

§4. Two important morphisms

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§5. Separated morphisms.

⊙ Separated morphisms

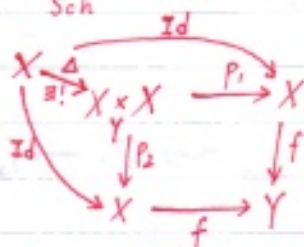
Recall that if X is a topological space, then

X is Hausdorff $\Leftrightarrow \Delta = \{(a, a) \in X \times X \mid a \in X\}$ is closed w.r.t. the product topology.
(The topology of a scheme $X \times X$ is generally stronger than the prod. top.)

$$\text{eg. } \mathbb{C} \times \mathbb{C} \supset \{(a, a) \mid a \in \mathbb{C}\} = V(x-y)$$

\mathbb{C}^2 closed w.r.t. Zariski top
not closed w.r.t. product top.

Def: $f \in \text{Hom}_{\text{Sch}}(X, Y)$



Δ is called the diagonal morphism.

Here, it is a relative version.

($Y = \text{Spec } k$, it is)

* U, V : affine
 $\Rightarrow U \cap V = \Delta(U \cap V)$
closed in $U \times V$
 \Rightarrow affine.

f is separated if Δ is a closed immersion.

Here, we say that X is separated over Y .

X is separated if it is separated over $\text{Spec } \mathbb{Z}$. ($\mathbb{Z} \rightarrow \mathcal{O}_X(U)/\mathcal{O}_X(V)$)

Prop 1: $f: X \rightarrow Y$ is separated $\Leftrightarrow \Delta(X)$ is a closed subset of $X \times_Y X$.

(pf): " \Rightarrow ": o.k.

" \Leftarrow ": $\because p_1 \circ \Delta = \text{Id}$

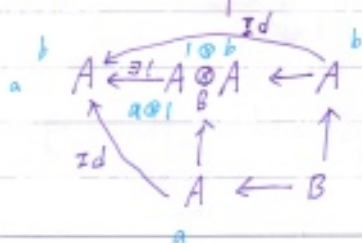
$\therefore X \rightarrow \Delta(X)$ is a homeomorphism.

Any $p \in X$, $f(p) \in \text{Spec } A \subset Y$ and take $\text{Spec } B \subset f^{-1}(\text{Spec } A)$.

$$\text{So } f: \underset{U}{\text{Spec } B} \rightarrow \underset{V}{\text{Spec } A}$$

Lemma 1: If $f: X \rightarrow Y$ with X, Y affine, then f is separated.

(pf): Let $X = \text{Spec } A$ and $Y = \text{Spec } B$.



$$A \otimes_B A \rightarrow A$$

$$a \otimes b \mapsto ab$$

$$\underset{X}{\text{Spec } A} \xrightarrow{\Delta} \underset{X \times_Y X}{\text{Spec } A \otimes_B A} \text{ is a closed immersion.}$$

" \Leftarrow ": For $x, y \in X$, $p = (x, y) \in X \times X$. $\therefore x \neq y \therefore p \notin \Delta \rightarrow p \in B \subseteq (X \times X) \setminus \Delta \rightarrow x \in U, y \in V$
 with $x \neq y$ $U \times V$ & $U \cap V = \emptyset$

" \Rightarrow ": Take $p \in X \times X \setminus \Delta$, say $p = (x, y)$ with $x \neq y$ and pick $U, V \subseteq X$ with $x \in U, y \in V$ & $U \cap V = \emptyset$
 open DATE

Then $p \in U \times V$ & $U \times V \subseteq X \times X \setminus \Delta$. Hence $X \times X \setminus \Delta$ is open $\Rightarrow \Delta$ is closed

Hence $\Delta_U : U \rightarrow U \times_U U$ is a closed immersion

$$\Rightarrow \Delta_U^* : \mathcal{O}_{U \times_U U} \rightarrow \Delta_U^* \mathcal{O}_U$$

$$\Rightarrow \mathcal{O}_{X \times X, \Delta(p)} \rightarrow \mathcal{O}_{X, p}$$

Therefore, $\mathcal{O}_{X \times X} \rightarrow \Delta_* \mathcal{O}_X$.

Eg 1: Any immersion is separated.

Qf: • $f: Y \rightarrow X$ is a closed immersion:

Let $U = \text{spec } A \subset X$ and thus $V = f^{-1}(U) \xrightarrow{f} \text{spec } A \subset \text{spec } A$.

$$\therefore \frac{A}{I} \otimes_A \frac{A}{I} \cong \frac{A}{I \cdot I} \cong \frac{A}{I} \quad \therefore \Delta|_V : V \xrightarrow{\sim} V \times_U V$$

$$\left(\begin{array}{l} M \otimes_A M \cong M \\ m \otimes n \mapsto mn \end{array} \right)$$

$$\Rightarrow \Delta : Y \xrightarrow{\sim} Y \times_X Y$$

An isomorphism is a closed immersion.

• $g: Z \rightarrow X$ is an open immersion:

Let $U = \text{spec } A \subset f(Z)$ and thus $V = f^{-1}(U) \xrightarrow{\sim} \text{spec } A$.

$$\text{Trivially, } \Delta|_V : V \xrightarrow{\sim} V \times_U V \Rightarrow \Delta : Z \xrightarrow{\sim} Z \times_X Z$$

$\Rightarrow \Delta$ is a closed immersion.

Eg 2: Let $X = \text{spec } \mathbb{C}[X]$, $Y = \text{spec } \mathbb{C}[Y]$ and

$$U = X \setminus \{0\} = \text{spec } \mathbb{C}[X, \frac{1}{X}], \quad V = Y \setminus \{0\} = \text{spec } \mathbb{C}[Y, \frac{1}{Y}].$$

We have $\varphi : U \xrightarrow{\sim} V$ via $\mathbb{C}[Y, \frac{1}{Y}] \xrightarrow{\sim} \mathbb{C}[X, \frac{1}{X}]$.

$$f(Y, \frac{1}{Y}) \mapsto f(X, \frac{1}{X}).$$

Let Z be the scheme obtained by gluing X and Y through φ .

Then Z is not separated over \mathbb{C} .

