114.2)

(2) 令 $\frac{x}{y}=u$. $z=f(x,u)$ . $u=u(x,y)=\frac{x}{y}$.

$\therefore \frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}=f_1+f_2\cdot\frac{1}{y}$

$\frac{\partial^2 z}{\partial x^2}=\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x}+(\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x})\frac{1}{y}$

$=f_{11}+f_{12}\frac{1}{y}+f_{21}\frac{1}{y}+f_{22}\frac{1}{y^2}=f_{11}+\frac{2}{y}f_{12}+f_{22}\frac{1}{y^2}$

$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}\frac{1}{y}-\frac{1}{y^2}f_2=-f_{12}\frac{x}{y^2}-f_{22}\frac{x}{y^3}-\frac{1}{y^2}f_2.$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}=-f_2\frac{x}{y^2}$

$\frac{\partial^2 z}{\partial y^2}=-(\frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y}\frac{x}{y^2}-\frac{2x}{y^3}f_2)=\frac{2x}{y^3}f_2+f_{22}\frac{x^2}{y^4}.$

(3) 令 $z=f(u)$, $u=x^2+y^2=u(x,y)$

$\therefore \frac{\partial z}{\partial x}=\frac{df}{du}\cdot\frac{\partial u}{\partial x}=f'2x.$

$\frac{\partial z}{\partial y}=f'2y$

$\frac{\partial^2 z}{\partial x^2}=4x^2f''+2f'$ $\quad\frac{\partial^2 z}{\partial y^2}=4y^2f''+2f'$

$\frac{\partial^2 z}{\partial x\partial y}=4xyf''$

(4) $z=f(u,v,w)$, $u=x+y$, $v=xy$, $w=\frac{x}{y}$.

$\frac{\partial z}{\partial x}=f_1+yf_2+\frac{1}{y}f_3$, $\frac{\partial z}{\partial y}=f_1+xf_2-\frac{x}{y^2}f_3.$

$\frac{\partial^2 z}{\partial x^2}=(f_{11}+yf_{12}+\frac{1}{y}f_{13})+y(f_{21}+yf_{22}+\frac{1}{y}f_{23})+\frac{1}{y}(f_{31}+yf_{32}+\frac{1}{y}f_{33})$

$=f_{11}+y^2f_{22}+\frac{1}{y^2}f_{33}+2yf_{12}+\frac{2}{y}f_{13}+2f_{23}$

$\frac{\partial^2 z}{\partial x\partial y}=f_{11}+xf_{12}-\frac{x}{y^2}f_{13}+f_2+y(f_{21}+xf_{22}-\frac{x}{y^2}f_{23})-\frac{1}{y^2}f_3$

$+\frac{1}{y}(f_{31}+xf_{32}-\frac{x}{y^2}f_{33})$ ? $-\frac{x}{y^2}f_3$ ?

$=f_{11}+xyf_{22}-\frac{x}{y^3}f_{33}+(x+y)f_{12}+(\frac{1}{y}-\frac{x}{y^2})f_{13}+f_2-\frac{1}{y^2}f_3.$

$\frac{\partial^2 z}{\partial y^2}=f_{11}+x^2f_{22}+\frac{x^2}{y^4}f_{33}+\frac{2x}{y^3}f_3+2xf_{12}-\frac{2x}{y^2}f_{13}-\frac{2x^2}{y^2}f_{23}.$

114.28

(2) $\frac{\partial F}{\partial x} = f(x^2y, e^{x^2y}) \cdot 2xy \qquad \frac{\partial F}{\partial y} = f(x^2y, e^{x^2y}) \cdot x^2.$

$\cancel{\frac{\partial F}{\partial x\partial y} = \frac{\partial F}{\partial x}}$ $\frac{\partial F}{\partial x\partial y} = (\frac{df}{dt}\frac{\partial t}{\partial y} + \frac{df}{dt}\frac{\partial e^t}{\partial y})2xy + 2x f(x^2y, e^{x^2y}).$

$= (x^2f' + f' \cdot x^2e^{x^2y})2xy + 2xf(x^2y, e^{x^2y})$

$= 2x^3y(f_1 + f_2e^{x^2y}) + 2xf(x^2y, e^{x^2y}).$

$x^2y = u, \ e^{x^2y} = v.$

114.29

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_1\cos\theta + u_2\sin\theta.$

$\frac{\partial^2 u}{\partial r^2} = (u_{11}\cos\theta + u_{12}\sin\theta)\cos\theta + (u_{21}\cos\theta + u_{22}\sin\theta)\sin\theta$

$= u_{11}\cos^2\theta + u_{12}\sin2\theta + u_{22}\sin^2\theta.$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_1r\sin\theta + u_2r\cos\theta$

$\frac{\partial^2 u}{\partial \theta^2} = -(-u_{11}r\sin\theta + u_{12}r\cos\theta)r\sin\theta - r\cos\theta\, u_1 + (-u_{21}r\sin\theta + u_{22}r\cos\theta)r\cos\theta$ $- r\sin\theta\, u_2.$

$= u_{11}r^2\sin^2\theta - r\cos\theta\, u_1 + u_{22}r^2\cos^2\theta - ru_2\sin\theta$ $- u_{21}r^2\sin2\theta.$

∴右边 $= u_{11} + u_{22} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$

114.30.

