Complex Analysis II

Minerva

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Chapter 1

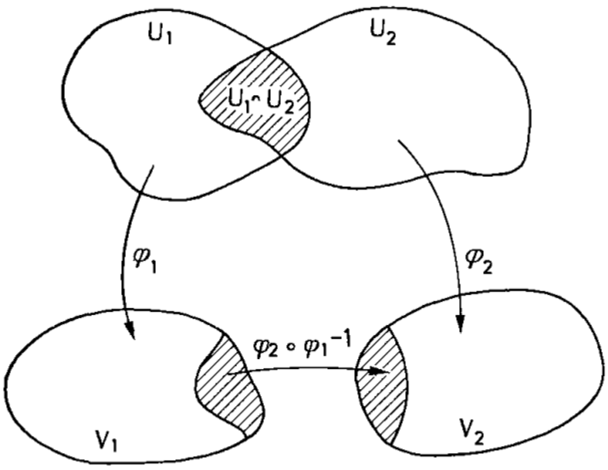
Covering spaces

1.1 Preliminary

1.1.1 Riemann surface

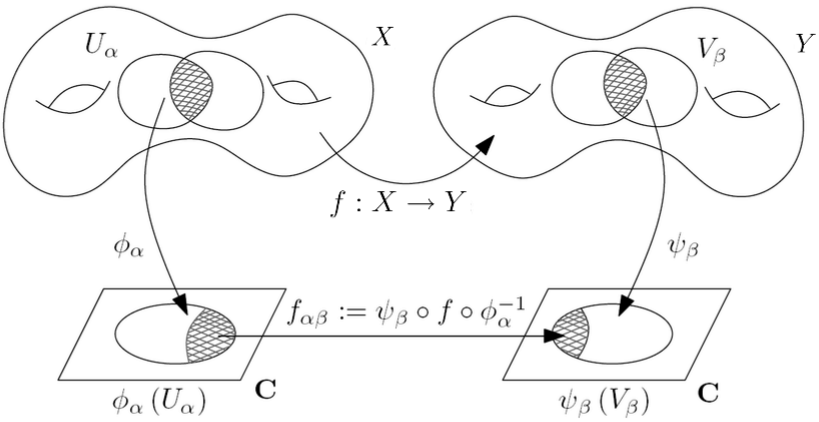
Definition 1.1.1.

- Let X be a Hausdorff topological space X is called a 2**-dimension manifold** if $\forall p \in X$, $\exists p \in U \subset X$ s.t. $U \xrightarrow{\sim} V \subset \mathbb{R}^2$ and U is called the **chart**.
- Let X be a 2-dimension manifold. A **complex atlas** on X is a system $\{\varphi_i\}_{i\in I}$ with $\varphi_i : U_i \simeq V_j \subset \mathbb{C}$ s.t. $X = \bigcup\limits_{i\in I} U_i$ and $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is biholomorphic $\forall i, j$.



- Two complex atlases $\{\varphi_i\}_{i\in I}$, $\{\psi_\lambda\}_{\lambda\in\Lambda}$ are **equivalent** if $\varphi_j\psi_\lambda^{-1}$, $\psi_\lambda\varphi_j^{-1}$ are biholomorphic when and where defined.
- A complex structure on X is an equivalence class of complex atlases on X.
- A **Riemann surface** is a pair (X, Σ), where X is connected 2-dimension manifold with complex structure Σ.
  - Y is a domain (connected open subset) in $X \rightsquigarrow Y$ is also Riemann surface $(Y, \Sigma|_Y)$.
  - Usually, we pick a maximal system $\{U_\alpha, z_\alpha\}_{\alpha\in\Lambda}$ representing Σ on X with $z_\alpha : U_\alpha \simeq V_\alpha \subseteq \mathbb{C}$.

- $Y \underset{\text{open}}{\subset} X$, $f : Y \to \mathbb{C}$ is **holomorphic** if $f \circ z_\alpha^{-1}$ is holomorphic $\forall \alpha$. Note that holomorphic is local property, which is independent on choice of atlases. Define $\mathcal{O}(Y)$ be all holomorphic functions on Y, which form a $\mathbb{C}$-algebra.
- Let X and Y be two Riemann surface. A continuous mapping $f : X \to Y$ is called holomorphic if all $f_{\alpha\beta}$ is holomorphic function in $\mathbb{C}$.



Theorem 1.1.1 (identity theorem). $f_1, f_2 : X \to Y$ holomorphic. If $f_1|_A = f_2|_A$ with $A \subset X$ having a limit point $a \in X$, then $f_1 \equiv f_2$.

Proof:

- $G := \{x \in X : \exists x \in W \subset X \text{ s.t. } f_1|_W = f_2|_W\}$ which is open.
- Let $b \in \overline{G} \rightsquigarrow f_1(b) = f_2(b)$ since f_1, f_2 is continuous. Choose $b \in U \subset X$, $V \subseteq Y$, $f_i(U) \subseteq V$. We may assume U is connected. Write $g_i = w \circ f_i \circ z^{-1} : z(U) \to w(V)$ $\forall i = 1, 2$. Since $b \in \overline{G} \cap U$, $U \cap G \neq \varnothing$ and $f_1|_{Z(U\cap G)} = f_2|_{Z(U\cap G)}$. By identity theorem for $\mathbb{C}$, $g_1 = g_2 \rightsquigarrow f_1|_U = f_2|_U \rightsquigarrow b \in G$, so G is closed.
- Since X is connected, $G = \varnothing$ or $G = X$.
- Choose $a \in W \subset X \rightsquigarrow f_1|_{A\cap W} = f_2|_{A\cap W}$ and $a \in \overline{A \cap W}$. By identity theorem on $\mathbb{C}$, $f_1|_W = f_2|_W \implies a \in G \implies G = X$ and thus $f_1 \equiv f_2$.

$\square$

Example 1.1.1. $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

- Topology : $\{\text{open set}\} = \{U \underset{\text{open}}{\subset} \mathbb{C}\} \cup \{V \cup \{\infty\} : V = \mathbb{C} \setminus K, K \underset{\text{cpt.}}{\subset} \mathbb{C}\}$
- Charts :

$$\begin{matrix} U_1 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C} \xrightarrow{z_1 = \mathrm{id}} \mathbb{C} \\ U_2 = \mathbb{P}^1 \setminus \{0\} = \mathbb{C} \xrightarrow{z_1 = z^{-1}} \mathbb{C} \end{matrix} \implies \begin{matrix} Z_1(U_1 \cap U_2) & \longrightarrow & Z_2(U_1 \cap U_2) \\ z & \longmapsto & z^{-1} \end{matrix}$$

Definition 1.1.2. f is **meromorphic function** if $\exists Y' \underset{\text{open}}{\subseteq} Y$.

- $f \in \mathcal{O}(Y')$
- $Y \setminus Y'$ contains only isolated points.

- $\forall p \in Y \setminus Y'$, $\lim\limits_{x\to p} |f(x)| = \infty$

Let $\mathscr{M}(Y)$ be all meromorphic function on Y.

Property 1.1.1. Property for $\mathscr{M}(X)$.

(1) For $f \in \mathscr{M}(X)$, define $f(p) = \infty$ $\forall p$: pole of f. Then $f : X \to \mathbb{P}^1$ is holomorphic :

- f is continuous : f is already continuous on Y'. For $p \in Y \setminus Y'$, by def, $\lim\limits_{p\to\infty} |f(x)| = \infty$.
- f is holomorphic : Just need to take care of a neighborhood of p. We can choose $p \in U$ s.t. U doesn't contain others pole (by isolated). Let $z_1 : U \simeq V$, $f(p) = \infty \in U' \xrightarrow{z_2 = z^{-1}} V' \ni 0$. Then $g := z_2 \circ f \circ z_1^{-1}$ is holomorphic on $V \setminus z_1(p)$ and g is bounded near $z_1(p) \rightsquigarrow z_1(p)$ is a removable singularity $\rightsquigarrow g$ is hol. on V.

(2) If $f : X \xrightarrow{\text{hol.}} \mathbb{P}^1$, then $f \equiv \infty$ or $f \in \mathscr{M}(X)$:

If $f \not\equiv \infty$, then by identity theorem, $f^{-1}(\infty)$ consists of isolated points and thus $f \in \mathscr{M}(X)$.

(3) $\mathscr{M}(X)$ is a field :

If $f \not\equiv 0$, then $f^{-1}(0)$ has only isolated points $\rightsquigarrow f$ has only isolated zero $\rightsquigarrow f^{-1} \in \mathscr{M}(X)$.

Property 1.1.2. Property for $f : X \to Y$ holomorphic.

(1) Let $f : X \to Y$ be non-constant holomorphic. Say $a \in U_1 \subseteq X$ and $b = f(a) \in U' \subseteq Y$. We may assume $z_1 : U_1 \xrightarrow{\sim} V_1 \subseteq \mathbb{C}$, $z_2 : U' \xrightarrow{\sim} V'$ s.t. $z_1(a) = 0$, $z_2(b) = 0$. Then $f_1 = z_1^{-1} \circ f \circ z_2 : V_1 \to V'$ is holomorphic and $f(0) = 0 \rightsquigarrow f_1(z) = z^m g(z)$ with $g(z) \in \mathcal{O}(V_1)$ and $g(0) \neq 0$. Then $\exists 0 \in V_2 \subset V_1$ and $g \in \mathcal{O}(V_2)$ s.t. $g = h^m$ on V_2. Let $z_2(z) = zh(z)$, then

$$\frac{d}{dz} zh(z)\bigg|_{z=0} = h'(0) \neq 0$$

By inverse function theorem, z_2 is biholmorphic near 0 (may assume V_2 satisfy). Let $U = z_1^{-1}(V_2)$ and $z = g \circ z_1 : U \to V$. Then $f_1' = z_2 \circ f \circ z^{-1} : (zh(z)) \mapsto (zh(z))^m$.

In this case, $U \setminus \{a\} \xrightarrow{f} U'$ is m to 1. The **multiplicity** of a is defined by $v_f(a) = m$ and the **branch number** of f at a is defined by $b_f(a) = m - 1$.

(2) Non-constant holomorphic map are open mapping since the image of a full circle under the map z^m is again a full circle.

(3) **Maximal principle** : If $f : X \to \mathbb{C}$ is non-constant holomorphic function, then $|f|$ doex not attain it max.

subproof : If not, say $R = |f(a)| = \max\{|f(x)| : x \in X\}$. Then $f(X) \subseteq \overline{B_R(0)}$. Since $f(X)$ is open, $f(X) \subset B_R(0)$ ($\rightarrow\!\leftarrow$). $\square$

(4) $f : X \xrightarrow[\text{hol.}]{\text{non const.}} Y$ and X is compact, then f is surjective and Y is compact.

subproof : Since $f(X)$ is compact, $f(X)$ is closed. Since $f(X)$ is open and closed and Y is connected, $f(X) = Y$. $\square$

(5) If X is compact, then $\mathcal{O}(X) = \mathbb{C}$.

subproof : By (4) and $\mathbb{C}$ is not compact, $f \in \mathcal{O}(X)$ must be constant. $\square$

(6) $\mathscr{M}(\mathbb{P}^1) = \mathbb{C}(z)$.

subproof : For $f \in \mathscr{M}(\mathbb{P}^1)$, f has only finitely many poles (by isolated and $\mathbb{P}^1$ is compact) $a_1, ..., a_n$. If $a_i = \infty$, consider $\widetilde{f} = \dfrac{1}{f-a}$, then we may assume $a_i \neq \infty$. Write $f(z) = h_1(z) + g_1(z)$, where $h_1(z) = \sum_{j=-m}^{-1} c_j(z-a_1)^j$ and g_1 is hol. at a. Since $\lim_{z\to\infty} h_1(z) = 0$, $h_1 \in \mathscr{M}(\mathbb{P}^1)$ by setting $h(\infty) = 0$. Continue the process by $g_1(z) = h_2(z) + g_2(z) \implies h_2(z) \in \mathscr{M}(\mathbb{P}^1)$. We have

$$f - h_1 - h_2 - \cdots - h_n \in \mathcal{O}(\mathbb{P}^1) = \mathbb{C} \implies f \in \mathbb{C}(z)$$

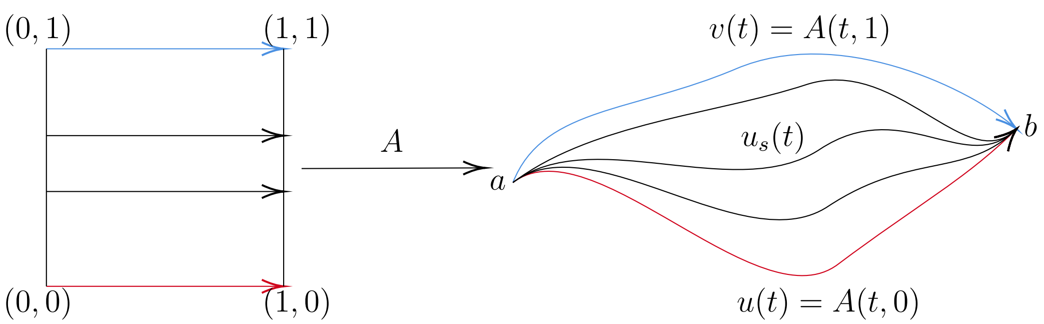
Example 1.1.2. Let $\omega_1, \omega \in \mathbb{C}$ is linearly independent over $\mathbb{R}$, $\Gamma := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. For $z_1, z_2 \in \mathbb{C}$, $z_1 \sim z_2 \iff z_1 - z_2 \in \Gamma$. Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be projection.

- Topology : $\{\text{open sets}\} = \{U \subset \mathbb{C}/\Gamma : \pi^{-1}(U) \underset{\text{open}}{\subset} \mathbb{C}\}$ which is Hausdorff, connected, compact. π is open mapping since $\pi^{-1}(\pi(V)) = \bigcup_{\omega\in\Gamma}(\omega + V)$ is open in $\mathbb{C} \implies \pi(V)$ is open in $\mathbb{C}/\Gamma$.
- Complex charts : Consider $V \underset{\text{open}}{\subset} \mathbb{C}$ s.t. no two points in V are equivalent $\rightsquigarrow U = \pi(V)$ is open in $\mathbb{C}/\Gamma$. $\varphi_V := (\pi|_V)^{-1} : U \to V \subseteq \mathbb{C}$ is a chart for $\mathbb{C}/\Gamma$. Now consider $\varphi_i : U_i \to V_i$ for $i = 1, 2$. Consider $\psi = \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$, then $\pi(z) = \pi(\psi(z)) \rightsquigarrow z - \psi(z) \in \Gamma$. Since $z - \psi(z)$ is continuous and Γ is discrete, $z - \psi(z) = \omega$ which is holomophic.
- $\mathcal{O}(\mathbb{C}/\Gamma) = \mathbb{C}$ since $\mathbb{C}/\Gamma$ is compact.
- $\mathscr{M}(\mathbb{C}/\Gamma) = \{f \in \mathbb{C} : f(z) = f(z+\omega_1) = f(z+\omega_2)\ \forall z \in \mathbb{C}\}$ with $F \longleftrightarrow f = F \circ \pi : \mathbb{C} \to \mathbb{P}^1$. If $F : \mathbb{C}/\Gamma \to \mathbb{P}^1$, then F is constant or surjective. For the latter, $f : \mathbb{C} \to \mathbb{P}^1$ is also surjective and $f(z) = f(z+\omega_1) = f(z+\omega_2)$.

1.1.2 Fundamental group

Let X be a topological space, a continuous map $u : I = [0,1] \to X$ give a curve from $u(0) = a$ to $u(1) = b$.

- u and v is **homotopic** (denoted by $u \sim v$) if $\exists$ continuous map $A : I \times I \to X$ s.t.



- " $\sim$ " is equivalent relation :
  - $u \sim u : A(t,s) = u(t)$
  - $u \underset{A}{\sim} v \implies v \sim u : B(t,s) = A(t, 1-s)$

- - $u \underset{A}{\sim} v,\ v \underset{B}{\sim} w : C(t,s) = \begin{cases} A(t,2s) & 0 \le s \le 1/2 \\ B(t,2s-1) & 1/2 \le s \le 1 \end{cases}$

- u : from a to b, v : from b to $c \rightsquigarrow u \cdot v(t) := \begin{cases} u(2t) & 0 \le t \le 1/2 \\ v(2t-1) & 1/2 \le t \le 1 \end{cases}$ from a to b.

- - $u \underset{A}{\sim} u',\ v \underset{B}{\sim} v' \implies u \cdot v \sim u' \cdot v' : C(t,s) = \begin{cases} A(2t,s) & 0 \le t \le 1/2 \\ B(2t-1,s) & 1/2 \le t \le 1 \end{cases}$
- - $u^{-1} := u(1-t)$ $(u \sim u' \implies u^{-1} \sim (u')^{-1} : B(t,s) = A(1-t,s))$

- a : constant curve via $u(t) = a$

- $\pi_1(X,a) = \{u : \text{closed curve with } u(0) = u(1) = a\}/\sim$ form a group, which is called the **fundamental group** of X with basis point a.

- - $u \cdot u^{-1} \sim a$:

$$A(t,s) = \begin{cases} u(2t) & 0 \le 2t \le s \\ u(s) & s \le 2t \le 2-s \\ u^{-1}(2t-1) & 2-s \le 2t \le 2 \end{cases}$$

- - $u \cdot a \sim u,\ a \cdot u \sim u$:

$$A(t,s) = \begin{cases} u\left(\dfrac{2t}{s+1}\right) & 0 \le t \le (s+1)/2 \\ a & (s+1)/2 \le t \le 1 \end{cases}$$

- - $(u \cdot v) \cdot w = u \cdot (v \cdot w)$:

$$A(t,s) = \begin{cases} u\left(\dfrac{4t}{s+1}\right) & 0 \le t \le (s+1)/4 \\ v(4t-s-1) & (s+1)/4 \le t \le (s+2)/4 \\ w\left(\dfrac{4t-s-2}{2-s}\right) & (s+2)/4 \le t \le 1 \end{cases}$$

Remark 1.1.1. If X is arcwise connected, then $\pi_1(X,a)$ is independent of a up to isomorphism. Indeed, for $a, b \in X$, a curve u from a to b induce the isomorphism

$$\begin{array}{cccc} \varphi : & \pi_1(X,a) & \longrightarrow & \pi_1(X,b) \\ & [v] & \longmapsto & [u^{-1}vu] \end{array}$$

Definition 1.1.3. An arcwise connected space X is **simply connected** if $\pi_1(X) = 0$.

Example 1.1.3. $\pi_1(\mathbb{D}) = 0$, since $A(t,s) = sa + (1-s)u(t) \implies u \sim a$.

1.1.3 Covering maps

Definition 1.1.4. Let X and Y be two topological space and $p : Y \to X$ continuous.

- p is called a **local homeomorphism** if $\forall y \in Y$, $\exists y \in V \subseteq Y$ s.t. $U = p(V)$ is open in X and $p|_V : V \simeq U$.

- p is called a **covering map** if $\forall x \in X$, $\exists x \in U \subset X$ s.t. $p^{-1}(U) = \bigsqcup_{j \in J} V_j$ and $p|_{V_j} : V_j \simeq U$.

Example 1.1.4.

- $k \geq 2$, $p_k : \mathbb{C}^\times \to \mathbb{C}^\times$ defined by $z \mapsto z^k$ is a covering map : $\forall a \in \mathbb{C}^\times$, $\exists b \in \mathbb{C}^\times$ s.t. $b^k = a$. $\exists a \in U \subset \mathbb{C}^\times$ and $b \in V_0 \subset \mathbb{C}^\times$ s.t. $\rho_k|_{V_0} : V_0 \simeq U$. Let $\omega = e^{2\pi i/k}$, then

$$p_k^{-1}(U) = V_0 \sqcup \omega V_0 \sqcup \cdots \sqcup \omega^{k-1} V_0$$

- $\exp : \mathbb{C} \to \mathbb{C}^\times$ is a covering map : Let $a \in U \subseteq \mathbb{C}^\times$, $\exists b \in V_0 \subseteq \mathbb{C}$ s.t. $e^b = a$ and $\exp|_{V_0} : V_0 \simeq U$. Then

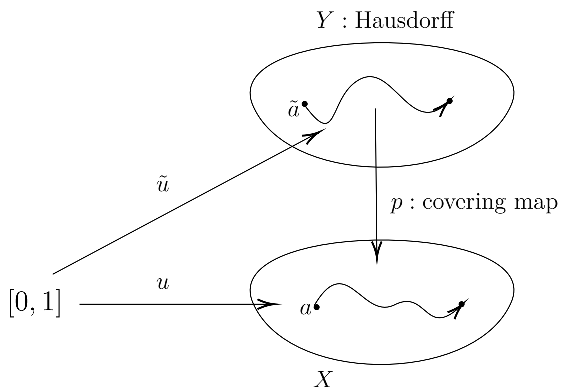
$$\exp^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} (2\pi i n + V_0)$$

- $\Gamma \subset \mathbb{C}$: a lattice and $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map : $V \underset{\text{open}}{\subset} \mathbb{C}$ s.t. no two points in V are equivalent, $\pi|_V : V \simeq \pi(V) =: U$ and

$$\pi^{-1}(U) = \bigsqcup_{\omega \in \Gamma} (\omega + V)$$

1.1.4 Theorem for liftings

Theorem 1.1.2. Let $p : Y \to X$ be a covering map. Given a curve $u(t) \subset X$, $x_0 = u(0)$ and $y_0 \in p^{-1}(x_0)$, $\exists \widetilde{u}(t) \subset Y$ s.t. $p \circ \widetilde{u} = u$.



Proof: Let

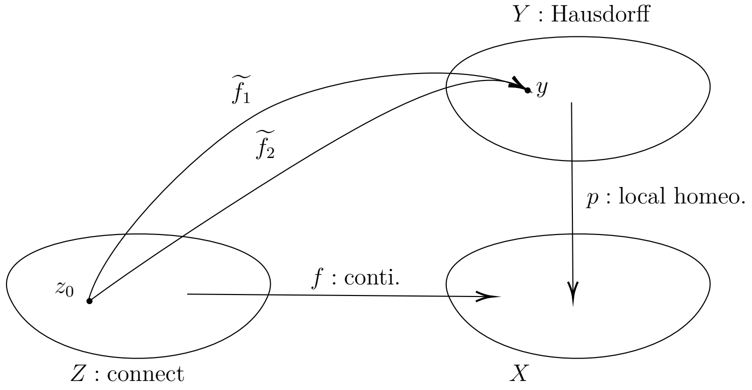
$$J = \left\{ t \in [0,1] \middle| \exists \widetilde{u} : [0,1] \to Y \text{ s.t. } \widetilde{u}(0) = y_0 \text{ and } p(\widetilde{u}|_{[0,t]}) = u|_{[0,t]} \right\}$$

- J is open : $t \in J$, $u(t) \in U \subset X$ with $p^{-1}(U) = \bigsqcup_{j \in \Lambda} V_j$. Let $\widetilde{u}(t) \in V_j$ for some j. Since u is continuous, $\exists [t, t+\varepsilon] \subset u^{-1}(U)$. Since $\varphi : U \xrightarrow{\sim} V_j$, $\widetilde{u}|_{[t,t+\varepsilon]} := \varphi \circ u|_{[t,t+\varepsilon]}$ and thus $t + \varepsilon \in J$.

- J is closed : choose $\{t_n\} \subset J$ s.t. $t_n \to t_0 \in [0,1]$. Choose $u(t_0) \in U \subset X$ with $\rho^{-1}(U) = \bigsqcup_{j \in \Lambda} V_j$. Assume that $u([t_n, t]) \subset U$ and $\widetilde{u}(t_n) \in V_j$ for some j. Then $\widetilde{u}|_{[t_n,t_0]} := \varphi \circ u|_{[t_n,t_0]}$ and thus $t_0 \in J$.

Since $[0,1]$ is connected, $J = [0,1]$. $\square$

Theorem 1.1.3 (uniqueness). Given the condition in below s.t. $\widetilde{f_1}(z_0) = \widetilde{f_2}(z_0)$ and $f = p \circ \widetilde{f_1} = p \circ \widetilde{f_2}$, then $\widetilde{f_1} \equiv \widetilde{f_2}$.



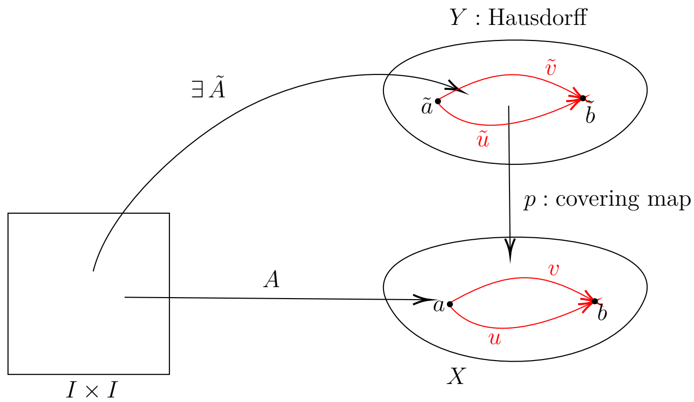
Proof: Let $G = \{z \in Z : \widetilde{f_1}(z) = \widetilde{f_2}(z)\}$.

- G is closed : Consider
$$\begin{array}{rccc} (\widetilde{f_1}, \widetilde{f_2}) : & Z & \longrightarrow & Y \times Y \\ & z & \longmapsto & (\widetilde{f_1}(z), \widetilde{f_2}(z)) \end{array}$$
Since Y is Hausdorff, $\Delta := \{(y, y) \in Y \times Y\}$ is closed in $Y \times Y$ and thus $G = (\widetilde{f_1}, \widetilde{f_2})^{-1}(\Delta)$ is closed.

- G is open : Let $z \in G$ and $y := \widetilde{f_1}(z_0) = \widetilde{f_2}(z_0)$. Choose $y \in U \subset Y$ s.t. $\varphi : V = p(U) \xrightarrow{\sim} U$. Since $\widetilde{f_1}, \widetilde{f_2}$ are continuous, $\exists z \in W \subset Z$ s.t. $\widetilde{f_i}(W) \subset U$ $\forall i$. By definition of liftings, $p \circ \widetilde{f_i} = f$, then $f(W) \subset V$ and $\widetilde{f_i}|_W = \varphi \circ f|_W$ i.e. $\widetilde{f_1}|_W = \widetilde{f_2}|_W \implies W \subset G$.

Since Z is connected and $z_0 \in G$, $G = Z$. $\square$

Theorem 1.1.4. Given the condition below and $u \underset{A}{\sim} v$ in X with initial point a and $\widetilde{a} \in p^{-1}(a)$. We can lift the curve to Y by Theorem 1.1.2, then $\widetilde{u} \underset{\widetilde{A}}{\sim} \widetilde{v}$.



Proof: For each curve $u_s(t)$, $\exists \widetilde{u_s}(t)$ in Y s.t. $\rho \circ \widetilde{u_s} = u_s$. Define $\widetilde{A}(t, s) = \widetilde{u_s}(t)$, then $p \circ \widetilde{A} = A$.

- $\widetilde{A}$ is continuous on $[0, \varepsilon_0] \times I$: We take $\widetilde{a} \in U \subset Y$ s.t. $\varphi : U = p(V) \xrightarrow{\sim} V$. Since $A(\{0\} \times I) = \{a\}$ and A is continuous, $\exists \varepsilon_0 > 0$ s.t. $A([0, \varepsilon_0] \times I) \subset V$ (by Lebesgue number) i.e. $u_s([0, \varepsilon_0]) \subset V$ $\forall s$. By the uniqueness, $\widetilde{u_s}|_{[0,\varepsilon_0]} = \varphi \circ u_s|_{[0,\varepsilon_0]}$ $\forall s$. By $\widetilde{A} = \varphi \circ A$ on $[0, \varepsilon_0] \times I$ and A is continuous, $\widetilde{A}$ is continuous on $[0, \varepsilon_0] \times I$.

- $\widetilde{A}$ is continuous on $I \times I$: Suppose not, $\exists (t_0, \sigma) \in I \times I$ at which $\widetilde{A}$ is not continuous. Let

$$\tau = \inf\{t \in [0,1] : \widetilde{A} \text{ is not continuous at } (t, \sigma)\}$$

  Then $\tau \geq \varepsilon_0$. Let $x = A(\tau, \sigma)$ and $y = \widetilde{A}(\tau, \sigma)$. There are neighborhoods of y and U of x s.t. $\varphi : U = p(V) \xrightarrow{\sim} V$. Since A is continuous, $\exists \varepsilon > 0$ s.t. $A(I_\varepsilon(\tau) \times I_\varepsilon(\sigma)) \subset U$, where

$$I_\varepsilon(\xi) = \{t \in I : |t - \xi| < \varepsilon\}$$

  In particular $u_\sigma(I_\varepsilon(\tau)) \subset U$. By uniqueness of lifting,

$$\widetilde{u_\sigma}|_{I_\varepsilon(\tau)} = \varphi \circ u_\sigma|_{I_\varepsilon(\tau)}$$

  Choose $t_1 \in I_\varepsilon(\tau)$ with $t_1 < \tau$. Since $\widetilde{A}$ is continuous at (t_1, σ), there exists $\delta > 0$, $\delta \leq \varepsilon$ s.t.

$$\widetilde{A}(t_1, s) = \widetilde{u_s}(t_1) \in V \ \forall s \in I_\delta(\sigma)$$

  By uniqueness of lifting,

$$\widetilde{u_s}|_{I_\varepsilon(\tau)} = \varphi \circ u_s|_{I_\varepsilon(\tau)} \ \forall s \in I_\delta(\sigma)$$

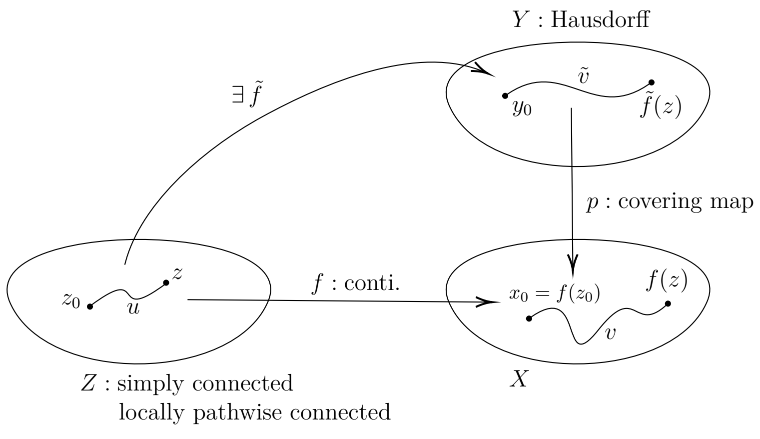
  Thus $\widetilde{A} = \varphi \circ A$ on $I_\varepsilon(\tau) \times I_\delta(\sigma)$. Then $\widetilde{A}$ is continuous in $I_\varepsilon(\tau) \times I_\delta(\sigma)$ which contradict that τ is infimum.

- The same end point L Since $A(\{1\} \times I) = b$ and $p \circ \widetilde{A} = A$, $\widetilde{A}(\{1\} \times I) \subset p^{-1}(b)$. Since $p^{-1}(b)$ is discrete and $\{1\} \times I$ is connected, $\widetilde{A}(\{1\} \times I)$ consists of a single point.

□

Remark 1.1.2. The condition of covering map is only for existence of lifting. If we already assume the lifting of $u_s(t)$ exists, then the theorem also holds when p is local homeomorphism.

Theorem 1.1.5. Given $f : Z \to X$ be continuous and Z is simply connected. Let $p : Y \to X$ be the covering map and Y is Hausdorff. Then $\exists \widetilde{f} : Z \to Y$ s.t. $f = p \circ \widetilde{f}$.



Proof: Fixed $z_0 \in Z$, let $x_0 \in f(z_0)$ and fixed $y_0 \in p^{-1}(x_0)$. $\forall z \in Z$, let u from z_0 to z, $v = f(u)$ from x_0 to $f(z)$. Then $\exists \widetilde{v}$ in Y with initial point y_0 and end point $\widetilde{f}(z)$.

- $\widetilde{f}$ is well-defined : For another u' from z_0 to z, $u \sim u' \implies f \circ u \sim f \circ u'$. By theorem 1.1.4, $\widetilde{f \circ u}(1) = \widetilde{f \circ u'}(1)$.

- $\widetilde{f}$ is continuous : Take the neighborhood U of $\widetilde{f}(z)$ and $V = p(U)$ of $f(z)$ s.t. $\varphi : V \xrightarrow{\sim} U$. Since f is continuous and Z is locally pathwise connected, $\exists$ a pathwise connected open subset $W \subset Z$ contained z s.t. $f(W) \subset V$.

  Claim : $\widetilde{f}(W) \subset U$ ($\rightsquigarrow \widetilde{f}$ is continuous)

  subproof :

  Let $z' \in W$ and u' be a curve from z to z' with $u' \subset W$. Then $v' = f(u') \subset V$ and $\widetilde{v'} = \varphi(v') \subset U$. Hence, $\widetilde{f}(z') = \widetilde{f \circ (u \cdot u')}(1) = \widetilde{v} \cdot \widetilde{v'}(1) \in U$. □

Theorem 1.1.6. $p : Y \to X$ is continuous map from Hausdorff space to a manifold. Then

$$\text{``}p \text{ is local homeo. + curve lifting property''} \iff \text{``}p \text{ is a covering map''}$$

Proof:

- $(\Leftarrow)$: OK!
- $(\Rightarrow)$: Let $x_0 \in X$ and U be a simply connected open chart contain x_0, which homeomorphic to $U' \subseteq \mathbb{R}^n$. Consider $p^{-1}(x_0) = \{y_i : j \in J\}$. Choose $f : U \to X$ be identity, by Remark 1.1.2 Theorem 1.1.5, $\exists \widetilde{f_j}$ s.t. $\widetilde{f_j}(x_0) = y_j$ and $f = p \circ \widetilde{f_j}$. Let $V_j = \widetilde{f_j}(U)$. Since $p \circ \widetilde{f_j} : U \xrightarrow{\text{id}} U$ and p is local homeomorphism, $\widetilde{f_j} : U \xrightarrow{\sim} V_j$.
  - $\{V_j\}$ is disjoint : If $y \in V_i \cap V_j$, let $x = p(y)$, then $\widetilde{f_i}, \widetilde{f_j}$ are two lifting of f with $\widetilde{f_i}(x) = \widetilde{f_j}(x) = y$ $(\rightarrow\!\leftarrow)$.
  - $p^{-1}(U) = \bigcup\limits_{j \in J} V_j$: $\forall y \in p^{-1}(U)$, $p(y) = x \in U$. Choose a curve u from x to x_0 in U and lift u to $\widetilde{u}$ with $\widetilde{u}(0) = y$, then $p \circ \widetilde{u}(1) = u(1) = x_0 \rightsquigarrow \widetilde{u}(1) = y_j$ for some j and thus $y \in V_j$.

□

1.2 Deck transformation

Definition 1.2.1. Denote $p : Y \rightharpoonup X$ if p is covering map.

Definition 1.2.2. Let X, Y : Hausdorff and $p : Y \to X$ be covering map.

- A **deck transformation** of p is homeomorphism $f : Y \to Y$ s.t. the diagram below commute.

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

- $\text{Deck}(Y/X) = \{\text{deck transformation of } p\}$ form a group.
- p is **Galois** if $\forall y_0, y_1 \in Y$, $\exists f \in \text{Deck}(Y/X)$ s.t. $f(y_0) = y_1$.

Fact 1.2.1.

(1) If covering map $p : Y \to X$ with $y_0 \mapsto x_0$, then

$$\begin{array}{cccc} p_* : & \pi_1(Y, y_0) & \hookrightarrow & \pi_1(X, x_0) \\ & [v] & \longmapsto & [p \circ v] \end{array}$$

subproof : If $[v] \mapsto [p \circ v] = [x_0]$, then $p \circ v \underset{A}{\sim} x_0 \rightsquigarrow v \underset{\widetilde{A}}{\sim} y_0 \rightsquigarrow [v] = [y_0]$.

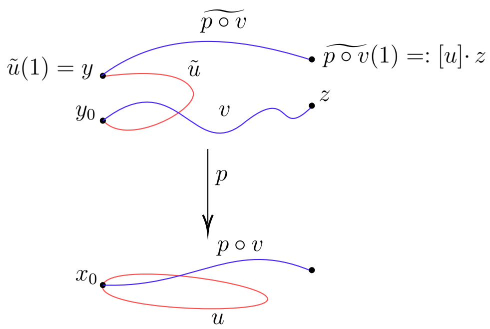
(2) If $[u] \in \pi_1(X, x_0)$ and $\widetilde{u}(0) = y_0$, then $\widetilde{u}(1) = y_0 \iff [u] \in \operatorname{Im} p_*$.
subproof : $(\Rightarrow)$: $[u] = p_*[\widetilde{u}] \in \operatorname{Im} p_*$. $(\Leftarrow)$: $\exists [v] \in \pi_1(Y, y_0)$ s.t. $p_*[v] = [u] \rightsquigarrow p \circ v \underset{A}{\sim} u$.
By lifting the curve and Y is Hausdorff, $v \underset{\widetilde{A}}{\sim} \widetilde{u} \implies \widetilde{u}(1) = v(1) = y_0$.

(3) Let u_1, u_2 be two curve from x_0 to x_1. Then $\widetilde{u_1}(1) = \widetilde{u_2}(1) \iff [u_1 \cdot u_2^{-1}] \in \operatorname{Im} p_*$.

Theorem 1.2.1. Let X, Y be connected manifold and a covering map $p : Y \to X$ with $y_0 \mapsto x_0$. If $p_*(\pi_1(Y, y_0)) \lhd \pi_1(X, x_0)$, then $\exists\, \overline{\varphi} : \pi_1(X, x_0)\big/ p_*(\pi_1(Y, y_0)) \simeq \operatorname{Deck}(Y/X)$.

Proof:

- Construct $\varphi : \pi_1(X, x_0) \to \operatorname{Deck}(Y/X)$: Given $[u] \in \pi_1(X, x_0)$ and $z \in Y$,



Let y be the end point of lifting curve of u with initial point y_0. Since Y is Hausdorff, y is independent on homotopic. Since Y is connected manifold, Y is path connected. Let v be the curve connected y_0 and z and. Let $\widetilde{p \circ v}$ be the lifting curve of $p \circ v$ with initial point y and define $[u] \cdot z$ be it end point.

  - $[u] \cdot z$ is independent on v : Let v' be another curve from y_0 to z. By Fact 1.2.1,

$$\begin{aligned}
\widetilde{p \circ v}(1) = \widetilde{p \circ v'}(1) &\iff \widetilde{u} \cdot \widetilde{p \circ v}(1) = \widetilde{u} \circ \widetilde{p \circ v'}(1) \\
&\iff [(u \cdot (p \circ v)) \cdot (u \cdot (p \circ v'))^{-1}] \in \operatorname{Im} p_* \\
&\iff [u] \cdot p_*[(v) \cdot (v')^{-1}] \cdot [u]^{-1} \in \operatorname{Im} p_* \\
&\iff \operatorname{Im} p_* \trianglelefteq \pi_1(X, x_0)
\end{aligned}$$

  - z and $[u] \cdot z$ lie in the same fiber of p.
  - $\pi_1(X, x_0) \curvearrowright Y$:
    - $([u_1] \cdot [u_2]) \cdot z = [u_1] \cdot ([u_2] \cdot z)$: Since $(u_1 \cdot u_2) \cdot (p \circ v) \sim u_1 \cdot (u_2 \cdot (p \circ v))$,

$$([u_1] \cdot [u_2]) \cdot z = (\widetilde{u_1 \cdot u_2}) \cdot \widetilde{p \circ v}(1) = \widetilde{u_1} \cdot (\widetilde{u_2} \cdot \widetilde{p \circ v})(1) = \widetilde{u_1} \cdot (p \circ (\widetilde{u_2} \cdot \widetilde{p \circ v}))^{\sim}(1)$$

    Note that $\widetilde{u_2} \cdot \widetilde{p \circ v}$ from y_0 to $[u_2] \cdot z$, then RHS $= [u_1] \cdot ([u_2] \cdot z)$.
    - $[x_0] \cdot z = z$: By construction.
  - $z \mapsto [u] \cdot z$ is continuous : Let $z_0 \in Y$ with $p(z_0) \in U \subseteq X$ and $p^{-1}(U) = \bigsqcup_{i \in \Lambda} V_i$ with $p : V_i \underset{\varphi_i}{\overset{\sim}{\rightleftarrows}} U$. Assume $z_0 \in V_0$ and $[u] \cdot z \in V_1$.

    Claim : $[u] \cdot V_0 \subset V_1$:
    subproof : Given $z \in V_0$, let $\alpha \subset V_0$ from z_0 to z. Then $[u] \cdot z$ is the end point of the lifting curve of $p \circ \alpha$ with initial point z_1 i.e. $[u] \cdot z = \widetilde{p \circ \alpha}(1) = \varphi_1 \circ p \circ \alpha(1) \in V_1$ $\square$

Hence, $z \mapsto [u] \cdot z$ is a homeomorphism with the inverse $z \mapsto [u^{-1}] \cdot z$.

- Now

$$\begin{array}{rccc} \varphi : & \pi_1(X, x_0) & \longrightarrow & \mathrm{Deck}(Y/X) \\ & [u] & \longmapsto & f_u : z \mapsto [u] \cdot z \end{array}$$

is a group homomorphism.

  - φ is onto : $\forall f \in \mathrm{Deck}(Y/X)$, $f(y_0) = y_1$. Pick β from y_0 to y_1. Let $[u] = [p \circ \beta] \in \pi_1(X, x_0)$, then $f_u(y_0) = y_1$. Since f and f_u are both the lifting in below diagram with $f_u(y_0) = f(y_0)$, $f = f_u$ by uniqueness.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Y \arrow[ur, "f"'] \arrow[ur, bend left, "f_u"] \arrow[r, "p"'] & X \end{tikzcd}$$

  - $\ker \varphi = p_*(\pi_1(Y, y_0))$: Since Y is connected,

$$f_u(y_0) = y_0 \iff \widetilde{u}(1) = y_0 \iff [u] \in p_*(\pi_1(Y, y_0))$$

Hence, $\widetilde{\varphi} : \pi_1(X, x_0) \big/ p_*(\pi_1(Y, y_0)) \simeq \mathrm{Deck}(Y/X)$.

$\square$

From now on, X and Y are connected manifolds.

Fact 1.2.2.

(1)

$$\begin{tikzcd} Y \arrow[rr, "y_0 \mapsto y_0'", "f:\text{homeo.}"'] \arrow[dr, "p"'] & & Y \arrow[dl, "p'"] \\ & X & \end{tikzcd} \iff p_*(\pi_1(Y, y_0)) = p'_*(\pi_1(Y', y_0'))$$

subproof : Immediately from Homework 5.3.1.

(2) $p : Y \to X$ with $y_0 \mapsto x_0$, $H = p_* \pi_1(Y, y_0)$, then by proof, $N(H)/H \simeq \mathrm{Deck}(Y/X)$ and

$$p \text{ is Galois} \iff H \trianglelefteq \pi_1(X, x_0) \text{ i.e. } \mathrm{Deck}(Y/X) \simeq \pi_1(X, x_0)/H$$

Definition 1.2.3. For convenience, we denote the curve u from a to b by $u : a \rightsquigarrow b$.

Theorem 1.2.2. If $H \leq \pi_1(X, x_0)$, then exists connected manifold Y and covering map $p : Y \to X$ with $y_0 \mapsto x_0$ s.t. $p_* \pi_1(Y, y_0) = H$.

Proof: Let $Y := \{(x, [[u]]) | x \in X, u : x_0 \rightsquigarrow x\}$, where $[[u]]$ denoted the equivalent classes defined by $u_1 \simeq u_2 \iff [u_1 \cdot u_2^{-1}] \in H$. Define

$$\begin{array}{rccc} p : & Y & \longrightarrow & X \\ & (x, [[u]]) & \longmapsto & x \end{array}$$

- Y inherits a manifold structure :

  - For $y = (x, [[u]]) \in Y$, $\exists x \in U_x \subseteq X$ with U_x homeomorphic to a small open ball in $\mathbb{R}^n$. Let
$$[U_x, y] := \{(z, [[u \cdot v]]) | z \in U_x,\ v : x \rightsquigarrow z,\ v \subset U_x\}$$
Since U_x is simply connected, $[[u \cdot v]]$ is independent on v. For $y_0 = (z_0, r_0) \in [U_{x_1}, y_1] \cap [U_{x_2}, y_2]$, say $y_i = (x_i, [[u_i]])$, then $z_0 \in W \subseteq U_{x_1} \cap U_{x_2}$ for some W homeomorphic to small ball in $\mathbb{R}^n$. Say $r_0 = [[u_i \cdot v_i]]$ for some $v_i : x_i \rightsquigarrow z_0$. Then we have $[W, y_0] \subset [U_{x_1}, y_1] \cap [U_{x_2}, y_2]$, since $\forall z \in W$, $W \supset w : z_0 \rightsquigarrow z \implies U_{x_1} \supset v_i \cdot w : x_1 \rightsquigarrow z_1$ and
$$[U_{x_1}, y_1] \ni (z, [[u_1 \cdot v_1 \cdot w]]) = (z, [[u_2 \cdot v_2 \cdot w]]) \in [U_{x_2}, y_2]$$
Then Y has topological space structure defined by the open base $[U_x, y]$.
  - Let $y = (x, [[u]])$, then
$$\begin{array}{rccc} p|_{[U_x,y]} : & [U_x, y] & \longrightarrow & U_x \\ & (z, [[u \cdot v]]) & \longmapsto & z \end{array}$$
is a homeomorphism and U_x homeomorphic to open ball in $\mathbb{R}^n$.

- Y is Hausdorff : Given distinct two points $y_i = (x_i, [[u_i]]) \in Y$.
  - If $x_1 \neq x_2$, choose $U_{x_1} \cap U_{x_2} = \varnothing$ by X is Hausdorff.
  - If $x_1 = x_2 = x$, then $[[u_1]] \neq [[u_2]]$. If $(z, r) \in [U_x, y_1] \cap [U_x, y_2]$, then $r = [[u_1 \cdot v]] = [[u_2 \cdot v]]$ for some $U_x \supset v : x \rightsquigarrow z$. Then
$$[u_1 \cdot u_2^{-1}] = [(u \cdot v)(u_2 \cdot v)^{-1}] \in H \ (\rightarrow\!\leftarrow)$$

- $p : Y \to X$ has the curve lifting property ($\rightsquigarrow p$ is covering map) : Given $u : I \to X$ with $u(0) = x_1$. Fix $(x_1, [[v]]) \in p^{-1}(x_1)$. Let $u_s(t) = u(st) : x_1 \rightsquigarrow u(s)$. Then
$$\begin{array}{rccc} \widetilde{u} : & I & \longrightarrow & Y \\ & t & \longmapsto & (u(t), [[v \cdot u_t]]) \end{array}$$
is the lifting of curve u with initial point $\widetilde{u}(0) = (x_1, [[v]])$.

- Since X is path connected and have curve lifting property, Y is path connected.

- $p_*\pi_1(Y, y_0) = H$, where $y_0 = (x_0, [[x_0]])$: Let $u : I \to X$ be a loop of x_0. Then $\widetilde{u} : [0,1] \to Y$ with $t \mapsto (u(t), [[u_t]])$.
$$u \in H \iff (u(1), [[u]]) = (x_0, [[x_0]]) \iff \widetilde{u} \in \pi_1(Y, y_0) \iff u \in p_*\pi_1(Y, y_0)$$

$\square$

Definition 1.2.4. $p : Y \rightharpoonup X$ is **universal** if $\forall p' : Y' \rightharpoonup X$ with $y_0' \mapsto x_0$, $\exists! f : Y \to Y'$ is continuous fiber preserving map s.t. $f(y_0) = y_0'$, denoted by $p : Y \overset{\wedge}{\rightharpoonup} X$.

Fact 1.2.3. If Y is simply connected, then p is universal, Galois and $\pi_1(X) \simeq \mathrm{Deck}(Y/X)$.

Proof: Since Y is simply connected, it immediately from Theorem 1.1.5. $\square$

Fact 1.2.4. If $p : Y \to X$ is universal covering, then Y is simply connected.

Proof: If not, $\pi_1(Y, y_0) \neq e$. By theorem 1.2.2, $\exists \widetilde{p} : \widetilde{Y} \rightharpoonup Y$ s.t. $\widetilde{p}_*(\pi_1(\widetilde{Y}, \widetilde{y}_0)) = e \lneq \pi_1(Y, y_0)$. By Fact 1.2.2, $\mathrm{Deck}(\widetilde{Y}/Y) \simeq \pi_1(Y, y_0) \neq e (\rightarrow\!\leftarrow)$. $\square$

Definition 1.2.5. Suppose X and Y are topological spaces, $p : Y \to X$ is a covering map and $G \leq \mathrm{Deck}(Y/X)$. Two points $y, y' \in Y$ are called **equivalent modulo** G, if $\exists \sigma \in G$ s.t. $\sigma(y) = y'$, and denoted by $X = Y/G$.

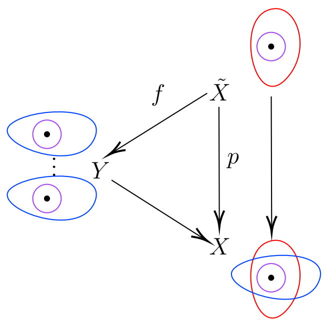
Theorem 1.2.3. If $p : \widetilde{X} \to X$ is universal covering and $q : Y \to X$ is covering map. Let $f : \widetilde{X} \to Y$ be a continuous fiber-preserving mapping by universal property.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "p"] \arrow[dr, "\exists! f"'] & X \\
& Y \arrow[u, "q"']
\end{tikzcd}$$

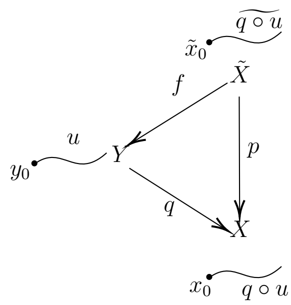
Then f is a covering map and exists $G \leq \mathrm{Deck}(\widetilde{X}/X)$ s.t. $Y = \widetilde{X}/G$. Moreover, $G \simeq \pi_1(Y)$.

Proof: To show that f is covering map, we suffice to show f is local homeomorphism and has curve lifting property.

- f is local homomorphism : Consider following diagram.



- f has curve lifting property :



  Note that $q \circ f \circ \widetilde{q \circ u} = p \circ \widetilde{q \circ u} = q \circ u$ i.e. $f \circ \widetilde{q \circ u}$ and u are two lifting curve of $q \circ u$ with same initial point via q. Hence, $u = f \circ \widetilde{q \circ u}$ i.e. $\widetilde{q \circ u}$ is the lifting curve of u via f.

- Let $G = \mathrm{Deck}(\widetilde{X}/Y) \leq \mathrm{Deck}(\widetilde{X}/X)$. Since $\widetilde{X}$ is simply connected, $f : \widetilde{X} \twoheadrightarrow Y$ is Galois and $G \simeq \pi_1(Y)/f_*\pi_1(\widetilde{X}) = \pi_1(Y)$.

$\square$

Definition 1.2.6.

- $p : Y \to X$, $y \in Y$ is a **branch point** of p if $\nexists V \ni y$ s.t. $p|_V$ is injective.
- p is **proper** if every preimage of compact set is compact.