Indeed, $X \times_{\mathbb{C}} X, X \times_{\mathbb{C}} Y, Y \times_{\mathbb{C}} X, Y \times_{\mathbb{C}} Y$ cover $Z \times_{\mathbb{C}} Z$

and the closed points of $Z \times_{\mathbb{C}} Z = \{(z_1, z_2) : z_1, z_2 \text{ closed points in } Z\}$
 $\Delta(Z) = \{(z, z) : z \text{ closed point of } Z\}$

and $\Delta(Z) \cap (X \times_c Y) = \{(x, x) \mid x \in X \cap Y = \bigcup_{\substack{\uparrow \\ \text{open}}} \}$ is not closed in $(X \times_c Y)$ since $(0_x, 0_y)$ is missing.

$\forall p \in Y, \exists V_p$ s.t. $f^{-1}(V_p)$ is quasi-compact.

Prop 2: Let $f: X \rightarrow Y$ be a quasi-compact morphisms of schemes.
Then $f(X)$ is a closed subset of Y
 $\Leftrightarrow f$ is stable under specialization.

($x_0 \in \overline{\{x_1\}}$, if $x_1 \in f(X)$, then $x_0 \in f(X)$.)
 \uparrow
a specialization of x_1

(pf): " \Rightarrow " $x_1 \in f(X)$: closed $\Rightarrow x_0 \in \overline{\{x_1\}} \subset f(X)$. $Y = f(X)$

" \Leftarrow ": • $X = \text{spec } A, Y = \text{spec } B$ with $(X) Y$ reduced and $\frac{B}{f(X)} = \overline{f(X)}$:

• $B \hookrightarrow A$:

Assume $q \mapsto \langle 0 \rangle \Rightarrow \frac{B}{q} \rightarrow A \Rightarrow \text{spec } A \rightarrow \text{spec } \frac{B}{q} = V(q)$

$f^\# : B \rightarrow A$:

$\text{spec } B = \overline{f(X)}$

minimal prime.

$\text{spec } A \rightarrow \text{spec } B$
 $\ni q \rightarrow \overline{q} \in \overline{\{p'\}} = V(p')$ $\Rightarrow \text{spec } \frac{B}{q} = \text{spec } B \Rightarrow q \subset \sqrt{0} = \langle 0 \rangle \Rightarrow q = \langle 0 \rangle$.

• If p' is a minimal prime, then $B_{p'}$ is a field:

First, we claim that $\forall b \in p', \exists x \in B \setminus p' \text{ s.t. } b^* x = 0$.

Indeed, consider $S = \{b^* x \mid k \in \mathbb{N}, x \in B \setminus p'\}$ which is multiplicatively closed. If $0 \notin S$, then $\exists p_0 \in \text{spec } B$ and $p_0 \cap S = \emptyset$.

$\Rightarrow p_0 \subset p'$ and $p_0 \neq p'$
 $\nearrow b \in p_0$

since if $\exists y \in p_0$ but $y \notin p'$ then $yb \in S$, however $yb \in p_0$.

Then it implies that $\forall b \in p', s \notin p' \Rightarrow \frac{b}{s} \in \text{nil}(B_{p'}) = \text{nil}(B)_{p'} = \{0\}$.

• $\forall p \in \text{spec } B, \exists q \in \text{spec } A$ s.t. $f(q) = p$:

Let p' be a minimal prime in B , i.e. $p \in \overline{\{p'\}} = V(p')$.

\uparrow
 $S = \{R \setminus q \mid q \subset p'\}$

$R \setminus q_i \in R \setminus q_i \subset S$
 $\Rightarrow q_i \supset q_1 \supset \dots$

$\Rightarrow \bigcap_{i=1}^{\infty} q_i \in \text{spec } B \Rightarrow \text{Zariski lemma } \exists \text{ max. element } R \setminus p'$.

$$\text{Then } B \hookrightarrow A \Rightarrow B_{p'} \hookrightarrow A_{p'} \cong A \otimes B_{p'} \hookrightarrow A$$

$$\begin{array}{ccc} \{0\} & \xleftarrow{\quad} & \begin{array}{c} q_0 \\ \parallel \\ q_{p'} \end{array} & \xrightarrow{\quad} & \begin{array}{c} q' \\ \parallel \\ \text{Max } A_{p'} \end{array} \end{array}$$

$$\text{So } \begin{array}{c} q_{p'} \cap B_{p'} = \{0\} \\ \parallel \\ (q' \cap B)_{p'} \end{array} \Rightarrow \begin{array}{c} q' \cap B = p' \\ \parallel \\ p_{p'} \end{array} \Rightarrow p' \in f(X)$$

By assumption, $p \in f(X)$.

• General case: $(X, (\mathcal{O}_X)_{\text{red}})$

associated the sheaf to $U \rightarrow \mathcal{O}_X(U)_{\text{red}}$

$$\text{Consider } (X_{\text{red}} \rightarrow X) \rightarrow \overline{f(X)} \rightarrow Y$$

with reduced induced structure.

so we can assume that $(X), Y$ (are) reduced and $Y = \overline{f(X)}$.
Hope " $\forall y \in Y \Rightarrow y \in f(X)$."

For $y \in U$, $p \in \text{spec } B \subset Y$, $f: f^{-1}(U) \rightarrow U$, so we can assume that Y is affine.

By assumption of quasi-compactness, $X = \bigcup_{i=1}^n \text{spec } A_i$, so $y \in \overline{f(X)} = \bigcup_{i=1}^n \overline{f(\text{spec } A_i)}$
 $\Rightarrow y \in f(\text{spec } A_t)$ for some t .

Hence we can reduce it to the key case.

U, V affine in X which is separated over an affine scheme $S \Rightarrow U \cap V = \Delta(X) \cap (U \times_S V)$. closed in $U \times_S V \leadsto$ affine

① Valuation Criterion

Recall that an integral domain R is called a valuation ring if
 \exists a valuation $v: K \rightarrow G$ s.t. $R = \{x \in K \mid v(x) \geq 0\}$
the g.f. of R totally ordered abelian group

$$\bullet v(xy) = v(x) + v(y)$$

$$\bullet v(x+y) \geq \min\{v(x), v(y)\}$$

• R is a local ring with $m = \{x \in K \mid v(x) > 0\}$

• R is a valuation ring $\Leftrightarrow \forall x \in K \setminus \{0\}, x \in R \text{ or } x^{-1} \in R$.

Def: R, S : local rings in a field K .

S is said to dominate R if $R \subset S$ and $m_S \cap R = m_R$.

