CHAPTER III

Cohomology

In this chapter we define the general notion of cohomology of a sheaf of abelian groups on a topological space, and then study in detail the cohomology of coherent and quasi-coherent sheaves on a noetherian scheme.

Although the end result is usually the same, there are many different ways of introducing cohomology. There are the fine resolutions often used in several complex variables—see Gunning and Rossi [1]; the Čech cohomology used by Serre [3], who first introduced cohomology into abstract algebraic geometry; the canonical flasque resolutions of Godement [1]; and the derived functor approach of Grothendieck [1]. Each is important in its own way.

We will take as our basic definition the derived functors of the global section functor (§1, 2). This definition is the most general, and also best suited for theoretical questions, such as the proof of Serre duality in §7. However, it is practically impossible to calculate, so we introduce Čech cohomology in §4, and use it in §5 to compute explicitly the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space $\mathbf{P}^r$. This calculation is the basis of many later results on projective varieties.

In order to prove that the Čech cohomology agrees with the derived functor cohomology, we need to know that the higher cohomology of a quasi-coherent sheaf on an affine scheme is zero. We prove this in §3 in the noetherian case only, because it is technically much simpler than the case of an arbitrary affine scheme ([EGA III, §1]). Hence we are bound to include noetherian hypotheses in all theorems involving cohomology.

As applications, we show for example that the arithmetic genus of a projective variety X, whose definition in (I, §7) depended on a projective embedding of X, can be computed in terms of the cohomology groups $H^i(X,\mathcal{O}_X)$, and hence is intrinsic (Ex. 5.3). We also show that the arithmetic genus is constant in a family of normal projective varieties (9.13).

Another application is Zariski's main theorem (11.4) which is important in the birational study of varieties.

The latter part of the chapter (§8–12) is devoted to families of schemes, i.e., the study of the fibres of a morphism. In particular, we include a section on flat morphisms and a section on smooth morphisms. While these can be treated without cohomology, it seems to be an appropriate place to include them, because flatness can be understood better using cohomology (9.9).

1 Derived Functors

In this chapter we will assume familiarity with the basic techniques of homological algebra. Since notation and terminology vary from one source to another, we will assemble in this section (without proofs) the basic definitions and results we will need. More details can be found in the following sources: Godement [1, esp. Ch. I, §1.1–1.8, 2.1–2.4, 5.1–5.3], Hilton and Stammbach [1, Ch. II,IV,IX], Grothendieck [1, Ch. II, §1,2,3], Cartan and Eilenberg [1, Ch. III,V], Rotman [1, §6].

Definition. An *abelian category* is a category $\mathfrak{A}$, such that: for each $A,B \in \operatorname{Ob} \mathfrak{A}$, $\operatorname{Hom}(A,B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. (Hilton and Stammbach [1, p. 78].)

The following are all abelian categories.

Example 1.0.1. $\mathfrak{Ab}$, the category of abelian groups.

Example 1.0.2. $\mathfrak{Mod}(A)$, the category of modules over a ring A (commutative with identity as always).

Example 1.0.3. $\mathfrak{Ab}(X)$, the category of sheaves of abelian groups on a topological space X.

Example 1.0.4. $\mathfrak{Mod}(X)$, the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X,\mathcal{O}_X)$.

Example 1.0.5. $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X (II, 5.7).

Example 1.0.6. $\mathfrak{Coh}(X)$, the category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X (II, 5.7).

Example 1.0.7. $\mathfrak{Coh}(\mathfrak{X})$, the category of coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $(\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ (II, 9.9).

In the rest of this section, we will be stating some basic results of homological algebra in the context of an arbitrary abelian category. However, in most books, these results are proved only for the category of modules over a ring, and proofs are often done by "diagram-chasing": you pick an element and chase its images and pre-images through a diagram. Since diagram-chasing doesn't make sense in an arbitrary abelian category, the conscientious reader may be disturbed. There are at least three ways to handle this difficulty. (1) Provide intrinsic proofs for all the results, starting from the axioms of an abelian category, and without even mentioning an element. This is cumbersome, but can be done—see, e.g., Freyd [1]. Or (2), note that in each of the categories we use (most of which are in the above list of examples), one can in fact carry out proofs by diagram-chasing. Or (3), accept the "full embedding theorem" (Freyd [1, Ch. 7]), which states roughly that any abelian category is equivalent to a subcategory of $\mathfrak{Ab}$. This implies that any category-theoretic statement (e.g., the 5-lemma) which can be proved in $\mathfrak{Ab}$ (e.g., by diagram-chasing) also holds in any abelian category.

Now we begin our review of homological algebra. A *complex* $A^{\cdot}$ in an abelian category $\mathfrak{A}$ is a collection of objects A^i, $i \in \mathbf{Z}$, and morphisms $d^i: A^i \to A^{i+1}$, such that $d^{i+1} \circ d^i = 0$ for all i. If the objects A^i are specified only in a certain range, e.g., $i \geqslant 0$, then we set $A^i = 0$ for all other i. A *morphism* of complexes, $f: A^{\cdot} \to B^{\cdot}$ is a set of morphisms $f^i: A^i \to B^i$ for each i, which commute with the coboundary maps d^i.

The ith *cohomology object* $h^i(A^{\cdot})$ of the complex $A^{\cdot}$ is defined to be $\ker d^i/\operatorname{im} d^{i-1}$. If $f: A^{\cdot} \to B^{\cdot}$ is a morphism of complexes, then f induces a natural map $h^i(f): h^i(A^{\cdot}) \to h^i(B^{\cdot})$. If $0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$ is a short exact sequence of complexes, then there are natural maps $\delta^i: h^i(C^{\cdot}) \to h^{i+1}(A^{\cdot})$ giving rise to a long exact sequence

$$\ldots \to h^i(A^{\cdot}) \to h^i(B^{\cdot}) \to h^i(C^{\cdot}) \xrightarrow{\delta^i} h^{i+1}(A^{\cdot}) \to \ldots$$

Two morphisms of complexes $f,g: A^{\cdot} \to B^{\cdot}$ are *homotopic* (written $f \sim g$) if there is a collection of morphisms $k^i: A^i \to B^{i-1}$ for each i (which need not commute with the d^i) such that $f - g = dk + kd$. The collection of morphisms, $k = (k^i)$ is called a *homotopy operator*. If $f \sim g$, then f and g induce the *same* morphism $h^i(A^{\cdot}) \to h^i(B^{\cdot})$ on the cohomology objects, for each i.

A covariant functor $F: \mathfrak{A} \to \mathfrak{B}$ from one abelian category to another is *additive* if for any two objects A,A' in $\mathfrak{A}$, the induced map $\operatorname{Hom}(A,A') \to \operatorname{Hom}(FA,FA')$ is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathfrak{A}$, the sequence

$$0 \to FA' \to FA \to FA''$$

is exact in $\mathfrak{B}$. If we can write a 0 on the right instead of the left, we say F is *right exact*. If it is both left and right exact, we say it is *exact*. If only the middle part $FA' \to FA \to FA''$ is exact, we say F is *exact in the middle*.

For a contravariant functor we make analogous definitions. For example, $F:\mathfrak{A} \to \mathfrak{B}$ is *left exact* if it is additive, and for every short exact sequence as above, the sequence

$$0 \to FA'' \to FA \to FA'$$

is exact in $\mathfrak{B}$.

Example 1.0.8. If $\mathfrak{A}$ is an abelian category, and A is a fixed object, then the functor $B \to \mathrm{Hom}(A,B)$, usually denoted $\mathrm{Hom}(A,\cdot)$, is a covariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$. The functor $\mathrm{Hom}(\cdot,A)$ is a contravariant left exact functor from $\mathfrak{A}$ to $\mathfrak{Ab}$.

Next we come to resolutions and derived functors. An object I of $\mathfrak{A}$ is *injective* if the functor $\mathrm{Hom}(\cdot,I)$ is exact. An *injective resolution* of an object A of $\mathfrak{A}$ is a complex $I^{\cdot}$, defined in degrees $i \geqslant 0$, together with a morphism $\varepsilon:A \to I^0$, such that I^i is an injective object of $\mathfrak{A}$ for each $i \geqslant 0$, and such that the sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to \cdots$$

is exact.

If every object of $\mathfrak{A}$ is isomorphic to a subobject of an injective object of $\mathfrak{A}$, then we say $\mathfrak{A}$ *has enough injectives*. If $\mathfrak{A}$ has enough injectives, then every object has an injective resolution. Furthermore, a well-known lemma states that any two injective resolutions are homotopy equivalent.

Now let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F:\mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor. Then we construct the *right derived functors* R^iF, $i \geqslant 0$, of F as follows. For each object A of $\mathfrak{A}$, choose once and for all an injective resolution $I^{\cdot}$ of A. Then we define $R^iF(A) = h^i(F(I^{\cdot}))$.

Theorem 1.1A. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F:\mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then*

(a) *For each $i \geqslant 0$, R^iF as defined above is an additive functor from $\mathfrak{A}$ to $\mathfrak{B}$. Furthermore, it is independent (up to natural isomorphism of functors) of the choices of injective resolutions made.*

(b) *There is a natural isomorphism $F \cong R^0F$.*

(c) *For each short exact sequence $0 \to A' \to A \to A'' \to 0$ and for each $i \geqslant 0$ there is a natural morphism $\delta^i:R^iF(A'') \to R^{i+1}F(A')$, such that we obtain a long exact sequence*

$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to R^{i+1}F(A) \to \cdots$$

(d) *Given a morphism of the exact sequence of* (c) *to another* $0 \to B' \to B \to B'' \to 0$, *the* δ*'s give a commutative diagram*

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B'). \end{array}$$

(e) *For each injective object* I *of* $\mathfrak{A}$, *and for each* $i > 0$, *we have* $R^iF(I) = 0$.

Definition. With $F: \mathfrak{A} \to \mathfrak{B}$ as in the theorem, an object J of $\mathfrak{A}$ is *acyclic* for F if $R^iF(J) = 0$ for all $i > 0$.

Proposition 1.2A. *With* $F: \mathfrak{A} \to \mathfrak{B}$ *as in* (1.1A), *suppose there is an exact sequence*

$$0 \to A \to J^0 \to J^1 \to \dots$$

where each J^i *is acyclic for* F, $i \geqslant 0$. (*We say* $J^{\cdot}$ *is an* F-acyclic resolution *of* A.) *Then for each* $i \geqslant 0$ *there is a natural isomorphism* $R^iF(A) \cong h^i(F(J^{\cdot}))$.

We leave to the reader the analogous definitions of projective objects, projective resolutions, an abelian category having enough projectives, and the left derived functors of a covariant right exact functor. Also, the right derived functors of a left exact contravariant functor (use projective resolutions) and the left derived functors of a right exact contravariant functor (use injective resolutions).

Next we will give a universal property of derived functors. For this purpose, we generalize slightly with the following definition.

Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be abelian categories. A (*covariant*) δ-*functor* from $\mathfrak{A}$ to $\mathfrak{B}$ is a collection of functors $T = (T^i)_{i \geqslant 0}$, together with a morphism $\delta^i: T^i(A'') \to T^{i+1}(A')$ for each short exact sequence $0 \to A' \to A \to A'' \to 0$, and each $i \geqslant 0$, such that:

(1) For each short exact sequence as above, there is a long exact sequence

$$0 \to T^0(A') \to T^0(A) \to T^0(A'') \xrightarrow{\delta^1} T^1(A') \to \dots$$

$$\dots \to T^i(A) \to T^i(A'') \xrightarrow{\delta^i} T^{i+1}(A') \to T^{i+1}(A) \to \dots;$$

(2) for each morphism of one short exact sequence (as above) into another $0 \to B' \to B \to B'' \to 0$, the δ's give a commutative diagram

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta^i} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\delta^i} & T^{i+1}(B'). \end{array}$$

Definition. The δ-functor $T = (T^i): \mathfrak{A} \to \mathfrak{B}$ is said to be *universal* if, given any other δ-functor $T' = (T'^i): \mathfrak{A} \to \mathfrak{B}$, and given any morphism of

functors $f^0: T^0 \to T'^0$, there exists a unique sequence of morphisms $f^i: T^i \to T'^i$ for each $i \geqslant 0$, starting with the given f^0, which commute with the δ^i for each short exact sequence.

Remark 1.2.1. If $F: \mathfrak{A} \to \mathfrak{B}$ is a covariant additive functor, then by definition there can exist at most one (up to unique isomorphism) universal δ-functor T with $T^0 = F$. If T exists, the T^i are sometimes called the *right satellite functors* of F.

Definition. An additive functor $F: \mathfrak{A} \to \mathfrak{B}$ is *effaceable* if for each object A of $\mathfrak{A}$, there is a monomorphism $u: A \to M$, for some M, such that $F(u) = 0$. It is *coeffaceable* if for each A there exists an epimorphism $u: P \to A$ such that $F(u) = 0$.

Theorem 1.3A. *Let $T = (T^i)_{i \geqslant 0}$ be a covariant δ-functor from $\mathfrak{A}$ to $\mathfrak{B}$. If T^i is effaceable for each $i > 0$, then T is universal.*

PROOF. Grothendieck [1, II, 2.2.1]

Corollary 1.4. *Assume that $\mathfrak{A}$ has enough injectives. Then for any left exact functor $F: \mathfrak{A} \to \mathfrak{B}$, the derived functors $(R^iF)_{i \geqslant 0}$ form a universal δ-functor with $F \cong R^0F$. Conversely, if $T = (T^i)_{i \geqslant 0}$ is any universal δ-functor, then T^0 is left exact, and the T^i are isomorphic to R^iT^0 for each $i \geqslant 0$.*

PROOF. If F is a left exact functor, then the $(R^iF)_{i \geqslant 0}$ form a δ-functor by (1.1A). Furthermore, for any object A, let $u: A \to I$ be a monomorphism of A into an injective. Then $R^iF(I) = 0$ for $i > 0$ by (1.1A), so $R^iF(u) = 0$. Thus R^iF is effaceable for each $i > 0$. It follows from the theorem that (R^iF) is universal.

On the other hand, given a universal δ-functor T, we have T^0 left exact by the definition of δ-functor. Since $\mathfrak{A}$ has enough injectives, the derived functors R^iT^0 exist. We have just seen that (R^iT^0) is another universal δ-functor. Since $R^0T^0 = T^0$, we find $R^iT^0 \cong T^i$ for each i, by (1.2.1).

2 Cohomology of Sheaves

In this section we define cohomology of sheaves by taking the derived functors of the global section functor. Then as an application of general techniques of cohomology we prove Grothendieck's theorem about the vanishing of cohomology on a noetherian topological space. To begin with, we must verify that the categories we use have enough injectives.

Proposition 2.1A. *If A is a ring, then every A-module is isomorphic to a submodule of an injective A-module.*

PROOF. Godement [1, I, 1.2.2] or Hilton and Stammbach [1, I, 8.3].

Proposition 2.2. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the category $\mathfrak{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

PROOF. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{x,X}$-module. Therefore there is an injection $\mathcal{F}_x \to I_x$, where I_x is an injective $\mathcal{O}_{x,X}$-module (2.1A). For each point x, let j denote the inclusion of the one-point space $\{x\}$ into X, and consider the sheaf $\mathcal{I} = \prod_{x \in X} j_*(I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$, and j_* is the direct image functor (II, §1).

Now for any sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) = \prod \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},j_*(I_x))$ by definition of the direct product. On the other hand, for each point $x \in X$, we have $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},j_*(I_x)) \cong \mathrm{Hom}_{\mathcal{O}_{x,X}}(\mathcal{G}_x,I_x)$ as one sees easily. Thus we conclude first that there is a natural morphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{I}$ obtained from the local maps $\mathcal{F}_x \to I_x$. It is clearly injective. Second, the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot,\mathcal{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathcal{G} \mapsto \mathcal{G}_x$, which is exact, followed by $\mathrm{Hom}_{\mathcal{O}_{x,X}}(\cdot,I_x)$, which is exact, since I_x is an injective $\mathcal{O}_{x,X}$-module. Hence $\mathrm{Hom}(\cdot,\mathcal{I})$ is an exact functor, and therefore $\mathcal{I}$ is an injective $\mathcal{O}_X$-module.

Corollary 2.3. *If X is any topological space, then the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

PROOF. Indeed, if we let $\mathcal{O}_X$ be the constant sheaf of rings $\mathbf{Z}$, then $(X,\mathcal{O}_X)$ is a ringed space, and $\mathfrak{Mod}(X) = \mathfrak{Ab}(X)$.

Definition. Let X be a topological space. Let $\Gamma(X,\cdot)$ be the global section functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We define the *cohomology functors* $H^i(X,\cdot)$ to be the right derived functors of $\Gamma(X,\cdot)$. For any sheaf $\mathcal{F}$, the groups $H^i(X,\mathcal{F})$ are the *cohomology groups* of $\mathcal{F}$. Note that even if X and $\mathcal{F}$ have some additional structure, e.g., X a scheme and $\mathcal{F}$ a quasi-coherent sheaf, we always take cohomology in this sense, regarding $\mathcal{F}$ simply as a sheaf of abelian groups on the underlying topological space X.

We let the reader write out the long exact sequences which follow from the general properties of derived functors (1.1A).

Recall (II, Ex. 1.16) that a sheaf $\mathcal{F}$ on a topological space X is *flasque* if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Lemma 2.4. *If $(X,\mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is flasque.*

PROOF. For any open subset $U \subseteq X$, let $\mathcal{O}_U$ denote the sheaf $j_!(\mathcal{O}_X|_U)$, which is the restriction of $\mathcal{O}_X$ to U, extended by zero outside U (II, Ex. 1.19). Now let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module, and let $V \subseteq U$ be open sets. Then we have an inclusion $0 \to \mathcal{O}_V \to \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. Since $\mathcal{I}$ is injective, we get a surjection $\mathrm{Hom}(\mathcal{O}_U,\mathcal{I}) \to \mathrm{Hom}(\mathcal{O}_V,\mathcal{I}) \to 0$. But $\mathrm{Hom}(\mathcal{O}_U,\mathcal{I}) = \mathcal{I}(U)$ and $\mathrm{Hom}(\mathcal{O}_V,\mathcal{I}) = \mathcal{I}(V)$, so $\mathcal{I}$ is flasque.

Proposition 2.5. *If $\mathscr{F}$ is a flasque sheaf on a topological space X, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$.*

PROOF. Embed $\mathscr{F}$ in an injective object $\mathscr{I}$ of $\mathfrak{Ab}(X)$ and let $\mathscr{G}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0.$$

Then $\mathscr{F}$ is flasque by hypothesis, $\mathscr{I}$ is flasque by (2.4), and so $\mathscr{G}$ is flasque by (II, Ex. 1.16c). Now since $\mathscr{F}$ is flasque, we have an exact sequence (II, Ex. 1.16b)

$$0 \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{I}) \to \Gamma(X,\mathscr{G}) \to 0.$$

On the other hand, since $\mathscr{I}$ is injective, we have $H^i(X,\mathscr{I}) = 0$ for $i > 0$ (1.1Ae). Thus from the long exact sequence of cohomology, we get $H^1(X,\mathscr{F}) = 0$ and $H^i(X,\mathscr{F}) \cong H^{i-1}(X,\mathscr{G})$ for each $i \geqslant 2$. But $\mathscr{G}$ is also flasque, so by induction on i we get the result.

Remark 2.5.1. This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X,\cdot)$. Hence we can calculate cohomology using flasque resolutions (1.2A). In particular, we have the following result.

Proposition 2.6. *Let $(X,\mathscr{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X,\cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ coincide with the cohomology functors $H^i(X,\cdot)$.*

PROOF. Considering $\Gamma(X,\cdot)$ as a functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, we calculate its derived functors by taking injective resolutions in the category $\mathfrak{Mod}(X)$. But any injective is flasque (2.4), and flasques are acyclic (2.5) so this resolution gives the usual cohomology functors (1.2A).

Remark 2.6.1. Let $(X,\mathscr{O}_X)$ be a ringed space, and let $A = \Gamma(X,\mathscr{O}_X)$. Then for any sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, $\Gamma(X,\mathscr{F})$ has a natural structure of A-module. In particular, since we can calculate cohomology using resolutions in the category $\mathfrak{Mod}(X)$, all the cohomology groups of $\mathscr{F}$ have a natural structure of A-module; the associated exact sequences are sequences of A-modules, and so forth. Thus for example, if X is a scheme over Spec B for some ring B, the cohomology groups of any $\mathscr{O}_X$-module $\mathscr{F}$ have a natural structure of B-module.

A Vanishing Theorem of Grothendieck

Theorem 2.7 (Grothendieck [1]). *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathscr{F}$ on X, we have $H^i(X,\mathscr{F}) = 0$.*

Before proving the theorem, we need some preliminary results, mainly concerning direct limits. If $(\mathscr{F}_\alpha)$ is a direct system of sheaves on X, indexed by a directed set A, then we have defined the direct limit $\varinjlim \mathscr{F}_\alpha$ (II, Ex. 1.10).

Lemma 2.8. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

PROOF. Let $(\mathscr{F}_\alpha)$ be a directed system of flasque sheaves. Then for any inclusion of open sets $V \subseteq U$, and for each α, we have $\mathscr{F}_\alpha(U) \to \mathscr{F}_\alpha(V)$ is surjective. Since $\varinjlim$ is an exact functor, we get

$$\varinjlim \mathscr{F}_\alpha(U) \to \varinjlim \mathscr{F}_\alpha(V)$$

is also surjective. But on a noetherian topological space, $\varinjlim \mathscr{F}_\alpha(U) = (\varinjlim \mathscr{F}_\alpha)(U)$ for any open set (II, Ex. 1.11). So we have

$$(\varinjlim \mathscr{F}_\alpha)(U) \to (\varinjlim \mathscr{F}_\alpha)(V)$$

is surjective, and so $\varinjlim \mathscr{F}_\alpha$ is flasque.

Proposition 2.9. *Let X be a noetherian topological space, and let $(\mathscr{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geqslant 0$*

$$\varinjlim H^i(X,\mathscr{F}_\alpha) \to H^i(X,\varinjlim \mathscr{F}_\alpha).$$

PROOF. For each α we have a natural map $\mathscr{F}_\alpha \to \varinjlim \mathscr{F}_\alpha$. This induces a map on cohomology, and then we take the direct limit of these maps. For $i = 0$, the result is already known (II, Ex. 1.11). For the general case, we consider the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ consisting of all directed systems of objects of $\mathfrak{Ab}(X)$, indexed by A. This is an abelian category. Furthermore, since $\varinjlim$ is an exact functor, we have a natural transformation of δ-functors

$$\varinjlim H^i(X,\cdot) \to H^i(X,\varinjlim \cdot)$$

from $\mathfrak{ind}_A(\mathfrak{Ab}(X))$ to $\mathfrak{Ab}$. They agree for $i = 0$, so to prove they are the same, it will be sufficient to show they are both effaceable for $i > 0$. For in that case, they are both universal by (1.3A), and so must be isomorphic.

So let $(\mathscr{F}_\alpha) \in \mathfrak{ind}_A(\mathfrak{Ab}(X))$. For each α, let $\mathscr{G}_\alpha$ be the sheaf of discontinuous sections of $\mathscr{F}_\alpha$ (II, Ex. 1.16e). Then $\mathscr{G}_\alpha$ is flasque, and there is a natural inclusion $\mathscr{F}_\alpha \to \mathscr{G}_\alpha$. Furthermore, the construction of $\mathscr{G}_\alpha$ is functorial, so the $\mathscr{G}_\alpha$ also form a direct system, and we obtain a monomorphism $u:(\mathscr{F}_\alpha) \to (\mathscr{G}_\alpha)$ in the category $\mathfrak{ind}_A(\mathfrak{Ab}(X))$. Now the $\mathscr{G}_\alpha$ are all flasque, so $H^i(X,\mathscr{G}_\alpha) = 0$ for $i > 0$ (2.5). Thus $\varinjlim H^i(X,\mathscr{G}_\alpha) = 0$, and the functor on the left-hand side is effaceable for $i > 0$. On the other hand, $\varinjlim \mathscr{G}_\alpha$ is also flasque by (2.8). So $H^i(X,\varinjlim \mathscr{G}_\alpha) = 0$ for $i > 0$, and we see that the functor on the right-hand side is also effaceable. This completes the proof.

Remark 2.9.1. As a special case we see that cohomology commutes with infinite direct sums.

Lemma 2.10. *Let Y be a closed subset of X, let $\mathscr{F}$ be a sheaf of abelian groups on Y, and let $j: Y \to X$ be the inclusion. Then $H^i(Y,\mathscr{F}) = H^i(X,j_*\mathscr{F})$, where $j_*\mathscr{F}$ is the extension of $\mathscr{F}$ by zero outside Y* (II, Ex. 1.19).

PROOF. If $\mathscr{J}^{\cdot}$ is a flasque resolution of $\mathscr{F}$ on Y, then $j_*\mathscr{J}^{\cdot}$ is a flasque resolution of $j_*\mathscr{F}$ on X, and for each i, $\Gamma(Y,\mathscr{J}^i) = \Gamma(X,f_*\mathscr{J}^i)$. So we get the same cohomology groups.

Remark 2.10.1. Continuing our earlier abuse of notation (II, Ex. 1.19), we often write $\mathscr{F}$ instead of $j_*\mathscr{F}$. This lemma shows there will be no ambiguity about the cohomology groups.

PROOF OF (2.7). First we fix some notation. If Y is a closed subset of X, then for any sheaf $\mathscr{F}$ on X we let $\mathscr{F}_Y = j_*(\mathscr{F}|_Y)$, where $j: Y \to X$ is the inclusion. If U is an open subset of X, we let $\mathscr{F}_U = i_!(\mathscr{F}|_U)$, where $i: U \to X$ is the inclusion. In particular, if $U = X - Y$, we have an exact sequence (II, Ex. 1.19)

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

We will prove the theorem by induction on $n = \dim X$, in several steps.

Step 1. Reduction to the case X irreducible. If X is reducible, let Y be one of its irreducible components, and let $U = X - Y$. Then for any $\mathscr{F}$ we have an exact sequence

$$0 \to \mathscr{F}_U \to \mathscr{F} \to \mathscr{F}_Y \to 0.$$

From the long exact sequence of cohomology, it will be sufficient to prove that $H^i(X,\mathscr{F}_Y) = 0$ and $H^i(X,\mathscr{F}_U) = 0$ for $i > n$. But Y is closed and irreducible, and $\mathscr{F}_U$ can be regarded as a sheaf on the closed subset $\bar{U}$, which has one fewer irreducible components than X. Thus using (2.10) and induction on the number of irreducible components, we reduce to the case X irreducible.

Step 2. Suppose X is irreducible of dimension 0. Then the only open subsets of X are X and the empty set. For otherwise, X would have a proper irreducible closed subset, and $\dim X$ would be $\geqslant 1$. Thus $\Gamma(X,\cdot)$ induces an equivalence of categories $\mathfrak{Ab}(X) \to \mathfrak{Ab}$. In particular, $\Gamma(X,\cdot)$ is an exact functor, so $H^i(X,\mathscr{F}) = 0$ for $i > 0$, and for all $\mathscr{F}$.

Step 3. Now let X be irreducible of dimension n, and let $\mathscr{F} \in \mathfrak{Ab}(X)$. Let $B = \bigcup_{U \subseteq X} \mathscr{F}(U)$, and let A be the set of all finite subset of B. For each $\alpha \in A$, let $\mathscr{F}_\alpha$ be the subsheaf of $\mathscr{F}$ generated by the sections in α (over various open sets). Then A is a directed set, and $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$. So by (2.9), it will be sufficient to prove vanishing of cohomology for each $\mathscr{F}_\alpha$. If α' is a subset of α, then we have an exact sequence

$$0 \to \mathscr{F}_{\alpha'} \to \mathscr{F}_\alpha \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a sheaf generated by $\#(\alpha - \alpha')$ sections over suitable open sets. Thus, using the long exact sequence of cohomology, and induction on $\#(\alpha)$, we reduce to the case that $\mathscr{F}$ is generated by a single section over some open set U. In that case $\mathscr{F}$ is a quotient of the sheaf $\mathbf{Z}_U$ (where $\mathbf{Z}$ denotes the constant sheaf $\mathbf{Z}$ on X). Letting $\mathscr{R}$ be the kernel, we have an exact sequence

$$0 \to \mathscr{R} \to \mathbf{Z}_U \to \mathscr{F} \to 0.$$

Again using the long exact sequence of cohomology, it will be sufficient to prove vanishing for $\mathscr{R}$ and for $\mathbf{Z}_U$.

Step 4. Let U be an open subset of X and let $\mathscr{R}$ be a subsheaf of $\mathbf{Z}_U$. For each $x \in U$, the stalk $\mathscr{R}_x$ is a subgroup of $\mathbf{Z}$. If $\mathscr{R} = 0$, skip to Step 5. If not, let d be the least positive integer which occurs in any of the groups $\mathscr{R}_x$. Then there is a nonempty open subset $V \subseteq U$ such that $\mathscr{R}|_V \cong d \cdot \mathbf{Z}|_V$ as a subsheaf of $\mathbf{Z}|_V$. Thus $\mathscr{R}_V \cong \mathbf{Z}_V$ and we have an exact sequence

$$0 \to \mathbf{Z}_V \to \mathscr{R} \to \mathscr{R}/\mathbf{Z}_V \to 0.$$

Now the sheaf $\mathscr{R}/\mathbf{Z}_V$ is supported on the closed subset $(U - V)^-$ of X, which has dimension $<n$, since X is irreducible. So using (2.10) and the induction hypothesis, we know $H^i(X,\mathscr{R}/\mathbf{Z}_V) = 0$ for $i \geqslant n$. So by the long exact sequence of cohomology, we need only show vanishing for $\mathbf{Z}_V$.

Step 5. To complete the proof, we need only show that for any open subset $U \subseteq X$, we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. Let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathbf{Z}_U \to \mathbf{Z} \to \mathbf{Z}_Y \to 0.$$

Now $\dim Y < \dim X$ since X is irreducible, so using (2.10) and the induction hypothesis, we have $H^i(X,\mathbf{Z}_Y) = 0$ for $i \geqslant n$. On the other hand, $\mathbf{Z}$ is flasque, since it is a constant sheaf on an irreducible space (II, Ex. 1.16a). Hence $H^i(X,\mathbf{Z}) = 0$ for $i > 0$ by (2.5). So from the long exact sequence of cohomology we have $H^i(X,\mathbf{Z}_U) = 0$ for $i > n$. q.e.d.

Historical Note: The derived functor cohomology which we defined in this section was introduced by Grothendieck [1]. It is the theory which is used in [EGA]. The use of sheaf cohomology in algebraic geometry started with Serre [3]. In that paper, and in the later paper [4], Serre used Čech cohomology for coherent sheaves on an algebraic variety with its Zariski topology. The equivalence of this theory with the derived functor theory follows from the "theorem of Leray" (Ex. 4.11). The same argument, using Cartan's "Theorem B" shows that the Čech cohomology of a coherent analytic sheaf on a complex analytic space is equal to the derived functor cohomology. Gunning and Rossi [1] use a cohomology theory computed by fine resolutions of a sheaf on a paracompact Hausdorff space. The equivalence of this theory with ours is shown by Godement [1, Thm. 4.7.1, p. 181 and Ex. 7.2.1, p. 263], who shows at the same time that both theories coincide with his theory which is defined by a canonical flasque resolution. Godement also shows [1, Thm. 5.10.1, p. 228] that on a paracompact Hausdorff space, his theory coincides with Čech cohomology. This provides a bridge to the standard topological theories with constant coefficients, as developed in the book of Spanier [1]. He shows that on a paracompact Hausdorff space, Čech cohomology and Alexander cohomology and singular cohomology all agree (see Spanier [1, pp. 314, 327, 334]).

