## §3. First properties of schemes

**Def**: A scheme $(X, \mathcal{O}_X)$ is
- connected if $X$ is connected
- irreducible if $X$ is irreducible
- reduced if $\mathcal{O}_X(U)$ is reduced $\forall U$
- integral if $\mathcal{O}_X(U)$ is an integral domain $\forall U$

**Prop 1**: $(X, \mathcal{O}_X) = (\operatorname{spec} R, \mathcal{O}_{\operatorname{spec} R})$

1. $(X, \mathcal{O}_X)$ is connected iff $R \not\cong R_1 \times R_2$ with $R_i \neq 0$
2. $(X, \mathcal{O}_X)$ is irreducible iff $\sqrt{0}$ is a prime in $R$ — e.g. $\frac{k[x,y]}{\langle xy \rangle}$ : reducible
3. $(X, \mathcal{O}_X)$ is reduced iff $\sqrt{0} = \{0\}$ in $R$ — e.g. $\frac{k[x]}{\langle x^2 \rangle}$ : non-reduced eg.
4. $(X, \mathcal{O}_X)$ is integral iff $R$ is an integral domain.

(pf):

1. "$\Rightarrow$": Assume that $R \cong R_1 \times R_2$ and $e_1 = (1,0)$, $e_2 = (0,1)$.
Given any $P \in \operatorname{Spec} R$, $e_1 e_2 = 0 \in P \Rightarrow e_1 \in P$ or $e_2 \in R$.
If $e_1 \in P$, then $P$ is of the form $R_1 \times I_2$.
and $P$ is a prime $\Rightarrow R/_{R_1 \times I_2}$ is an integral domain $\Rightarrow I_2 \in \operatorname{Spec} R_2$.
Similarly, if $e_2 \in P$, then $P = P_1 \times R_2$.

Also, we find that $e_1 \in P \Rightarrow e_2 \notin P$ and $e_2 \in P \Rightarrow e_1 \notin P$.
Hence $\operatorname{Spec} R = D(e_1) \cup D(e_2)$ and $D(e_1) \cap D(e_2) = \phi$
i.e. $\operatorname{Spec} R$ is not connected $\times$.

"$\Leftarrow$": If $X = U_1 \cup U_2$ with $U_1 \cap U_2 = \phi$, then $R = \Gamma(X, \mathcal{O}_X) \cong \underbrace{\Gamma(U_1, \mathcal{O}_X)}_{R_1} \times \underbrace{\Gamma(U_2, \mathcal{O}_X)}_{R_2}$ $\times$

Here, we use that $D(f) = \phi \Leftrightarrow f \in \sqrt{0}$

2. "$\Rightarrow$": $ab \in \sqrt{0} \Rightarrow \operatorname{Spec} R = V(ab) = V(a) \cup V(b) \Rightarrow \operatorname{spec} R = V(a)$ or $\operatorname{spec} R = V(b)$
$D(ab) = \phi$
$\Rightarrow a \in \sqrt{0}$ or $b \in \sqrt{0}$

"$\Leftarrow$": Assume that $\operatorname{Spec} R = V(I) \cup V(J) = V(I \cap J)$ with $V(I) \subsetneq \operatorname{Spec} R$, $V(J) \subsetneq \operatorname{Spec} R$
Let $P_1 \supset I$, $P_1 \not\supset J$ and $P_2 \supset J$, $P_2 \not\supset I$ and choose $a \in J \setminus P_1$, $b \in I \setminus P_2$
Then $ab \in I \cap J \Rightarrow ab \in P \ \forall P \in \operatorname{Spec} R \Rightarrow ab \in \sqrt{0} \Rightarrow a \in \sqrt{0}$ or $b \in \sqrt{0}$
$\Downarrow$ $a \in P \ \forall P$ $\times$ ; $\Downarrow$ $b \in P \ \forall P$ $\times$

$\times$

"⇒" $R = \mathcal{O}_X(X)$ is reduced

"⇐" $\mathcal{O}_X(D(f)) = R_f$ is reduced

finite cover $\mathcal{U}$ $\Rightarrow \mathcal{O}_X(U)$ is reduced

3. (We find that $\mathcal{O}_X(U)$ is reduced $\forall U \Leftrightarrow \mathcal{O}_{X,p}$ is reduced $\forall p$.)

Hence $(X, \mathcal{O}_X)$ is reduced $\Leftrightarrow \operatorname{nil} R_p = 0 \quad \forall p \in X$

$\Leftrightarrow (\operatorname{nil} R)_p = 0 \quad \forall p \in X$

$\Leftrightarrow \operatorname{nil} R = 0.$

4. follows from 2, 3 and prop 2. ※

(no need: quasi-compact)

<u>Prop 2</u>: For a general scheme $(X, \mathcal{O}_X)$, integral $\Leftrightarrow$ reduced + irreducible

(pf): "⇒": • $\forall U$, $\mathcal{O}_X(U)$ is an integral domain $\Rightarrow$ reduced

• If $X = V_1 \cup V_2$ with $V_i \subsetneq X$,


then $U_1 \cap U_2 = \emptyset$ where $U_i = X \setminus V_i$ is open in $X$.

Now $\mathcal{O}_X(U_1 \cup U_2) \cong \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is not an integral domain.

