149.60

(1) $\frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(\frac{dy}{dx})}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\sin^2 t\cos t}{-3\cos^2 t\sin t} = -\frac{\sin t}{\cos t} = -\tan t$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{\cos^2 t}\cdot\frac{1}{3a\cos^2 t\sin t} = \frac{1}{3\cos^4 t\sin t\cdot a}$

(2) $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d(\frac{dy}{dx})}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{1+t^2}}{1-\frac{2t}{1+t^2}} = \frac{1}{(1-t)^2}$ $\therefore$ 原式 $= \frac{2}{(1-t)^3}\cdot\frac{1+t^2}{(1-t)^2} = \frac{2+2t^2}{(1-t)^5}$

$\frac{d^2x}{dy^2} = \frac{d}{dy}(\frac{dx}{dy}) = \frac{d(\frac{dx}{dy})}{dt}\frac{dt}{dy} = (2-2t)\cdot(1+t^2)$

(3) $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d(\frac{dy}{dx})}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{tf''(t)}{f''(t)} = t$ $\frac{dt}{dx} = \frac{1}{f''(t)}$

$\therefore$ 原式 $= \frac{1}{f''(t)}$

208.4

$f(1)=f(2)=f(3)=f(4)=0$ $\therefore \exists x_1\in(1,2), x_2\in(2,3), x_3\in(3,4)$

使 $f'(x_i)=0$, $i=1,2,3$

208.5

$f(a)=f(c)$ $\therefore \exists \xi_1\in(a,c)$, 使 $f'(\xi_1)=0$

同理, $\exists \xi_2\in(c,b)$, 使 $f'(\xi_2)=0$ $\therefore \exists \xi_3\in(\xi_1,\xi_2)$, 使 $f''(\xi_3)=0$

208.6

(1) 令 $f(x)=x^5+x-1$

∵ $f(0)=-1$, $f(1)=1>0$ ∴ $\exists$ ~~中~~ $\xi\in(0,1)$ 使 $f(\xi)=0$

又∵ $f(x)$ 在 R 上单调 ∴ $\exists$ 唯一的 ξ.

(2) 令 $g(x)=x^3-3x+c$

$g'(x)=3x^2-3=3(x+1)(x-1)$ 当 $x\in(0,1)$ 时. $g'(x)\neq 0$

若在 $(0,1)$ 上存在 x_1, x_2, 则 $\exists\xi\in(x_1,x_2)$, 使 $g'(\xi)=0$ 矛盾.

∴ 不存在两个.

208.7

令 $F(x)=\frac{a_0}{n+1}x^{n+1}+\frac{a_1}{n}x^n+\cdots+a_nx$

∵ $F(0)=0$, $F(1)=0$ ∴ $\exists\xi\in(0,1)$, 使 $F'(\xi)=a_0\xi^n+a_1\xi^{n-1}+\cdots+a_n=0$

208.8

令 $f(a), f(b)>0$ ∴ $f(\frac{a+b}{2})<0<f(a)$, $f(\frac{a+b}{2})<f(b)$

又∵ $f(x)\in C[a,b]$. ∴ $\exists c\in(a,b)$, 使 $f(c)=\min\limits_{a\le x\le b}f(x)<0$

~~又~~ 令 $F(x)=f(x)\cdot e^{-x}$ ∴ ~~$\exists F(c$~~

∴ $\exists\xi_1\in(a,c)$, 使 $F(\xi_1)=0$, $\exists\xi_2\in(c,b)$, $F(\xi_2)=0$

∴ $\exists\xi_3\in(\xi_1,\xi_2)$, $F'(\xi_3)=e^{-\xi_3}(f'(\xi_3)-f(\xi_3))=0$

又∴ $f'(\xi_3)=f(\xi_3)$

208.10

对 $\forall G>0$, $\exists\delta$, 当 $x\in(a,a+\delta)$ 时. $f(x)>G$. 同理有 $x\in(b-\delta,b)$ 时 $f(x)>G$