Prop 3: A local ring R with g.f. K is a valuation ring

$\Leftrightarrow R$ is max with respect to the domination order in $\{S \mid S \text{ local ring in } K\}$

(pf) \Rightarrow : If a local ring $S \subset K$ dominates R , then $\exists a \in S \setminus R$, $a^{-1} \in R$ and thus $a^{-1} \in S$. (Note: $m_S \cap R = m_R$)

Since a is not a unit in R , $a^{-1} \in m_R = m_S \cap R$. We claim that $a^{-1} \in m_R$ and thus $a^{-1} \in m_S \Rightarrow 1 = a \cdot a^{-1} \in m_S$.
However, $1 = a \cdot a^{-1} \in m_S$. Indeed, if $a^{-1} \notin m_R$, then a^{-1} is a unit in R , i.e. $a = (a^{-1})^{-1} \in R$.
"⇐": Claim: $x \in K \setminus \{0\} \Rightarrow$ either $m[x] \neq R[x]$ or $m[x^{-1}] \neq R[x]$.
Let L be the algebraic closure of R/m and $g: R \rightarrow L$.
If $(R', g') \geq (R, g)$, i.e. $R' \supset R$, $g'|_R = g$, where m, n are chosen as small as possible.
Assume that $m \geq n$. (similar for $m < n$)
then $p' = f \circ g'$
 $\Rightarrow p' \cap R = m$
 $\Rightarrow R_{p'}'$ dominates R
($R \subset R' \Rightarrow R_p \subset R_{p'}$)
 $\Rightarrow R_{p'}' = R$
 $\Rightarrow R' = R$.

where $m = m_R$

(pf): If not, \Rightarrow

$$u_0 + u_1 x + \dots + u_m x^m = 1 \quad (1) \quad u_i \in m$$

$$v_0 + v_1 x^{-1} + \dots + v_n x^{-n} = 1 \quad (2) \quad v_j \in m$$

where m, n are chosen as small as possible.

Assume that $m \geq n$. (similar for $m < n$)

$$(2) \times x^m: (1 - v_0) x^m = v_1 x^{m-1} + \dots + v_n x^{m-n} \Rightarrow x^m = v_1' x^{m-1} + \dots + v_n' x^{m-n} \quad (2')$$

We plug (2') into (1) to get a relation with smaller degree of x .

For $x \in K \setminus \{0\}$, by the above claim, we assume that $m[x] \neq R[x]$.

\exists max ideal m' in $R[x]$, s.t. $m[x] \subseteq m'$.

$R[x]_{m'}$ dominates $R_m = R$.

Since $m' \cap R \supseteq m$, $m' \cap R = m$. Now we have $\left(\begin{matrix} R_m \hookrightarrow R[x]_{m'} \\ R \end{matrix} \right)$

By the max of R , $R[x]_{m'} = R_m$, i.e. $x \in R$.

Similarly, if $m[x^{-1}] \neq R[x]$, then $x^{-1} \in R$.

xx.

Then $\text{Hom}_{\text{Sch}}(\text{Spec } R, X)$

$$\text{spec } K \rightarrow X$$

$$\Leftrightarrow \exists x \in X \text{ \& } K(x) \hookrightarrow K$$

$$\{ \{x_0, x_1\} \subset X \mid x_0 \in \overline{\{x_1\}}, \quad k(x_1) \leq K, \quad R \text{ dominates } \mathcal{O}_{\overline{\{x_1\}}, x_0} \text{ on } \overline{\{x_1\}} \text{ with its reduced induced structure} \}$$

(pf): " \downarrow ": Given $f: T = \text{spec } R \rightarrow X$

closed point: $t_0 = m \mapsto x_0$; $\mathcal{O}_{T, t_0} = R_m = R \Rightarrow f_{x_0}^\#: \mathcal{O}_{x_0, x_1} \rightarrow k$
 generic point: $t_1 = \langle 0 \rangle \mapsto x_1$; $\mathcal{O}_{T, t_1} = R_{\langle 0 \rangle} = K$.
 a local homo.
 i.e. $m_{x_0, x_1} \mapsto x(x)$

Observe that $f^{-1}(\overline{\{x_i\}})$ is closed and $t_i \in f^{-1}(\overline{\{x_i\}}) \Rightarrow f(x_i) \subset K$
 $\Rightarrow \overline{\{t_i\}} \subseteq f^{-1}(\overline{\{x_i\}}) \Rightarrow T = f^{-1}(\overline{\{x_i\}})$ and $x_0 \in f(T) = \overline{\{x_i\}}$
 and T is reduced $\Rightarrow f: T_{\text{red}} \rightarrow Z = \overline{\{x_i\}}$ with reduced induced structure

It induces the local homomorphisms:

the function field of Z $\xrightarrow{\quad} K(X_i) = \mathcal{O}_{Z, X_i} \xrightarrow{\quad} \mathcal{O}_{T, t_i} = K$
 $\mathcal{O}_{Z, X_0} \xrightarrow{\quad} \mathcal{O}_{T, t_0} = R$

" \uparrow ": $\mathcal{O}_Z(Z) \xrightarrow{\mathcal{O}_{Z, X_0}} R \Rightarrow \text{spec } R \rightarrow Z \xrightarrow{\text{closed immersion}} X_{\text{red}} \xrightarrow{\text{ex}} X$

Main theorem (Valuative criterion of separatedness)

Let $f \in \text{Hom}_{\text{Sch}}(X, Y)$ and X be noeth.

Then f is separated $\Leftrightarrow \forall$ field K and \forall valuation ring R with g.f. K

$$\begin{array}{ccc} \text{Spec } K = \mathbb{A}^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } R = T & \longrightarrow & Y \end{array}$$

\exists at most one morphism $T \rightarrow X$
s.t. the whole diagram commutes.

$\text{spec } \mathbb{Z}/12\mathbb{Z} \times \text{spec } \mathbb{Z}/12\mathbb{Z} \cong \text{spec } \mathbb{Z}/12\mathbb{Z}$ $\xrightarrow{\sim} \mathbb{Z}/12\mathbb{Z}$, $\text{spec } \mathbb{Z}/12\mathbb{Z} \rightarrow \text{spec } \mathbb{Z}$
 $\text{spec } \mathbb{C}[X] \times \text{spec } \mathbb{C}[Y] = \text{spec } \mathbb{C}[X, Y]$ not the same
 $\text{spec } \mathbb{C}[X] \times \text{spec } \mathbb{C}[Y] \xrightarrow{\sim} \text{spec } \mathbb{C}[X, Y]$
 $\text{spec } \mathbb{Z}/12\mathbb{Z} \times \text{spec } \mathbb{Z}/12\mathbb{Z} = \text{spec } \mathbb{Z}/12\mathbb{Z}$