"⇐": Let $f, g \in \mathcal{O}_X(U)$ s.t. $fg = 0$ on $U$.

Then $(fg)_p = 0 \quad \forall p \in U \Rightarrow f_p \cdot g_p = 0 \in m_p \overset{\forall p \in U}{\Rightarrow} f_p \in m_p$ or $g_p \in m_p \quad \forall p \in U$

$(fg)_p = f_p \cdot g_p$ (Since $\mathcal{O}_{X,p}$ ← ring homomorphism from $\mathcal{O}_X(U)$)

$m_p$: the unique max ideal in $\mathcal{O}_{X,p}$

This implies that $U = \underbrace{\{p \mid f_p \in m_p\}}_{Y} \cup \underbrace{\{p \mid g_p \in m_p\}}_{Z}$

<u>Claim</u>: $Y, Z$ are closed in $U$.

(pf): Given any $\operatorname{Spec} B \subset U$, $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(\operatorname{Spec} B)$

$f \longmapsto \bar{f}$

Then $Y \cap \operatorname{Spec} B = \{p \in \operatorname{Spec} B \mid \frac{\bar{f}}{1} \in P_p \subseteq B_p\} = V(\bar{f})$ is closed in $\operatorname{Spec} B$.

We conclude that $Y$ is closed in $U$.

Similarly, $Z$ is closed in $U$. ※

By assumption, $X$ is irr $\Rightarrow U$ is irr $\Rightarrow U = Y$ or $U = Z$.

If $U = Y$, then $\forall \operatorname{Spec} B \subset U \Rightarrow \operatorname{Spec} B = V(\bar{f}) \Rightarrow \bar{f} \in \sqrt{0} = 0$ in $B \Rightarrow f|_{\operatorname{Spec} B} = 0$

$\Rightarrow f = 0$ in $U$ ※

Noetherian property and fibered products

~~§4. Noetherian properties of schemes.~~ 1st

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**Def:** $(X, \mathcal{O}_X)$ is locally noetherian if $X = \bigcup_{i\in\Lambda} \operatorname{Spec} R_i$ with $R_i$ noetherian

noetherian if $(X, \mathcal{O}_X)$ is "locally noeth." + "quasi-compact"

**Remark:**
- $R$ is noetherian $\Rightarrow$ $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is noetherian.
- $(X, \mathcal{O}_X)$ is a noetherian scheme. $\underset{\not\Leftarrow}{\Rightarrow}$ $X$ is a noetherian topological space.

(pf): "$\Rightarrow$": Let $X = \bigcup_{i=1}^{n} \operatorname{Spec} R_i$ with $R_i$ noeth.

Given a descending chain of closed sets in $X$,

$$V_1 \supset V_2 \supset V_3 \supset \cdots$$

$$\Rightarrow \forall i, \quad \underset{\displaystyle V(I_{i1})}{\overset{\shortparallel}{V_1 \cap \operatorname{Spec} R_i}} \supset \underset{\displaystyle V(I_{i2})}{\overset{\shortparallel}{V_2 \cap \operatorname{Spec} R_i}} \supset \cdots \quad \text{in } \operatorname{Spec} R_i$$

$$\Rightarrow \quad I_{i1} \subset I_{i2} \subset \cdots \quad \text{in } R_i$$

Choose $m$ large enough s.t.

$$V_m \cap \operatorname{Spec} R_i = V_{m+1} \cap \operatorname{Spec} R_i = \cdots \quad , \forall i$$

Then $V_m = V_{m+1} = \cdots$

"$\not\Leftarrow$": Let $R = \dfrac{k[X_1, X_2, \cdots]}{\langle X_i X_j : i,j = 1,2,\cdots\rangle} = k[\bar{X}_1, \bar{X}_2, \cdots]$

which is not noetherian since $\langle \bar{X}_1 \rangle \subsetneq \langle \bar{X}_1, \bar{X}_2 \rangle \subsetneq \cdots$.

But $\bar{X}_i^2 = 0 \Rightarrow \bar{X}_i \in \sqrt{0} \quad \forall i \Rightarrow \operatorname{Spec} R = \operatorname{Spec} R/\sqrt{0} = \operatorname{Spec} k = \{(0)\}$.
is a noetherian top. space.

Here, $\operatorname{Spec} R$ is not a noetherian scheme by prop 3.

**Prop3:** $(X, \mathcal{O}_X)$ is locally noeth. $\Leftrightarrow$ Any $U = \operatorname{Spec} R \subset X$, $R$ is noeth.
In particular,

$\operatorname{Spec} R$ is a noeth. scheme $\Leftrightarrow$ $R$ is a noeth. ring.

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(pf): "$\Leftarrow$": $X = \bigcup_{i\in\Lambda} \operatorname{Spec} R_i$, $R_i$ : noeth.

"$\Rightarrow$":

Let $X = \bigcup_{i\in\Lambda} \operatorname{Spec} R_i$ with $R_i$ noeth.

Then $\operatorname{Spec} R = U = \bigcup_{i\in\Lambda} \underbrace{(U \cap \operatorname{Spec} R_i)}_{\| \\ \bigcup_j \operatorname{Spec}(R_i)_{g_{ij}}}$ , $(R_i)_{g_{ij}}$ : noeth

We may assume that $\operatorname{Spec} R = \bigcup_{i\in\Lambda'} \underbrace{\operatorname{Spec} A_i}_{\| \\ \bigcup_j \operatorname{Spec} R_{f_{ij}}}$ with $A_i$ noeth.

