PPT讲过的问题

两种定义等价

$|A| = \sum_{j_1 \cdots j_n} (-1)^{\tau(j_1 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$

$|A| = \sum_{j_1 \cdots j_n} (-1)^{\tau(j_1 \cdots j_n) + \tau(i_1 \cdots i_n)} a_{i_1 j_1} \cdots a_{i_n j_n}$

取一组 a_{ij}，对定义2交换 i 的位置，排为定义1的顺序

∵ 每次交换 i，$\tau(i_1 \cdots i_n) \pm 1$，同时 j 也交换，$\tau(j_1 \cdots j_n) \pm 1$

∴ $\tau(i_1 \cdots i_n) + \tau(j_1 \cdots j_n)$ ±2 或不变 ∴

∵ 每次交换都变偶数次 ∴ $(-1)^{\tau(i_1 \cdots i_n) + \tau(j_1 \cdots j_n)}$ 在交换后不变

∴ 对每组 a_{ij}，对定义2交换后 有 $(-1)^{\tau(j_1 \cdots j_n)} a_{1j_1} \cdots a_{nj_n}$ = 定义

∴ 定义1，2 等价

2 $|AB| = |A||B|$

构造 $\begin{vmatrix} A & O \\ -E & B \end{vmatrix} = |A| \cdot |B| \cdot (-1)^{2(1+\cdots+n)} = |A| \cdot |B|$

$\begin{vmatrix} A & O \\ -E & B \end{vmatrix} \xlongequal{r_1 + Ar_2} \begin{vmatrix} O & AB \\ -E & B \end{vmatrix} = |AB| (-1)^n (-1)^{1+\cdots+n+n+1+\cdots+n+n}$

$= (-1)^{n + \frac{1}{2} 2n(2n+1)} |AB| = |AB| = |A| \cdot |B|$

$$|A_4| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= (-1)^{2+3+1+2} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix}$$

$$+ (-1)^{2+3+1+3} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{12} & a_{14} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ (-1)^{2+3+1+4} \begin{vmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{44} \end{vmatrix}$$

$$+ (-1)^{2+3+2+3} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}$$

$$+ (-1)^{2+3+2+4} \begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix}$$

$$+ (-1)^{2+3+3+4} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}$$

有 $A_m = \begin{pmatrix} B_n & C_n & \cdots \\ B'_n & C'_n & \cdots \\ \vdots & \vdots & \end{pmatrix}$

$$\xrightarrow[i\in[1,n]]{r_i + r_{i+n}} \begin{pmatrix} B_n + B'_n & C_n + C'_n & \cdots \\ B'_n & C'_n & \cdots \\ \vdots & & \end{pmatrix}$$

$$\xrightarrow[i\in[1,n]]{r_i \cdot k} \begin{pmatrix} kB_n & kC_n & \cdots \\ B'_n & C'_n & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

$$\xrightarrow[i\in[1,n]]{r_i \leftrightarrow r_{i+n}} \begin{pmatrix} B'_n & C'_n & \cdots \\ B_n & C_n & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

对 $\begin{pmatrix} A & 0 \\ E & B \end{pmatrix} \xrightarrow{r_1 - Ar_2} \begin{pmatrix} 0 & -AB \\ E & B \end{pmatrix}$

有 $\begin{pmatrix} A_n & O_n \\ E_n & B_n \end{pmatrix} \xrightarrow[\substack{k\in[1,n] \\ i\in[1,n]}]{r_k - a_{ki}r_{n+i}} \begin{pmatrix} 0 & -A_nB_n \\ E_n & B_n \end{pmatrix}$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots & 0 \\ \vdots \\ a_{n1} & \cdots & & a_{nn} & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots & \vdots \\ \vdots & & & 1 & b_{n1} & \cdots & b_{nn} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \cdots & 0 & -\sum_{k=1}^{n} a_{1k}b_{k1} & \cdots \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \vdots \\ 0 & & & \end{pmatrix}$$

$$\xrightarrow[i\in[1,n]]{r_i - a_{ii}r_{i+n}} \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} & -a_{11}b_{11} & -a_{11}b_{12} & \cdots & -a_{11}b_{1n} \\ a_{21} & 0 & \cancel{a_{23}} & \cdots \\ \end{pmatrix}$$

$$\xrightarrow[i\in[1,n-1]]{r_i - a_{i,i+1}r_{i+n+1}} \begin{pmatrix} 0 & 0 & a_{13} & \cdots & -a_{11}b_{11} - a_{12}b_{21} & \cdots \\ a_{21} & 0 & 0 & a_{24} & \cdots & -a_{22}b_{21} - a_{23}b_{31} & \cdots \\ \end{pmatrix}$$

