§7. Projective morphisms

⊙ Morphisms to $\mathbb{P}^n_A$:

Recall that $\mathbb{P}^n_A = \operatorname{Proj} A[X_0, \cdots, X_n] = \bigcup_{i=0}^{n} \underbrace{D_+(X_i)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong \operatorname{Spec} A[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]$

and $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(1))$

$\qquad = \{$all homogeneous polys of deg 1 in $A[X_0, \cdots, X_n]\}$

$\qquad = \langle X_0, X_1, \cdots, X_n\rangle_A$

When $A = k$ is an alg closed field, the space of closed points in $\operatorname{Proj} k[X_0, \cdots, X_n]$

$$\mathbb{P}^n_k = \{[a_0 : a_1 : \cdots : a_n] \mid a_i \in k\} = \bigcup_{i=0}^{n} \{(\tfrac{a_0}{a_i}, \cdots, \tfrac{a_n}{a_i}) \mid a_i \neq 0\}$$

$$U_i \cong k^n$$

X_i is a linear form and $\frac{X_j}{X_i}$ is a $_\wedge$function on U_i.
(product)

Let X be a projective variety over k.

$$\varphi : X \longrightarrow \mathbb{P}^n_k$$

$$[\varphi_0 : \varphi_1 : \cdots : \varphi_n] \quad p \longmapsto [\varphi_0(p) : \cdots : \varphi_n(p)] \qquad \text{with } x_i(\varphi(p)) = \varphi_i(p)$$

$\varphi_i \in k(X)$

$$\varphi^* x_i = \varphi_i$$

- In general, X is a scheme over A

$$\begin{array}{ccc} \varphi : X & \longrightarrow & \mathbb{P}^n_A \\ & \searrow \ \ \swarrow & \\ & \operatorname{Spec} A & \end{array}$$

For $q \in \mathbb{P}^n_A$, $\mathcal{O}(1)_q = S(1)_{(q)}$
$= (X_0)_q S_{(q)} + \cdots + (X_n)_q S_{(q)}$.

If $q = \varphi(p)$, then by applying φ^*_p

$$\mathcal{L}_p = (s_0)_p \mathcal{O}_{X,p} + \cdots + (s_n)_p \mathcal{O}_{X,p}$$

Note: $S_{(q)} = \mathcal{O}_{\mathbb{P}^n_A, q} \xrightarrow{\varphi^*_{qp}}$

- $\mathcal{L} = \varphi^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \in \operatorname{Pic}(X)$
- $s_i := \varphi^* x_i \in \Gamma(X, \mathcal{L})$, $i = 0, \cdots, n$ generate $\mathcal{L}$

i.e. $\mathcal{L}$ is finitely g.b.g.s.

- Conversely, given $\mathcal{L} \in \operatorname{Pic}(X)$ and $s_0, \cdots, s_n \in \Gamma(X, \mathcal{L})$ which generate $\mathcal{L}$,

$$\exists! \ \varphi : X \longrightarrow \mathbb{P}^n_A \quad \text{s.t. } \mathcal{L} = \varphi^* \mathcal{O}_{\mathbb{P}^n_A}(1) \text{ and } s_i = \varphi^* x_i$$

(pf): By assumption, $\forall p \in X$, $\exists s_i$ s.t. $(s_i)_p \notin m_p \mathcal{L}_p$, i.e. $p \in X_{s_i}$

$\Rightarrow \{X_{s_i}\}$ covers X $\qquad \Leftrightarrow (s_i)_p \mathcal{O}_p = \mathcal{L}_p$

where $X_{s_i} := \{p \in X \mid (s_i)_p \notin m_p \mathcal{L}_p\}$ $\qquad (s_i)_p \mathcal{L}_p$

Note that $\mathcal{L}|_{X_{s_i}} \cong s_i \mathcal{O}_{X_{s_i}} \xrightarrow{\sim} \mathcal{O}_{X_{s_i}}$

$$s_i \longmapsto 1$$

$$s_j = s_i \frac{s_j}{s_i} \longmapsto \frac{s_j}{s_i}$$

Define a A-linear ring homo : $A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}})$

$$\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i}$$

which induces $\varphi_i : X_{s_i} \longrightarrow D_+(x_i)$

We glue $\{\varphi_i\}$ to get $\varphi : X \longrightarrow \mathbb{P}^n_A$. $\divideontimes$

• φ is a closed immersion iff (1) X_{s_i} is affine (2) $\underbrace{A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]}_{= A_i} \twoheadrightarrow \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}})$

Prop1: Let X be a projective scheme over $k = \bar{k}$. $X \xrightarrow{\text{projective}} \operatorname{Spec} k$

$$\varphi : X \longrightarrow \mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$$

s.t. $\mathcal{L} = \varphi^* \mathcal{O}(1)$ and $s_i = \varphi^* x_i$.

If $V = \langle s_0, \dots, s_n \rangle_k \subset \Gamma(X, \mathcal{L})$, then φ is a closed immersion iff

(1) Separate points : $\forall\, q \neq p$: closed pts in X, $\exists\, s \in V$ s.t. $\begin{cases} s_p \in m_p \mathcal{L}_p \\ s_q \notin m_q \mathcal{L}_q \end{cases}$

(2) Separate tangents : $\forall\, p$: closed pt in X, $\{ s \in V \mid s_p \in m_p \mathcal{L}_p \}$ spans $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$ as k-v.s.

(pf):
"$\Rightarrow$": Assume $X \overset{\varphi}{\cong} \operatorname{Proj} \frac{k[x_0, \dots, x_n]}{I}$, $\mathcal{L} = \mathcal{O}_X(1)$, $s_i = \bar{x}_i$, $V = \langle \bar{x}_0, \dots, \bar{x}_n \rangle_k$.

(1) "$q \neq p \Rightarrow \varphi(q) \neq \varphi(p)$" $\Rightarrow$ $\exists$ a hyperplane $H = V(\sum a_i x_i)$ s.t. $\varphi(p) \in H$, $\varphi(q) \notin H$

$\Rightarrow p \in \underbrace{H|_X}_{= V(\sum a_i \bar{x}_i)}$, $q \notin H|_X$ $\Rightarrow s_p \in m_p \mathcal{L}_p$, $s_q \notin m_q \mathcal{L}_q$.

$\phantom{V(\sum a_i \bar{x}_i)}\ \overset{\shortparallel}{s} \in V$

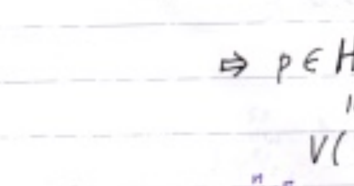
$T_p X \to k$, $\vec{v} \in T_pX$: $d_p f(\vec{v}) = \vec{v}(f)$. $\quad (T_p X)^*$

$V\left(\sum_{i=0}^{n} \frac{\partial F_j}{\partial \bar{x}_i}(p) \bar{x}_i,\ j = 1, \dots, r\right)$ with $I = \langle F_1, \dots, F_r \rangle$

(2) $d\varphi : T_p X \longrightarrow T_q \mathbb{P}^n_k$ $(\mathbb{P}^n \cong \mathbb{P}^n_k)$ is injective $\Leftrightarrow$ $m_{\mathbb{P}^n, q} \longrightarrow m_{\mathbb{P}^n, q} / m^2_{\mathbb{P}^n, q} \twoheadrightarrow m_{X,p} / m^2_{X,p} \xrightarrow{d_p} (T_pX)^*$

$\searrow k$

$y^2 = x^2 + x^3$
$y^2 - x^2 = (y-x)(y+x) = 0$



$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^n, q} & \twoheadrightarrow & \mathcal{O}_{X,p} \\ \cup & & \cup \\ m_{\mathbb{P}^n, q} & \longrightarrow & m_{X,p} = m_{\mathbb{P}^n, q} / \mathcal{I}_X \leftarrow \divideontimes \end{array}$$

$\langle F \mid F$ linear form & $F(q) = 0 \rangle$ $\quad \Downarrow$ $\quad m_{X,p} = \langle s \in V \mid s_p \in m_p \mathcal{L}_p \rangle$

$F \longmapsto s \in V$

$(y-b+b)^2 = (x-a+a)^2 + (x-a+a)^3$ ✓ 看低一次式, smooth.

$(y-b)^2 + 2b(y-b) + b^2 = (x-a)^2 + 2a(x-a) + a^2 + \cdots$

"$\Leftarrow$":

- φ is injective :

  (1) $\Rightarrow$ $\exists$ a hyperplane $\overset{V(s)}{\overset{\shortparallel}{H}}$ s.t. $\varphi(p) \in H$, $\varphi(q) \notin H$ i.e. $\varphi(p) \neq \varphi(q)$.

- φ is closed :

$$\begin{tikzcd} X \arrow[r, "\text{proj} \Rightarrow \text{proper}"] \arrow[d] & \operatorname{spec} k \\ \mathbb{P}^n_k \arrow[ur, "\text{Proj} \Rightarrow \text{proper}"'] & \end{tikzcd}$$

  so it is proper $\Rightarrow$ closed. $\nearrow^{\mathbb{P}^n_k}_{\text{closed}}$

- φ is a homeomorphism : $X \to \varphi(X)$: "φ is conti" + "φ is closed"

- $\varphi_p^{\#} : \mathcal{O}_{\mathbb{P}^n_k, \underset{q}{\underset{\shortparallel}{\varphi(p)}}} \twoheadrightarrow \mathcal{O}_{X,p}$ $\forall$ closed point $p \in X$: $\underset{\text{max ideal}}{\updownarrow}$ $\left(\begin{array}{l} M \twoheadrightarrow N \\ \Leftrightarrow M_m \twoheadrightarrow N_m \ \forall m \end{array} \right)$

$\because$ (2) $\Rightarrow$

$$\begin{tikzcd} \mathfrak{m}_{\mathbb{P}^n, q} \arrow[rr, two heads] \arrow[dr] & & \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \\ & \mathfrak{m}_{X,p} \arrow[ur] & \end{tikzcd}$$

By Nakayma lemma, $\overline{\varphi_p^{\#}(t_1)}, \cdots, \overline{\varphi_p^{\#}(t_m)}$ generate $\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$

$\Rightarrow \langle \varphi_p^{\#}(t_1), \cdots, \varphi_p^{\#}(t_m) \rangle_{\mathcal{O}_{X,p}} = \mathfrak{m}_{X,p}$

So $\varphi_p^{\#}(\mathfrak{m}_{\mathbb{P}^n, q}) = \mathfrak{m}_{X,p}$.

