ p_1\circ g$

$x_0 = g(t_0)$
$\quad = \Delta(p_1\circ g(t_0))$

$$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\otimes_{\mathbb{R}}\left(\mathbb{R}[x]/(x^2+1)\right) \cong \mathbb{C}[x]/(x^2+1) \cong \frac{\mathbb{C}[x]}{x+i} \times \frac{\mathbb{C}[x]}{x-i} \cong \mathbb{C}\times\mathbb{C}$$

$$\begin{array}{c}\operatorname{Spec} \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} = \operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C} \\ \wr\| \\ \operatorname{Spec}\mathbb{C}\times_{\mathbb{R}}\operatorname{Spec}\mathbb{C}\end{array}$$

Coro: 1. Separated morphisms are stable under base change.

2. A composition of two separated morphisms is separated

3. $f: X\to Y$, $f': X'\to Y'$ are separated $\Rightarrow$ $f\times f': X\times_S X' \to Y\times_S Y'$ is separated

4. $g\circ f$ is separated $\Rightarrow$ f is separated

pf) 1.

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X' \arrow[r, "p_1"] \arrow[d, "f'"] & X \arrow[d, "f\text{: separated}"] \\
T \arrow[ur, shift left, "\text{If } \exists h_1"] \arrow[ur, shift right, "h_2"'] \arrow[urr, shift left, "p_1\circ h_1"] \arrow[urr, shift right, "p_1\circ h_2"'] \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

If $\exists\, h_1, h_2$,

then f: separated $\Rightarrow$ $p_1\circ h_1 = p_1\circ h_2$

So

$$\begin{tikzcd}
T \arrow[rr, bend left, "p_1\circ h_1 = p_1\circ h_2"] \arrow[r, shift left, "h_1"] \arrow[r, shift right, "\exists !\, h_2"'] \arrow[dr, bend right] & X' = X\times_Y Y' \arrow[r] \arrow[d] & X \arrow[d] \\
& Y' \arrow[r] & Y
\end{tikzcd}$$

$\Rightarrow h_1 = h_2$

2.

$$\begin{tikzcd}
U \arrow[r] \arrow[ddd] & X \arrow[d, "f\text{: sep.}"] \arrow[dd, bend left=60, "?\text{ sep.}"] \\
& Y \arrow[d, "g\text{: sep}"] \\
& Z \\
T \arrow[uuur, shift left, "\exists h_1"] \arrow[uuur, shift right, "h_2"'] \arrow[uur, shift left, "f\circ h_2"] \arrow[uur, shift right, "f\circ h_1"'] \arrow[ur]
\end{tikzcd}$$

g: sep $\Rightarrow$ $f\circ h_1 = f\circ h_2$

f: sep $\Rightarrow$ $h_1 = h_2$

3.

$$\begin{tikzcd}
U \arrow[r] \arrow[dd] & X\times_S X' \arrow[rr, "p_2"] \arrow[dr, "p_1"] \arrow[dd] & & X' \arrow[dr] \arrow[dd, "f'"] & \\
& & X \arrow[rr] \arrow[dd, "f"] & & S \\
T \arrow[r] \arrow[ur, shift left, "\text{If } \exists h_1"] \arrow[ur, shift right, "h_2"'] \arrow[uurrr, shift left, "p_2\circ h_1"] \arrow[uurrr, shift right, "p_2\circ h_1"'] \arrow[urr, shift left, "p_1\circ h_1"] \arrow[urr, shift right, "p_1\circ h_2"'] & Y\times_S Y' \arrow[rr] \arrow[dr] & & Y' \arrow[dr] & \\
& & Y \arrow[rr] & & S
\end{tikzcd}$$

f: sep $\Rightarrow$ $p_1\circ h_1 = p_1\circ h_2$

f': sep $\Rightarrow$ $p_2\circ h_1 = p_2\circ h_2$

So $h_1 = h_2$ by the universal property of $X\times_S X'$.

4.

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
T \arrow[ur, shift left, "\text{If } \exists h_1"] \arrow[ur, shift right, "h_2"'] \arrow[r] \arrow[dr] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}$$

$g\circ f$: sep $\Rightarrow$ $h_1 = h_2$.