(pf): " \Rightarrow ":

$\{t_i\} = \sqcup \begin{array}{ccc} & \xrightarrow{\quad} & X \\ \downarrow h_1 & \nearrow h_2 & \downarrow f \\ T & \xrightarrow{\quad} & Y \end{array}$

$h_{1,2} = \Delta \circ h_{1,2}$
 $x_i' \in \Delta(X)$

$X \times X \begin{array}{ccc} \xrightarrow{\quad} & \xrightarrow{\quad} & X \\ \downarrow h_1 & \nearrow h_2 & \downarrow f \\ X & \xrightarrow{\quad} & Y \end{array}$

$h_1(t_i) = h_2(t_i) = x_i \Rightarrow h(t_i) \in \Delta(X)$

$\therefore \Delta(X)$ is closed $\because h(t_0) \in \Delta(X)$ i.e. $h_1(t_0) = h_2(t_0) = x_0$

Also, $\mathcal{O}_{Z, x_i} = k(x_i) \xrightarrow{h_1} K$ and the same, by prop 4, $h_1 = h_2$.
 $\mathcal{O}_{Z, x_0} \xrightarrow{h_1} K$

" \Leftarrow ":

By prop 1, it is sufficient to show " $\Delta(X)$ is a closed subset of $X \times X$ ".

Since X is noeth, Δ is quasi-compact.

By prop 2, it is sufficient to show " Δ is stable under specialization".

Let $x_i \in \Delta(X)$ and $x_0 \in \overline{\{x_i\}}$.

Let $Z = \overline{\{x_i\}}$ with the reduced induced structure.

We know that $K := k(Z) = \mathcal{O}_{Z, x_i}$ is the function field of Z

and $\mathcal{O}_{Z, x_0} \hookrightarrow K$. {local rings in K containing \mathcal{O}_{Z, x_0} }

By prop 3, $R = \max \{ S \mid \mathcal{O}_{Z, x_0} \subset S \}$ and $S \subset K$ is a valuation ring.

i.e. $\mathcal{O}_{Z, x_0} \hookrightarrow R \hookrightarrow K$ with R a valuation ring.

By prop 4,

$T \xrightarrow{\exists g} X \times X \begin{array}{ccc} \xrightarrow{p_1} & X & \xrightarrow{f} \\ \downarrow p_2 & \searrow & \downarrow f \\ X & \xrightarrow{\quad} & Y \end{array}$
 $t_0 \mapsto x_0$
 $t_1 \mapsto x_1$

$\therefore p_1(x_1) = p_1 \Delta(a) = a$
 $p_2(x_1) = p_2 \Delta(a) = a$

$\{t_i\} = \sqcup \begin{array}{ccc} & \xrightarrow{\quad} & X \\ \downarrow p_1 \circ g & \nearrow p_2 \circ g & \downarrow f \\ T & \xrightarrow{\quad} & Y \end{array} \Rightarrow p_1 \circ g = p_2 \circ g$

So $p_1 g(t_0) = p_2 g(t_0) \Rightarrow p_1(x_0) = p_2(x_0) \Rightarrow x_0 \in \Delta(X)$.

$T \xrightarrow{g} X \times X \begin{array}{ccc} \xrightarrow{p_1} & X & \xrightarrow{f} \\ \downarrow p_2 & \searrow & \downarrow f \\ X & \xrightarrow{\quad} & Y \end{array}$

$T \xrightarrow{g} X \times X \begin{array}{ccc} \xrightarrow{p_1} & X & \xrightarrow{f} \\ \downarrow p_2 & \searrow & \downarrow f \\ X & \xrightarrow{\quad} & Y \end{array}$

$g = \Delta \circ p_1 \circ g$
 $x_0 = g(t_0)$
 $= \Delta(p_1 g(t_0))$

$X \times X \begin{array}{ccc} \xrightarrow{p_2} & X & \xrightarrow{f} \\ \downarrow p_1 & \searrow & \downarrow f \\ X & \xrightarrow{\quad} & Y \end{array}$

$$\mathbb{C}_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}_{\mathbb{R}}(\mathbb{R}[X]/(X^2+1)) \cong \mathbb{C}[X]/(X^2+1) \cong \frac{\mathbb{C}[X]}{X^2+1} \cong \frac{\mathbb{C}[X]}{X^2-1} \cong \mathbb{C} \times \mathbb{C}$$

$$\text{Spec } \mathbb{C}_{\mathbb{R}} \mathbb{C} = \text{Spec } \mathbb{C} \sqcup \text{Spec } \mathbb{C}$$

$$\text{Spec } \mathbb{C}_{\mathbb{R}} \mathbb{C} \cong \text{Spec } \mathbb{C} \times \text{Spec } \mathbb{C}$$

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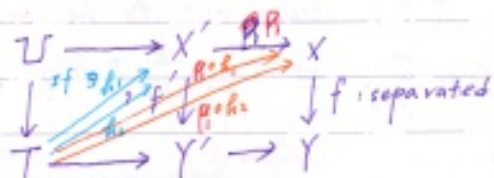
Coro: 1. Separated morphisms are stable under base change.

2. A composition of two separated morphisms is separated.

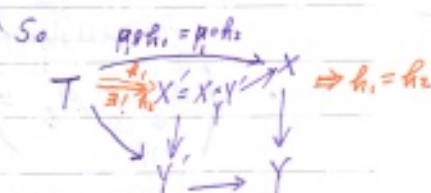
3. $f: X \rightarrow Y$, $f': X' \rightarrow Y'$ are separated $\Rightarrow f \times f': X \times_S X' \rightarrow Y \times_S Y'$ is separated

4. $g \circ f$ is separated $\Rightarrow f$ is separated

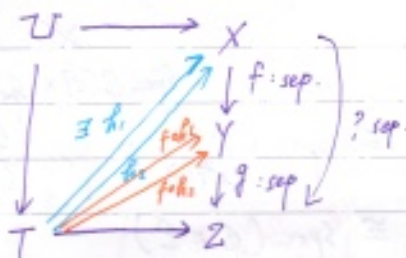
(pf): 1.