補§5 : $f: X \to Y$ projective, X, Y: of finite type over k

$\mathcal{F} \in Coh(X) \Rightarrow f_*\mathcal{F} \in Coh(Y)$

(pf) :

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & \mathbb{P}^n_Y \arrow[ur] & \end{tikzcd}$$

, assume $Y = \operatorname{spec} A$, with A : f.g k-alg.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \operatorname{spec} A \\ & \mathbb{P}^n_A \arrow[ur] & \end{tikzcd}$$

Already know $f_*\mathcal{F} \in Qco(Y)$.

So $f_*\mathcal{F} = \widetilde{\Gamma(Y, f_*\mathcal{F})} = \widetilde{\Gamma(X, \mathcal{F})}_{\text{f.g } A\text{-module}}$

✗.

$\mathcal{O}_{X,p}/m_{X,p} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \, \mathcal{O}_{P_k^n,q}/m_{P_k^n,q}$

$\Downarrow$

$= \langle x_1, \dots, x_n \rangle \mathcal{O}_{P_k^n,q}$

Hence $\varphi_* \mathcal{O}_X \in \mathrm{Coh}(P_k^n) \Rightarrow \mathcal{O}_{X,p}$ is a f.g. $\mathcal{O}_{P_k^n,q}$ - module.

Since $\mathcal{O}_{P^n,q}/m_{P^n,q} \simeq \mathcal{O}_{X,p}/m_{X,p} \simeq k$, $\leftarrow$ since $\bar{k} = k$, otherwise, $\mathcal{O}_{X,p}/m_{X,p}$ is alg over k.

by Nakayama lemma, $\mathcal{O}_{X,p}$ is generated by 1 as $\mathcal{O}_{P^n,q}$-module $\Rightarrow \varphi_p^\#$ is surjective $\#$

eg: $\{k\text{-automorphisms of } P_k^n\} = PGL(n,k) := GL(n+1,k)/k \setminus \{0\}$

(pf): "$\supset$": Let $[a_{ij}] \in GL(n+1,k)$.

$$\begin{aligned} k[x_0', \dots, x_n'] &\longrightarrow k[x_0, \dots, x_n] \\ x_i' &\longmapsto \sum a_{ij} x_j \end{aligned}$$

induces an auto : $P_k^n \to P_k^n$.

$\forall \lambda \in k \setminus \{0\}$, $[\lambda a_{ij}]$ determines the same auto. $(x_i' \mapsto \lambda \sum a_{ij} x_j)$

"$\subset$": Given $\varphi : P_k^n \longrightarrow P_k^n$ an k-auto,

$\varphi^* \mathcal{O}(1) \in \mathrm{Pic}(P_k^n) = \langle \mathcal{O}(1) \rangle$.

$\nwarrow$ cyclic group

φ is auto, so $\varphi^* \mathcal{O}(1)$ must be a generator of $\mathrm{Pic}(P_k^n)$

i.e. $\varphi^* \mathcal{O}(1) = \mathcal{O}(1)$ or $\mathcal{O}(-1)$

$\downarrow$ impossible since $\Gamma(X, \mathcal{O}(-1)) = \phi$

$S(-1)$: no (-1)-term in S.

$\{\underbrace{\varphi^* x_0}_{s_0}, \dots, \underbrace{\varphi^* x_n}_{s_n}\}$ is a basis for $\Gamma(P_k^n, \mathcal{O}(1))$ over k.

Write $s_i = \sum a_{ij} x_j$, $[a_{ij}] \in GL(n+1,k)$

$\rightsquigarrow \overline{[a_{ij}]} \in PGL(n+1,k)$.

$\#$

PERFECTION

projective integral scheme over k

⊚ Linear systems

X : nonsingular projective variety over $k=\bar{k}$ $\rightsquigarrow$ $\mathcal{O}_{X,p}$: UFD $\forall p$.

$\mathcal{O}_{X,p}$ is a regular local ring $\forall p$

$\Rightarrow \mathrm{Cl}\,X \cong \mathrm{CaCl}\,X \cong \mathrm{Pic}\,X$.

Def: Given a divisor D_0 on X,

$$|D_0| := \{ D \geq 0 \mid D \sim D_0 \}$$

is called a complete linear system

Prop 2: $$\underset{\text{給定一個 projective space 的結構}}{|D_0|} \longleftrightarrow \mathbb{P}(\overbrace{\Gamma(X, \underbrace{\mathcal{L}(D_0)}_{\in \mathrm{Coh}(X),\ \text{proj. over } k})}^{\because \text{ f.g. } k\text{-v.s}}) := \Gamma(X, \mathcal{L}(D_0)) \setminus \{0\} \Big/ k - \{0\} = k^*$$

(pf): $$\Gamma(X, \mathcal{L}(D_0)) \setminus \{0\} \longrightarrow |D_0|$$
$$s \longmapsto (s)_0$$

$\mathcal{L}(D_0) \hookrightarrow \mathcal{K}$: the constant sheaf $K = K(X)$

$s \longleftrightarrow f \in K^*$

(Here, X is irr $\Rightarrow \mathcal{K}(U) = K$)

- To construct $(s)_0$: let $D_0 = \{ (U_i, f_i) \}$. $f_i \in \mathcal{K}^*(U_i) = K^*$

Then $$\mathcal{L}(D_0)|_{U_i} = \frac{1}{f_i} \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$$
$$f|_{U_i} \longmapsto f f_i$$

We define $(s)_0 = D = \{ (U_i, f f_i) \}$ is effective

$$= D_0 + (f) \sim D_0$$

(Note: we can also write $\Gamma(X, \mathcal{L}(D_0)) = \{ f \in K^* \mid (f) + D_0 \geq 0 \}$)

- Onto: For $D \in |D_0|$, say $D = D_0 + (f) \geq 0$

$$= \{ (U_i, f f_i) \}, \quad f f_i \in \mathcal{O}_X(U_i)$$

$$\Downarrow$$

$$f|_{U_i} \in \frac{1}{f_i} \mathcal{O}_X(U_i) \Rightarrow \overset{s}{f} \in \mathcal{L}(D_0)(X) \Rightarrow D = (s)_0$$

PAGE

DATE

$$\begin{array}{cc} f & f' \\ \updownarrow & \updownarrow \end{array}$$

- $(s)_0 = (s')_0 \iff D_0 + (f) = D_0 + (f') \iff (f'/f) = 0$

$$\iff f'/f \in \Gamma(X, \mathcal{O}_X^*)$$

$$\left(\begin{array}{l} \because \Gamma(X, \mathcal{O}_X) = k \\ \therefore \Gamma(X, \mathcal{O}_X^*) = k^* \end{array} \right) \qquad \iff f'/f = \lambda, \ \lambda \in k^*$$

$$\iff s' = \lambda s, \ \lambda \in k^*$$

$\divideontimes$

Let V be a subspace of $\Gamma(X, \mathcal{L}(D_0))$.

- $L := \{ (s)_0 \mid s \in V \setminus \{0\} \} \subset |D_0|$ is called a linear system.
- $\dim L := \dim V - 1$
- Assume that $V = \langle s_0, \cdots, s_n \rangle_k$.

As before, $\quad X_{s_i} \longrightarrow D_+(x_i) \subset \mathbb{P}^n_k \quad \forall i$

$$\Rightarrow \underbrace{\bigcup_{i=0}^{n} X_{s_i}}_{= U} \longrightarrow \mathbb{P}^n_k$$

$\because$ $s_0, \cdots, s_n$ generate $\mathcal{L}(D_0)$ over U

$\because$ $\forall p \in X \setminus U$, "$\forall s_i$, $(s_i)_p \in \mathfrak{m}_p \mathcal{L}(D_0)_p$" $\iff$ "$\forall s \in V \setminus \{0\}$, $(s)_p \in \mathfrak{m}_p \mathcal{L}(D_0)_p$"

$$\Updownarrow$$

"$p \in \operatorname{supp} D$, $\forall D \in L$"

$$\operatorname{supp} D = \bigcup D_i, \qquad D = \sum a_i D_i$$

We call such p a base point of L

- $s_0, \cdots, s_n$ generate $\mathcal{L}(D_0)$ over X

$\iff X = \bigcup_{i=0}^{n} X_{s_i} \iff L$ has no base point $\iff \phi = \bigcap_{D \in L} \operatorname{supp} D$

i.e. base-point-free

- a base-point-free linear system $L \iff \varphi : X \longrightarrow \mathbb{P}^n_k$

"$\Rightarrow$": Just done! "

"$\Leftarrow$": $\mathcal{L} = \varphi^* \mathcal{O}(1) \in \operatorname{Pic} X$, $\mathcal{L} \cong \mathcal{L}(D_0)$ for some D_0

$s_i = \varphi^* x_i$, $s_0, \cdots, s_n$ generate $\mathcal{L}(D_0)$

$V = \langle s_0, \cdots, s_n \rangle_k \subset \Gamma(X, \mathcal{L})$

Note: different choice of basis for V induces an auto of $\mathbb{P}^n_k$

b.p.f.

- a linear system L separates points $\Leftrightarrow$ for $p \neq q$: closed pts in X, $\exists\, s \in V$ s.t. $\begin{cases} s_p \in m_p \mathcal{L}(D)_p \\ s_q \notin m_q \mathcal{L}(D)_q \end{cases}$

$(s)_0 = D$

$\Leftrightarrow$ for $p \neq q$; closed pts in X, $\exists\, D \in L$ s.t $p \in \operatorname{supp} D$ and $q \notin \operatorname{supp} D$.

