P238. 8

(1) $1 * 1 = \int_0^t 1 \cdot 1\, dt = t$.

(2) $\sin t * \cos t = \int_0^t \sin x \cdot \cos(t-x)\, dx = \frac{1}{4}(\cos t + 2t\sin t - \overset{\cos t}{\cancel{\cos t}}) = \frac{1}{2} t\sin t$.

(4). $t * \sinh t = \frac{1}{2}\int_0^t t(e^{t} - e^{-t})\, d\tau = \frac{1}{2}((t-1)(e^{t} + e^{t}) + 2)$

P238. 9

(1) $\frac{a}{p(p^2+a^2)} = \frac{a}{p}\cdot\frac{1}{p^2+a^2} \Rightarrow \mathcal{L}^{-1}(原式) = \mathcal{L}^{-1}(\frac{a}{p}) * \mathcal{L}^{-1}(\frac{1}{p^2+a^2})$

$= a * \frac{1}{a}\sin at = \int_0^t \sin a\tau\, d\tau = \frac{1}{a}(1 - \cos at)$.

(3). $\frac{1}{p(p-1)(p-2)} = \frac{1}{p}\cdot\frac{1}{p-1}\cdot\frac{1}{p-2} \Rightarrow \mathcal{L}^{-1}(原式) = \cancel{e^{t} * e^{2t}}\ \cancel{\Rightarrow}\ 1 * (e^{2t} - e^{t})$.

$= \int_0^t e^{2\tau} - e^{\tau}\, d\tau = \frac{1}{2}e^{2t} - e^{t} + \frac{1}{2}$

P238. 10

(3). 两边 Laplace. ~~$p^3Y(p) - p^2y(0) - py'(0) - py''(0)$.~~

$p^3Y(p) - p^2y(0) - py(0) - y'(0) = \frac{6}{(p+1)} \Rightarrow Y(p) = \frac{6}{p^3(p+1)}$

$\Rightarrow y(t) = 3(t-1)^2 - 6e^{-t}$.

(5). Laplace $\Rightarrow$ $p^2Y(p) - py(0) - y'(0) + 9Y(p) = \frac{p}{p^2+4}$

$\Rightarrow Y(p) = \frac{1}{p^2+9}(\frac{p}{p^2+4} + p + y'(0))$.

$\therefore y(t) = \frac{-1}{10}(e^{3ti} + e^{-3ti} - e^{-2ti} - e^{2ti}) + \frac{1}{2}(e^{-3ti} + e^{3ti}) + y'(0)(\frac{1}{6}e^{3ti} - \frac{1}{6}e^{-3ti})$.

$y(\frac{\pi}{2}) = -1 \Rightarrow y'(0) = \frac{12}{5i}$ $\therefore y(t) = \frac{4}{5}\cos 3t + \frac{1}{2}\cos 2t + \frac{4}{5}\sin 3t$.

P239. 12.

(1) $\int_0^{+\infty} \frac{e^{-t} - e^{-2t}}{t}\, dt = \int_0^{+\infty} \frac{e^{-t}}{t}\, dt \int_0^{+\infty} \frac{e^{-2t}}{t}\, dt$

$\mathcal{L}[\frac{e^{-t}}{t}] = \int_p^{+\infty} \frac{1}{s+1}\, ds$, $\mathcal{L}[\frac{e^{-2t}}{t}] = \int_p^{+\infty} \frac{1}{s+2}\, ds$.

$\therefore$ 原式 $= \lim_{p\to 0} \cdots = \ln 2$.

(3). $\int_0^{+\infty} e^{-3t} \cos 2t \, dt$

$\mathcal{L}[\cos 2t] = \dfrac{p}{p^2+4}$, $\mathcal{L}[e^{-3t}\cos 2t] = \dfrac{p+3}{(p+3)^2+4} = \int_0^{+\infty} e^{-3t}\cos 2t \, e^{-pt} dt$.

