P210.8

$E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n} E[(X_i-\mu)^2]$, $E[(X_i-\mu)^2] = E(X_i^2) - 2\mu E(X_i) + \mu^2$

$= E[(X_i-\mu)^2] = \sigma^2$ ∴ ——— $= \mu^2+\sigma^2-2\mu^2+\mu^2 = \sigma^2$.

P210.9

$E[C\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2] = C\sum_{i=1}^{n-1} E[(X_{i+1}-X_i)^2]$. $(X_{i+1}-X_i) \sim N(0, 2\sigma^2)$.

$= (n-1)\cdot C \cdot 2\sigma^2 = \sigma^2 \Rightarrow C = \frac{1}{2(n-1)}$

P210.10

~~E(λ)~~ $E(X)=\lambda$, $D(X)=\lambda$. $E(X^2) = \lambda+\lambda^2$.

$E(X^2) = E(M_2) = \mu_2$, $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. ∴ $\lambda^2 = E(X^2) - E(\lambda)$

$= E(M_2) - E(\bar{X})$.

∴ $M_2 - \bar{X}$ 为一个无偏——

P210.13

(1) $E(X) = \int_{-\infty}^{+\infty} x f(x;\theta)dx = \theta + \frac{1}{2} = \bar{X} \Rightarrow \hat{\theta}_1 = \bar{X} - \frac{1}{2}$

$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = 2^n e^{-2n(X-\theta)}$ ∴ $\ln L(\theta) = n\ln 2 - 2n(X-\theta)$

显然. $\hat{\theta}_2 = X_{(1)}$

(2) $E(\hat{\theta}_1) = E(X) - \frac{1}{2}$. $\theta = E(X) - \frac{1}{2}$ ∴ $\hat{\theta}_1$ 无偏. 自右

$E(\hat{\theta}_2) = E(X_{(1)})$, 又 ~~P(X(1)≤x)≤~~ $X_{(1)} \sim E(2n)$ 移动 θ.

∴ $f(X_{(1)};\theta) = \begin{cases} 2ne^{-2n(x-\theta)} & x>\theta \\ 0 & o.w. \end{cases}$ ∴ $E(X_{(1)}) = \frac{1}{2n}+\theta$ ∴ $\hat{\theta}_2 - \frac{1}{2n}$ 为无偏.

(3) $D(\hat{\theta}_1) = D(\bar{X}) = \frac{1}{n}D(X) = \frac{1}{4n}$. $D(\hat{\theta}_2 - \frac{1}{2n}) = D(\hat{\theta}_2) = \frac{1}{4n^2}$

∴ $\hat{\theta}_2 - \frac{1}{2n}$ 更有效.

~~P211.14.~~

$\cancel{D(S_1^2) = \frac{n}{n^2} D[(X_i-\mu)^2] = \frac{1}{n} D[(X_i-\mu)^2]. \quad (X_i-\mu) \sim N(0, \sigma^2).}$

$\cancel{\text{记 } Y_i = X_i - \mu \quad \therefore D(Y_i^2) = E(Y_i^4) - E(Y_i^2)} \quad \therefore \frac{X_i-\mu}{\sigma} \sim N(0,1).$

$\cancel{D[(\frac{X_i-\mu}{\sigma})^2] = 1 \Rightarrow D[(X_i-\mu)^2] = \sigma^4}$

P211.14.

$S_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \Rightarrow \frac{nS_1^2}{\sigma^2} = \sum_{i=1}^{n}(\frac{X_i-\mu}{\sigma})^2 \sim \chi^2(n).$

$\therefore D(\frac{nS_1^2}{\sigma^2}) = 2n \Rightarrow D(S_1^2) = \frac{2\sigma^4}{n}.$

$S_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 \Rightarrow \frac{(n-1)S_2^2}{\sigma^2} = \sum_{i=1}^{n}(\frac{X_i-\bar{X}}{\sigma})^2 \sim \chi^2(n-1).$

$\therefore D(\frac{(n-1)S_2^2}{\sigma^2}) = 2n-2 \Rightarrow D(S_2^2) = \frac{2\sigma^4}{n-1} > D(S_1^2).$

$\therefore S_1^2$ 更有效.

