P204.4

(2) $\mathcal{F}(f(t)) = \int_{-\infty}^{+\infty} u(t)e^{-t}\cos t\, e^{-i\omega t}\,dt = \int_{0}^{+\infty} e^{-t}\cos t\, e^{-i\omega t}\,dt$

$= \frac{1}{2}\int_{0}^{+\infty}\left(e^{-t(1+\omega i-i)} + e^{-t(1+\omega i+i)}\right)dt$

$= \frac{1+\omega i}{(1+\omega i)^2+1}$

(4) $\mathcal{F}(f(t)) = \frac{1}{2}\int_{-\infty}^{+\infty}\sin 2t\, e^{-i\omega t}\,dt = \frac{1}{2}\cdot\int_{-\infty}^{+\infty}\frac{e^{2it}-e^{-2it}}{2i}e^{-i\omega t}\,dt$.

又: $\int_{-\infty}^{+\infty} e^{i\omega t}\,d\omega = 2\pi\delta(t) \Rightarrow \int_{-\infty}^{+\infty} e^{-i\omega t}\,dt = 2\pi\delta(\omega)$.

∴ 原式 $= \frac{\pi i}{2}[\delta(\omega+2)-\delta(\omega-2)]$

(5). $\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty}\frac{e^{-i\omega t}}{a^2+t^2}\,dt = 2\int_{-\infty}\frac{\cos\omega t}{a^2+t^2}\,dt \cancel{= \omega\int_{-\infty}^{+\infty}\frac{\cos x}{a^2\omega^2+x^2}dx}$.

$\cancel{= \mathrm{Re}\int_{-\infty}^{+\infty}\frac{2\omega}{a^2+} = \int_{-\infty}^{+\infty}}$

$t=\pm ai$ 为一阶极点 ∴ 原式 $= 2\pi i\,\mathrm{Res}\left[\frac{e^{-i\omega t}}{a^2+t^2}, ai\right]$

$\boxed{\mathcal{F}(f(t)) = \frac{\pi}{a}e^{-a|\omega|}\ (\forall\omega)}$ $\mathcal{F}(f(t)) = \frac{e^{\omega a}}{2ai}\cdot 2\pi i = \frac{\pi}{a}e^{\omega a}\ (\omega<0)$.

$\cancel{\mathcal{F}[f(t)] = \frac{\pi}{a}\exp(\ \ )\ \forall\omega}$

P204.5.

$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt$. $F(-\omega) = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}\,dt$

$\mathcal{F}(f(-t)) = \int_{-\infty}^{+\infty} f(-t)e^{-i\omega t}\,dt = -\int_{-\infty}^{+\infty} f(-t)e^{i\omega t}\,d(-t)$

$= -\int_{+\infty}^{-\infty} f(t)e^{i\omega t}\,dt = \int_{-\infty}^{+\infty} f(t)e^{i\omega t}\,dt$

∴ — — —

P204.7

(1). $\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\delta(\omega+2)-\delta(\omega-2)]e^{i\omega t}\,d\omega$

$= \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty}\delta(\omega+2)e^{i\omega t}\,d\omega - \int_{-\infty}^{+\infty}\delta(\omega-2)e^{i\omega t}\,d\omega\right)$

$= \frac{1}{2\pi}e^{-2it} - e^{2it} = -\frac{i}{\pi}\sin 2t$.

P204. 7.

(3) $\mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos 2\omega\, e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{2i\omega}+e^{-2i\omega}}{2} e^{i\omega t} d\omega.$

$= \frac{1}{4\pi}\int_{-\infty}^{+\infty} e^{i(2+t)\omega} + e^{i(t-2)\omega} d\omega.$

$= \frac{2\pi}{4\pi}[\delta(2+t) + \delta(t-2)] = \frac{1}{2}(\delta(2+t)+\delta(t-2))$

(6). $\mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sin\omega}{\omega} e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\sin\omega\cos\omega t}{\omega} d\omega.$

$= \frac{1}{2\pi} \mathrm{Im}[\int_{-\infty}^{+\infty} \frac{e^{i\omega(1+t)}}{\omega} + \frac{e^{i\omega(1-t)}}{\omega} d\omega]$

$\therefore$ ~~|t|<1时~~, ~~原式 = $\frac{1}{2\pi}$ Im[Res[, 0] 2πi]~~

注意到 $f(t)=\begin{cases}1 & |t|<1\\ 0 & |t|>1\end{cases}$. 有 $\mathscr{F}[f(t)] = \frac{2\sin\omega}{\omega}$.

$\therefore \mathscr{F}^{-1}(F(\omega)) = f(t)$ [写全若是]

(4). $\mathscr{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2+\omega^2} e^{i\omega t} d\omega = \frac{1}{\pi}\int_{0}^{+\infty} \frac{\cos\omega t}{2+\omega^2} d\omega.$

~~$= \frac{1}{2\pi} \times \frac{1}{2} 2\pi i \,\mathrm{Res}[\frac{e^{i\omega t}}{2+\omega^2}$~~

$t>0, = \frac{1}{\pi} \times \frac{1}{2} \times \mathrm{Re}[2\pi i\, \mathrm{Res}[\frac{e^{i\omega t}}{2+\omega^2}, \sqrt{2}i]].$

$= \frac{\sqrt{2}}{4} e^{-\sqrt{2}t}$

~~t<0~~ $t=0$, 原式 $= \frac{\sqrt{2}}{4}$.