以此类推. $\xrightarrow[\substack{k\in[1,n] \\ i\in[1,n]}]{r_k - a_{ki}r_{n+i}} \begin{pmatrix} O_n & -A_nB_n \\ E_n & B_n \end{pmatrix}$

对一个 $\begin{pmatrix} A_2 & O_2 \\ E_2 & B_2 \end{pmatrix}$

$$= \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix}$$

$$\xrightarrow[r_2 - a_{22}r_4]{r_1 - a_{11}r_3} \begin{pmatrix} 0 & a_{12} & -a_{11}b_{11} & -a_{11}b_{12} \\ a_{21} & 0 & -a_{22}b_{21} & -a_{22}b_{22} \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix}$$

$$\xrightarrow[r_2 - a_{21}r_3]{r_1 - a_{12}r_4} \begin{pmatrix} 0 & 0 & -a_{11}b_{11} - a_{12}b_{21} & -a_{11}b_{12} - a_{12} \cdot b_{22} \\ 0 & 0 & -a_{22}b_{21} - a_{21}b_{11} & -a_{22}b_{22} - a_{21}b_{12} \\ 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} O_2 & -A_2B_2 \\ E_2 & B_2 \end{pmatrix}$$

①

$A=(a_{ij})_{4\times4}$ ，A有 $2^3\times2^3=64$ 种分块

②

$A=(A_{ij})_{m\times n}$ ∴ $B=(B_{ij})_{n\times m}$ $1\le m\le4, 1\le n\le4$ ~~m+n=4，3≥m≥1，3≥n≥1~~

① $m=1, n=4$ ∵ $A\cdot B$ 与 $B\cdot A$ 存在 ∴ $A_{m\times n}, B_{n\times m}$

∴ $A=\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \end{pmatrix}$ $B=\begin{pmatrix} B_{11} \\ B_{21} \\ B_{31} \\ B_{41} \end{pmatrix}$

$A_{ij}=\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{pmatrix}$ $B_1=\begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \end{pmatrix}$ 等.

∴ 一共 $4\times4=16$ 种.

PPT例题

$$A=\begin{pmatrix} 0 & 1 & 2 & -1 & 4 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & -1 & -2 & 6 & -7 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2-2r_1} \begin{pmatrix} 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 5 & -3 \end{pmatrix}$$

$$\xrightarrow{r_3-r_2} \begin{pmatrix} 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[r_1+r_2]{\frac{1}{5}r_2} \begin{pmatrix} 0 & 1 & 2 & 0 & \frac{17}{5} \\ 0 & 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[c_3-2c_2]{c_5+\frac{3}{5}c_4} \begin{pmatrix} 0 & 1 & 0 & 0 & \frac{17}{5} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{c_5-\frac{17}{5}c_2} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[c_2\leftrightarrow c_4]{c_1\leftrightarrow c_2} \begin{pmatrix} E_2 & O_{2\times3} \\ O & O_{1\times3} \end{pmatrix}$$

$$A=\left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right)=\begin{pmatrix} O_{2\times3} & \begin{matrix}1&2\\2&3\end{matrix} \\ \begin{matrix}1&1&0\\0&1&2\\0&0&3\end{matrix} & O_{3\times2} \end{pmatrix}=\begin{pmatrix} O_{2\times3} & A_{12} \\ A_{21} & O_{3\times2} \end{pmatrix}$$

$$\therefore A^{-1}=\begin{pmatrix} & A_{21}^{-1} \\ A_{12}^{-1} & \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} \\ -3 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$A=\begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$，$|B|\neq0$，$|C|\neq0$

证 A 可逆，并求 A^{-1}

$|A|=|B||C|\neq0$ ∴ A^{-1} 存在.

$$A^{-1}=\begin{pmatrix} B^{-1} & -B^{-1}DC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

$A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 3 & 0 \end{pmatrix}$ 判断 A 是否可逆，求 A 的逆

$|A|=(0+-4+3-6-0)=-7\neq0$

∴ A^{-1} 存在

$$A^{-1}=\frac{1}{|A|}A^{*}=\cancel{-\frac{1}{7}\begin{pmatrix} -6 & 3 & -4 \\ -2 & -1 & -3 \\ 3 & -5 & -2 \end{pmatrix}}\quad -\frac{1}{7}\begin{pmatrix} -6 & 3 & 4 \\ -2 & 1 & -1 \\ 3 & -5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -5 \\ -1 & 4 \end{pmatrix} X = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & -5 \\ -1 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -4 & -5 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -17 & -28 \\ -4 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} X \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 1 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^{-1}$$

$A^{-1}BA = 6A + BA$, $A = \begin{pmatrix} \frac{1}{2} & & \\ & \frac{1}{4} & \\ & & \frac{1}{7} \end{pmatrix}$ 求 B.