∴ $\exists c\in[a+\delta,b-\delta]$, 使 $f(c)=\min\limits_{a<x<b}f(x)$

∴ $f'(c)=0$

20[illegible].11

~~$0 \le f(0) \le 0 \Rightarrow f(0)=0$~~

$\ln\frac{2x+1}{x+\sqrt{1+x^2}} = \ln\frac{(2x+1)(\sqrt{1+x^2}-x)}{1} = \ln\frac{2+\frac{1}{x}}{1+\sqrt{\frac{1}{x^2}+1}}$

$\because \lim\limits_{x\to 0}\ln\frac{2x+1}{x+\sqrt{1+x^2}}=0$, $\lim\limits_{x\to 0}\ln\frac{2x+1}{x+\sqrt{1+x^2}}=0$

$\therefore \exists x_0 \to 0^+$, 使 $0 \le f(x_0) \le 0 \Rightarrow f(x_0)=0$

$\exists \bar{X} \to +\infty$, 使 $0 \le f(\bar{X}) \le 0 \Rightarrow$ 有 $f(\bar{X})=0$

使 $F(x)=f(x)-\ln\frac{2x+1}{x+\sqrt{1+x^2}}$ $\therefore F(x_0)=F(\bar{X})=0$

$\therefore \exists \xi \in (x_0, \bar{X}) \subset (0,+\infty)$ 使 $F'(\xi)=f'(\xi)-\frac{2}{2\xi+1}+\frac{1}{\sqrt{1+\xi^2}}=0$

209.12

(1) 即证 $na^{n-1} < \frac{b^n-a^n}{b-a} < nb^{n-1}$，$f(x)=x^n$

又 $\exists\, \xi \in (a,b)$，使 $f'(\xi)=\frac{b^n-a^n}{b-a}$　$f'(x)=nx^{n-1}$，$f''(x)=n(n-1)x^{n-2}$

$\therefore f'(x)\nearrow$ $\therefore$ $f'(a)<f'(\xi)<f'(b)$ $\therefore$得证.

(2) 即证 $\frac{1}{1+x} < \frac{\ln(1+x)-\ln 1}{x+1-1} < 1$.

$\because \exists\, \xi \in (1,1+x)$，使 $f'(\xi)=\frac{\ln(1+x)-\ln 1}{x+1-1}$ $\therefore$易证.

209.13

$f(x)=\arctan x+\arctan\frac{1}{x}$.　$(x\neq 0)$

$\therefore f'(x)=\frac{1}{1+x^2}+\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}=0$

$\therefore f(x)\equiv C$ 在 $(-\infty,0)$，$(0,+\infty)$ 上分别成立.

$f(1)=\frac{\pi}{2}$，$f(-1)=-\frac{\pi}{2}$ $\therefore f(x)=\begin{cases}\frac{\pi}{2} & x>0\\ -\frac{\pi}{2} & x<0\end{cases}$.

209.14

$\begin{vmatrix} f(a) & f(b)\\ g(a) & g(b)\end{vmatrix}=f(a)g(b)-f(b)g(a)$

$(b-a)\begin{vmatrix} f(a) & f'(\xi)\\ g(a) & g'(\xi)\end{vmatrix}=(b-a)\left(f(a)g'(\xi)-g(a)f'(\xi)\right)$

即证 $f(a)g(b)-f(b)g(a)=(b-a)\left[f(a)g'(\xi)-g(a)f'(\xi)\right]$

~~即证~~ $\cancel{\frac{\frac{g(b)}{f(b)}-\frac{g(a)}{f(a)}}{b-a}=\frac{g'(\xi)}{f(b)}}$

令 $F(x)=f(a)g(x)-g(a)f(x)-(x-a)\frac{f(a)g(b)-f(b)g(a)}{b-a}$

$\because F(a)=0$，$F(b)=0$ $\therefore \exists\, \xi\in(a,b)$.

使 $F'(\xi)=f(a)g'(\xi)-g(a)f'(\xi)-\frac{f(a)g(b)-f(b)g(a)}{b-a}=0$ 成立.

$\therefore$得证

209.15

$f'(x)$ 存在 ∴ $f'(x)$ 存在 ∴ $f(x) \in C(a,b) \cap D(a,b)$. 令 $x < x_0$

∴ $\exists \xi \in (x, x_0)$, 使 $f'(\xi) = \frac{f(x)-f(x_0)}{x-x_0}$ 又 $f''(x)>0$

∴ $f'(x)$ 在 (a,b) 上单增. ∴ 只存在唯一 ξ.

