294.8

(1) $\lim\limits_{n\to\infty}\int_0^{\frac{1}{2}}\frac{x^n}{1+x}dx = \frac{\xi^n}{1+\xi}\left(\frac{1}{2}-0\right) = \frac{1}{2}\frac{\xi^n}{1+\xi} = 0$

(2) $\lim\limits_{n\to\infty}\int_n^{n+1}x^2e^{-x}dx = \lim\limits_{n\to\infty}\int_n^{n+1}e^{-x}dx\cdot\xi^2 \quad \xi\in(n,n+1)$

$e^{-(n+1)}n^2 \le e^{-(n+1)}\xi^2 \le \lim\limits_{n\to\infty}\int_n^{n+1}e^{-x}dx\,\xi^2 \le e^{-n}\xi^2 \le e^{-n}(n+1)^2$

又∵ $\lim\limits_{n\to\infty} n^2e^{-(n+1)} = \frac{n^2}{e^{n+1}} = \lim\limits_{n\to\infty}\frac{(n+1)^2}{e^n} = 0$ ∴ $\lim\limits_{n\to\infty}\int_n^{n+1}x^2e^{-x}dx = 0$

294.9.

证明: $\exists\xi\in(\frac{2}{3},1)$, $f(\xi)(1-\frac{2}{3}) = \int_{\frac{2}{3}}^{1}f(x)dx = \frac{1}{3}f(\xi) = \frac{1}{3}f(0)$

∴ $f(0)=f(\xi)$ ∴ $\exists\eta\in(0,\xi)$, $f'(\eta)=0$

294.10

令 $F(x)=e^{1-x^2}f(x)$ ∴ $F(1)=f(1)$, 又∵ $\int_0^{\frac{1}{3}}e^{1-x^2}f(x) = F(\xi)\frac{1}{3}$, $\xi\in(0,\frac{1}{3})$

∴ $F(\xi)=F(1)$ ∴ $\exists\eta\in(\xi,1)$, $F'(\eta)=0$ ∴ 得证.

294.11

(2) $y=-\int_0^x\sqrt{1+t^2}\,dt$ ∴ $y'=-\sqrt{1+x^2}$

(4). $y=\int_{\cos x}^{\sin x}e^{-t^2}dt$ ∴ $y'=e^{-\sin^2x}\cos x+\sin x\,e^{-\cos^2x}$

(6) $y=\int_0^{2x}x(t-1)^2dt = x\int_0^{2x}(t-1)^2dt$ ∴ $y'=\int_0^{2x}(t-1)^2dt+2x(2x-1)^2$

(8) $x'=\ln t$, $y'=t\ln t$ ∴ $\frac{dy}{dx}=t$

299.12

(1) $\lim\limits_{x\to0}\frac{\int_{\cos x}^{1}e^{-t^2}dt}{x^2} = \frac{\sin x\,e^{-\cos^2x}}{2x} = \frac{e^{-1}}{2} = \frac{1}{2e}$

(2) $\lim\limits_{x\to0}\frac{\int_0^x\sin t^2dt}{\sin x^3} = \frac{\sin x^2}{3x^2} = \frac{1}{3}$

(3) $\lim\limits_{x\to0^+}\frac{\int_0^{x^2}t^{\frac{3}{2}}dt}{\int_0^x t(t-\sin t)dt} = \frac{2x\,x^3}{x(x-\sin x)} = 2\frac{x^3}{x-\sin x} = 2\frac{x^3}{x-(x-\frac{x^3}{3!}+o(x^3))} = 12$

(4) $\lim\limits_{x\to0}\frac{\int_0^x\ln(1+t)dt}{\ln\frac{\sin x}{x}} = \frac{\ln(1+x)}{\frac{x}{\sin x}\cdot\frac{x\cos x-\sin x}{x^2}} = \frac{x^3\sin x}{x(x\cos x-\sin x)} = \frac{x^3}{x\cos x-\sin x}$

$= \frac{x^3}{x(1-\frac{x^2}{2}+o(x^3))-x+\frac{x^3}{3!}+o(x^3)} = -3$

299.13

$\Phi x = \int_0^1 f(t)dt + \int_1^x f(t)dt$

$= \int_0^1 \cancel{t}^2 dt + \int_1^x f(t)dt = \int_1^x f(t)dt + \frac{1}{3}$

2≥x≥1时 $\Phi x = \frac{1}{2}x^2 - 2x + \frac{1}{3} + \frac{3}{2}$

$0 \le x < 1$时. $\Phi x = \frac{1}{3}x^2$

299.14

(1) $F'(x) = f(x) + \frac{1}{f(x)}$ 又 $\because f(x) > 0$ $\therefore F'(x) \ge 2$. $f(x)=1$时取等.