$t<0$. 原式 $= \frac{1}{\pi}\int_0^{+\infty} \frac{\cos(-\omega t)}{2+\omega^2} d\omega = \frac{1}{2\pi}\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{i\omega(-t)}}{2+\omega^2} d\omega.$

$= \frac{\sqrt{2}}{4} e^{\sqrt{2}t}.$

$\therefore \mathscr{F}^{-1}[F(\omega)] = \frac{\sqrt{2}}{4} e^{-\sqrt{2}|t|}.\ \forall(t).$

P204.4

(1) $\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} \delta(t-1)(t-2)^2 \sin t \, e^{-i\omega t} dt = \sin 1 \cdot e^{-i\omega}$.

(3) $\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} (\delta(t) + 2\delta'(t) + 3\delta''(t)) e^{-i\omega t} dt = \mathcal{F}[\delta(t)] + 2\mathcal{F}[\delta'(t)] + 3\mathcal{F}[\delta''(t)]$.

$= 1 + 2i\omega + 3(i\omega)^2 = 1 + 2i\omega - 3\omega^2$

(6). $\mathcal{F}[t\sin t \cdot e^{it}] = F(\omega+1)$. $F(\omega) = \mathcal{F}[t\sin t]$.

$\mathcal{F}[t f(t)] = iF'(\omega)$, $\mathcal{F}[\sin t] = i\pi[\delta(\omega+1) - \delta(\omega-1)]$

$\therefore$ 原式 $= \pi[\delta'(\omega) - \delta'(\omega+2)]$

P204.7 答案有误。

(2). $\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} -2\pi\delta''(\omega) e^{i\omega t} d\omega = -\int_{-\infty}^{+\infty} \delta''(\omega) e^{i\omega t} d\omega = t^2$.

(5). $\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{-2i\omega e^{i\omega t}}{(\omega^2+1)(\omega^2+4)} d\omega = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\omega}{(\omega+i)(\omega-i)(\omega+2i)(\omega-2i)} d\omega \cdot e^{i\omega t}$

$= \frac{1}{\pi i} \cdot 2\pi i (\mathrm{Res}[\cdots, i] + \mathrm{Res}[\cdots, 2i])$

$= \begin{cases} \frac{1}{3}(e^{-|t|} - e^{-2|t|}) & t \neq 0 \\ 0 & t = 0 \end{cases} = \begin{cases} \frac{1}{3}(e^{-t} - e^{-2t}) & t \geq 0 \\ \frac{1}{3}(e^{-2t} - e^{-t}) & t < 0 \end{cases}$

P204.8

(1) $\mathcal{F}[g(t)] = \mathcal{F}[t f(2t)] = \frac{1}{2}\mathcal{F}[2t f(2t)] = i(\frac{1}{2}F(\frac{\omega}{2}))'$

$= \frac{i}{4} F'(\frac{\omega}{2})$

(4) $\mathcal{F}[t f'(t)] = \mathcal{F}[(t f(t))' - f(t)] = \mathcal{F}[(t f(t))'] - \mathcal{F}[f(t)]$.

$= i\omega \cdot iF'(\omega) - F(\omega) = -\omega F'(\omega) - F(\omega)$

~~P204.9~~

~~$f*g(t) = \int_{-\infty}^{+\infty} f(s) g(s-t) ds = \int_{t}^{\frac{\pi}{2}+t} f(s) \cdot \sin(\frac{t-s}{2}) ds$~~

~~$= \begin{cases} \int_t^{\frac{\pi}{2}+t} e^{-s} \sin(\frac{t-s}{2}) ds & t \geq 0 \\ \int_0^{\frac{\pi}{2}+t} e^{-s} \sin(\frac{t-s}{2}) ds & -\frac{\pi}{2} \leq t < 0 \\ 0 & t < -\frac{\pi}{2} \end{cases} = \begin{cases} \frac{1}{2}e^{-t} - \frac{1}{2}e^{-\frac{\pi}{2}-t} & t \geq 0 \\ \frac{1}{2}(\cos t - \sin t) - \frac{1}{2}e^{-\frac{\pi}{2}-t} & -\frac{\pi}{2} \leq t < 0 \\ 0 & t < -\frac{\pi}{2} \end{cases}$~~

P204. 10. (1)

$\mathcal{F}[X''(t)] = -\omega^2 F(\omega) = -\omega^2 X(\omega)$, $\mathcal{F}[X(t)] = X(\omega)$.

$\therefore \quad -(\omega^2+1)X(\omega) = 1 \Rightarrow X(\omega) = -\frac{1}{\omega^2+1}$

$\therefore \quad X(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} -\frac{e^{i\omega t}}{\omega^2+1}\,d\omega = -\frac{1}{2}e^{-|t|}.$

P204. 9.

$f * g(t) = \int_{-\infty}^{+\infty} f(s)\,g(t-s)\,ds = \int_{t-\frac{\pi}{2}}^{t} f(s)\cdot\sin(t-s)\,ds.$

$$= \begin{cases} \int_{t-\frac{\pi}{2}}^{t} e^{-s}\sin(t-s)\,ds & t \geq \frac{\pi}{2} \\ \int_{0}^{t} e^{-s}\sin(t-s)\,ds & 0 \leq t < \frac{\pi}{2} \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^{-t}(1+e^{\frac{\pi}{2}}) & t \geq \frac{\pi}{2} \\ \frac{1}{2}(\sin t - \cos t + e^{-t}) & 0 \leq t < \frac{\pi}{2} \\ 0 & t < 0. \end{cases}$$