$\therefore$ 原式 $= \lim\limits_{p\to 0} \cdots = \dfrac{3}{13}$

(5). $\mathcal{L}[\sin 2t] = \dfrac{2}{p^2+4}$, $\mathcal{L}[e^{-3t}\sin 2t] = \dfrac{2}{(p+3)^2+4}$

$\mathcal{L}[t e^{-3t}\sin 2t] = \dfrac{4(p+3)}{((p+3)^2+4)^2}$ $\therefore$ 原式 $= \lim\limits_{p\to 0} \cdots = \dfrac{12}{169}$

(8). $\mathcal{L}[\sin t] = \dfrac{1}{p^2+1}$, $\mathcal{L}[e^{-t}\sin t] = \dfrac{1}{(p+1)^2+1}$

$(-6p^2-12p-4)$

$\mathcal{L}[t^3 e^{-t}\sin t] = -\left[\dfrac{1}{(p+1)^2+1}\right]^{(3)} = \dfrac{(-12p-12)[(p+1)^2+1]^3 - 3[(p+1)^2+1]^2 2(p+1)}{[(p+1)^2+1]^6}$.

$\therefore$ 原式 $= 0$

P238. 10

(7). $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}1\\ t\end{pmatrix} \Rightarrow \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}\frac{1}{2}(1-t)\\ \frac{1}{2}(1+t)\end{pmatrix}$.

$\Rightarrow \begin{cases} x=(t+1)^2\frac{1}{4} \\ y=-(1-t)^2\frac{1}{4}\end{cases} \Rightarrow \begin{cases} x=\frac{1}{4}(t+1)^2+a-\frac{1}{4} \\ y=-\frac{1}{4}(t-1)^2+b+\frac{1}{4}\end{cases}$.

(8). Laplace 先，有 $\begin{cases} pX(p)-x(0)+Y(p)=\frac{1}{p} \\ X(p)-pY(p)+y(0)=\frac{1}{p^2}\end{cases}$ 代入 $\begin{cases} x(0)=0 \\ y(0)=1\end{cases}$

$\therefore \begin{pmatrix}p & 1\\ 1 & -p\end{pmatrix}\begin{pmatrix}X(p)\\ Y(p)\end{pmatrix}=\begin{pmatrix}\frac{1}{p}\\ \frac{1}{p^2}-1\end{pmatrix}. \Rightarrow \begin{cases} X(p)=\frac{1}{p^2}-\frac{1}{1+p^2} \\ Y(p)=\frac{p}{1+p^2}\end{cases}$

$\therefore \begin{cases} x(t)=t-\sin t \\ y(t)=\cos t\end{cases}$.

P238. 11

(1) 先 laplace. 有 $Y(p)+\frac{1}{p}Y(p)=\frac{1}{1+p} \Rightarrow Y(p)=\frac{p}{(1+p)^2}$

$\therefore y(t)=(1-t)e^{-t}$

(3). 先Laplace, 有 $\cancel{2}pY(p)-\cancel{2}y(0)+\frac{1}{p}Y(p)=\frac{1}{p} \Rightarrow Y(p)=\frac{1-p}{p^2+1}$

$\therefore \cancel{2}y(t)=\sin t-\cos t$.

(5) 先Laplace, 有 $\cancel{\frac{2}{p}Y}\ pY(p)-y(0)+3Y(p)+\frac{2}{p}Y(p)=\frac{1}{p}e^{-p}-\frac{1}{p}e^{-2p}$.

$\Rightarrow Y(p)=\frac{1}{(p+1)(p+2)}(p+e^{-p}-e^{-2p}) \cancel{\Rightarrow 2y(t)=-e^{-t}+3e^{-2t}-2e^{-2t}+e^{-4t}}$.

$\Rightarrow y(t)=2e^{-2t}-e^{-t}+u(t-1)(e^{1-t}-e^{2-2t})-u(t-2)[e^{2-t}-e^{4-2t}]$.

