§6. Divisors.

◎ Weil divisors

X : noetherian integral separated scheme which is regular in codim 1.

• $X = \operatorname{spec} A$: $\quad X \longleftrightarrow A$

$$\text{integral subschemes of } X \begin{cases} \dim 0^{=\operatorname{codim} n}, & \{p\}\,(\text{point}) & , \ m_p\ (\text{max ideal}),\ ht(m_p)=n \\ & \cap & \quad \cup \\ \dim 1^{=\operatorname{codim} n-1}, & \operatorname{spec} A/p_1 & , \ p_1 \qquad , ht(p_1)=n-1 \\ & \cap & \quad \cup \\ \dim 2^{=\operatorname{codim} n-2}, & \operatorname{spec} A/p_2 & , \ p_2 \qquad , ht(p_2)=n-2 \\ & \vdots & \quad \vdots \qquad\qquad \vdots \\ & \cap & \quad \cup \\ \dim n-1^{=\operatorname{codim} 1}, & \operatorname{spec} A/p_{n-1} & , \ p_{n-1} \qquad , ht(p_{n-1})=1 \end{cases}$$

Here, $\overline{\{p_i\}} = \operatorname{spec} A/p_i$.

※ p_{n-1} is called a point of codim 1.

regular in codim 1 $\equiv \mathcal{O}_{X,p_{n-1}}$ is a regular local ring $\forall p_{n-1}$.

(A local ring (R, m) is regular if $\dim R = \dim_{R/m} m/m^2$)

Here, $\dim \mathcal{O}_{X,p_{n-1}} = 1$.

Def : • A prime divisor on X is a closed integral subscheme of codim one.

• A weil divisor $D = \sum n_i Y_i \in \operatorname{Div} X$

$n_i \in \mathbb{Z}$, Y_i a prime divisor, with $\#\{n_i \neq 0\} < \infty$

$\operatorname{Div} X$: free abelian gp generated by prime divisors.

motivation for the assumption of X

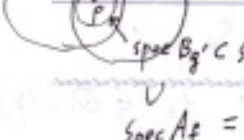
• D is effective if $n_i \geq 0 \quad \forall i$.

Remark : • integral :

•• $\exists$ the function field of $X = K :=$ q.f of A with $\operatorname{spec} A \subseteq X$:

for another $\operatorname{spec} B \subseteq X$, as before, $\exists f \in A, g \in B$ s.t. $A_f \cong B_g$

$\Rightarrow$ q.f of $A \cong$ q.f of B.



$\operatorname{spec} B_{g'} \subset \operatorname{spec} A$

$\cup$

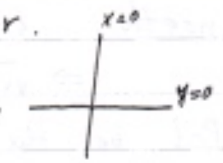
$\operatorname{spec} A_f = \operatorname{spec}(B_{g'})_{\bar f} \Rightarrow A_f \cong B_{g'\bar f}$

$\Gamma(\operatorname{spec} A, \mathcal{O}_{\operatorname{spec} A}) \to \Gamma(\operatorname{spec} B_{g'}, \mathcal{O}_{\operatorname{spec} B_{g'}})$

$f \longmapsto \bar f$

- A zero set defined by a function is a divisor.

  (otherwise, eg. $X = \operatorname{spec} A = \operatorname{spec} \frac{k[x,y]}{(xy)}$,



  $\{x=0\}$ is a component of X, not a divisor of X)

- separated :

  For a prime divisor Y, $Y = \overline{\{\eta\}}$ and $\mathcal{O}_{X,\eta}$ is dim 1, noeth, regular local domain

  $\Rightarrow \mathcal{O}_{X,\eta}$ is a discrete valuation ring (DVR)

  i.e. $v_Y : K^* \to \mathbb{Z}$

  (Recall: Under local, noeth, dim 1, DVR $\equiv$ regular $\equiv$ integrally closed)

  (eg. $X = \operatorname{Spec} A$, $Y = \overline{\{p\}}$ with $\underbrace{ht(p) = 1}_{\dim A/p = \dim A - 1}$, $\mathcal{O}_{X,p} = \underbrace{A_p}_{\dim A_p = 1}$: regular $\Rightarrow$ DVR

