PAGE
DATE

§5. Coherent sheaves

<u>Def</u>: • A sheaf $\mathcal{F}$ on $(X, \mathcal{O}_X)$ is called a sheaf of $\mathcal{O}_X$-modules if $\forall\, U \in \mathrm{Top}(X)$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module

and for $V \subset U$, $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

$$\left(\begin{tikzcd} \mathcal{O}(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(U) \arrow[d] \\ \mathcal{O}(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}\right)$$

• $\varphi : \mathcal{F} \to \mathcal{F}' \rightsquigarrow \varphi(U) : \mathcal{F}(U) \to \mathcal{F}'(U)$: a $\mathcal{O}_X(U)$-module homo.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}'(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{F}'(V) \end{tikzcd}$$

$\Rightarrow$ They form a category $\mathrm{Mod}(X)$

◎ Basic example : $X = \operatorname{spec} A$, M : A-module (i.e. $M \in \mathrm{Mod}_A$)

• For $D(f) \subset X$ with $f \in A$,

$$\widetilde{M}(D(f)) := M_f \text{ which is a } \underset{\mathcal{O}_X(D(f))}{\overset{\shortparallel}{A_f}}\text{-module.}$$

∵ $D(g) \subset D(f)$, say $g^n = hf$,
$$\begin{array}{ccc} M_f & \longrightarrow & M_g \\ \frac{m}{f^r} & \longmapsto & \frac{mh^r}{g^{nr}} \end{array}$$

Spec A

• For any $U \in \mathrm{Top}(X)$, $\widetilde{M}(U) := \{ s : U \to \coprod_{p \in U} M_p \mid \forall p \in U$, ~~s can be locally represented at p as a/g, a∈M, g∈R~~, $\exists\, p \in D(f_p) \subset U$ and $\frac{a_p}{f_p} \in M_{f_p}$ s.t. $s(q) = \frac{a_p}{f_p} \in M_q$ $\forall q \in D(f_p)\}$

$$\widetilde{M}(U) = \varprojlim_{D(f) \subset U} M_f \text{ which is a } \underset{\mathcal{O}_X(U)}{\overset{\shortparallel}{\varprojlim_{D(f) \subset U} A_f}} \text{-module}$$

∵ For $V \subset U$,

$$\begin{tikzcd} \varprojlim_{D(f) \subset U} M_f \arrow[r, "\exists !"] \arrow[dr, "\exists"'] & \varprojlim_{D(f) \subset V} M_f \arrow[d, "\forall"] \\ & M_f, \ D(f) \subset V \subset U \end{tikzcd}$$

by the universal property of $\varprojlim$

Similar to the case of $\mathcal{O}_{\operatorname{spec} A}$,
$$\begin{cases} \widetilde{M}(D(f)) \cong M_f \\ \widetilde{M}_p \cong M_p \\ \widetilde{M}(X) = M. \end{cases}$$

• $\widetilde{M}_p = \varinjlim_{p \in D(f)} M_f = M_p$,

$$\begin{tikzcd} \underset{\text{unit in}}{\overset{\forall}{B}} & A_p \arrow[l, "\exists !"'] \\ f \in A_f \arrow[u] \arrow[ur] & \\ f \notin p, \ f \in A \arrow[u] & \end{tikzcd}$$

$\Rightarrow A_p = \varinjlim_{p \in D(f)} A_f$

so $M_p = M \otimes \varinjlim_{p \in D(f)} A_f$

$= \varinjlim_{p \in D(f)} M_f$.

($M_S \cong M \otimes A_S$)

• $\widetilde{M}(\underset{D(1)}{\overset{\shortparallel}{X}}) = M$. i.e. $\Gamma(X, \widetilde{M}) \cong M$.

PERFECTION

PAGE
DATE

Def: $\mathcal{F}, \mathcal{G} \in Mod(X)$

- $\mathcal{F} \oplus \mathcal{G}$ is the sheaf : $U \to \mathcal{F}(U) \oplus \mathcal{G}(U)$
- $\mathcal{F} \otimes \mathcal{G}$ is the associated sheaf to the presheaf : $U \to \mathcal{F}(U) \otimes \mathcal{G}(U)$.

Remark: The tensor product presheaf need not be a sheaf.

eg. Let X be a topological space $\{*, P_1, P_2\}$ where $*$ is the generic point and P_1, P_2 are closed points.

$\{\langle 0\rangle, \langle x-1\rangle, \langle x\rangle\}$

(eg: $A = \{ f(x)/g(x) \in \mathbb{Q}(x) \mid g(0) \neq 0, g(1) \neq 0 \}$, $\overset{\shortparallel}{\text{Spec}}\, A$ is an example)

open: $U_1 = \{*, P_1\}$, $U_2 = \{*, P_2\}$, $U = U_1 \cap U_2 = \{*\}$, X, ϕ.

Define $\mathcal{O}_X$: $\mathcal{O}_X(U) = \mathbb{Q}[X_1, X_2]$, $\mathcal{O}_X(U_1) = \mathcal{O}_X(U_2) = \mathcal{O}_X(X) = \mathbb{Q}$, $\mathcal{O}_X(\phi) = 0$.

rest := inclusion , $\rho_{W,\phi} = 0$

$\mathcal{F}$: $\mathcal{F}(U) = \mathbb{Q}[X_1, X_2]$, $\mathcal{F}(U_i) = \mathbb{Q}[X_i]$, $\mathcal{F}(X) = \mathbb{Q}$, $\mathcal{F}(\phi) = 0$

$\mathcal{G}$: $\mathcal{G}(U) = \mathbb{Q}[X_1, X_2]$, $\mathcal{G}(U_1) = \mathbb{Q}[X_2]$, $\mathcal{G}(U_2) = \mathbb{Q}[X_1]$, $\mathcal{G}(X) = \mathbb{Q}$, $\mathcal{G}(\phi) = 0$.

P is the presheaf : $U \to \mathcal{F}(U) \otimes \mathcal{G}(U)$, then $P(U) = \mathbb{Q}[X_1, X_2]$,

$P(U_1) = \mathbb{Q}[X_1 \otimes 1, 1 \otimes X_2]$, $P(U_2) = \mathbb{Q}[X_2 \otimes 1, 1 \otimes X_1]$, $P(X) = \mathbb{Q}$, $P(\phi) = 0$

Find $X_i \otimes 1 \big|_U = 1 \otimes X_i \big|_U = X_i \quad \forall i = 1, 2.$ but $P(U_1 \cup U_2) = P(X) = \mathbb{Q}$

i.e. we can't glue $X_i \otimes 1$ and $1 \otimes X_i$ to a global section of P.

Properties:

- $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_P \cong \mathcal{F}_P \otimes_{\mathcal{O}_P} \mathcal{G}_P \quad \forall P \in X$: $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_P \overset{\mathcal{F}_P = \mathcal{F}_P^+}{=} \underbrace{\varinjlim_{P \in U} \mathcal{F}(U) \otimes \mathcal{G}(U)}_{(s \otimes t) \xrightarrow{\exists 6} (s_P \otimes t_P)} \cong \mathcal{F}_P \otimes \mathcal{G}_P$

$(s, t) \to s_P \otimes t_P$ is bi-linear

- $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})|_U \cong \mathcal{F}|_U \otimes_{\mathcal{O}|_U} \mathcal{G}|_U \quad \forall U \in Top(X)$:

$\varinjlim_{P \in U} \mathcal{F}(U) \otimes \mathcal{G}(U) \to \mathcal{F}_P \otimes \mathcal{G}_P$

Conversely, $\mathcal{F}_P \otimes \mathcal{G}_P \to LHS$

$(s_P \otimes t_P) \mapsto (s|_{U \cap V} \otimes t|_{U \cap V})$

$s \in \mathcal{F}(U)$ $t \in \mathcal{G}(V)$

Let P be the presheaf $U \to \mathcal{F}(U) \otimes \mathcal{G}(U)$.

It is clear that $(P^+)|_U \cong (P|_U)^+$

Hence $$(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})|_U = (P^+)|_U = (P|_U)^+ = \mathcal{F}|_U \otimes_{\mathcal{O}|_U} \mathcal{G}|_U$$

Prop 1 : $\sim : Mod_A \longrightarrow Mod(X)$ is exact, fully faithful and preserves $\oplus, \otimes$. $M \longmapsto \widetilde{M}$

pf: Given $\varphi : M \to N$ an A-mod homo, we have

$$\varphi_f : M_f \to N_f \quad \forall f \in A$$

and $\forall U \in Top(X)$, $\varphi(U) : \varprojlim_{D(f)\subset U} M_f \longrightarrow \varprojlim_{D(f)\subset U} N_f$. universal property of

$$\begin{tikzcd} \varprojlim_{D(f)\subset U} M_f \arrow[r] \arrow[d] \arrow[dr] & \varprojlim_{D(f)\subset U} N_f \arrow[d] \\ M_f \arrow[r] & N_f \end{tikzcd}$$

$\Rightarrow \tilde{\varphi} : \widetilde{M} \to \widetilde{N}$

- exact : $0 \to M' \to M \to M'' \to 0$ exact

$\Rightarrow 0 \to M'_p \to M_p \to M''_p \to 0$ exact $\forall p$ ($M'_p = \widetilde{M'}_p$, $M_p = \widetilde{M}_p$, $M''_p = \widetilde{M''}_p$)

$\Rightarrow 0 \to \widetilde{M'} \to \widetilde{M} \to \widetilde{M''} \to 0$ exact.