209.16

$\exists \xi_1 \in (a,c)$, $f'(\xi_1) = \frac{f(a)-f(c)}{a-c} > 0$

$\exists \xi_2 \in (c,b)$, $f'(\xi_2) = \frac{f(b)-f(c)}{b-c} < 0$

∴ $\exists \xi_3 \in (\xi_1, \xi_2)$, $f''(\xi_3) = \frac{f'(\xi_1)-f'(\xi_2)}{\xi_1-\xi_2} < 0$.

209.17

$f(x) \in D(a,b)$. $f(x)$ 在 (a,b) 内无界 ∴ $\exists c \in (a,b)$, 使 $f(c) > a$, $\forall a>0$

~~$\exists d \in (a,b)$, $f(d)$~~ 记

∴ $\exists \xi \in (c,d)$, 使 $f'(\xi) = \frac{f(c)-f(d)}{c-d}$ $\cancel{> \frac{f(c)-f(a)}{c}}$

$> \frac{1}{c}(f(c)-f(d))$

令 $f(d) < \frac{a}{2}$.

∴ $f'(\xi) > \frac{a}{2c}$ 无界

反之的反例: $f(x) = \frac{\sin x}{x}$, $x \in (0,1)$. $f'(x)$ 无界 $0 < f(x) < 1$

209.18.

令 $g(x) = e^x$, $F(x) = e^x f(x)$.

∴ $\exists \xi \in (a,b)$, 使 $g'(\xi) = \frac{g(a)-g(b)}{a-b} = \frac{e^a-e^b}{a-b} = e^{\xi}$.

$\exists \eta \in (a,b)$, 使 $F'(\eta) = \frac{F(a)-F(b)}{a-b} = \frac{e^a-e^b}{a-b} = e^{\eta}(f(\eta)+f'(\eta))$

∴ 得证

209.20

证明 $\exists\, \xi \in (a,b)$ 使 $\xi f'(\xi) = \dfrac{f(b)-f(a)}{\ln b - \ln a}$

令 $g(x) = \ln x$ $\therefore$ $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(\xi)}{g'(\xi)} = \xi f'(\xi)$

209.21

$$\frac{f(x)-f(0)}{x^n-0} = \frac{f'(\xi)}{nx^{n-1}} = \frac{f'(\xi_1)-f'(0)}{nx^{n-1}-0} = \frac{f''(\xi_2)}{n(n-1)x^{n-1}} = \frac{f''(\xi_2)-f''(0)}{n(n-1)x^{n-1}-0}$$

$= \cdots = \dfrac{f^{(n)}(\xi_n)}{n!}$ $\quad \xi_n \in (0, x)$ 记 $\theta = \dfrac{\xi_n}{x}$.

$\therefore \dfrac{f(x)}{x^n} = \dfrac{f^{(n)}(\theta x)}{n!} \quad (0<\theta<1)$

209.22

$$f'(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & x>0 \\ 2ax+b & x\le 0 \end{cases}$$

$$\lim_{\Delta x\to 0^+} \frac{f(\Delta x)-f(0)}{\Delta x} = \frac{\frac{\sin\Delta x}{\Delta x}-1}{\Delta x} = \frac{\sin\Delta x - \Delta x}{\Delta x^2} \overset{\text{L.Hospital}}{=} \frac{\cos\Delta x - 1}{2\Delta x} = \frac{-\sin\Delta x}{2} = 0$$

$\therefore \lim\limits_{\Delta x\to 0^-} f'(x) = b$ $\therefore b = 0$.

$$\cancel{f''(x) = \begin{cases} -\dfrac{x^3\sin x + 2x^2\cos x + 2x\sin x}{x^4} & x\ge 0 \\ 2a & x\le 0 \end{cases}}$$

$$\lim_{\Delta x\to 0^+} \frac{f'(\Delta x)}{\Delta x} = \frac{\Delta x\cos\Delta x - \sin\Delta x}{\Delta x^3} = \frac{-\Delta x\sin x}{3\Delta x^2} = -\frac{1}{3}$$

