



KTH Computer Science  
and Communication

## DT2119 Lab2: Hidden Markov Models with Gaussian Emissions

### 1 Objective

The objective is to implement the algorithms for evaluation and decoding of Hidden Markov Models (HMMs) combined with Gaussian emission probability distributions. The lab is designed in Python, but the same functions can be obtained in Matlab/Octave or using the Hidden Markov Toolkit (HTK).

### 2 Task

The overall task is to implement and test methods for isolated word recognition:

- combine phonetic HMMs into word HMMs using a lexicon
- implement the *forward-backward* algorithm,
- use it to compute the log likelihood of spoken utterances given a Gaussian HMM
- perform isolated word recognition
- implement the *Viterbi algorithm*, and use it to compute Viterbi path and likelihood
- compare and comment Viterbi and Forward likelihoods
- implement the *Baum-Welch algorithm* to update the parameters of the emission probability distributions

In order to pass the lab, you will need to follow the steps described in this document, and present your results to a teaching assistant in the designated time slots.

### 3 Data and model set

The speech data used in this lab is similar but not the same as in Lab 1. You can load the array containing speech utterances with the commands:

```
>>> import numpy as np
>>> data = np.load('lab2_data.npz')['data']
```

The data contains also MFCC features (`1mfcc` key), but you are welcome to test how the algorithms perform on the MFCCs computed with your own code from Lab 1. Refer to the instructions to Lab 1 for more information about the data structures.

Additionally, the `lab2_models.npz` file contains the parameters of the models you are going to use to test your functions and the `lab2_example.npz` file contains an example that can be used for debugging.

### 3.1 The phonetic models

Load the model file with:

```
>>> phoneHMMs = np.load('lab2_models.npz')['phoneHMMs'].item()
```

`phoneHMMs` is a dictionary with 21 keys each corresponding to a phonetic model. You can list the model names with:

```
>>> list(sorted(phoneHMMs.keys()))
['ah', 'ao', 'ay', 'eh', 'ey', 'f', 'ih', 'iy', 'k', 'n', 'ow', 'r', 's',
 'sil', 'sp', 't', 'th', 'uw', 'v', 'w', 'z']
```

Special cases are `sil` and `sp` that model silence and short pauses. For this exercise you can ignore the `sp` model, that will become important only in Lab 3. Note that the list is a subset of the phonemes used in the English language limited to the pronunciation of the 11 digits.

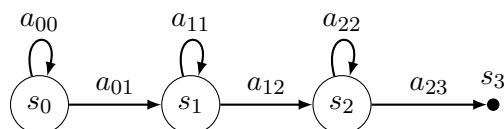
Each model is an HMM with a single Gaussian emission probability distribution per state and diagonal covariance matrices stored as vectors. The models were trained on the training data of the TIDIGITS database, using the 13 MFCC feature vectors computed as in Lab 1.

Each model contains the following keys:

```
>>> phoneHMMs['ah'].keys()
['name', 'startprob', 'transmat', 'means', 'covars']
```

key	symbol	description
<code>name</code>		phonetic symbol, <code>sil</code> or <code>sp</code>
<code>startprob</code>	$\pi_i = P(z_0 = s_i)$	probability to start in state $i$
<code>transmat</code> :	$a_{ij} = P(z_n = s_j   z_{n-1} = s_i)$	transition probability from state $i$ to $j$
<code>means</code> :	$\mu_{id}$	array of mean vectors (rows correspond to different states)
<code>covars</code> :	$\sigma_{id}^2$	array of variance vectors (rows correspond to different states)

If you ignore the `sp` model, all models have three emitting states. Consequently, the `means` and `covars` arrays will be both  $3 \times 13$  in size. Note, however, that both the `startprob` and `transmat` arrays have sizes corresponding to four states (`startprob` is length 4 and `transmat` is  $4 \times 4$ ). The reason for this is that we will concatenate these models to form word models, and we therefore want to be able to express the probability to stay ( $a_{22}$ ) or leave ( $a_{23}$ ) the last (third) emitting state in each phonetic model. This is illustrated in the following figure and will be clearer in Section 3.2.