  $\quad = V(p) = \operatorname{spec} A/p$

  $P_p = \langle \eta \rangle$, $\forall f \in \underset{\text{integral domain}}{A}$, $\langle f \rangle_{A_p} = \langle \eta^{\ell} \rangle_{A_p}$, $v_Y(f) := \ell$

  $\forall \frac{f}{g} \in K^*$, $v_Y(\frac{f}{g}) = v_Y(f) - v_Y(g)$.)

$$\begin{tikzcd} \operatorname{spec} K \arrow[r] \arrow[d] & X \supset \overline{\{\eta\}} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{X,\eta} \arrow[r] \arrow[ur, "\exists!"] & \operatorname{spec} \mathbb{Z} \end{tikzcd}$$

, ~~if possible, then~~ exists a DVR will correspond to a unique prime divisor.

- noetherian :

  Fact: If $f \in K \setminus \{0\}$, then $\exists$ only finitely many prime divisors Y s.t. $v_Y(f) \neq 0$.

  pf: Let $U = \operatorname{spec} A \subseteq X$ s.t. f is regular on U i.e. $f \in A$.

  Observe that X : noeth $\Rightarrow \begin{cases} Z = X \setminus U = \underset{\text{Some of them are prime divisors}}{Y_1 \cup \cdots \cup Y_r}, & \text{if } \operatorname{codim} Z = 1 \\ Z \text{ contains no prime divisors}, & \text{otherwise} \end{cases}$

  $\Rightarrow$ almost all prime divisors $Y \cap U \neq \phi$.

  $\Rightarrow$ we can reduce it to show " $\exists$ finitely many prime divisors Y in U s.t. $v_Y(f) \neq 0$ ".

Note that if f is a unit in A, then $f \notin \mathfrak{p} \ \forall \mathfrak{p} \in \operatorname{Spec} A$

$\Rightarrow v_Y(f) = 0 \quad \forall Y.$

Assume that f is not a unit.

Since A is noeth, $\langle f \rangle \overset{\text{primary decomposition}}{=} \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_m$

$\Rightarrow \sqrt{\langle f \rangle} = \sqrt{\mathfrak{q}_1} \cap \sqrt{\mathfrak{q}_2} \cap \cdots \cap \sqrt{\mathfrak{q}_m} = P_1 \cap \cdots \cap P_m$

Let $\{P_1, \cdots, P_r\}$ be the set of minimal ~~primes~~ elements among $P_1, \cdots, P_m$.

which is also the set of all minimal primes of A containing f.

Recall that for a noeth ring A, if f is ~~not~~ neither a unit nor a zero divisor, then any minimal prime $\mathfrak{p}$ containing f has height 1.

~~Here,~~ Now, $v_Y(f) > 0 \Leftrightarrow f \in \mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$ $\Leftrightarrow$ $\mathfrak{p}$ is minimal over f

($\Leftarrow$ by; $\Rightarrow$: $(\mathfrak{p} \supsetneq \mathfrak{p}_0 \supsetneq \langle f \rangle \supsetneq \{0\} \Rightarrow \operatorname{ht}(\mathfrak{p}) = 2.)$)

$\Updownarrow$ $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) = 1$

$\Leftrightarrow \mathfrak{p} \in \{P_1, \cdots, P_r\}.$ #.

Def:
- a principal divisor $= (f) := \sum v_Y(f) Y \in \operatorname{Div} X$. for some $f \in K \setminus \{0\}$
- the divisor class group $\operatorname{Cl} X := \operatorname{Div} X / \sim$

where $D \sim D'$ iff $D - D' = (f)$ with $f \in K \setminus \{0\}$

($\sim$: linearly equivalent.)

i.e. $K \setminus \{0\} \longrightarrow \operatorname{Div} X \longrightarrow \operatorname{Cl} X \longrightarrow 0$ exact.

$f \longmapsto (f)$

(multiplicative group → additive group)

here, $(fg) = (f) + (g)$ since $v_Y(fg) = v_Y(f) + v_Y(g)$

$\left(\frac{f}{g}\right) = (f) - (g)$

⊙ Cartier divisors

~ to extend the notion of divisors to an arbitrary scheme.

