P54. 3

(1) $\lim\limits_{z\to 0} f(z) = \frac{(x-y)(x^2+xy+y^2)}{x^2+y^2} + i\frac{(x+y)(x^2-xy+y^2)}{x^2+y^2}$

$= (x-y)(1+\frac{xy}{x^2+y^2}) + i(x+y)(1-\frac{xy}{x^2+y^2})$ 而 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2} \in (-\frac{1}{2}, \frac{1}{2})$.

又 ∴ $\lim\limits_{z\to 0} f(z) = 0 = f(0)$ ∴ $f(z) \in C(z=0)$.

~~(2). $\lim\limits_{\Delta z\to 0} \frac{f(\Delta z)-f(0)}{\Delta z} = \frac{\Delta x^3-\Delta y^3+i(\Delta x^3+\Delta y^3)}{\Delta x^2+\Delta y^2} = 0$ ∴ $f'(z)|_{z=0}$ 存在.~~

∴ — —

(2). ∴ $f(z) = 0+i0$ $(z=0)$ ∴ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ∴ — —

~~(3). $\lim\limits_{\Delta z\to 0} \frac{f(\Delta z)-f(0)}{\Delta z} = \lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x^3-\Delta y^3+i(\Delta x^3+\Delta y^3)}{\Delta x^2+\Delta y^2} \frac{1}{\Delta x+i\Delta y} = \lim\limits_{\substack{\Delta x\to 0\\ \Delta y=k\Delta x}} \frac{(1-k^3)+i(1+k^3)}{(1+k^2)(1+ki)}$ 不定.~~

∴ 不可导.

P54. 4

(1) $z\neq -1$, $z\neq \pm i$ 时可导, $f'(z) = \frac{-2z^5-7z^2-4z-1}{(z+1)^2(z^2+1)^2}$

(2). $z\neq 0$ 时可导 $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} + i\frac{y^2-x^2+2xy}{(x^2+y^2)^2}$

$= -\frac{1+i}{z^2}$

P54. 5

(1). $u, v \in C^1(D)$ ∴ ~~$f(z)$ 在 D 上可导解析~~. $\frac{\partial u}{\partial x} = y^2$, $\frac{\partial v}{\partial y} = x^2$ ∴ 仅 $x=y=0$ 时.

~~$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = y^2+2xyi$~~ $\lim\limits_{\Delta z\to 0} \frac{f(\Delta z)-f(0)}{\Delta z} = \frac{xy^2+xy i}{x+yi} = 0$ ∴ 在 $z=0$ 可导 不解析.

(3). 设 $z = x+yi$ ∴ $f(z) = (x+yi)y - x = xy - x + y^2 i$

∴ ~~$f(z)$ 在 D 上可导可解析~~

~~$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = y + \dots$~~

$\frac{\partial u}{\partial x} = y-1$, $\frac{\partial v}{\partial y} = 2y$ 当 $y=-1$ 时. $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(-i+\Delta z)-f(-i)}{\Delta z}$

$\frac{\partial u}{\partial y} = x$, $\frac{\partial v}{\partial x} = 0 \Rightarrow x=0$ $= -2$.

不解析

P54.6

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$x = r\cos\theta, \quad y = r\sin\theta \quad \therefore \quad r^2 = x^2+y^2, \quad \tan\theta = \frac{y}{x}.$

$\therefore \quad 2r\,dr = 2x\,dx + 2y\,dy \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$

$\frac{d\theta}{\cos^2\theta} = \frac{-y\,dx + x\,dy}{x^2} \Rightarrow \frac{\partial \theta}{\partial x} = \frac{-\cos^2\theta\, y}{x^2}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{x}.$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$

$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$

$$\therefore \begin{cases} \frac{\partial u}{\partial r} - \frac{\partial u}{\partial \theta}\frac{1}{r} = \frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta}\frac{1}{r} \\ \frac{\partial u}{\partial r} + \frac{\partial u}{\partial \theta}\cdot\frac{1}{r} = -\frac{\partial v}{\partial r} + \frac{\partial v}{\partial \theta}\cdot\frac{1}{r} \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

P34.7.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow 2nxy = 2lxy, \quad 3y^2 m + nx^2 = -(3x^2 + ly^2).$

$$\Rightarrow \begin{cases} n = l = -3 \\ 3m = -l \end{cases} \Rightarrow \begin{cases} m = 1 \\ n = -3 \\ l = -3. \end{cases}$$

P34.8.

