Complex Analysis I

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2021-1st

Contents

Chapter 1

Not sure yet

1.1 Basics of analytic functions

1.1.1 Cauchy-Riemann equation

Let Ω be a connect open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$.

Definition 1.1.1. For $a \in \Omega$,

- $\lim\limits_{z \in a} f(z) = A \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall z \in \Omega$ and $0 < |z - a| < \delta \rightsquigarrow |f(z) - A| < \varepsilon$.
- $f(z)$ is **continuous** at a if $\lim\limits_{z \to a} f(z) = f(a)$.
- $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ provide the limit exists.

Observation : $f(z) = u(z) + iv(z)$ can be regard as $f(x, y) = u(x, y) + iv(x, y)$ where $z = x + iy$, then $f : (x, y) \mapsto (u(x, y), v(x, y))$. Recall that

- f is conti. at $z_0 = (x_0, y_0) \iff u, v$ are conti. at z_0
- f is differentiable at $z_0 \implies f$ is conti. at z_0.

Also, $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2 = d((x, y), (x_0, y_0))^2$, so we have same result what we learn in calculus in $\mathbb{R}^2$.

Now we see some different between $\mathbb{C}$ and $\mathbb{R}^2$:

Theorem 1.1.1 (Cauchy Riemann equation)**.** Let $u, v \in C^1(\Omega)$. Then

$$f \text{ is differentiable } \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}$$

Proof: Let $z = x + iy \in \Omega$. Since Ω is open, we can that the line segment $\overline{z(z + \Delta z)} \subseteq \Omega$, where $\Delta z = \Delta x + i\Delta y$.

- $(\Rightarrow)$: Since $f'(z)$ exists, we have

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - (u(x, y) + iv(x, y))}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$

and

$$f'(z) = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - (u(x, y) + iv(x, y))}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{1}{i}\frac{\partial f}{\partial y}$$

Hence we have $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$ or $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$.

- $(\Leftarrow)$: Since u, v are differentiable,

$$\begin{cases} \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + |\Delta z|\psi_1(\Delta z) \\ \Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + |\Delta z|\psi_2(\Delta z) \end{cases}$$

where $\psi_1(\Delta), \psi_2(\Delta z) \to 0$ as $\Delta z \to 0$. Combine with assumption we have

$$\begin{aligned} \frac{\Delta f}{\Delta z} = \frac{\Delta u}{\Delta z} + i\frac{\Delta v}{\Delta z} &= \frac{1}{\Delta z}\left(\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(\Delta x + i\Delta y) + |\Delta z|(\psi_1(\Delta z) + \psi_2(\Delta z))\right) \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + \underbrace{\frac{|\Delta z|}{\Delta z}(\psi_1(\Delta z) + \psi_2(\Delta z))}_{\to 0 \text{ as } \Delta z \to 0} \end{aligned}$$

Hence $\lim\limits_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ exists.

□

Definition 1.1.2.

- $f(z)$ is said to be **analytic** in open set Ω if it has derivative at each point of Ω.
- $f(z)$ is called an **entire function** if it is analytic in $\mathbb{C}$.

Example 1.1.1. $f(z) = \mathrm{Re}z$ is continuous but nowhere analytic since $\frac{\partial u}{\partial x} = 1 \neq 0 = \frac{\partial v}{\partial y}$.

Corollary 1.1.1. If $f'(z) = 0$ in open connected subset Ω, then f is constant in Ω.

Proof: By $f'(z) = 0$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ and thus $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Given point $z, z' \in \Omega$, use line segment $\overline{z_k z_{k+1}}$ which parallel x-axis or y-axis to connect $z := z_0, z' := z_n$. By partial derivative of u, v with respect to x, y are all zero we have

$$f(z) = f(z_1) = \cdots = f(z_{n-1}) = f(z')$$

□

1.1.2 Change of coordinate

1. complex conjugate

If $z = x + iy$, $\overline{z} = x - iy$, then $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2} \rightsquigarrow f(x,y) = f(z,\overline{z})$.
By chain rule,

$$\begin{cases} \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \overline{z}} = \frac{\partial x}{\partial \overline{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \overline{z}}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \end{cases} \implies \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \\ \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right) \end{cases}$$

Hence, f is differentiable $\iff \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \iff \frac{\partial f}{\partial \overline{z}} = 0$. So $f'(z) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}$.

Example 1.1.2. $f(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z] \rightsquigarrow f'(z) = n a_n z^{n-1} + \cdots + a_1$

Theorem 1.1.2 (Lucas's theorem). The half plane that contains the zeros of $f(z)$ also contains the zeros of $f'(z)$.

Proof: Recall that give two point a, b on line ℓ with $b \neq 0$, then we can write ℓ by $z = a + tb$ with $t \in \mathbb{R} \iff \mathrm{Im}\left(\frac{z-a}{b}\right) = 0$. So two half plane cut by ℓ are

$$H^+ := \mathrm{Im}\left(\frac{z-a}{b}\right) > 0 \text{ and } H^- := \mathrm{Im}\left(\frac{z-a}{b}\right) > 0$$

Let $f(z) = a_n(z-\alpha_1)\cdots(z-\alpha_n)$ with $\alpha_i \in H^-$ $\forall i = 1, ..., n$. Notice that

$$\frac{f'(z)}{f(z)} = \frac{1}{z-\alpha_1} + \cdots + \frac{1}{z-\alpha_n}$$

Assume $z_0 \in H^+ \cup \ell$ i.e. $\mathrm{Im}(\frac{z_0-a}{b}) \geq 0$, then

$$\mathrm{Im}\left(\frac{z_0-\alpha_i}{b}\right) = \mathrm{Im}\left(\frac{z_0-a}{b}\right) - \mathrm{Im}\left(\frac{\alpha_i-a}{b}\right) > 0 \implies \mathrm{Im}\,\frac{b}{z_0-\alpha_i} < 0$$

Hence $\mathrm{Im}\,\frac{bf'(z_0)}{f(z_0)} = \sum_{i=1}^{n} \mathrm{Im}\,\frac{b}{z_0-\alpha_i} < 0$ i.e. $bf'(z_0) \neq 0 \rightsquigarrow f'(z_0) \neq 0$. □

2. polar coordinate

Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{cases} \rightsquigarrow f(x,y) = f(r,\theta).$

$$\begin{cases} \frac{\partial r}{x} = \frac{x}{r} = \cos\theta \\ \frac{r}{y} = \frac{y}{r} = \sin\theta \end{cases} \text{ and } \begin{cases} \frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r} \\ \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r} \end{cases}$$

$$\implies \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \text{ and } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Hence

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x} \end{cases} \implies \begin{cases} \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta} & (1) \\ \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta} = -\cos\theta\frac{\partial v}{\partial r} + \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta} & (2) \end{cases}$$

$$\begin{cases} (1)\times\cos\theta + (2)\times\sin\theta & \implies r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial\theta} \\ -(1)\times\sin\theta + (2)\times\cos\theta & \implies r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial\theta} \end{cases}$$

and

$$f'(z) = (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \left(\frac{\cos\theta - i\sin\theta}{r}\right)\left(\frac{\partial v}{\partial\theta} - i\frac{\partial v}{\partial\theta}\right)$$

1.1.3 Power series

Recall :

- $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, let $a_n = \max\{\alpha_1, ..., \alpha_n\} \rightsquigarrow a_n \nearrow$ (non-decrasing), so $\exists A_1$ s.t. $\lim_{n\to\infty} a_n = A_1$ (**least upper bound** or **supremum**). Let $A_k = \sup\{\alpha_n\}_{n=k}^{\infty} \rightsquigarrow A_k \searrow$ (non-increasing), so we can define **limit superior**

$$\overline{\lim}\alpha_n = \lim_{k\to\infty} A_k = A \in \mathbb{R} \text{ or } \pm\infty$$

If $A \in \mathbb{R}$, then by definition, $\forall \varepsilon > 0$, $\exists n_0$ s.t. $n \geq n_0$, $A_n < A + \varphi \rightsquigarrow \alpha_n < A + \varepsilon$. Similarly, we can define **limit inferior** by

$$\underline{\lim} \alpha_n = \lim_{k \to \infty} \inf_{n \geq k} \alpha_n = B$$

If $B \in \mathbb{R}$, then $\forall \varepsilon > 0$, $\exists n_0$ s.t. $n \geq n_0$, $\alpha_n > B - \varepsilon$.

- $\{\alpha_n\}$ converge ($\overline{\lim} \alpha = \underline{\lim} \alpha$) $\iff$ $\{\alpha_n\}$ is a Cauchy sequence :

  Proof:

  - $(\Rightarrow)$: $\forall \varepsilon > 0$, $\exists n_0$ s.t. $\forall n \geq n_0, |\alpha_n - A| < \varepsilon/2 \rightsquigarrow \forall m, n \geq n_0, |\alpha_n - \alpha_m| < \varepsilon$.
  - $(\Leftarrow)$: Assume $A = \overline{\lim} \alpha_n > \underline{\lim} \alpha = B$. Let $\varepsilon = \frac{A-B}{3}$, then $\exists n_0$ s.t.

  $$\begin{cases} \forall n \geq n_0, \ B - \varepsilon < \alpha_n < A + \varepsilon \\ \forall n, m \geq n_0, \ |\alpha_n - \alpha_m| < \varepsilon \end{cases}$$

  Then $\forall n, m \geq n_0$

  $$3\varepsilon = |A - B| \leq |A - \alpha_n| + |\alpha_n - \alpha_m| + |\alpha_m - B| < 3\varepsilon \ (\rightarrow\leftarrow)$$

  □

- Let $S_n = \sum_{k=1}^{n} \alpha_k$. $\sum_{n=1}^{\infty} \alpha_n$ converges $\iff$ $\{S_n\}$ converges $\iff$ $\{S_n\}$: Cauchy. Especially $|\alpha_n| < \varepsilon$ i.e. $\lim_{n \to \infty} \alpha_n = 0$.

- Since $|\alpha_n + \cdots + \alpha_{n+p}| \leq |\alpha_n| + \cdots + |\alpha_{n+p}|$, $\sum_{n=1}^{\infty} |\alpha_n|$ converges $\implies \sum_{n=1}^{\infty} \alpha_n$ converges, which is call **absolutely convergent**.

- **Uniformly converge** : $f_n(x) \xrightarrow{\text{unif.}} f(x)$ on Ω if $\forall \varepsilon > 0, \exists n_0$ s.t. $\forall n \geq n_0$

  $$|f(x) - f_n(x)| < \varepsilon \ \forall x \in \Omega$$

- **Weierstrass M-test** : If $\forall n >> 0$, $|f_n(x)| \leq M_n \ \forall x \in \Omega$. Then

  $$\sum_{n=1}^{\infty} M_n \text{ conv.} \implies \sum_{n=1}^{\infty} f_n(x) \text{ unif. conv.}$$

Definition 1.1.3. $\sum_{n=0}^{\infty} a_n z^n$ ($a_n \in \mathbb{C}$) is called a **power series**.

Theorem 1.1.3 (Abel's 1st theorem)**.** Given $\sum_{n=0}^{\infty} a_n z^n$, $\exists \ 0 \leq R \leq \infty$ s.t.

(1) if $|z| < R$, then $\sum_{n=0}^{\infty} a_n z^n$ absolutely. converge and for $0 \leq \rho < R$, the converge is uniform for $|z| < \rho$

(2) if $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ is diverge

(3) if $|z| < R$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ converge for $|z| < R$.

Hadamard's formula : $R^{-1} = \overline{\lim} \sqrt[n]{|a_n|}$

Proof:

(1) For z with $|z| < R$, $\exists \rho$ s.t. $|z| < \rho < R \rightsquigarrow 1/\rho > 1/R$. By def of $\overline{\lim}$, $\exists n_0$ s.t. $\forall n \geq n_0$, $\sqrt[n]{|a_n|} < 1/\rho \rightsquigarrow |a_n| < \rho^{-n}$. So $|a_n z^n| < (|z|/\rho)^n$ $\forall n \geq n_0$. Since $\sum \underbrace{(|z|/\rho)}_{<1}{}^n$ converge, $\sum_{n=0}^{\infty} |a_n z^n|$ converge.

For $0 \leq \rho < R$, pick ρ' s.t. $\rho < \rho' < R \rightsquigarrow \exists n_0$ s.t. $n \geq n_0$, $|a_n| < (1/\rho')^n$. So $|a_n z^n| < (\rho/\rho')^n$ $\forall |z| \leq \rho$ and thus $\sum a_n z^n$ conv. unif. by Weierstrass M-test.

(2) For z with $|z| > R$, $\exists \rho$ s.t. $|z| > \rho > R \rightsquigarrow 1/\rho < 1/R$. Then exists infinitely n s.t. $\sqrt[n]{|a_n|} > \rho^{-1} \rightsquigarrow |a_n z^n| > \underbrace{(|z|/\rho)}_{>1}{}^n$ which is unbound i.e. $\lim_{n\to\infty} a_n z^n \neq 0$.

(3) For $|z| < R$, write $f(z) = S_n(z) + R_n(z)$ with $S_n(z) = \sum_{k=0}^{n-1} a_k z^k$. Let $f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} S_n'(z)$.

- Let $|z| < \rho < R$. $\exists n_0$ s.t. $n \geq n_0$, $|a_n| < \rho^{-n}$, then

$$|n a_n z^{n-1}| < \frac{n}{\rho} \left(\frac{|z|}{\rho} \right)^{n-1}$$

  Let $r = |z|/\rho < 1$, then by ratio test, $\sum n r^{n-1}/\rho$ converges and thus $f_1(z)$ converges in $|z| < R$.

- **Claim** : $f'(z_0) = f_1(z_0)$ for $|z_0| < R$
  subproof : For $n \geq n_0$,

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) = \left(\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right) + (S_n'(z_0) - f_1(z)) + \underbrace{\frac{R_n(z) - R_n(z_0)}{z - z_0}}_{(3)}$$

where $z \neq z_0$, $|z|, |z_0| < \rho < R$. Also

$$|(3)| = \left| \sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + \cdots + z_0^{k-1}) \right| \leq \sum_{k=n}^{\infty} \frac{k}{\rho} r^{k-1} : \text{converge, where } r = \max \left\{ \frac{|z|}{\rho}, \frac{|z_0|}{\rho} \right\}$$

$\forall\, \varepsilon > 0$,

$$\begin{cases} \exists n_1 \text{ s.t. } \forall n \geq n_1,\ |(3)| < \varepsilon/3 \\ \exists n_2 \text{ s.t. } \forall n \geq n_2,\ |S_n'(z_0) - f_1(z_0)| < \varepsilon/3 \end{cases}$$

Choose a fixed $n \geq n_0, n_1, n_2$, $\exists \delta$ s.t. $0 < |z - z_)| < \delta$,

$$\left| \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right| < \frac{\varepsilon}{3}$$

Hence, $f'(z_0)$ exists and equal to $f_1(z_0)$.

□

Theorem 1.1.4 (Abel's 2nd theorem)**.** If the convergence radius R of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is 1 and the series converges at $z = 1$, then $f(z) \to f(1)$ as $z \to 1$ in a such way that $|1-z| \big/ 1 - |z|$ is bounded.

Proof: Let $|1-z|/(1-|z|) < M$ If $\sum_{n=0}^{\infty} a_n = C$, then we consider $(a_0 - C) + \sum_{n=1}^{\infty} a_n z^n$. So we may assume "$f(1) = 0$". Write $S_n = \sum_{k=0}^{n} a_k$.

$$\begin{aligned} S_n(z) &= \sum_{k=0}^{n} a_k z^k = S_0 + \sum_{k=1}^{n} (S_k - S_{k-1}) z^k = \sum_{k=0}^{n-1} S_k (z^k - z^{k+1}) + S_n z^n \\ &= (1-z) \sum_{k=0}^{n-1} S_k z^k + S_n z^n \end{aligned}$$

$S_n z^n \to 0$ as $n \to \infty$ since $|z| < 1$ and $S_n \to 0$. For $|z| < 1$, $f(z) = \lim_{n\to\infty} S_n(z) = (1-z) \sum_{n=0}^{\infty} S_n z^n$. Let $n \geq n_0$, $|S_n| < \varepsilon$. Then

$$\begin{aligned} f(z)| &\leq |1-z| \left| \sum_{k=0}^{n_0 - 1} s_k z^k \right| + \varepsilon |1-z| \sum_{k=n_0}^{\infty} |z^k| \\ &= \leq |1-z| \left| \sum_{k=0}^{n_0 - 1} s_k z^k \right| + \underbrace{\frac{\varepsilon |1-z| |z|^{n_0}}{1-|z|}}_{<\varepsilon M} \end{aligned}$$

As $z \to 1$ subject to $|1-z|/(1-|z|) < M$, $f(z) \to 0 = f(1)$. □

1.1.4 Basic example

Problem : Solve $f'(z) = f(z)$ with $f(0) = 1$.
Ans : Write $f(z) = \sum_{n=0}^{\infty} a_n z^n \rightsquigarrow f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. By assumption, $a_{n-1} = n a_n$ and thus $a_n = 1/n! \rightsquigarrow f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $(0! = 1)$.

Definition 1.1.4. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

- $R = \overline{\lim} \sqrt[n]{n!} = \infty$:

  $$(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \geq n^n \implies \sqrt[n]{n!} \geq \sqrt{n} \ \forall n$$

  So $f(z)$ is entire.

- $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$: $(e^z \cdot e^{c-z})' = e^z e^{c-z} - e^z e^{c-z} = 0 \rightsquigarrow e^z e^{c-z}$ is a constant. Substitute $z = 0 \rightsquigarrow e^z e^{c-z} = e^c$.

- $e^z e^{-z} = e^0 = 1 \rightsquigarrow e^z \neq 0 \ \forall z$.

For $z = iy \in$ imaginary axis,

$$\begin{cases} e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\ \overline{e^{iy}} = 1 - iy + \frac{(iy)^2}{2!} - \frac{(iy)^3}{3!} + \cdots = e^{-iy} \end{cases}$$

$|e^{iy}|^2 = e^{iy}e^{-iy} = 1 \implies |e^{iy}| = 1 \implies |e^{x+iy}| = e^x$.

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right) = \cos y + i \sin y$$

$$\implies e^z = e^x(\cos y + i \sin y) \text{ and } \begin{cases} \cos y = \frac{e^{iy}+e^{-iy}}{2} \\ \sin y = \frac{e^{iy}-e^{-iy}}{2i} \end{cases}$$

Definition 1.1.5. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
($\rightsquigarrow \cos^2 z + \sin^2 z = 1$, $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$)

Definition 1.1.6. $f(z)$ has the period ω if $f(z + \omega) = f(z)$ $\forall z \in \Omega$ and $z + \omega \in \Omega$.