P316.3

(1) $u(t,x)=\frac{1}{2}\{(x+at)^2+(x-at)^2\}+\frac{1}{2a}\int_{x-at}^{x+at}\sin s\,ds$

$=x^2+a^2t^2+\frac{1}{2a}\{\cos(x-at)-\cos(x+at)\}$

$=x^2+a^2t^2+\frac{1}{a}\sin x\sin at.$

P318.12

(1) $u(t,x)=\frac{1}{2}\{\varphi(at+x)+\varphi(x-at)\}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)ds+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\tau,s)\,ds\,d\tau$

$=\sin x\sin t+\cancel{t\sin x(1-\cos t)}\,\cancel{=t(\sin t}\sin x(t-\sin t).$

$=\sin x\cdot t$

P318.3B

令 $\mathcal{F}[u(t,x)]=U(t,\omega)$，$\mathcal{F}[\varphi(x)]=\Phi(\omega)$，$\mathcal{F}[f(x,t)]=F(t,\omega)$

$\therefore\begin{cases}\frac{\partial U}{\partial t}(t,\omega)+i\omega a U(t,\omega)=F(t,\omega) & (1)\\ U(0,\omega)=\Phi(\omega). & (2)\end{cases}$

~~由(1)得~~ $\cancel{U(t,\omega)=C_1e^{-i\omega at}+}$

~~令~~ $\cancel{U=V+W.}$ $\cancel{V(t,\omega)=\Phi(\omega)e^{-i\omega at}}$

~~对W.有~~ $\cancel{\begin{cases}\frac{\partial W}{\partial t}(t,\omega)+i\omega a W(t,\omega)=0\\ W(0,\omega)\end{cases}}$

由(1)(2)解得 $U(t,\omega)=e^{-i\omega at}\left(\int_0^t F(\tau,\omega)e^{i\omega a\tau}d\tau+\Phi(\omega)\right)$

$\therefore\ \mathcal{F}^{-1}[U(t,\omega)]=\varphi(x-at)+\mathcal{F}^{-1}[F(t,\omega)*e^{i\omega at}\cdot\cancel{e^{-i\omega at}}].$

$\cancel{=\varphi(x-at)+2\pi\,\Phi(t,x)\,(f(t,x-at)\cdot\frac{\delta(t)}{2\pi})}$

$\mathcal{F}^{-1}[F(t,\omega)*e^{i\omega at}]=2\pi f(t,x)\cdot\frac{1}{2\pi}$

P318. 3B

$\mathcal{F}[u(t,x)] = U(t,\omega)$, $\mathcal{F}[\varphi(x)] = \Phi(\omega)$. $\mathcal{F}[f(t,x)] = F(t,\omega)$

$$\therefore \begin{cases} \dfrac{\partial U}{\partial t}(t,\omega) + i\omega a U(t,\omega) = F(t,\omega) \\ U(0,\omega) = \Phi(\omega) \end{cases}$$

$$\Rightarrow \quad U(t,\omega) = \Phi(\omega) e^{-i\omega a t} + \int_0^t F(\omega,\tau) e^{-i\omega a(t-\tau)} d\tau.$$

$$\therefore u(t,x) = \cancel{\varphi(t-\omega a)}\, \varphi(x-at) + \mathcal{F}^{-1}\left[\int_0^t F(\omega,\tau) e^{-i\omega a(t-\tau)} d\tau\right]$$

$\cancel{\mathcal{F}}$ 后者 $= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t F(\omega,\tau) e^{-i\omega a(t-\tau)} d\tau \cdot e^{i\omega x} d\omega$

$$= \frac{1}{2\pi}\int_0^t \int_{-\infty}^{+\infty} F(\omega,\tau) e^{i\omega(x-a(t-\tau))} d\omega\, d\tau.$$

$$= \int_0^t f(t,x) * \delta(x-a(t-\tau))\, d\tau$$

$$= \cancel{\int_0^t \int_{-\infty}^{+\infty} f(t,s)\, \delta[x+a(t-\tau)-s]\, ds\, d\tau}$$

$$= \int_0^t \int_{-\infty}^{+\infty} f(t, x-(a(t-\tau)+s))\, \delta(s)\, ds\, d\tau$$

$$= \int_0^t f[t, x-a(t-\tau)]\, d\tau.$$

$$\therefore u(t,x) = \varphi(x-at) + \int_0^t f[\tau, x-a(t-\tau)]\, d\tau.$$