- fully faithful : $Hom_A(M, N) \xrightarrow{\sim} Hom_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$

$$\varphi \longmapsto \tilde{\varphi}$$

$$\theta(X) \longleftarrow\!\shortmid\ \theta$$
$$\theta \overset{?}{=} \widetilde{\theta(X)}$$

$$\begin{tikzcd} M \arrow[r, "\theta(X)"] \arrow[d] & N \arrow[d] \\ M_f \arrow[r, "\theta(D(f))"] \arrow[r, phantom, "\theta(X)_f"', shift right=3] & N_f \end{tikzcd}$$

$\theta(D(f)) = \theta(X)_f$ by the universal property of $(M_f, M \to M_f)$

- $\oplus$: $M_f \oplus N_f \cong (M \oplus N)_f \Rightarrow \varprojlim_{D(f)\subset U} M_f \oplus \varprojlim_{D(f)\subset U} N_f \cong \varprojlim_{D(f)\subset U} (M \oplus N)_f$

$\Rightarrow \widetilde{M}(U) \oplus \widetilde{N}(U) \cong \widetilde{(M \oplus N)}(U)$

$\qquad = (\widetilde{M} \oplus \widetilde{N})(U)$

- $\otimes$: $M_f \otimes_{A_f} N_f \cong (M \otimes_A N)_f \Rightarrow \widetilde{M}(U) \otimes \widetilde{N}(U) \cong \widetilde{M \otimes_A N}(U)$

In particular, $\rho \xrightarrow{\text{presheaf}} \widetilde{M \otimes_A N}$ as a presheaf morphism $\Rightarrow \widetilde{M} \otimes \widetilde{N} = \rho^+ \longrightarrow \widetilde{M \otimes_A N}$

Now $\forall p$, $(\widetilde{M} \otimes \widetilde{N})_p \cong \widetilde{M}_p \otimes \widetilde{N}_p \cong M_p \otimes_{A_p} N_p \cong (M \otimes_A N)_p \cong \widetilde{M \otimes_A N}_p$

$\Rightarrow \widetilde{M} \otimes \widetilde{N} \cong \widetilde{M \otimes_A N}$.

via $Hom_{Sh(X)}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X) \cong Hom_{Sh(Y)}(\mathcal{O}_Y, f_*\mathcal{O}_X)$

⊙ Operations on sheaves

$f : X \to Y, \quad f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X \rightsquigarrow f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$

$\mathcal{F} \in Mod(X), \quad f_*\mathcal{F} : f_*\mathcal{O}_X\text{-module} \rightsquigarrow \mathcal{O}_Y\text{-module}.$

the direct image of $\mathcal{F}$

$\mathcal{G} \in Mod(Y), \quad f^{-1}\mathcal{G} : f^{-1}\mathcal{O}_Y\text{-module}$

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X : \mathcal{O}_X\text{-module}.$$

If $\varphi : A \to B$ and

the inverse image of $\mathcal{F}$

<u>Fact 1</u>: $f : \operatorname{spec} B \to \operatorname{spec} A, \quad N \in Mod_B, \quad M \in Mod_A,$

then $f_*\tilde{N} \cong {}_A\tilde{N}$ (as an A-module) and $f^*\tilde{M} = \widetilde{M \otimes_A B}$.

(pf):

- For $f \in A$, $\varphi(f) \in B$, $f^{-1}(\underbrace{D(f)}_{U}) = D(\varphi(f))$

$$\underset{\varphi_f}{\underset{\parallel}{f^{\#}_U}} : \underset{A_f}{\underset{\parallel}{\mathcal{O}_{\operatorname{spec} A}(U)}} \longrightarrow f_*\mathcal{O}_{\operatorname{spec} B}(U) = \underset{B_{\varphi(f)}}{\underset{\parallel}{\mathcal{O}_{\operatorname{spec} B}(f^{-1}(U))}}$$

$$f_*\tilde{N}(U) = \tilde{N}(f^{-1}(U)) = \underset{N_f \text{ as } A_f\text{-module}}{\underset{\parallel}{N_{\varphi(f)}}} : B_{\varphi(f)}\text{-module} \rightsquigarrow A_f\text{-module}$$

- Let h be the A-mod homo. $M \xrightarrow{(A \to B \rightsquigarrow M \otimes_A A \xrightarrow{1_M \otimes \varphi} M \otimes_A B)} {}_A(M \otimes_A B)$. which induces $\tilde{h} : \tilde{M} \to {}_A\widetilde{(M \otimes_A B)}$ on spec A

$\Rightarrow \underset{\mathcal{O}_{\operatorname{spec} A}\text{-module}}{\tilde{h}} : \tilde{M} \longrightarrow f_*(\widetilde{M \otimes_A B}) \rightsquigarrow \hat{h} : f^{-1}\tilde{M} \longrightarrow \widetilde{M \otimes_A B}$

$f^{-1}\mathcal{O}_{\operatorname{spec} A}$-module $\qquad \mathcal{O}_{\operatorname{spec} B}$-module

$$\rightsquigarrow \bar{h} : \underset{\widetilde{f^* \tilde{M}}}{\underset{\parallel}{f^{-1}\tilde{M} \otimes_{f^{-1}\mathcal{O}_{\operatorname{spec} A}} \mathcal{O}_{\operatorname{spec} B}}} \longrightarrow \underset{\widetilde{M \otimes_A B}}{\underset{\cong}{\widetilde{M \otimes_A B} \otimes_{f^{-1}\mathcal{O}_{\operatorname{spec} A}} \mathcal{O}_{\operatorname{spec} B}}}$$

$$\forall \underset{\operatorname{spec} B}{p \in}, \quad \bar{h}_p : \overset{M_q}{\overset{\cong}{(f^{-1}\tilde{M})_p}} \otimes_{A_q} B_p \xrightarrow{\sim} (M \otimes_A B)_p \quad \Rightarrow \bar{h} \text{ is an isomorphism}$$

$f(p) = q$

<u>Remark</u>: $Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \quad Hom_{\mathcal{O}_X\text{-presheaf}}(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F}) \cong Hom_{\mathcal{O}_Y\text{-presheaf}}(\mathcal{G}, f_*\mathcal{F})$

$\cong Hom_{\mathcal{O}_Y\text{-sheaf}}(\mathcal{G}, f_*\mathcal{F})$

Def: $\mathcal{F} \in Mod(X)$ is called quasi-coherent (resp. coherent) if $\exists \{\operatorname{spec} A_i\}_{i \in \Lambda}$ covers X ($\operatorname{spec} A_i = U_i$)

s.t. $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some A_i-mod M_i.
(resp. f.g. A_i-mod M_i)

$\Rightarrow$ They form a category $Qco(X)$ (resp. $Coh(X)$)

Fact 1: If $X = \operatorname{spec} A$ and $\mathcal{F} \in Qco(X)$, then $\exists \{D(f_i)\}_{i=1,\dots,n}$ covers X

s.t. $\mathcal{F}|_{D(f_i)} \cong \widetilde{M_i}$ for some $M_i \in Mod_{A_{f_i}}$.

(pf): $\forall p \in X$, $\exists$ $p \in U_p = \operatorname{spec} B \subset X$ s.t. $\mathcal{F}|_{U_p} = \widetilde{M}$ with $M \in Mod_B$

Now, if $i : \operatorname{spec} A_f \hookrightarrow U_p = \operatorname{spec} B$, $p \in \operatorname{spec} A_f$, then $\mathcal{F}|_{\operatorname{spec} A_f} = i^* \widetilde{M} = \widetilde{M \otimes_B A_f} \in Mod_{A_f}$

Hence $X \overset{\text{quasi-compact}}{=} \bigcup_{i=1}^{n} \operatorname{spec} A_{f_i}$ and $\mathcal{F}|_{\operatorname{spec} A_{f_i}} \cong \widetilde{M_i}$, $M_i \in Mod_{A_{f_i}}$.