Proposition 1.1.1. The smallest positive period of e^{iz} is 2π. (then for $\cos z$, $\sin z$)

Proof:

- $e^{i(z+\omega)} = e^{iz} \implies e^{i\omega} = 1 \rightsquigarrow \omega \in \mathbb{R}$.
- Let $\varphi : (\mathbb{R}, +, 0) \to$ unit circle in $\mathbb{C}$ define by $y \mapsto e^{iy} = \cos y + i \sin y$, then $\ker \varphi = \langle 2\pi \rangle_{\mathbb{Z}}$.

$\square$

Now we consider the inverse function of e^z, denoted by $\log z$. $z = e^\omega$, where $z = re^{i\theta}$ and $\omega = u + iv$, then $r = e^u$ and $v = \arg z + 2k\pi$, so $\log z$ is a multiple-valued function. Note that $\arg z$ is discontinuous on the negative real axis. Let $\log z = \ln |z| + i \arg z$, $-\pi < \arg z < \pi$ which is called **principal branch**.

- $\log z$ is analytic on $\mathbb{C} \setminus \mathbb{R}^-$: $z = re^{i\theta}$, $-\pi < \theta < \pi \rightsquigarrow \log z = \ln r + i\theta$.

$$\begin{cases} r\frac{\partial \ln r}{\partial r} = \frac{\partial \theta}{\partial \theta} \\ r\frac{\partial \theta}{\partial r} = -\frac{\partial \ln r}{\partial \theta} \end{cases} \quad \text{and } \frac{1}{r}, 1, 0, 0 \text{ are conti.}$$

$$\implies (\log z)' = (\cos\theta - i \sin\theta)\left(\frac{\partial \ln r}{\partial r} + i\frac{\partial \theta}{\partial r}\right) = \frac{1}{r(\cos\theta + i \sin\theta)} = \frac{1}{z}$$

1.2 Cauchy theorem

1.2.1 Line integral

Definition 1.2.1. Let $f : \begin{array}{ccc} (a,b) & \longrightarrow & \mathbb{C} \\ t & \longmapsto & u(t) + iv(t) \end{array}$, then define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Property 1.2.1. $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Proof: Let $\theta = \arg\left(\int_a^b f(t)dt\right)$, then

$$\begin{aligned}\left|\int_a^b f(t)dt\right| &= \text{Re}\left(e^{i\theta}\int_a^b f(t)dt\right) = \int_a^b \text{Re}\left(e^{i\theta}f(t)\right)dt \\ &\leq \int_a^b |e^{i\theta}f(t)|dt = \int_a^b |f(t)|dt\end{aligned}$$

□

Definition 1.2.2.

- γ is a **piecewise smooth curve (arc)** in $\mathbb{C}$ if γ is parameterized by $z(t) = x(t) + iy(t)$, $t \in [\alpha, \beta]$ and exists a partition $\{[\alpha_i, \beta_i]\}$ of $[\alpha, \beta]$ s.t. $z|_{[\alpha_i,\beta_i]} \in C^1$.
- Let f be continuous on Ω and $\gamma \subset \Omega$, define

$$\int_\gamma f(z)dz := \sum_{i=1}^n \int_{\alpha_i}^{\beta_i} f(z(t))z'(t)dt$$

  By chain rule, the definition is independent of the choice of parameters of γ.

  - $dz = z'(t)dt = (x'(t) + iy'(t))dt$, $\overline{dz} = (x'(t) - iy'(t))dt$
  - $|dz| = \sqrt{x'(t)^2 + y'(t)^2}dt = ds$

Property 1.2.2.

$$\left|\int_\gamma fdz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \leq \int_a^b |f(z(t))||z'(t)|dt = \int_\gamma |f||dz|$$

Observation : If $f = u + iv$, then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u+iv)(x'+iy')dt = \int_\gamma (ux'-vy')dt + i\int_\gamma (vx'+uy')dt \\ &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy)\end{aligned}$$

Recall : Ω : open connected in $\mathbb{R}^2$, $A, B \in C^1(\Omega)$. Then $\int Adx+Bdy$ is only determined by P, Q in Ω and is independent of arcs connecting P and Q $\iff$ $\exists U \in C^1(\Omega)$ s.t. $dU = Adx + Bdy$ i.e. $\frac{\partial U}{\partial x} = A$, $\frac{\partial U}{\partial y} = B$. Actually, $U(x,y) = \inf_\gamma Adx+Bdy$, where $\gamma \subset \Omega$ is any curve connected P, Q.

Proposition 1.2.1. Let f be continuous on Ω. Then $\int_\gamma fdz$ depends only on the end points of γ $\iff$ f is the derivative of an analytic function F on Ω.

Proof:

- $(\Leftarrow)$: Say $F = U + iV$, then $f = F' = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} - i\frac{\partial U}{\partial y} = u + iv$. Then

$$udx - vdy = dU \;\; vdx + udy = dV$$

  By observation, $\int_\gamma fdz$ is independent on arcs.

- $(\Rightarrow)$: By recall, $\exists U, V \in C^1(\Omega)$ s.t.

$$dU = udx - vdy \; dV = vdx + udy$$

Let $F = U + iV \rightsquigarrow F$ is analytic and $F' = f$.

$\square$

Example 1.2.1.

- $\forall n \in \mathbb{N}$, $\int_\gamma (z-a)^n dz = 0 \; \forall \gamma$: closed arc in $\mathbb{C}$, since $\left(\frac{(z-a)^{n+1}}{n+1}\right)' = (z-a)^n$.
- Let $C_r(a) := \{z \in \mathbb{C} : |z-a| = r\}$, then

$$\begin{aligned}\int_{C_r(a)} \frac{dz}{z-a} &= \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}} d\theta \; (z = a + re^{i\theta}) \\ &= 2\pi i\end{aligned}$$

We can't apply proposition 1.2.1 since we can't define a single-valued branch of $\log(z-a)$ in $B_r(a) := \{z : |z-a| < r\}$.

Theorem 1.2.1 (Cauchy theorem for a rectangle)**.** Let f be analytic in a rectangle R (i.e analytic in an open set containing R). Then

$$\int_{\partial R} f(z)dz = 0$$

where the preset orientation of ∂R is counterclockwise.

Proof: Divide R to four small rectangle $R_1^{(1)}, ..., R_1^{(4)}$. Define $\Gamma(R) := \int_{\partial R} f(z)dz$, then

$$\Gamma(R) = \Gamma(R_1^{(1)}) + \cdots + \Gamma(R_4^{(1)})$$

Then exists $R_1^{(k)}$ for some $k \in \{1, ..., 4\}$ s.t. $|\Gamma(R_1^{(k)})| \geq \frac{1}{4}|\Gamma(R)|$. Say $R_1 = R_1^{(k)}$ and define $R_2, ...$ by same method. Let d_i, L_i be the diameter, perimeter of R_i respectively. We obtain a nested rectangles $R \supset R_1 \supset R_2 \supset \cdots$ with $d_i = d_{i-1}/2$, then $\exists! \; z^* \in R_i \; \forall i$ i.e. $\forall \delta > 0$, $\exists n_0 \geq 0$ s.t. $\forall n \geq n_0$, $R_n \subset B_\delta(z^*)$. Since f is analytic at z^*, $\forall \varepsilon > 0$, $\exists \delta > 0 (\rightsquigarrow \exists n_0)$ s.t.

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*)\right| < \varepsilon \; \forall z \in R_n, \; n \geq n_0$$

$$\implies |f(z) - f(z^*) - (z - z^*)f'(z^*)| < \varepsilon |z - z^*| \; \forall z \in R_n, \; n \geq n_0$$

By example 1.2.1, $\int_{\partial R_n} dz = 0$ and $\int_{\partial R_n} zdz = 0$, then

$$\begin{aligned}\Gamma(R_n) &= \int_{\partial R_n} (f(z) - f(z') - (z - z^*)f'(z^*))dz \\ \implies \frac{1}{4^n}|\Gamma(R)| &\leq |\Gamma(R_n)| \leq \varepsilon \int_{\partial R_n} |z - z^*||dz| \leq \varepsilon L_n d_n \;\; \forall n \geq n_0 \\ \implies |\Gamma(R)| &\leq \varepsilon 4^n L_n d_n = \varepsilon L d \; \forall \varepsilon \implies \Gamma(R) = 0\end{aligned}$$

$\square$

Theorem 1.2.2 (Stronger form). Let $R' = R \setminus \{\xi_1, ..., \xi_n\}$ with $\xi_i \in (R \setminus \partial R) =: R^o$ and f be analytic in R'. If $\lim\limits_{z \to \xi_i} (z - \xi_i) f(z) = 0$, then

$$\int_{\partial R} f(z) dz = 0$$

Proof:

- $n = 1 : \forall \varepsilon > 0, \exists \delta > 0$ s.t. $|z - \xi| < \delta \implies |z - \xi||f(z)| < \varepsilon$. Choose a square R_0 with center ξ s.t. $R \subset B_\delta(\xi)$. Extend the side length of R_0 and cut R into nine rectangle $R_0, R_1, ..., R_8$. We already know $\Gamma(R_i) = 0$, so $\Gamma(R) = \Gamma(R_0)$.

$$|\Gamma(R)| = |\Gamma(R_0)| = \left| \int_{\partial R_0} f(z) dz \right| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z - \xi|} \leq \varepsilon \frac{8}{L_0} L_0 = 8\varepsilon$$

  where L_0 is the perimeter of R_0. Hence, $\Gamma(R) = 0$.

- In general n, we just cut R into several rectangle and apply Cauchy theorem for a rectangle and the case of $n = 1$.

□

Theorem 1.2.3 (local existence of primitives). Any analytic function f in $B_\rho(a)$ has a primitive (antiderivatives) in $B_\rho(a)$.

Proof: For $z \in B_r(a)$, let γ_z connected a and z by one horizontal line first and one vertical line. Define $F(z) = \int_{\gamma_z} f(u) du$.
Claim : F is analytic in $B_\rho(a)$ and $F'(z) = f(z)$.
subproof : Apply Cauchy theorem for a rectangle we have

$$F(z + \Delta z) - F(z) = \int_{\overline{z(z+\Delta x)}} f(u) du + \int_{\overline{(z+\Delta x)(z+\Delta z)}} f(u) du$$

Since f is continuous at z, we can write $f(u) = f(z) + \delta(u)$, where $\delta(u) \to 0$ as $u \to z$.

$$\int_{\overline{z(z+\Delta x)}} f(z) du + \int_{\overline{(z+\Delta x)(z+\Delta z)}} f(z) du = f(z) \Delta z$$

$$\int_{\overline{z(z+\Delta x)}} |\delta(u)||du| + \int_{\overline{(z+\Delta x)(z+\Delta z)}} |\delta(u)||du| \leq (\sup |\delta(u)|) \underbrace{(|\Delta x| + |\Delta y|)}_{\leq 2|\Delta z|}$$

where $\sup |\delta(u)|$ is consider $u \in \overline{z(z + \Delta x)} \cup \overline{(z + \Delta x)(z + \Delta z)}$.

$$\implies \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) + \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left(\quad \bigstar \quad \right)$$

where $|\bigstar| \leq \frac{2|\Delta| \sup \delta(u)}{|\Delta u|} \to 0$ as $\Delta z \to 0$. Hence, $F'(z) = f(z)$. □

Theorem 1.2.4 (Cauchy theorem for a disk). If f is a analytic in $B_\rho(a)$, then

$$\int_\gamma f(z) dz = 0$$

for all closed arc $\gamma \subset B_\rho(a)$.

Proof: Since f has a primitive in $B_\rho(a)$, by proposition 1.2.1 the statement will holds. $\square$

Corollary 1.2.1. If f is analytic in Ω and $B_\rho(a) \subset \Omega$, then

$$\int_{C_\rho(a)} f(z)dz = 0$$

Proof: Choose a larger $B_{\rho'}(a')$ s.t. $B_\rho(a) \subsetneq B_{\rho'}(a') \subseteq \Omega$. Then $\gamma = C_\rho(a) \subseteq B_{\rho'}(a')$ and apply Theorem 1.2.4 $\square$

Theorem 1.2.5 (Stronger form)**.** Let $B = B_\rho(a) \setminus \{\xi_1, ..., \xi_n\}$ and f be analytic in B. If $\lim\limits_{z\to\xi_i}(z-\xi_i)f(z) = 0$, then f has a primitive in B. Moreover,

$$\int_\gamma f(z)dz = 0$$

for all closed arc in B.

Proof: For $z \in B$, define

$$F(z) = \int_{\gamma_z} f(u)du = \int_{\gamma'_z} f(u)du$$

where γ_z, γ'_z connected a and z and composed by finite horizontal line and vertical line not pass $\{\xi_1, ..., \xi_n\}$. The red equation will holds since $\gamma'_z - \gamma_z = \sum\limits_{\text{finite}} \pm\partial R_i$ and by stronger form of Cauchy theorem for a rectangle $\Gamma(R_i) = 0$. By the similar argument, $F'(z) = f(z)$. $\square$

1.2.2 Winding number

Theorem 1.2.6 (winding number)**.** Let γ be a closed arc and $a \notin \gamma$. Then

$$\int_\gamma \frac{dz}{z-a} = 2\pi i n$$

for some nonnegative integer n.

Proof: Let $z : [\alpha, \beta] \to \gamma$ with $t \mapsto z(t)$ and $z|_{[\alpha_i,\beta_i]}$: smooth. Consider

$$p(x) = \int_\alpha^x \frac{z'(t)}{z(t)-a}dt$$

Then we have

$$\begin{cases} p(x) \text{ is continuous on } [\alpha, \beta] \\ p'(x) = \dfrac{z'(x)}{z(x)-a} \text{ on } (\alpha, \beta) \setminus \{t_1, ..., t_{n-1}\} \\ p(\beta) = \displaystyle\int_\gamma \frac{dz}{z-a} \end{cases}$$

Notice that $2\pi i$ is the period of e^x, so hope that $e^{p(\beta)} = 1$. Now

$$\left(e^{-p(x)}\right)' = -p'(x)e^{-p(x)} = \frac{-z'(x)}{z(x)-a}e^{-p(x)} \implies -p'(x)(z(x)-a)e^{-p(x)} + z'(x)e^{-p(x)} = 0$$

$$\implies \left(e^{-p(x)}(z(x)-a)\right)' = 0 \implies e^{-p(x)}(z(x)-a) = \text{constant} = e^{-p(\alpha)}(z(\alpha)-a) = z(\alpha)-a$$

Hence, $e^{p(x)} = \dfrac{z(x)-a}{z(\alpha)-a} \rightsquigarrow e^{p(\beta)} = 1 \implies p(\beta) = (2\pi i)n$ for some $n \in \mathbb{N}$. $\square$

Observation : Define

$$n(\gamma, a) := \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$

be the **winding number** of γ around a. It will be the number of turns of γ around a.

- $n(-\gamma, a) = -n(\gamma, a)$
- If $\gamma \subseteq B_\rho(a) \subseteq \Omega$, then $\forall a \in \Omega \setminus B_\rho(z)$, $n(\gamma, a) = 0$:
  Since $(z-a)^{-1}$ is analytic in $B_\rho(z)$.
- $n(\gamma, a)$ is constant for each region cut by γ and $n(\gamma, a) = 0$ $\forall a$ in unbounded region.
  Claim : If $\gamma \cap \overline{aa'} = \varnothing$, then $n(\gamma, a) = n(\gamma, a')$.
  subproof : for $z \in \overline{aa'}$, $\dfrac{z-a}{z-a'} \in \mathbb{R}_{\leq 0}$ and $\dfrac{z-a}{z-a'} \notin \mathbb{R}_{\leq 0}$ for all $z \notin \overline{aa'}$. Then $\log\left(\dfrac{z-a}{z-a'}\right)$ is analytic on $\mathbb{C} \setminus \overline{aa'}$. Hence,

  $$0 = \int_\gamma \left(\log\left(\frac{z-a}{z-a'}\right)\right)' dz = \int_\gamma (\log(z-a) - \log(z-a'))' dz = \int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-a'}\right) dz$$

  Hence, $n(\gamma, a) = n(\gamma, a')$. □

  In the same region, we can connected by polyline, hence $n(\gamma, z)$ is constant on same region. Choose a open ball $B_\rho(a)$ that cover γ, and choose a point $b \in \mathbb{C} \setminus B_\rho(a)$, then $n(\gamma, b) = 0$. Hence, $n(\gamma, z) = 0$ on unbound region.
- Let γ be the simple curve around 0, then $n(\gamma, 0) = 1$:
  Let $C = C_\rho(0)$ for some ρ s.t. $C \cap \gamma = \varnothing$. Choose $a_1, a_2 \in \gamma$, $b_1, b_2 \in C$ s.t. $a_2, b_2, 0, b_1, a_1$ collinear as this order. Let γ, C be cut by this line into $\gamma_1 \cup \gamma_2, C_1 \cup C_2$ respectively and C_1, γ_1 are in same side w.r.t. this line. Let $\sigma_1 = \gamma_1 + \overline{a_1 b_1} - c_1 - \overline{a_2 b_2}$, $\sigma_2 = \gamma_2 + \overline{a_2 b_2} - C_1 - \overline{a_1 b_1}$. By definition of winding number,

  $$n(\gamma, 0) = n(C, 0) + n(\sigma_1, 0) + n(\sigma_2, 0) = 1$$

  where $n(\sigma_1, 0) = n(\sigma_2, 0) = 0$ by 0 is in the unbounded region w.r.t. to σ_1, σ_2.

Let $f(z)$ be analytic in $B_\rho(b)$, $\gamma \subset B_\rho(b)$ and $a \in B_\rho(b) \setminus \gamma$. Then $F(z) = \dfrac{f(z) - f(a)}{z-a}$ is analytic for $z \neq a$ and $\lim\limits_{z \to a}(z-a)F(a) = \lim\limits_{z \to a}(f(z) - f(a)) = 0$. By Theorem 1.2.5,

$$\int_\gamma \frac{f(z) - f(a)}{z-a} dz = 0 \implies \int_\gamma \frac{f(z)}{z-a} = f(a) \int_\gamma \frac{dz}{z-a}$$

Then we have Cauchy integral formula :

$$f(a) = \frac{1}{2\pi i \cdot n(r, a)} \int_\gamma \frac{f(z)}{z-a} dz$$

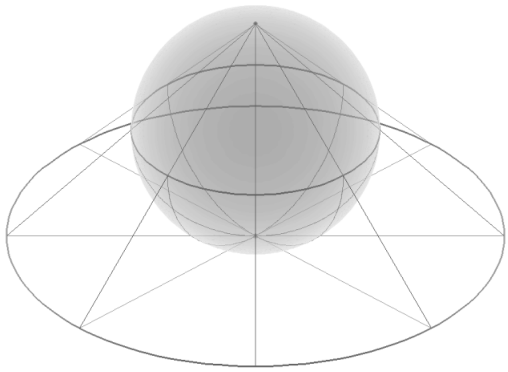
In particular, if $n(\gamma, z) = 1$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z} d\xi$$

1.2.3 Simply connected

Set up

- extended complex plane (**Riemann sphere**) : $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ with $a + \infty = \infty + 0 = \infty$, $b \cdot \infty = \infty \cdot b = \infty$ for $b \neq 0$. $a/0 = \infty$ for $a \neq 0$, $b/\infty = 0$ for $b \neq \infty$.