Observe that

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) & \longrightarrow & \Gamma(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} R}|_{\operatorname{Spec} A_i}) \\ \psi & & \psi \\ f_{ij} \in R & \longmapsto & \bar{f}_{ij} \in A_i \end{array}$$

Since $D(f_{ij}) \subseteq \operatorname{Spec} A_i$ , $\underset{\operatorname{Spec}(A_i)_{\bar{f}_{ij}}}{D(\bar{f}_{ij})} = \underset{\operatorname{Spec} R_{f_{ij}}}{D(f_{ij})}$ $\Rightarrow$ $R_{f_{ij}} \xrightarrow{\sim} (A_i)_{\bar{f}_{ij}}$ is noeth.

Hence we can further assume that $\operatorname{Spec} R = \bigcup_{i\in\Lambda} \operatorname{Spec} \underset{\text{noeth}}{R_{f_i}} \overset{\text{quasi-compact}}{=} \bigcup_{i=1}^{n} \operatorname{Spec} \underset{\text{noeth}}{R_{f_i}}$

and $\varphi_i : R \to R_{f_i}$.

claim: For $I$: ideal in $R$, $I = \bigcap_{i=1}^{n} \varphi_i^{-1}(\varphi_i(I)\cdot R_{f_i})$

(pf): "$\subset$": o.k. "$\supset$": $b \in$ RHS, write $\underset{\frac{b}{1}}{\varphi_i(b)} = \frac{a_i \in I}{f_i^n}$ $\leftarrow$ the same $n$ $\forall i$

Then $\exists$ $m$ (the same $m$) s.t. $f_i^m(f_i^n b - a) = 0$ $\forall i$ $\Rightarrow$ $f_i^{\overset{N}{m+n}} b \in I$

From $\operatorname{Spec} R = \bigcup_{i=1}^{n} D(f_i^N)$ , $1 = \sum_{i=1}^{n} h_i f_i^N$ $\Rightarrow$ $b = \sum_{i=1}^{n} h_i f_i^N b \in I$.

By the claim,

$I_1 \subseteq I_2 \subseteq \cdots \Rightarrow \varphi_i(I_1)\cdot R_{f_i} \subseteq \varphi_i(I_2) R_{f_i} \subseteq \cdots$ stable $\forall i$ since $R_{f_i}$ is Noeth.

$\Rightarrow \bigcap_{i=1}^{n} \varphi_i^{-1}(\varphi_i(I_r) R_{f_i}) = \bigcap_{i=1}^{n} \varphi_i^{-1}(\varphi_i(I_{r+1}) R_{f_i}) = \cdots$ $\forall i$

$\Rightarrow I_r = I_{r+1} = \cdots$

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<u>Def</u> (Finite conditions)

$f : X \to Y$ is locally of finite type if $\forall\, q \in Y \ \exists \ \operatorname{spec} R \subset Y$ ($\ni q$)
s.t. $f^{-1}(\operatorname{spec} R) = \bigcup_{i \in \Lambda} \operatorname{spec} A_i$
and $A_i$ is a f.g. $R$-alg.

of finite type if $\forall\, q \in Y$, $\exists$ $(\forall)$ $\operatorname{spec} R \subset Y$ ($\ni q$) s.t.
$f^{-1}(\operatorname{spec} R) = \bigcup_{i=1}^{n} \operatorname{spec} A_i$ & $A_i$ is a f.g. $R$-alg.

finite if $\forall\, q \in Y$, $\exists$ $(\forall)$ $\operatorname{spec} R \subset A$ ($\ni q$) s.t. $f^{-1}(\operatorname{spec} R) = \operatorname{spec} A$
& $A$ : f.g. $R$-module.

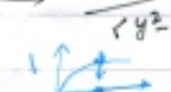
<u>Eg</u>: $\mathbb{C}[x] \hookrightarrow \dfrac{\mathbb{C}[x,y]}{\langle xy^2 - x - 1\rangle} \rightsquigarrow f : \operatorname{spec} \dfrac{\mathbb{C}[x,y]}{\langle xy^2 - x - 1\rangle} \longrightarrow \operatorname{spec} \mathbb{C}[x]$

$k$: Reduced, irreducible scheme of finite type over $k$.
$f : X \to \operatorname{spec} k$
$X = \bigcup_{i=1}^{n} \operatorname{spec} R_i$   $R_i$ : f.g. reduced $k$-alg with no nilpotent ... affine variety

$\dfrac{\mathbb{C}[x,y]}{\langle xy^2 - x - 1\rangle}$ is a f.g. $\mathbb{C}[x]$-alg but not f.g. $\mathbb{C}[x]$-module.

$\Rightarrow$ $f$ is of finite type (but not a finite morphism)

$\mathbb{C}[x] \hookrightarrow \dfrac{\mathbb{C}[x,y]}{\langle y^2 - x\rangle} \rightsquigarrow f : \operatorname{spec} \underbrace{\dfrac{\mathbb{C}[x,y]}{\langle y^2 - x\rangle}}_{\cong\ \mathbb{C}[x] \oplus \bar{y}\,\mathbb{C}[x]} \longrightarrow \operatorname{Spec} \mathbb{C}[x]$



$\mathbb{C}[x] \oplus \bar{y}\,\mathbb{C}[x] \Rightarrow f$ is finite.