Fact 1.2.5.

(1) p has no branch point $\iff$ p is a local homeomorphism.

subproof : $(\Leftarrow)$: OK! $(\Rightarrow)$: $\forall y \in Y$, $\exists y \in V \subseteq Y$ s.t. $p|_V$ is injective. Since p is continuous and open, $p|_V : V \simeq p(V)$. □

(2) If p is proper, then

- $\forall x \in X$, $p^{-1}(x)$ is finite, since $p^{-1}(x)$ is discrete and compact.
- p is a closed map : for all V closed in X and for all compact subset E of Y,

$$P(V) \cap E = P(\underbrace{V \cap p^{-1}(E)}_{\text{cpt. in } X}) \underset{\text{cpt.}}{\subset} Y$$

Hence, $P(V)$ is closed.

- $\forall x \in X$, $p^{-1}(x) \subseteq V \underset{\text{open}}{\subset} Y$, then $\exists x \in U \underset{\text{open}}{\subset} X$ s.t. $p^{-1}(U) \subset V$:

$A := P(Y \setminus V)$ is closed in X. Set $U = X \setminus A \underset{\text{open}}{\subset} X$ and $p^{-1}(A) \subset V$.

(3) p is proper local homeomorphism, then p is covering map.

subproof : Let $x \in X$ and $p^{-1}(x) = \{y_1, ..., y_n\} \subset Y$, $\exists y_j \in W_j \subset Y$, $U_j \subset X$ s.t. $p : W_j \simeq U_j$. Since Y is Hausdorff, we may assume $\{W_j\}$ pairwise disjoint. Now $V = W_1 \cup \cdots \cup W_n \supset p^{-1}(x) \rightsquigarrow \exists U \subset U_1 \cap \cdots \cap U_n$ s.t. $p^{-1}(U) \subset V$. If we set $V_j = W_j \cap p^{-1}(U)$, then $p^{-1}(U) = V_1 \sqcup \cdots \sqcup V_n$ and $p : V_i \simeq U_j$.

(4) $A = \{\text{branch points of } p\}$ is closed and discrete in Y.

subproof : Recall that we may assume the chart map is $z \mapsto z^k$ in the neighborhood U of y_0 to U' of x_0. Then $p : U \setminus \{y_0\} \to U'$ with $\#\{p^{-1}(x) \cap U\} = k$ for $x \neq x_0$. Hence, if $k > 1 \rightsquigarrow y_0$ is branch point. Also, if $k = 1 \rightsquigarrow y_0$ is unbranch point, which is open condition.

(5) If p is proper, $B = p(A)$ is closed and discrete is called **critical values** of p. Then

$$p : Y \setminus p^{-1}(B) \to X \setminus B$$

is a unbranch holomorphic covering. Moreover, for any $a_0, a_1 \in X \setminus B$, $u : a_0 \rightsquigarrow a_1$, then

$$\begin{array}{rccc} \varphi : & p^{-1}(a_0) & \longrightarrow & p^{-1}(a_1) \\ & \widetilde{u}(0) = y_0 & \longmapsto & \widetilde{u}(1) \end{array}$$

is injective, where $\widetilde{u}$ is lifting curve of u with initial point y_0. Then $\#p^{-1}(a_0) \leq \#p^{-1}(a_1)$. By symmetric, $\#p^{-1}(a_1) \leq \#p^{-1}(a_0)$. We call $p : Y \setminus p^{-1}(B) \to X \setminus B$ be n-**sheeted covering map** if the constant $\#p^{-1}(a_0) = n$.

(6) If p is finite sheeted covering map, then p is proper.

subproof : It is suffices to show that if $p : Y \to X$ is finite sheeted covering map and X is compact, then Y is compact. Given a open cover $\Gamma = \{W_\alpha\}$ of Y. For $x \in X$, let $p^{-1}(x) = \{b_1, ..., b_{n(x)}\}$ and exists a neighborhood U_x of x s.t. $p^{-1}(U_x) = \bigsqcup_{v=1}^{n(x)} V_{x,v}$ with $b_v \in V_{x,v} \simeq U_x$. Also say $b_v \in W_{x,v} \in \Gamma$. Since p is open mapping, $p(V_{x,v} \cap W_{x,v})$ is open and thus $U'_x := \bigcap_{v=1}^{n(x)} p(V_{x,v} \cap W_{x,v})$ is open neighborhood of x. By compactness, X can be covered by finite number of U'_{x_i}, $x \in \Lambda$. Note that $p^{-1}(U'_x) = \bigcup_{v=1}^{n(x)} A_{x,v}$ with $A_{x,v} \subseteq V_{x,v} \cap W_{x,v} \subset W_{x,v}$. Hence, Y has finite subcover $\{W_{x,v} : x \in \Lambda, v = 1, ..., n(x)\}$. □

Theorem 1.2.4 (Main theorem)**.** Let X be the Riemann surface, $\mathbb{D} = B_1(0)$, $p : X \to \mathbb{D}$ be proper, non-constant and unbranch over $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Then $\exists k > 1$ s.t.

$$\begin{tikzcd} X \arrow[rr, "\exists\varphi", "\sim"'] \arrow[dr, "p"'] & & \mathbb{D} \arrow[dl, "p_k"] \\ & \mathbb{D} & \end{tikzcd}$$

where $p_k : z \mapsto z^k$.

Proof: Let $X^* = p^{-1}(\mathbb{D}^*)$. Then $p : X^* \to \mathbb{D}^*$ is unbranch covering map. Let $H = \{z \in \mathbb{C} : \mathrm{Re} z < 0\}$, then

$$\begin{array}{rccc} \exp : & \mathbb{H} & \rightharpoonup & \mathbb{D}^* \\ & z & \longmapsto & \exp(z) \end{array}$$

is universal covering, since $\mathbb{H}$ is simply connected. Then $\mathrm{Deck}(\mathbb{H}/\mathbb{D}^*) = \pi_1(\mathbb{D}^*)$ and it is clear that $\mathrm{Deck}(\mathbb{H}/\mathbb{D}^*) = \{\tau_n : z \mapsto z + 2\pi i n | \forall n \in \mathbb{Z}\} \simeq \mathbb{Z}$. By Theorem 1.2.3, $\exists f : \mathbb{H} \rightharpoonup X^*$ s.t. $\exp = p \circ f$ and $\exists G \leq \mathrm{Deck}(H/\mathbb{D}^*)$ s.t. $\pi_1(X^*) = G$.

- $G = \{\mathrm{id}\}$: $f : \mathbb{H} \to X^*$ is biholomorphic and $\mathrm{Card}(p^{-1}(x)) = \mathrm{Card}(\mathbb{Z}) = \infty$ ($\rightarrow\!\leftarrow$).
- $G = k\mathbb{Z}$: For $\sigma \in G = \langle \tau_k \rangle$. Define $g : \mathbb{H} \to \mathbb{D}^*$ by $g(z) = \exp(z/k)$. Then $g(z) = g(z')$ precisely if z and z' are equivalent modulo G. Hence there exists a bijective mapping $\varphi : X \to \mathbb{D}^*$ such that $g = \varphi \circ f$. Since f and g are locally biholomorphic, φ is also. We can check that

$$\begin{tikzcd} & \mathbb{H} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\varphi"'] \arrow[dr, "p"'] & & \mathbb{D} \arrow[dl, "p_k"] \\ & \mathbb{D} & \end{tikzcd}$$

  will commute.

Now we claim that $|p^{-1}(0)| = 1$. To the contrary suppose $f^{-1}(0) = \{b_1, ...b_n\}$ with $n \geq 2$. Then there exists disjoint open neighborhoods V_i of b_i and a disk $D(r) = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \leq 1$ s.t. (by Fact (2))

$$p^{-1}(D(r)) \subseteq V_1 \cup \cdots \cup V_n \tag{$*$}$$

Let $\mathbb{D}^*(r) = D(r) \setminus \{0\}$. Since

$$p^{-1}(D^*(r)) \simeq p_k^{-1}(D^*(r)) = D^*(\sqrt[k]{r})$$

which is connected and every point b_i is accumulation point of $p^{-1}(D^*(r))$, $p^{-1}(D(r))$ is also connected, which contradicts to $(*)$. Thus $p^{-1}(0)$ consists of a single point $b \in X$. Hence by defining $\varphi(b) = 0$, we can continue the mapping $\varphi : X^* \to \mathbb{D}^*$ to a biholomorphic mapping $\varphi : X \to \mathbb{D}$ s.t. $p = p_k \circ \varphi$ on X. $\square$

Remark 1.2.1. In the proof above, we also show that every covering map of $\mathbb{D}^*$ is either isomorphic to the covering given by the logarithm or by the kth root.

1.3 Riemann surface of algebraic function

1.3.1 Sheaf

Definition 1.3.1. Let X be a Riemann surface and $a \in X$.

- $\mathcal{O}$ is the sheaf of holomorphic function :
  - $\forall U \underset{\text{open}}{\subset} X$, $\mathcal{O}(U)$ is the ring of holomorphic function on U.
  - for $V \subset U$, $\rho_V^U : \mathcal{O}(U) \xrightarrow{\text{Res}} \mathcal{O}(V)$ with $\rho_U^U = \text{id}$, $\rho_W^V \circ \rho_V^U = \rho_W^U$.
  - If $f, g \in \mathcal{O}(U)$ and $U = \bigcup_i U_i$ s.t. $f|_{U_i} = g|_{U_i}$, then $f = g$.
  - If $U = \bigcup_i U_i$ and $f_i \in \mathcal{O}(U_i)$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then $\exists f \in \mathcal{O}(U)$ s.t. $f|_{U_i} = f_i$.
- The **stalk** of $\mathcal{O}$ at a is defined to be

$$\mathcal{O}_a = \left(\bigsqcup_{a \in U} \mathcal{O}(U) \right) \Big/ \sim$$

  where $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(V)$, $f \sim g \iff \exists a \in W \subseteq U \cap V$ s.t. $f|_W = g|_W$.
- $\begin{array}{rccc} \rho_a : & \mathcal{O}(U) & \longrightarrow & \mathcal{O}_a \\ & f & \longmapsto & [f] \end{array}$, $\rho_a(f)$ is called the germ of f at a.

Definition 1.3.2. Suppose X is a topological space and $\mathscr{F}$ is a presheaf on X. Define the **topological space associated to a presheaf** by

$$|\mathscr{F}| := \bigsqcup_{x \in X} \mathscr{F}_x$$

- Define the topology induces by the open base

$$[U, f] := \{\rho_x(f) : x \in U\} \ \forall U \underset{\text{open}}{\subset} X, \ f \in \mathscr{F}(U)$$

  This form a open base by the definition of stalk.
- $p : |\mathscr{F}| \to X$ send the element in $\mathscr{F}_x$ to x, which is a local homeomorphism since $p([U, f]) = U$. Hence $|\mathscr{F}|$ inherits a complex structure from X s.t. p is holomorphism.

Definition 1.3.3. A presheaf $\mathscr{F}$ on a topological space X is said to satisfy the **identity theorem** if the following holds. If $Y \subset X$ is a domain and $f, g \in \mathscr{F}(Y)$ s.t. $\rho_a(f) = \rho_a(g)$ for some $a \in Y$, then $f = g$.

Theorem 1.3.1. Suppose X is a locally connected Hausdorff space and $\mathscr{F}$ is a presheaf on X which satisfies the identity theorem. Then $|\mathscr{F}|$ is Hausdorff.

Proof: Given distinct element $\varphi_1, \varphi_2 \in |\mathscr{F}|$, let $x_i = p(\varphi_i)$.

- If $x_1 \neq x_2$. Since X is Hausdorff, there exists disjoint neighborhood U of x and x_1 of x_2. Then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint neighborhood of φ_1 and φ_2, respectively.

- If $x = x_1 = x_2$. Suppose $\varphi_i = p_x(f_i)$ for some $f_i \in \mathscr{F}(U_i)$, where U_i are open neighborhood of x. Let $U \subset U_1 \cap U_2$ be a connected open neighborhood of x. Then $[U, f_i|_U]$ are open neighborhood of φ_i. Suppose exists $\psi \in [U, f_1|_U] \cap [U, f_2|_U]$, say $y = p(\psi)$, then $\rho_y(f_1) = \psi = \rho_y(f_2)$. By identity theorem, $f_1|_U = f_2|_U$ i.e. $\varphi_1 = \varphi_2$ ($\rightarrow\!\leftarrow$). Hence $[U, f_1|_U]$ and $[U, f_2|_U]$ are disjoint.

$\square$

Remark 1.3.1. In particular, since $\mathcal{O}$ satisfy identity theorem, $|\mathcal{O}|$ is Hausdorff.

1.3.2 Analytic continuation

Definition 1.3.4. Suppose X is a Riemann surface, u be the curve from a to b in X. The holomorphic function germ $\psi \in \mathcal{O}_b$ is said to result from the **analytic continuation** along the curve u if the holomorphic function germ $\varphi \in \mathcal{O}_a$ if the following holds. For $t \in [0,1]$, $\exists \varphi_t \in \mathcal{O}_{u(t)}$ s.t. $\varphi_0 = \varphi_1$ and $\varphi_1 = \psi$ satisfy that $\forall \tau \in [0,1]$, there exists a neighborhood $T \subset [0,1]$ of τ and open subset $U \subset X$ with $u(T) \subset U$ and a function $f \in \mathcal{O}(U)$ s.t.

$$\rho_{u(t)}(f) = \varphi_t \text{ for every } t \in T$$

Since $[0,1]$ is compact, this condition is equivalent to exists a partition $\{t_i\}_{i=0}^n$ of $[0,1]$, domains $U_i \subset X$ with $u([t_{i-1}, t_i]) \subset U_i$ and $f_i \in \mathcal{O}(U_i)$ for $i = 1, ..., n$ s.t.

- $\varphi = \rho_a(f_1)$, $\psi = \rho_b(f_n)$
- $f_i|_{V_i} = f_{i+1}|_{V_i}$ for $i = 1, ..., n-1$, where V_i denotes the connected component of $U_i \cap U_{i+1}$ containing the point $u(t_i)$.

Lemma 1.3.1. Suppose X is a Riemann surface and u is a curve from a to b in X. Then a function germ $\psi \in \mathcal{O}_b$ is the analytic continuation of a function germ $\varphi \in \mathcal{O}_a$ along u precisely if there exists a lifting $\widetilde{u}$ of u from φ to ψ in $|\mathcal{O}|$.

Proof:

- $(\Rightarrow)$: Let $\varphi_t \in \mathcal{O}_{u(t)}$ given by definition. Then $\widetilde{u}(t) := \varphi_t$ is continuous by definition and is the lifting of u from φ to ψ.
- $(\Leftarrow)$: For $t \in [0,1]$, define $\varphi_t = \widetilde{u}(t) \in \mathcal{O}_{u(t)}$. Let $\tau \in [0,1]$ and suppose $[U, f] \subset |\mathcal{O}|$ is open neighborhood of $\widetilde{u}(\tau)$, then exists a neighborhood of $T \subset [0,1]$ of τ such that $\widetilde{u}(T) \subset [U, f]$. This implies $u(T) \subset U$ and $\varphi_t = \widetilde{u}(t) = \rho_{u(t)}(f)$ $\forall t \in T$, which means ψ is the analytic continuation of φ along u.

$\square$

Remark 1.3.2. Since $p : |\mathcal{O}| \to X$ is local homeomorphism and $|\mathcal{O}|$ is Hausdorff, by uniqueness of lifting, if the analytic continuation of a function germ exists, then it is unique determined. Also, if u_0, u_1 are homotopic curves from a to b in X and assume $\varphi \in \mathcal{O}_a$ is a function germ which admits an analytic continuation along every deformation curve u_s from φ into ψ. Then by Theorem 1.1.4, the analytic continuations of φ along u_0 and u_0 yield the same function germ $\psi \in \mathcal{O}_b$.

Corollary 1.3.1. Suppose X is a simply connected Riemann surface, $a \in X$ and $\varphi \in \mathcal{O}_a$ admits an analytic continuation along every curve starting at a. Then $\exists f \in \mathcal{O}(X)$ such that $\rho_a(f) = \varphi$.

Proof: For any $x \in X$, let $\psi_x \in \mathcal{O}_x$ be the function germ from the analytic continuation along any curve from a to x. Since X is simply connected, ψ is well-defined. Set $f(x) := \psi_x(x)$. Then f is holomorphic function on X s.t. $\rho_a(f) = \varphi$. $\square$

Suppose X and Y are Riemann surface and $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the sheaves of holomorphic functions on them. Suppose $p : Y \to X$ is an unbranch holomorphic map. Since p is local homeomorphic, for each $y \in Y$ it induces an isomorphism

$$\begin{array}{cccc} p^* : & \mathcal{O}_{X,p(y)} & \longrightarrow & \mathcal{O}_{Y,y} \\ & [f \in \mathcal{O}_X(U)] & \longmapsto & [f \circ p \in \mathcal{O}_Y(p^{-1}(U))] \end{array}$$

and let $p_* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,p(y)}$ be the inverse of p^*.

Definition 1.3.5. Suppose X is a Riemann surface, $a \in X$ is a point and $\varphi \in \mathcal{O}_a$ is a function germ. A quadrupel (Y, p, f, b) is called an **analytic continuation** of φ if

- Y is a Riemann surface and $p : Y \to X$ is an unbranch holomorphic map.
- f is a holomorphic function on Y.
- $b \in p^{-1}(a)$ s.t. $p_*(\rho_b(f)) = \varphi$.

An analytic continuation (Y, p, f, b) of φ is said to be **maximal** if it has the following universal property. If (Z, q, g, c) is any other analytic continuation of φ, then there exists a fiber-preserving holomorphic mapping $F : Z \to Y$ such that $F(c) = b$ and $F^*(f) = g$.

Remark 1.3.3. A maximal analytic continuation is unique up to isomorphism. If (Y, p, f, b) and (Z, q, g, c) are two maximal analytic continuations of φ, then there exists a fiber-preserving holomorphic map $F : Z \to Y$, $G : Y \to Z$ s.t. $F(c) = b$, $G(b) = c$ and $F^*(f) = g$, $G^*(g) = f$. Then $F \circ G$ is a fiber preserving holomorphic mapping from Y to Y which fixed b. By uniqueness of lifting, $F \circ G = \mathrm{id}_Y$. Similarly, $G \circ F = \mathrm{id}_Z$ and thus $G : Y \to Z$ is biholomorphhic.

Lemma 1.3.2. Suppose X is a Riemann surface, $a \in X$, $\varphi \in \mathcal{O}_a$ and (Y, p, f, b) is an analytic continuation of φ. Then if v is a curve from b to y in Y, then the function germ $\psi = p_*(\rho_y(f)) \in \mathcal{O}_{p(y)}$ is an analytic continuation of φ along the curve $u = p \circ v$.

Proof: For $t \in [0, 1]$ let $\varphi_t := p_*(\rho_{v(t)}(f)) \in \mathcal{O}_{u(t)}$. Then $\varphi_0 = \varphi$ and $\varphi_1 = p_*(f_y) =: \psi$. Suppose $t_0 \in [0, 1]$. Since $p : Y \to X$ is local homeomorphism, there exists neighborhoods $V \subset Y$ of $v(t_0)$ and $U \subset X$ of $u(t_0)$ s.t. $p|_V : V \underset{q}{\overset{\sim}{\rightleftarrows}} U$. Let $g := q^*(f|_V) \in \mathcal{O}(U)$, then $p_*(\rho_\eta(f)) = \rho_{p(\eta)}(g)\ \forall \eta \in V$. There exists a neighborhood $T \subset [0, 1]$ of t_0 s.t. $v(T) \subset V$ i.e. $u(T) \subset U$. For every $t \in T$,

$$\rho_{u(t)}(g) = p_*(p_{v(t)}(f)) = \varphi_t$$

This prove that ψ is an analytic continuation of φ along u. $\square$

Theorem 1.3.2. Suppose X is a Riemann surface, $a \in X$ and $\varphi \in \mathcal{O}_a$. Then there exists a maximal analytic continuation (Y, p, f, b) of φ.

Proof:

- Let Y be the connected component of $|\mathcal{O}|$ containing φ. Let p also denote the restriction of the mapping $p : |\mathcal{O}| \to X$ to Y, then $p : Y \to X$ is local homeomorphism. Then exists a complex structure on Y s.t. it becomes a Riemann surface and $p : Y \to X$ so holomorphic. Define $f : Y \to \mathbb{C}$ by $f(\eta) = \eta(p(\eta))$, then it is clear that f is holomorphic and $p_*(\rho_\eta(f)) = \eta\ \forall \eta \in Y$. If set $b := \varphi$, then (Y, p, f, b) is an analytic continuation of φ.

- If (Z, q, g, c) is another another analytic continuation of φ. Define the map $F : Z \to Y$ as follows. Suppose $\zeta \in Z$ and $x := q(\zeta)$. By Lemma 1.3.2, $q_*(\rho_\zeta(g)) \in \mathcal{O}_{X,x}$ arises by analytic continuation along a some curve from a to x from φ (since Z is path connected). By Lemma 1.3.1, Y consists of all function germs which obtained by the analytic continuation of φ along curves. Hence $q_*(\rho_\zeta(g)) \in Y$ and define $F(\zeta) = q_*(\rho_\zeta(g))$. Then $F : Z \to Y$ is a fiber-preserving holomorphic map s.t. $F(c) = b$ and

$$F^*(f)(\zeta) = f(F(\zeta)) = f(q_*(\rho_\zeta(g))) = q_*(\rho_\zeta(g))(q(\zeta)) = g(\zeta)$$

  implies $F^* f = g$.

□

1.3.3 Algebraic functions

Suppose X and Y are Riemann surfaces, $\pi : Y \to X$ is an n-sheeted unbranched holomorphic covering map and f is a meromorphic function on Y. Every point $x \in X$ has an open neighborhood U s.t. $\pi^{-1}(U) = \bigsqcup_{v=1}^{n} V_v$ with $\pi : V_v \underset{\tau_v}{\overset{\sim}{\rightleftarrows}} U$. Let $f_v := \tau_v^* f = f \circ \tau_v$. Suppose T is an indeterminate and consider

$$\prod_{v=1}^{n} (T - f_v) = T^n + c_1 T^{n-1} + \cdots + c_n$$

Then the c_v are meromorphic functions in U and

$$c_v = (-1)^v s_v(f_1, ..., f_n)$$

where s_v denoted the vth elementary symmetric function in n variables. We can construct c'_v on neighborhood U' of another point x', then c_v and c'_v are compatible on the intersection. So we can glue it to get $c_1, ..., c_n \in \mathscr{M}(X)$, which call the **elementary symmetric functions** of f w.r.t. the covering $Y \to X$.

Theorem 1.3.3. Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is an n-sheeted branch holomorphic covering map. Suppose $A \subset X$ is a closed discrete subset which contains all the critical values of π and let $B = \pi^{-1}(A)$. Suppose f is a holomorphic (resp meromorphic) function on $Y \setminus B$ and $c_1, ..., c_n \in \mathcal{O}(X \setminus A)$ (resp. $\mathscr{M}(X \setminus A)$) are the elementary symmetric functions of f. Then f may be continued holomorphically (resp. meromorphically) to Y precisely if all the c_v may be continued holomorphically (resp. meromorphically) to X.

Proof: Suppose $a \in A$ and $b_1, ..., b_m$ are the preimages of a. Suppose (U, z) is a relatively compact chart of a with $z(a) = 0$ and $U \cap A = \{a\}$. Then $V = \pi^{-1}(U)$ is a relatively compact chart of each of b_μ.

(1) If $f \in \mathcal{O}(Y \setminus B)$:

- $(\Rightarrow)$: Note that f is bounded on $V \setminus \{b_1, ..., b_m\}$, which implies all the c_v are bounded on $U \setminus \{a\}$, and thus a is removable singularity, which can continued holomorphically to a for all c_v.
- $(\Leftarrow)$: Note that c_v are all bounded on $U \setminus \{a\}$, which implies f is bounded on $V \setminus \{b_1, ..., b_m\}$. Indeed, for $y \in V \setminus \{b_1, ..., b_m\}$ and let $x = \pi(y)$, then

$$f(y)^n + c_1(x) f(y)^{n-1} + \cdots + c_n(x) = 0$$

  Hence, b_μ are removable singularity, which can continued holomorphically to every b_μ.

(2) If $f \in \mathscr{M}(Y \setminus B)$:

- $(\Rightarrow)$: The function $\varphi := \pi^* z \in \mathcal{O}(V)$ vanishes at all points b_μ. Thus $\varphi^k f$ may continued holomorphically to all the point b_μ if k is sufficiently large. The elementary symmetric functions of $\varphi^k f$ are $z^{kv} c_v$ and by (1) they may continued holomorphically to a. Thus all the c_v may be continued meromorphically to a.
- $(\Leftarrow)$: Similarly, $z^{kv} c_v$ admit holomorphic continuations to a as k sufficiently large. Thus $\varphi^k f$ admits a holomorphic continuation to all the points b_μ. This implies that f may be continued meromorphically to all of the points b_μ.

□

Remark 1.3.4. Note that the proof doesn't use the fact Y is connected. Thus the theorem will also holds when Y is a disjoint union of finitely many Riemann surfaces.

If $\pi : Y \to X$ is a non-constant holomorphic map between Riemann surfaces X and Y, then for any meromorphic function f on X the function $\pi^* f$ is a meromorphic function on Y. Thus there is a map

$$\pi^* : \mathscr{M}(X) \to \mathscr{M}(Y)$$

which is a monomorphism of fields.

Theorem 1.3.4. Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is a branched holomorphic n-sheeted covering map. If $f \in \mathscr{M}(Y)$ and $c_1, ..., c_n \in \mathscr{M}(X)$ are the elementary symmetric functions of f, then

$$f^n + (\pi^* c_1) f^{n-1} + \cdots + (\pi^* c_{n-1}) f + \pi^* c_n = 0 \tag{$*$}$$

The monomorphism $\pi^* : \mathscr{M}(X) \to \mathscr{M}(Y)$ is an algebraic field extension of degree $\leq n$. Moreover, if there exists an $f \in \mathscr{M}(X)$ and an $x \in X$ with preimages $y_1, ..., y_n \in Y$ such that the valued $f(y_v)$ for $v = 1, ..., n$ are all distinct, then the field extension $\pi^* : \mathscr{M}(X) \to \mathscr{M}(Y)$ has degree n.

Proof:

- $(*)$ holds immediately by the definition of elementary symmetric functions of f.
- Let $L = \mathscr{M}(Y)$ and $K = \pi^* \mathscr{M}(X) \subset L$. Then every $f \in L$ is algebraic over K and $\deg m_{f,K} \leq n$. Choose $f_0 \in L$ s.t.

  $$n_0 := \max_{f \in L} \deg m_{f,K} = \deg m_{f_0,K}$$

  Then $\forall f \in L$, consider the intermediate field $K(f, f_0)$ of L/K. By primitive element theorem, $K(f, f_0) = K(g)$ for some $g \in L$. Then

  $$n_0 = [K(f_0) : K] \leq [K(f, f_0) : K] = \deg m_{g,K} \leq n_0$$

  which implies that $K(f, f_0) = K(f_0)$ i.e. $f \in K(f_0)$ and thus $L = K(f_0)$. Hence, $[L : K] \leq n$.
- If $\deg m_{f,K} = m \leq n$, then f would be able to take at most m different values over every point $x \in X$. □

Theorem 1.3.5. Suppose X is a Riemann surface, $A \subset X$ is a closed discrete subset and let $X' = X \setminus A$. Suppose Y' is another Riemann surface and $\pi' : Y' \to X'$ is a proper unbranch holomorphic covering. Then π' extends to a branch covering of X, i.e. there exists a Riemann surface Y, a proper holomorphic mapping $\pi : Y \to X$ and a fiber-preserving biholomorphic mapping

$$\varphi : Y \setminus \pi^{-1}(A) \to Y'$$

Proof:

- For any $a \in A$, choose a coordinate neighborhood (U_a, z_a) on X such that $z_a(a) = 0$, $z_a(U_a)$ is the unit disk in $\mathbb{C}$ and $U_a \cap U_{a'} = \varnothing$ if $a \neq a'$. Let $U_a^* = U_a \setminus \{a\}$. Since $\pi' : Y' \to X'$ is proper, $\pi'^{-1}(U_a^*)$ consists of a finite number of connected components V_{av}^*, $v = 1, ..., n(a)$. For every v the mapping $\pi' : V_{av}^* \to U_a^*$ is an unbranched covering, say k_{av} be the covering number. By Remark 1.2.1, there exists biholomorphic mappings $\zeta_{av} : V_{av}^* \to \mathbb{D}^*$ such that the diagram

$$\begin{tikzcd} V_{av}^* \arrow[r, "\zeta_{av}"] \arrow[d, "\pi'"'] & \mathbb{D}^* \arrow[d, "\pi_{av}"] \\ U_a^* \arrow[r, "z_a"] & \mathbb{D}^* \end{tikzcd}$$

  is commute, where $\pi_{av}(\zeta) = \zeta^{k_{av}}$.

- Now choose "ideal point" p_{av} and define the topology

$$Y = Y' \cup \{p_{av} : a \in A, v = 1, ..., n(a)\}$$

  as follows. If W_i $(i \in I)$ is a neighborhood basis of a, then let

$$\{p_{av}\} \cup (\pi'^{-1}(W_i) \cap V_{av}^*) \ i \in I$$

  be a neighborhood basis of p_{av} and on Y' induces the given topology. It is clear that Y is Hausdorff. Define $\pi : Y \to X$ by $\pi(y) = \pi'(y)$ for $y \in Y'$ and $\pi(p_{av}) = a$, then it is clear that π is proper.

- In order to make Y into a Riemann surface, it suffices give a complex structure on Y'. Let $V_{av} = V_{av}^* \cup \{p_{av}\}$. By Theorem 1.2.4, we can extend $\zeta_{av} : V_{av}^* \to \mathbb{D}^*$ to $\zeta_{av} : V_{av} \to \mathbb{D}$. Note that $\zeta_{av} : V_{av}^* \to \mathbb{D}$ is biholomorphic w.r.t. the complex structure on Y', the new charts $\zeta_{av} : V_{av} \to D$ are holomorphically compatible with the charts of the complex charts of Y'. Hence, $\pi : Y \to X$ is holomorphic. Since $Y \setminus \pi^{-1}(A) = Y'$ by construction, we may choose $\varphi : Y \setminus \pi^{-1}(A) \to Y'$ to be the identity mapping.

□

Theorem 1.3.6. Suppose X, Y and Z are Riemann surfaces and $\pi : Y \to X$, $\tau : Z \to X$ are proper holomorphic covering maps. Let $A \subseteq X$ be a closed discrete subset and let $X' := X \setminus A$, $Y' = \pi^{-1}(X')$ and $Z' = \tau^{-1}(X')$. Then every fiber-preserving biholomorphic mapping $\sigma' : Y' \to Z'$ can be extended to a fiber-preserving biholomorphic mapping $\sigma : Y \to Z$. In particular, every $\sigma' \in \mathrm{Deck}(Y'/X')$ can be extend to $\sigma \in \mathrm{Deck}(Y/X)$.

Proof:

- Suppose $a \in A$ and (U, z) is a chart of a s.t. $z(a) = 0$ and $z(U) = \mathbb{D}$. Let $U^* = U \setminus \{a\}$. We may assume that U is small enough s.t. π and τ are unbranched over U^*. Let $V_1, ..., V_n$ (resp. $W_1, ..., W_m$) be the connected components of $\pi^{-1}(U)$ (resp. $\tau^{-1}(U)$). Then $V_v^* = V_v \setminus \pi^{-1}(a)$ (resp. $W_\mu^* := W_\mu \setminus \tau^{-1}(a)$) are connected components of $\pi^{-1}(U^*)$ (rsep. $\tau^{-1}(U^*)$).

- Since $\sigma' : \pi^{-1}(U^*) \to \tau^{-1}(U^*)$ is biholomorphic, $n = m$ and may assume $\sigma'(V_v^*) = W_v^*$. Since $\pi : V_v^* \to U^*$ is finite sheeted unbranched covering, then $V_v \cap \pi^{-1}(a)$ (resp. $W_v \cap \tau^{-1}(a)$) consists exactly one point b_v (resp. c_v) by Theorem 1.2.4. Then we may extend $\sigma' : \pi^{-1}(U^*) \to \tau^{-1}(U^*)$ to a bijective mapping $\pi^{-1}(U) \to \tau^{-1}(U)$ by $b_v \mapsto c_v$. It is clear that this map is biholomorphic.

□

Definition 1.3.6. Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is a branch holomorphic covering. Let $A \subset X$ be the set of critical values of π and let $X' := X \setminus A$ and $Y' := \pi^{-1}(X')$. Then the covering $Y \to X$ is called **Galois** if the covering $Y' \to X'$ is Galois.

Lemma 1.3.3. Suppose $c_1, ..., c_n$ are holomorphic functions on the disk $D(R) = \{z \in \mathbb{C} : |z| < R\}$. Suppose $w_0 \in \mathbb{C}$ is a simple zero of the polynomial

$$T^n + c_1(0)T^{n-1} + \cdots + c_n(0) \in \mathbb{C}[T]$$

Then $\exists r \in (0, R]$ and $\varphi \in \mathcal{O}(D(r))$ s.t. $\varphi(0) = w_0$ and

$$\varphi^n + c_1\varphi^{n-1} + \cdots + c_n = 0 \text{ on } D(r)$$

Proof: For $z \in D(R)$ and $w \in \mathbb{C}$ let

$$F(z, w) = w^n + c_1(z)w^{n-1} + \cdots + c_n(z)$$

Then $\exists \varepsilon > 0$ s.t. $w \mapsto F(0, w)$ has a unique zero w_0 in $\overline{B_{w_0}(\varepsilon)}$. Since F is continuous, $\exists r \in (0, R]$ s.t. $F(z, w)$ is not vanish on

$$\{(z, w) \in \mathbb{C}^2 : |z| < r, |w - w_0| = \varepsilon\}$$

For fixed $z \in D(r)$,

$$n(z) = \frac{1}{2\pi i} \int_{|w-w_0|=\varepsilon} \frac{F_w(z, w)}{F(z, w)} dw$$

is the number of zeros of the function $w \mapsto F(z, w)$ in $B_\varepsilon(w_0)$. Since $n(0) = 1$ and $n(z)$ is continuous w.r.t. z, which shows that $n(z) = 1$ for every $z \in D(r)$. Define

$$\varphi(z) = \frac{1}{2\pi i} \int_{|w-w_0|=\varepsilon} w \cdot \frac{F_w(z, w)}{F(z, w)} dw$$

By the residue theorem, $\varphi(z)$ is the unique zero of $w \mapsto F(z, w)$. It is clear that $\varphi(z)$ is holomorphic on $D(r)$. □

Corollary 1.3.2. Let X be a Riemann surface, $x \in X$ and

$$P(T) = T^n + c_1T^{n-1} + \cdots + c_n \in \mathcal{O}_x[T]$$

Suppose that the polynomial

$$p(T) := T^n + c_1(x)T^{n-1} + \cdots + c_n(x) \in \mathbb{C}[T]$$

has n distinct zeros $w_1, ..., w_n$. Then there exist elements $\varphi_1, ..., \varphi_n \in \mathcal{O}_x$ such that $\varphi_v(x) = w_v$ and

$$P(T) = \prod_{v=1}^{n} (T - \varphi_v)$$

Theorem 1.3.7. Suppose X is a Riemann surface and

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathscr{M}(X)[T]$$

is an irreducible polynomial. Then exists a Riemann surface Y and a branch holomorphic n-sheeted covering $\pi : Y \to X$ and a meromorphic function $F \in \mathscr{M}(Y)$ s.t. $(\pi^* P)(F) = 0$. The triple (Y, π, F) is uniquely determined in the following sense. If (Z, τ, G) has the corresponding properties, then there exists exactly one fiber-preserving biholomorphic mapping $\sigma : Z \to Y$ s.t. $G = \sigma^* F$. Such (Y, π, F) is called the **algebraic function** defined by the polynomial $P(T)$.

Proof:

- (existence) Let $\Delta \in \mathscr{M}(X)$ be the discriminant of $P(T)$. Then $\Delta \not\equiv 0$ on X, since P is irreducible. Then there exists a closed discrete subset $A \subset X$ s.t. $c_1, ..., c_n$ are holomorphic on $X' := X \setminus A$ and $\Delta(x) \neq 0$ $\forall x \in X'$. Then $\forall x \in X'$, the polynomial

$$p_x(T) := T^n + c_1(x) T^{n-1} + \cdots + c_n(x) \in \mathbb{C}[T]$$

has n distinct zeros. Define

$$Y' = \{\varphi \in \mathcal{O}_x : x \in X', P(\varphi) = 0\} \subset |\mathcal{O}|$$

and let $\pi' : Y' \to X'$ be the canonical projective. For each $x \in X'$, apply Corollary 1.3.2 on $p_x(T)$, exists open neighborhood $U \subset X$ of x and $f_1, ..., f_n \in \mathcal{O}(U)$ s.t.

$$P(T) = \prod_{v=1}^{n} (T - f_v) \text{ on } U$$

Then $\pi'^{-1}(U) = \bigsqcup_{v=1}^{n} [U, f_v]$, which shows that $\pi' : Y' \to X'$ is a n-sheeted covering map. The connected components of Y' are Riemann surfaces which also admit covering maps over X'. Define $f : Y' \to \mathbb{C}$ by $f(\varphi) = \varphi(\pi'(\varphi))$. Then f is holomorphic and

$$f(y)^n + c_1(\pi'(y)) f(y)^{n-1} + \cdots + c_n(\pi^{-1}(y)) = 0 \ \forall y \in Y'$$

By Theorem 1.3.6, $\pi' : Y' \to X'$ can be extend to a proper holomorphic covering $\pi : Y \to X$, where we identify Y' with $\pi^{-1}(X')$. By Theorem 1.3.3, f may be extended to $F \in \mathscr{M}(Y)$ and thus

$$(\pi^* P)(F) = F^n + (\pi^* c_1) F^{n-1} + \cdots + \pi^* c_n = 0$$

Now we will show that Y is connected and thus is Riemann surface. To the contrary, Y has finitely many connected components $Y_1, ..., Y_k$ and $\pi|_{Y_i} : Y_i \to X$ is a proper holomorphic n_i-sheeted covering, where $\sum n_i = n$. Using the elementary symmetric functions of $F|_{Y_i}$ one gets polynomials $P_i(T) \in \mathscr{M}(X)[T]$ of degree n_i s.t.

$$P(T) = P_1(T) \cdots P_k(T)$$

which is contradict to $P(T)$ is irreducible.

- (uniqueness) Let $B \subset Z$ be the union of the poles of G and the branch points of τ. Let $A' := \tau(B)$ and define

$$X'' := X' \setminus A', \ Y'' := \pi^{-1}(X''), \ Z := \tau^{-1}(X'')$$

- Let $z \in Z''$, then $\tau_* G_z \in \mathcal{O}_{\tau(z)}$. By $(\tau^* P)(G) = 0$, $P(\tau_* G_z) = 0$. By construction of Y'', $\tau_* G_z \in Y''$. Define
$$\begin{aligned} \sigma'' : \quad Z'' &\longrightarrow \quad Y" \\ z &\longmapsto \quad \tau_* G_z \end{aligned}$$

- σ'' is continuous : Given $[U, f] \subset Y''$ with $U \subset X"$. For $x \in U$, exists neighborhood $U_0 \subset U$ of x s.t. $\tau^{-1}(U_0) = \bigsqcup_{j=1}^{n} V_j'$. Since and $f(x)$ is zero of
$$T^n + c_1(x)T^{n-1} + \cdots + c_n(x) = \prod_{j=1}^{n}(T - \tau_* G_z(x))$$
$f(x) = \tau_* G_z(x)$ for some $z \in V_j'$. Then $\exists U_0' \subset U_0$ s.t. $\tau_* G_{z'} = \rho_{\tau(z')}(f)$ $\forall z' \in \tau^{-1}(U_0')$. Then
$$\sigma''(\tau^{-1}(U_0') \cap V_j') = [U_0', f|_{U_0'}] \subset [U, f]$$

- By definition σ'' is continuous and fiber-preserving and thus is holomorphic. Moreover, σ'' is proper, since $\pi : Y'' \to X''$ is continuous and $\tau : Z'' \to X''$ is proper. Hence σ'' is surjective. Since $Y'' \to X''$ and $Z'' \to X''$ have the same number of sheets, $\sigma'' : Z'' \to Y''$ is biholomorphic.

- By definition of σ'', $G|_{Z''} = (\sigma'')^*(F|_{Y''})$. By Theorem 1.3.6, σ'' can be extended to a fiber-preserving biholomorphic mapping $\sigma : Z \to Y$, then $G = \sigma^* F$.

- The mapping σ is in fact uniquely determined by the property of $G = \sigma^* F$. Otherwise there would exist id $\neq \alpha \in \mathrm{Deck}(Y/X)$ s.t. $\alpha^* F = F$. But $\forall x \in X'$ and distinct $y_1, y_2 \in \pi^{-1}(x)$, $F(y_1) \neq F(y_2)$.

□

Example 1.3.1. Suppose $f(z) = (z - a_1)\cdots(z - a_n)$ is a polynomial with distinct roots $a_1, ..., a_n \in \mathbb{C}$. Consider f as a meromorphic function on the Riemann sphere $\mathbb{P}^1$. The polynomial $P(T) = T^2 - f$ is irreducible over $\mathscr{M}(\mathbb{P}^1)$ and defines an algebraic function which is usually denoted by $\sqrt{f(z)}$. To describe its Riemann surface $\pi : Y \to X$. Let $A = \{a_1, ..., a_n, \infty\}$, $X' = \mathbb{P}^1 \setminus A$ and $Y' = \pi^{-1}(X')$. Then $\pi : Y' \to X'$ is an unbranched holomorphic 2-sheeted covering. This implies every $\varphi \in \mathcal{O}_x$ with $x \in X'$ s.t. $\varphi^2 = f$ can be analytically continued along every curve lying in X'.

- For each $a_1, ..., a_n$, choose $r_i > 0$ s.t. $B_{r_i}(a_i) \cap A = \{a_i\}$. Since the function $g(z) = \prod_{i \neq j}(z - a_i)$ has no zero in U_j and U_j is simply connected, there exists a holomorphic function $h : U_j \to \mathbb{C}$ s.t. $h^2 = g$. Thus $f(z) = (z - a_j)h(z)^2$ on U_j. Suppose $0 < \rho < r_j$, $\theta \in \mathbb{R}$ and let $\zeta = a_j + \rho e^{i\theta}$. By Lemma 1.3.3, $\exists \varphi_\zeta \in \mathcal{O}_\zeta$ s.t. $\varphi_\zeta^2 = f$ and
$$\varphi_\zeta(\zeta) = \sqrt{\rho} e^{i\theta/2} h(\zeta)$$
If one continues this function germ along the closed curve $u(t) = a_j + \rho e^{2\pi i t}$, then one obtain the negative of the original germ. Let $U_j^* = U_j \setminus \{a_j\}$ and $V_j^* = \pi^{-1}(U_j^*)$. Then $\pi : V_j^* \to U_j^*$ is a connected 2-sheeted covering isomorphic to $z \mapsto z^2$. For otherwise $\pi : V_j^* \to U_j^*$ would split into two single-sheeted coverings and the analytic continuation of φ_ζ along the curve $u(t)$ will lead back to the same function germ. Hence the Riemann surface Y has exactly one point over the point a_j.

- Suppose $r > \max_i |a_i|$ and let $U^* = \{z \in \mathbb{C} : |z| > r\}$. Then $U^* \cup \{\infty\}$ is a neighborhood of ∞, which is isomorphic to a disk, and contains no other points of A. On U one can write $f = z^n F$, where F is a holomorphic function having no zeros in U

(1) If n is odd, then $\exists h \in \mathscr{M}(U)$ s.t. $f(z) = zh(z)^2$.

(2) If n is even, then $\exists h \in \mathscr{M}(U)$ s.t. $f(z) = h(z)^2$.

Let $V^* = \pi^{-1}(U^*)$. Similarly to above, in the case (1), $\pi : V^* \to U^*$ is a connected two-sheeted coveing and thus Y has precisely one point over ∞. But in case (2), $\pi : V^* \to U^*$ splits into two single-sheeted covering and thus when n is even, Y has two point over ∞.

If X and Y are Riemann surfaces and $\pi : Y \to X$ is a branched holomorphic covering map, then $\mathrm{Deck}(Y/X)$ has a representation into $\mathrm{Aut}(\mathscr{M}(Y))$ defined in the following way. For $\sigma \in \mathrm{Deck}(Y/X)$, let $\sigma f := f \circ \sigma^{-1}$. Clearly $(f \mapsto \sigma f) \in \mathrm{Aut}(\mathscr{M}(Y))$. The mapping

$$\begin{array}{ccc} \mathrm{Deck}(Y/X) & \longrightarrow & \mathrm{Aut}(\mathscr{M}(Y)) \\ \sigma & \longmapsto & (f \mapsto \sigma f) \end{array}$$

is a group homomorphism, since $\forall \sigma, \tau \in \mathrm{Deck}(Y/X)$ and $f \in \mathscr{M}(Y)$

$$(\sigma\tau)f = f \circ \tau^{-1} \circ \sigma^{-1} = \sigma(\tau f)$$

Notice that for $f \in \mathscr{M}(X)$ and $\sigma \in \mathrm{Deck}(Y/X)$,

$$\sigma(\pi^* f) = f \circ \pi \circ \sigma^{-1} = f \circ \pi = \pi^* f$$

which means $\mathrm{Deck}(Y/X)$ leaves invariant the functions of the subfield $\pi^*\mathscr{M}(X) \subset \mathscr{M}(Y)$ and thus is the element of $\mathrm{Aut}(\mathscr{M}(Y)/\pi^*\mathscr{M}(X))$.

Theorem 1.3.8. Suppose X is a Riemann surface, $K = \mathscr{M}(X)$ and $P(T) \in K[T]$ is an irreducible monic polynomial of degree n. Let (Y, π, F) be the algebraic function defined by $P(T)$ and $L = \mathscr{M}(Y)$. Regard K as the subfield of L via $\pi^* : K \to L$. Then L/K is a field extension of degree n and $L \simeq K[T]/(p(T))$ and the group homomorphism

$$\varphi : \mathrm{Deck}(Y/X) \to \mathrm{Aut}(L/K)$$

is isomorphism. The covering $Y \to X$ is Galois precisely if the field extension L/K is Galois.

Proof:

- By Theorem 1.3.4, L/K is field extension of degree n. Since $P(F) = 0$, there is a homomorphism $K[T]/(P(T)) \to L$. Since both these fields are of degree n over K, this is an isomorphism.

- φ is injective, since $\sigma F \neq F$ $\forall \sigma \in \mathrm{Deck}(Y/X) \setminus \{\mathrm{id}\}$.

- φ is surjective : Given $\alpha \in \mathrm{Aut}(L/K)$, $(Y, \pi, \alpha F)$ is also an algebraic function defined by the polynomial $P(T)$. By uniqueness, $\exists \tau \in \mathrm{Deck}(Y/X)$ s.t. $\alpha F = \tau^* F$. If $\sigma = \tau^{-1}$, then

  $$\sigma F = F \circ \sigma^{-1} = F \circ \tau = \tau^* F = \alpha F$$

  Since L is generated by F over K, α is the map $f \mapsto \sigma f$.

- Y is Galois over X (resp. L/K is Galois) precisely when $\mathrm{Deck}(Y/X)$ (resp. $\mathrm{Aut}(L/K)$) contains n elements.

$\square$

Example 1.3.2 (**Puiseux expansions**). Denote $\mathbb{C}\{\{z\}\}$ be the field of all Laurent series with finite principal part

$$\varphi(z) = \sum_{v=k}^{\infty} c_v z^v,\ k \in \mathbb{Z},\ c_v \in \mathbb{C}$$

converging in some punctured disk $\{0 < |z| < r\}$, where $r > 0$ depend on φ. Then $\mathbb{C}\{z\}$ is isomorphic to the stalk $\mathscr{M}_0$ of the sheaf $\mathscr{M}$ of meromorphic functions in the complex plane and is the quotient field of $\mathbb{C}\{z\}$. Consider an irreducible polynomial

$$F(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) \in \mathbb{C}\{\{z\}\}[w]$$

of degree n over the field $\mathbb{C}\{\{z\}\}$. Then $\exists r > 0$ s.t. all a_v are meromorphic functions on $D(r) = \{z \in \mathbb{C} : |z| < r\}$ and thus F may also be consider as an element of $\mathscr{M}(D(r))[w]$. It is clear that F is also irreducible over the field $\mathscr{M}(D(r))$. Now suppose that r has been chosen small enough s.t. for every $a \in D(r)\setminus\{0\}$ the polynomial $F(a, w) \in \mathbb{C}[w]$ has no multiple roots. Let (Y, π, f) be the algebraic function defined by $F(z.w) \in \mathscr{M}(D(r))[w]$. Then $\pi : Y \to D(r)$ is an n-sheeted proper holomorphic map which is ramified only over the origin. By Theorem 1.2.4, there exists an isomorphism $\alpha : D(\rho) \to Y$, $\rho = \sqrt[n]{r}$ s.t.

$$\pi(\alpha(\xi)) = \xi^n\ \forall \xi \in D(\rho)$$

Since $F(\pi, f) = 0$, it follows that

$$F(\zeta^n, \varphi(\zeta)) = 0,\ \text{where } \varphi = f \circ \alpha$$

This proves the following theorem.

Theorem 1.3.9 (Puiseux). Let

$$F(z, w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) \in \mathbb{C}\{\{z\}\}[w]$$

be an irreducible polynomial of degree n over the field $\mathbb{C}\{\{z\}\}$. Then there exists a Laurent series

$$\varphi(\zeta) = \sum_{v=k}^{\infty} c_v \zeta^v \in \mathbb{C}\{\{\zeta\}\}$$

such that

$$F(\zeta^n, \varphi(\zeta)) = 0$$

as an element of $\mathbb{C}\{\{\zeta\}\}$.

Remark 1.3.5.

- If all a_v are holomorphic, i.e. $a_v \in \mathbb{C}\{z\}$, then $\varphi \in \mathbb{C}\{\zeta\}$ as well. Since f is holomorphic on Y.
- Another way of expressing the assertion of the Theorem is to say that the equation can be solved by a **Puiseux series**

$$w = \varphi(\sqrt[n]{z}) = \sum_{v=k}^{\infty} c_v z^{v/n}$$

- We can interpret the Theorem of Puiseux in the following algebraic way. By means of the map

$$\mathbb{C}\{\{z\}\} \to \mathbb{C}\{\{\zeta\}\},\quad z \mapsto \zeta^n$$

$\mathbb{C}\{\{\zeta\}\}$ becomes an extension field of $\mathbb{C}\{\{z\}\}$ of degree n. A basis of $\mathbb{C}\{\{\zeta\}\}$ over $\mathbb{C}\{\{z\}\}$ is given by $1, \zeta, ..., \zeta^{n-1}$. The series $\varphi(\zeta)$ is a root of F in this extension field. Let ε be a primitive nth root of unity, e.g. $\varepsilon = e^{2\pi i/n}$. Then for $k = 0, 1, ..., n-1$ we have $(\varepsilon^k\zeta)^n = \zeta^n$ and hence

$$F(\zeta^n, \varphi(\varepsilon^k\zeta)) = 0$$

Thus $\varphi(\varepsilon^k\zeta) \in \mathbb{C}\{\{\zeta\}\}$ is also a root of the polynomial F. It is easy to see that the series $\varphi(\varepsilon^k\zeta)$, $k = 0, ..., n-1$ are distinct. Thus $\mathbb{C}\{\{\zeta\}\}$ is a splitting field of the polynomial $F \in \mathbb{C}\{\{z\}\}[w]$.