- γ : piecewise-smooth curve ($z|_{[\alpha_i,\beta_i]}$: smooth i.e. $z'(t)$ continuous and $z'(t) \neq 0$)
  - **simple** : $z(t_1) = z(t_2) \iff t_1 = t_2$
  - **closed** : $z(\alpha) = z(\beta)$
  - **Jordan curve** : simple closed curve
  - **opposite arc** : $-\gamma$ define by $z(-t)$.
- Let $\gamma_1, ..., \gamma_n$ be arcs in Ω. A **form sum** $\gamma_1 + \cdots + \gamma_n$ is called a **chain** in Ω.
- Define $\gamma_1 + \cdots + \gamma_n \sim \gamma_1' + \cdots + \gamma_m' \iff \int_{\gamma_1+\cdots+\gamma_n} f dz = \int_{\gamma_1'+\cdots+\gamma_m'} f dz \ \forall f$ on Ω, where

$$\int_{\gamma_1+\cdots+\gamma_n} f dz = \sum_{i=1}^{n} \int_{\gamma_i} f dz$$

  In general, we can write a chain $\gamma = b_1\gamma_1 + \cdots + b_n\gamma_n$, where γ_i are distinct arcs and $b_i \in \mathbb{Z}$.
- γ is a **cycle** if $\forall \gamma_i$ is closed

$$\implies \int_\gamma dF = 0 \text{ and define } n(\gamma, a) := \sum b_i n(r_i, a)$$

Definition 1.2.3. A region $\Omega \subseteq \mathbb{C}$ is simply connected if $\widetilde{\mathbb{C}} \setminus \Omega$ is connected.

Property 1.2.3. Ω is simply connected $\iff n(\gamma, a) = 0 \ \forall \gamma$: cycle in Ω and $a \in \widetilde{\mathbb{C}} \setminus \Omega$.

Proof:

- $(\Rightarrow)$ Since $\widetilde{\mathbb{C}} \setminus \Omega$ is connected and $\gamma \subseteq \Omega$, $\widetilde{\mathbb{C}} \setminus \Omega$ is contained in the unbounded region determined by $\gamma \rightsquigarrow n(\gamma, a) = 0 \ \forall a \in \widetilde{\mathbb{C}} \setminus \Omega$.
- $(\Leftarrow)$: Assume $\widetilde{\mathbb{C}} \setminus \Omega = A \sqcup B$ with A, B closed and assume A is bouneded. Let δ be the shortest distance between A and B. Let

$$\{Q : Q \text{ is a square of side } = \frac{\delta}{2\sqrt{2}}\}$$

covers A and a be a center of some Q in A. Let $\gamma = \sum_{Q_j \cap A \neq \varnothing} \partial Q_j$, then $\gamma \in \Omega$ and

$$n(\gamma, a) = \sum_{Q_j \cap A \neq \varnothing} n(\partial Q_j, a) = n(\partial Q, a) = 1 \ (\rightarrow\!\leftarrow)$$

since ∂Q is a simple curve around a and a is in the unbounded region w.r.t. others ∂Q_j.

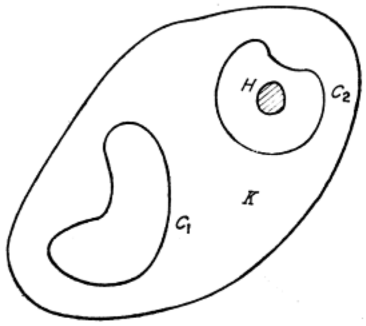
□

Remark 1.2.1. If Ω is not simply connected, then $\exists \gamma$ in Ω s.t. $n(\gamma, a) \neq 0$ for some $a \in \widetilde{\mathbb{C}} \setminus \Omega$. Now $(z-a)^{-1}$ is analytic in Ω, but

$$\int_\gamma \frac{1}{z-a} dz = n(\gamma, a) \neq 0$$

i.e. Cauchy theorem doesn't hold in this case.

Definition 1.2.4. γ in Ω is said to be **homologous to** 0 w.r.t. Ω if $n(\gamma, a) = 0 \ \forall a \in \widetilde{\mathbb{C}} \setminus \Omega$, and denoted by $\gamma \sim 0 \pmod{\Omega}$.



For example, $C_1 \sim 0 \pmod{K \setminus H}$, but $C_2 \not\sim 0 \pmod{K \setminus H}$.

Theorem 1.2.7 (Cauchy theorem)**.** If $f(z)$ is analytic in Ω, then $\int_\gamma f(z) dz = 0$ for all cycle $\gamma \sim 0$ in Ω.

Corollary 1.2.2. Let Ω be simply connected and f be analytic in Ω. Then

- $\int_\gamma f dz = 0$ for all cycle γ in Ω, since $r \sim 0$.
- $\int f dz$ is independent of the path connecting P and Q. Which means $f dz = dF$ for some F i.e. f has a primitive.

Proof: (Cauchy theorem)

- Ω is bounded : For $\delta > 0$, let $\{S_i : i \in I\}$ be a subset of closed squares of side δ which are contained in Ω (Ω : bounded $\rightsquigarrow |I| < \infty$). Let $\Gamma_\delta = \sum_{i \in I} \partial S_i$, $\Omega_\delta = \left(\bigcup_{i \in I} S_i \right)^o$. Choose δ s.t. $\gamma \subset \Omega_\delta$. Let $\xi \in \Gamma_\delta \subseteq \Omega \setminus \Omega_\delta$, then exists a square $S \notin \{S_i : i \in I\}$ s.t. $\xi \in S$. Let $\xi_0 \in S \setminus \Omega \rightsquigarrow \overline{\xi\xi_0} \subset S \rightsquigarrow \overline{\xi_0 \xi} \cap \Omega = \varnothing$. Since $\gamma \sim 0 \pmod{\Omega}$, $n(\gamma, \xi_0) = 0$ and thus $n(\gamma, \xi) = n(\gamma, \xi_0) = 0$ since they are in same region w.r.t. Ω.

- If $z \in S_k^o$, then

$$\frac{1}{2\pi i}\int_{\partial S_i}\frac{f(\xi)}{\xi - z}d\xi = \begin{cases} f(z) & ,\text{if } i = j \\ 0 & , \text{ if } i \neq j \end{cases}$$

since $\dfrac{f(\xi)}{\xi - z}$ is analytic on S_i when $i \neq j$ and by Cauchy integral formula when $i = j$. Then

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi \quad \forall z \in \bigcup_{j\in I} S_j^o \underset{\text{by conti.}}{\Longrightarrow} f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi \quad \forall z \in \Omega_\delta$$

Hence,

$$\begin{aligned}\int_\gamma f(z)dz &= \frac{1}{2\pi i}\int_\gamma\int_{\Gamma_\delta}\frac{f(\xi)}{\xi - z}d\xi dz = \frac{1}{2\pi i}\int_{\Gamma_\delta}\int_\gamma\frac{f(\xi)}{\xi - z}dzd\xi \text{ (since it conti. on } \Gamma_\delta, \gamma)\\ &= \frac{1}{2\pi i}\int_{\Gamma_\delta}\left(f(\xi)\int_\gamma\frac{-1}{z - \delta}dz\right)d\xi = \frac{-1}{2\pi i}\int_{\Gamma_\delta} f(\xi)\underbrace{n(\gamma, \xi)}_{=0}d\xi = 0\end{aligned}$$

- If Ω is unbound : We replace Ω by $\Omega' := \Omega \cap B_R(0)$ for R large enough to get $\gamma \subset \Omega'$. Then $\forall a \in \widetilde{\mathbb{C}} \setminus \Omega'$,

$$\begin{cases} a \in \widetilde{\mathbb{C}} \setminus \Omega \implies n(\gamma, a) = 0 & \text{, since } \gamma \simeq 0 \pmod{\Omega} \\ a \in \widetilde{\mathbb{C}} \setminus B_R(0) \implies n(\gamma, a) = 0 & \text{, since } \frac{1}{z-a} \text{ is analytic on } B_R(0) \end{cases}$$

Hence, $\gamma \sim 0 \pmod{\Omega'}$, which is follow from the bounded case.

□

1.3 Cauchy's integral theorem

Recall : Let f : analytic in $B_\rho(a)$, γ : closed arc in $B_\rho(a)$. For $z \neq \gamma$,

$$f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

In fact we can have more relax assumption : f : analytic in $B_\rho(a)\setminus\{a\}$ with $\lim_{z\to a}(z-a)f(z) = 0$. Then

$$f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

Proof: For $z \in B_\rho(a)$, let $F(\xi) = \dfrac{f(\xi) - f(z)}{\xi - z}$, then F analytic on $B_\rho(a) \setminus \{a, z\}$. Also

$$\begin{cases} \lim_{\xi\to z}(\xi - z)F(\xi) = 0 \\ \lim_{\xi\to a}(\xi - a)F(\xi) = \lim_{\xi\to a}\frac{(\xi-a)(f(\xi)-f(z))}{\xi-z} = 0 \end{cases}$$

By stronger Cauchy theorem,

$$\int_\gamma F(\xi)d\xi = 0 \implies f(z) = \frac{1}{2\pi i n(\gamma, z)}\int_\gamma\frac{f(\xi)}{\xi - z}d\xi$$

□

Lemma 1.3.1 (key lemma). If $\varphi(\xi)$ is continuous on γ, then

$$F_n(z) := \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n} d\xi$$

is analytic in each regions determined by γ and $F_n'(z) = nF_{n+1}(z)$.

Proof:

- $n = 1$:
  - continuous : $\forall z_0 \notin \gamma$, pick $\delta > 0$ s.t. $B_\delta(z_0) \cap \gamma = \varnothing$. If $|z - z_0| < \delta/2 \rightsquigarrow |\xi - z| > \delta/2 \; \forall \xi \in \gamma$. So

$$|F_1(z) - F_1(z_0)| = \left| \int_\gamma \varphi(\xi) \frac{z - z_0}{(\xi - z)(\xi - z_0)} dz \right| \leq |z - z_0| M \frac{2}{\delta^2} L$$

    where $M = \max\limits_{\xi \in \gamma} \varphi(\xi)$ and L be the length of γ. Hence, $F_1(z)$ is continuous.
  - differentiable :

$$\frac{F(z) - F(z_0)}{z - z_0} = \int_\gamma \frac{\varphi(\xi)}{(\xi - z)(\xi - z_0)} d\xi \xrightarrow{z \to z_0} F'(z_0) = \int_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^2} d\xi = F_2(z_0)$$

    where red approaching is by

$$\left| \int_\gamma \frac{\varphi(\xi)}{(\xi - z)(\xi - z_0)} d\xi - \int_\gamma \frac{\varphi(\xi)}{(\xi - z_0)^2} d\xi \right| \leq \int_\gamma \left| \frac{\varphi(\xi)(z - z_0)}{(\xi - z)(\xi - z_0)^2} \right| |d\xi| \leq \frac{4ML}{\delta^3} |z - z_0|$$

- By induction on $n > 1$:
  - continuous :

$$\begin{aligned} |F_n(z) - F_n(z_0)| &= \left| \int_\gamma \varphi(\xi) \left(\frac{1}{(\xi - z)^n} - \frac{1}{(\xi - z_0)^n} \right) dz \right| \\ &= \left| \int_\gamma \varphi(\xi) \left(\frac{\xi - z + z - z_0}{(\xi - z)^n (\xi - z_0)} - \frac{1}{(\xi - z_0)^n} \right) dz \right| \\ &= \left| \int_\gamma \frac{\varphi(\xi)}{\xi - z_0} \left(\frac{1}{(\xi - z)^{n-1}} - \frac{1}{(\xi - z_0)^{n-1}} \right) d\xi + (z - z_0) \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \right| \end{aligned}$$

    By induction hypothesis on $\varphi(\xi)/(\xi - z_0)$, we have

$$\left| \int_\gamma \frac{\varphi(\xi)}{\xi - z_0} \left(\frac{1}{(\xi - z)^{n-1}} - \frac{1}{(\xi - z_0)^{n-1}} \right) d\xi \right| \to 0$$

    and similar method,

$$\left| (z - z_0) \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \right| \leq \frac{2^n ML}{\delta^{n+1}} |z - z_0|$$

    Hence, $F_n(z)$ is continuous.
  - differentiable :

$$\begin{aligned} \frac{F_n(z) - F_n(z_0)}{z - z_0} &= \frac{1}{z - z_0} \int_\gamma \left(\frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z)^{n-1}} - \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^{n-1}} \right) d\xi + \int_\gamma \frac{\varphi(\xi)}{(\xi - z)^n (\xi - z_0)} d\xi \\ &\longrightarrow (n-1) \int_\gamma \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^{n-1}} d\xi + \int_\gamma \frac{\varphi(\xi)/(\xi - z_0)}{(\xi - z_0)^{n-1}} d\xi = nF_{n+1}(z) \end{aligned}$$

    the former is by induction hypothesis.

$\square$

Corollary 1.3.1. Let f be analytic in Ω. For $a \in \Omega$, $\exists B_\rho(a) \subset \Omega$, if $\gamma = C_\rho(a)$, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi \text{ and } f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)}{(z-\xi)^{n+1}}d\xi$$

Hence, if f is analytic in Ω, then $f \in C^\infty(\Omega)$.

Theorem 1.3.1 (Removable singularities)**.** If f is analytic in $\Omega' = \Omega \setminus \{\xi_1, ..., \xi_n\}$ and $\lim\limits_{z\to\xi_i}(z-\xi_i)f(z) = 0\ \forall i$, then $\exists!$ analytic function $\widetilde{f}$ in Ω s.t. $\widetilde{f}|_\Omega = f$.

Proof: Let $a = \xi_i \rightsquigarrow \exists B_\rho(a) \setminus \{a\} \subseteq \Omega'$. If $\widetilde{f}$ exists, then

$$\begin{cases} \widetilde{f}(a) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \frac{\widetilde{f}(\xi)}{\xi - a}d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{\xi - a}d\xi \\ \widetilde{f}(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \frac{\widetilde{f}(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{\xi - z}d\xi = f(z)\ \forall z \neq a \end{cases}$$

So we should define $\widetilde{f}(a) = \dfrac{1}{2\pi i}\displaystyle\int_{C_\rho(a)} \frac{f(\xi)}{\xi - a}d\xi$ $\square$

Observation : Let $F(z) = \dfrac{f(z) - f(a)}{z-a}$. $\because \lim\limits_{z\to a}(z-a)F(z) = 0 \therefore \exists!$ analytic function s.t.

$$f_1(z) = \begin{cases} F(z) & \text{, for } z \neq a \\ f'(z) & \text{, for } z = a \end{cases}$$

$\because \lim\limits_{z\to a}(z-a)\dfrac{f_1(z) - f_1(a)}{z-a} = 0 \therefore \exists!$ analytic function s.t.

$$f_2(z) = \begin{cases} \frac{f_1(z) - f_1(a)}{z-a} & \text{, for } z \neq a \\ f_1'(z) & \text{, for } z = a \end{cases}$$

That is $f_{k-1}(z) = f(a) + (z-a)f_k(z)\ \forall k = 1, ..., n$, where $f_0(z) = f(z)$.

$$\implies f(z) = f(a) + (z-a)f_1(a) + \cdots + (z-a)^{n-1}f_{n-1}(a) + (z-a)^n f_n(z)$$

Differentiate n times and evaluation $z = a$, we can get $f^{(n)}(z) = n!f_n(a)$. Then we have

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + (z-a)^n f_n(z)$$

Here for $z \in B_\rho(a)$,

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f_n(\xi)}{\xi - z}d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{1}{\xi - z}\left(\frac{f(\xi)}{(\xi-a)^n} - \sum_{k=0}^{n-1}\frac{f^k(a)}{k!(\xi-a)^{n-k}}\right)d\xi$$

Let $G_k(u) = \displaystyle\int_{C_\rho(a)} \frac{1/(\xi - z)}{(\xi - u)^k}d\xi$ for $u \in B_\rho(a)$. By key lemma, $G_{k+1}(u) = G_1^k(u)/k!$.

$$\begin{aligned} G_1(u) = \int_{C_\rho(a)} \frac{d\xi}{(\xi-u)(\xi-z)} &= \frac{1}{u-z}\int_{C_\rho(a)}\left(\frac{1}{\xi-u} - \frac{1}{\xi-z}\right)d\xi \\ &= \frac{2\pi i(n(C_\rho(a), u) - n(C_\rho(a), z))}{u-z} = 0 \end{aligned}$$

Hence, $G_k(u) = 0\ \forall k$ and thus

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - z)(\xi - a)^n} d\xi$$

Theorem 1.3.2 (Cauchy's estimate). Let $M = \max_{\xi \in C_\rho(a)} |f(\xi)|$, then

$$|f^n(a)| = \left| \frac{n!}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \right| \leq \frac{n!}{2\pi} \frac{M \cdot 2\pi\rho}{\rho^{n+1}} \leq n! M\rho^{-n}$$

Theorem 1.3.3 (Liouville's theorem). If f is bounded entire function, then f is a constant.

Proof: Say $|f(z)| \leq M\ \forall z \in \mathbb{C}$. For all $a \in \mathbb{C}$, $|f'(a)| \leq M\rho^{-1} \to 0$ as $\rho \to \infty$. Hence, $f'(a) = 0\ \forall a \in \mathbb{C}$ which means f is constant. □

Theorem 1.3.4 (Fundamental theorem of algebra). Given $p(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n \neq 0$, $n \geq 1$, then $\exists \alpha \in \mathbb{C}$ s.t. $p(\alpha) = 0$.