(1) $f(z) = u(x,y). \quad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$

$\therefore u = C \quad \therefore f(z) = C$

(2) $f(z) = u(x,y) - iv(x,y) \quad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} = 0.$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} = 0$

$\therefore f(z) = C$

(6). $a\,du + b\,dv = 0$. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$.

$\therefore a\,du + b\,dv = (a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y})dx + (a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x})dy = 0$

$\therefore \begin{cases} a\frac{\partial u}{\partial x} = b\frac{\partial u}{\partial y} \\ a\frac{\partial u}{\partial y} = -b\frac{\partial u}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial y} = 0 \end{cases}$ $\therefore u = C_1$, $v = C_2$. $\therefore f(z) = C$

P54.9

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

记 $au + bv = w$ $\therefore \frac{\partial w}{\partial x} = a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}$, $\frac{\partial w}{\partial y} = a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}$.

记 h 为 w 的--- $\therefore \frac{\partial w}{\partial x} = \frac{\partial h}{\partial y}$, $\frac{\partial w}{\partial y} = -\frac{\partial h}{\partial x}$.

$\therefore \frac{\partial h}{\partial y} = a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}$, $\frac{\partial h}{\partial x} = -a\frac{\partial u}{\partial y} - b\frac{\partial v}{\partial y}$

$\therefore h = \int_{(x_0,y_0)}^{(x,y)} (-a\frac{\partial u}{\partial y} - b\frac{\partial v}{\partial y})dx + (a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x})dy$

$\frac{\partial h}{\partial y} = a\frac{\partial v}{\partial y} - b\frac{\partial u}{\partial y} \Rightarrow h = av - bu + c(x)$.

$\frac{\partial h}{\partial x} = a\frac{\partial v}{\partial x} - b\frac{\partial u}{\partial x} \Rightarrow h = av - bu + c(y)$.

$\therefore h = av - bu + \text{Const}$.

P54. 11.

显然 $(u_y - u_x)$ 和 $(u_x + u_y) \in C'(D)$，则证满足 C-R

$\frac{\partial(u_y - u_x)}{\partial x} = u_{yx} - u_{xx}$, $\frac{\partial(u_x + u_y)}{\partial y} = u_{xy} + u_{yy}$.

又 $\because u_{xx} + u_{yy} = 0$ $\therefore u_{yx} - u_{xx} = u_{xy} + u_{yy}$.

同理有 $\frac{\partial(u_y - u_x)}{\partial y} = -\frac{\partial(u_x + u_y)}{\partial x}$

$\therefore$ 得证.

P34.12

(1). $u_{xx}=2$, $u_{yy}=-2$ ∴ $\nabla^2 u=0$

$\frac{\partial u}{\partial x}=2x+y=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=x-2y=-\frac{\partial v}{\partial x}$

$\Rightarrow v=\frac{1}{2}y^2+2xy+C_1(x)=2xy+\frac{1}{2}x^2+C_2(y)$

$\Rightarrow v=\frac{1}{2}x^2+2xy+\frac{1}{2}y^2+C$. 又∵ $f(i)=-1+i$

∴ $f(z)=x^2+xy-y^2+i(\frac{1}{2}x^2+2xy+\frac{1}{2}y^2+\frac{1}{2})$

(2) $u_{xx}=0$, $u_{yy}=0$ ∴ $\nabla^2 u=0$

$\frac{\partial u}{\partial x}=2y=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=2(x-1)=-\frac{\partial v}{\partial x}$

$\Rightarrow v=y^2+C_1(x)=-(x-1)^2+C_1(y) \Rightarrow v=y^2-(x-1)^2+C$.