(2) $F'(x) > 0$ $x \in [a,b]$, $\therefore F(x)$ 在 $[a,b]$上严格单增.

$F(a) = 0 - \int_a^b \frac{dt}{f(t)} < 0$

$F(b) = \int_a^b f(t)dt + 0 > 0$ $\therefore \exists \xi \in (a,b)$, $F(\xi) = 0$. 且ξ唯一.

299.15.

令 $F(\xi) = \int_a^\xi f(x)dx - \int_\xi^b f(x)dx$ $\xi \in [a,b]$.

$F'(\xi) = f(\xi) + f(\xi) = 2f(\xi) > 0$ $F(a) = -\int_a^b f(x)dx < 0$, $F(b) = \int_a^b f(x)dx$

$\therefore \exists \eta \in [a,b]$, $F(\eta) = 0$ $\therefore$ 左边得证.

$\frac{1}{2}\int_a^b f(x)dx = \frac{1}{2}\int_a^\xi f(x)dx + \frac{1}{2}\int_\xi^b f(x)dx$

要. 则 ~~式中~~ $\frac{1}{2}\int_a^\xi f(x)dx + \frac{1}{2}\int_\xi^b f(x)dx = \int_\xi^b f(x)dx$

则 $\int_\xi^b f(x)dx = \int_a^\xi f(x)dx$ $\therefore$ 得证.

3.2.16

(2) $\int (x^a + a^x) dx = \frac{x^{a+1}}{a+1} + \frac{a^x}{\ln a} + C$

(4) $\int \frac{1}{\cos x}(\frac{1}{\cos x} - \frac{\sin x}{\cos x}) dx = \int \frac{1-\sin x}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x} dx$

$= \tan x + \int \frac{d(\cos x)}{\cos^2 x} = \tan x - \frac{1}{\cos x} + C.$

(6) $\int (x+\frac{1}{x})^2 dx = \int (x^2 + 2 + \frac{1}{x^2}) dx = \frac{1}{3}x^3 + 2x - \frac{1}{x} + C.$

(8) $\int (\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}}) dx = 3\arctan x - 2\arcsin x + C.$

(10) $\int \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}} dx = \int \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

(12) $\int \frac{\cos 2x}{\cos x + \sin x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} dx = \int (\cos x - \sin x) dx = \sin x + \cos x + C.$

(14) $\int \frac{1+\cos^2 x}{1+\cos 2x} dx = \frac{1}{2}\int \frac{1+\cos^2 x}{\cos^2 x} dx = \frac{1}{2}\int (1 + \frac{1}{\cos^2 x}) dx = \frac{x}{2} + \frac{\tan x}{2} + C.$

3.2.17

$f(e) = -1, \quad f'(x) = \frac{1}{x} \quad \therefore f(x) = \int f'(x) dx + C = \ln|x| + C$

$f(e) = 1 + C = -1 \Rightarrow C = -2 \quad \therefore f(x) = \ln|x| - 2.$

3.2.18.

(2) $\int \frac{dx}{(3-2x)^{\frac{1}{3}}} \xlongequal{u=(3-2x)^{\frac{1}{3}}} \int \frac{d(\frac{3-u^3}{2})}{u} = \int \frac{-\frac{3}{2}u^2 du}{u} = -\frac{3}{2}\int u\, du = -\frac{3}{4}u^2 + C$

$\xlongequal{u=(3-2x)^{\frac{1}{3}}} -\frac{3}{4}(3-2x)^{\frac{2}{3}} + C.$

(4) $\int \cos^2 5x\, dx = \frac{1}{2}\int (1+\cos 10x) dx = \frac{1}{2}x + \frac{1}{20}\sin 10x + C$

(6) $\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x}{e^{2x}-1} dx \xlongequal{u=e^x} \int \frac{u}{u^2-1} d(\ln u) = \int \frac{u}{u^2-1}\frac{1}{u} du = \int \frac{1}{u^2-1} du$

$= \frac{1}{2}\int (\frac{1}{u-1} - \frac{1}{u+1}) du$

$= \frac{1}{2}\int \frac{1}{u-1} d(u-1) - \frac{1}{2}\int \frac{1}{u+1} d(u+1) = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C$

$= \frac{1}{2}\ln|\frac{u-1}{u+1}| + C \xlongequal{u=e^x} \frac{1}{2}\ln|\frac{e^x-1}{e^x+1}| + C.$

(8) $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx \xlongequal{u=\sqrt{x}} \int \frac{\cos u}{u} d(u^2) = \int \frac{\cos u}{u} 2u\, du = 2\int \cos u\, du = 2\sin u + C$

$= 2\sin\sqrt{x} + C.$

300.18.