Not that the last state  $s_3$  is non-emitting, meaning that it does not have any state to output probability distribution associated with it.

### 3.2 Pronunciation dictionary and word models

The mapping between digits and phonetic models can be obtained with the pronunciation dictionary that you can find in `prondict.py`, and that looks like this:

```
prondict = {}
prondict['o'] = ['ow']
prondict['z'] = ['z', 'iy', 'r', 'ow']
prondict['1'] = ['w', 'ah', 'n']
...
```

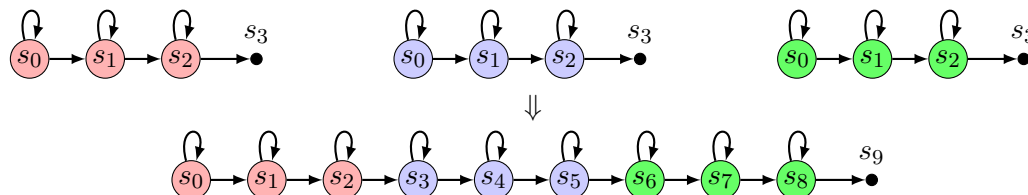
Because we are working with recordings of isolated digits, a model of each utterance should also contain initial and final silence:

```
>>> modellist = {}
>>> for digit in prondict.keys():
>>>     modellist[digit] = ['sil'] + prondict[digit] + ['sil']
```

Write the `concatHMMs` function from `proto2.py` that, given the set of HMM models and a list of model names, will return a combined model created by the concatenation of the corresponding models. For example:

```
>>> wordHMMs['o'] = concatHMMs(phoneHMMs, modellist['o'])
```

Remember that each model in `phoneHMMs` has an extra state to simplify the concatenation. An illustration of the process in the case of three models is in the following figure:



The following figure tries to explain how the transition matrices are combined to form the resulting transition matrix.

$a_{00}$	$a_{10}$	$a_{20}$	$a_{30}$	$a_{40}$	$a_{50}$	$a_{60}$	$a_{70}$	$a_{80}$	$a_{90}$
$a_{01}$	$a_{11}$	$a_{21}$	$a_{31}$	$a_{41}$	$a_{51}$	$a_{61}$	$a_{71}$	$a_{81}$	$a_{91}$
$a_{02}$	$a_{12}$	$a_{22}$	$a_{32}$	$a_{42}$	$a_{52}$	$a_{62}$	$a_{72}$	$a_{82}$	$a_{92}$
$a_{03}$	$a_{13}$	$a_{23}$	$a_{33}$	$a_{43}$	$a_{53}$	$a_{63}$	$a_{73}$	$a_{83}$	$a_{93}$
$a_{04}$	$a_{14}$	$a_{24}$	$a_{34}$	$a_{44}$	$a_{54}$	$a_{64}$	$a_{74}$	$a_{84}$	$a_{94}$
$a_{05}$	$a_{15}$	$a_{25}$	$a_{35}$	$a_{45}$	$a_{55}$	$a_{65}$	$a_{75}$	$a_{85}$	$a_{95}$
$a_{06}$	$a_{16}$	$a_{26}$	$a_{36}$	$a_{46}$	$a_{56}$	$a_{66}$	$a_{76}$	$a_{86}$	$a_{96}$
$a_{07}$	$a_{17}$	$a_{27}$	$a_{37}$	$a_{47}$	$a_{57}$	$a_{67}$	$a_{77}$	$a_{87}$	$a_{97}$
$a_{08}$	$a_{18}$	$a_{28}$	$a_{38}$	$a_{48}$	$a_{58}$	$a_{68}$	$a_{78}$	$a_{88}$	$a_{98}$
$a_{09}$	$a_{19}$	$a_{29}$	$a_{39}$	$a_{49}$	$a_{59}$	$a_{69}$	$a_{79}$	$a_{89}$	$a_{99}$

To be precise, we should remove the last row from the transition matrix of the red and blue models. However, in each original transition matrix the last row is  $[0.0, 0.0, 0.0, 1.0]$ . If we copy the values in into the new transition matrix in the natural order (red, blue, green), the first element of new transition matrix will overwrite the last of the previous one, and we would obtain the same affect. The last row of the combined transition matrix will also be all zeros but one ( $a_{99}$  in this case).