X : any scheme

$\mathcal{K}$: the sheaf of total quotient rings of $\mathcal{O}_X$

$\forall U \subset X, \ \mathcal{K}(U) = \mathcal{O}_X(U)_{S(U)}$ with $S(U) = \mathcal{O}_X(U) \setminus \{\text{zero divisors of } \mathcal{O}_X(U)\}$

i.e. $(\forall\, U = \operatorname{Spec} A \subset X, \ \mathcal{K}(U) := A_S$ with $S = A \setminus \{\text{divisors of } A\})$

$\mathcal{K}^*$: the sheaf of invertible elements in $\mathcal{K}$

$\mathcal{O}_X^*$: the sheaf of invertible elements in $\mathcal{O}_X$.

Def : • $\{\text{Cartier divisors}\} = \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$

$s \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*) \leftrightarrow \{(U_i, \bar{f}_i \in \mathcal{K}^*(U_i)/\mathcal{O}^*(U_i)) \mid \underbrace{\bar{f}_i|_{U_i \cap U_j} = \bar{f}_j|_{U_i \cap U_j}}\}$

$\Updownarrow$

$(f_i \mathcal{O}^*(U_i \cap U_j) = f_j \mathcal{O}^*(U_i \cap U_j) \Leftrightarrow) \ f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$

$\leftrightarrow \{(U_i, f_i \in \mathcal{K}^*(U_i)) \mid f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)\}$

• $\{\text{Principal Cartier divisors}\} = \operatorname{Im}(\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*))$

• two Cartier divisors are linearly equivalent $(\sim)$ if their difference is principal.

• $\operatorname{CaCl} X := \dfrac{\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)}{\sim}$

• $\{(U_i, f_i)\} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ is effective if $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ $\forall i$

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Invertible sheaves

<u>Def</u>: • $\mathcal{F}\in Mod(X)$ is free if $\mathcal{F}\cong \bigoplus \mathcal{O}_X$ (finite) ← no need!

• $\mathcal{F}\in Mod(X)$ is locally free if $\exists \{U_i\}_{i\in\Lambda}$ covers X and $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module.

• A locally free sheaf of rank 1 is called an invertible sheaf.

<u>Fact</u>: • If $\mathcal{L}, \mathcal{U}$ are invertible, then $\mathcal{L}\otimes\mathcal{U}$ is also invertible.

• If $\mathcal{L}$ is invertible, then $\exists \mathcal{L}^{-1}$ s.t. $\mathcal{L}\otimes\mathcal{L}^{-1}\cong\mathcal{O}_X$.

(pf) • We can choose $\{U_\lambda\}_{\lambda\in\Lambda}$ s.t. $\mathcal{L}|_{U_\lambda}\cong\mathcal{O}_{U_\lambda}$ and $\mathcal{U}|_{U_\lambda}\cong\mathcal{O}_{U_\lambda}$.

Then $(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{U})|_{U_\lambda}\cong \mathcal{L}|_{U_\lambda}\otimes_{\mathcal{O}_{U_\lambda}}\mathcal{U}|_{U_\lambda}\cong\mathcal{O}_{U_\lambda}\otimes_{\mathcal{O}_{U_\lambda}}\mathcal{O}_{U_\lambda}\cong\mathcal{O}_{U_\lambda}$.

• $\mathcal{L}^{-1} := \mathcal{H}om(\mathcal{L},\mathcal{O}_X)$ ← the dual sheaf of $\mathcal{L}$.

$U\longmapsto Hom_{\mathcal{O}_X|_U}(\mathcal{L}|_U, \mathcal{O}_X|_U)$

In fact, $\mathcal{L}\otimes\mathcal{L}^{-1}\cong\mathcal{O}_X$ follows from $\mathcal{L}(U)\otimes\mathcal{L}(U)^*\cong\mathcal{O}_X(U)$, $s\otimes s^*\mapsto s^*(s)$

•• $\mathcal{H}om(\mathcal{L},\mathcal{O}_X)|_{U_\lambda}\cong\mathcal{H}om(\mathcal{L}|_{U_\lambda},\mathcal{O}_X|_{U_\lambda})\cong\mathcal{H}om(\mathcal{O}_{U_\lambda},\mathcal{O}_{U_\lambda})\cong\mathcal{O}_{U_\lambda}$

so $\mathcal{L}^{-1}$ is invertibe.