(10) $\int \frac{x^2}{4+x^6}dx = \frac{1}{3}\int\frac{d(x^3)}{4+x^6} \overset{u=x^3}{=\!=} \frac{1}{3}\int\frac{du}{4+u^2} = \frac{1}{12}\int\frac{du}{1+(\frac{u}{2})^2} = \frac{1}{6}\int\frac{d(\frac{u}{2})}{1+(\frac{u}{2})^2}$

$= \frac{1}{6}\arctan\frac{u}{2}+C \overset{u=x^3}{=\!=} \frac{1}{6}\arctan\frac{x^3}{2}+C.$

(12) $\int\frac{dx}{x^2-2x+2} = \int\frac{d(x-1)}{(x-1)^2+1} \overset{u=x-1}{=\!=} \int\frac{du}{u^2+1} = \arctan u + C = \arctan(x-1)+C$

$\cancel{(14)\ \int\frac{x\,dx}{\sqrt{5+x-x^2}} = \frac{1}{2}\int\frac{d(x^2)}{\sqrt{5+x-x^2}} \overset{u=\sqrt{5+x-x^2}}{=\!=} \frac{1}{2}\int\frac{d(5+x-u^2)}{u}}$

$\cancel{(14)\ \int\frac{x}{\sqrt{5+x-x^2}}dx = \int\frac{1}{\sqrt{\frac{5}{x^2}+\frac{1}{x}-1}}dx \overset{t=\frac{1}{x}}{=\!=} \int\frac{1}{\sqrt{5t^2+t-1}}d(\frac{1}{t})}$

$\cancel{= \int\frac{-\frac{1}{t^2}}{\sqrt{5t^2+t-1}}dt}$

(16) $\int\frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx = \int\frac{d(\sin x-\cos x)}{\sqrt[3]{\sin x-\cos x}} \overset{u=\sin x-\cos x}{=\!=} \int\frac{du}{u^{\frac{1}{3}}} = \frac{3}{2}u^{\frac{2}{3}}+C.$

$\overset{u=\sin x-\cos x}{=\!=} \frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}}+C$

(18) $\int\frac{dx}{1+e^x} \overset{u=e^x}{=\!=} \int\frac{d(\ln u)}{1+u} = \int\frac{1}{1+u}\frac{1}{u}du = \int(\frac{1}{u}-\frac{1}{1+u})du$

$= \ln|u| - \ln|1+u| + C = \ln\frac{e^x}{1+e^x}+C.$

(20) $\int\frac{x\tan\sqrt{1+x^2}}{\sqrt{1+x^2}}dx \overset{\cancel{u=\sqrt{1+x^2}}}{=\!=} \frac{1}{2}\int\frac{\tan\sqrt{1+x^2}}{\sqrt{1+x^2}}d(x^2) \overset{u=\sqrt{1+x^2}}{=\!=} \frac{1}{2}\int\frac{\tan u}{u}d(u^2-1)$

$= \frac{1}{2}\int\frac{\tan u}{u}du\cdot 2u = \int\tan u\,du = \int\frac{\sin u}{\cos u}du = -\int\frac{d(\cos u)}{\cos u}$

$= -\ln|\cos u| + C = \cancel{-\ln|\sqrt{1+x^2}|+C}\ -\ln|\cos\sqrt{1+x^2}|+C$

(14). $\int\frac{x}{\sqrt{5+x-x^2}}dx \overset{u=x-\frac{1}{2}}{=\!=} \int\frac{u+\frac{1}{2}}{\sqrt{\frac{21}{4}-u^2}}du = \frac{1}{2}\int\frac{du^2}{\sqrt{\frac{21}{4}-u^2}} + \frac{1}{2}\int\frac{du}{\sqrt{\frac{21}{4}-u^2}}$

$= -(\frac{21}{4}-u^2)^{\frac{1}{2}} + \frac{1}{2}\int\frac{du}{\sqrt{\frac{21}{4}-u^2}}+C$