### 3.3 The example

Load the example file with:

```
example = np.load('lab2_example.npz')['example'].item()
```

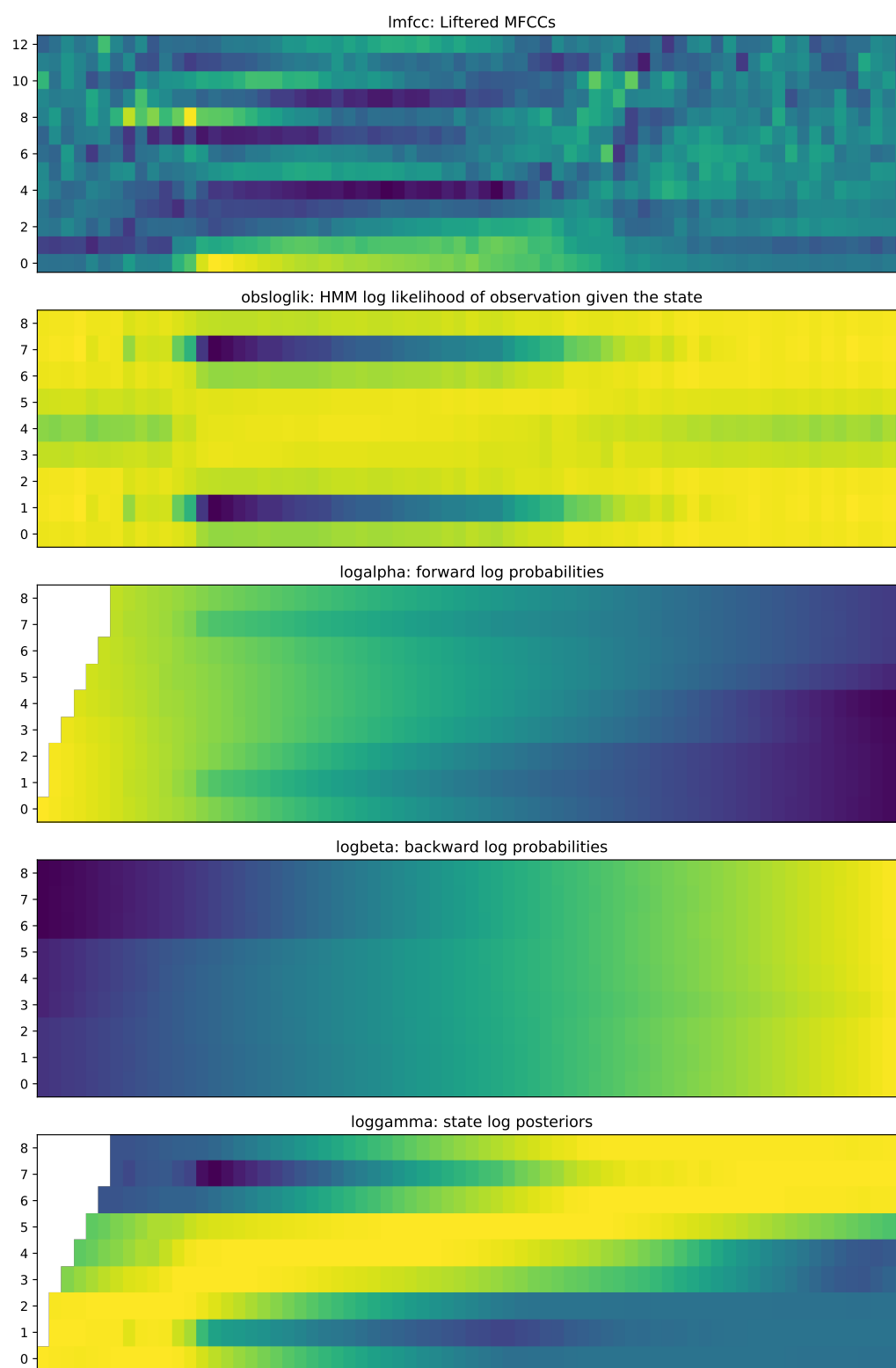
This is a dictionary containing all the fields as in the `data` array, plus the following additional fields:

```
list(example.keys())
[... , 'obsloglik', 'logalpha', 'loglik', 'vloglik', 'loggamma', 'logxi']
```