5. $f: X\to Y$ is separated $\Leftrightarrow$ for any cover $\{U_i\}$ of Y, $f^{-1}(U_i)\to U_i$ is sep. $\forall i$

pf) "$\Rightarrow$" ~~Ey~~,

$$\begin{tikzcd}
T \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & f^{-1}(U_i) \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\
& U_i \arrow[r, hook] & Y
\end{tikzcd}$$

"$\Leftarrow$" Recall that $f^{-1}(U_i)\times_{U_i} f^{-1}(U_i) = f^{-1}(U_i)\times_Y f^{-1}(U_i) = f^{-1}(U_i)\times_Y X$ cover $X\times_Y X$.

And $\Delta(X)\cap(f^{-1}(U_i)\times_{U_i} f^{-1}(U_i)) = \Delta(f^{-1}(U_i))$ is closed in $f^{-1}(U_i)\times_{U_i} f^{-1}(U_i)$

✱

(修正):

As sets,

$$\begin{tikzcd} \{(x,y)\in X\times Y \mid f(x)=g(y)\} \arrow[d, equal] & & X\times Y \arrow[d, equal] & \\ X\times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] & X\times_{\operatorname{Spec}k} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & S & X \arrow[r] & \operatorname{Spec}k \end{tikzcd}$$

However, as schemes, $X\times_S Y$ turns out to be more complicated.

eg. • $\operatorname{Spec}\mathbb{Z}/12\mathbb{Z} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/18\mathbb{Z} \cong \operatorname{Spec} \mathbb{Z}/12\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/18\mathbb{Z} = \operatorname{Spec}\mathbb{Z}/6\mathbb{Z}$

$\mathbb{Z}\to\mathbb{Z}/12\mathbb{Z}$

$3\mathbb{Z}/12\mathbb{Z}$

$\mathbb{C}$ with Zariski topology.

• $\left(\mathbb{A}^1_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} =\right) \operatorname{Spec}\mathbb{C}[x]\times_{\operatorname{Spec}\mathbb{C}}\operatorname{Spec}\mathbb{C}[y] \cong \operatorname{Spec}\mathbb{C}[x]\otimes_{\mathbb{C}}\mathbb{C}[y] = \operatorname{Spec}\underset{\langle x-y\rangle}{\underset{\cup}{\mathbb{C}[x,y]}} \left(=\mathbb{A}^2_{\mathbb{C}}\right)$

$\not\cong$

$\operatorname{Spec}\mathbb{C}[x]\times\operatorname{Spec}\mathbb{C}[x] \not\nearrow$

• $\operatorname{Spec}\mathbb{C}\times_{\operatorname{Spec}\mathbb{R}}\operatorname{Spec}\mathbb{C} = \operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} \cong \operatorname{Spec}\frac{\mathbb{C}[x]}{\langle x^2+1\rangle} \cong \operatorname{Spec}\left(\frac{\mathbb{C}[x]}{\langle x+i\rangle}\times\frac{\mathbb{C}[x]}{\langle x-i\rangle}\right)$

$\cong \operatorname{Spec}(\mathbb{C}\times\mathbb{C})$

$\cong \operatorname{Spec}\mathbb{C} \underset{\uparrow\ \text{disconnected}}{\dot{\sqcup}} \operatorname{Spec}\mathbb{C}$

We conclude that $p_1(a)=p_2(a) \not\Rightarrow a\in\Delta(X)$.

Go back to the proof of valuative criterion of separatedness:

($f: X\to Y$ is separated $\Leftrightarrow$ $\forall K$, $\forall R$: v.r. with q.f. K

s.t.

$$\begin{tikzcd} \{t_1\}=\operatorname{Spec}K=U \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}R=T \arrow[r] \arrow[ur, shift left, "h_1"] \arrow[ur, shift right, "h_2"'] & Y \end{tikzcd} \Rightarrow h_1=h_2$$)

"⇒":

$$\begin{tikzcd} X\times_Y X \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\ & T \arrow[ul, "h"', "\exists!"] \arrow[r, "h_1"] \arrow[d, "h_2"] & X \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

$h_1(t_1)=h_2(t_1) \Rightarrow p_1h(t_1)=p_2h(t_1) \overset{?}{\underset{1}{\Rightarrow}} h(t_1)\in\Delta(X)$

$\rightsquigarrow$

$$\begin{tikzcd} X\times_Y X \arrow[rrdd, bend left, "p_2"] \arrow[rddd, bend right, "p_1"'] & & \\ & X \arrow[ul, "\Delta"'] & \\ & U \arrow[uul, "h|_U"] \arrow[u, "h_1|_U"] \arrow[r, "h_1|_U"] \arrow[d, "h_2|_U"] & X \arrow[d] \\ & X \arrow[r] & Y \end{tikzcd}$$