The vanishing theorem (2.7) was proved by Serre [3] for coherent sheaves on algebraic curves and projective algebraic varieties, and later [5] for abstract algebraic varieties. It is analogous to the theorem that singular cohomology on a (real) manifold of dimension n vanishes in degrees $i > n$.

Exercises

2.1. (a) Let $X = \mathbf{A}^1_k$ be the affine line over an infinite field k. Let P,Q be distinct closed points of X, and let $U = X - \{P,Q\}$. Show that $H^1(X,\mathbf{Z}_U) \neq 0$.

*(b) More generally, let $Y \subseteq X = \mathbf{A}^n_k$ be the union of $n+1$ hyperplanes in general position, and let $U = X - Y$. Show that $H^n(X,\mathbf{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

2.2. Let $X = \mathbf{P}^1_k$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X,\mathcal{O}) = 0$ for all $i > 0$.

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathcal{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X,\mathcal{F})$ denote the group of sections of $\mathcal{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X,\cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X,\cdot)$ by $H^i_Y(X,\cdot)$. They are the *cohomology groups* of X *with supports* in Y, and coefficients in a given sheaf.

(b) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves, with $\mathcal{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathcal{F}') \to \Gamma_Y(X,\mathcal{F}) \to \Gamma_Y(X,\mathcal{F}'') \to 0$$

is exact.

(c) Show that if $\mathcal{F}$ is flasque, then $H^i_Y(X,\mathcal{F}) = 0$ for all $i > 0$.

(d) If $\mathcal{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathcal{F}) \to \Gamma(X,\mathcal{F}) \to \Gamma(X - Y,\mathcal{F}) \to 0$$

is exact.

(e) Let $U = X - Y$. Show that for any $\mathcal{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0_Y(X,\mathcal{F}) \to H^0(X,\mathcal{F}) \to H^0(U,\mathcal{F}|_U) \to \\ &\to H^1_Y(X,\mathcal{F}) \to H^1(X,\mathcal{F}) \to H^1(U,\mathcal{F}|_U) \to \\ &\to H^2_Y(X,\mathcal{F}) \to \dots . \end{aligned}$$

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathcal{F}$,

$$H^i_Y(X,\mathcal{F}) \cong H^i_Y(V,\mathcal{F}|_V).$$

2.4. *Mayer–Vietoris Sequence.* Let Y_1,Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\begin{aligned} \dots \to H^i_{Y_1 \cap Y_2}(X,\mathcal{F}) &\to H^i_{Y_1}(X,\mathcal{F}) \oplus H^i_{Y_2}(X,\mathcal{F}) \to H^i_{Y_1 \cup Y_2}(X,\mathcal{F}) \to \\ &\to H^{i+1}_{Y_1 \cap Y_2}(X,\mathcal{F}) \to \dots . \end{aligned}$$

2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j: X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all i, $\mathscr{F}$, we have

$$H^i_P(X,\mathscr{F}) = H^i_P(X_P,\mathscr{F}_P).$$

2.6. Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathscr{R} \subseteq \mathbf{Z}_U$, and for every map $f: \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.
(a) Show that $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, using our definition of cohomology.
(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1,\mathscr{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

In this section we will prove that if $X = \operatorname{Spec} A$ is a noetherian affine scheme, then $H^i(X,\mathscr{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves $\mathscr{F}$ of $\mathcal{O}_X$-modules. The key point is to show that if I is an injective A-module, then the sheaf $\tilde{I}$ on $\operatorname{Spec} A$ is flasque. We begin with some algebraic preliminaries.

Proposition 3.1A (Krull's Theorem). *Let A be a noetherian ring, let $M \subseteq N$ be finitely generated A-modules, and let $\mathfrak{a}$ be an ideal of A. Then the $\mathfrak{a}$-adic topology on M is induced by the $\mathfrak{a}$-adic topology on N. In particular, for any $n > 0$, there exists an $n' \geqslant n$ such that $\mathfrak{a}^n M \supseteq M \cap \mathfrak{a}^{n'} N$.*

PROOF. Atiyah–Macdonald [1, 10.11] or Zariski–Samuel [1, vol. II, Ch. VIII, Th. 4].

Recall (II, Ex. 5.6) that for any ring A, and any ideal $\mathfrak{a} \subseteq A$, and any A-module M, we have defined the submodule $\Gamma_{\mathfrak{a}}(M)$ to be $\{m \in M \mid \mathfrak{a}^n m = 0$ for some $n > 0\}$.

Lemma 3.2. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.*

PROOF. To show that J is injective, it will be sufficient to show that for any ideal $\mathfrak{b} \subseteq A$, and for any homomorphism $\varphi: \mathfrak{b} \to J$, there exists a homomorphism $\psi: A \to J$ extending φ. (This is a well-known criterion for an injective module—Godement [1, I, 1.4.1]). Since A is noetherian, $\mathfrak{b}$ is finitely generated. On the other hand, every element of J is annihilated by some

power of $\mathfrak{a}$, so there exists an $n > 0$ such that $\mathfrak{a}^n\varphi(\mathfrak{b}) = 0$, or equivalently, $\varphi(\mathfrak{a}^n\mathfrak{b}) = 0$. Now applying (3.1A) to the inclusion $\mathfrak{b} \subseteq A$, we find that there is an $n' \geqslant n$ such that $\mathfrak{a}^n\mathfrak{b} \supseteq \mathfrak{b} \cap \mathfrak{a}^{n'}$. Hence $\varphi(\mathfrak{b} \cap \mathfrak{a}^{n'}) = 0$, and so the map $\varphi:\mathfrak{b} \to J$ factors through $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$. Now we consider the following diagram:

$$\begin{tikzcd}
A \arrow[r] & A/\mathfrak{a}^{n'} \arrow[drr, dashed, "\psi'"] & & \\
\mathfrak{b} \arrow[u, hook] \arrow[r] \arrow[rrr, bend right, "\varphi"'] & \mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'}) \arrow[u, hook] \arrow[r] & J \arrow[r] & I
\end{tikzcd}$$

Since I is injective, the composed map of $\mathfrak{b}/(\mathfrak{b} \cap \mathfrak{a}^{n'})$ to I extends to a map $\psi':A/\mathfrak{a}^{n'} \to I$. But the image of ψ' is annihilated by $\mathfrak{a}^{n'}$, so it is contained in J. Composing with the natural map $A \to A/\mathfrak{a}^{n'}$, we obtain the required map $\psi:A \to J$ extending φ.

Lemma 3.3. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

PROOF. For each $i > 0$, let $\mathfrak{b}_i$ be the annihilator of f^i in A. Then $\mathfrak{b}_1 \subseteq \mathfrak{b}_2 \subseteq \ldots$, and since A is noetherian, there is an r such that $\mathfrak{b}_r = \mathfrak{b}_{r+1} = \ldots$. Now let $\theta:I \to I_f$ be the natural map, and let $x \in I_f$ be any element. Then by definition of localization, there is a $y \in I$ and an $n \geqslant 0$ such that $x = \theta(y)/f^n$. We define a map φ from the ideal (f^{n+r}) of A to I by sending f^{n+r} to $f^r y$. This is possible, because the annihilator of f^{n+r} is $\mathfrak{b}_{n+r} = \mathfrak{b}_r$, and $\mathfrak{b}_r$ annihilates $f^r y$. Since I is injective, φ extends to a map $\psi:A \to I$. Let $\psi(1) = z$. Then $f^{n+r}z = f^r y$. But this implies that $\theta(z) = \theta(y)/f^n = x$. Hence θ is surjective.

Proposition 3.4. *Let I be an injective module over a noetherian ring A. Then the sheaf $\tilde{I}$ on $X = \operatorname{Spec} A$ is flasque.*

PROOF. We will use noetherian induction on $Y = (\operatorname{Supp} \tilde{I})^-$. See (II, Ex. 1.14) for the notion of support. If Y consists of a single closed point of X, then $\tilde{I}$ is a skyscraper sheaf (II, Ex. 1.17) which is obviously flasque.

In the general case, to show that $\tilde{I}$ is flasque, it will be sufficient to show, for any open set $U \subseteq X$, that $\Gamma(X,\tilde{I}) \to \Gamma(U,\tilde{I})$ is surjective. If $Y \cap U = \varnothing$, there is nothing to prove. If $Y \cap U \neq \varnothing$, we can find an $f \in A$ such that the open set $X_f = D(f)$ (II, §2) is contained in U and $X_f \cap Y \neq \varnothing$. Let $Z = X - X_f$, and consider the following diagram:

$$\begin{tikzcd}
\Gamma(X,\tilde{I}) \arrow[r] & \Gamma(U,\tilde{I}) \arrow[r] & \Gamma(X_f,\tilde{I}) \\
\Gamma_Z(X,\tilde{I}) \arrow[r] \arrow[u, hook] & \Gamma_Z(U,\tilde{I}), \arrow[u, hook] &
\end{tikzcd}$$

where Γ_Z denotes sections with support in Z (II, Ex. 1.20). Now given a section $s \in \Gamma(U,\tilde{I})$, we consider its image s' in $\Gamma(X_f,\tilde{I})$. But $\Gamma(X_f,\tilde{I}) = I_f$ (II, 5.1), so by (3.3), there is a $t \in I = \Gamma(X,\tilde{I})$ restricting to s'. Let t' be the restriction of t to $\Gamma(U,\tilde{I})$. Then $s - t'$ goes to 0 in $\Gamma(X_f,\tilde{I})$, so it has support in Z. Thus to complete the proof, it will be sufficient to show that $\Gamma_Z(X,\tilde{I}) \to \Gamma_Z(U,\tilde{I})$ is surjective.

Let $J = \Gamma_Z(X,\tilde{I})$. If $\mathfrak{a}$ is the ideal generated by f, then $J = \Gamma_{\mathfrak{a}}(I)$ (II, Ex. 5.6), so by (3.2), J is also an injective A-module. Furthermore, the support of J is contained in $Y \cap Z$, which is strictly smaller than Y. Hence by our induction hypothesis, J is flasque. Since $\Gamma(U,\tilde{J}) = \Gamma_Z(U,\tilde{I})$ (II, Ex. 5.6), we conclude that $\Gamma_Z(X,\tilde{I}) \to \Gamma_Z(U,\tilde{I})$ is surjective, as required.

Theorem 3.5. *Let $X = \operatorname{Spec} A$ be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathscr{F}$ on X, and for all $i > 0$, we have $H^i(X,\mathscr{F}) = 0$.*

PROOF. Given $\mathscr{F}$, let $M = \Gamma(X,\mathscr{F})$, and take an injective resolution $0 \to M \to I^\cdot$ of M in the category of A-modules. Then we obtain an exact sequence of sheaves $0 \to \tilde{M} \to \tilde{I}^\cdot$ on X. Now $\mathscr{F} = \tilde{M}$ (II, 5.5) and each $\tilde{I}^i$ is flasque by (3.4), so we can use this resolution of $\mathscr{F}$ to calculate cohomology (2.5.1). Applying the functor Γ, we recover the exact sequence of A-modules $0 \to M \to I^\cdot$. Hence $H^0(X,\mathscr{F}) = M$, and $H^i(X,\mathscr{F}) = 0$ for $i > 0$.

Remark 3.5.1. This result is also true without the noetherian hypothesis, but the proof is more difficult [EGA III, 1.3.1].

Corollary 3.6. *Let X be a noetherian scheme, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then $\mathscr{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathscr{G}$.*

PROOF. Cover X with a finite number of open affines $U_i = \operatorname{Spec} A_i$, and let $\mathscr{F}|_{U_i} = \tilde{M}_i$ for each i. Embed M_i in an injective A_i-module I_i. For each i, let $f: U_i \to X$ be the inclusion, and let $\mathscr{G} = \bigoplus f_*(\tilde{I}_i)$. For each i we have an injective map of sheaves $\mathscr{F}|_{U_i} \to \tilde{I}_i$. Hence we obtain a map $\mathscr{F} \to f_*(\tilde{I}_i)$. Taking the direct sum over i gives a map $\mathscr{F} \to \mathscr{G}$ which is clearly injective. On the other hand, for each i, $\tilde{I}_i$ is flasque (3.4) and quasi-coherent on U_i. Hence $f_*(\tilde{I}_i)$ is also flasque (II, Ex. 1.16d) and quasi-coherent (II, 5.8). Taking the direct sum of these, we see that $\mathscr{G}$ is flasque and quasi-coherent.

Theorem 3.7 (Serre [5]). *Let X be a noetherian scheme. Then the following conditions are equivalent:*

(i) *X is affine;*
(ii) *$H^i(X,\mathscr{F}) = 0$ for all $\mathscr{F}$ quasi-coherent and all $i > 0$;*
(iii) *$H^1(X,\mathscr{I}) = 0$ for all coherent sheaves of ideals $\mathscr{I}$.*

PROOF. (i) $\Rightarrow$ (ii) is (3.5). (ii) $\Rightarrow$ (iii) is trivial, so we have only to prove (iii) $\Rightarrow$ (i). We use the criterion of (II, Ex. 2.17). First we show that X can

be covered by open affine subsets of the form X_f, with $f \in A = \Gamma(X,\mathcal{O}_X)$. Let P be a closed point of X, let U be an open affine neighborhood of P, and let $Y = X - U$. Then we have an exact sequence

$$0 \to \mathcal{I}_{Y\cup\{P\}} \to \mathcal{I}_Y \to k(P) \to 0,$$

where $\mathcal{I}_Y$ and $\mathcal{I}_{Y\cup\{P\}}$ are the ideal sheaves of the closed sets Y and $Y \cup \{P\}$, respectively. The quotient is the skyscraper sheaf $k(P) = \mathcal{O}_P/\mathfrak{m}_P$ at P. Now form X_f. By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \ldots, f_r \in A$.

$$\Gamma(X,\mathcal{I}_Y) \to \Gamma(X,k(P)) \to H^1(X,\mathcal{I}_{Y\cup\{P\}}) = 0.$$

So there is an element $f \in \Gamma(X,\mathcal{I}_Y)$ which goes to 1 in $k(P)$, i.e., $f_P \equiv 1 \pmod{\mathfrak{m}_P}$. Since $\mathcal{I}_Y \subseteq \mathcal{O}_X$, we can consider f as an element of A. Then by construction, we have $P \in X_f \subseteq U$. Furthermore, $X_f = U_{\bar{f}}$, where $\bar{f}$ is the image of f in $\Gamma(U,\mathcal{O}_U)$, so X_f is affine.

Thus every closed point of X has an open affine neighborhood of the form X_f. By quasi-compactness, we can cover X with a finite number of these, corresponding to $f_1, \ldots, f_r \in A$.

Now by (II, Ex. 2.17), to show that X is affine, we need only verify that $f_1, \ldots, f_r$ generate the unit ideal in A. We use $f_1, \ldots, f_r$ to define a map $\alpha: \mathcal{O}_X^r \to \mathcal{O}_X$ by sending $\langle a_1, \ldots, a_r\rangle$ to $\sum f_i a_i$. Since the X_{f_i} cover X, this is a surjective map of sheaves. Let $\mathcal{F}$ be the kernel:

$$0 \to \mathcal{F} \to \mathcal{O}_X^r \xrightarrow{\alpha} \mathcal{O}_X \to 0.$$

We filter $\mathcal{F}$ as follows:

$$\mathcal{F} = \mathcal{F} \cap \mathcal{O}_X^r \supseteq \mathcal{F} \cap \mathcal{O}_X^{r-1} \supseteq \ldots \supseteq \mathcal{F} \cap \mathcal{O}_X$$

for a suitable ordering of the factors of $\mathcal{O}_X^r$. Each of the quotients of this filtration is a coherent sheaf of ideals in $\mathcal{O}_X$. Thus using our hypothesis (iii) and the long exact sequence of cohomology, we climb up the filtration and deduce that $H^1(X,\mathcal{F}) = 0$. But then $\Gamma(X,\mathcal{O}_X^r) \xrightarrow{\alpha} \Gamma(X,\mathcal{O}_X)$ is surjective, which tells us that $f_1, \ldots, f_r$ generate the unit ideal in A. q.e.d.

Remark 3.7.1. This result is analogous to another theorem of Serre in complex analytic geometry, which characterizes Stein spaces by the vanishing of coherent analytic sheaf cohomology.

Exercises

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II, Ex. 2.3) is affine. [*Hint*: Use (3.7), and for any coherent sheaf $\mathcal{F}$ on X, consider the filtration $\mathcal{F} \supseteq \mathcal{N}\cdot\mathcal{F} \supseteq \mathcal{N}^2\cdot\mathcal{F} \supseteq \ldots$, where $\mathcal{N}$ is the sheaf of nilpotent elements on X.]

3.2. Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.

(a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdot)$.

(b) Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \tilde{M}),$$

where $H^i_Y(X, \cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

(c) For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

3.4. *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A-module, then $\mathit{depth}_{\mathfrak{a}} M$ is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geqslant 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

(b) Show inductively, for M finitely generated, that for any $n \geqslant 0$, the following conditions are equivalent:

(i) $\operatorname{depth}_{\mathfrak{a}} M \geqslant n$;

(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i) $\operatorname{depth} \mathcal{O}_P \geqslant 2$;

(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

3.6. Let X be a noetherian scheme.

(a) Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

*(b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines . . .]

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

3.7. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subsetneq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.

(a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \tilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

(b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, x_2, \ldots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \ldots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

4 Čech Cohomology

In this section we construct the Čech cohomology groups for a sheaf of abelian groups on a topological space X, with respect to a given open covering of X. We will prove that if X is a noetherian separated scheme, the sheaf is quasi-coherent, and the covering is an open affine covering, then these Čech cohomology groups coincide with the cohomology groups defined in §2. The value of this result is that it gives a practical method for computing cohomology of quasi-coherent sheaves on a scheme.

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \ldots \cap U_{i_p}$ by $U_{i_0, \ldots, i_p}$.

Now let $\mathscr{F}$ be a sheaf of abelian groups on X. We define a complex $C^{\cdot}(\mathfrak{U}, \mathscr{F})$ of abelian groups as follows. For each $p \geqslant 0$, let

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \ldots < i_p} \mathscr{F}(U_{i_0, \ldots, i_p}).$$

Thus an element $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$ is determined by giving an element

$$\alpha_{i_0, \ldots, i_p} \in \mathscr{F}(U_{i_0, \ldots, i_p}),$$

for each $(p+1)$-tuple $i_0 < \ldots < i_p$ of elements of I. We define the coboundary map $d: C^p \to C^{p+1}$ by setting

$$(d\alpha)_{i_0, \ldots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}} | U_{i_0, \ldots, i_{p+1}}.$$

Here the notation $\hat{i}_k$ means omit i_k. Then since $\alpha_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}}$ is an element of $\mathscr{F}(U_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}})$, we restrict to $U_{i_0, \ldots, i_{p+1}}$ to get an element of $\mathscr{F}(U_{i_0, \ldots, i_{p+1}})$. One checks easily that $d^2 = 0$, so we have indeed defined a complex of abelian groups.

Remark 4.0.1. If $\alpha \in C^p(\mathfrak{U}, \mathscr{F})$, it is sometimes convenient to have the symbol $\alpha_{i_0, \ldots, i_p}$ defined for *all* $(p+1)$-tuples of elements of I. If there is a repeated index in the set $\{i_0, \ldots, i_p\}$, we define $\alpha_{i_0, \ldots, i_p} = 0$. If the indices are all distinct, we define $\alpha_{i_0, \ldots, i_p} = (-1)^\sigma \alpha_{\sigma i_0, \ldots, \sigma i_p}$, where σ is the permutation for which $\sigma i_0 < \ldots < \sigma i_p$. With these conventions, one can check that the formula given above for $d\alpha$ remains correct for any $(p+2)$-tuple $i_0, \ldots, i_{p+1}$ of elements of I.

Definition. Let X be a topological space and let $\mathfrak{U}$ be an open covering of X. For any sheaf of abelian groups $\mathscr{F}$ on X, we define the pth *Čech cohomology group of* $\mathscr{F}$, *with respect to the covering* $\mathfrak{U}$, to be

$$\check{H}^p(\mathfrak{U},\mathscr{F}) = h^p(C^{\cdot}(\mathfrak{U},\mathscr{F})).$$

Caution 4.0.2. Keeping X and $\mathfrak{U}$ fixed, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves of abelian groups on X, we do *not* in general get a long exact sequence of Čech cohomology groups. In other words, the functors $\check{H}^p(\mathfrak{U},\cdot)$ do not form a δ-functor (§1). For example, if $\mathfrak{U}$ consists of the single open set X, then this results from the fact that the global section functor $\Gamma(X,\cdot)$ is not exact.

Example 4.0.3. To illustrate how well suited Čech cohomology is for computations, we will compute some examples. Let $X = \mathbf{P}^1_k$, let $\mathscr{F}$ be the sheaf of differentials Ω (II, §8), and let $\mathfrak{U}$ be the open covering by the two open sets $U = \mathbf{A}^1$ with affine coordinate x, and $V = \mathbf{A}^1$ with affine coordinate $y = 1/x$. Then the Čech complex has only two terms:

$$C^0 = \Gamma(U,\Omega) \times \Gamma(V,\Omega)$$
$$C^1 = \Gamma(U \cap V,\Omega).$$

Now

$$\Gamma(U,\Omega) = k[x]\,dx$$
$$\Gamma(V,\Omega) = k[y]\,dy$$
$$\Gamma(U \cap V,\Omega) = k\left[x, \frac{1}{x}\right] dx,$$

and the map $d: C^0 \to C^1$ is given by

$$x \mapsto x$$
$$y \mapsto \frac{1}{x}$$
$$dy \mapsto -\frac{1}{x^2}\,dx.$$

So $\ker d$ is the set of pairs $\langle f(x)\,dx, g(y)\,dy\rangle$ such that

$$f(x) = -\frac{1}{x^2}\,g\left(\frac{1}{x}\right).$$

This can happen only if $f = g = 0$, since one side is a polynomial in x and the other side is a polynomial in $1/x$ with no constant term. So $\check{H}^0(\mathfrak{U},\Omega) = 0$.

To compute H^1, note that the image of d is the set of all expressions

$$\left(f(x) + \frac{1}{x^2}\,g\left(\frac{1}{x}\right)\right) dx,$$

where f and g are polynomials. This gives the subvector space of $k[x,1/x]\,dx$ generated by all $x^n\,dx$, $n \in \mathbf{Z}$, $n \neq -1$. Therefore $\check{H}^1(\mathfrak{U},\Omega) \cong k$, generated by the image of $x^{-1}\,dx$.

Example 4.0.4. Let S^1 be the circle (in its usual topology), let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$, and let $\mathfrak{U}$ be the open covering by two connected open semi-circles U, V, which overlap at each end, so that $U \cap V$ consists of two small intervals. Then

$$C^0 = \Gamma(U,\mathbf{Z}) \times \Gamma(V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$
$$C^1 = \Gamma(U \cap V,\mathbf{Z}) = \mathbf{Z} \times \mathbf{Z}$$

and the map $d: C^0 \to C^1$ takes $\langle a,b\rangle$ to $\langle b-a, b-a\rangle$. Thus $\check{H}^0(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$ and $\check{H}^1(\mathfrak{U},\mathbf{Z}) = \mathbf{Z}$. Since we know this is the right answer (Ex. 2.7), this illustrates the general principle that Čech cohomology agrees with the usual cohomology provided the open covering is taken fine enough so that there is no cohomology on any of the open sets (Ex. 4.11).

Now we will study some properties of the Čech cohomology groups.

Lemma 4.1. *For any $X,\mathfrak{U},\mathscr{F}$ as above, we have $\check{H}^0(\mathfrak{U},\mathscr{F}) \cong \Gamma(X,\mathscr{F})$.*

PROOF. $\check{H}^0(\mathfrak{U},\mathscr{F}) = \ker(d: C^0(\mathfrak{U},\mathscr{F}) \to C^1(\mathfrak{U},\mathscr{F}))$. If $\alpha \in C^0$ is given by $\{\alpha_i \in \mathscr{F}(U_i)\}$, then for each $i < j$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. So $d\alpha = 0$ says the sections α_i and α_j agree on $U_i \cap U_j$. Thus it follows from the sheaf axioms that $\ker d = \Gamma(X,\mathscr{F})$.

Next we define a "sheafified" version of the Čech complex. For any open set $V \subseteq X$, let $f: V \to X$ denote the inclusion map. Now given $X,\mathfrak{U},\mathscr{F}$ as above, we construct a complex $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ of sheaves on X as follows. For each $p \geqslant 0$, let

$$\mathscr{C}^p(\mathfrak{U},\mathscr{F}) = \prod_{i_0 < \ldots < i_p} f_*(\mathscr{F}|_{U_{i_0},\ldots,i_p}),$$

and define

$$d: \mathscr{C}^p \to \mathscr{C}^{p+1}$$

by the same formula as above. Note by construction that for each p we have $\Gamma(X,\mathscr{C}^p(\mathfrak{U},\mathscr{F})) = C^p(\mathfrak{U},\mathscr{F})$.

Lemma 4.2. *For any sheaf of abelian groups $\mathscr{F}$ on X, the complex $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ is a resolution of $\mathscr{F}$, i.e., there is a natural map $\varepsilon: \mathscr{F} \to \mathscr{C}^0$ such that the sequence of sheaves*

$$0 \to \mathscr{F} \xrightarrow{\varepsilon} \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \to \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \to \ldots$$

is exact.

PROOF. We define $\varepsilon: \mathscr{F} \to \mathscr{C}^0$ by taking the product of the natural maps $\mathscr{F} \to f_*(\mathscr{F}|_{U_i})$ for $i \in I$. Then the exactness at the first step follows from the sheaf axioms for $\mathscr{F}$.

To show the exactness of the complex $\mathscr{C}^\cdot$ for $p \geqslant 1$, it is enough to check exactness on the stalks. So let $x \in X$, and suppose $x \in U_j$. For each $p \geqslant 1$, we define a map

$$k: \mathscr{C}^p(\mathfrak{U},\mathscr{F})_x \to \mathscr{C}^{p-1}(\mathfrak{U},\mathscr{F})_x$$

as follows. Given $\alpha_x \in \mathscr{C}^p(\mathfrak{U},\mathscr{F})_x$, it is represented by a section $\alpha \in \Gamma(V,\mathscr{C}^p(\mathfrak{U},\mathscr{F}))$ over a neighborhood V of x, which we may choose so small that $V \subseteq U_j$. Now for any p-tuple $i_0 < \ldots < i_{p-1}$, we set

$$(k\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{ji_0,\ldots,i_{p-1}},$$

using the notational convention of (4.0.1). This makes sense because $V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{ji_0,\ldots,i_{p-1}}$. Then take the stalk of $k\alpha$ at x to get the required map k. Now one checks that for any $p \geqslant 1$, $\alpha \in \mathscr{C}^p_x$,

$$(dk + kd)(\alpha) = \alpha.$$

Thus k is a homotopy operator for the complex $\mathscr{C}^\cdot_x$, showing that the identity map is homotopic to the zero map. It follows (§1) that the cohomology groups $h^p(\mathscr{C}^\cdot_x)$ of this complex are 0 for $p \geqslant 1$.

Proposition 4.3. *Let X be a topological space, let $\mathfrak{U}$ be an open covering, and let $\mathscr{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$.*

PROOF. Consider the resolution $0 \to \mathscr{F} \to \mathscr{C}^\cdot(\mathfrak{U},\mathscr{F})$ given by (4.2). Since $\mathscr{F}$ is flasque, the sheaves $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are flasque for each $p \geqslant 0$. Indeed, for any $i_0,\ldots,i_p$, $\mathscr{F}|_{U_{i_0,\ldots,i_p}}$ is a flasque sheaf on $U_{i_0,\ldots,i_p}$; f_* preserves flasque sheaves (II, Ex. 1.16d), and a product of flasque sheaves is flasque. So by (2.5.1) we can use this resolution to compute the usual cohomology groups of $\mathscr{F}$. But $\mathscr{F}$ is flasque, so $H^p(X,\mathscr{F}) = 0$ for $p > 0$ by (2.5). On the other hand, the answer given by this resolution is

$$h^p(\Gamma(X,\mathscr{C}^\cdot(\mathfrak{U},\mathscr{F}))) = \check{H}^p(\mathfrak{U},\mathscr{F}).$$

So we conclude that $\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$ for $p > 0$.

Lemma 4.4. *Let X be a topological space, and $\mathfrak{U}$ an open covering. Then for each $p \geqslant 0$ there is a natural map, functorial in $\mathscr{F}$,*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \to H^p(X,\mathscr{F}).$$

PROOF. Let $0 \to \mathscr{F} \to \mathscr{I}^\cdot$ be an injective resolution of $\mathscr{F}$ in $\mathfrak{Ab}(X)$. Comparing with the resolution $0 \to \mathscr{F} \to \mathscr{C}^\cdot(\mathfrak{U},\mathscr{F})$ of (4.2), it follows from a general result on complexes (Hilton and Stammbach [1, IV, 4.4]) that there is a morphism of complexes $\mathscr{C}^\cdot(\mathfrak{U},\mathscr{F}) \to \mathscr{I}^\cdot$, inducing the identity map on $\mathscr{F}$, and unique up to homotopy. Applying the functors $\Gamma(X,\cdot)$ and h^p, we get the required map.

Theorem 4.5. *Let X be a noetherian separated scheme, let $\mathfrak{U}$ be an open affine cover of X, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then for all $p \geqslant 0$, the natural maps of* (4.4) *give isomorphisms*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^p(X,\mathscr{F}).$$

PROOF. For $p = 0$ we have an isomorphism by (4.1). For the general case, embed $\mathscr{F}$ in a flasque, quasi-coherent sheaf $\mathscr{G}$ (3.6), and let $\mathscr{R}$ be the quotient:

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

For each $i_0 < \ldots < i_p$, the open set $U_{i_0,\ldots,i_p}$ is affine, since it is an intersection of affine open subsets of a separated scheme (II, Ex. 4.3). Since $\mathscr{F}$ is quasi-coherent, we therefore have an exact sequence

$$0 \to \mathscr{F}(U_{i_0,\ldots,i_p}) \to \mathscr{G}(U_{i_0,\ldots,i_p}) \to \mathscr{R}(U_{i_0,\ldots,i_p}) \to 0$$

of abelian groups, by (3.5) or (II, 5.6). Taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^{\cdot}(\mathfrak{U},\mathscr{F}) \to C^{\cdot}(\mathfrak{U},\mathscr{G}) \to C^{\cdot}(\mathfrak{U},\mathscr{R}) \to 0$$

is exact. Therefore we get a long exact sequence of Čech cohomology groups. Since $\mathscr{G}$ is flasque, its Čech cohomology vanishes for $p > 0$ by (4.3), so we have an exact sequence

$$0 \to \check{H}^0(\mathfrak{U},\mathscr{F}) \to \check{H}^0(\mathfrak{U},\mathscr{G}) \to \check{H}^0(\mathfrak{U},\mathscr{R}) \to \check{H}^1(\mathfrak{U},\mathscr{F}) \to 0$$

and isomorphisms

$$\check{H}^p(\mathfrak{U},\mathscr{R}) \xrightarrow{\sim} \check{H}^{p+1}(\mathfrak{U},\mathscr{F})$$

for each $p \geqslant 1$. Now comparing with the long exact sequence of usual cohomology for the above short exact sequence, using the case $p = 0$, and (2.5), we conclude that the natural map

$$\check{H}^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is an isomorphism. But $\mathscr{R}$ is also quasi-coherent (II, 5.7), so we obtain the result for all p by induction.