Here is a description of each field. You will see how to use this information in the remainder of these instructions. All the probabilities described below were obtained using the HMM model `wordHMMs['o']` (that is, the models for digit 'o') on the sequence of MFCC vectors in `example['lmfcc']`:

key	idxs	symbol	description
obsloglik	[i,j]	$\log \phi_j(x_i)$	observation log likelihood for each Gaussians in <code>wordHMMs['o']</code> , shape: (n_timesteps, n_states)
logalpha	[i,j]	$\log \alpha_i(j)$	alpha log probabilities, see definition later, shape: (n_timesteps, n_states)
logbeta	[i,j]	$\log \beta_i(j)$	beta log probabilities, see definition later, shape: (n_timesteps, n_states)
loglik	-	$\log P(X \theta_{\text{HMM}})$	log likelihood of the observations sequence $X$ given the HMM model, scalar
vloglik	-	$\log P(X, S_{\text{opt}} \theta_{\text{HMM}})$	Viterbi log likelihood of the observations sequence $X$ and the best path given the HMM model, scalar
loggamma	[i,j]	$\log \gamma_i(j)$	gamma log probabilities, see definition later, shape: (n_timesteps, n_states)

Figure 1 shows some of the relevant fields in `example`.



**Figure 1.** Step-by-step probability calculations for the utterance in the example. The utterance contains the digit “oh” spoken by a female speaker. The model parameters are obtained by concatenating the HMM models for `sil`, `ow` and `sil`.

## 4 HMM Likelihood and Recognition

### 4.1 Gaussian emission probabilities

The function `log_multivariate_normal_density(..., 'diag')` from `sklearn.mixture` can be used to compute

$$\text{obsloglik}[i, j] = \log \phi_j(x_i) = \log N(x_i, \mu_j, \Sigma_j) = \log P(x_i | \mu_j, \Sigma_j),$$

that is the log likelihood for each observation  $x_i$  and each term in a multivariate Gaussian density function with means  $\mu_j$  and diagonal covariance matrices  $\Sigma_j$ . In the case of Gaussian HMMs,  $\phi_j$  corresponds to the emission probability model for a single state  $j$ .

Verify that you get the same results as in `example['obsloglik']` when you apply the `log_multivariate_normal_density` function to the `example['lmfcc']` data using the Gaussian distributions defined by the `wordHMMs['o']` that you have created with your `concatHMMs` function. Note that in these cases we are not using the time dependency structure of the HMM, but only the Gaussian distributions.

Plot the log likelihood for Gaussians from HMMs models corresponding to the same digit on a test utterance of your choice. What can you say about the order of the Gaussians, and the time evolution of the likelihood along the utterance? Remember that each utterance starts and ends with silence.