$h|_U = \Delta\circ h_1|_U$

$\Rightarrow h(t_1)=\Delta(h_1(t_1)) \in \Delta(X)$

$x_0 \in \{\overline{x_1}\}$ & $x_1 \in \Delta(X) \not\Rightarrow x_0 \in \Delta(X)$

"$\Leftarrow$":

$$\begin{tikzcd}
 & T \arrow[r, "g"] & X \times X \\
 & t_1 \arrow[r, mapsto] & x_1 \\
 & t_0 \arrow[r, mapsto] & x_0
\end{tikzcd}$$

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] \arrow[ur, shift left, "p_1 \circ g"] \arrow[ur, shift right, "p_2 \circ g"'] & Y
\end{tikzcd}$$

$\Rightarrow p_1 \circ g = p_2 \circ g$

(之前 say $p_1 \circ g(t_0) = p_2 \circ g(t_0)$ $\Rightarrow g(t_0) \in \Delta(X)$, $\overset{\shortparallel}{x_0}$)

wrong.

$$\begin{tikzcd}
X \times_Y X \arrow[rrd, bend left, "p_1"] \arrow[rdd, bend right, "p_2"'] & \cancel{X} \arrow[l, "\exists ! \Delta"'] & \\
 & T \arrow[ul, "\exists ! g"'] \arrow[u, "p_1 \circ g"] \arrow[r, "p_1 \circ g"] \arrow[d, "p_2 \circ g"'] & X \arrow[d] \\
 & X \arrow[r] & Y
\end{tikzcd}$$

$\exists !$

$g \overset{\downarrow}{=} \Delta \circ (p_1 \circ g)$

$\Rightarrow x_0 = g(t_0) = \Delta(p_1 \circ g(t_0)) \in \Delta(X)$.

$$\left(\begin{tikzcd}
X \times_Y X \arrow[rrd, bend left, "p_1"] \arrow[rdd, bend right, "p_2"'] & & \\
 & T \arrow[ul, "\Delta \circ (p_1 \circ g)"', "\cancel{\Delta \circ (p_2 \circ g)}"] \arrow[r, "p_1 \circ g"] \arrow[d, "(p_2 \circ g)"'] & X \arrow[d] \\
 & X \arrow[r] & Y
\end{tikzcd}\right)$$

§6. Properness.

◎ Proper morphisms

Recall that a Hausdorff space X is compact $\Leftrightarrow$ $\forall$ top. space Z, the projection $\pi : X \times Z \to Z$ is a closed map.

Def: $f \in \mathrm{Hom}_{Sch}(X, Y)$.

f is said to be proper if it is separated, of finite type, and universally closed. ($\forall Z \to Y$, $X \times_Y Z \to Z$ is a closed morphism)

eg: $\overset{\operatorname{spec}\mathbb{C}[x]}{\mathbb{A}^1_{\mathbb{C}}} \longrightarrow \operatorname{spec}\mathbb{C}$ is separated, of finite type and closed but not universally closed.

$$\left(\overset{\operatorname{Spec}\mathbb{C}[x] \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[y]}{\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}}} \xrightarrow[\pi]{} \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[y] \ : Y = V(xy-1) \longmapsto \overset{\operatorname{Spec}\mathbb{C}[y] \setminus V(y)}{\underset{\overset{\shortparallel}{D(y)}}{\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}}} \text{ which is open} \right)$$

$\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x, y]$

Valuative criterion of properness

Let $f: X \to Y$ be of finite type with X noeth.

Then f is proper $\Leftrightarrow$ $\forall$ field K and $\forall$ valuation ring R with q.f. K

$$\begin{tikzcd}
\operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R = T \arrow[r] \arrow[ur, "\exists !"] & Y
\end{tikzcd}$$

Pf: "$\Rightarrow$": Separatedness $\Rightarrow$ uniqueness.

Existence:

$Z = \overline{\{x_1\}}$ with reduced induced structure

$$\begin{tikzcd}
 & X \underset{Y}{\times} T =: X_T \arrow[drr, bend left] \arrow[ddr, bend right, "f'"'] & & \\
x_1 & t_1 \arrow[l] & U \arrow[ul, "\exists !"'] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & & T \arrow[r] & Y
\end{tikzcd}$$

$\because$ f is proper $\Rightarrow$ f' is closed $\therefore$ $f'(Z)$ is closed in T.