- a b.p.f. linear system L separates tangents

$\Leftrightarrow$ $\forall p$: closed pt in X, $\Gamma(X, \mathcal{L} \otimes m_p) \twoheadrightarrow \mathcal{L}_p \otimes m_p/m_p^2$

$$\Updownarrow$$

$$\mathcal{L}_p^\vee \otimes \Gamma(X, \mathcal{L}\otimes m_p) \twoheadrightarrow m_p/m_p^2$$

$$\Updownarrow$$

$$\left(m_p/m_p^2\right)^\vee \hookrightarrow \mathcal{L}_p \otimes \Gamma(X, \mathcal{L}\otimes m_p)^\vee$$

$$\operatorname{Hom}(m_p/m_p^2, k) \qquad \operatorname{Hom}(\Gamma(X, \mathcal{L}\otimes m_p), \mathcal{L}_p \otimes k)$$

$$\Updownarrow$$

$\Gamma(X, \mathcal{L}\otimes m_p) \ni f \longleftrightarrow D = D_0 + \operatorname{div}(f)$ with $p \in \operatorname{supp} D$

locally principal closed subscheme

$\bar{t} \neq 0$

$$\begin{tikzcd} & \Gamma(X, \mathcal{L}\otimes m_p) \arrow[d] \arrow[dr] & \\ 0 \neq t : & m_p/m_p^2 \arrow[r] & k \otimes \mathcal{L}_p \cong k \end{tikzcd}$$

$t \in T_pX$, $df \in m_p/m_p^2$

$t(df) \neq 0$ i.e. $df(t) \neq 0$, $df(t) = t(f)$ a derivation

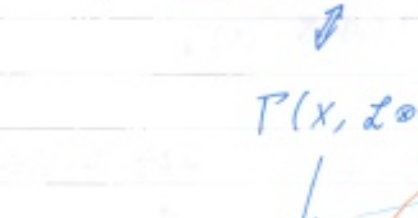
$\because$ f is a defining equation of D

$\therefore$ $t \notin T_p D$

$\Updownarrow$ $\forall p \in X$, $\forall t \in T_pX$, $\exists\, D \in$ ~~|D_0|~~ L s.t. $t \notin T_p D$

eg: $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$ is not an embedding.

$t \longmapsto (t^3, t^2)$



$t^3 = 2t^2 \Rightarrow t = 2$ or $t^2 = 0$

$D = 2p \leftrightarrow \operatorname{spec} \dfrac{k[x]}{(x^2)} \rightsquigarrow (T_pD)' \rightsquigarrow \dim T_pD = 1$

$= x/\langle x^2\rangle$, $m_{p,D} = x/\langle x^2 \rangle$

$m_{p,D}^2 = 0$

Then $\dim T_pD = \dim T_pX = 1$

so it is impossible to find $t \in T_pX$ but $t \notin T_pD$

◉ Ample sheaves

($\mathcal{L} = \mathcal{L}(D)$, $|D|$ = the set of all hyperplane sections.)

- $\mathcal{L}$ is very ample if $\exists$ an immersion $i : X \to \mathbb{P}^n_A$ s.t. $\mathcal{L} = i^*\mathcal{O}(1) =: \mathcal{O}_X(1)$.

Def: Let X be noeth and $\mathcal{L} \in \operatorname{Pic} X$.

- $\mathcal{L}$ is ample if $\forall\ \mathcal{F} \in \operatorname{Coh}(X)$, $\exists\ n_0 > 0$ s.t. $n \geq n_0$
$\mathcal{F} \otimes \mathcal{L}^n$ is g.b.g.s.

eg: A very ample sheaf $\mathcal{L}$ ($\cong \mathcal{O}_X(1)$ via $X \hookrightarrow \mathbb{P}^m_A$) on a proj. scheme X over a noeth ring A is ample. (By Serre thm, $\mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$ is g.b.g.s.)

- Let $\mathcal{L} \in \operatorname{Pic} X$ with X noeth.

TFAE (1) $\mathcal{L}$ is ample (2) $\mathcal{L}^m$ is ample $\forall\ m > 0$ (3) $\mathcal{L}^m$ is ample for some $m > 0$.

(pf): (1) $\Rightarrow$ (2): $\mathcal{F} \otimes (\mathcal{L}^m)^n$ is g.b.g.s $\forall\ mn \geq n_0$

(2) $\Rightarrow$ (3): O.K

(3) $\Rightarrow$ (1):
$$\underbrace{\mathcal{F} \otimes \mathcal{L}^i}_{\in \operatorname{Coh}(X)} \otimes (\mathcal{L}^m)^n \quad \text{is g.b.g.s} \quad \forall\ n \geq n_i \ , i = 0, 1, \cdots, m-1$$

Let $N = \max\{n_i m + i\}$.

Then $\mathcal{F} \otimes \mathcal{L}^n$ is g.b.g.s $\forall\ n \geq N$. ($n \geq N$, $n \equiv i \pmod m$, $n = mn' + i \geq n_i m + i$, $n' \geq n_i$)

\#

Prop 3: Let X be a scheme of finite type over A with A noeth and $\mathcal{L} \in \operatorname{Pic} X$.

Then $\mathcal{L}$ is ample $\Leftrightarrow$ $\mathcal{L}^m$ is very ample for some $m > 0$

(pf): "$\Leftarrow$": $\exists$ an immersion $i : X \hookrightarrow \mathbb{P}^n_A$ s.t. $\mathcal{L}^m = i^*\mathcal{O}(1)$ ($=: \mathcal{O}_X(1)$)

open ∩ closed

$\Rightarrow$ $\overline{X}$ is a projective scheme over A

By Serre thm, $\mathcal{O}_{\overline{X}}(1)$ is ample on $\overline{X}$.

$\forall\ \mathcal{F} \in \operatorname{Coh}(X)$ (ex 5.15) $\exists\ \overline{\mathcal{F}} \in \operatorname{Coh}(\overline{X})$ s.t. $\overline{\mathcal{F}}|_X \cong \mathcal{F}$

Then $\overline{\mathcal{F}} \otimes \mathcal{O}_{\overline{X}}(n)$ is g.b.g.s $\forall\ n \geq n_0$

大的 sections 會 generate sheaf

$\Rightarrow$ $\mathcal{F} \otimes \mathcal{O}_X(n)$ is g.b.g.s $\forall\ n \geq n_0$

限制在 open set 当然会

$\Rightarrow$ $\mathcal{L}^m$ is ample $\Rightarrow$ $\mathcal{L}$ is ample

"$\Rightarrow$": <u>Step 1</u>: $\exists\, s_1, \dots, s_k \in \Gamma(X, \mathcal{L}^{n_1})$ s.t. X_{s_i} is affine and $\{X_{s_i}\}$ covers X:

For $p \in X$, let $U \subset X$ and $\mathcal{L}|_U$ ~~be free~~ $\cong \mathcal{O}_U$.
$p \in \overset{\shortparallel}{\operatorname{spec} A}$ (wish $X_s \subset U$ i.e. s vanishes outside U)

If $Y := X - U$ with the reduced induced structure, then $\mathcal{I}_Y \in \operatorname{Coh}(X)$.

Hence $\mathcal{I}_Y \otimes \mathcal{L}^m$ is g.b.g.s for some $m > 0$.

i.e. $\exists\, s \in \underset{\Gamma(X, \mathcal{L}^m)}{\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^m)}$ s.t. $s_p \notin m_p (\cancel{\mathcal{I}_Y \otimes} \mathcal{L}^m)_p$

$\mathcal{I}_Y \hookrightarrow \mathcal{O}_X$
$\mathcal{I}_Y \otimes \mathcal{L}^m \hookrightarrow \mathcal{O}_X \otimes \mathcal{L}^m \cong \mathcal{L}^m$

Then $p \in X_s = \{ q \in X \mid s_q \notin m_q \mathcal{L}^m_q \} \subset U$ and

$\mathcal{L}^m|_U \cong \mathcal{O}_U \Rightarrow X_s = \operatorname{spec} A_f$.
$s|_U \leftrightarrow f \in A$

Conclusion: $\forall p \in X$, $\exists\, m > 0$ and $s \in \Gamma(X, \mathcal{L}^m)$ s.t. $p \in X_s$ and X_s is affine

$(s_i \in \Gamma(X, \mathcal{L}^{m_i}),\ s_i^{\ell} \in \Gamma(X, \mathcal{L}^{\ell m_i}),\ X_{s_i} = X_{s_i^{\ell}})$

Since X is quasi-compact, $\exists\, s_i \in \Gamma(X, \mathcal{L}^{n_1})$ $\forall i = 1, \dots, k$ s.t. X_{s_i} is affine & $\{X_{s_i}\}$ covers X.

<u>Step 2</u>: To construct an immersion $\varphi: X \to \mathbb{P}^N_A$:

By assumption, $X = \bigcup_{i=1}^{k} \underset{X_{s_i}}{\overset{\shortparallel}{\operatorname{spec} B_i}} \to \operatorname{spec} A$ is of finite type,

so $B_i = A[b_{i1}, \dots, b_{ik_i}]$ $\subset \Gamma(X_{s_i}, \mathcal{L}^{\ell n_1})$

Now, $b_{ij} \in \Gamma(X_{s_i}, \mathcal{O}_X)$, by key lemma,

$\exists\, c_{ij} \in \Gamma(X, \mathcal{L}^{n})$ s.t. $c_{ij}|_{X_{s_i}} = s_i^{n_2} b_{ij}$

Since $\bigcup_{i=1}^{k} X_{s_i^{n_2}} = X$, $s_1^{n_2}, \cdots, s_k^{n_2}$ generate $\mathcal{L}^{\overset{n}{\overset{\parallel}{n_1 n_2}}}$.

and thus $\{c_{ij}, s_i^{n_2}\}_{i,j}$ generate $\mathcal{L}^n$,

$\exists \ \varphi : X \longrightarrow P_A^N = \mathrm{Proj}\, A[\{x_i\}_{i=1,\cdots,k}, \{x_{ij}\}_{\substack{i=1,\cdots,k \\ j=1,\cdots,k_i}}]$

s.t. $\mathcal{L}^n = \varphi^* \mathcal{O}(1)$.