<u>Def</u>:
- $(U, \mathcal{O}_U)$ is an open subscheme of $(X, \mathcal{O}_X)$ if $U$ is open in $X$ and $\mathcal{O}_U \cong \mathcal{O}_X|_U$.
- $f : Y \to X$ is called an open immersion if $(f(Y), f_*\mathcal{O}_Y|_{f(Y)})$ which is an open subscheme of $(X, \mathcal{O}_X)$. $\cong (Y, \mathcal{O}_Y)$
- $(V, \mathcal{O}_V)$ is a closed subscheme of $(X, \mathcal{O}_X)$ if $V$ is closed in $X$ and $\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_V$ with $i : V \hookrightarrow X$.
- $f : Y \to X$ is called a closed immersion if it induces an isom. of $(Y, \mathcal{O}_Y)$ onto a closed subscheme of $(X, \mathcal{O}_X)$.

<u>eg.</u>: Affine : $X = \operatorname{spec} R$, a closed set $= V(I) \Leftrightarrow \operatorname{spec} R/I$ , $R \twoheadrightarrow R/I$
conversely, $\varphi : R \twoheadrightarrow R'$ , $R/\ker\varphi \cong R'$
$\updownarrow$
$V(\ker\varphi) = $

<u>Remark</u>: A closed immersion is finite. $\operatorname{spec} R/I \to \operatorname{spec} R$   $R/I = \langle \bar{1} \rangle_R$.
In general, a open immersion is rarely finite. <u>eg</u>. $\underset{\mathbb{C} - \{0\}}{\mathbb{C}^*} \hookrightarrow \mathbb{C} = \operatorname{spec} \mathbb{C}[x]$   $\operatorname{spec} \mathbb{C}[x, x^{-1}]$   $\mathbb{C}[x, x^{-1}]$ is not a f.g. $\mathbb{C}[x]$-module.

Remark: • Every open subset $\mathcal{U}$ of a scheme $(X, \mathcal{O}_X)$ carries a unique structure of open subscheme, i.e. $(\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}})$.

• Any closed subset $Y$ of a scheme $(X, \mathcal{O}_X)$ will have many possible closed subscheme structure.
However, there is one which is smaller than any other, called "the reduced induced closed subscheme structure".

(Construction): (證明): • $X = \operatorname{Spec} R$, $Y = V(I) \hookleftarrow \operatorname{Spec} R/I$

Note that $V(I) = V(J) \Leftrightarrow \sqrt{I} = \sqrt{J}$. So $(Y = \operatorname{Spec} \underbrace{R/\sqrt{I}}_{\text{reduced ring}}, \mathcal{O}_{\operatorname{Spec} R/\sqrt{I}})$ is the desired one.

$\begin{matrix} I & & J \\ \updownarrow & & \updownarrow \\ R/I & & R/J \end{matrix}$

• For general $X$, $Y$, any affine open $\underset{\operatorname{Spec} R}{\mathcal{U}} \subset X$, $\underset{V(I)}{Y \cap \mathcal{U}}$ is closed in $\mathcal{U}$.

Take $\mathcal{O}_Y|_{Y\cap\mathcal{U}} := \mathcal{O}_{\operatorname{Spec} R/\sqrt{I}}$

"$f \in I \Rightarrow V(I) \cap D(f) = \phi$"

Now for $\underset{\operatorname{Spec} R_f}{D(f)}^{V} \subset \underset{\operatorname{Spec} R}{\mathcal{U}}$ with $f \notin I$ !!!, $\mathcal{O}_{\operatorname{Spec} R/\sqrt{I}}\Big|_{V \cap Y} = \mathcal{O}_{\operatorname{Spec} (R/\sqrt{I})_{\bar{f}}}$

$V \cap Y = D(f) \cap V(I) = D(\bar{f})$ in $\operatorname{Spec} R/\sqrt{I}$

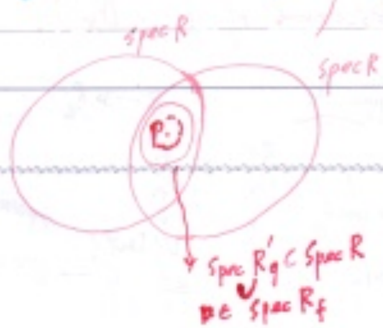
For $\underset{\operatorname{Spec} R_f}{\overset{V}{D(f)}} \subset X$, $Y \cap V = V(I \cdot R_f)$, $\mathcal{O}_Y|_{Y\cap V} := \mathcal{O}_{\operatorname{Spec} R_f / \sqrt{I R_f}}$ $\cong \left(R/\sqrt{I}\right)_{\bar{f}}$, $\sqrt{I R_f} = \sqrt{I}_f$

$\left( \begin{matrix} R & \longrightarrow & R_f \\ \cup & & \\ I & \longmapsto & I \cdot R_f \end{matrix} \right.$ & $V(I) \cap \operatorname{Spec} R_f = V(I \cdot R_f)$ in $\operatorname{Spec} R_f$. $\Big)$

• Let $X = \bigcup \mathcal{U}_i$ and $Y_i = Y \cap \mathcal{U}_i$.