1.4 Linear differential equations

Theorem 1.4.1 (local solution). If $A \in M_n(\mathcal{O}(B_R(0)))$ with $0 < R \leq \infty$, then $\forall w_0 \in \mathbb{C}^n$,

$$\exists! w = \begin{pmatrix} w_1(z) \\ \vdots \\ w_n(z) \end{pmatrix} \in M(n \times 1, \mathcal{O}(B_R(0))) \text{ s.t. } \begin{cases} w'(z) = A(z)w(z) \forall z \in B_R(0) \\ w(0) = w_0 \end{cases}$$

Proof:

- By Taylor expansion, $A(z) = \sum_{\nu=0}^{\infty} A_\nu z^\nu$ with $A_\nu = (a_{ij\nu}) \in M_n(\mathbb{C})$ in $B_R(0)$. Suppose the solution $w(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$, where $c_\nu = (c_{i\nu}) \in \mathbb{C}^n$. Then

$$w'(z) = A(z)w(z) \iff (k+1)c_{k+1} = \sum_{\nu=0}^{k} A_{k-\nu}c_\nu \ \forall k \in \mathbb{N}$$

By $c_0 = w(0) = w_0$, we can determined all c_k recursively, which shows the uniqueness.

- For $0 < r < R$, the series $\sum_{\nu=0}^{\infty} |a_{ij\nu}|r^\nu$ converge, then $\exists N \in \mathbb{N}$ s.t. $|a_{ij\nu}| \leq \frac{N}{r^{\nu+1}}$. Define $B(z) = (b_{ij}(z))$ with

$$b_{ij}(z) = \frac{N}{r}\left(1 - \frac{z}{r}\right)^{-1} = \frac{N}{r}\sum_{\nu=0}^{\infty} \frac{z^r}{r^\nu} \text{ holomorphic on } |z| < r \text{ i.e. } b_{ij\nu} = \frac{N}{r^{\nu+1}}$$

Let $w_0 = \begin{pmatrix} w_{10} & \cdots & w_{n0} \end{pmatrix}^T$ and $K = \max_{1 \leq i \leq n} |w_{i0}|$. Define

$$\psi(z) = K\left(1 - \frac{z}{r}\right)^{-nN} = \sum_{\nu=0}^{\infty} f_\nu z^\nu \implies \psi'(z) = \frac{KnN}{r}\left(1 - \frac{z}{r}\right)^{-nN-1} = n\frac{N}{r}\left(1 - \frac{z}{r}\right)^{-1}\psi(z)$$

Then $v(z) = (\psi(z), ..., \psi(z))^T$ is the solution of

$$\begin{cases} v'(z) = B(z)v(z) \\ v(0) = (K, ..., K)^T \end{cases}$$

Claim : $|c_{ik}| \leq f_k \ \forall k \in \mathbb{Z}_{\geq 0}$.

subproof : We induct on k. $k = 0 : |c_{i0}| \leq K = f_0$. Notice that $|a_{ij\nu}| \leq b_{ij\nu} \ \forall i, j, \nu$.

$$|c_{i1}| = \left|\sum_{j=1}^{n} a_{ij0}c_{j0}\right| \leq \sum_{j=1}^{n} |a_{ij0}||c_{j0}| \leq \sum_{j=1}^{n} b_{ij0}\delta_0 = f_1$$

Assume that $|c_{i\nu}| \leq f_\nu$ $\forall i = 1, ..., n$ and $\forall \nu = 0, ..., k$. Then

$$(k+1)|c_{i,k+1}| = \left|\sum_{\nu=0}^{k}\sum_{j=1}^{n} a_{ij(k-\nu)}c_{j\nu}\right| \leq \sum_{\nu=0}^{k}\sum_{j=1}^{n} |a_{ij(k-\nu)}|\cdot|c_{j\nu}| \leq \sum_{\nu=0}^{k}\sum_{j=1}^{n} b_{ij(k-\nu)}f_\mu = (k+1)f_{k+1}$$

$\square$

Since $\sum_{k=0}^{\infty} f_k z^k = \psi(z)$ converge for $|z| < r$, $w_i(z) = \sum_{k=0}^{\infty} c_{ik}z^k$ converge for $|z| < r$. Then $w(z)$ is converge for all $r < R$, $w(z)$ converge on $B_R(0)$.

$\square$

Definition 1.4.1. Define $\Omega(X)$ be the holomorphic 1-forms on X. For $\omega \in \Omega(X)$, exists open cover U_i of X s.t. $\omega|_{U_i} = f_i dz$ with $f_i \in \mathcal{O}(U_i)$.

Theorem 1.4.2 (global solution)**.** Suppose X is a simply connected Riemann surface, $A \in M_n(\Omega(X))$ and $x_0 \in X$. Then $\forall c \in \mathbb{C}^n$, $\exists! w \in \mathcal{O}(X)^n$ is a solution of $dw = Aw$ with $w(x_0) = c$.

Proof: By Theorem 1.4.1, there exists a connected open neighborhood U_0 of x_0 and a solution $f \in \mathcal{O}(U_0)^n$ of the differential equation $df = Af$ with $f(x_0) = c$. Let $\alpha : [0,1] \to X$ with $\alpha(0) = x_0$. If $\alpha([0,t]) \subset U_0$, $x_1 = \alpha(t_1) \in U_0$. Then exists neighborhood U_1 of x_1 and exists $f_1 \in \mathcal{O}(U_1)^n$ s.t. f_1 is a local solution on U_1 with initial value $f(x_1)$. By uniqueness, exists neighborhood $W \subset U_0 \cap U_1$ s.t. $f|_W = f_1|_W$ and by identity theorem, $f = f_1$ on the connected component of $U_1 \cap U_2$ containing x_1. Since $[0,1]$ is compact, by finite steps, f can be analytic continuation along α. By Corollary 1.3.1, $\exists w \in \mathcal{O}(X)^n$ s.t. $dw = Aw$. $\square$

Corollary 1.4.1. Suppose X is a Riemann surface, $p : \widetilde{X} \to X$ is its universal covering with $y_0 \mapsto x_0$. Suppose $A \in M_n(\Omega(X))$ and $c \in \mathbb{C}^n$. Then $\exists! w \in \mathcal{O}(\widetilde{X})^n$ is a solution of $dw = (p^*A)w$ satisfying $w(y_0) = c$.

Example 1.4.1. Under the assumption in above. Define

$$L_A = \left\{ w \in \mathcal{O}(\widetilde{X})^n : dw = (p^*A)w \right\}$$

As the theory of ODE, one can show that L_A is an n-dimensional vector space over $\mathbb{C}$ and that $w_1, ..., w_n \in L_A$ are linearly independent precisely if for any arbitrary point $a \in \widetilde{X}$ the vectors $w_1(a), ..., w_n(a) \in \mathbb{C}^n$ are linearly independent. Therefore a basis $w_1, ..., w_n$ of L_A defines an invertible matrix

$$\Phi = (w_1, ..., w_n) \in \mathrm{GL}_n(\mathcal{O}(\widetilde{X}))$$

such that $d\Phi = (p^*A)\Phi$. Such a matrix is called a **fundamental system of solutions** of the differential equation $dw = Aw$.

- Let $G = \mathrm{Deck}(\widetilde{X}/X) \simeq \pi_1(X)$. $\forall \sigma \in G$, define the action $\sigma\Phi := \Phi \circ \sigma^{-1}$, then $d(\sigma\Phi) = (p^*A)(\sigma\Phi)$. Hence, $\exists T_\sigma \in \mathrm{GL}_n(\mathbb{C})$ s.t. $\sigma\Phi = \Phi T_\sigma$, this matrix is called the **factor of automorphy** of Φ. Then

$$\begin{array}{ccc} \mathrm{Deck}(\widetilde{X}/X) & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ \sigma & \longmapsto & T_\sigma \end{array}$$

is a group homomorphism, since

$$\Phi T_{\tau\sigma} = \tau\sigma\Phi = \tau(\Phi T_\sigma) = (\tau\Phi)T_\sigma = \Phi T_\tau T_\sigma \implies T_{\tau\sigma} = T_\tau T_\sigma$$

- Conversely, if $T : \mathrm{Deck}(\widetilde{X}/X) \to \mathrm{GL}_n(\mathbb{C})$ is a group homomorphism and $\Phi : \widetilde{X} \to \mathrm{GL}_n(\mathbb{C})$ s.t. $\sigma\Phi = \Phi T_\sigma\ \forall \sigma \in \mathrm{Deck}(\widetilde{X}/X)$. The matrix $(d\Phi)\Phi^{-1} \in M_n(\Omega(\widetilde{X}))$ is the invariant under $\mathrm{Deck}(\widetilde{X}/X)$, since

$$\sigma(d\Phi \cdot \Phi^{-1}) = (d\Phi \cdot T_\sigma)(\Phi T_\sigma)^{-1} = d\Phi \cdot \Phi^{-1}$$

By Galois correspondent, $\exists A \in M_n(\Omega(\widetilde{X}))$ s.t. $p * A = d\Phi \cdot \Phi^{-1}$ and thus Φ is a fundamental system of solution of differential equation $dw = (p^*A)w$.

Example 1.4.2.

Chapter 2

Compact Riemann surface

2.1 Vanishing theorems

Definition 2.1.1. For a open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of X and $\mathscr{F}$ be a sheaf of abelian group.

- 0th cochain group of $\mathscr{F}$: $C^0(\mathfrak{U},\mathscr{F}) = \prod\limits_{i\in I} \mathscr{F}(U_i)$
- 1st cochain group of $\mathscr{F}$: $C^1(\mathfrak{U}\mathscr{F}) = \prod\limits_{i,j} \mathscr{F}(U_i\cap U_j)$
- 2nd cochain group of $\mathscr{F}$: $C^2(\mathfrak{U}\mathscr{F}) = \prod\limits_{i,j,k} \mathscr{F}(U_i\cap U_j\cap U_k)$
- boundary map :

$$\begin{array}{rccc} \delta: & C^0(\mathfrak{U},\mathscr{F}) & \longrightarrow & C^1(\mathfrak{U},\mathscr{F}) \\ & (f_i)_{i\in I} & \longmapsto & \big((f_j - f_i)|_{U_i\cap U_j}\big) \end{array} \qquad \begin{array}{rccc} \delta: & C^1(\mathfrak{U},\mathscr{F}) & \longrightarrow & C^2(\mathfrak{U},\mathscr{F}) \\ & (f_{ij})_{i,j} & \longmapsto & \big((f_{jk} - f_{ik} + f_{ij})|_{U_i\cap U_j\cap U_k}\big) \end{array}$$

- 1-cocycle : $Z^1(\mathfrak{U},\mathscr{F}) = \ker(C^1(\mathfrak{U},\mathscr{F}) \to C^2(\mathfrak{U},\mathscr{F}))$
  1-coboundary : $B^1(\mathfrak{V},\mathscr{F}) = \mathrm{Im}(C^0(\mathfrak{U},\mathscr{F}) \to C^1(\mathfrak{U},\mathscr{F}))$
  - $(f_{ij}) \in Z^1(\mathfrak{U},\mathscr{F}) \iff f_{ij} + f_{jk} = f_{ik}$ on $U_i\cap U_j\cap U_k \implies f_{ii} = 0, f_{ij} = -f_{ij}$.
  - $(f_{ij}) \in B^1(\mathfrak{U},\mathscr{F}) \iff \exists (g_i) \in C^0$ s.t. $f_{ij} = g_i - g_j$ on $U_i\cap U_j$.
- 1st cohomology group $H^1(X,\mathscr{F})$:
  - Define $H^1(\mathfrak{U},\mathscr{F}) = Z^1(\mathfrak{U},\mathscr{F})\big/B^1(\mathfrak{U},\mathscr{F})$.
  - Directed partial order on open cover $\{\mathfrak{U}\}$:
    - $\mathfrak{V} = (V_k)_{k\in K} < \mathfrak{U} = (U_i)_{i\in I} \iff \exists \tau : K \to I$ s.t. $V_k \subset U_{\tau(k)}$.
    - $\mathfrak{U} = (U_k)_{k\in K}$, $\mathfrak{U}' = (V_{k'})_{k'\in K'}$. Define the open covering $\mathfrak{V}$ by $I = K\times K'$ and $W_{(k,k')} = U_k \cap V_{k'}$. Then $\mathfrak{V} < \mathfrak{U},\mathfrak{U}'$
    - For $\mathfrak{V} < \mathfrak{U}$, define

$$\begin{array}{rccc} t^{\mathfrak{U}}_{\mathfrak{V}}: & H^1(\mathfrak{U},\mathscr{F}) & \longrightarrow & H^1(\mathfrak{V},\mathscr{F}) \\ & (\overline{f_{ij}}) & \longrightarrow & \left(\overline{f_{\tau(k)\tau(\ell)}|_{V_k\cap V_\ell}}\right) \end{array}$$

      - independent on τ : For another $\tau' : K \to I$: On $V_k\cap V_\ell$,

$$\begin{aligned} f_{\tau(k)\tau(\ell)} - f_{\tau'(k)\tau'(\ell)} &= f_{\tau(k)\tau(\ell)} + f_{\tau(\ell)\tau'(k)} - f_{\tau(\ell)\tau'(k)} - f_{\tau'(k)\tau'(\ell)} \\ &= \underbrace{f_{\tau(k)\tau'(k)}}_{=:h_k\in\mathscr{F}(V_k)} - \underbrace{f_{\tau(\ell)\tau'(\ell)}}_{=:h_\ell\in\mathscr{F}(V_\ell)} \end{aligned}$$

- - - - injective : Suppose $(f_{\tau(k)\tau(\ell)}) = (g_k - g_\ell) \in B^1(\mathfrak{V}, \mathscr{F})$. Note that $(U_i \cap V_k)_{k\in K}$ is an open covering of U_i and on $U_i \cap V_k \cap V_\ell$

$$g_k - g_\ell = f_{\tau(k)\tau(\ell)} = f_{\tau(k)i} + f_{i\tau(\ell)} = f_{i\tau(\ell)} - f_{i\tau(k)}$$

$$\implies g_k + f_{i\tau(k)} = g_\ell + f_{i\tau(\ell)} \implies \exists h_i \in \mathscr{F}(U_i) \text{ s.t. } h_i|_{U_i\cap V_k} = g_k + f_{j\tau(k)}$$

and on $U_i \cap U_j \cap V_k$,

$$f_{ij} = f_{i\tau(k)} + g_k - g_k - f_{i\tau(k)} = h_i - h_j \ \forall k$$

Hence, $f_{ij} = h_i - h_j$ on $U_i \cap U_j$ and thus $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{F})$.

- - - $\mathcal{W} < \mathfrak{V} < \mathfrak{U} \implies t^{\mathfrak{V}}_{\mathcal{W}} \circ t^{\mathfrak{U}}_{\mathfrak{V}} = t^{\mathfrak{U}}_{\mathcal{W}}$ since we can use

$$\begin{tikzcd} J \arrow[r, "\rho"] \arrow[rr, bend right, "\sigma=\tau\circ\rho"'] & K \arrow[r, "\tau"] & I \end{tikzcd}$$

- - - Define $H^1(X, \mathscr{F}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) = \bigsqcup_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) \big/ \sim$, where

$$\xi \in H^1(\mathfrak{U}, \mathscr{F}),\ \xi' \in H^1(\mathfrak{U}', \mathscr{F}),\ \xi \sim \xi' \iff \exists \mathfrak{V} < \mathfrak{U}, \mathfrak{U}' \text{ s.t. } t^{\mathfrak{U}}_{\mathfrak{V}}(\xi) = t^{\mathfrak{U}'}_{\mathfrak{V}}(\xi')$$

$x = [\xi]$, $y = [\xi']$, define $x + y = \left[t^{\mathfrak{U}}_{\mathfrak{V}}(\xi) + t^{\mathfrak{U}'}_{\mathfrak{V}}(\xi')\right]$.

- - - $\begin{array}{ccc} H^1(\mathfrak{U}, \mathscr{F}) & \hookrightarrow & H^1(X, \mathscr{F}) \\ \xi & \longmapsto & [\xi] \end{array}$, so $H^1(X, \mathscr{F}) = 0 \iff H^1(\mathfrak{U}, \mathscr{F}) = 0\ \forall \mathfrak{U}$.

- - $H^0(X, \mathscr{F})$:

$$Z^0(\mathfrak{U}, \mathscr{F}) = \ker\left(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})\right) \simeq \mathscr{F}(X)$$

Hence, we define $H^0(X, \mathscr{F}) = \mathscr{F}(X)$.

Theorem 2.1.1. Suppose X is a Riemann surface, then $H^1(X, \mathscr{E}) = 0$ also

$$H^1(X, \mathscr{E}^{(1)}) = H^1(X, \mathscr{E}^{1,0}) = H^1(X, \mathscr{E}^{0,1}) = H^1(X, \mathscr{E}^2) = 0$$

Proof: For an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of X, $\exists \psi_i \in \mathscr{E}(X)\ \forall i \in I$ s.t.

- $\operatorname{supp}(\psi_i) \subset U_i$
- $\forall x \in X$, $\exists x \in W \subset X$ s.t. $W \cap \operatorname{supp}(\psi_i) \neq \varnothing$ only for finitely many ψ_i.
- $\sum_{i\in I} \psi_i = 1$.

Now, given $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{E})$. On $U_i \cap U_j$,

$$f_{ij} = \sum_k \psi_k f_{ij} = \sum_k \psi_k (f_{ik} + f_{kj}) = \underbrace{\sum_k \psi_k f_{ik}}_{:=g_i \in \mathscr{E}(U_i)} - \sum_k \psi_k f_{jk}$$

Hence, $(f_{ij}) \in B^1(\mathfrak{U}, \mathscr{E})$. □

Proposition 2.1.1. Suppose X is a Riemann surface, $\omega \in \mathscr{E}^{(1)}(X)$ is closed. Then $\exists p : \widehat{X} \rightharpoonup X$ with Riemann surface $\widehat{X}$ and $F \in \mathscr{E}(\widehat{X})$ s.t. $dF = p^*\omega$.

Proof:

- For $U \underset{\text{open}}{\subset} X$, define $\mathscr{F}(U) = \{f \in \mathscr{E}(U) : df = \omega\}$. $\forall a \in X$, $\rho_a : \mathscr{F}(U) \to \mathscr{F}_a$. Then $\mathscr{F}$ satisfy identity theorem, since any two elements $f_1, f_2 \in \mathscr{F}(U)$ differ by a constant, where U is a domain in X. By Theorem 1.3.1, $|\mathscr{F}|$ is Hausdorff. For $a \in X$, there exists a connected open neighborhood U and a primitive $f \in \mathscr{F}(U)$ of ω, since ω is closed. Then $p^{-1}(U) = \bigsqcup_{c \in \mathbb{C}} [U, f + c] \rightsquigarrow p$ is covering map.

- Let $\widehat{X}$ be a connected component of $|\mathscr{F}|$ and $p|_{\widehat{X}} : \widehat{X} \to X$ is also the covering map. Define $F(\varphi) = \varphi(p(\varphi))$ $\forall \varphi \in \widehat{X} \implies F \in \mathcal{E}(\widehat{X})$ is a primitive of $p^*\omega$.

$\square$

Corollary 2.1.1. If X is simply connected Riemann surface, then every closed form is exact.

Theorem 2.1.2. Suppose X is simply connected Riemann surface, then $H^1(X, \mathbb{C}) = 0$.

Proof: For $\mathfrak{U} = (U_i)_{i \in I}$ and $(c_{ij}) \in Z^1(\mathfrak{U}, \mathbb{C}) \subset Z^1(\mathfrak{U}, \mathscr{E})$. By theorem 2.1.1, $\exists (f_i) \in C^0(\mathfrak{U}, \mathscr{E})$ s.t. $c_{ij} = f_i - f_j$ on $U_i \cap U_j$. Then

$$0 = dc_{ij} = df_i - df_j \text{ on } U_i \cap U_j \implies df_i = df_j \text{ on } U_i \cap U_j$$

Then $\exists \omega \in \mathscr{E}^{(1)}(X)$ s.t. $\omega|_{U_i} = df_i$ and $d\omega|_{U_i} = ddf_i = 0$ $\forall i \implies d\omega = 0$ on X. Since X is simply connected, $\exists f \in \mathscr{E}(X)$ s.t. $\omega = df$. Then $c_{ij} = (f_i - f) - (f_j - f)$, where $c_i := f_i - f \in \mathbb{C}$ since $d(f_i - f) = 0$. Hence, $(c_{ij}) \in B^1(\mathfrak{U}, \mathbb{C})$. $\square$

Theorem 2.1.3. Suppose X is simply connected Riemann surface, then $H^1(X, \mathbb{Z}) = 0$.

Proof: For $\mathfrak{U} = (U_i)_{i \in I}$ and $(a_{ij}) \in Z^1(\mathfrak{U}, \mathbb{Z}) \subset Z^1(\mathfrak{U}, \mathbb{C})$. By Theorem 2.1.2, $a_{ij} = c_i - c_j$ on $U_i \cap U_j$.

$$1 = e^{2\pi i a_i} = e^{2\pi i (c_i - c_j)} \implies e^{2\pi i c_i} = e^{2\pi i e_j} \text{ on } U_i \cap U_j$$

Since X is connected, $\exists b \in \mathbb{C}^\times$ s.t. $b = e^{2\pi i c_i}$ $\forall i$. Then exists $b = e^{2\pi i c}$ for some c and thus

$$a_{ij} = \underbrace{(c_i - c)}_{:= a_i \in \mathbb{Z}} - (c_j - c)$$

since $e^{2\pi i (c_i - c)} = 0$. Hence, $(a_{ij}) \in B^1(\mathfrak{U}, \mathbb{Z})$. $\square$

Theorem 2.1.4. Let $X = B_R(0)$ with $0 < R \leq \infty$, then $H^1(X, \mathcal{O}) = 0$.

Proof: For $\mathfrak{U} = (U_i)_{i \in I}$ and $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}) \subset Z^1(\mathfrak{U}, \mathscr{E})$. By Theorem 2.1.1, $\exists (f_i) \in C^0(\mathfrak{U}, \mathscr{E})$ s.t. $f_{ij} = f_i - f_j$ on $U_i \cap U_j$. Let $\overline{\partial} = \dfrac{\partial}{\partial \overline{z}}$, then

$$0 = \overline{\partial} f_{ij} = \overline{\partial} f_i - \overline{\partial} f_j \implies \overline{\partial} f_i = \overline{\partial} f_j \text{ on } U_i \cap U_j$$

Then $\exists h \in \mathscr{E}(X)$ s.t. $h|_{U_i} = \overline{\partial} f_i$. By below lemma, $\exists g \in \mathscr{E}(X)$ s.t. $h = \overline{\partial} g$ and thus

$$f_{ij} = \underbrace{(f_i - g)}_{\in \mathcal{O}(U_i)} - (f_j - g) \implies (f_{ij}) \in B^1(\mathfrak{U}, \mathcal{O})$$

$\square$

Lemma 2.1.1 (Dolbeault's lemma). Suppose $X = B_R(0)$ with $0 \le R \le \infty$ and $g \in \mathscr{E}(X)$. Then $\exists f \in \mathscr{E}(X)$ s.t $\overline{\partial} f = g$.

Proof:

- Case 1.g has compact support. We may assume $X = \mathbb{C}$. For $\xi \in \mathbb{C}$,

$$f(\xi) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{z-\xi} dz \wedge d\overline{z}$$

Since supp(g) is compact,

$$f(\xi) = \frac{1}{2\pi i} \iint\limits_{\substack{0 \le r \le K \\ 0 \le \theta \le 2\pi}} \frac{g(\xi + re^{i\theta})}{re^{i\theta}}(-2ir) dr \wedge d\theta = \frac{-1}{\pi} \iint g(\xi + re^{i\theta}) e^{-i\theta} dr \wedge d\theta$$

for some $K < \infty$ sufficiently large and $g|_{B_K(0)} \equiv 0$. This implies $f \in \mathscr{E}(\mathbb{C})$.

$$\frac{\partial f(\xi)}{d\overline{\xi}} = \frac{-1}{\pi} \iint \frac{\partial g(\xi + re^{i\theta})}{\partial \overline{\xi}} e^{-i\theta} dr d\theta = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \iint\limits_{\varepsilon \le |z| \le K} \frac{\partial g(\xi + z)}{\partial \overline{\xi}} \frac{1}{z} dz \wedge d\overline{z}$$

Let $\omega(z) = \dfrac{1}{2\pi i} \dfrac{g(\xi + z)}{z} dz$ and thus

$$= -\lim_{\varepsilon \to 0} \iint\limits_{\varepsilon \le |z| \le K} d\omega$$

By Stoke's theorem,

$$\begin{aligned} &= -\lim_{\varepsilon \to 0} \left(\int_{|z|=K} \omega - \int_{|z|=\varepsilon} \omega \right) = \lim_{\varepsilon \to 0} \int_{B_\varepsilon(0)} \omega \\ &= \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_0^{2\pi} g(\xi + \varepsilon e^{i\theta}) d\theta = g(\xi) \end{aligned}$$

Then $f(\xi)$ as desired.

- Case 2. Arbitrary g : Let $0 < R_0 < \cdots < R_n < \cdots$ s.t. $\lim\limits_{n \to \infty} R_n = R$. Let $X_n = B_{R_n}(0)$. $\forall n$, $\exists \psi_n \in \mathscr{E}(X)$ with supp$(\psi_n) \subset B_{R_{n+1}(0)}$ and $\psi_n|_{B_{R_n}(0)} = 1$. Then $\psi_n g$ is compact support. By Case 1. $\forall n \in \mathbb{N}$, $\exists f_n \in \mathscr{E}(X)$ s.t. $\overline{\partial} f_n = \psi_n g$ on X. Now we construct $\{\widetilde{f}_n\}_{n=1}^\infty$ by induction which satisfies

$$\begin{cases} \overline{\partial} \widetilde{f}_n = g \text{ on } X_n \\ \|\widetilde{f}_{n+1} - \widetilde{f}_n\|_{X_n} \le 2^{-n} \end{cases}$$

where $\|f\|_K = \sup\limits_{x \in K} |f(x)|$. Set $\widetilde{f}_1 = f_1$. Suppose $\widetilde{f}_1, ..., \widetilde{f}_n$ are already constructed. Then

$$\overline{\partial}(f_{n+1} - \widetilde{f}_n) = 0 \text{ on } X_n$$

and thus $(f_{n+1} - \widetilde{f}_n) \in \mathcal{O}(X_n)$. Hence exists a Taylor polynomial P s.t. $\|f_{n+1} - \widetilde{f}_n - P\|_{X_n} \le 2^{-n}$. If we set $\widetilde{f}_{n+1} = f_{n+1} - P$, then

$$\overline{\partial} \widetilde{f}_{n+1} = \overline{\partial} f_{n+1} = \psi_n g = g \text{ on } X_{n+1}$$

Define $f(z) := \lim\limits_{n \to \infty} \widetilde{f}_n(z)$ $\forall z \in X$. On X_n,

$$f(z) = \widetilde{f}_n(z) + \sum_{k=n}^{\infty} (\widetilde{f}_{k+1}(z) - \widetilde{f}_k(z))$$

Note that $(\widetilde{f}_{k+1}(z) - \widetilde{f}_k(z)) \in \mathcal{O}(X_n)$ and converge uniformly, $\sum_{k=n}^{\infty} (\widetilde{f}_{k+1}(z) - \widetilde{f}_k(z)) \in \mathcal{O}(X_n)$ and thus $f \in \mathcal{O}(X_n)$ $\forall n$. Hence, $f \in \mathcal{O}(X)$ which satisfies $\overline{\partial} f = g$ on X.

$\square$

Theorem 2.1.5 (Leray theorem)**.** Suppose $\mathscr{F}$ is a sheaf of abelian groups on topological space X and $\mathfrak{U} = (U_i)_{i\in I}$ is an open covering of X s.t. $H^1(U_i, \mathscr{F}) = 0$ $\forall i \in I$. Then

$$H^1(X, \mathscr{F}) = H^1(\mathfrak{U}, \mathscr{F})$$

Such $\mathfrak{U}$ is called **Leray covering**.

Proof:

- **Claim** : $\forall \mathfrak{V} < \mathfrak{U}$ via $\tau : K \to I$, then $t^{\mathfrak{U}}_{\mathfrak{V}} : H^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^1(\mathfrak{V}, \mathscr{F})$.

  subproof : It suffices to show surjective. For $(f_{k\ell}) \in Z^1(\mathfrak{V}, \mathscr{F})$. $(U_i \cap V_k)_{k\in K}$ is an open covering of U_i. By assumption, $H^1(U_i \cap \mathfrak{V}, \mathscr{F}) = 0$ and thus

  $$f_{k\ell} = g_{ik} - g_{i\ell} \text{ on } U_i \cap V_k \cap V_\ell \text{ with } g_{ik} \in \mathscr{F}(U_i \cap V_k) \ \forall i \in I$$

  On $U_i \cap U_j \cap V_k \cap V_\ell$,

  $$g_{ik} - g_{i\ell} = f_{k\ell} = g_{jk} - g_{j\ell} \implies \underbrace{g_{ik} - g_{jk}}_{\in U_i \cap U_j \cap V_k} = g_{i\ell} - g_{j\ell}$$

  Then $\exists F_{ij} \in \mathscr{F}(U_i \cap U_j)$ s.t. $F_{ij}|_{U_i \cap U_j \cap V_k} = g_{jk} - g_{ik}$ $\forall k$. On $U_i \cap U_j \cap V_k \cap V_\ell$,

  $$F_{ij} + F_{jt} = g_{jk} - g_{ik} + g_{tk} - g_{jk} = F_{it} \implies (F_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$$

  Now on $V_k \cap V_\ell$,

  $$F_{\tau(k)\tau(\ell)} - f_{k\ell} = (g_{\tau(\ell)\ell} - g_{\tau(k)\ell}) - (g_{\tau(k)k} - g_{\tau(k)\ell}) = \underbrace{g_{\tau(\ell)\ell}}_{\in \mathscr{F}(V_\ell)} - g_{\tau(k)k} \in B^1(\mathfrak{V}, \mathscr{F})$$

  Hence, $(f_{k\ell}) = (F_{\tau(k)\tau(\ell)}) - (g_{\tau(\ell)\ell} - g_{\tau(k)k})$.

$\square$

Example 2.1.1. $H^1(\mathbb{C}^\times, \mathbb{Z}) = \mathbb{Z}$, $H^1(\mathbb{C}^\times, \mathbb{C}) = \mathbb{C}$.

Proof:

- Let $U_1 = \mathbb{C}^\times \setminus \mathbb{R}^-$ and $\mathbb{C}^\times \setminus \mathbb{R}^+$. Since U_1, U_2 are simply connected, $H^1(U_i, \mathbb{Z}) = 0$ by Theorem 2.1.3. Then $\mathfrak{U} = \{U_1, U_2\}$ is a Leray covering of $\mathbb{C}^\times$. By Leray theorem, $H^1(X, \mathbb{Z}) = H^1(\mathfrak{U}, \mathbb{Z})$.

- For $(a_{ij}) \in Z^1(U_1, \mathbb{Z})$, say $(a_{ij}) = (0, a_{12}, -a_{12}, 0) \rightsquigarrow Z^1(\mathfrak{U}, \mathbb{Z}) \simeq \mathbb{Z}(U_1 \cap U_2) \simeq \mathbb{Z} \times \mathbb{Z}$ since $U_1 \cap U_2$ has two connected components. Also, $C^0(\mathfrak{U}, \mathbb{Z}) = \mathbb{Z}(U_1) \times \mathbb{Z}(U_2) \simeq \mathbb{Z} \times \mathbb{Z}$.

  $$\begin{array}{rccc} \delta: & C^0(\mathfrak{U}, \mathbb{Z}) & \longrightarrow & Z^1(\mathfrak{U}, \mathbb{Z}) \\ & (b_1, b_2) & \longrightarrow & (b_2 - b_1, b_1 - b_2) \end{array}$$

  Then $\operatorname{Im} \delta = \mathbb{Z}(1, -1) \simeq \mathbb{Z} \implies H^1(\mathfrak{U}, \mathbb{Z}) \simeq \mathbb{Z}(1, -1) \times \mathbb{Z} / \mathbb{Z} \simeq \mathbb{Z}$.

- Similarly, $H^1(\mathbb{C}^\times, \mathbb{C}) = \mathbb{C}$.

□

Example 2.1.2. $H^1(\mathbb{P}, \mathcal{O}) = 0$.

Proof:

- Let $U_1 = \mathbb{P}^1 \setminus \{\infty\} \simeq \mathbb{C}$ and $U_2 = \mathbb{P}^1 \setminus \{0\} \simeq \mathbb{C}$. By theorem 2.1.4, $H^1(U_i, \mathcal{O}) = 0$. Then $\mathfrak{U} = \{U_1, U_2\}$ is Leray covering and thus $H^1(X, \mathscr{F}) \simeq H^1(\mathfrak{U}, \mathscr{F})$.
- For $(f_{ij}) \in Z^1(\mathfrak{U}, \mathbb{P})$, $U_1 \cap U_2 = \mathbb{C}^\times$. Let $f_{1,2}(z) = \sum\limits_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{C}^\times$. Let $f_1(z) = \sum_{n=0}^{\infty} c_n z^n \in \mathcal{O}(U_1)$ and $f_2(z) := -\sum\limits_{n=-\infty}^{-1} c_n z^n \in \mathcal{O}(U_2)$. $f_{12} = f_1 - f_2$ on $U_1 \cap U_2 \implies (f_{ij}) \in B^1(\mathfrak{U}, \mathcal{O})$.

□

2.2 Finiteness theorem

Definition 2.2.1. Suppose $D \subset \mathbb{C}$ is an open set. Given a holomorphic function $f \in \mathcal{O}(D)$ define its L^2**-norm** by

$$\|f\|_{L^2(D)} := \left(\iint_D |f(x+iy)|^2 dxdy \right)^{1/2} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

If $\|f\|_{L^2(D)} < \infty$, then f is called **square integrable**, and denoted $L^2(D, \mathcal{O})$ be the vector space of all square integrable holomorphic functions on D. If

$$\mathrm{Vol}(D) := \iint_D dxdy < \infty$$

then for every bounded function $f \in \mathcal{O}(D)$ we has

$$\|f\|_{L^2(D)} \leq \sqrt{\mathrm{Vol}(D)} \|f\|_{L^\infty(D)}$$

Define the inner product on $L^2(D, \mathcal{O})$ by

$$\langle f, g \rangle = \iint_D f\overline{g} dxdy$$

Note that the integral exists since

$$|f(z)\overline{g(x)}| \leq \frac{1}{2} \left(|f(z)|^2 + |g(x)|^2 \right)$$

Suppose $D = B_r(a)$, then $\{\psi_n(z) = (z-a)^n\}_{n \in \mathbb{N}}$ form a orthogonal system in $L^2(B_r(a), \mathcal{O})$.

$$\langle \psi_n, \psi_m \rangle = \int_0^{2\pi} \int_0^r r^n e^{ni\theta} r^m e^{-mi\theta} r dr d\theta = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{\pi r^{2n+2}}{n+1} & \text{if } n = m \end{cases}$$

If $f \in L^2(B_r(a), \mathcal{O})$ and $f(z) = \sum\limits_{n=0}^{\infty} c_n (z-a)^n$, then

$$\|f\|^2_{L^2(B_r(a))} = \sum_{n=0}^{\infty} \frac{\pi r^{2n+2}}{n+1} |c_n|^2$$

Definition 2.2.2. Suppose X is a Riemann surface and $\mathfrak{U}' = (U_i', z_i)_{i=1}^n$ is a finite set of charts on X s.t. $z_i(U_i') \simeq B_R(0) \subset \mathbb{C}$. Suppose $U_i \subset U_i'$ and $\mathfrak{U} = (U_i)_{i=1}^n$. Define $|\mathfrak{U}| = \bigcup_{i=1}^n U_i$ and

- For $\eta = (f_i) \in C^0(\mathfrak{U}, \mathcal{O})$, define

$$\|\eta\|_{L^2(\mathfrak{U})}^2 = \sum_{i=1}^n \|f_i\|_{L^2(U_i)}^2 \text{ where } \|f_i\|_{L^2(U_i)} := \|f_i \circ z_i^{-1}\|_{L^2(z_i(U_i))}$$

- For $\xi = (f_{ij}) \in C^1(\mathfrak{U}, \mathcal{O})$, define

$$\|\xi\|_{L^2(\mathfrak{U})}^2 = \sum_{i,j} \|f_{ij}\|_{L^2(U_i \cap U_j)}^2 \text{ where } \|f_{ij}\|_{L^2(U_i \cap U_j)} := \|f_{ij} \circ z_i^{-1}\|_{L^2(z_i(U_i \cap U_j))}$$

- Let $C_{L^2}^i(\mathfrak{U}, \mathcal{O})$ be the set of cochain having finite L^2-norm contain in $C^i(\mathfrak{U}, \mathcal{O})$ $(i = 1, 2)$, which is a Hilbert space. Indeed, let $D_r = \{z \in D : B_r(z) \subset D\}$ and for all $a \in D_r$, $f(z) = \sum c_n (z-a)^n$. Then

$$|f(a)| = |c_0| \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(B_r(0))} \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)} \implies \|f\|_{L^\infty(D_r)} \leq \frac{1}{\sqrt{\pi} r} \|f\|_{L^2(D)}$$

  This shows that every Cauchy sequence $\{f_i\} \subseteq L^2(D, \mathcal{O})$ converge uniformly on each compact subset of D, so the limit function is holomorphic on D. Hence $L^2(D, \mathcal{O})$ is complete.

- We denote $\mathfrak{V} \ll \mathfrak{U}$ if $\mathfrak{V} = (V_i)_{i=1}^n$ with $V_i \Subset U_i$.

Fact 2.2.1 (key fact)**.** If $\mathfrak{V} \ll \mathfrak{U}$, then $\forall \xi \in C^1(\mathfrak{U}, \mathcal{O})$, $\|\xi\|_{L^2(\mathfrak{V})} < \infty$. $\forall \varepsilon > 0$, there exists a closed vector subspace $A \subset Z_{L^2}^1(\mathfrak{U}, \mathcal{O})$ of finite codimension s.t.

$$\|\xi\|_{L^2(\mathfrak{V})} \leq \varepsilon \|\xi\|_{L^2(\mathfrak{U})} \; \forall \xi \in A$$

Proof:

- Since $\mathrm{Vol}(V_i \cap V_j) := \displaystyle\iint_{z_i(V_i \cap V_j)} dxdy < \infty$ and f_{ij} is bounded on $\overline{V_i \cap V_j}$,

$$\|f_{ij}\|_{L^2(V_i \cap V_j)}^2 \leq \mathrm{Vol}(V_i \cap V_j) \cdot \|f_{ij}\|_{V_i \cap V_j} < \infty$$

- We do the case for $D' \Subset D$ first. Since $\overline{D'}$ is compact, $\exists r > 0$ and $a_1, ..., a_k \in D'$ s.t.
  - $B_r(a_i) \subset D$
  - $D' \subset \bigcup_{j=1}^k B_{r/2}(a_j)$

  Pick n s.t. $2^{-n-1} k \leq \varepsilon$. Let A be the set of all functions $f \in L^2(D, \mathcal{O})$ which vanish at every point a_j at least of order n. Then A is a closed vector subspace of $L^2(D, \mathcal{O})$ of codimension $\leq kn$. Consider the Taylor series of f at a_j, $f(z) = \sum_{i=n}^\infty c_{ij} (z - a_j)^i$. For every $\rho \leq r$, we have

$$\|f\|_{L^2(B_\rho(a_j))}^2 = \sum_{i=n}^\infty \frac{\pi \rho^{2i+2}}{i+1} |c_{ij}|^2$$

from which it follows that

$$\|f\|^2_{L^2(B_{r/2}(a_j))} = \sum_{i=n}^{\infty} \frac{\pi r^{2i+2}}{i+1} \frac{1}{2^{2i+2}} |c_{ij}|^2 \le \frac{1}{2^{2n+2}} \|f\|^2_{L^2(B_r(a_j))} \le \frac{1}{2^{2n+2}} \|f\|^2_{L^2(D)}$$

$$\implies \|f\|_{L^2(D')} \le \sum_{j=1}^{k} \|f\|_{L^2(B_{r/2}(a_j))} \le \frac{k}{2^{n+1}} \|f\|_{L^2(D)} \le \varepsilon \|f\|_{L^2(D)}$$

- Fixed $\varepsilon > 0$, $\forall i, j$, $\overline{V_i \cap V_j} \subset \overline{V_i} \cap \overline{V_j} \subset U_i \cap U_j$. By above exists closed subspace $A_{ij} \subseteq Z^1_{L^2}(U_i \cap U_j, \mathcal{O})$ of finite codimension s.t.

$$\|\xi\|_{L^2(V_i \cap V_j)} \le \varepsilon \|\xi\|_{L^2(U_i \cap U_j)}$$

Choose

$$A = \left(\prod_{1 \le i,j \le n} A_{ij} \right) \cap Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$$

then A is closed finite codimension subspace of $Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$. So $\forall \xi \in A$,

$$\|\xi\|^2_{L^2(\mathfrak{V})} = \sum_{i,j} \|f_{ij}\|_{L^2(V_i \cap V_j)} \le \varepsilon^2 \sum_{i,j} \|f_{ij}\|_{L^2(U_i \cap U_j)} = \varepsilon^2 \|\xi\|^2_{L^2(\mathfrak{U})}$$

□

Definition 2.2.3. Given $f \in \mathscr{E}(X)$, define $d'f = \dfrac{\partial f}{\partial z} dz$ and $d''f = \dfrac{\partial f}{\partial \bar{z}} d\bar{z}$ i.e. $df = d'f + d''f$.

Lemma 2.2.1 (key lemma)**.** For $\mathfrak{W} \ll \mathfrak{V} \ll \mathfrak{U} \ll \mathfrak{U}'$, $\exists c > 0$ s.t. $\forall \xi \in Z^1_{L^2}(\mathfrak{V}, \mathcal{O})$, $\exists \zeta \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$, $\eta \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ s.t. $\zeta = \xi + \delta\eta$ and

$$\max \left\{ \|\zeta\|_{L^2(\mathfrak{U})}, \|\eta\|_{L^2(\mathfrak{W})} \right\} \le c \|\xi\|_{L^2(\mathfrak{V})}$$

Proof: By the fact $H^1(|\mathfrak{V}|, \mathscr{E}) = 0$, $\exists g_i \in C^0(|\mathfrak{V}|, \mathscr{E})$ s.t. $f_{ij} = g_j - g_i$ on $V_i \cap V_j$. $d''f_{ij} = 0 \rightsquigarrow d''g_i = d''g_j$ on $V_i \cap V_j$. $\exists w \in \mathscr{E}^{(0,1)}(|\mathfrak{V}|)$ s.t. $w|_{V_i} = d''g_i$. Since $|\mathfrak{W}| \Subset |\mathfrak{V}|$, $\exists \psi \in \mathscr{E}(X)$ s.t. $\mathrm{supp}(\psi) \subset |\mathfrak{V}|$ and $\psi|_{|\mathfrak{W}|} \equiv 1 \rightsquigarrow \psi w \in \mathscr{E}(|\mathfrak{U}'|)$. By Dolbeault's lemma, $\exists h_j \in \mathscr{E}(U'_i)$ s.t. $d''h_i = \psi w$ on U'_i. Since $d''(h_i - h_j) = 0$ on $U'_i \cap U'_j$, $F_{ij} = (h_j - h_i) \in \mathcal{O}(U'_i \cap U'_j) \rightsquigarrow (F_{ij}) \in Z^1(\mathfrak{U}', \mathcal{O})$ and $\xi := (F_{ij}|_{U_i \cap U_j}) \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$. On W_i, $d''h_i = \psi w = w = d''g_i$ i.e. $(h_i - g_i) \in \mathcal{O}(W_i) \rightsquigarrow \eta = (h_i - g_i)_i \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$, then

$$F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i) \text{ on } W_{ij} \implies \zeta - \xi = \delta\eta \text{ on } \mathfrak{W}$$

Consider the Hilbert space $H := Z^1_{L^2}(\mathfrak{U}, \mathcal{O}) \times Z^1_{L^2}(\mathfrak{V}, \mathcal{O}) \times C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ with

$$\|(\cdots)\|_H = \left(\|\cdot\|^2_{L^2(\mathfrak{U})} + \|\cdot\|^2_{L^2(\mathfrak{V})} + \|\cdot\|^2_{L^2(\mathfrak{W})} \right)^{1/2}$$

Let $L := \{(\zeta, \xi, \eta) \in H : \zeta = \xi + \delta\eta \text{ on } \mathfrak{W}\}$ which is a closed Hilbert subspace of H. By Banach-Schauder theorem, $\pi : (\zeta, \xi, \eta) \mapsto \xi$ is surjective $\implies$ π is open mapping. Let $U = \{x \in H : \|x\|_H < 1\}$, $\exists \varepsilon > 0$ s.t. $\pi(U) \supset V = \{\xi \in Z^1_{L^2}(\mathfrak{V}, \mathcal{O}) : \|\xi\|_{L^2(\mathfrak{V})} < \varepsilon\}$. Let $c = 2/\varepsilon$. If $\xi = 0$, choose $x = 0$, otherwise $\lambda := \|\xi\| > 0$, $\xi_1 = \dfrac{1}{\lambda c} \xi \in V$ since $\|\xi_1\| = \dfrac{1}{\lambda c} \cdot \lambda = \dfrac{\varepsilon}{2}$. Then $\exists x_1 \in U$ s.t. $\pi(x_1) = \xi_1$. Then $\pi(\lambda c x_1) = \xi$ and

$$\|\lambda c x_1\|_H = \lambda c \|x_1\| \le \lambda c = c \|\xi\|_{L^2(\mathfrak{V})}$$

Hence,

$$\max \left\{ \|\zeta\|_{L^2(\mathfrak{U})}, \|\eta\|_{L^2(\mathfrak{W})} \right\} \le c \|\xi\|_{L^2(\mathfrak{V})}$$

□

Proposition 2.2.1 (key prop.). $\exists$ finite dimensional vector space $S \subset Z^1(\mathfrak{U}, \mathcal{O})$ s.t. $\forall \xi \in Z^1(\mathfrak{U}, \mathcal{O})$, $\exists \sigma \in S$, $\eta \in C(w, \mathcal{O})$ s.t. $\sigma = \xi + \delta\eta$ on $\mathfrak{W}$.

Proof: Let c be given in key lemma, and $\varepsilon = (2c)^{-1}$. By key fact, exists a finite codimension closed vector space $A \subset Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$ s.t. $\|\xi\|_{L^2(\mathfrak{V})} \leq \varepsilon \|\xi\|_{L^2(\mathfrak{U})}$ $\forall \xi \in A$. Pick S s.t. $A \oplus S = Z^1_{L^2}(\mathfrak{U}, \mathcal{O}) \rightsquigarrow \dim S < \infty$.

- For $\xi \in Z^1(\mathfrak{U}, \mathcal{O})$, $M := \|\xi\|_{L^2(\mathfrak{V})} < \infty$ since $\mathfrak{V} \ll \mathfrak{U}$. By key lemma, $\exists \zeta_0 \in Z^1(\mathfrak{U}, \mathcal{O})$, $\eta_0 \in C^0(\mathfrak{W}, \mathcal{O})$ s.t. $\zeta_0 = \xi + \delta\eta_0$ on W and $\|\zeta_0\|_{L^2(\mathfrak{U})}, \|\eta_0\|_{L^2(\mathfrak{W})} \leq cM$. Suppose $\zeta_0 = \xi_0 + \sigma_0$ for $\xi_0 \in A$, $\sigma_0 \in S$ (by $Z^1(\mathfrak{U}, \mathcal{O}) = A \oplus S$).
- We now construct the element by induction,

  $$\zeta_n \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O}),\ \eta_n \in C^0_{L^2}(\mathfrak{W}, \mathcal{O}),\ \xi_n \in A,\ \sigma_n \in S$$

  with the following properties :

  (1) $\zeta_n = \xi_{n-1} + \delta\eta_n$ on $\mathfrak{W}$

  (2) $\zeta_n = \xi_n + \sigma_n$

  (3) $\|\zeta_n\|_{L^2(\mathfrak{U})} \leq 2^{-n}cM$, $\|\eta_n\|_{L^2(\mathfrak{W})} \leq 2^{-n}cM$

  Consider the induction step from n to $n+1$. Since $\zeta_n = \xi_n + \sigma_n$ is an orthogonal decomposition,

  $$\|\xi_n\|_{L^2(\mathfrak{U})} \leq \|\zeta_n\|_{L^2(\mathfrak{U})} \leq 2^{-n}cM$$

  Since $\xi_n \in A$,

  $$\|\xi_n\|_{L^2(\mathfrak{B})} \leq \varepsilon \|\xi_n\|_{L^2(\mathfrak{U})} \leq \varepsilon \cdot 2^{-n}cM = 2^{-n-1}M$$

  By key Lemma, $\exists \zeta_{n+1} \in Z^1_{L^2}(\mathfrak{U}, \mathcal{O})$, $\eta_{n+1} \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$ s.t. $\zeta_{n+1} = \xi_n + \delta\eta_{n+1}$ on $\mathfrak{W}$ and

  $$\max(\|\zeta_{n+1}\|_{L^2(\mathfrak{U})}, \|\eta_{n+1}\|_{L^2(\mathfrak{W})}) \leq c\|\xi_n\|_{L^2(\mathfrak{V})} \leq 2^{-(n+1)}cM$$

  Since $Z^1_{L^2}(\mathfrak{U}, \mathcal{O}) = A \oplus S$, $\zeta_{n+1} = \xi_{n+1} + \sigma_{n+1}$, where $\xi_{n+1} \in A$, $\sigma_{n+1} \in S$ and thus the induction is complete.
- Note that

  $$\sum_{n=1}^{k}(\xi_n + \sigma_n) = \sum_{n=1}^{k} \zeta_n = \sum_{n=1}^{k}(\xi_{n-1} + \delta\eta_n) \implies \xi_k + \sum_{n=1}^{k} \sigma_n = \xi_{-1} + \delta\left(\sum_{n=1}^{k} \eta_n\right) \qquad (*)$$

  where $\xi_{-1} = \xi$. By (2), (3), we have

  $$\max\left(\|\xi_n\|_{L^2(\mathfrak{U})}, \|\sigma_n\|_{L^2(\mathfrak{U})}, \|\eta_n\|_{L^2(\mathfrak{B})}\right) \leq 2^{-n}cM$$

  Then we have $\lim_{n\to\infty} \xi_n = 0$ and the series

  $$\sigma := \sum_{n=0}^{\infty} \sigma_n \in S,\ \eta := \sum_{n=1}^{\infty} \eta_n \in C^0_{L^2}(\mathfrak{W}, \mathcal{O})$$

  converge. Combine with $(*)$, we have $\sigma = \xi + \delta\eta$.

$\square$

Remark 2.2.1. We have

$$\dim \operatorname{Im}\Big(H^1(\mathfrak{U},\mathcal{O}) \to H^1(\mathfrak{W},\mathcal{O})\Big) < \infty$$

Indeed, given $\xi \in Z^1(\mathfrak{U},\mathcal{O})$, let σ and η be given in above. Then $\overline{\xi} = \overline{\sigma}$ on $\mathfrak{W}$. Hence the image can be presented by S.