And $t_1 \in f'(Z) \Rightarrow \underset{\overset{\shortparallel}{T}}{\overline{\{t_1\}}} \subseteq f'(Z) \subseteq T \Rightarrow f'(Z) = T$

So $\exists$ $x_0 \in \overset{\overline{\{x_1\}}}{\cancel{X_T}}$ s.t. $f'(x_0) = t_0$ and

$$\begin{tikzcd}
o = q \arrow[r, mapsto] & \{0\} \arrow[ddddd, bend left] \\
\cap & \cap \\
R \arrow[r, hook] \arrow[d] & \mathcal{O}_{Z,x_0} \arrow[d] \\
K & \mathcal{O}_{Z,x_1} = k(x_1) \arrow[l, hook'] \\
\cup & \cup \\
\{0\} & \{0\} \arrow[l, mapsto]
\end{tikzcd}$$

$R \to \mathcal{O}_{Z,x_0}$ is a local homo.

$\mathcal{O}_{Z,x_0}$ is a local ring in K

By prop 3, R is max in {local rings in K}

so $R \overset{\leq}{\cong} \mathcal{O}_{Z,x_0}$ $\longrightarrow$ i.e. R dominates $\mathcal{O}_{Z,x_0}$,

By prop 4, $\exists$

$$\begin{tikzcd}
T \arrow[r] & X_T \arrow[r] & X \\
t_0 \arrow[r, mapsto] & x_0 & \\
t_1 \arrow[r, mapsto] & x_1 &
\end{tikzcd}$$

※.

"$\Leftarrow$": Uniqueness $\Rightarrow$ separatedness

Universally closed:

$$\begin{tikzcd}
 & \bigcup_{i=1}^{n} \operatorname{Spec} A_i \underset{R}{\otimes} S \arrow[drr, bend left] \arrow[ddd, bend right] & & \\
Z \arrow[r, phantom, "\subseteq"] \arrow[r, phantom, "\text{closed}"'] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \arrow[r, phantom, "\supseteq"] & \bigcup_{i=1}^{n} \operatorname{Spec} \underset{\underset{R[a_1, a_n]}{\shortparallel}}{A_i} \arrow[ddl, bend left] \\
 & Y' \arrow[r] \arrow[d, phantom, "\cup"] & Y \arrow[d, phantom, "\cup"] & \\
 & \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R &
\end{tikzcd}$$

with reduced induced structure

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"To show that $f'(Z)$ is closed" $\Leftarrow$ locally of finite type & quasi-compact.

Since f is of finite type $\Rightarrow f'$ is of finite type $\Rightarrow f'$ is quasi-compact
$\Rightarrow f'|_Z$ is quasi-compact,

it suffices to show that $f'|_Z$ is stable under specialization.

Let $y_0 \in \overline{\{y_1\}}$ and $y_1 = f'(z_1)$ for some $z_1 \in Z$.

Consider $W = \overline{\{y_1\}}$ with reduced induced structure.

$$\begin{tikzcd} z_1 \arrow[r, mapsto] & y_1 \arrow[r, Rightarrow] & \mathcal{O}_{Y',y_1} \arrow[r] & \mathcal{O}_{X',z_1} \arrow[r] & k(z_1) \arrow[r, Rightarrow] & K(W) = k(y_1) \arrow[r, hook] & k(z_1) \\ & & m_{Y',y_1} \arrow[u, phantom, "\cup"] & m_{X',z_1} \arrow[u, phantom, "\cup"] & k(y_1) \hookrightarrow k(z_1) & \mathcal{O}_{W,y_0} \arrow[u, hook] & K \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

Find that $z_1 \longmapsto y_1 \Rightarrow \mathcal{O}_{Y',y_1} \longrightarrow \mathcal{O}_{X',z_1} \longrightarrow k(z_1) \Rightarrow K(W) = k(y_1) \hookrightarrow k(z_1)$, with $K(W) = \mathcal{O}_{W,y_1}$ and $k(z_1) = K$.

$y_1 \in \operatorname{Spec} R \subset Y'$, $\overline{\{y_1\}} = V(p) = \operatorname{Spec} R/p$

p

$\mathcal{O}_{Y,y_1} = R_p$ $\quad R_p/pR_p \leq (R/p)_0$

Let R be max in {local rings containing $\mathcal{O}_{W,y_0}$ in K}, which is a valuation ring by prop 3.