EXERCISES

4.1. Let $f: X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,

$$H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F}).$$

[*Hint*: Use (II, 5.8).]

4.2. Prove Chevalley's theorem: Let $f: X \to Y$ be a finite surjective morphism of noetherian separated schemes, ~~with~~ X affine. Then Y is affine.

(a) Let $f: X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha: \mathscr{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta:\mathscr{F}^r \to f_*\mathscr{G}$ which is an isomorphism at the generic point of Y. [*Hint*: Apply $Hom(\cdot,\mathscr{F})$ to α and use (II, Ex. 5.17e).]

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider ker β and coker β, and use noetherian induction on Y.

4.3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U,\mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^iy^j | i,j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j\in J}$, together with a map $\lambda: J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i: \check{H}^i(\mathfrak{U},\mathscr{F}) \to \check{H}^i(\mathfrak{V},\mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U},\mathscr{F}).$$

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U},\mathscr{F}) \to H^i(X,\mathscr{F})$$

are compatible with the refinement maps above.

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U},\mathscr{F}) \to H^1(X,\mathscr{F})$$

is an isomorphism. [*Hint*: Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$, so that we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

Define a complex $D^{\cdot}(\mathfrak{U})$ by

$$0 \to C^{\cdot}(\mathfrak{U},\mathscr{F}) \to C^{\cdot}(\mathfrak{U},\mathscr{G}) \to D^{\cdot}(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D^{\cdot}(\mathfrak{U}) \to C^{\cdot}(\mathfrak{U},\mathscr{R}),$$

and see what happens under refinement.]

4.5. For any ringed space $(X,\mathcal{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves (II, §6). Show that Pic $X \cong H^1(X,\mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U,\mathcal{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathcal{L}$ on X, cover X by open sets U_i on which $\mathcal{L}$ is free, and fix isomorphisms $\varphi_i:\mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U},\mathcal{O}_X^*)$. Now use (Ex. 4.4).]

4.6. Let $(X,\mathcal{O}_X)$ be a ringed space, let $\mathcal{I}$ be a sheaf of ideals with $\mathcal{I}^2 = 0$, and let X_0 be the ringed space $(X,\mathcal{O}_X/\mathcal{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathcal{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathcal{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathcal{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X,\mathcal{I}) \to \text{Pic } X \to \text{Pic } X_0 \to H^2(X,\mathcal{I}) \to \ldots .$$

4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0,x_1,x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that (1,0,0) is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U,\mathcal{O}_X) \oplus \Gamma(V,\mathcal{O}_X) \to \Gamma(U \cap V,\mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X,\mathcal{O}_X) = 1,$$

$$\dim H^1(X,\mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\text{cd}(X)$, to be the least integer n such that $H^i(X,\mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\text{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\text{cd}(X) \leqslant \dim X$.

(a) In the definition of $\text{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

(c) Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\text{cd}(X) \leqslant r$.

*(d) If X is a quasi-projective variety of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude (independently of (2.7)) that $\text{cd}(X) \leqslant \dim X$.

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\text{cd}(X - Y) \leqslant r - 1$.

4.9. Let $X = \text{Spec } k[x_1,x_2,x_3,x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure

$\bar{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y,\mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P,\mathcal{O}_X) \neq 0$.]

***4.10.** Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X,\mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II,§8). [*Hint*: Use (II, Ex. 8.6) and (4.5).]

4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V,\mathcal{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps

$$\check{H}^p(\mathfrak{U},\mathcal{F}) \to H^p(X,\mathcal{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

5 The Cohomology of Projective Space

In this section we make explicit calculations of the cohomology of the sheaves $\mathcal{O}(n)$ on a projective space, by using Čech cohomology for a suitable open affine covering. These explicit calculations form the basis for various general results about cohomology of coherent sheaves on projective varieties.

Let A be a noetherian ring, let $S = A[x_0, \ldots, x_r]$, and let $X = \operatorname{Proj} S$ be the projective space $\mathbf{P}_A^r$ over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre (II, §5). For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\Gamma_*(\mathcal{F})$ the graded S-module $\bigoplus_{n \in \mathbf{Z}} \Gamma(X,\mathcal{F}(n))$ (see II, §5).

Theorem 5.1. *Let A be a noetherian ring, and let $X = \mathbf{P}_A^r$, with $r \geqslant 1$. Then:*

(a) *the natural map $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbf{Z}} H^0(X,\mathcal{O}_X(n))$ is an isomorphism of graded S-modules;*

(b) *$H^i(X,\mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbf{Z}$;*

(c) *$H^r(X,\mathcal{O}_X(-r-1)) \cong A$;*

(d) *The natural map*

$$H^0(X,\mathcal{O}_X(n)) \times H^r(X,\mathcal{O}_X(-n-r-1)) \to H^r(X,\mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbf{Z}$.

PROOF. Let $\mathcal{F}$ be the quasi-coherent sheaf $\bigoplus_{n \in \mathbf{Z}} \mathcal{O}_X(n)$. Since cohomology commutes with arbitrary direct sums on a noetherian topological space (2.9.1), the cohomology of $\mathcal{F}$ will be the direct sum of the cohomology of the sheaves $\mathcal{O}(n)$. So we will compute the cohomology of $\mathcal{F}$, and keep track

of the grading by n, so that we can sort out the pieces at the end. Note that all the cohomology groups in question have a natural structure of A-module (2.6.1).

For each $i = 0, \ldots, r$, let U_i be the open set $D_+(x_i)$. Then each U_i is an open affine subset of X, and the U_i cover X, so we can compute the cohomology of $\mathscr{F}$ by using Čech cohomology for the covering $\mathfrak{U} = (U_i)$, by (4.5). For any set of indices $i_0, \ldots, i_p$, the open set $U_{i_0, \ldots, i_p}$ is just $D_+(x_{i_0} \cdots x_{i_p})$, so by (II, 5.11) we have

$$\mathscr{F}(U_{i_0, \ldots, i_p}) \cong S_{x_{i_0} \cdots x_{i_p}},$$

the localization of S with respect to the element $x_{i_0} \cdots x_{i_p}$. Furthermore, the grading on $\mathscr{F}$ corresponds to the natural grading of $S_{x_{i_0} \cdots x_{i_p}}$ under this isomorphism. Thus the Čech complex of $\mathscr{F}$ is given by

$$C^{\cdot}(\mathfrak{U}, \mathscr{F}): \prod S_{x_{i_0}} \to \prod S_{x_{i_0} x_{i_1}} \to \ldots \to S_{x_0 \cdots x_r},$$

and the modules all have a natural grading compatible with the grading on $\mathscr{F}$.

Now $H^0(X, \mathscr{F})$ is the kernel of the first map, which is just S, as we have seen earlier (II, 5.13). This proves (a).

Next we consider $H^r(X, \mathscr{F})$. It is the cokernel of the last map in the Čech complex, which is

$$d^{r-1}: \prod_k S_{x_0 \cdots \hat{x}_k \cdots x_r} \to S_{x_0 \cdots x_r}.$$

We think of $S_{x_0 \cdots x_r}$ as a free A-module with basis $x_0^{l_0} \cdots x_r^{l_r}$, with $l_i \in \mathbf{Z}$. The image of d^{r-1} is the free submodule generated by those basis elements for which at least one $l_i \geqslant 0$. Thus $H^r(X, \mathscr{F})$ is a free A-module with basis consisting of the "negative" monomials

$$\{x_0^{l_0} \cdots x_r^{l_r} | l_i < 0 \text{ for each } i\}.$$

Furthermore the grading is given by $\sum l_i$. There is only one such monomial of degree $-r-1$, namely $x_0^{-1} \cdots x_r^{-1}$, so we see that $H^r(X, \mathcal{O}_X(-r-1))$ is a free A-module of rank 1. This proves (c).

To prove (d), first note that if $n < 0$, $H^0(X, \mathcal{O}_X(n)) = 0$ by (a), and $H^r(X, \mathcal{O}_X(-n-r-1)) = 0$ by what we have just seen, since in that case $-n-r-1 > -r-1$, and there are no negative monomials of that degree. So the statement is trivial for $n < 0$. For $n \geqslant 0$, $H^0(X, \mathcal{O}_X(n))$ has a basis consisting of the usual monomials of degree n, i.e., $\{x_0^{m_0} \cdots x_r^{m_r} | m_i \geqslant 0$ and $\sum m_i = n\}$. The natural pairing with $H^r(X, \mathcal{O}_X(-n-r-1))$ into $H^r(X, \mathcal{O}_X(-r-1))$ is determined by

$$(x_0^{m_0} \cdots x_r^{m_r}) \cdot (x_0^{l_0} \cdots x_r^{l_r}) = x_0^{m_0 + l_0} \cdots x_r^{m_r + l_r},$$

where $\sum l_i = -n-r-1$, and the object on the right becomes 0 if any $m_i + l_i \geqslant 0$. So it is clear that we have a perfect pairing, under which $x_0^{-m_0-1} \cdots x_r^{-m_r-1}$ is the dual basis element of $x_0^{m_0} \cdots x_r^{m_r}$.

It remains to prove statement (b), which we will do by induction on r. If $r = 1$ there is nothing to prove, so let $r > 1$. If we localize the complex $C^{\cdot}(\mathfrak{U},\mathscr{F})$ with respect to x_r, as graded S-modules, we get the Čech complex for the sheaf $\mathscr{F}|_{U_r}$ on the space U_r, with respect to the open affine covering $\{U_i \cap U_r | i = 0, \ldots, r\}$. By (4.5), this complex gives the cohomology of $\mathscr{F}|_{U_r}$ on U_r, which is 0 for $i > 0$ by (3.5). Since localization is an exact functor, we conclude that $H^i(X,\mathscr{F})_{x_r} = 0$ for $i > 0$. In other words, every element of $H^i(X,\mathscr{F})$, for $i > 0$, is annihilated by some power of x_r.

To complete the proof of (b), we will show that for $0 < i < r$, multiplication by x_r induces a bijective map of $H^i(X,\mathscr{F})$ into itself. Then it will follow that this module is 0.

Consider the exact sequence of graded S-modules

$$0 \to S(-1) \xrightarrow{x_r} S \to S/(x_r) \to 0.$$

This gives the exact sequence of sheaves

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

on X, where H is the hyperplane $x_r = 0$. Twisting by all $n \in \mathbf{Z}$ and taking the direct sum, we have

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}_H \to 0,$$

where $\mathscr{F}_H = \bigoplus_{n \in \mathbf{Z}} \mathcal{O}_H(n)$. Taking cohomology, we get a long exact sequence

$$\ldots \to H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F}) \to H^i(X,\mathscr{F}_H) \to \ldots .$$

Considered as graded S-modules, $H^i(X,\mathscr{F}(-1))$ is just $H^i(X,\mathscr{F})$ shifted one place, and the map $H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ of the exact sequence is multiplication by x_r.

Now H is isomorphic to $\mathbf{P}^{r-1}_A$, and $H^i(X,\mathscr{F}_H) = H^i(H,\bigoplus \mathcal{O}_H(n))$ by (2.10). So we can apply our induction hypothesis to $\mathscr{F}_H$, and find that $H^i(X,\mathscr{F}_H) = 0$ for $0 < i < r - 1$. Furthermore, for $i = 0$ we have an exact sequence

$$0 \to H^0(X,\mathscr{F}(-1)) \to H^0(X,\mathscr{F}) \to H^0(X,\mathscr{F}_H) \to 0$$

by (a), since $H^0(X,\mathscr{F}_H)$ is just $S/(x_r)$. At the other end of the exact sequence we have

$$0 \to H^{r-1}(X,\mathscr{F}_H) \xrightarrow{\delta} H^r(X,\mathscr{F}(-1)) \xrightarrow{x_r} H^r(X,\mathscr{F}) \to 0.$$

Indeed, we have described $H^r(X,\mathscr{F})$ above as the free A-module with basis formed by the negative monomials in $x_0, \ldots, x_r$. So it is clear that x_r is surjective. On the other hand, the kernel of x_r is the free submodule generated by those negative monomials $x_0^{l_0} \cdots x_r^{l_r}$ with $l_r = -1$. Since $H^{r-1}(X,\mathscr{F}_H)$ is the free A-module with basis consisting of the negative monomials in $x_0, \ldots, x_{r-1}$, and δ is division by x_r, the sequence is exact. In particular, δ is injective.

Putting these results all together, the long exact sequence of cohomology shows that the map multiplication by $x_r: H^i(X,\mathscr{F}(-1)) \to H^i(X,\mathscr{F})$ is bijective for $0 < i < r$, as required. q.e.d.

Theorem 5.2 (Serre [3]). *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over* Spec A. *Let $\mathscr{F}$ be a coherent sheaf on X. Then:*

(a) *for each $i \geqslant 0$, $H^i(X,\mathscr{F})$ is a finitely generated A-module;*

(b) *there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X,\mathscr{F}(n)) = 0$.*

PROOF. Since $\mathcal{O}_X(1)$ is a very ample sheaf on X over Spec A, there is a closed immersion $i: X \to \mathbf{P}^r_A$ of schemes over A, for some r such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbf{P}^r}(1)$—cf. (II, 5.16.1). If $\mathscr{F}$ is coherent on X, then $i_*\mathscr{F}$ is coherent on $\mathbf{P}^r_A$ (II, Ex. 5.5), and the cohomology is the same (2.10). Thus we reduce to the case $X = \mathbf{P}^r_A$.

For $X = \mathbf{P}^r_A$, we observe that (a) and (b) are true for any sheaf of the form $\mathcal{O}_X(q)$, $q \in \mathbf{Z}$. This follows immediately from the explicit calculations (5.1). Hence the same is true for any finite direct sum of such sheaves.

To prove the theorem for arbitrary coherent sheaves, we use descending induction on i. For $i > r$, we have $H^i(X,\mathscr{F}) = 0$, since X can be covered by $r + 1$ open affines (Ex. 4.8), so the result is trivial in this case.

In general, given a coherent sheaf $\mathscr{F}$ on X, we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E}$, which is a finite direct sum of sheaves $\mathcal{O}(q_i)$, for various integers q_i (II, 5.18). Let $\mathscr{R}$ be the kernel,

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Then $\mathscr{R}$ is also coherent. We get an exact sequence of A-modules

$$\ldots \to H^i(X,\mathscr{E}) \to H^i(X,\mathscr{F}) \to H^{i+1}(X,\mathscr{R}) \to \ldots.$$

Now the module on the left is finitely generated because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$, as remarked above. The module on the right is finitely generated by the induction hypothesis. Since A is a noetherian ring, we conclude that the one in the middle is also finitely generated. This proves (a).

To prove (b), we twist and again write down a piece of the long exact sequence

$$\ldots \to H^i(X,\mathscr{E}(n)) \to H^i(X,\mathscr{F}(n)) \to H^{i+1}(X,\mathscr{R}(n)) \to \ldots.$$

Now for $n \gg 0$, the module on the left vanishes because $\mathscr{E}$ is a sum of $\mathcal{O}(q_i)$. The module on the right also vanishes for $n \gg 0$ because of the induction hypothesis. Hence $H^i(X,\mathscr{F}(n)) = 0$ for $n \gg 0$. Note since there are only finitely many i involved in statement (b), namely $0 < i \leqslant r$, it is sufficient to determine n_0 separately for each i. This proves (b).

Remark 5.2.1. As a special case of (a), we see that for any coherent sheaf $\mathscr{F}$ on X, $\Gamma(X,\mathscr{F})$ is a finitely generated A-module. This generalizes, and gives another proof of (II, 5.19).

As an application of these results, we give a cohomological criterion for an invertible sheaf to be ample (II, §7).

Proposition 5.3. *Let A be a noetherian ring, and let X be a proper scheme over* Spec A. *Let $\mathscr{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is ample;*

(ii) *For each coherent sheaf $\mathscr{F}$ on X, there is an integer n_0, depending on $\mathscr{F}$, such that for each $i > 0$ and each $n \geqslant n_0$, $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$.*

PROOF. (i) $\Rightarrow$ (ii). If $\mathscr{L}$ is ample on X, then for some $m > 0$, $\mathscr{L}^m$ is very ample on X over Spec A, by (II, 7.6). Since X is proper over Spec A, it is necessarily projective (II, 5.16.1). Now applying (5.2) to each of the sheaves $\mathscr{F}, \mathscr{F} \otimes \mathscr{L}, \mathscr{F} \otimes \mathscr{L}^2, \ldots, \mathscr{F} \otimes \mathscr{L}^{m-1}$ gives (ii). Cf. (II, 7.5) for a similar technique of proof.

(ii) $\Rightarrow$ (i). To show that $\mathscr{L}$ is ample, we will show that for any coherent sheaf $\mathscr{F}$ on X, there is an integer n_0 such that $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections for all $n \geqslant n_0$. This is the definition of ampleness (II, §7).

Given $\mathscr{F}$, let P be a closed point of X, and let $\mathscr{I}_P$ be the ideal sheaf of the closed subset $\{P\}$. Then there is an exact sequence

$$0 \to \mathscr{I}_P\mathscr{F} \to \mathscr{F} \to \mathscr{F} \otimes k(P) \to 0,$$

where $k(P)$ is the skyscraper sheaf $\mathscr{O}_X/\mathscr{I}_P$. Tensoring with $\mathscr{L}^n$, we get

$$0 \to \mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{F} \otimes \mathscr{L}^n \otimes k(P) \to 0.$$

Now by our hypothesis (ii), there is an n_0 such that $H^1(X, \mathscr{I}_P\mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n \geqslant n_0$. Therefore

$$\Gamma(X, \mathscr{F} \otimes \mathscr{L}^n) \to \Gamma(X, \mathscr{F} \otimes \mathscr{L}^n \otimes k(P))$$

is surjective for all $n \geqslant n_0$. It follows from Nakayama's lemma over the local ring $\mathscr{O}_P$, that the stalk of $\mathscr{F} \otimes \mathscr{L}^n$ at P is generated by global sections. Since it is a coherent sheaf, we conclude that for each $n \geqslant n_0$, there is an open neighborhood U of P, depending on n, such that the global sections of $\mathscr{F} \otimes \mathscr{L}^n$ generate the sheaf at every point of U.

In particular, taking $\mathscr{F} = \mathscr{O}_X$, we find there is an integer $n_1 > 0$ and an open neighborhood V of P such that $\mathscr{L}^{n_1}$ is generated by global sections over V. On the other hand, for each $r = 0, 1, \ldots, n_1 - 1$, the above argument gives a neighborhood U_r of P such that $\mathscr{F} \otimes \mathscr{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_P = V \cap U_0 \cap \ldots \cap U_{n_1-1}.$$

Then over U_P, all of the sheaves $\mathscr{F} \otimes \mathscr{L}^n$, for $n \geqslant n_0$, are generated by global sections. Indeed, any such sheaf can be written as a tensor product

$$(\mathscr{F} \otimes \mathscr{L}^{n_0+r}) \otimes (\mathscr{L}^{n_1})^m$$

for suitable $0 \leqslant r \leqslant n_1$ and $m \geqslant 0$.

Now cover X by a finite number of the open sets U_P, for various closed points P, and let the new n_0 be the maximum of the n_0 corresponding to those points P. Then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections over all of X, for all $n \geqslant n_0$. q.e.d.

EXERCISES

5.1. Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X,\mathscr{F}).$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

5.2. (a) Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbf{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathcal{O}_X(1)$. [*Hints*: Use induction on dim Supp $\mathscr{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.]$$

(b) Now let $X = \mathbf{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

5.3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X,\mathcal{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X,\mathcal{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X,\mathcal{O}_X).$$

[*Hint*: Use (I, 3.4).]

(b) If X is a closed subvariety of $\mathbf{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P: K(X) \to \mathbf{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

(b) Now let $X = \mathbf{P}^r_k$. For each $i = 0,1,\ldots,r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0,\ldots,r\}$, and

(2) the map $P: K(X) \to \mathbf{Q}[z]$ is injective.

[*Hint*: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on n, using (II, Ex. 6.10c).]

5.5. Let k be a field, let $X = \mathbf{P}^r_k$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then:

(a) for all $n \in \mathbf{Z}$, the natural map

$$H^0(X,\mathcal{O}_X(n)) \to H^0(Y,\mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

(b) Y is connected;

(c) $H^i(Y,\mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;

(d) $p_a(Y) = \dim_k H^q(Y,\mathcal{O}_Y)$.

[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

5.6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.

(a) Use the special cases $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

(1) if $|a - b| \leqslant 1$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;

(2) if $a,b < 0$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;

(3) If $a \leqslant -2$, then $H^1(Q,\mathcal{O}_Q(a,0)) \neq 0$.

(b) Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a,b), with $a,b > 0$, then Y is connected;

(2) now assume k is algebraically closed. Then for any $a,b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.1.8).

(3) an irreducible nonsingular curve Y of type (a,b), $a,b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type (1,3), which is just the rational quartic curve (I, Ex. 3.18).

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab - a - b + 1$. [*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.]

5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i: Y \to X$ is the inclusion.

(b) $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes \mathcal{O}_{X_{\text{red}}}$ is ample on X_{red}.

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

(d) Let $f: X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X. [*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.

(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f: \tilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show $\operatorname{Pic} X \to \bigoplus \operatorname{Pic} X_i$ is surjective. Then use (Ex. 5.7c) to show X is projective.

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that $\operatorname{Pic} X \to \operatorname{Pic} X_{\text{red}}$ is surjective. Then use (Ex. 5.7b) to show X is projective.

5.9. *A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbf{P}^2_k$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω^1_X, and since $\dim X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\ldots \to H^1(X,\omega) \to \operatorname{Pic} X' \to \operatorname{Pic} X \xrightarrow{\delta} H^2(X,\omega) \to \ldots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X,\omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X,\omega) = 0$, we see that $\operatorname{Pic} X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

Note. In fact, this result can be generalized to show that for any nonsingular projective surface X over an algebraically closed field k of characteristic 0, there is an infinitesimal extension X' of X by ω, such that X' is not projective over k.

Indeed, let D be an ample divisor on X. Then D determines an element $c_1(D) \in H^1(X,\Omega^1)$ which we use to define X', as above. Then for any divisor E on X one can show that $\delta(\mathscr{L}(E)) = (D.E)$, where $(D.E)$ is the intersection number (Chapter V), considered as an element of k. Hence if E is ample, $\delta(\mathscr{L}(E)) \neq 0$. Therefore X' has no ample divisors.

On the other hand, over a field of characteristic $p > 0$, a proper scheme X is projective if and only if X_{red} is!

5.10. Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \ldots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\Gamma(X,\mathscr{F}^1(n)) \to \Gamma(X,\mathscr{F}^2(n)) \to \ldots \to \Gamma(X,\mathscr{F}^r(n))$$

is exact.

6 Ext Groups and Sheaves

In this section we develop the properties of Ext groups and sheaves, which we will need for the duality theorem. We work on a ringed space $(X,\mathcal{O}_X)$, and all sheaves will be sheaves of $\mathcal{O}_X$-modules.

If $\mathscr{F}$ and $\mathscr{G}$ are $\mathcal{O}_X$-modules, we denote by $\operatorname{Hom}(\mathscr{F},\mathscr{G})$ the group of $\mathcal{O}_X$-module homomorphisms, and by $\mathscr{H}om(\mathscr{F},\mathscr{G})$ the sheaf Hom (II, §5). If necessary, we put a subscript X to indicate which space we are on: $\operatorname{Hom}_X(\mathscr{F},\mathscr{G})$. For fixed $\mathscr{F}$, $\operatorname{Hom}(\mathscr{F},\cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, and $\mathscr{H}om(\mathscr{F},\cdot)$ is a left exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Since $\mathfrak{Mod}(X)$ has enough injectives (2.2) we can make the following definition.

Definition. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We define the functors $\operatorname{Ext}^i(\mathscr{F},\cdot)$ as the right derived functors of $\operatorname{Hom}(\mathscr{F},\cdot)$, and $\mathscr{E}xt^i(\mathscr{F},\cdot)$ as the right derived functors of $\mathscr{H}om(\mathscr{F},\cdot)$.

Consequently, according to the general properties of derived functors (1.1A) we have $\operatorname{Ext}^0 = \operatorname{Hom}$, a long exact sequence for a short exact sequence in the second variable, $\operatorname{Ext}^i(\mathscr{F},\mathscr{G}) = 0$ for $i > 0$, $\mathscr{G}$ injective in $\mathfrak{Mod}(X)$, and ditto for the $\mathscr{E}xt$ sheaves.

Lemma 6.1. *If $\mathscr{I}$ is an injective object of $\mathfrak{Mod}(X)$, then for any open subset $U \subseteq X$, $\mathscr{I}|_U$ is an injective object of $\mathfrak{Mod}(U)$.*

PROOF. Let $j: U \to X$ be the inclusion map. Then given an inclusion $\mathscr{F} \subseteq \mathscr{G}$ in $\mathfrak{Mod}(U)$, and given a map $\mathscr{F} \to \mathscr{I}|_U$, we get an inclusion $j_!\mathscr{F} \subseteq j_!\mathscr{G}$, and a map $j_!\mathscr{F} \to j_!(\mathscr{I}|_U)$, where $j_!$ is extension by zero (II, Ex. 1.19). But $j_!(\mathscr{I}|_U)$ is a subsheaf of $\mathscr{I}$, so we have a map $j_!\mathscr{F} \to \mathscr{I}$. Since $\mathscr{I}$ is injective in $\mathfrak{Mod}(X)$, this extends to a map of $j_!\mathscr{G}$ to $\mathscr{I}$. Restricting to U gives the required map of $\mathscr{G}$ to $\mathscr{I}|_U$.

Proposition 6.2. *For any open subset $U \subseteq X$ we have*

$$\mathscr{E}xt^i_X(\mathscr{F},\mathscr{G})|_U \cong \mathscr{E}xt^i_U(\mathscr{F}|_U,\mathscr{G}|_U).$$

PROOF. We use (1.3A). Both sides give δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(U)$. They agree for $i = 0$, both sides vanish for $i > 0$ and $\mathscr{G}$ injective, by (6.1), so they are equal.

Proposition 6.3. *For any $\mathscr{G} \in \mathfrak{Mod}(X)$, we have:*

(a) $\mathscr{E}xt^0(\mathscr{O}_X,\mathscr{G}) = \mathscr{G}$;

(b) $\mathscr{E}xt^i(\mathscr{O}_X,\mathscr{G}) = 0$ *for* $i > 0$;

(c) $\mathrm{Ext}^i(\mathscr{O}_X,\mathscr{G}) \cong H^i(X,\mathscr{G})$ *for all* $i \geqslant 0$.

PROOF. The functor $\mathscr{H}om(\mathscr{O}_X,\cdot)$ is the identity functor, so its derived functors are 0 for $i > 0$. This proves (a) and (b). The functors $\mathrm{Hom}(\mathscr{O}_X,\cdot)$ and $\Gamma(X,\cdot)$ are equal, so their derived functors (as functors from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$) are the same. Then use (2.6).

Proposition 6.4. *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence in $\mathfrak{Mod}(X)$, then for any $\mathscr{G}$ we have a long exact sequence*

$$0 \to \mathrm{Hom}(\mathscr{F}'',\mathscr{G}) \to \mathrm{Hom}(\mathscr{F},\mathscr{G}) \to \mathrm{Hom}(\mathscr{F}',\mathscr{G}) \to$$
$$\to \mathrm{Ext}^1(\mathscr{F}'',\mathscr{G}) \to \mathrm{Ext}^1(\mathscr{F},\mathscr{G}) \to \ldots,$$

and similarly for the $\mathscr{E}xt$ sheaves.

PROOF. Let $0 \to \mathscr{G} \to \mathscr{I}^{\cdot}$ be an injective resolution of $\mathscr{G}$. For any injective sheaf $\mathscr{I}$, the functor $\mathrm{Hom}(\cdot,\mathscr{I})$ is exact, so we get a short exact sequence of complexes

$$0 \to \mathrm{Hom}(\mathscr{F}'',\mathscr{I}^{\cdot}) \to \mathrm{Hom}(\mathscr{F},\mathscr{I}^{\cdot}) \to \mathrm{Hom}(\mathscr{F}',\mathscr{I}^{\cdot}) \to 0.$$

Taking the associated long exact sequence of cohomology groups h^i gives the sequence of Ext^i.

Similarly, using (6.1) we see that $\mathscr{H}om(\cdot,\mathscr{I})$ is an exact functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. Thus the same argument gives the exact sequence of $\mathscr{E}xt^i$.

Proposition 6.5. *Suppose there is an exact sequence*

$$\ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

in $\mathfrak{Mod}(X)$, where the $\mathscr{L}_i$ are locally free sheaves of finite rank (in this case we say $\mathscr{L}.$ is a locally free resolution of $\mathscr{F}$). Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$ we have

$$\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) \cong h^i(\mathscr{H}om(\mathscr{L}.,\mathscr{G})).$$

PROOF. Both sides are δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$. For $i = 0$ they are equal, because then $\mathscr{H}om(\cdot,\mathscr{G})$ is contravariant and left exact. Both sides vanish for $i > 0$ and $\mathscr{G}$ injective, because then $\mathscr{H}om(\cdot,\mathscr{G})$ is exact. So by (1.3A) they are equal.

Example 6.5.1. If X is a scheme, which is quasi-projective over Spec A, where A is a noetherian ring, then by (II, 5.18), any coherent sheaf $\mathscr{F}$ on X is a quotient of a locally free sheaf. Thus any coherent sheaf on X has a locally free resolution $\mathscr{L}. \to \mathscr{F} \to 0$. So (6.5) tells us that we can calculate $\mathscr{E}xt$ by taking locally free resolutions in the first variable.

Caution 6.5.2. The results (6.4) and (6.5) do *not* imply that $\mathscr{E}xt$ can be construed as a derived functor in its first variable. In fact, we cannot even define the right derived functors of Hom or $\mathscr{H}om$ in the first variable because the category $\mathfrak{Mod}(X)$ does not have enough projectives (Ex. 6.2). However, see (Ex. 6.4) for a universal property.

Lemma 6.6. *If $\mathscr{L} \in \mathfrak{Mod}(X)$ is locally free of finite rank, and $\mathscr{I} \in \mathfrak{Mod}(X)$ is injective, then $\mathscr{L} \otimes \mathscr{I}$ is also injective.*

PROOF. We must show that the functor $\mathrm{Hom}(\cdot, \mathscr{L} \otimes \mathscr{I})$ is exact. But it is the same as the functor $\mathrm{Hom}(\cdot \otimes \mathscr{L}^\vee, \mathscr{I})$ (II, Ex. 5.1), which is exact because $\cdot \otimes \mathscr{L}^\vee$ is exact, and $\mathscr{I}$ is injective.

Proposition 6.7. *Let $\mathscr{L}$ be a locally free sheaf, and let $\mathscr{L}^\vee = \mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$ be its dual. Then for any $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$ we have*

$$\mathrm{Ext}^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \cong \mathrm{Ext}^i(\mathscr{F}, \mathscr{L}^\vee \otimes \mathscr{G}),$$

and for the sheaf $\mathscr{E}xt$ we have

$$\mathscr{E}xt^i(\mathscr{F} \otimes \mathscr{L}, \mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{L}^\vee \otimes \mathscr{G}) \cong \mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) \otimes \mathscr{L}^\vee.$$

PROOF. The case $i = 0$ follows from (II, Ex. 5.1). For the general case, note that all of them are δ-functors in $\mathscr{G}$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ (respectively, $\mathfrak{Mod}(X)$), since tensoring with $\mathscr{L}^\vee$ is an exact functor. For $i > 0$ and $\mathscr{G}$ injective they all vanish, by (6.6), so by (1.3A) they are equal.

