Ming Hin Cheung

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CSC 32200 Final Project Written Report: Heat Equation

Abstract

The heat equation is of fundamental importance in diverse scientific fields. Heat is a type of energy that exists in all matter. For instance, the temperature in an object changes with time and the position within the object. Moreover, the heat equation can be applied in solving the heat flow that is related to science and engineering. We can use numerical methods to solve the problems which involve the heat equation in science or engineering fields. The accuracy in using a numerical method is more reliable rather than using other methods. In this paper, I will implement the three finite difference methods- explicit method, implicit method, and the Crank-Nicolson method to solve parabolic partial differential equations ,and discuss and analyze each of them. Besides, I will use python with matplotlib to solve the solution and graph the equation. Other than that, the heat model will be discussed in this paper. Some examples of the heat question will be solved too.

History of Heat Equation

The Heat equation was initially developed and solved by Joseph Fourier in 1822 to describe Heat Flow. Heat Equation is a partial differential equation which defines how the distribution of heat evolves with time in a solid medium and how and it flows from places where it's higher to where it’s lower. It is a special case of the diffusion equation.

Importances

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation.In statistics, the heat equation is connected with the study of Brownian motion via the Fokker-Planck equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes. It also describes the distribution of heat in a given region over time. The numerical solution methods for the parabolic partial differential equations are important in such fields as molecular diffusion, heat transfer, nuclear reactor analysis, and fluid flow.

Partial differential equations

Differential equation is an equation involving derivative of an unknown function of one or more variables. There are two topics always discussed in differential equations which are ordinary differential equations that depend on only one variable and partial differential equations which depend on more than one variable. Partial differential equations are important in many branches of science and engineering because there are always more than one independent variables involved in physical problems. The applications of the subject are many, and the types of equations that arise have a great deal of variety. There are very few partial differential equations that can be solved, mostly linear equations and some nonlinear equations. The order of the partial differential equations is the order of the highest derivative occurring in the equations. If a function $u(x,y)$ satisfies the equation, then it is the solution for the partial differential equation.

General Information about the Heat Equation

The equation for the heat equation for a function u, variable x and variable t, is as follow:

$$\frac{u}{t} = c^2 \frac{u}{t^2}$$

u = Temperature at a position x

t = time

where c is a constant thermal conductivity with

Boundary conditions : u(0,t) = 0, u(l,t) = 0,

Initial condition : u(x,0) = f(x).

We can use the heat equation to determine the change over time as heat spreads throughout space. We need to determine the change in the function temperature at a position u over time. It can be predicted that if a hot body is placed in a box of cold water, the temperature of the body will decrease and after the past few times the temperature in the box will equalize. One of the interesting properties of the heat equation is the maximum principle which says that the maximum value of u is either earlier in time than the region of concern or on the edge of the region of concern. This is essentially saying that temperature comes either from some source or from earlier in time because heat permeates but is not created from nothingness. This is a property of the parabolic partial differential equation.

The heat equation can be solved by many methods. In the partial differential equation, there is the easy way to solve the equation which is by using a separable method. The equation will be separate and put them with the same variable. Easy to solve using a separable method since it only needs to know and apply integrating factor and some derivative factor to get the exact solution. This solution also needs to apply a series solution to find un. This method is a direct solution and just needs knowledge about differential equations and calculus since it will be applied in this solution.

For numerical methods, the finite difference methods- explicit method, implicit method and Crank- Nicolson method can be used to solve the heat equation. The Crank-Nicolson method is invented by John Crank (1961) and Phyllis Nicolson (1917-1968) based on numerical approximations for solutions at the point u $u\ (x,t+\frac{k}{2})$ that lies between the rows in the grid. Specifically, the approximation used for $u_t\ (x,t+\frac{k}{2})$ is obtained from the central- difference formula

$$u_t\ (x,t+\frac{k}{2}) = \frac{u(x,t+k) - u(x,t)}{k} + 0\ (k^2)$$

Initial Condition

We have to consider the condition given to solve its solution. The condition can be initial, boundary or both initial and boundary condition given for the model. Different conditions will give a different answer, it's because the condition is important to the solution. For ordinary differential equations it is easier to solve rather than partial differential equations since partial got more condition than ordinary. For example, specifying initial conditions for a temperature requires giving the temperature at each point in the material at the initial time. In the case of the rod this means that we give a function f(x) is defined for $0 \leq x \leq L$ and solution to the heat equation also satisfies

