$X=(x_1\ x_2\ \cdots\ x_n)^T \quad X^TX=1 \quad E_n,\ H=E-2XX^T$

证明: $H^T=H$, $HH^T=E$.

证: $H^T=(E-2XX^T)^T=E^T-2XX^T=E-2XX^T=H$

$2XX^T=E-H^T=E-H$

$\therefore (E-H^T)(E-H)=4XX^T\cdot XX^T=4XX^T=2E-2H$

$=E^2-EH-H^TE+H^TH=E-2H+HH^T=2E-2H$

$\Rightarrow HH^T=E$

$A_3=(a_{ij})_3 \quad \alpha=\begin{pmatrix}0\\1\\0\end{pmatrix},\ \beta=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$\alpha^TA\alpha=(0\ 1\ 0)\left[\begin{pmatrix}a_{11} & & \\ & & \\ & & a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\cancel{(0\ 1\ 0)}\right]$

$=(0\ 1\ 0)\begin{pmatrix}a_{12}\\a_{22}\\a_{23}\end{pmatrix}=a_{22}$

$\beta^TA\beta=(1\ 1\ 0)\left[\begin{pmatrix}a_{11} & & \\ & & a_{13}\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\right]$

$=(1\ 1\ 0)\begin{pmatrix}a_{11}+a_{12}\\a_{12}+a_{22}\\a_{13}+a_{23}\end{pmatrix}=a_{11}+a_{12}+a_{12}+a_{22}=a_{11}+2a_{12}+a_{22}$

当 $A^T=-A$ 时. $\therefore a_{ii}=0$ 且 $\alpha^TA\alpha=0$, $\beta^TA\beta=2a_{12}$

$A=\alpha\beta^T$，$\alpha=(2\ -3\ 1)^T$　$\beta=(-1\ 0\ 5)^T$

$tr(A)=tr(\alpha\beta^T)=tr(\beta^T\alpha)=\beta^T\alpha=(-1\ 0\ 5)\begin{pmatrix}2\\-3\\1\end{pmatrix}$

$=3$

$A^T=A$，$A^2=0$，证 $A=0$

令 $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{12}&a_{22}&a_{23}\\a_{13}&a_{23}&a_{33}\end{pmatrix}$ ∴ $A^2=\begin{pmatrix}a_{11}^2+a_{12}^2+a_{13}^2 & \cdots & -- \\ -- & a_{12}^2+a_{22}^2+a_{23}^2 & -- \\ -- & --\cdots & a_{13}^2+a_{23}^2+a_{33}^2\end{pmatrix}=0$

∴ $\begin{cases}a_{11}^2+a_{12}^2+a_{13}^2=0\\a_{12}^2+a_{22}^2+a_{23}^2=0\\a_{13}^2+a_{23}^2+a_{33}^2=0\end{cases} \Rightarrow \begin{cases}a_{11}=0\\a_{12}=0\\a_{13}=0\\a_{22}=0\\a_{23}=0\\a_{33}=0\end{cases} \Rightarrow A=0$

$A_{m\times n}$，$AA^T=0$　证 $A=0$.

$\Rightarrow tr(AA^T)=0$

$AA^T=B$.

∴ $b_{ii}=\sum_{k=1}^{n}a_{ik}^2=0$ ∴ $a_{ik}=0$ ∴ $A=0$

A_n，$A^2=AA^T$，证 A 为实对称阵.

令 $A^2=B$，$AA^T=C$.

$b_{ij}=\sum_{k=1}^{n}a_{ik}a_{kj} = c_{ij}=\sum_{k=1}^{n}a_{ik}a_{jk}$.

∴ $\sum_{k=1}^{n}a_{ik}(a_{kj}-a_{jk})=0$ 又∵ $\forall a_{ij}\in A_n$ ∴ $a_{kj}=a_{jk}$

∴ A 为实对~~阵~~称阵

证：$tr(AB) = tr(BA)$　($A_{n\times m}$，$B_{m\times n}$)

~~证 $tr(AB)$~~ 记 $AB = C_n$，$BA = D_m$

∴ $tr(C) = \sum_{i=1}^{n} c_{ii}$，$c_{ii} = \sum_{k=1}^{m} a_{im} b_{mi}$

∴ $tr(C) = \sum_{i=1}^{n} \left(\sum_{k=1}^{m} a_{ik} b_{ki} \right)$

同理，$tr(D) = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} b_{ik} \cdot a_{ki} \right)$

将 $tr(C)$ 各个 $a_{ik}b_{ki}$ 制为 $n\times m$ 的矩阵.

$$\begin{pmatrix} a_{11}b_{11} & a_{12}b_{21} & \cdots & a_{1m}b_{m1} \\ a_{21}b_{12} & & & \\ \vdots & & & \\ a_{n1}b_{1n} & - & - & a_{nm}b_{mn} \end{pmatrix} = E_{n\times m}.$$

同理，对 $tr(D)$，制为 $n\times m$ 的矩阵

$$\begin{pmatrix} b_{11}a_{11} & b_{21}a_{12} & - & b_{m1}a_{1m} \\ \vdots & & & \\ \vdots & & & \\ b_{1n}a_{n1} & - & - & b_{mn}a_{nm} \end{pmatrix} = F_{n\times m}.$$

∴ $E = F$ ∴ $tr(C) = tr(D)$

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证 $(AB)^T = B^T A^T$ ($A_{n\times m}$, $B_{m\times n}$)

记 $(AB)^T = C^T_n$, $B^T A^T = D_n$

$\therefore$ $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$ $\therefore$ $C^T_{ij} = \sum_{k=1}^{m} a_{jk} b_{ki}$

$d_{ij} = \cancel{\sum_{k=1}^{m} b_{jk} a_{ki}} \sum_{k=1}^{m} b_{ki} a_{jk} = C^T_{ij}$ $\therefore (AB)^T = B^T A^T$

当 A的秩为1时. 当且仅当 $A = BC^T$, B, C为非零列向量

充分性: 记 $B = (b_1 \ b_2 \ \cdots \ b_n)^T$ $C = (c_1 \ c_2 \ \cdots \ c_m)^T$

$\therefore$ $A_{n\times m} = (a_{ij})_{n\times m}$

$a_{ij} = \sum_{k=1}^{m} b_i c_j$ 当 i 给定, j 取[1, m]时.

$(a_{i1} \ a_{i2} \ \cdots \ a_{im}) = (b_i c_1 \ b_i c_2 \ \cdots \ b_i c_m)$

$\therefore$ A的每一行成比例, 为 $\frac{b_i}{b_{i2}}$ $\therefore$ 化为简化阶梯型后 A只有一个非零行

$\therefore$ A的秩为1

必要性: 记最简的 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & \vdots & 0\ 0\ 0 & 0 \\ 0 & & 0\ 0\ 0 & 0 \\ \vdots & 0 & - - - & 0 \end{pmatrix}$

$\therefore$ A经过逆变换, 可为 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ k_2 a_{11} & k_2 a_{12} & \cdots & k_2 a_{1m} \\ \vdots & & & \\ k_n a_{11} & - & - & k_n a_{1m} \end{pmatrix} = (1 \ k_2 \ \cdots \ k_n)^T (a_{11} \ a_{12} \ \cdots \ a_{1m}$

此时 $B = (1 \ k_2 \ \cdots \ k_n)^T$, $C = (a_{11} \ a_{12} \ \cdots \ a_{1m})^T$

$\therefore$ 得证.

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$tr(ABC) = tr(CAB) = tr(BCA) = \cancel{tr(ABC)}$

$\therefore$ 三种