又∵ f - - - 写着若题]

(3) $u_{xx}=e^x(x\cos y-y\sin y+2\cos y)$, $u_{yy}=-e^x x\cos y-e^x(2\cos y-y\sin y)$

∴ $u_{xx}+u_{yy}=0 \Rightarrow \nabla^2 u=0$

$\frac{\partial u}{\partial x}=e^x(x\cos y-y\sin y+\cos y)=\frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y}=e^x(-x\sin y-\sin y-y\cos y)=-\frac{\partial v}{\partial x}$

$\Rightarrow v=e^x(x\sin y+\sin y+y\cos y-\sin y)+C_1(x)$

$=e^x(x\sin y-\cancel{\sin y}+\sin y+y\cos y)+C_2(y)$

∴ $v=e^x(x\sin y+y\cos y)+C$ ∴ $f(z)=e^x(x\cos y-y\sin y)+ie^x(x\sin y+y\cos y)$

(4). $v_{xx}=6x$, $v_{yy}=-6x$ ∴ $\nabla^2 v=0$

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=-6xy$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=3y^2-3x^2$

$\Rightarrow u=-3x^2y+C_1(y)=y^3-3x^2y+C_2(x)$

$\Rightarrow u=y^3-3x^2y+C$ $f(i)=2 \Rightarrow u=y^3-3x^2y+2$

∴ $f(z)=(y^3-3x^2y+2)+i(x^3-3xy^2)$.

P55 13

(1) $e^{\frac{2-\pi i}{3}} = e^{\frac{2}{3}} e^{-\frac{\pi}{3}i} = e^{\frac{2}{3}}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = e^{\frac{2}{3}}(\frac{1}{2} - \frac{\sqrt{3}}{2}i)$

(3) $Ln(-3+4i) = \ln 5 + i(\arg z + 2\pi k) = \ln 5 + (\arccos -\frac{3}{5} + 2\pi k)i$, $k \in \mathbb{Z}$

(5) $3^i = e^{iLn3}$, $iLn3 = i(\ln|3| + 2\pi k i) = i\ln 3 - 2\pi k$, $k \in \mathbb{Z}$.

$\therefore 3^i = e^{i\ln 3} / e^{2\pi k} = (\cos\ln 3 + i\sin\ln 3)e^{-2\pi k}$, $k \in \mathbb{Z}$

(7) $\sin(1+i) = \frac{e^{i(1+i)} - e^{-i(1+i)}}{2i} = -\frac{1}{2}\left[(\frac{1}{e}\cos 1 - e\cos 1)i - \sin 1(e + \frac{1}{e})\right]$

$= \frac{1}{2}\sin 1\,(e + \frac{1}{e}) - \frac{i}{2}(\frac{1}{e} - e)\cos 1$

$= \cosh 1 \sin 1 + i\sinh 1 \cos 1$

P55. 14

(1) $Ln z^2 = \ln|z^2| + i(\arg z^2 + 2\pi k) \neq 2Ln z = 2[\ln|z| + i(\arg z + 2\pi k)]$ 不正

(3) e^{Lnz}, $Lnz = \ln|z| + i(\arg z + 2\pi k)$, $e^{Lnz} = |z|(\cos\arg z + i\sin\arg z)$

$z = re^{i\theta}$. $e^{Lnz} = r(\cos\theta + i\sin\theta) = z$ ∴ 正确

(4). $Ln e^z = \ln|e^z| + i(\arg e^z + 2\pi k)$, $z = x + yi$

$\therefore Ln e^z = \ln e^x + i(y + \pi k_1 + 2\pi k_2) = x + iy + i(\pi k_1 + 2\pi k_2) \neq z$ 不对.

P55. 16

(1) 令 $z = x + iy$ ∴ $e^{\bar{z}} = e^{x-iy} = e^x(\cos y - i\sin y)$

∴ $u(x,y) = e^x\cos y$, $v(x,y) = -e^x\sin y$.

$u, v \in C'(\mathbb{C})$. $\frac{\partial u}{\partial x} = e^x\cos y \neq \frac{\partial v}{\partial y} = -e^x\cos y$ ∴ 不满足 C-R.

∴ 不解析.