$\therefore 2a = -\frac{1}{3} \Rightarrow a = -\frac{1}{6}$

$$f''(x) = \begin{cases} -\dfrac{x^2\sin x + 2x\cos x - 2\sin x}{x^3} & x>0. \\ -\dfrac{1}{3} & x\le 0 \end{cases}$$

2.10.23

(2) $\lim\limits_{x\to0}\frac{x\cos x-\sin x}{x^3}=\lim\limits_{x\to0}\frac{\cos x-\sin x\cdot x-\cos x}{3x^2}=-\lim\limits_{x\to0}\frac{\sin x}{3x}=-\frac{1}{3}$

(4) $\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}\le\frac{x}{x}=1$. $\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}\ge\lim\limits_{x\to\infty}\frac{x-1}{x+1}=1$ $\therefore\lim\limits_{x\to\infty}\frac{x-\sin x}{x+\sin x}=1$

(6) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{(\pi-2x)^2}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\frac{\cos x}{\sin x}}{4(2x-\pi)}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{4(2x-\pi)\sin x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{\cos x}{4(2x-\pi)}=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\sin x}{8}=\cancel{-\frac{1}{4}}\ -\frac{1}{8}$

(7) $\lim\limits_{x\to0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=e\lim\limits_{x\to0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}=e\lim\limits_{x\to0}\frac{\frac{1}{x}\ln(1+x)-1}{x}=e\lim\limits_{x\to0}\frac{\ln(1+x)-x}{x^2}=e\,\frac{\frac{1}{x+1}-1}{2x}$

$=e\,\frac{-x}{2x(x+1)}=-\frac{e}{2}$

(8) $\lim\limits_{x\to0}\frac{\ln(1+x+x^2)+\ln(1-x+x^2)}{\frac{1}{\cos x}-\cos x}=\frac{\frac{2x+1}{x^2+x+1}+\frac{2x-1}{x^2-x+1}}{\frac{\sin x}{\cos^2 x}+\sin x}=\frac{\frac{2x+1}{x^2+x+1}+\frac{2x-1}{x^2-x+1}}{x(1+\frac{1}{\cos^2 x})}=\frac{2x(2x^2+1)}{2x(x^2+x+1)(x^2-x+1)}$

$=\frac{1}{1}=1$

(10) $\lim\limits_{x\to1^-}\ln x\cdot\ln(1-x)=\frac{\ln x}{\frac{1}{\ln(1-x)}}=\cancel{\frac{\frac{1}{x}}{-\frac{1}{\ln^2(1-x)(1-x)}}}=\frac{\ln(x-1+1)}{\frac{1}{\ln(1-x)}}=\frac{1}{-}(1-x)\ln(1-x)$.

$=-\frac{\ln(1-x)}{\frac{1}{1-x}}=\frac{\frac{1}{1-x}}{\frac{1}{(1-x)^2}}=1-x=0$

$-\lim\limits_{x\to0}\frac{\tan x}{x}=1\ \cancel{=}-1$ $\therefore$

(12) $\lim\limits_{x\to0}\left(\frac{1}{x^2}-\cot^2 x\right)=\lim\limits_{x\to0}\frac{\tan^2 x-x^2}{x^2\tan^2 x}=\lim\limits_{x\to0}\frac{(\tan x+x)(\tan x-x)}{x^2\tan^2 x}=\lim\limits_{x\to0}\frac{2x(\tan x-x)}{x^4}$

$=2\,\frac{\tan x-x}{x^3}$ ~~$=2\frac{\frac{\sin x}{\cos x}-x}{x^3}=2\frac{\frac{\sin x}{x}\cdot\frac{1}{\cos x}-\frac{x}{\sin x}}{x^2}=2\frac{1-\cos x}{x^2\cos x}$~~

$=2\,\frac{\frac{1}{\cos^2 x}-1}{3x^2}=2\,\frac{1-\cos^2 x}{3x^2\cos^2 x}=2\,\frac{(1+\cos x)(1-\cos x)}{3x^2}=\frac{4}{3}\,\frac{1-\cos x^2}{x^2}$

$=\frac{2}{3}$

(14) $\lim\limits_{x\to0^+}(\cot x)^{\sin x}=\lim\limits_{x\to0^+}\left(\frac{1}{\tan x}\right)^{\sin x}=e^{\lim\limits_{x\to0^+}\sin x\ln\frac{1}{\tan x}}$