If $\mathcal{O}_Y|_{Y_i}$ is given and $\mathcal{O}_Y|_{Y_i}\Big|_{Y_i \cap Y_j} \cong \mathcal{O}_Y|_{Y_j}\Big|_{Y_i \cap Y_j}$, (~~also it is~~ they are also compatible on $Y_i \cap Y_j \cap Y_k$)

then we can glue them to get $\mathcal{O}_Y$.




$\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \longrightarrow \Gamma(\operatorname{Spec} R'_g, \mathcal{O}_{\operatorname{Spec} R}|_{\operatorname{Spec} R'_g})$

$f \longmapsto \bar{f}$

$D(f) \subset \operatorname{Spec} R'_g \Rightarrow D(f) = D(\bar{f})$

$\Rightarrow (R'_g)_{\bar{f}} \cong R_f$

$R'_{gf'} \cong$ $\varphi: R' \to R'_g$, $f' \to \bar{f}$

⊙ Fibered products

$X, Y, S$ : sets,

$$\begin{tikzcd} \forall Z \arrow[dr, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X\times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "\psi"] \\ & X \arrow[r, "\varphi"'] & S \end{tikzcd}$$

As sets, $X\times_S Y := \{ (x,y) \in X\times Y : \varphi(x) = \psi(y) \}$

Thm 4: Let $X, Y$ be schemes over a scheme $S$, i.e. $\exists\, X \to S$ and $Y \to S$. Then $X \times_S Y$ exists and is unique up to isomorphisms.

(pf): Step 1: $X = \operatorname{spec} A$, $Y = \operatorname{spec} B$, $S = \operatorname{spec} R$, $X\times_S Y = \operatorname{spec} A \otimes_R B$ :

$$\begin{tikzcd} Z \arrow[d] \arrow[dd, bend right] \arrow[r, bend left] & \\ \operatorname{Spec} A\otimes_R B \arrow[r] \arrow[d] & \operatorname{spec} B \arrow[d] \\ \operatorname{spec} A \arrow[r] & \operatorname{spec} R \end{tikzcd} \rightsquigarrow \begin{tikzcd} \Gamma(Z, \mathcal{O}_Z) & \\ A\otimes_R B \arrow[u, "\exists"] & B \arrow[l] \arrow[ul, bend right] \\ A \arrow[u] \arrow[uu, bend left] & R \arrow[l] \arrow[u] \end{tikzcd}$$

by universal property of ⊗

Step 2: If $U \underset{\text{open}}{\hookrightarrow} X$, then "$X\times_S Y$ exists, $p_1 : X\times_S Y \to X$, $p_2 : X\times_S Y \to Y$ $\Rightarrow$ $p_1^{-1}(U) \cong U \times_S Y$":

① •
$$\begin{tikzcd} p_1^{-1}(U) \arrow[r, hook] \arrow[d, "\exists"] \arrow[rr, bend left, "\exists"] & X\times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\ U \arrow[r, hook] \arrow[rr, bend right] & X \arrow[r] & S \end{tikzcd}$$

②
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, "\exists\theta"] & Y \arrow[d] \\ U \arrow[r] & S \end{tikzcd}$$

③
$$\begin{tikzcd} \exists Z \arrow[r] \arrow[dr, "\exists\theta"] \arrow[d, bend right] & Y \arrow[d] \\ U \arrow[r] & S \end{tikzcd} \qquad p_1^{-1}(U) \to Y$$

•
$$\begin{tikzcd} p_1^{-1}(U) \arrow[r, hook] & X\times_S Y \arrow[rr, bend left] & Z \arrow[l, "\exists\theta"'] \arrow[r] \arrow[d] & Y \arrow[d] \\ & & U \arrow[r] \arrow[d, hook] & S \\ & & X \arrow[ur] & \end{tikzcd}$$

key : $\theta(Z) \subset p_1^{-1}(U)$

④
$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\rho"'] \arrow[rr, bend left, "\exists ! \theta"] & p_1^{-1}(U) \arrow[r, hook] \arrow[dl, "p_1"] & X\times_S Y \arrow[dl, "p_1"] \\ U \arrow[r, hook] & X & \end{tikzcd}$$

$p_1 \circ \theta = \rho$

$p_1\circ\theta(Z) = \rho(Z)$

$\Downarrow$ $U$ $\Rightarrow \theta(Z) \subset p_1^{-1}(U)$

$U_{ji}$

$U^i_j = p_1^{(i)^{-1}}(X_{ij}) \subset X_j \times_S Y$

$\downarrow \quad \downarrow p_1^{(i)}$

$X_{ij} \subset X_j$

$\cong$ $X_{ij}\times_S Y$ $\cong$ $\varphi_{ij}$ $\varphi_{ij}$

Step 3: $\{X_i\}$ : an open covering of $X$

$X_i \times_S Y$ exists $\forall i$ $\Rightarrow$ $X\times_S Y$ exists :

Put $X_{ij} = X_i \cap X_j$ if $X_i \cap X_j \neq \phi$. Let ~~$U_{ij} = (p_1^{(i)})^{-1}(X_{ij}) \cong X_{ij}\times_S Y \subset X_i \times_S Y$.~~