Hence assume that $X = \bigcup_{i \in \Lambda} \operatorname{spec} B_i$ with $\mathcal{F}|_{\operatorname{spec} B_i} = \widetilde{M_i'}$, $M_i' \in Mod_{B_i}$

$= \bigcup_{i,j} \operatorname{spec} A_{f_{ij}}$ with $\mathcal{F}|_{\operatorname{spec} A_{f_{ij}}} = \widetilde{M_i' \otimes_{B_i} A_{f_{ij}}} \in Mod_{A_{f_{ij}}}$

quasi-compact $\Rightarrow$ $= \bigcup_{j=1}^{n} \operatorname{spec} A_{f_j}$ with $\mathcal{F}|_{\operatorname{spec} A_{f_j}} = \widetilde{M_i}$, $M_i \in Mod_{A_{f_j}}$

Fact 2: If $X = \operatorname{spec} A$, $M \in Mod_A$, $\mathcal{F} \in Mod(X)$, then $Hom_A(M, \Gamma(X, \mathcal{F})) \cong Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F})$

(pf): For any $f \in A$, given $\varphi : M \longrightarrow \Gamma(X, \mathcal{F})$

$\Rightarrow \varphi' : M \to \Gamma(X, \mathcal{F}) \xrightarrow{res_{X, D(f)}} \mathcal{F}(D(f))$

$\Rightarrow \varphi'_f : \underset{\cong M_f}{M \otimes_A A_f} \longrightarrow \mathcal{F}(D(f)) \otimes_A A_f \cong \mathcal{F}(D(f))$ since it is a A_f-module.

$\Rightarrow \widetilde{\varphi} : \widetilde{M} \longrightarrow \mathcal{F}$

Conversely, given $\widetilde{\varphi} : \widetilde{M} \longrightarrow \mathcal{F}$

$\Rightarrow \varphi = \widetilde{\varphi}(X) : M \to \mathcal{F}(X)$.

Prop 2: $\mathcal{F} \in Qco(X) \Leftrightarrow \forall\, U = \operatorname{spec} A \subset X$, $\mathcal{F}|_U = \widetilde{M}$ with $M \in mod_A$.
(X: noeth, resp. $Coh(X)$) (M: f.g)

(pf): " $\Leftarrow$ " : O.K.

" $\Rightarrow$ ": Assume that $X = \bigcup \operatorname{spec} B_i$ with $\mathcal{F}|_{\operatorname{spec} B_i} \cong \widetilde{M_i}$

Then $\underset{\underset{\operatorname{spec} A_f}{\cup}}{\operatorname{spec} B_i \cap U} \subset \underset{\underset{\operatorname{spec} A}{\shortparallel}}{U} \Rightarrow \mathcal{F}|_{\operatorname{spec} A_f} \cong \widetilde{M_i \otimes_{B_i} A_f}$

This says that $\mathcal{F}|_U \in Qco(U)$.

and thus we may assume that $X = \operatorname{spec} A$.

Now set $M = \Gamma(X, \mathcal{F})$.

$\exists\ \alpha : \widetilde{M} \to \mathcal{F}$ and

By fact 2, $\alpha_f : M_f \longrightarrow \mathcal{F}(D(f))$: A_f-homo.

$\frac{s}{f^r} \longmapsto \frac{s|_{D(f)}}{f^r}$

If α_f is (we want that) $\begin{cases} \text{1-1} \\ \text{onto} \end{cases}$, then we need that

Key lemma: $X = \operatorname{spec} A$, $\mathcal{F} \in Qco(X)$, $f \in A$.
(1) $\forall\, s \in \mathcal{F}(X)$, $s|_{D(f)} = 0 \Rightarrow f^n s = 0$ for some $n > 0$
(2) $\forall\, t \in \mathcal{F}(D(f))$, $\exists\, n \in \mathbb{N}$ & $s \in \mathcal{F}(X)$ s.t. $s|_{D(f)} = f^n t$

pf): By fact 1, we can Assume that $X = \bigcup_{i=1}^{m} D(f_i)$ and $\mathcal{F}|_{D(f_i)} = \widetilde{M_i}$ $M_i \in mod_{A_{f_i}}$

(1)

$$\begin{tikzcd}
 & s \in \mathcal{F}(X) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, bend left] & & \\
0 \in \mathcal{F}(D(f)) \arrow[drr] \arrow[ddrr, bend right] & \mathcal{F}(D(f_i)) = M_i \arrow[dr] & & \ni s_i \arrow[ddl, bend left] \\
 & & \mathcal{F}(D(ff_i)) = (M_i)_f = (A_{f_i})_f & \\
 & & \ni 0 &
\end{tikzcd}$$

i.e. $\exists n$ s.t. $f^n s_i = 0 \quad \forall i$

$\shortparallel$

$f^n s|_{D(f_i)}$

$\Rightarrow f^n s = 0$

(2)

$$\begin{tikzcd}
t \in \mathcal{F}(D(f)) \arrow[dr] \arrow[ddr, bend right] & & \mathcal{F}(D(f_i)) = M_i \ni ? \arrow[dl] \arrow[ddl, bend left] \\
 & \mathcal{F}(D(ff_i)) = (M_i)_f & \\
 & \ni t|_{D(ff_i)} &
\end{tikzcd}$$

$\exists\, l$ s.t. $t|_{D(ff_i)} = \frac{t_i}{f^l} \quad \forall i \Rightarrow f^l t|_{D(ff_i)} = t_i|_{D(ff_i)} \quad \forall i$

$t_i|_{D(ff_if_j)} = f^l t|_{D(ff_if_j)} = t_j|_{D(ff_if_j)}$

$\Rightarrow (t_i - t_j)|_{D(ff_if_j)} = 0 \underset{\text{by (1)}}{\Rightarrow} f^{l'}(t_i - t_j)|_{D(f_if_j)} = 0$

Summary, $X = \operatorname{Spec} A$

$$\begin{array}{ccc} \sim : \mathrm{Mod}_A & \xrightarrow[\text{an equivalence}]{\sim} & \mathrm{Qco}(X) \\ \cup & & \cup_{A:\text{noeth}} \\ \{\text{f.g. } A\text{-modules}\} & \xrightarrow{\sim} & \mathrm{Coh}(X) \end{array} \qquad \begin{array}{c} M \longmapsto \widetilde{M} \\ \Gamma(X, \mathcal{F}) \longleftarrow \mathcal{F} \end{array}$$

We find that $f^{\ell'} t_i|_{D(f_i f_j)} = f^{\ell'} t_j|_{D(f_i f_j)}$ and $f^{\ell'} t_i|_{D(ff_i)} = f^{\overset{n}{\overset{\|}{\ell+\ell'}}} t|_{D(ff_i)}$

$\Rightarrow \exists\, s \in \mathcal{F}(X)$ s.t. $s|_{D(f_i)} = f^{\ell'} t_i$ and $s|_{D(ff_i)} = f^{n} t|_{D(ff_i)}$ $\forall i$

$\Rightarrow s|_{D(f)} = f^{n} t$.

$\#$

By the key lemma, α_f is an isom $\forall f$ $\Rightarrow \alpha_p$ is an isom $\forall p \Rightarrow \alpha$ is an isom.

(X : noeth, $M_{f_i} \overset{\alpha_{f_i}}{\cong} M_i$: f.g. A_{f_i}-module and A_{f_i} : noeth
$\Rightarrow M_{f_i}$: noeth, $\forall i \Rightarrow M$: noeth
Similar to the proof of prop 3 in §3.)

Prop 3
~~Core~~: Let $\mathcal{F}, \mathcal{G} \in \mathrm{Qco}(X)$ (resp. $\mathrm{Coh}(X)$ with X: noeth) and $\varphi : \mathcal{F} \to \mathcal{G}$.
Then • $\ker\varphi$, $\operatorname{coker}\varphi$, $\operatorname{Im}\varphi \in \mathrm{Qco}(X)$. (resp. $\mathrm{Coh}(X)$)

• $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \mathrm{Qco}(X)$ (resp. $\mathrm{Coh}(X)$) $\qquad \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \widetilde{(M \otimes_A N)}$
• $\mathcal{F}$ ~~has~~ locally has finite presentation $\Rightarrow$ $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{Qco}(X)$ (resp $\mathrm{Coh}(X)$)

$A^m \to A^n \to M \to 0 \rightsquigarrow 0 \to (\mathrm{Hom}_A(M,N))_p \to (\mathrm{Hom}_A(A^n, \ldots$
$\Downarrow$ so $\qquad \Downarrow$
$0 \to \mathrm{Hom}_{A_p}(M_p, N_p) \to \mathrm{Hom}_{A_p}(A_p^n, \ldots$

Similarly, $f \in A$
$(\mathrm{Hom}_A(M,N))_f \cong \mathrm{Hom}_{A_f}(M_f, N_f)$

pf):
The question is local, so we may assume $X = \operatorname{Spec} A$ and $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$.