Proof: If $\forall z \in \mathbb{C}$, $p(z) \neq 0$, then $1/p(z)$ is entire. Also

$$\left| \frac{1}{p(z)} \right| \leq \frac{1}{|a_n||z|^n - |a_{n-1}||z|^{n-1} - \cdots - |a_1||z| - |a_0|} \to 0 \text{ as } |z| \to \infty$$

Then $\exists R \in \mathbb{R}$ s.t. $|z| > R$, $|1/p(z)| \leq 1$. Since $|z| \leq R$ is compact set, $M := \max_{|z| \leq R} |1/p(z)|$ exists. Then $|1/p(z)| \leq \max\{1, M\}\ \forall z \in \mathbb{C}$. By Liouville's theorem, $1/p(z) = c$, which contradict to $n \geq 1$. □

Theorem 1.3.5 (Morera's theorem). If $\displaystyle\int_\gamma f(z)dz = 0\ \forall \gamma$: closed arc in Ω, then $f(z)$ is analytic in Ω.

Proof: Since the line integral is independent on path, there exists F : analytic s.t. $F'(z) = f(z)$. Then $f'(z) = F''(z)$ i.e. f is analytic. □

Theorem 1.3.6 (zero order). If $\exists a$ s.t. $f(a) = 0$ and $f^{(k)}(a) = 0\ \forall k \neq \mathbb{N}$, then $f \equiv 0$.

Proof: $\forall n \in \mathbb{N}$, $f(z) = f_n(z)(z-a)^n$, for some analytic function $f_n(z)$ in Ω. For $z \in B_\rho(a)$

$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{(\xi - a)^n(\xi - z)} d\xi$$

$$\implies |f(z)| = |z-a|^n |f_n(z)| \leq \frac{|z-a|^n}{2\pi} \frac{M \cdot 2\pi\rho}{\rho^n(\rho - |a-z|)} = \left(\frac{|z-a|}{\rho}\right)^n \frac{M\rho}{\rho - |a-z|} \to 0 \text{ as } n \to \infty$$

Define

$$A_1 = \{z \in \Omega | f(z) = 0, f^{(k)}(z) = 0\ \forall k \geq 1\} \text{ which is open}$$

$$A_2 = \{z \in \Omega | f(z) \neq 0 \text{ or } f^k(z) \neq 0 \text{ for some } k \geq 1\} \text{ which is also open}$$

Since $\Omega = A_1 \sqcup A_2$ is open connected and $A_1 \neq 0$, $A_2 = \varnothing$ and thus $f \equiv 0$ in Ω. □

Definition 1.3.1. If $f \neq 0$ and $f(a) = 0$, then $\exists$ the smallest $m \in \mathbb{N}$ s.t. $f^m(a) \neq 0$. This m is called the **zero order** of a. Since we can write $f(z) = (z-a)^m f_m(z)$ with $f_m(z)$ is analytic and $f_m(a) = \frac{1}{m!} f^m(a) \neq 0$.

1.4 Singularity

Recall : If $f \not\equiv 0$ in Ω, m : the zero order of $a \rightsquigarrow f(z) = (z-a)^m f_m(z)$ and $f_m(a) = \frac{1}{m!}f^m(a) \neq 0 \rightsquigarrow \exists$ a neighborhood of a s.t. $f_m(a) \neq 0$ in this neighborhood $\rightsquigarrow f(z) \neq 0$ in this neighborhood except a. Then $z = a$ is an isolated zero.

Proposition 1.4.1. If f, g are analytic in Ω and $U \subset \Omega$ with an accumulation point $a \in U$, then $f = g$ on $U \implies f = g$ on Ω.

Proof: Assume $f \neq g$ on Ω and $(f-g)(a) = 0 \implies a$ is isolated zero ($\rightarrow\!\leftarrow$). $\square$

Corollary 1.4.1. $f \equiv 0$ in a subregion of $\Omega \rightsquigarrow f \equiv 0$ in Ω.

Corollary 1.4.2. $f \equiv 0$ on an arc $\rightsquigarrow f \equiv 0$ in Ω.

Corollary 1.4.3. Let f be analytic in Ω and $f(z) = f(a) + \frac{f'(a)}{1!})(z-a) + \cdot$ in $B_\rho(a)$. Let R be the radius of convergence, then $f(z) = f(a) + \frac{f'(a)}{1!} + \cdots$ in $\Omega \cap B_R(a)$.

Definition 1.4.1. Let f be analytic in $0 < |z-a| < \delta$ except perhaps at a itself. We call a an isolated singularity.

- removable : $\lim\limits_{z\to a}(z-a)f(z) = 0 \rightsquigarrow f(a)$ can be define s.t. f is analytic in $|z-a| < \delta$.
- pole : $\lim\limits_{z\to a} f(z) = \infty \rightsquigarrow \exists\delta' \le \delta$ s.t. $f(z) \neq 0$ for $0 < |z-a| < \delta' \rightsquigarrow g(z) = f(z)^{-1}$ is analytic for $0 < |z-a| < \delta' \rightsquigarrow \lim\limits_{z\to a} g(z) = \frac{1}{\infty} = 0 \rightsquigarrow g(z)$ has removable singularity. $\because g(z) \not\equiv 0$ in $B_{\delta'}(a) \setminus \{a\} \therefore g(z) = (z-a)^m g_m(z)$ with $g_m(a) \neq 0 \rightsquigarrow f(z) = g(z)^{-1} = (z-a)^{-m}\dfrac{1}{g_m(z)}$ and define m be the **order of pole** a.
- $f(z)$ is analytic in Ω except for removable singularity or poles $\rightsquigarrow f$: **meromorphic** in Ω.

Property 1.4.1. f, g : analytic in Ω with $g \neq 0 \rightsquigarrow \frac{f}{g}$ is meromorphic, since the only possible pole are the zero of g and a common zero of f, g is pole or removable.

Definition 1.4.2.

- An isolated singularity is called an **essential singularity** if is not a removable singularity or a pole.
- $f(\infty)$ is always not defined so ∞ is regard as an isolated singularity

$$\infty \text{ is a } \begin{cases} \text{removable} \\ \text{pole} \\ \text{essential} \end{cases} \quad \text{if } g(z) = f(z^{-1}) \text{ is a } \begin{cases} \text{removable} \\ \text{pole} \\ \text{essential} \end{cases} \quad \text{at } 0$$

Example 1.4.1. Classify singularity for

(a) $\dfrac{\sin z}{z}$:

$z = 0$ is singularity and $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \rightsquigarrow \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \rightsquigarrow$ removable

(b) $e^{1/z}$:

$z=0$ is singularity : It is clear that is not removable. If it is a pole, then $z^m e^{1/z}$ is analytic at $z=0$ ($\rightarrow\!\leftarrow$). Hence $z=0$ is essential and thus $z=\infty$ is essential singularity for e^z.

(c) $\dfrac{1}{z^3(z-2)^2}$:

$z=0$ is pole of order 3 and $z=2$ is pole of order 2

(d) $\dfrac{\sin z}{z^4}$

$z=0$ is pole of order 4

(e) $\dfrac{z+1}{z^{1/2}-1}$

$z=1$ is pole of order 1, since $\dfrac{z+1}{z^{1/2}-1}=\dfrac{(z+1)(z^{1/2}+1)}{z-1}$.

Theorem 1.4.1 (Weierstrass-Casorati)**.** If $z=a$ is an essential singularity of f, then $\forall B_\rho(a)$, f comes arbitrary close to any complex value in $B_\rho(a)$.

Proof: If not, $\exists B_\rho(a)$, $A\in\mathbb{C}$, $\exists\delta>0$ s.t. $|f(z)-A|>\delta$ for $0<|z-a|<\rho$, then

$$\lim_{z\to a}\frac{f(z)-A}{z-a}=\infty$$

We can write $\dfrac{f(z)-A}{z-a}=(z-a)^{-m}\dfrac{1}{g_m(z)}$ with $g_m(z)\neq 0$.

- If $m=1 \rightsquigarrow f(z)-A=\dfrac{1}{g_m(z)} \rightsquigarrow f$ is analytic at $z=a$
- If $m\geq 2 \rightsquigarrow f(z)=A+(z-a)^{-(m-1)}\dfrac{1}{g_m(z)} \rightsquigarrow \lim\limits_{z\to a} f(z)=\infty \rightsquigarrow z=a$ is a pole of $f(z)$.

$\square$

1.5 Analytic function as mappings

f : analytic in Ω, $f:\Omega\to\mathbb{C}$. We say that $\Omega\subseteq\mathbb{C}$ is **z-plane** and $\mathbb{C}$ is **w-plane**. Given a curve γ in Ω and a parameterize $z:[\alpha,\beta]\to\gamma$ with $t\mapsto z(t)$, Γ be the image of γ via f can be parameterized by $t\mapsto f(z(t))=:w(t)$.

- $w'(t)=z'(t)f'(z(t))$: If $z'(t_0)\neq 0$, then $f'(z_0)\neq 0 \implies w'(t_0)\neq 0$. Then we have

$$\arg w'(t_0)=\arg f'(z_0)+\arg z'(t_0)$$

  So if $f'(z_0)\neq 0$, then f is conformal in the neighborhood of z_0.

- Now we still not know the image of analytic function, so we may ask that
  - whether $w_0\in \operatorname{Im} f$ or not.
  - and how many such z_0? i.e. find the zero order of $f(z)-w_0$ at $z=z_0$
- Now we consider $f\not\equiv 0$:

- f has zero $z_1, ..., z_m$ (count multiplicity) in $B_\rho(a)$ and $\gamma \subset B_\rho(a)$ with $f \neq 0$ on γ.
  Write $f(z) = (z - z_1) \cdots (z - z_m) g(z)$ with $g(z) : \begin{cases} \text{analytic in } B_\rho(a) \\ \text{no zero in } B_\rho(a) \end{cases}$, then

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_m} + \frac{g'(z)}{g(z)}$$

$$\implies \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_k) + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz$$

  Since $g'(z)/g(z)$ is analytic in $B_\rho(a)$, by Cauchy theorem, $\displaystyle\int_\gamma \frac{g'(z)}{g(z)} dz = 0$ and thus

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_k)$$

- f has infinite zero in $B_\rho(a)$: Let $\gamma \subset B_{\rho'}(a) \subsetneq B_\rho(a)$
  Claim : exists only finite many $z_{i_1}, ..., z_{i_m}$ in $B_{\rho'}(a)$
  subproof : If $\exists$ infinitely many $z_j's$ in $B\rho'(a)$, then by Bolzano-Weiestrass theorem, exists an accumulation point of $z_j's$ in $\overline{B_{\rho'}(a)} \subseteq B_\rho(a) \rightsquigarrow f \equiv 0$ on $\overline{B_{\rho'}(a)} \rightsquigarrow f \equiv 0$ in Ω ($\rightarrow\!\leftarrow$).
  $\square$

  For $z_k's$ outside $B_{\rho'}(a)$, $n(\gamma, z_k)$. Hence,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(\gamma, z_{i_k}) = \sum_i n(\gamma, z_i) \text{ (finite sum)}$$

  This formula is called **argument principal**.

- If $\gamma = C_\rho(b)$, then $n(\gamma, z_i) = 0$ or 1, then $\displaystyle\int_\gamma \frac{f'(z)}{f(z)} dz = \#$of zeros inside γ. Moreover we have it equal to

$$\frac{1}{2\pi i} \int_\alpha^\beta \frac{f'(z(t))}{f(z(t))} z'(t) dt = \frac{1}{2\pi i} \int_\alpha^\beta \frac{w'(t)}{w(t)} dt = \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w} = n(\Gamma, 0)$$

- Let $f(z) \neq w_0$ on γ. If $\{z_j(w_0) : j = 1, ...\}$ is the set of zeros of $f(z) = w_0$, then

$$\sum_j n(\gamma, z_j(w_0)) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w_0} dz = \frac{1}{2\pi i} \int_\Gamma \frac{w}{w - w_0} dw = n(\Gamma, w_0)$$

  In particular, choose γ be the circle $C_\rho(b)$, then # of zeros of $f(z) - w_0$ inside $\gamma = n(\Gamma, w_0)$. Hence, if w_1, w_2 lie in the same region determined by Γ, then $\#f^{-1}(w_1) = \#f^{-1}(w_2)$ inside γ.

Property 1.5.1 (key result). If $f(z) - w_0$ has a zero with order being n, then for small $\varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < |w - w_0| < \delta$, $f(z) = w$ has exactly n roots in $|z - z_0| < \varepsilon$.

Proof: Pick $\varepsilon > 0$ s.t.

$$\begin{cases} f \text{ is analytic for } |z - z_0| < \varepsilon \\ z_0 \text{ is the only zero of } f(z) = w_0 \quad \text{(since } z_0 \text{ is isolated)} \\ f'(z_0) \neq 0 \text{ for } 0 < |z - z_0| < \varepsilon \end{cases}$$

We can get the third since

$$\begin{cases} \text{if } f'(z_0) = 0 : \text{since zero point is isolated or } f' \equiv 0 \text{ i.e. } f \text{ is constant} \rightsquigarrow n = \infty \\ \text{if } f'(z_0) \neq 0 : \text{we can choose it by continuous} \end{cases}$$

Choose ε suitable s.t. $B_\delta(w_0) \cap \Gamma = \varnothing$. Then $n(\Gamma, w_0) = \sum_{n \text{ times}} n(\gamma, z_0) = n$. Now $\forall w$ with $0 < |w - w_0| < \delta$, $\sum n(\gamma, z_i(w)) = n(\Gamma, w) = n(\Gamma, w_0) = n$ and $\because f'(z_i(w)) \neq 0 \therefore z_i(w)$ is simple zero of $f(z) = w$. □

Corollary 1.5.1.

- f is an open mapping : for U open in Ω, $\forall z_0 \in U$, $\exists N_\rho(z_0) \subseteq \Omega$, choose smaller ε with $0 < \varepsilon < \rho$ s.t. if $w_0 = f(z_0)$, $\exists \delta > 0$ s.t. $B_\delta(w_0) \subseteq f(B_\varepsilon(z_0)) \subseteq f(U) \implies f(U)$ is open in $\mathbb{C}$.
- f is analytic at z_0 and $f'(z_0) \neq 0$. For small ε (in above), there exists δ satisfy key result. We will prove that $B_\varepsilon(z_0)$ is homeomorphic to it image. Given a open ball $B_\rho(w)$ in $f(B_\varepsilon(z_0))$, say $w = f(z)$, then $f^{-1}(B_\delta(w)) \overset{f}{\twoheadrightarrow} B_\delta(w)$. Since $f'(z) \neq 0$, f will be $1-1$ and thus f will be homeomorphism (topologically).

  Conversely, if $f'(z_0) = 0$, then f is not topologically. Thus $f'(z_0) \neq 0$.

Theorem 1.5.1 (maximal principal)**.** If $f(z)$ is analytic and non-constant, then $|f(z)|$ has no max in Ω.

Proof: For $z \in \Omega$, if $f(z) = w_0$, then $\exists \delta > 0$ s.t. $B_\delta(w_0) \subset f(\Omega)$, and there exists w s.t. $|w| > |w_0|$, so $|w_0|$ is not max. □

We have another form of maximal principal.

Theorem 1.5.2. If f is analytic in a bounded region Ω and continuous on $\partial\Omega$, then $|f(z)|$ attains its max on $\partial\Omega$.

Proof: Since $\Omega \cup \partial\Omega$ is a compact set, $M = \max_{z \in \Omega \cup \partial\Omega} |f(z)|$ exists. If $f \neq$ constant, then $M \notin \{|f(z)| : z \in \Omega\} \implies M \in \{|f(z)| : z \in \partial\Omega\}$ □

Proof: (Another proof for maximal principal) If not, $\exists z_0 \in \Omega$ s.t. $|f(z)| \leq |f(z_0)|$ $\forall \Omega$. Let $\gamma = C_\rho(z_0) \subseteq \Omega \rightsquigarrow z = z_0 + \rho e^{i\theta}$, $\theta \in (-\pi, \pi)$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z_0 + \rho e^{i\theta}) \cdot i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + \rho e^{i\theta}) d\theta$$

- If $\exists z \in \gamma$ s.t. $|f(z)| < |f(z_0)| \rightsquigarrow \exists [\theta_1, \theta_2] \subseteq [-\pi, \pi]$ s.t. $|f(z_0 + \rho e^{i\theta})| < |f(z_0)|$ for all $\theta \in [\theta_1, \theta_2]$. So

$$|f(z_0)| = \left| \frac{1}{2\pi} \left(\int_{-\pi}^{\theta_1} + \int_{\theta_1}^{\theta_2} + \int_{\gamma_2}^{\pi} \right) f(z_0 + \rho e^{i\theta}) d\theta \right| < |f(z_0)| \; (\rightarrow\!\leftarrow)$$

- $\forall z \in \gamma$, $|f(z)| = |f(z_0)|$. Since ρ is arbitrary (only need $C_\rho(z_0) \subseteq \Omega$), $|f(z)| = |f(z_0)|$ $\forall z \in \overline{B_\rho(z_0)}$. Set

$$S = \{z \in \Omega : |f(z)| = |f(z_0)|\}$$

which is open, since for all $z' \in S$, we can replace z_0 in above argument to get $|f(z)| = |f(z')| = |f(z_0)|\ \forall z \in B_\rho(z')$. Also $\Omega \setminus S = \{z \in \Omega : |f(z)| < |f(z_0)|\}$ is open. Since Ω is simply connect and $z_0 \in S$, $\Omega = S$ i.e. $|f(z)|$ is constant. Since

$$|f(z)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z+\rho e^{i\theta})d\theta\right| \leq \int_{-\pi}^{\pi} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0+\rho e^{0\theta})|d\theta = |f(z)|$$

the equality holds, which means $f(z)$ is constant on $C_\rho(z)$ for all $z \in \Omega$ and suitable ρ i.e. $f(z)$ is constant in Ω.

□

Theorem 1.5.3 (Schwarz lemma)**.**

- If f is analytic in $B_1(0)$ and $\begin{cases} f(B_1(0)) \subseteq \overline{B_1(0)} \\ f(0) = 0 \end{cases}$, then $|f(z)| \leq |z|$ and $|f'(0)| = 1$
- If $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ with $|c| = 1$.