P211.15.

$E(\hat{\theta}_1) = \theta, \ E(\hat{\theta}_2) = \theta. \quad \therefore a+b=1.$

$D(\hat{\theta}_1) = D(\hat{\theta}_2). \quad D(a\hat{\theta}_1 + b\hat{\theta}_2) = a^2 D(\hat{\theta}_1) + b^2 D(\hat{\theta}_2) = D(\hat{\theta}_1)(a^2+b^2).$

$\therefore a = b = \frac{1}{2}$ 时. - - -

P211. 17

$\bar{X}=4.364.\quad S^2=2.93\times10^{-3},\ S=0.054$

(1) $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1)$ $\therefore$ $P(-N_{\frac{0.05}{2}}<\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}<N_{\frac{0.05}{2}})=1-0.05$

$\therefore\quad \mu\in(4.269,4.458)$

$P(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}<N_{0.05})=0.95 \Rightarrow \mu\in(4.284,+\infty) \Rightarrow \underline{\mu}=4.284$

(2). $\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t(n-1)$. $t(4)$. $P(-t_{\frac{0.05}{2}}<\frac{\bar{X}-\mu}{S/\sqrt{n}}<t_{\frac{0.05}{2}})=1-0.05 \rightarrow 2.776$

$\Rightarrow\ \mu\in(4.297,4.431)$

$P(-t_{0.05}(4)<\frac{\bar{X}-\mu}{\sigma/\sqrt{n}})=0.95\Rightarrow \mu\in(-\infty,4.415).\ \bar{\mu}=4.415.$

P211. 18.

$\bar{X}=14.72.\quad S^2=1.91.\ S=1.38$

(1) $\frac{\bar{X}-\mu}{S/\sqrt{n}}\sim t(n-1).\quad n=30.\ \alpha=0.1.$

$P(-t_{0.05}(29)<\frac{\bar{X}-\mu}{S/\sqrt{n}}<t_{0.05}(29))=0.9\Rightarrow \mu\in(14.29,15.15)$

$\cancel{\bar{X}-\mu}$

$\frac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)\Rightarrow P(\chi^2(29)_{0.95}<\frac{(n-1)S^2}{\sigma^2}<\chi^2_{0.05}(29))=0.9$

$\Rightarrow\ \sigma^2\in(1.3,\ 3.13)$

(2). $P(\frac{\bar{X}-\mu}{S/\sqrt{n}}<t_{0.1}(29))=0.9\Rightarrow \mu\in(14.39,+\infty)\Rightarrow \underline{\mu}=14.39$

$P(-t_{0.1}(29)<\frac{\bar{X}-\mu}{S/\sqrt{n}})=0.9\Rightarrow \mu\in(-\infty,15.05)\Rightarrow\bar{\mu}=15.05$

(3) $P(\frac{(n-1)S^2}{\sigma^2}<\chi^2_{0.1}(29))=0.9\Rightarrow \sigma^2\in(1.42,+\infty).\Rightarrow \underline{\sigma^2}=1.42$

$P(\chi^2_{0.9}(29)<\frac{(n-1)S^2}{\sigma^2})=0.9\Rightarrow\sigma^2\in(0,2.8)\Rightarrow\bar{\sigma}^2=2.8.$

P211. 19.

(1) 记 $Y = \ln X \Rightarrow Y \sim N(\mu, 1)$.

$$E(X) = E(e^Y) = \int_{-\infty}^{+\infty} e^y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2}} dy = e^{\mu+\frac{1}{2}}.$$

(2). ~~$\bar{X} = \ln X$~~ $\bar{Y} = 0$

~~$\frac{\bar{X}-\mu}{S/\sqrt{n}}$~~ $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$, $n=4$. $\sigma = 1$.

$$P\left(-N_{0.025} < \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} < N_{0.025}\right) = 0.95 \Rightarrow \mu \in (-0.98, 0.98)$$

(3). $E(X) = e^{\mu+\frac{1}{2}}$ $\quad P(a < e^{\mu+\frac{1}{2}} < b) = 0.95$

$$\Rightarrow P\left(\ln a - \tfrac{1}{2} < \mu < \ln b - \tfrac{1}{2}\right) = 0.95$$

$\Rightarrow \ln a - \frac{1}{2} = -0.98 \Rightarrow a = e^{-0.48}$, $b = e^{1.48}$ $\therefore \mu \in (e^{-0.48}, e^{1.48})$.