• $\varphi : \mathcal{F} \to \mathcal{G} \Rightarrow \varphi(X) : M \to N \Rightarrow 0 \to \ker(\varphi(X)) \to M \to N$ $\quad$ ($\ker(\varphi(X)) \in \mathrm{mod}_A$; f.g. module)

$$\Rightarrow \begin{array}{ccccccc} 0 \to & \widetilde{\ker(\varphi(X))} & \to & \widetilde{M} & \xrightarrow{\widetilde{\varphi(X)}} & \widetilde{N} \\ & \text{so} \to \| & & \| & & \| \\ 0 \to & \ker\varphi & \to & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} \end{array}$$

Note: $\sim$ is exact, fully faithful.

• $M \xrightarrow{\varphi(X)} N \to \operatorname{coker}(\varphi(X)) \to 0 \Rightarrow$
$$\begin{array}{ccccccc} \widetilde{M} & \xrightarrow{\widetilde{\varphi(X)}} & \widetilde{N} & \to & \widetilde{\operatorname{coker}(\varphi(X))} & \to & 0 \\ \| & & \| & & \text{so}\ \| & & \\ \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \to & \operatorname{coker}\varphi & \to & 0 \end{array}$$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(D(f))$
$\| $
$\mathrm{Hom}_{\mathcal{O}_X|_{D(f)}}(\mathcal{F}|_{D(f)}, \mathcal{G}|_{D(f)})$
$\cong$
$\mathrm{Hom}_{A_f}(M_f, N_f)$
$\cong$
$(\mathrm{Hom}_A(M,N))_f$
$\cong$
$\widetilde{\mathrm{Hom}_A(M,N)}(D(f))$

• $0 \to \operatorname{Im}(\varphi(X)) \to N \to \operatorname{coker}(\varphi(X)) \to 0$ $\quad$ ($\operatorname{Im}(\varphi(X))$: f.g module; over noeth A; noeth module)
$$\Rightarrow \begin{array}{ccccccc} 0 \to & \widetilde{\operatorname{Im}(\varphi(X))} & \to & \widetilde{N} & \to & \widetilde{\operatorname{coker}(\varphi(X))} & \to 0 \\ & \text{so}\ \| & & \| & & \| & \\ 0 \to & \operatorname{Im}\varphi & \to & \mathcal{G} & \to & \operatorname{coker}\varphi & \to 0 \end{array}$$

Φ: $\mathcal{F}'(U) \xrightarrow{\varphi(U)} \mathcal{F}(U)$ is injective because otherwise $\ker(\varphi(U)) = (\ker\varphi)(U) \neq 0$. Φ $\psi(U)\circ\varphi(U) = 0$.

Φ: $s \in \mathcal{F}(U)$ s.t. $\psi(U)(s) = 0 \rightsquigarrow s \in \ker(\psi(U)) = (\ker\psi)(U) = (\operatorname{Im}\varphi)(U) \rightsquigarrow \exists \{E_p\}_{p\in U}$ & $s(p) \in \mathcal{F}(E_p)$ with $s(p) = s|_{E_p}$ and $s(p) \in \operatorname{Im}(\varphi(E_p))$

a nbd of p

$\mathcal{F}(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$

$s \to \psi(s) \to \psi\varphi(s) = 0$

$s_p \to \psi(s_p) \to 0$

$\because \varphi$ is injective $\therefore \exists!\, t(p) \in \mathcal{F}'(E_p)$ s.t. $\varphi(E_p)(t(p)) = s(p)$

and $t(p)|_{E_p\cap E_q} = t(q)|_{E_p\cap E_q}$ since their image under φ are the same

~~Hence $\exists\, t \in \mathcal{F}'(U)$ s.t. $t|_{E_p} = t(p)$ and $\varphi(t)|_{E_p} = \varphi(E_p)(t(p)) = s(p) \rightsquigarrow \varphi(t) = s$.~~ exact

Prop Φ: If $X = \operatorname{spec} A$ and $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ in $\operatorname{Mod}(X)$ with $\mathcal{F}' \in \operatorname{Qco}(X)$,

then $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ exact

(pf): Given $s'' \in \Gamma(X, \mathcal{F}'')$, $\exists \{D(f_i)\}_{i=1,\dots,m}$ covers X and $t_i \in \mathcal{F}(D(f_i))$ s.t. $t_i \longmapsto s''|_{D(f_i)}$.

Claim: $\exists\, n$ and $s_i \in \mathcal{F}(X)$ s.t. $s_i \longmapsto f_i^n s''$ $\forall i$.

If the claim is true, then $X = \bigcup_{i=1}^m D(f_i) = \bigcup_{i=1}^m D(f_i^n)$, say $1 = \sum_{i=1}^m a_i f_i^n$

$\Rightarrow s = \sum a_i s_i \longmapsto \sum a_i f_i^n s'' = s''$

Proof of the claim: Fixing i, let $f = f_i$, $t = t_i$.

$$f^{n_1}\omega_j \longmapsfrom \underset{\mathcal{F}'(D(ff_j))}{\omega_j} \longrightarrow (t - t_j)|_{D(ff_j)} \longmapsto 0$$

$$f^{n_1}\omega_j \longmapsto f^{n_1}(t-t_j)|_{D(ff_j)} \longmapsto 0$$

$\because \mathcal{F}' \in \operatorname{Qco}(X) \rightsquigarrow$ $u_j' \mapsto f^{n_1}\omega_j$

$$u_j' \in \mathcal{F}'(D(f_j)) \longrightarrow \underset{\mathcal{F}(D(f_j))}{u_j} \longmapsto 0 \quad \text{exact}$$

$$u_j \mapsto f^{n_1}(t-t_j)|_{D(ff_j)}$$

$(N^m)_p \to (\operatorname{Hom}_A(A^m, N))_p$

$\cong\downarrow$

$N_p^m \to \operatorname{Hom}_{A_p}(A_p^m, N_p)$

$$\underset{t_j'}{\overset{\parallel}{u_j + f^{n_1}t_j}}\Big|_{D(ff_j)} = f^{n_1}t\,|_{D(ff_j)}$$

$t_j' \longmapsto f^{n_1}s''|_{D(f_j)}$

$t_k' \longmapsto f^{n_1}s''|_{D(f_k)}$ $\forall j$ $\Big\} \Rightarrow t_j' - t_k' \in \mathcal{F}'(D(f_j f_k))$

$\Rightarrow (t_j' - t_k')|_{D(ff_jk)} = 0$

$\Downarrow$

$$\Rightarrow t_j'|_{D(ff_jf_k)} = t_k'|_{D(ff_jf_k)} \quad \Rightarrow \exists\, n_2 \text{ s.t. } f^{n_2}(t_j' - t_k')|_{D(f_jf_k)} = 0$$

So $\exists\, t'' \in \mathcal{F}(X)$ s.t. $t''|_{D(f_j)} = f^{n_2}t_j'|_{D(f_j)} \longmapsto f^n s''|_{D(f_j)}$

$$\overset{\parallel}{f^{n_2}u_j + f^{\overset{n_1+n_2}{\underset{n}{\parallel}}}t_j}\Big|_{D(f_j)}$$

$\phi: \tilde{M}|_U \to \tilde{N}|_U$

$\rightsquigarrow \delta(\phi): U \longrightarrow \prod_{p\in\operatorname{Spec}A} (\operatorname{Hom}_A(M,N))_p$

$p \longmapsto$ the image of $\phi_p: M_p \to N_p$ under $\operatorname{Hom}_{A_p}(M_p, N_p) \cong (\operatorname{Hom}_A(M,N))_p$

$$\Rightarrow \underset{s_i}{\overset{\uparrow}{t''}} \longmapsto f^n s''$$

$\delta(\phi)$ is a section in $(\widetilde{\operatorname{Hom}_A(M,N)})(U)$: $U = \cup D(f)$

$\phi|_{D(f)}$ is a section in $\mathcal{H}om_{\mathcal{O}_X}(\tilde{M}, \tilde{N})(D(f))$

$\cong$

$(\operatorname{Hom}_A(M,N))_f$

we get a morphism δ: on stalk p is an isom.

<u>Coro</u>: If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact and $\mathcal{F}', \mathcal{F}'' \in Qco(X)$,
then $\mathcal{F} \in Qco(X)$.
$(Coh(X), X: \text{noeth})$
$(Coh(X))$

(pf): Assume that $X = \operatorname{spec} A$.
$\because \mathcal{F}' \in Qco(X)$

$$
\therefore\ 0 \to \Gamma(X,\mathcal{F}') \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}'') \to 0
$$

$$
\underset{\text{f.g noeth}}{M'} \qquad \underset{\text{f.g so noeth}}{M} \qquad \underset{\text{f.g noeth}}{M''}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \widetilde{M'} \arrow[r] \arrow[d, "\wr"] & \widetilde{M} \arrow[r] \arrow[d, "\wr \leftarrow \text{so}"] & \widetilde{M''} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}
$$

※.