$\{ k(y_1) \hookrightarrow K$

Now, $\mathcal{O}_{W,y_0} \hookrightarrow R \hookrightarrow K \Rightarrow \operatorname{Spec} K \to \operatorname{Spec} R \longrightarrow W \hookrightarrow Y' \Rightarrow$

$U = \operatorname{Spec} K$, $T = \operatorname{Spec} R$

$\mathcal{O}_W(W) \to \mathcal{O}_{W,y_0}$ $\quad y_0 \in \overline{\{y_1\}}$

$K := k(z_1)$

$$\begin{tikzcd} & z_1 \arrow[d, phantom, "\wedge"] & \\ t_1 \in U \arrow[ur, bend left] \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ T \arrow[r, "\exists g"'] \arrow[ur, "\exists! h'"] \arrow[urr, "\exists! h"'] \arrow[d, phantom, "\Downarrow"] & Y' \arrow[r] \arrow[d, phantom, "\Downarrow"] & Y \\ t_1 \arrow[r] & y_1 & \\ t_0 \arrow[r] & y_0 & \end{tikzcd}$$

by the universal property of X'

$X' = Y' \times_Y X$

$\because t_1 \in h'^{-1}(Z)$

$\therefore T = \overline{\{t_1\}} \leq h'^{-1}(Z)$

$\Rightarrow h'(T) \leq Z$

$\Rightarrow h'(t_0) = z_0 \in Z$

$\Rightarrow y_0 = g(t_0) = f' \circ \underbrace{h'(t_0)}_{\in f'(Z)}$ *

All schemes are noetherian.

Cor: 1. Proper morphisms are stable under base change.

2. A composition of proper morphisms is proper.

3. Product of proper morphisms are proper

4. $g \circ f$: proper + g: separated $\Rightarrow f$: proper
(of finite type) (quasi-compact) (of finite type)

~~5.~~ 6. $f: X \to Y$ is proper $\Leftrightarrow Y = \cup U_i$ s.t. $f^{-1}(U_i) \to U_i$ is proper $\forall i$

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \supset U \text{spec} \arrow[d, "f"] \\ T \arrow[r, "p_2"] \arrow[ur, "\exists!"] \arrow[dr] & Y \supset U \text{spec} \arrow[d, "g"] \\ & Z \supset \operatorname{Spec} R \end{tikzcd}$$

$\because g$ is separated $\therefore f \circ h = p$

i.e. $\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, "\exists"] & Y \end{tikzcd}$

5. A closed ~~scheme~~ immersion is proper.

- finite $\Rightarrow$ of finite type.
- A closed immersion is stable under base change & a closed immersion $\Rightarrow$ a closed map

$B_{ij} = A_i \otimes_{A_i} R[x_1, \cdots, x_n]$

$$\begin{tikzcd} t_1 \in U \arrow[r] \arrow[d] & Z \ni z_1 \arrow[d, "i"] \\ T \arrow[r, "g"] \arrow[ur, "g'"] & X \end{tikzcd}$$

$t_1 \in g^{-1}(i(Z))$

$\overline{\{t_1\}} \subset g^{-1}(i(Z))$

$T \xrightarrow{\exists} i(Z)$

$R \to A_i \to R[x_1, \cdots, x_n]$

$g^{-1}(i(Z))$ is closed in T

$B_{ij} = $ f.g. R-alg.

$B_{ij} = R[x_1, \cdots, x_n] = A_i[x_1, \cdots, x_n]$

PERFECTION

$f\,h(T) \subseteq U_i$

$h(T) \subseteq f^{-1}(U_i)$

$\forall i$ $\cong$ ~~$X \times_Y U_i$~~

(pf) : "⇒" :

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & f^{-1}(U_i) \arrow[r] \arrow[d, "f_i"] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[urr, "\exists ! h"] & U_i \arrow[r, hook] & Y
\end{tikzcd}$$

f_i is proper by 1.

"⇐" :

- of finite type : $\forall\, y \in Y$, say $y \in U_i$, take $y \in \operatorname{Spec} R \subset U_i \subset Y$ and $\underset{f^{-1}(\operatorname{Spec} R)}{\overset{\shortparallel}{f_i^{-1}(\operatorname{Spec} R)}} = \bigcup_{j=1}^{m} \operatorname{Spec} A_j$ with A_j : f.g. R-alg.

- separated : Recall that $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i) = f^{-1}(U_i) \times_Y f^{-1}(U_i) = f^{-1}(U_i) \times_Y X$ cover $X \times_Y X$

And $\Delta(X) \cap (f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)) = \Delta(f^{-1}(U_i))$ is closed in $f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$

$\Rightarrow \Delta(X)$ is closed in $X \times_Y X$.