P242. 20.

$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$. $P\left(-N_{\frac{\alpha}{2}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < N_{\frac{\alpha}{2}}\right) = 1-\alpha$

$$\Rightarrow \bar{X} - N_{\frac{\alpha}{2}} \cdot \sigma/\sqrt{n} < \mu < \bar{X} + N_{\frac{\alpha}{2}} \sigma/\sqrt{n}.$$

$\Rightarrow$ $L = \frac{2\sigma}{\sqrt{n}} \cdot N_{\frac{\alpha}{2}}$ $\quad \le l \Rightarrow n \geqslant \frac{4\sigma^2 N_{\frac{\alpha}{2}}^2}{l^2}$ $\quad$ $N_{\frac{\alpha}{2}}$ 为 $N(0,1)$ 的上 $\frac{\alpha}{2}$ 分位数

补 1. $f(x;\theta) = \begin{cases} \theta & 0 < x < 1 \\ 1-\theta & 1 \le x < 2 \\ 0 & o.w. \end{cases}$

(1) $E(X) = \int_{-\infty}^{+\infty} x f(x;\theta) dx = \int_0^1 x \cdot \theta dx + \int_1^2 x \cdot (1-\theta) dx = \frac{3}{2} - \theta = \bar{X}$

$\therefore \hat{\theta}_{矩估} = \frac{3}{2} - \bar{X}$, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

(2) $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^N \cdot (1-\theta)^{n-N} \Rightarrow \ln L(\theta) = N\ln\theta + (n-N)\ln(1-\theta)$

$$\frac{d\ln L(\theta)}{d\theta} = \frac{N}{\theta} - \frac{n-N}{1-\theta} = 0 \Rightarrow \hat{\theta}_{MLE} = \frac{N}{n}$$

补2.

(1) $L(\sigma)=\prod_{i=1}^{n} f(x_i;\sigma)=(\frac{1}{2\sigma})^n \prod_{i=1}^{n} e^{-\frac{|x_i|}{\sigma}}$

$\ln L(\sigma)= -n(\ln 2+\ln\sigma)-\sum_{i=1}^{n}\frac{|x_i|}{\sigma} = -n\ln\sigma-\frac{1}{\sigma}\sum_{i=1}^{n}|x_i|-n\ln 2.$

$\frac{d\ln L(\sigma)}{d\sigma} = -\frac{n}{\sigma}+\frac{1}{\sigma^2}\sum_{i=1}^{n}|x_i| =0 \Rightarrow \hat{\sigma}_{MLE}=\frac{1}{n}\sum_{i=1}^{n}|x_i|.$

(2). $D(\hat{\sigma}_{MLE})= \frac{1}{n^2}\sum_{i=1}^{n}D(|X_i|)=\frac{1}{n}D(|X|)$

$E(|X|)=\int_{-\infty}^{+\infty}|x|\frac{1}{2\sigma}e^{-\frac{|x|}{\sigma}}dx = \int_0^{+\infty}\frac{x}{\sigma}e^{-\frac{x}{\sigma}}dx \overset{\frac{x}{\sigma}=t}{=} \sigma\int_0^{+\infty}te^{-t}dt$ $\sim E(1).$

$=\sigma$, $|X|\sim E(\frac{1}{\sigma})$. ∴ $D(|X|)=\sigma^2.$

∴ $D(\hat{\sigma}_{MLE})=\frac{\sigma^2}{n}$

(3) $E(\hat{\sigma}_{MLE}) = E(|X|)=\sigma$ 又 $\lim_{n\to\infty}D(\hat{\sigma}_{MLE})=0$ ∴ 符合一致性.

补3. $X\sim N(\mu,\sigma^2)$.