<u>Prop ~~4~~ 5</u>: Let $f \in \operatorname{Hom}_{Sch}(X, Y)$.

(1) $\mathcal{G} \in Qco(Y) \Longrightarrow f^*\mathcal{G} \in Qco(X)$
$(Coh(Y), X, Y: \text{noeth})$ $\quad (Coh(X))$

(2) Assume that X is noeth or f is quasi-compact & separated
$\mathcal{F} \in Qco(X) \Rightarrow f_*\mathcal{F} \in Qco(Y)$ $\quad \forall p \in X \rightsquigarrow f(p) \in \operatorname{spec} A \subset Y$

(pf): (1) Note that $f^*\mathcal{G}(\operatorname{spec} B) = \varinjlim_{\operatorname{spec} A \supset V \supset f(\operatorname{spec} B)} \mathcal{G}(V)$.

$f^{-1}(\operatorname{spec} A) \supset \operatorname{spec} B \ni p$, $\operatorname{spec} B \to \operatorname{spec} A \subset Y$; $\operatorname{spec} B \subset f^{-1}(\operatorname{spec} A)$, p; $\operatorname{spec} C \cap f^{-1}(\operatorname{spec} A) \supset \operatorname{spec} C_g$

Assume that $f: \operatorname{spec} B \to \operatorname{spec} A$ and $\mathcal{G} = \widetilde{M}$ ✓ Y: noeth [illegible]
f.g A-module.

Then $f^*\mathcal{G}|_{\operatorname{spec} B} = f^*\widetilde{M} = \widetilde{M \otimes_A B}$: f.g B-module.

(2) Assume that $f: X \xrightarrow[\text{assume}]{= f^{-1}(\operatorname{spec} A)} \operatorname{spec} A = Y$
$\bigcup_{i=1}^{m} U_i$ (X is noeth or f is quasi-compact), $U_i = \operatorname{spec} B_i$

- f is separated $\Rightarrow U_{ij} = U_i \cap U_j$ is still affine
- X is noeth $\Rightarrow U_i \cap U_j = \bigcup_{k=1}^{r} U_{ijk}$, U_{ijk}: affine

PAGE
DATE

We have

$$0 \longrightarrow f_*\mathcal{F}(V) \longrightarrow \oplus f_*(\mathcal{F}|_{U_i})(V) \xrightarrow{\varphi} \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}})(V)$$

U_{ijk} ← noeth case, or U_{ij} ← second case

$$s \longmapsto (\cdots, s|_{f^{-1}(V)\cap U_i}, \cdots)$$

$$(\cdots, s_i, \cdots, s_j, \cdots) \longmapsto (\cdots, s_i|_{f^{-1}(V)\cap U_{ijk}} - s_j|_{f^{-1}(V)\cap U_{ijk}}, \cdots)$$

and $f: \underset{U_{ijk}}{\overset{\shortparallel}{\operatorname{Spec} B_{ijk}}} \longrightarrow \operatorname{Spec} A$, $\mathcal{F}|_{U_{ijk}} = \widetilde{N_{ijk}}$

Then $f_*(\mathcal{F}|_{U_{ijk}}) = \widetilde{{}_A N_{ijk}} \Rightarrow \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{ijk}}) \in Qco(Y)$

Similarly, $\oplus f_*(\mathcal{F}|_{U_i}) \in Qco(Y)$

Hence $f_*\mathcal{F} = \ker\varphi \in Qco(Y)$. ※

Def: • A sheaf of ideals on X is a subsheaf of $\mathcal{O}_X$ in $Mod(X)$.

• $i: Y \hookrightarrow X$ closed subscheme, the ideal sheaf of $Y = \mathcal{I}_Y := \ker i^\#$ with $i^\#: \mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Y$.
(i.e. $\mathcal{O}_X/\mathcal{I}_Y \cong i_*\mathcal{O}_Y$)

Prop 5b: (1) If $i: Y \hookrightarrow X$ closed subscheme, then $\mathcal{I}_Y \in Qco(X)$
(X: noeth) (Coh(X))

(2) Any quasi-coherent sheaf of ideals on $X \rightleftarrows \mathcal{I}_Y$ via $i: Y \hookrightarrow X$ for some closed subscheme.

(pf): (1) i is quasi-compact and separated $\Rightarrow i_*\mathcal{O}_Y \in Qco(X)$

So $\mathcal{I}_Y = \ker i^\# \in Qco(X)$.

(X: noeth, let $\underset{U}{\overset{\shortparallel}{\operatorname{Spec} A}} \subset X$ with A noeth, $U \cap Y \neq \phi$. Then $I = \Gamma(\overset{U}{\overset{\shortparallel}{\operatorname{Spec} A}}, \mathcal{I}_Y|_U) \subset \Gamma(\operatorname{Spec} A, \mathcal{O}_X|_U) = A$
$\Rightarrow \mathcal{I}_Y|_U \cong \tilde{I}$ f.g. A-mod)

$f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Z$ with $\ker f^{\#} = \mathcal{J}$ i.e. $\mathcal{O}_X/\mathcal{J} \cong f_*\mathcal{O}_Z$

We have $\operatorname{supp}(f_*\mathcal{O}_Z) = f(Z) \rightsquigarrow \operatorname{supp}(\mathcal{O}_X/\mathcal{J}) = f(Z)$.

(2). Given $\mathcal{J} \in Qco(X)$ and $\mathcal{J} \hookrightarrow \mathcal{O}_X$, $Y := \operatorname{supp} \mathcal{O}_X/\mathcal{J} = \{p \in X \mid (\mathcal{O}_X/\mathcal{J})_p \neq 0\}$

<u>Claim</u>: $(Y, \mathcal{O}_X/\mathcal{J}) \underset{closed}{\hookrightarrow} (X, \mathcal{O}_X)$

(pf): Assume that $X = \operatorname{spec} A$.

$\because \mathcal{J} \in Qco(X) \quad \therefore \mathcal{J} = \tilde{I}$ $\quad \mathfrak{q}_p = \mathfrak{J}_p$

Thus $\underset{I}{\underset{\|}{\Gamma(X, \mathcal{J})}} \underset{ideal}{\hookrightarrow} \underset{A}{\underset{\|}{\Gamma(X, \mathcal{O}_X)}}$ $\quad \underset{\|}{(\mathcal{O}_X/\mathcal{J})_p}$ $\left(\begin{array}{l} 0 \to I \to A \to A/I \to 0 \\ 0 \to \tilde{I} \to \tilde{A} \to \widetilde{A/I} \to 0 \\ \quad\ \ \wr\| \quad\ \ \wr\| \quad\ \ \wr\| \\ 0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0 \end{array}\right)$

and $Y = \operatorname{supp} \mathcal{O}_X/\mathcal{J} = \{p \in \operatorname{spec} A \mid (A/I)_p \neq 0\} = V(I)$

since if $p \not\supset I$, then $\exists\, a \in I \setminus P$, $\underset{\bar{1} \cdot a = \bar{0}}{(1+I)a = I} \Rightarrow (A/I)_p = 0$

if $p \supset I$, then $P_p \supset I_p \Rightarrow (A/I)_p \neq 0$

<u>Remark</u>: $\cancel{X} = \operatorname{spec} A$,

$$\{ I \underset{ideal}{\hookrightarrow} A \} \leftrightarrow \{ Y \underset{\text{closed subscheme}}{\hookrightarrow} X \}$$

prop 2 $\quad$ prop 56

$$\{ \text{quasi-coherent sheaves of ideals on } X \}$$

$\tilde{M}$: a sheaf of $\mathcal{O}_{\operatorname{proj} S}$ - module :

$\tilde{M}(U) = \{ s : U \to \coprod_{p \in \operatorname{proj} S} M_{(p)}$
$\quad p \mapsto s(p) \in M_{(p)}$, $\exists_{p \in D_+(f_p) \subseteq U}$, $m_p \in M$, s.t. $s(q) = \frac{m_p}{f_p} \in M_{(q)}$, $\forall q \in D_+(f_p)$ $\}$

We have

$\forall p \in \operatorname{proj} S$, $(\tilde{M})_p \cong M_{(p)}$

$\underset{\text{homo elt}}{f \in S}$, $(\tilde{M})|_{D_+(f)} \cong \widetilde{M_{(f)}}$

⊙ Quasi-coherent sheaves on Proj S

Let $X = \operatorname{proj} S$ with S a graded ring

and $M = \bigoplus_{d \in \mathbb{Z}} M_d$ a graded S-module with $S_e \cdot M_d \subseteq M_{e+d}$

Given $f \in S_+$, $\quad \varphi_f : D_+(f) \xrightarrow{\sim} \operatorname{Spec} S_{(f)}$.