Next we will give some properties which are more particular to the case of schemes.

Proposition 6.8. *Let X be a noetherian scheme, let $\mathscr{F}$ be a coherent sheaf on X, let $\mathscr{G}$ be any $\mathcal{O}_X$-module, and let $x \in X$ be a point. Then we have*

$$\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_x}(\mathscr{F}_x, \mathscr{G}_x)$$

for any $i \geqslant 0$, where the right-hand side is Ext *over the local ring $\mathcal{O}_x$.*

PROOF. Of course, Ext over a ring A is defined as the right derived functor of $\mathrm{Hom}_A(M, \cdot)$ for any A-module M, considered as a functor from $\mathfrak{Mod}(A)$ to $\mathfrak{Mod}(A)$. Or, by considering a one-point space with the ring A attached, it becomes a special case of the Ext of a ringed space defined above.

Our question is local, by (6.2), so we may assume that X is affine. Then $\mathscr{F}$ has a locally free (or even a free) resolution $\mathscr{L}. \to \mathscr{F} \to 0$, which on the stalks at x gives a free resolution $(\mathscr{L}.)_x \to \mathscr{F}_x \to 0$. So by (6.5) we can calculate both sides by these resolutions. Since $\mathscr{H}om(\mathscr{L},\mathscr{G})_x = \mathrm{Hom}_{\mathscr{O}_x}(\mathscr{L}_x,\mathscr{G}_x)$ for a locally free sheaf $\mathscr{L}$, and since the stalk functor is exact, we get the equality of Ext's.

Note that even the case $i = 0$ is not true without some special hypothesis on $\mathscr{F}$, such as $\mathscr{F}$ coherent.

Proposition 6.9. *Let X be a projective scheme over a noetherian ring A, let $\mathscr{O}_X(1)$ be a very ample invertible sheaf, and let $\mathscr{F},\mathscr{G}$ be coherent sheaves on X. Then there is an integer $n_0 > 0$, depending on $\mathscr{F}$ and $\mathscr{G}$, such that for every $n \geqslant n_0$ we have*

$$\mathrm{Ext}^i(\mathscr{F},\mathscr{G}(n)) \cong \Gamma(X,\mathscr{E}xt^i(\mathscr{F},\mathscr{G}(n))).$$

Proof. If $i = 0$, this is true for any $\mathscr{F},\mathscr{G},n$. If $\mathscr{F} = \mathscr{O}_X$, then the left-hand side is $H^i(X,\mathscr{G}(n))$ by (6.3). So for $n \gg 0$ and $i > 0$ it is 0 by (5.2). On the other hand, the right-hand side is always 0 for $i > 0$ by (6.3), so we have the result for $\mathscr{F} = \mathscr{O}_X$.

If $\mathscr{F}$ is a locally free sheaf, we reduce to the case $\mathscr{F} = \mathscr{O}_X$ by (6.7).

Finally, if $\mathscr{F}$ is an arbitrary coherent sheaf, write it as a quotient of a locally free sheaf $\mathscr{E}$ (II, 5.18), and let $\mathscr{R}$ be the kernel:

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Since $\mathscr{E}$ is locally free, by the earlier results, for $n \gg 0$, we have an exact sequence

$$0 \to \mathrm{Hom}(\mathscr{F},\mathscr{G}(n)) \to \mathrm{Hom}(\mathscr{E},\mathscr{G}(n)) \to \mathrm{Hom}(\mathscr{R},\mathscr{G}(n)) \to \mathrm{Ext}^1(\mathscr{F},\mathscr{G}(n)) \to 0$$

and isomorphisms, for all $i \geqslant 0$

$$\mathrm{Ext}^i(\mathscr{R},\mathscr{G}(n)) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(\mathscr{F},\mathscr{G}(n)),$$

and similarly for the sheaf $\mathscr{H}om$ and $\mathscr{E}xt$. Now by (Ex. 5.10), the sequence of global sections of the sheaf sequence is exact after twisting a little more, so from the case $i = 0$, using (6.7), we get the case $i = 1$ for $\mathscr{F}$. But $\mathscr{R}$ is also coherent, so by induction we get the general result.

Remark 6.9.1. More generally, on any ringed space X, the relation between the global Ext and the sheaf $\mathscr{E}xt$ can be expressed by a spectral sequence (see Grothendieck [1] or Godement [1, II, 7.3.3]).

Now, for future reference, we recall the notion of projective dimension of a module over a ring. Let A be a ring, and let M be an A-module. A *projective resolution* of M is a complex $L.$ of projective A-modules, such that

$$\ldots \to L_2 \to L_1 \to L_0 \to M \to 0$$

is exact. If $L_i = 0$ for $i > n$, and $L_n \neq 0$, we say it has *length n*. Then we define the *projective dimension* of M, denoted $\mathrm{pd}(M)$, to be the least length of a projective resolution of M (or $+\infty$ if there is no finite projective resolution).

Proposition 6.10A. *Let A be a ring, and M an A-module. Then:*

(a) *M is projective if and only if $\mathrm{Ext}^1(M,N) = 0$ for all A-modules N;*

(b) *$\mathrm{pd}(M) \leqslant n$ if and only if $\mathrm{Ext}^i(M,N) = 0$ for all $i > n$ and all A-modules N.*

PROOF. Matsumura [2, pp. 127–128].

Proposition 6.11A. *If A is a regular local ring, then:*

(a) *for every M, $\mathrm{pd}(M) \leqslant \dim A$;*

(b) *If $k = A/\mathfrak{m}$, then $\mathrm{pd}(k) = \dim A$.*

PROOF. Matsumura [2, Th. 42, p. 131].

Proposition 6.12A. *Let A be a regular local ring of dimension n, and let M be a finitely generated A-module. Then we have*

$$\mathrm{pd}\, M + \mathrm{depth}\, M = n.$$

PROOF. Matsumura [2, p. 113, Ex. 4] or Serre [11, IVI , Prop. 21].

EXERCISES

6.1. Let $(X,\mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}',\mathcal{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathcal{F}''$ by $\mathcal{F}'$ is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathcal{F}'$ and $\mathcal{F}''$. Given an extension as above consider the long exact sequence arising from $\mathrm{Hom}(\mathcal{F}'',\cdot)$, in particular the map

$$\delta : \mathrm{Hom}(\mathcal{F}'',\mathcal{F}'') \to \mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}'),$$

and let $\xi \in \mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$ be $\delta(1_{\mathcal{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathcal{F}''$ by $\mathcal{F}'$, and elements of the group $\mathrm{Ext}^1(\mathcal{F}'',\mathcal{F}')$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

6.2. Let $X = \mathbf{P}^1_k$, with k an infinite field.

(a) Show that there does not exist a projective object $\mathcal{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathcal{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathcal{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, where $j: V \to X$ is the inclusion.]

(b) Show that there does not exist a projective object $\mathcal{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathcal{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathcal{L} \to \mathcal{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathcal{L}$ is an invertible sheaf on X.]

6.3. Let X be a noetherian scheme, and let $\mathscr{F}\mathscr{P} \in \mathfrak{Mod}(X)$.
(a) If $\mathscr{F},\mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is coherent, for all $i \geqslant 0$.
(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F},\mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

6.4. Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees*. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathscr{E}xt^i(\cdot,\mathscr{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathscr{E}xt^i(\cdot,\mathscr{G})$ is coefficeable (§1) for $i > 0$.]

6.5. Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\operatorname{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:
(a) $\mathscr{F}$ is locally free $\Leftrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;
(b) $\operatorname{hd}(\mathscr{F}) \leqslant n \Leftrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;
(c) $\operatorname{hd}(\mathscr{F}) = \sup_x \operatorname{pd}_{\mathscr{O}_X} \mathscr{F}_x$.

6.6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.
(a) M is projective if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > 0$. [*Hint*: Use (6.11A) and descending induction on i to show that $\operatorname{Ext}^i(M,N) = 0$ for all $i > 0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p. 129]).]
(b) Use (a) to show that for any n, $\operatorname{pd} M \leqslant n$ if and only if $\operatorname{Ext}^i(M,A) = 0$ for all $i > n$.

6.7. Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\tilde{M},\tilde{N}) \cong \operatorname{Ext}^i_A(M,N)$$

and

$$\mathscr{E}xt^i_X(\tilde{M},\tilde{N}) \cong \operatorname{Ext}^i_A(M,N)^{\sim}.$$

6.8. Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).
(a) First show that open sets of the form X_s, for various $s \in \Gamma(X,\mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X. [*Hint*: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathscr{L},s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathscr{O}_x$, $f \notin \mathscr{O}_\zeta$. Let $D = (f)_\infty$, and let $\mathscr{L} = \mathscr{L}(D)$, $s \in \Gamma(X,\mathscr{L}(D))$ correspond to D (II, §6).]
(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

6.9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is *regular* if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a

short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon: K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathscr{E}_n \to \ldots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}. \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum(-1)^i \gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

6.10. *Duality for a Finite Flat Morphism.*

(a) Let $f: X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{G}$ (II, Ex. 5.17e).

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

(c) For each $i \geqslant 0$, there is a natural map

$$\varphi_i: \mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

[*Hint*: First construct a map

$$\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}).$$

Then compose with a suitable map from $f_*f^!\mathscr{G}$ to $\mathscr{G}$.]

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat—see §9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y. [*Hints*: First do $i = 0$. Then do $\mathscr{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathscr{F}$ locally free. Do the general case by induction on i, writing $\mathscr{F}$ as a quotient of a locally free sheaf.]

7 The Serre Duality Theorem

In this section we prove the Serre duality theorem for the cohomology of coherent sheaves on a projective scheme. First we do the case of projective space itself, which follows easily from the explicit calculations of §5. Then on an arbitrary projective scheme X, we show that there is a coherent sheaf ω_X°, which plays a role in duality theory similar to the canonical sheaf of a nonsingular variety. In particular, if X is Cohen–Macaulay, it gives a duality theorem just like the one on projective space. Finally, if X is a nonsingular variety over an algebraically closed field, we show that the dualizing sheaf ω_X° coincides with the canonical sheaf ω_X. At the end of the section, we mention the connection between duality and residues of differential forms.

Let k be a field, let $X = \mathbf{P}^n_k$ be the n-dimensional projective space over k, and let $\omega_X = \wedge^n \Omega_{X/k}$ be the canonical sheaf on X (II, §8).

Theorem 7.1 (Duality for $\mathbf{P}^n_k$). *Let $X = \mathbf{P}^n_k$ over a field k. Then:*

(a) $H^n(X,\omega_X) \cong k$. *Fix one such isomorphism;*

(b) *for any coherent sheaf $\mathscr{F}$ on X, the natural pairing*

$$\operatorname{Hom}(\mathscr{F},\omega) \times H^n(X,\mathscr{F}) \to H^n(X,\omega) \cong k$$

is a perfect pairing of finite-dimensional vector spaces over k;

(c) *for every $i \geqslant 0$ there is a natural functorial isomorphism*

$$\operatorname{Ext}^i(\mathscr{F},\omega) \xrightarrow{\sim} H^{n-i}(X,\mathscr{F})',$$

where $'$ denotes the dual vector space, which for $i = 0$ is the one induced by the pairing of (b).

PROOF.

(a) It follows from (II, 8.13) that $\omega_X \cong \mathcal{O}_X(-n-1)$ (see II, 8.20.1). Thus (a) follows from (5.1c).

(b) A homomorphism of $\mathscr{F}$ to ω induces a map of cohomology groups $H^n(X,\mathscr{F}) \to H^n(X,\omega)$. This gives the natural pairing. If $\mathscr{F} \cong \mathcal{O}(q)$ for some $q \in \mathbf{Z}$, then $\operatorname{Hom}(\mathscr{F},\omega) \cong H^0(X,\omega(-q))$, so the result follows from (5.1d). Hence (b) holds also for a finite direct sum of sheaves of the form $\mathcal{O}(q_i)$. If $\mathscr{F}$ is an arbitrary coherent sheaf, we can write it as a cokernel $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$ of a map of sheaves $\mathscr{E}_i$, each $\mathscr{E}_i$ being a direct sum of sheaves $\mathcal{O}(q_i)$. Now $\operatorname{Hom}(\cdot,\omega)$ and $H^n(X,\cdot)'$ are both left-exact contravariant functors, so by the 5-lemma we get an isomorphism $\operatorname{Hom}(\mathscr{F},\omega) \xrightarrow{\sim} H^n(X,\mathscr{F})'$.

(c) Both sides are contravariant δ-functors, for $\mathscr{F} \in \mathfrak{Coh}(X)$, indexed by $i \geqslant 0$. For $i = 0$ we have an isomorphism by (b). Thus to show they are isomorphic, by (1.3A), it will be sufficient to show both sides are coeffaceable for $i > 0$. Given $\mathscr{F}$ coherent, it follows from (II, 5.18) and its proof that we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E} = \bigoplus_{i=1}^N \mathcal{O}(-q)$, with $q \gg 0$. Then $\operatorname{Ext}^i(\mathscr{E},\omega) = \bigoplus H^i(X,\omega(q)) = 0$ for $i > 0$ by (5.1). On the other hand, $H^{n-i}(X,\mathscr{E})' = \bigoplus H^{n-i}(X,\mathcal{O}(-q))'$, which is 0 for $i > 0$, $q > 0$, as we see again from (5.1) by inspection. Thus both sides are coeffaceable for $i > 0$, so the δ-functors are universal, hence isomorphic.

Remark 7.1.1. One may ask, why bother phrasing (7.1) with the sheaf ω_X, rather than simply writing $\mathcal{O}_X(-n-1)$, which is what we use in the proof? One reason is that this is the form of the theorem which generalizes well. But a more intrinsic reason is that when written this way, the isomorphism of (a) can be made independent of the choice of basis of $\mathbf{P}^n$, hence stable under automorphisms of $\mathbf{P}^n$. Thus it is truly a *natural* isomorphism. To do this, consider the Čech cocycle

$$\alpha = \frac{x_0^n}{x_1 \cdots x_n} d\left(\frac{x_1}{x_0}\right) \wedge \cdots \wedge d\left(\frac{x_n}{x_0}\right)$$

in $C^n(\mathfrak{U},\omega)$, where $\mathfrak{U}$ is the standard open covering. Then one can show that α determines a generator of $H^n(X,\omega)$, which is stable under change of variables.

To generalize (7.1) to other schemes, we take properties (a) and (b) as our guide, and make the following definition.

Definition. Let X be a proper scheme of dimension n over a field k. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace* morphism $t: H^n(X,\omega_X^\circ) \to k$, such that for all coherent sheaves $\mathscr{F}$ on X, the natural pairing

$$\operatorname{Hom}(\mathscr{F},\omega_X^\circ) \times H^n(X,\mathscr{F}) \to H^n(X,\omega_X^\circ)$$

followed by t gives an *isomorphism*

$$\operatorname{Hom}(\mathscr{F},\omega_X^\circ) \xrightarrow{\sim} H^n(X,\mathscr{F})'.$$

Proposition 7.2. *Let X be a proper scheme over k. Then a dualizing sheaf for X, if it exists, is unique. More precisely, if ω° is one, with its trace map t, and if ω',t' is another, then there is a unique isomorphism $\varphi:\omega^\circ \xrightarrow{\sim} \omega'$ such that $t = t' \circ H^n(\varphi)$.*

PROOF. Since ω' is dualizing, we get an isomorphism $\operatorname{Hom}(\omega^\circ,\omega') \cong H^n(\omega^\circ)'$. So there is a unique morphism $\varphi:\omega^\circ \to \omega'$ corresponding to the element $t \in H^n(\omega^\circ)'$, i.e., such that $t' \circ H^n(\varphi) = t$. Similarly, using the fact that ω° is dualizing, there is a unique morphism $\psi:\omega' \to \omega^\circ$ such that $t \circ H^n(\psi) = t'$. It follows that $t \circ H^n(\psi \circ \varphi) = t$. But again since ω° is dualizing, this implies that $\psi \circ \varphi$ is the identity map of ω°. Similarly $\varphi \circ \psi$ is the identity map of ω', so φ is an isomorphism. (This proof is a special case of the uniqueness of an object representing a functor (see Grothendieck [EGA I, new ed., Ch. 0, §1]). For by definition (ω°,t) represents the functor $\mathscr{F} \mapsto H^n(X,\mathscr{F})'$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(k)$.)

The question of existence of dualizing sheaves is more difficult. In fact they exist for any X proper over k, but we will prove the existence here only for projective schemes. First we need some preliminary results.

Lemma 7.3. *Let X be a closed subscheme of codimension r of $P = \mathbf{P}_k^N$. Then $\mathscr{E}xt_P^i(\mathcal{O}_X,\omega_P) = 0$ for all $i < r$.*

PROOF. For any i, the sheaf $\mathscr{F}^i = \mathscr{E}xt_P^i(\mathcal{O}_X,\omega_P)$ is a coherent sheaf on P (Ex. 6.3), so after twisting by a suitably large integer q, it will be generated by global sections (II, 5.17). Thus to show $\mathscr{F}^i$ is zero, it will be sufficient to show that $\Gamma(P,\mathscr{F}^i(q)) = 0$ for all $q \gg 0$. But by (6.7) and (6.9) we have

$$\Gamma(P,\mathscr{F}^i(q)) \cong \operatorname{Ext}_P^i(\mathcal{O}_X,\omega_P(q))$$

for $q \gg 0$. On the other hand, by (7.1) this last Ext group is dual to $H^{N-i}(P,\mathcal{O}_X(-q))$. For $i < r$, $N - i > \dim X$, so this group is 0 by (2.7) or (Ex. 4.8d).

Lemma 7.4. *With the same hypotheses as* (7.3), *let* $\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X,\omega_P)$. *Then for any* $\mathcal{O}_X$*-module* $\mathcal{F}$, *there is a functorial isomorphism*

$$\mathrm{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \mathrm{Ext}_P^r(\mathcal{F},\omega_P).$$

PROOF. Let $0 \to \omega_P \to \mathcal{I}^\cdot$ be an injective resolution of ω_P in $\mathfrak{Mod}(P)$. Then we calculate $\mathrm{Ext}_P^i(\mathcal{F},\omega_P)$ as the cohomology groups h^i of the complex $\mathrm{Hom}_P(\mathcal{F},\mathcal{I}^\cdot)$. But since $\mathcal{F}$ is an $\mathcal{O}_X$-module, any morphism $\mathcal{F} \to \mathcal{I}^i$ factors through $\mathcal{J}^i = \mathcal{H}om_P(\mathcal{O}_X,\mathcal{I}^i)$. Thus we have

$$\mathrm{Ext}_P^i(\mathcal{F},\omega_P) = h^i(\mathrm{Hom}_X(\mathcal{F},\mathcal{J}^\cdot)).$$

Now each $\mathcal{J}^i$ is an *injective* $\mathcal{O}_X$-module. Indeed, for $\mathcal{F} \in \mathfrak{Mod}(X)$, $\mathrm{Hom}_X(\mathcal{F},\mathcal{J}^i) = \mathrm{Hom}_P(\mathcal{F},\mathcal{I}^i)$, so $\mathrm{Hom}_X(\cdot,\mathcal{J}^i)$ is an exact functor. Furthermore, by (7.3) we have $h^i(\mathcal{J}^\cdot) = 0$ for $i < r$, so the complex $\mathcal{J}^\cdot$ is exact up to the rth step. Since the $\mathcal{J}^i$ are injective, it is actually split exact up to the rth step. This implies that we can write the complex as a direct sum of two injective complexes, $\mathcal{J}^\cdot = \mathcal{J}^\cdot_1 \oplus \mathcal{J}^\cdot_2$, where $\mathcal{J}^\cdot_1$ is in degrees $0 \leqslant i \leqslant r$ and is exact, and $\mathcal{J}^\cdot_2$ is in degrees $i \geqslant r$. It follows that $\omega_X^\circ = \ker(d^r : \mathcal{J}^r_2 \to \mathcal{J}^{r+1}_2)$, and that for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathrm{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \mathrm{Ext}_P^r(\mathcal{F},\omega_P).$$

(It also follows that $\mathrm{Ext}_P^i(\mathcal{F},\omega_P) = 0$ for $i < r$, which we won't need.)

Proposition 7.5. *Let* X *be a projective scheme over a field* k. *Then* X *has a dualizing sheaf.*

PROOF. Embed X as a closed subscheme of $P = \mathbf{P}_k^N$ for some N, let r be its codimension, and let $\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X,\omega_P)$. Then by (7.4) we have an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\mathrm{Hom}_X(\mathcal{F},\omega_X^\circ) \cong \mathrm{Ext}_P^r(\mathcal{F},\omega_P).$$

On the other hand, when $\mathcal{F}$ is coherent, the duality theorem for P (7.1) gives an isomorphism

$$\mathrm{Ext}_P^r(\mathcal{F},\omega_P) \cong H^{N-r}(P,\mathcal{F})'.$$

But $N - r = n$, the dimension of X, and $\mathcal{F}$ is a sheaf on X, so we obtain a functorial isomorphism, for $\mathcal{F} \in \mathfrak{Coh}(X)$,

$$\mathrm{Hom}_X(\mathcal{F},\omega_X^\circ) \cong H^n(X,\mathcal{F})'.$$

In particular, taking $\mathcal{F} = \omega_X^\circ$, the element $1 \in \mathrm{Hom}(\omega_X^\circ,\omega_X^\circ)$ gives us a homomorphism $t : H^n(X,\omega_X^\circ) \to k$, which we take as our trace map. Then it is clear by functoriality that (ω_X°,t) is a dualizing sheaf for X.

Now we can prove the duality theorem for a projective scheme X. Recall that a scheme is *Cohen–Macaulay* if all of its local rings are Cohen–Macaulay rings (II, §8).

Theorem 7.6 (Duality for a Projective Scheme). *Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω_X° be a dualizing sheaf on X, and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:*

(a) *for all $i \geqslant 0$ and $\mathcal{F}$ coherent on X, there are natural functorial maps*

$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})',$$

such that θ^0 is the map given in the definition of dualizing sheaf above;

(b) *the following conditions are equivalent:*

(i) *X is Cohen–Macaulay and equidimensional* (i.e., *all irreducible components have the same dimension*);

(ii) *for any $\mathcal{F}$ locally free on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$;*

(iii) *the maps θ^i of* (a) *are isomorphisms for all $i \geqslant 0$ and all $\mathcal{F}$ coherent on X.*

PROOF.

(a) As in the proof of (7.1c), we can write any coherent sheaf $\mathcal{F}$ as a quotient of a sheaf $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$, with $q \gg 0$. Then $\operatorname{Ext}^i(\mathcal{E}, \omega_X^\circ) \cong \bigoplus H^i(X, \omega_X^\circ(q))$, which is 0 for $i > 0$ and $q \gg 0$ by (5.2). Thus the functor $\operatorname{Ext}^i(\cdot, \omega_X^\circ)$ is coeffaceable for $i > 0$, so we have a universal contravariant δ-functor by (1.3A). On the right-hand side we have a contravariant δ-functor, indexed by $i \geqslant 0$, so there is a unique morphism of δ-functors (θ^i) reducing to the given θ^0 for $i = 0$.

(b) (i) $\Rightarrow$ (ii). Embed X as a closed subscheme of $P = \mathbf{P}_k^N$. Then for any $\mathcal{F}$ locally free on X, and any closed point $x \in X$, we have depth $\mathcal{F}_x = n$, since X is Cohen–Macaulay and equidimensional of dimension n. Let $A = \mathcal{O}_{P,x}$ be the local ring of x on P. Then A is a regular local ring of dimension N. (Since k is algebraically closed, x is rational over k, so the fact that A is regular can be seen directly. Or it follows from the fact that P is a nonsingular variety over k (II, §8).) Now depth $\mathcal{F}_x$ is the same, whether calculated over $\mathcal{O}_{X,x}$ or over A. Thus we conclude from (6.12A) that $\operatorname{pd}_A \mathcal{F}_x = N - n$. Therefore by (6.8) and (6.10A) we have

$$\mathcal{E}xt_P^i(\mathcal{F}, \cdot) = 0$$

for $i > N - n$.

On the other hand, using (7.1), we find that $H^i(X, \mathcal{F}(-q))$ is dual to $\operatorname{Ext}_P^{N-i}(\mathcal{F}, \omega_P(q))$. For $q \gg 0$, this Ext is isomorphic to $\Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q)))$ by (6.9). But this is 0 for $N - i > N - n$, as we have just seen. In other words, $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$.

(ii) $\Rightarrow$ (i). Running the above argument backwards, using condition (ii) with $\mathcal{F} = \mathcal{O}_X$, we find that

$$\mathcal{E}xt_P^i(\mathcal{O}_X, \omega_P) = 0$$

for $i > N - n$. This implies that over a local ring $A = \mathcal{O}_{P,x}$ as above, we have $\operatorname{Ext}^i_A(\mathcal{O}_{X,x},A) = 0$ for all $i > N - n$. Therefore by (Ex. 6.6) we have $\operatorname{pd}_A \mathcal{O}_{X,x} \leqslant N - n$, and so by (6.12A), depth $\mathcal{O}_{X,x} \geqslant n$. But since $\dim X = n$, we must have equality for every closed point of X. This shows, using (II, 8.21Ab), that X is Cohen–Macaulay and equidimensional.

(ii) $\Rightarrow$ (iii). Since we have already seen that $\operatorname{Ext}^i(\cdot,\omega_X^\circ))$ is a universal contravariant δ-functor, to show that the θ^i are isomorphisms, it will be sufficient to show that the δ-functor $(H^{n-i}(X,\cdot)')$ is universal also. For this it suffices by (1.3A) to show that $H^{n-i}(X,\cdot)'$ is coeffaceable for $i > 0$. So given a coherent sheaf $\mathcal{F}$, write $\mathcal{F}$ as a quotient of $\mathcal{E} = \bigoplus \mathcal{O}(-q)$ with $q \gg 0$. Then $H^{n-i}(X,\mathcal{E})' = 0$ for $i > 0$ by (ii), so the functor is coeffaceable.

(iii) $\Rightarrow$ (ii). If θ^i is an isomorphism, then for any $\mathcal{F}$ locally free, we have

$$H^i(X,\mathcal{F}(-q)) \cong \operatorname{Ext}^{n-i}(\mathcal{F}(-q),\omega_X^\circ)'.$$

But this Ext is isomorphic to $H^{n-i}(X,\mathcal{F}^\vee \otimes \omega_X^\circ(q))$ by (6.3) and (6.7), so it is 0 for $n - i > 0$ and $q \gg 0$ by (5.2). q.e.d.

Remark 7.6.1. In particular, if X is nonsingular over k, or more generally a local complete intersection, then X is Cohen–Macaulay (II, 8.21A) and (II, 8.23), so the θ^i are isomorphisms. In these two cases, one can show directly (cf. proof of (7.11) below) that $\operatorname{pd}_P \mathcal{O}_X = N - n$, and thus avoid use of the algebraic results (6.12A) and (Ex. 6.6).

Corollary 7.7. *Let X be a projective Cohen–Macaulay scheme of equidimension n over k. Then for any locally free sheaf $\mathcal{F}$ on X there are natural isomorphisms*

$$H^i(X,\mathcal{F}) \cong H^{n-i}(X,\mathcal{F}^\vee \otimes \omega_X^\circ)'.$$

PROOF. Use (6.3) and (6.7).

Corollary 7.8 (Lemma of Enriques–Severi–Zariski (Zariski [4])). *Let X be a normal projective scheme of dimension $\geqslant 2$. Then for any locally free sheaf $\mathcal{F}$ on X,*

$$H^1(X,\mathcal{F}(-q)) = 0$$

for $q \gg 0$.

PROOF. Since X is normal of dimension $\geqslant 2$, we have depth $\mathcal{F}_x \geqslant 2$ for every closed point $x \in X$ by (II, 8.22A). So the result follows by the same method as the proof of (i) $\Rightarrow$ (ii) in (7.6b).

Corollary 7.9. *Let X be an integral, normal projective variety of dimension $\geqslant 2$ over an algebraically closed field k. Let Y be a subvariety of codimension 1 which is the support of an effective ample divisor. Then Y is connected.*

(not subst

PROOF. By (II, 7.6) we may assume that Y is the support of a very ample divisor D. Let $\mathcal{O}(1)$ be the corresponding very ample invertible sheaf. For each $q > 0$, let Y_q be the closed subscheme supported on Y corresponding to the divisor qD (II, 6.17.1). Then we have an exact sequence (II, 6.18)

$$0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to \mathcal{O}_{Y_q} \to 0.$$

Taking cohomology and applying (7.8), we find that for $q \gg 0$,

$$H^0(X,\mathcal{O}_X) \to H^0(Y,\mathcal{O}_{Y_q}) \to 0$$

is surjective. But $H^0(X,\mathcal{O}_X) = k$ (I, 3.4a), and $H^0(Y,\mathcal{O}_{Y_q})$ contains k, so we conclude that $H^0(Y,\mathcal{O}_{Y_q}) = k$. Hence Y is connected. (If not, there would be at least one copy of k for each connected component.)

Remark 7.9.1. This implies that the schemes $H \cap X$ mentioned in Bertini's theorem (II, 8.18) are in fact *irreducible* and nonsingular when $\dim X \geqslant 2$. Indeed, they are connected by (7.9). On the other hand, they are regular by (II, 8.18). Hence the local rings are all integral domains, so we could not have two irreducible components meeting at a point.

Now that we have proved the duality theorem (7.6), our next task is to give more information about the dualizing sheaf ω_X° in some special cases. Again we need some algebraic preliminaries.

Let A be a ring, and let $f_1, \ldots, f_r \in A$. We define the *Koszul complex* $K.(f_1, \ldots, f_r)$ as follows: K_1 is a free A-module of rank r with basis $e_1, \ldots, e_r$. For each $p = 0, \ldots, r$, $K_p = \wedge^p K_1$. We define the boundary map $d: K_p \to K_{p-1}$ by its action on the basis vectors:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum (-1)^{j-1} f_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_p}.$$

Thus $K.(f_1, \ldots, f_r)$ is a (homological) complex of A-modules. If M is any A-module, we set $K.(f_1, \ldots, f_r; M) = K.(f_1, \ldots, f_r) \otimes_A M$.

Proposition 7.10A. *Let A be a ring, $f_1, \ldots, f_r \in A$, and let M be an A-module. If the f_i form a regular sequence for M, then*

$$h_i(K.(f_1, \ldots, f_r; M)) = 0 \qquad \text{for } i > 0$$

and

$$h_0(K.(f_1, \ldots, f_r; M)) \cong M/(f_1, \ldots, f_r)M.$$

PROOF. Matsumura [2, Th. 43, p. 135] or Serre [11, IV.A].