If $\exists h_1, h_2$,
then f separated $\Rightarrow p_1 \circ h_1 = p_1 \circ h_2$



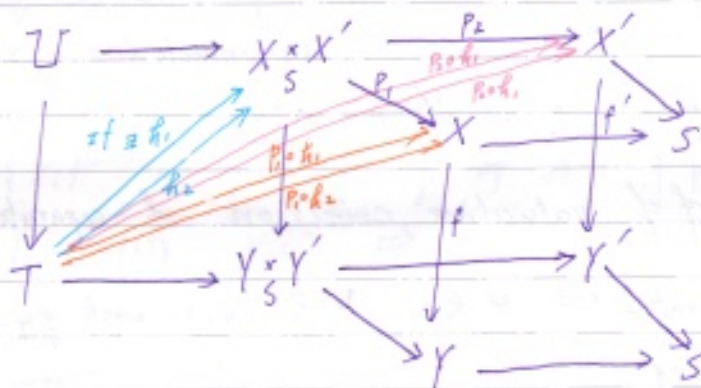
2.



g sep $\Rightarrow f \circ h_1 = f \circ h_2$

f sep $\Rightarrow h_1 = h_2$

3.

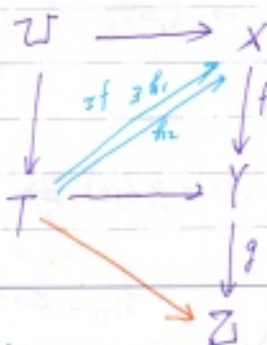


f sep $\Rightarrow p_1 \circ h_1 = p_1 \circ h_2$

f' sep $\Rightarrow p_2 \circ h_1 = p_2 \circ h_2$

So $h_1 = h_2$ by the universal property of $X \times_S X'$

4.



$g \circ f$ sep $\Rightarrow h_1 = h_2$

5. $f: X \rightarrow Y$ is separated \Leftrightarrow for any cover $\{U_i\}$ of Y , $f^{-1}(U_i) \rightarrow U_i$ is sep. $\forall i$

(pf) \Rightarrow Recall that $f^{-1}(U_i) \times_Y f^{-1}(U_j) = f^{-1}(U_i \times_Y U_j) = f^{-1}(U_i \times_Y U_j) \times_X X$ covers $X \times_Y X$.

And $\Delta(X) \cap (f^{-1}(U_i) \times_Y f^{-1}(U_j)) = \Delta(f^{-1}(U_i \times_Y U_j))$ is closed in $f^{-1}(U_i \times_Y U_j)$

✱

(修正): $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$

As sets,

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array}$$

$$\begin{array}{ccc} X \times Y & & \\ \parallel & & \\ X \times_{\text{Spec } \mathbb{Z}} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \text{Spec } \mathbb{Z} \end{array}$$

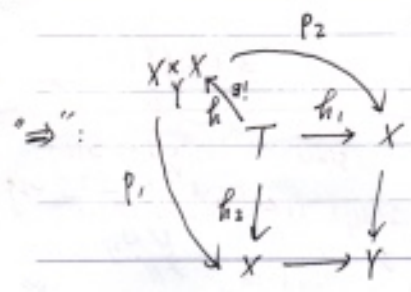
However, as schemes, $X \times_S Y$ turns out to be more complicated.

- eg. • $\text{Spec } \frac{\mathbb{Z}}{12\mathbb{Z}} \times_{\text{Spec } \frac{\mathbb{Z}}{18\mathbb{Z}}} \text{Spec } \frac{\mathbb{Z}}{18\mathbb{Z}} \cong \text{Spec } \frac{\mathbb{Z}}{12\mathbb{Z}} \otimes_{\frac{\mathbb{Z}}{18\mathbb{Z}}} \frac{\mathbb{Z}}{18\mathbb{Z}} = \text{Spec } \frac{\mathbb{Z}}{12\mathbb{Z}}$
- $\left(\begin{array}{c} \mathbb{A}_{\mathbb{C}}^1 \\ \text{Spec } \mathbb{C} \end{array} \right) \times_{\text{Spec } \mathbb{C}} \left(\begin{array}{c} \mathbb{A}_{\mathbb{C}}^1 \\ \text{Spec } \mathbb{C} \end{array} \right) = \text{Spec } \mathbb{C}[X] \times_{\text{Spec } \mathbb{C}} \text{Spec } \mathbb{C}[Y] \cong \text{Spec } \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y] = \text{Spec } \mathbb{C}[X, Y] = \mathbb{A}_{\mathbb{C}}^2$
- $\text{Spec } \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \text{Spec } \mathbb{C} \otimes_{\mathbb{R}} \frac{\mathbb{R}[X]}{\langle X^2 + 1 \rangle} \cong \text{Spec } \frac{\mathbb{C}[X]}{\langle X^2 + 1 \rangle} \cong \text{Spec } \left(\frac{\mathbb{C}[X]}{\langle X + i \rangle} \times \frac{\mathbb{C}[X]}{\langle X - i \rangle} \right)$
- $\cong \text{Spec } (\mathbb{C} \times \mathbb{C})$
- $\cong \text{Spec } \mathbb{C} \sqcup \text{Spec } \mathbb{C}$
- disconnected

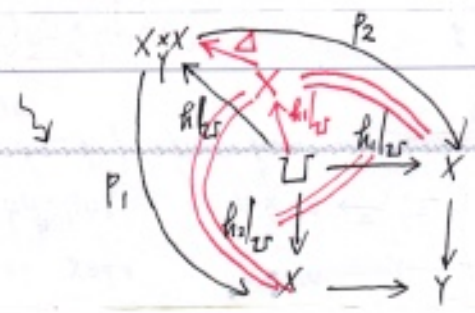
We conclude that $p_1(a) = p_2(a) \nRightarrow a \in \Delta(X)$.

Go back to the proof of valuative criterion of separatedness:

$$\left(\begin{array}{l} f: X \rightarrow Y \text{ is separated} \Leftrightarrow \forall K, \forall R: \text{u.r. with } g = f \circ K \\ \text{s.t.} \\ \begin{array}{ccc} \{t_i\} = \text{Spec } K = \mathcal{U} & \longrightarrow & X \\ \downarrow h_1 & \nearrow h_2 & \downarrow \\ \text{Spec } R = T & \longrightarrow & Y \end{array} \Rightarrow h_1 = h_2 \end{array} \right)$$



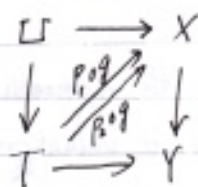
$$h_1(t_1) = h_2(t_1) \Rightarrow p_1 h(t_1) = p_2 h(t_1) \Rightarrow h(t_1) \in \Delta(X)$$



$$\begin{aligned} h|_{\mathcal{U}} &= \Delta \circ h|_{\mathcal{U}} \\ \Rightarrow h(t_1) &= \Delta(h(t_1)) \in \Delta(X) \end{aligned}$$

$$x_0 \in \overline{\Delta(X)} \text{ \& } x_1 \in \Delta(X) \not\Rightarrow x_0 \in \Delta(X)$$