- universally closed

$$\begin{tikzcd}
\underset{Z_i \times_Y X}{\overset{f'^{-1}(Z_i)}{\shortparallel}} = Z_i \times_{U_i} X_i \subset Z \times_Y X \arrow[r] \arrow[d, "f'"] \arrow[dd, bend right, "f_i'"'] & X \supset X_i \arrow[d, "f"] \arrow[dd, bend left, "f_i"] \\
Z \arrow[r, "g"] & Y \\
Z_i \arrow[u, phantom, "\cup"] \arrow[r] & U_i \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

f_i' : closed map

f_i' : closed $\forall i$

$\Rightarrow f'$: closed

(A : closed in $Z \times_Y X \Rightarrow A \cap (Z_i \times_Y X)$: closed in $Z_i \times_Y X$

$\Rightarrow f_i'(A \cap (Z_i \times_Y X)) = f'(A) \cap Z_i$: closed in Z_i $\forall i$

$\Rightarrow f'(A)$ is closed in Z)

◉ Projective morphisms

Def : $\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \cdots, x_n]$: projective n-space over A

Recall that $\underset{\text{homo. prime ideal}}{\mathfrak{p}} \subset k[x_0, \cdots, x_n]$, the $\underset{V(\mathfrak{p})}{\text{zero}}$ set of $\mathfrak{p}$ is called a projective variety.

$k^{n+1} \setminus \{0\} / \sim$

$\shortparallel$

$\mathbb{P}^n_k \rightsquigarrow \operatorname{Proj} k[x_0, \cdots, x_n]$

generalization

$\operatorname{Proj} k[x_0, \cdots, x_n] / \mathfrak{p}$

Fact 1: If $A \to B$, then $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B$.

(pf): Since $\forall\, P \in \operatorname{Proj} \underbrace{A[x_0, \dots, x_n]}_{\overset{\shortparallel}{S}}$, $\exists\, x_i$ s.t. $x_i \notin P$ otherwise $P \supseteq \underset{\not\to}{S_+}$,

$$\operatorname{Proj} S = \bigcup_{i=0}^{n} D_+(x_i) \cong \bigcup_{i=0}^{n} \operatorname{Spec} S_{(x_i)} = \bigcup_{i=0}^{\infty} \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

For each i,

$$\begin{aligned} \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \times_{\operatorname{Spec} A} \operatorname{Spec} B &\cong \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \otimes_A B \\ &\cong \operatorname{Spec} B\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \\ &= \operatorname{Spec} B[x_0, \dots, x_n]_{(x_i)} \end{aligned}$$

$\Big\downarrow$ glue together $\qquad\qquad$ $\Big\downarrow$ glue together

$\operatorname{Proj} A[x_0, \dots, x_n] \times_{\operatorname{Spec} A} \operatorname{Spec} B \qquad \operatorname{Proj} B[x_0, \dots, x_n]$ ※

Def:

- $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$: projective n-space over Y

∴ Fibre over y: $\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} k(y) = \mathbb{P}^n_{k(y)}$ ← projective space over $k(y)$

$$\begin{tikzcd} \mathbb{P}^n_{k(y)} \arrow[r, hook] \arrow[d] & \mathbb{P}^n_Y \arrow[r] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\ \operatorname{Spec} k(y) \arrow[r] & Y \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

- $f : X \to Y$ is projective if

$$\begin{tikzcd} f^{-1}(y) \arrow[r, phantom, "\subset"] \arrow[ddrr, bend right] & X \arrow[rr, "f"] \arrow[dr, "i"', "\text{a closed immersion}"] & & Y \arrow[r, phantom, "\ni"] & y \arrow[llll, bend right] \arrow[ddll, bend left] \\ & & \mathbb{P}^n_Y \arrow[ur, "\text{projection}"'] & & \\ & & \mathbb{P}^n_{k(y)} \arrow[u, phantom, sloped, "\subset"] & & \end{tikzcd}$$

analogue: projective variety in $\mathbb{P}^n_{k(y)}$

- $f : X \to Y$ is quasi-projective if

$$\begin{tikzcd} U_y \arrow[r, phantom, "\subset"] \arrow[ddrr, bend right] & X \arrow[rr, "f"] \arrow[dr, "\text{an open immersion}"'] & & Y \arrow[r, phantom, "\ni"] & y \arrow[llll, bend right] \arrow[ddll, bend left] \\ & & X' \arrow[ur, "\text{projective morphism}"'] & & \\ & & V_y \arrow[u, phantom, sloped, "\subset"] & & \end{tikzcd}$$

quasi-projective variety

projective variety

eg: $\mathbb{A}^2_{\mathbb{C}} - \{(0,0)\}$ ← enlarge the category we are interested in.