~~$\lim\limits_{x\to0^+}\sin x\ln\frac{1}{\tan x}=\frac{\ln\frac{1}{\tan x}}{\frac{1}{\sin x}}=-\sin x\ln\tan x=-\sin x(\ln\sin x-\ln\cos x)=-\sin x\ln\sin x+\sin x\ln\cos x$~~

$\lim\limits_{x\to0^+}-\sin x\ln\tan x=-\frac{\ln\tan x}{\frac{1}{\sin x}}=\frac{\frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}}{\frac{\cos x}{\sin^2 x}}=\frac{\tan x}{\cos x}=0$

$\therefore\lim\limits_{x\to0^+}$ 原式 $=1$

210.24

$$\lim_{x\to 0^-} f(x) = e^{-\frac{1}{2}}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}} = e^{\frac{1}{x}\ln\frac{(1+x)^{\frac{1}{x}}}{e}}$$

$$\lim_{x\to 0^+}\frac{1}{x}\ln\frac{(1+x)^{\frac{1}{x}}}{e} = \frac{\frac{1}{x}\ln(1+x)-1}{x} = \frac{\ln(1+x)-x}{x^2} = \frac{\frac{1}{x+1}-1}{2x} = -\frac{x}{2x(x+1)}$$
$$= -\frac{1}{2}$$

$\therefore \lim_{x\to 0^+} f(x) = e^{-\frac{1}{2}} = \lim_{x\to 0^-} f(x)$ ∴ 连续.

210.25

(1) $\cancel{g'(x)=}$ $x \neq a$ 时 $g'(x) = \cancel{\lim_{\Delta x\to 0}\frac{f(x+\Delta x)}{\Delta x}}\ \frac{f'(x)(x-a)-f(x)}{(x-a)^2}$

$$\lim_{x\to a^+} g'(x) = g'_+(a) = \frac{f''(x)(x-a)+f'(x)-f'(x)}{2(x-a)} = \frac{f''(x)}{2}$$

$$\lim_{x\to a^-} g'(x) = g'_-(a) = \frac{f''(x)}{2}.$$

$$\therefore g'(x) = \begin{cases} \frac{f'(x)(x-a)-f(x)}{(x-a)^2} & x \neq a \\ \frac{f''(a)}{2} & x = a \end{cases}$$

$$\lim_{\Delta x\to 0}\frac{g(a+\Delta x)-g(a)}{\Delta x} = \frac{f(a+\Delta x)-\Delta x f'(a)}{(\Delta x)^2} = \frac{f'(a+\Delta x)-f'(a)}{2\Delta x} = \frac{f''(a+\Delta x)}{2} = \frac{f''(a)}{2}.$$

(2) 由(1)知, $\lim_{x\to a^+} g'(x) = \lim_{x\to a^-} g'(x) = g'(a)$

∴ 连续.

210.26

$$\lim_{h\to 0}\frac{f(a+h)-f(a)+f(a-h)-f(a)}{h^2} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h^2} + \lim_{h\to 0}\frac{f(a-h)-f(a)}{h^2}$$

$$= \frac{f'(a+h)}{2h} - \frac{f'(a-h)}{2h} = \frac{f'(a+h)-f'(a-h)}{2h} = \frac{f''(a+h)+f''(a-h)}{2} = f''(a)$$

210.27

$AB = R\alpha$ 过 C 作 $CH \perp AE$ ∴ $CH = R\cdot\sin\alpha$

$AH = R(1-\cos\alpha)$, $\frac{AE}{HE} = \frac{AB}{CH} = \frac{AE}{AE-AH}$

$$\Rightarrow AE = \frac{\alpha R(1-\cos\alpha)}{\alpha-\sin\alpha} \quad \therefore \lim_{\alpha\to 0} AE = R\,\frac{\alpha(1-\cos\alpha)}{\alpha-\sin\alpha}$$
$$= R\,\frac{1-\cos\alpha+\alpha\sin\alpha}{1-\cos\alpha}$$
$$= R\,\frac{\sin\alpha+\sin\alpha+\alpha\cos\alpha}{\sin\alpha} = 3R$$