$\varphi_U : \mathcal{L}(U)\otimes Hom_{\mathcal{O}_U}(\mathcal{L}|_U,\mathcal{O}_X|_U)\longrightarrow Hom_{\mathcal{O}_U}(\mathcal{L}|_U,\mathcal{L}|_U)$ $\quad \mathcal{L}|_U\cong\mathcal{O}_U$

presheaf: $s\otimes\varphi \longmapsto s\varphi$ $\quad \beta_U\cdot\mathcal{O}_U$, $\beta_U\mapsto 1$

$\overset{\text{sheaf}}{\rightsquigarrow}$ $\varphi: \mathcal{L}\otimes\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{O}_X)\xrightarrow{\sim}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})$

$\beta_U\cdot\mathcal{O}_U$ $\quad \beta_U\cdot\mathcal{O}_U$, $\beta_U\mapsto\beta_U\cdot\alpha_U$

•• $\psi_U : \mathcal{O}_X(U)\longrightarrow Hom_{\mathcal{O}_U}(\mathcal{L}|_U,\mathcal{L}|_U)$

$\alpha\longmapsto\psi_U(\alpha): \mathcal{L}(W)\longrightarrow\mathcal{L}(W)\quad \forall W\subset U$

$\beta\longmapsto\alpha|_W\cdot\beta$

$\rightsquigarrow \psi: \mathcal{O}_X\cong\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L},\mathcal{L})$

<u>Def</u>: $\mathrm{Pic}\,X := \left(\dfrac{\{\text{invertible sheaves on } X\}}{\cong}, \otimes\right)$

regular in codim 1. $\Uparrow$ ($\mathcal{O}_p$ is integrally closed) $\Uparrow$

◉ Correspondence between them (i.e. $\forall p \in X$, $\mathcal{O}_p$ is a UFD)

<u>Thm 1</u>: If X is noeth, integral, separated, locally factorial,

then $\quad \mathrm{Cl}\, X \cong \mathrm{Ca\,Cl}\, X$.

(pf): "To construct

$$\begin{array}{ccc} \mathrm{Div}\, X & \xrightarrow{\sim} & \Gamma(X, \mathcal{K}^*/\mathcal{O}^*) \\ \cup & & \cup \\ \{\text{primicipal}\} & \xrightarrow{\sim} & \{\text{principal}\} \end{array}$$ "

"$\Rightarrow$": Let $D \in \mathrm{Div}\, X$.

We may assume that $X = \mathrm{spec}\, A$ and $D = \sum_{i=1}^{r} n_i \underbrace{\mathrm{Spec}\, A/P_i}_{= Y_i}$ with $ht(P_i) = 1$

For any $\underset{}{x} \in X$ ($x = q$), if $x \notin Y_i \ \forall i$, then $D_x := 0$.

Otherwise, say $\begin{cases} P_i \subset q, & i = 1, \dots, t \\ P_i \not\subset q, & i = t+1, \dots, r, \end{cases}$ we define a divisor D_x on $\mathrm{Spec}\, \mathcal{O}_{X,x}$:

$$D_x = \sum_{i=1}^{t} n_i \underbrace{\mathrm{Spec}\, A_q/P_i A_q}_{= Y_i'}, \quad ht(P_i A_q) = 1$$

no need here!