Theorem 7.11. *Let X be a closed subscheme of $P = \mathbf{P}_k^N$ which is a local complete intersection of codimension r. Let $\mathcal{I}$ be the ideal sheaf of X. Then $\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathcal{I}/\mathcal{I}^2)^\vee$. In particular, ω_X° is an invertible sheaf on X.*

PROOF. We have to calculate $\omega_X^\circ = \mathscr{E}xt_P^r(\mathcal{O}_X,\omega_P)$. Let U be an open affine subset over which $\mathscr{I}$ can be generated by r elements $f_1,\ldots,f_r \in A = \Gamma(U,\mathcal{O}_U)$. Then, because X has codimension r, and A is locally Cohen–Macaulay, $f_1,\ldots,f_r$ form a regular sequence for A (II, 8.21A). Therefore the Koszul complex $K.(f_1,\ldots,f_r)$ gives a free resolution of $A/(f_1,\ldots,f_r)$ over A. Taking associated sheaves, we get a free resolution of $\mathcal{O}_X$ over U, which we denote by $\tilde{K}.(f_1,\ldots,f_r;\mathcal{O}_P)$. We can use this resolution to calculate $\mathscr{E}xt_P^r(\mathcal{O}_X,\omega_P)$ over U (6.5). We get

$$h^r(\mathscr{H}om(K.(f_1,\ldots,f_r;\mathcal{O}_P),\omega_P)) \cong \omega_P/(f_1,\ldots,f_r)\omega_P.$$

In other words,

$$\mathscr{E}xt_P^r(\mathcal{O}_X,\omega_P) \cong \omega_P \otimes \mathcal{O}_X$$

over U. However, this isomorphism depends on the choice of basis $f_1,\ldots,f_r$ for $\mathscr{I}$. If $g_i = \sum c_{ij}f_j$, $i = 1,\ldots,r$, is another basis, then the exterior powers of the matrix $\|c_{ij}\|$ give an isomorphism of Koszul complexes. In particular, we have a factor of $\det|c_{ij}|$ on K_r, so our isomorphism of $\mathscr{E}xt^r$ changes by $\det|c_{ij}|$.

To remedy this situation, we consider the sheaf $\mathscr{I}/\mathscr{I}^2$ on X, which is locally free of rank r (II, 8.21A). In particular, it is free over U, with basis $f_1,\ldots,f_r$. Therefore $\wedge^r(\mathscr{I}/\mathscr{I}^2)$ is free of rank 1, with basis $f_1 \wedge \ldots \wedge f_r$. If we change to the basis $g_1,\ldots,g_r$, this element changes by $\det|c_{ij}|$. Therefore, we can obtain an intrinsic isomorphism above by tensoring with this free sheaf of rank 1 (check variance!)

$$\mathscr{E}xt_P^r(\mathcal{O}_X,\omega_P) \cong \omega_P \otimes \mathcal{O}_X \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee.$$

This isomorphism, defined over U, is independent of the choice of basis. Therefore when we cover P with such open sets, these isomorphisms glue together, and we obtain the required isomorphism $\omega_X^\circ \cong \omega_P \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$.

Corollary 7.12. *If X is a projective nonsingular variety over an algebraically closed field k, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.*

PROOF. Embed X in $P = \mathbf{P}_k^N$. Then X is a local complete intersection in P (II, 8.17), and $\omega_X \cong \omega_P \otimes \wedge^r(\mathscr{I}/\mathscr{I}^2)^\vee$ by (II, 8.20).

Remark 7.12.1. Thus for a projective nonsingular variety X, the duality theorem (7.6) and its corollary (7.7) hold with ω_X in place of ω_X°. In particular, we obtain an isomorphism $H^n(X,\omega_X) \cong k$, whose existence is by no means obvious a priori.

Remark 7.12.2. If X is a projective nonsingular curve, we find that $H^1(X,\mathcal{O}_X)$ and $H^0(X,\omega_X)$ are dual vector spaces. Hence the arithmetic genus $p_a = \dim H^1(X,\mathcal{O}_X)$ and the geometric genus $p_g = \dim \Gamma(X,\omega_X)$ are equal—cf. (Ex. 5.3a) and (II. 8.18.2).

Remark 7.12.3. If X is a projective nonsingular surface, then $H^0(X,\omega)$ is dual to $H^2(X,\mathcal{O}_X)$, so $p_g = \dim H^2(X,\mathcal{O}_X)$. On the other hand $p_a = \dim H^2(X,\mathcal{O}_X) - \dim H^1(X,\mathcal{O}_X)$ by (Ex. 5.3a). Thus $p_g \geqslant p_a$. The difference, $p_g - p_a = \dim H^1(X,\mathcal{O}_X)$ is usually denoted by q, and is called the *irregularity* of X. For example, the surface of (II, Ex. 8.3c) has irregularity 2.

Corollary 7.13. *Let X be a nonsingular projective variety of dimension n. For any $p = 0,1,\ldots,n$, let $\Omega^p = \wedge^p\Omega_{X/k}$ be the sheaf of differential p-forms. Then for each $p,q = 0,1,\ldots,n$, we have a natural isomorphism*

$$H^q(X,\Omega^p) \cong H^{n-q}(X,\Omega^{n-p})'.$$

PROOF. Indeed, for any p, $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes \omega$ (II, Ex. 5.16b). Then use (7.7).

Remark 7.13.1. The numbers $h^{p,q} = \dim H^q(X,\Omega^p)$ are important biregular invariants of the variety X.

Remark 7.14 (Residues of Differentials on Curves). A weakness of the duality theorem as we have proved it is that even for a nonsingular projective variety X, we don't have much information about the trace map $t: H^n(X,\omega) \to k$. We know only that it exists. In the case of curves, there is another way of proving the duality theorem, using residues, which improves this situation.

Let X be a complete nonsingular curve over an algebraically closed field k, and let K be the function field of X. Let Ω_X be the sheaf of differentials of X over k, and for a closed point $P \in X$, let Ω_P be its stalk at P. Let Ω_K be the module of differentials of K over k. Then one first proves:

Theorem 7.14.1 (Existence of Residues). *For each closed point $P \in X$, there is a unique k-linear map $\operatorname{res}_P: \Omega_K \to k$ with the following properties:*

(a) $\operatorname{res}_P(\tau) = 0$ *for all* $\tau \in \Omega_P$;

(b) $\operatorname{res}_P(f^n df) = 0$ *for all* $f \in K^*$, *all* $n \neq -1$;

(c) $\operatorname{res}_P(f^{-1}df) = v_P(f) \cdot 1$, *where v_P is the valuation associated to P.*

From these properties we see immediately how to calculate the residue of any differential. Indeed, let $t \in \mathcal{O}_P$ be a uniformizing parameter. Then dt is a generator for Ω_K as a K-vector space, so we can write any $\tau \in \Omega_K$ as gdt for some $g \in K$. Furthermore, since $\mathcal{O}_P$ is a valuation ring, we can write $g = \sum_{i<0} a_i t^i + h$ with $a_i \in k$, $h \in \mathcal{O}_P$, and the sum finite. Thus $\tau = \sum a_i t^i dt + hdt$. Now from linearity and (a), (b), (c) we find

(d) $\operatorname{res}_P \tau = a_{-1}$.

Thus the uniqueness of res_P is clear.

The existence is more difficult. One approach by Serre [7, Ch. II] is to take (d) as the definition of the residue. Then one has an awkward time proving that it is independent of the choice of the uniformizing parameter t, especially in the case of characteristic $p > 0$. Another approach by Tate [2]

gives an intrinsic construction of the residue map by a clever use of certain k-linear transformations of K.

The basic result about residues is:

Theorem 7.14.2 (Residue Theorem). *For any* $\tau \in \Omega_K$, *we have* $\sum_{P \in X} \operatorname{res}_P \tau = 0$.

In Serre's approach this theorem is first proved on $\mathbf{P}^1$, by explicit calculation. Then the general case is obtained by using a finite morphism $X \to \mathbf{P}^1$ and studying the relationship between the residues in both places. In Tate's approach the residue theorem follows directly from the construction of the residue map.

Once one has the theory of residues, the duality theorem for X can be proved by a method of Weil using repartitions. We refer to the lucid expositions of Serre and Tate mentioned above for the details of this classic story.

The connection with our approach can be explained as follows. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0,$$

where $\mathcal{K}_X$ is the constant sheaf K_X, is a flasque resolution of $\mathcal{O}_X$ (cf. Ex. 2.2). Furthermore,

$$\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{P \in X} i_*(K_X/\mathcal{O}_P)$$

where we consider $K_X/\mathcal{O}_P$ as an $\mathcal{O}_P$-module, and $i: \{P\} \to X$ is the inclusion map. Tensoring with Ω_X, we get a flasque resolution of Ω_X:

$$0 \to \Omega_X \to \Omega_X \otimes \mathcal{K}_X \to \bigoplus_{P \in X} i_*(\Omega_K/\Omega_P) \to 0.$$

Taking cohomology, we get an exact sequence

$$\Omega_K \to \bigoplus_{P \in X} \Omega_K/\Omega_P \to H^1(X, \Omega_X) \to 0.$$

We define a map

$$\bigoplus_{P \in X} \Omega_K/\Omega_P \to k$$

by taking the sum of all the maps $\operatorname{res}_P: \Omega_K/\Omega_P \to k$. Then by (7.14.2) this map vanishes on the image of Ω_K, hence it passes to the quotient and gives a map $t: H^1(X, \Omega_X) \to k$. This is the trace map of our duality theorem, which appears now in a much more explicit form.

Remark 7.15 (The Kodaira Vanishing Theorem). Our discussion of the cohomology of projective varieties would not be complete without mentioning the Kodaira vanishing theorem. It says if X is a projective nonsingular variety of dimension n over $\mathbf{C}$, and if $\mathcal{L}$ is an ample invertible sheaf on X, then:

(a) $H^i(X, \mathcal{L} \otimes \omega) = 0$ for $i > 0$;

(b) $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < n$.

Of course (a) and (b) are equivalent to each other by Serre duality. The theorem is proved using methods of complex analytic differential geometry. At present there is no purely algebraic proof. On the other hand, Raynaud has recently shown that this result does not hold over fields of characteristic $p > 0$.

The first proof was given by Kodaira [1]. For other proofs, including the generalization by Nakano, see Wells [1, Ch. VI, §2], Mumford [3], and Ramanujam [1]. For a relative version of the theorem, see Grauert and Riemenschneider [1].

References for the Duality Theorem. The duality theorem was first proved by Serre [2] (in the form of (7.7)) for locally free sheaves on a compact complex manifold, and in the case of abstract algebraic geometry by Serre [1]. Our proof follows Grothendieck [5] and Grothendieck [SGA 2, exp. XII], with some improvements suggested by Lipman. The duality theorem and the theory of residues have been generalized to the case of an arbitrary proper morphism by Grothendieck—see Grothendieck [4] and Hartshorne [2]. Deligne has given another proof of the existence of a dualizing sheaf, and Verdier [1] has shown that this one agrees with the sheaf ω for a nonsingular variety. Kunz [1] gives another construction, using differentials, of the dualizing sheaf ω_X° for an integral projective scheme X over k.

The duality theorem has also been generalized to the case of a proper morphism of complex analytic spaces—see Ramis and Ruget [1] and Ramis, Ruget, and Verdier [1]. For a generalization to noncompact complex manifolds, see Suominen [1].

In the case of curves, the duality theorem is the most important ingredient in the proof of the Riemann–Roch theorem (IV, §1). See Serre [7, Ch. II] for the history of this approach, and also Gunning [1] for a proof in the language of compact Riemann surfaces.

Exercises

7.1. Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X,\mathscr{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

7.2. Let $f: X \to Y$ be a finite morphism of projective schemes over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f'\omega_Y^\circ$ is a dualizing sheaf for X, where f' is defined as in (Ex. 6.10).

(b) If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t: f_*\omega_X \to \omega_Y$.

7.3. Let $X = \mathbf{P}_k^n$. Show that $H^q(X,\Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leqslant p, q \leqslant n$.

***7.4.** *The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n - p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology $H^{n-p}(X,\Omega_X^{n-p}) \to H^{n-p}(Y,\Omega_Y^{n-p})$. Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf

for Y, so we have the trace map $t_Y: H^{n-p}(Y,\Omega_Y^{n-p}) \to k$. Composing, we obtain a linear map $H^{n-p}(X,\Omega_X^{n-p}) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X,\Omega_X^p)$, which we call the *cohomology class* of Y.

(a) If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X,\Omega^n)$ and t_X is the trace map.

(b) If $X = \mathbf{P}^n$, identify $H^p(X,\Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y)\cdot 1$, where $\deg Y$ is its *degree* as a projective variety (I, §7). [*Hint*: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

(c) For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log: \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X,\mathcal{O}_X^*) \to H^1(X,\Omega_X)$ which we denote by c—see (Ex. 4.5).

(d) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathcal{L}(Y))$, where $\mathcal{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

8 Higher Direct Images of Sheaves

For the remainder of this chapter we will be studying families of schemes. Recall (II, §3) that a family of schemes is simply a morphism $f: X \to Y$, and the members of the family are the fibres $X_y = X \times_Y \operatorname{Spec} k(y)$ for various points $y \in Y$. To study a family, we need some form of "relative cohomology of X over Y," or "cohomology along the fibres of X over Y." This notion is provided by the higher direct image functors R^if_* which we define below. The precise relationship between these functors and the cohomology of the fibres X_y will be studied in §11, 12.

Definition. Let $f: X \to Y$ be a continuous map of topological spaces. Then we define the *higher direct image* functors $R^if_*: \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_* (II, §1).

This makes sense because f_* is obviously left exact, and $\mathfrak{Ab}(X)$ has enough injective (2.3).

Proposition 8.1. *For each $i \geqslant 0$ and each $\mathcal{F} \in \mathfrak{Ab}(X)$, $R^if_*(\mathcal{F})$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

on Y.

PROOF. Let us denote the sheaf associated to the above presheaf by $\mathcal{H}^i(X,\mathcal{F})$. Then, since the operation of taking the sheaf associated to a presheaf is exact, the functors $\mathcal{H}^i(X,\cdot)$ form a δ-functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}(Y)$. For $i = 0$ we have $f_*\mathcal{F} = \mathcal{H}^0(X,\mathcal{F})$ by definition of f_*. For an injective object $\mathcal{I} \in \mathfrak{Ab}(X)$ we have $R^if_*(\mathcal{I}) = 0$ for $i > 0$ because R^if_* is a derived functor.

On the other hand, for each V, $\mathscr{I}|_{f^{-1}(V)}$ is injective in $\mathfrak{Ab}(f^{-1}(V))$ by (6.1) (think of X as a ringed space with the constant sheaf $\mathbf{Z}$), so $\mathscr{H}^i(X,\mathscr{I}) = 0$ for $i > 0$ also. Hence there is a unique isomorphism of δ-functors $R^if_*(\cdot) \cong \mathscr{H}^i(X,\cdot)$ by (1.3A).

Corollary 8.2. *If $V \subseteq Y$ is any open subset, then*

$$R^if_*(\mathscr{F})|_V = R^if'_*(\mathscr{F}|_{f^{-1}(V)})$$

where $f':f^{-1}(V) \to V$ is the restricted map.

PROOF. Obvious.

Corollary 8.3. *If $\mathscr{F}$ is a flasque sheaf on X, then $R^if_*(\mathscr{F}) = 0$ for all $i > 0$.*

PROOF. Since the restriction of a flasque sheaf to an open subset is flasque, this follows from (2.5).

Proposition 8.4. *Let $f:X \to Y$ be a morphism of ringed spaces. Then the functors R^if_* can be calculated on $\mathfrak{Mod}(X)$ as the derived functors of $f_*:\mathfrak{Mod}(X) \to \mathfrak{Mod}(Y)$.*

PROOF. To calculate the derived functors of f_* on $\mathfrak{Mod}(X)$, we use resolutions by injective objects of $\mathfrak{Mod}(X)$. Any injective of $\mathfrak{Mod}(X)$ is flasque by (2.4), hence acyclic for f_* on $\mathfrak{Ab}(X)$ by (8.3), so they can be used to calculate R^if_* by (1.2A).

Proposition 8.5. *Let X be a noetherian scheme, and let $f:X \to Y$ be a morphism of X to an affine scheme $Y = \operatorname{Spec} A$. Then for any quasi-coherent sheaf $\mathscr{F}$ on X, we have*

$$R^if_*(\mathscr{F}) \cong H^i(X,\mathscr{F})^{\sim}.$$

PROOF. By (II, 5.8), $f_*\mathscr{F}$ is a quasi-coherent sheaf on Y. Hence $f_*\mathscr{F} \cong \Gamma(Y,f_*\mathscr{F})^{\sim}$. But $\Gamma(Y,f_*\mathscr{F}) = \Gamma(X,\mathscr{F})$. So we have an isomorphism for $i = 0$.

Since $^{\sim}$ is an exact functor from $\mathfrak{Mod}(A)$ to $\mathfrak{Mod}(Y)$, both sides are δ-functors from $\mathfrak{Qco}(X)$ to $\mathfrak{Mod}(Y)$. Furthermore, by (3.6), any quasi-coherent sheaf $\mathscr{F}$ on X can be embedded in a flasque, quasi-coherent sheaf. Hence both sides are effaceable for $i > 0$. We conclude from (1.3A) that there is a unique isomorphism of δ-functors as above, reducing to the given one for $i = 0$.

Note that we must work in the category $\mathfrak{Qco}(X)$, because already the case $i = 0$ fails if $\mathscr{F}$ is not quasi-coherent.

Corollary 8.6. *Let $f:X \to Y$ be a morphism of schemes, with X noetherian. Then for any quasi-coherent sheaf $\mathscr{F}$ on X, the sheaves $R^if_*(\mathscr{F})$ are quasi-coherent on Y.*

PROOF. The question is local on Y, so we may use (8.5).

Proposition 8.7. *Let $f: X \to Y$ be a morphism of separated noetherian schemes. Let $\mathscr{F}$ be a quasi-coherent sheaf on X, let $\mathfrak{U} = (U_i)$ be an open affine cover of X, and let $\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})$ be the Čech resolution of $\mathscr{F}$ given by* (4.2). *Then for each $p \geqslant 0$,*

$$R^p f_*(\mathscr{F}) \cong h^p(f_*\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F})).$$

PROOF. For any open affine subset $V \subseteq Y$, the open subsets $U_i \cap f^{-1}(V)$ of X are all affine (check!—cf. (II, Ex. 4.3)). Hence we may reduce to the case Y affine. The sheaves $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ are all quasi-coherent, so we have

$$f_*\mathscr{C}^{\cdot}(\mathfrak{U},\mathscr{F}) \cong C^{\cdot}(\mathfrak{U},\mathscr{F})^{\sim}$$

by (II, 5.8). Now the result follows from (4.5) and (8.5).

Theorem 8.8. *Let $f: X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over Y, and let $\mathscr{F}$ be a coherent sheaf on X. Then:*

(a) *for all $n \gg 0$, the natural map $f^*f_*(\mathscr{F}(n)) \to \mathscr{F}(n)$ is surjective;*
(b) *for all $i \geqslant 0$, $R^i f_*(\mathscr{F})$ is a coherent sheaf on Y;*
(c) *for $i > 0$ and $n \gg 0$, $R^i f_*(\mathscr{F}(n)) = 0$.*

PROOF. Since Y is quasi-compact, the question is local on Y, so we may assume Y is affine, say $Y = \operatorname{Spec} A$. Then, using (8.5), (a) says that $\mathscr{F}(n)$ is generated by global sections, which is (II, 5.17). (b) says that $H^i(X,\mathscr{F})$ is a finitely generated A-module, which is (5.2a). Finally, (c) says that $H^i(X,\mathscr{F}(n)) = 0$, which is (5.2b).

Remark 8.8.1. Part (b) of this theorem is true more generally for a proper morphism of noetherian schemes—see Grothendieck [EGA III, 3.2.1]. The analogous theorem for a proper morphism of complex analytic spaces was proved by Grauert [1].

EXERCISES

8.1. Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F}).$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

8.2. Let $f: X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X,\mathscr{F}) \cong H^i(Y,f_*\mathscr{F})$ for each $i \geqslant 0$. (This gives another proof of (Ex. 4.1).)

8.3. Let $f: X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the *projection formula* (cf. (II, Ex. 5.1))

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$

8.4. Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geqslant 1$. Let $X = \mathbf{P}(\mathscr{E})$ (II, §7), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi: X \to Y$.

(a) Then $\pi_*(\mathcal{O}(l)) \cong S^l(\mathscr{E})$ for $l \geqslant 0$, $\pi_*(\mathcal{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbf{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.

(b) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0,$$

cf. (II, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ (cf. (7.1.1)).

(c) Now show, for any $l \in \mathbf{Z}$, that

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee.$$

(d) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

(e) In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).

9 Flat Morphisms

In this section we introduce the notion of a flat morphism of schemes. By taking the fibres of a flat morphism, we get the notion of a flat family of schemes. This provides a concise formulation of the intuitive idea of a "continuous family of schemes." We will show, through various results and examples, why flatness is a natural as well as a convenient condition to put on a family of schemes.

First we recall the algebraic notion of a flat module. Let A be a ring, and let M be an A-module. We say that M is *flat* over A if the functor $N \mapsto M \otimes_A N$ is an exact functor for $N \in \mathfrak{Mod}(A)$. If $A \to B$ is a ring homomorphism, we say that B is *flat* over A if it is flat as a module.

Proposition 9.1A.

(a) *An A-module M is flat if and only if for every finitely generated ideal $\mathfrak{a} \subseteq A$, the map $\mathfrak{a} \otimes M \to M$ is injective.*

(b) Base extension: *If M is a flat A-module, and $A \to B$ is a homomorphism, then $M \otimes_A B$ is a flat B-module.*

(c) Transitivity: *If B is a flat A-algebra, and N is a flat B-module, then N is also flat as an A-module.*

(d) Localization: *M is flat over A if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.*

(e) *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* A*-modules. If* M' *and* M'' *are both flat then* M *is flat; if* M *and* M'' *are both flat, then* M' *is flat.*

(f) *A finitely generated module* M *over a local noetherian ring* A *is flat if and only if it is free.*

PROOFS. Matsumura [2, Ch. 2, §3] or Bourbaki [1, Ch. I.].

Example 9.1.1. If A is a ring and $S \subseteq A$ is a multiplicative system, then the localization $S^{-1}A$ is a flat A-algebra. If $A \to B$ is a ring homomorphism, if M is a B-module which is flat over A, and if S is a multiplicative system in B, then $S^{-1}M$ is flat over A.

Example 9.1.2. If A is a noetherian ring and $\mathfrak{a} \subseteq A$ an ideal, then the $\mathfrak{a}$-adic completion $\hat{A}$ is a flat A-algebra (II, 9.3A).

Example 9.1.3. Let A be a principal ideal domain. Then an A-module M is flat if and only if it is torsion-free. Indeed, by (9.1Aa) we must check that for every ideal $\mathfrak{a} \subseteq A$, $\mathfrak{a} \otimes M \to M$ is injective. But $\mathfrak{a}$ is principal, say generated by t, so this just says that t is not a zero divisor in M, i.e., M is torsion-free.

Definition. Let $f: X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is *flat over* Y *at a point* $x \in X$, if the stalk $\mathscr{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module, where $y = f(x)$ and we consider $\mathscr{F}_x$ as an $\mathcal{O}_{y,Y}$-module via the natural map $f^{\#}: \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. We say simply $\mathscr{F}$ is *flat over* Y if it is flat at every point of X. We say X is *flat* over Y if $\mathcal{O}_X$ is.

Proposition 9.2.

(a) *An open immersion is flat.*

(b) Base change: *let* $f: X \to Y$ *be a morphism, let* $\mathscr{F}$ *be an* $\mathcal{O}_X$*-module which is flat over* Y*, and let* $g: Y' \to Y$ *be any morphism. Let* $X' = X \times_Y Y'$*, let* $f': X' \to Y'$ *be the second projection, and let* $\mathscr{F}' = p_1^*(\mathscr{F})$*. Then* $\mathscr{F}'$ *is flat over* Y'*.*

(c) Transitivity: let $f: X \to Y$ *and* $g: Y \to Z$ *be morphisms. Let* $\mathscr{F}$ *be an* $\mathcal{O}_X$*-module which is flat over* Y*, and assume also that* Y *is flat over* Z*. Then* $\mathscr{F}$ *is flat over* Z*.*

(d) *Let* $A \to B$ *be a ring homomorphism, and let* M *be a* B*-module. Let* $f: X = \operatorname{Spec} B \to Y = \operatorname{Spec} A$ *be the corresponding morphism of affine schemes, and let* $\mathscr{F} = \tilde{M}$*. Then* $\mathscr{F}$ *is flat over* Y *if and only if* M *is flat over* A*.*

(e) *Let* X *be a noetherian scheme, and* $\mathscr{F}$ *a coherent* $\mathcal{O}_X$*-module. Then* $\mathscr{F}$ *is flat over* X *if and only if it is locally free.*

PROOF. These properties all follow from the corresponding properties of modules, taking into account that the functor $\sim$ is compatible with $\otimes$ (II, 5.2).

Next, as an illustration of the convenience of flat morphisms, we show that "cohomology commutes with flat base extension":

Proposition 9.3. *Let $f: X \to Y$ be a separated morphism of finite type of noetherian schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Let $u: Y' \to Y$ be a flat morphism of noetherian schemes.*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"] & Y \end{tikzcd} \quad \mathscr{F}$$

Then for all $i \geqslant 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^* \mathscr{F}).$$

PROOF. The question is local on Y and on Y', so we may assume they are both affine, say $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Then by (8.5) what we have to show is that

$$H^i(X, \mathscr{F}) \otimes_A A' \cong H^i(X', \mathscr{F}').$$

Since X is separated and noetherian, and $\mathscr{F}$ is quasi-coherent, we can calculate $H^i(X, \mathscr{F})$ by Čech cohomology with respect to an open affine cover $\mathfrak{U}$ of X (4.5). On the other hand, $\{v^{-1}(U) \mid U \in \mathfrak{U}\}$ forms an open affine cover $\mathfrak{U}'$ of X', and clearly the Čech complex $C^{\cdot}(\mathfrak{U}', v^* \mathscr{F})$ is just $C^{\cdot}(\mathfrak{U}, \mathscr{F}) \otimes_A A'$. Since A' is flat over A, the functor $\cdot \otimes_A A'$ commutes with taking cohomology groups of the Čech complex, so we get our result. Note that g is also separated and of finite type by base extension, so X' is also noetherian and separated, allowing us to apply (4.5) on X'.

Remark 9.3.1. Even if u is not flat, this proof shows that there is a natural map $u^* R^i f_*(\mathscr{F}) \to R^i g_*(v^* \mathscr{F})$.

Corollary 9.4. *Let $f: X \to Y$ and $\mathscr{F}$ be as in (9.3), and assume Y affine. For any point $y \in Y$, let X_y be the fibre over y, and let $\mathscr{F}_y$ be the induced sheaf. On the other hand, let $k(y)$ denote the constant sheaf $k(y)$ on the closed subset $\{y\}^-$ of Y. Then for all $i \geqslant 0$ there are natural isomorphisms*

$$H^i(X_y, \mathscr{F}_y) \cong H^i(X, \mathscr{F} \otimes k(y)).$$

PROOF. First let $Y' \subseteq Y$ be the reduced induced subscheme structure on $\{y\}^-$, and let $X' = X \times_Y Y'$, which is a closed subscheme of X. Then both sides of our desired isomorphism depend only on the sheaf $\mathscr{F}' = \mathscr{F} \otimes k(y)$ on X'. Thus we can replace $X, Y, \mathscr{F}$ by $X', Y', \mathscr{F}'$, i.e., we can assume that Y is an integral affine scheme and that $y \in Y$ is its generic point. In that case, $\operatorname{Spec} k(y) \to Y$ is a flat morphism, so we can apply (9.3) and conclude that

$$H^i(X_y, \mathscr{F}_y) \cong H^i(X, \mathscr{F}) \otimes k(y).$$

But after our reduction, $H^i(X,\mathscr{F})$ is already a $k(y)$-module, so tensoring with $k(y)$ has no effect, and we obtain the desired result. (This result is used in §12.)

Flat Families

For many reasons it is important to have a good notion of an algebraic family of varieties or schemes. The most naive definition would be just to take the fibres of a morphism. To get a good notion, however, we should require that certain numerical invariants remain constant in a family, such as the dimension of the fibres. It turns out that if we are dealing with nonsingular (or even normal) varieties over a field, then the naive definition is already a good one. Evidence for this is the theorem (9.13) that in such a family, the arithmetic genus is constant.

On the other hand, if we deal with nonnormal varieties, or more general schemes, the naive definition will not do. So we consider a flat family of schemes, which means the fibres of a flat morphism, and this is a very good notion. Why the algebraic condition of flatness on the structure sheaves should give a good definition of a family is something of a mystery. But at least we will justify this choice by showing that flat families have many good properties, and by giving necessary and sufficient conditions for flatness in some special cases. In particular, we will show that a family of closed subschemes of projective space (over an integral scheme) is flat if and only if the Hilbert polynomials of the fibres are the same.

Proposition 9.5. *Let $f: X \to Y$ be a flat morphism of finite type of noetherian schemes. For any point $x \in X$, let $y = f(x)$. Then*

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

Here for any scheme X and any point $x \in X$, by $\dim_x X$ we mean the dimension of the local ring $\mathcal{O}_{x,X}$.

Proof. First we make a base change $Y' \to Y$ where $Y' = \operatorname{Spec} \mathcal{O}_{y,Y}$, and consider the new morphism $f': X' \to Y'$ where $X' = X \times_Y Y'$. Then f' is also flat by (9.2), x lifts to X', and the three numbers in question are the same. Thus we may assume that y is a closed point of Y, and $\dim_y Y = \dim Y$.

Now we use induction on $\dim Y$. If $\dim Y = 0$, then X_y is defined by a nilpotent ideal in X, so we have $\dim_x(X_y) = \dim_x X$, and $\dim_y Y = 0$.

If $\dim Y > 0$, we make a base extension to Y_{red}. Nothing changes, so we may assume that Y is reduced. Then we can find an element $t \in \mathfrak{m}_y \subseteq \mathcal{O}_{y,Y}$ such that t is not a zero divisor. Let $Y' = \operatorname{Spec} \mathcal{O}_{y,Y}/(t)$, and make the base extension $Y' \to Y$. Then $\dim Y' = \dim Y - 1$ by (I, 1.8A) and (I, 1.11A). Since f is flat, $f^{\#}t \in \mathfrak{m}_x$ is also not a zero divisor. So for the same reason, $\dim_x X' = \dim_x X - 1$. Of course the fibre X_y does not change under base extension, so we have only to prove our formula for $f': X' \to Y'$. But this follows from the induction hypothesis, so we are done.

Corollary 9.6. *Let $f: X \to Y$ be a flat morphism of-schemes of finite type over a field k, and assume that Y is irreducible. Then the following conditions are equivalent:*

(i) *every irreducible component of X has dimension equal to* $\dim Y + n$;
(ii) *for any point $y \in Y$ (closed or not), every irreducible component of the fibre X_y has dimension n.*

PROOF.

(i) $\Rightarrow$ (ii). Given $y \in Y$, let $Z \subseteq X_y$ be an irreducible component, and let $x \in Z$ be a closed point, which is not in any other irreducible component of X_y. Applying (9.5) we have

$$\dim_x Z = \dim_x X - \dim_y Y.$$

Now $\dim_x Z = \dim Z$ since x is a closed point (II, Ex. 3.20). On the other hand, since Y is irreducible and X is equidimensional, and both are of finite type over k, we have (II, Ex. 3.20)

$$\dim_x X = \dim X - \dim\{x\}^-$$
$$\dim_y Y = \dim Y - \dim\{y\}^-$$

Finally, since x is a closed point of the fibre X_y, $k(x)$ is a finite algebraic extension of $k(y)$ and so

$$\dim\{x\}^- = \dim\{y\}^-.$$

Combining all these, and using (i) we find $\dim Z = n$.

(ii) $\Rightarrow$ (i). This time let Z be an irreducible component of X, and let $x \in Z$ be a closed point which is not contained in any other irreducible component of X. Then applying (9.5), we have

$$\dim_x(X_y) = \dim_x X - \dim_y Y.$$

But $\dim_x(X_y) = n$ by (ii), $\dim_x X = \dim Z$, and $\dim_y Y = \dim Y$, since $y = f(x)$ must be a closed point of Y. Thus

$$\dim Z = \dim Y + n$$

as required.

