Pluripotential-theoretic methods in K-stability and the space of Kähler metrics

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Abstract

It is a natural problem, dating back to Calabi, to find canonical metrics on complex manifolds. In the case of polarized compact Kähler manifolds, a natural candidate is a metric with constant scalar curvature (cscK metric). Since the 80s, Yau, Tian, Donaldson among others proposed that the existence of these special metrics are equivalent to an algebrico-geometric notion of K-stability.

There are several known approaches to the study of K-stability and canonical metrics, using various tools from the theory of PDEs, algebraic geometry and non-Archimedean geometry for example. In this thesis, we study a different approach, based on pluripotential theory. In geometric terms, pluripotential theory is the study of positively curved metrics on vector bundles. For the purpose of K-stability, we only need pluripotential theory on an ample line bundle. In this case, pluripotential theory can be identified with the study of quasi-plurisubharmonic functions on the manifold.

The application of pluripotential theory in K-stability is not completely new, but previously, people are principally interested in the *regular* (or mildly singular) quasi-plurisubharmonic functions. In this thesis, we put more emphasis on the role of *singular* quasi-plurisubharmonic functions and their singularities.

In Paper 1 and Paper 2, we prove a criterion for the existence of canonical metrics on Fano manifolds in terms of quasi-plurisubharmonic functions. In Paper 3, we are concerned with the case when there are no canonical metrics, we prove that there is always an optimal destabilizer to the K-stability.

Keywords: K-stability, quasi-plurisubharmonic functions, cscK metrics.

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Mingchen Xia
Göteborg, July 2022

List of Publications

This thesis is based on the following appended papers:

Paper 1: Tamás Darvas and Mingchen Xia. *The closures of test configurations and algebraic singularity types.* Advances in Mathematics (2022).

Paper 2.: Mingchen Xia. *Pluripotential-theoretic stability thresholds.* IMRN (2022).

Paper 3.: Mingchen Xia. *On sharp lower bounds for Calabi-type functionals and destabilizing properties of gradient flows.* Analysis & PDE (2021).

Other publications.

- Mingchen Xia. *On Liu morphisms in non-Archimedean geometry.* Israel journal of mathematics (2022).
- Mingchen Xia. *Analytic Bertini theorem.* Mathematische Zeitschrift (2022).

Preprints.

- Mingchen Xia. *Integration by parts formula for non-pluripolar product.* (2019).
- Mingchen Xia. *Mabuchi geometry of big cohomology classes with prescribed singularities.* (2019).
- Tamás Darvas and Mingchen Xia. *The volume of pseudoeffective line bundles and partial equilibrium.* (2021).
- Mingchen Xia. *Partial Okounkov bodies and Duistermaat–Heckman measures of non-Archimedean metrics.* (2021).
- Mingchen Xia. *Non-pluripolar products on vector bundles and Chern–Weil formulae on mixed Shimura varieties.* (2022).

Contents

Part 1

Introductory chapters

CHAPTER 1

General introduction

This thesis is devoted to the problem of Kähler–Einstein metrics. As the name suggests, these metrics have roots in both physics and mathematics. These metrics are closely related to Einstein's theory of general relativity. More precisely, they are special vacuum solutions of Einstein's field equations enriched by a special mathematical structure, called the Kähler structure. In this chapter, we give a short introduction to both the physics and the mathematics underlying the problem of Kähler–Einstein metrics.

1.1. A quick recap of general relativity

In 1687, Issac Newton published the celebrated book *Philosophiae Naturalis Principia Mathematica*. In *Principia*, Newton developed a theory of mechanics, a theory later known as *Newton mechanics*. According to Newton, particles propagate in the 3-dimensional space $\mathbb{R}^3$. In the absence of external forces, particles move in straight lines. In general, the acceleration of a particle is proportional to the force. In addition to the three spacial axes, there is a fourth axis: time. Time and space are irrelevant to each other.

According to Newton's theory, the speed of light depends on the observer. If one observer moves in the direction of the propagation of light, he will observe a lower speed of light than one observer who move in the opposite direction.

In the 19th century, with the development of electromagnetism, or more precisely electricity and magnetism according to the pre-Maxwell terminology, people begin to understand that light is nothing but the electromagnetic wave. In 1873, James Clerk Maxwell published *A Treatise on Electricity and Magnetism*, proposing the laws governing electromagnetism. These laws are written in a number of equations, called *Maxwell equations*. A surprising consequence of Maxwell equations is that the speed of light is constant, independent of the choice of the observer! Hence time and space must be dependent. This observation is in stark contradiction with Newton mechanics. It suggests the need of a new theory of mechanics.

