P117. 4

(1) $f(z)=\frac{1}{2+z-2}=\frac{1}{2}\frac{1}{1+\frac{1}{2}(z-2)}=\frac{1}{2}\sum_{n=0}^{+\infty}[-\frac{1}{2}(z-2)]^n$

$=\sum_{n=0}^{+\infty}(-1)^n(\frac{1}{2})^{n+1}(z-2)^n$ $|z-2|<2$ $(R=2)$

(3). $f(z)=\frac{z+1-2}{z+1}=1-\frac{2}{z+1}=1-\frac{2}{1+z}=1-2\sum_{n=0}^{+\infty}(-z)^n$

$=\cancel{1-\sum_{n=0}^{+\infty}}$

$=\frac{-2}{2+(z-1)}+1=1-\frac{1}{1+(z-1)/2}=1-\sum_{n=0}^{+\infty}[-\frac{1}{2}(z-1)]^n$

$=1-\sum_{n=0}^{+\infty}(-1)^n(\frac{1}{2})^n(z-1)^n$ $|z-1|<2$ $(R=2)$.

$=\sum_{n=0}^{+\infty}\frac{(-1)^{n-1}}{2^n}(z-1)^n$

(5).

$\sinh z=\frac{e^z-e^{-z}}{2}$, $e^z=e^{\pi i}e^{z-\pi i}=e^{\pi i}\sum_{n=0}^{+\infty}\frac{1}{n!}(z-\pi i)^n$

$=-1\sum_{n=0}^{+\infty}\frac{1}{n!}(z-\pi i)^n$ $R=+\infty$

同理 $e^{-z}=\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n!}(z-\pi i)^n$ $R=+\infty$

$\therefore \sinh z=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n!}(z-\pi i)^n-\frac{1}{n!}(z-\pi i)^n$ $R=+\infty$

$=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}-1}{n!}(z-\pi i)^n$

P117. 5

(1) 令 $(z-1)=t$ $\therefore f(t)=\frac{1}{t^2}=-\frac{d\frac{1}{t}}{dt}$

$\cancel{\frac{1}{t}=\sum_{n=0}^{+\infty}t}$ $f(z)=d(\frac{1}{1-z})/dt$.

$\frac{1}{1-z}=\sum_{n=0}^{+\infty}z^n$ $\therefore f(z)=\sum_{n=0}^{+\infty}(z^n)'=\sum_{n=1}^{+\infty}nz^{n-1}$. $R=1$. $|z|<1$.

(2) $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$, $\sin^2 z=-\frac{1}{4}(e^{2iz}+e^{-2iz})+\frac{1}{2}$

$e^{2iz}=\sum_{n=0}^{+\infty}\frac{1}{n!}z^n\cdot(2i)^n$, $e^{-2iz}=\sum_{n=0}^{+\infty}\frac{1}{n!}z^n(-2i)^n$

$\therefore \sin^2 z=\frac{1}{2}+\sum_{n=0}^{+\infty}(\quad)$ ---

$(\sin^2 z)'=\sin 2z=\sum_{n=0}^{+\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\cdot 2^{2n+1}$ $R=+\infty$.

$\therefore \sin^2 z=\sum_{n=0}^{+\infty}(-1)^n 2^{2n+1}\frac{z^{2n+2}}{(2n+2)(2n+1)!}=\sum_{n=1}^{+\infty}(-1)^{n}2^{2n-1}\frac{z^{2n}}{(2n)!}$ $|z|<+\infty$

(5). $f'(z) = e^{-z^2} = \sum_{n=0}^{+\infty} \frac{1}{n!}(-z^2)^n = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!} z^{2n}$

$\therefore f(z) = \left(\sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!} \frac{1}{2n+1} z^{2n+1}\right) + C$

$f(0) = 0 \Rightarrow f(z) = \sum_{n=0}^{+\infty} (-1)^n \frac{1}{n!} \frac{1}{2n+1} z^{2n+1} \quad |z| < +\infty,\ R = +\infty$

P117.6

令 $z = re^{i\theta}$

$\therefore f(z) = \frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n = \sum_{n=0}^{+\infty} r^n e^{in\theta}$

$= \sum_{n=0}^{+\infty} \left(r^n \cos n\theta + i \cdot r^n \sin n\theta\right)$

$\frac{1}{1-z} = \frac{1}{1-re^{i\theta}} = \frac{1-r\cos\theta + ir\sin\theta}{1-2r\cos\theta + r^2} = \sum_{n=0}^{+\infty} r^n \cos n\theta + i r^n \sin n\theta$

$\because 0 < r < 1 \therefore \sum r^n e^{in\theta}$ 收敛.