Theorem 2.2.1 (Main theorem)**.** Suppose X is a Riemann surface. $Y_1 \Subset Y_2 \subset X$. Then

$$\dim \operatorname{Im}\Big(H^1(Y_2,\mathcal{O}) \to H^1(Y_1,\mathcal{O})\Big) < \infty$$

In particular, if X is a compact Riemann surface, then $\dim H^1(X,\mathcal{O}) < \infty$.

Proof: Choose charts $(U_i', z_i)_{i=1}^n$ on X and $W_i \Subset U_i \Subset U_i'$ s.t. $z_i(W_i)$, $z_i(U_i)$, $z_i(U_i')$ are disks in $\mathbb{C}$ and

$$Y_1 \subset \underbrace{\bigcup_{i=1}^n W_i}_{:=Y'} \subset \underbrace{\bigcup_{i=1}^n U_i}_{:=Y''} \subset Y_2$$

By Theorem 2.1.4, $H^1(W_i,\mathcal{O}) = H^1(U_i,\mathcal{O}) = 0$. By Leray theorem, $H^1(Y',\mathcal{O}) = H^1(\mathfrak{W},\mathcal{O})$, $H^1(Y'',\mathcal{O}) = H^1(\mathfrak{U},\mathcal{O})$. Then we can split the map $H^1(Y_2,\mathcal{O}) \to H^1(Y_2,\mathcal{O})$ to

$$\begin{tikzcd}
H^1(Y_2,\mathcal{O}) \arrow[r] & H^1(Y'',\mathcal{O}) \arrow[r] \arrow[d, equal] & H^1(Y',\mathcal{O}) \arrow[r] \arrow[d, equal] & H^1(Y_1,\mathcal{O}) \\
& H^1(\mathfrak{U},\mathcal{O}) \arrow[r, "\text{finite image}"] & H^1(\mathfrak{W},\mathcal{O}) &
\end{tikzcd}$$

and thus have finite dimensional image. $\square$

Corollary 2.2.1. Suppose X is a Riemann surface, $Y \Subset X$ and $a \in Y \rightsquigarrow \exists f \in \mathscr{M}(Y)$ has a pole at a and $f \in \mathcal{O}(Y \setminus \{a\})$.

Proof: Let $z : U_1 \xrightarrow{\sim} B_1(0) \subset \mathbb{C}$ and $U_2 = X \setminus \{a\}$. Let $\mathfrak{U} = \{U_1, U_2\}$. Consider

$$\begin{array}{ccc}
\dim\big(H^1(X,\mathcal{O}) & \longrightarrow & H^1(Y,\mathcal{O})\big) = k \\
\updownarrow & & \updownarrow \\
\dim\big(H^1(\mathfrak{U},\mathcal{O}) & \longrightarrow & H^1(\mathfrak{U}\cap Y,\mathcal{O})\big) < k+1
\end{array}$$

Observe that $z^{-j} \in \mathcal{O}(U_1 \cap U_2) \rightsquigarrow \xi_j \in H^1(\mathfrak{U},\mathcal{O})$. Then $\xi_j|_Y \in H^1(\mathfrak{U}\cap Y,\mathcal{O})$ $(j = 1, ..., k+1)$ are linearly dependent, then $\exists c_1, ..., c_{k+1}$ are not all zero and $\eta = (f_1, f_2) \in C^0(\mathfrak{U}\cap Y,\mathcal{O})$ s.t. $\sum\limits_{j=1}^{k+1} c_j z^{-j} = f_2 - f_1$ on $U_1 \cap U_2 \cap Y$, then

$$\mathcal{O}(U_2) \ni f_2 = f_1 + \sum_{j=1}^{k+1} c_j z^{-j} \in \mathcal{O}(U_1)$$

Hence, $\exists f \in \mathscr{M}(Y)$ s.t. $f|_{U_1\cap Y} = f_1 + \sum\limits_{j=1}^{k+1} c_j z^{-j}$ has a pole at a, $f|_{U_2\cap Y} = f_2 \in \mathcal{O}(Y\setminus\{a\})$. $\square$

Corollary 2.2.2. Suppose X is compact Riemann surface, $a_1, ..., a_n \in X$, $c_1, ..., c_n \in \mathbb{R}$. Then $\exists f \in \mathscr{M}(X)$ s.t. $f(a_i) = c_i$.

Proof: Let $f_i \in \mathscr{M}(X)$ has a pole at a_i and $f_i \in \mathcal{O}(X \setminus \{a_i\})$. Choose $\lambda_{ij} \in \mathbb{C}^\times$ s.t. $\lambda_{ij} \neq f_i(a_j) - f_i(a_k)$ $\forall k = 1, ..., n$. Define

$$g_{ij} = \frac{f_i - f_i(a_j)}{f_i - f_i(a_j) + \lambda_{ij}} \in \mathscr{M}(X) \implies \begin{cases} g_{ij} \text{ is hol. at } a_k \\ g_{ij}(a_i) = 1 \\ g_{ij}(a_j) = 0 \end{cases}$$

Set $h_i = \prod\limits_{j \neq i} g_{ij} \rightsquigarrow h_i(a_j) = \delta_{ij}$ and let $f = \sum\limits_{i=1}^n c_i h_i$. □

Fact 2.2.2. Suppose X is a non-compact Riemann surface and $Y \Subset X$. Then $\exists f \in \mathcal{O}(Y)$ s.t. f is not constant on each connected component of Y.

Proof: Choose a domain Y_1 s.t. $Y \Subset Y_1 \Subset X$ and $a \in Y_1 \setminus Y$ (since X is non-compact) $\rightsquigarrow \exists f \in \mathcal{O}(Y \setminus \{a\})$ and f_1 has a pole at $a \rightsquigarrow f = f_1|_Y$ (see the singular part). □

Lemma 2.2.2 (General Dolbeault lemma)**.** Suppose X is a non-compact Riemann surface and $Y \Subset Y' \Subset X$. $\omega \in \mathscr{E}^{0,1}(Y')$, $\exists g \in \mathscr{E}(Y)$ s.t. $d''g = \omega|_Y$.

Proof:

- $\operatorname{Im}(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})) = 0$: By Theorem 2.2.1,

  $$L = \operatorname{Im}\Big(H^1(Y', \mathcal{O}) \to H^1(Y, \mathcal{O})\Big)$$

  is finite dimensional. Let $\xi_1, ..., \xi_n \in H^1(Y', \mathcal{O})$ s.t. $\langle \xi_1|_Y, ..., \xi_n|_Y \rangle = L$. Let $f \in \mathcal{O}(Y')$ in the above fact, then $f\xi_i|_Y \in L$ $(i = 1, ..., n)$. Say

  $$f\xi_i = \sum_{j=1}^n c_{ij}\xi_j \text{ on } Y \ \forall i = 1, ..., n$$

  By the determinant trick, $F\xi_i|_Y = 0$ $\forall i = 1, ..., n$, where $F = \det(f\delta_{ij} - c_{ij}) \in \mathcal{O}(Y')$. Since f is non-constant on each connected component of Y, F also. For any $\zeta \in H^1(Y', \mathcal{O})$, say $(f_{ij}) \in H^1(\mathfrak{U}, \mathcal{O})$, $\mathfrak{U}$ can be refined s.t. each zero of F is contained at most on one U_i. Then $F|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$. Define $g_{ij} = F^{-1}|_{U_i \cap U_j} f_{ij} \in \mathcal{O}(U_i \cap U_j)$, let $\xi = (g_{ij}) \in H^1(\mathfrak{U}, \mathcal{O}) \hookrightarrow H^1(Y', \mathcal{O})$, then $\zeta|_Y = F\xi|_Y = 0$.

- By Dolbeault's lemma, $\exists \mathfrak{U} = (U_{ij})$ cover Y' and $f_i \in \mathscr{E}(U_i)$ s.t. $d''f_i = \omega|_{U_i} \rightsquigarrow (f_j - f_i) \in \mathcal{O}(U_i \cap U_j) \rightsquigarrow (f_j - f_i) \in Z^1(\mathfrak{U}, \mathcal{O})$. As above, $\exists g_i \in \mathcal{O}(U_i \cap Y)$ s.t. $f_j - f_i = g_j - g_i$ on $U_i \cap U_j \cap Y \implies f_i - g_i = f_j - g_j \rightsquigarrow \exists g \in \mathscr{E}(Y)$ s.t. $g|_{U_i \cap Y} = f_i - g_i$. Then

  $$d''g|_{U_i \cap Y} = d''f_i|_{U_i \cap Y} = \omega|_{U_i \cap Y} \implies d''g = \omega|_Y$$

□

2.3 The exact cohomology sequence

Suppose

$$0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$$

is an exact sequence of sheaves on the topological space X. A **connecting homomorphism**

$$\delta^* : H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F})$$

is defined as follows. Suppose $h \in H^0(X, \mathscr{H})$. Since $\mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$ are surjective $\forall x$, there exists an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of X and a cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{F})$ such that $\beta(g_i) = h|_{U_i}$. Hence, $\beta(g_j - g_i) = 0$ on $U_i \cap U_j \rightsquigarrow \exists f_{ij} \in \mathscr{F}(U_i \cap U_j)$ s.t. $\alpha(f_{ij}) = g_j - g_i$. On $U_i \cap U_j \cap U_k$,

$$\alpha(f_{ij} + f_{jk} - f_{ki}) = 0 \implies f_{ij} + f_{jk} = f_{ik} \implies (f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$$

Define $\delta^* h = \overline{(f_{ij})} \in H^1(X, \mathscr{F})$. It is clear that the definition is independent on the choice.

Now we check that the short exact sequence of sheave induces the exact sequence of cohomology groups :

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathscr{F}) &\xrightarrow{\alpha^0} H^0(X, \mathscr{G}) \xrightarrow{\beta^0} H^0(X, \mathscr{H}) \\ \xrightarrow{\delta^*} H^1(X, \mathscr{F}) &\xrightarrow{\alpha^1} H^1(X, \mathscr{G}) \xrightarrow{\beta^1} H^1(X, \mathscr{H}) \end{aligned}$$

- It is clear that α^0 is injective and $\operatorname{Im} \alpha^0 = \ker \beta^0$.
- $\operatorname{Im} \beta^0 \subset \ker \delta^*$: Suppose $g \in H^0(X, \mathscr{G})$ and $h = \beta^0(g)$, we may choose $g_i = g|_{U_i}$ and thus $f_{ij} = 0 \rightsquigarrow \delta^* h = 0$.
- $\ker \delta^* \subset \operatorname{Im} \beta^0$: Suppose $h \in \ker \delta^*$, say $\delta^* h = (f_{ij})$ as we say in above. Then $\exists f_i \in \mathscr{F}(U_i)$ s.t. $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. Then

$$g_j - g_i = \alpha(f_{ij}) = \alpha(f_j) - \alpha(f_i) \implies g_j - \alpha(f_j) = g_i - \alpha(f_i) \text{ on } U_i \cap U_j$$

  Then $\exists g \in \mathscr{G}(X)$ s.t. $g|_{U_i} = g_i - \alpha(f_i)$. Then we have

$$\beta(g) = \beta(g_i - \alpha(f_i)) = \beta(g_i) = h|_{U_i} \text{ on } U_i \implies h \in \operatorname{Im} \beta^0$$

- $\operatorname{Im} \delta^* \subset \ker \alpha^1$: By construction.
- $\ker \alpha^1 \subset \operatorname{Im} \delta^*$: Suppose $\xi \in \ker \alpha^1$ and $\xi = (f_{ij}) \in H^1(\mathfrak{U}, \mathscr{F})$. Since $\alpha(\xi) = 0$, $\exists (g_i) \in C^0(\mathfrak{U}, \mathscr{G})$ s.t. $\alpha(f_{ij}) = g_j - g_i$ on $U_i \cap U_j$. This implies

$$0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i) \implies \exists h \in \mathscr{H}(X) \text{ s.t. } h|_{U_i} = \beta(g_i)$$

  Then $\xi = \delta^* h$ by construction.
- $\operatorname{Im} \alpha^1 \subset \ker \beta^1$: This follows from the fact that

$$\mathscr{F}(U_i \cap U_j) \xrightarrow{\alpha} \mathscr{G}(U_i \cap U_j) \xrightarrow{\beta} \mathscr{H}(U_i \cap U_j)$$

  is exact.
- $\ker \beta^1 \subset \operatorname{Im} \alpha^1$: Suppose $\eta \in \ker \beta^1$, say $\eta = (g_{ij}) \in Z^1(\mathfrak{U}, \mathscr{G})$. Then $\exists (h_i) \in C^0(\mathfrak{U}, \mathscr{H})$ s.t. $\beta(g_{ij}) = h_j - h_i$. For every $x \in X$, say $x \in U_{\tau x}$. Since $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ is surjective, $\exists V_x \subset U_{\tau x}$ and $g_x \in \mathscr{G}(V_x)$ s.t. $\beta(g_x) = h_{\tau x}|_{V_x}$. Let $\mathfrak{V} = (V_x)_{x\in X}$ and $\widetilde{g}_{xy} = g_{\tau x, \tau y}|_{V_x \cap V_y} \rightsquigarrow (\widetilde{g}_{xy}) \in Z^1(\mathfrak{V}, \mathscr{G})$ and let $\eta = \overline{(\widetilde{g}_{xy})} \in H^1(X, \mathscr{G})$. Let $\psi_{xy} := \widetilde{g}_{xy} - g_y + g_x$, then $\eta = \overline{(\psi_{xy})} \in H^1(X, \mathscr{G})$ and

$$\beta(\psi_{xy}) = h_{\tau y} - h_{\tau x} - (h_{\tau y} - h_{\tau x}) = 0 \text{ on } V_x \cap V_y$$

  Thus $\exists f_{xy} \in \mathscr{F}(V_x \cap V_y)$ s.t. $\alpha(f_{xy}) = \psi_{xy}$. Since

$$\alpha : \mathscr{F}(V_x \cap V_y \cap V_z) \to \mathscr{G}(V_x \cap V_y \cap V_z)$$

  is injective, $(f_{xy}) \in Z^1(\mathfrak{V}, \mathscr{F})$. Let $\zeta = \overline{(f_{xy})} \in H^1(X, \mathscr{F})$, then $\alpha^1(\zeta) = \eta$.

Example 2.3.1.

- $0 \to \mathcal{O} \to \mathscr{E} \xrightarrow{d''} \mathscr{E}^{0,1} \to 0$: is a exact sequence, since the surjective part holds by Doblault's lemma in the local chart which homeomorphic to a disk. Then we have

$$0 \to H^0(X,\mathcal{O}) \to H^0(X,\mathscr{E}) \to H^0(X,\mathscr{E}^{0,1}) \xrightarrow{\delta^*} H^1(X,\mathcal{O}) \to \underbrace{H^1(X,\mathscr{E})}_{=0} \to \underbrace{H^1(X,\mathscr{E}^{0,1})}_{=0}$$

  Then we have **Dolbeault theorem** :

$$H^1(X,\mathcal{O}) \simeq \mathscr{E}^{0,1}(X)\Big/d''\mathscr{E}(X)$$

- $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} \Omega \to 0$ is exact, since X is locally simply connected, every closed form have primitive. Then

$$0 \to H^0(X,\mathbb{C}) \to H^0(X,\mathcal{O}) \to H^0(X,\Omega) \xrightarrow{\delta^*} H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O}) \to H^1(X,\Omega)$$

  If X is simply connected, $H^1(X,\mathbb{C}) = 0$. Then we have

$$H^0(X,\mathcal{O}) \twoheadrightarrow H^0(X,\Omega) \implies \text{every global closed form is exact form}$$

  and $H^1(X,\mathcal{O}) \hookrightarrow H^1(X,\Omega)$.

- $0 \to \Omega \to \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^{(2)} \to 0$ is exact sequence, since Dolbeault's lemma guarantee the surjective. Then

$$0 \to H^0(X,\Omega) \to H^0(X,\mathscr{E}^{1,0}) \to H^0(X,\mathscr{E}^{(2)}) \xrightarrow{\delta^*} H^1(X,\Omega) \to \underbrace{H^1(X,\mathscr{E}^{1,0})}_{=0} \to \underbrace{H^1(X,\mathscr{E}^{(2)})}_{=0}$$

  Then we have **Dolbeault theorem** :

$$H^1(X,\Omega) \simeq \mathscr{E}^{(2)}(X)\Big/d\mathscr{E}^{(1,0)}(X)$$

- $0 \to \mathbb{C} \to \mathscr{E} \xrightarrow{d} \mathscr{E}^{(1)} \xrightarrow{d} \mathscr{E}^{(2)}$ is a complex. Let $L = \ker(\mathscr{E}^{(1)} \to \mathscr{E}^{(2)})$, then

$$0 \to \mathbb{C} \to \mathscr{E} \to L \to 0$$

  since locally closed differential 1-form has a primitive. Then

$$0 \to H^0(X,\mathbb{C}) \to H^0(X,\mathscr{E}) \to H^0(X,L) \to H^1(X,\mathbb{C}) \to \underbrace{H^1(X,\mathscr{E})}_{=0} \to H^1(X,L)$$

  Then we have **de Rham theorem**

$$H^1(X,\mathbb{C}) \simeq L(X)\Big/d\mathscr{E}(X) = \text{closed 1 form}\Big/\text{exact 1 form} \simeq \mathrm{Rh}^1(X)$$

2.4 Riemann Roch theorem and Serre duality

2.4.1 Divisor and Riemann Roch

Suppose X is compact Riemann surface,

- Define the **group of divisor** be $\operatorname{Div}(X) = \bigoplus_{P \in X} \mathbb{Z}P$ be an free abelian group. We can regard $D = \sum_{i=1}^{n} k_i P_i \in \operatorname{Div}(X)$ as the map

$$\begin{array}{rccc} D: & X & \longrightarrow & \mathbb{Z} \\ & P_i & \longmapsto & k_i \\ & \text{other point} & \longmapsto & 0 \end{array}$$

  Define the degree of divisor D by $\deg D = \sum_{i=1}^{n} k_i$, and $\operatorname{Div}_0(X) := \{D \in \operatorname{Div}(X) : \deg D = 0\}$.

- For $D_1, D_2 \in \operatorname{Div}(X)$, $D_1 \geq D_2$ if $D_1(x) \geq D_2(x)\ \forall x \in X$.

- For $f \in \mathscr{M}(X)$, define the **principal divisor** $(f) := \sum \operatorname{ord}_x(f)x$. By argument principle and triangulation, we can show that $\deg(f) = 0$. Let $\operatorname{Div}_p(X)$ be the group of all principal divisor on X, and $\operatorname{Pic}(X) = \operatorname{Div}(X)/\operatorname{Div}_p(X)$ is called **Picard group** of X.

- We say D_1, D_2 are linearly equivalent ($D_1 \sim D_2$) if $D_1 = (f) + D_2$ for some $f \in \mathscr{M}(X)$.

- For $\omega \in \mathscr{M}^{(1)}(X)$, $\forall x \in X$, $\exists x \in U \subset X$ s.t. $\omega|_U = f dz$ with $f \in \mathscr{M}(U)$. Define $\operatorname{ord}_x(\omega) = \operatorname{ord}_x(f)$, if $\omega|_{U'} = f' dz$, then $f/f' \in \mathcal{O}^*(U \cap U')$ and thus the $\operatorname{ord}_x(\omega)$ is independent on chart. Define $(\omega) = \sum(\operatorname{ord}_x(\omega))x$.

- For $\omega_1, \omega_2 \in \mathscr{M}^{(1)}(X)\backslash\{0\}$, $\omega_i|_U = f_i dz$, then $f_U := f_1/f_2 \in \mathscr{M}(U)\backslash\{0\} \implies \omega_1|_U = f_U \omega_2|_U$. We can glue all f_U to a global section $f \in \mathscr{M}(X)\backslash\{0\}$ s.t. $\omega_1 = f\omega_2$ and thus $(\omega_1) = (f) + (\omega_2)$. This shows that every principle divisor of differential 1-form are linearly equivalent. Define the **Canonical divisor** be (ω) for any $\omega \in \mathscr{M}^{(1)}(X)$.

- For $D \in \operatorname{Div}(X)$, $U \underset{\text{open}}{\subset} X$, define

$$\mathcal{O}_D(U) = \{f \in \mathscr{M}(U) : \operatorname{ord}_x(f) + D(x) \geq 0\ \forall x \in U\}$$

  Then $\mathcal{O}_D$ is a sheaf. Similarly, define a sheaf Ω_D by

$$\Omega_D(U) = \{\omega \in \mathscr{M}^{(1)}(U) : \operatorname{ord}_x(\omega) + D(x) \geq 0\ \forall x \in U\}$$

  If $K = (\omega_0)$ be the Canonical divisor. For $\omega \in \mathscr{M}^{(1)}(U)$, say $\omega = f_U \omega_0$ for some $f_U \in \mathscr{M}(U)$, then

$$0 \leq \operatorname{ord}_x(\omega) + D(x) = \operatorname{ord}_x(f_U) + K(x) + D(x) \implies f_U \in \mathcal{O}_{K+D}(U)$$

  Hence, $\Omega_D \simeq \mathcal{O}_{K+D}(U)$.

- $D_1 \sim D_2 \implies D_1 = D_2 + (g)$ induces a isomorphism

$$\begin{array}{ccc} \mathcal{O}_{D_1}(U) & \longrightarrow & \mathcal{O}_{D_2}(U) \\ f & \longmapsto & fg \end{array}$$

  Note that $f \in \mathcal{O}_{D_1}(U) \rightsquigarrow 0 \leq (f) + D_1 = (fg) + D_2$, so the map is well defined.

Definition 2.4.1 (skyscraper sheaf). For $P \in X$, define the **skyscraper sheaf**

$$\mathbb{C}_P(U) = \begin{cases} \mathbb{C} & \text{if } P \in U \\ 0 & \text{if } P \notin U \end{cases}$$

Then we have following properties.

- $(\mathbb{C}_P)_P = \mathbb{C}$ and $(\mathbb{C}_P)_Q = 0$ for $Q \neq P$.
- $H^0(X, \mathbb{C}_P) = \mathbb{C}$
- $H^1(X, \mathbb{C}_P) = 0$: $\forall \xi = \overline{(f_{ij})} \in H^1(X, \mathbb{C}_P)$ with $(f_{ij}) \in Z^1(\mathfrak{U}, \mathbb{C}_P)$. We can find a refinement $\mathfrak{V}$ of $\mathfrak{U}$ s.t. $\exists! V_{\ell_0}$ s.t. $P \in V_{\ell_0}$. Then $p \notin V_k \cap V_\ell$ for $k \neq \ell$. Then

$$\begin{array}{ccc} Z^1(\mathfrak{U}, \mathbb{C}_P) & \longrightarrow & Z^1(\mathfrak{V}, \mathbb{C}_P) \\ (f_{ij}) & \longmapsto & 0 \end{array}$$

this implies $\xi = 0$.

Now given $D \in \mathrm{Div}(X)$ and $P \in X$, define the morphism $\beta : \mathcal{O}_{D+P} \to \mathbb{C}_P$ as follows.

- $U \subset X$, $P \notin U$, $\beta_U : \mathcal{O}_{D+P}(U) \to \mathbb{C}_P(U) = 0$ be zero morphism.
- $U \subset X$, $P \in U$, say (U_0, z) be the chart s.t. $z : U_0 \xrightarrow{\sim} B_R(0)$ and $z(P) = 0$. Given $f \in \mathcal{O}_{D+P}(U)$, say $f|_{U_0} = \sum\limits_{n=-k-1}^{\infty} c_n z^n$, where $k = D(P)$, since $(f) + D + P \geq 0$. Define

$$\begin{array}{cccc} \beta_U : & \mathcal{O}_{D+P}(U) & \longrightarrow & \mathbb{C}_P(U) = \mathbb{C} \\ & f & \longmapsto & c_{-k-1} \end{array}$$

- $\ker \beta = \mathcal{O}_D$: If $P \notin U$, then $\ker \beta_U = \mathcal{O}_{D+P}(U) = \mathcal{O}_D(U)$.
  If $P \in U$, then $\ker \beta_U = \{f \in \mathcal{O}_{D+P}(U) : \mathrm{ord}_P(f) \geq -k\} = \mathcal{O}_D(U)$.
- Then we have the exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P$$

Choose U small enough s.t. $P \in U$ and $D|_{U \setminus \{P\}} = 0$, then $cz^{-k-1} \mapsto c$, which shows that β is surjective. Hence,

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P \to 0$$

Theorem 2.4.1 (Riemann Roch theorem)**.** Suppose X is compact Riemann surface of genus g $(= \dim H^1(X, \mathcal{O}))$, $D \in \mathrm{Div}(X)$. Then $\dim H^0(X, \mathcal{O}_D), \dim H^1(X, \mathcal{O}_D) < \infty$ and satisfy

$$\dim H^1(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D \tag{$*$}$$

Proof:

- For $D = 0$, $\dim H^0(X, \mathcal{O}) = \dim \mathbb{C} = 1$, $\dim H^1(X, \mathcal{O}) = g < \infty$, $\deg D = 0$. Then $(*)$ holds.
- Consider the relation between D and $D' = D + P$. By $0 \to \mathcal{O}_D \to \mathcal{O}_{D'} \to \mathbb{C}_P \to 0$,

$$0 \to \mathcal{O}_D(X) \to \mathcal{O}_{D'}(X) \xrightarrow{\beta_X} \underbrace{\mathbb{C}_P(X)}_{=\mathbb{C}} \xrightarrow{\delta^*} H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to H^1(\mathbb{C}_P, X) = 0$$

Split it to three exact sequences of $\mathbb{C}$ vector space in below.

$$\begin{cases} 0 \to \mathcal{O}_D(X) \to \mathcal{O}_{D'}(X) \to \mathrm{Im}\, \beta_X \to 0 \\ 0 \to \mathrm{Im}\, \delta^* \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0 \\ 0 \to \mathrm{Im}\, \beta_X \to \mathbb{C} \to \mathrm{Im}\, \delta^* \to 0 \end{cases}$$

$$\implies \begin{cases} \dim \mathcal{O}_{D'}(X) = \dim \mathcal{O}_D(X) + \dim \operatorname{Im} \beta_X \\ \dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}_{D'}) + \dim \operatorname{Im} \delta^* \\ 1 = \dim \operatorname{Im} \beta_X + \dim \operatorname{Im} \delta^* \end{cases}$$

This show that

$$\dim H^i(X, \mathcal{O}_{D'}) < \infty \iff \dim H^i(X, \mathcal{O}_D) < \infty \text{ for } i = 0, 1$$

and

$$\dim H^0(X, \mathcal{O}_{D'}) - \dim H^1(X, \mathcal{O}_{D'}) = \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) + 1$$

Hence, D satisfies $(*) \iff D'$ satisfies $(*)$. By induction, for all $D \in \operatorname{Div}(X)$ satisfy $(*)$.

□

Corollary 2.4.1. Suppose X is a compact Riemann surface of genus g and $P \in X$. Let $D = (g+1)P$. By Riemann Roch,

$$\dim H^0(X, \mathcal{O}_D) \geq 1 - g + (1 + g) = 2$$

So $\mathcal{O}_D(X)$ contain a non-constant function f. Then f has a pole at P of order $\leq g + 1$ and holomorphic on $X \setminus \{P\} \rightsquigarrow f : X \to \mathbb{P}^1$ is surjective since X is compact and is a holomorphic covering with at most $g + 1$ sheets. In particular, if $g = 0$, then $f : X \xrightarrow{\sim} \mathbb{P}^1$.

2.4.2 Serre duality

Theorem 2.4.2 (Serre duality)**.** Let X be a compact Riemann surface and $D \in \operatorname{Div}(X)$, then

$$\exists \iota_D : H^0(X, \Omega_{-D}) \xrightarrow{\sim} H^1(X, \mathcal{O}_D)^\vee$$

First, we construct dual pairing

$$\langle \ , \ \rangle : H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \Omega) \xrightarrow{\text{Res}} \mathbb{C}$$

- For $\omega \in H^0(X, \Omega_{-D})$, $\xi = [\overline{(f_{ij})}]$ with $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_D)$, $[\overline{(f_{ij}\omega|_{U_i \cap U_j})}] \in H^1(X, \Omega)$ since

$$(f_{ij}\omega|_{U_i \cap U_j}) = (\omega|_{U_i \cap U_j}) - D|_{U_i \cap U_j} + (f_{ij}) + D|_{U_i \cap U_j} \geq 0$$

  It is clear that the map $(\omega, \xi) \mapsto [\overline{f_{ij}\omega|_{U_i \cap U_j}}]$ is well-defined and is bilinear form.

- By Dolbeault theorem,

$$\begin{array}{ccc} H^1(X, \Omega) & \xrightarrow{\sim} & \mathscr{E}^{(2)}(X)/d\mathscr{E}^{1,0}(X) \\ \eta & \longmapsto & \overline{\omega} \end{array}$$

  Define $\operatorname{Res}(\eta) := \dfrac{1}{2\pi i} \displaystyle\iint_X \omega = \dfrac{1}{2\pi i} \sum_{k=1}^n \iint_X f_k \omega$, where $\{f_k\}_{k=1}^n \subset \mathscr{E}(X)$ is the partition of unity on chart $\{U_k\}_{k=1}^n$. Note that Res is well defined since

$$\iint_X d\omega = \sum_{k=1}^n \iint_X d\omega_k = \sum_{k=1}^n \iint_{\partial U_k} \omega_k = 0$$

  where $\omega_k = f_k \omega$ and $\operatorname{supp}(f_k) \subset U_k$.

- This pairing is non-degenerate : Given $\omega \in H^0(X, \Omega_{-D})$, we need find ξ s.t. $\langle \omega, \xi \rangle \neq 0$. For $a \in X$ with $D(a) = 0$, choose the chart (U_0, z) s.t. $z : U_0 \xrightarrow{\sim} B_\varepsilon(0) \subset \mathbb{C}$, $z(a) = 0$ and

$$\begin{cases} D|_{U_0} = 0 \implies (\omega)|_{U_0} \geq 0 \\ \omega|_{U_0} = f dz \text{ s.t. } f \in \mathcal{O}(U_0) \text{ has no zero in } U_0 \setminus \{a\} \end{cases}$$

Consider $\mathfrak{U} = \{U_0, U_1\}$ with $U_1 = X \setminus \{a\}$ and $\eta = (f_0, f_1) = ((zf)^{-1}, 0) \in C^0(\mathfrak{U}, \mathscr{M})$. Then $\delta\eta \in Z^1(\mathfrak{U}, \mathcal{O}_D)$ and let $\xi = [\overline{\delta\eta}] \in H^1(X, \mathcal{O}_D)$. Also

$$\omega\eta = \left(\frac{dz}{z}, 0\right) \in C^0(\mathfrak{U}, \mathscr{M}^{(1)})$$

On $U_0 \cap U_1$, $\dfrac{dz}{z} - 0$ is holomorphic 1 form, then $\delta(\omega\eta) \in Z^1(\mathfrak{U}, \Omega) \rightsquigarrow [\overline{\delta(\omega\eta)}] \in H^1(X, \Omega)$. Hence, $\omega\xi = [\overline{\delta(\omega\eta)}]$. By the Lemma 2.4.1 in below, we have

$$\langle \omega, \xi \rangle = \operatorname{Res}([\overline{\delta(\omega\eta)}]) = \operatorname{Res}(\omega\eta) = \operatorname{Res}_a(dz/z) = 1$$

By non-degenerate bilinear form, we have

$$\iota_D : H^0(X, \Omega_{-D}) \hookrightarrow H^1(X, \mathcal{O}_D)^\vee$$

Definition 2.4.2. Suppose X is a Riemann surface with open covering $\mathfrak{U} = \{U_i\}_{i \in I}$. A cochain $\mu = (\omega_i) \in C^0(\mathfrak{U}, \mathscr{M}^{(1)})$ is called a **Mittag-Leffler distribution** if $\omega_j - \omega_i \in \Omega(U_i \cap U_j)$ i.e. $\delta\mu \in Z^1(\mathfrak{U}, \Omega)$.

- For $a \in X$, define
$$\operatorname{Res}_a(\mu) = \operatorname{Res}_a(\omega_i)$$
for $a \in U_i$, which is independent of choice of i since $\omega_j - \omega_i$ is holomorphic on $U_i \cap U_j$.
- Moreover, if X is compact, then $\operatorname{Res}_a(\mu) \neq 0$ for only finite many points a. Then we can define
$$\operatorname{Res}(\mu) = \sum_{a \in X} \operatorname{Res}_a(\omega)$$

Lemma 2.4.1. Continuing the notation of Definition in above,

$$\operatorname{Res}(\mu) = \operatorname{Res}[\overline{\delta\mu}]$$

where $[\overline{\delta\mu}] \in H^1(X, \Omega)$.

Proof:

- First we find $\tau \in \mathscr{E}^{(2)}(X)$ correspond to $[\overline{\delta\mu}]$ via $H^1(X, \Omega) \simeq \mathscr{E}^{(2)}(X)/d\mathscr{E}^{1,0}(X)$. Note that

$$(\omega_j - \omega_i) \in Z^1(\mathfrak{U}, \Omega) \subset Z^1(\mathfrak{U}, \mathscr{E}^{1,0}) = B^1(\mathfrak{U}, \mathscr{E}^{1,0})$$

$\exists (\sigma_i) \in C^0(\mathfrak{U}, \mathscr{E}^{1,0})$ s.t. $\omega_j - \omega_i = \sigma_j - \sigma_i$.

$$d(\omega_j - \omega_i) = d''(\omega_j - \omega_i) = 0 \implies d\sigma_j = d\sigma_i \implies \exists \tau \in \mathscr{E}^{(2)}(X) \text{ s.t. } \tau|_{U_i} = d\sigma_i$$

Hence, $\operatorname{Res}[\overline{\delta\mu}] = \dfrac{1}{2\pi i} \displaystyle\iint_X \tau$.

- Suppose $a_1, ..., a_n$ are poles of μ and $X' = X \setminus \{a_1, ..., a_n\}$. Since $\sigma_j - \omega_j = \sigma_i - \omega_i$ on $X' \cap U_i \cap U_j$, $\exists \sigma \in \mathscr{E}^{1,0}(X')$ s.t. $\sigma|_{X' \cap U_i} = \sigma_i - \omega_i$. On $X' \cap U_i$,
$$d\sigma = d\sigma_i - d\omega_i = \tau$$
since $\omega_i \in \Omega(X' \cap U_i)$. Hence, $\tau = d\sigma$ on X'.

- For each a_k, say $a_k \in U_{i(k)}$. Let (z_k, V_k) be the chart s.t. $V_k \subset U_{i(k)}$, $z_k(V_k) = B_R(0)$, $z_k(a_k) = 0$ and $\{V_k\}$ are disjoint. Then $\exists f_i \in \mathscr{E}(X)$ s.t.
$$\begin{cases} \mathrm{supp}(f_i) \subset V_k \\ f_k|_{V_k'} = 1 \text{ with } V_k' \subset V_k \end{cases}$$
Since $d(f_k\sigma) = d\sigma = d(\sigma_{i(k)} - \omega_{i(k)}) = d\sigma_{i(k)}$ on $V_k' \setminus \{a\}$ and vanish on $X \setminus V_k$, we can extend it to X. Hence $d(f_k\sigma) \in \mathscr{E}^{(2)}(X)$ and we have
$$\begin{aligned} \iint_X d(f_k\sigma) = \iint_{V_k} d(f_k\sigma) &= \lim_{\varepsilon\to 0} \iint_{\varepsilon \le |z| \le R} d(f_k\sigma_{i(k)} - f_k\omega_{i(k)}) \\ &= -\lim_{\varepsilon\to 0} \iint_{\varepsilon \le |z| \le R} d(f_k\omega_{i(k)}) \qquad (f_k\sigma_{i(k)} \in \mathscr{E}^{1,0}(U_{i(k)})) \end{aligned}$$
By Stokes' theorem.
$$= \lim_{\varepsilon\to 0} \left(\int_{|z|=\varepsilon} - \int_{|z|=R} \right) f_k\omega_{i(k)} = \lim_{\varepsilon\to 0} \int_{|z|=\varepsilon} \omega_{i(k)} = 2\pi i \operatorname{Res}_{a_k} \omega_{i(k)}$$

- Set $g = 1 - \sum\limits_{k=1}^{n} f_k$. Since $g|_{V_k'} = 0 \ \forall k$, $g\sigma \in \mathscr{E}^{1,0}(X)$. Then $\displaystyle\iint_X d(g\sigma) = 0$ and
$$d\sigma = dg + \sum_{k=1}^{n} d(f_k\sigma) \in \mathscr{E}^{(2)}(X) \implies \tau = dg + \sum_{k=1}^{n} d(f_k\sigma) \text{ on X}$$
Hence,
$$\operatorname{Res}[\overline{\delta\mu}] = \frac{1}{2\pi i} \iint_X \tau = \frac{1}{2\pi i} \iint_X d(g\sigma) + \frac{1}{2\pi i} \sum_{k=1}^{n} \iint_X d(f_k\sigma) = \sum_{k=1}^{n} \operatorname{Res}_{a_k} \omega_{i(k)} = \operatorname{Res}(\mu)$$

□

Lemma 2.4.2. Let $D' \le D$ be divisors on a compact Riemann surface X. Then the inclusion map $\mathcal{O}_{D'} \to \mathcal{O}_D$ induces an epimorphism
$$H^1(X, \mathcal{O}_{D'}) \to H^1(X, \mathcal{O}_D) \to 0$$

Proof: If $D = D' + P$, then it hold immediately by the map induced by $0 \to \mathcal{O}_{D'} \to \mathcal{O}_D \to \mathbb{C}_P \to 0$. In general, $D = D' + P_1 + \cdots + P_m$ and holds by induction. □

Remark 2.4.1. If $D' \le D$, the map of cohomology group induces
$$0 \to H^1(X, \mathcal{O}_D)^\vee \to H^1(X, \mathcal{O}_{D'})^\vee$$
It is clear clear that the diagram
$$\begin{tikzcd} 0 \arrow[r] & H^1(X, \mathcal{O}_D)^\vee \arrow[r, "i_{D'}^D"] & H^1(X, \mathcal{O}_{D'})^\vee \\ 0 \arrow[r] & H^0(X, \Omega_{-D}) \arrow[r] \arrow[u, "\iota_D"] & H^0(X, \Omega_{-D'}) \arrow[u, "\iota_{D'}"] \end{tikzcd}$$
commute.

Lemma 2.4.3. Using the notation same above. Suppose $\lambda \in H^1(X, \mathcal{O}_D)^\vee$ and $\omega \in H^0(X, \Omega_{-D'})$ satisfy

$$i_{D'}^D(\lambda) = \iota_{D'}(\omega)$$

Then $\omega \in H^0(X, \Omega_{-D})$ and $\lambda = \iota_D(\omega)$.

Proof: If $\omega \notin H^0(X, \Omega_{-D})$, then $\exists a \in X$ s.t. $\text{ord}_a(\omega) < D(a)$. Let (U_0, z) be the chart of a s.t. $z(U_0)$ is a disk in $\mathbb{C}$, $z(a) = 0$ and

$$\begin{cases} D|_{U_0\setminus\{a\}} = 0, \ D'|_{U_0\setminus\{a\}} = 0 \\ \omega|_{U_0} = f dz \text{ with } f \in \mathscr{M}(U_0) \text{ s.t. } (f)|_{U_0\setminus\{a\}} = 0 \end{cases}$$

Set $U_1 = X \setminus \{a\}$, $\mathfrak{U} = \{U_0, U_1\}$ and $\eta = ((zf)^{-1}, 0) \in C^0(\mathfrak{U}, \mathscr{M})$. Note that

$$\text{ord}_a((zf)^{-1}) = -1 - \text{ord}_a(\omega) \geq -D(a) \implies \eta \in C^0(\mathfrak{U}, \mathcal{O}_D)$$

$$\implies \delta\eta \in Z^1(\mathfrak{U}, \mathcal{O}_D) = Z^1(\mathfrak{U}, \mathcal{O}) = Z^1(\mathfrak{U}, \mathcal{O}_{D'})$$

Let ξ, ξ' be the cohomology class of $\delta\eta$ in $H^1(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_{D'})$, then $\xi = 0$. By assumption,

$$\langle \omega, \xi' \rangle = \iota_{D'}(\omega)(\xi') = i_{D'}^D(\lambda)(\xi') = \lambda(\xi) = 0$$

On the other hand, since $\omega\eta = (dz/z, 0)$, by Lemma 2.4.1 we have

$$\langle \omega, \xi' \rangle = \text{Res}(\overline{\delta(\omega\eta)}) = \text{Res}(\omega\eta) = 1 \ (\rightarrow\!\leftarrow)$$

$\square$

Fact 2.4.1. If $D, B \in \text{Div}(X)$, then $\psi \in H^0(X, \mathcal{O}_B)$ induces the map

$$\begin{array}{ccc} \mathcal{O}_{D-B} & \xrightarrow{\psi} & \mathcal{O}_D \\ f & \longrightarrow & f\psi \end{array}$$

If $\psi \neq 0$, then $H^1(X, \mathcal{O}_D)^\vee \hookrightarrow H^1(X, \mathcal{O}_{D-B})^\vee$.

Proof: Let $A = (\psi) \rightsquigarrow A + B \geq 0$, then $\mathcal{O}_{D-B} \xrightarrow{\psi} \mathcal{O}_D$ factor thought $\mathcal{O}_{D+A}$

$$\mathcal{O}_{D-B} \hookrightarrow \mathcal{O}_{D+A} \xrightarrow[\sim]{\psi} \mathcal{O}_D$$

By Lemma 2.4.2,

$$H^1(X, \mathcal{O}_D)^\vee \xrightarrow[\sim]{\psi} H^1(X, \mathcal{O}_{D+A})^\vee \hookrightarrow H^1(X, \mathcal{O}_{D-B})^\vee$$

$\square$

Back to the proof of surjective in Serre duality.

- Now, given $0 \neq \lambda \in H^1(X, \mathcal{O}_D)^\vee$. For P fixed in X, let $D_n := D - nP$. Let

$$\Lambda = \{\psi\lambda : \psi \in H^0(X, \mathcal{O}_{nP})\}$$

be the subspace of $H^1(X, \mathcal{O}_{D_n})^\vee$. If $\psi\lambda = 0$ and $\psi \neq 0$. By Fact 2.4.1, $\lambda = 0$ $(\rightarrow\!\leftarrow)$. Hence, $\Lambda \simeq H^0(X, \mathcal{O}_{nP})$. By Riemann Roch theorem,

$$\dim \Lambda = \dim H^0(X, \mathcal{O}_{nP}) \geq 1 - g + n$$

On the other hand, $\mathrm{Im}(\iota_{D_n}) \subset H^1(X, \mathcal{O}_{D_n})^\vee$. Since ι_{D_n} is injective,

$$\begin{aligned}\dim \mathrm{Im}(\iota_{D_n}) &= \dim H^0(X, \Omega_{-D_n}) = \dim H^0(X, \mathcal{O}_{K-D_n}) \\ &\geq 1 - g + \deg(K - D_n) = n + (1 - g + \deg(K - D))\end{aligned}$$

For $n > \deg D$, $H^0(X, \mathcal{O}_{D_n}) = 0$. Indeed, if $(f) + D_n \geq 0$, then $0 \leq \deg(f) + \deg D_n = \deg D_n$ $(\rightarrow\leftarrow)$. By Riemann Roch theorem again,

$$\dim H^1(X, \mathcal{O}_{D_n})^\vee = -(1 - g + \deg D_n) = n + (g - 1 - \deg D)$$

We can pick $n \gg 0$ s.t.

$$\dim \Lambda + \dim \mathrm{Im}(\iota_{D_n}) > \dim H^1(X, \mathcal{O}_{D_n})^\vee$$

Then $\Lambda \cap \mathrm{Im}(\iota_{D_n}) \neq 0 \implies \exists \psi\lambda = \iota_{D_n}(\omega_n)$ with $\omega_n \in H^0(X, \Omega_{D_n})$.

- Let $A = (\psi)$ and $D' = D_n - A \rightsquigarrow \psi^{-1} \in H^0(X, \mathcal{O}_A)$. Since $\psi \in H^0(X, \mathcal{O}_{nP})$, $A + nP \geq 0 \rightsquigarrow D' \leq D$. By Fact 2.4.1 and Remark 2.4.1, we have

$$\begin{tikzcd} 0 \arrow[r] & H^1(X, \mathcal{O}_D)^\vee \arrow[r, "i_{D'}^D"] & H^1(X, \mathcal{O}_{D'})^\vee \arrow[r, "\psi", "\sim"'] & H^1(X, \mathcal{O}_{D_n})^\vee \\ 0 \arrow[r] & H^0(X, \Omega_{-D}) \arrow[r] \arrow[u, "\iota_D"] & H^0(X, \Omega_{-D'}) \arrow[r] \arrow[u, "\iota_{D'}"] & H^0(X, \Omega_{-D_n}) \arrow[u, "\iota_{D_n}"'] \end{tikzcd}$$

Then
$$i_{D'}^D(\lambda) = \frac{1}{\psi}(\psi\lambda) = \frac{1}{\psi}\iota_{D_n}(\omega_n) = \iota_{D'}(\omega_n/\psi)$$

From Lemma 2.4.3, $\omega_n/\psi \in H^0(X, \mathcal{O}_{-D})$ and $\iota_D(\omega_n/\psi) = \lambda$. Hence, ι_D is surjective.

Corollary 2.4.2.

- $H^0(X, \mathcal{O}_{-D}) \simeq H^1(X, \Omega_D)^\vee$: Let K be canonical divisor, then $\Omega_D \simeq \mathcal{O}_{K+D}$. Hence,

$$H^1(X, \Omega_D)^\vee \simeq H^1(X, \mathcal{O}_{K+D})^\vee \simeq H^0(X, \Omega_{-(K+D)}) \simeq H^0(X, \mathcal{O}_{-D})$$

The second isomorphism is by Serre duality.

- The Residue map $\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}$ is isomorphism, since

$$\dim H^1(X, \Omega) = \dim H^0(X, \mathcal{O}) = 1$$

- Let K be canonical divisor. Then

$$\begin{aligned}1 - g + \deg K &= \dim H^0(X, \mathcal{O}_K) - \dim H^1(X, \mathcal{O}_K) && \text{(Riemann Roch)} \\ &= \dim H^0(X, \Omega) - \dim H^1(X, \Omega) \\ &= \dim H^1(X, \mathcal{O})^\vee - \dim H^0(X, \mathcal{O})^\vee = g - 1 && \text{(Serre duality)}\end{aligned}$$

Hence, $\deg K = 2g - 2$.

- If $X = \mathbb{C}/\Gamma$, then $\pi : \mathbb{C} \to X$ is a universal covering map. Note that dz is invariant under $\mathrm{Deck}(\mathbb{C}/X) \simeq \mathbb{Z} \times \mathbb{Z}$, dz can be regard as 1-form on X. Then $K = (dz) = 0$. Hence,

$$0 = \deg K = 2g - 2 \implies g = 1$$

2.4.3 Application

Theorem 2.4.3 (Riemann Hurwitz formula)**.** Let X, Y be the compact Riemann surface with genus g, g' respective. If $f : X \to Y$ is n-sheeted holomorphic covering map, then

$$g = \frac{b}{2} + n(g' - 1) + 1$$

where $b = b(f) = \sum\limits_{x \in X} b_f(x)$ is **total branch order** of f.

Proof: For fixed $\omega \in \mathscr{M}^{(1)}(X)$. For $x \in X$, $y = f(x) \in Y$. Let (U, z), (V, z') be the chart of x, y s.t. $z(U) \to z'(V)$ is w^k. Say $\omega|_V = g(z')dz'$, then

$$f^*\omega = g(z^k) = kz^{k-1}g(z^k)dz \implies \mathrm{ord}_x(f^*\omega) = k - 1 + k \cdot \mathrm{ord}_y(\omega) = b_f(x) + v_f(x) \cdot \mathrm{ord}_y(\omega)$$

Sum over $x \in f^{-1}(y)$, note that $\sum\limits_{x \in f^{-1}(y)} v_f(x) = n$ and thus

$$\implies \sum_{x \in f^{-1}(y)} \mathrm{ord}_x(f^*\omega) = \sum_{x \in f^{-1}(y)} b_f(x) + n\mathrm{ord}_y(\omega)$$

Sum over $y \in Y$, we have

$$\deg(f^*\omega) = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \mathrm{ord}_x(f^*\omega) = \sum_{y \in Y} \sum_{x \in f^{-1}} b_f(y) + n \sum_{y \in Y} \mathrm{ord}_y(\omega) = b + n \deg \omega$$

$$2g - 2 = b + n(2g' - 2) \implies g = \frac{b}{2} + n(g - 1) + 1$$

□

Property 2.4.1.

- $\deg D > 2g - 2 \rightsquigarrow H^1(X, \mathcal{O}_D) = 0 : H^1(X, \mathcal{O}_D) \simeq H^0(X, \mathcal{O}_{K-D})^\vee = 0$ since $\deg(K - D) < 0$.
- $H^1(X, \mathscr{M}) = 0$: For $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{M})$ (since X is compact, we may assume $\mathfrak{U}$ is finite cover), then $\{f_{ij}\}$ involves finite numbers of pole, then $\exists D \in \mathrm{Div}(X)$ s.t. $\deg D > 2g - 2$ and $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_D) \hookrightarrow Z^1(\mathfrak{V}, \mathcal{O}_D) = B^1(\mathfrak{V}, \mathcal{O}_D)$. Hence $\overline{[(f_{ij})]} = 0$ in $H^1(X, \mathcal{O}_D) \hookrightarrow H^1(X, \mathscr{M})$.
- $H^1(X, \mathscr{M}^{(1)}) = 0$: For fixed $\omega \in \mathscr{M}^{(1)}(X)$, we have the corresponding $(f_{ij}) \longleftrightarrow (f_{ij}\omega|_{U_i \cap U_j})$ and by $H^1(X, \mathscr{M}) = 0$.
- Consider $\mathrm{Res} : H^1(X, \Omega) \to \mathbb{C}$ and $\xi = (\omega_{ij}) \in Z^1(\mathfrak{U}, \Omega) \subset Z^1(\mathfrak{U}, \mathscr{M}^{(1)}) = B^1(\mathfrak{U}, \mathscr{M}^{(1)})$, there exists Mittag-Leffler distribution $\mu \in C^0(\mathfrak{U}, \mathscr{M}^{(1)})$ s.t. $\delta\mu = \xi$. Then

$$\mathrm{Res}(\xi) = \mathrm{Res}(\delta\mu) = \mathrm{Res}(\mu)$$

Definition 2.4.3. $\mathcal{O}(D)$ is called **globally generated** (or called **generated by global section**, g.b.g.s.) if $\forall x \in X$, $\exists f \in H^0(X, \mathcal{O}(D))$ s.t. $\mathcal{O}(D)_x = \mathcal{O}_X \cdot f \iff \mathrm{ord}_x(f) = -D(x)$.

Lemma 2.4.4. $\deg D \geq 2g$, then $\mathcal{O}(D)$ is globally generated.