$p_1^{(i)^{-1}}(X_{ij}) \subset X_i \times_S Y$

$U^i_i = U_{ij}$ $\quad \downarrow \quad \downarrow p_1^{(i)}$

$X_{ij} \subset X_i$

$$\left(\begin{tikzcd} X_{ji}\times_S Y \arrow[r, shift left, "\varphi'_{ji}"] \arrow[d, "p_1^{(j)}"'] & X_{ij}\times_S Y \arrow[l, shift left, "\varphi'_{ij}"] \arrow[r, "p_2^{(i)}"] \arrow[d, "p_1^{(i)}"] & Y \arrow[d] \\ X_{ji} \arrow[r, equal] & X_{ij} \arrow[r] & S \end{tikzcd} \quad \begin{gathered} \varphi_{ij} : U_{ij} \cong X_{ij}\times_S Y \xrightarrow{\varphi'_{ij}} X_{ji}\times_S Y \cong U_{ji} \\ \Rightarrow \varphi_{ji} = \varphi_{ij}^{-1} \end{gathered}\right)$$

If $X_i \cap X_j \cap X_k \neq \phi$, then $U_{ij} \cap U_{ik} = p_1^{(i)^{-1}}(X_i \cap X_j) \cap p_1^{(i)^{-1}}(X_i\cap X_k) = p_1^{(i)^{-1}}(X_i \cap X_j \cap X_k)$

~~$U^i_j \cap U^i_k$~~

$$\Big\} \varphi_{ij}$$

~~$U^j_i \cap U^j_k$~~

$U_{ji} \cap U_{jk} = p_1^{(j)^{-1}}(X_j \cap X_i) \cap p_1^{(j)^{-1}}(X_j \cap X_k) = p_1^{(j)^{-1}}(X_i \cap X_j \cap X_k)$

Hence $\varphi_{jk}\circ\varphi_{ij}(U_{ij}\cap U_{ik}) = \varphi_{jk}(U_{ji}\cap U_{jk}) = U_{ki}\cap U_{kj} = \varphi_{ik}(U_{ij}\cap U_{ik})$.

We can glue $\{X_i \times_S Y\}$ by $\varphi_{ij}$ to get $X \times_S Y$ and get $p_1, p_2$

s.t. $p_1|_{X_i \times_S Y} = p_1^{(i)}$ and $p_2|_{X_i \times_S Y} = p_2^{(i)}$. (Note: $\varphi_{ij}$ preserves the projection)

• Universal property of $X \times_S Y$ :

$$\begin{tikzcd}
\underset{U_{ij}}{(X_{ij} \times_S Y)} \arrow[r, hook] & X_i \times_S Y & & & \\
Z_i \cap Z_j \arrow[u, "\theta_{ij}"] \arrow[r, hook] \arrow[d] & Z_i \arrow[u, "\theta_i"'] \arrow[r, hook] \arrow[d] & Z \arrow[r] \arrow[d] & Y \arrow[d] \\
X_i \cap X_j \arrow[r, hook] & X_i \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
\underset{U_{ji}}{(X_{ji} \times_S Y)} \arrow[r, hook] & X_j \times_S Y & & & \\
Z_j \cap Z_i \arrow[u, "\theta_{ji}"] \arrow[r, hook] \arrow[d] & Z_j \arrow[u, "\theta_j"'] \arrow[r, hook] \arrow[d] & Z \arrow[r] \arrow[d] & Y \arrow[d] \\
X_j \cap X_i \arrow[r, hook] & X_j \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd}$$

$U_{ij} \xrightarrow{\varphi_{ij}} U_{ji}$

$\theta_i|_{Z_i \cap Z_j}$

$\varphi_{ij}^{\#} \circ \theta_{ij} = \theta_{ji} \Rightarrow$ We can glue $\{\theta_i\}$ to get $\theta: Z \to X \times_S Y$

Step 4: Arbitrary $X, Y, S$.

• Any $X$ ; $Y, S$ : affine : $\quad X_i \times_S Y$ exists $\Rightarrow X \times_S Y$ exists

$X = \bigcup X_i$, $X_i$ : affine

• Any $X, Y$ ; $S$ : affine : $\quad X \times_S Y_i$ exists $\Rightarrow X \times_S Y$ exists

$Y = \bigcup Y_j$, $Y_j$ : affine

• Any $X, Y, S$ :

$S = \bigcup S_i$, $S_i$ : affine.

$\because X_i \times_S Y_i = X_i \times_{S_i} Y_i$ :

$$\begin{tikzcd}
Z \arrow[r, "\exists"] \arrow[rr, bend left] \arrow[ddr, bend right] & X_i \times_{S_i} Y_i \arrow[r, bend left] \arrow[dd, bend right] & Y \supset Y_i \arrow[d] \arrow[dd, bend left] \\
 & X \arrow[r] & S \\
 & X_i \arrow[u, phantom, "\cup"] \arrow[r] \arrow[ur] & S_i \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\because X_i \times_S Y = X_i \times_S Y_i$ :

$$\begin{tikzcd}
Z \arrow[dr, "\exists"] \arrow[rr, bend left] \arrow[rrr, bend left] \arrow[dddr, bend right] & & & \\
 & X_i \times_S Y_i \arrow[r] \arrow[dd] & Y_i \arrow[r, hook] \arrow[d] & Y \arrow[ddl] \\
 & & S_i \arrow[d] \arrow[dl, equal] & \\
 & X_i \arrow[r] & S_i \arrow[r, hook] & S
\end{tikzcd}$$

$Z \to S_i$

$\Rightarrow Z \to Y_i$

$\therefore \{X_i \times_S Y\}$ exist $\Rightarrow X \times_S Y$ exists.