### 4.2 Forward Algorithm

Write the function `forward` following the prototypes in `proto2.py`. The function should take as input a lattice of emission probabilities as in the previous section, that is an array containing:

$$\text{obsloglik}[i, j] = \log \phi_j(x_i)$$

and the model parameters  $a_{ij}$  and  $\pi_i$ . When you compute the log of  $a_{ij}$  and  $\pi_i$ , some of the values will become negative infinity ( $\log(0)$ ). This should not be a problem, because all the formulas should remain consistent<sup>1</sup>.

The output is the array:

$$\text{logalpha}[i, j] = \log \alpha_i(j)$$

where  $i$  corresponds to time steps and  $j$  corresponds to states in the model.

Remember that the forward probability is defined as:

$$\alpha_n(j) = P(x_0, \dots, x_n, z_n = s_j | \theta), \quad (1)$$

where  $\theta = \{\Pi, A, \Phi\}$  are the model parameters. There is a recursion formula (see Appendix A) to obtain  $\alpha_n(j)$  from  $\alpha_{n-1}(i)$  for all the previous states. In the recursion formulae for the forward (and backward) probabilities there is an expression involving the log of a sum of exponents ( $\log \sum \exp(\cdot)$ ). Make use of the `logsumexp` function from `tools2.py` to calculate those cases.

Apply your function to the `example['obsloglik']` utterance using the model parameters in your `wordHMMs['o']` model and verify that you obtain the same  $\log \alpha$  array as in `example['logalpha']`.

Remembering the definition of  $\alpha$  in Eqn. 1, derive the formula to compute the likelihood  $P(X | \theta)$  of the whole sequence  $X = \{x_0, \dots, x_{N-1}\}$ , given the model parameters  $\theta$  in terms of  $\alpha_n(j)$ .

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<sup>1</sup>An alternative would be to use the function `numpy.ma.log()` which will mask  $\log(0)$  elements, but at the time of writing I could not get this to work properly, so I do not recommend it at this point.

**Hint:** if the events  $B_j, j \in [0, M)$  form a disjoint partition of the sure event, that is:

$$\begin{aligned} P(B_i, B_j) &= 0, \quad \forall i, j, \quad i \neq j, \quad \text{and} \\ \sum_{j=0}^{M-1} P(B_j) &= 1, \end{aligned}$$

then it is true that  $P(A) = \sum_{j=0}^{M-1} P(A, B_j)$ , that is, we can *marginalize out* the variable  $B_j$ .

Convert the formula you have derived into log domain. Verify that the log likelihood obtained this way using the model parameters in `wordHMMs['o']` and the observation sequence in `example['lmfcc']` is the same as `example['loglik']`.

Using your formula, score all the 44 utterances in the `data` array with each of the 11 HMM models in `wordHMMs`. Do you see any mistakes if you take the maximum likelihood model as winner?

### 4.3 Viterbi Approximation

Here you will compute the log likelihood of the observation  $X$  given a HMM model and the best sequence of states. This is called the Viterbi approximation. Implement the function `viterbi` in `proto2.py`.

In order to recover the best path, you will also need an array storing the best previous path for each time step and state (this needs to be defined only for  $n \in [1, N)$ , that is not for the first time step):

$$B_n(j) = \arg \max_{i=0}^{M-1} \left( \log V_{n-1}(i) + \log a_{ij} \right)$$

Consult the course book [1], Section 8.2.3, to see the details of the implementation (note that the book uses indices from 1, instead of 0).

Compute the Viterbi log likelihood for `wordHMMs['o']` and `example['lmfcc']`, and verify that you obtain the same result as in `example['vloglik']`.

Plot the alpha array that you obtained in the previous Section and overlay the best path obtained by Viterbi decoding. Can you explain the reason why the path looks this way?

Using the Viterbi algorithm, score all the 44 utterances in the `data` with each of the 11 HMM models in `wordHMMs`. How many mistakes can you count if you take the maximum likelihood model as winner? Compare these results with the results obtained in previous section.