Proof: $\forall x \in X$, consider $D' = D - x$. Since $\deg D' = \deg D - 1 > 2g - 2$, $H^1(X, \mathcal{O}_D) = H^1(X, \mathcal{O}_{D'}) = 0$. By Riemann Roch theorem, $\dim H^0(X, \mathcal{O}_D) > \dim H^0(X, \mathcal{O}_{D'})$. $\exists f \in H^0(X, \mathcal{O}_D) \setminus H^0(X, \mathcal{O}_{D'})$, then

$$\begin{cases} \operatorname{ord}_x(f) + D(x) \geq 0 \\ \operatorname{ord}_x(f) + D(x) - 1 < 0 \end{cases} \implies \operatorname{ord}_x(f) = -D(x)$$

□

Definition 2.4.4. If X is compact Riemann surface and $F : X \to \mathbb{P}^N$ is a continuous map. Let $U_j = \{x_j \neq 0\} \subseteq \mathbb{P}^N$ and $W_j = F^{-1}(U_j)$. Let $\varphi_j : U_j \simeq \mathbb{C}^N$ as usual. Consider the map $F_j = \varphi_j \circ F : W_j \to \mathbb{C}^N$, say $F_j = (F_{j1}, ..., F_{jN})$.

- F is called a **holomorphic** if F_{jn} are all holomorphic.
- F is called an **immersion** if it is holomorphic and $\forall x \in X$, $\exists F_{jn}$ s.t. $x \in W_j$ and $dF_{jn}(x) \neq 0$.
- F is called an **embedding** if inclusion immersion.

Theorem 2.4.4 (Embedding theorem)**.** If $\deg D \geq 2g + 1$ and $f_1, ..., f_N \in H^0(X, \mathcal{O}_D)$ form a basis, then $F = (f_0 : \cdots : f_N) : X \to \mathbb{P}^N$ is a embedding.

Proof:

- Separate points : For $x_1 \neq x_2$ in X, consider $D' = D - x_2$. Since $\deg D' = \deg D - 1 \geq 2g$, $\mathcal{O}_{D'}, \mathcal{O}_D$ are g.b.g.s. $\rightsquigarrow \exists f \in H^0(X, \mathcal{O}_{D'}) \setminus H^0(X, \mathcal{O}_D)$ s.t.

$$\begin{cases} \operatorname{ord}_{x_1}(f) = -D'(x_1) = -D(x_1) \\ \operatorname{ord}_{x_2}(f) \geq -D'(x_2) = -D(x_2) + 1 \end{cases}$$

  Say $f = \sum_{i=0}^{N} \lambda_i f_i$ and let

$$\begin{cases} k_1 = \min_j \operatorname{ord}_{x_1}(f_j) = -D(x_1) \\ k_2 = \min_j \operatorname{ord}_{x_2}(f_j) = -D(x_2) \end{cases}$$

  since $\mathcal{O}_D$ is g.b.g.s. Write $f_j = z_j^{k_i} g_{ij}$ $(i = 1, 2)$ and $\sum_{j=0}^{N} \lambda_j g_{ij} = g_i$ i.e. $f = z_j^{k_i} g_i$. Then $g_1(x_1) \neq 0$ and $g_2(x_2) = 0 \rightsquigarrow F(x_1) \neq F(x_2)$.

- Separate tangents : Let $x_0 \in X$, consider $D' = D - x_0$. Since $\mathcal{O}_{D'}$ is g.b.g.s., $\exists f \in H^0(X, \mathcal{O}_{D'})$ s.t. $\operatorname{ord}_{x_0}(f) = -D(x_0) + 1$. As before, $f = \sum \lambda_j f_j$ and let $k = \min_J \operatorname{ord}_{x_0}(f_j) = -D(x_0)$. We may assume $k = \operatorname{ord}_{x_0}(f_0)$. Let $f_j = z^k g_j$ and $f = z^k g$.

$$F(x_0) = (g_0(x_0) : \cdots : g_N(x_0)) = \left(1 : \frac{g_1(x_0)}{g_0(x_0)} : \cdots : \frac{g_N(x_0)}{g_0(x_0)}\right) \in U_0$$

  Then $F_0 : W_0 = F^{-1}(U_0) \to \mathbb{C}^N$ and

$$\sum_{j=1}^{N} \lambda_j \cdot \frac{g_j}{g_0} = \frac{g}{g_0} - \lambda_0 \implies \sum_{j=1}^{N} \lambda_j \cdot d\left(\frac{g_j}{g_0}\right) = d\left(\frac{g}{g_0}\right)$$

  Since $g_0(x_0) \neq 0$ and $\operatorname{ord}_{x_0}(g) = 1$, $d\left(\frac{g}{g_0}\right)(x_0) \neq 0$, then $\exists j$ s.t. $d\left(\frac{g_j}{g_0}\right)(x_0) \neq 0$. Hence F is embedding.

□

2.5 Mittag-Leffler problem and Weierstrass point

Definition 2.5.1. Let X be a Riemann surface and $\mathfrak{U}$ be the open covering of X. Given a Mittag-Leffler distribution of meromorphic function $\mu = (f_i) \in C^0(\mathfrak{U}, \mathscr{M})$. f is called a solution of μ if $f|_{U_i} - f_i \in \mathcal{O}(U_i)\ \forall i \in I$. Similarly, we can define the solution of Mittag-Leffler distribution of 1-form $\mu \in C(\mathfrak{U}, \mathscr{M}^{(1)})$.

Theorem 2.5.1.

(1) $\mu \in C^0(\mathfrak{U}, \mathscr{M})$ has a solution $\iff$ $[\overline{\delta\mu}] = 0$ in $H^0(X, \mathcal{O})$

(2) If X is compact Riemann surface, then $\mu \in C^0(\mathfrak{U}, \mathscr{M}^{(1)})$ has a solution $\iff$ $\operatorname{Res}(\mu) = 0$.

Proof:

(1)
- $(\Rightarrow)$: Set $g_i = f_i - f \in \mathcal{O}(U_i)$. On $U_i \cap U_j$, $f_j - f_i = g_j - g_i \implies \delta\mu \in B^1(\mathfrak{U}, \mathcal{O})$.
- $(\Leftarrow)$: $\exists g_i \in \mathcal{O}(U_i)$ s.t. $f_j - f_i = g_j - g_i$ on $U_i \cap U_j \rightsquigarrow f_j - g_j = f_i - g_i$ on $U_i \cap U_j \implies \exists f \in \mathscr{M}(X)$ s.t. $f|_{U_i} = f_i - g_i$. Then $f|_{U_i} - f_i = -g_i \in \mathcal{O}(U_i)$.

(2) Since X is compact Riemann surface, $\operatorname{Res}(\mu) = \operatorname{Res}[\overline{\delta\mu}]$. Since $\operatorname{Res} : H^1(X, \Omega) \to \mathbb{C}$, $\operatorname{Res}[\overline{\delta\mu}] = 0 \iff \overline{\delta\mu} = 0$.

□

Example 2.5.1.

- Since $H^1(\mathbb{P}^1, \mathcal{O}) = 0$, for each Mittag-Leffler distribution $\mu \in C^0(\mathfrak{U}, \mathscr{M})$ has a solution.
- Recall that $H^1(X, \mathscr{M}) = 0$ when X is compact Riemann surface. If $g \geq 1$, $\forall \xi \in H^1(X, \mathcal{O}) \setminus 0$, say $\xi = (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}) \subset Z^1(\mathfrak{U}, \mathscr{M}) = B^1(\mathfrak{U}, \mathscr{M}) \implies \exists \mu = (f_i) \in C^0(\mathfrak{U}, \mathscr{M})$ s.t. $\delta\mu = \xi$. Then μ is Mittag-Leffler distribution and $\xi = [\overline{\delta\mu}] \neq 0 \rightsquigarrow \mu$ has no solution.

Definition 2.5.2.

- $W(f_1, ..., f_g) = \det(f_j^{(i-1)})_{g\times g}$ is called **Wronskian determinant** of $f_1, ..., f_g$.
- If X is compact Riemann surface of genus $g \geq 1$, $\omega_1, ..., \omega_g \in \Omega(X)$ form a basis. Let (U, z) be the chart of X and $\omega_k = f_k dz$ on U, define $W_z(\omega_1, ..., \omega_g) := W(f_1, ..., f_g)$.

Fact 2.5.1. If $f_1, ..., f_g \in \mathcal{O}(U)$ are linearly independent over $\mathbb{C}$. Then $W(f_1, ..., f_g) \not\equiv 0$.

Proof: By induction on g. For $g = 1$, $f_1 \neq 0 \rightsquigarrow W(f_1) \not\equiv 0$. Assume $W(f_1, ..., f_{g-1}) \not\equiv 0$. Consider the linear differential equation

$$W(f_1, ..., f_{g-1}, w) = \det \begin{pmatrix} f_1 & \cdots & f_{g-1} & w \\ f_1^{(1)} & & & w^{(1)} \\ \vdots & & & \vdots \\ f_1^{(g-1)} & \cdots & f_{g-1}^{(g-1)} & w^{(g-1)} \end{pmatrix} = W(f_1, ..., f_{g-1}) w^{(g-1)} + \cdots + c_{g-1} w$$

has solution $f_1, ..., f_{g-1}$. If $W(f_1, ..., f_g) \equiv 0$, then f_g is another solution for ODE, then

$$f_g = \sum_{i=1}^{g-1} a_i f_i \text{ on } U' = \{z \in U : W(f_1, ..., f_{g-1}) \neq 0\} \underset{\text{open}}{\subset} U$$

By identity theorem, $f_g = \sum_{i=1}^{g-1} a_i f_i$ on U $(\rightarrow\!\leftarrow)$. □

Fact 2.5.2. If (U, z), $(\widetilde{U}, \widetilde{z})$ are two charts, then on $U \cap \widetilde{U}$

$$W_z(\omega_1, ..., \omega_g) = \left(\frac{d\widetilde{z}}{dz}\right)^N W_{\widetilde{z}}(\omega_1, ..., \omega_g)$$

where $N = \dfrac{g(g+1)}{2}$.

Proof: On $U \cap \widetilde{U}$, $\omega_k = f_k dz = \widetilde{f}_k d\widetilde{z} \rightsquigarrow f_k = \psi \widetilde{f}_k$ with $\psi = \dfrac{d\widetilde{z}}{dz} \in \mathcal{O}^*(U \cap \widetilde{U})$. Then

$$\frac{df_k}{dz} = \psi^2 \frac{d\widetilde{f}_k}{d\widetilde{z}} + \frac{d\psi}{dz}\widetilde{f}_k \implies \frac{d^m f_k}{dz^m} = \psi^{m+1}\frac{d^m \widetilde{f}_k}{d\widetilde{z}^m} + \sum_{j=0}^{m-1} \varphi_{mj} \frac{d^j \widetilde{f}_k}{d\widetilde{z}^j}$$

for some $\varphi_{mj} \in \mathcal{O}(U \cap \widetilde{U})$ independent of k. Then

$$W_z(f_1, ..., f_g) = \psi^{1+\cdots+g} W_{\widetilde{z}}(\widetilde{f}_1, ..., \widetilde{f}_g)$$

□

Fact 2.5.3. If $\widetilde{\omega}_1$,..., $\widetilde{\omega}_g$ is another basis of $\Omega(X)$, then $\exists c \in \mathbb{C}^\times$ s.t.

$$W_z(\omega_1, ..., \omega_g) = cW(\widetilde{\omega}_1, ..., \widetilde{\omega}_g)$$

Proof: Write $\omega_j = \sum\limits_{k=1}^g c_{jk}\widetilde{\omega}_k$. On (U, z), $f_j dz = \sum\limits_{k=1}^g c_{jk}\widetilde{f}_k dz \rightsquigarrow f_j^{(m)} = \sum\limits_{k=1}^g c_{jk}\widetilde{f}_k^{(m)}$. Then

$$W_z(\omega_1, ..., \omega_g) = W(f_1, ..., f_g) = \det(c_{jk})W(\widetilde{f}_1, ..., \widetilde{f}_g) = \det(c_{jk})W_z(\widetilde{\omega}_1, ..., \widetilde{\omega}_g)$$

□

Definition 2.5.3. Suppose X is compact Riemann surface of genus $g \geq 1$. A point $p \in X$ is called **Weierstrass point** if for a basis $\omega_1, ..., \omega_g \in \Omega(X)$ and a chart (U, z) of p, $W_z(\omega_1, ..., \omega_g) = 0$. The order of zero is called the **weight** of the Weierstrass point.

Remark 2.5.1. By fact, the definition of Weierstrass point and weight are independent of the choice of basis and coordinate.

Theorem 2.5.2. Suppose X is compact Riemann surface of genus $g \geq 1$ and $p \in X$. Then $\dim H^0(X, \mathcal{O}_{gp}) > 1 \iff p$ is Weierstrass point.

Proof: Suppose (U, z) is chart of p s.t. $z(p) = 0$ and $z(U)$ is a disk in $\mathbb{C}$. The non-constant function $f \in H^0(X, \mathcal{O}_{gp})$ has the principal part at p of the form $h = \sum\limits_{i=0}^{g-1} \dfrac{c_i}{z^{i+1}}$ with $(c_0, ..., c_{g-1}) \neq (0, ..., 0)$, and thus is a solution of M-L distribution

$$\mu = (h, 0) \in C^0(\mathfrak{U}, \mathscr{M}) \text{ where } \mathfrak{U} = (U, X \setminus \{p\})$$

By Theorem 2.5.1 and Serre duality,

$$\begin{aligned} \mu \text{ has a solution } &\iff \delta\mu = 0 \text{ in } H^1(X, \mathcal{O}) \\ &\iff \lambda(\delta\mu) = 0 \ \forall \lambda \in H^1(X, \mathcal{O})^\vee \\ &\iff 0 = \langle \omega, [\overline{\delta\mu}] \rangle = \operatorname{Res}([\overline{\delta\omega\mu}]) = \operatorname{Res}(\omega\mu) \ \forall \omega \in H^0(X, \Omega) \end{aligned}$$

Now let $\omega_1, ..., \omega_g \in \Omega(X)$ form a basis. Choose U smaller if necessary, $\omega_k = f_k dz = \left(\sum_{i=0}^{\infty} a_{ki} z^i\right) dz$ on U. Now

$$\operatorname{Res}(\omega_k \mu) = \operatorname{Res}_p(\omega_k h) = \sum_{i=0}^{g-1} a_{ki} c_i \ \forall k$$

Thus $\operatorname{Res}(\omega\mu) = 0 \ \forall \omega \in H^0(X, \Omega) \iff (a_{ki})_{g\times g}$ has nontrivial solution $(c_0, ..., c_{g-1}) \iff W_z(\omega_1, ..., \omega_g)(p) = 0$. □

Theorem 2.5.3. On a compact Riemann surface X of $g \geq 1$. The total weight of Weierstrass points is $g(g-1)(g+1)$.

Proof: Let $(U_i, z_i)_{i\in I}$ be a covering of X and $\psi_{ij} = \dfrac{dz_j}{dz_i} \in \mathcal{O}^*(U_i \cap U_j)$. Fixed a basis $\omega_1, ..., \omega_g$ of $\Omega(X)$. Let $W_i = W_{z_i}(\omega_1, ..., \omega_g) \in \mathcal{O}(U_i)$. By Fact 2.5.2, on $U_i \cap U_j$, $W_i = \psi_{ij}^N W_j$, where $N = g(g+1)/2$. Set $D(x) = \operatorname{ord}_x(W_i)$ for $x \in U_i \rightsquigarrow D \in \operatorname{Div}(X)$. If $\omega_1 = f_{1i} dz_i$ on U_i, then $f_{1i} = \psi_{ij} f_{1j}$ on $U_i \cap U_j$ and thus

$$W_i f_{1i}^{-N} = W_j f_{1j}^{-N} \text{ on } U_i \cap U_j \implies \exists f \in \mathscr{M}(X) \text{ s.t. } f|_{U_i} = W_i f_{1i}^{-N}$$

So $D - NK = (f)$ and thus $\deg D = (2g-2)N = g(g-1)(g+1)$. □

Corollary 2.5.1. Every compact Riemann surface X of genus $g \geq 2$ admits a holomorphic covering mapping $f : X \to \mathbb{P}^1$ having at most g sheets. In particular every compact Riemann surface of genus 2 is hyperelliptic.

Property 2.5.1.

- Let X be a compact Riemann surface, $p \in X$ and $n \geq 2$. Then $\exists \omega \in \mathscr{M}^{(1)}(X)$ s.t. $\operatorname{ord}_p(\omega) = -n$ and $\omega \in \Omega(X \setminus \{p\})$.

  subproof : Let (U, z) be the chart of p s.t. $p(z) = 0$ and $p(U)$ be the disk in $\mathbb{C}$. Then $\mu = (z^{-n} dz, 0)$ is M-L distribution and $\operatorname{Res}(\mu) = 0$. □

- Let X be a compact Riemann surface and distinct point $p_1, p_2 \in X$. Then exists $\omega \in \mathscr{M}^{(1)}(X)$ s.t. ω has pole of first order at p_1 and p_2 with residues 1 and -1.

  subproof : Let $(U_1, z_1), (U_2, z_2)$ be the chart of p_1, p_2 as usual with $U_1 \cap U_2 = \varnothing$. Let $\mathfrak{U} = (U_1, U_2, X \setminus \{p_1, p_2\})$, then $\mu = (z_1^{-1} dz_1, -z_2^{-1} dz_2, 0)$ is M-L distribution with $\operatorname{Res}(\mu) = 0$. □

- Let $r_1, r_2 \in \mathbb{C}$ are linearly indep. over $\mathbb{R}$. Let $\mathcal{P} = \{t_1 r_1 + t_2 r_2 : 0 \leq r_1, r_2 < 1\}$, $\Gamma = \mathbb{Z} r_1 + \mathbb{Z} r_2$. Let $a_1, ..., a_n \in \mathcal{P}$, $h_j(z) = \sum_{k=-r_j}^{-1} c_{kj} (z - a_j)^k \ \forall j$. Then $\exists f \in \mathscr{M}(\mathbb{C})$ is doubly periodic w.r.t. Γ with the prescribed principal points at $a_1, ..., a_n \iff \sum_{j=1}^{n} c_{-1j} = 0$.

  subproof : Note that $\omega = dz$ is a basis of $\Omega(\mathbb{C}/\Gamma)$, then $\operatorname{Res}(\omega\mu) = \sum_{j=-1}^{n} c_{-1j} = 0$, where

  $$\mu = (h_1, ..., h_n, 0) \in C^0(\mathfrak{U}, \mathscr{M}) \text{ with } \mathfrak{U} = (U_1, ..., U_n, X \setminus \{a_1, ..., a_n\})$$

  and $U_i \cap U_j = \varnothing$. □

Definition 2.5.4. Suppose X is compact Riemann surface of genus $g \geq 1$. The **Weierstrass gap** for $p \in X$ is a value of k s.t. no function has exactly k pole at p only.

Fact 2.5.4.

- Every point has exactly g gap numbers.
- p is not Weierstrass point $\iff$ $1, ..., g$ are gap number.

Theorem 2.5.4. If $n_1, ..., n_g$ are the gap number of Weierstrass points $p \in X$. Then the weight of p is

$$\sum_{i=1}^{g}(n_i - i)$$

Proof: By definition, for each i, $H^0(X, \mathcal{O}_{n_i p}) = H^0(X, \mathcal{O}_{(n_i-1)p})$. By Serre duality and Riemann Roch theorem,

$$H^1(X, \Omega_{-n_i p}) = H^1(X, \Omega_{-(n_i-1)p}) \implies H^0(X, \Omega_{-(n_i-1)p}) \supsetneq H^0(X, \Omega_{-n_i p})$$

Let $\omega_i \in H^0(X, \Omega_{-(n_i-1)p}) \setminus H^0(X, \Omega_{-n_i p})$ and (U, z) be the chart of p as usual s.t. $\omega_i = z^{n_i-1} f_i(z)dz$ for some $f_i \in \mathcal{O}(U)$ with $f_i(0) \neq 0$. To calculate the order of Wronskian, we only need take cure the leading term u_{ij} of each entry. Then

$$u_{ij} = \begin{cases} P_{i-1}^{n_i-1} z^{n_j-1-(i-1)} h_j(0) & \text{if } i \leq n_j \\ (n_i-1)! h_j^{(i-1-(n_j-1))}(0) & \text{if } i > n_j \end{cases}$$

where $P_m^n = \dfrac{n!}{(n-m)!}$. Then

$$W_z(\omega_1, ..., \omega_n) = \det u_{ij} = z^{\sum(n_i - i)} \det(u_{ij} \cdot z^{i-n_j})$$

We claim that $\det(u_{ij})|_{z=0} \neq 0$ i.e. $\det(P_{i-1}^{n_j-1}) \neq 0$. Consider the polynomial of g variables with degree $g(g-1)/2$

$$D(x_1, ..., x_g) = \det(P_{i-1}^{x_j})$$

Then $D(x_1, ..., x_k, ..., x_k, ..., x_n) = 0 \implies (x_i - x_j) \big| D(x_1, ..., x_n)$ and thus $\prod\limits_{i<j}(x_i - x_j) \big| D(x_1, ..., x_n)$. Counting degree we have

$$D(x_1, ..., x_n) = C \prod_{i<j}(x_i - x_j)$$

for some constant C. Note that $D(0, ..., g-1) = \prod\limits_{i=1}^{g} i! \neq 0 \implies C \neq 0$. Hence $\det(P_{i-1}^{n_j-1}) \neq 0$ and thus W_z has zero of order $\sum(n_i - i)$. $\square$

2.6 deRham Hodge theorem

Definition 2.6.1.

- For $\omega \in \mathscr{E}^{(1)}(X)$, locally $\omega|_U = f dz + g d\bar{z}$, define $\overline{\omega}|_U = \overline{f} d\bar{z} + \overline{g} dz \rightsquigarrow \overline{\omega} \in \mathscr{E}^{(1)}(X)$.
- $\omega \in \Omega(X) \rightsquigarrow \overline{\omega} \in \overline{\Omega}(X)$.

- Define **Hodge star operator**

$$\begin{array}{rccccl} * : & \mathscr{E}^{(1)}(X) & & & \longrightarrow & \mathscr{E}^{(1)}(X) \\ f = & \underbrace{\omega_1}_{\in \mathscr{E}^{1,0}(X)} & + & \underbrace{\omega_2}_{\in \mathscr{E}^{0,1}(X)} & \longmapsto & i(\overline{\omega_1} - \overline{\omega_2}) \end{array}$$

- If X is compact Riemann surface, define inner product on $\mathscr{E}^{(1)}(X)$ by

$$\langle \omega_1, \omega_2 \rangle = \iint_X \omega_1 \wedge *\omega_2$$

Locally, $\omega = fdz + gd\overline{z} \rightsquigarrow \omega \wedge *\omega = 2(|f|^2 + |g|^2)dx \wedge dy$.

- ω is **harmonic** ($\omega \in \mathrm{Harm}^1(X)$) if $d\omega = d(*\omega) = 0$. Locally $\omega = fdz + gd\overline{z}$ and thus

$$\omega \in \mathrm{Harm}^1(X) \iff \begin{cases} \left(-\dfrac{\partial f}{\partial \overline{z}} + \dfrac{\partial g}{\partial z}\right) dz \wedge d\overline{z} = 0 \\ i\left(-\dfrac{\partial \overline{f}}{\partial z} - \dfrac{\partial \overline{g}}{\partial \overline{z}}\right) dz \wedge d\overline{z} = 0 \end{cases} \iff \begin{cases} \overline{\partial} f = \partial g \\ \partial \overline{f} = -\overline{\partial} \overline{g} \end{cases} \iff \overline{\partial} f = \partial g = 0$$

Property 2.6.1. TFAE

(1) $\omega \in \mathrm{Harm}^1(X)$

(2) $d'\omega = d''\omega = 0$

(3) $\omega = \omega_1 + \omega_2$ for some $\omega_1 \in \Omega(X)$, $\omega_2 \in \overline{\Omega}(X)$.

(4) $\forall a$, $\exists U_a \subset X$ s.t. $\omega = df$ for some harmonic function f on U_a.

2.7 Abel theorem

Definition 2.7.1. Suppose X is compact Riemann surface and $D \in \mathrm{Div}(X)$, define

$$X_D = \{x \in X : D(X) \geq 0\}$$

$f \in \mathscr{E}(X_D)$ is a **weak solution** of D if $\forall a \in X$, $\exists (U, z)$ with $z(a) = 0$ and $\psi \in \mathscr{E}(U)$ with $\psi(a) \neq 0$ s.t. $f = \psi z^k$ on $U \cap X_D$ with $k = D(a)$.

Lemma 2.7.1 (key formula)**.** Let $D = \sum k_i a_i \in \mathrm{Div}(X)$. If f is a weak solution of D, then $\forall g \in \mathscr{E}(X)$ with compact support,

$$\frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge dg = \sum_{i=1}^n k_i g(a_i)$$

Proof: Choose disjoint $(U_j, z_j)_{j=1}^n$ with $z_j(a_j) = 0$, $z_j(U_j) = D \subset \mathbb{C}$ s.t. $f = \psi_j z_j^{k_j}$ with $\psi_j \in \mathscr{E}(U_j)$ and $\psi_j(x) \neq 0 \forall x \in U_j$. Suppose $0 < r < r' < 1$, $\exists \varphi_j \in \mathscr{E}(X)$ s.t. $\mathrm{supp}(\varphi_j) \subset \{|z_j| < r'\}$ and $\varphi_j|_{\{|z_j|<r\}} = 1$. Let $g_j = \varphi_j g \in \mathscr{E}(X)$ and $g_0 = g - \sum\limits_{j=1}^n g_j$. Then $g_0|_{\{|z_j| \leq r\}} = 0 \rightsquigarrow \mathrm{supp}(g_0)$ is compact in $X' = X \setminus \{a_1, ..., a_n\}$. Then

$$\frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge dg_0 = -\frac{1}{2\pi i} \iint_{X'} d\left(g_0 \frac{df}{f}\right) = 0$$

and thus

$$\frac{1}{2\pi i}\iint_X \frac{df}{f}\wedge dg = \frac{1}{2\pi i}\sum_{j=1}^n \iint_{U_j}\frac{df}{f}\wedge dg_j = \frac{1}{2\pi i}\sum_{j=1}^n k_j\iint_{U_j}\frac{dz_j}{z_j}\wedge dg_j$$

since $\dfrac{d\psi_j}{\psi_j}\wedge dg_j = -d\left(g_j\dfrac{d\psi_j}{\psi_j}\right)$ on U_j. By Stoke theorem and $g_j\in\mathscr{E}(U_j)$,

$$\iint_{U_j}\frac{dz_j}{z_j} = -\lim_{\varepsilon\to 0}\iint_{\varepsilon\le|z|\le r'} d\left(g_j\frac{dz_j}{z_j}\right) = \lim_{\varepsilon\to 0}\int_{|z|=\varepsilon} g_j\frac{dz_j}{z_j} = 2\pi i g_j(a_j) = 2\pi i g(a_j)$$

□

Lemma 2.7.2 (key lemma). If $c:[0,1]\to X$ and $U\Subset X$ with $c([0,1])\subset U$, then exists a weak solution of ∂C with $f|_{X\setminus U}=1$ s.t. $\forall\omega\in\mathscr{E}^{(1)}(X)$ with $d\omega=0$ we have

$$\int_c\omega = \frac{1}{2\pi i}\iint_X\frac{df}{f}\wedge\omega$$

Proof:

- Case 1. $U\simeq D\subset\mathbb{C}$: Assume $a=c(0)$, $b=c(1)$, $c([0,1])\subset B_r(0)$ with $0<r<1$. Then $\dfrac{z-b}{z-a}$ is a nowhere vanishing holomorphic function on $\{r<|z|<1\}$ and $\log\left(\dfrac{z-b}{z-a}\right)$ has a well-defined branch on $\{r<|z|<1\}$ since $\dfrac{z-b}{z-a}\notin\mathbb{R}_{\le 0}$. Choose $\psi\in\mathscr{E}(U)$ s.t. $\psi|_{\{|z|\le r\}}=1$ and $\operatorname{supp}\psi\subset\{|z|<r'\}$ with $r<r'<1$. Define

$$f_0(z) := \begin{cases}\exp\left(\psi\log\dfrac{z-b}{z-a}\right) & \text{if } r<|z|<1\\ \dfrac{z-b}{z-a} & \text{if } |z|\le r\end{cases}$$

  Then $f_0\equiv 1$ on $r'<|z|<1$. Extend f_0 to f by 1 outside U, then f is a weak solution of ∂C with $f|_{X\setminus U}=1$. For $\omega\in\mathscr{E}^{(1)}(X)$ with $d\omega=0$, let ω, let $\omega|_U=dh$ with $h\in\mathscr{E}(U)$. Choose $\varphi\in\mathscr{E}(X)$ s.t. $\varphi|_{\{|z|<r'\}}=1$ and $\operatorname{supp}\varphi\subseteq\{|z|<r''\}$ with $r'<r''<1$. Then $g=\varphi h\in\mathscr{E}(X)$ with compact support $\subset U$ and $g=h$ on $\{|z|\le r'\}\rightsquigarrow\omega=dg$ on $\{|z|\le r'\}$. Since $\operatorname{supp}(df/f)\subseteq\{|z|\le r\}$, we have

$$\frac{1}{2\pi i}\iint_X\frac{df}{f}\wedge\omega = \frac{1}{2\pi i}\iint_{|z|\le r'}\frac{df}{f}\wedge dg = g(b)-g(a) = \int_c\omega$$

- Case 2. In general case, exists a partition $0=t_0<t_1<\cdots<t_n=1$ s.t. $c([t_{j-1},t_j])\subset U_j\subset U$ with $z_j(U_j)=D\subset\mathbb{C}$. Let $c_j=c|_{[t_{j-1},t_j]}$. By Case 1, $\exists$ a weak solution f_j of ∂c_j s.t. $f_j|_{X\setminus U_j}=1$ and

$$\int_{c_j}\omega = \frac{1}{2\pi i}\iint_X\frac{df_j}{f_j}\wedge\omega\ \forall\omega\in\mathscr{E}^{(1)}(X)\text{ with } d\omega=0$$

  Then $f=f_1f_2\cdots f_n$ is a weak solution of $\sum\limits_{j=1}^n\partial c_j=c$ and

$$\frac{1}{2\pi i}\iint_X\frac{df}{f}\wedge\omega = \frac{1}{2\pi i}\sum_{j=1}^n\int\frac{df_j}{f_j}\wedge\omega = \sum_{j=1}^n\int_{c_j}\omega = \int_c\omega$$

□

Theorem 2.7.1 (Abel theorem)**.** Suppose X is compact Riemann surface and $D \in \mathrm{Div}(X)$ with $\deg D = 0$. Then D has a solution (i.e. $\exists f \in \mathscr{M}(X)$ s.t. $D = (f)$) $\iff$ $\exists$ 1-chain $c = \sum n_i c_i \in C_1(X) = \bigoplus_{\gamma:\text{curve}} \mathbb{Z}\gamma$ with $\partial C = D$ s.t.

$$\int_c \omega = 0 \ \forall \omega \in \Omega(X)$$

Proof:

- $(\Leftarrow)$: By key lemma, $\exists$ a weak solution f_j of ∂c_j s.t.

$$\int_{c_j} \omega = \frac{1}{2\pi i} \iint_X \frac{df_j}{f_j} \wedge \omega \ \forall \omega \in \mathscr{E}^{(1)}(X) \text{ with } d\omega = 0$$

  Set $f = \prod_{j=1}^{m} f_j^{n_j}$, then f is a weak solution of ∂c and

$$\frac{df}{f} = \sum_{j=1}^{m} n_j \frac{df_j}{f_j} \implies \int_c \omega = \sum_{j=1}^{m} n_j \int_{c_j} \omega = \frac{1}{2\pi i} \iint_X \frac{df}{f} \wedge \omega$$

  By assumption, $\forall \omega \in \Omega(X)$,

$$0 = \iint_X \frac{df}{f} \wedge \omega = \iint_X \frac{d'' f}{f} \wedge *(-i\omega)$$

  Note that $\mathscr{E}^{0,1}(X) = d''\mathscr{E}(X) \oplus \overline{\Omega}(X) \implies \dfrac{d''f}{f} = d''g$ for some $g \in \mathscr{E}(X)$. Consider $e^{-g} f$, which is also a weak solution of ∂C (since e^{-g} is nowhere vanish) and

$$d''(e^{-g} f) = e^{-g}(d'' f - f d'' g) = 0 \implies e^{-g} f \in \mathcal{O}(X)$$

- $(\Rightarrow)$: If $D = 0$, then for all closed curve c, $\partial c = 0 \rightsquigarrow \displaystyle\int_c \omega = 0 \ \forall \omega \in \Omega(X)$.

  If $D \neq 0$ and $f \in \mathscr{M}(X)$ s.t. $(f) = D \rightsquigarrow f : X \to \mathbb{P}^1$ is a branch n-sheeted covering map for some $n \geq 1$.For $\omega \in \Omega(X)$, we define $\mathrm{trace}(\omega) \in \Omega(\mathbb{P}^1)$ as follows. Let Y be the regular value of f. $\forall y \in Y$, $\exists y \in V \subset Y$ s.t. $f^{-1}(V) = \bigcup_{i=1}^{n} U_i$ with $\{U_j\}$ disjoint and $U_j \underset{\varphi_i}{\overset{f}{\rightleftarrows}} V$. Define

$$\mathrm{trace}(\omega)|_V = \varphi_1^* \omega + \cdots + \varphi_n^* \omega$$

  Then we can glue it to holomorphic 1-form $\mathrm{trace}(\omega)$ on Y as y run though all point in Y. Similarly to Theorem 1.3.3, we may extend $\mathrm{trace}(\omega)$ to $\mathbb{P}^1$. Since $\Omega(\mathbb{P}^1) = 0$, $\mathrm{trace}(\omega) = 0$. Now choose $\gamma : 0 \rightsquigarrow \infty$ s.t. $\gamma \setminus \partial\gamma \subset Y$. Then $f^{-1}\gamma$ is the union of n curves, $c_1, ..., c_n$, from the zero of f to the pole of f. Set $c := c_1 + \cdots + c_n$, then $\partial c = D$ and

$$\int_c \omega = \int_\gamma \mathrm{trace}(\omega) = 0$$

$\square$

2.8 Abel Jacobi map

Let X be compact Riemann surface of genus $g \geq 1$ and $\omega_1, ..., \omega_g$ be a basis of $\Omega(X)$.

- Define $C_1(X) = \bigoplus_{c_i:\text{curve}} \mathbb{Z}c_i$. Consider

$$\begin{array}{rccc} \partial: & C_1(X) & \longrightarrow & \mathrm{Div}(X) \\ & \sum n_i c_i & \longmapsto & \sum n_i \partial c_i \end{array}$$

  where $\partial c_i = c_i(1) - c_i(0)$. Let $Z_1(X) = \ker(C_1(X) \to \mathrm{Div}(X))$ and $H_1(X) = Z_1(X) / \sim$, where equivalent relation on $Z_1(X)$ is defined by

$$c \sim c' \iff \int_c \omega = \int_{c'} \omega \quad \forall \omega \in \mathscr{E}^{(1)}(X) \text{ with } d\omega = 0$$

  If $c - c' = \partial S$, then $\displaystyle\int_{\partial S} \omega = \int_S d\omega = 0$.

- The **period lattice** of X w.r.t. $\omega_1, ..., \omega_g$ is

$$\mathrm{Per}(\omega_1, ..., \omega_g) = \left\{ \left(\int_\alpha \omega_1, \cdots, \int_\alpha \omega_g \right) : \alpha \in H_1(X) \right\}$$

Theorem 2.8.1. $\Lambda := \mathrm{Per}(\omega_1, ..., \omega_g)$ is a lattice in $\mathbb{C}^g \simeq \mathbb{R}^{2g}$.

To prove Λ is lattice, we need the following lemma.

Lemma 2.8.1 (Lattice criterion)**.** Let V be n-dimensional vector space over $\mathbb{R}$ and an additive group $\Gamma \subseteq V$. Then $\Gamma = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_n$, where $r_1, ..., r_n$ linearly independent over $\mathbb{R}$ $\iff$

(1) exists a neighborhood of 0 s.t. $\Gamma \cap U = \{0\}$

(2) $\Gamma \not\subseteq W \subsetneq V$ for any proper subspace V of W.

Proof:

- $(\Rightarrow)$: Consider the coordinate change to $e_1, ..., e_n$, then it is clear.

- $(\Leftarrow)$: By induction on n. $n = 0$: trivial. Now for $n > 0$, by (2), $\exists x_1, ..., x_n \in \Gamma$ which are linearly independent. Let $V_1 = \langle v_1, ..., v_{n-1} \rangle_{\mathbb{R}}$ and $\Gamma_1 = \Gamma \cap V_1$. Then Γ_1 satisfy conditions (1), (2). By induction hypothesis, $\exists r_1, ..., r_{n-1} \in \Gamma$ are linearly independent over $\mathbb{R}$ s.t. $\Gamma_1 = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_{n-1} \rightsquigarrow r_1, ..., r_{n-1}, x$ are linearly independent over $\mathbb{R}$. $\forall x \in V$, say

$$x = c_1(x)r_1 + \cdots + c_{n-1}(x)r_{n-1} + c(x)x_n$$

  for some $c_i(x) \in \mathbb{R}$. Consider

$$\mathcal{P} = \{\lambda_1 r_1 + \cdots + \lambda_{n-1} r_{n-1} + \lambda x_n : \lambda \in [0, 1]\}$$

  By (1), $\exists r_n \in \Gamma \cap \mathcal{P} \setminus V_1$ s.t. $c(r_n) = \min\{c(x) : x \in (\Gamma \cap \mathcal{P}) \setminus V_1\} \in (0, 1]$. We claim that

$$\Gamma = \mathbb{Z}r_1 + \cdots + \mathbb{Z}r_{n-1} + \mathbb{Z}r_n$$

For $x \in \Gamma$, $\exists n, \lambda$ s.t. $c(x) = nc(r_n) + \lambda$ with $0 \le \lambda < c(r_n)$. Say $c_i(x) - nc_i(r_n) = n_i + \lambda_i$, where $n_i \in \mathbb{Z}$ and $\lambda_i \in [0,1)$. Then

$$x - nr_n - \sum_{i=1}^{n-1} n_i r_i = \sum_{i=1}^{n-1} \lambda_i r_i + \lambda r_n \in \Gamma \cap \mathcal{P}$$

Since $\lambda < c(r_n)$, $\lambda = 0$ and thus $x - nr_n - \sum\limits_{i=1}^{n-1} n_i r_i \in V_1 \implies x \in \bigoplus\limits_{i=1}^{n} \mathbb{Z} r_i$.

□

Proof: (Theorem 2.8.1)

- $\exists a_1, ..., a_g$ distinct in X s.t. if $\omega \in \Omega(X)$ with $\omega(a_i) = 0\ \forall i$, then $\omega = 0$:

  subproof : For $a \in X$, $H_a := \{\omega \in \Omega(X) : \omega(a) = 0\}$, then $\operatorname{codim}_{\Omega(X)} H_a \le 1$. Since $\bigcap\limits_{a \in X} H_a = 0$, $\exists a_1 \in X$ s.t. $\operatorname{codim}_{\Omega(X)} Ha_1 = 1 \rightsquigarrow \exists a_2$ s.t. $\operatorname{codim}_{Ha_1} Ha_1 \cap Ha_2 = 1$. Choose a_i similarly, we may find $a_1, ..., a_g$ s.t. $Ha_1 \cap \cdots \cap Ha_g = 0$. □

- Let $a_i \in (U_i, z_i)$ s.t. $z_i(U_i) = D \subset \mathbb{C}$, $U_i \cap U_j = \varnothing\ \forall i \neq j$, and $\omega_i|_{U_j} = \varphi_{ij} dz_j$ with $\varphi_{ij} \in \mathcal{O}(U_{ij})$. Then $(\varphi_{ij}(a_j))_{1 \le i,j \le g}$ is invertible. If not, $\exists r_1, ..., r_g \in \mathbb{C}$ not all zero s.t.

$$\sum_{i=1}^{g} r_i \varphi_{ij}(a_j) = 0\ \forall j \implies \underbrace{\left(\sum_{i=1}^{g} r_i \omega_i\right)}_{\neq 0}(a_j) = 0\ \forall j\ (\rightarrow\!\leftarrow)$$

- Define

$$\begin{array}{rrcl} (F_1, ..., F_g) = F: & U_1 \times \cdots U_g & \longrightarrow & \mathbb{C}^g \\ & (x_1, ..., x_g) & \longmapsto & \left(\sum\limits_{j=1}^{g} \int_{a_j}^{x_j} \omega_1, ..., \sum\limits_{j=1}^{g} \int_{a_j}^{x_j} \omega_g\right) \end{array}$$

  which is well-defined since U_i are simply connected. Then

$$J_F(x) = \left(\frac{\partial F_i}{\partial z_j}(x)\right) = (\varphi_{ij}) \implies \det J_F(a) = \det(\varphi_{ij}(a_j)) \neq 0$$

  where $a = (a_1, ..., a_g)$. Hence we can choose U_j smaller s.t. $U = F(U_1 \times \cdots \times U_g) \underset{\text{open}}{\subset} \mathbb{C}^g$ is neighborhood of 0.

- $\Gamma \cap U = \varnothing$: If not, say $F(x_1, ..., x_g) \in \Gamma \cap (U \setminus 0)$ with $x_j \neq a_j\ \forall j = 1, ..., k$ and $x_j = a_j\ \forall j = k+1, ..., g$. By Abel theorem, $\exists f \in \mathscr{M}(X)$ s.t. $(f) = \sum\limits_{j=1}^{k} (x_j - a_j)$. Then $\operatorname{Res}_{a_j} f = c_j \neq 0\ \forall j = 1, ..., k$. By Residue theorem,

$$0 = \operatorname{Res}(f\omega_i) = \sum_{i=1}^{k} c_i \varphi_{ij}(a_j) \implies \operatorname{rank}(\varphi_{ij}(a_j)) < g\ (\rightarrow\!\leftarrow)$$

- Now we show that Γ will not contain in any proper subspace. If not,

$$\Gamma \subseteq \left\{(x_1, y_1, ..., x_g, y_g) \subseteq \mathbb{R}^{2g} : \sum_{j=1}^{g} r_j x_j + \sum_{j=1}^{g} r_j y_j' = 0\right\}$$

for some $r_1, ..., r_g, r'_1, ..., r'_g$ not all zero. Let $c_j = r_j - ir'_j$, then the condition becomes to

$$0 = \mathrm{Re}\sum_{j=1}^{g} c_j(x_j + iy_j) = \mathrm{Re}\left(\sum_{j=1}^{g} c_j \int_\alpha \omega_j\right) = \int_\alpha \mathrm{Re}\left(\sum_{j=1}^{g} c_j\omega_j\right) \ \forall\alpha \in H_1(X)$$

Let $\omega = \sum_{j=1}^{g} c_j\omega_j$, then $\mathrm{Re}\omega$ is harmonic and $\mathrm{Re}\omega$ is exact $\implies \mathrm{Re}\omega = 0$. Locally, $\omega = f dz = df$ for some $f \in \mathcal{O}(U)$. By $\mathrm{Re}\omega = 0$, $d(f + \overline{f}) = 0 \implies f + \overline{f} = c$. Then $\mathrm{Re}f = c/2$ and $f \in \mathcal{O}(U) \implies f$ is constant on $U \rightsquigarrow \omega = df = 0$. Hence $\sum c_j\omega_j = 0 \rightsquigarrow c_j = 0\ \forall j$ ($\rightarrow\!\leftarrow$).

$\square$

Definition 2.8.1. $\mathrm{Jac}(X) := \mathbb{C}^g/\Lambda$ is called the **Jacobi variety** of X.

- $0 \to \mathrm{Div}_0(X) \to \mathrm{Div}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$. Since $\mathrm{Div}_p(X) \subset \mathrm{Div}_0(X)$, we have

$$0 \to \mathrm{Pic}_0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

where $\mathrm{Pic}_0(X) = \mathrm{Div}_0(X)/\mathrm{Div}_p(X)$.

- Define

$$\begin{array}{rccc} \phi: & \mathrm{Div}_0(X) & \longrightarrow & \mathrm{Jac}(X) \\ & D & \longrightarrow & \left(\int_c \omega_1, ..., \int_c \omega_g\right) + \Lambda \end{array}$$

where $\partial c = D$. If $\partial c' = D$, then $\partial(c - c') = 0 \rightsquigarrow [c - c'] \in H_1(X)$. Hence ϕ is well-defined.

$$\begin{aligned} D \in \ker\phi &\iff \left(\int_c \omega_1, ..., \int_c \omega_g\right) = \left(\int_\alpha \omega_1, ..., \int_\alpha \omega_g\right) \text{ for some } \alpha = [c_0] \in H_1(X) \\ &\iff \int_{c-c_0} \omega = 0\ \forall\omega \in \Omega(X) \\ &\iff \exists f \in \mathscr{M}(X) \text{ s.t. } (f) = \partial(c - c_0) = \partial c = D \qquad \text{(Abel theorem)} \\ &\iff D \in \mathrm{Div}_p(X) \end{aligned}$$

Hence $j : \mathrm{Pic}_0(X) \hookrightarrow \mathrm{Jac}(X)$.

Now our goal is surjective of j, i.e. $j : \mathrm{Pic}_0(X) \xrightarrow{\sim} \mathrm{Jac}(X)$.

- Let $a_1, ..., a_g$ be chosen in the proof of Λ is lattice. Consider

$$\begin{array}{rccc} \psi: & X^g & \longrightarrow & \mathrm{Pic}_0(X) \\ & (x_1, ..., x_g) & \longrightarrow & \sum_{j=1}^{g}(x_j - a_j) + \mathrm{Pic}_0(X) \end{array}$$

and $J = j \circ \psi$.

- In particular for $g = 1$, pick $a \in X$, $\omega \in \Omega(X) \setminus \{0\}$, and $\Gamma = \mathrm{Per}(\omega)$. Then

$$\begin{array}{rccc} J: & X & \longrightarrow & \mathbb{C}/\Gamma \\ & x & \longmapsto & \int_a^x \omega + \Gamma \end{array}$$

is holomorphic. Since X is compact and J is non-constant, J is surjective.

Theorem 2.8.2. $j : \mathrm{Pic}_0(x) \xrightarrow{\sim} \mathrm{Jac}(X)$.

Proof: Continue the proof of Λ is a lattice. For $p = [\xi] \in \mathrm{Jac}(X)$, let $N \gg 0$ s.t. $\dfrac{1}{N}\xi \in F(U_1 \times \cdots \times U_g)$. Then $\exists x_j \in U_j$ $\forall j$ s.t.

$$\left(\int_c \omega_1, ..., \int_c \omega_g\right) = \frac{1}{N}\xi$$

where $\gamma_j : a_j \rightsquigarrow x_j$ and $c = r_1 + \cdots + r_g$. Let $D = \partial C$, then $\phi(D) = \dfrac{1}{N}\xi$. Take $\theta = ND + \mathrm{Pic}_p(X)$, then $j(\theta) = p$. $\square$

Corollary 2.8.1. $J : X^g \to \mathrm{Jac}(X)$ is surjective.

Proof: It suffices to show $\psi : X^g \to \mathrm{Pic}_0(X)$ is surjective. Let $D \in \mathrm{Pic}_0(X)$, let $D' = D + \sum\limits_{i=1}^{g} a_i \rightsquigarrow \deg D' = g$ and thus $\dim H^0(X, \mathcal{O}_{D'}) \geq 1$. Then $\exists f \in \mathscr{M}(X)$ s.t. $(f) + D' \geq 0$. Also $\deg((f) + D') = g$, say $(f) + D' = \sum\limits_{i=1}^{n} x_i$. Then

$$D + (f) = \sum_{i=1}^{n}(x_i - a_i) \implies \psi(x_1, ..., x_g) = D + \mathrm{Pic}_p(X)$$

$\square$

Chapter 3

Non-compact Riemann surface

3.1 Countable topology

Theorem 3.1.1 (Radó). Every Riemann surface has a countable topology.

Proof: Let $(U, z) \subset X$ and K_0, K_1 be two compact disks in U with $K_0 \cap K_1 = \varnothing$. Set $Y = X \setminus (K_0 \cup K_1)$. Then via Dirichlet boundary value problem for K_0, K_1,

$$\exists u : \overline{Y} \to \mathbb{R} \text{ s.t. } \begin{cases} u|_Y \text{ is harmonic} \\ u|_{\partial K_0} = 0, \ u|_{\partial K_1} = 1 \end{cases}$$

Then $\omega := du$ is a nontrivial holomorphic 1-form on Y $(d''\omega = d''d'u + d''d''u = 0)$. Consider $p : \widetilde{Y} \xrightarrow{\simeq} Y$. Then $\exists f \in \mathcal{O}(\widetilde{Y})$ s.t. $df = p^*\omega$. Then $f : \widetilde{Y} \to \mathbb{C}$ is non-constant. Now we need the following two lemmas.

Lemma 3.1.1 (Poincaré-Volterra). If $f : Z' \to Z$ is discrete (i.e. fiber of singleton is discrete). If Z is Hausdorff with countable topology, Z' is connected manifold. Then Z' has a countable topology.

Proof: Let $\mathfrak{U}$ be a countable basis for the topology on Z. Set

$$\mathfrak{B} = \{V \underset{\text{open}}{\subset} Z' : V \text{ is a connected component of } f^{-1}U \text{ with } U \in \mathfrak{U} \text{ with } V \text{ has countable topology}\}$$

- $\mathfrak{B}$ is a basis for the topology on Z' and $\mathfrak{B}$ is non-empty :

  Let $x \in D \underset{\text{open}}{\subset} Z'$. Since $f^{-1}(f(x))$ is discrete, $\exists x \in W \Subset D$ s.t. $\partial W \cap f^{-1}(f(x)) = \varnothing$. Also $f(\partial W)$ is compact on Z with $f(x) \notin f(\partial W)$, $\exists U \in \mathfrak{U}$ s.t. $f(x) \in U$ and $U \cap f(\partial W) = \varnothing$. Let V be the connected component of $f^{-1}(U)$ containing x. Then $V \cap \partial W = \varnothing \rightsquigarrow V \subset W$. Since $\overline{W}$ is compact, W has a countable topology, so does V and $V \in \mathfrak{B}$, $V \subset D$.

- $\mathfrak{B}$ is countable :

  Fixed $V^* \in \mathfrak{B}$ and $\forall n \in \mathbb{N}$, define

  $$\beta_n = \{V \in \mathfrak{B} : \exists V^* = V_0, V_1, ..., V_n = V \text{ distinct in } \mathfrak{B} \text{ s.t. } V_{k-1} \cap V_k = \varnothing \ \forall k\}$$

  Claim : $\mathfrak{B} = \bigcup\limits_{n \in \mathbb{N}} \beta_n$.

  subproof : Fix $x_0 \in V^*$, for $V \in \mathfrak{B}$ and $x \in V$. By connectedness, let $c : x_0 \rightsquigarrow x$ which is compact and thus can cover by finite element in $\mathfrak{B}$ $\square$

Claim : $\forall V' \in \mathfrak{B}$, exists at most countable many $V \in \mathfrak{B}$ with $V' \cap V \neq \varnothing$.

subproof : $\forall U \in \mathfrak{U}$, the connected component of $f^{-1}(U)$ are disjoint. Since V' has countable topology, exists at most countably many V which are connected components of $f^{-1}(U)$ with $V' \cap V \neq \varnothing$. Then the claim holds by $\mathfrak{U}$ is countable. □

Now we can prove by induction that β_n are countable and thus $\mathfrak{B}$ is countable.