$= -(\frac{21}{4}-u^2)^{\frac{1}{2}}+C + \frac{1}{\sqrt{21}}\int\frac{du}{\sqrt{1-(\frac{2u}{\sqrt{21}})^2}}$

$= -(\frac{21}{4}-u^2)^{\frac{1}{2}}+C + \frac{1}{\sqrt{21}}\cdot\frac{\sqrt{21}}{2}\int\frac{d(\frac{2u}{\sqrt{21}})}{\sqrt{1-(\frac{2u}{\sqrt{21}})^2}}$

$= -(\frac{21}{4}-u^2)^{\frac{1}{2}} + \frac{1}{2}\arcsin\frac{2u}{\sqrt{21}} + C$

$= -(5+x-x^2)^{\frac{1}{2}} + \frac{1}{2}\arcsin\frac{2}{\sqrt{21}}(x-\frac{1}{2}) + C.$

301.19

(2) $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} \overset{x=\sin t}{=} \int \frac{d(\sin t)}{\cos^3 t} = \int \frac{\cos t}{\cos^3 t} d(t) = \tan t + C \overset{t=\arcsin x}{=} \tan\arcsin x + C$

$= \frac{x}{\sqrt{1-x^2}} + C.$

(4) $\cancel{\int \frac{dx}{x\sqrt{a^2-x^2}} \overset{t=\frac{1}{x}}{=} \int \frac{t}{\sqrt{a^2-\frac{1}{t^2}}} d(\frac{1}{t}) = -\int \frac{1}{\sqrt{a^2t^2-1}} d(t) = -\frac{1}{a}\int \frac{d(at)}{\sqrt{a^2t^2-1}}}$

$\cancel{\overset{u=at}{=} -\frac{1}{a}\int \frac{du}{\sqrt{u^2-1}} \overset{u=\frac{1}{\cos v}}{=} -\frac{1}{a}\int \frac{\sin v}{\tan v \cos^2 v} dv =}$

(4) $\int \frac{dx}{x\sqrt{a^2-x^2}} \overset{x=a\sin t}{=} a\int \frac{\cos t}{a^2\sin t\cos t} dt = \frac{1}{a}\int \frac{1}{\sin t} dt = \frac{1}{a}\int \frac{\sin t}{\sin^2 t} dt$

$= -\frac{1}{a}\int \frac{d(\cos t)}{1-\cos^2 t} \overset{u=\cos t}{=} -\frac{1}{a}\int \frac{du}{1-u^2} = -\frac{1}{2a}\int (\frac{1}{1-u}+\frac{1}{1+u})du$

$(\frac{x}{a})^2+u^2=1$

$= \frac{\ln|1-u|}{2a} - \frac{\ln|1+u|}{2a} + C \overset{u=\cos t}{=} \frac{1}{2a}(\ln|1-\cos t| + \ln|1+\cos t|) + C$

$\overset{t=\arcsin\frac{x}{a}}{=} \frac{1}{2a}(|\ln|1-\cos\arcsin\frac{x}{a}| + \ln|1+\cos\arcsin\frac{x}{a}|) + C$

(6) $\int \frac{\sqrt{x^2-9}}{x} dx = \int \sqrt{1-\frac{9}{x^2}} dx \overset{x=\frac{3}{\sin t}}{=} \int \cos t \, d(\frac{3}{\sin t}) = -3\int \cot^2 t \, d(t)$

$\cancel{= -3(\cos^2 t \frac{t}{\tan^2 t} - \int t}$

$\overset{\tan t=u}{=} \int \frac{du}{u^2(1+u^2)} = \int (\frac{1}{u^2} - \frac{1}{1+u^2}) du = -\frac{1}{u} - \arctan u + C$

$= -\frac{1}{\tan t} - t + C \overset{t=\arcsin\frac{3}{x}}{=} -\frac{1}{\tan\arcsin\frac{3}{x}} - \arcsin\frac{3}{x} + C$

(8) $\int \frac{dx}{1+\sqrt{1-x^2}} = \int \frac{1-\sqrt{1-x^2}}{1-1+x^2} dx = \int (\frac{1}{x^2} - \frac{\sqrt{1-x^2}}{x^2}) dx = -\frac{1}{x} - \int \frac{\sqrt{1-x^2}}{x^2} dx + C$

$\cancel{= -\frac{1}{x} + C + (-\frac{1}{x}\sqrt{1-x^2} - \frac{1}{2}\int x^3 \frac{x}{\sqrt{1-x^2}} dx}$