### 4.4 Backward Algorithm

Write the function `backward` following the prototypes in `proto2.py`. Similarly to the function `forward` in the previous section, the function should take as input a lattice of emission probabilities as in the previous section, that is an array containing:

$$\text{obsloglik}[i, j] = \log \phi_j(x_i)$$

The output is the array:

$$\text{logbeta}[i, j] = \log \beta_i(j),$$

where  $i$  corresponds to time steps and  $j$  corresponds to states in the model. See Appendix A for the recursion formulae.

In all the cases where there is an expression involving the log of a sum of exponents ( $\log \sum \exp(\cdot)$ ), make use of the `logsumexp` function in `tools2.py`.

Apply your function to the `example['lmfcc']` utterance using the model parameters in `wordHMMs['o']` and verify that you obtain the  $\log \beta$  arrays as in `example['logbeta']`.

The definitions of  $\beta_n(j)$  in terms of probabilities of events are defined below (where  $\theta = \{\Pi, A, \Phi\}$  are the model parameters):

$$\beta_n(i) = P(x_{n+1}, \dots, x_{N-1} | z_n = s_i, \theta)$$

Optional: Derive the formula that computes  $P(X|\theta)$  using the betas  $\beta_n(i)$  instead of the alphas.

**Hint 1:** only the  $\beta_0(i)$  are needed for the calculation.

**Hint 2:** note that the definition of  $\alpha_n(j)$  is a *joint* probability of observations *and* state at time step  $n$  whereas  $\beta_n(i)$  is a *conditional* probability of the observations *given* the state at time step  $n$ .

**Hint 3:** the calculation will not only involve the betas, but also some of the observation likelihoods  $\phi_j(x_n)$  and some of the model parameters,  $\pi_i$  or  $a_{ij}$ .

Verify that also this method gives you the same log likelihood and that they both are equal to `example['loglik']`.

## 5 HMM Retraining (emission probability distributions)

### 5.1 State posterior probabilities

Implement the `statePosteriors` function in `proto2.py` that calculates the state posteriors  $\gamma_n(i) = P(z_n = s_i | X, \theta)$  given the observation sequence, also called gamma probabilities. See Appendix A for the formulas in log domain. Calculate the state posteriors for the utterance in the example. Verify that for each time step the state posteriors sum to one (in linear domain). Now sum the posteriors (in linear domain) for each state along the time axis. What is the meaning of the values you obtain? What about summing over both states and time steps? Compare this number to the length of the observation sequence.

### 5.2 Retraining the emission probability distributions

Write the function `updateMeanAndVar` in `proto2.py` that, given a sequence of feature vectors and the array of state posteriors probabilities, estimates the new mean and variance vectors for each state. Note that this function has an input argument called `varianceFloor` with default value 5.0. The reason is that, the more we tend to maximise the likelihood, the narrower the Gaussian distributions will become (variance that tends to zero), especially if a Gaussian component is associated with very few data points. To prevent this, after the update, you should substitute any value of the variances that falls below `varianceFloor`, with the value of `varianceFloor`. In theory the variance floor should be different for each element in the feature vector. In this exercise, for simplicity we use a single variance floor.

Consider the utterance in `data[10]` containing the digit “four” spoken by a male speaker. Consider also the model in `wordHMMs['4']`, with the model parameters for the same digit trained



on utterances from all the training speakers. First of all estimate the **log likelihood** of the data given the current model ( $\log P(X|\theta)$ ), where the model parameters are, as usual:

- $\pi_i$  a priori probability of state  $s_i$ ,
- $a_{ij}$  transition probabilities from state  $s_i$  to state  $s_j$ ,
- $\mu_{ik}$  Gaussian mean parameter for state  $s_j$  and feature component  $k$
- $\sigma_{ik}^2$  Gaussian variance parameter for state  $s_j$  and feature component  $k$

Keeping  $\pi_j$  and  $a_{ij}$  constant, repeat the following steps until convergence (the increase in log likelihood falls below a threshold):

1. Expectation: compute the alpha, beta and gamma probabilities for the utterance, given the current model parameters (using your functions `forward()`, `backward()` and `statePosteriors()`)
2. Maximization: update the means  $\mu_{jk}$  and variances  $\sigma_{ik}^2$  given the sequence of feature vectors and the gamma probabilities (using `updateMeanAndVar()`)
3. estimate the likelihood of the data given the new model, if the likelihood has increased, go back to 1

You can use a max number of iterations, for example 20, and a threshold of 1.0 on the increase in log likelihood, as stopping criterion.