- $\tilde{M}|_{D_+(f)} := \widetilde{M_{(f)}}$, $\quad M_{(f)} : \underset{\mathcal{O}_{D_+(f)}(D_+(f))}{\underset{\|}{S_{(f)}}}$ - module $\overset{\text{glue (ex. 1.22)}}{\Longrightarrow}$ $\tilde{M} \in Qco(X)$.
- $\tilde{M}_p = M_{(p)}$
- S : noeth, M : f.g. S-module $\Rightarrow \tilde{M} \in Coh(X)$ $\quad$ Proj S is Noetherian

$\Downarrow$

$X = \bigcup D_+(f)$ $\quad M_{(f)}$: f.g. $S_{(f)}$-module.

- $\mathcal{O}_X = \tilde{S}$.

Def: • $n \in \mathbb{Z}$, $S(n)_d := S_{n+d}$ $\forall d$ $\rightsquigarrow$ $S(n)$ is a graded S-module

$M(n)_d := M_{n+d}$ $\forall d$ $\rightsquigarrow$ $M(n)$ ... $S_\ell \cdot S(n)_d = S_\ell \cdot S_{n+d} \subseteq S_{n+\ell+d} = S(n)_{\ell+d}$

• $n \in \mathbb{Z}$, $\mathcal{O}_X(n) := \widetilde{S(n)}$;

• $\mathcal{L} \in Mod(X)$ is said to be invertible if it is locally isomorphic to $\mathcal{O}_X$ (i.e. $\exists \{U_\lambda\}$ covers X & $\mathcal{L}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda}$)

Remark: Let $\theta_\lambda : \mathcal{O}_{U_\lambda} \longrightarrow \mathcal{L}|_{U_\lambda}$ be an isomorphism.

$$\theta_\lambda(U_\lambda) : \mathcal{O}_{U_\lambda}(U_\lambda) \longrightarrow \mathcal{L}(U_\lambda) \Rightarrow \mathcal{L}(U_\lambda) = \beta_\lambda \cdot \mathcal{O}_{U_\lambda}(U_\lambda)$$
$$1 \longmapsto \beta_\lambda$$

We write $\mathcal{L}_{U_\lambda} = \mathcal{O}_{U_\lambda} \cdot \beta_\lambda$ via $\mathcal{L}_{U_\lambda}(W) = \mathcal{O}_{U_\lambda}(W) \cdot \beta_\lambda|_W$, $W \subset U_\lambda$.

Fact 4: $\widetilde{M} \otimes_S \widetilde{N} = \widetilde{M \otimes_S N}$ on $X = \operatorname{Proj} S$. $M_f \otimes N_f \cong (M \otimes_S N)_f$

$M \otimes_{\mathbb{Z}} N = (\bigoplus_d M_d) \otimes_{\mathbb{Z}} (\bigoplus_e N_e) = \bigoplus_n (\bigoplus_{d+e=n} M_d \otimes_{\mathbb{Z}} N_e)$

$M \otimes_S N = M \otimes_{\mathbb{Z}} N / \langle sm \otimes n - m \otimes sn : s \in S, m \in M, n \in N \rangle$.

Pf: $\forall f \in S_m, m > 0$

$$M_{(f)} \otimes_{S_{(f)}} N_{(f)} \cong (M \otimes_S N)_{(f)}$$
$$\frac{a}{f^d} \otimes \frac{b}{f^e} \longmapsto \frac{a \otimes b}{f^{d+e}}$$

which induces a presheaf morphism and thus a sheaf morphism

$$\widetilde{M} \otimes \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

Since $M_{(p)} \otimes_{S_{(p)}} N_{(p)} \cong (M \otimes_S N)_{(p)}$, $\widetilde{M} \otimes \widetilde{N} \cong \widetilde{M \otimes_S N}$

⊛ Let S be generated by S_1 as S_0-algebra.

Prop 6: (1) $\mathcal{O}_X(n)$ is an invertible sheaf. $\rightsquigarrow$ twisting sheaf.

(2) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(m+n)$

(3) $\widetilde{M}(n) := \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{M(n)}$.

Pf: (1) Let $x \in S_1$. $\mathcal{O}_X(n)|_{D_+(x)} = \widetilde{S(n)_{(x)}}$

$$\mathcal{O}_X(n)(D_+(x)) = S(n)_{(x)} \xrightarrow{\sim} S_{(x)} = \mathcal{O}_X(D_+(x)) \Rightarrow \widetilde{S(n)_{(x)}} \xrightarrow{\sim} \widetilde{S_{(x)}} = \mathcal{O}_{D_+(x)}$$

$\widetilde{S(n)_{(x)}} = \mathcal{O}_X(n)|_{D_+(x)}$

$$g \longmapsto \frac{g}{x^n}$$

deg n in S_x; deg 0 in S_x

$x^n h \longleftarrow h$

$\mathcal{O}_X(n)|_{D_+(x)} = x^n \mathcal{O}_{D_+(x)}$

(2) $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \widetilde{S(n)} \otimes \widetilde{S(m)} = \widetilde{S(n) \otimes S(m)} \cong \widetilde{S(n+m)} = \mathcal{O}_X(n+m)$

graded ring 中 ⊗ degree 相加

(3) $\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \widetilde{M} \otimes \widetilde{S(n)} = \widetilde{M \otimes S(n)} = \widetilde{M(n)}$ ✱.

$\mathcal{U} = \operatorname{Proj} T - \{ p \in \operatorname{Proj} T \mid p \supseteq Q \}$ where Q is the smallest homo. ideal containing $\varphi(S_+)$

$f(p) = \varphi^{-1}(p)$: homo prime & $\because$ $p \not\supseteq \varphi(S^+) \rightsquigarrow \varphi^{-1}(p) \not\supseteq S^+$

$f^{-1}(V(I)) = \{ p \in \operatorname{Proj} T \mid p \not\supseteq \varphi(S_+),\ p \supseteq \varphi(I) \} = \{ p \in \operatorname{Proj} T \mid p \supseteq \langle \varphi(I) \rangle \} \cap \mathcal{U}$ is closed in $\mathcal{U}$.

<u>Remark</u>: $\underset{Y = V(Q)}{\varphi : S} \longrightarrow \underset{X}{T}$ where both are generated by deg 1 as deg 0-alg.

$\Rightarrow f : \mathcal{U} \longrightarrow \operatorname{proj} S$

$\{ p \in \operatorname{Proj} T \mid p \not\supseteq \varphi(S_+) \}$ open in $\operatorname{proj} T$.

If $N \in \mathrm{GrMod}_T$ and $M \in \mathrm{GrMod}_S$, then

$$f^* \widetilde{M} \cong (\widetilde{M \otimes_S T})\big|_{\mathcal{U}} \quad \text{and} \quad f_* (\widetilde{N}|_{\mathcal{U}}) \cong \widetilde{({}_S N)}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f^*(\mathcal{O}_X(n)) = f^*(\widetilde{S(n)}) = \widetilde{S(n) \otimes_S T}\big|_{\mathcal{U}} \qquad f_*(\mathcal{O}_Y(n)|_{\mathcal{U}}) = f_*(\widetilde{T(n)}|_{\mathcal{U}}) \cong \widetilde{({}_S T(n))}$$

$$\cong \widetilde{T(n)}\big|_{\mathcal{U}} = \mathcal{O}_Y(n)\big|_{\mathcal{U}} \qquad \cong {}_S\widetilde{T}(n) \cong f_*(\widetilde{T}|_{\mathcal{U}})(n) = (f_* \mathcal{O}_{\mathcal{U}})(n).$$

• The graded S-module associated to $\mathcal{F}$: $\Gamma_*(\mathcal{F}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{F}(n)(X)$.

$\Gamma_*(\mathcal{O}_X) \cong S$

Eg: $\Gamma(\underset{X}{P_A^n}, \mathcal{O}_{P_A^n}(m)) \cong \{$ all homogeneous polynomials of deg m in $\underset{A[x_0, \dots, x_n]}{S}$ $\}$.

$$\mathcal{O}_X(m)\big|_{D_+(x_i)} = x_i^m \cdot \mathcal{O}_X\big|_{D_+(x_i) = \operatorname{spec} A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]}$$

$0 \to \mathcal{O}(n)(X) \to \prod_{i=0}^{n} \mathcal{O}(n)(D_+(x_i)) \longrightarrow \prod_{i,j=0}^{n} \mathcal{O}(n)(D_+(x_i) \cap D_+(x_j))$ $\Rightarrow$ Taking $\bigoplus_{n=-\infty}^{\infty}$: $0 \to \Gamma_*(\mathcal{O}_X) \to \prod_{i=0}^{n} A[x_0, \dots, x_n]_{x_i} \longrightarrow \prod_{i,j=0}^{n} A[x_0, \dots, x_n]_{x_i x_j}$

$s \longmapsto (s|_{D_+(x_i)})$, $(s_i) \longmapsto (s_i|_{D_+(x_i) \cap D_+(x_j)} - s_j|_{D_+(x_i) \cap D_+(x_j)})$

• Any $\varphi \in \Gamma(X, \mathcal{O}_X(m)) \rightsquigarrow \{ \varphi|_{D_+(x_i)} = x_i^m \cdot \varphi_i(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) \mid x_i^m \cdot \varphi_i = x_j^m \cdot \varphi_j$ on $D_+(x_i x_j) \}$

$(A[x_0, \dots, x_n]_{x_i} \hookrightarrow A[x_0, \dots, x_n]_{x_i x_j} \hookrightarrow A[x_0, \dots, x_n]_{x_0 \cdots x_n}$

$\because$ $\varphi_i = \varphi_0 \cdot (\frac{x_0}{x_i})^m$ has deg m, $\therefore$ a homo. poly. of degree m.