By assumption, $A_q = \mathcal{O}_{X,x}$ is a UFD $\Rightarrow \begin{cases} A_q \text{ is integrally closed} \\ \forall \text{ prime ideal of } ht\ 1 \text{ in } A_q \text{ is principal} \end{cases}$

$\Rightarrow D_x = (f_x)$ for some $f_x \in K \setminus \{0\}$

$P_{i_q} = \langle \frac{f}{1} \rangle_{A_q}$

In fact, say $P_i A_q = \langle f_i \rangle_{A_q}$, $\quad Y_i' = V(f_i)$, $\quad f_x = f_1^{n_1} f_2^{n_2} \cdots f_t^{n_t}$

$v_Y(f_x) = \sum n_i v_Y(f_i)$

$\frac{f}{1} = \frac{s}{1}\frac{f}{s} \in P_q$

i.e. $\quad \underset{V(f_i) \text{ in } X = \mathrm{spec}\, A}{(f_i)_X}\Big|_{\mathrm{spec}\, A_q} = V(P_i)\Big|_{\mathrm{spec}\, A_q} = \mathrm{spec}\, A_q/P_i A_q = Y_i'$

$\langle \frac{f}{1} \rangle \subseteq \langle \frac{f}{s} \rangle$

Hence $\quad (f_x)_X\Big|_{\mathrm{spec}\, A_q} = D_x =: D\Big|_{\mathrm{spec}\, A_q}$ ← prime divisors pass through x.

and other prime divisors in $(f_x)_X$ do not pass through x.
$\{Z_1, \dots, Z_\ell\}$

Since $x \notin \bigcup_{i=1}^{t} Z_i$, $x \in X \setminus (\bigcup_{i=1}^{t} Z_i) \cup (\bigcup_{i=t+1}^{r} Y_i)$: open

$$\Rightarrow \exists\, U_x \subset X \setminus (\bigcup_{i=1}^{t} Z_i) \cup (\bigcup_{i=t+1}^{r} Y_i)$$

$$\text{s.t. } (f_x)_X \big|_{U_x} = D\big|_{U_x}$$

<u>Claim</u>: $\{(U_x, f_x)\}_{x \in X} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$

(pf): f_x, f_y give the same Weil divisor on $U_x \cap U_y$.

If necessary, we can assume that $U_x = D(a)$, $U_y = D(b)$.

By assumption,

$\because$ $\forall p \in \operatorname{Spec} A$, A_p is integrally closed $\Rightarrow$ A is integrally closed

$\Rightarrow A_{ab}$ is integrally closed.

For any prime devisor Y of $U_x \cap U_y$, $v_Y(f_x) = v_Y(f_y)$,

$$\Rightarrow f_x/f_y,\ f_y/f_x \in \bigcap_{ht(p)=1} (A_{ab})_p = A_{ab} = \Gamma(U_x \cap U_y, \mathcal{O}_X)$$

$$\Rightarrow f_x/f_y \in \Gamma(U_x \cap U_y, \mathcal{O}_X^*)$$

"$\Leftarrow$": $\because$ X is integral $\therefore$ $\mathcal{K}$ = the constant sheaf corr. to K

Given $\{(U_i, f_i)\} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, $f_i \in \Gamma(U_i, \mathcal{K}^*) = K \setminus \{0\}$

define $D = \sum n_Y Y$ where $n_Y := v_Y(f_i)$ for $U_i \cap Y \neq \phi$

- well-defined: for other $U_j \cap Y \neq \phi$, since $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$, $v_Y(f_i/f_j) = 0 \Rightarrow v_Y(f_i) = v_Y(f_j)$ for $Y \cap U_i \cap U_j \neq \phi$.

- finite: $\because$ X is noeth $\therefore$ we can assume that $\{U_i\}$ is a finite set i.e. $\{f_i\}$ is a finite set

Only finitely many $v_Y(f_i) \neq 0$, sum through finite terms.

It is easy to see that these two constructions are inverse to each other and the principal divisors correspond to each other $\star$.