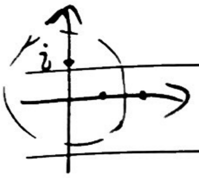
$\sum r^n \cos n\theta < \sum r^n$ 后者收敛 $\therefore \sum r^n \cos n\theta,\ \sum r^n \sin n\theta$ 均收敛.

$\therefore \sum r^n e^{in\theta} = \sum r^n \cos n\theta + i \sum r^n \sin n\theta$

$\therefore \begin{cases} \sum_{n=0}^{+\infty} r^n \cos n\theta = \mathrm{Re}\left(\frac{1}{1-z}\right) \\ \sum_{n=0}^{+\infty} r^n \sin n\theta = \mathrm{Im}\left(\frac{1}{1-z}\right) \end{cases}$ 得证.

P117. 7

(2) $1<|z|<2$ 时.



$$\cancel{f(z)=\frac{1}{z^2+1}\cdot\frac{1}{z-2}=\frac{1}{4i-2}\frac{1}{z+i}-\frac{1}{4i-2}\frac{1}{z-i}+\frac{1}{5}\frac{1}{z-2}}$$

$$\cancel{=-\frac{1}{2i+4}\sum_{n=0}^{+\infty}i^n\left[1+(-1)^n\right]z^n-\frac{1}{10}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n z^n}$$

$f(z)=\frac{1}{4i-2}\frac{1}{z+i}-\frac{1}{4i+2}\frac{1}{z-i}+\frac{1}{5}\frac{1}{z-2}.$

$$\frac{1}{z+i}=\frac{1}{z}\frac{1}{1+\frac{i}{z}}=\frac{1}{z}\sum_{n=0}^{+\infty}(-1)^n\frac{i^n}{z^n},\quad \frac{1}{z-i}=\frac{1}{z}\cdot\sum_{n=0}^{+\infty}\left(\frac{i}{z}\right)^n.$$

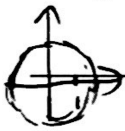
$$\frac{1}{z-2}=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}=-\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n$$

$$\therefore f(z)=\frac{1}{4i-2}\sum_{n=0}^{+\infty}\frac{i^{3n}}{z^{n+1}}-\frac{1}{4i+2}\sum_{n=0}^{+\infty}\frac{i^n}{z^{n+1}}-\frac{1}{10}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n z^n.$$

$|z|>2$ 时.

$$f(z)=\frac{1}{4i-2}\sum_{n=0}^{+\infty}\frac{i^{3n}}{z^{n+1}}-\frac{1}{4i+2}\sum_{n=0}^{+\infty}\frac{i^n}{z^{n+1}}+\frac{1}{5}\sum_{n=0}^{+\infty}\frac{2^n}{z^{n+1}}$$

(4).



$$e^z=\sum_{n=0}^{+\infty}\frac{1}{n!}z^n$$

$$z=t-1\quad\therefore e^{t-1}=\sum_{n=0}^{+\infty}\frac{1}{n!}(t-1)^n.\quad\therefore e^z=e\sum_{n=0}^{+\infty}\frac{1}{n!}(z-1)^n.$$

$$e^{\frac{1}{1-z}}=\sum_{n=0}^{+\infty}e\cdot\frac{1}{n!}\left(\frac{z}{1-z}\right)^n=\sum_{n=0}^{+\infty}\frac{e}{n!}\cdot\left(\frac{-1}{1-\frac{1}{z}}\right)^n$$

$$\frac{1}{1-z}=\frac{-1}{z}\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^{n+1}\quad\therefore e^{\frac{1}{1-z}}=\prod_{n=0}^{+\infty}e^{-\left(\frac{1}{z}\right)^{n+1}}$$

$$e^{-\left(\frac{1}{z}\right)^{n+1}}=\sum_{k=0}^{+\infty}\frac{1}{k!}(-1)^k\left(\frac{1}{z}\right)^{(n+1)k}.$$

$$\therefore e^{\frac{1}{1-z}}=1-\frac{1}{z}-\frac{1}{2z^2}-\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{19}{120z^5}+\cdots$$

$$=1-\frac{1}{z}\left(\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n\right)+\frac{1}{2!}\frac{1}{z}\left(\sum_{n=0}^{+\infty}\left(\frac{1}{z}\right)^n\right)+\cdots$$