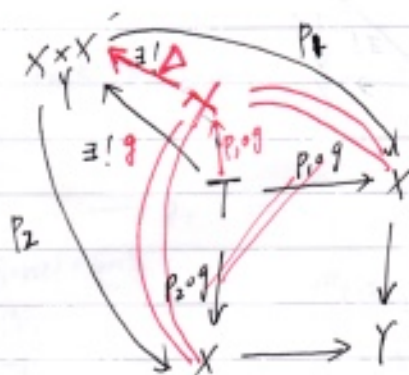
" \Leftarrow ":



$$\Rightarrow p_1 \circ g = p_2 \circ g$$

$$\begin{array}{ccc} T & \xrightarrow{g} & X \times X \\ t_1 \mapsto & & x_1 \\ t_0 \mapsto & & x_0 \end{array}$$

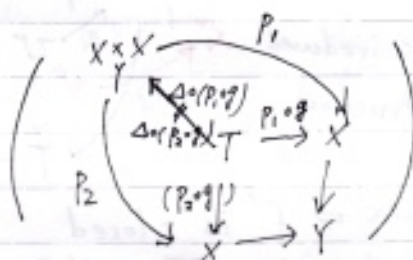
(\nexists say $p_1 \circ g(t_0) = p_2 \circ g(t_0)$)
 $\Rightarrow g(t_0) \in \Delta(X)$
 x_0
 wrong.



$\exists!$

$$g \downarrow = \Delta \circ (p_1 \circ g)$$

$$\Rightarrow x_0 = g(t_0) = \Delta(p_1 \circ g(t_0)) \in \Delta(X)$$



§6. Properness.

⊙ Proper morphisms

Recall that a Hausdorff space X is compact $\Leftrightarrow \forall$ top. space Z ,

the projection $\pi: X \times Z \rightarrow Z$ is a closed map.

Def: $f \in \text{Hom}_{\text{Sch}}(X, Y)$.

f is said to be proper if it is separated, of finite type,

and universally closed. ($\forall g: Z \rightarrow Y$, $X \times_Y Z \rightarrow Z$ is a closed morphism)

eg: $A'_C \rightarrow \text{Spec } \mathbb{C}$ is separated, of finite type and closed but not

universally closed.

$$\left(\begin{array}{ccc} \text{Spec } \mathbb{C}[t] \times \text{Spec } \mathbb{C}[t] & \xrightarrow{\pi} & \text{Spec } \mathbb{C}[t] \\ \downarrow & & \downarrow \\ A'_C & \xrightarrow{\pi} & A'_C \end{array} : Y = V(xy-1) \mapsto A'_C \setminus \{0\} \text{ which is open} \right)$$

Valuative criterion of properness

Let $f: X \rightarrow Y$ be of finite type with X noeth.

Then f is proper $\Leftrightarrow \forall$ field K and \forall valuation ring R with g.f. K

$$\begin{array}{ccc} \text{Spec } K = \sqcup & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \exists! & \downarrow f \\ \text{Spec } R = T & \xrightarrow{\quad} & Y \end{array}$$

(Pf): " \Rightarrow ": Separatedness \Rightarrow uniqueness.

Existence:

$Z = \overline{\{x_i\}}$ with reduced induced structure

$$\begin{array}{ccc} X \times T = X_T & \xrightarrow{\quad} & X \\ \downarrow f' & \nearrow \exists! & \downarrow f \\ T & \xrightarrow{\quad} & Y \end{array}$$

$\therefore f$ is proper $\Rightarrow f'$ is closed $\therefore f'(Z)$ is closed in T .

And $t_i \in f'(Z) \Rightarrow \overline{\{t_i\}} \subseteq f'(Z) \subseteq T \Rightarrow f'(Z) = T$

So $\exists x_0 \in \overline{\{x_i\}} \subseteq X_T$ s.t. $f'(x_0) = t_0$ and

$$\begin{array}{ccc} R & \xrightarrow{\quad} & \mathcal{O}_{Z, x_0} \\ \downarrow & & \downarrow \\ K & \xleftrightarrow{\quad} & \mathcal{O}_{Z, x_0} = K(x_0) \end{array}$$

\mathcal{O}_{Z, x_0} is a local ring in K

By prop 3, R is max in {local rings in K }

so $R \cong \mathcal{O}_{Z, x_0}$ i.e. R dominates \mathcal{O}_{Z, x_0}

$$\begin{array}{ccccc} \text{By prop 4,} & \exists & T & \longrightarrow & X_T \longrightarrow X \\ & & t_0 & \longmapsto & x_0 \\ & & t_i & \longmapsto & x_i \end{array}$$

" \Leftarrow ": Uniqueness \Rightarrow separatedness

Universally closed:

$$\begin{array}{ccc} Z \subseteq X' & \xrightarrow{\quad} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{\quad} & Y \\ \downarrow & & \downarrow \\ \text{Spec } S & \xrightarrow{\quad} & \text{Spec } R \end{array}$$

"To show that $f'(Z)$ is closed"

\Rightarrow locally of finite type & quasi-compact.

Since f is of finite type $\Rightarrow f'$ is of finite type $\Rightarrow f'$ is quasi-compact
 $\Rightarrow f'_Z$ is quasi-compact.
 it suffices to show that f'_Z is stable under specialization.

Let $y_0 \in \overline{\{y_1\}}$ and $y_1 = f'(z_1)$ for some $z_1 \in Z$.