<u>Thm 2</u>: For any scheme X,

$\exists$ an injective homo. $\mathrm{CaCl}\,X \hookrightarrow \mathrm{Pic}\,X$

$D \longmapsto \mathcal{L}(D)$

pf:

- To construct $\mathcal{L}(D)$ for $D = \{(U_i, f_i)\}$:

$\mathcal{L}(D)|_{U_i} := f_i^{-1}\mathcal{O}_{U_i} \hookrightarrow \mathcal{K}|_{U_i}$ (Invertible subsheaf of $\mathcal{K}$)

$(W \subset U_i \mapsto f_i|_W^{-1}\cdot \mathcal{O}_{U_i}(W))$

Since $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$, $\frac{1}{f_i} = \frac{1}{f_j}\cdot \text{unit}$

$\Rightarrow f_i^{-1}\mathcal{O}_{U_i\cap U_j} = f_j^{-1}\mathcal{O}_{U_i\cap U_j}$

We can glue $\{\mathcal{L}(D)|_{U_i}\}$ to get $\mathcal{L}(D) \in \mathrm{Pic}\,X$.

- 1-1 : $(\Gamma(X, \mathcal{K}^*/\mathcal{O}^*) \leftrightarrow \{\text{invertible subsheaves of } \mathcal{K}\})$:

Given any invertible subsheaf $\mathcal{L}$ of $\mathcal{K}$, say $\mathcal{L}|_{U_i} \underset{\varphi_i}{\cong} \mathcal{O}_{U_i}$, $g_i \longleftrightarrow 1$, $g_i \in \mathcal{K}^*(U_i)$

we have that $\mathcal{L}|_{U_i} = g_i\mathcal{O}_{U_i}$

<u>Claim</u>: $\{(U_i, \frac{1}{g_i})\} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$

pf: $\underbrace{g_i|_{U_i\cap U_j}}_{=g_i'}\mathcal{O}_{U_i\cap U_j} = \mathcal{L}|_{U_i\cap U_j} = \underbrace{g_j|_{U_i\cap U_j}}_{=g_j'}\cdot\mathcal{O}_{U_i\cap U_j}$

$\Rightarrow \exists\, u_{ij} \in \mathcal{O}(U_i\cap U_j)$ s.t. $g_i' = g_j' u_{ij}$

By def, $\underset{\text{on } U_i}{u_{ii} = 1}$, $\underset{\text{on } U_i\cap U_j}{u_{ij}u_{ji} = 1}$, $u_{ik} = u_{ij}\cdot u_{jk}$ on $U_i\cap U_j\cap U_k$

In particular, u_{ij} is a unit in $\mathcal{O}(U_i\cap U_j)$ $\star$

- Group homomorphism, i.e. $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$:

Let $D_1 = \{(U_i, f_i)\}$ and $D_2 = \{(U_i, g_i)\}$.

$\mathcal{O}_X$

Then $D_1 - D_2 = \{(U_i, f_i/g_i)\}$ $\mathcal{L}(D_1)$ $\mathcal{L}(D_2)^{-1}$

$\Rightarrow \mathcal{L}(D_1 - D_2)|_{U_i} := f_i^{-1} g_i \, \mathcal{O}_{U_i} \hookrightarrow \mathcal{K}|_{U_i}$

$= \mathcal{L}(D_1)|_{U_i} \cdot \mathcal{L}(D_2)^{-1}|_{U_i}$

滿足 universal property

$\Rightarrow \mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

$= \mathcal{L}(D_1) \cdot \mathcal{L}(D_2)^{-1}$

- well-defined for classes i.e $D_1 \sim D_2 \Leftrightarrow \mathcal{L}(D_1) \cong \mathcal{L}(D_2)$

i.e D is principal $\Leftrightarrow \mathcal{L}(D) \cong \mathcal{O}_X$: $D = D_1 - D_2$: $\mathcal{L}(D_1 - D_2) \cong \mathcal{O}_X$ $\| \; \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

"$\Rightarrow$" $D = \{(X, f)\}$, $\mathcal{L}(D) = f^{-1} \mathcal{O}_X \cong \mathcal{O}_X$

$f^{-1} \mapsto 1$

"$\Leftarrow$": $\varphi : \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}(D)$, $\mathcal{O}_X(X) \to \mathcal{L}(D)(X)$, $D = \{(X, g^{-1})\}$ $\divideontimes$.

$1 \longmapsto g$

Thm 3: If X is integral, then $\mathrm{CaCl}\,X \xrightarrow{\sim} \mathrm{Pic}\,X$.