Moreover, for $y_i = \frac{x_i}{x_t}$, $y_{ij} = \frac{x_{ij}}{x_t}$, $A[\{y_i\}, \{y_{ij}\}] \twoheadrightarrow B_t$

$$y_{tj} \longmapsto \frac{c_{tj}}{s_t^{n_2}} = b_{tj}$$

$$\rightsquigarrow \quad X_{s_t^{n_2}} \underset{\text{closed}}{\hookrightarrow} D_+(x_t) \qquad \forall\, t = 1, \cdots, k.$$

$$\Rightarrow \quad \varphi : X \underset{\text{closed}}{\hookrightarrow} \bigcup_{t=1}^{k} D_+(x_t) \underset{\text{open}}{\hookrightarrow} P_A^N$$

$\Rightarrow \ \varphi$ is an immersion ※.

PAGE
DATE

◎ Proj $\mathscr{F}$ X : noetherian scheme.

- $\mathscr{F} = \bigoplus_{d\geq 0} \mathscr{F}_d$: a sheaf of graded algebras over X

$$(*)\begin{cases} \cdot\cdot\ \mathscr{F}_0 = \mathcal{O}_X \\ \cdot\cdot\ \mathscr{F}_1 \in Coh(X) \\ \cdot\cdot\ \mathscr{F} \text{ is locally generated by } \mathscr{F}_1 \text{ as } \mathcal{O}_X\text{-alg}. \end{cases}$$

$(\Rightarrow \mathscr{F}_d \in Coh(X))$

- Construction :

for $p \in X$, $p \in \underset{\text{Spec } A}{\overset{\shortparallel}{U}} \subset X$, $A \longrightarrow \mathscr{F}(U) = A[s_0, \cdots, s_n]$, $\mathscr{F}_1(U) = \underbrace{\langle s_0, \cdots, s_n \rangle_A}_{\overset{\shortparallel}{M}}$

$\Rightarrow \pi_U : \text{Proj } \mathscr{F}(U) \longrightarrow U$

$M_f \Rightarrow \mathscr{F}(D(f))$

$\mathscr{F}(U_f)$ $\because \mathscr{F} \in Qco(X)$ $\quad \mathscr{F}_1(U_f) \cong M_f = \langle \bar{s}_0, \cdots, \bar{s}_n \rangle_{A_f}$

$\cdot\cdot\ \forall f \in A$, $U_f = \text{Spec } A_f$,

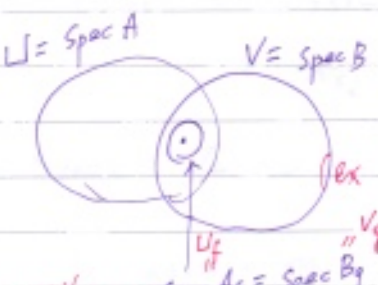
$$\begin{tikzcd} A_f \arrow[r] & \mathscr{F}(U)_f = A_f[\bar{s}_0, \cdots, \bar{s}_n] \\ A \arrow[u] \arrow[r] & \mathscr{F}(U) \arrow[u] \end{tikzcd}$$

, $\bar{s}_i \in M_f$

$\text{Proj } \mathscr{F}(U)_f \hookrightarrow \text{Proj } \mathscr{F}(U)$
$\quad \cong$
$\text{Proj } \mathscr{F}(U_f)$

$$\Rightarrow \begin{tikzcd} \text{Proj } \mathscr{F}(U_f) \arrow[r] \arrow[d, "i_f"] & U_f \arrow[d, hook] \\ \text{Proj } \mathscr{F}(U) \arrow[r] & U \end{tikzcd}$$

, so $\pi_U^{-1}(U_f)$
$\cong \text{Proj } \mathscr{F}(U_f)$



$\forall p \in U \cap V$, $p \in \text{Spec } A_f = \text{Spec } B_g$
$A_f \cong B_g$

(ex 2.12.) $\Rightarrow \pi_U^{-1}(U \cap V) \overset{\varphi_{UV}}{\cong} \pi_V^{-1}(U \cap V)$

$\pi_U^{-1}(U_f) \cong \text{Proj } \mathscr{F}(U_f)$
$\shortparallel$
$\pi_V^{-1}(V_g) \cong \text{Proj } \mathscr{F}(V_g)$

We can glue $\{\text{Proj } \mathscr{F}(U)\}$ to get a scheme $\text{Proj } \mathscr{F}$ and $\pi : \text{Proj } \mathscr{F} \longrightarrow X$.

$\cdot\cdot\ \because \mathcal{O}_{\text{Proj } \mathscr{F}(U_f)}(1) = i_f^* \mathcal{O}_{\text{Proj } \mathscr{F}(U)}(1)$

$\therefore$ We can also glue $\{\mathcal{O}_{\text{Proj } \mathscr{F}(U)}(1)\}$ to get an

invertible sheaf $\mathcal{O}(1)$ on $\text{Proj } \mathscr{F}$, i.e. $\mathcal{O}(1)\big|_{\text{Proj } \mathscr{F}(U)} = \mathcal{O}_{\text{Proj } \mathscr{F}(U)}(1)$

PERFECTION

eg. If $\mathcal{F} = \mathcal{O}_X[x_0, \dots, x_n]$, i.e. $\mathcal{F}(\overset{\operatorname{Spec} A}{\overset{\parallel}{U}}) = A[x_0, \dots, x_n]$,

then $\operatorname{Proj} \mathcal{F}(U) = \mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec} A = \mathbb{P}^n_{\mathbb{Z}} \times U$

$\rightsquigarrow \mathbf{Proj}\, \mathcal{F} = \mathbb{P}^n_{\mathbb{Z}} \times X = \mathbb{P}^n_X$

- Important ~~special~~ case :
  - X noeth, $\mathcal{E}$: locally free coherent sheaf on X

    ($S^d(\mathcal{E})$: ~~the d-th symmetric product~~ the dth symmetric power of $\mathcal{E}$) $= \mathcal{E}\otimes\cdots\otimes\mathcal{E} / \langle x_1\otimes\cdots\otimes x_d - x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(d)}\rangle$

    $\mathcal{F} = \underset{\substack{\uparrow \\ \text{the symmetric algebra of } \mathcal{E}}}{S(\mathcal{E})} = \bigoplus_{d \geq 0} S^d(\mathcal{E})$

    ... $S^0(\mathcal{E}) := \mathcal{O}_X$

    ... $S^1(\mathcal{E}) = \mathcal{E} \in \operatorname{Coh}(X)$

    ... $S(\mathcal{E})$ is locally g.b. $S^1(\mathcal{E})$ as $\mathcal{O}_X$-alg.

    ($\mathcal{E}|_U = s_0\mathcal{O}_U \oplus \cdots \oplus s_n\mathcal{O}_U$, $U = \operatorname{Spec} A$

    $S(\mathcal{E})(U) = A[s_0, \dots, s_n] \cong A[x_0, \dots, x_n]$)

    $\mathbb{P}(\mathcal{E}) := \mathbf{proj}\, S(\mathcal{E})$

    $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow X$

    (If $\mathcal{E}$ is free of rank $n+1$ over U

    then $\pi^{-1}(U) \cong \mathbb{P}^n_U$: $\mathcal{E}|_U = s_0\mathcal{O}_U \oplus \cdots \oplus s_n\mathcal{O}_U$

    $\Rightarrow S(\mathcal{E})|_U = \mathcal{O}_U[x_0, \dots, x_n]$)

  - For any $U = \operatorname{Spec} A$, with $\mathcal{E}|_U$ free,

    $\pi_*\mathcal{O}(1)(U) = \mathcal{O}(1)(\pi^{-1}(U)) = \mathcal{O}(1)(\operatorname{Proj} \underset{A[s_0, \dots, s_n]}{\underset{\parallel}{S(\mathcal{E})(U)}}) = \langle s_0, \dots, s_n \rangle_A = \mathcal{E}(U)$

    $\rightsquigarrow \pi_*\mathcal{O}(1) = \mathcal{E}$

  - $S(\mathcal{E}) = \bigoplus_{d \geq 0} \pi_*\mathcal{O}(d)$ for rank $\mathcal{E} \geq 2$

  - $\underset{\pi^*\pi_*\mathcal{O}(1)}{\overset{\parallel}{\pi^*\mathcal{E}}} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}(1)$:

    (since $\operatorname{Hom}(\pi_*\mathcal{O}(1), \pi_*\mathcal{O}(1)) = \operatorname{Hom}(\pi^*\pi_*\mathcal{O}(1), \mathcal{O}(1))$

    $\operatorname{Id} \longleftrightarrow \varphi : \pi^*\pi_*\mathcal{O}(1) \longrightarrow \mathcal{O}(1)$) $\Leftarrow$ no need!

    If $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus(n+1)}$, then $\pi^*\mathcal{E}|_{\pi^{-1}(U)} \cong \pi^*(\mathcal{E}|_U) \cong \mathcal{O}_{\pi^{-1}(U)}^{\oplus(n+1)} \longrightarrow\!\!\!\!\rightarrow \mathcal{O}(1)|_{\pi^{-1}(U)}$

    $e_i \longmapsto s_i$

Another important application is $Bl_Y X := \operatorname{Proj}(\bigoplus_{d \geq 0} \mathcal{I}_Y^d)$ *

Prop 4: Let $\mathcal{L} \in \operatorname{Pic} X$. and $\mathcal{F}$ be a sheaf of graded algebras over X satisfying $(*)$.
If $\underbrace{\mathcal{F}' = \bigoplus_{d \geq 0} (\mathcal{F}_d \otimes \mathcal{L}^d)}_{\mathcal{F} * \mathcal{L}}$, then $\exists\ \varphi : \operatorname{Proj} \mathcal{F}' \xrightarrow{\sim} \operatorname{Proj} \mathcal{F}$

$$\begin{tikzcd} \operatorname{Proj} \mathcal{F}' \arrow[rr, "\sim"] \arrow[dr, "\pi'"'] & \circlearrowleft & \operatorname{Proj} \mathcal{F} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

s.t. $\mathcal{O}_{\operatorname{Proj} \mathcal{F}'}(1) \cong \varphi^* \mathcal{O}_{\operatorname{Proj} \mathcal{F}}(1) \otimes \pi'^* \mathcal{L}$.