Definition. A point x of a scheme X is an *associated point* of X if the maximal ideal $\mathfrak{m}_x$ is an associated prime of 0 in the local ring $\mathcal{O}_{x,X}$, or in other words, if every element of $\mathfrak{m}_x$ is a zero divisor.

Proposition 9.7. *Let $f: X \to Y$ be a morphism of schemes, with Y integral and regular of dimension 1. Then f is flat if and only if every associated point $x \in X$ maps to the generic point of Y. In particular, if X is reduced, this says that every irreducible component of X dominates Y.*

PROOF. First suppose that f is flat, and let $x \in X$ be a point whose image $y = f(x)$ is a closed point of Y. Then $\mathcal{O}_{y,Y}$ is a discrete valuation ring. Let $t \in \mathfrak{m}_y - \mathfrak{m}_y^2$ be a uniformizing parameter. Then t is not a zero divisor in $\mathcal{O}_{y,Y}$. Since f is flat, $f^{\#}t \in \mathfrak{m}_x$ is not a zero divisor, so x is not an associated point of X.

Conversely, suppose that every associated point of X maps to the generic point of Y. To show f is flat, we must show that for any $x \in X$, letting $y = f(x)$, the local ring $\mathcal{O}_{x,X}$ is flat over $\mathcal{O}_{y,Y}$. If y is the generic point, $\mathcal{O}_{y,Y}$ is a field, so there is nothing to prove. If y is a closed point, $\mathcal{O}_{y,Y}$ is a discrete valuation ring, so by (9.1.3) we must show that $\mathcal{O}_{x,X}$ is a torsion-free module. If it is not, then $f^{\#}t$ must be a zero divisor in $\mathfrak{m}_x$, where t is a uniformizing parameter of $\mathcal{O}_{y,Y}$. Therefore $f^{\#}t$ is contained in some associated prime ideal $\mathfrak{p}$ of (0) in $\mathcal{O}_x$ (Matsumura [2, Cor. 2, p. 50]). Then $\mathfrak{p}$ determines a point $x' \in X$, which is an associated point of X, and whose image by f is y, which is a contradiction.

Finally, note that if X is reduced, its associated points are just the generic points of its irreducible components, so our condition says that each irreducible component of X dominates Y.

Example 9.7.1. Let Y be a curve with a node, and let $f: X \to Y$ be the map of its normalization to it. Then f is not flat. For if it were, then $f_*\mathcal{O}_X$ would be a flat sheaf of $\mathcal{O}_Y$-modules. Since it is coherent, it would be locally free by (9.2e). And finally, since its rank is 1, it would be an invertible sheaf on Y. But there are two points P_1, P_2 of X going to the node Q of Y, so $(f_*\mathcal{O}_X)_Q$ needs two generators as an $\mathcal{O}_Y$-module, hence it cannot be locally free.

Example 9.7.2. The result of (9.7) also fails if Y is regular of dimension >1. For example, let $Y = \mathbf{A}^2$, and let X be obtained by blowing up a point. Then X and Y are both nonsingular, and X dominates Y, but f is not flat, because the dimension of the fibre over the blown-up point is too big (9.5).

Proposition 9.8. *Let Y be a regular, integral scheme of dimension* 1, *let $P \in Y$ be a closed point, and let $X \subseteq \mathbf{P}^n_{Y-P}$ be a closed subscheme which is flat over $Y - P$. Then there exists a unique closed subscheme $\bar{X} \subseteq \mathbf{P}^n_Y$, flat over Y, whose restriction to $\mathbf{P}^n_{Y-P}$ is X.*

PROOF. Take $\bar{X}$ to be the scheme-theoretic closure of X in $\mathbf{P}^n_Y$ (II, Ex. 3.11d). Then the associated points of $\bar{X}$ are just those of X, so by (9.7), $\bar{X}$ is flat over Y. Furthermore, $\bar{X}$ is unique, because any other extension of X to $\mathbf{P}^n_Y$ would have some associated points mapping to P.

Remark 9.8.1. This proposition says that we can "pass to the limit," when we have a flat family of closed subschemes of $\mathbf{P}^n$ over a punctured curve. Hence it implies that "the Hilbert scheme is proper." The Hilbert scheme is a

scheme H which parametrizes all closed subschemes of $\mathbf{P}^n_k$. It has the property that to give a closed subscheme $X \subseteq \mathbf{P}^n_T$, flat over T, for any scheme T, is equivalent to giving a morphism $\varphi: T \to H$. Here, naturally, for any $t \in T$, $\varphi(t)$ is the point of H corresponding to the fibre $X_t \subseteq \mathbf{P}^n_{k(t)}$.

Now once one knows that the Hilbert scheme exists (see Grothendieck [5, exp. 221]) then the question of its properness can be decided using the valuative criterion of properness (II, 4.7). And the result just proved is the essential point needed to show that each connected component of H is proper over k.

Example 9.8.2. Even though the dimension of the fibres is constant in a flat family, we cannot expect properties such as "irreducible" or "reduced" to be preserved in a flat family. Take for example, the families given in (II, 3.3.1) and (II, 3.3.2). In each case the total space X is integral, the base Y is a nonsingular curve and the morphism $f: X \to Y$ is surjective, so the family is flat. Also most fibres are integral in both families. However, the special fibre in one is a doubled line (not reduced), and the special fibre in the other is two lines (not irreducible).

Example 9.8.3 (Projection from a Point). We get some new insight into the geometric process of projection from a point (I, Ex. 3.14) using (9.8). Let $P = (0,0,\ldots,0,1) \in \mathbf{P}^{n+1}$, and consider the projection $\varphi: \mathbf{P}^{n+1} - \{P\} \to \mathbf{P}^n$, which is defined by $(x_0,\ldots,x_{n+1}) \mapsto (x_0,\ldots,x_n)$. For each $a \in k$, $a \neq 0$, consider the automorphism σ_a of $\mathbf{P}^{n+1}$ defined by $(x_0,\ldots,x_{n+1}) \mapsto (x_0,\ldots,x_n,ax_{n+1})$. Now let X_1 be a closed subscheme of $\mathbf{P}^{n+1}$, not containing P. For each $a \neq 0$, let $X_a = \sigma_a(X_1)$. Then the X_a form a flat family parametrized by $\mathbf{A}^1 - \{0\}$. It is flat, because the X_a are all isomorphic as abstract schemes, and in fact, the whole family is isomorphic to $X_1 \times (\mathbf{A}^1 - \{0\})$ if we forget the embedding in $\mathbf{P}^{n+1}$.

Now according to (9.8) this family extends uniquely to a flat family defined over all of $\mathbf{A}^1$, and clearly the fibre X_0 over 0 agrees, at least set-theoretically, with the projection $\varphi(X_1)$ of X_1. Thus we see that there is a flat family over $\mathbf{A}^1$, whose fibres for all $a \neq 0$ are isomorphic to X_1, and whose fibre at 0 is some scheme with the same underlying space as $\varphi(X_1)$.

Example 9.8.4. We will now calculate the flat family just described in the special case where X_1 is a twisted cubic curve in $\mathbf{P}^3$, φ is a projection to $\mathbf{P}^2$, and $\varphi(X_1)$ is a nodal cubic curve in $\mathbf{P}^2$. The remarkable result of this calculation is that the special fibre X_0 of our flat family consists of the curve $\varphi(X_1)$ together with some nilpotent elements at the double point! We say that X_0 is a scheme with an embedded point. It seems as if the scheme X_0 is retaining the information that it is the limit of a family of space curves, by having these nilpotent elements which point out of the plane. In particular, X_0 is not a closed subscheme of $\mathbf{P}^2$ (Fig. 11).

Figure 11. A flat family of subschemes of $\mathbf{P}^3$.

Now for the calculation. We are just interested in what happens near the double point, so we will use affine coordinates x, y in $\mathbf{A}^2$ and x, y, z in $\mathbf{A}^3$. Let X_1 be given by the parametric equations

$$\begin{cases} x = t^2 - 1 \\ y = t^3 - t \\ z = t. \end{cases}$$

Then since $t = z$, $t^2 = x + 1$, $t^3 = y + z$, we recognize this as a twisted cubic curve in $\mathbf{A}^3$ (I, Ex. 1.2).

Now for any $a \neq 0$, the scheme X_a is given by

$$\begin{cases} x = t^2 - 1 \\ y = t^3 - t \\ z = at. \end{cases}$$

To get the ideal $I \subseteq k[a,x,y,z]$ of the total family $\bar{X}$ extended over all of $\mathbf{A}^1$, we eliminate t from the parametric equations, and make sure a is not a zero divisor in $k[a,x,y,z]/I$, so that $\bar{X}$ will be flat. We find

$$I = (a^2(x + 1) - z^2, a^3y + a^2z - z^3, xz - ay, y^2 - x^2(x + 1)).$$

From this, setting $a = 0$, we obtain the ideal $I_0 \subseteq k[x,y,z]$ of X_0, which is

$$I_0 = (z^2, z^3, xz, y^2 - x^2(x + 1)).$$

So we see that X_0 is a scheme with support equal to the nodal cubic curve $y^2 = x^2(x + 1)$. At any point where $x \neq 0$, we get z in the ideal, so X_0 is reduced there. But in the local ring at the node (0,0,0), we have the element z with $z^2 = 0$, a nonzero nilpotent element.

So here we have an example of a flat family of curves, whose general member is nonsingular, but whose special member is singular, with an embedded point. See also (9.10.1) and (IV, Ex. 3.5).

Example 9.8.5 (Algebraic Families of Divisors). Let X be a scheme of finite type over an algebraically closed field k, let T be a nonsingular curve over k, and let D be an effective Cartier divisor on $X \times T$ (II, §6). Then we can think of D as a closed subscheme of $X \times T$, which is locally described on a small open set U as the zeros of a single element $f \in \Gamma(U, \mathcal{O}_U)$ such that f is not a zero divisor. For any closed point $t \in T$, let $X_t (\cong X)$ be the fibre of $X \times T$ over t. We say that the intersection divisor $D_t = D.X_t$ *is defined* if at every point of X_t, the image $\bar{f} \in \Gamma(U \cap X_t, \mathcal{O}_{X_t})$ of a local equation f of D is not a zero divisor. In that case the covering $\{U \cap X_t\}$ and the elements $\bar{f}$ define a Cartier divisor D_t on X_t. If D_t is defined for all t, we say that the divisors $\{D_t | t \in T\}$ form an *algebraic family of divisors on X parametrized by T.*

This definition, which is natural in the context of Cartier divisors, is connected with flatness in the following way: the original Cartier divisor D, considered as a scheme over T, is flat over T if and only if $D_t = D.X_t$ is defined for each $t \in T$. Indeed, let $x \in D$ be any point, let $A = \mathcal{O}_{x, X \times T}$ be the local ring of x on $X \times T$, let $f \in A$ be a local equation for D, let $p_1(x) = t$, and let $u \in \mathcal{O}_{t,T}$ be a uniformizing parameter. Then $D.X_t$ is defined at x if and only if $\bar{f} \in A/uA$ is not a zero divisor. Since u is automatically not a zero divisor in A, this is equivalent to saying that (u, f) is a regular sequence (II, §8) in A. On the other hand, D is flat over T at x if and only if $\mathcal{O}_{x,D}$ is flat over $\mathcal{O}_{t,T}$. By (9.1.3) this is equivalent to $\mathcal{O}_{x,D}$ being torsion-free, i.e., u not being a zero divisor in $\mathcal{O}_{x,D}$. But $\mathcal{O}_{x,D} \cong A/fA$, so this says that (f, u) is a regular sequence in A. Since the property of being a regular sequence is independent of the order of the sequence (Matsumura [2, Th. 28, p. 102]), the two conditions are equivalent.

Theorem 9.9. *Let T be an integral noetherian scheme. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme. For each point $t \in T$, we consider the Hilbert polynomial $P_t \in \mathbf{Q}[z]$ of the fibre X_t considered as a closed subscheme of $\mathbf{P}^n_{k(t)}$. Then X is flat over T if and only if the Hilbert polynomial P_t is independent of t.*

Proof. Recall that the Hilbert polynomial was defined in (I, §7), and computed another way in (Ex. 5.2). We will use the defining property that

$$P_t(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))$$

for all $m \gg 0$.

First we generalize, replacing $\mathcal{O}_X$ by any coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_T$, and using the Hilbert polynomial of $\mathcal{F}_t$. Thus we may assume $X = \mathbf{P}^n_T$. Second, the question is local on T. In fact, by comparing any point to the generic point, we see that it is sufficient to consider the case $T = \operatorname{Spec} A$, with A a *local* noetherian ring.

So now let $T = \operatorname{Spec} A$ with A a local noetherian domain, let $X = \mathbf{P}^n_T$, and let $\mathcal{F}$ be a coherent sheaf on X. We will show that the following conditions are equivalent:

(i) $\mathcal{F}$ is flat over T;

(ii) $H^0(X, \mathcal{F}(m))$ is a free A-module of finite rank, for all $m \gg 0$;

(iii) the Hilbert polynomial P_t of $\mathscr{F}_t$ on $X_t = \mathbf{P}^n_{k(t)}$ is independent of t, for any $t \in T$.

(i) $\Rightarrow$ (ii). We compute $H^i(X,\mathscr{F}(m))$ by Čech cohomology using the standard open affine cover $\mathfrak{U}$ of X. Then

$$H^i(X,\mathscr{F}(m)) = h^i(C^{\cdot}(\mathfrak{U},\mathscr{F}(m))).$$

Since $\mathscr{F}$ is flat, each term $C^i(\mathfrak{U},\mathscr{F}(m))$ of the Čech complex is a flat A-module. On the other hand, if $m \gg 0$, then $H^i(X,\mathscr{F}(m)) = 0$ for $i > 0$, by (5.2). Thus the complex $C^{\cdot}(\mathfrak{U},\mathscr{F}(m))$ is a resolution of the A-module $H^0(X,\mathscr{F}(m))$: we have an exact sequence

$$0 \to H^0(X,\mathscr{F}(m)) \to C^0(\mathfrak{U},\mathscr{F}(m)) \to C^1(\mathfrak{U},\mathscr{F}(m)) \to \ldots \to C^n(\mathfrak{U},\mathscr{F}(m)) \to 0.$$

Splitting this into short exact sequences, using (9.1Ae) and the fact that the C^i are all flat, we conclude that $H^0(X,\mathscr{F}(m))$ is a flat A-module. But it is also finitely generated (5.2), and hence free of finite rank by (9.1Af).

(ii) $\Rightarrow$ (i). Let $S = A[x_0, \ldots, x_n]$, and let M be the graded S-module

$$M = \bigoplus_{m \geq m_0} H^0(X,\mathscr{F}(m)),$$

where m_0 is chosen large enough so that the $H^0(X,\mathscr{F}(m))$ are all free for $m \geq m_0$. Then $\mathscr{F} = \tilde{M}$ by (II, 5.15). Note that M is the same as $\Gamma_*(\mathscr{F})$ in degrees $m \geq m_0$, so $\tilde{M} = \Gamma_*(\mathscr{F})^{\sim}$. Since M is a free (and hence flat) A-module, we see that $\mathscr{F}$ is flat over A (9.1.1).

(ii) $\Rightarrow$ (iii). It will be enough to show that

$$P_t(m) = \operatorname{rank}_A H^0(X,\mathscr{F}(m))$$

for $m \gg 0$. To prove this we will show, for any $t \in T$, that

$$H^0(X_t,\mathscr{F}_t(m)) \cong H^0(X,\mathscr{F}(m)) \otimes_A k(t)$$

for all $m \gg 0$.

First we let $T' = \operatorname{Spec} A_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal corresponding to t, and we make the flat base extension $T' \to T$. Thus by (9.3) we reduce to the case where t is the closed point of T. Denote the closed fibre X_t by X_0, $\mathscr{F}_t$ by $\mathscr{F}_0$, and $k(t)$ by k. Take a presentation of k over A,

$$A^q \to A \to k \to 0.$$

Then we get an exact sequence of sheaves on X,

$$\mathscr{F}^q \to \mathscr{F} \to \mathscr{F}_0 \to 0.$$

Now by (Ex. 5.10) for $m \gg 0$ we get an exact sequence

$$H^0(X,\mathscr{F}(m)^q) \to H^0(X,\mathscr{F}(m)) \to H^0(X_0,\mathscr{F}_0(m)) \to 0.$$

On the other hand, we can tensor the sequence $A^q \to A \to k \to 0$ with $H^0(X,\mathscr{F}(m))$. Comparing, we deduce that

$$H^0(X_0,\mathscr{F}_0(m)) \cong H^0(X,\mathscr{F}(m)) \otimes_A k$$

for all $m \gg 0$, as required.

(iii) $\Rightarrow$ (ii). According to (II, 8.9) we can check the freeness of $H^0(X, \mathscr{F}(m))$ by comparing its rank at the generic point and the closed point of T. Hence the argument of (ii) $\Rightarrow$ (iii) above is reversible.

Corollary 9.10. *Let T be a connected noetherian scheme, and let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme which is flat over T. For any $t \in T$, let X_t be the fibre, considered as a closed subscheme of $\mathbf{P}^n_{k(t)}$. Then the dimension of X_t, the degree of X_t, and the arithmetic genus of X_t are all independent of t.*

PROOF. By base extension to the irreducible components of T with their reduced induced structure, we reduce to the case T integral. Now the result follows from the theorem and the facts (I, §7) and (Ex. 5.3) that

$$\dim X_t = \deg P_t,$$
$$\deg X_t = (r!) \cdot (\text{leading coefficient of } P_t),$$

where $r = \dim X$, and

$$p_a(X_t) = (-1)^r(P_t(0) - 1).$$

Definition. Let k be an algebraically closed field, let $f: X \to T$ be a surjective map of varieties over k, and assume that for each closed point $t \in T$, we have

(1) $f^{-1}(t)$ is irreducible, and
(2) if $\mathfrak{m}_t \subseteq \mathcal{O}_{t,T}$ is the maximal ideal, and if $\zeta \in f^{-1}(t)$ is the generic point, then $f^{\#}\mathfrak{m}_t$ generates the maximal ideal $\mathfrak{m}_\zeta \subseteq \mathcal{O}_{\zeta,X}$.

Under these circumstances, we let $X_{(t)}$ be the variety $f^{-1}(t)$ (with the reduced induced structure) and we say that the $X_{(t)}$ form an *algebraic family of varieties*, parametrized by T. The second condition is necessary to be sure that $X_{(t)}$ occurs with "multiplicity one" in the family. It is equivalent to saying that the scheme-theoretic fibre X_t is reduced at its generic point.

Example 9.10.1. In the flat family of (9.8.4), if we take the fibres with their reduced induced structures, we get an algebraic family of varieties $X_{(t)}$ parametrized by $\mathbf{A}^1$. For $t \neq 0$ it is a nonsingular rational curve, and for $t = 0$ it is the plane nodal cubic curve. Note that the arithmetic genus is not constant in this family: $p_a(X_{(t)}) = 0$ for $t \neq 0$ and $p_a(X_{(0)}) = 1$. This accounts for the appearance of nilpotent elements in the scheme-theoretic fibre X_0, since in a flat family of schemes p_a is constant by (9.10). The embedded point at 0 alters the constant term of the Hilbert polynomial so that we get $p_a(X_0) = 0$.

Theorem 9.11. *Let $X_{(t)}$ be an algebraic family of normal varieties parametrized by a nonsingular curve T over an algebraically closed field k. Then $X_{(t)}$ is a flat family of schemes.*

PROOF. Let $f: X \to T$ be the defining morphism of the family. Then f is a flat morphism by (9.7). So we have only to show that for each closed point $t \in T$, the scheme-theoretic fibre X_t coincides with the variety $X_{(t)}$. In other words, we must show that X_t is reduced. For any point $x \in X$, let $A_{\mathfrak{p}} = \mathcal{O}_{x,X}$ be its local ring, let $f(x) = t$, and denote also by t a uniformizing parameter in the local ring $\mathcal{O}_{t,T}$. Then A/tA is the local ring of x on X_t. By hypothesis X_t is irreducible, so t has a unique minimal prime ideal $\mathfrak{p}$ in A. Furthermore, t generates the maximal ideal of the local ring of the generic point of X_t on X, which says that t generates the maximal ideal of $A_{\mathfrak{p}}$. Finally, the local ring of x on $X_{(t)}$ is $A/\mathfrak{p}$, so our hypothesis says that $A/\mathfrak{p}$ is normal. Now our result is a consequence of the following lemma, which tells us that $\mathfrak{p} = tA$, so $X_{(t)} = X_t$.

Lemma 9.12 (Lemma of Hironaka [1]). *Let A be a local noetherian domain, which is a localization of an algebra of finite type over a field k. Let $t \in A$, and assume*

(1) *tA has only one minimal associated prime ideal $\mathfrak{p}$,*
(2) *t generates the maximal ideal of $A_{\mathfrak{p}}$,*
(3) *$A/\mathfrak{p}$ is normal.*

Then $\mathfrak{p} = tA$ and A is normal.

PROOF. Let $\tilde{A}$ be the normalization of A. Then $\tilde{A}$ is a finitely generated A-module by (I, 3.9A). We will show that the maps

$$\varphi: A/tA \to \tilde{A}/t\tilde{A}$$

and

$$\psi: A/tA \to A/\mathfrak{p}$$

are both isomorphisms.

First we localize at $\mathfrak{p}$. Then ψ is an isomorphism by hypothesis. Therefore $A_{\mathfrak{p}}$ is a discrete valuation ring, hence normal. So $\tilde{A}_{\mathfrak{p}} = A_{\mathfrak{p}}$ and φ is also an isomorphism.

Now suppose that at least one of φ, ψ is not an isomorphism. Then, after localizing A at a suitable prime ideal, we may assume that φ and ψ are isomorphisms at every localization $A_{\mathfrak{q}}$ with $\mathfrak{q} \neq \mathfrak{m}$, but that at least one of φ, ψ is not an isomorphism at $\mathfrak{m}$. By the previous step, we have $\mathfrak{p} < \mathfrak{m}$, so $\dim A \geqslant 2$. Now $\tilde{A}$ is normal of dimension $\geqslant 2$, so it has depth $\geqslant 2$ (II, 8.22A), so $\tilde{A}/t\tilde{A}$ has depth $\geqslant 1$. Therefore it does not have $\mathfrak{m}$ as an associated prime. On the other hand, $\tilde{A}/t\tilde{A}$ agrees with $A/\mathfrak{p}$ outside of $\mathfrak{m}$, so we conclude that $\tilde{A}/t\tilde{A}$ is an integral domain. Thus we have a natural map $(A/tA)_{\text{red}} \to \tilde{A}/t\tilde{A}$. But $(A/tA)_{\text{red}} \cong A/\mathfrak{p}$ since ψ is an isomorphism outside $\mathfrak{m}$. Thus $\tilde{A}/t\tilde{A}$ is a finitely generated $(A/\mathfrak{p})$-module with the same quotient field. Since $A/\mathfrak{p}$ is normal by hypothesis, we conclude that $\tilde{A}/t\tilde{A} \cong A/\mathfrak{p}$. Therefore φ is surjective, so we can write $\tilde{A} = A + t\tilde{A}$. By Nakayama's lemma, this implies that $A = \tilde{A}$. Thus $A/tA \cong A/\mathfrak{p}$ and both φ and ψ are isomorphisms. But this is a contradiction, so we conclude that φ and ψ were already isomorphisms on the original ring A before localization.

To conclude, we find that $\mathfrak{p} = tA$ because ψ is an isomorphism. Since φ is an isomorphism, we have $\tilde{A} = A + t\tilde{A}$, so by Nakayama's lemma as before, we find that $A = \tilde{A}$, so A is normal.

Corollary 9.13 (Igusa [1]). *Let $X_{(t)}$ be an algebraic family of normal varieties in $\mathbf{P}^n_k$, parametrized by a variety T. Then the Hilbert polynomial of $X_{(t)}$, and hence also the arithmetic genus $p_a(X_{(t)})$, are independent of t.*

PROOF. Any two closed points of T lie in the image of a morphism $g: T' \to T$, where T' is a nonsingular curve, or can be connected by a finite number of such curves, so by base extension, we reduce to the case where T is a nonsingular curve. Then the result follows from (9.10) and (9.11).

Example 9.13.1 (Infinitesimal Deformations). Now that we have seen that flatness is a natural condition for algebraic families of varieties, we come to an important nonclassical example of flatness in the category of schemes. Let X_0 be a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the *ring of dual numbers* over k. An *infinitesimal deformation* of X_0 is a scheme X', flat over D, and such that $X \otimes_D k \cong X_0$.

These arise geometrically in the following way. If $f: X \to T$ is any flat family, having a point $t \in T$ with $X_t \cong X_0$, we say that X is a (*global*) *deformation* of X_0. Now given an element of the Zariski tangent space of T at t, we obtain a morphism $\operatorname{Spec} D \to T$ (II, Ex. 2.8). Then by base extension we obtain an X' flat over $\operatorname{Spec} D$ with closed fibre X_0. Thus the study of the infinitesimal deformations of X_0 ultimately will help in the study of global deformations.

Example 9.13.2. Continuing the same ideas, it is often possible to classify the infinitesimal deformations of a scheme X. In particular, if X is nonsingular over an algebraically closed field k, we will show that the set of infinitesimal deformations of X, up to isomorphism, is in one-to-one correspondence with the elements of the cohomology group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf.

Indeed, given X' flat over D, we consider the exact sequence

$$0 \to k \xrightarrow{t} D \to k \to 0$$

of D-modules. By flatness, we obtain an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. Thus X' is an infinitesimal extension of the scheme X by the sheaf $\mathcal{O}_X$, in the sense of (II, Ex. 8.7). Conversely, such an extension gives X' flat over D. Now these extensions are classified by $H^1(X, \mathcal{T}_X)$ by (Ex. 4.10).

Remark 9.13.3. There is a whole subject called *deformation theory* devoted to the study of deformations of a given scheme (or variety) X_0 over a field k.

It is closely related to the moduli problem. There one attempts to classify all varieties, and put them into algebraic families. Here we study only those that are close to a given one X_0.

Deformation theory is one area of algebraic geometry where the influence of schemes has been enormous. Because even if one's primary interest is in a variety X_0 over k, by working in the category of schemes, one can consider flat families over arbitrary Artin rings with residue field k, whose closed fibre is X_0. Taking the limit of Artin rings, one can study flat families over a complete local ring. Both of these types of families are intermediate between X_0 itself and a global deformation $f: X \to T$ where T is another variety. Thus they form a powerful tool for studying all deformations of X_0. For some references on deformation theory see Schlessinger [1], or Morrow and Kodaira [1, Ch. 4].

Exercises

9.1. A flat morphism $f: X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X$, $f(U)$ is open in Y. [*Hint*: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

9.2. Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

9.3. Some examples of flatness and nonflatness.

(a) If $f: X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

(b) Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x,y]$ and $X = \operatorname{Spec} k[x,y,z,w]/(z,w) \cap (x+z, y+w)$.

(c) Again let $Y = \operatorname{Spec} k[x,y]$, but take $X = \operatorname{Spec} k[x,y,z,w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{\text{red}} \cong Y$, X has no embedded points, but that f is not flat.

9.4. *Open Nature of Flatness.* Let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X \mid f \text{ is flat at } x\}$ is an open subset of X (possibly empty)—see Grothendieck [EGA IV_3, 11.1.1].

9.5. *Very Flat Families.* For any closed subscheme $X \subseteq \mathbf{P}^n$, we denote by $C(X) \subseteq \mathbf{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k[x_0, \ldots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k[x_0, \ldots, x_{n+1}]$.

(a) Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbf{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbf{P}^{n+1}$.

(b) To remedy this situation, we make the following definition. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$, let $I_t \subseteq S_t = k(t)[x_0, \ldots, x_n]$ be the homogeneous ideal of X_t in $\mathbf{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is *very flat* if for all $d \geqslant 0$,

$$\dim_{k(t)} (S_t/I_t)_d$$

is independent of t. Here $(\ \ )_d$ means the homogeneous part of degree d.

(c) If $\{X_t\}$ is a very flat family in $\mathbf{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbf{P}^{n+1}$, and hence flat.

(d) If $\{X_{(t)}\}$ is an algebraic family of projectively normal varieties in $\mathbf{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_{(t)}\}$ is a very flat family of schemes.

9.6. Let $Y \subseteq \mathbf{P}^n$ be a nonsingular variety of dimension $\geqslant 2$ over an algebraically closed field k. Suppose $\mathbf{P}^{n-1}$ is a hyperplane in $\mathbf{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbf{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbf{P}^n$ if and only if Y' is a complete intersection in $\mathbf{P}^{n-1}$. [*Hint*: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

9.7. Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an *infinitesimal deformation* of Y *as a closed subscheme* of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathcal{N}_{Y/X})$, where

$$\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

***9.8.** Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\mathrm{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

9.9. A k-algebra A is said to be *rigid* if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = \mathrm{Spec}\, k[x,y,z,w]/(x,y) \cap (z,w)$, and show that A is rigid. This corresponds to two planes in $\mathbf{A}^4$ which meet at a point.

9.10. A scheme X_0 over a field k is *rigid* if it has no infinitesimal deformations.

(a) Show that $\mathbf{P}^1_k$ is rigid, using (9.13.2).

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f: X \to \mathbf{A}^1$ over k algebraically closed, such that $X_0 \cong \mathbf{P}^1_k$, but there is no open neighborhood U of 0 in $\mathbf{A}^1$ for which $f^{-1}(U) \cong U \times \mathbf{P}^1$.

*(c) Show, however, that one can trivialize a global deformation of $\mathbf{P}^1$ after a flat base extension, in the following sense: let $f: X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbf{P}^1_k$. Then there exists a nonsingular curve T', and a flat morphism $g: T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f': X' \to T'$ is isomorphic to $\mathbf{P}^1_{T'} \to T'$.

9.11. Let Y be a nonsingular curve of degree d in $\mathbf{P}^n_k$, over an algebraically closed field k. Show that

$$0 \leqslant p_a(Y) \leqslant \tfrac{1}{2}(d-1)(d-2).$$

[*Hint*: Compare Y to a suitable projection of Y into $\mathbf{P}^2$, as in (9.8.3) and (9.8.4).]

10 Smooth Morphisms

The notion of smooth morphism is a relative version of the notion of nonsingular variety over a field. In this section we will give some basic results about smooth morphisms. As an application, we give Kleiman's elegant proof of the characteristic 0 Bertini theorem. For further information about smooth and étale morphisms, see Altman and Kleiman [1, Ch. VI, VII], Matsumura [2, Ch. 11], and Grothendieck [SGA 1, exp. I, II, III].

For simplicity, we assume that all schemes in this section are of finite type over a field k.

Definition. A morphism $f: X \to Y$ of schemes of finite type over k is *smooth of relative dimension* n if:

(1) f is flat;
(2) if $X' \subseteq X$ and $Y' \subseteq Y$ are irreducible components such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y' + n$;
(3) for each point $x \in X$ (closed or not),

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n.$$

Example 10.0.1. For any Y, $\mathbf{A}^n_Y$ and $\mathbf{P}^n_Y$ are smooth of relative dimension n over Y.

Example 10.0.2. If X is integral, then condition (3) is equivalent to saying $\Omega_{X/Y}$ is locally free on X of rank n (II, 8.9).

Example 10.0.3. If $Y = \operatorname{Spec} k$ and k is algebraically closed, then X is smooth over k if and only if X is regular of dimension n. In particular, if X is irreducible and separated over k, then it is smooth if and only if it is a nonsingular variety. Cf. (II, 8.8) and (II, 8.15).

Proposition 10.1.