$= -\frac{1}{x} + C + \frac{\sqrt{1-x^2}}{x} + \int \frac{1}{\sqrt{1-x^2}} dx = -\frac{1}{x} + \frac{\sqrt{1-x^2}}{x} + \arcsin x + C$

(10) $\int \frac{dx}{\sqrt{1+e^x}} = \int \frac{e^x}{e^x\sqrt{1+e^x}} dx \overset{u=e^x}{=} \int (\frac{du}{u\sqrt{1+u}}) \cancel{= 2\int \frac{du}{u\sqrt{1+u^2}}}$

$\cancel{= 2\int u}$ $= 2\int \frac{1}{u} d(\sqrt{1+u}) \overset{\sqrt{1+u}=t}{=} 2\int \frac{1}{t^2-1} d(t)$

$= \int (\frac{1}{t-1} - \frac{1}{t+1}) dt = \ln|t-1| - \ln|t+1|$

代错了

$\cancel{= \ln|e^x-1| - \ln(e^x+1)}$ $= \ln|\sqrt{1+u}-1| - \ln|\sqrt{1+u}+1| = ---$

(12) $\int \frac{dx}{\sqrt{(x-a)(b-x)}} \cancel{= 2\int \frac{d(\sqrt{x-a})}{\sqrt{b-x}} = 2(\frac{\sqrt{x-a}}{\sqrt{b-x}} - \int \sqrt{x-a}}$

$= \int \frac{dx}{\sqrt{\frac{(b-a)^2}{4} - (x-\frac{a+b}{2})^2}} = \frac{2}{|b-a|}\int \frac{dx}{\sqrt{1-(\cdots)^2}} = \int \frac{d(\frac{2}{b-a}x)}{\sqrt{1-(\cdots)^2}} = \arcsin(\frac{2}{b-a}x - \frac{a+b}{b-a})$

$= 2\arctan\sqrt{\frac{x-a}{b-x}} + C$

301.20.

(2) $\int x\ln(x-1)\,dx = \frac{1}{2}\int \ln(x-1)\,dx^2 = \frac{1}{2}\left(x^2\ln(x-1) - \int \frac{x^2}{x-1}\,dx\right)$

$= \frac{1}{2}x^2\ln(x-1) + C - \frac{1}{2}\int \frac{x^2-2x+1+2x-2+1}{x-1}\,dx$

$= \frac{1}{2}x^2\ln(x-1) + C - \frac{1}{2}\int\left((x-1)+2+\frac{1}{x-1}\right)dx$

$= \frac{1}{2}x^2\ln|x-1| - \frac{1}{4}(x-1)^2 - x - \frac{1}{2}\ln|x-1| + C.$

(4) $\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx + C$

$= x\arctan x + C - \frac{1}{2}\int \frac{d(x^2)}{1+x^2} = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

(6) $\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^2\,d(x) = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$

(8) $\int \frac{\arcsin x}{\sqrt{1-x}}\,dx = -2\left(\arcsin x\sqrt{1-x} - \int \frac{1}{\cancel{\sqrt{1+x}}}\frac{\sqrt{1-x}}{\sqrt{1-x^2}}\,dx\right)$

$= \cancel{-\frac{1}{2}\arcsin x} - 2\sqrt{1-x}\arcsin x + 2\int \frac{\sqrt{1-x}}{(1-x)\sqrt{1+x}}\,dx.$

$= -2\sqrt{1-x}\arcsin x + 4\int \frac{d(\sqrt{1+x})}{(1-x)} \overset{u=\sqrt{1+x}}{=} \int \frac{1}{2-u^2}\,du = \frac{\sqrt{2}}{2}\arcsin\frac{\sqrt{1+x}}{\sqrt{2}}$

$= -2\sqrt{1-x}\arcsin x \cancel{+ \frac{\sqrt{2}}{2}\arcsin\sqrt{\frac{1+x}{2}} + C} + 4\sqrt{1+x} + C.$

(10) $\int \ln(x+\sqrt{1+x^2})\,dx = x\ln(x+\sqrt{1+x^2}) - \int x\frac{1}{\sqrt{1+x^2}}\,dx$

$= x\ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int \frac{d(x^2)}{\sqrt{1+x^2}} \overset{u=x^2}{=} -\frac{1}{2}\int \frac{du}{\sqrt{1+u}} = -\sqrt{1+u} = -\sqrt{1+x^2}.$

$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

(12) $\int \frac{\ln(\cos x)}{\cos^2 x}\,dx \cancel{\overset{u=\cos x}{=} \int \frac{\ln u}{u^2}} \tan x\ln(\cos x) + \int \tan^2 x\,dx$