Proof:

- Define $g(z) = \begin{cases} \frac{f(z)}{z} & \text{for } z \neq 0 \\ f'(0) & \text{for } z = 0 \end{cases}$. For $0 < \rho < 1$, $\forall z \in B_\rho(0)$, by maximal principal,
$$|g(z)| \leq \max_{z \in C_\rho(0)} |g(z)| = \frac{|f(z)|}{\rho} < \frac{1}{\rho}$$
As $\rho \to 1$, $|g(z)| < 1$ on $B_1(0)$ i.e. $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.
- If $|f(z)| = |z|$ for some $z \neq 0$ in $B_1(0)$, then $|g(z)| = 1 \rightsquigarrow |g(z)|$ attains a max in $B_1(0) \rightsquigarrow g(z) = c$ is a constant function. Since $|f(z)| = |z|$, $|c| = 1$. □

1.6 Automorphism of unit disk and half plane

1.6.1 Automorphism of unit disk

- $f(z) = e^{i\theta}z : B_1(0) \to B_1(0)$ is a rotation $\rightsquigarrow f \in \mathrm{Aut}(B_1(0))$, the group of **bianalytic map** from $B_1(0)$ to $B_1(0)$.
- For $0 \neq a \in B_1(0)$, $T_a(z) = \dfrac{a-z}{1-\overline{a}z} \rightsquigarrow T_a \in \mathrm{Aut}(B_1(0))$:
  - $|a| < 1 \rightsquigarrow |1/\overline{a}| > 1 \rightsquigarrow T_a$ is analytic in $B_1(0)$
  - $\forall |z| = 1$,
  $$|T_a(z)| = \frac{|a-z|}{|1-\overline{a}z|}\frac{1}{|\overline{z}|} = \frac{|a-z|}{|\overline{z}-\overline{a}|} = 1$$
  By maximal principal, $|T_a(z)| < 1\ \forall z \in B_1(0)$.
  - $T_a(0) = a, T_a(a) = 0 \rightsquigarrow T_a \circ T_a(0) = 0, T_a \circ T_a(a) = a \rightsquigarrow T_a \circ T_a(0) = 0, T_a \circ T_a(a) = a$. By Schwarz lemma, $T_a \circ T_a(z) = cz$. Evaluate a, then $c = 1 \rightsquigarrow T_a \circ T_a = \mathrm{id}$ on $B_1(0)$
  - $\mathrm{Aut}(B_1(0)) = \{e^{i\theta} \circ T_a : \theta \in \mathbb{R}, a \in B_1(0)\}$:
  $\forall f \in \mathrm{Aut}(B_1(0))$, let a s.t. $f(a) = 0$. Define $g := f \circ T_a \in \mathrm{Aut}(B_1(0)) \rightsquigarrow g(0) = 0, g^{-1}(0) = 0$. By Schwarz lemma, $\begin{cases} |g'(0)| \leq 1 \\ |(g^{-1})'(0)| \leq 1 \end{cases} \implies |g'(0)| = 1 \rightsquigarrow g(z) = e^{i\theta}z$ i.e. $f = e^{i\theta} \circ T_a$.

1.6.2 Automorphism of half plane

Since we already knew all element in $\mathrm{Aut}(B_1(0))$, our idea is construct the bianalytic between $B_1(0)$ and half plane $\mathbb{H}$. Construct

$$\begin{array}{rccc} S: & \mathbb{H} & \longrightarrow & B_1(0) \\ & z & \longmapsto & \dfrac{i-z}{i+z} \end{array}$$

which is analytic in $\mathbb{H}$. $|i+z|<|i-z|$ $\forall z\in\mathbb{H}$, since z in the half plane divide by perpendicular bisector of $i,-i$ which contain i. Also $S^{-1}(z)=i\left(\dfrac{1-z}{1+z}\right)$ will sent $B_1(0)$ to $\mathbb{H}$, since

$$\operatorname{Im} S^{-1}(z)=\frac{(1-r^2)\cos^2\theta+(1+r^2)\sin^2\theta}{(1+r\cos\theta)^2+(r\sin\theta)^2}>0 \text{ if } z=re^{i\theta} \text{ with } r<1$$

Hence, S is bianalytic from $B_1(0)$ to $\mathbb{H}$, which will induce the group homomorphism

$$\begin{array}{rccc} \varphi: & \mathrm{Aut}(B_1(0)) & \longrightarrow & \mathrm{Aut}(\mathbb{H}) \\ & f & \longmapsto & s^{-1}\circ f\circ s \end{array}$$

and it clear that the inverse is

$$\begin{array}{rccc} \varphi^{-1}: & \mathrm{Aut}(\mathbb{H}) & \longrightarrow & \mathrm{Aut}(B_1(0)) \\ & g & \longmapsto & s\circ f\circ s^{-1} \end{array}$$

Hence, $\mathrm{Aut}(B_1(0))\simeq\mathrm{Aut}(\mathbb{H})$.

Definition 1.6.1. A **linear functional transformation** is $F_A(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$, where $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{C})$. It is clear that $F_B\circ F_A=F_{AB}$.

Theorem 1.6.1. $\mathrm{Aut}(\mathbb{H})\simeq\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}\simeq\{A\in\mathrm{SL}_2(\mathbb{R}):\det A>0\}=:\overline{\mathrm{SL}_2(\mathbb{R})}$

Proof:

- $A\in\overline{\mathrm{SL}_2(\mathbb{R})}\rightsquigarrow F_A\in\mathrm{Aut}(\mathbb{H})$:
  - pole is $z=-d/c\in\mathbb{R}$ which is not in $\mathbb{H}\rightsquigarrow F_A$ is analytic in $\mathbb{H}$
  - $\operatorname{Im}\dfrac{az+b}{cz+d}=\operatorname{Im}\dfrac{(az+b)(c\overline{z}+d)}{|cz+d|^2}=\operatorname{Im}\dfrac{adz+bc\overline{z}}{|cz+d|^2}=\dfrac{(ad-bc)\operatorname{Im} z}{|cz+d|^2}>0$
  - $F_A^{-1}=F_{A^{-1}}\rightsquigarrow F_A:\mathbb{H}\xrightarrow{\sim}\mathbb{H}$
- If $g\in\mathrm{Aut}(\mathbb{H})$ with $g(i)=i\rightsquigarrow\varphi^{-1}(g)(0)=s\circ g\circ s^{-1}(0)=0$. By Schwarz lemma, $|(\varphi^{-1}(g))'(0)|\leq 1$. Similar we have $|\varphi(g)'(0)|\leq 1$. By Schwarz lemma, $\varphi(g)=e^{i\theta}$. Let

$$B=\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\in\overline{\mathrm{SL}_2(\mathbb{R})}\implies\begin{cases}F_B(i)=i\\ F_B'(i)=e^{i\theta}\end{cases}\implies\varphi^{-1}(F_B)(z)=cz$$

  Differentiate in both side and substitute $z=0$ we have $e^{i\theta}=c$. Hence, $\varphi^{-1}(F_B)=\varphi^{-1}(g)\rightsquigarrow g=F_B$.

- $\forall z_0 \in \mathbb{H}$, $\exists D \in \overline{\mathrm{SL}_2(\mathbb{R})}$ s.t. $F_D(i) = z_0$:
  Let $D_1 = \begin{pmatrix} \sqrt{\operatorname{Im} z_0} & 0 \\ 0 & \sqrt{\operatorname{Im} z_0}^{-1} \end{pmatrix} \rightsquigarrow F_{D_1}(i) = i \operatorname{Im} z_0$. Let $D_2 = \begin{pmatrix} 1 & \mathrm{Re} z_0 \\ 0 & 1 \end{pmatrix}$, then $F_{D_2} \circ F_{D_1}(i) = z_0$. Let $D = D_2 D_1$, then $F_D(i) = z_0$.
- $\forall f \in \mathrm{Aut}(\mathbb{H})$, $\exists z_0$ s.t. $f(z_0) = i \rightsquigarrow \exists D \in \overline{\mathrm{SL}_2(\mathbb{R})}$ s.t $F_D(i) = z_0 \rightsquigarrow g = f \circ F_D$ and $g(i) = i \rightsquigarrow g = F_B$ for some $B \in \overline{\mathrm{SL}_2(\mathbb{R})} \rightsquigarrow f = F_D^{-1} \circ F_B = F_{D^{-1}B}$

$\square$

1.7 Residue

1.7.1 Laurent series

Recall : $f(z)$ is analytic in Ω and $z_0 \in \Omega$.

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^n(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$$

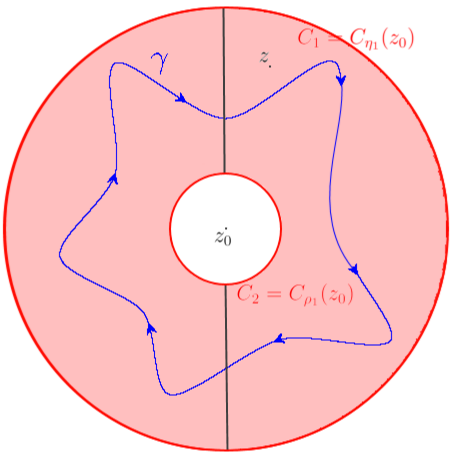
where

$$f_{n+1}(z) = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}(\xi - z)} d\xi$$

Then we can general the Taylor expansion.

Theorem 1.7.1 (Laurent series). f : analytic in Ω : $\rho < |z - z_0| < \eta$. Then $f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$ converge in Ω with $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}}$, where γ : simple closed arc in Ω, z_0 lies inside γ. The part of $\sum_{-\infty}^{-1} a_n (z - z_0)^n$ is called **singular part**.

Proof: Given $z \in \Omega$, choose $\rho < \rho_1 < \eta_1 < \eta$ such that $\rho_1 < |z - z_0| < \eta$, C_1 contain γ and C_2 inside γ. Choose two line (black) connected C_1 and C_2 do not pass z and let L_1, L_2 be the right and left half curve in below, then $L_1 + L_2 = C_1 - C_2$. WLOG z inside L_1.



$$f(z) = \frac{1}{2\pi i} \int_{L_1} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{L_2} \frac{f(\xi)}{\xi - z} d\xi = \underbrace{\frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi}_{(1)} - \underbrace{\frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{\xi - z} d\xi}_{(2)}$$

$$(1) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{(\xi - z_0)\left(1 - \dfrac{z-z_0}{\xi - z_0}\right)} = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z_0}\left(1 + \frac{z-z_0}{\xi - z_0} + \cdots + \frac{\left(\dfrac{z-z_0}{\xi-z_0}\right)^n}{1 - \dfrac{z-z_0}{\xi-z_0}}\right)$$

$$\implies a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi$$

and the error is

$$R_n = \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\xi)d\xi}{(\xi - z_0)^n(z-z_0)} \implies |R_n| \leq \frac{|z-z_0|^n}{2\pi}\frac{M\cdot 2\pi\eta_1}{\eta_1^n(\eta_1 - |z-z_0|)} \to 0$$

Similarly, we have

$$-(2) = \int_{C_2}\frac{f(z)}{z-z_0}\left(\frac{1}{\dfrac{\xi - z_0}{z-z_0} - 1}\right)d\xi \implies a_{-m} = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{(\xi - z_0)^{-m+1}}d\xi$$

□

Remark 1.7.1. z_0 : isolated singularity $\rightsquigarrow$ f analytic in $\Omega : 0 < |z-z_0| < \eta \implies f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$

- $a_n = 0\ \forall n < 0 \rightsquigarrow z_0$: removable
- $a_n = 0\ \forall n < -m$ but $a_m \neq 0 \rightsquigarrow z_0$: pole of order m
- $a_n \neq 0\ \forall n \to \infty \rightsquigarrow z_0$: essential singularity

If z_0 is a pole of order m, then

$$\frac{1}{2\pi i}\int_{C_\rho(z_0)} f(z)dz = \underbrace{\frac{1}{2\pi i}\int_{C_0(z_0)}\frac{a_{-m}}{(z-z_0)^m}dz}_{\text{have primitive}} + \cdots + \frac{1}{2\pi i}\int_{C_0(z_0)}\frac{a_{-1}}{(z-z_0)}dz + \underbrace{\frac{1}{2\pi i}\int_{C_0(z_0)} p(z)dz}_{\text{analytic}}$$
$$= a_{-1}n(C_\rho(z_0), z_0) = a_{-1} =: \operatorname{Res}_{z=z_0} f(z)$$

Definition 1.7.1.

- A region Ω is called **multiply connected** if it is not simply connected
- Ω has the finite **connectivity** n if $\widetilde{C} \setminus \Omega$ has exactly n connected components $A_1, A_2, ..., A_n$ and usually let $\infty \in A_n$.

Recall $\forall \gamma$: cycle in Ω, $n(\gamma, a)$ is constant in A_i $\forall i$ and $n(\gamma, a) = 0$ on A_n. For $i = 1, ..., n-1$, since A_i is bounded, as in the proof of fact about simply connectivity, $\exists \gamma_i \subset \Omega$ s.t. $n(\gamma_i, a) = 1\ \forall a \in A_i$ and $n(\gamma_i, b) = 0\ \forall b \in A_j \neq A_i$. $\forall \gamma$: cycle in Ω, let $c_i = n(\gamma, a)\ \forall a \in A_i$. Since $\forall a \in \widetilde{\mathbb{C}} \setminus \Omega$, say $a \in A_i$, then

$$n(\gamma - c_1\gamma_1 - \cdots - c_{n-1}\gamma_{n-1}, a) = n(\gamma, a) - c_i n(\gamma_i, a) = 0$$

i.e. $\gamma \sim c_1\gamma_1 + \cdots + c_{n-1}\gamma_{n-1}$ w.r.t. Ω. Hence, if f is analytic in Ω, then

$$\int_\gamma f dz = c_1\int_{\gamma_1} f dz + \cdots + c_{n-1}\int_{\gamma_{n-1}} f dz$$

$f(z)$ is analytic in Ω except for isolated singularities $a_1, ..., a_n$. Let $\Omega' = \Omega \setminus \{a_1, ..., a_n\}$ and $\gamma_i = C_{\rho_i}(a_i)$ with $\begin{cases} 0 < |z-a| < \rho_i \subset \Omega' \\ \gamma_i \sim 0 \text{ w.r.t. } \Omega \end{cases}$. Let γ be a cycle in Ω' with $\gamma \sim 0$ w.r.t. Ω. Since

$$n(\sum_{i=1}^{n} n(\gamma, a_i)\gamma_i, a_j) = \sum_{i=1}^{n} n(\gamma, a_i) n(\gamma_i, a_j) = n(\gamma, a_j) \ \forall j$$

and $\gamma \sim \sum_{i=1}^{n} n(\gamma, a_i)\gamma_i$ w.r.t. Ω, $\gamma \sim \sum_{i=1}^{n} n(\gamma, a_i)\gamma_i$ w.r.t. Ω'. Hence, we have

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{i=1}^{n} n(\gamma, a_i)\left(\frac{1}{2\pi i}\int_{\gamma_i} f(z)dz\right)$$

If a_i is pole, then $\frac{1}{2\pi i}\int_{\gamma_i} f(z)dz = \operatorname{Res}_{z=a_i} f(z)$. If all a_i are pole, we can rewrite it as

$$\sum_{i=1}^{n} n(\gamma, a_i) \operatorname{Res}_{z=a_i} f(z)$$

Property 1.7.1 (key fact). If z_0 is a pole of f of order m, then

$$a_{-1} = \operatorname{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1} (z-z_0)^m f(z)$$

Proof: Since $(z-z_0)^{m-1} f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + p(z)(z-z_0)^m$. $\square$

1.7.2 Evaluation definite integrals

(1) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta : z = e^{i\theta} \rightsquigarrow dz = ie^{i\theta} = izd\theta$, $\sin\theta = \frac{z-z^{-1}}{2}$, $\cos\theta = \frac{z+z^{-1}}{2}$.

$$L = \frac{-1}{4i}\int_{|z|=1} \underbrace{\frac{(z^2-1)^2}{z^2(2z^2+5z+2)}}_{=f} dz = \frac{-2\pi i}{4i}(\operatorname{Res}_{z=-1/2} f + \operatorname{Res}_{z=0} f) = \frac{-5}{4}$$

(2) $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+1} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos 2x}{x^2+1} dx$: Consider the curve Γ consists the segment from $-R$ to R and counterclockwise circular arc γ from R to $-R$ with radius R and center in 0.

$$\int_\Gamma \frac{1-e^{2\pi iz}}{z^2+1} dz = \int_{-R}^{R} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1} dx + \int_\gamma \frac{1-e^{2\pi iz}}{z^2+1} dz$$

Calculate residue, it will be $2\pi i \operatorname{Res}_{z=i} f = \pi(1-e^{-2})$. Also,

$$\left|\int_\gamma f dz\right| \le \int_\gamma \frac{1+|e^{2zi}|}{|z|^2-1}|dz| \le \int_\gamma \frac{1+|e^{-2\operatorname{Im} z}|}{|z|^2-1}|dz| \le \frac{2\pi R}{R^2-1} \to 0 \text{ as } R \to \infty$$

Hence,

$$\begin{aligned}\pi(1-e^{-2}) &= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1} dx + \int_\gamma \frac{1-e^{2\pi iz}}{z^2+1} dz\right) \\ &= \int_{-\infty}^{\infty} \frac{1-(\cos 2x + i\sin 2x)}{x^2+1} dx\end{aligned}$$

Consider the real part and thus $L = \pi(1-e^{-2})$.

(3) $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}dx,\ a>0$: First we check that will converge.