$A^{-1}BA = (6E + B)A$

$A^{-1}B = 6E + B$

$\therefore (A^{-1} - E)B = 6E$

$\therefore B = 6(A^{-1} - E)^{-1} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

证: $|A^*| = |A|^{n-1}$ (A_n)

$A \cdot A^* = |A| \cdot E$ 取行列式得 $|A| \cdot |A^*| = |A|^n$

① $|A| \neq 0$ $\therefore$ $|A^*| = |A|^{n-1}$

② $|A| = 0$ $\therefore$ 证 $|A^*| = 0$ 若 $|A^*| \neq 0$

$\cancel{A^{-1} = |A|}$ $A \cdot A^* = 0$ 又 $\because |A^*| \neq 0$ $\therefore (A^*)^{-1} \cdot A^* = E$

$\therefore$ $\cancel{A \cdot A^* (A^*)^{-1} = |A| E = |A| \cdot (A^*)^{-1}}$

$A = A \cdot A^* (A^*)^{-1} = |A| \cdot E (A^*)^{-1} = 0$ $\therefore$ $|A^*| = 0$ $\therefore$ 矛盾 $\therefore$ $|A^*| =$

A_n, B_n, C_n, $ABC=E$.

则 $A^{-1}=BC$ ∴ $BCA=E$

$B^{-1}=CA$ ∴ $CAB=E$

$A^2-A-2E=0$. 试求 A^{-1}, $(A+2E)^{-1}$

$A(A-E)=2E \Rightarrow A\cdot\frac{A-E}{2}=E$ ∴ $A^{-1}=\frac{A-E}{2}$.

$(A+2E)(A-3E)=-4E \Rightarrow (A+2E)^{-1}=\frac{3E-A}{4}$

$A^3=2E$, 证 $(A+2E)^{-1}$ 存在 并求

$|A|\cdot|A^2|=|A|^3=2^n$

∴ $|A|=2^{\frac{n}{3}}\neq 0$ ∴ ~~由A~~ A^{-1} 存在.

∴ $A^3\cdot A^{-1}=2E\cdot A^{-1} \Rightarrow A^2=2A^{-1}$

$A=2(A^{-1})^2=2(A^2)^{-1}$

$A+2E=2(A^2)^{-1}+2E=2[(A^2)^{-1}+E]$

$A+2E=A+A^3=A(E+A^2)$

∴ $(A^{-1})(A+2E)=E+A^2=E+2A^{-1}$

$A^3+8E^3=10E$.

$\Rightarrow (A+2E)(A^2-2A+4E)=10E$.

又∵ $\cancel{A^2=}\ 4E=2A^3$

∴ $\cancel{A^2-2A+2A^3=A(A-2E+2A^2)=A(A+2A^2-A^3)}$

$\cancel{=A^2(\frac{A^3}{2}-A^2+2A)=A^3(\frac{1}{2}A^3+\frac{A^2}{2}-A)}$

∴ $(A+2E)\frac{1}{10}(A^2-2A+4E)=E$.

∴ $(A+2E)^{-1}=\frac{1}{10}(A^2-2A+4E)$

A_n，$A^k=0$，证 $(E-A)^{-1}=E+A+\cdots+A^{k-1}$

$\because$ ~~$(E-A)(E-A)^{-1}=E$.~~

$(E-A)(E+A+\cdots+A^{k-1}) = \cancel{A}E+A+\cdots+A^{k-1}-A-\cdots-A^k$

$= E-A^k = E$

$\therefore (E+A+\cdots+A^{k-1}) = \cancel{EA}(E-A)^{-1}$

$$\begin{cases} 3x_1+5x_2+2x_3+x_4=3 \\ 3x_2+4x_4=4 \\ x_1+x_2+x_3+x_4=\frac{11}{6} \\ x_1-x_2-3x_3+2x_4=\frac{5}{6} \end{cases}$$

$A=\begin{pmatrix} 3 & 5 & 2 & 1 \\ 0 & 3 & 0 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -3 & 2 \end{pmatrix}$ $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ $B=\begin{pmatrix} 3 \\ 4 \\ \frac{11}{6} \\ \frac{5}{6} \end{pmatrix}$