□

Lemma 3.1.2. If $f : Z' \to Z$ is continuous open surjective, then Z' has a countable topology implies Z has a countable topology.

Proof: Let $\mathfrak{U}$ be a countable basis for the topology on Z', then $\mathfrak{B} = \{f(U) : U \in \mathfrak{U}\}$ is a countable topology on Z. Indeed, for $x \in D \underset{\text{open}}{\subset} Z$, let $y \in f^{-1}(x)$, $\exists U \in \mathfrak{U}$ s.t. $y \in U \subset f^{-1}(D)$, then $x \in f(U) \subset D$. □

Now back to the proof of Radó. By identity theorem, f is discrete. By Lemma 3.1.1, $\widetilde{Y}$ has a countable topology. By Lemma 3.1.2, Y has a countable topology. Since $K_0 \cup K_1$ is compact, $K_0 \cup K_1$ has countable topology, and so does X. □

3.2 Weyl's lemma

Definition 3.2.1.

- $X \underset{\text{open}}{\subset} \mathbb{C}$, define

$$\mathscr{D}(X) = \{f \in \mathscr{E}(X) : f \text{ has compact support}\}$$

- For $(f_n)_{n\in\mathbb{N}} \subset \mathscr{D}$, $f_n \underset{\mathscr{D}}{\to} f$ if
  - $\exists K \underset{\text{cpt.}}{\subset} X$ s.t. $\text{supp}(f_n) \subset K$ $\forall n$ and $\text{supp}(f) \subset K$
  - $\forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$, define $D^\alpha = \dfrac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$. Then

$$D^\alpha f_n \to D^\alpha f \text{ uniformly on } K$$

  - A **distribution** on X is a continuous linear mapping

$$\begin{array}{rccc} T: & \mathscr{D}(X) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & T[f] \end{array}$$

    By continuous, $f_n \underset{\mathscr{D}}{\to} f \implies T[f_n] \to T[f]$.
  - $\mathscr{D}'(X) = \{\text{all distribution on } X\}$

Remark 3.2.1.

- $\forall h \in C(X)$, we can define $T_h \in \mathscr{D}'(X) : \forall f \in \mathscr{D}(X)$,

$$T_h[f] = \iint_X h(z)f(z)dxdy \text{ with } z = x + iy$$

- $C(X) \hookrightarrow \mathscr{D}'(X)$ by $h \mapsto T_h$: If $h \neq 0$ and satisfy

$$\iint_X h(z)f(z)dxdy = 0 \ \forall f \in \mathscr{D}(X) \text{ then } h(z) = 0$$

Say $h(a) \neq 0$ and $\exists a \in U \subset X$ s.t. $\operatorname{Re} h(z) > 0 \ \forall z \in U$. Let $\varphi \in \mathscr{D}(X)$ with $\operatorname{supp} \varphi \subset U$. Let $\varphi \in \mathscr{D}(X)$ with $\operatorname{supp} \varphi \subset U$ and $\varphi|_A = 1$, $\varphi \geq 0$ on U, then

$$\iint_X h(z)\varphi(z)dxdy > 0 \ (\rightarrow\!\leftarrow)$$

For $h \in \mathscr{E}(X)$, $f \in \mathscr{D}(X)$,

$$d(hfdy) = d(hf) \wedge dy = \frac{\partial h}{\partial x} f dx \wedge dy + h \frac{\partial f}{\partial x} dx \wedge dy$$

Then we have the integration by part,

$$0 = \iint_X d(hfdy) \implies \iint_X h \frac{\partial f}{\partial x} dxdy = -\iint_X f \frac{\partial h}{\partial x} dxdy$$

Apply more times, we have

$$\iint_X h(z) D^\alpha f(z) dxdy = (-1)^{\alpha_1+\alpha_2} \iint_X f(x) D^\alpha h(z) dxdy$$

By imitating the above conclusion, we can define the differential of $T \in \mathscr{D}'$.

Definition 3.2.2. For $T \in \mathscr{D}'(X)$

$$(D^\alpha T)[f] = (-1)^{|\alpha|} T[D^\alpha f] \ \forall f \in \mathscr{D}(X)$$

Now we choose $\rho \in \mathscr{D}(X)$ s.t.

- $\operatorname{supp}(\rho) \subset B_1(0)$
- $\rho(z) = \rho(|z|) \ \forall z \in \mathbb{C}$
- $\iint_{\mathbb{C}} \rho(z)dxdy = 1$

For $\varepsilon > 0$, construct the approximation to identity $\rho_\varepsilon(z) = \dfrac{1}{\varepsilon^2}\rho(z/\varepsilon)$, then

- $\operatorname{supp}(\rho_\varepsilon) \subset B_\varepsilon(0)$
- $\iint_X \rho_\varepsilon(z)dxdy = \iint_{\mathbb{C}} \rho_\varepsilon(z/\varepsilon)d(d/\varepsilon)d(y/\varepsilon) = \iint_{\mathbb{C}} \rho(z)dxdy = 1$

Define

$$X^{(\varepsilon)} := \{z \in X | \overline{B_\varepsilon(z)} \subset X\} \underset{\text{open}}{\subset} X$$

For $f \in C(X)$ and $z \in X^{(\varepsilon)}$, we can define

$$(\operatorname{sm}_\varepsilon f)(z) := \iint_X \rho_\varepsilon(z-\zeta) f(\zeta) d\xi d\eta$$

- $\operatorname{sm}_\varepsilon f \in \mathscr{E}(X)$:
$$\frac{d}{dz}\operatorname{sm}_\varepsilon f = \iint_X \frac{d\rho_\varepsilon(z-\zeta)}{dz} f(\zeta)d\xi d\eta$$

- $f \in \mathscr{E}(X)$, $\alpha \in \mathbb{Z}^2_{\geq 0}$, $D^\alpha(\operatorname{sm}_\varepsilon f) = \operatorname{sm}_\varepsilon(D^\alpha f)$:
subproof : For $z \in X^{(\varepsilon)}$,
$$(\operatorname{sm}_\varepsilon f)(z) = \iint_{B_\varepsilon(z)} \rho_\varepsilon(\zeta - z)f(\zeta)d\xi d\eta = \iint_{|w|<\varepsilon} \rho_\varepsilon(w)f(z+w)dudv$$
$$D^\alpha(\operatorname{sm}_\varepsilon f)(z) = \iint_{|w|<\varepsilon} \rho_\varepsilon(w)D^\alpha f(z+w)dudv = \iint_{|\zeta-z|<\varepsilon} \rho_\varepsilon(\zeta-z)D^\alpha(\zeta)d\xi d\eta = \operatorname{sm}_\varepsilon(D^\alpha f)$$

- For $z \in X^{(\varepsilon)}$, f : harmonic on $B_\varepsilon(z)$, then $(\operatorname{sm}_\varepsilon f)(z) = f(z)$:
subproof : For $0 \leq r < \varepsilon$, by mean value property,
$$f(z) = \frac{1}{2\pi}\int_0^r f(z+re^{i\theta})d\theta$$
Then
$$\begin{aligned}(\operatorname{sm}_\varepsilon f)(z) &= \iint_{|w|<\varepsilon} \rho_\varepsilon(w)f(z+w)dudv = \int_0^\varepsilon \int_0^{2\pi} \rho_\varepsilon(w)f(z+re^{i\theta})rd\theta dr \\ &= \int_0^\varepsilon r\rho_\varepsilon(r)\int_0^{2\pi} f(z+re^{i\theta})d\theta dr = 2\pi\int_0^\varepsilon r\rho_\varepsilon(r)f(z)dr \\ &= f(z)\int_0^{2\pi}\int_0^\varepsilon \rho_\varepsilon(r)rdrd\theta = f(z)\iint_{B_\varepsilon(0)} \rho_\varepsilon(\zeta)d\xi d\eta = f(z)\end{aligned}$$

Theorem 3.2.1 (Weyl's lemma). If $T \in \mathscr{D}'(X)$ and $\Delta T = 0$, then T is a smooth function.

Proof:

- For $\varepsilon > 0$ and $z \in X^{(\varepsilon)}$, $\operatorname{supp}(\rho_\varepsilon(\zeta - z)) \subseteq B_\varepsilon(z) \subset X \rightsquigarrow \rho_\varepsilon(\zeta - z) \in \mathscr{D}(X)$. Set
$$h(z) = T[\rho_\varepsilon(\zeta - z)]$$
where T is a distribution with ζ as a variable. For convenience, say $\rho_\varepsilon(\zeta - z) = g(\zeta, z)$.

- **Claim** : $h(z) \in \mathscr{E}(X^{(\varepsilon)})$:
subproof :
$$\begin{aligned}\frac{\partial h}{\partial x}(z) &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\Big(T[g(\zeta, x+\Delta x+iy)] - T[g(\zeta, x+iy)]\Big) \\ &= \lim_{\Delta x \to 0} T\underbrace{\left[\frac{g(\zeta, x+\Delta x+iy) - g(\zeta, x+iy)}{\Delta x}\right]}_{:=f_{\Delta x}(\zeta)}\end{aligned}$$
Then $f_{\Delta x}(\zeta) \in \mathscr{D}(X)$ and $f_{\Delta x} \xrightarrow[\mathscr{D}]{} \frac{\partial g}{\partial x}(\zeta, x+iy)$. Since T is continuous,
$$\lim_{\Delta x \to 0} T[f_{\Delta x}(\zeta)] = T\left[\frac{\partial g(\zeta, x+iy)}{\partial x}\right]$$
Hence $h \in \mathscr{E}(X^{(\varepsilon)})$.

- $T[\operatorname{sm}_\varepsilon f] = \iint_{X^{(\varepsilon)}} h(z)f(z)dxdy$: By definition, it suffices to show

$$T\left[\iint_X \rho_\varepsilon(\zeta - z)f(z)dxdy\right] = \iint_{X^{(\varepsilon)}} T[\rho_\varepsilon(\zeta - z)]f(z)dxdy$$

Let

$$G_n(\zeta) = \frac{\text{Area}(R)}{n^2}\sum_{i=1}^{n^2} \rho_\varepsilon(\zeta - z_i)f(z_i)$$

be the Riemann sum of $\iint_X \rho_\varepsilon(\zeta - z)f(z)dxdy$, where $\operatorname{supp}(\rho_\varepsilon(\zeta - z)f(z))$ contain in a rectangle R and we partition R into n^2 sub-rectangle with $z_i \in R_i$. Then

$$G_n(\zeta) \to \iint_X \rho_\varepsilon(\zeta - z)f(z)dxdy \text{ as } n \to \infty$$

Since T is continuous,

$$T[G_n(\xi)] \to T\left[\iint_X \rho_\varepsilon(\zeta - z)f(z)dxdy\right]$$

On the other hand,

$$TG_n(\zeta) = \frac{\text{Area}(R)}{n^2}\sum_{i=1}^{n^2} T[\rho_\varepsilon(\zeta - z)]f(z_i) \to \iint_{X^{(\varepsilon)}} T[\rho_\varepsilon(\zeta - z)]f(z)dxdy$$

where $\varepsilon < \operatorname{dist}(\operatorname{supp} f, \partial X)$.

- $T[f] = T[\operatorname{sm}_\varepsilon f]$: By Dolbeault's lemma two times, $\exists \psi \in \mathscr{E}(\mathbb{C})$ s.t. $\Delta\psi = f$. Then ψ is harmonic on $V = \mathbb{C} \setminus \operatorname{supp} f$. By above, $\psi = \operatorname{sm}_\varepsilon \psi$ on $V^{(\varepsilon)} \rightsquigarrow \varphi = \psi - \operatorname{sm}_\varepsilon \psi \in \mathscr{D}(X)$. Hence

$$0 = \Delta T(\varepsilon) = T(\Delta\varepsilon) = T[\Delta\psi - \Delta \operatorname{sm}_\varepsilon \psi]$$

and thus

$$T[f] = T[\Delta\psi] = T[\Delta \operatorname{sm}_\varepsilon \psi] = T[\operatorname{sm}_\varepsilon(\Delta\psi)] = T[\operatorname{sm}_\varepsilon f]$$

- Finally,

$$T[f] = \iint_{X^{(\varepsilon)}} h(z)f(z)dxdy \to \iint_X h(z)f(z)dxdy$$

and thus $T = T_h$.

□

Corollary 3.2.1. If $\dfrac{\partial T}{\partial \overline{z}} = 0$, then T is holomorphic function on X.

Proof: Since $\Delta T = 4\dfrac{\partial}{\partial z}\dfrac{\partial T}{\partial \overline{z}} = 0$, $T \in \mathscr{E}(X)$. Combine with $\dfrac{\partial T}{\partial z} = 0$, $T \in \mathcal{O}(X)$. □

3.3 Runge approximation

In this section, we need assume some knowledge in functional analysis to proof our main theorem.

3.3.1 Topological vector space

In this subsection, all vector space is over $\mathbb{C}$.

Definition 3.3.1. A **topological vector space** is a vector space E with a topology s.t. addition and scalar multiplication are continuous map.

Remark 3.3.1. For $a \in E$, the translation $E \to E$ is a homeomorphism. Thus the topology of E can be determined by the neighborhood basis of zero. If $\mathfrak{B}$ is a neighborhood basis of zero, then $a + \mathfrak{B}$ form a neighborhood basis of a.

Definition 3.3.2. The **semi-norm** on a vector space is a mapping $p : E \to \mathbb{R}$ s.t.

- $p(x + y) \leq p(x) + p(y)\ \forall x, y \in E$
- $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{C}$, $x \in E$

By those condition we have $p(x) \geq 0\ \forall x \in E$. p is called a **norm** if $p(x) = 0 \iff x = 0$.

A family of semi-norm $\{p_i\}_{i\in I}$ on a vector space E induces a topology on E. Define the open neighborhood basis of zero by the set of form

$$U(p_{i_1}, ..., p_{i_m}, \varepsilon) := B(p_{i_1}, \varepsilon) \cap \cdots \cap B(p_{i_m}) \text{ where } B(p_i, \varepsilon) = \{x \in E : p_i(x) < \varepsilon\}$$

with $m \in \mathbb{N}$, $i_1, ..., i_m \in I$. Note that this topology is Hausdorff if

$$p_i(x) = 0\ \forall i \in I \implies x = 0$$

Definition 3.3.3. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in a topological vector space is called a **Cauchy sequence** if for every neighborhood U of zero there exists an $n_0 \in \mathbb{N}$ s.t.

$$x_n - x_m \in U \text{ for every } n, m \geq n_0$$

A topological vector space E is called a **Fréchet space** if the following hold :

- The topological of E is Hausdorff and can be defined by a countable family of semi-norms.
- E is complete i.e. every Cauchy sequence in E is convergent.

Remark 3.3.2. A Fr echet space E is metrizable. Suppose $p_n \in \mathbb{Z}_{\geq 0}$ is a family of semi-norms which defines the topology on E. If for $x, y \in E$ one sets

$$d(x, y) := \sum_{n=0}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}$$

Then $d : E \times E \to \mathbb{R}$ is a metric on E which induces the same topology as the semi-norm p_n, $n \in \mathbb{Z}_{\geq 0}$.

Definition 3.3.4. A **Banach space** is a complete normed vector space.

Example 3.3.1.

- $C_{\mathbb{R}}([0, 1])$ is a Banach space with

$$\|f\|_0 = \sup_{x\in[0,1]} |f(x)|$$

- $C^k_{\mathbb{R}}([0,1])$ is a Banach space with

$$\|f\| = \|f\|_0 + \cdots + \|f\|_k \text{ where } \|f\|_k = \sup_{x\in[0,1]} |f^{(k)}(x)|$$

Theorem 3.3.1 (Theorem of Banach)**.** Suppose E and F are Fréchet spaces and $f : E \to F$ is a continuous linear surjective mapping. Then f is open.

Corollary 3.3.1. Suppose E and F are Banach spaces and $f : E \to F$ is a continuous linear surjective mapping. Then $\exists C > 0$ s.t. $\forall y \in F$, $\exists x \in f^{-1}(y)$ s.t.

$$\|x\| \leq C\|y\|$$

Proof: Already proof in the proof of Lemma 2.2.1. □

Theorem 3.3.2 (Hahn-Banach)**.** Suppose E is a locally convex topological vector space, $E_0 \subset E$ is a vector subspace and $\varphi_0 : E_0 \to \mathbb{C}$ is a continuous linear functional. Then there exists a continuous linear functional $\varphi : E \to \mathbb{C}$ such that $\varphi|_{E_0} = \varphi_0$.

Corollary 3.3.2. Suppose E is a locally convex topological vector space and $A \subset B \subset E$ are vector subspaces. If every continuous linear functional $\varphi : E \to \mathbb{C}$ s.t. $\varphi|_A = 0$ satisfies $\varphi|_B = 0$, then A is dense in B.

Proof: Suppose A is not dense in B. Then there exists $b_0 \in B$ s.t. $b_0 \notin \overline{A}$. Let $E_0 := \overline{A} \oplus \mathbb{C}b_0$ and define a linear functional

$$\begin{array}{rccc} \varphi_0 : & E_0 & \longrightarrow & \mathbb{C} \\ & a + \lambda b_0 & \longmapsto & \lambda \end{array} \quad \text{for } a \in \overline{A},\ \lambda \in \mathbb{C}$$

It is clear that φ_0 is continuous. By Hahn-Banach theorem, φ_0 can extends to $\varphi : E \to \mathbb{C}$. Then $\varphi|_A = 0$, but $\varphi|_B \not\equiv 0$ ($\rightarrow\!\leftarrow$). □

Definition 3.3.5. A linear mapping $\psi : E \to F$ between two topological space vector E and F is called **compact** or **completely continuous**, if there exists a neighborhood U of zero in E s.t. $\psi(U)$ is relatively compact in F. In particular, a compact linear mapping is continuous.

Theorem 3.3.3 (Schwartz)**.** Suppose E and F are Fréhet spaces and φ, $\psi : E \to F$ are continuous linear mapping s.t. φ is surjective and ψ is compact. Then the image of the mapping $\varphi - \psi : E \to F$ has finite codimension in F.

3.3.2 Runge approximation

Given a Riemann surface X and $Y \underset{\text{open}}{\subset} X$, we intend to a Fréchet space structure on $\mathscr{E}(Y)$. Since Y has a countable topology, we may choose countable family of compact subset $K_j \subset Y$, $j \in J$, with $\bigcup K_j^\circ = Y$ and each K_j is contained in some chart (U_j, z_j). For $j \in J$ and $\alpha \in \mathbb{Z}^2_{\geq 0}$, define a semi-norm $p_{j\alpha} : \mathscr{E}(Y) \to \mathbb{R}$ by

$$p_{j\alpha} := \sup_{a\in K_j} |D^\alpha_j f(a)| \text{ where } D^\alpha_j = \left(\frac{\partial}{\partial x_j}\right)^{\alpha_1} \left(\frac{\partial}{\partial y_j}\right)^{\alpha_2} \text{ and } z_j = x_j + iy_j$$

These countable many semi-norm $p_{j\alpha}$ define a topology on $\mathscr{E}(Y)$. Then the convergence $f_n \to f$ w.r.t. this topology means uniform convergence of the functions and all of their derivatives on every K_j. This show that $\mathscr{E}(Y)$ is a Fréchet space. One can check that this topology is independent of the choice of K_j and (U_j, z_j). Similarly, we can define the structure of a Fréchet space on $\mathscr{E}^{0,1}(Y)$. For $\omega \in \mathscr{E}^{0,1}(Y)$, say $\omega = f_j d\overline{z_j}$, where $f_j \in \mathscr{E}(U_i \cap Y)$. Define semi-norm by

$$p_{j\alpha}(\omega) := \sup_{a \in K_j} |D_j^\alpha f_j(a)|$$

Then $\mathscr{E}^{0,1}(Y)$ is a Fréchet space define by semi-norm $\{p_{j\alpha} : j \in J, \alpha \in \mathbb{Z}^2_{\geq 0}\}$.

Lemma 3.3.1. Suppose Y is an open subset of a Riemann surface X. Then every continuous linear map $T : \mathscr{E}(Y) \to \mathbb{C}$ has compact support, i.e. $\exists K \underset{\text{cpt.}}{\subset} Y$ s.t.

$$T[f] = 0 \ \forall f \in \mathscr{E}(Y) \text{ with } \operatorname{supp}(f) \subset Y \setminus K$$

An analogous result is also true for $\mathscr{E}^{0,1}(Y)$.

Proof: Since T is continuous, $\exists j_1, ..., j_m \in J$, $\alpha_1, ..., \alpha_m \in \mathbb{Z}^2_{\geq 0}$ and $\varepsilon > 0$ s.t

$$|T[f]| < 1 \ \forall f \in B(p_{j_1\alpha_1}, \varepsilon) \cap \cdots \cap B(p_{j_m\alpha_m}, \varepsilon)$$

Let $K = K_{j_1} \cup \cdots \cup K_{j_m}$. Given $f \in \mathscr{E}(Y)$ with $\operatorname{supp}(f) \subset Y \setminus K$, then $f \equiv 0$ on K and thus $nf \equiv 0$ on K. Thus

$$nf \in B(p_{j_1\alpha_1}, \varepsilon) \cap \cdots \cap B(p_{j_m\alpha_m}, \varepsilon) \implies |T[nf]| < 1 \ \forall n \implies T[f] = 0$$

□

Lemma 3.3.2. If $S : \mathscr{E}^{0,1}(X) \to \mathbb{C}$ is a continuous linear map s.t. $S[d''g] = 0 \ \forall g \in \mathscr{E}(X)$ with $\operatorname{supp}(g) \Subset Y$. Then $\exists \sigma \in \Omega(X)$ s.t.

$$S[\omega] = \int_Y \sigma \wedge \omega \ \forall \omega \in \mathscr{E}^{0,1}(Y) \text{ with } \operatorname{supp}(\omega) \subset Y$$

Proof:

- Let $(U, z) \subset Y$. For $\varphi \in \mathscr{D}(U)$, define

$$\widetilde{\varphi} := \begin{cases} \varphi d\overline{z} & \text{on } U \\ 0 & \text{on } X \setminus U \end{cases}$$

  Then $\overline{\varphi} \in \mathscr{E}^{0,1}(X)$. Define a distribution on U by

$$\begin{array}{rccc} S_U : & \mathscr{D}(U) & \longrightarrow & \mathbb{C} \\ & \varphi & \longmapsto & S[\widetilde{\varphi}] \end{array}$$

  By assumption,

$$0 = S[d''g] = S\left[\frac{\partial g}{\partial \overline{z}} dz\right] = S_U\left(\frac{\partial g}{\partial \overline{z}}\right) = \frac{\partial S_U}{\partial \overline{z}}(g) \ \forall g \in \mathscr{D}(U)$$

  Hence $\dfrac{\partial S_U}{\partial \overline{z}} = 0 \rightsquigarrow \Delta S_U = 0$. By Weyl's lemma, $\exists h \in \mathcal{O}(U)$ s.t.

$$S[\widetilde{\varphi}] = S_U[\varphi] = \iint_U h(z)\varphi(z) dz \wedge d\overline{z} \ \forall \varphi \in \mathscr{D}(U)$$

  Setting $hdz = \varphi_U$, we get

$$S[\omega] = \iint_U \sigma_U \wedge \omega \ \forall \omega \in \mathscr{E}^{0,1}(U) \text{ with } \operatorname{supp}(\omega) \Subset U$$

- For another $(U', z') \subset Y$, $\exists \sigma_{U'} \in \Omega(U')$. Since
$$\iint_U \sigma_U \wedge \omega = \iint_{U'} \sigma_{U'} \wedge \omega \ \forall \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset U \cap U'$$
Then $\sigma_U = \sigma_{U'}$ on $U \cap U'$. We can glue $\{\sigma_U\}$ to get $\sigma \in \Omega(Y)$ s.t. $\sigma|_U = \sigma_U$. $\forall \omega \in \mathscr{E}^{0,1}(X)$ with $\operatorname{supp}(\omega) \Subset Y$. By partition of unity, $\omega = \omega_1 + \cdots + \omega_n$ for $\operatorname{supp}(\omega_j) \Subset U_j$. Hence
$$S[\omega] = \sum_{j=1}^{n} S[\omega_j] = \sum_{j=1}^{n} \iint_Y \sigma \wedge \omega_j = \iint_Y \sigma \wedge \omega$$

$\square$

Definition 3.3.6. Let X be a Riemann surface.

- If Y is a subset of X, define
$$h_X(Y) := Y \cup \left(\bigcup D\right)$$
where the union is run thought all compact connected component D of (X/Y).

- $Y \underset{\text{open}}{\subset} X$ is called **Runge** if $h_X(Y) = Y$.

Proposition 3.3.1. Let X is non-compact Riemann surface and $Y \Subset X$ is Runge. Then $\forall Y'$ with $Y' \subset Y' \Subset X$, the image of restriction map $\beta : \mathcal{O}(Y') \to \mathcal{O}(Y)$ is dense, where the topology is uniform convergence on each compact subset.

Proof: By Corollary 3.3.2, reduction to the following statement :
$$\text{“}\forall \text{ continuous map } T : \mathscr{E}(X) \to \mathbb{C} \text{ with } T|_{\beta(\mathcal{O}(Y'))} \implies T|_{\mathcal{O}(Y)} = 0\text{”}$$

- By general Dolbeault lemma on $Y \Subset Y \Subset X$, $\forall \omega \in \mathscr{E}^{0,1}(X)$, $\exists f \in \mathscr{E}(Y')$ s.t. $\omega|_{Y'} = d''f$. Define
$$\begin{array}{rccc} S : & \mathscr{E}^{0,1}(X) & \longrightarrow & \mathbb{C} \\ & \omega & \longrightarrow & T[f|_Y] \end{array}$$
which is independent of the choice of f. Indeed, if $d''f = d''g$ on $Y' \rightsquigarrow f - g \in \mathcal{O}(Y')$ and thus
$$S[(f-g)|_Y] = S[\beta(f-g)] = 0 \implies S[f|_Y] = S[g|_Y]$$
Now we show that S is continuous. Consider the vector space
$$V = \{(\omega, f) \in \mathscr{E}^{0.1}(X) \times \mathscr{E}(Y') : d''f = \omega|_{Y'}\}$$
Since $d'' : \mathscr{E}(Y') \to \mathscr{E}^{0,1}(Y')$ is continuous, V is closed subspace of $\mathscr{E}^{0,1}(X) \times \mathscr{E}(Y')$ and thus is Fréchet space. By construction, the following diagram commute.
$$\begin{tikzcd} V \arrow[r, "\beta\circ\pi_2"] \arrow[d, "\pi_1"'] & \mathscr{E}(Y) \arrow[d, "T"] \\ \mathscr{E}^{0,1}(X) \arrow[r, "S"] & \mathbb{C} \end{tikzcd}$$
where π_1, π_2 be the projection map. Since π_1 is surjective and by Theorem of Banach, it is open. Then S is continuous, since $\beta \circ \pi_2$ and T are continuous.

- By Lemma 3.3.1, $\exists K \underset{\text{cpt.}}{\subset} Y$ and $L \underset{\text{cpt.}}{\subset} X$ s.t.

$$T[f] = 0 \forall f \in \mathscr{E}(Y) \text{ with } \operatorname{supp}(f) \subset Y \setminus K \tag{1}$$
$$S[\omega] = 0 \forall \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \subset X \setminus L \tag{2}$$

Claim : $S[\omega] = 0\ \forall \omega \in \mathscr{E}^{0,1}(X)$ with $\operatorname{supp}(\omega) \Subset X \setminus h(K)$.

subproof : If $f \in \mathscr{E}(X)$ with $\operatorname{supp}(f) \Subset X \setminus K$, then $S[d''f] = T[f|_Y] = 0$ by (1). By Lemma 3.3.2, $\exists \sigma \in \Omega(X \setminus K)$ s.t.

$$S[\omega] = \iint_{X \setminus K} \sigma \wedge \omega\ \forall \omega \in \mathscr{E}^{0,1}(X) \text{ with } \operatorname{supp}(\omega) \Subset X \setminus K$$

By (2), $\sigma|_{X \setminus (K \cup L)} = 0$. For any connected component U of $X \setminus h(K)$ is not relative compact, $U \cap (X \setminus (K \cup L)) \neq \varnothing$. Otherwise, $U \subset K \cup L$, but $K \cup L$ is compact ($\rightarrow\!\leftarrow$). Thus by identity theorem, $\sigma|_U = 0$ and thus $\sigma|_{X \setminus h(K)} = 0$. □

- Now suppose $f \in \mathcal{O}(Y)$. Since Y is Runge, $h(K) \subset h(Y) = Y$. Hence $\exists g \in \mathscr{E}(X)$ with $f = g$ in a neighborhood of $h(K)$ and $\operatorname{supp}(g) \Subset Y$. Since $\operatorname{supp}(f - g) \subset Y \setminus K$, by (1) we have

$$T[f] = T[g|_Y] = S[d''g] = 0$$

the last equality is by (2), since $d''g = d''f = 0 \rightsquigarrow \operatorname{supp}(d''g) \Subset X \setminus h(K)$.

□

3.3.3 Runge approximation

Observation :

- If K is closed, then $h(K)$ is closed : Let C_j, $j \in J$ be the connected component of $X \setminus K \rightsquigarrow C_j$ are open. Let $J_0 = \{j \in J : c_j \Subset \overline{X \setminus K}\} \rightsquigarrow X \setminus h(K) = \bigcup_{j \in J \setminus J_0} C_j$ is open $\rightsquigarrow h(K)$ is closed.

- If K is compact, then $h(K)$ is compact : Let C_j be as above and $K \subset U \Subset X$.
  - $C_j \cap \overline{U} = \varnothing\ \forall j$: If $\exists C_\ell \subset X \setminus \overline{U}$, then $C_\ell \subset \overline{C_\ell} \subset X \setminus U \subset X \setminus K \rightsquigarrow C_\ell = \overline{C_\ell}$. Then C_ℓ is clopen, which contradict to X is connected.
  - Only finitely many C_j meet ∂U, since ∂U is compact and $\{C_j\}$ disjoint.
  - Let $\{C_{j_1}, .., C_{j_n}\} \subset \{C_j : j \in J_0\}$ s.t. $C_{j_t} \cap \partial U \neq \varnothing$. Then

$$h(K) = K \cup \left(\bigcup_{j \in J_0} C_j \right) \subset U \cup \left(\bigcup_{i=1}^{m} C_{j_i} \right)$$

  is relative compact. By above, $h(K)$ is closed and thus compact.

- Since X has countable topology, $\{K_n'\}_{n=0}^\infty$ be the ascending chain of compact subset s.t. $\bigcup_{i=0}^\infty K_i' = X$. Now we define K_n by induction as follows. Let $K_0 = h(K_0')$. Now suppose we have $K_0, ..., K_n$. Choose a compact set M s.t. $K_n \cup K_n' \subset M^\circ$ and let $K_{n+1} = h(M)$. Then $\{K_n\}_{n=0}^\infty$ satisfy

$$(1)\ K_n = h(K_n)\ (2)\ K_n \nearrow X\ (3)\ K_n \subset K_{n+1}^\circ$$

Theorem 3.3.4 (Existence of special exhaustion)**.** Let X be a non-compact Riemann surface. Then $\exists Y_0 \Subset Y_1 \Subset \cdots$ with Y_i are Runge domain s.t. $X = \bigcup_{i=0}^{\infty} Y_i$ and each Y_i has a regular boundary, i.e. can solve Dirichlet problem.

Proof: It suffices to show $\forall K \underset{\text{cpt.}}{\subset} X$, exists Runge domain $Y \Subset X$ s.t. $K \subset Y$ and Y has regular boundary.

- Choose a connected compact set $K_1 \supset K$ and a compact set K_2 with $K_1 \subset K_2^\circ$. Let $K_2' = h(K_2)$. $\forall x \in \partial K_2'$, choose the chart (U, z) near x s.t. $U \cap K_1 = \varnothing$ and $D_x = \overline{B_{r_x}(x)} \subset U$. Since $\partial K_2'$ is compact, $\exists D_{x_1}, ..., D_{x_k}$ cover $\partial K_2'$. Set
$$Y_1 = K_2' \setminus (D_{x_1} \cup \cdots \cup D_{x_k})$$
which is open and has regular boundary, $K_1 \subset Y_1 \subset K_2'$. Let C_j, $j \in J$ be the connected component of $X \setminus K_2'$. Since $h(K_2') = K_2'$, $C_j \not\Subset X \setminus K_2'$. Each connected component C of $X \setminus Y_1$, $C \cap D_{x_j} \neq \varnothing$ for some j. Since C is connected component, $D_{x_j} \subset C$. Note that $D_{x_j} \cap C_i \neq \varnothing$ for some $i \rightsquigarrow C_i \cap C \neq \varnothing \rightsquigarrow C_i \subset C$. Then C is not relative compact. Hence $h(Y_1) = Y_1$.
- **Claim** : The connected component of Runge open set Y is also Runge.

  subproof : Suppose Y_i $(i \in I)$ are the connected components of Y, which are open. Let $A = X \setminus Y$ and A_k $(k \in K)$ are the connected components of A, which is closed but not compact by assumption. Then $\overline{Y_i} \cap A \neq \varnothing$ $\forall i \in I$. Otherwise $\overline{Y_i} \subset Y$. Then
$$\overline{Y_i} \cap \bigcup_{j \neq i} Y_j = \varnothing \implies \overline{Y_i} = Y_i$$
which contradict to X is connected. Next we claim that $C \cap A \neq \varnothing$ for every connected component C of $X \setminus Y_i$. Otherwise $\exists j \neq i$ s.t. $C \cap Y_j \neq \varnothing$. Since C is closed connected component of $X \setminus Y_i$ and Y_j is connected, $\overline{Y_j} \subset C$. As above, this show that $\overline{Y_j} = Y_j$ ($\rightarrow\!\leftarrow$). Finally, say $C \cap A_k \neq \varnothing$ for some $k \in K \rightsquigarrow C \supset A_k$. Since A_k is not compact, C is also not compact. Hence Y_i is Runge.
- Let Y be the connected component of Y_1 which contain K_1. Then by Claim, Y is the desired Runge domain.

□

Theorem 3.3.5 (Runge Approximation)**.** Suppose X is non-compact Riemann surface, Y is Runge open subset of X. Then $\forall f \in \mathcal{O}(Y)$, $\forall K \underset{\text{cpt.}}{\subset} Y$, $\forall \varepsilon > 0$, $\exists g \in \mathcal{O}(X)$ s.t. $\|g - f\|_K < \varepsilon$.

Proof: By replacing f by $f|_V$ with $K \subset V \Subset X$, we may assume that $Y \Subset X$. Let $Y = Y_0 \Subset Y_1 \Subset \cdots$ be a special exhaustion of X by Runge domain. By Proposition 3.3.1, $\exists f_1 \in \mathcal{O}(Y_1)$ s.t. $\|f_1 - f\|_K < \varepsilon/2$, $\overline{Y_{-1}} = K \subset Y_0$. By induction (note that $Y_{n-2} \Subset Y_{n-1}$.), we get $f_n \in \mathcal{O}(Y_n)$ s.t.
$$\|f_n - f_{n-1}\|_{\overline{Y_{n-2}}} \leq 2^{-n}\varepsilon \ \forall n \geq 2$$
Then $\forall n \in \mathbb{N}, (f_j)_{j \geq n} \to g_n \in \mathcal{O}(Y_n)$ uniformly on $\overline{Y_{n-1}}$ and thus $g_{n+1}|_{Y_n} = g_n$. Then $\exists g \in \mathcal{O}(X)$ s.t. $g|_{Y_n} = g_n$. Then $g|_k = g_{-1} = \lim_{n \to \infty} f_n$ and thus
$$\|g - f\|_K = \|(g - f_N) + (f_N - f_{N-1}) + \cdots + (f_1 - f)\| \leq \|g - f_N\| + \sum_{k=1}^{N} 2^{-k}\varepsilon \xrightarrow{N \to \infty} \varepsilon$$

□

We have followed promotion theorem by the same trick in the proof of Dolbeault lemma.

Theorem 3.3.6 (General form (II) for Dolbeault lemma)**.** Suppose X is non-compact Riemann surface. $\forall \omega \in \mathscr{E}^{0,1}(X)$, $\exists f \in \mathscr{E}(X)$ s.t. $d''f = \omega$.

Proof: Let $Y_0 \Subset Y_1 \Subset \cdots$ be a special exhaustion by Runge domains. By induction on n, we will construct $f_n \in \mathscr{E}(Y_n)$ s.t.

$$\begin{cases} d''f_n = \omega|_{Y_n} \\ \|f_{n+1} - f_n\|_{\overline{Y_{n-1}}} \le 2^{-n} \end{cases}$$

$n = 0$: By General Dolbeault lemma (I), $\exists f_0 \in \mathscr{E}(Y_0)$ s.t. $d''f_0 = \omega|_{Y_0}$. Suppose that $f_0, ..., f_n$ have been constructed. By (I) again, $\exists g \in \mathscr{E}(Y_{n+1})$ s.t. $d''g = \omega|_{Y_{n+1}}$. On Y_n,

$$d''g = d''f_n \implies g - f_n \in \mathcal{O}(Y_n)$$

By Runge approximation, $\exists h \in \mathcal{O}(Y_{n+1})$ s.t. $\|(g - f_n) - h\|_{\overline{Y_{n-1}}} \le 2^{-n}$. Set $f_{n+1} = g - h \in \mathscr{E}(Y_{n+1})$. Then

$$d''f_{n+1} = d''g = \omega|_{Y_{n+1}} \text{ and } \|f_{n+1} - f_n\|_{\overline{Y_{n-1}}} \le 2^{-n}$$

Define $f(x) = \lim_{n\to\infty} f_n(x)$. Then

$$f = \underbrace{f_n}_{\in\mathscr{E}(Y_n)} + \sum_{k=n}^{\infty} \underbrace{(f_{k+1} - f_k)}_{\in\mathcal{O}(Y_n)} \implies f \in \mathscr{E}(Y_n)\ \forall n \implies f \in \mathscr{E}(X)$$

and

$$d''f = d''f_n + \sum_{k=n}^{\infty} d''(f_{k+1} - f_k) = \omega|_{Y_n}\ \forall n \implies d''f = \omega$$

□

Problem 3.3.1. Suppose X is non-compact Riemann surface. The divisor D on X is the infinite formal sum s.t. $D|_K$ is finite sum for $K \underset{\text{cpt.}}{\subset} X$. The group of divisor is denoted by $\mathrm{Div}(X)$.

Theorem 3.3.7 (Weierstrass product theorem)**.** Suppose X is non-compact Riemann surface, $D \in \mathrm{Div}\, X \rightsquigarrow \exists f \in \mathscr{M}^*(X)$ s.t. $(f) = D$. Hence $\mathrm{Div}_p(X) = \mathrm{Div}(X)$.

Proof: Let $K_0 \subset K_1 \subset \cdots$ be a sequence of compact subset s.t.

$$(1)\ K_j = h(K_j)\ (2)\ K_{j-1} \subset K_j^\circ\ (3)\ K_j \nearrow X$$

- For $a_0 \in X \setminus K_{j_0}$, exists a weak solution f of a_0 with $f|_{K_{j_0}} = 1$:
  Since $K_{j_0} = h(K_{j_0})$, $\exists U$: connected component of $X \setminus K_{j_0}$ contain a with $U \not\Subset X \setminus K_{j_0}$. $\exists a_1 \in U \setminus K_{j_0+1}$ and a curve $c_0 : a_1 \rightsquigarrow a_0$ in U. By the proof in key lemma in Abel theorem, $\exists$ a weak solution f_0 of ∂c_0 with $f_0|_{X\setminus U} = 1 \implies f_0|_{K_{j_0}} = 1$. Similarly, for $a_\ell \in X \setminus K_{j_0+\ell}$, $c_\ell : a_{\ell+1} \rightsquigarrow a_\ell$, there exists a weak solution f_ℓ of ∂c_ℓ with $f_\ell|_{K_{j_0+\ell}} = 1$. Then $f_0 f_1 \cdots f_n$ is a weak solution of $\sum_{\ell=0}^{n} \partial c_\ell = a_0 - a_{n+1}$. Let $f = \prod_{\ell=0}^{\infty} f_\ell$, which is well-defined, since it is finite product on each compact subset. Then f is a weak solution of a.

- For $D \in \mathrm{Div}(X)$, $j \in \mathbb{Z}_{\geq 0}$, define

$$D_j(x) := \begin{cases} D(x) & \text{if } x \in K_j \setminus K_{j-1} \\ 0 & \text{otherwise} \end{cases} \quad \text{where } K_{-1} = \varnothing$$

  Since K_j is compact, $D_j(x) = \sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j$. Let f_i, f_j' be the weak solution of a_i, b_j in above. Then $g_j := \prod f_j / \prod f_j'$ is a weak solution of $D_j(x)$ s.t. $g_j|_{K_{j-1}} = 1$. Since $D(x) = \sum_{j=0}^{\infty} D_j(x)$, $g = \prod_{j=0}^{\infty} g_j$ is a weak solution of D.

- $\exists f \in \mathscr{M}^*(X)$ s.t. $(f) = D$:
  $\forall x \in X$, $\exists U_x \Subset X$ s.t. U_x homeomorphic to the unit disk in $\mathbb{C}$ and $D|_{U_x}$ is finite, say $D|_{U_x} = \sum_{i=1}^{n} r_i a_i$ with $a_i \in \mathbb{Z}$. Define $f_x(z) := \prod_{i=1}^{n} (z - a_i)^{r_i} \in \mathscr{M}^*(U_x) \rightsquigarrow (f_x) = D|_{U_x}$. Then

$$(f_x)|_{U_x \cap U_y} = D|_{U_x \cap U_y} = (f_y)|_{U_x \cap U_y} \implies f_x/f_y \in \mathcal{O}^*(U_x \cap U_y)$$

  Let $\psi \in \mathscr{E}(X_D)$ be a weak solution of $D \rightsquigarrow \psi|_{U_x} = f_x \psi_x$ for some $\psi_x \in \mathscr{E}(U_x)$ having no zeros. Since U_x is simply connected, $\exists \varphi_x \in \mathscr{E}(U_x)$ s.t. $\varphi_x = \log \psi_x$. Then

$$f_x e^{\varphi_x} = \psi = f_y e^{\varphi_y} \text{ on } U_x \cap U_y$$

$$\implies e^{\varphi_y - \varphi_c} = \frac{f_x}{f_y} \in \mathcal{O}^*(U_x \cap U_y) \rightsquigarrow \varphi_{xy} := \varphi_y - \varphi_x \in \mathcal{O}(U_x \cap U_y) \rightsquigarrow (\varphi_{xy}) \in Z^1(\mathfrak{U}, \mathcal{O}) = B^1(\mathfrak{U}, \mathcal{O})$$

  Then $\exists g_x \in \mathcal{O}(U_x)$ s.t. $\varphi_{xy} = g_y - g_x$ on $U_x \cap U_y$. Then

$$e^{g_y - g_x} = \frac{f_x}{f_y} \text{ on } U_x \cap U_y \implies f_x e^{g_x} = f_y e^{g_y} \text{ on } U_x \cap U_y$$

  and thus $\exists f \in \mathscr{M}^*(X)$ s.t. $f|_{U_x} = f_x e^{g_x} \rightsquigarrow (f) = (f_x) = D|_{U_x}$ on $U_x \implies (f) = D$.

□

Corollary 3.3.3. If X is a non-compact Riemann surface, then $\exists \omega \in \Omega(X)$ nowhere vanish.

Proof: Take $g \in \mathscr{M}(X) \setminus \mathbb{C}$, $\exists g \in \mathscr{M}^*(X)$ s.t. $(f) = -(dg)$. Define $\omega = f dg \in \Omega(X) \rightsquigarrow (\omega) = 0$. □

Corollary 3.3.4. If $(a_i)_{i \in \mathbb{Z}}$ is discrete, $(c_j)_{j \in \mathbb{N}} \subset \mathbb{C} \implies \exists f \in \mathcal{O}(X)$ s.t. $f(a_j) = c_j$.

Proof: $\exists h \in \mathcal{O}(X)$ s.t. $(h) = \sum_{j=1}^{\infty} a_j$. Let $U_j = X \setminus \{a_\ell : \ell \neq j\}$, then $\{U_j\}$ is a cover of X. Define $g_j = c_j / h \in \mathscr{M}(U_j) \rightsquigarrow g_j - g_i = \dfrac{c_j - c_i}{h} \in \mathcal{O}(U_i \cap U_j)$. By $H^1(X, \mathcal{O}) = 0$, $\exists g$ s.t. $g|_{U_i} - g_i \in \mathcal{O}(U_i)$. Let $f = gh$. On U_j,

$$f = g_j h + (g - g_j) h = c_j + (g - g_j) h \in \mathcal{O}(U_i)$$

Hence $f \in \mathcal{O}(X)$ and $f(a_i) = c_i$. □

3.4 Riemann mapping theorem

Theorem 3.4.1 (Riemann mapping theorem)**.** If X is Riemann surface with $\mathrm{Rh}^1_{\mathcal{O}}(X)=0$, then X homeomorphic to one of $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{D}$.

Observation : Any simply connected Riemann surface X, $\mathrm{Rh}^1_{\mathcal{O}}(X) := \Omega(X)\big/ d\mathcal{O}(X) = 0$.

- $\forall f\in\mathcal{O}^*(X) \rightsquigarrow \exists g,h\in\mathcal{O}(X)$ s.t. $e^g=h^2=f$:

  subproof : $\dfrac{df}{f}\in\Omega(X)\rightsquigarrow \exists g_1\in\mathcal{O}(X)$ s.t. $dg_1=\dfrac{df}{f}\rightsquigarrow d(fe^{-g})=0$. Hence $f=ce^{g_1}=e^g$ for some $c\in\mathbb{C}^\times$. Let $h=e^{g/2}\rightsquigarrow h^2=f$.

- $\forall$ harmonic function $u:X\to\mathbb{R}$, $\exists f\in\mathcal{O}(X)$ s.t. $u\in\mathrm{Re}f$. □

  subproof : $\exists\omega\in\Omega(X)$ s.t. $\mathrm{Re}\omega=du\rightsquigarrow\exists g\in\mathcal{O}(X)$ s.t. $\omega=dg\rightsquigarrow u=\mathrm{Re}g+c=\mathrm{Re}(g+c)$ for some $c\in\mathbb{C}$. □

Definition 3.4.1. Let $\widetilde{X}$ is the universal cover of Riemann surface X.

- X is called **elliptic** if $\widetilde{X}\simeq\mathbb{P}^1$.
- X is called **parabolic** if $\widetilde{X}\simeq\mathbb{C}$.
- X is called **hyperbolic** if $\widetilde{X}\simeq\mathbb{D}$.

Theorem 3.4.2 (Classification)**.**

- elliptic : $\mathbb{P}^1$
- parabolic : $\mathbb{C}^\times,\mathbb{C},\mathbb{C}/\Gamma$
- hyperbolic : others.

Proof: If X is not hyperbolic.

- If X is elliptic. Recall since $p:\widetilde{X}\to X$ is Galois, $G=\mathrm{Deck}(\widetilde{X}/X)\curvearrowright\widetilde{X}$ and by uniqueness, $\forall\varphi\neq\mathrm{id}$, φ has no fixed point. Also $X\simeq\widetilde{X}/G$. However

$$\mathrm{Deck}(\widetilde{X}/X)\le\mathrm{Aut}(\mathbb{P}^1)=\left\{z\mapsto\frac{az+b}{cz+d}:ad-bc\neq0\right\}$$

  and $\dfrac{az+b}{cz+d}=z$ always have solution when $(a,b,c,d)\neq(a,b,0,a)$. But $z+b\notin G$ since ∞ is a fixed point. Hence $G=\{\mathrm{id}\}$ and thus $X\simeq\widetilde{X}=\mathbb{P}^1$.

- If X is parabolic. Similarly

$$\mathrm{Deck}(\widetilde{X}/X)\le\mathrm{Aut}(\mathbb{C})=\{az+b:a\in\mathbb{C}^\times,b\in\mathbb{C}\}$$

  $\implies G\le\{z+b:b\in\mathbb{Z}\}$ and $X\simeq\widetilde{X}/G$. Let Γ be the orbit of 0 under G. Then Γ is discrete additive subgroup of $\mathbb{C}$, since $p:\widetilde{X}\to X$ is a covering. Let $V=\Gamma\otimes_{\mathbb{Z}}\mathbb{R}$ be the smallest real vector space in $\mathbb{C}$ containing Γ.

  - If $\dim V=0\rightsquigarrow G=\{\mathrm{id}\}\rightsquigarrow X\simeq\mathbb{C}$.

- If $\dim V = 1 \rightsquigarrow G = \{z \mapsto z + b : b \in \langle \eta \rangle_{\mathbb{Z}}\} \rightsquigarrow X \simeq \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^\times$ since

$$\begin{array}{rccc} \pi : & \mathbb{C} & \longrightarrow & \mathbb{C}^\times \\ & z & \longmapsto & \exp\left(\dfrac{2\pi i z}{\eta}\right) \end{array} \text{ with } \pi^{-1}(0) = \langle \eta \rangle_{\mathbb{Z}} \simeq \mathbb{Z}$$

- If $\dim V = 2 \rightsquigarrow G = \{z \mapsto z + b : b \in \langle \omega_1, \omega \rangle_{\mathbb{Z}}\}$ with $\omega_1/\omega \notin \mathbb{R} \rightsquigarrow X \simeq \mathbb{C}/\Gamma$.

□

Theorem 3.4.3 (simple form). Suppose X is non-compact Riemann surface and $Y \Subset X$ with $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$ and having regular boundary, then $Y \simeq \mathbb{D}$.

Proof: For $a \in Y$, by Weierstrass theorem, $\exists g \in \mathcal{O}(X)$ s.t. $(g) = a$. $\exists u \in C(\overline{Y})$ s.t.

$$\begin{cases} u|_Y \text{ is harmonic} \\ u|_{\partial Y} = \log|g| \end{cases} \implies u = \mathrm{Re} h \text{ for some } h \in \mathcal{O}(Y)$$

Claim : Set $f = e^{-h} g \in \mathcal{O}(Y)$, then $f : Y \xrightarrow{\sim} \mathbb{D}$.
subproof :

- $f(Y) \subset \mathbb{D}$: Note that $|f|$ can define on ∂Y and

$$|f| = |e^{-h + \log|g|}| = \exp(-u + \log|g|) = 1 \text{ on } \partial Y$$

  By maximal principle, $|f| < 1$ on Y

- $r < 1$, $f^{-1}(\overline{B_r(0)}) = \{y \in Y : |f(y)| \le r\} = \{y \in \overline{Y} : |f(y)| \le r\}$ which is closed in $\overline{Y}$ and thus compact. Hence f is proper.

- Since f has a zero at a of order 1, f is 1-sheeted covering map.

□

Recall : If X is compact, $\forall f \in \mathcal{O}(X)$, f is constant and thus $df = 0$. So $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0 \implies \Omega(X) = 0$. Then $g = 0$ and thus $X = \mathbb{P}^1$. From now on, we assume that X is non-compact and $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. Notice that $Y \underset{\text{open}}{\subset} X$, $\mathcal{O}(Y)$ is metrizable Fréchet space and thus sequential compactness $\iff$ compactness.

Lemma 3.4.1. If Y is a Runge domain in X, then $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$.