- $\displaystyle\int_1^\infty \frac{\ln x}{x^2+a^2}$ will converge since $\displaystyle\frac{\ln x}{x^2+a^2} < \frac{1}{x^{1.5}}$ for x sufficiently large.
- $\displaystyle\int_0^1 \frac{\ln x}{x^2+a^2}dx$ will converge since in

$$\int_0^1 \frac{-\ln x dx}{x^2+a^2}dx \leq \int_0^1 \frac{-\ln x dx}{a^2} = \frac{-1}{a^2}\left(x\ln x - x\right)\Big|_0^1 = \frac{1}{a^2}$$

Let C_ρ be the curve $\{\rho e^{i\theta} : 0\leq\theta\leq\pi\}$ and $\gamma = \overline{(-R)(-r)} - C_r + \overline{rR} + C_R$. We define $\ln x$ by branch $-\pi/2$. By residue,

$$\int_\gamma \frac{\log z}{z^2+a^2}dz = 2\pi i\,\mathrm{Res}_{z=ai}\frac{\log z}{z^2+a^2} = 2\pi i\lim_{z\to ai}\frac{\log z}{z+ai} = \frac{\pi}{a}\left(\ln a + \frac{\pi}{2}\right)$$

On C_R, $z=Re^{i\theta}$, $0\leq\theta\leq\pi$

$$\left|\frac{\log z}{z^2+a^2}\right| = \left|\frac{\ln R+\theta i}{R^2-a^2}\right| \leq \frac{\ln R+\pi}{R^2-a^2} \implies \left|\int_{C_R}\frac{\log z}{z^2+a^2}\right| \leq \frac{\pi R(\ln R+\pi)}{R^2-a^2} \to 0 \text{ as } R\to 0$$

On $-C_r$,

$$\left|\int_{-C_r}\frac{\log z}{z^2+a^2}dz\right| \leq \left(\frac{-\ln r+\pi}{a^2-r^2}\right)\pi r \to 0 \text{ as } r\to 0$$

Hence,

$$\frac{\pi}{a}\left(\ln a+\frac{\pi}{2}i\right) = \lim_{\substack{r\to 0\\ R\to\infty}}\left(\int_{-R}^{-r}\frac{\ln z}{z^2+a^2}dz + \int_r^R\frac{\ln z}{z^2+a^2}dz + \int_{C_R} f dz + \int_{-C_r} f dz\right)$$
$$= \int_0^\infty \frac{\ln x}{x^2+a^2}dx + \int_{-\infty}^0 \frac{\ln(-x)+\pi i}{x^2+a^2}dx = 2\int_0^\infty \frac{\ln x}{x^2+a^2}dx + \pi i\int_0^\infty\frac{dx}{x^2+a^2}$$

Hence, $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi\ln a}{2a}$

1.7.3 Rouche's theorem

Recall : Argument principle : Let $f\not\equiv 0$ be analytic in $B_\rho(a)$ and $\gamma\subseteq B_\rho(a)$ with $f\neq 0$ on γ. Let z_i be the roots of $f(z)=0$, then

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

Theorem 1.7.2 (General form)**.** Let $f(z)$ be meromorphic in Ω with zeros a_i's and the poles b_k's. Then $\forall\gamma\sim 0$ w.r.t. Ω and $a_j, b_k\notin\gamma$,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)dz}{f(z)} = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

Proof: Rearrange a_i's and b_k's s.t. there is h_i $\widetilde{a}_i$'s in $\{a_1, a_2, ...\}$ and ℓ_j $\widetilde{b}_j$'s in $\{b_1, b_2, ...\}$, where $\widetilde{a}_i \neq \widetilde{a}_j, \widetilde{b}_i \neq \widetilde{b}_j$ $\forall i \neq j$. Since zero and pole are isolated, $\exists \gamma_i$ and γ_i' s.t.

$$\begin{cases} n(\gamma_i, \widetilde{a}_j) = n(\gamma_i', \widetilde{b}_j) = \delta_{ij} \\ n(\gamma_i, \widetilde{b}_k) = n(\gamma_i', \widetilde{a}_k) = 0 \\ \gamma_i \sim 0, \gamma_i' \sim 0 \text{ w.r.t. } \Omega \end{cases}$$

We have known that $\gamma \sim \sum\limits_i n(\gamma, \widetilde{a}_i)\gamma_i + \sum\limits_j n(\gamma, \widetilde{b})j)\gamma_j'$ w.r.t. $\Omega' = \Omega \setminus \{a_i, b_j \ \forall i, j\}$. Observe that $\dfrac{f'(z)}{f(z)}$ is meromorphic. By residue formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, \widetilde{a}_i)\operatorname{Res}_{z=\widetilde{a}_i} f + \sum_j n(\gamma, \widetilde{b}_j)\operatorname{Res}_{z=\widetilde{b}_j} f$$

Now, for $a = \widetilde{a}_i$, $h = h_i$, write $f(z) = (z-a)^h f_h(z)$, $\dfrac{f'(z)}{f(z)} = \dfrac{h}{z-a} + \dfrac{f_h'(z)}{f_h(z)} \rightsquigarrow \operatorname{Res}_{z=a}\dfrac{f'(z)}{f(z)} = h$. For $b = \widetilde{b}_j$, $\ell = \ell_j$, write $f(z) = (z-b)^{-\ell}g_\ell(z)$, $\dfrac{f'(z)}{f(z)} = \dfrac{-\ell}{z-b} + \dfrac{g_\ell'(z)}{g_\ell(z)} \rightsquigarrow \operatorname{Res}_{z=b}\dfrac{f'(z)}{f(z)} = -\ell$. Hence,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, \widetilde{a}_i)h_i - \sum_j n(\gamma, \widetilde{b}_j)\ell_j = \sum_i n(\gamma, a_i) - \sum_k n(\gamma, b_k)$$

□

Remark 1.7.2. More general, if $g(z)$ is analytic in Ω, then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}dz = \sum_i n(\gamma, a_i)g(a_i) - \sum_j n(\gamma, b_j)g(b_j)$$

which can prove by same method in above.

Theorem 1.7.3 (Rouche's theorem)**.** Let $\gamma \sim 0$ w.r.t. Ω and $n(\gamma, z) = 0$ or 1 $\forall z \notin \gamma$. Let f and g be analytic in Ω. If $|f(z) - g(z)| < |f(z)|$ on γ, then f and g have the same number of zeros inside γ.

Proof: By assumption, $f \neq 0$ and $g \neq 0$ on γ, $\left|\dfrac{g(z)}{f(z)} - 1\right| < 1$ on γ. Let $\omega = F(z) = \dfrac{g(z)}{f(z)} \rightsquigarrow$ $\Gamma := \operatorname{Im} F|_\gamma \subset B_1(1) \rightsquigarrow n(\Gamma, 0) = 0$. Then

$$\begin{aligned} 0 = n(\Gamma, 0) &= \frac{1}{2\pi i}\int_\Gamma \frac{d\omega}{\omega} = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)dz}{F(z)} \\ &= \#\text{ zero of } F - \#\text{ pole of } F \text{ inside } \gamma \\ &= \#\text{ zero of } g - \#\text{ zero of } f \text{ inside } \gamma \end{aligned}$$

□

Example 1.7.1. Show that $e^z = az^n$ has exactly n solution in $|z| < 1$, where $a > e$.

Proof: $|e^z| = e^x \leq e \ \forall z \in C_1(0)$, $|-az^n| = a > e \ \forall z \in C_1(0) \implies |-az^n| > |e^z| \ \forall z \in C_1(0) \rightsquigarrow e^z = az^n$ and $-az^n$ have the same number of zeros in $|z| < 1 \rightsquigarrow e^z = az^n$ has exactly n solution in $|z| < 1$. $\square$

Question 1 : Let $w = f(z)$ be analytic in Ω. For $w_0 \in f(\Omega)$, find $z_j(w_0) \in \Omega$ s.t. $f(z_j(w_0)) = w_0$.
Assume that for $|w - w_0| < \delta$, $f - w$ has exactly n roots $z_j(w)$ in $|z - z_0| < \varepsilon$.

- Set $g(z) = z \rightsquigarrow\rightsquigarrow \sum_{j=1}^{n} z_j(w) = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z) - w} z dz$. In particular, $n = 1$ we have
$$f^{-1}(w) = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z) - w} z dz$$
- Set $g(z) = z^m \rightsquigarrow \sum_{j=1}^{n} z_j(w)^m = \frac{1}{2\pi i}\int_{C_\varepsilon(z_0)} \frac{f'(z)}{f(z) - w} z^m dz$ which is analytic w.r.t. w. Since the elementary symmetric polynomial of $z_1(w), ..., z_n(w)$ is in the $\mathbb{C}$-algebra generated by k-power sum $z_k(w)$ of $z_1(w), ..., z_n(w)$ which is also analytic in Ω. Hence, we can calculate the roots of $z^n - s_1(w)z^{n-1} + \cdots + (-1)^n s_n(w)$ which is $z_1(w), ..., z_n(w)$.

Question 2 : Find the number of zero of f in $|z| < R$.
Write $f(z) = P_{n-1} + z^n f_n(z)$. If we can choose n s.t. $R^n|f_n(z)| < |P_{n-1}(z)|$ on $|z| = R$, then # zero of $f(z) =$ # zero of $P_{n-1}(z)$ in $|z| < R$.

1.8 Sum and product

Definition 1.8.1. We write $f_n \xrightarrow{\overline{\text{unif}}} f$ in Ω if f_n converge uniform on each compact subset in Ω.

Theorem 1.8.1 (Weierstrass theorem)**.** f_n : analytic in Ω_n with $(\Omega_n \subset \Omega_{n+1})$ and $f_n \xrightarrow{\overline{\text{unif}}} f$ in $\Omega = \bigcup_n \Omega_n$. Then f is analytic and $f_n' \xrightarrow{\overline{\text{unfi}}} f'$.

Proof:

- For a fixed $\overline{B_\rho} \subset \Omega$, $\exists n_0$ s.t. $\overline{B_\rho(a)} \subseteq \Omega \ \forall n \geq n_0$. By Cauchy integral formula,
$$f_n(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f_n(\xi)}{\xi - z} d\xi \text{ in } B_\rho(a)$$
By assumption, since $f_n \to f$ uniformly converge in $B_\rho(a)$
$$f(z) = \lim_{n\to\infty} f_n(z) = \int_{C_\rho(a)} \frac{\lim_{n\to\infty} f_n(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f(\xi)}{\xi - z} d\xi$$
which is analytic.
- $\forall n \geq n_0$, $f_n'(z) = \frac{1}{2\pi i}\int_{C_\rho(a)} \frac{f_n(\xi)}{(\xi - z)^2} d\xi$ in $B_\rho(a)$. For all $\delta < \rho$, choose $\delta' \in (\delta, \rho)$. Since f_n uniformly converge to f in $\overline{B_{\delta'}(a)}$, for sufficiently large n, we have
$$|f'(z) - f_n'(z)| \leq \frac{1}{2\pi}\int_{C_{\delta'}(a)} \frac{|f(\xi) - f_n(\xi)|}{|\xi - z|^2} |d\xi| \leq \frac{\varepsilon\delta\pi}{2\pi|\delta' - \delta|^2} \ \forall z \in \overline{B_\delta(a)}$$

and thus f_n' is uniformly converge to f in $\overline{B_\delta(a)}$. Since any compact subset of Ω can be covered by $\{\overline{B_(\delta_1)(a_1)}, ..., \overline{B_(\delta_k)(a_k)}\}$, the result follow.

□

Theorem 1.8.2 (Mittag-Leffler theorem)**.** Let $\{b_n\} \subset \mathbb{C} \setminus \{0\}$ with $\lim_{n\to\infty} b_n = \infty$ and let $P_m(z) \in \mathbb{C}[z]$ with $P_m(0) \neq 0$. Then $\exists$ a meromorphic function $f(z)$ in $\mathbb{C}$ with pole at $b_m{}'s$ is $P_m(1/(z-b_m))$.

Proof:

- Since $P_m\left(\dfrac{1}{z-b_m}\right)$ is analytic for $|z| < |b_m|$, consider the Taylor series at $z = 0$

$$F_m(z) := P_m\left(\frac{1}{z-b_m}\right) = a_0^m + a_1^m + \cdots + \left(\frac{1}{2\pi i}\int_{C_{|b_m|/2}(0)} \frac{F_m(\xi)}{\xi^{n+1}(\xi - z)} d\xi\right) z^{m+1}$$

Let $M_m := \max_{|z|=|b_m|/2} |F_m(z)|$, $q_m(z) = a_0^m + \cdots + a_{n_m}^m z^{n_m}$ and choose n_m s.t. $2^{n_m} \geq M^m \cdot 2^m$. Then

$$|F_m(z) - q_m(z)| \leq \frac{M_m}{2\pi} \frac{|z|^{n_m+1}}{(|b_m|/2)^{n_m+1}} \frac{2\pi |b_m|/2}{|b_m|/4} = \frac{M_m}{2^{n_m}} \leq \frac{1}{2^m}$$

$\forall N \in \mathbb{N}$, $\exists n_N > 0$ s.t. $n \geq n_N$, $|b_n| > N$. So for $|z| \leq N/4 < |b_n|/4$ $\forall n \geq n_N$,

$$|F_n(z) - q_n(z)| \leq \left(\frac{1}{2}\right)^n \ \forall n \geq n_N,\ |z| \leq \frac{|N|}{4}$$

By Weierstrass M-test, $g_N(z) := \sum_{n=n_N}^{\infty} (F_n(z) - q_n(z))$ converge uniformly on $|z| \leq N/4$ and thus is analytic in $|z| < N/4$. Define

$$f_N(z) = \sum_{n=1}^{N_n - 1} (F_n(z) - q_n(z)) + g_N(z) : \text{meromorphic } |z| < N/4 \text{ with poles } b_1, ..., b_{n_N - 1}$$

Notice that

$$|g_{N+1} - g_N| = \left|\sum_{n_N+1}^{n_{N+1}-1} (F_n(z) - q_n(z))\right| \leq \sum_{n_N+1}^{n_{N+1}-1} \frac{1}{2^n} \leq \frac{1}{2^{n_N}} \to 0 \text{ as } N \to \infty$$

Then g_N is Cauchy sequence and thus $g_N \to g$ uniformly in $\mathbb{C}$.

$$|f_{N+1} - f_N| \leq \left|\sum_{n=n_N}^{n_{N+1}-1} (F_n(z) - q_n(z))\right| + |g_{N+1} - g_N| \leq \frac{2}{2^{n_N}} \text{ for } |z| \leq N/4$$

Then f_N is Cauchy sequence and thus

$$f_n \to f = \sum_{n=1}^{\infty} (F_n(z) - q_n(z)) + g(z) \text{ as } N \to \infty$$

where $g(z)$ is entire function.

□

Remark 1.8.1. If we consider $\{b_m\}_{m\in\mathbb{N}} \cup \{0, ..., 0\}$, then $\widetilde{f}(z) = f(z) + \sum_{i=1}^{\ell} \overline{P}_i(1/z)$.

Example 1.8.1. $f(z) = \dfrac{\pi^2}{\sin^2 \pi z}$ has pole when $z \in \mathbb{Z}$.

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\left(\sum_{n=1}^{\infty} \frac{(-\pi z)^{2n-1}}{(2n-1)!}\right)} = \frac{1}{z^2}\left(1 - \frac{(\pi z)^2}{3!} + \cdots\right) = \frac{1}{z^2}\left(1 - \left(\frac{(\pi z)^2}{3!} + \cdots\right) + \cdots\right)$$

The singularity part at 0 is $\frac{1}{z^2}$. Since $\sin^2 \pi(z-n) = \sin^2 \pi z$ $\therefore$ the singularity part at n is $\dfrac{1}{(z-n)^2}$. Then

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z)$$

where $g(z)$ analytic in $\mathbb{C}$. Claim $g(z) = 0$:
subproof : g has period $\omega = 1$.

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

and thus

$$\frac{\pi^2}{|\sin^2 \pi z|} \leq \frac{\pi^2}{|\cosh^2 \pi y| - \cos^2 \pi x} \leq \frac{\pi^2}{|\cosh^2 \pi y| - 1} \xrightarrow[0\leq x\leq 1]{\text{unif.}} 0 \text{ as } |y| \to \infty$$

and $\sum_{n=-\infty}^{\infty} \dfrac{1}{(z-n)^2} \xrightarrow[0\leq x\leq 1]{\text{unif.}} 0$ as $|y| \to \infty \implies g(z) \xrightarrow[0\leq x\leq 1]{\text{unif.}} 0$ as $|y| \to \infty$. Then $|g(z)|$ is bounded in $0 \leq x \leq 1$ and thus bounded in $\mathbb{C}$. By Lioville's theorem, $g = c$ is constant. Since $\lim_{|y|\to\infty} g(z) = 0 \rightsquigarrow g = 0$.

Definition 1.8.2.

- $p_n \neq 0\ \forall n$, $q_n = p_1 p_2 \cdots p_n$, $\prod_{n=0}^{\infty} p_n = \lim_{n\to\infty} q_n$.
- In general, $\prod_{n=1}^{\infty} p_n$ converge $\iff \#\{p_i | p_i = 0\} < \infty$ and $\prod_{p_n \neq 0} p_n$ exists.

Fact 1.8.1.

- $\prod_{n=1}^{\infty} p_n$ converge $\implies \lim_{n\to\infty} p_n = 1 \rightsquigarrow \prod_{n=1}^{\infty}(1 + a_n)$ with $\lim_{n\to\infty} a_n = 0$.
- $\prod(1 + a_n)$ with $(1 + a_n) \neq 0 \iff \sum \log(1 + a_n)$ converge (principal branch).
- $\prod(1 + a_n)$ absolutely converge $\iff \sum |a_n|$ converge :

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1 \rightsquigarrow \forall \varepsilon > 0,\ (1-\varepsilon)|a_n| < |\log(1 + a_n)| < (1+\varepsilon)|\varepsilon|$$

- $g(z)$: entire $\rightsquigarrow f(z) = e^{g(z)}$: entire and $\neq 0$.

- $f(z)$: entire and never zero, for a fixed z_0

$$g(z) := \int_{\gamma_z} \frac{f'(\xi)}{f(\xi)} d\xi + c_0,\ e^{c_0} = f(z_0)$$

where γ_z. Then $g'(z) = \dfrac{f'(z)}{f(z)} \rightsquigarrow \dfrac{d}{dz}\left(f(z)e^{-g(z)}\right) = 0 \rightsquigarrow f(z)e^{-g(z)} = c \xrightarrow{z=z_0} c = 1 \rightsquigarrow f(z) = e^{g(z)}$.

Theorem 1.8.3 (Weierstrass)**.** Given $\{a_n\} \subseteq \mathbb{C} \setminus \{0\}$ with $\lim_{n\to\infty} a_n = \infty$, there exists entire functions with zeros $= \{a_n\}$. possibly including 0.

Proof:

- # of $\{a_n\} < \infty$: $f(z) = z^m e^{g(z)} \prod_{i=1}^{n} \left(1 - \frac{z}{a_i}\right)$, $g(z)$: entire function.
- # of $\{a_n\} = \infty$:
  - $\sum |a_n|^{-1}$ converge $\iff \sum \dfrac{|z|}{|a_n|}$ converge $\forall |z| \le R \iff \prod \left(1 - \dfrac{z}{a_n}\right)$ converge uniform $\forall |z| < R$. So $f(z) = z^m \prod_{n=1}^{\infty} \left(1 - \dfrac{z}{a_n}\right)$ analytic in $\mathbb{C}$.
  - In general, $\exists$ polynomial $p_n(z)$ s.t. $\prod \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$ converge to entire function.
    subproof : For $R > 0$, say $|a_n| > R\ \forall n > N$. For $|z| \le R$, $\forall n \in \mathbb{N}$

$$\log\left(1 - \frac{z}{a_n}\right) = \frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \frac{1}{3}\left(\frac{z}{a_n}\right)^3 - \cdots$$

    Let $p_n(z) = \sum_{k=1}^{m_n} \dfrac{1}{k}\left(\dfrac{z}{a_n}\right)^k$ for some $m_n \in \mathbb{Z}_{\ge 0}$. If $R_n(z) = \log\left(1 - \dfrac{z}{a_n}\right) + P_n(z)$, then

$$|R_n(z)| \le \frac{1}{m_n + 1}\left(\frac{R}{|a_n|}\right)\left(1 - \frac{R}{|a_n|}\right)^{-1}$$

    Choose $m_n = n$, then by root test, $\sum_{n=N}^{\infty} \dfrac{1}{m_n + 1}\left(\dfrac{R}{|a_n|}\right)^{m_n+1}$ converge. Then
    - $R_n(z) \to 0 \rightsquigarrow -\pi < \operatorname{Im} R_n(z) < \pi\ \forall n \ge N_0 > N$.
    - $\sum_{n=N_0}^{\infty}$ converge absolutely convergent and uniformly for $|z| \le R \rightsquigarrow \prod_{n=N_0}^{\infty} \left(1 - \dfrac{z}{a_n}\right) e^{p_n(z)}$ is analytic for $|z| \le R$ and thus

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$$

      is analytic for $|z| \le R$, where $p_n(z) \in \mathbb{C}[z]$. By a similar argument for Mittag-Leffler theorem, $\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$ is analytic in $\mathbb{C}$.