Pf: Let $\theta : \mathcal{L}|_U \cong \mathcal{O}_U$ i.e. $\mathcal{L}|_U = g\mathcal{O}_U$ with $U = \operatorname{spec} A$.
$g \mapsto 1$

- $\mathcal{F}'_1|_U = \mathcal{F}_1|_U \otimes g\mathcal{O}_U$ & $\mathcal{F}'_d = \mathcal{F}_d|_U \otimes g^d \mathcal{O}_U \Rightarrow \mathcal{F}'$ satisfies $(*)$

Then $\cdot\ \mathcal{F}'|_U = \bigoplus_{d \geq 0} (\mathcal{F}_d|_U \otimes \mathcal{L}|_U^{\,d}) \cong \bigoplus_{d \geq 0} (\mathcal{F}_d|_U \otimes \mathcal{O}_U^{\,d}) \cong \bigoplus_{d \geq 0} \mathcal{F}_d|_U = \mathcal{F}|_U$

For another $\theta' : \mathcal{L}|_U \cong \mathcal{O}_U$, observe that
$g' \mapsto 1$

$$\mathcal{F}'_d|_U = \mathcal{F}_d|_U \otimes \mathcal{L}|_U^{\,d} = \begin{cases} \mathcal{F}_d|_U \otimes g^d \mathcal{O}_U \xrightarrow[\sim]{\theta} \mathcal{F}_d|_U \\ \qquad\qquad\qquad\qquad \big\downarrow \leftarrow \text{multiply by } (g/g')^d \in \mathcal{O}_U^*(U) \\ \mathcal{F}_d|_U \otimes g'^d \mathcal{O}_U \xrightarrow[\theta']{\sim} \mathcal{F}_d|_U \end{cases}$$

and $\operatorname{Proj} \mathcal{F}(U) = \operatorname{Proj} A[s_0, \dots, s_n] = \operatorname{Proj} A[(\tfrac{g}{g'})s_0, \dots, (\tfrac{g}{g'})s_n]$ & the same structure sheaf.
($\tfrac{g}{g'} \in \{\text{units of } A\}$)

$= \operatorname{Proj} \bigoplus_{d \geq 0} (\tfrac{g}{g'})^d \mathcal{F}_d(U)$

We conclude that $\exists\ \theta_U : \operatorname{Proj} \mathcal{F}'(U) \xrightarrow{\sim} \operatorname{Proj} \mathcal{F}(U)$
which is indep. of θ.

Hence we can glue $\{\theta_U\}$ to get $\varphi : \operatorname{Proj} \mathcal{F}' \xrightarrow{\sim} \operatorname{Proj} \mathcal{F}$

~~and~~ $\because\ \mathcal{F}'_1 = \mathcal{F}_1 \otimes \mathcal{L}$

$\therefore\ \mathcal{O}_{\operatorname{Proj} \mathcal{F}'}(1) \cong \varphi^* \mathcal{O}_{\operatorname{Proj} \mathcal{F}}(1) \otimes \pi'^* \mathcal{L}$.

Prop 5: (1) $\pi : \mathrm{Proj}\, \mathcal{J} \longrightarrow X$ is proper

(2) $\pi : \mathrm{Proj}\, \mathcal{J} \longrightarrow X$ is projective if $\exists$ ample $\mathcal{L}$ on X.

In this case, $\exists\, n$ s.t. $\mathcal{O}_{\mathrm{Proj}\,\mathcal{J}}(1) \otimes \pi^* \mathcal{L}^n$ is very ample ~~on~~ over X.

(pf) (1) $\forall p \in X$, let U_p be affine s.t. $\mathcal{J}(U_p) = \mathcal{O}_X(U_p)[s_0, \cdots, s_n]$ with $\mathcal{J}_1(U_p) = \langle s_0, \cdots, s_n \rangle_{\mathcal{O}_X(U_p)}$

Then $\pi_{U_p} : \mathrm{Proj}\, \mathcal{J}(U_p) \longrightarrow U_p$ is projective $\Rightarrow$ proper

Hence $\pi : \mathrm{Proj}\, \mathcal{J} \longrightarrow X$ is proper.

(2) Since X is noeth, $\mathcal{J}_1 \otimes \mathcal{L}^n \in \mathrm{Coh}(X)$ $\forall n$ and by def, $\mathcal{J}_1 \otimes \mathcal{L}^n$ is g.b.g.s for some $n > 0$,

$\exists \{ \underset{\mathrm{spec}\, A_i}{\overset{\shortparallel}{U_i}} \}_{i=1}^{k}$ covers X s.t. $\mathcal{J}_1 \otimes \mathcal{L}^n|_{U_i} = \widetilde{M_i}$

with $M_i = \langle s_{i1}|_{U_i}, \cdots, s_{ik_i}|_{U_i} \rangle$ for $s_{ij} \in \Gamma(X, \mathcal{J}_1 \otimes \mathcal{L}^n)$

Then $\mathcal{J}_1 \otimes \mathcal{L}^n$ is generated by $\{ s_{ij} \}_{\substack{i=1,\cdots,k \\ j=1,\cdots,k_i}} = \{ s_0, \cdots, s_N \}$

$\Rightarrow \mathcal{J}_1 \otimes \mathcal{L}^n|_{U_i} = \langle s_0|_{U_i}, \cdots, s_N|_{U_i} \rangle_{A_i}$ $\forall i$, for some $s_0, \cdots, s_N \in \Gamma(X, \mathcal{J}_1 \otimes \mathcal{L}^n)$

$\Rightarrow \mathcal{J} * \mathcal{L}^n|_{U_i} = A_i[s_0|_{U_i}, \cdots, s_N|_{U_i}]$ $\forall i$

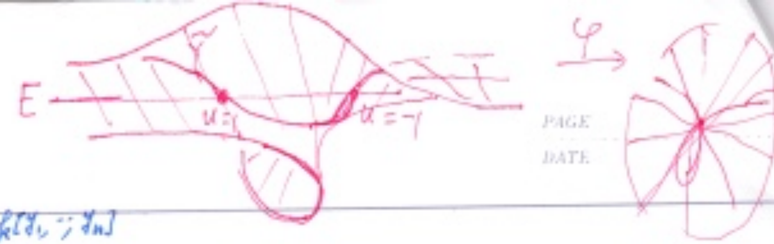
Consider $A_i \to A_i[x_0, \cdots, x_N] \twoheadrightarrow \mathcal{J} * \mathcal{L}^n|_{U_i}$

We get $\mathrm{Proj}\, \mathcal{J} * \mathcal{L}^n(U_i) \xhookrightarrow[\text{closed subscheme}]{} \mathrm{Proj}\, A_i[x_0, \cdots, x_N] \to \mathrm{spec}\, A_i$, $\forall i$.

Glue them, $\underset{\mathrm{Proj}\,\mathcal{J}}{} \cong \mathrm{Proj}\, \mathcal{J} * \mathcal{L}^n \hookrightarrow P_X^N \to X$

$\mathcal{O}_{\mathrm{Proj}\,\mathcal{J}}(1) \otimes \pi^* \mathcal{L}^n \cong \mathcal{O}_{\mathrm{Proj}\,\mathcal{J} * \mathcal{L}^n}(1) \longleftarrow \mathcal{O}_{P_X^n}(1)$

*



◎ Blow-up.

$$\begin{array}{c} A \\ \shortparallel \\ \operatorname{Spec} k[x_1,\dots,x_n] \end{array} \qquad \operatorname{proj} k[y_1,\dots,y_n]$$

• $Bl_0 \mathbb{A}^n_k = \{(x,y) \in \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \mid x_i y_j - x_j y_i = 0,\ i,j=1,\dots,n\} = \operatorname{Proj} \dfrac{A[y_1,\dots,y_n]}{\langle x_i y_j - x_j y_i : i,j=1,\dots,n\rangle}$

$\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k = \operatorname{Proj} A[y_1,\dots,y_n]$

$\updownarrow$

$I = \langle x_1,\dots,x_n\rangle \subset A$.

Consider the Rees algebra $S = \bigoplus_{d \geq 0} I^d \left(= A \oplus It \oplus I^2 t^2 \oplus \cdots\right)$
$= A \oplus I \oplus I^2 \oplus \cdots$

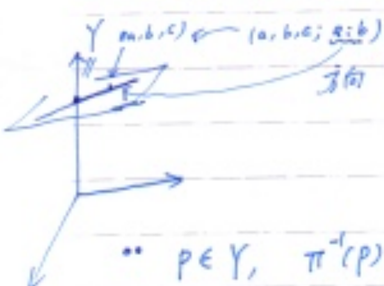
The surjective map of graded A-algebras $\varphi: A[y_1,\dots,y_n] \twoheadrightarrow S$
$y_i \longmapsto x_i t$

(Note: $y_i^2 \longmapsto x_i^2 t^2$
$x_i y_i \longmapsto x_i^2 t$)

induces $\dfrac{A[y_1,\dots,y_n]}{\langle x_i y_j - x_j y_i \mid i,j=1,\dots,n\rangle} \cong S$

$\Rightarrow \operatorname{Proj} S = \operatorname{Proj} \dfrac{A[y_1,\dots,y_n]}{\langle x_i y_j - x_j y_i \mid i,j=1,\dots,n\rangle}$

• $X = \mathbb{A}^n_k$, $\quad Y = V(x_1,\dots,x_m)$, $\quad \operatorname{codim} Y = m$



$Bl_Y X = \{(x,y) \in \mathbb{A}^n_k \times \mathbb{P}^{m-1}_k \mid x_i y_j - x_j y_i = 0,\ i,j=1,\dots,m\}$
$\downarrow \pi$
X

•• $p \in Y$, $\pi^{-1}(p) = \{p\} \times \mathbb{P}^{m-1}_k$ $\quad$ i.e. $\pi^{-1}(Y) = \underbrace{\overset{\dim n-m}{Y} \times \mathbb{P}^{m-1}_k}$ ← dim $n-1$ ⇝ a divisor.

•• $Bl_Y X \setminus \pi^{-1}(Y) \cong X \setminus Y$
$((a_1,\dots,a_n), [a_1:\cdots:a_m]) \leftrightarrow (\underbrace{a_1,\dots,a_m}_{\text{at least one } a_i \neq 0},\dots,a_n)$

•• $I_Y = \langle x_1,\dots,x_m\rangle$, $\quad S = \bigoplus_{d\geq 0} I^d$, $\quad \varphi: A[y_1,\dots,y_m] \to S$
$y_i \longmapsto x_i t$, $i=1,\dots,m$

$\Rightarrow \operatorname{Proj} S = Bl_Y X$.