Repeat the same procedure on the same utterance `data[10]`, but starting with the model for nine: `wordHMMs['9']`. Does it take longer to converge? Does this converge to the same likelihood?

## A Recursion Formulae in Log Domain

### A.1 Forward probability

The recursion formulae for the forward probabilities in *log domain* are given here, where we have used Python convention with array indices going from 0 to len-1:

$$\begin{aligned}\log \alpha_0(j) &= \log \pi_j + \log \phi_j(x_0) \\ \log \alpha_n(j) &= \log \left( \sum_{i=0}^{M-1} \exp \left( \log \alpha_{n-1}(i) + \log a_{ij} \right) \right) + \log \phi_j(x_n)\end{aligned}$$

### A.2 Backward probability

The recursion formulae for the backward probabilities in *log domain* are given here, where we have used Python convention with array indices going from 0 to len-1:

$$\begin{aligned}\log \beta_{N-1}(i) &= 0 \\ \log \beta_n(i) &= \log \left( \sum_{j=0}^{M-1} \exp \left( \log a_{ij} + \log \phi_j(x_{n+1}) + \log \beta_{n+1}(j) \right) \right)\end{aligned}$$

Note also that the initialization of the  $\beta_{N-1}(i)$  is different from the one defined in the course book [1] (Section 8.2.4) but corresponds to the one used in [2].

### A.3 Viterbi approximation

The Viterbi recursion formulas are as follows:

$$\begin{aligned}\log V_0(j) &= \log \pi_j + \log \phi_j(x_0) \\ \log V_n(j) &= \max_{i=0}^{M-1} \left( \log V_{n-1}(i) + \log a_{ij} \right) + \log \phi_j(x_n)\end{aligned}$$

### A.4 State posteriors (gamma)

The gamma probability is the posterior of the state given the whole observation sequence and the model parameters:  $\gamma_n(i) = P(z_n = s_i | X, \theta)$  This can be easily computed from the forward and backward probabilities. In log domain:

$$\log \gamma_n(i) = \log \alpha_n(i) + \log \beta_n(i) - \sum_i \exp(\log \alpha_{N-1}(i))$$

where we have used the Python convention of starting indices from 0.

### A.5 Pair of states posteriors (xi, not used in this exercise)

The xi probabilities are the posteriors of being in a two subsequent states given the whole observation sequence and the model parameters:  $\xi_n(i, j) = P(z_n = s_j, z_{n-1} = s_i | X, \theta)$ . In order to compute them you will need the forward and backward probabilities, but also the emission probabilities  $\phi_j(x_n)$  for each state and feature vector and the state transition probabilities  $a_{ij}$ . In log domain:

$$\log \xi_n(i, j) = \log \alpha_{n-1}(i) + \log a_{ij} + \log \phi_j(x_n) + \log \beta_n(j) - \sum_i \exp(\log \alpha_{N-1}(i))$$

Note that you can only compute this from the second time step ( $n = 1$ ) because you need the alpha probabilities for the previous time step.

## References

- [1] Xuedong Huang, Alex Acero, and Hsiao-Wuen Hon. *Spoken Language Processing: A Guide to Theory, Algorithm and System Development*. Prentice Hall PTR, 2001.
- [2] L. R. Rabiner. “A tutorial on hidden Markov models and selected applications in speech recognition”. In: *Proceedings of the IEEE* 77.2 (Feb. 1989), pp. 257–286.