Hence, in general,

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{m_n}\right)^{m_n}}$$

□

Definition 1.8.3.

- $m_n = n$, $E_n(z/m) = \left(1 - \frac{z}{a_m}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right)$ is called **canonical factor**
- h is called the **genus of canonical product** of f if h is the smallest integer s.t.

$$\sum \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1} \text{ converge i.e. } \sum \frac{1}{|a_n|^{h+1}} \text{ converge}$$

Example 1.8.2. $f(z) = \sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}$, since $\sum n^{-1}$ diverge and $\sum n^{-2}$ converge $\rightsquigarrow m_n = 1\ \forall n$. Consider $f'(z)/f(z)$, we have

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \underbrace{\sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right)}_{(1)} \rightsquigarrow g'(z) = 0 \rightsquigarrow g(z) \text{ is constant}$$

$\because \lim_{z \to 0} \frac{\sin \pi z}{z} = \pi \therefore e^{g(z)} = \pi$. Now we check that (1) is converge for all $z \in \mathbb{C}$.

$$\sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right) = z \sum_{n \neq 0} \frac{1}{n(z-n)}$$

which will converge by comparison test with $\sum n^{-2}$.

Proposition 1.8.1. If $f(z)$ is meromorphic in $\mathbb{C}$, then $F(z) = f(z)/g(z)$, where $f(z), g(z)$: entire.

Proof: Let $g(z)$ be an entire function with zero = poles of $F(z) \rightsquigarrow g(z)F(z)$ is an entire function $f(z) \rightsquigarrow F(z) = f(z)/g(z)$. □

1.9 Gamma function

Recall that

$$\frac{\sin \pi z}{\pi} = z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n} \underbrace{\prod_{n \neq 0} \left(1 + \frac{z}{n}\right) e^{z/n}}_{:=G(z)}$$

Observation : zero of $G(z-1) = \{0, -1, -2, ...\} \rightsquigarrow G(z-1) = zG(z)e^{g(z)}$ for some $g(z)$: entire. Consider $\frac{(\,\cdot\,)'}{(\,\cdot\,)}$, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{n}\right) + g'(z) \implies g'(z) = 0 \text{ i.e. } g(z) = c$$

Let $z = 1$, $1 = G(0) = e^c \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n} \rightsquigarrow e^{-c} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n} = \lim_{n \to \infty} (n+1) e^{-(1+\cdots+1/n)}$, then $c = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)\right) = \gamma$ which is Euler's constant. Then $G(z-1) = zG(z)e^c$.

- $H(z) := e^{cz}G(z) \rightsquigarrow H(z-1) = e^{c(z-1)}G(z-1) = ze^{cz}G(z) = zH(z)$.

- $\Gamma(z) := 1/zH(z)$, then $\Gamma(z-1) = \frac{1}{(z-1)H(z-1)} = \frac{1}{z(z-1)H(z)} = \frac{1}{z-1}\Gamma(z)$

$$\implies \Gamma(z) = (z-1)\Gamma(z-1)$$

In particular, $\Gamma(1) = 1/H(1) = 1/e^c G(1) = 1$, $\Gamma(2) = 1$, $\cdots \Gamma(n) = (n-1)!$.

•• $\Gamma(z) = 1/ze^{cz}G(z) = z^{-1}e^{-cz}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$ which is meromorphic function with poles $0, -1, -2, ...$ and no zero.

•• $\Gamma(1-z) = (1-z)^{-1}e^{cz-z}\prod_{n=1}^{\infty}\left(1+\frac{1-z}{n}\right)^{-1}e^{(1-z)/n}$. Then

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= z^{-1}(1-z)^{-1}e^{-c}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}\left(1+\frac{1-z}{n}\right)^{-1}e^{1/n} \\
&= \frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)\left(1+\frac{1-z}{n}\right)\left(\frac{n}{n+1}\right)\right)^{-1} \\
&= \frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{n}\right)\left(1-\frac{z}{n+1}\right)\right)^{-1} \\
&= \frac{1}{z(1-z)}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}(e^{z/n})^{-1}\prod_{n=1}^{\infty}\left(1-\frac{z}{n+1}\right)^{-1}(e^{-z/(n+1)}) \\
&= \frac{1}{z(1-z)}\frac{1}{G(z)}\cdot\frac{1-z}{G(-z)} = \frac{\pi}{\sin \pi z}
\end{aligned}$$

In particular, $\Gamma(1/2)^2 = \dfrac{\pi}{\sin(\pi/2)} \implies \Gamma(1/2) = \sqrt{\pi}$.

•• **Legendre's duplication formula** : $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$:

••• $\dfrac{\Gamma'(z)}{\Gamma(z)} = \dfrac{-1}{z} - c + \sum_{n=1}^{\infty}\left(\dfrac{-1}{z+n}+\dfrac{1}{n}\right)$ and $\dfrac{d}{dz}\left(\dfrac{\Gamma'(z)}{\Gamma(z)}\right) = \dfrac{1}{z^2} + \sum_{n=1}^{\infty}\dfrac{1}{(z+n)^2} = \sum_{n=0}^{\infty}\dfrac{1}{(z+n)^2}$ and thus

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) = 4\left(\sum_{n=0}^{\infty}\frac{1}{(2z+2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2z+1+n)^2}\right) = 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)$$

Integral in both side we have

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = \frac{\Gamma'(2z)}{\Gamma(2z)} + a$$

Integral in both side we have

$$\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)e^{az+b}$$

Substitute $z = 1, 1/2$, we have

$$\begin{cases}\Gamma(1)\Gamma(3/2) = \Gamma(2)e^{a+b} \\ \Gamma(1/2)\Gamma(1) = \Gamma(1)e^{a/2+b}\end{cases} \implies \begin{cases}e^{a+b} = \sqrt{\pi}/2 \\ e^{a/2+b} = \sqrt{\pi}\end{cases} \implies \begin{cases}e^a = 1/4 \\ e^b = 2\sqrt{\pi}\end{cases}$$

Hence, $2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \Gamma(2z)\sqrt{\pi}$.

1.10 Entire function

Definition 1.10.1. $u : \mathbb{C} \to \mathbb{R}$ is **harmonic** if u_{xx}, u_{yy} continuous and $\Delta u = u_{xx} + u_{yy} = 0$.

Fact 1.10.1.

(1) $f = u + iv$: analytic $\rightsquigarrow u, v \in \mathcal{H}$:

By Cauchy Riemann equation, $u_x = v_y, u_y = -v_x \rightsquigarrow u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

(2) $u \in \mathcal{H}(\Omega)$ with Ω : simply connected $\rightsquigarrow \exists v \in \mathcal{H}(\Omega)$ s.t. $f = u + iv$ is analytic in Ω :

- $g = u_x - iu_y$ is analytic, since $\begin{cases} u_{xx} = (-u_y)_y & \text{, since } u \in \mathcal{H} \\ u_{xy} = -(-u_y)_x \end{cases}$
- Since Ω is simply connected, g has a primitive $f(z) = \displaystyle\int_{z_0}^{z} g(z)dz + u(x_0, y_0)$
- $f = U + iV \rightsquigarrow f' = U_x - iU_y$ and equal to $g = u_x - iu_y \implies U_x = u_x, U_y = u_y$ and thus $U = u + c$
- $f(z_0) = u(x_0, y_0) = U(x_0, y_0) + iV(x_0, y_0) \rightsquigarrow c = 0$.

(3) **Mean-value property** : $u \in \mathcal{H}$, let $v \in \mathcal{H}$ s.t. $f = u + iv$ is analytic.

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \implies u(x_0, y_0) = \frac{1}{2\pi i}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

(4) **Poisson's formula** : $u \in \mathcal{H}(\overline{B_R(0)})$, $\forall z \in B_R(0)$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} u(Re^{i\theta})d\theta = \frac{1}{2\pi}\mathrm{Re}\left(\frac{Re^{i\theta} + z}{Re^{i\theta} - z}\right) u(Re^{i\theta})d\theta$$

proof : For $a \in B_R(0)$, i.e. $|a| < R$. Consider $w = T(\xi) = \dfrac{R(R\xi + a)}{R + \overline{a}\xi}$, then

$$\begin{aligned} T: \quad |\xi| \le 1 \quad &\longrightarrow \quad |w| \le R \\ \xi = 0 \quad &\longmapsto \quad w = a \end{aligned}$$

Then $u(T(\xi)) \in \mathcal{H}(|\xi| < 1)$, then

$$u(a) = u(T(0)) = \frac{1}{2\pi}\int_{|\xi|=1} u(T(\xi))d\arg\xi$$

Notice that $\xi = |\xi|e^{i\arg\xi} \rightsquigarrow d\xi = i|\xi|e^{i\arg\xi}d\arg\xi \rightsquigarrow d\arg\xi = \dfrac{d\xi}{i\xi}$. Also

$$\xi = \frac{R(w-a)}{R^2 - \overline{a}w} \rightsquigarrow \frac{1}{\xi} = \frac{R^2 - \overline{a}w}{R(w-a)} \text{ and } d\xi = \frac{R(R^2 - |a|^2)}{(R^2 - \overline{a}w)^2}dw$$

Also, for $|\xi| = 1$, $|T(\xi)|^2 = \dfrac{R^2(R\xi + a)(R\overline{\xi} + \overline{a})}{(R + \overline{a}\xi)(R + a\overline{\xi})} = R^2$. Let $w = Re^{i\theta}$, then $-idw = wd\theta$.

$$\begin{aligned} \frac{d\xi}{i\xi} &= \frac{(R^2 - |a|^2)dw}{i(w-a)(R^2 - \overline{a}w)} = -i\left(\frac{1}{w-a} + \frac{\overline{a}}{R^2 - \overline{a}w}\right)dw = \left(\frac{w}{w-a} + \frac{\overline{a}w}{w\overline{w} - \overline{a}w}\right)d\theta \\ &= \left(\frac{a}{w-a} + \frac{\overline{a}}{\overline{w} - \overline{a}}\right) = \frac{R^2 - |a|^2}{|w-a|^2}d\theta \end{aligned}$$

Also,

$$\mathrm{Re}\left(\frac{w+a}{w-a}\right)=\frac{1}{2}\left(\frac{w+a}{w-a}+\frac{\overline{w}+\overline{a}}{\overline{w}-\overline{a}}\right)=\frac{R^2-|a|^2}{|w-a|^2}$$

which proved the equation.

Notice that $\log(z)$ is analytic in $\mathbb{C}$ except one line, and $\log|z|$ be the real part of $\log(z)$ which is harmonic. Then we have below formula.

Theorem 1.10.1 (Jensen's formula)**.** f : analytic for $|z|\le\rho$ with $f(0)\neq 0$. Then

$$\log|f(0)|=-\sum_{i=1}^{n}\log\left(\frac{\rho}{|a_i|}\right)+\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|d\theta$$

where $a_1,...,a_n$ are zero of f in $\overline{B_\rho(0)}$.

Proof:

- If $f\neq 0$ in $|z|\le\rho$: OK!
- $\forall i,\ |a_i|=\rho$: By induction on n :

  $n=1$: Let $g=\dfrac{f}{z-a_1}\rightsquigarrow g\neq 0$ in $|z|\le\rho$ and thus

  $$\log|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|g(\rho e^{i\theta})|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left(\log|f(\rho e^{i\theta})|-\log\rho|e^{i\theta}-e^{i\theta_0}|\right)d\theta$$

  where $a_1=\rho e^{i\theta_0}$. We can calculate that

  $$\frac{1}{2\pi}\int_0^{2\pi}\log|1-e^{i\theta}|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log 2\left|\sin\frac{\theta}{2}\right|d\theta=\log 2+\frac{2}{\pi}\underbrace{\int_0^{\pi/2}\log\sin x dx}_{:=I}\qquad(1)$$

  Notice that $x^{1/2}\log\sin x|=x^{1/2}\log x+x^{1/2}\log(\sin x/x)\to 0$ as $x\to 0$, so I converge. Consider $x=\pi/2-\theta$, then

  $$I=\int_{\pi/2}^{0}\log\cos x(-dx)=\int_0^{\pi}\log\cos x dx$$

  and thus

  $$2I=\int_0^{\pi}\log\sin x dx=2\int_0^{\pi/2}\log\sin 2\theta d\theta=\frac{1}{2}\int_0^{\pi/2}\log(2\sin\theta\cos\theta)d\theta=\pi\log 2+4I$$

  So we have $I=\dfrac{-\pi}{2}\log 2$ and thus $(1)=0$. So

  $$\log|f(0)|-\log|\rho|=\log|g(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta-\log|\rho|\implies\log|f(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta$$

  For $n>1$, we can do same argument.
- In general, let $F(z)=f(z)\prod_{i=1}^{n}\dfrac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\neq 0$ in $|z|<\rho$, since $|\rho^2/\overline{a_i}|\ge\rho$. Also, $|F(z)|=|f(z)|$ on $z=\rho$, so

  $$\log|f(0)|+\sum_{i=1}^{n}\log\frac{\rho}{|a_i|}=\log|F(0)|=\frac{1}{2\pi}\int_0^{2\pi}\log|F(\rho e^{i\theta})|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|d\theta$$

$\square$

Theorem 1.10.2 (Poisson-Jensen's formula)**.** For $z \in B_\rho(0)$ with $f(z) \neq 0$, by Possion formula for $\log|F(z)|$,

$$\log|f(z)| + \sum_{i=1}^{m} \log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| = \log|F(z)| = \int_0^{2\pi} \mathrm{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)\log|f(\rho e^{i\theta})|d\theta$$

Definition 1.10.2. Let f be an entire function. The **order** of f is defined by

$$\lambda := \limsup_{r\to\infty} \frac{\log\log M(\rho)}{\log \rho} \text{ where } M(\rho) = \max_{|z|=\rho}|f(z)|$$

Fact 1.10.2. λ is the smallest number s.t. $M(\rho) \leq e^{\rho^{\lambda+\varepsilon}}$ for any $\varepsilon > 0$ as soon as large enough.

Proof:

- $\lambda = \lim\limits_{\delta\to\infty}\sup\limits_{\rho\geq\delta} \dfrac{\log\log M(\rho)}{\log\rho} \rightsquigarrow \forall\varepsilon > 0,\ \exists\delta_0 > 0$ s.t. for all $\delta > \delta_0$

$$\left|\sup_{\rho\geq\delta}\frac{\log\log M(\rho)}{\log\rho} - \lambda\right| < \varepsilon \implies \frac{\log\log M(\rho)}{\log\rho} < \lambda + \varepsilon\ \forall\rho \geq \delta_0$$

  and thus $M(\rho) \leq e^{\rho^{\lambda+\varepsilon}}$.

- For $\mu < \lambda$, let $\varepsilon = (\lambda-\mu)/3 \rightsquigarrow \exists\rho > 0$ s.t. $\dfrac{\log\log M(\rho)}{\log\rho} > \lambda - \varepsilon = \mu + 2\varepsilon$ i.e. $M(\rho) > e^{\mu+2\varepsilon}$.

$\square$

Theorem 1.10.3 (Main theorem)**.** Let $f(z)$ be the entire function with order $\lambda < \infty$ and h be the largest integer $\leq \lambda$ i.e. $h \leq \lambda < h+1$. If $a_1, a_2, ...$ be the zero of $f(z)$ and $0 \neq a_i\ \forall i$, then

- $\sum |a_n|^{-(h+1)}$ converge
- $f(z) = e^{g(z)}\prod\limits_{n=1}^{\infty}\left(1 - \dfrac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h}$ with $g(z)$ is a polynomial with deg $\leq h$.

Proof:

- Assume $\mu(\rho)$ be the number of a_i's with $|a_i| \leq \rho$, then $n \leq \mu(|a_n|)$. By Jensen's formula,

$$\log|f(0)| = -\sum_{i=1}^{\mu(2\rho)}\log\left|\frac{2\rho}{a_i}\right| + \frac{1}{2\pi}\int_0^{2\pi}|f(2\rho e^{i\theta})|d\theta$$

  Observer that if $\rho \leq |a_i| \leq 2\rho \rightsquigarrow 0 \leq \log\left|\dfrac{2\rho}{a_i}\right| \leq \log 2$. Then

$$\mu(\rho)\log 2 \leq \sum_{|a_i|\leq\rho}\log\frac{2\rho}{|a_i|} \leq \sum_{i=1}^{\mu(2\rho)} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(2\rho e^{i\theta})|d\theta - \log|f(0)|$$

  and $\log|f(2\rho e^{i\theta})| \leq \log M(2\rho) < (2\rho)^{\lambda+\varepsilon}$.

$$\implies \mu(\rho) \leq \frac{1}{\log 2}(2^{\lambda+\varepsilon}\rho^{\lambda+\varepsilon} - \log|f(0)|) < K(2\rho)^{\lambda+\varepsilon}$$

for some constant $K > 0$. So $n \le \mu(|a_n|) < K(2|a_n|)^{\lambda+\varepsilon}$. Choose $\varepsilon > 0$ s.t. $\lambda + \varepsilon < h + 1$ and thus

$$|a_n|^{-(h+1)} = (n^{-(\lambda+\varepsilon)})^{\frac{h+1}{\lambda+\varepsilon}} \le \frac{2^{h+1}K^{\frac{h+1}{\lambda+\varepsilon}}}{n^{\frac{h+1}{\lambda+\varepsilon}}}$$

Since $\frac{h+1}{\lambda+\varepsilon} > 1$, $\sum |a_n|^{-(h+1)}$ converge.