$\overset{u=\tan x}{=} \tan x\ln(\cos x) + \int \frac{u^2+1-1}{u^2+1}\,du = u - \arctan u$

$= \tan x\ln(\cos x) + \tan x - x + C$

(14) $\int \sin x\ln(\tan x)\,dx = -\cos x\ln(\tan x) + \int \cancel{\cos x}\frac{1}{\sin x}\,dx$

$= -\cos x\ln(\tan x) + \int \frac{\sin x}{\sin^2 x}\,dx = -\cos x\ln(\tan x) - \int \frac{d(\cos x)}{1-\cos^2 x}$

$= -\cos x\ln(\tan x) + \frac{1}{2}\ln|1-\cos x| - \frac{1}{2}\ln|1+\cos x| + C$

301.20

(16) $\int \sqrt{x}\, e^{\sqrt{x}}\, dx \overset{u=\sqrt{x}}{=\!=} \int u e^u du^2 = 2\int u^2 e^u du \cancel{= 2(\frac{1}{3}u^3 e^u - \int \frac{1}{3}u^3 e^u du)}$.

∴ $\cancel{I = \frac{1}{6}u^3 e^u = \frac{1}{6}\sqrt{x}^3 e^{\sqrt{x}}} \cancel{= \frac{2}{3}(u^3 e^u - \int u^3 e^u du)}$

$= \cancel{ue^u}\ 2\int u^2 e^u du = 2(u^2 e^u - 2\int u e^u du)$

$= 2u^2 e^u - 4\int u e^u du = 2u^2 e^u - 4(u-1)e^u + C$.

$= 2x e^{\sqrt{x}} - 4(\sqrt{x}-1)e^{\sqrt{x}} + C$

(18) $\int \frac{\arcsin x}{x^2} dx = -\frac{1}{x}\arcsin x + \int \frac{1}{x\sqrt{1-x^2}} dx + C$.

$= -\frac{1}{x}\arcsin x + C + \int \frac{x}{x^2\sqrt{1-x^2}} dx \cancel{= \frac{1}{2}\int \frac{d(x^2)}{x^2\sqrt{1-x^2}} \overset{u=x^2}{=} \frac{1}{2}\int \frac{du}{u\sqrt{1-u}}}$

$= -\frac{1}{x}\arcsin x + C - \int \frac{d(\sqrt{1-x^2})}{x^2} \overset{u=\sqrt{1-x^2}}{=} -\int \frac{du}{1-u^2}$

$= -\frac{1}{x}\arcsin x + C + \frac{1}{2}\ln|1-\sqrt{1-x^2}| - \frac{1}{2}\ln|1+\sqrt{1-x^2}|$

(20) $\int \ln(\sqrt{1+x}+\sqrt{1-x})dx = x\ln(\sqrt{1+x}+\sqrt{1-x}) - \frac{1}{2}\int x \frac{\frac{1}{\sqrt{1+x}} - \frac{1}{\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}} dx + C$

$= x\ln(\sqrt{1+x}+\sqrt{1-x}) + C - \frac{1}{4}\int \frac{(\sqrt{1-x}-\sqrt{1+x})^2}{\sqrt{1+x}\sqrt{1-x}} dx$ $\quad \int(\frac{1}{\sqrt{1-x^2}} - \frac{1}{2})dx$

$= x\ln(\sqrt{1+x}+\sqrt{1-x}) + C - \frac{1}{4}\int(\sqrt{\frac{1+x}{1-x}} - 2 + \sqrt{\frac{1-x}{1+x}})dx$

$\overset{\sqrt{\frac{1+x}{1-x}}=t}{=\!=} x\ln(\sqrt{1+x}+\sqrt{1-x}) + C - \frac{1}{4}\int(t+\frac{1}{t})d(\frac{t^2-1}{t^2+1}) + \frac{1}{2}x$

$= x\ln(\sqrt{1+x}+\sqrt{1-x}) \cancel{+ \frac{1}{2}x - \arctan\sqrt{\frac{1+x}{1-x}} + C}$ $\quad (\frac{1}{2}\arcsin x + \arctan\sqrt{\frac{1+x}{1-x}})'$

$+ \frac{1}{2}\arcsin x - \frac{1}{2}x \cancel{\frac{1}{2}\arcsin x}$ $\quad = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{\frac{1}{2}(\sqrt{\frac{1-x}{1+x}})\cdot\frac{2}{(1-x)^2}}{1+\frac{1+x}{1-x}}$