<u>Fact 2</u>: If S is a graded ring with $S_0 = A$, which is finitely generated as an A-algebra by S_1, then $\operatorname{Proj} S \longrightarrow \operatorname{spec} A$ is a projective morphism.

$$\begin{tikzcd} \operatorname{spec} A \\ \operatorname{spec} A/m \arrow[u] \end{tikzcd} \qquad A \to A/m$$

pf: By assumption, $S \cong A[x_0, \cdots, x_n]/I$ for some homo. ideal I $\rightsquigarrow$ $A[x_0, \cdots, x_n]/I \otimes_A A/m$

$\left(A[x_0, \cdots, x_n] \twoheadrightarrow A[\overset{S}{\overset{\shortparallel}{S_1}}] \right)$

Let $S' = A[x_0, \cdots, x_n]$.

Then $S' \twoheadrightarrow S \Rightarrow$ a closed immersion $\operatorname{Proj} S \hookrightarrow \operatorname{Proj} S' \overset{\mathbb{P}^n_A}{\shortparallel}$

$$\begin{tikzcd} \operatorname{Proj} S \arrow[r, hook] \arrow[d] & \operatorname{Proj} S' \arrow[dl] \\ \operatorname{Spec} A & \end{tikzcd}$$

$\Rightarrow \forall i$, $S'_{(x_i)} \twoheadrightarrow S_{(\bar{x}_i)} \Rightarrow$ a closed immersion $\operatorname{Spec} S_{(\bar{x}_i)} \hookrightarrow \operatorname{Spec} S'_{(x_i)}$ *

$$\begin{tikzcd} S'_{(x_i)} \arrow[r, two heads] & S_{(\bar{x}_i)} \\ & A \arrow[ul] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} S_{(\bar{x}_i)} \arrow[r, hook] \arrow[d] & \operatorname{Spec} S'_{(x_i)} \arrow[dl] \\ \operatorname{Spec} A & \end{tikzcd}$$

<u>Main theorem</u>:

A projective morphism of noeth. schemes is proper.
(resp. quasi-projective) (resp. $\left\{ \begin{matrix} \text{of finite type} \\ \text{separated} \end{matrix} \right.$).

$\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{spec} \mathbb{Z}} Y \longrightarrow \operatorname{spec} \mathbb{Z} \times_{\operatorname{spec} \mathbb{Z}} Y$

pf:

- $$\begin{tikzcd} X \arrow[rr, "\therefore \text{ proper}"] \arrow[dr, "i"', "\text{proper}"] & & Y \\ & \mathbb{P}^n_Y \arrow[ur, "\text{proper}"'] & \end{tikzcd}$$ $\Leftarrow$ "$\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{spec} \mathbb{Z}$ is proper"

- $$\begin{tikzcd} X \arrow[rr, "\therefore \text{ separated}"] \arrow[dr, "\text{open}"', "\Downarrow \text{ separated}"] & & Y \\ & X' \arrow[ur, "\text{proper}"', "\Downarrow \text{ separated}"] & \end{tikzcd}$$ and $$\begin{tikzcd} X \arrow[rr, "\therefore \text{ of finite type}"] \arrow[dr, "i", "\text{open + noeth}"', "\Downarrow * \text{ of finite type}"] & & Y \\ & X' \arrow[ur, "\text{proper}"', "\Downarrow \text{ of finite type}"] & \end{tikzcd}$$

$S \times_S X \cong X$

$$\begin{tikzcd} Z \arrow[r, bend left] \arrow[dr] \arrow[d, bend right] & \\ & X \arrow[r, Rightarrow] \arrow[d] & X \arrow[d] \\ S & S \arrow[r, dash] & S \end{tikzcd}$$

*: $\underbrace{i^{-1}(\operatorname{spec} A)}_{\overset{\shortparallel\wr}{\bigcup_{i=1}^{n} \operatorname{Spec} A_{f_i}}} \hookrightarrow \operatorname{spec} A \subset X'$, $A_{f_i} = A[\frac{1}{f_i}]$: f.g A-alg

The remaining thing is to show "$\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \operatorname{spec} \mathbb{Z}$ is proper".