(a) *An open immersion is smooth of relative dimension* 0.

(b) Base change. *If $f: X \to Y$ is smooth of relative dimension n, and $g: Y' \to Y$ is any morphism, then the morphism $f': X' \to Y'$ obtained by base extension is also smooth of relative dimension n.*

(c) Composition. *If $f: X \to Y$ is smooth of relative dimension n, and $g: Y \to Z$ is smooth of relative dimension m, then $g \circ f: X \to Z$ is smooth of relative dimension $n + m$.*

(d) Product. *If X and Y are smooth over Z, of relative dimensions n and m, respectively, then $X \times_Z Y$ is smooth over Z of relative dimension $n + m$.*

PROOFS.

(a) is trivial.

(b) f' is flat by (9.2). According to (9.6), the condition (2) in the definition of smoothness is equivalent to saying that every irreducible component of every fibre X_y of f has dimension n. This condition is preserved under base extension. Finally, $\Omega_{X/Y}$ is stable under base extension (II, 8.10), so the number $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x))$ is also. Hence f' is smooth.

(c) $g \circ f$ is flat by (9.2). If $X' \subseteq X$, $Y' \subseteq Y$, and $Z' \subseteq Z$ are irreducible components such that $f(X') \subseteq Y'$ and $g(Y') \subseteq Z'$, then clearly $\dim X' = \dim Z' + n + m$ by hypothesis. For the last condition, we use the exact sequence of (II, 8.11)

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Tensoring with $k(x)$ we have

$$f^*\Omega_{Y/Z} \otimes k(x) \to \Omega_{X/Z} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0.$$

Now the first has dimension m, and the last has dimension n, by hypothesis. So the middle one has dimension $\leqslant n + m$.

On the other hand, let $z = g(f(x))$. Then

$$\Omega_{X/Z} \otimes k(x) = \Omega_{X_z/k(z)} \otimes k(x),$$

since relative differentials commute with base extension. Let X' be an irreducible component of X_z containing x, with its reduced induced structure. Then we have a surjective map

$$\Omega_{X_z/k(z)} \otimes k(x) \to \Omega_{X'/k(z)} \otimes k(x) \to 0$$

by (II, 8.12). But X' is an integral scheme of finite type over $k(z)$, of dimension $n + m$, by (9.6), so $\Omega_{X'/k(z)}$ is a coherent sheaf of rank $\geqslant n + m$ by (II, 8.6A). Hence it requires at least $n + m$ generators at every point, so

$$\dim_{k(x)}(\Omega_{X'/k(z)} \otimes k(x)) \geqslant n + m.$$

Combining our inequalities, we find that

$$\dim_{k(x)}(\Omega_{X/Z} \otimes k(x)) = n + m$$

as required.

(d) This statement is a consequence of (b) and (c) since we can factor into $X \times_Z Y \xrightarrow{p_2} Y \to Z$.

Theorem 10.2. *Let $f: X \to Y$ be a morphism of schemes of finite type over k. Then f is smooth of relative dimension n if and only if:*

(1) *f is flat; and*

(2) *for each point $y \in Y$, let $X_{\bar{y}} = X_y \otimes_{k(y)} k(y)^-$, where $k(y)^-$ is the algebraic closure of $k(y)$. Then $X_{\bar{y}}$ is equidimensional of dimension n and*

regular. (*We say "the fibres of f are* geometrically regular of equidimension n.")

PROOF. If f is smooth of relative dimension n, so is any base extension. In particular, X_y is smooth of relative dimension n over $k(y)^-$, so is regular (10.0.3).

Conversely, suppose (1) and (2) satisfied. Then f is flat by (1). From (2) we conclude that every irreducible component of X_y has dimension n, which gives condition (2) of the definition of smoothness by (9.6). Finally, since $k(y)^-$ is algebraically closed, regularity of X_y implies that $\Omega_{X_y/k(y)^-}$ is locally free of rank n (10.0.3). This in turn implies that $\Omega_{X_y/k(y)}$ is locally free of rank n (see e.g. Matsumura [2, (4.E), p. 29]), and so for any $x \in X$,

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = \dim_{k(x)}(\Omega_{X_y/k(y)} \otimes k(x)) = n$$

as required.

Next we will study when a morphism of nonsingular varieties is smooth. Recall (II, Ex. 2.8) that for a point x in a scheme X we define the *Zariski tangent space* T_x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. If $f: X \to Y$ is a morphism, and $y = f(x)$, then there is a natural induced mapping on the tangent spaces

$$T_f: T_x \to T_y \otimes_{k(y)} k(x).$$

Before stating our criterion, we recall an algebraic fact.

Lemma 10.3.A. *Let $A \to B$ be a local homomorphism of local noetherian rings. Let M be a finitely generated B-module, and let $t \in A$ be an element that is not a zero divisor. Then M is flat over A if and only if:*

(1) *t is not a zero divisor in M; and*
(2) *M/tM is flat over A/tA.*

PROOF. This is a special case of the "Local criterion of flatness." See Bourbaki [1, III, §5] or Altman and Kleiman [1, V, §3].

Proposition 10.4. *Let $f: X \to Y$ be a morphism of nonsingular varieties over an algebraically closed field k. Let $n = \dim X - \dim Y$. Then the following conditions are equivalent:*

(i) *f is smooth of relative dimension n;*
(ii) *$\Omega_{X/Y}$ is locally free of rank n on X;*
(iii) *for every closed point $x \in X$, the induced map on the Zariski tangent spaces $T_f: T_x \to T_y$ is surjective.*

PROOF.

(i) $\Rightarrow$ (ii) follows from the definition of smoothness, since X is integral (10.0.2).

(ii) $\Rightarrow$ (iii). From the exact sequence of (II, 8.11), tensoring with $k(x)$, we have

$$f^*\Omega_{Y/k} \otimes k(x) \to \Omega_{X/k} \otimes k(x) \to \Omega_{X/Y} \otimes k(x) \to 0.$$

Now X and Y are both smooth over k, so the dimensions of these vector spaces are equal to $\dim Y$, $\dim X$, and n respectively. Therefore the map on the left is injective. But for a closed point x, $k(x) \cong k$, so using (II, 8.7) we see that this map is just the natural map

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$$

induced by f. Taking dual vector spaces over k, we find that T_f is surjective.

(iii) $\Rightarrow$ (i). First we show f is flat. For this, it is enough to show that $\mathcal{O}_x$ is flat over $\mathcal{O}_y$ for every *closed* point $x \in X$, where $y = f(x)$, by localization of flatness. Since X and Y are nonsingular, these are both regular local rings. Furthermore, since T_f is surjective, we have $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ injective as above. So let $t_1, \ldots, t_r$ be a regular system of parameters for $\mathcal{O}_y$. Then their images in $\mathcal{O}_x$ form part of a regular system of parameters of $\mathcal{O}_x$. Since $\mathcal{O}_x/(t_1, \ldots, t_r)$ is automatically flat over $\mathcal{O}_y/(t_1, \ldots, t_r) = k$, we can use (10.3A) to show by descending induction on i that $\mathcal{O}_x/(t_1, \ldots, t_i)$ is flat over $\mathcal{O}_y/(t_1, \ldots, t_i)$ for each i. In particular, for $i = 0$, $\mathcal{O}_x$ is flat over $\mathcal{O}_y$. Thus f is flat.

Now we can read the argument of (ii) $\Rightarrow$ (iii) backwards to conclude that

$$\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$$

for each closed point $x \in X$. On the other hand since f is flat, it is dominant, so for the generic point $\zeta \in X$, we have

$$\dim_{k(\zeta)}(\Omega_{X/Y} \otimes k(\zeta)) \geqslant n$$

by (II, 8.6A). We conclude that $\Omega_{X/Y}$ is coherent of rank $\geqslant n$, so it must be locally free of rank $= n$ by (II, 8.9). Therefore $\Omega_{X/Y} \otimes k(x)$ has dimension n at every point of X, so f is smooth of relative dimension n.

Next we will give some special results about smoothness which hold only in characteristic zero.

Lemma 10.5. *Let $f: X \to Y$ be a dominant morphism of integral schemes of finite type over an algebraically closed field k of characteristic 0. Then there is a nonempty open set $U \subseteq X$ such that $f: U \to Y$ is smooth.*

PROOF. Replacing X and Y by suitable open subsets, we may assume that they are both nonsingular varieties over k (II, 8.16). Next, since we are in characteristic 0, $K(X)$ is a separably generated field extension of $K(Y)$ (I, 4.8A). So by (II, 8.6A), $\Omega_{X/Y}$ is free of rank $n = \dim X - \dim Y$ at the generic point of X. Therefore it is locally free of rank n on some nonempty open set $U \subseteq X$. We conclude that $f: U \to Y$ is smooth by (10.4).

Example 10.5.1. Let k be an algebraically closed field of characteristic p, let $X = Y = \mathbf{P}^1_k$, and let $f: X \to Y$ be the Frobenius morphism (I, Ex. 3.2). Then f is not smooth on any open set. Indeed, since $d(t^p) = 0$, the natural map $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is the zero map, and so $\Omega_{X/Y} \cong \Omega_{X/k}$ is locally free of rank 1. But f has relative dimension 0, so it is nowhere smooth.

Proposition 10.6. *Let $f: X \to Y$ be a morphism of schemes of finite type over an algebraically closed field k of characteristic 0. For any r, let*

$$X_r = \{\text{closed points } x \in X \mid \text{rank } T_{f,x} \leqslant r\}.$$

Then

$$\dim \overline{f(X_r)} \leqslant r.$$

PROOF. Let Y' be any irreducible component of $\overline{f(X_r)}$, and let X' be an irreducible component of $\bar{X}_r$ which dominates Y'. We give X' and Y' their reduced induced structures, and consider the induced dominant morphism $f': X' \to Y'$. Then by (10.5) there is a nonempty open subset $U' \subseteq X'$ such that $f': U' \to Y'$ is smooth. Now let $x \in U' \cap X_r$, and consider the commutative diagram of maps of Zariski tangent spaces

$$\begin{tikzcd} T_{x,U'} \arrow[r] \arrow[d, "T_{f',x}"] & T_{x,X} \arrow[d, "T_{f,x}"] \\ T_{y,Y'} \arrow[r] & T_{y,Y} \end{tikzcd}$$

The horizontal arrows are injective, because U' and Y' are locally closed subschemes of X and Y, respectively. On the other hand, rank $T_{f,x} \leqslant r$ since $x \in X_r$, and $T_{f',x}$ is surjective because f' is smooth (10.4). We conclude that $\dim T_{y,Y'} \leqslant r$, and therefore $\dim Y' \leqslant r$.

Corollary 10.7 (Generic Smoothness). *Let $f: X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic 0, and assume that X is nonsingular. Then there is a nonempty open subset $V \subseteq Y$ such that $f: f^{-1}V \to V$ is smooth.*

PROOF. We may assume Y is nonsingular by (II, 8.16). Let $r = \dim Y$. Let $X_{r-1} \subseteq X$ be the subset defined in (10.6). Then $\dim \overline{f(X_{r-1})} \leqslant r - 1$ by (10.6), so removing it from Y, we may assume that rank $T_f \geqslant r$ for every closed point of X. But since Y is nonsingular of dimension r, this implies that T_f is surjective for every closed point of X. Hence f is smooth by (10.4).

Note that if the original f was not dominant, then $V \subseteq Y - \overline{f(X)}$, and $f^{-1}V$ will be empty.

For the next results, we recall the notion of a group variety (I, Ex. 3.21). A *group variety* G over an algebraically closed field k is a variety G, together

with morphisms $\mu: G \times G \to G$ and $\rho: G \to G$, such that the set $G(k)$ of k-rational points of G (which is just the set of all closed points of G, since k is algebraically closed) becomes a group under the operation induced by μ, with ρ giving the inverses.

We say that a group variety G *acts* on a variety X if we have a morphism $\theta: G \times X \to X$ which induces a homomorphism $G(k) \to \operatorname{Aut} X$ of groups.

A *homogeneous space* is a variety X, together with a group variety G acting on it, such that the group $G(k)$ acts transitively on the set $X(k)$ of k-rational points of X.

Remark 10.7.1. Any group variety is a homogeneous space if we let it act on itself by left multiplication.

Example 10.7.2. The projective space $\mathbf{P}_k^n$ is a homogeneous space for the action of $G = \mathrm{PGL}(n)$—cf. (II, 7.1.1).

Example 10.7.3. A homogeneous space is necessarily a nonsingular variety. Indeed, it has an open subset which is nonsingular by (II, 8.16). But we have a transitive group of automorphisms acting, so it is nonsingular everywhere.

Theorem 10.8 (Kleiman [3]). *Let X be a homogeneous space with group variety G over an algebraically closed field k of characteristic 0. Let $f: Y \to X$ and $g: Z \to X$ be morphisms of nonsingular varieties Y, Z to X. For any $\sigma \in G(k)$, let Y^σ be Y with the morphism $\sigma \circ f$ to X. Then there is a nonempty open subset $V \subseteq G$ such that for every $\sigma \in V(k)$, $Y^\sigma \times_X Z$ is nonsingular and either empty or of dimension exactly*

$$\dim Y + \dim Z - \dim X.$$

PROOF. First we consider the morphism

$$h: G \times Y \to X$$

defined by composing f with the group action $\theta: G \times X \to X$. Now G is nonsingular since it is a group variety (10.7.3), and Y is nonsingular by hypothesis, so $G \times Y$ is nonsingular by (10.1). Since char $k = 0$, we can apply generic smoothness (10.7) to h, and conclude that there is a nonempty open subset $U \subseteq X$ such that $h: h^{-1}(U) \to U$ is smooth. Now G acts on $G \times Y$ by left multiplication on G; G acts on X by θ, and these two actions are compatible with the morphism h, by construction. Therefore, for any $\sigma \in G(k)$, $h: h^{-1}(U^\sigma) \to U^\sigma$ is also smooth. Since the U^σ cover X, we conclude that h is smooth everywhere.

Next, we consider the fibred product

$$W = (G \times Y) \times_X Z,$$

with maps g' and h' to $G \times Y$ and Z as shown.

$$\begin{tikzcd} W \arrow[r, "h'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\ G \times Y \arrow[r, "h"] \arrow[d, "p_1"] & X \\ G & \end{tikzcd}$$

Since h is smooth, h' is also smooth by base extension (10.1). Since Z is nonsingular, it is smooth over k, so by composition (10.1), W is also smooth over k, so W is nonsingular.

Now we consider the morphism

$$q = p_1 \circ g' : W \to G.$$

Applying generic smoothness (10.7) again, we find there is a nonempty open subset $V \subseteq G$ such that $q : q^{-1}(V) \to V$ is smooth. Therefore, if $\sigma \in V(k)$ is any closed point, the fibre W_σ will be nonsingular. But W_σ is just $Y^\sigma \times_X Z$, so this is what we wanted to show. Note that W_σ may not be irreducible, but our result shows that each connected component is a nonsingular variety.

To find the dimension of W_σ, we first note that h is smooth of relative dimension

$$\dim G + \dim Y - \dim X.$$

Hence h' has the same relative dimension, and we see that

$$\dim W = \dim G + \dim Y - \dim X + \dim Z.$$

If W is nonempty, then q on $q^{-1}(V)$ has relative dimension equal to $\dim W - \dim G$, so for each σ,

$$\dim W_\sigma = \dim Y + \dim Z - \dim X.$$

Corollary 10.9 (Bertini). *Let X be a nonsingular projective variety over an algebraically closed field k of characteristic 0. Let $\mathfrak{d}$ be a linear system without base points. Then almost every element of $\mathfrak{d}$, considered as a closed subscheme of X, is nonsingular (but maybe reducible).*

PROOF. Let $f : X \to \mathbf{P}^n$ be the morphism to $\mathbf{P}^n$ determined by $\mathfrak{d}$ (II, 7.8.1). We consider $\mathbf{P}^n$ as a homogeneous space under the action of $G = \mathrm{PGL}(n)$ (10.7.2). We apply the theorem taking $g : H \to \mathbf{P}^n$ to be the inclusion map of a hyperplane $H \cong \mathbf{P}^{n-1}$. We conclude that for almost all $\sigma \in G(k)$, $X \times_{\mathbf{P}^n} H^\sigma = f^{-1}(H^\sigma)$ is nonsingular. But the divisors $f^{-1}(H^\sigma)$ are just the elements of the linear system $\mathfrak{d}$, by construction of f. Thus almost all elements of $\mathfrak{d}$ are nonsingular.

Remark 10.9.1. We will see later (Ex. 11.3) that if $\dim f(X) \geqslant 2$, then all the divisors in $\mathfrak{d}$ are connected. Hence almost all of them are irreducible and nonsingular.

Remark 10.9.2. The hypothesis "X projective" is not necessary if we talk about a finite-dimensional linear system $\mathfrak{d}$. In particular, if X was projective, and $\mathfrak{d}$ was a linear system with base points Σ, then by considering the base-point-free linear system $\mathfrak{d}$ on $X - \Sigma$ we obtain the more general statement that "a general member of $\mathfrak{d}$ can have singularities only at the base points."

Remark 10.9.3. This result fails in characteristic $p > 0$. For example, in (10.5.1) the morphism f corresponds to the one-dimensional linear system $\{pP | P \in \mathbf{P}^1\}$. Thus every divisor in $\mathfrak{d}$ is a point with multiplicity p.

Remark 10.9.4. Compare this result to the earlier Bertini theorem (II, 8.18).

Exercises

10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq \mathbf{A}^2_k$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

10.2. Let $f: X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f: f^{-1}(U) \to U$ is smooth.

10.3. A morphism $f: X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;
(ii) f is flat, and $\Omega_{X/Y} = 0$;
(iii) f is flat and unramified.

10.4. Show that a morphism $f: X \to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x \in X$, let $y = f(x)$. Let $\hat{\mathcal{O}}_x$ and $\hat{\mathcal{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x) \subseteq \hat{\mathcal{O}}_x$ and $k(y) \subseteq \hat{\mathcal{O}}_y$ so that $k(y) \subseteq k(x)$ via the natural map $\hat{\mathcal{O}}_y \to \hat{\mathcal{O}}_x$. Then our condition is that for every $x \in X$, $k(x)$ is a separable algebraic extension of $k(y)$, and the natural map

$$\hat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \hat{\mathcal{O}}_x$$

is an isomorphism.

10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f: U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathcal{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f: U \to X$ for which $f^*\mathcal{F}$ is a free $\mathcal{O}_U$-module, then $\mathcal{F}$ is locally free on X.

10.6. Let Y be the plane nodal cubic curve $y^2 = x^2(x + 1)$. Show that Y has a finite étale covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Fig. 12).

Figure 12. A finite étale covering.

10.7. (*Serre*). *A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbf{P}^2_k$ be the seven points of the projective plane over the prime field $\mathbf{F}_2 \subseteq k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

(a) $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbf{P}^2$.

(b) Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a 1-1 correspondence between $\mathfrak{d}$ and $\mathbf{P}^2$. Thus the singular points of elements of $\mathfrak{d}$ move all over.

10.8. *A linear system with moving singularities contained in the base locus* (*any characteristic*). In affine 3-space with coordinates x, y, z, let C be the conic $(x - 1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0,0,t)$ on the z-axis. Let Y_t be the closure in $\mathbf{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

10.9. Let $f: X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen–Macaulay, and that every fibre of f has dimension equal to $\dim X - \dim Y$. Then f is flat. [*Hint*: Imitate the proof of (10.4), using (II, 8.21A).]

11 The Theorem on Formal Functions

In this section we prove the so-called theorem on formal functions, and its important corollaries, Zariski's Main Theorem, and the Stein factorization theorem. The theorem itself compares the cohomology of the infinitesimal neighborhoods of a fibre of a projective morphism to the stalk of the higher direct image sheaves. While the corollaries use only the case $i = 0$ of the theorem (which could be stated without cohomology), the proof is by descending induction on i, and thus makes essential use of the cohomological machinery. Zariski's first proof [2] of his "Main Theorem" was by an entirely different method which did not use cohomology.

Let $f: X \to Y$ be a projective morphism of noetherian schemes, let $\mathscr{F}$ be a coherent sheaf on X, and let $y \in Y$ be a point. For each $n \geqslant 1$ we

define

$$X_n = X \times_Y \operatorname{Spec} \mathcal{O}_y/\mathfrak{m}_y^n.$$

Then for $n = 1$, we get the fibre X_y, and for $n > 1$, we get a scheme with nilpotent elements having the same underlying space as X_y. It is a kind of "thickened fibre" of X over the point y.

$$\begin{tikzcd} X_n \arrow[r, "v"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}_y/\mathfrak{m}_y^n \arrow[r] & Y \end{tikzcd}$$

Let $\mathcal{F}_n = v^*\mathcal{F}$, where $v: X_n \to X$ is the natural map. Then by (9.3.1) we have natural maps, for each n,

$$R^if_*(\mathcal{F}) \otimes \mathcal{O}_y/\mathfrak{m}_y^n \to R^if'_*(\mathcal{F}_n).$$

Since $\operatorname{Spec} \mathcal{O}_y/\mathfrak{m}_y^n$ is affine, concentrated at one point, the right-hand side is just the group $H^i(X_n,\mathcal{F}_n)$ by (8.5). As n varies, both sides form inverse systems (see (II, §9) for generalities on inverse systems and inverse limits). Thus we can take inverse limits and get a natural map

$$R^if_*(\mathcal{F})_y^\wedge \to \varprojlim H^i(X_n,\mathcal{F}_n).$$

Theorem 11.1 (Theorem on Formal Functions). *Let $f: X \to Y$ be a projective morphism of noetherian schemes, let $\mathcal{F}$ be a coherent sheaf on X, and let $y \in Y$. Then the natural map*

$$R^if_*(\mathcal{F})_y^\wedge \to \varprojlim H^i(X_n,\mathcal{F}_n)$$

is an isomorphism, for all $i \geqslant 0$.

PROOF. As a first step, we embed X in some projective space $\mathbf{P}_Y^N$, and consider $\mathcal{F}$ as a coherent sheaf on $\mathbf{P}_Y^N$. Thus we reduce to the case $X = \mathbf{P}_Y^N$.

Next we let $A = \mathcal{O}_y$, and make the flat base extension $\operatorname{Spec} A \to Y$. Thus, using (9.3), we reduce to the case where Y is affine, equal to the spectrum of a local noetherian ring A, and y is the closed point of Y. Then using (8.5) again, we can restate our result as an isomorphism of A-modules,

$$H^i(X,\mathcal{F})^\wedge \xrightarrow{\sim} \varprojlim H^i(X_n,\mathcal{F}_n).$$

Now suppose $\mathcal{F}$ is a sheaf of the form $\mathcal{O}(q)$ on $X = \mathbf{P}_A^N$, for some $q \in \mathbf{Z}$. Then $\mathcal{F}_n$ is just $\mathcal{O}(q)$ on $X_n = \mathbf{P}_{A/\mathfrak{m}^n}^N$. So by the explicit calculations of (5.1) we see that

$$H^i(X_n,\mathcal{F}_n) \cong H^i(X,\mathcal{F}) \otimes_A A/\mathfrak{m}^n$$

for each n. Therefore by definition of completion, we have

$$H^i(X,\mathcal{F})^\wedge \cong \varprojlim H^i(X_n,\mathcal{F}_n)$$

in this case. Clearly the same calculation holds for any finite direct sum of sheaves of the form $\mathcal{O}(q_i)$.

We will now prove the theorem for an arbitrary coherent sheaf $\mathscr{F}$ on X, by descending induction on i. For $i > N$, both sides are 0, because X can be covered by $N + 1$ open affine subsets (Ex. 4.8). So we assume the theorem has been proved for $i + 1$, and for all coherent sheaves.

Given $\mathscr{F}$ coherent on X, it follows from (II, 5.18) that we can write $\mathscr{F}$ as a quotient of a sheaf $\mathscr{E}$ which is a finite direct sum of sheaves $\mathcal{O}(q_i)$ for suitable $q_i \in \mathbf{Z}$. Let $\mathscr{R}$ be the kernel:

$$0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0. \tag{1}$$

Now unfortunately, tensoring with $\mathcal{O}_{X_n}$ is not an exact functor—it is only right exact, so we have an exact sequence

$$\mathscr{R}_n \to \mathscr{E}_n \to \mathscr{F}_n \to 0$$

of sheaves on X_n for each n. We introduce the image $\mathscr{T}_n$ and the kernel $\mathscr{S}_n$ of the map $\mathscr{R}_n \to \mathscr{E}_n$, so that we have exact sequences

$$0 \to \mathscr{S}_n \to \mathscr{R}_n \to \mathscr{T}_n \to 0 \tag{2}$$

and

$$0 \to \mathscr{T}_n \to \mathscr{E}_n \to \mathscr{F}_n \to 0. \tag{3}$$

We now consider the following diagram:

$$\begin{tikzcd}
H^i(X,\mathscr{R})\hat{} \arrow[r] \arrow[d, "\alpha_1"] & H^i(X,\mathscr{E})\hat{} \arrow[r] \arrow[dd, "\alpha_2"] & H^i(X,\mathscr{F})\hat{} \arrow[r] \arrow[dd, "\alpha_3"] & H^{i+1}(X,\mathscr{R})\hat{} \arrow[r] \arrow[d, "\alpha_4"] & H^{i+1}(X,\mathscr{E})\hat{} \arrow[dd, "\alpha_5"] \\
\varprojlim H^i(X_n,\mathscr{R}_n) \arrow[d, "\beta_1"] & & & \varprojlim H^{i+1}(X_n,\mathscr{R}_n) \arrow[d, "\beta_2"] & \\
\varprojlim H^i(X_n,\mathscr{T}_n) \arrow[r] & \varprojlim H^i(X_n,\mathscr{E}_n) \arrow[r] & \varprojlim H^i(X_n,\mathscr{F}_n) \arrow[r] & \varprojlim H^{i+1}(X_n,\mathscr{T}_n) \arrow[r] & \varprojlim H^{i+1}(X_n,\mathscr{E}_n)
\end{tikzcd}$$

The top row comes from the cohomology sequence of (1) by completion. Since they are all finitely generated A-modules (5.2), completion is an exact functor (II, 9.3A). The bottom row comes from the cohomology sequence of (3) by taking inverse limits. These groups are all finitely generated $A/\mathfrak{m}^n$-modules, and so satisfy d.c.c. for submodules. Therefore the inverse systems all satisfy the Mittag–Leffler condition (II, 9.1.2), and so the bottom row is exact (II, 9.1). The vertical arrows $\alpha_1, \ldots, \alpha_5$ are the maps of the theorem. We have α_2 an isomorphism because $\mathscr{E}$ is a sum of sheaves $\mathcal{O}(q_i)$, and α_4 and α_5 are isomorphisms by the induction hypothesis. Finally, β_1 and β_2 are maps induced from the sequence (2). We will show below that β_1 and β_2 are isomorphisms.

Admitting this for the moment, it follows from the subtle 5-lemma that α_3 is surjective. But this is then true for any coherent sheaf on X, so α_1 must also be surjective. This in turn implies α_3 is an isomorphism, which is what we want.

It remains to prove that β_1 and β_2 are isomorphisms. Taking the cohomology sequence of (2), and passing to the inverse limit, using (II, 9.1) again, it will be sufficient to show that

$$\varprojlim H^i(X_n, \mathscr{S}_n) = 0$$

for all $i \geqslant 0$. To accomplish this, we will show that for any n, there is an $n' > n$ such that the map of sheaves $\mathscr{S}_{n'} \to \mathscr{S}_n$ is the zero map. By quasi-compactness, the question is local on X, so we may assume that X is affine, $X = \operatorname{Spec} B$. We denote by R, E, S_n the B-modules corresponding to the sheaves $\mathscr{R}, \mathscr{E}, \mathscr{S}_n$, and we denote by $\mathfrak{a}$ the ideal $\mathfrak{m}B$.

Recall that R is a submodule of E, and that

$$S_n = \ker(R/\mathfrak{a}^n R \to E/\mathfrak{a}^n E).$$

Thus

$$S_n = (R \cap \mathfrak{a}^n E)/\mathfrak{a}^n R.$$

But by Krull's theorem (3.1A), the $\mathfrak{a}$-adic topology on R is induced by the $\mathfrak{a}$-adic topology on E. In other words, for any n, there is an $n' > n$ such that

$$R \cap \mathfrak{a}^{n'} E \subseteq \mathfrak{a}^n R.$$

In that case the map $S_{n'} \to S_n$ is zero. q.e.d.

Remark 11.1.1. This theorem is proved more generally for a proper morphism in Grothendieck [EGA III, §4].

Remark 11.1.2. Many applications of this theorem use only the case $i = 0$. In that case the right-hand side is equal to $\Gamma(\hat{X}, \hat{\mathscr{F}})$, where $\hat{X}$ is the formal completion of X along X_y, and $\hat{\mathscr{F}} = \mathscr{F} \otimes \mathscr{O}_{\hat{X}}$ (II, 9.2). In particular, if $\mathscr{F} = \mathscr{O}_X$, we have $\Gamma(\hat{X}, \mathscr{O}_{\hat{X}})$, which is the ring of *formal-regular functions* (also called *holomorphic functions*) on X along X_y. Hence the name of the theorem.

Remark 11.1.3. One can also introduce the cohomology $H^i(\hat{X}, \hat{\mathscr{F}})$ of $\hat{\mathscr{F}}$ on the formal scheme $\hat{X}$, and prove that it is isomorphic to the two other quantities in the theorem [EGA III, §4].

Corollary 11.2. *Let $f: X \to Y$ be a projective morphism of noetherian schemes, and let $r = \max\{\dim X_y \mid y \in Y\}$. Then $R^i f_*(\mathscr{F}) = 0$ for all $i > r$, and for all coherent sheaves $\mathscr{F}$ on X.*

PROOF. For any $y \in Y$, X_n is a scheme whose underlying topological space is the same as X_y. Hence

$$H^i(X_n, \mathscr{F}_n) = 0$$

for $i > r$ by (2.7). If follows that $R^i f_*(\mathscr{F})_y^{\wedge} = 0$ for all $y \in Y$, $i > r$, and therefore since $R^i f_*(\mathscr{F})$ is coherent (8.8) it must be 0.

Corollary 11.3. *Let $f: X \to Y$ be a projective morphism of noetherian schemes, and assume that $f_* \mathscr{O}_X = \mathscr{O}_Y$. Then $f^{-1}(y)$ is connected, for every $y \in Y$.*

PROOF. Suppose to the contrary that $f^{-1}(y) = X' \cup X''$, where X' and X'' are disjoint closed subsets. Then for each n, we would have

$$H^0(X_n, \mathcal{O}_{X_n}) = H^0(X'_n, \mathcal{O}_{X_n}) \oplus H^0(X''_n, \mathcal{O}_{X_n}).$$

By the theorem, we have

$$\hat{\mathcal{O}}_y = (f_*\mathcal{O}_X)\hat{}_y = \varprojlim H^0(X_n, \mathcal{O}_{X_n}).$$

Therefore $\hat{\mathcal{O}}_y = A' \oplus A''$, where

$$A' = \varprojlim H^0(X'_n, \mathcal{O}_{X_n})$$

and

$$A'' = \varprojlim H^0(X''_n, \mathcal{O}_{X_n}).$$

But this is impossible, because a local ring cannot be a direct sum of two other rings. Indeed, let e', e'' be the unit elements of A' and A''. Then $e' + e'' = 1$ in $\hat{\mathcal{O}}_y$. On the other hand, $e'e'' = 0$, so e', e'' are nonunits, hence contained in the maximal ideal of $\hat{\mathcal{O}}_y$, so their sum cannot be 1 (cf. (II, Ex. 2.19)).