P117. 8

(1).

$$f(z)=\frac{1}{a-b}\frac{1}{z-a}+\frac{1}{b-a}\frac{1}{z-b}.$$

$$\therefore \frac{1}{z-a}=\frac{1}{z}\frac{1}{1-\frac{a}{z}}=\frac{1}{z}\sum_{n=0}^{+\infty}\left(\frac{a}{z}\right)^n. \quad \frac{1}{z-b}=-\frac{1}{b}\frac{1}{1-\frac{z}{b}}=-\frac{1}{b}\sum_{n=0}^{+\infty}\left(\frac{z}{b}\right)^n.$$

$$\therefore f(z)=\frac{1}{a-b}\sum_{n=0}^{+\infty}\frac{a^n}{z^{n+1}} \pm \frac{1}{b-a}\sum_{n=0}^{+\infty}\frac{z^n}{b^{n+1}}$$

(3). $$f(z)=\frac{1}{z-a}\cdot\frac{1}{a-b+z-a}=\frac{1}{z-a}\cdot\frac{1}{a-b}\frac{1}{1-\frac{a-z}{a-b}}$$

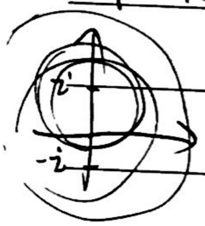
$$=\frac{1}{z-a}\frac{1}{a-b}\cdot\sum_{n=0}^{+\infty}\left(\frac{a-z}{a-b}\right)^n=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(a-b)^{n+1}}(z-a)^{n-1}.$$

$$=-\sum_{n=0}^{+\infty}\frac{1}{(b-a)^{n+1}}(z-a)^{n-1}. \quad 0<|z-a|<|z-b|.$$

$$f(z)=\frac{1}{z-a}\cdot\frac{1}{z-a}\frac{1}{1+\frac{a-b}{z-a}}=\frac{1}{(z-a)^2}\sum_{n=0}^{+\infty}\frac{(b-a)^n}{(z-a)^n}=\sum_{n=0}^{+\infty}\frac{(b-a)^n}{(z-a)^{n+2}} \quad \text{o.w.}$$

P118. 10.(3)

$$f(z)=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{2i}\frac{1}{z+i}.$$



① $0<|z-i|<2.$

$$f(z)=\frac{1}{z-i}\cdot\frac{1}{z+i}$$

$$\frac{1}{z+i}=\frac{1}{2i+z-i}=\frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}}=\frac{1}{2i}\sum_{n=0}^{+\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}.$$

$$\therefore f(z)=\sum_{n=0}^{+\infty}\frac{i^{n-1}}{2^{n+1}}(z-i)^{n-1}.$$

② $2<|z-i|<+\infty$, $f(z)=\sum_{n=0}^{+\infty}\frac{2^n i^{3n}}{(z-i)^{n+2}}$

③ $1<|z|<+\infty.$

$$\frac{1}{z-i}=\frac{1}{z}\frac{1}{1-\frac{i}{z}}=\sum_{n=0}^{+\infty}\frac{i^n}{z^{n+1}}, \quad \frac{1}{z+i}=\frac{1}{z}\frac{1}{1+\frac{i}{z}}=\sum_{n=0}^{+\infty}(-1)^n\frac{i^n}{z^{n+1}}$$

$$\therefore f(z)=\frac{1}{2i}\left(\sum_{n=0}^{+\infty}\frac{i^n}{z^{n+1}}+(-1)^{n+1}\frac{i^n}{z^{n+1}}\right).=\sum_{n=0}^{+\infty}(-1)^n\frac{1}{z^{2n+2}}$$

P118. 11

$\tan z = \int \frac{1}{1+z^2} dz$

$\frac{1}{1+z^2} = \sum_{n=0}^{+\infty} (-1)^n z^{2n}$ 在 $|z|<1$ 收敛.

∴ $\tan z$ 的收敛半径为 $R=1$.

对 $\tan\frac{1}{z-1}$，当 $z_0=1$ 时，$U(z_0)$ 本内不收敛 ∴ 不行