$\frac{x_\cdot}{x_i}$, $\frac{x_\cdot}{x_0}$

分母部需被消光

$(\frac{x_1}{x_0})^{\alpha_1} (\frac{x_2}{x_0})^{\alpha_2} \cdots (\frac{x_n}{x_0})^{\alpha_n} (\frac{x_0}{x_i})^m = (\frac{x_1}{x_i})^{\alpha_1} (\frac{x_2}{x_i})^{\alpha_2} \cdots (\frac{x_i}{x_i})^{\alpha_i} \cdots (\frac{x_n}{x_i})^{\alpha_n} \cdot (\frac{x_0}{x_i})^{m - (\alpha_1 + \cdots + \alpha_n)}$, $\alpha_1 + \cdots + \alpha_n \le m$

• For any homo. poly F of deg m, so $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{n} A[x_0, \dots, x_n]_{x_i}$

$k_j \ge 0$ $\forall j \ne i$

$x_0^{k_0} \cdots x_n^{k_n} \cdot f$ ← m factors x_i, $k_i \in \mathbb{Z}$

$$\text{define} \quad \varphi\big|_{D_+(x_i)} := F(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) \cdot x_i^m \in \mathcal{O}_X(m)\big|_{D_+(x_i)}.$$

$k_i \ge 0$.

so $= A[x_0, \dots, x_n]$.

※

Def: • $Y \in \mathrm{Sch}$, $\mathbb{P}_Y^n := \mathbb{P}_{\mathbb{Z}}^n \times Y \xrightarrow{\pi} \mathbb{P}_{\mathbb{Z}}^n$, $\mathcal{O}_{\mathbb{P}_Y^n}(1) := \pi^* \mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^n}(1)$

the twisting sheaf of $\mathbb{P}_{\mathbb{Z}}^n$

• If $X \longrightarrow Y$ is projective, then

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "i"', "\text{closed immersion}"] & & Y \\ & \mathbb{P}_Y^n \arrow[ur] & \end{tikzcd}.$$

We define $\mathcal{O}_X(1) = i^*(\mathcal{O}_{\mathbb{P}_Y^n}(1))$.

$\ast$: $X \underset{\text{closed}}{\hookrightarrow} \mathbb{P}^m_A$, by prop ?, $\mathcal{I}_X$ is Qcoh & $\mathcal{I}_X \cong \widetilde{\Gamma_*(\mathcal{I}_X)}$. Since $\underset{\substack{\updownarrow \\ \mathfrak{a}:\text{ homo. ideal in}}}{\Gamma_*(\mathcal{I}_X)} \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^m_A}) \cong A[x_0, \dots, x_m]$

$X \cong \operatorname{Proj} A[x_0, \dots, x_m]/\mathfrak{a}$.

$\ast$ A scheme X over $\operatorname{Spec} A$ is projective i.e.
$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} A \\ \mathbb{P}^n_A \arrow[ur] & \end{tikzcd}$$
$\rightsquigarrow X \cong \operatorname{Proj} A[x_0, \dots, x_n]/\mathfrak{a} = \operatorname{Proj} \dfrac{A[x_0, \dots, x_n]}{\mathfrak{a} \cap (x_0, \dots, x_n)}$

<u>Def</u>: $\mathcal{F} \in \operatorname{Mod}(X)$ is generated by global sections if

$$\exists \{s_i\}_{i \in \Lambda} \subset \Gamma(X, \mathcal{F}) \text{ s.t. } \forall p \in X,\ \mathcal{F}_p = \langle (s_i)_p : i \in \Lambda \rangle_{\mathcal{O}_p}.$$

$\Rightarrow \bigoplus \mathcal{O}_X \twoheadrightarrow \mathcal{F}$, $e_i \mapsto s_i$

(eg: $x_0, \dots, x_n \in \Gamma(X, \mathcal{O}_X(1)) \Rightarrow \mathcal{O}_X(1)$ is g.b.g.s.) i.e. $X = \operatorname{Proj} A[x_0, \dots, x_n]/I \leftarrow$ homo.

<u>Serre theorem</u>: Let X be a projective scheme over A with A: noeth and $\mathcal{F} \in \operatorname{Coh}(X)$. Then $\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes m}$ is (finitely) g.b.g.s $\forall m >> 0$

$X \xrightarrow{\text{projective}} \operatorname{Spec} A \rightsquigarrow$
$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} A \\ \mathbb{P}^n_A \arrow[ur] & \end{tikzcd}$$

<u>Stronger key lemma</u>: Let X be a scheme and $\mathcal{L}$ be an invertible sheaf. For $f \in \Gamma(X, \mathcal{L})$, $X_f := \{p \in X \mid f_p \notin m_p \mathcal{L}_p\}$. Let $\mathcal{F} \in \operatorname{Qco}(X)$.

$\mathcal{L}|_{U = \operatorname{Spec} A} \cong \mathcal{O}_U$, $X_f \cap U = D(f')$

$f|_U \longleftrightarrow f'$

(1) If X is quasi-compact and $s \in \Gamma(X, \mathcal{F})$, then $s|_{X_f} = 0 \Rightarrow \underset{\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)}{f^n s} = 0$

(2) If $X = \bigcup_{i=1}^m \underbrace{\operatorname{Spec} A_i}_{\overset{\shortparallel}{U_i}}$ s.t. $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$ and $U_i \cap U_j$ is quasi-compact,

then $\forall t \in \Gamma(X_f, \mathcal{F}) \Rightarrow \exists n \in \mathbb{N}$ & $s \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ s.t. $s|_{X_f} = f^n t$.

(Note : we need $(\mathcal{F} \otimes \mathcal{L}^{\otimes \ell})|_{U_i} \cong \mathcal{F}|_{U_i} \otimes \mathcal{L}|_{U_i}^{\otimes \ell} \cong \mathcal{F}|_{U_i} \cong \widetilde{M_i}$)

(proof of Serre thm): $i : X \hookrightarrow \mathbb{P}^n_A$ s.t. $\mathcal{O}_X(1) = i^*(\mathcal{O}_{\mathbb{P}^n_A}(1))$

$\underset{\text{ex}}{\Rightarrow} i_* \mathcal{F} \in \operatorname{Coh}(\mathbb{P}^n_A)$, $i_*(\mathcal{F}(m)) = (i_*\mathcal{F})(m)$, $\mathcal{F}(m)$: finitely g.b.g.s $\Leftrightarrow i_*(\mathcal{F}(m))$: f.g.b.g.s.

We can assume that $X = \mathbb{P}^n_A$.

Let $\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$ with $M_i = \langle t_{ij} \mid j = 1, \dots, r_i \rangle_{S_{(x_i)}}$.