Corollary 11.4 (Zariski's Main Theorem). *Let $f: X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume that Y is normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.* (*See also* (V, 5.2).)

PROOF. By the previous result, we have only to verify that $f_*\mathcal{O}_X = \mathcal{O}_Y$. The question is local on Y, so we may assume Y is affine, equal to Spec A. Then $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, so $B = \Gamma(Y, f_*\mathcal{O}_X)$ is a finitely generated A-module. But A and B are integral domains with the same quotient field, and A is integrally closed, so we must have $A = B$. Thus $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Corollary 11.5 (Stein Factorization). *Let $f: X \to Y$ be a projective morphism of noetherian schemes. Then one can factor f into $g \circ f'$, where $f': X \to Y'$ is a projective morphism with connected fibres, and $g: Y' \to Y$ is a finite morphism.*

PROOF. Let $Y' = \mathbf{Spec}\, f_*\mathcal{O}_X$ (II, Ex. 5.17). Then since $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, the natural map $g: Y' \to Y$ is finite. On the other hand f clearly factors through g, so we get a morphism $f': X \to Y'$. Since g is separated, we conclude that f' is projective by (II, Ex. 4.9).

EXERCISES

11.1. Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbf{A}^n_k$, let $P = (0, \ldots, 0)$, let $U = X - P$, and let $f: U \to X$ be the inclusion. Then the fibres of f all have dimension 0, but $R^{n-1}f_*\mathcal{O}_U \neq 0$.

11.2. Show that a projective morphism with finite fibres (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

11.3. Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is *not composite with a pencil*, which means that if $f: X \to \mathbf{P}^n_k$ is the morphism

determined by $\mathfrak{d}$, then $\dim f(X) \geqslant 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1). [*Hints*: Use (11.5), (Ex. 5.7) and (7.9).]

11.4. *Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbf{P}_k^n$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subseteq T$, such that for all closed points $t \in U$, X_t is connected. Then prove that X_t is connected for all $t \in T$.

***11.5.** Let Y be a hypersurface in $X = \mathbf{P}_k^N$ with $N \geqslant 4$. Let $\hat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is an isomorphism. [*Hint*: Use (II, Ex. 9.6), and then study the maps $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ for each n using (Ex. 4.6) and (Ex. 5.5).]

11.6. Again let Y be a hypersurface in $X = \mathbf{P}_k^N$, this time with $N \geqslant 2$.

(a) If $\mathscr{F}$ is a locally free sheaf on X, show that the natural map

$$H^0(X,\mathscr{F}) \to H^0(\hat{X},\hat{\mathscr{F}})$$

is an isomorphism.

(b) Show that the following conditions are equivalent:

(i) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there exists a coherent sheaf $\mathscr{F}$ on X such that $\mathfrak{F} \cong \hat{\mathscr{F}}$ (i.e., $\mathfrak{F}$ is *algebraizable*);

(ii) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geqslant n_0$.

[*Hint*: For (ii) $\Rightarrow$ (i), show that one can find sheaves $\mathscr{E}_0, \mathscr{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\hat{\mathscr{E}}_1 \to \hat{\mathscr{E}}_0 \to \mathfrak{F} \to 0$ on $\hat{X}$. Then apply (a) to the sheaf $\mathscr{H}om(\mathscr{E}_1,\mathscr{E}_0)$.]

(c) Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \hat{X}$ is an isomorphism.

Note. In fact, (i) and (ii) always hold if $N \geqslant 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [SGA 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbf{P}_k^N$ with $N \geqslant 4$, then $\operatorname{Pic} Y \cong \mathbf{Z}$, and it is generated by $\mathcal{O}_Y(1)$. See Hartshorne [5, Ch. IV] for more details.

11.7. Now let Y be a curve in $X = \mathbf{P}_k^2$.

(a) Use the method of (Ex. 11.5) to show that $\operatorname{Pic} X \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.

(b) Conclude that there is an invertible sheaf Ω on $\hat{X}$ which is not algebraizable.

(c) Conclude also that there is a locally free sheaf $\mathfrak{F}$ on $\hat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections. Cf. (II, 9.9.1).

11.8. Let $f: X \to Y$ be a projective morphism, let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y,\mathscr{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathscr{F})$ is 0 in a neighborhood of y.

12 The Semicontinuity Theorem

In this section we consider a projective morphism $f: X \to Y$ and a coherent sheaf $\mathscr{F}$ on X, flat over Y. We ask, how does the cohomology along the fibre $H^i(X_y,\mathscr{F}_y)$ vary as a function of $y \in Y$? Our technique is to find some

relation between these groups and the sheaves $R^i f_*(\mathscr{F})$. The main results are the semicontinuity theorem (12.8), and the theorem on cohomology and base change (12.11).

Since the question is local on Y, we will usually restrict our attention to the case $Y = \operatorname{Spec} A$ is affine. Then we compare the A-modules $H^i(X,\mathscr{F})$ and $H^i(X_y,\mathscr{F}_y)$. Using (9.4), the cohomology of the fibre is equal to $H^i(X,\mathscr{F} \otimes k(y))$. Grothendieck's idea is to study more generally $H^i(X,\mathscr{F} \otimes_A M)$ for any A-module M, and consider it as a functor on A-modules.

Definition. Let A be a noetherian ring, let $Y = \operatorname{Spec} A$, let $f: X \to Y$ be a projective morphism, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. (This data will remain fixed throughout this section.) Then for each A-module M, define

$$T^i(M) = H^i(X,\mathscr{F} \otimes_A M)$$

for all $i \geqslant 0$.

Proposition 12.1. *Each T^i is an additive, covariant functor from A-modules to A-modules which is exact in the middle. The collection $(T^i)_{i \geqslant 0}$ forms a δ-functor* (§1).

PROOF. Clearly each T^i is an additive, covariant functor. Since $\mathscr{F}$ is flat over Y, for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of A-modules, we get an exact sequence

$$0 \to \mathscr{F} \otimes M' \to \mathscr{F} \otimes M \to \mathscr{F} \otimes M'' \to 0$$

of sheaves on X. Now the long exact sequence of cohomology shows that each T^i is exact in the middle, and that together they form a δ-functor.

We reduce the calculation of the functors T^i to a process involving only A-modules, by the following result.

Proposition 12.2. *With the hypotheses above, there exists a complex $L^{\cdot}$ of finitely generated free A-modules, bounded above* (i.e., $L^n = 0$ *for* $n \gg 0$), *such that*

$$T^i(M) \cong h^i(L^{\cdot} \otimes_A M)$$

for any A-module M, any $i \geqslant 0$, and this gives an isomorphism of δ-functors.

PROOF. For any A-module M, the sheaf $\mathscr{F} \otimes_A M$ is quasi-coherent on X, so we can use Čech cohomology to compute $H^i(X,\mathscr{F} \otimes_A M)$. Let $\mathfrak{U} = (U_i)$ be an open cover of X. Let $C^{\cdot} = C^{\cdot}(\mathfrak{U},\mathscr{F})$ be the Čech complex of $\mathscr{F}$ (§4). Then for any $i_0, \ldots, i_p$, we have

$$\Gamma(U_{i_0,\ldots,i_p},\mathscr{F} \otimes_A M) = \Gamma(U_{i_0,\ldots,i_p},\mathscr{F}) \otimes_A M,$$

so

$$C^{\cdot}(\mathfrak{U},\mathscr{F} \otimes_A M) = C^{\cdot} \otimes_A M.$$

Hence we have

$$T^i(M) = h^i(C^{\cdot} \otimes_A M)$$

for each M.

This is a step in the right direction, since $C^{\cdot}$ is a bounded complex of A-modules. However, the C^i will almost never be finitely generated A-modules. But the complex $C^{\cdot}$ does have the good properties that for each i, C^i is a *flat* A-module (since $\mathscr{F}$ is flat over Y), and for each i, $h^i(C^{\cdot}) = H^i(X,\mathscr{F})$ is a finitely generated A-module, since $\mathscr{F}$ is coherent and f is projective. Now the result of the proposition is a consequence of the following algebraic lemma.

Lemma 12.3. *Let A be a noetherian ring, and let $C^{\cdot}$ be a complex of A-modules, bounded above, such that for each i, $h^i(C^{\cdot})$ is a finitely generated A-module. Then there is a complex $L^{\cdot}$ of finitely generated free A-modules, also bounded above, and a morphism of complexes $g:L^{\cdot} \to C^{\cdot}$, such that the induced map $h^i(L^{\cdot}) \to h^i(C^{\cdot})$ is an isomorphism for all i. Furthermore, if each C^i is a flat A-module, then the map*

$$h^i(L^{\cdot} \otimes M) \to h^i(C^{\cdot} \otimes M)$$

is an isomorphism for any A-module M.

PROOF. First we fix our notation. For any complex $N^{\cdot}$, we let

$$Z^n(N^{\cdot}) = \ker(d^n: N^n \to N^{n+1})$$

and

$$B^n(N^{\cdot}) = \operatorname{im}(d^{n-1}: N^{n-1} \to N^n).$$

Thus we have

$$h^n(N^{\cdot}) = Z^n(N^{\cdot})/B^n(N^{\cdot}).$$

Now for large n, we have $C^n = 0$, so we define $L^n = 0$ there also. Suppose inductively that the complex $L^{\cdot}$ and the morphism of complexes $g:L^{\cdot} \to C^{\cdot}$ has been defined in degrees $i > n$ in such a way that

$$h^i(L^{\cdot}) \xrightarrow{\sim} h^i(C^{\cdot}) \text{ for all } i > n + 1, \text{ and} \tag{1}$$

$$Z^{n+1}(L^{\cdot}) \to h^{n+1}(C^{\cdot}) \text{ is surjective.} \tag{2}$$

Then we will construct L^n, $d:L^n \to L^{n+1}$, and $g:L^n \to C^n$ to propagate these properties one step further.

Choose a set of generators $\bar{x}_1, \ldots, \bar{x}_r$ of $h^n(C^{\cdot})$, which is possible since $h^n(C^{\cdot})$ is finitely generated. Lift them to a set of elements $x_1, \ldots, x_r \in Z^n(C^{\cdot})$. On the other hand, let $y_{r+1}, \ldots, y_s$ be a set of generators of $g^{-1}(B^{n+1}(C^{\cdot}))$, which is a submodule of L^{n+1}, hence finitely generated. Let $g(y_i) = \bar{y}_i \in B^{n+1}(C^{\cdot})$, and lift the $\bar{y}_i$ to a set of elements $x_{r+1}, \ldots, x_s$ of C^n.

Now take L^n to be a free A-module on s generators $e_1, \ldots, e_s$. Define $d:L^n \to L^{n+1}$ by $de_i = 0$ for $i = 1, \ldots, r$, and $de_i = y_i$ for $i = r + 1, \ldots, s$. Define $g:L^n \to C^n$ by $ge_i = x_i$ for all i.

Then one checks easily that g commutes with d, that $h^{n+1}(L^{\cdot}) \to h^{n+1}(C^{\cdot})$ is an isomorphism, and that $Z^n(L^{\cdot}) \to h^n(C^{\cdot})$ is surjective. So inductively, we construct the complex $L^{\cdot}$ required.

Now suppose that each C^i is a flat A-module. Then we will prove, by descending induction on i, that

$$h^i(L^{\cdot} \otimes M) \to h^i(C^{\cdot} \otimes M)$$

is an isomorphism for all A-modules M. For $i \gg 0$, both L^i and C^i are 0, so both sides are 0. So suppose this is true for $i + 1$. It is sufficient to prove the result for finitely generated A-modules, because any A-module is a direct limit of finitely generated ones, and both $\otimes$ and h^i commute with direct limits. So given M finitely generated, write it as a quotient of a free finitely generated A-module E, and let R be the kernel:

$$0 \to R \to E \to M \to 0.$$

Since each C^i is flat by hypothesis, and each L^i is flat, because it is free, we get an exact, commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & L^{\cdot} \otimes R \arrow[r] \arrow[d] & L^{\cdot} \otimes E \arrow[r] \arrow[d] & L^{\cdot} \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{\cdot} \otimes R \arrow[r] & C^{\cdot} \otimes E \arrow[r] & C^{\cdot} \otimes M \arrow[r] & 0.
\end{tikzcd}$$

Applying h^i, we get a commutative diagram of long exact sequences. Since the result holds for $i + 1$ by induction, and for E, since E is free, and $h^i(L^{\cdot}) \to h^i(C^{\cdot})$ is an isomorphism, the result for any M follows from the subtle 5-lemma.

Now we will study conditions under which one of the functors T^i is left exact, right exact, or exact. For any complex $N^{\cdot}$, we define

$$W^i(N^{\cdot}) = \operatorname{coker}(d^{i-1}: N^{i-1} \to N^i)$$

so that we have an exact sequence

$$0 \to h^i(N^{\cdot}) \to W^i(N^{\cdot}) \to N^{i+1}.$$

Proposition 12.4. *The following conditions are equivalent:*

(i) *T^i is left exact;*
(ii) *$W^i = W^i(L^{\cdot})$ is a projective A-module;*
(iii) *there is a finitely generated A-module Q, such that*

$$T^i(M) = \operatorname{Hom}_A(Q, M)$$

for all M.

Furthermore the Q in (iii) *is unique.*

PROOF. Since tensor product is right exact, we have

$$W^i(L^{\cdot} \otimes M) = W^i(L^{\cdot}) \otimes M,$$

for any A-module M. We will write simply W^i for $W^i(L^{\cdot})$. Hence

$$T^i(M) = \ker(W^i \otimes M \to L^{i+1} \otimes M).$$

Let $0 \to M' \to M$ be an inclusion. Then we obtain an exact, commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
0 \arrow[r] & T^i(M') \arrow[r] \arrow[d, "\alpha"] & W^i \otimes M' \arrow[r] \arrow[d, "\beta"] & L^{i+1} \otimes M' \arrow[d] \\
0 \arrow[r] & T^i(M) \arrow[r] & W^i \otimes M \arrow[r] & L^{i+1} \otimes M.
\end{tikzcd}$$

The third vertical arrow is injective, since L^{i+1} is free. A simple diagram chase shows that α is injective if and only if β is. Since this is true for any choice of $0 \to M' \to M$, we see that T^i is left exact if and only if W^i is flat. (Recall that in any case T^i is exact in the middle (12.1).) But since W^i is finitely generated, this is equivalent to W^i being projective (9.1A). This shows (i) $\Leftrightarrow$ (ii).

(iii) $\Rightarrow$ (i) is obvious.

To prove (ii) $\Rightarrow$ (iii), let $\check{L}^{i+1}$ and $\check{W}^i$ be the dual projective modules. Define

$$Q = \operatorname{coker}(\check{L}^{i+1} \to \check{W}^i).$$

Then for every A-module M, we have

$$0 \to \operatorname{Hom}(Q,M) \to \operatorname{Hom}(\check{W}^i,M) \to \operatorname{Hom}(\check{L}^{i+1},M).$$

But the last two groups are $W^i \otimes M$, and $L^{i+1} \otimes M$, respectively, so $\operatorname{Hom}(Q,M) = T^i(M)$.

To see the uniqueness of Q, let Q' be another module such that $T^i(M) = \operatorname{Hom}(Q',M)$ for all M. Then

$$\operatorname{Hom}(Q,M) = \operatorname{Hom}(Q',M)$$

for all M. In particular, the elements

$$1 \in \operatorname{Hom}(Q,Q) = \operatorname{Hom}(Q',Q)$$

and

$$1' \in \operatorname{Hom}(Q',Q') = \operatorname{Hom}(Q,Q')$$

give isomorphisms of Q and Q', inverse to each other, and canonically defined.

Remark 12.4.1. There is a general theorem to the effect that any left-exact functor T on A-modules, which commutes with direct sums, is of the form $\mathrm{Hom}(Q,\cdot)$ for some A-module Q. But even if T takes finitely generated modules into finitely generated modules, Q need not be finitely generated. Thus the fact that our Q in (iii) above is finitely generated is a strong fact about the functor T^i.

For example, let A be a noetherian ring with infinitely many maximal ideals $\mathfrak{m}_i$. Let $Q = \sum A/\mathfrak{m}_i$, and let T be the functor $\mathrm{Hom}(Q,\cdot)$. Then Q is not finitely generated, but for any finitely generated A-module M, $T(M)$ is finitely generated, because $\mathrm{Hom}(A/\mathfrak{m}_i,M) \neq 0$ if and only if $\mathfrak{m}_i \in \mathrm{Ass}\, M$, which is a finite set.

Proposition 12.5. *For any M, there is a natural map*

$$\varphi: T^i(A) \otimes M \to T^i(M).$$

Furthermore, the following conditions are equivalent:

(i) *T^i is right exact;*
(ii) *φ is an isomorphism for all M;*
(iii) *φ is surjective for all M.*

PROOF. Since T^i is a functor, we have a natural map, for any M,

$$M = \mathrm{Hom}(A,M) \xrightarrow{\psi} \mathrm{Hom}(T^i(A),T^i(M)).$$

This gives φ, by setting

$$\varphi(\textstyle\sum a_i \otimes m_i) = \sum \psi(m_i)a_i.$$

Since T^i and $\otimes$ commute with direct limits, it will be sufficient to consider finitely generated A-modules M. Write

$$A^r \to A^s \to M \to 0.$$

Then we have a diagram

$$\begin{tikzcd}
T^i(A) \otimes A^r \arrow[r] \arrow[d] & T^i(A) \otimes A^s \arrow[r] \arrow[d] & T^i(A) \otimes M \arrow[r] \arrow[d, "\varphi"] & 0 \\
T^i(A^r) \arrow[r] & T^i(A^s) \arrow[r] & T^i(M) &
\end{tikzcd}$$

where the bottom row is not necessarily exact. The first two vertical arrows are isomorphisms. Thus, if T^i is right exact, φ is an isomorphism. This proves (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is obvious, so we have only to prove (iii) $\Rightarrow$ (i). We must show if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of A-modules, then

$$T^i(M') \to T^i(M) \to T^i(M'') \to 0$$

is exact. By (12.1) it is exact in the middle, so we have only to show that $T^i(M) \to T^i(M'')$ is surjective. This follows from the diagram

$$\begin{array}{ccccc} T^i(A) \otimes M & \longrightarrow & T^i(A) \otimes M'' & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi(M)} & & \downarrow{\scriptstyle \varphi(M'')} & & \\ T^i(M) & \longrightarrow & T^i(M'') & & \end{array}$$

and the fact that $\varphi(M'')$ is surjective.

Corollary 12.6. *The following conditions are equivalent:*

(i) *T^i is exact;*
(ii) *T^i is right exact, and $T^i(A)$ is a projective A-module.*

PROOF. In any case, T^i is right exact, so by (12.5) we have $T^i(M) \cong T^i(A) \otimes M$ for all A-modules M. Therefore T^i is exact if and only if $T^i(A)$ is flat. But $T^i(A)$ is a finitely generated A-module, so this is equivalent to being locally free (9.1A), hence projective.

Now we wish to localize the above discussion. For any point $y \in Y = \operatorname{Spec} A$, we denote by T^i_y the restriction of the functor T^i to the category of $A_\mathfrak{p}$-modules, where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to y. Then we say "T^i is left exact at y" to mean T^i_y is left exact, and similarly for right exact, or exact. Note that for any $A_\mathfrak{p}$-module N, $T^i_y(N) = H^i(L^\cdot_\mathfrak{p} \otimes N)$. Also note that T^i is left exact if and only if it is left exact at all points $y \in Y$; similarly for right exact, or exact. Finally, since cohomology commutes with flat base extension (9.3), we see that T^i_y is the functor T^i associated with the morphism $f': X' \to Y'$ obtained from f by the flat base extension $Y' = \operatorname{Spec} \mathcal{O}_y \to Y$. So we can apply the results (12.4), (12.5), (12.6) locally to each T^i_y.

Proposition 12.7. *If T^i is left exact (respectively, right exact, exact) at some point $y_0 \in Y$, then the same is true for all points y in a suitable open neighborhood U of y_0.*

PROOF. T^i is left exact at y_0 if and only if $\tilde{W}^i_{y_0}$ is free, by (12.4). But since $\tilde{W}^i$ is a coherent sheaf on Y, this implies that $\tilde{W}^i$ is locally free in some neighborhood U of y_0, and so T^i is left exact at all points of U.

T^i is right exact at a point y if and only if T^{i+1} is left exact there, by the long exact sequence (12.1). So the second statement follows from the first, applied to T^{i+1}.

T^i is exact at a point if and only if it is both left exact and right exact, so the third statement is the conjunction of the first two.

Definition. Let Y be a topological space. A function $\varphi: Y \to \mathbf{Z}$ is *upper semicontinuous* if for each $y \in Y$, there is an open neighborhood U of y, such that for all $y' \in U$, $\varphi(y') \leqslant \varphi(y)$. Intuitively, this means that φ may get bigger at special points.

Remark 12.7.1. A function $\varphi: Y \to \mathbf{Z}$ is upper semicontinuous if and only if for each $n \in \mathbf{Z}$, the set $\{y \in Y | \varphi(y) \geqslant n\}$ is a closed subset of Y.

Example 12.7.2. Let Y be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf on Y. Then the function

$$\varphi(y) = \dim_{k(y)}(\mathscr{F} \otimes k(y))$$

is upper semicontinuous. Indeed, by Nakayama's lemma, $\varphi(y)$ is equal to the minimal number of generators of the $\mathcal{O}_y$-module $\mathscr{F}_y$. But if $s_1, \ldots, s_r \in \mathscr{F}_y$ form a minimal set of generators, they extend to sections of $\mathscr{F}$ in some neighborhood of y, and they generate $\mathscr{F}$ in some neighborhood, because $\mathscr{F}$ is coherent. So if y' is in that neighborhood, then $\varphi(y')$, which is the minimal number of generators of $\mathscr{F}_{y'}$, is $\leqslant r = \varphi(y)$.

Theorem 12.8 (Semicontinuity). *Let $f: X \to Y$ be a projective morphism of noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then for each $i \geqslant 0$, the function*

$$h^i(y, \mathscr{F}) = \dim_{k(y)} H^i(X_y, \mathscr{F}_y)$$

is an upper semicontinuous function on Y.

PROOF. The question is local on Y, so we may assume $Y = \operatorname{Spec} A$ is affine, with A noetherian. Thus we can apply the earlier results of this section. By (9.4) we have

$$h^i(y, \mathscr{F}) = \dim_{k(y)} T^i(k(y)).$$

As in the proof of (12.4), we have

$$T^i(k(y)) = \ker(W^i \otimes k(y) \to L^{i+1} \otimes k(y)).$$

On the other hand, there is an exact sequence

$$W^i \to L^{i+1} \to W^{i+1} \to 0,$$

so tensoring with $k(y)$, we obtain a four-term exact sequence

$$0 \to T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0.$$

Therefore, counting dimensions, we have

$$h^i(y, \mathscr{F}) = \dim_{k(y)} W^i \otimes k(y) + \dim_{k(y)} W^{i+1} \otimes k(y) - \dim_{k(y)} L^{i+1} \otimes k(y).$$

Now W^i and W^{i+1} are finitely generated A-modules, so by (12.7.2) the first two terms of this sum are upper semicontinuous functions of y. But L^{i+1} is a free A-module, so the last term is a constant function of y. Combining, we see that $h^i(y, \mathscr{F})$ is upper semicontinuous.

Corollary 12.9 (Grauert). *With the same hypotheses as the theorem, suppose furthermore that Y is integral, and that for some i, the function $h^i(y, \mathscr{F})$ is constant on Y. Then $R^i f_*(\mathscr{F})$ is locally free on Y, and for every y the natural*

map

$$R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$$

is an isomorphism.

PROOF. As above, we may assume that Y is affine. Using the expression for $h^i(y,\mathscr{F})$ in the proof of the theorem, we conclude that the functions dim $W^i \otimes k(y)$ and dim $W^{i+1} \otimes k(y)$ must both be constant. But this implies (II, 8.9) that $\tilde{W}^i$ and $\tilde{W}^{i+1}$ are both locally free sheaves on Y. So by (12.4), T^i and T^{i+1} are both left exact, so T^i is exact, so by (12.6), $T^i(A)$ is a projective A-module. But $R^i f_*(\mathscr{F})$ is just $T^i(A)^\sim$, so it is a locally free sheaf. Finally by (12.5) we see that

$$R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$$

is an isomorphism for all $y \in Y$.

Example 12.9.1. Let $\{X_t\}$ be a flat family of integral curves in $\mathbf{P}^n_k$, with k algebraically closed. Then for every closed point $t \in T$, $H^0(X_t, \mathcal{O}_{X_t}) = k$. On the other hand, the arithmetic genus $p_a = 1 - \chi(\mathcal{O}_{X_t})$ is constant, by (9.10). So we see that in this case the functions $h^0(t,\mathcal{O}_X)$ and $h^1(t,\mathcal{O}_X)$ are both constant on T.

Example 12.9.2. In the flat family of (9.8.4), we have $h^0(X_t,\mathcal{O}_{X_t}) = 1$ if $t \neq 0$, and 2 if $t = 0$, because of the nilpotent elements. On the other hand, $h^1(X_t,\mathcal{O}_{X_t}) = 0$ for $t \neq 0$, since X_t is rational, and $h^1(X_0,\mathcal{O}_{X_0}) = h^1(X_0,(\mathcal{O}_{X_0})_{\text{red}}) = 1$, since $(X_0)_{\text{red}}$ is a plane cubic curve. So in this case the functions h^0, h^1 both jump up at $t = 0$.

Example 12.9.3. If $\{X_t\}$ is an algebraic family of nonsingular projective varieties over $\mathbf{C}$, parametrized by a variety T, then the functions $h^i(X_t,\mathcal{O}_{X_t})$ are actually constant for all i. The proof of this result requires transcendental methods, namely the degeneration of the Hodge spectral sequence—cf. Deligne[4].

Now we wish to give some more precise information about when the map

$$T^i(A) \otimes k(y) \to T^i(k(y))$$

is an isomorphism. And here we will use a new ingredient in our proof, namely the theorem on formal functions (11.1).

Proposition 12.10. *Assume that for some i, y, the map*

$$\varphi: T^i(A) \otimes k(y) \to T^i(k(y))$$

is surjective. Then T^i is right exact at y (and conversely, by (12.5)*).*

PROOF. By making a flat base extension Spec $\mathcal{O}_y \to Y$ if necessary (9.3), we may assume that y is a closed point of Y; A is a local ring, with maximal ideal

$\mathfrak{m}$, and $k(y) = k = A/\mathfrak{m}$. By (12.5), it is sufficient to show that

$$\varphi(M): T^i(A) \otimes M \to T^i(M)$$

is surjective for all A-modules M. Since T^i and tensor product commute with direct limits, it is sufficient to consider finitely generated M.

First, we consider A-modules M of finite length, and we show that $\varphi(M)$ is surjective, by induction on the length of M. If the length is 1, then $M = k$, and $\varphi(k)$ is surjective by hypothesis. In general, write

$$0 \to M' \to M \to M'' \to 0,$$

where M' and M'' have length less than length M. Then using (12.1) we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} T^i(A) \otimes M' & \longrightarrow & T^i(A) \otimes M & \longrightarrow & T^i(A) \otimes M'' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ T^i(M') & \longrightarrow & T^i(M) & \longrightarrow & T^i(M'') & & \end{array}$$

The two outside vertical arrows are surjective, by the induction hypothesis, so the middle one is surjective also.

Now let M be any finitely generated A-module. For each n, $M/\mathfrak{m}^n M$ is a module of finite length, so that by the previous case,

$$\varphi_n: T^i(A) \otimes M/\mathfrak{m}^n M \to T^i(M/\mathfrak{m}^n M)$$

is surjective. Note that $\ker \varphi_n$ is an A-module of finite length, so the inverse system $(\ker \varphi_n)$ satisfies the Mittag–Leffler condition (II, 9.1.2). Hence by (II, 9.1) the map

$$\varprojlim \varphi_n: (T^i(A) \otimes M)^\wedge \to \varprojlim T^i(M/\mathfrak{m}^n M)$$

is also surjective. But by the theorem on formal functions (11.1), applied to the sheaf $\mathscr{F} \otimes_A M$ on X, the right hand side is just $T^i(M)^\wedge$. So we have a surjection

$$(T^i(A) \otimes M)^\wedge \to T^i(M)^\wedge.$$

Since completion is a faithful exact functor for finitely generated A-modules, it follows that

$$\varphi(M): T^i(A) \otimes M \to T^i(M)$$

is surjective, so we are done.

Combining this with our earlier results, we obtain the following theorem.

Theorem 12.11 (Cohomology and Base Change). *Let $f: X \to Y$ be a projective morphism of noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Let y be a point of Y. Then:*

(a) *if the natural map*

$$\varphi^i(y): R^i f_*(\mathscr{F}) \otimes k(y) \to H^i(X_y, \mathscr{F}_y)$$

is surjective, then it is an isomorphism, and the same is true for all y' in a suitable neighborhood of y;

(b) *Assume that $\varphi^i(y)$ is surjective. Then the following conditions are equivalent:*

(i) *$\varphi^{i-1}(y)$ is also surjective;*
(ii) *$R^i f_*(\mathscr{F})$ is locally free in a neighborhood of y.*

PROOF. (a) follows from (12.10), (12.7), and (12.5). (b) follows from (12.10), (12.6), and (12.5), using the fact that T^i is exact if and only if T^{i-1} and T^i are both right exact.

References for §12. The semicontinuity theorem was first proved by Grauert [1] in the complex-analytic case. These theorems in the algebraic case are due to Grothendieck [EGA III, 7.7]. Our proof follows the main ideas of Grothendieck's proof, with simplifications due to Mumford [5, II, §5].

EXERCISES

12.1. Let Y be a scheme of finite type over an algebraically closed field k. Show that the function

$$\varphi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$$

is upper semicontinuous on the set of closed points of Y.

12.2. Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbf{P}_k^n$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

12.3. Let $X_1 \subseteq \mathbf{P}_k^4$ be the *rational normal quartic curve* (which is the 4-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^4$). Let $X_0 \subseteq \mathbf{P}_k^3$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbf{P}^4$, parametrized by $T = \mathbf{A}^1$, with the given fibres X_1 and X_0 for $t = 1$ and $t = 0$.

Let $\mathscr{I} \subseteq \mathcal{O}_{\mathbf{P}^4 \times T}$ be the ideal sheaf of the total family $X \subseteq \mathbf{P}^4 \times T$. Show that $\mathscr{I}$ is flat over T. Then show that

$$h^0(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

and also

$$h^1(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$

This gives another example of cohomology groups jumping at a special point.

12.4. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f: X \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathscr{L}, \mathscr{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathscr{L}_y \cong \mathscr{M}_y$ on the fibre X_y. Then show that there is an invertible sheaf $\mathscr{N}$ on Y such that $\mathscr{L} \cong \mathscr{M} \otimes f^*\mathscr{N}$. [*Hint*: Use the results of this section to show that $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is locally free of rank 1 on Y.]

12.5. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathscr{E}$ be a locally free sheaf on Y, and let $X = \mathbf{P}(\mathscr{E})$—see (II, §7). Then show that $\operatorname{Pic} X \cong (\operatorname{Pic} Y) \times \mathbf{Z}$. This strengthens (II, Ex. 7.9).

***12.6.** Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X,\mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.

(a) If $\mathcal{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathcal{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.

(b) Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$. (Do *not* assume that T is reduced!) Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\operatorname{Pic}(X \times T) \neq \operatorname{Pic} X \times \operatorname{Pic} T$. [*Hint*: Apply (12.11) with $i = 0,1$ for suitable invertible sheaves on $X \times T$.]