~~$D=|A|=$~~

$D=|A|=\cancel{3}\,67$，$D_1=\begin{vmatrix} 3 & 5 & 2 & 1 \\ 4 & 3 & 0 & 4 \\ \frac{11}{6} & 1 & 1 & 1 \\ \frac{5}{6} & -1 & -3 & 2 \end{vmatrix}=\frac{67}{3}$

$D_2=\begin{vmatrix} 3 & 3 & 2 & 1 \\ 0 & 4 & 0 & 4 \\ 1 & \frac{11}{6} & 1 & 1 \\ 1 & \frac{5}{6} & -3 & 2 \end{vmatrix}=0$ $D_3=\begin{vmatrix} 3 & 5 & 3 & 1 \\ 0 & 3 & 4 & 4 \\ 1 & 1 & \frac{11}{6} & 1 \\ 1 & -1 & \frac{5}{6} & 2 \end{vmatrix}=\frac{67}{2}$

$D_4=\begin{vmatrix} 3 & 5 & 2 & 3 \\ 0 & 3 & 0 & 4 \\ 1 & 1 & 1 & \frac{11}{6} \\ 1 & 1 & -3 & \frac{5}{6} \end{vmatrix}=67$ $\therefore X=\begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$

3)

$A_n, B_n, (A+B)$ 均可逆，证 $A^{-1}+B^{-1}$ 可逆

且 $(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B = B(B+A)^{-1}A$

$A(A^{-1}+B^{-1})B = (E+AB^{-1})B = B+A = A+B$

$\therefore [A(A^{-1}+B^{-1})B]^{-1} = (A+B)^{-1}$

$\Rightarrow B^{-1}(A^{-1}+B^{-1})^{-1}A^{-1} = (A+B)^{-1}$

$\therefore BB^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}A = B(A+B)^{-1}A$

$\therefore (A^{-1}+B^{-1})^{-1} = B(A+B)^{-1}A = B(B+A)^{-1}A$

~~再证 $A(A+B)^{-1}B = B(B+A)^{-1}A$~~

~~A^{-1} 左边 $B^{-1} = (A+B)^{-1}$~~

~~$\therefore (A^{-1}$ 左边 $B^{-1})^{-1} = B$ 左边$^{-1}$ A~~

~~$B^{-1}(B+A)^{-1}A$ B^{-1} 右边 $A^{-1} = (B+A)^{-1} = A^{-1}$ 左边 B^{-1}~~

~~$\therefore B$ 左边$^{-1}$ $A = A$ 右边$^{-1}$ B~~

$[B(B^{-1}+A^{-1})A] = (E+BA^{-1})A = A+B$

$\therefore [B(B^{-1}+A^{-1})A]^{-1} = (A+B)^{-1} = A^{-1}(B^{-1}+A^{-1})^{-1}B^{-1}$

$\therefore A(A+B)^{-1}B = (B^{-1}+A^{-1})^{-1}$

PPT证明.

$A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{n\times s}$, $AB=0$

证 B的列向量为齐次线性方程组 $AX=0$ 的解

$A=(A_{m\times n})$, $B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

$\therefore A\times B=\begin{pmatrix} A_{m\times n}b_1 \\ A_{m\times n}b_2 \\ \vdots \\ A_{m\times n}b_n \end{pmatrix}=0$

$\therefore A_{m\times n}b_1=0, A_{m\times n}b_2=0, \cdots A_{m\times n}b_n=0$

$\therefore X=b_i \ (i=1 \text{ to } n)$

$|A|\neq 0 \Leftrightarrow A^{-1}$ 存在.

证明: $A\cdot A^*=|A|E \Rightarrow A\cdot\frac{A^*}{|A|}=E$

$\therefore \frac{1}{|A|}A^*=A^{-1}$ $\therefore |A|\neq 0$, A^{-1} 存在.

又 A^{-1} 存在, $\therefore A^{-1}$ 唯一 $=\frac{1}{|A|}A^*$ $\therefore |A|\neq 0$

$\therefore |A|\neq 0 \Leftrightarrow \exists A^{-1}$

① A可逆, A^{-1} 也可逆, 且 $(A^{-1})^{-1}=A$

$A\cdot A^{-1}=E$, $(A^{-1})^{-1}=A$

② $\exists A^{-1}$; $\lambda\neq 0 \Rightarrow \lambda A$ 可逆 且 $(\lambda A)^{-1}=\frac{1}{\lambda}A^{-1}$

$\lambda A\cdot\frac{1}{\lambda}A^{-1}=E \Rightarrow (\lambda A)^{-1}=\frac{1}{\lambda}A^{-1}$

A_n, B_n, 可A，$E+AB$ 可逆（B不知）证：$E+BA$ 可逆，并求.