$\mathcal{K}$ = the constant sheaf K.

(pf) Given $\mathcal{L} \in \mathrm{Pic}\,X$, say $\mathcal{L}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda}$, we have that

$(\mathcal{L} \otimes \mathcal{K})|_{U_\lambda} \cong \mathcal{L}|_{U_\lambda} \otimes \mathcal{K}|_{U_\lambda} \cong \mathcal{O}_{U_\lambda} \otimes \mathcal{K}|_{U_\lambda} \cong \mathcal{K}|_{U_\lambda}$: constant sheaf on U_λ

since X is irreducible, $\mathcal{L} \otimes \mathcal{K}$ is a constant sheaf

and thus $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$

Now $\mathcal{O}_X \hookrightarrow \mathcal{K} \Rightarrow \mathcal{L} \cong \underset{\text{locally free}}{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{O}_X \hookrightarrow \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$

$\Rightarrow \mathcal{L}$ is a subsheaf of $\mathcal{K}$

Coro1: X : noeth, integral, separated, locally factorial
$\Rightarrow \ Cl\, X \cong CaCl\, X \cong Pic\, X$.

Coro2: If $D = \{(U_i, f_i)\}$ is effective, i.e. $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$,

define $\mathcal{I}|_{U_i} := f_i \mathcal{O}_{U_i} \hookrightarrow \mathcal{O}_{U_i}$, i.e. $\mathcal{I} \hookrightarrow \mathcal{O}_X$

then $\mathcal{I} = \mathcal{I}_Y$ for some closed subscheme $Y \subset X$

and $\mathcal{I}_Y \cong \mathcal{L}(\underbrace{-D}_{\{(U_i,\, f_i^{-1})\}})$.

⊙ Examples

Prop4: $\underset{\text{closed}}{Z \subsetneq X}$, $U = X \setminus Z$

Then
- $\exists\ Cl\, X \twoheadrightarrow Cl\, U$
- $\operatorname{codim}(Z, X) \geq 2$, $Cl\, X \xrightarrow{\sim} Cl\, U$
- Z : irr of codim 1, $\mathbb{Z} \to Cl\, X \to Cl\, U \to 0$ exact
  $1 \longmapsto Z$

(pf): •
$$\mathrm{Div}\, X \longrightarrow \mathrm{Div}\, U$$
$$\sum n_i Y_i \longmapsto \sum n_i (Y_i \cap U)$$
$$(f) \longmapsto (f)_U \qquad \text{since } v_Y(f) = v_{\underbrace{Y\cap U}_{\neq \phi}}(f_U)$$

consider f rational on U

$\rightsquigarrow Cl\, X \longrightarrow Cl\, U$

surj : $\forall$ prime divisor $Y \subset U$, $Y = \overline{Y} \cap U$

prime divisor in X.

- Removing a closed subset Z of codim ≥ 2 doesn't change anything.

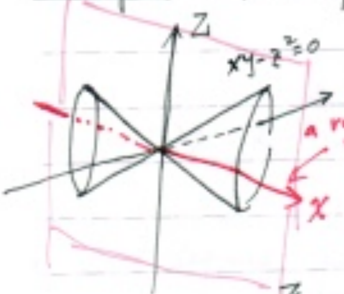
$$\sum n_i Y_i \longmapsto \sum n_i (Y_i \cap U) = 0$$

$$\Downarrow$$

$$\underset{\forall i}{Y_i \cap U = \phi} \Rightarrow Y_i \subset Z \ , \forall i \Rightarrow Y_i = Z \quad \forall i$$

irr. of codim 1

<u>Example</u> : $X = \operatorname{Spec} A$ where $A = {k[x,y,z]}/{\langle xy - z^2 \rangle}$.