$$u\ (x,0) = f(x), \text{ for } 0 \leq x \leq L$$

Types of Boundary Conditions

In addition to specifying the initial temperature, it will be necessary to specify conditions on the boundary of the material. Example, the temperature may be fixed at one endpoint of the rod as the result of the material being set in a source of heat kept at a constant temperature. There will be a different temperature in the two ends of the rod. Thus if the temperature at x = 0 is T0 and that at x = L is TL, then the temperature u(x,t) satisfies

$$u(0,t) = T0 \text{ and } u(l,t) = TL \text{ for all } t.$$

Boundary conditions of the form in (3.1) specifying If the value of the temperature at the boundary is specified, they are called Dirichlet Conditions. In other situations one or both ends of the rod might be insulated. This means that there is no flow of heat into or out of the rod at these points.

$$\frac{\partial u}{\partial x} = 0$$

at an insulated point.

This type of condition is called Neumann condition. For the experiment in python, I will show the example of the Neumann condition. A rod could satisfy a Dirichlet at one boundary point and a Neumann condition at the other. There is a third condition that occurs, for example, when one end of the rod is poorly insulated from the exterior. According to Newton’s law of cooling, the flow of heat across the insulation is proportional to the difference in the temperature on the two sides of the insulation.

Initial/Boundary Value Problems

Put everything together, we can see that the temperature u(x,t) in an insulated rod with Dirichlet boundary conditions must satisfy the heat equation together with initial and boundary conditions.

Thus, the complete function u(x,t) of the heat equation is

$$u_t(x,t) = ku_{xx}, \text{ for } 0 \leq x \leq L \text{ and } t > 0,$$

$$u(0,t) = T_0, \text{ and } u(L,t) = T_L, \text{ for } t > 0,$$

$$u(x,0) = f(x), \text{ for } 0 \leq x \leq L$$

The function f(x) is the initial temperature distribution. We can replace the Dirichlet boundary with Neumann or a Robin condition.

Solving the Heat Equation:

Most real mathematical problems do not have analytical solutions. However, they do have real answers for each of the problems. To obtain these solutions we can use other methods such as graphical representations or numerical analysis. Numerical analysis is the mathematical method that also considers the accuracy of an approximation. The solution in partial differential equations arising in science and engineering problems are very important to conclude the study of a physical situation. In the case of an analytic solution of the partial differential equation also, numerical values at different intervals of variables are required. Therefore the numerical solution of a problem is very useful and important.

We will use numerical analysis to solve the parabolic partial differential equations. Heat equation, diffusion problems are examples of parabolic equations. There are two methods in solving these problems which are the explicit method and implicit method.

Explicit Method:

Given the heat equation for One-dimensional,

$$\frac{u}{t} = c\frac{^2u}{t^2}$$

We also can write the equation in other way when (xi,tj) as

$$\frac{\partial}{\partial t}u(x_i,t_j) = c\ \frac{\partial^2}{\partial x^2}u(x_i,t_j) \quad \text{or} \quad \frac{\partial u_{i,j}}{\partial t} = c\ \frac{\partial^2 u_{i,j}}{\partial x^2}.$$

Using Forward Difference we can replace the original heat equation which $\frac{u}{t}$ by $\frac{u_{i,j+1} - u_{i,j}}{\Delta t}$,

where $u_{i,j}$ denotes the temperature in the rod at x_i = x0 + iΔx and after the j interval of times t_j = jΔt ,and replace $\frac{^2u}{t^2}$ at x_i by $\frac{u_{i+1,j} - 2u_{i,j}+u_{i-1,j}}{(\Delta x)^2}$, then the heat equation can be written as

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = c\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}\right)$$

The explicit scheme is the least accurate and can be unstable, but it's the easiest to implement and the least numerically intensive.