- By Poisson-Jensen's formula,

$$\log|f(z)| = -\sum_{i=1}^{\mu(\rho)} \log\left|\frac{\rho^2 - \overline{a_i}z}{\rho(z-a_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)\log|f(\rho e^{i\theta})|d\theta$$

Note $f = u + iv$, then $f' = 2\dfrac{\partial u}{\partial z}$, $\dfrac{f'(z)}{f(z)} = (\log f(z))' = 2\dfrac{\partial}{\partial z}\log|f(z)|$

  - $\dfrac{\partial}{\partial z}\left(\sum 2\log\left|\dfrac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\right|\right) = \sum \dfrac{\partial}{\partial z}\log\left(\dfrac{\rho^2-\overline{a_i}z}{\rho(z-a_i)}\right)\left(\dfrac{\rho^2-a_i\overline{z}}{\rho(\overline{z}-\overline{a_i})}\right) = -\sum\left(\dfrac{1}{z-a_i}+\dfrac{\overline{a_i}}{\rho^2-\overline{a_i}z}\right)$
  - $2\dfrac{\partial}{\partial z}\operatorname{Re}\left(\dfrac{\rho e^{i\theta}+z}{\rho e^{i\theta - z}}\right) = \left(\dfrac{\rho e^{i\theta}+z}{\rho e^{i\theta - z}}\right)' = 2\cdot\dfrac{2\rho e^{i\theta}}{(\rho e^{i\theta}-z)^2}$

Hence,

$$\frac{f'(z)}{f(z)} = \sum_{i=1}^{\mu(\rho)}\frac{1}{z-a_i} + \sum_{i=1}^{\mu(\rho)}\frac{\overline{a_i}}{\rho^2-\overline{a_i}z} + \frac{2}{\pi}\int_0^{2\pi}\frac{\rho e^{i\theta}}{(\rho e^{i\theta}-z)^2}\log|f(\rho e^{i\theta})|d\theta$$

Differentiate h times, we have

$$\left(\frac{f'(z)}{f(z)}\right)^{(h)} = \sum_{i=1}^{\mu(\rho)}\frac{-h!}{(a_i-z)^{h+1}} + \underbrace{\sum_{i=1}^{\mu(\rho)}\frac{h!\cdot\overline{a_i}^{h+1}}{(\rho^2-\overline{a_i}z)^{h+1}}}_{(2)} + \underbrace{\frac{2}{\pi}\int_0^{2\pi}\frac{(h+1)!\rho e^{i\theta}}{(\rho e^{i\theta}-z)^{h+2}}\log|f(\rho^{i\theta})|d\theta}_{(3)}$$

  - (3) : If $\rho > 2|z|$, then

$$|(3)| \le \frac{(h+1)!\cdot 2}{\pi}\int_0^{2\pi}\frac{\log M(\rho)}{(\rho-|z|)^{h+2}}d\theta = \frac{4(h+1)!\log M(\rho)}{\rho^{h+1}(1-|z|/\rho)^{h+2}}$$

    since $\log M(\rho) \le \rho^{\lambda+\varepsilon}\ \forall\varepsilon > 0 \rightsquigarrow \rho^{-(h+1)}\log M(\rho) \le \rho^{\lambda-h-1+\varepsilon}$. Choose ε s.t. $\lambda - h - 1 + \varepsilon < 0$. Hence, $(3) \to 0$ as $\rho \to \infty$.
  - (2) : $\rho > 2|z|$, $|a_i| \le \rho$, then

$$|(2)| \le h!\sum_{i=1}^{\mu(\rho)}\frac{\rho^{h+1}}{\rho^{2h+2}}(\rho^2/2)^{h+1} = h!\cdot\mu(\rho)2^{h+1}\rho^{-(h+1)} < h!\cdot K2^{h+1+\lambda+\varepsilon}\rho^{\lambda+\varepsilon-h-1} \to 0$$

    as $\rho \to \infty$.

Therefore,

$$\left(\frac{f'(z)}{f(z)}\right)^{(h)} = -h!\sum_{i=1}^{\infty}\frac{1}{(a_i-z)^{h+1}}$$

Let $p(z) = \prod\left(1-\dfrac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h}$ and $f(z) = e^{g(z)}p(z)$. Then

$$\left(\frac{p'(z)}{p(z)}\right)^{(h)} = \sum_{n=1}^{\infty}\frac{-h!}{(a_n-z)^{h+1}} = \left(\frac{f'(z)}{f(z)}\right)^{(h)}$$

and $g^{(h+1)}(z) = \left(\dfrac{f'(z)}{f(z)}\right)^{(h)} - \left(\dfrac{p'(z)}{p(z)}\right)^{(h)} = 0 \implies g(z) \in \mathbb{C}[z]$ and $\deg g \le h$. □

Definition 1.10.3. f has genus h if h is the smallest integer s.t.

$$\begin{cases} \sum |a_n|^{-(h+1)} \text{ converge} \\ \deg g(x) \le h \end{cases}$$

Theorem 1.10.4. Let h be the genus of f and λ be the order of f, then $h \le \lambda \le h+1$.

Proof:

- If h is finite, then $\lambda \le h+1$:
  Claim : $\log |E_h(z)| \le (2h+1)|z|^{h+1}$:
  subproof :

  - If $|z| \le 1$, $\log(1-z) = -z - \frac{z^2}{2} - \cdots$, then $E_n(z) = e^{\log(1-z)+z+\cdots+\frac{1}{h}z^h} = e^{-\sum\limits_{n=h+1}^{\infty} \frac{z^n}{n}}$

  $$\implies |E_n(z)| \le e^{\sum\limits_{n=h+1}^{\infty} \frac{|z|^n}{n}} \implies \log |E_n(z)| \le \sum_{n=h+1}^{\infty} \frac{|z|^n}{n} \le \frac{1}{h+1} \cdot \frac{|z|^{h+1}}{1-|z|}$$

  and thus $(1-|z|) \log |E_n(z)| \le \dfrac{|z|^{h+1}}{h+1} \le |z|^{h+1}$
  - $h = 0$: $\log |E_0(z)| = \log |1-z| \le \log(1+|z|) \le |z|$
  - For $h \ge 1$: We induction on h. $\log |E_h(z)| \le \log |E_{h-1}(z)| + \dfrac{|z|^h}{h} \le \log |E_{h-1}(z)| + |z|^h$
    - $|z| \ge 1$: $\log |E_h(z)| \le (2h-1)|z|^h + |z|^h \le (2h+1)|z|^{h+1}$
    - $|z| \le 1$: $\log |E_h(z)| \le |z| \log |E_h(z)| + |z|^{h+1} \le |z|(2h|z|^h) = (2h+1)|z|^{h+1}$ □

  By Claim,

  $$\begin{aligned} \log |f(z)| &\le \log |e^{g(z)}| + \log |p(z)| \le |g(z)| + \sum_n \log |E_h(z/a_n)| \\ &\le |z|^{h+1} \left(\frac{|b_q|}{|z|^{h+1-q}} + \cdots + \frac{|b_0|}{|z|^{h+1}} \right) + (2h+1)|z|^{h+1} \sum_n |a_n|^{-(h+1)} \end{aligned}$$

  Hence,

  $$\log \log |f|(z)| \le (h+1) \log |z| + \log \left(\frac{|b_q|}{|z|^{h+1-q}} + \cdots + \frac{|b_0|}{|z|^{h+1}} + (2h+1) \sum_n |a_n|^{-(h+1)} \right)$$

  When z on $C_r(0)$:

  $$\frac{\log \log M(r)}{\log r} \le h+1 + \log \left(O(|r|^{q-h-1}) + (2h+1) \sum_n |a_n|^{-(h+1)} \right) \Big/ \log r$$

  As $r \to \infty$, we have $\lambda \le h+1$ and hence λ is finite.

- If λ is finite, let h_0 be the smallest integer h_0 s.t. $h_0 \le \lambda$. By Theorem 1.10.3, $\begin{cases} \sum |a_n|^{-(h_0+1)} \text{ converge} \\ \deg g \le h_0 \end{cases}$
  By definition of genus, $h \le h_0 \le \lambda$.

Theorem 1.10.5. Let $f(z) = \sum_{n=1}^{\infty} c_n z^n$ and $\alpha = \liminf \dfrac{\log(1/|c_n|)}{n \log n}$. Then

- $\alpha > 0 \implies f$ is entire of order α
- $\alpha = 0 \implies f$ has infinite order

Also, if $f(z)$ is entire of finite order λ, then $\lambda = 1/\alpha$.

Proof:

- $\alpha > 0 : \forall \varepsilon > 0, \exists n_0$ s.t. $\forall n > n_0$, $\log(1/|c_n|) > (\alpha - \varepsilon) n \log n$ i.e. $|c_n| \leq n^{-n(\alpha-\varepsilon)}$, then $\sum c_n z^n$ converge for all $z \in \mathbb{C} \rightsquigarrow f$ is entire.
  - α is finite : Notice that $|c_n|$ is bounded, say $|c_n| \leq A$ with $A > 1$. $\forall r > 1$, for $|z| \leq r$,

    $$|f(z)| \leq A r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n n^{-n(\alpha-\varepsilon)}$$

    where $n_0 = \lfloor (2r)^{1/(\alpha-\varepsilon)} \rfloor$. Then $\forall n > n_0$, $n \geq (2r)^{1/(\alpha-\varepsilon)} \rightsquigarrow r n^{-(\alpha-\varepsilon)} \leq 1/2$ and thus

    $$\sum_{n=n_0+1}^{\infty} r^n n^{-n(\alpha-\varepsilon)} \leq \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \leq 1$$

    Hence,

    $$|f(z)| \leq 2A r^{n_0} \implies \log|f(z)| \leq \log 2 + n_0 \log r \leq (2r)^{1/\alpha-\varepsilon} \log r$$
    $$\implies \frac{\log\log M(r)}{\log r} \leq \frac{\frac{1}{\alpha-\varepsilon}\log r + \log\log r}{\log r}$$

    as $r \to \infty$, we have $\lambda \leq \frac{1}{\alpha-\varepsilon}$ for all $\varepsilon > 0 \implies \lambda \leq \frac{1}{\alpha}$.
  - $\alpha = \infty$: By definition, $\forall N > 0, \varepsilon > 0$, n_0 s.t. $\forall n > n_0$,

    $$\log(1/|c_n|) > (N - \varepsilon) n \log n \implies \cdots \implies \lambda \leq \frac{1}{N} \xrightarrow{N\to\infty} \lambda = 0 = \frac{1}{\infty}$$

  - If $0 < \alpha < \infty \rightsquigarrow \lambda \geq \frac{1}{\alpha}$: $\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $(\alpha+\varepsilon) n_\varepsilon \log n_\varepsilon > \log(1/|c_{n_\varepsilon}|)$ i.e.

    $$|c_{n_\varepsilon}| > n^{-n_\varepsilon(\alpha+\varepsilon)} \implies |c_{n_\varepsilon}| r^{n_\varepsilon} > (r n_\varepsilon^{-(\alpha+\varepsilon)})^{n_\varepsilon}$$

    Choose $r = (2n)^{(\alpha+\varepsilon)}$, then

    $$|c_{n_\varepsilon}| r^{n_\varepsilon} > 2^{n_\varepsilon(\alpha+\varepsilon)} \implies \log|c_{n_\varepsilon}| r^{n_\varepsilon} > \frac{r^{1/(\alpha+\varepsilon)}(\alpha+\varepsilon)}{2} \log 2$$

    By Cauchy estimate,

    $$|c_{n_\varepsilon}| = \frac{|f^{(n_\varepsilon)}(0)|}{n!} \leq M(r) r^{-n_\varepsilon} \implies \log\log M(r) \geq \frac{1}{\alpha+\varepsilon}\log r + \log\frac{(\alpha+\varepsilon)\log 2}{2}$$

    Hence, $\lambda \geq \frac{1}{\alpha+\varepsilon}$ $\forall \varepsilon > 0 \rightsquigarrow \lambda \geq \frac{1}{\alpha}$.

- If $\alpha = 0 : \forall \dfrac{1}{N}$, $\lambda \geq \dfrac{1}{1/N + \varepsilon} \xrightarrow{\varepsilon\to 0} \lambda \geq N \xrightarrow{N\to\infty} \lambda = \infty$.

$\square$

Chapter 2

Homework

2.1

Problem 2.1.1. Show that

$$|\cos z|^2 = \sinh^2 y + \cos^2 x = \cosh^2 y - \sin^2 x = \frac{1}{2}(\cosh 2y + \cos 2x)$$

and

$$|\sin z|^2 = \sinh^2 y + \sin^2 x = \cosh^2 y - \cos^2 x = \frac{1}{2}(\cosh 2y - \cos 2x).$$

Problem 2.1.2. Determine all values of $2^i, i^i, (-1)^{2i}$.

Problem 2.1.3. Express $\arctan w$ in terms of the logarithm.

Problem 2.1.4. Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when :

(a) $a_n = (\log n)^2$

(b) $a_n = n!$

(c) $a_n = \dfrac{n^2}{4^n + 3n}$

(d) $a_n = (n!)^3/(3n)!$

(e) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha, \beta, \gamma : z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n.$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma \neq 0, -1, -2, ...$

(f) Find the radius of convergence of the Bessel function of order r :

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n}$$

Problem 2.1.5. Expand $(1-z)^{-m}$ in powers of z. Here m is a fixed positive integer. Also show that if

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,$$

the one obtains the following asymptotic relation for the coefficients :

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \text{ as } n \to \infty.$$

Problem 2.1.6. Show that for $|z| < 1$, one has

$$\sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^{n+1}}} = \frac{z}{1-z},$$

and

$$\sum_{n=0}^{\infty} \frac{2^n z^{2^n}}{1 + z^{2^n}} = \frac{z}{1-z}.$$

justify any change in the order of summation.

2.2

Problem 2.2.1. Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a,b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reserve orientation. Prove that for any continuous function f on γ

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

Problem 2.2.2. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

(a) Evaluate the integrals

$$\int_\gamma z^n dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Problem 2.2.3. It is possible to define $n(\gamma, a)$ for any continuous closed curve γ that does not pass through a, whether piecewise differentiable or not. For this purpose γ is divided into subarces $\gamma_1, ..., \gamma_n$, each contained in a disk that does not include a. Let σ_k be the directed line segment from the initial to the terminal point of γ_k, and set $\sigma = \sigma_1 + \cdots + \sigma_n$. We define $n(\gamma, a)$ to be the value of $n(\sigma, a)$. To justify the definition, prove the following :

(a) the result is independent of the subdivision;

(b) if γ is piecewise differentiable the new definition is equivalent to the old;

(c) If γ lies inside of a circle, then $n(\gamma, a) = 0$ for all points a outside of the same circle. As a function of a the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.

Problem 2.2.4. The **Jordan cure theorem** asserts that every Jordan curve in the plane determines exactly two regions. The notion of winding number leads to a quick proof of one part of the theorem, namely that the complement of a Jordan curve γ has at least two components. This will be so if there exists a point a with $n(\gamma, a) \neq 0$.

We may assume that $\mathrm{Re} z > 0$ on γ, and that there are points $z_1, z_2 \in \gamma$ with $\mathrm{Im}\, z_1 < 0$, $\mathrm{Im}\, z_2 > 0$. These point may be chosen so that there are no other points of γ on the line segments from 0 to z_1 and from 0 to z_2. Let γ_1 and γ_2 be the arcs of γ from z_1 to z_2 (excluding the end points).

Let σ_1 be the closed curve that consists of the line segment from 0 to z_1 followed by γ_1 and the segment from z_2 to 0, and let σ_2 be constructed in the same way with γ_2 in the place of γ_1. Then $\sigma_1 - \sigma_2 = \gamma$ or γ.

The positive real axis intersects both γ_1 and γ_2. Choose the notation so that the intersection x_2 farthest to the right is with γ_2.

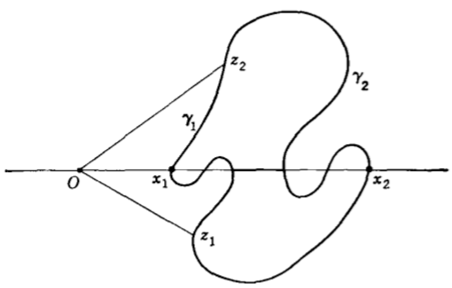


FIG. 4-6. Part of the Jordan curve theorem.

Prove the following :

(a) $n(\sigma_1, x_2) = 0$, hence $n(\sigma_1, z) = 0$ for $z \in \gamma_2$;

(b) $n(\sigma_1, x) = n(\sigma_2, x) = 1$ for small $x > 0$;

(c) the first intersection x_1 of the positive real axis with γ lies on γ_1;

(d) $n(\sigma_2, x_1) = 1$, hence $n(\sigma_2, z) = 1$ for $z \in \gamma_1$;

(e) there exists a segment of the positive real axis with one end point on γ_1, the other on γ_2, and no other points on γ. The points x between the end points satisfy $n(\gamma, x) = 1$ or -1.

2.3

Example 2.3.1. Compute

$$\int_{|z|=1} \frac{e^z}{z} dz.$$

Example 2.3.2. Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}$$

by decomposition of the integrand in partial fractions.

Example 2.3.3. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$.

2.4

Problem 2.4.1. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

Problem 2.4.2. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

Problem 2.4.3. Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$. Formulate a sharper theorem of the same kind.

Problem 2.4.4. Let the function $\varphi(z,t)$ be continuous as a function of both variables when z lies in a region Ω and $\alpha \leq t \leq \beta$. Suppose further that $\varphi(z,t)$ is analytic as a function of $z \in \Omega$ for any fixed t. Then

$$F(z) = \int_\alpha^\beta \varphi(z,t)dt$$

is analytic in z and

$$F'(z) = \int_\alpha^\beta \frac{\partial \varphi(z,t)}{\partial z} dt. \tag{1}$$

to prove this represent $\varphi(z,t)$ as a Cauchy integral

$$\varphi(z,t) = \frac{1}{2\pi i} \int_C \frac{\varphi(\xi,t)}{\xi - z} d\xi$$

Fill in the necessary details to obtain

$$F(z) = \int_C \left(\frac{1}{2\pi i} \int_\alpha^\beta \varphi(\xi,t)dt \right) \frac{d\xi}{\xi - z}$$

and use Lemma 3 to prove (1).

Problem 2.4.5. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

is equal to 0. Prove that f is a polynomial.

2.5

Problem 2.5.1. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that fg has the order $h + k$, f/g the order $h - k$, and $f + g$ an order which does not exceed $\max(h, k)$.

Problem 2.5.2. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

Problem 2.5.3. Show that any function which is meromorphic in the extended plane is rational.

Problem 2.5.4. Prove that an isolated singularity of $f(z)$ is removable as soon as either $\mathrm{Re} f(z)$ or $\mathrm{Im}\, f(z)$ is bounded above or below.

Problem 2.5.5. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.