Consider $W = \overline{\{y_1\}}$ with reduced induced structure. \mathcal{O}_{W, y_1}

$$\text{Find that } z_1 \mapsto y_1 \Rightarrow \mathcal{O}_{Y', y_1} \xrightarrow{m_{Y', y_1}} \mathcal{O}_{X', z_1} \xrightarrow{m_{X', z_1}} k(z_1) \Rightarrow k(W) = k(y_1) \hookrightarrow k(z_1)$$

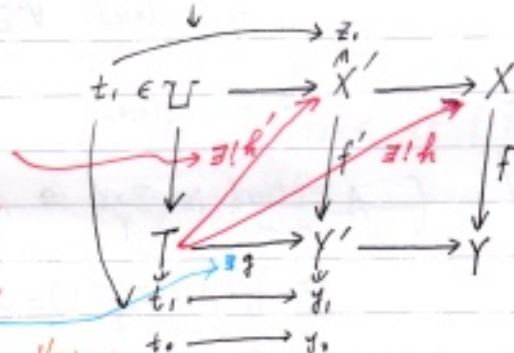
$y_1 \in \text{Spec } R[Y']$, $\overline{\{y_1\}} = V(\mathfrak{p}) = \text{Spec } R_{\mathfrak{p}}$
 $\mathcal{O}_{Y', y_1} \cong R_{\mathfrak{p}}$
 $R_{\mathfrak{p}} \subseteq \mathcal{O}_{W, y_1}$

Let R be max in $\{\text{local rings containing } \mathcal{O}_{W, y_0} \text{ in } k\}$, which is a valuation ring by prop 3.

$$\text{Now, } \mathcal{O}_{W, y_0} \hookrightarrow R \hookrightarrow K \xrightarrow{\text{Spec } R} \text{Spec } R \rightarrow W \hookrightarrow Y' \Rightarrow$$

$$\mathcal{O}_{W(W)}$$

by the universal property of X'



$$\begin{aligned} & \therefore t_1 \in h'^{-1}(Z) \\ & \therefore \overline{\{t_1\}} \subseteq h'^{-1}(Z) \\ & \Rightarrow h'(T) \subseteq Z \\ & \Rightarrow h'(t_0) = z_0 \in Z \\ & \Rightarrow y_0 = g(t_0) = f'(h'(t_0)) \in f'(Z) \end{aligned}$$

Cor: 1. Proper morphisms are stable under base change.

2. A composition of proper morphisms is proper.

3. Product of proper morphisms are proper

4. $g \circ f$: proper + g : separated $\Rightarrow f$: proper (of finite type)

5. $f : X \rightarrow Y$ is proper $\Leftrightarrow Y = \bigcup U_i$ s.t. $f^{-1}(U_i) \rightarrow U_i$ is proper (of finite type)

6. A closed immersion is proper.

• finite \Rightarrow of finite type.

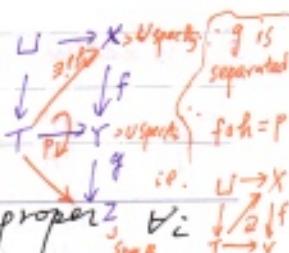
• A closed immersion is stable under base change & a closed immersion \Rightarrow a closed map

$$B_i = A_i \otimes_R R[x_1, \dots, x_n] \xrightarrow{t_i \in U} Z \xrightarrow{g} X$$

$$\begin{aligned} & t_i \in g^{-1}(i(Z)) \\ & \overline{\{t_i\}} \subseteq g^{-1}(i(Z)) \\ & R \rightarrow A_i \rightarrow R[x_1, \dots, x_n] \\ & g^{-1}(i(Z)) \text{ is closed in } T \end{aligned}$$

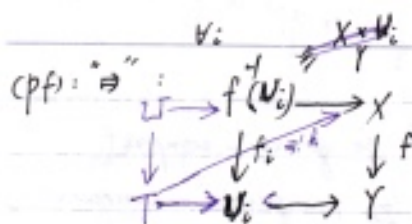
$$B_i = f.g. R\text{-alg.}$$

$$B_i = R[x_1, \dots, x_n] = A_i[x_1, \dots, x_n]$$



$$f^{-1}(T) \subseteq \bigcup_i U_i$$

$$f^{-1}(T) \subseteq f^{-1}(U_i)$$



f_i is proper by 1.

" \Leftarrow ":

• of finite type: $\forall y \in Y$, say $y \in U_i$, take $y \in \text{Spec } R \subset U_i \subset Y$

$$\text{and } f_i^{-1}(\text{Spec } R) = \bigcup_{j=1}^m \text{Spec } A_j \text{ with } A_j: f \cdot g \cdot R \text{ -alg.}$$

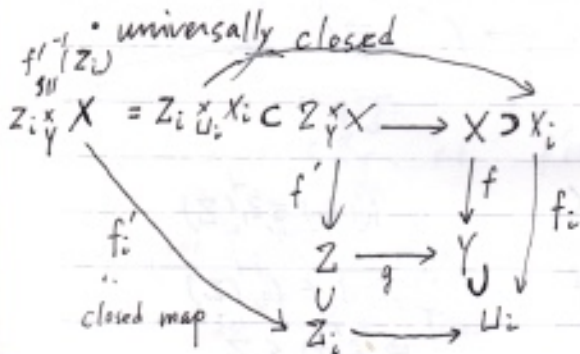
$$f^{-1}(\text{Spec } R)$$

• separated:

Recall that $f^{-1}(U_i) \times_Y f^{-1}(U_i) = f^{-1}(U_i) \times_Y f^{-1}(U_i) = f^{-1}(U_i) \times_Y X$ cover $X \times_Y X$

And $\Delta(X) \cap (f^{-1}(U_i) \times_Y f^{-1}(U_i)) = \Delta(f^{-1}(U_i))$ is closed in

$\Rightarrow \Delta(X)$ is closed in $X \times_Y X$.



$f_i': \text{closed } \forall i$

$\Rightarrow f': \text{closed}$

($A: \text{closed in } Z \times_Y X \Rightarrow A \cap (Z_i \times_Y X): \text{closed}$

$\Rightarrow f'_i(A \cap (Z_i \times_Y X)) = f'_i(A) \cap Z_i: \text{closed in } Z_i$

$\Rightarrow f'(A)$ is closed in Z)

Projective morphisms

Def: $\mathbb{P}_A^n := \text{Proj } A[x_0, \dots, x_n]$: projective n -space over A

Recall that $p \in k[x_0, \dots, x_n]$, the zero set of p is called a projective variety.

$$\mathbb{P}_k^n \rightsquigarrow \text{Proj } k[x_0, \dots, x_n]$$

generalization

Fact 1: If $A \rightarrow B$, then $\mathbb{P}_B^n \cong \mathbb{P}_A^n \times_{\text{Spec } A} \text{Spec } B$.