Implicit Method:

In implicit method second partial derivative $\frac{{}^2u}{t^2}$ can be expressed using Central Difference at x_i

and t_j and for □,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)\text{ij} = (1-\alpha)\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \alpha\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2}$$

With a = 1/2 then it is called Crank-Nicolson method for each i,j = 0,1,2,3,…

Therefore the heat equation will become

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left(\frac{c\Delta t}{\Delta x^2}\right)\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2}\right)$$

Or

$$u_{i,j+1} - u_{i,j} = \frac{r}{2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}).$$

Therefore the heat equation for implicit can be written as

$$u_{i,j+1} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i+1,j+1} + u_{i-1,j+1}).$$

The implicit Method is more accurate than the explicit method, it requires solving a system of numerical equations on each time step. Therefore, it's more numerically intensive than the explicit method.

Heat Equation Results:

We can use the Finite-difference methods to solve the heat equation below using the methods below.

- The Forward Time Centered Space Method(Explicit);
- The Backward Time Centered Space Method(Implicit);
- Crank-Nicolson(Implicit);

Heat equation:

$$\partial_t u = \partial_x^2 u, \quad 0 < x < 1, \quad t > 0$$

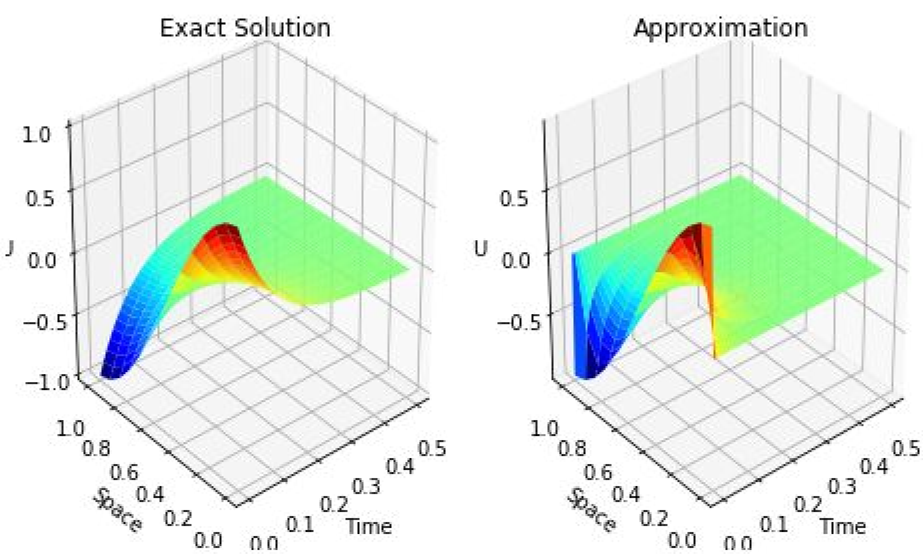
And conditions

$$\partial_x u(0,t) = 0, \quad \partial_x u(1,t) = 0$$

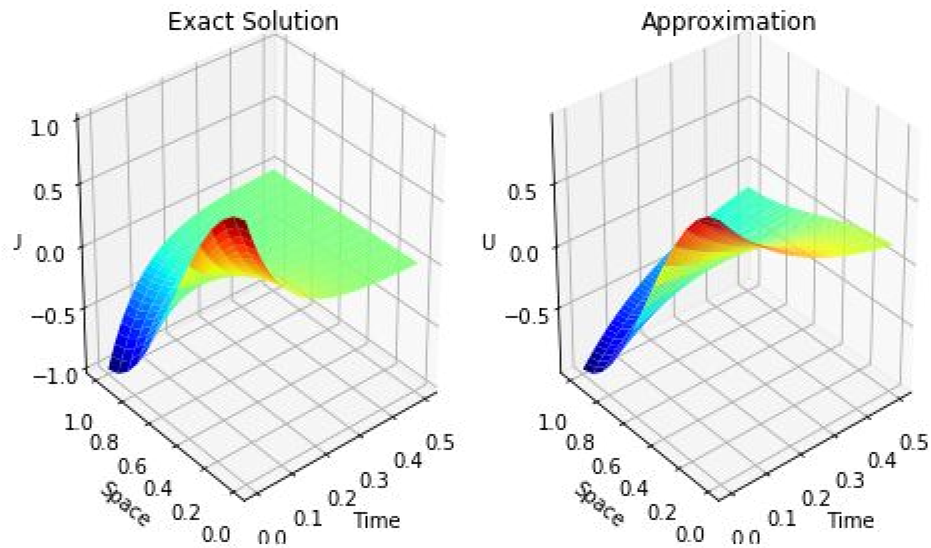
$$u(x,0) = cos(\pi x)$$

Below are the results of the methods using these three Finite-difference methods.