~~$A^{-1}(E+AB)B^{-1} = (A^{-1}+B)^{-1}B^{-1} = A^{-1}B^{-1}+E$~~

~~$A\cdot A^{-1}(E+AB)B^{-1}B = A(A^{-1}B^{-1}+E)B = E+A$~~

$A^{-1}(E+AB)A = (A^{-1}+EB)A = E+BA.$

$A^{-1}(E+AB)A = (A^{-1}(E+AB)^{-1}A)^{-1} = (E+BA)$

$\therefore (E+BA)^{-1} = A^{-1}(E+AB)^{-1}A$

A_5，$|A|=3$，求 $|2A^*|$，$|2A^*-7A^{-1}|$

由 $A\cdot A^* = |A|\cdot E$

$\therefore |A|\cdot|A^*| = |A|^n \Rightarrow |A^*| = A^4 \quad \therefore |2A^*| = 2^5\times 3^4$

$A^{-1} = \frac{1}{|A|}A^*$

$|2A^*-\frac{7}{3}A^*| = \cancel{(\frac{5}{3})^5\cdot 3^4} = \frac{5^5}{3}\ (-\frac{1}{3})^5\times 3^4 = -\frac{1}{3}$

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(1)
$$\left(\begin{array}{cc|cc} 3 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ \hline -2 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \end{array}\right)\left(\begin{array}{cc} 2 & 1 \\ 0 & 2 \\ \hline -1 & 0 \\ 0 & 3 \end{array}\right)$$

$$= \begin{pmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 6 & 7 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} \\ \begin{pmatrix} -4 & 6 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -4 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 7 \\ 3 & 5 \\ -4 & 9 \\ -2 & 6 \end{pmatrix}$$

(2)
$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 3 & -1 \\ 0 & 2 & 1 & 4 \end{array}\right)$$

$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -6 \end{pmatrix}$$

③ A_n, B_n, ~~AB~~ $\exists A^{-1}, B^{-1}$, ~~$(AB)^{-1}$~~ $\Rightarrow \exists (AB)^{-1}$, $(AB)^{-1} = B^{-1}A^{-1}$

$A \cdot B \cdot B^{-1} A^{-1} = A E \cdot A^{-1}$ ~~$A^{-1}(A \cdot B)B^{-1} =$~~

$= E \cdot A \cdot A^{-1} = E \quad \therefore (AB)(B^{-1}A^{-1}) = E$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

④ $\exists A^{-1} \Rightarrow \exists (A^T)^{-1}$, 且 $(A^T)^{-1} = (A^{-1})^T$

$A \cdot A^{-1} = E \quad \therefore (A \cdot A^{-1})^T = E^T = E = (A^{-1})^T A^T$

$\therefore (A^T)^{-1} = (A^{-1})^T$

⑤ $\exists A^{-1}$, ~~$\exists (A^k)^{-1}$~~ $\Rightarrow \exists (A^k)^{-1}$ 且 $(A^k)^{-1} = (A^{-1})^k$.

$A \cdot A^{-1} = E$ 左乘 A^{k-1}，右乘 $(A^{-1})^{k-1}$

$A^k (A^{-1})^k = E \quad \therefore (A^{-1})^k = (A^k)^{-1}$

⑥ $\exists A^{-1} \Rightarrow |A^{-1}| = \frac{1}{|A|}$

$A \cdot A^{-1} = E$ 取行列式 $|A| \cdot |A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$

85.41

(1) $\begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ -3 & -5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 3 & 2 \end{pmatrix} = \begin{pmatrix} A_{2\times2} & O_{2\times3} \\ O_{3\times2} & B_{3\times3} \end{pmatrix}$

$\therefore$ 原式$^{-1} = \begin{pmatrix} A_{2\times2}^{-1} & 0 \\ 0 & B_{3\times3}^{-1} \end{pmatrix} = \begin{pmatrix} 5 & -3 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 & 8 \end{pmatrix}$

(2) $\begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} O_{2\times3} & A_{2\times2} \\ B_{3\times3} & O_{3\times2} \end{pmatrix}$

原式$^{-1} = \begin{pmatrix} 0 & B_{3\times3}^{-1} \\ A_{2\times2}^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ -3 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \end{pmatrix}$