PAGE
DATE

- $X = \operatorname{Spec} A$, $Y = V(f_1, \cdots, f_m)$, $\{f_1, \cdots, f_m\}$: a regular sequence in A.

$I = \langle f_1, \cdots, f_m \rangle$

$$S = \bigoplus_{d\geq 0} I^d \cong \frac{A[y_1, \cdots, y_m]}{\langle f_i y_j - f_j y_i : i,j = 1, \cdots, m \rangle} \quad \text{via} \quad \begin{aligned} A[y_1, \cdots, y_m] &\to S \\ y_i &\mapsto f_i t \end{aligned}$$

$\Rightarrow \operatorname{Proj} S = Bl_Y X$.

Def: X : noeth ; Y : closed subscheme of X, $\mathcal{I}_Y \in \operatorname{coh}(X)$.

Define $\tilde{X} = Bl_Y X := \operatorname{proj}(\underbrace{\bigoplus_{d\geq 0} \mathcal{I}_Y^d t^d}_{= \mathcal{G}})$ and $\pi : \tilde{X} \to X$ be the associated proper map.

Remark : For $p \in X$, $p \in U = \operatorname{Spec} A \subset X$, $\mathcal{G}(U) = A[s_0, \cdots, s_n]$, with $\mathcal{G}_1(U) = \langle s_0, \cdots, s_n \rangle_A$

$Y \cap U = \operatorname{Spec} A/I$, $\mathcal{I}_Y(U) = I = \langle s_0, \cdots, s_n \rangle_A$, $(\bigoplus_{d\geq 0} \mathcal{I}_Y^d)(U) = A[s_0, \cdots, s_n]$

$\mathcal{G}_p = A_p[(s_0)_p, \cdots, (s_n)_p]$

Now if $p \in Y = V(I)$, i.e. $p \supset I$, then $I_p \subset P_p$ & $I_p = \langle (s_0)_p, \cdots, (s_n)_p \rangle_{A_p}$

but if $p \notin Y = V(I)$, i.e. $p \not\supset I$, then $I_p = A_p$ & $\bigoplus_{d\geq 0} I_p^d t^d = A_p \oplus A_p t \oplus A_p t^2 \oplus \cdots$
$= A_p[t]$

$\Rightarrow \operatorname{Proj} A_p[t] = \operatorname{Spec} A_p$

Prop 6 : (1) $\pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$.

(pf) : Note that $\mathcal{O}_X \xrightarrow{\pi^\#} \pi_* \mathcal{O}_{\tilde{X}} \rightsquigarrow \pi^{-1}\mathcal{O}_X \to \mathcal{O}_{\tilde{X}}$

so $\pi^{-1}\mathcal{I}_Y \hookrightarrow \pi^{-1}\mathcal{O}_X \Rightarrow \underbrace{\pi^{-1}\mathcal{I}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{\tilde{X}}}_{= \pi^*\mathcal{I}_Y} \longrightarrow \pi^{-1}\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$

$$\left(\begin{aligned} A \otimes_A B &\to B \\ a \otimes b &\mapsto ab \end{aligned} \right)$$

the image of $\pi^*\mathcal{I}_Y$ to $\mathcal{O}_{\tilde{X}} := \pi^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{X}}$.

For $\underset{\text{spec} A}{\overset{\shortparallel}{U}} \subset X$, $\quad Y \cap U = \text{spec}\, A/I$,

$$\text{Proj}\, \mathcal{J}(U) = \text{Proj}\,(\bigoplus_{d \geq 0} I^d) \xrightarrow{\pi_U} U$$

Here, $\mathcal{J}_Y|_U = \tilde{I}$, $\quad \pi_U^* \mathcal{J}_Y|_U = \widetilde{I \otimes_A \mathcal{J}(U)}$

$$\Rightarrow \pi_U^{-1} \mathcal{J}_Y|_U \cdot \mathcal{O}_{\text{Proj}\, \mathcal{J}(U)} = \widetilde{I \cdot \mathcal{J}(U)} = \widetilde{\bigoplus_{d \geq 0} I^{d+1}} = \widetilde{\mathcal{J}(U)(1)}$$

$$= \mathcal{O}_{\text{Proj}\, \mathcal{J}(U)}(1)$$

$\Rightarrow \pi^{-1}\mathcal{J}_Y \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(1)$ is invertible on $\tilde{X}$. $\quad *$

(2) $\pi^{-1}(X \setminus Y) \simeq X \setminus Y$

(pf): Since $\mathcal{J}|_{X \setminus Y} \hookrightarrow \mathcal{O}_{X \setminus Y}$ and $\forall p \in X \setminus Y$, $\exists\, s \in \Gamma(X \setminus Y, \mathcal{J}|_{X \setminus Y})$ s.t. $s_p \notin m_p$,

$\mathcal{J}_p \cong \mathcal{O}_p$ and thus $\mathcal{J}|_{X \setminus Y} \cong \mathcal{O}_{X \setminus Y}$.

$\quad (\bigoplus_k \mathcal{J}|_{X \setminus Y} t^k = \mathcal{O}_{X \setminus Y} \oplus \mathcal{O}_{X \setminus Y} t \oplus \mathcal{O}_{X \setminus Y} t^2 \oplus \cdots)$

Hence $\pi^{-1}(X \setminus Y) = \text{Proj}\, \mathcal{O}_{X \setminus Y}[T] \cong X \setminus Y$. $\quad *$

(3) If $f: Z \to X$ s.t. $\underset{\mathcal{L}}{\overset{\shortparallel}{f^{-1}\mathcal{J}_Y \cdot \mathcal{O}_Z}} \in \text{Pic}\, Z$, then $\exists!$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

(pf): <u>Uniqueness</u>: $f^*\mathcal{J}_Y = g^*\pi^*\mathcal{J}_Y \rightsquigarrow f^{-1}\mathcal{J}_Y \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}\mathcal{J}_Y \cdot \mathcal{O}_{\tilde{X}}) \cdot \mathcal{O}_Z = g^{-1}(\mathcal{O}_{\tilde{X}}(1)) \cdot \mathcal{O}_Z$

By def, $\quad g^*\mathcal{O}_{\tilde{X}}(1) \twoheadrightarrow \underset{f^{-1}\mathcal{J}_Y \cdot \mathcal{O}_Z}{\overset{\shortparallel}{g^{-1}(\mathcal{O}_{\tilde{X}}(1)) \cdot \mathcal{O}_Z}}$

Both are invertible, so surj $\Rightarrow$ isom. $\quad \left(\mathcal{L} \twoheadrightarrow \mathcal{L}',\ \mathcal{L}_p \to \mathcal{L}'_p \to 0,\ \mathcal{L}_p \cong \mathcal{O}_p,\ \mathcal{L}'_p \cong \mathcal{O}_p \Rightarrow \mathcal{L}_p \cong \mathcal{L}'_p \right)$

Hence g is uniquely determined by $\mathcal{L}$ ~~and $\{s_i = g^* x_i\}$ where $\{x_i\}$ is the generating set of global sections of $\mathcal{O}_{\tilde{X}}(1)$.~~

as follows.

Existence: By the uniqueness of g, we can assume that $X = \operatorname{Spec} A$ & $\mathcal{I}_Y = \tilde{I}$.

Let $I = \langle a_0, \dots, a_n \rangle$ and $S = \bigoplus_{d \geq 0} I^d$.

Then
$$\begin{aligned} \varphi : A[x_0, \dots, x_n] &\xrightarrow{\text{graded}} S \\ x_i &\longmapsto a_i \end{aligned} \qquad \text{induces} \quad \tilde{X} = \operatorname{Proj} S \hookrightarrow P^n_A .$$

Note that $\operatorname{Ker} \varphi = \{ F \in A[x_0, \dots, x_n] \mid F(a_0, \dots, a_n) = 0 \}$ a homogeneous ideal

and $a_i \in \Gamma(X, \mathcal{I}_Y) \rightsquigarrow s_i \in \Gamma(Z, f^{-1}\mathcal{I}_Y \mathcal{O}_Z)$

$\qquad \qquad \qquad \qquad \qquad \qquad \parallel$ the image of $f^* a_i$ under $f^* \mathcal{I}_Y \to \mathcal{O}_Z$.

We find that $a_0, \dots, a_n$ generate $\mathcal{I}_Y \Rightarrow s_0, \dots, s_n$ generate $\mathcal{L}$.

Hence $\exists ! \ Z \xrightarrow{\bar{g}} P^n_A$ s.t. $\mathcal{L} \cong \bar{g}^* \mathcal{O}_{P^n_A}(1)$ & $s_i = \bar{g}^* x_i$

Now,
$$A \to A\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\right] \to \Gamma(Z_{s_i}, \mathcal{O}_{Z_{s_i}}) \rightsquigarrow A \to A\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_n}{x_i}\right] / \operatorname{Ker} \varphi_{(i)} \to \Gamma(Z_{s_i}, \mathcal{O}_{Z_{s_i}})$$

$$\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i}$$

dehomogenize w.r.t. x_i

$F(a_0, \dots, a_n) = 0$ in $A \Rightarrow F(s_0, \dots, s_n) = 0$ in $\Gamma(Z, \mathcal{L}^d)$

of degree d

$D_+(x_i)$ in P^n_A $\qquad$ $D_+(\bar{x}_i)$ in $\operatorname{Proj} S$

$$\Rightarrow \quad \begin{tikzcd} Z \arrow[r, "g"] \arrow[rr, bend left, "\bar{g}"] \arrow[dr, "f"'] & \operatorname{Proj} S \arrow[r, hook] \arrow[d, "\pi"] & P^n_A \arrow[dl] \\ & X = \operatorname{Spec} A & \end{tikzcd}$$

$\star$.