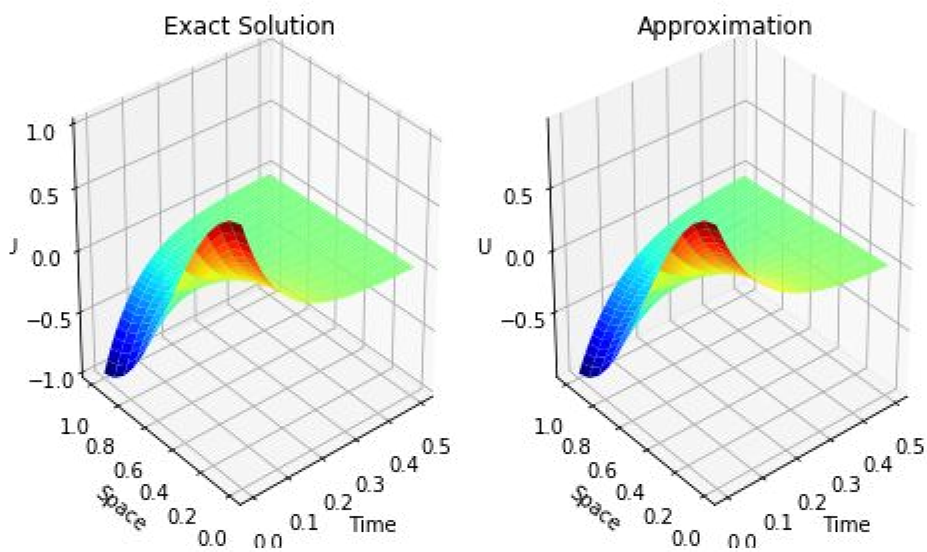
The Forward Time Centered Space Method(Explicit):



The Backward Time Centered Space Method(Implicit):



Crank-Nicolson Method (Implicit):



Absolute errors and relative errors analysis

Error tests results are in the appendix.

The explicit method has the most absolute error, and relative error. When the time elapses, the error doesn't decrease, it will stay the same.

The implicit method has less absolute error, and relative error. Unlike using the explicit method, when the time elapses, the error does decrease.

The Crank-Nicolson method has the least absolute error, and relative error.

Advantages and Disadvantages of the methods used

We can see that, Crank–Nicolson method is the most accurate scheme for small time steps. For larger time steps, the implicit scheme works better since it is less computationally demanding. The explicit scheme is the least accurate and can be unstable but is also the easiest to implement and the least numerically intensive.

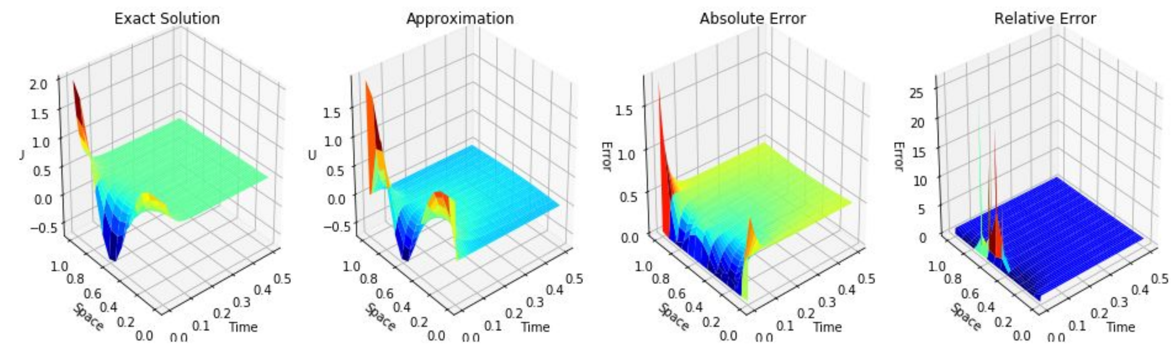
Conclusion

In this paper, the examples given are only for one-dimensional heat equation problems. As we know, in the real world, the problem can be much more complex. Using more complex equations can solve these problems.