※.

## ◎ Applications of fibered products

- Fibres of a morphism

**Def:** Let $f \in \mathrm{Hom}_{Sch}(X, Y)$ and $y \in Y$.
The fibre of $f$ over $y$ is defined to be by $X_y = X \times_Y \mathrm{Spec}\, k(y)$
where $k(y) := \mathcal{O}_{Y,y} / m_{Y,y}$ is the residue field of $y$.
called

$$\begin{array}{ccc} X_y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\, k(y) & \longrightarrow & Y \\ \langle 0 \rangle & \longmapsto & y \end{array}$$

$\mathcal{O}_{Y,y} \twoheadrightarrow \mathcal{O}_{Y,y}/m_{Y,y}$

$\left( \begin{array}{l} \mathrm{spec}\, k \longrightarrow y \\ \mathcal{O}_{Y,y} \longrightarrow k \text{ : local homo} \\ m_{Y,y} \longmapsto 0 \quad k(y) \hookrightarrow k \end{array} \right)$

Note that
- $X_y$ is a scheme over $k(y)$
- $X_y \underset{\text{homeomorphic}}{\simeq} f^{-1}(y)$.

**prop5:** If $f: X \to Y$ is a finite morphism, then $\forall y \in Y$, $f^{-1}(y)$ is a finite set.

(pf): By def, $\exists\ \mathrm{Spec} R \subset Y$ s.t. $f^{-1}(\mathrm{Spec} R) = \mathrm{Spec} A$ and $A$: f.g $R$-mod. ($y \leftrightarrow p$, $p \in \mathrm{Spec} R$)

We get $f|_{\mathrm{Spec} A} : \mathrm{Spec} A \to \mathrm{Spec} R$ and $\varphi: R \to A$.

So $f^{-1}(y) \simeq \mathrm{Spec} A \times_{\mathrm{Spec} R} \mathrm{Spec}\, k(y) = \mathrm{Spec}\, A \otimes_R k(p)$.

And $A$: f.g $R$-mod $\Rightarrow A \otimes_R k(p)$ : finite dimensional vector space over $k(p)$
$\Rightarrow A \otimes_R k(p)$ is Artinian
$\Rightarrow \#$ of $\mathrm{spec}\, A \otimes_R k(p) < \infty$ $\quad$ //.

**Eg:** $\varphi: \mathbb{C}[x, y] \longrightarrow \mathbb{C}[u, v]$
$x \longmapsto u$
$y \longmapsto uv$

$\Rightarrow f: \underset{\mathbb{C}^2}{\overset{X}{\mathrm{spec}\, \mathbb{C}[u, v]}} \longrightarrow \underset{\mathbb{C}^2}{\overset{Y}{\mathrm{spec}\, \mathbb{C}[x, y]}}$

$\dfrac{\mathbb{C}[u,v]}{\langle x-a, y-b \rangle \mathbb{C}[u,v]} \leftarrow \mathbb{C}[x,y]$ acts on $\mathbb{C}[u,v]$. ($x - a \mapsto u-a$, $y-b \mapsto uv-b$)

Let $\langle x-a, y-b \rangle$ be a closed point in $Y$. ($\leftrightarrow (a,b)$)

$X_{(a,b)} = \mathrm{Spec}\, \mathbb{C}[u, v] \otimes_{\mathbb{C}[x,y]} \left( \dfrac{\mathbb{C}[x,y]}{\langle x-a, y-b \rangle} \right) \cong \mathrm{Spec}\, \dfrac{\mathbb{C}[u,v]}{\langle u-a, uv-b \rangle}$

$M \otimes_R R/I \cong M/IM$

- $a \neq 0$, $X_{(a,b)} = \mathrm{Spec}\, \dfrac{\mathbb{C}[u,v]}{\langle u-a, v-\frac{b}{a} \rangle} \cong \mathrm{Spec}\, \mathbb{C} \leftarrow$ point
- $a = 0, b \neq 0$, $\langle u, uv-b \rangle = \langle 1 \rangle \Rightarrow X_{(0,b)} = \phi$
- $a = 0, b = 0$, $X_{(0,0)} = \mathrm{Spec}\, \dfrac{\mathbb{C}[u,v]}{\langle u, uv \rangle} \cong \mathrm{Spec}\, \mathbb{C}[v] \leftarrow$ line

$$\begin{array}{ccc} \# & \longleftarrow & \dfrac{\mathbb{C}[x,y]}{\langle x-a, y-b \rangle} \\ \uparrow & & \uparrow \\ \mathbb{C}[u,v] & \longleftarrow & \mathbb{C}[x,y] \\ u & \longleftarrow & x \\ uv & \longleftarrow & y \end{array}$$