By lemma, $\exists s_{ij} \in \Gamma(X, \mathcal{F}(m_0))$ s.t. $x_i^{m-m_0} s_{ij}|_{D_+(x_i)} = \overset{x_i^m}{x_i^{m_0}} t_{ij}$

$\mathcal{F}(m_0)|_{D_+(x_i)} \cong \widetilde{M}_i(m_0) \cong \widetilde{M_i(m_0)}$, $M_i(m_0) = \langle \underbrace{x_i^{m_0} t_{ij}}_{\overset{\shortparallel}{s_{ij}|_{D_+(x_i)}}} \mid j = 1, \dots, r_i \rangle_{S_{(x_i)}}$

Hence $\mathcal{F}(m_0)$ is generated by $\{\underset{x_i^{m-m_0}}{s_{ij}}\}$

(for $m \geq m_0$,)

$\diamondsuit$

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\Gamma_*(\mathcal{F})}(U) \arrow[r] \arrow[d, "\varphi(U)"', "\wr"] & \prod_{i\in I} \widetilde{\Gamma_*(\mathcal{F})}(D_+(f_i)) \arrow[r] \arrow[d, "\wr"] & \prod_{i,j} \widetilde{\Gamma_*(\mathcal{F})}(D_+(f_i)\cap D_+(f_j)) \arrow[d, "\wr"] \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \prod_{i\in I} \mathcal{F}(D_+(f_i)) \arrow[r] & \prod_{i,j} \mathcal{F}\left(\underset{A[\bar{x}_0,\dots,\bar{x}_n]}{D_+(f_i)\cap D_+(f_j)}\right)
\end{tikzcd}$$

$\{D_+(f_i)\}_{i\in I}$: an open cover of U

so

Prop 7: If $X = \mathrm{Proj}\, S$ with $S = \frac{A[x_0,\dots,x_n]}{I}$ and $\mathcal{F} \in Qco(X)$, then $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}$.

$$\Gamma_*(\mathcal{F}) = \bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathcal{F}(m))$$

(pf): Define $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ via $\beta_{\bar{x}_i} : \Gamma_*(\mathcal{F})_{(\bar{x}_i)} \longrightarrow \mathcal{F}(D_+(\bar{x}_i))$

$\frac{s}{\bar{x}_i^d} \in \Gamma(X, \mathcal{F}(d)),\ d\geq 0 \longmapsto \frac{1}{\bar{x}_i^d} \otimes s|_{D_+(\bar{x}_i)}$

$s' \in \mathcal{F}(-1)$, $s'\cdot\bar{x}_i^d \in \mathcal{F}(1)$, $\frac{s'\bar{x}_i}{\bar{x}_i}$

$\bar{x}_i \in \mathcal{O}_X(1)$

$\mathcal{O}(-d)\otimes\mathcal{F}(d) \cong \mathcal{F}$

The strong key lemma

(It is easy to see that) $\beta_{\bar{x}_i}$ is an isom $\forall i$.

$\therefore \beta$ is an isom.

Thm: If A is a f.g. k-alg, X is a projective scheme over A ($\subseteq \mathrm{Proj}\, S$) and $\mathcal{F} \in Coh(X)$, then $\Gamma(X, \mathcal{F})$ is a f.g. A-module.

(pf): Step 1: $\Gamma(X, \mathcal{O}_X(m))$ is a f.g. A-mod with $X = \mathrm{Proj}\, \overset{S}{\overbrace{\frac{A[x_0,\dots,x_n]}{P}}}$ (P: homo. prime), $\forall m \in \mathbb{Z}$.

(pf): Since S is an integral domain, we have

$$0 \to S(l) \to S(l+1)$$
$$g \mapsto \bar{x}_0\cdot g$$

$$\Rightarrow\ 0 \to \widetilde{S(l)} \to \widetilde{S(l+1)}\ \Rightarrow\ 0 \to \Gamma(X, \mathcal{O}_X(l)) \to \Gamma(X, \mathcal{O}_X(l+1)),\ \forall l \in \mathbb{Z}.$$

Hence "A: noeth" + "$\Gamma(X, \mathcal{O}_X(m))$ is f.g. $\forall m \geq 0$" $\Rightarrow$ Step 1 ($m\in\mathbb{Z}$)

noeth module $\Rightarrow$ $\forall$ submodule is f.g.

Observe that $\forall t \in \Gamma(X, \mathcal{O}_X(m))$, $t|_{D_+(\bar{x}_i)} = t_i$ is an element of deg m in $S_{\bar{x}_i}$.

and $\bar{t}_i = \bar{t}_j$ in $S_{\bar{x}_i\bar{x}_j}$

Since S is an integral domain, $S \hookrightarrow S_{\bar{x}_i} \hookrightarrow S_{\bar{x}_i\bar{x}_j} \hookrightarrow S_{\bar{x}_0\bar{x}_1\cdots\bar{x}_n}$

We can regard t as an element in $\bigcap_{i=0}^n S_{\bar{x}_i} \hookrightarrow S_{\bar{x}_0\bar{x}_1\cdots\bar{x}_n}$.

Let $S' = \bigoplus_{m=0}^{\infty} \Gamma(X, \mathcal{O}_X(m))$. and thus $S \subset S' \subset \bigcap_{i=0}^{n} S_{\bar{x}_i}$.

($s \mapsto \frac{s}{1} \in S_{\bar{x}_i}$)

Claim: S' is integral over S

(pf): $\because t \in \bigcap_{i=0}^n S_{\bar{x}_i}$ $\therefore$ $\bar{x}_i^{l_0} t \in S$ $\forall i$ $\Rightarrow g t \in S_{\geq l}$ $\forall g \in S_{\geq l}$, $l_0(n+1)$

$\Rightarrow g t^q \in S_{\geq l}$ $\forall q \in \mathbb{N}$, $\forall g \in S_{\geq l}$

Let $g = \bar{x}_0^{l}$. Then $\bar{x}_0^{l} t^q \in S \Rightarrow t^q \in \frac{1}{\bar{x}_0^{l}} S$ $\forall q \in \mathbb{N}$

$$\Rightarrow\ S[t] \hookrightarrow \underbrace{\tfrac{1}{x_0^k} S}_{\text{f.g. } S\text{-module}} \hookrightarrow \text{q.f of } S \quad \Rightarrow\ t \text{ is integral over } S \ \#$$

When S is a f.g. k-alg, by finiteness of integral closure,
the integral closure of S in its q.f is a f.g. S-module.

$\because$ S is noeth $\therefore$ $\overset{= Sa_1 + Sa_2 + \cdots + Sa_l}{S'}$ is a f.g. S-module $\Rightarrow \Gamma(X, \mathcal{O}_X(m))$ is a f.g. $\overset{S_0}{\overset{\parallel}{A}}$-mod.

<u>Step 2</u>: M: graded f.g. $\bar{S}$-module ($\bar{S} = \frac{A[x_0, \dots, x_n]}{I}$) $\Rightarrow \Gamma(X, \widetilde{M})$ is a f.g A-mod.

(pf): Let $0 = M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r = M$ with $M^i/M^{i-1} \cong \bar{S}/P_i\,(n_i)$ (I. §7.4)

$$0 \to \widetilde{M^{i-1}} \to \widetilde{M^i} \to \widetilde{M^i/M^{i-1}} \to 0$$

$$\Rightarrow 0 \to \Gamma(X, \widetilde{M^{i-1}}) \to \Gamma(X, \widetilde{M^i}) \to \Gamma(X, \widetilde{M^i/M^{i-1}})$$

By step 1, $\Gamma(X, \widetilde{\bar{S}/P_i(n_i)})$ is a f.g. A-module

Then $\widetilde{M^1}$ ok, $\widetilde{M^2/M^1}$ o.k $\Rightarrow \widetilde{M^2}$ o.k
$\left.\begin{matrix}\widetilde{M^2} \text{ o.k} \\ \widetilde{M^3/M^2} \text{ o.k}\end{matrix}\right\} \Rightarrow \widetilde{M^3}$ o.k $\Rightarrow \cdots \Rightarrow \widetilde{M}$ o.k.

<u>Step 3</u>: By prop 7, set $M = \Gamma_*(\mathcal{F})$, then $\widetilde{M} \cong \mathcal{F}$

By Serre's thm, $\mathcal{F}(m)$ is finitely g.b.g.s $\forall\, m \gg 0$
say $s_1, \dots, s_l \in M$

Then $M' = \langle s_1, \dots, s_l \rangle_S \hookrightarrow M \ \Rightarrow\ \widetilde{M'} \hookrightarrow \widetilde{M} = \mathcal{F} \ \overset{\otimes \mathcal{O}_X(m)}{\underset{\text{locally free} \Rightarrow \text{flat}}{\Rightarrow}}\ \widetilde{M'}(m) \hookrightarrow \mathcal{F}(m)$ (g.b. $s_1, \dots, s_l \in M'$)

$$\Rightarrow \widetilde{M'}(m) \simeq \mathcal{F}(m)$$

$$\Rightarrow \widetilde{M'}(m) \otimes \mathcal{O}_X(-m) \simeq \mathcal{F}(m) \otimes \mathcal{O}_X(-m)$$

$$\Rightarrow \widetilde{M'} \simeq \mathcal{F}$$

By step 2, $\Gamma(X, \mathcal{F})$ is a f.g. A-module.

<u>Coro</u>: If X, Y are of finite type over k and $f: X \to Y$ is projective,
then $\forall\, \mathcal{F} \in Coh(X) \Rightarrow f_* \mathcal{F} \in Coh(Y)$

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & \operatorname{Spec} A \\ \mathbb{P}^n_A \arrow[ur] & \end{tikzcd}$$

(pf). Assume $Y = \operatorname{spec} A$, A: f.g. k-alg. Already know $f_* \mathcal{F} \in Qco(Y)$

So $f_* \mathcal{F} = \widetilde{\Gamma(Y, f_* \mathcal{F})} = \widetilde{\Gamma(X, \mathcal{F})}$, f.g. A-mod.