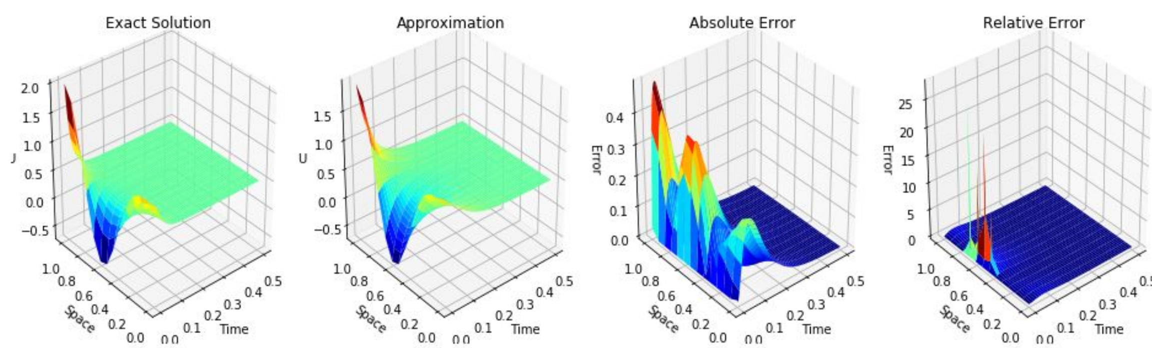
Many equations can be solved in numerical methods. Besides solving heat equation problems other equations can also be faced by computer scientists or engineers in real-world problems. One of the other equations is the wave equation, it's widely used because it is related to the vibration that always occurs in the real world. We can see many applications of different equations occurring in the science and engineering fields. Other than that, using programming languages like python and Matlab, can save time but also gives accurate solutions.

Appendix

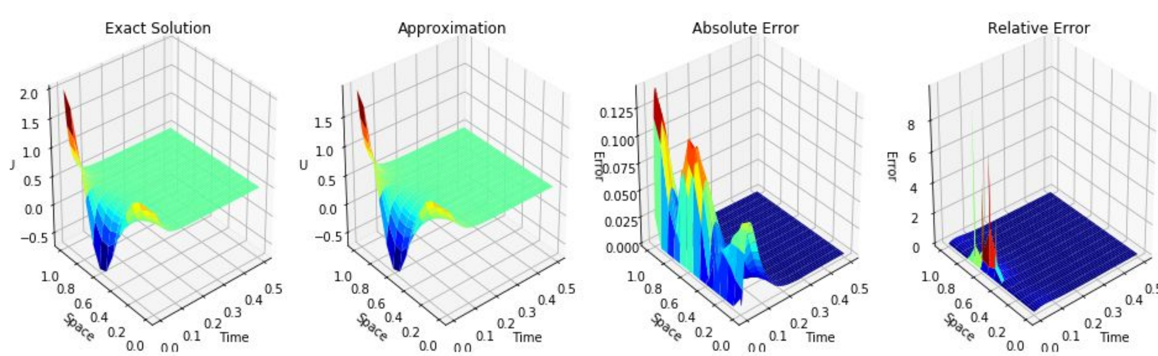
Explicit Method Error:



Implicit Method Error:



Crank-Nicolson Method Error:



Reference

NJOGU, S. K. (n.d.). HEAT EQUATIONS AND THEIR APPLICATIONS I. 2009

L.C. Evans, Partial Differential Equations, Grad. Stud. Math., Amer. Math. Soc., Providence, RI, 1998

Edmund Milne, William, Numerical Solution for Differential Equation, Dover Publications, New York, 2000

Shintani, Hisayoshi, “Explicit and Implicit Difference Formulas Of Higher Order Accuracy For One-Dimensional Heat Equation”,, 1970

Burden, Richard L, J D. Faires, and Annette M. Burden. Numerical Analysis. , 2016.

Cannon, John Rozier (1984), The one–dimensional heat equation, Encyclopedia of Mathematics and its Applications, 1984.