$z = f(u), \ u = u(x, y) = e^x\sin y.$

$\frac{\partial z}{\partial x} = \frac{df}{du}\cdot\frac{\partial u}{\partial x} = e^x\sin y\frac{df}{du}. \qquad \frac{\partial z}{\partial y} = \frac{df}{du}e^x\cos y.$

$\frac{\partial^2 z}{\partial x^2} = \sin y(e^x\frac{df}{du} + e^{2x}\frac{d^2f}{du^2}\cdot\sin y) \qquad \frac{\partial^2 z}{\partial y^2} = e^x(\frac{d^2f}{du^2}e^x\cos^2 y - \sin y\frac{df}{du}).$

$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{2x}f'' = e^{2x}z \Rightarrow f'' - f = 0 \quad \therefore \lambda = \pm 1$

$y = C_1e^x + C_2e^{-x} \quad \therefore f(u) = C_1e^u + C_2e^{-u}.$

114.31

(2) $z=f(u,v)$, $u=u(x,y)=xy$, $v=v(x,y)=\frac{x}{y}$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=f_1 y+f_2\frac{1}{y}$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=f_1 x-\frac{x}{y^2}f_2$.

$\frac{\partial^2 z}{\partial x^2}=(f_{11}y+f_{12}\frac{1}{y})y+(f_{21}y+f_{22}\frac{1}{y})\frac{1}{y}$

$\frac{\partial^2 z}{\partial y^2}=(f_{11}x-\frac{x}{y^2}f_{12})x+\frac{2x}{y^3}f_2-\frac{x}{y^2}(f_{21}x-\frac{x}{y^2}f_{22})$.

$\therefore$ 原式 $\Rightarrow 4uvf_{12}-2vf_2=0 \Rightarrow 2uf_{12}-f_2=0$

114.32.

(2) $z=f(u,v)$, $u=x-2y$, $v=x+ay$.

$\therefore \frac{\partial z}{\partial x}=f_1+f_2$, $\frac{\partial z}{\partial y}=-2f_1+af_2$.

$\frac{\partial^2 z}{\partial x^2}=f_{11}+f_{12}+f_{21}+f_{22}$, $\frac{\partial^2 z}{\partial y^2}=-2(f_{11}(-2)+f_{12}\cdot a)+a(-2f_{21}+af_{22})$

$=f_{11}+f_{22}+2f_{12}$ $\qquad=4f_{11}+a^2f_{22}-4af_{12}$.

$\frac{\partial^2 z}{\partial x\partial y}=-2f_{11}+af_{12}-2f_{21}+af_{22}=-2f_{11}+(a-2)f_{21}+af_{22}$.

$\Rightarrow a^2-a-6=0 \Rightarrow a=-2$ 或 $3$ $\quad a=-2$ 时, $u=v$ $\therefore a\neq-2$ ?

114.33.

(2) $\arctan\frac{y}{x}-\frac{1}{2}\ln(x^2+y^2)=0$ 两边 d.

$\therefore \frac{1}{1+\frac{y^2}{x^2}}d\frac{y}{x}-\frac{1}{2}\frac{1}{x^2+y^2}d(x^2+y^2)=0$

$\Rightarrow \frac{dy}{dx}=\cancel{1\ (x\neq y)}\cdot\frac{x+y}{x-y}$ $(x\neq y)$

(3) $x\ln y=y\ln x$ 两边导.

$\therefore \ln y+\frac{y'}{y}x=y'\ln x+\frac{y}{x} \Rightarrow y'=\frac{\ln y-\frac{y}{x}}{\ln x-\frac{x}{y}}$

115.33

(4) $\sin(xy) - \ln\frac{x+1}{y} - 1 = 0$ 两边d.

$\cos xy\,(x\,dy + y\,dx) - \frac{y}{x+1}\cdot\frac{y\,dx - (x+1)\,dy}{y^2} = 0$ $\quad x=0$ 时 $y = e$.

$\Rightarrow$ $e\,dx - dx + \frac{1}{e}dy = 0 \Rightarrow \frac{dy}{dx}\Big|_{x=0} = e(1-e)$.

115.34

(2) $e^z - xyz = 0$ $\quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{yz}{e^z - xy} = \frac{yz}{xy(z-1)} = \frac{z}{x(z-1)}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{xz}{e^z - xy} = \frac{xz}{xyz - xy} = \frac{z}{y(z-1)}$

(4) $x^2 + y^2 + z^2 - 2z = 0$ 两边d.

$\cancel{2x\,dy +}\ 2x\,dx + 2y\,dy + 2z\,dz - 2dz = 0$ $\cancel{\Rightarrow dz = -\frac{x}{z}dx - \frac{y}{z}dy + 1}$

$\Rightarrow dz = -\frac{x}{z-1}dx - \frac{y}{z-1}dy$