(pf): Since $\forall P \in \text{Proj } A[x_0, \dots, x_n]$, $\exists x_i$ s.t. $x_i \notin P$ otherwise $P \supseteq S_+$

$$\text{Proj } S = \bigcup_{i=0}^n D_+(x_i) \cong \bigcup_{i=0}^n \text{Spec } S_{(x_i)} = \bigcup_{i=0}^n \text{Spec } A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$$

For each i , $\text{Spec } A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \times_{\text{Spec } A} \text{Spec } B \cong \text{Spec } A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \otimes_A B$
 $\cong \text{Spec } B[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$
 glue together $= \text{Spec } B[x_0, \dots, x_n]_{(x_i)}$

$\text{Proj } A[x_0, \dots, x_n] \times_{\text{Spec } A} \text{Spec } B$
 glue together $\text{Proj } B[x_0, \dots, x_n]$

Def: $\bullet \mathbb{P}_Y^n := \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} Y$: projective n -space over Y

$$\begin{array}{c} \mathbb{P}_{\mathbb{Z}}^n \hookrightarrow \mathbb{P}_Y^n \rightarrow \mathbb{P}_{\mathbb{Z}}^n \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{Spec } \mathbb{Z} \rightarrow Y \rightarrow \text{Spec } \mathbb{Z} \end{array}$$

$\bullet f: X \rightarrow Y$ is projective if $f^{-1}(y) \subset X \xrightarrow{f} Y \ni y$

analogue: projective variety in $\mathbb{P}_{k(y)}^n$

a closed immersion

\bullet Fibre over y : $\mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} \text{Spec } k(y) \xrightarrow{\pi} \text{Spec } k(y)$
 $\mathbb{P}_{k(y)}^n \xrightarrow{\pi} \text{Spec } k(y)$
 projection space over $k(y)$

$\bullet f: X \rightarrow Y$ is quasi-projective if $U_y \subset X \xrightarrow{f} Y \ni y$

eg: $A_{\mathbb{C}}^2 - \{(0,0)\}$

quasi-projective variety
enlarge the category we are interested in.

an open immersion

projective morphism

projective variety

Fact 2: If S is a graded ring with $S_0 = A$, which is finitely generated as an A -algebra by S_1 , then $\text{Proj } S \rightarrow \text{Spec } A$ is a projective morphism.

(pf) By assumption, $S \cong A[x_0, \dots, x_n] / I$ for some homo. ideal I . ($A[x_0, \dots, x_n] \xrightarrow{S} A[S_1]$)

Let $S' = A[x_0, \dots, x_n]$.

Then $S' \rightarrow S \Rightarrow$ a closed immersion $\text{Proj } S \hookrightarrow \text{Proj } S'$
 $\Rightarrow \forall i, S'_{(x_i)} \rightarrow S_{(x_i)} \Rightarrow$ a closed immersion $\text{Spec } S_{(x_i)} \hookrightarrow \text{Spec } S'_{(x_i)}$

Main theorem:

A projective morphism of noeth. schemes is proper.
 (resp. quasi-projective) ($\mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec } \mathbb{Z}} Y \rightarrow \text{Spec } \mathbb{Z} \times_{\text{Spec } \mathbb{Z}} Y$) (resp. of finite type, separated)

(pf): $X \xrightarrow{\text{proper}} Y$

$\hookrightarrow \mathbb{P}^n_Y \xrightarrow{\text{proper}} Y \Leftarrow \mathbb{P}^n_{\mathbb{Z}} \rightarrow \text{Spec } \mathbb{Z}$ is proper

$X \xrightarrow{\text{proper}} Y$
 $\swarrow \text{open}$
 $X' \xrightarrow{\text{proper}} Y$
 $\swarrow \text{separated}$

and $X \xrightarrow{\text{of finite type}} Y$
 $\swarrow \text{open + noeth.}$
 $X' \xrightarrow{\text{proper}} Y$
 $\swarrow \text{of finite type}$

*: $j^{-1}(\text{Spec } A) \hookrightarrow \text{Spec } A \subset X'$
 $\downarrow \text{isom.}$
 $\bigcup_{i=1}^n \text{Spec } A_{f_i}, \quad A_{f_i} = A[\frac{1}{f_i}] : \text{f.g. } A\text{-alg}$

The remaining thing is to show " $\mathbb{P}^n_{\mathbb{Z}} \rightarrow \text{Spec } \mathbb{Z}$ is proper".

- of finite type: $\mathbb{P}_{\mathbb{Z}}^n = \bigcup_{i=0}^n D_+(X_i) \cong \bigcup_{i=0}^n \text{Spec } \mathbb{Z}[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$
f.g. \mathbb{Z} -algebra

- Valuative criterion:

$$\begin{array}{ccc} t_i \in \mathcal{U} & \xrightarrow{\quad} & \mathbb{P}_{\mathbb{Z}}^n \ni P_i \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad} & \text{Spec } \mathbb{Z} \end{array}$$

By induction on n , $n=0$, $\text{Proj } \mathbb{Z}[X_0] = \text{Spec } \mathbb{Z}[X_0]_{(X_0)} = \text{Spec } \mathbb{Z}$ O.K.

We can assume that $P_i \in D_+(X_i) \forall i$

(otherwise, say $P_i \in \mathbb{P}_{\mathbb{Z}}^n \setminus D_+(X_i) \cong \mathbb{P}_{\mathbb{Z}}^{n-1}$, then it is done by induction hypothesis)

So $P_i \in \bigcap_{i=0}^n D_+(X_i)$, i.e. $\frac{X_i}{X_j}$, i,j are invertible elements in \mathcal{O}_{P_i}
(分子,分母在 P_i 类皆不为 0)

$$\begin{array}{ccc} \text{and } \text{Spec } K & \xrightarrow{\quad} & \mathbb{P}_{\mathbb{Z}}^n \\ t_i & \mapsto & P_i \end{array} \Rightarrow \begin{array}{ccc} k(P_i) & \hookrightarrow & K \\ \frac{X_i}{X_j} & \mapsto & f_{ij} \end{array}$$

Let $v: K \rightarrow G$ be the valuation w.r.t. R .

Choose k s.t. $v(f_{k0}) = \min \{v(f_{00}), v(f_{10}), \dots, v(f_{n0})\}$

Then $v(f_{ik}) = v(\frac{f_{i0}}{f_{k0}}) = v(f_{i0}) - v(f_{k0}) \geq 0 \Rightarrow f_{ik} \in R \forall i$.

Define $\varphi: \mathbb{Z}[\frac{X_0}{X_k}, \dots, \frac{X_n}{X_k}] \rightarrow R$

For Uniqueness, if $v(f_{k0}) = v(f_{k0})$ then $v(f_{k0}) = 0 \Rightarrow f_{k0} \notin \mathfrak{m}$ is a unit in R .

Since the following diagram commutes

on $D_+(X_k) \cap D_+(X_k)$, the same

$$t_0 \mapsto P_0$$