(1) $Y_i = X_i-\frac{1}{n}\sum_{j=1}^{n}X_j = \frac{n-1}{n}X_i-\frac{1}{n}\sum_{j\neq i}X_j$

$Y_1+Y_n = \frac{n-1}{n}X_1-\frac{1}{n}\sum_{j\neq 1}X_j+\frac{n-1}{n}X_n-\frac{1}{n}\sum_{j\neq n}X_j = \frac{n-2}{n}X_1+\frac{n-2}{n}X_n-\frac{2}{n}\sum_{j\neq 1,n}X_j$

∴ $(Y_1+Y_n)\sim N(0,\frac{2n-4}{n}\sigma^2)$.

∴ $\frac{1}{\sigma}\sqrt{\frac{n}{(2n-4)}}(Y_1+Y_n)\sim N(0,1)$, $\frac{n}{(2n-4)\sigma^2}(Y_1+Y_n)^2\sim\chi^2(1)$

∴ $E(\frac{n}{2n-4}(Y_1+Y_n)^2)=\sigma^2 \Rightarrow C=\frac{n}{2n-4}$

(2) $P((Y_1+Y_n)\le 0) = \Phi(\frac{0-0}{\sqrt{\frac{2n-4}{n}\sigma^2}}) = \Phi(\cancel{\sqrt{\frac{n}{2n-4}}\sigma}0)=\frac{1}{2}$.

补4.

$E(\hat{\sigma}_1^2) = E(\frac{1}{4}\sum_{k=1}^{4} X_k^2)$ 矩估计无偏

$E(\hat{\sigma}_2^2) = \frac{1}{3}E(\sum_{k=1}^{4}(X_k-\overline{X})^2)$ ， $\frac{\sum_{k=1}^{4}(X_k-\overline{X})^2}{\sigma^2} \sim \chi^2(3)$.

$\therefore E(\sum_{k=1}^{4}(X_k-\overline{X})^2) = 3\sigma^2$ $\therefore E(\hat{\sigma}_2^2) = \sigma^2$ 无偏.

$D(\hat{\sigma}_1^2) = \frac{1}{16}D(\sum_{k=1}^{4}X_k^2) = \frac{1}{4}D(X^2)$

$\frac{1}{\sigma}X \sim N(0,1)$， $\frac{1}{\sigma^2}X^2 \sim \chi^2(1)$ $D(\frac{X^2}{\sigma^2}) = 2 \Rightarrow D(X^2) = 2\sigma^4$.

$\therefore D(\hat{\sigma}_1^2) = \frac{1}{2}\sigma^4$

$D(\hat{\sigma}_2^2)$， $\hat{\sigma}_2^2 = \frac{1}{3}\sum_{k=1}^{4}(X_k-\overline{X})^2 \Rightarrow 3\frac{\hat{\sigma}_2^2}{\sigma^2} = \frac{\sum_{k=1}^{4}(X_k-\overline{X})^2}{\sigma^2} \sim \chi^2(3)$.

$\therefore \frac{9}{\sigma^4}D(\hat{\sigma}_2^2) = 6 \Rightarrow D(\hat{\sigma}_2^2) = \frac{2}{3}\sigma^4 > D(\hat{\sigma}_1^2)$

$\therefore$ $\hat{\sigma}_1^2$ 更有效.

补5.

$X \sim N(\mu, \sigma_0^2)$. $\frac{(\overline{X}-\mu)}{\sigma_0/\sqrt{n}} \sim N(0,1)$

$\therefore P(-u_{0.05} < \frac{\overline{X}-\mu}{\sigma_0/\sqrt{n}} < u_{0.05}) = 0.9 \Rightarrow \overline{X} - u_{0.05}\sigma_0/\sqrt{n} < \mu < \overline{X} + u_{0.05}\cdot\sigma_0/\sqrt{n}$

$\Rightarrow L = \frac{2\sigma_0}{\sqrt{n}}\cdot u_{0.05} \le \frac{\sigma_0}{2} \Rightarrow n \ge 16u_{0.05}^2 = 43.29$

$\therefore n \ge 43$ 时.