$u = a \leftarrow a$, $uv = b \leftarrow b$

Hence $\mathrm{Im}\, \varphi^{f} = (\mathbb{C}^2 \setminus y\text{-axis}) \cup \{(0,0)\}$. ($\langle u \rangle$)

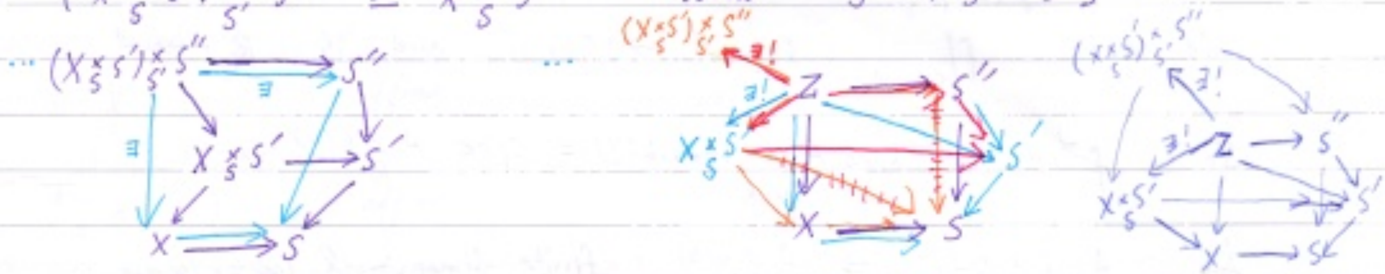
## • Base change.

•• Let $X$ be a scheme over a base scheme $S$.
If $S'$ is another base scheme and $\exists\ S' \to S$,
then $X \times_S S'$ is a scheme over $S'$.

eg.


Analogue :
$M : R\text{-mod}\ ;\ R' : R\text{-alg}$
$\Rightarrow M \otimes_R R' : R'\text{-mod}.$

•• $(X \times_S S') \times_{S'} S'' \cong X \times_S S''$ with $S'' \to S' \to S$ :




•• The property of a morphism being of finite type is stable under base change :

$$
\begin{tikzcd}
\bigcup_{i=1}^{n} \operatorname{Spec} A_i \otimes_R R' \subset X' \arrow[r] \arrow[d, "f'"] & X \supset \bigcup_{i=1}^{n} \operatorname{Spec} A_i \arrow[d, "f"] \\
S' \arrow[r] & S \supset \\
t' \in U \arrow[r] & t \in \operatorname{Spec} R \\
U = \bigcup_{i\in\Lambda} \operatorname{Spec} R_i \supset t' \in \operatorname{Spec} R' \arrow[ur]
\end{tikzcd}
$$

$A_i$ : f.g. $R$-alg.
$= R[a_1, \dots, a_m]$

Here $A_i \otimes_R R' = R'[a_1, \dots, a_m]$ is a f.g. $R'$-algebra.

•• The property of (a morphism) being a closed immersion is stable under base change :

It is also a fibered product of $\operatorname{spec} B$ and $\operatorname{spec} A'$ over $\operatorname{spec} A$.

$$f'^{-1}(\operatorname{spec} A') = \operatorname{spec} B' \subset Y' \longrightarrow Y \supset \operatorname{spec} B = f^{-1}(\operatorname{spec} A)$$

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow f' & & \downarrow f \\ X' & \xrightarrow{g} & X \\ \cup & & \cup \\ \forall\, x' & \longrightarrow & x \in \operatorname{spec} A \\ \cap & \nearrow & \\ \operatorname{spec} A' & & \\ \cap & & \\ g^{-1}(\operatorname{spec} A) & & \end{array}$$

$f$: closed immersion $\Leftrightarrow$ $A \twoheadrightarrow B$ finite.

$\Rightarrow A' \cong A \otimes_A A' \twoheadrightarrow B \otimes_A A' = B'$.

$\rightsquigarrow \mathcal{O}_{X'} \twoheadrightarrow f'_* \mathcal{O}_{Y'}$

$f'$ is a closed immersion. $\Leftarrow$ s.t. it is a closed immersion

•• The property of a scheme being integral is not stable under base change:

eg: $xy - t$ is irr in $\mathbb{C}[x, y, t]$,

so $\dfrac{\mathbb{C}[x,y,t]}{\langle xy-t \rangle}$ is an integral domain.

and $X = \operatorname{spec} \dfrac{\mathbb{C}[x,y,t]}{\langle xy-t \rangle}$ is integral over $\operatorname{spec} \mathbb{C}$.

But $X_0 = \operatorname{spec} \dfrac{\mathbb{C}[x,y]}{\langle xy \rangle}$ is not integral.

$$\begin{array}{ccc} \operatorname{Spec} \dfrac{\mathbb{C}[x,y]}{\langle xy \rangle} & \longrightarrow & \operatorname{Spec} \dfrac{\mathbb{C}[x,y,t]}{\langle xy-t \rangle} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} = \operatorname{Spec} \dfrac{\mathbb{C}[t]}{\langle t \rangle} & \longrightarrow & \operatorname{Spec} \mathbb{C}[t] \end{array}$$

not irr.

"$t=0$"

$\mathbb{C}[t] \to \mathbb{C}$

$t \mapsto 0$