Prop 7 (Commute with base change)

Let $f: Z \to X$ and $\mathcal{I}' = f^{-1}\mathcal{I}_Y \cdot \mathcal{O}_Z$
$\wr$
$\mathcal{I}_W$ with $W \hookrightarrow Z$, $W = f^{-1}(Y)$
closed subscheme.

Then

$$\begin{tikzcd}
Z \times_X \tilde{X} \arrow[rr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & \overline{p_1^{-1}(Z \setminus W)} = \tilde{Z} = \mathrm{Bl}_W Z \arrow[l] \arrow[r, "\bar{f}"] \arrow[d, "\pi_2"'] & \mathrm{Bl}_Y X = \tilde{X} \arrow[d, "\pi_1"] \\
& Z \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}$$

$\exists ! \, \bar{f}$ & if f is a closed immersion then so is $\bar{f}$.

(pf):

- $$(f\pi_2)^{-1}\mathcal{I}_Y \cdot \mathcal{O}_{\tilde{Z}} = \pi_2^{-1}(f^{-1}\mathcal{I}_Y \cdot \mathcal{O}_Z)\cdot \mathcal{O}_{\tilde{Z}} = \pi_2^{-1}\mathcal{I}_W \cdot \mathcal{O}_{\tilde{Z}}$$
  is invertible, so $\exists ! \, \bar{f}$.
- $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/J$, $\mathcal{I}_Y = \tilde{I}$.
  $W = V(J) \cap V(I) = V(I+J)$, $\mathcal{I}_W = \widetilde{I+J/J}$
  So $$I \twoheadrightarrow \frac{I+J}{J} \rightsquigarrow \bigoplus_{d \geq 0} I^d \twoheadrightarrow \bigoplus_{d\geq 0}\left(\frac{I+J}{J}\right)^d$$
  $$\rightsquigarrow \tilde{Z} \hookrightarrow \tilde{X}$$

#

Def: $Z \hookrightarrow X$, $\tilde{Z}$ is called the strict transform of Z under π_1.

⊙ $Bl_Y X$, X: a variety, $Y \underset{\neq}{\hookrightarrow} X$
(integral, separated, of finite type, over $k = \bar{k}$)

Prop 8:
- $\tilde{X}$ is also a variety
- π is birational, proper, surjective

If X is quasi-projective (resp. projective) over k, then ~~so is $\tilde{X}$~~ & π is projective & $\tilde{X}$ is also ~~quasi-projective~~ (resp. projective) (pf): • X is integral $\Rightarrow \mathcal{O}_X$ is a sheaf of integral domains

$(\Rightarrow \mathcal{J}_Y)$ "

$\Rightarrow \bigoplus_{d\geq 0} \mathcal{J}_Y^d$ "

$\Rightarrow \tilde{X}$ is integral.

• By prop 8, π is proper i.e.

proper $\Rightarrow$ separated, of finite type

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[dr] & X \arrow[d, "\text{sep. of finite type}"] \\ & \operatorname{Spec} k \end{tikzcd}$$

so it is sep. of finite type.

Hence $\tilde{X}$ is a variety.

• $\pi: \pi^{-1}(X \setminus Y) \simeq \underbrace{X \setminus Y}_{\text{open in } X} \Rightarrow \pi$ is birational

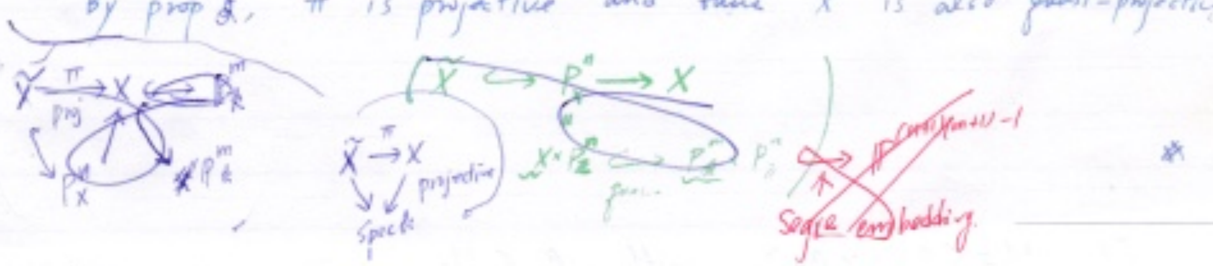
• π is proper $\Rightarrow \pi(\tilde{X})$ is closed in X and $\pi(\tilde{X}) \supset U$ ($U = X \setminus Y \subset X$)

$\Rightarrow \pi(\tilde{X}) \supset \overline{U} = X \Rightarrow \pi(\tilde{X}) = X$

irr $\Rightarrow$ U: dense in X.

• X is quasi-projective $\Rightarrow \exists$ ample sheaf on X

By prop 8, π is projective and thus $\tilde{X}$ is also quasi-projective.

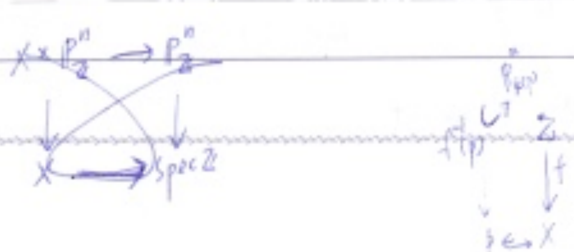


<u>Thm 9</u>: Let X be quasi-proj over k.

If $f: \underset{\text{any variety}}{Z} \to X$ is birational projective, then $\exists\, Y \underset{\text{closed}}{\hookrightarrow} X$

s.t. $Z \cong Bl_Y X$ and

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, phantom, "\cong"] & X \\ Bl_Y X \arrow[ur, "\pi"'] & \end{tikzcd}$$



PAGE

DATE

Lemma 1: Let S be a finitely generated graded (by S_1) A-algebra and $S^{(e)}$ be a graded A-alg defined by $(S^{(e)})_d = S_{ed} \quad \forall\, d \geq 0$.

Then $\operatorname{Proj} S^{(e)}$ is homeomorphic to $\operatorname{Proj} S$.

(pf):
$$i : S^{(e)} \hookrightarrow S \Rightarrow \bar{i} : \operatorname{proj} S \longrightarrow \operatorname{Proj} S^{(e)}$$

Since, any $\alpha \in S_1$, $\alpha \notin p$ with $p \in \operatorname{Proj} S \Rightarrow \alpha^e \notin \bar{i}(p)$,
$$\bar{i}(D_+(\alpha)) \subseteq D_+(\alpha^e).$$

Also,
$$S^{(e)}_{(\alpha^e)} \cong S_{(\alpha)}$$
$$\frac{f\cdot\alpha^{(e-1)d}}{(\alpha^e)^d} \longleftarrow \frac{f}{\alpha^d}$$

In fact, $p' \in \operatorname{Proj} S^{(e)}$,

$P = \bigoplus_{d\geq 0} P_d$, $P_d = \{x \in S_d : x^e \in p'\}$

Hence
$$\operatorname{Proj} S \xrightarrow{\sim} \operatorname{Proj} S^{(e)}$$

$\operatorname{Proj} S \ni P$ & $P \cap S^{(e)} = p'$.

$D_+(\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_r^{e_r}) = D_+((\alpha_1^{e_1}\cdots\alpha_r^{e_r})^e) = D_+(\alpha_1^e)\cap\cdots\cap D_+(\alpha_r^e)$

Lemma 2: Let $X = \operatorname{proj} S$ with S as above and M : f.g. graded S-module.

Then
$$M_d \xrightarrow{\sim} \Gamma(X, \widetilde{M}(d)) \quad \text{for } d \gg 0.$$

(pf): Note : $M_0 \longrightarrow M_{(\alpha)} \rightsquigarrow M_0 \to \Gamma(X, \widetilde{M})$; $M_d = M(d)_0 \longrightarrow M(d)_{(\alpha)} \rightsquigarrow M_d \to \Gamma(X, \widetilde{M}(d))$
$$x \longmapsto \frac{x}{1} \qquad\qquad x \longmapsto \frac{x}{1}$$

Let $M = \langle a_1, \dots, a_n\rangle_S$ with $a_i \in M_{\ell_i}$.

If $L = S(-\ell_1) \oplus \cdots \oplus S(-\ell_n)$ and $\varphi : L \to M$
$$(r_1, \dots, r_n) \longmapsto \underbrace{r_1 a_1}_{\deg \ell_1} + \cdots + r_n a_n$$

graded free S-module.

$\deg r_1 \Leftrightarrow \deg \ell_1$

then φ preserves degree and surjective.

Hence by repeating the same procedure to $\ker\varphi$ (f.g. S-module), we have
$$L' \longrightarrow L \longrightarrow M \longrightarrow 0 \quad \text{exact}$$

graded free S-module.
$$\Rightarrow \widetilde{L'} \to \widetilde{L} \to \widetilde{M} \to 0$$

Serre

(Vanishing theorem : $X = \mathrm{proj}\, S$, $\mathcal{F} \in \mathrm{Coh}(X)$.

$\Rightarrow H^i(X, \mathcal{F}(m)) = 0 \quad \forall i > 0 \ \& \ m \gg 0$)

$\mathcal{F} \cong \tilde{L}'$

For $d \gg 0$,

$$\begin{tikzcd} \Gamma(X, \tilde{L}'(d)) \arrow[r] & \Gamma(X, \tilde{L}(d)) \arrow[r] & \Gamma(X, \tilde{M}(d)) \arrow[r] & 0 \\ L'_d \arrow[r] \arrow[u, "\phi'"] & L_d \arrow[r] \arrow[u, "\phi"] & M_d \arrow[r] \arrow[u, "\psi"] & 0 \end{tikzcd}$$

Then ϕ', ϕ : isom $\Rightarrow \psi$ is an isom.

It suffices to show this lemma for $M = S(l)$, $l \in \mathbb{Z}$.