- of finite type: $\mathbb{P}^n_{\mathbb{Z}} = \bigcup_{i=0}^{n} D_+(X_i) \cong \bigcup_{i=0}^{n} \operatorname{spec} \underbrace{\mathbb{Z}[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]}_{\text{f.g. } \mathbb{Z}\text{-algebra}}$

- Valuative criterion:

$$\begin{tikzcd}
t_1 \in U \arrow[r] \arrow[d] \arrow[r, bend left] & \mathbb{P}^n_{\mathbb{Z}} \ni P_1 \arrow[d] \\
T \arrow[r] & \operatorname{spec}\mathbb{Z}
\end{tikzcd}$$

By induction on n, $n=0$, $\operatorname{Proj} \mathbb{Z}[X_0] = \operatorname{Spec} \mathbb{Z}[X_0]_{(X_0)} = \operatorname{spec}\mathbb{Z}$ O.K.

We can assume that $P_1 \in D_+(X_i)$ $\forall i$

(otherwise, say $P_1 \in \mathbb{P}^n_{\mathbb{Z}} \setminus D_+(X_i) \cong \mathbb{P}^{n-1}_{\mathbb{Z}}$, then it is done by induction hypothesis)

$V(X_i) = \operatorname{Proj} \frac{\mathbb{Z}[X_0, \cdots, X_n]}{\langle X_i \rangle}$

So $P_1 \in \bigcap_{i=0}^{n} D_+(X_i)$, i.e. X_i / X_j, i, j are invertible elements in $\mathcal{O}_{P_1}$

(分子，分母在P_1處皆不為0)

and $\operatorname{spec} K \longrightarrow \mathbb{P}^n_{\mathbb{Z}}$ $\Rightarrow$ $k(P_1) \hookrightarrow K$

$t_1 \longmapsto P_1$ $\qquad \frac{X_i}{X_j} \longmapsto f_{ij}$

Let $\nu : K \to G$ be the valuation w.r.t. R.

Choose k s.t. $\nu(f_{k0}) = \min\{\nu(f_{00}), \nu(f_{10}), \cdots, \nu(f_{n0})\}$

Then $\nu(f_{ik}) = \nu\left(\frac{f_{i0}}{f_{k0}}\right) = \nu(f_{i0}) - \nu(f_{k0}) \geq 0 \Rightarrow f_{ik} \in R$ $\forall i$.

Define

$$\begin{tikzcd}
& \mathbb{Z} \arrow[dl] \arrow[dr] & \\
\mathbb{Z}[\frac{X_0}{X_k}, \cdots, \frac{X_n}{X_k}] \arrow[rr, "\varphi"] \arrow[d, hook] & & R \arrow[d] \\
k(P_1) \arrow[rr, hook] & & K
\end{tikzcd}$$

$\varphi : \frac{X_i}{X_k} \longmapsto f_{ik}$

unique for commuting this diagram

$$\Rightarrow \begin{tikzcd}
\operatorname{spec} K \arrow[r] \arrow[d] & \operatorname{spec} k(P_1) \arrow[r] & D_+(X_k) \subset \mathbb{P}^n_{\mathbb{Z}} \arrow[d] \\
\operatorname{spec} R \arrow[urr, "\exists"] \arrow[rr] & & \operatorname{spec}\mathbb{Z}
\end{tikzcd}$$

For Uniqueness, if $\nu(f_{k0}) = \nu(f_{h0})$ then $\nu(f_{hk}) = 0 \Rightarrow f_{hk} \notin m$ is a unit in R.

Since the following diagram commutes on $D_+(X_k) \cap D_+(X_h)$,

$t_0 \longmapsto \overset{\text{the same}}{P_0}$

$$\begin{tikzcd}
P_0 \subset \mathbb{Z}[\frac{X_0}{X_k}, \cdots, \frac{X_n}{X_k}]_{\frac{X_h}{X_k}} \arrow[r, "\overset{\text{Identified}}{\sim}"] \arrow[dddd] & \mathbb{Z}[\frac{X_0}{X_h}, \cdots, \frac{X_n}{X_h}]_{\frac{X_k}{X_h}} \supset q_0 \arrow[ddddl, bend left] \\
\frac{X_i}{X_k} \arrow[r, mapsto] \arrow[dd, bend right] & \frac{X_i / X_h}{X_k / X_h} \arrow[ddl, bend left] \\
\frac{X_i / X_k}{X_h / X_k} & \frac{X_i}{X_h} \arrow[l, mapsto] \\
f_{ik} = \frac{f_{ih}}{f_{hk}} & \\
m \subset R \arrow[uuuu, bend left] \arrow[uuuur, bend right] &
\end{tikzcd}$$