Proof: Let $\omega_0 \in \Omega(X)$ and ω_0 never vanish. $\forall \omega \in \Omega(Y)$, $\exists h \in \mathscr{M}(Y)$ s.t. $\omega = h\omega_0|_Y$. Then

$$0 \le (\omega) = (h) + (\omega_0) = (h) \text{ on } Y \implies h \in \mathcal{O}(Y)$$

Let γ be a closed curve in Y. By Runge approximation theorem, $\forall \gamma \subset K \underset{\text{cpt.}}{\subset} Y$, $\exists \{h_n\} \in \mathcal{O}(X)$ s.t. $h_n \to h$ uniformly on K. Then $h_n\omega_0 \in \Omega(X) = d\mathcal{O}(X)$ and thus

$$0 = \int_\gamma h_n\omega_0 \to \int_\gamma h\omega_0 = \int_\gamma \omega \implies \int_\gamma \omega = 0$$

So ω has a primitive on Y. Hence $\Omega(Y) = d\mathcal{O}(Y) \rightsquigarrow \mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$ □

Lemma 3.4.2. If $0 \in Y \subsetneq B_R(0)$ with $0 < R \le \infty$ and $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$, then $\exists r < R$, $f : Y \xrightarrow{\text{hol.}} B_r(0)$, $f(0) = 0$, $f'(0) = 1$.

Proof:

- $R = \infty$: Let $a \in \mathbb{C} \setminus Y$ and

$$\begin{array}{rccc} \varphi: & \mathbb{C} & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & z-a \end{array} \implies 0 \notin \varphi(Y)$$

  Since $\mathrm{Rh}^1_{\mathcal{O}}(Y) = 0$, $\exists g \in \mathcal{O}(Y)$ s.t. $g^2 = \varphi|_Y$. Let $g(0) = b \neq 0$ and $\psi(z) = \dfrac{z-b}{z-\bar{b}}$. Set $h = \psi \circ g$, then $h(0) = 0$ and

$$h'(0) = \psi'(b)g'(0) = \frac{1}{b-\bar{b}} \cdot \frac{1}{b} \neq 0$$

  Define $f = h/h'(0) \rightsquigarrow f(0) = 0$, $f'(0) = 1$ and $r = |h'(0)|^{-1}$.

- $R < \infty$: We may assume $R = 1$ i.e. $Y \subsetneq \mathbb{D} \rightsquigarrow \exists a \in \mathbb{D} \setminus Y$. Consider $T_a(z) = \dfrac{z-a}{1-\bar{a}z} \rightsquigarrow 0 \notin T_a(Y) \rightsquigarrow \exists g \in \mathcal{O}(Y)$ s.t. $g^2 = T_a|_Y \rightsquigarrow g(Y) \subset \mathbb{D}$. If $g(0) = b \neq 0$, consider $T_b = \dfrac{z-b}{1-\bar{b}z}$ and $h = T_b \circ g \rightsquigarrow h(0) = 0$ and $T_b'(z) = \dfrac{1-|b|^2}{(1-\bar{b}z)^2} \implies T_b'(b) = \dfrac{1}{1-|b|^2}$. Then

$$h'(0) = T_b'(b)g'(0) = \frac{1}{1-|b|^2} \cdot \frac{1-|a|^2}{2b} = \frac{1+|b|^2}{2b} \text{ since } b^2 = -a$$

  Hence $|h'(0)| > 1$. Set $f = h/h'(0) \rightsquigarrow f(0) = 0$, $f'(0) = 1$ and $r = |h'(0)|^{-1} < 1$.

□

Lemma 3.4.3. Let Z be the domain of $\mathbb{C}$ s.t. $(\mathbb{C} \setminus Z)^\circ \neq \varnothing$, $\omega_0 \in Z$. Then

$$\mathcal{F} = \{f \in \mathcal{O}(D) : f(D) \subset Z, f(0) = \omega_0\}$$

is compact in $\mathcal{O}(D)$.

Proof: Recall Montel's theorem show that any locally bounded sequence $\{f_n\} \subset \mathcal{O}(D)$ has a subsequence which converges on each compact subset on D to some $f \in \mathcal{O}(D)$. Let $a \in \mathbb{C} \setminus Z$ and $B_r(a) \subset \mathbb{C} \setminus Z$ with some $r > 0$. Consider

$$\begin{array}{rccc} \varphi: & Z & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & (z-a)^{-1} \end{array} \implies Z \xrightarrow{\sim} \varphi(Z) \subset B_{1/r}(0)$$

Hence $f(D) \subset Z$ is bounded $\forall f \in \mathcal{F}$. □

Fact 3.4.1. If $f : B_r(0) \to B_{r'}(0)$ is holomorphic, then

$$|f'(0)| \leq \frac{r'}{r}$$

Proof: By Cauchy's integral formula,

$$f'(0) = \frac{1}{2\pi i} \int_{\partial B_{r-\varepsilon}(0)} \frac{f(z)}{z^2} dz$$

Then

$$|f'(0)| \leq \frac{1}{2\pi} \int_{\partial B_{r-\varepsilon}} \frac{|f(z)|}{|z|^2} dz \leq \frac{1}{2\pi} \frac{r'}{(r-\varepsilon)^2} \cdot 2\pi(r-\varepsilon) = \frac{r'}{r-\varepsilon}$$

as desired when $\varepsilon \to 0$. □

Proposition 3.4.1 (key prop.).

$$\mathcal{F} := \{f \in \mathcal{O}(D) : f \text{ is injective, } f(0) = 0,\ f'(0) = 1\}$$

is compact in $\mathcal{O}(D)$.

Proof: Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{F}$.

- If $B_r(0) \subset f_n(D)$ and $\varphi = f_n^{-1}|_{B_r(0)} : B_r(0) \to D$. By Fact 3.4.1, $1 = |\varphi'(0)| \le \dfrac{1}{r} \implies r \le 1$. Let $r_n = \sup\{r : B_r(0) \subset f_n(D)\}\ \forall n \in \mathbb{N} \rightsquigarrow r_n \le 1$. Since it is open condition, we may choose $a_n \in \partial B_{r_n}(0)$ s.t. $a_n \notin f_n(D)$. Set $g_n = f_n/a_n$. Then $\forall |z| < 1$,

$$|a_n z| < r_n \implies a_n z = f_n(\omega) \text{ for some } \omega \in D \rightsquigarrow z = \frac{f_n(\omega)}{a_n} = g_n(\omega) \in g_n(D)$$

  That is $D \subset g_n(D)$ and $1 \notin g_n(D)$, since $a_n \notin f_n(D)$.

- Since $g_n(D) \simeq D$ is simply connected, $\exists \psi \in \mathcal{O}^*(g_n(D))$ s.t.

$$\begin{cases} \psi^2 = z - 1 \\ \psi(0) = i \end{cases}$$

  Set $h_n = \psi \circ g_n$ i.e. $h_n^2 = g_n - 1$.

  Claim : If $\omega \in h_n(D)$, then $-\omega \notin h_n(D)$.

  subproof : If not, $\exists z_1, z_2 \in D$ s.t. $h_n(z_1) = \omega$, $h_n(z_2) = -\omega$.

$$\implies g_n(z_1) = \omega^2 + 1 = g_n(z_2) \implies z_1 = z_2 \implies \omega = -\omega \implies \omega = 0$$

  i.e. $1 \in g_n(D)$ $(\rightarrow\!\leftarrow)$ □

- Since $D \subset g_n(D)$, $U := \psi(D) \subset h_n(D)$. By Claim, $(-U) \cap h_n(D) = \varnothing$.

$$\implies -U \subset \mathbb{C} \setminus h_n(D)$$

  By Lemma 3.4.3, we may assume (h_n) is compactly convergence sequence. Since

$$f_n = a_n(1 + h_n^2) \text{ with } |a_n| \le 1\ \forall n$$

  f_n is locally bounded. Hence (f_n) has a compactly convergence subsequence $\{f_{n_k}\}$ and $f_{n_k} \xrightarrow{\text{unif}} f \in \mathcal{O}(D) \rightsquigarrow 1 = f'_{n_k} \to f'$. So $f(0) = 0$, $f'(0) = 1$.

- f is injective : If not, $\exists a \in \mathbb{C}$ s.t. $f - a$ has a t least two zeros in $D \rightsquigarrow \exists r < 1$ s.t. $f - a$, $f_{n_k} - a$ does not vanish on $\partial B_r(0)\ \forall k$, then

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{z - a} dz \ge 2$$

$$\implies \frac{1}{2\pi i}\int_{|z|=r} \frac{f'_{n_k}(z)}{f_{n_k}(z) - a} dz \ge 2 \text{ when } n_k \gg 0$$

  which contradict f_{n_k} is injective.

□

Proof: (Riemann mapping theorem)

- Let $Y_0 \Subset Y_1 \Subset \cdots$ be a special exhaustion of X. By Lemma 3.4.1 and Y_i are Runge domain, $\mathrm{Rh}^1_{\mathcal{O}}(Y_i) = 0$. Since $\mathrm{Rh}^1_{\mathcal{O}}(Y_i) = 0$ and Y_i has regular boundary, by 3.4.3, each Y_i is homeomorphic to open disk $\forall i$. Choose $a \in Y_0$, $\exists f_i \in \mathcal{O}(Y_i)$ s.t.

$$f_i(a) = 0,\ f_i'(a) = 1,\ f_i : Y_i \xrightarrow{\sim} B_{r_i}(0) \text{ for some } r_i > 0$$

- **Claim** : $r_n \le r_{n+1}$ $\forall n$.

  subproof : Consider the diagram

$$\begin{tikzcd} Y_n \arrow[r, "f_n", "\sim"'] \arrow[d, hook] & B_{r_n}(0) \arrow[d, "h=f_{n+1}\circ f_n^{-1}"] \\ Y_{n+1} \arrow[r, "f_{n+1}", "\sim"'] & B_{r_{n+1}}(0) \end{tikzcd}$$

  Then $h(0) = 0$ and $h'(0) = f_{n+1}'(a)(f_n^{-1})'(0) = 1$. By Fact 3.4.1, $1 = |h'(0)| \le \dfrac{r_{n+1}}{r_n}$. □

- Define $R = \lim_{n\to\infty} r_n \in (0, \infty]$. Now we want to show that $X \simeq B_R(0)$. $\begin{pmatrix} R = \infty \implies X \simeq \mathbb{C} \\ R < \infty \implies X \simeq D \end{pmatrix}$.

$$\begin{tikzcd} D \arrow[r, "z\mapsto f_0^{-1}(r_0 z)"] \arrow[dr, "\widetilde{g}_n"'] & Y_0 \arrow[d, hook, "f_n|_{Y_0}"] \\ & \mathbb{C} \end{tikzcd}$$

  Define $\widetilde{g}_n$ as above. Then $\widetilde{g}_n(0) = 0$ and

$$\widetilde{g}_n'(0) = f_n'(a) \cdot \left(\frac{df_0(z)}{dz}\bigg|_{z=0} \right)^{-1} \cdot r_0 = r_0$$

  Set $g_n = \widetilde{g}_n / r_0$. Then $g_n(0) = 0$, $g_n'(0) = 1$ and g_n is injective. By Proposition 3.4.1, $\exists$ a compactly convergent subsequence $(g_{n_{0k}}) \longleftrightarrow (f_{n_{0k}})$. By induction on ℓ, apply similarly argument by replace Y_0 to Y_ℓ and $f_n|_{Y_0}$ to $f_{n_{(\ell-1)k}}|_{Y_\ell}$, there exists a compactly convergent subsequence $(f_{n_{\ell k}})$. Set $f_{n_k} = f_{n_{kk}}$ and $f = \lim f_{n_k}$. By uniformly convergence, $f(a) = 0$, $f'(a) = 0$ and f is injective.

- $f : X \xrightarrow{\sim} B_R(0)$.
  - $\forall x \in X$, $x \in Y_i$ for some i, $f(x) = \lim_{k \ge i} f_{n_i}(x) \in B_{r_i}(0) \subset B_R(0)$.
  - f is surjective : Notice that $\mathrm{Rh}^1_{\mathcal{O}}(f(X)) = 0$, since $f : X \xrightarrow{\sim} f(X)$ and $\mathrm{Rh}^1_{\mathcal{O}}(X) = 0$. If f is not surjective, by Lemma 3.4.2, $\exists r < R$ and $g : f(X) \to B_r(0)$, $g(0) = 0$, $g'(0) = 1$. Choose n s.t. $r_n > r$ and set $h = g \circ f \circ f_n^{-1} : B_{r_n}(0) \to B_r(0)$, $h(0) = 0$, $h'(0) = 1$. By Fact 3.4.1,

$$1 = h'(0) \le \frac{r}{r_n} \implies r_n \le r \ (\rightarrow\!\leftarrow)$$

□

Corollary 3.4.1 (little Picard theorem)**.** If $f : \mathbb{C} \to \mathbb{C}$ is non-constant holomorphic map, then f takes every value $c \in \mathbb{C}$ with at most one exception.

Proof: Suppose $a \neq b$ and $a, b \notin \operatorname{Im} f$. Then $X = \mathbb{C} \setminus \{a, b\}$ is hyperbolic.

$$\begin{tikzcd} & \widetilde{X} = D \arrow[d, "\pi"] \\ D \arrow[ur, "\exists \widetilde{f}"] \arrow[r, "f"'] & Y_0 \end{tikzcd}$$

Then $\widetilde{f}$ is bounded. By Liouville theorem, $\widetilde{f}$ is constant and thus f is constant ($\rightarrow\!\leftarrow$). $\square$

3.5 Functions with prescribed summands of automorphy

If $\varphi : \pi_1(X) \to \mathbb{C}$, $\sigma \mapsto a_\sigma$ is a group homomorphism i.e.

$$a_{\sigma\tau} = a_\sigma + a_\tau \tag{$*$}$$

Now we consider $\pi_1(X) \curvearrowright \mathbb{C}$ trivially i.e. $\sigma \cdot a = a$, then $(*)$ can be written to

$$a_{\sigma\tau} = a_\sigma + \sigma a_\tau$$

If φ satisfy above equality, then φ is called a **crossed homomorphism**.

Theorem 3.5.1. If $p : Y \to X$ is holomorphic unbranched Galois covering with X, Y are non-compact Riemann surface. Let $G = \operatorname{Deck}(Y/X)$. Given any crossed homomorphism

$$\begin{array}{rccc} \varphi : & G & \longrightarrow & \mathcal{O}(Y) \\ & \sigma & \longmapsto & f_\sigma \end{array}$$

$\exists f \in \mathcal{O}(Y)$ s.t. $f - \sigma f = f_\sigma$ $\forall \sigma \in G$, where $G \curvearrowright \mathcal{O}(Y)$ by $\sigma \cdot f = f \circ \sigma^{-1}$.

Before proof the theorem, we introduce a convenient notation. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be the cover of X s.t. $Y_i := p^{-1}(U_i) = \bigsqcup\limits_{j \in \Lambda} V_{ij}$ with $p : V_{ij} \xleftarrow{\sim} U_i$. For fixed V_{ij_0}. Since p is Galois, $\forall V_{ij}$, $\exists! \sigma \in G$ s.t. $\sigma(V_{ij_0}) = V_{ij}$. Then we denote $\eta_i(V_{ij}) = \sigma$. Then

$$\begin{array}{rccc} \psi = (p, \eta_i) : & p^{-1}(U_i) & \longrightarrow & U_i \times G \\ & y & \longmapsto & (p(y), \eta_i(y)) \end{array}$$

is a homeomorphism, where G is discrete topology. Indeed, $V_{ij} \mapsto V_{ij} \times \sigma$ shows that is homeomorphism. Moreover, ψ is compatible with the action G i.e.

$$\psi(\tau y) = (x, \tau\sigma)$$

Proof: (Theorem 3.5.1)

- Define $f_i : Y_i \to \mathbb{C}$ by $y \mapsto f_{\eta_i(y)}(y) \rightsquigarrow f_i \in \mathcal{O}(Y_i)$.

  Claim : $f_i - \sigma f_i = f_\sigma$ on Y_i $\forall \sigma \in G$.

  subproof : $\forall y \in Y_i$,

  $$(\sigma f_i)(y) = f_i(\sigma^{-1} y) = f_{\eta_i(\sigma^{-1}(y))}(\sigma^{-1}(y)) = f_{\sigma^{-1}\eta_i(y)}(\sigma^{-1}(y)) = \sigma f_{\sigma^{-1}\eta_i(y)}(y)$$

  Since φ is crossed homomorphism,

  $$f_i(y) = f_{\eta_i(y)}(y) = f_\sigma(y) + \sigma f_{\sigma^{-1}\eta_i(y)}(y) \implies f_i - \sigma f_i = f_\sigma$$

$\square$

- Let $g_{ij} := f_j - f_i = (\sigma f_j + f_\sigma) - (\sigma f_i + f_\sigma) = \sigma(f_j - f_i) \in \mathcal{O}(Y_i \cap Y_j)\ \forall \sigma \in G$. Then g_{ij} is G-invariant on $Y_i \cap Y_j \rightsquigarrow g_{ij} \in \mathcal{O}(U_i \cap U_j)$ and $g_{ij} + g_{jk} = g_{ik}$. So $(g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}) = B^1(\mathfrak{U}, \mathcal{O})$, since $H^1(X, \mathcal{O}) = 0 \rightsquigarrow \exists g_i \in \mathcal{O}(U_i)$ s.t. $g_{ij} = g_j - g_i$. Let $\widetilde{f_i} = f_i - g_i \in \mathcal{O}(Y_i)$ (where g_i regard as $p^* g_i$). Then

$$\widetilde{f_i} - \sigma \widetilde{f_i} = f_i - \sigma f_i \underbrace{- g_i + \sigma g_i}_{=0} = f_\sigma\ \forall \sigma \in G$$

since $p \circ \sigma = p$. On $Y_i \cap Y_j$,

$$\widetilde{f_i} - \widetilde{f_j} = (f_i - f_j) - (g_i - g_j) = g_{ij} - g_{ij} = 0$$

$\implies \exists f \in \mathcal{O}(Y)$ s.t. $f|_{Y_i} = \widetilde{f_i}\ \forall i \in I$ and $f - \sigma f = f_\sigma\ \forall \sigma \in G$.

□

Theorem 3.5.2 (Behnke-Stein)**.** If X is non-compact Riemann surface, $\varphi : \pi_1(X) \to \mathbb{C}$, $\sigma \mapsto a_\sigma$ is a group homomorphism, then $\exists \omega \in \Omega(X)$ s.t.

$$\int_\sigma \omega = a_\sigma\ \forall \sigma \in \pi_1(X)$$

Proof: Let $p : \widetilde{X} \to X$ be the universal covering of X, then $\mathrm{Deck}(\widetilde{X}/X) = \pi_1(X)$. Let $\pi_1(X) \curvearrowright \mathbb{C}$ trivially, then φ is crossed homomorphism. By Theorem 3.5.1, $\exists F \in \mathcal{O}(\widetilde{X})$ s.t. $F - \sigma F = a_\sigma\ \forall \pi_1(X)$. Then

$$dF - \sigma(dF) = 0 \implies dF \in \Omega(X)$$

For $\sigma \in \pi_1(X)$ represented by closed curve u. Let v be the lifting of u, then $\sigma(v(0)) = v(1)$ and

$$\int_\sigma dF = \int_v dF = F(v(1)) - F(v(0)) = F(v(1)) - F(\sigma^{-1}(v(1))) = a_\sigma$$

□

Theorem 3.5.3. If X is compact Riemann surface, $\pi_1(X) \to \mathbb{C}$, $\sigma \mapsto a_\sigma$ is a group homomorphism, then $\exists! \omega \in \mathrm{Harm}^1(X)$ s.t.

$$\int_\sigma \omega = a_\sigma\ \forall \sigma \in \pi_1(X)$$

Proof: Modify Theorem 3.5.1 for general Riemann surface X by replace $\mathcal{O}(Y)$ to $\mathscr{E}(Y)$. Since $H^1(X, \mathscr{E}) = 0$, it also work. So $\exists \widetilde{\omega} \in \mathscr{E}^{(1)}(X)$ s.t.

$$\int_\gamma \widetilde{\omega} = a_\sigma\ \forall \sigma \in \pi_1(X)$$

Write $\widetilde{\omega} = \omega + df$, where $\omega \in \mathrm{Harm}^1(X)$ and $f \in \mathscr{E}(X)$, then

$$\int_\sigma \omega = \int_\sigma \widetilde{\omega} - \int_\sigma df = a_\sigma$$

For another $\omega' \in \mathrm{Harm}^1(X)$ satisfy condition, then

$$\int_\sigma (\omega - \omega') = a_\sigma - a_\sigma = 0\ \forall \sigma \in \pi_1(X)$$

Hence $\sigma = \sigma'$. □

3.6 The triviality of vector bundles

Definition 3.6.1. Suppose X is a Riemann surface.

- $p : E \to X$ is a **holomorphic vector bundle** of rank n if $\exists \mathfrak{U} = (U_i)_{i\in I}$: an open cover of X and a **local trivialization** φ_i of E over U_i s.t.

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i", "\sim"'] \arrow[dr, "p"'] & & U_i \times \mathbb{C}^n \arrow[dl, "\pi_1"] \\ & U_i & \end{tikzcd}$$

In particular, $\forall x \in X$, $p^{-1}(X) \simeq \mathbb{C}^n$. On $U_i \cap U_j$,

$$\begin{tikzcd} & (U_i \cap U_j) \times \mathbb{C}^n \arrow[dd, "g_{ij} := \varphi_i \circ \varphi_j^{-1}", "\wr"'] & \supset \{x\} \times \mathbb{C} \arrow[dd, "g_{ij}(x) \in \mathrm{GL}_n(\mathbb{C})", "\wr"'] \\ p^{-1}(U_i \cap U_j) \arrow[ur, "\varphi_j"] \arrow[dr, "\varphi_i"'] & & \\ & (U_i \cap U_j) \times \mathbb{C}^n & \supset \{x\} \times \mathbb{C} \end{tikzcd}$$

where $g_{ij} \in \mathrm{GL}_n(\mathcal{O}(U_i \cap U_j))$ is called **transition function**, which satisfy

$$g_{ij} g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k$$

Then $(g_{ij}) \in Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$. Hence

$$Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O})) \iff p : E \to X \text{ a holomorphic vector bundle of rank } n$$

- A **holomorphic section** f of E if $p \circ f = \mathrm{id}_X$ and

$$\begin{tikzcd} & p^{-1}(U_i) \arrow[d, "\varphi_i", "\wr"'] \\ f|_{U_i} : U_i \arrow[r] \arrow[ur] & U_i \times \mathbb{C}^n \\ x \arrow[r, mapsto] & (x, f_i(x)) \end{tikzcd}$$

where $f_i \in \mathcal{O}(U_i)^n$ s.t.

$$\begin{tikzcd} & (U_i \cap U_j) \times \mathbb{C}^n \arrow[dd, "g_{ij}", "\wr"'] & \supset \{x\} \times \mathbb{C} \arrow[dd, "g_{ij}(x)", "\wr"'] \\ U_i \cap U_j \arrow[ur, "f|_{U_j}"] \arrow[dr, "f|_{U_i}"'] & & \\ & (U_i \cap U_j) \times \mathbb{C}^n & \supset \{x\} \times \mathbb{C} \end{tikzcd}$$

i.e. $f_i = g_{ij} f_j$ on $U_i \cap U_j$. Define $\mathcal{O}_E$, the sheaf of holomorphic sections of E

$$\mathcal{O}_E(U) = \{f|_U : U \to p^{-1}(U) \text{ is holomorphic section}\}$$

- A **meromorphic section** of E having a pole at a of order m if $p \circ f = \mathrm{id}$ and

$$\begin{array}{rccc} f|_{U_i} : & U_i & \longrightarrow & U_i \times \mathbb{C}^n \\ & x & \longmapsto & (x, f_i(x)) \end{array}$$

s.t. $f_i = g_{ij} f_j$, where $f_i = (f_{i1}, ..., f_{in}) \in \mathcal{O}(U_i \setminus \{a\})^n$ s.t. $\min\limits_{1 \le j \le n} \mathrm{ord}_a f_{ij} = -m$.

Proposition 3.6.1. Let $p : E \to X$ be a holomorphic vector bundle of rank n and $Y \Subset X$. Then $\dim H^1(X, \mathcal{O}_E) < \infty$.

Application : $\forall a \in Y$, let U_1 be the trivialization near a which homeomorphic to disk, and let $U_2 = X \setminus \{a\}$. Then $\mathfrak{U} = \{U_1, U_2\}$ is a open cover of X. Define

$$g_{1i}(z) = \underbrace{(z^{-1}, ..., z^{-i})}_{n \text{ times}} \text{ on } U_1 \text{ and } g_{2i}(z) = \underbrace{(0, ..., 0)}_{n \text{ times}} \text{ on } U_2$$

Then $f_i(z) := g_{1i}(z) - g_{2i}(z) \in \mathcal{O}_E(U_1 \cap U_2) \implies f_i|_Y \in Z^1(\mathfrak{U} \cap Y, \mathcal{O}_E)$. Let $\dim H^1(Y, \mathcal{O}_E) = k$, then $\exists a_1, ..., a_{k+1}$ are not all zero s.t. $\sum_{i=1}^{k+1} a_i f_i = 0$ in $H^1(Y, \mathcal{O}_E)$ i.e. $\exists h_i \in \mathcal{O}_E(U_i \cap Y)$ s.t.

$$\sum_{i=1}^{k+1} a_i f_i = h_2 - h_1 \text{ on } U_1 \cap U_2 \cap Y$$

Then exists $f \in \mathcal{O}_E(Y)$ s.t. $f|_{U_1 \cap Y} = \sum_{i=1}^{k+1} a_i z^i + h_1$ and $f|_{U_2 \cap Y} = h_2$. Then f is a non-trivial meromorphic section of E over Y with a pole at a. In particular, if X is compact Riemann surface, then exists a global meromorphic section which does not vanish identically.

Definition 3.6.2. E is **trivial** if $E \simeq X \times \mathbb{C}^n$.

Theorem 3.6.1 (Criterion)**.** T.F.A.E.

(1) E is trivial, $\exists n$ holomorphic sections $f_1, ..., f_n$ of E s.t. $\forall x \in X$, $f_1(x), ..., f_n(x)$ are linearly independent over $\mathbb{C}$.

(2) For the transition functions $(g_{ij}) \in Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$, $\exists (g_i) \in C^0(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$ s.t. $g_{ij} = g_i g_j^{-1}$ on $U_i \cap U_j$.

Proof:

- (1) $\implies$ (2) : Define $X \to X \times \mathbb{C}^n$ by $f_i : x \mapsto (x, e_i)$, then f_i as desired.
- (2) $\implies$ (3) : Let $\{(U_i, \varphi_i)\}$ be the trivialization of E and $f_s|_{U_i} = (f^i_{s1}, ..., f^i_{sn})$. By assumption, $g_i := (f^i_{st})_{s,t=1,...,n} \in \mathrm{GL}_n(\mathcal{O}(U_i))$. Note that $f^i_s = g_{ij} f^j_s$ $\forall s = 1, ..., n$, which implies

$$\begin{pmatrix} f^i_{s1} \\ \vdots \\ f^i_{sn} \end{pmatrix} = g_{ij} \begin{pmatrix} f^j_{s1} \\ \vdots \\ f^j_{sn} \end{pmatrix} \implies g_i = g_{ij} g_j$$

- Suppose $\varphi_i : p^{-1}(U_i) \simeq U_i \times \mathbb{C}^n$ $\forall i$. If $y \in E$, say $x = p(y)$, let $\varphi_i(y) = (x, v_i)$ if $y \in p^{-1}(U_i)$. By compatible on intersection,

$$v_i = g_{ij} v_j = g_i g_j^{-1} v_j \implies g_i^{-1} v_i = g_j^{-1} v_j$$

Then we have

$$\begin{array}{rccc} \varphi : & E & \xrightarrow{\sim} & X \times \mathbb{C}^n \\ & y & \longmapsto & (x, g_i^{-1} v_i) \end{array}$$

□

Theorem 3.6.2 (Main theorem). Every holomorphic vector bundle E on a non-compact Riemann surface X is trivial.

Proof: By induction on the rank n of E.

- $n = 1$, i.e. E is a line bundle : Let $\varnothing \neq V_0 \Subset V_1 \Subset \cdots$ be a special exhaustion of X by Runge domains. Over Y_i $(\forall i)$, let f_i be a non-trivial meromorphic section of E.

  - E is trivial on Y_i $\forall i \in I$: By Criterion in above, our goal is to find a holomorphic section on Y_i has no zero on Y_i. Let A_i be the discrete subset of Y_i consists of zeros and poles of f_i. For $a \in A_i$ and the $a \in U \subset Y_i$ s.t. (U, φ) is a trivialization. Say

  $$\varphi \circ f_i|_U : \quad \longrightarrow \quad U \times \mathbb{C} \quad x \quad \mapsto \quad (x, h_i(x)) \text{ where } h_i \in \mathscr{M}(U)$$

  By Weierstrass theorem, $\exists g \in \mathscr{M}(Y_i)$ s.t.

  $$(g_i) = -\sum_{a \in A_i} \mathrm{ord}_a(h_i)a \implies (f_i g_i) = 0$$

  i.e. $f_i g_i \in \mathcal{O}_E(Y_i)$ and $f_i g_i$ has no zeros i.e. $f_i g_i(x)$ are linearly independent $\forall x \in Y_i$.

  - E is trivial over X : Since E is trivial over Y_i $\forall i$, $\mathcal{O}_E(Y_i) \simeq \mathcal{O}(Y_i)$ $\forall i$. For $a \in Y_0$, $g_0 \in \mathcal{O}(Y_0)$ with $g_0(a) \neq 0$. Let $Y_{-1} := K \underset{\text{cpt.}}{\subset} Y_0$. By Proposition **??**, $\exists g_i \in \mathcal{O}(Y_i)$ s.t. $\|g_i - g_{i-1}\|_{\overline{Y_{i-2}}} \leq 2^{-i}\varepsilon$. Then

  $$(g_j)_{j \geq n} \to f_n \in \mathcal{O}(Y_n) \text{ uniformly on } \overline{Y_{n-1}} \implies f_{n+1}|_{Y_n} = f_n \text{ and } f_n(a) = \lim_{j \to \infty} g_j(a) \neq 0$$

  Then $\exists f \in \mathcal{O}(X)$ s.t. $f|_{Y_n} = f_n \rightsquigarrow f(a) \neq 0$ i.e. $f \not\equiv 0$. By Weierstrass theorem, $\exists \widetilde{f} \in \mathscr{M}(X)$ s.t.

  $$(\widetilde{f}) = -(f) \implies (\widetilde{f} f) = 0$$

  Then $\widetilde{f} f \in \mathcal{O}(X)$ and has no zeros.

- $n > 1$:

  - Case 1. $\exists$ a global holomorphic section $F_n \in \mathcal{O}_E(X)$ does not vanish anywhere :
  Since E is locally trivial, exists the trivialization $\{(U_i, \varphi_i)\}_{i \in I}$ and local frames $F_1^i, ..., F_{n-1}^i \in \mathcal{O}_E(U_i)$ s.t. $F_1^i(x), ..., F_{n-1}^i(x), F_n(x)$ are linearly independent over $\mathbb{C}$. On $U_i \cap U_j$,

  $$\begin{pmatrix} F_1^i \\ \vdots \\ F_{n-1}^i \\ F_n \end{pmatrix} = \left(\begin{array}{c:c} G^{ij} & a^{ij} \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} F_1^j \\ \vdots \\ F_{n-1}^j \\ F_n \end{pmatrix}$$

  where $G^{ij} \in \mathrm{GL}_{n-1}(\mathcal{O}(U_i \cap U_j))$ and $a^{ij} \in \mathcal{O}(U_i \cap U_j)^{n-1}$ is column vectors. On $U_i \cap U_j \cap U_k$, we have

  $$\left(\begin{array}{c:c} G^{ik} & a^{ik} \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} F_1^k \\ \vdots \\ F_{n-1}^k \\ F_n \end{pmatrix} = \begin{pmatrix} F_1^i \\ \vdots \\ F_{n-1}^i \\ F_n \end{pmatrix} = \left(\begin{array}{c:c} G^{ij} & a^{ij} \\ \hdashline 0 & 1 \end{array}\right) \left(\begin{array}{c:c} G^{jk} & a^{jk} \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} F_1^k \\ \vdots \\ F_{n-1}^k \\ F_n \end{pmatrix}$$

Then $G^{ij}G^{jk} = G^{ik} \rightsquigarrow (G^{ij}) \in Z^1(\mathfrak{U}, \mathrm{GL}_{nn-1}(\mathcal{O}))) \longleftrightarrow$ a holomorphic vector bundle of rank $n-1$. By induction hypothesis, it is trivial. By Criterion of triviality, $\exists G^i \in \mathrm{GL}_{n-1}(\mathcal{O}(U_i))$ s.t. $G^{ij} = G^i(G^j)^{-1}$ on $U_i \cap U_j$. Set

$$\begin{pmatrix} \widetilde{F}_1^i \\ \vdots \\ \widetilde{F}_{n-1}^i \end{pmatrix} = (G^i)^{-1} \begin{pmatrix} F_1^i \\ \vdots \\ F_{n-1}^i \end{pmatrix} \implies \begin{pmatrix} \widetilde{F}_1^i \\ \vdots \\ \widetilde{F}_{n-1}^i \end{pmatrix} = \left(\begin{array}{c:c} I_{n-1} & b^{jk} \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} \widetilde{F}_1^j \\ \vdots \\ \widetilde{F}_{n-1}^j \end{pmatrix}$$

for some $b^{ij} \in \mathcal{O}(U_i \cap U_j)^{n-1}$. Then

$$\begin{pmatrix} \widetilde{F}_1^i \\ \vdots \\ \widetilde{F}_{n-1}^i \end{pmatrix} = \begin{pmatrix} \widetilde{F}_1^j \\ \vdots \\ \widetilde{F}_{n-1}^j \end{pmatrix} + \begin{pmatrix} b_1^{ij} \\ \vdots \\ b_{n-1}^{ij} \end{pmatrix} F_n \implies b^{ij} + b^{jk} = b^{ik} \rightsquigarrow (b^{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}^{n-1})$$

Since $H^1(X, \mathcal{O}^{n-1}) = H^1(X, \mathcal{O})^{\oplus(n-1)} = 0$, $\exists b^i \in \mathcal{O}(U_i)^{n-1}$ s.t. $b^{ij} = b^i - b^j$ on $U_i \cap U_j$. Set

$$\begin{pmatrix} \overline{F}_1^i \\ \vdots \\ \overline{F}_{n-1}^i \end{pmatrix} = \begin{pmatrix} F_1^i \\ \vdots \\ F_{n-1}^i \end{pmatrix} - \begin{pmatrix} b_1^i \\ \vdots \\ b_{n-1}^i \end{pmatrix} F_n \rightsquigarrow \begin{pmatrix} \overline{F}_1^i \\ \vdots \\ \overline{F}_{n-1}^i \end{pmatrix} = \begin{pmatrix} \overline{F}_1^j \\ \vdots \\ \overline{F}_{n-1}^j \end{pmatrix} \text{ on } U_i \cap U_j$$

Then $\exists F_1, ..., F_{n-1} \in \mathcal{O}_E(X)$ s.t. $F_k|_{U_i} = \overline{F}_k^i$. Hence E is trivial.

- Case 2. Recall that

$$\begin{aligned} Y \Subset X &\implies \exists \text{ a non-trivial meromorphic section of } E \text{ over } Y \\ &\implies \exists \text{ a holomorphic section of } E \text{ over } Y \text{ which has no zeros. (By Weierstrass thm.)} \\ &\implies E \text{ is trivial over } Y \text{ (By Case 1.)} \end{aligned}$$

Suppose $Y_0 \Subset Y_1 \Subset Y_2 \Subset \cdots$ is a special exhaustion, then E is trivial on Y_i $\forall i \rightsquigarrow \mathcal{O}_E(Y_i) \simeq \mathcal{O}(Y_i)^n$. By using the similar argument for the case of $n = 1$, by Proposition 3.3.1, $\exists$ a non-trivial holomorphic section of E over X. By Weierstrass theorem, $\exists$ a global holomorphic section does not vanish anywhere. By Case 1, E is trivial.

□

Corollary 3.6.1. If X is non-compact Riemann surface, then $H^1(X, \mathrm{GL}_n(\mathcal{O})) = 0$. In particular, $n = 0 \implies H^1(X, \mathcal{O}^*) = 0$.

3.6.1 Proof of finiteness

Lemma 3.6.1. If $Y_0 \underset{\text{open}}{\subset} Y$, then the restriction map $H^1(Y, \mathcal{O}_E) \twoheadrightarrow H^1(Y_0, \mathcal{O}_E)$.

Proof: Let $\{(U_i, \varphi_i)\}_{i=1}^m$ be the trivialization over Y s.t. U_i homeomorphic to open disk in $\mathbb{C}$. Note that $\forall V \underset{\text{open}}{\subset} U_i$, $\mathcal{O}_E|_V \simeq \mathcal{O}_V^{\oplus n}$ and thus $H^1(V, \mathcal{O}_E) \simeq H^1(V, \mathcal{O}_V)^{\oplus n} = 0$.

Claim : Let $Y_k = Y_0 \cup \bigcup_{i=1}^k U_i$, then $H^1(Y_k, \mathcal{O}_E) \twoheadrightarrow H^1(Y_{k-1}, \mathcal{O}_E)$ $\forall k = 1, ..., m$.
subproof : Fix k, we construct the Leray cover of Y_{k-1}, Y_k by

$$\begin{cases} V_i = U_i \cap Y_{k-1},\ i = 1, ..., m & \implies \mathfrak{V} = \{V_i\} \text{ is a Leray cover of } Y_{k-1} \\ V_i' = Y_i \text{ for } i \neq k,\ V_k' = U_k & \implies \mathfrak{V}' = \{V_i'\} \text{ is a Leray cover of } Y_k \end{cases}$$

We find $V_i \cap V_j = V_i \cap V_j' \ \forall i \neq j$, since

$$V_i \cap V_k = V_i \cap U_k \cap Y_{k-1} = V_i \cap V_k' = V_i' \cap V_k'$$

Hence $Z^1(\mathfrak{V}, \mathcal{O}_E) = Z^1(\mathfrak{V}', \mathcal{O}_E)$. Then

$$\begin{array}{ccc} C^0(\mathfrak{V}', \mathcal{O}_E) & \longrightarrow & C^0(\mathfrak{V}, \mathcal{O}_E) \\ \big\downarrow \delta & & \big\downarrow \delta \\ C^1(\mathfrak{V}', \mathcal{O}_E) & \longrightarrow & C^1(\mathfrak{V}, \mathcal{O}_E) \end{array} \implies B^1(\mathfrak{V}', \mathcal{O}_E) \hookrightarrow B^1(\mathfrak{B}, \mathcal{O}_E)$$

Hence

$$\begin{aligned} H^1(Y_k, \mathcal{O}_E) = H^1(\mathfrak{V}', \mathcal{O}_E) = & Z^1(\mathfrak{V}', \mathcal{O}_E) \big/ B^1(\mathfrak{V}', \mathcal{O}_E) \\ & \twoheadrightarrow Z^1(\mathfrak{V}, \mathcal{O}_E) \big/ B^1(\mathfrak{V}, \mathcal{O}_E) = H^1(\mathfrak{V}, \mathcal{O}_E) = H^1(Y_{k-1}, \mathcal{O}_E) \end{aligned}$$

Hence by Claim, we have

$$H^1(Y, \mathcal{O}_E) = H^1(Y_m, \mathcal{O}_E) \twoheadrightarrow H^1(Y_{m-1}, \mathcal{O}_E) \twoheadrightarrow \cdots \twoheadrightarrow H^1(Y_1, \mathcal{O}_E) \twoheadrightarrow H^1(Y_0, \mathcal{O}_E)$$

□

Proposition 3.6.2. Let $p : E \to X$ be a holomorphic vector bundle of rank n and $Y \Subset X$. Then $\dim H^1(X, \mathcal{O}_E) < \infty$.

Proof:

- Choose $Y' \Subset X$ s.t. $Y \Subset Y'$ and $Y = \bigcup_{i=1}^{m} V_i$, $Y' = \bigcup_{i=1}^{m} U_i$ s.t. $\{(U_i, \varphi_i)\}_{i=1}^m$ is trivialization over Y', $V_i \Subset U_i$ and $U_i \simeq D \subset \mathbb{C}$. Since $H^1(V_i, \mathcal{O}_E) = 0$ and $H^1(U_i, \mathcal{O}_E) = 0$, $\mathfrak{V} = \{V_i\}$ and $\mathfrak{U} = \{U_i\}$ is a Leray cover of Y and Y' respectively. By Lemma,

  $$H^1(Y', \mathcal{O}_E) = H^1(\mathfrak{U}, \mathcal{O}_E) \twoheadrightarrow H^1(\mathfrak{V}, \mathcal{O}_E) = H^1(Y, \mathcal{O}_E)$$

  Note that $\mathcal{O}(U_i)$, $\mathcal{O}(U_i \cap U_j)$ are Fréchet space $\implies$ $\mathcal{O}_E(U) \simeq \mathcal{O}(U_i)^{\oplus n}$, $\mathcal{O}_E(U_i \cap U_j)$ are Fréchet space. Also

  $$C^0(\mathfrak{U}, \mathcal{O}_E) = \prod_i \mathcal{O}_E(U_i), \ C^1(\mathfrak{U}, \mathcal{O}_E) = \prod_{i,j} \mathcal{O}_E(U_i \cap U_j)$$

  are all Fréchet space. $Z^1(\mathfrak{U}, \mathcal{O}_E)$ is also Fréchet space, since it is closed subset of $C^1(\mathfrak{U}, \mathcal{O}_E)$. It is clear that

  $$\delta : C^0(\mathfrak{V}, \mathcal{O}_E) \to C^1(\mathfrak{B}, \mathcal{O}_E) \text{ and } \beta : Z^1(\mathfrak{U}, \mathcal{O}_E) \to Z^1(\mathfrak{V}, \mathcal{O}_E)$$

  are continuous map.

- If $V \Subset U$, then $\beta : \mathcal{O}(U) \to \mathcal{O}(V)$ is compact :
  Since $\overline{V} \underset{\text{cpt.}}{\subset} U$,

  $$W := \{f \in \mathcal{O}(U) : \sup_{x \in \overline{V}} |f(x)| < 1\}$$

  is a neighborhood of zero in $\mathcal{O}(U)$. By Montel's theorem,

  $$W' := \{g \in \mathcal{O}(V) : \sup_{y \in V} |g(y)| \leq 1\}$$

  is compact in $\mathcal{O}(V)$ and $\beta(W) \subset W'$. Hence β is compact.

- By above, we get $\beta : Z^1(\mathfrak{U}, \mathcal{O}_E) \to Z^1(\mathfrak{V}, \mathcal{O}_{\mathfrak{E}})$ is a compact map. Consider to map

$$\begin{array}{rccc} \varphi : & C^0(\mathfrak{V}, \mathcal{O}_E) \times Z^1(\mathfrak{U}, \mathcal{O}_E) & \longrightarrow & Z^1(\mathfrak{V}, \mathcal{O}_E) \\ & (\eta, \xi) & \longmapsto & \delta\eta + \beta(\xi) \\ \psi : & C^0(\mathfrak{V}, \mathcal{O}_E) \times Z^1(\mathfrak{U}, \mathcal{O}_E) & \longrightarrow & Z^1(\mathfrak{V}, \mathcal{O}_E) \\ & (\eta, \xi) & \longmapsto & \beta(\xi) \end{array}$$

Then by Lemma 3.6.1, φ is surjective. By β is compact, ψ is also compact. By Schwartz theorem, the image of $\varphi - \psi$ $(= B^1(\mathfrak{V}, \mathcal{O}_E))$ has finite codimension in $Z^1(\mathfrak{V}, \mathcal{O}_E)$. Hence $H^1(\mathfrak{V}, \mathcal{O}_E)$ has finite dimension.

□

3.6.2 $H^1(X, \mathcal{O}^*)$

Recall that if X is non-comapct Riemann surface, $H^1(X, \mathcal{O}^*) = 0$.

Theorem 3.6.3. If X is compact Riemann surface, then $H^1(X, \mathcal{O}^*) \simeq \mathrm{Pic}(X) = \mathrm{Div}(X) / \mathrm{Div}_p(X)$.

Proof:

- Define $\varphi : H^1(X, \mathcal{O}^*) \to \mathrm{Pic}(X)$ as follows. Given $\xi = [(g_{ij})] \in H^1(X, \mathcal{O}^*)$ with $(g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$, then it associate a line bundle $p : L \to X \rightsquigarrow \exists$ a non-trivial meromorphic section of L, say $f_i \in \mathscr{M}^*(U_i)$, $f_I = g_{ij} f_j$ on $U_i \cap U_j$. Let $D \in \mathrm{Div}(X)$ s.t. $\forall x \in X$, $D(x) = \mathrm{ord}_x(f_i)$ if $x \in U_i$, which is well-define, since $f_i/f_j = g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$. Then define $\varphi(\xi) = D + \mathrm{Div}_p(X)$.
- Now we check that φ is well-defined : For another $\widetilde{f_i} \in \mathscr{M}^*(U_i)$ with $\widetilde{f_i} = g_{ij} \widetilde{f_j}$. Then

$$\frac{\widetilde{f_i}}{\widetilde{f_j}} = g_{ij} = \frac{f_i}{f_j} \implies \frac{\widetilde{f_i}}{f_i} = \frac{\widetilde{f_j}}{f_j} \text{ on } U_i \cap U_j$$

  Then $\exists F \in \mathscr{M}^*(X)$ s.t. $F|_{U_i} = \widetilde{f_i}/f_i \implies \widetilde{D} = D + (F)$ and thus $\widetilde{D} + \mathrm{Div}_p(X) = D + \mathrm{Div}_p(X)$. If $\xi = [(g'_{st})]$ with $(g'_{st}) \in Z^1(\mathfrak{U}', \mathcal{O}^*)$, where $\mathfrak{U}'$ is refinement of $\mathfrak{U}$, then...
- It is clear that φ is group homomorphism.
- φ is surjective : Given $D \in \mathrm{Div}(X)$, $\exists \mathscr{U} = \{U_i\}_{i \in I}$ and $f_i \in \mathscr{M}(U_i)$ s.t. $(f_i) = D|_{U_i} \rightsquigarrow g_{ij} = f_i/f_j \in \mathcal{O}^*(U_i \cap U_j) \rightsquigarrow (g_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}^*)$.
- φ is injective : Let $\varphi(\xi) \in \mathrm{Div}_p(X)$, i.e. $(f_i)_{i \in I} = (f)$ for some $f \in \mathscr{M}(X)$. Then $g_i = f_i/f \in \mathcal{O}^*(U_i)$. Hence

$$g_{ij} = \frac{f_i}{f_j} = \frac{g_i}{g_j} \implies (g_{ij}) \in B^1(\mathfrak{U}, \mathcal{O}^*)$$

□

3.7 Riemann Hilbert problem

Recall : $p : \widetilde{X} \to X$, $A \in M_{n \times n}(\Omega(X))$, $G = \mathrm{Deck}(\widetilde{X}/X) = \pi_1(X)$, $\Psi = (\omega_1 \cdots \omega_n) \in \mathrm{GL}_n(\mathcal{O}(\widetilde{X}))$ with $d\Phi = (p^* A)\Phi$. $\forall \sigma \in G$, $\sigma\Phi = \Psi(\sigma^{-1})$ is another fundamental system of solution. So $\exists T_\sigma \in \mathrm{GL}_n(\mathbb{C})$ s.t. $\sigma\Phi = \Phi T_\sigma$ and

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ \sigma & \longmapsto & T_\sigma \end{array}$$

is a group homomorphism. How about converse statement.

Theorem 3.7.1 (Riemann Hilbert problem)**.** Suppose X is a non-compact Riemann surface, S be the discrete subset of X and $X' = X \setminus S$. $T : \pi_1(X') \to \mathrm{GL}_n(\mathbb{C})$, $\sigma \mapsto T_\sigma$ is a group homomorphism. Then $\exists A \in M_n(\Omega(X'))$ s.t. $d\omega = A\omega$ has a regular singular point at each $a \in S$ and a fundamental system Φ of solution with $\sigma\Phi = \Phi T_\sigma$ $\forall \sigma \in \pi_1(X')$.

Proof of $S = \varnothing$ **:**

First we improve the Theorem 3.5.1.

Lemma 3.7.1. Let X, Y are non-compact Riemann surface, $p : Y \to X$ holomorphic unbranch Galois covering, $G = \mathrm{Deck}(Y/X)$. If $T : G \to \mathrm{GL}_n(\mathbb{C})$, $\sigma \mapsto T_\sigma$ is a group homomorphism, then $\exists \Phi \in \mathrm{GL}_n(\mathcal{O}(Y))$ s.t. $\sigma\Phi = \Phi T_\sigma$ $\forall \sigma \in G$.

Proof:

- Local solutions : Choose $\mathcal{U} = (U_i)_{i\in I}$ be a cover of X s.t. $Y_i := p^{-1}(U_i) = \bigsqcup_{j\in\Lambda} V_{ij}$, $p|_{V_{ij}} : V_{ij} \xrightarrow{\sim} U_i$. Fix V_{ij_0}, then $\forall V_{ij}$, $\exists! \sigma_j \in G$ s.t. $\sigma_j(V_{ij_0}) = V_{ij}$. Consider

$$(p, \eta_i) : \begin{array}{ccc} Y_i & \xrightarrow{\sim} & U_i \times G \\ y & \longmapsto & (p(y), \sigma_j) \end{array} \text{ if } y \in V_{ij}$$

  Define

$$\Psi_i \quad \begin{array}{ccc} Y_i & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) \\ y & \longmapsto & T_{\eta_i(y)^{-1}} \end{array}$$

  Then Ψ_i is constant on each V_{ij} and thus $\Psi_i \in \mathrm{GL}_n(\mathcal{O}(Y_i))$. $\forall y \in Y_i$, $\sigma \in G$,

$$\sigma\Psi_i(y) = \Psi_i(\sigma^{-1}(y)) = T_{\eta_i(\sigma^{-1}(y))^{-1}} = T_{(\sigma^{-1}\eta_i(y))^{-1}} = T_{\eta_i(y)^{-1}\sigma} = \Psi_i(y)T_\sigma$$

- Global solution : Define $F_{ij} = \Psi_i\Psi_j^{-1} \in \mathrm{GL}_n(\mathcal{O}(Y_i \cap Y_j))$, then

$$\sigma(F_{ij}) = (\sigma\Psi_i)(\sigma\Psi_j)^{-1} = \Psi_i T_\sigma T_\sigma^{-1}\Psi_j^{-1} = \Psi_i\Psi_j^{-1} = F_{ij}$$

  i.e. F_{ij} is G-invariant $\rightsquigarrow$ F_{ij} can be regard as $\mathrm{GL}_n(\mathcal{O}(U_i \cap U_j))$. Since $(F_{ij}) \in Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$ and $H^1(X, \mathrm{GL}_n(\mathcal{O})) = 0$, $\exists F_i \in \mathrm{GL}_n(\mathcal{O}(U_i))$ s.t. $F_{ij} = F_iF_j^{-1}$ on $U_i \cap U_j$. Consider $\Phi_i = (p^*F_i)^{-1}\Psi_i \in \mathrm{GL}_n(\mathcal{O}(Y_i))$, then

$$\sigma\Phi_i = (\sigma F_i)^{-1}(\sigma\Psi_i) = F_i^{-1}\Psi_i T_\sigma = \Psi_i T_\sigma$$

$$\Phi_i^{-1}\Phi_j = \Psi_i^{-1}(p^*F_i)(p^*F_j)^{-1}\Psi_j = \Psi_i^{-1}F_{ij}\Psi_j = I$$

  Hence $\exists \Phi \in \mathrm{GL}_n(\mathcal{O}(Y))$ s.t. $\Phi|_{Y_i} = \Phi_i$.