, $U = X - Y$.

$$\mathbb{Z} \to Cl\, X \to Cl\, U \to 0$$

$$1 \longmapsto Y$$

- $U = D(y) = \operatorname{Spec} A_y = \operatorname{Spec} \dfrac{k[x,y,y^{-1},z]}{\underset{x = y^{-1}z^2}{\langle xy - z^2 \rangle}} = \operatorname{Spec} \underbrace{k[y, y^{-1}, z]}_{\text{is a UFD}}$

$\Downarrow$ $\forall$ prime ideal of ht 1 is principal

$\Rightarrow \forall\ D \in \operatorname{Div} U$ is principal

$\Rightarrow Cl\, U = 0$

- $2Y$ is principal :

$Y = V(\langle \bar{y}, \bar{z} \rangle)$ with $\underset{\overset{\parallel}{p}}{\langle \bar{y}, \bar{z} \rangle} \in \operatorname{Spec} A$ since ${A}/{\langle \bar{y}, \bar{z} \rangle} \cong k[x]$.

•• p is minimal over $\bar{y}$:

$\dim A = 2$, $\dim k[x] = 1 \Rightarrow ht(p) = 1$

and $p \supset \langle \bar{y} \rangle \supset \langle \bar{0} \rangle \Rightarrow p$ is minimal.

•• p is unique :

$\langle \bar{y} \rangle$ is primary since ${A}/{\langle \bar{y} \rangle} = {k[x,z]}/{\langle z^2 \rangle}$

"zero divisor $\Rightarrow$ nilpotent"

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•• $v_Y(\bar{y}) = 2$:

$\langle \bar{y}, \bar{z} \rangle A_{\langle \bar{y}, \bar{z}\rangle} = \langle \bar{z} \rangle A_{\langle \bar{y}, \bar{z}\rangle}$ since $xy = z^2 \Rightarrow y = x^{-1}z^2$ and x is a unit in $A_{\langle \bar{y}, \bar{z}\rangle}$ (A_P)

$PA_P = \langle \bar{z} \rangle A_P$

and $\bar{y} \in \langle \bar{z}^2 \rangle_{A_P}$

Hence $2Y = (\bar{y})$.

Here, Y is not Cartier since $\langle \bar{y}, \bar{z}\rangle$ is not principal in $\operatorname{Spec} A_m$. (A is not a UFD).

• Y is not principal :
$\Updownarrow$ under A being integrally closed (ex 6.4)
P is not principal

$m = \langle \bar{x}, \bar{y}, \bar{z} \rangle$, $m/m^2 = k\bar{x} \oplus k\bar{y} \oplus k\bar{z}$

$\bar{y}, \bar{z}$ are linearly indep in m/m^2

$\Rightarrow P$ cannot be principal. ※

Therefore, $Cl\, X \cong \mathbb{Z}/2\mathbb{Z}$.

Lemma : A : integrally closed, $X = \operatorname{spec} A$.
$\forall$ prime ideal of ht 1 is principal $\Leftrightarrow$ $Cl\, X = 0$.
(pf) "$\Rightarrow$" Given $Y \subset \operatorname{Spec} A$ prime divisor, $Y = \operatorname{spec} A/P$ with $ht(P) = 1$
$P = \langle f \rangle_A \Rightarrow Y = V(P) = V(f)$, i.e. $(f) = Y$
$\Rightarrow Y = 0$ in $Cl\, X$

"$\Leftarrow$": Let $p \in X$ with $ht(p) = 1$.
Consider $Y = \operatorname{spec} A/P$.
$Cl\, X = 0 \Rightarrow Y = (f)$ for some $f \in K \setminus \{0\}$ $\Rightarrow \begin{cases} v_Y(f) = 1 & \Rightarrow \langle f \rangle_{A_P} = PA_P \\ v_{Y'}(f) = 0 \ \forall Y' \neq Y & \Rightarrow f \in A_{P'} \end{cases}$

under ...

$\Rightarrow \begin{cases} f \in \bigcap_{ht(p)=1} A_P = A \\ \langle f \rangle_A = P \ ※ \end{cases}$

※: $g \in P \Rightarrow v_Y(g) \geq 1$, $v_{Y'}(g) \geq 0 \ \forall Y' \neq Y$

$\Rightarrow v_{Y'}(g/f) \geq 0 \ \forall Y' \Rightarrow g/f \in \bigcap_{ht(p)=1} A_P = A \Rightarrow g \in \langle f \rangle_A$. ※

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