$= 0$

302.21

(2) $\int (\cos 2x + \tan^2 x) d(\sin^2 x) = 2\int(\cos 2x + \tan^2 x)\sin x \cos x\, dx$

$= 2\int(\cos^2 x - \sin^2 x + \frac{\sin^2 x}{\cos^2 x})\sin x\cos x\, d$

$\overset{\cos^2 x = u}{=} 2\int(2u + \frac{1}{u} - 2)d(1-u) = \cancel{-2\int(2u^2 - 2u + 1)du}$

$\cancel{= -2(\frac{2}{3}u^3 - u^2 + u + C) \overset{u=\cos^2 x}{=} -\frac{4}{3}\cos^6 x + 2\cos^4 x - 2\cos^2 x + C}$

∴ $\cancel{f(x)=}$ $= -\int(2u + \frac{1}{u} - 2)du = -(u^2 + \ln u - 2u) + C$

∴ $f(\sin^2 x) = -(\cos^4 x + \ln\cos^2 x - 2\cos^2 x) + C$

∴ $f(x) = \cancel{-\ln|1-x^2|}\ -\ln|1-x| - x^2 + C$.

302.22

$f'(x)=\frac{\cos x-\sin^2x}{(1+x\sin x)^2}$ $\therefore$ $\cancel{\int f(x)f'(x)dx=\int\frac{\sin x(\cos x-\sin^2x)}{(1+x\sin x)^3}dx}$

$$\int f(x)f'(x)dx=\int f(x)d(f(x))=\frac{1}{2}f^2(x)=\cancel{\frac{1}{2}\frac{\sin^2x}{(1+x\sin x)^2}+C}$$
$$=\frac{1}{2}\frac{(\cos x-\sin^2x)^2}{(1+x\sin x)^4}+C.$$

302.23

$\cancel{\int f(\ln x)d(\ln x)}$ 令 $\ln x=t\Rightarrow x=e^t$.

$\therefore$ $\cancel{f(x)}$ $f(t)=\frac{\ln(1+e^t)}{e^t}$ $\therefore f(x)=\frac{\ln(1+e^x)}{e^x}$

$\therefore\int f(x)dx=-\frac{\ln(1+e^x)}{e^x}-\ln(1+e^x)+C.$

302.24

$I_1=\int\frac{\cos x}{\sin x+\cos x}dx=\cancel{\int\frac{\frac{1}{2}\sin 2x}{\sin^2x+\sin x\cos x}dx=-\frac{1}{4}\int\frac{d(\cos 2x)}{\frac{1-\cos 2x}{2}+\frac{1}{2}\sin 2x}}$

$\cancel{=-\frac{1}{2}\int\frac{d(\cos 2x)}{1+\sin 2x-\cos 2x}=\frac{1}{2}\int\frac{d(\cos(x+\frac{\pi}{4}))}{1+\sin(\cos x+\frac{\pi}{4})}}$

$\cancel{I_1=\int\frac{1}{\tan x+1}dx\overset{\tan x=u}{=\!=}\int\frac{1}{u+1}d(\arctan u)=\int}$

$I_1+I_2=\int 1dx=x+C;\ I_1-I_2=\int\frac{\cos x-\sin x}{\cos x+\sin x}dx=\int\frac{d(\sin x+\cos x)}{\sin x+\cos x}=\ln|\sin x+\cos x|+C$

$\therefore\ I_1=\frac{x}{2}+\frac{1}{2}\ln|\sin x+\cos x|+C,\ I_2=\frac{x}{2}-\frac{1}{2}\ln|\sin x+\cos x|+C$

302.25

(2) $I_1=\int\tan x\,dx=\frac{1}{2}\ln(1+\tan^2x)$.

$\cancel{I_n=\int\tan^n x\,dx=x\tan^n x-n\int\frac{x\tan^{n-1}x}{\cos^2x}}dx$

$\cancel{=x\tan^n x-n\int}$

$I_n=\int\tan x\tan^{n-1}x\,dx=\frac{1}{2}\ln(1+\tan^2x)\tan^{n-1}x-\int\frac{\tan^{(n-1)}x}{\cos^2x}dx$

$=\frac{1}{2}\ln(1+\tan^2x)\tan^{n-1}x-(n-1)\int\tan^{n-1}x\,d(\tan x)$

$=\frac{1}{2}\ln(1+\tan^2x)\tan^{n-1}x-\cancel{(n-1)\tan}\ \frac{n-1}{n}\tan^n+C$