□

Applying lemma 1 to $p : \widetilde{X} \to X$, we get $\Phi \in \mathrm{GL}_n(\mathcal{O}(\widetilde{X}))$ with $\sigma\Phi = \Phi T_\sigma$. Since

$$\sigma(d\Phi \cdot \Phi^{-1}) = d(\sigma\Phi) \cdot (\sigma\Phi)^{-1} = d\Phi \cdot T_\sigma(T_\sigma^{-1}\Phi) = d\Phi \cdot \Phi^{-1}$$

Then $d\Phi \cdot \Phi^{-1}$ is $\mathrm{Deck}(\widetilde{X}/X)$ invariant. Hence $\exists A \in M_{n\times n}(\Omega(X))$ s.t. $p^*A = d\Phi \cdot \Phi^{-1}$ and thus $d\Phi = (p^*A)\Phi$.

Proof of $S \neq \varnothing$:

Lemma 3.7.2. Let $p' : \widetilde{X'} \to X'$ is the universal covering of X'. For $a \in S$, let $a \in (U, z) \subset X' \cup \{a\}$ with $z(U) = D \subset \mathbb{C}$ and $z(a) = 0$. If z is any connected component of $p'^{-1}(U \setminus \{a\})$, then $p' : Z \to U \setminus \{a\}$ is the universal covering.

Proof: If $p'|_Z$ is not the universal covering, by Remark 1.2.1,

$$\begin{tikzcd} Z \arrow[rr, "\sim"] \arrow[dr, "p'"'] & & D \arrow[dl, "z^k"] \\ & D & \end{tikzcd}$$

with $k \geq 1$. By Weierstrass product theorem, $\exists h \in \mathcal{O}(X)$ s.t. $(h) = a \in \mathrm{Div}(X)$, then $\omega = dh/h \in \Omega(X')$. Let $\gamma : |z| = 1/2$ in D^*, we can lift the curve γ to $\gamma_1 : z_1 \rightsquigarrow z_2$ in Z, and by induction we can lift γ to $\gamma_i : z_i \rightsquigarrow z_{i+1}$ in Z with $z_{k+1} = z_1$. Then $c = \gamma_1 \cdots \gamma_k$ is a closed curve in Z. Since $\widetilde{X'}$ is simply connected, $p^*\omega$ has primitive on $\widetilde{X'}$. Hence

$$0 = \int_c p^*\omega = \sum_{i=1}^{k} \int_{\gamma_i} p^*\omega = \sum_{j=1}^{k} \int_\gamma \omega = 2\pi i k \implies k = 0 \ (\rightarrow\!\times\!\leftarrow)$$

□

Proof:

- Suppose $S = \{a_i : i \in I\}$ with $0 \notin I$. Choose $a_i \in (U_i, z_i) \subset X' \cup \{a_i\}$ with $z_i(U_i) = D \subset \mathbb{C}$ and $z_i(a_i) = 0$. Let $J = I \cup \{0\}$ and $U_0 = X' \rightsquigarrow \mathfrak{U} = (U_j)_{j \in J}$ is a cover of X. Let $Y_0 = \widetilde{X'}$ and $Y_i = p'^{-1}(U_i \setminus \{a_i\})$ $\forall i \in I$.

- By the case of $S = \varnothing$, $\exists \Psi_0 \in \mathrm{GL}_n(\mathcal{O}(Y_0))$ s.t. $\sigma\Psi_0 = \Psi_0 T_\sigma$ $\forall \sigma \in \pi_1(X')$.

- $\forall j \in I$, for any connected component Z of Y_j, $p' : Z \to U_i \setminus \{a_i\} \simeq D^*$ is the universal covering of D^* and $\mathrm{Deck}(Z/D^*) \simeq \pi^*(D^*) \simeq \mathbb{Z} = \langle \tau \rangle \subset \pi_1(X')$, where $\tau \log = \log + 2\pi i$. Let $B \in M_{n\times n}()$ s.t. $\exp(2\pi i B) = T_\tau$. Then $\Psi_j := \exp(B \log) \in \mathrm{GL}_n(\mathcal{O}(Z))$ be a fundamental system of solution of $w = \dfrac{1}{z} Bw$ s.t. $\tau\Psi_j = \Psi_j T_\tau$. $\forall \sigma \in \mathrm{Deck}(Z/D^*)$, say $\sigma = \tau^k$ and thus

$$\sigma\Psi_j = \tau^k \Psi_j = \Psi_j T_\tau^k = \Psi_j T_\sigma$$

  Since Y_j is disjoint union of connected component, there exists $\Psi_j \in \mathrm{GL}_n(\mathcal{O}(Y_j))$ s.t. $\Psi_j|_Z$ is want we find in above.

- Note that $F_{ij} := \Psi_i \Psi_j^{-1} \in \mathrm{GL}_n(\mathcal{O}(Y_i \cap Y_j))$ is invariant under covering transformation. So $F_{ij} \in \mathrm{GL}_n(\mathcal{O}(U_i \cap U_j))$. Then $(F_{ij}) \in Z^1(\mathfrak{U}, \mathrm{GL}_n(\mathcal{O}))$. By $H^1(X, \mathrm{GL}_n(\mathcal{O})) = 0$, $\exists F_i \in \mathrm{GL}_n(\mathcal{O}(U_i))$ s.t. $F_{ij} = F_i F_j^{-1}$ $\forall i, j$. Then we can glue Φ_j to $\Phi \in \mathrm{GL}_n(\mathcal{O}(Y))$ and $\sigma\Phi = \Phi T_\sigma$ $\forall \sigma \in \pi_1(X)$. On $U_i \setminus \{a_i\}$ one has $\Phi = F_i^{-1}\Psi_i$. Since Ψ_i has regular singular points and F_i^{-1} is holomorphic on all of U_i, it follows that Φ also has regular singular points. Since Φ is a fundamental system of solution of $dw = Aw$, where $A := d\Phi \cdot \Phi^{-1} \in M_{n\times n}(\Omega(X'))$, since it is invariant under covering transformation.

□

Chapter 4

More Jacobi variety

4.1 Riemann condition

Recall the Jacobi variety is $X = \mathbb{C}^n/\Lambda$, where $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i$ is a lattice. Let $x^1, ..., x^{2n}$ be the real coordinate on X.

- Let $p : \mathbb{C}^n \to X$ be the universal cover of X, then $\Lambda = \text{Deck}(\mathbb{C}^n/X) \simeq \pi_1(X) \simeq H_1(X)$, since H_1 is abelianization of $\pi_1(X)$ and Λ is abelian.
- $dz^1, ..., dz^n, d\overline{z}^1, ..., d\overline{z}^n$ are Λ-invariant $\rightsquigarrow$ global 1-form on X.
- $H^1(X, \mathbb{Z}) = \mathbb{Z}\{dx^1, ..., dx^{2n}\}$ via
$$\int_{\lambda_i} dx^j = \delta_{ij}$$
where λ_i is regard as the element in $H_1(X)$. By basis change,
$$H^1(X, \mathbb{Z}) \simeq \mathbb{Z}\{dz^1, ..., dz^n, d\overline{z}^1, ..., d\overline{z}^n\}$$
- $H^k(X, \mathbb{Z}) = Z(dx^I)_{|I|=k}$: Since $X \simeq (S^1)^n$, by Künneth formula,
$$H^k(X, \mathbb{Z}) \simeq \bigwedge^k(\Lambda^*) = \bigwedge^k(H^1(X, \mathbb{Z}))$$

Theorem 4.1.1 (Kodaira embedding theorem)**.** Suppose X is complex manifold. X is algebraic $\iff$ $\exists$ an integral closed positive $(1,1)$-form.

In general, the $(1,1)$-form on X is
$$\omega = \frac{\sqrt{-1}}{2} \sum h_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$$
Let $dx = \pi dz + \overline{\pi} d\overline{z}$ (since dx is real), where
$$dx = \begin{pmatrix} dx^1 \\ \vdots \\ dx^{2n} \end{pmatrix}, \quad dz = \begin{pmatrix} dz^1 \\ \vdots \\ dz^n \end{pmatrix} \text{ and } \pi \in M_{2n\times n}(\mathbb{C})$$
and let $\widetilde{\pi} = (\pi, \overline{\pi}) \in M_{2n\times 2n}(\mathbb{C})$.

Definition 4.1.1. $\omega = \frac{1}{2}\sum q_{ij} dx^i \wedge dx^j$ is **integral** if $q_{ij} \in \mathbb{Z}$. If ω is integral, then $Q = (q_{ij})$ is an integral skew-symmetric $2n \times 2n$ matrix.

Represent dx^i as $dz^\alpha, d\overline{z}^\alpha$ and omit the summation symbol if the index appear in both superscript and subscript, we have

$$\begin{aligned}
\omega =& \frac{1}{2}\sum q_{ij}\left(\pi_{i\alpha}dz^\alpha + \overline{\pi_{i\alpha}}d\overline{z}^\alpha\right)\wedge\left(\pi_{j\beta}dz^\beta + \overline{\pi_{j\beta}}d\overline{z}^\beta\right)\\
=& \frac{1}{2}\sum q_{ij}\left(\pi_{i\alpha}\pi_{j\beta}dz^\alpha\wedge dz^\beta + \pi_{i\alpha}\overline{\pi_{j\beta}}dz^\alpha\wedge d\overline{z}^\beta + \overline{\pi_{i\alpha}}\pi_{j\beta}d\overline{z}^\alpha\wedge dz^\beta + \overline{\pi_{i\alpha}\pi_{j\beta}}d\overline{z}^\alpha\wedge d\overline{z}^\beta\right)
\end{aligned}$$

Then

$$\begin{aligned}
\omega \text{ is } (1,1)\text{-type} &\iff 0 = \left(\sum q_{ij}\pi_{i\alpha}\pi_{j\beta}\right)_{\alpha,\beta} = {}^t\pi Q\pi = 0\\
\omega \text{ is positive-type} &\iff 0 < \frac{1}{2\sqrt{-1}}\left(\sum q_{ij}(\pi_{i\alpha}\overline{\pi_{j\beta}} - \overline{\pi_{i\beta}\pi_{j\alpha}})\right)_{\alpha,\beta} = -\sqrt{-1}\,{}^t\pi Q\overline{\pi}
\end{aligned}$$

Those two condition are called **1st-Riemann condition**. Now if we write $dz^\alpha = \omega_{\alpha i}dx^i$, then $d\overline{z}^\alpha = \overline{\omega_{\alpha i}}dx^i$. Let $\Omega = (\omega_{\alpha i}) \rightsquigarrow \widetilde{\Omega} = \begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}$ is the inverse of $\widetilde{\pi}$. Then the 1st-Riemann condition equivalent to

$$\begin{aligned}
&-\sqrt{-1}\,{}^t\widetilde{\pi}Q\overline{\widetilde{\pi}} = -\sqrt{-1}\begin{pmatrix}{}^t\pi Q\overline{\pi} & {}^t\pi Q\pi\\ {}^t\overline{\pi}Q\overline{\pi} & {}^t\overline{\pi}Q\pi\end{pmatrix} = \begin{pmatrix}H & O\\ O & -{}^tH\end{pmatrix} \text{ with } H > 0\\
\iff &\sqrt{-1}\overline{\widetilde{\Omega}}Q^{-1}\,{}^t\widetilde{\Omega} = \begin{pmatrix}H^{-1} & O\\ O & -{}^tH^{-1}\end{pmatrix} \text{ with } H > 0
\end{aligned}$$

By Kodaira embedding theorem, X is algebraic $\iff$ $\exists$ skew-symmetric integral $2n \times 2n$ matrix s.t.

$$\Omega Q^{-1}\,{}^t\Omega = 0 \text{ and } -\sqrt{-1}\Omega Q^{-1}\,{}^t\overline{\Omega} > 0 > 0$$

This is called **2nd Riemann condition**.

Lemma 4.1.1. If Q is integral skew-symmetric quadratic form on $\Lambda \simeq \mathbb{Z}^{2n}$, then there exists a basis $\lambda_1, ..., \lambda_{2n}$ for Λ in terms of which Q is given by the matrix

$$Q = \begin{pmatrix}O & \Delta_\delta\\ -\Delta_\delta & O\end{pmatrix} \text{ with } \Delta_\delta = \begin{pmatrix}\delta_1 & & 0\\ & \ddots & \\ 0 & & \delta_n\end{pmatrix} \in M_{n\times n}(\mathbb{Z})$$

with $\delta_1|\cdots|\delta_n$.

Proof: For fixed $\lambda \in \Lambda$, $\{Q(\lambda,\lambda') : \lambda' \in \Lambda\}$ is a principal ideal $d_\lambda\mathbb{Z}$ in $\mathbb{Z}$ for some $d_\lambda \geq 0$. Let $\delta_1 = \min(d_\lambda : \lambda \in \Lambda, d_\lambda \neq 0)$, and take λ_1 and λ_{n+1} s.t. $Q(\lambda_1, \lambda_{n+1}) = \delta_1$. Then $\forall \lambda \in \Lambda$,

$$\delta_1 \big| Q(\lambda,\lambda_1), Q(\lambda,\lambda_{n+1})$$

Then we have

$$\lambda + \frac{Q(\lambda,\lambda_1)}{\delta_1}\lambda_{n+1} - \frac{Q(\lambda,\lambda_{n+1})}{\delta_1}\lambda_1 \in \mathbb{Z}\{\lambda_1,\lambda_{n+1}\}^\perp$$

i.e.

$$\Lambda = \mathbb{Z}\{\lambda_1,\lambda_{n+1}\} \oplus \mathbb{Z}\{\lambda_1,\lambda_{n+1}\}^\perp$$

By induction, we may find $\lambda_1, ..., \lambda_{2n}$ s.t.

$$\Lambda = Z\{\lambda_1, \lambda_{n+1}\} \oplus \cdots \oplus \mathbb{Z}\{\lambda_n, \lambda_{2n}\}$$

with $\delta_i = Q(\lambda_i, \lambda_{n+i})$. It remains to show that $\delta_1 | \delta_2$. If not, $\exists k \in \mathbb{Z}$ s.t.

$$0 < Q(k\lambda_1 + \lambda_2, \lambda_{n+1} + \lambda_{n+2}) < \delta_1$$

contradict with the definition of δ_1. $\square$

By lemma, in terms of the new basis,

$$\omega = \sum_{i=1}^{n} \delta_i dx^i \wedge dx^{n+i}$$

If ω is non-degenerate, then $\delta_i \neq 0\ \forall i$, and we can take $e_\alpha = \delta^{-1}\lambda_\alpha,\ \alpha = 1, ..., n$ be the basis of $\mathbb{C}^n$. Then period matrix of $\Lambda \subset X$ will then be of the form

$$\Omega = (\Delta_\delta, Z)$$

which is called **normalized**. Then 2nd Riemann condition becomes to

$$\Omega Q^{-1}\,{}^t\Omega = 0 \iff 0 = \begin{pmatrix} \Delta_\delta & Z \end{pmatrix} \begin{pmatrix} O & -\Delta^{-1} \\ \Delta^{-1} & O \end{pmatrix} \begin{pmatrix} \Delta_\delta \\ {}^tZ \end{pmatrix} = Z - {}^tZ \iff Z \text{ is symmetric}$$

$$-\sqrt{-1}\Omega Q^{-1}\,{}^t\overline{\Omega} > 0 \iff 0 < -\sqrt{-1} \begin{pmatrix} \Delta_\delta & Z \end{pmatrix} \begin{pmatrix} O & -\Delta_\delta^{-1} \\ \Delta_\delta^{-1} & O \end{pmatrix} \begin{pmatrix} \Delta_\delta \\ {}^t\overline{Z} \end{pmatrix} = -\sqrt{-1}(Z - {}^t\overline{Z}) = 2\operatorname{Im} Z$$

the last equation need Z is symmetric. This is called **3nd Riemann condition**.

Theorem 4.1.2. *S is compact Riemann surface of $g \geq 1$, then* Jac(S) *is a variety.*

Proof: Consider the fundamental region Δ with boundary labeled by $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$. We may normalize a basis $\Omega_1, ..., \Omega_g$ of $\Omega^1(X)$ s.t.

$$\int_{\alpha_j} \omega_i = \delta_{ij}$$

Let ω be holomorphic 1-form, since Δ is simply connected, $\omega = d\varphi$ with holomorphic function

$$\varphi(z) = \int_{p_0}^{z} \omega$$

Let p, p' be the point on α_i, α_i^{-1} be the same point on S, then

$$\varphi(p') - \varphi(p) = \int_p^{p'} \omega = \left(\int_p^{\beta_1(0)} + \int_{\beta_1(0)}^{\beta_1(1)} + \int_{\beta_1(1)}^{p'} \right) \omega = \int_{\beta_1} \omega$$

Similarly, if q, q' be the point on β_i, β_i^{-1} be the same point on S, then

$$\varphi(q') - \varphi(q) = -\int_{\alpha_1} \omega$$

For any closed 1-form η,

$$\int_S \omega \wedge \eta = \int_\Delta (d\varphi \wedge \eta - \varphi \wedge d\eta) = \int_\Delta d(\varphi \wedge \eta) = \int_{\partial\Delta} \varphi\eta = \sum_{i=1}^{g} \left(\int_{\alpha_i + \alpha_i^{-1}} \varphi\eta + \int_{\beta_i + \beta_i^{-1}} \varphi\eta \right)$$

Let $s : [0,1] \to \alpha_i$, then

$$\int_{\alpha_i+\alpha_i^{-1}} \varphi\eta = \int_{\alpha_i} \varphi\eta - \int_{(\alpha_i^{-1})^{-1}} \varphi\eta = \int_0^1 (\varphi(s(t)) - \varphi(s(t)'))\,\eta(s(t))dt = \int_{\alpha_i} \left(-\int_{\beta_i} \omega\right)\eta$$

Similarly, we have

$$\int_{\beta_i+\beta_i^{-1}} \varphi\eta = \int_{\beta_i} \left(\int_{\alpha_i} \omega\right)\eta$$

Recall the entry of Ω is the coefficient of base change form dual basis of $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$ to $\omega_1, ..., \omega_g$, we have $\Omega = (I_g\ Z)$ with

$$z_{ij} = \int_{\beta_i} \omega_j$$

Since $\omega_i \wedge \omega_j = 0$,

$$0 = \int_S \omega_i \wedge \omega_j = \sum_{k=1}^{g} \left(-\int_{\alpha_k} \omega_j \int_{\beta_k} \omega_i + \int_{\alpha_k} \omega_i \int_{\beta_k} \omega_j\right) = z_{ij} - z_{ji}$$

which shows that Z is symmetric. Moreover, since $i\displaystyle\int_S \omega \wedge \overline{\omega} > 0$,

$$0 < \left(i\int_S \omega_i \wedge \overline{\omega_j}\right)_{i,j} = \left(i(-z_{ji} + \overline{z_{ij}})\right) = \left(\frac{z_{ij} - \overline{z_{ij}}}{2}\right) = 2\,\mathrm{Im}\,Z$$

Hence Z satisfy 3nd Riemann condition and thus $\mathrm{Jac}(S)$ is algebraic. $\square$

Chapter 5

Homework

5.1

Problem 5.1.1.

(a) **One point compactification** of $\mathbb{R}^n$. For $n \geq 1$ let ∞ be a symbol not belonging to $\mathbb{R}^n$. Introduce the following topology on the set $X := \mathbb{R} \cup \{\infty\}$. A set $U \subset X$ is open. By definition, if one of the following two condition is satisfied :

(1) $\infty \notin U$ and U is open in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$.

(2) $\infty \in U$ and $K := X \setminus U$ is compact in $\mathbb{R}^n$ with respect to the usual topology on $\mathbb{R}^n$.

Show that X is a compact Hausdorff topological space.

(b) **Stereographic projection**. Consider the unit n-sphere

$$S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

and the stereographic projection

$$\sigma : S^n \to \mathbb{R}^n \cup \{\infty\}$$

given by

$$\sigma(x_1, ..., x_{n+1}) := \begin{cases} \dfrac{1}{1 - x_{n+1}}(x_1, ..., x_n) & \text{if } x_{n+1} \neq 1 \\ \infty & \text{if } x_{n+1} = 1 \end{cases}$$

Show that σ is a homeomorphism of S^n onto X.

Problem 5.1.2. Identify $\mathbb{P}^1$ with the unit sphere in $\mathbb{R}^3$ using the stereographic projection

$$\sigma : S^2 \to \mathbb{C} \cup \{\infty\} \simeq \mathbb{P}^1$$

Let SO(3) be the group of orthogonal 3×3 matrices having determinant 1, i.e.

$$\mathrm{SO}(3) = \{A \in \mathrm{GL}(3, \mathbb{R}) : A^T A = I, \det A = 1\}.$$

For every $A \in \mathrm{SO}(3)$, show that the map

$$\sigma \circ A \circ \sigma^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$$

is biholomorphic.

Problem 5.1.3. Suppose X is a Riemann surface and $f : X \to \mathbb{C}$ is non-constant holomorphic function. Show that $\mathrm{Re}(f)$ does not attain its maximum.

Problem 5.1.4. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, whose real part is bounded from above. Then f is constant.

Problem 5.1.5. Suppose $f : X \to Y$ is a non-constant holomorphic map and

$$\begin{array}{rccc} f^* : & \mathcal{O}(Y) & \longrightarrow & \mathcal{O}(X) \\ & \varphi & \longmapsto & \varphi \circ f \end{array}$$

Show that f^* is a ring monomorphism.

5.2

Problem 5.2.1.

(a) Suppose X is a manifold and $U_1, U_2 \subset X$ are two open, connected and simply connected subsets such that $U_1 \cap U_2$ is connected. Show that $U_1 \cup U_2$ is simply connected.

(b) Using (a) show that S^n for $n \geq 2$ is simply connected.

Problem 5.2.2. Suppose X and Y are arcwise connected topological spaces. Prove $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

Problem 5.2.3. Let (X, a) and (Y, b) be topological spaces with base points $a \in X$ and $b \in Y$. Let $f, g : X \to Y$ be two continuous map with $f(a) = g(a) = b$. Then f and g are called homotopic if there exists a continuous map

$$F : X \times [0, 1] \to Y$$

such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for every $x \in X$ and $F(a, t) = b$ for every $t \in [0, 1]$. Consider the induced maps

$$f_*, g_* : \pi_1(X, a) \to \pi_1(Y, b)$$

Show that $f_* = g_*$ if f and g are homotopic.

Problem 5.2.4. Let $X := \mathbb{C} \setminus \{\pm 1\}$, $Y := \mathbb{C} \setminus \{(\pi/2) + k\pi : k \in \mathbb{Z}\}$. Show that

$$\sin : Y \to X$$

is a topological covering map. Consider the following curves in X.

$$u : [0, 1] \to X,\ u(t) := 1 - e^{2\pi i t}$$
$$v : [0, 1] \to X,\ v(t) := -1 + e^{2\pi i t}$$

Let $w_1 : [0, 1] \to Y$ be the lifting of $u \cdot v$ with $w_1(0) = 0$ and $w_2 : [0, 1] \to Y$ be the lifting of $v \cdot u$ with $w_2(0) = 0$. Show that

$$w_1(1) = 2\pi$$
$$w_1(1) = -2\pi$$

Conclude that $\pi_1(X)$ is not abelian.

Problem 5.2.5. Let X and Y be arcwise connected Hausdorff topological spaces and $f : Y \to X$ be a covering map. Show that the induced map

$$f_* : \pi_1(Y) \to \pi_1(X)$$

is injective.

5.3

Problem 5.3.1. Let X and Y be Hausdorff space and $p : Y \to X$ be a covering map. Let Z be a connected, locally arcwise connected topological space and $f : Z \to X$ a continuous map. Let $c \in Z$, $a := f(c)$ and $b \in Y$ such that $p(b) = a$. Prove that there exists a lifting $\widetilde{f} : Z \to Y$ of f with $\widetilde{f}(c) = b$ if and only if $f_*\pi_1(Z, c) \subset p_*\pi_1(Y, b)$.

Problem 5.3.2.

(a) Show that

$$\tan : \mathbb{C} \to \mathbb{P}^1$$

is a local homeomorphism.

(b) Show that $\tan(\mathbb{C}) = \mathbb{P}^1 \setminus \{\pm i\}$. Show that for every $k \in \mathbb{Z}$ there exists a unique holomorphic function $\arctan_k : X \to \mathbb{C}$ with

$$\tan \circ \arctan_k = \mathrm{id}_X$$

and

$$\arctan_k(0) = k$$

the kth branch of arctan.

Problem 5.3.3. Let $X = \mathbb{C} \setminus \{\pm 1\}$, and $Y = \mathbb{C} \setminus \{\pi/2 + k\pi : k \in \mathbb{Z}\}$. Prove that $\mathrm{Deck}(Y \xrightarrow{\sin} X)$ consists of the following transformations

(i) $f_k(z) = z + 2k\pi$, $k \in \mathbb{Z}$

(ii) $g_k(z) = -z + (2k + z)\pi$, $k \in \mathbb{Z}$

Calculate the products $f_k \circ f_\ell$, $f_k \circ g_\ell$, $g_\ell \circ f_k$, $g_k \circ g_\ell$.

Problem 5.3.4. Determine the covering transformations of

$$\tan : \mathbb{C} \to \mathbb{P}^1 \setminus \{i, -i\}$$

5.4

Problem 5.4.1. Let $\Gamma, \Gamma' \subset \mathbb{C}$ be lattices and

$$f : \mathbb{C} \setminus \Gamma \to \mathbb{C} \setminus \Gamma'$$

a non-constant holomorphic map with $f(0) = 0$ and the following diagram is commutative

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}/\Gamma & \xrightarrow{f} & \mathbb{C}/\Gamma' \end{array}$$

where $F(z) = \alpha z$ and π and π' are the canonical projection. Prove that f is an unbranched covering map and

$$\mathrm{Deck}(\mathbb{C} \setminus \Gamma \xrightarrow{f} \mathbb{C}/\Gamma') \simeq \Gamma'/\alpha\Gamma$$

Problem 5.4.2. Let $X := \mathbb{C} \setminus \{\pm 2\}$, $Y = \mathbb{C} \setminus \{\pm 1, \pm 2\}$, and let $p : Y \to X$ be the map

$$p(z) := z^3 - 3z$$

Prove that p is an unbranched 3-sheeted holomorphic covering map. Calculate $\mathrm{Deck}(Y/X)$ and show that the covering $Y \to X$ is not Galois.

Problem 5.4.3. Let $X := \mathbb{C} \setminus \{0, 1\}$, $Y := \mathbb{C} \setminus \{0, \pm i, \pm i\sqrt{2}\}$ and let $p : Y \to X$ be the map

$$p(z) := (z^2 + 1)^2$$

Prove that p is an unbranched 4-sheeted covering map, which is not Galois and that

$$\mathrm{Deck}(Y/X) = \{\mathrm{id}, \varphi\}$$

where $\varphi(z) := -z$.

Problem 5.4.4. Suppose X and Y are connected Hausdorff spaces. Show that every 2-sheeted covering map $p : Y \to X$ is Galois.

5.5

Problem 5.5.1. Suppose X is a Riemann surface and $a \in X$. Suppose $\varphi \in \mathcal{O}_a$ admits an analytic continuation along every curve in X which starts at a. Let (Y, p, f, b) be the maximal analytic continuation of φ. Prove that $p : Y \to X$ is a covering map.

5.6

Problem 5.6.1. Suppose $p : Y \to X$ is a holomorphic mapping of Riemann surfaces, $a \in X$, $b \in p^{-1}(a)$ and k is the multiplicity of p ar b. Given any holomorphic 1-form ω on $X \setminus \{a\}$ show that

$$\mathrm{Res}_b(p^*\omega) = k\,\mathrm{Res}_a(\omega)$$

5.7

Problem 5.7.1. Let X be a Riemann surface and ω be a holomorphic 1-form on X. Suppose φ is a primitive of ω on a neighborhood of a point $a \in X$ and (Y, p, f, b) is a maximal analytic continuation of φ. Prove

(a) $p : Y \to X$ is a covering map.

(b) f is a primitive of $p^*\omega$.

(c) The covering $p : Y \to X$ is a Galois and $\mathrm{Deck}(Y/X)$ is abelian.

Problem 5.7.2. Suppose X is a Riemann surface and $\omega \in \mathfrak{M}^{(1)}(X)$ is a meromorphic 1-form on X which has residue zero at every pole. Show that there is a covering $p : Y \to X$ and a meromorphic function $F \in \mathfrak{M}(X)$ such that $dF = p^*\omega$.

Problem 5.7.3.

(a) Let X be a manifold, $U \subset X$ open and $V \Subset U$. Show that V meets only a finite number of connected components of U.

(b) Let X be a compact manifold and $\mathcal{U} = (U_i)_{i \in I}, \mathcal{B} = (V_i)_{i \in I}$ be two finite open coverings of X such that $V_i \Subset U_i$ for every $i \in I$. Prove that

$$\operatorname{Im}(Z^1\left(\mathcal{B}, \mathbb{C}) \to Z^1(\mathcal{B}, \mathbb{C})\right)$$

is a finite-dimensional vector space.

(c) Let X be a compact Riemann surface. Prove that $H^1(X, \mathbb{C})$ is a finite-dimensional vector space.

5.8

Problem 5.8.1.

(a) Let X be a compact Riemann surface. Prove that the map

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$ is injective.

(b) Let X be a compact Riemann surface. Show that $H^1(X, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module.

Problem 5.8.2. Let $X = \{z \in \mathbb{C} : |z| < R\}$, where $0 < R \leq \infty$. Denoted by $\mathcal{H}$ the sheaf of harmonic function on X, i.e.

$$\mathcal{H}(U) = \{f : U \to \mathbb{C} : f \text{ is harmonic}\}$$

for $U \subset X$ open. Prove

$$H^1(X, \mathcal{H}) = 0.$$

Problem 5.8.3.

(a) Show that $\mathcal{U} = (\mathbb{P}^1 \setminus \infty, \mathbb{P}^1 \setminus 0)$ is a Leray covering for the sheaf Ω of holomorphic 1-forms on $\mathbb{P}^1$.

(b) Prove that

$$H^1(\mathbb{P}^1, \Omega) \simeq H^1(\mathcal{U}, \Omega) \simeq \mathbb{C}$$

and that the cohomology class of

$$\frac{dz}{z} \in \Omega(U_1 \cap U_2) \simeq Z^1(\mathcal{U}, \Omega)$$

is a basis of $H^1(\mathbb{P}^1, \Omega)$.

Problem 5.8.4. Suppose $g \in \mathscr{E}(\mathbb{C})$ is a function with compact support. Prove that there is a solution $f \in \mathscr{E}(\mathbb{C})$ of the equation
$$\frac{\partial f}{\partial \overline{z}} = g$$
having compact suppoet if and only if
$$\iint_{\mathbb{C}} z^n g(z) dz \wedge d\overline{z} = 0 \text{ for every } n \in \mathbb{N}.$$

5.9

Problem 5.9.1. Let $X = \{z \in \mathbb{C} : r < |z| < R\}$, where $0 < r < R < \infty$. Determine an orthonormal basis of $L^2(X, \mathcal{O})$ consisting of function of the form
$$\varphi_n(z) = c_n z^n,\ n \in \mathbb{N}$$

Problem 5.9.2. Let $X \subset \mathbb{C}$ be a bounded open subset, $p_1, ..., p_k \in X$ and $X' = X \setminus \{p_1, ..., p_k\}$. Show that the restriction map
$$L^2(X, \mathcal{O}) \to L^2(X', \mathcal{O})$$
is an isomorphism.

5.10

Problem 5.10.1. Let X be a Riemann surface and $\mathscr{H}$ be the sheaf of harmonic functions on X. Verify that the sequence
$$0 \to \mathscr{H} \to \mathscr{E} \xrightarrow{d'd''} \mathscr{E}^{(2)} \to 0$$
is exact.

Problem 5.10.2. Show that on any Riemann surface the sequence
$$0 \to \mathbb{C}^* \to \mathcal{O}^* \xrightarrow{d\log} \Omega \to 0$$
is exact, where $(d\log)f := f^{-1}df$.

Problem 5.10.3. On a Riemann surface X let $\mathscr{L} \subset \mathscr{M}^{(1)}$ be the sheaf of meromorphic 1-forms which have residue 0 at every point. Show that the sequence
$$0 \to \mathbb{C} \to \mathscr{M} \xrightarrow{d} \mathscr{L} \to 0$$
is exact.

Problem 5.10.4. Let $X = \mathbb{C}/\Gamma$ be a tours. Prove that
$$H^1(X, \mathbb{C}) \simeq \mathrm{Rh}^1(X) \simeq \mathbb{C}^2$$
and that the classes of dz and $d\overline{z}$ form a basis of $\mathrm{Rh}^1(X)$.

5.11

Problem 5.11.1. Let $X \to \mathbb{P}^1$ be the Riemann surface of the algebraic function $\sqrt[n]{1-z^n}$ i.e. the algebraic function defined by the polynomial

$$P(T) = T^n + z^n - 1 \in \mathscr{M}(\mathbb{P}^1)[T]$$

where $z \in \mathscr{M}(\mathbb{P}^1)$ is the canonical coordinate function. Show that the genus of X is

$$g = \frac{(n-1)(n-2)}{2}$$

Problem 5.11.2. Let X be a compact Riemann surface. Let $\mathscr{Q}(X) \subset \mathscr{M}^{(1)}(X)$ be the space of all meromorphic 1-forms on X whose residues vanish at every point. Show that

$$H^1(X, \mathbb{C}) \simeq \mathscr{Q}(X)/d\mathscr{M}(X)$$

Problem 5.11.3. Let $X = \mathbb{C}/\Gamma$ be a torus. Show that the classes of dz and $\wp_\Gamma dz$ form a basis of $\mathscr{Q}(X)$ mod $d\mathscr{M}(X)$.

Problem 5.11.4. Let D be a divisor on the compact Riemann surface X of genus g. Show

$$\begin{cases} \dim H^0(X, \mathcal{O}_D) = 0 & \text{for } \deg D \leq -1 \\ 0 \leq \dim H^0(X, \mathcal{O}_D) \leq 1 + \deg D & \text{for } -1 \leq \deg D \leq g-1 \\ 1 - g + \deg D \leq \dim H^0(X, \mathcal{O}_D) \leq g & \text{for } g-1 \leq \deg D \leq 2g-1 \\ \dim H^0(X, \mathcal{O}_D) = 1 - g + \deg D & \text{for } \deg D \geq 2g-1 \end{cases}$$

5.12

Problem 5.12.1. Let D be a divisor on the Riemann sphere $\mathbb{P}^1$. Prove

(a) $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = \max(0, 1 + \deg D)$

(b) $\dim H^1(\mathbb{P}^1, \mathcal{O}_D) = \max(0, -1 - \deg D)$

Problem 5.12.2. Let $X = \mathbb{C}/\Gamma$ be a torus, $x_0 \in X$ be a point and P be the divisor

$$P(x) = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{if } x \neq x_0 \end{cases}$$

Show

$$\dim H^0(X, \mathcal{O}_{nP}) = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n = 0 \\ n & \text{for } n \geq 1 \end{cases}$$

Problem 5.12.3. Let X be a compact Riemann surface, D be a divisor on X and $\mathfrak{U} = (U_i)$ be an open covering of X such that every U_i is isomorphic to a disk. Show that $\mathfrak{U}$ is a Leray covering for the sheaf $\mathcal{O}_D$.

Problem 5.12.4.

(a) On a Riemann surface X let $\mathfrak{D}$ be the sheaf of divisors, i.e., for $U \subset X$ open $\mathfrak{D}(U)$ consists of all maps
$$D : U \to \mathbb{Z}$$
such that for every compact set $K \subset U$ there are only finitely many $x \in K$ with $D(x) \neq 0$. Show that $\mathfrak{D}$ together with the natural restriction morphisms is actually a sheaf and that
$$H^1(X, \mathfrak{D}) = 0$$

(b) Let $\beta : \mathscr{M}^* \to \mathfrak{D}$ be the map which assigns to every meromorphic function $f \in \mathscr{M}^*(U)$ its divisor $(f) \in \mathfrak{D}(U)$ and let $\alpha : \mathcal{O}^* \to \mathscr{M}^*$ be the natural inclusion map. Show that
$$0 \to \mathcal{O}^* \xrightarrow{\alpha} \mathscr{M}^* \xrightarrow{\beta} \mathfrak{D} \to 0$$
is an exact sequence of sheaves and thus that there is an exact sequence of groups
$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathscr{M}^*) \to \mathrm{Div}(X) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathscr{M}^*) \to 0$$

5.13

Problem 5.13.1. Let K be a canonical divisor on a compact Riemann surface X of genus > 0, and let $D \geq K$ be a divisor with $\deg D = \deg K + 1$. Show that the sheaf $\mathcal{O}_K$ is globally generated, but $\mathcal{O}_D$ is not.

Problem 5.13.2. Let $\Gamma \subset \mathbb{C}$ be a lattice and let $\wp$ be the Weierstrass $\wp$-function with respect to Γ. Interpret $\wp$ and its derivative $\wp'$ as meromorphic functions on $\mathbb{C}/\Gamma$. Show that
$$(1 : \wp, \wp') : \mathbb{C}/\Gamma \to \mathbb{P}^2$$
is an embedding.

Problem 5.13.3. Let X be a compact Riemann surface of genus two. Suppose ω_1 and ω_2 form a basis of $H^0(X, \Omega)$ and define $f \in \mathscr{M}(X)$ by $\omega_1 = f\omega_2$. Show that $f : X \to \mathbb{P}^1$ is a 2-sheeted (branched) covering map.

5.14

Problem 5.14.1. Let $U := \{z \in \mathbb{C} : |z| < r\}$, $r > 0$, and let $f : U \to \mathbb{C}$ be a holomorphic function with $f(0) \neq 0$.

(a) Define $f_j(z) = z^{j-1} f(z)$ for $j = 1, ..., g$. Prove that the Wronskian determinant $W(f_1, ..., f_g)$ does not vanish at the origin.

(b) Define $\varphi_j(z) := z^{2j-2} f(z)$ for $j = 1, ..., g$. Prove that the Wronskian determinant $W(\varphi_1, ..., \varphi_g)$ has a zero of order $g(g-1)/2$ at the origin.

Problem 5.14.2. Let $\pi : X \to \mathbb{P}^1$ be a hyperelliptic Riemann surface of genus $g \geq 2$.

(a) Show that all the ramification points $p_1, ..., p_{2g+2} \in X$ of π are Weierstrass points of X.

(b) Prove that there are no other Weierstrass points and that every Weierstrass point p_j has weight $g(g-1)/2$.

Problem 5.14.3. Let X be a compact Riemann surface of genus $g \geq 1$ and suppose $\omega_1, ..., \omega_g$ os a basis of $\Omega(X)$. Let $D \geq 0$ be a non-negative divisor on X. Denote by M_D the of all Mittag-Leffler distributions $\mu \geq -D$ on X, i.e. the set of all Mittag-Leffler distributions lying in $C^0(\mathfrak{U}, \mathcal{O}_D)$ for some open covering $\mathfrak{U}$ of X. Define a linear map

$$R : M_D \to \mathbb{C}^g$$

by

$$R(\mu) = (\mathrm{Res}(\mu\omega_1), ..., \mathrm{Res}(\mu\omega_g))$$

Prove

$$\dim H^1(X, \mathcal{O}_D) = g - \dim R(M_D)$$

5.15

Problem 5.15.1. Let X be a compact Riemann surface. Prove

(a) $d\mathscr{E}^{0,1}(X) = d'd''\mathscr{E}(X) \subset \mathscr{E}^{(2)}(X)$.

(b) Let $\mathscr{H}$ be the sheaf of harmonic functions on X. Then

$$H^1(X, \mathscr{H} \simeq \mathscr{E}^{(2)}(X)/d'd''\mathscr{E}(X) \simeq \mathbb{C}$$

(c) Let $\omega \in \mathscr{E}^{(2)}(X)$. Prove that there exists a function $f \in \mathscr{E}(X)$ such that

$$d'd''f = \omega$$

if and only if

$$\iint_X \omega = 0$$

Problem 5.15.2. Let $X = \mathbb{C}/\Gamma$ be a tours. For a function $f \in \mathscr{E}(X)$ define its mean value $M(f)$ by

$$M(f) = \left(\iint_X f dz \wedge d\overline{z}\right)\left(\iint_X dz \wedge d\overline{z}\right)^{-1}$$

For $\omega = f dz + g d\overline{z} \in \mathscr{E}^{(1)}(X)$ let $M(\omega) := M(f)dz + M(g)d\overline{z}$. Show

(a) If $\omega \in \mathscr{L}(X) := \ker(\mathscr{E}^{(1)}(X) \xrightarrow{d} \mathscr{E}^{(2)}(X))$, then ω and $M(\omega)$ are cohomologus.

(b) The mapping

$$M : \mathscr{L}(X) \to \mathrm{Harm}^1(X)$$

induces an isomorphism

$$\mathrm{Rh}^1(X) \xrightarrow{\sim} \mathrm{Harm}^1(X)$$

5.16

Problem 5.16.1. Let X be a compact Riemann surface and $Y \subset X$ be an open subset such that $X \setminus Y$ has non-empty interior. Let D be a divisor on X. Show that there exists a function $f \in \mathscr{M}^*(X)$ such that

$$\mathrm{ord}_x(f) = D(x) \text{ for every } x \in Y$$

Problem 5.16.2.

(a) Show that the polynomial

$$F(z) := 4z^3 - g_2 z - g_3,\ \ g_2, g_3 \in \mathbb{C}$$

has 3 distinct roots if and only if

$$g_2^3 - 27g_3^2 \neq 0$$

(b) Let $\Gamma \subset \mathbb{C}$ be a lattice and

$$\wp(z) = \frac{1}{z^3} + \sum_{\omega \in \Gamma \backslash 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

be the associated Weierstrass $\wp$-function. Show that $\wp$ satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

where

$$g_2 = 60 \sum_{\omega \in \Gamma \backslash 0} \frac{1}{\omega^4} \text{ and } g_3 = 140 \sum_{\omega \in \Gamma \backslash 0} \frac{1}{\omega^6}$$

and that the tours $\mathbb{C}/\Gamma$ is isomorphic to the Riemann surface $X \to \mathbb{P}^1$ of the algebraic function $\sqrt{4z^3 - g_2 z - g_3}$.

(c) Given $g_2, g_3 \in \mathbb{C}$ with $g_2^3 \neq 27g_3^2$, show that there is a lattice $\Gamma \subset \mathbb{C}$ such that

$$g_2 = \sum_{\omega \in \Gamma \backslash 0} \frac{1}{\omega^4} \text{ and } g_3 = 140 \sum_{\omega \in \Gamma \backslash 0} \frac{1}{\omega^6}$$

5.17

Problem 5.17.1. Let X be a compact Riemann surface, α and β be closed curves in X and σ_α and σ_β be the harmonic 1-forms associated to α and β according Corollary. Show that

$$\iint_X \sigma_\alpha \wedge \sigma_\beta$$

is an integer.

Problem 5.17.2. Let $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 \subset \mathbb{C}$ be a lattice, $X = \mathbb{C}/\Gamma$ and

$$\alpha_j : [0, 1] \to X$$

be the closed curves defined by

$$\alpha_j(t) := \pi(t\gamma_j)$$

where $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is the canonical projection. Find the harmonic forms σ_{α_j}.

Problem 5.17.3. Let X be a compact Riemann surface of genus g. Show that there exists closed curves $\alpha_1, ..., \alpha_{2g}$ on X such that

$$\operatorname{Harm}^1(X) = \sum_{j=1}^{2g} \mathbb{C}\sigma_{\alpha_j}$$

5.18

Problem 5.18.1. Let X be an open subset of $\mathbb{C}$ and $T_v \in \mathscr{D}'(X)$, $v \in \mathbb{Z}_{\geq 0}$, a sequence of distributions in X. A sequence $(T_v)_{v\in\mathbb{Z}_{\geq 0}}$ is said to converge to a distribution $T \in \mathscr{D}'(X)$ if

$$T_v[\varphi] \to T[\varphi] \text{ for every } \varphi \in \mathscr{D}(X)$$

Denote this by $T_v \underset{\mathscr{D}}{\to} T$. Show that if $T_v \underset{\mathscr{D}}{\to} T$, then

$$D^\alpha T_v \underset{\mathscr{D}}{\to} D^\alpha T$$

for every differential operator

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2}$$

Problem 5.18.2. Let $Y \subset \mathbb{C}$ be open. A sequence of continuous function $f_v : Y \to \mathbb{C}$ is said to converge weakly to a continuous function $f : Y \to \mathbb{C}$ if

$$\iint_Y f_v \varphi dx dy \to \iint_Y f\varphi \text{ for every } \varphi \in \mathscr{D}(X)$$

Show that if all the f_v are harmonic (resp. holomorphic) and converge weakly to f, then f is also harmonic (resp. holomorphic).

5.19

Problem 5.19.1. Suppose X is a Riemann surface and $\widetilde{X} \to X$ is its universal covering. Show that $\mathrm{Deck}(\widetilde{X}/X)$ is countable.

Problem 5.19.2. Let $Y \subset \mathbb{C}$ be open and K a compact connected component of $\mathbb{C} \setminus Y$. Let $(f_v)_{v\in\mathbb{Z}_{\geq 0}}$ be a sequence of polynomials which converges uniformly on every compact subset of Y. Show that (f_v) converges uniformly on K.

Problem 5.19.3. Suppose $Y \subset \mathbb{C}$ is an open subset such that every holomorphic function $f \in \mathcal{O}(Y)$ can be approximated uniformly on every compact subset of Y by polynomials. Conclude that $Y = h_{\mathbb{C}}(Y)$.

5.20

Problem 5.20.1. Let X be a non-compact Riemann surface. Prove

$$H^1(X, \Omega) = 0$$

Problem 5.20.2. Let X be a non-compact Riemann surface.

(a) Given any $\omega \in \mathscr{E}^{(2)}(X)$ show that there exists $f \in \mathscr{E}(X)$ with

$$d'd''f = \omega$$

(b) Let $\mathscr{H}$ be the sheaf of harmonic functions on X. Show that

$$H^1(X, \mathscr{H}) = 0$$

Problem 5.20.3. Let X be a non-compact Riemann surface and suppose $f, g \in \mathcal{O}(X)$ are holomorphic function which have no common zero.

(a) Show that the following sequence of sheaves is exact

$$0 \to \mathcal{O} \xrightarrow{\alpha} \mathcal{O}^2 \xrightarrow{\beta} \mathcal{O} \to 0$$

where

$$\alpha(\psi) := (\psi g, -\psi f)$$
$$\beta(\varphi_1, \varphi_2) := \varphi_1 f + \varphi_2 g$$

(b) Show that there exists holomorphic function $\Phi, \Psi \in \mathcal{O}(X)$ such that

$$\Phi f + \Psi g = 1$$

Problem 5.20.4. Let X be a non-compact Riemann surface and let

$$\mathfrak{U} = \sum_{j=1}^{k} \mathcal{O}(X) f_j, \ \ f_j \in \mathcal{O}(X)$$

be a finitely generated ideal in $\mathcal{O}(X)$. Prove that $\mathfrak{U}$ is a principal ideal.

5.21

Problem 5.21.1. Let X be a Riemann surface. Prove

$$H^1(X, \mathbb{C}) \simeq \mathrm{Hom}(\pi_1(X), \mathbb{C})$$

Problem 5.21.2. Let X be a non-compact Riemann surface and let

$$\mathrm{Rh}^1_{\mathcal{O}}(X) := \Omega(X) \big/ d\mathcal{O}(X)$$

be the "holomorphic" deRham group. Prove

$$H^1(X, \mathbb{C}) \simeq \mathrm{Rh}^1_{\mathcal{O}}(X)$$

5.22

Problem 5.22.1. Let X be a Riemann surface and

$$f_v : X \to \mathbb{C} \setminus \{0, 1\}$$

be a sequence of holomorphic function which do not take the values 0 and 1. Suppose there exists a point $x_0 \in X$ such that $\{f_v(x_0)\}_{v \in \mathbb{N}}$ converges to a point $c \in \mathbb{C} \setminus \{0, 1\}$. Prove that there exists a subsequence $(f_{v_k})_{k \in \mathbb{N}}$ which converges uniformly on every compact subset of X to a holomorphic function

$$f : X \to \mathbb{C} \setminus \{0, 1\}$$

Problem 5.22.2. Prove the "Big theorem of Picard" : Let

$$U = \{z \in \mathbb{C} : 0 < |z| < r\} \text{ where } r > 0$$

and

$$f : U \to \mathbb{C}$$

a holomorphic function having an essential singularity at the origin. Then f attains every value $c \in \mathbb{C}$ with at most one exception.

5.23

Problem 5.23.1. Show that on any Riemann surface X (compact or not) one has $H^1(X, \mathscr{M}^*) = 0$.

Problem 5.23.2. Let X be a Riemann surface. For $U \subset X$ open, let $\mathrm{SL}_n(\mathcal{O}(U))$ be the group of all $n \times n$ matrices of determinant 1 with coefficients in $\mathcal{O}(U)$. Together with the natural restriction maps this defines a sheaf $\mathrm{SL}_n(\mathcal{O})$ on X. Prove that on a non-compact Riemann surface X

$$H^1(X, \mathrm{SL}_n(\mathcal{O})) = 0$$

5.24

Problem 5.24.1. Let X be a Riemann surface and $\pi : E \to X$ be a holomorphic vector bundle of rank n on X. A holomorphic subbundle $F \subset D$ of rank k is a subset such that the following holds. For every $x \in X$, there exists a holomorphic local trivialization

$$h; E_U \to U \times \mathbb{C}^n, \; E_U = \pi^{-1}(U)$$

of E with $x \in U$ such that

$$h(F_U) = U \times (\mathbb{C}^k \times 0)$$

where $F_U = E_U \cap F$.

(a) Let $f : X \to E$ be a holomorphic section of E which never vanishes. For $x \in X$ define $F_x := \mathbb{C} \cdot f(x) \subset E_x$. Show that

$$F := \bigcup_{x \in X} F_x \subset E$$

is a holomorphic subbundle of E of rank 1.

(b) Let f be a meromorphic section of E over X. Show that there exists a unique subbundle $F \subset E$ of rank 1 such that f is a meromorphic section of F.

Problem 5.24.2. Let $L \to X$ be a line bundle on a compact Riemann surface X. Then degree of L is defined as $\deg(L) := \deg(D)$, where D is the divisor of a meromorphic section s of L over X.

(a) Show that $\deg L$ is well-defined, i.e. it is independent of the choice of the meromorphic section s.

(b) On $\mathbb{P}^1$ let $\mathfrak{U} = (U_1, U_2)$ be the covering given by

$$U_1 := \{z \in \mathbb{C} : |z| < 1 + \varepsilon\}, \; U_2 := \{z \in \mathbb{P}^1 : |z| > 1 - \varepsilon\}, \; 0 < \varepsilon < 1$$

Let L be the holomorphic line bundle on $\mathbb{P}^1$ defined by some given transition function

$$g_{12} : U_1 \cap U_2 \to \mathbb{C}^\times$$

Prove that

$$\deg L = \frac{1}{2\pi i} \int_{|z|=1} \frac{g_{12}'(z)}{g_{12}(z)} dz$$