Let $S = A[a_0, \cdots, a_N]$ and $A[X_0, \cdots, X_N] \twoheadrightarrow S \rightsquigarrow X \xhookrightarrow{j} \mathbb{P}^N_A$

$X_i \longmapsto a_i$

We have $\mathcal{O}_X(d) = j^*(\mathcal{O}_{\mathbb{P}^N_A}(d))$ and $j_*(\mathcal{F}(d)) = j_*\mathcal{F}(d)$ $\forall d$

$j_*(\mathcal{F} \otimes j^*(\mathcal{O}_{\mathbb{P}^N_A}(d))) = j_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^N_A}(d)$

$$\Rightarrow \Gamma(X, \underbrace{\mathcal{O}_X}_{=\tilde{S}}(d)) = \Gamma(\mathbb{P}^N_A, (\underbrace{j_*\mathcal{O}_X}_{=\tilde{S} \text{ as } A[X_0, \cdots, X_N]\text{-mod}})(d)) \quad \forall d$$

$$\uparrow \wr \leftarrow \text{for } d \gg 0$$

$$S_d$$

Hence we can reduce ~~this~~ it to $S = A[X_0, \cdots, X_N]$.

And ~~since~~ it follows from the well-known fact

$\Gamma(X, \mathcal{O}_X(d)) = S_d$ ※.

$$\begin{tikzcd} \mathcal{L} = i^*\mathcal{O}(1) \arrow[d] & & \\ Z \arrow[r, hook, "i"] \arrow[dr, "f"] \arrow[ddr, bend right] & P^n_X \arrow[d] \\ & X \\ & f_*\mathcal{L} \end{tikzcd}$$

- $Z \xrightarrow{f} X$ projective

$\mathcal{F} \in Coh(Z) \Rightarrow f_*\mathcal{F} \in Coh(X)$

$X = \operatorname{spec} A$, $f_*\mathcal{F} \in Qcoh(X)$

$f_*\mathcal{F} = \widetilde{\Gamma(X, f_*\mathcal{F})} = \widetilde{\Gamma(Z, \mathcal{F})}$

$\Gamma(Z, \mathcal{F})$: f.g. A-mod

$\therefore \mathcal{F} = \mathcal{O}_X \oplus \bigoplus_{d\geq 1} f_*\mathcal{L}^d \in Qco(X)$.

- Claim: $\mathcal{F}^{(e)} = \bigoplus_{d\geq 0} \mathcal{F}_d^{(e)}$, $\mathcal{F}_d^{(e)} := \mathcal{F}_{de}$ is g.b. $\mathcal{F}_1^{(e)}$ as $\mathcal{O}_X$-alg.

(pf): X is quasi-compact, ~~so we can~~ i.e. $X = \underbrace{\bigcup_{i=1}^n \operatorname{Spec} A_i}_{= U_i}$, A_i : f.g. k-alg.

Assume $X = \operatorname{spec} A$, A : f.g. k-alg.

Then $Z = \operatorname{Proj} \underbrace{\frac{A[x_0, \cdots, x_n]}{I_Z}}_{= T} \hookrightarrow P^n_A$

and $\mathcal{L} = \mathcal{O}_Z(1)$, $\mathcal{F} = \widetilde{\Gamma(X, \mathcal{F})} = \widetilde{\underbrace{A \oplus \bigoplus_{d\geq 1} \Gamma(Z, \mathcal{O}_Z(d))}_{= S}}$

Lemma 2 $\Rightarrow$ $T_d = \Gamma(Z, \mathcal{O}_Z(d))$ $\forall d \geq e$ $\quad P^n_k \simeq P^N_k \; P^n_k \hookrightarrow P^N_k$

$T^{(e)}$ is g.b. $T_1^{(e)}$ as A-alg. $\quad \operatorname{proj} A[x_0, \cdots, x_n] \to \operatorname{proj} A[x_0, \cdots, x_n]^{(e)}$

$\Rightarrow S^{(e)}$ is g.b. $S_1^{(e)}$ as A-alg. $\quad \operatorname{Proj} S = \operatorname{Proj} S^{(e)}$

i.e. $\mathcal{G}^{(e)}$ is g.b. $\mathcal{F}_1^{(e)}$ as $\mathcal{O}_X$-alg.

- $\mathcal{G}^{(e)} \hookrightarrow \mathcal{F}$ & $\operatorname{Proj} \mathcal{F} \xrightarrow{\sim} \operatorname{Proj} \mathcal{G}^{(e)}$

e-uple embedding.

$\operatorname{Proj} T^{(e)} \subset \operatorname{Proj} A[x_0, \cdots, x_n]^{(e)}$

$A[y_0, \cdots, y_N] \to A[x_0, \cdots, x_n]^{(e)}$

$y_i \to x_0^{i_0} \cdots x_n^{i_n}$, $i_0 + \cdots + i_N = e$

$\operatorname{Proj} A[x_0, \cdots, x_n]^{(e)} \hookrightarrow P^N_A$

$\operatorname{Proj} A[x_0, \cdots, x_n] = P^n_A$

$$\left\{ \begin{array}{ll} P \in \operatorname{Proj} T, & P \cap T^{(e)} \in \operatorname{Proj} T^{(e)} \\ P' \in \operatorname{Proj} T^{(e)}, & P = \bigoplus_{n\geq 0} P_n, \; P_n = \{x \in T_n : x^e \in P'\} \\ & \in \operatorname{Proj} T \;\&\; P \cap T^{(e)} = P' \end{array} \right\}$$

$$\begin{tikzcd} \mathcal{L} = i^*\mathcal{O}(1) & & \mathcal{O}(1) \arrow[d, "\text{e-uple embedding}"] \\ Z \arrow[r, hook] \arrow[dr, "f"] \arrow[d, bend right] & P^n_X \arrow[r, hook] & P^N_X \arrow[dl] \\ \mathcal{F} \arrow[r, squiggly] & X & \end{tikzcd}$$

$\mathcal{F}$: g.b. $\mathcal{F}_1$ as $\mathcal{O}_X$-alg.

& $Z \cong \operatorname{Proj} \mathcal{F}$

$Z = \operatorname{Proj} T \cong \operatorname{Proj} T^{(e)} = \operatorname{Proj} \mathcal{F}^{(e)}(X) \cong \operatorname{Proj} \mathcal{F}(X)$

$\because f$ is birational
$\Downarrow$

- Z : integral, $\quad \mathcal{L} \hookrightarrow K_Z \rightsquigarrow f_*\mathcal{L} \hookrightarrow f_* K_Z = K_X$

Let $\mathcal{M}$ be ample on X. $\quad \mathcal{J}(U) := \{ s \in \mathcal{O}(U) : s \cdot f_*\mathcal{L}(U) \subseteq \mathcal{O}(U) \}$

$\Rightarrow \mathcal{J} \in \mathrm{Coh}(X)$ since $f_*\mathcal{L} \in \mathrm{Coh}(X)$

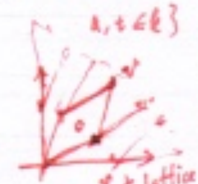
$\Downarrow$ locally, $\langle s_1, \cdots, s_m \rangle$ $\uparrow$ 12.88

Then $\mathcal{J} \otimes \mathcal{M}^n$ is g.b.g.s $\quad \forall n \gg 0$

Consider $\quad \mathcal{O}_X \hookrightarrow \mathcal{J} \otimes \mathcal{M}^n \rightsquigarrow \mathcal{M}^{-n} \hookrightarrow \mathcal{J}$
$\quad 1 \longmapsto s$

$$\mathcal{M}^{-n} \cdot f_*\mathcal{L} \subseteq \mathcal{O}_X$$

$\{ (x, y, w, z) = (u^4, u^3 t, u t^3, t^4) \mid u, t \in k \}$



Now $\quad Z = \mathrm{Proj}\, \mathcal{F} \cong \mathrm{Proj}\, \mathcal{F} * \mathcal{M}^{-n}$

$$(\mathcal{F} * \mathcal{M}^{-n})_1 = f_*\mathcal{L} \otimes \mathcal{M}^{-n} = f_*\mathcal{L} \cdot \mathcal{M}^{-n} =: \mathcal{I} \hookrightarrow \mathcal{O}_X$$

$$\& \ (\mathcal{F} * \mathcal{M}^{-n})_d = f_*\mathcal{L}^d \otimes \mathcal{M}^{-dn} \cong \mathcal{I}^d$$

$u^?, u^? t,$

$$\text{i.e.} \quad Z \cong \mathrm{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d$$

$\mathcal{I}^d \nearrow \overset{\shortparallel}{f_*\mathcal{L}^d \cdot \mathcal{M}^{-dn}}$ generated by $f_*\mathcal{L}$

$\because \mathcal{I}^d \hookrightarrow K_X$
$f_*\mathcal{L}^d \cdot \mathcal{M}^{-dn} \hookrightarrow K_X$
$\therefore$ it is injective.

Prop 10. $\quad X \longrightarrow \mathrm{Spec}\, A$, $\quad$ ~~A: noeth~~ X: noeth

$\mathcal{L}$: invertible on X, $\quad s_0, \cdots, s_n \in \Gamma(X, \mathcal{L})$

$$\varphi : \ U = \bigcup_{i=0}^{n} X_{s_i} \xrightarrow{\ \varphi\ } \mathbb{P}^n_A$$

$\mathcal{L}|_V \xrightarrow{\sim} \mathcal{O}_V$

$s_i|_V \to s_i^{(V)}$ $\quad \mathcal{I}|_V$

$\langle s_0|_V, \cdots, s_n|_V \rangle \leftrightarrow \overset{\shortparallel}{\langle s_0^{(V)}, \cdots, s_n^{(V)} \rangle} \hookrightarrow \mathcal{O}(V)$

$\exists \tilde{\varphi}$

glue $\Downarrow$

$Y = X - U$, $\quad \mathcal{I} \to X$

$\mathcal{I} = \langle s_0, \cdots, s_n \rangle \quad \mathcal{I} = \mathcal{I}_Y$ since

$\pi \uparrow$

$\mathcal{O}_{\tilde{X}}(1) = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}} \quad \tilde{X}$
invertible

$\mathcal{I}|_V = \mathcal{I}_{Y \cap V}$

& g.b. $\pi^* s_0, \cdots, \pi^* s_n$

In case X is a nonsingular proj var over k,
The set of the base points of $|D_0| = Y$, i.e. $\forall p \in Y \quad p \in \mathrm{supp}\, D \quad \forall D \in |D_0|$

$\tilde{X}$ $\quad$ extend $|D_0|$ to a base point-free linear system