$I_n=\int\tan^{n-2}x(\frac{1}{\cos^2x}-1)dx=\int\tan^{n-2}x\,d(\tan x)-I_{n-2}$

$=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}.$

$$\cancel{(4)\ I_1 = \int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}+C}$$

$$\cancel{I_n = \int \frac{x^n}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}x^{n-1}dx = -\sqrt{1-x^2}\,x^{n-1}+(n-1)\int\sqrt{1-x^2}\,x^{n-2}dx}$$

$$\cancel{\overset{x=\sin t}{=} \int \frac{\sin^n t}{\cos t}d(\sin t) = \int \sin^n t\, dt = -\cos t\sin^{n-1}t+\int\cos t}$$

$$\cancel{= \int \sin^{n-2}t(1-\cos^2 t)dt = I_{n-2} - \int \sin^{n-2}t\cos^2 t\,dt}$$

$$(4)\ I_n = \int \frac{x^n}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2}\,x^{n-1}+(n-1)\int x^{n-2}\sqrt{1-x^2}\,dx$$

$$= -\sqrt{1-x^2}\,x^{n-1}+(n-1)\int\frac{x^{n-2}(1-x^2)}{\sqrt{1-x^2}}dx$$

$$= -\sqrt{1-x^2}\,x^{n-1}+(n-1)(I_{n-2}-I_n)$$

$$\therefore\ I_n = -\frac{1}{n}x^{n-1}\sqrt{1-x^2}+\left(1-\frac{1}{n}\right)I_{n-2}$$

302, 26

$$(2)\ \int\frac{dx}{(x-1)(x+1)^2} = \frac{1}{4}\int\left(\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{(x+1)^2}\right)dx = \frac{1}{4}\left(\ln|x-1|-\ln|x+1|+\frac{2}{x+1}\right)+C$$

$$\cancel{(4)\ \int\frac{x^5+x^4-8}{x^3+x}dx = \int\left(x^2-\frac{8}{x^3+x}\right)dx = \frac{1}{3}x^3-8\int\frac{1}{x(x^2+1)}dx}$$

$$\cancel{= \frac{1}{3}x^3-8\int\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx = \frac{1}{3}x^3-8\ln|x|+4\int\frac{dx^2}{x^2+1}}$$

$$\cancel{= \frac{1}{3}x^3-8\ln|x|+4\ln|x^2+1|+C}$$

$$(6)\ \int\frac{x^3}{(1+x^8)^2}dx = \frac{1}{4}\int\frac{dx^4}{(1+x^8)^2} \overset{u=x^4}{=} \frac{du}{(1+u^2)^2} \overset{u=\tan t}{=} \frac{1}{4}\int \cos^4 t\, d(\tan t)$$

$$= \frac{1}{4}\int\cos^2 t\,dt = \frac{1}{4}\int\frac{1+\cos 2t}{2}dt = \frac{1}{4}\left(\frac{t}{2}+\frac{1}{4}\sin 2t+C\right)$$

$$= \frac{1}{4}\left(\frac{1}{2}\arctan u+\frac{1}{2}\frac{u}{u^2+1}\right)+C = \frac{1}{8}\arctan x^4+\frac{1}{8}\frac{x^4}{x^8+1}+C$$

$$(8)\ \int\frac{x^2+1}{x^4+1}dx \cancel{\overset{u=x^2}{=} \int\frac{(u+1)}{u^2+1}du} = \frac{1}{2}\int\left(\frac{1}{x^2-\sqrt{2}x+1}+\frac{1}{x^2+\sqrt{2}x+1}\right)dx$$

$$= \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x-1)+\frac{\sqrt{2}}{2}\arctan(\sqrt{2}x+1)+C.$$

$$(4)\ \int\frac{x^5+x^4-8}{x^3+x}dx = \int\left(x^2+\frac{x^4-x^3-8}{x^3+x}\right)dx = \int\left(x^2+x-\frac{x^3+x^2+8}{x^3+x}\right)dx$$

$$= \frac{1}{3}x^3+\frac{1}{2}x^2-\int\frac{x^3+x+x^2-x+8}{x^3+x}dx = \frac{1}{3}x^3+\frac{1}{2}x^2-x-\int\frac{x^2-x-8}{x^3+x}dx$$

$$= \frac{1}{3}x^3+\frac{x^2}{2}-x+\frac{7}{2}\ln(x^2+1)-\arctan x+C.$$