The solution was proposed in the beginning of 20th century by Minkowski, Poincaré and Einstein among others, leading to the theory of special relativity. In special relativity, space and time are no longer independent axes. Two observers moving at different speeds feel different time and space simultaneously. The inseparable object consisting of space and time is known as the *spacetime*. In the mathematical terminology, the spacetime in special relativity is a flat 4-dimensional space with a Lorentz metric. Particles propagate along geodesics, the shortest path in spacetime. A prominent feature of relativity is that the speed of objects are bounded from above: nothing can move faster than light.

It turns out that the gravity does not fit into the theory of relativity, as the propogators of the gravity force would have to move infinitely fast. The disastrous situation was solved by Albert Einstein in 1915, leading to a vast generalization of the theory of relativity, known as *general relativity*. Roughly speaking, in general relativity, the spacetime is no longer flat, it is curved instead. Matters in the spacetime make the spacetime curved. Gravity is no longer a force, instead it is the curvature of the spacetime. In the curved spacetime, particles still tend to move along the shortest paths, namely the geodesics, which are no longer straight lines. This movement is effectively the gravity.

When no matter is presented, the spacetime is a *vacuum*. The equation describing the vacuum is the celebrated *vacuum Einstein's field equation*. In mathematical terms, this means the *Ricci curvature* of the spacetime vanishes. We will explain the notion of Ricci curvature in detail in later chapters. Here we stress that a space with vanishing Ricci curvature is not necessarily flat. In fact, there is an abundance of such spaces. More generally, in the presence of a cosomological constant, the vacuum is described by the fact that the Ricci curvature is constant.

The spacetimes in general relativity are 4-dimensional with the Lorentzian signature. However, the quest for quantum gravity suggests that higher dimensional manifolds (spacetimes) with the Riemannian signature are also important. Moreover, in the presence of supersymmetry, one needs to put an extra structure on the manifold, namely, the Kähler structure.

In short, this thesis concerns the existence of vacua of the Riemannian signature in all even dimensions when an extra Kähler structure is presented. In mathematical terms, we are studying the *Kähler–Einstein metrics*.

1.2. A general introduction to Kähler geometry

As already indicated in the previous part, the study of relativity requires mathematical knowledge of curved spacetime. These curved objects are known as *manifolds*. A manifold is a space which locally looks like domains in the Euclidean space $\mathbb{R}^n$. In this thesis, we will be interested in a special class of manifolds, the *complex* manifolds. A complex manifold is a manifold that locally looks like a domain in $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

In general, complex manifolds behave better compared to general manifolds. For example, all complex manifolds are orientable. This excludes pathological examples like the Klein bottle. In 1933, Erich Kähler introduced a condition on complex manifolds, singling out the narrower class of Kähler manifolds. In modern terminology, Kähler defined a condition on a metric g on the complex manifold. A metric satisfying Kähler's condition is called a *Kähler metric*. A complex manifold admitting a Kähler metric is called a *Kähler manifold*. We will not spell out the technical definition here, see the next chapter for a more detailed introduction. Kähler manifolds have the surprisingly nice behaviours both analytically and topologically. Moreover, from the point of view of algebraic geometry, this class is large enough to include all projective manifolds.

The modern study of complex geometry is usually divided into Kähler and non-Kähler geometry, according to the objects of interest are Kähler or not. For the purpose of this thesis, we will restrict our attention to only the Kähler case, even though certain aspects of this thesis work in the non-Kähler setting as well.

1.3. Kähler–Einstein metrics

Having the Einstein's field equation and the Kähler condition at disposal, we can define a Kähler–Einstein metric. A metric g on a (compact) complex manifold X is *Kähler–Einstein* if g is both Einstein (satisfying the vacuum Einstein's field equation or equivalently, the Ricci curvature is constant) and Kähler.

It turns out that the existence of a Kähler–Einstein metric puts strong constraints on the topology of X. For instance, the first Chern class of X has to be definite. There are three distinct classes of manifolds satisfying this condition: Calabi–Yau manifolds, anti-Fano manifolds and Fano manifolds. Conversely, assuming that $c_1(X)$ is definite, one seeks to find the Kähler–Einstein metrics on X.

In the Calabi–Yau case, Kähler–Einstein metrics are also called Ricci flat metrics. In this case, the existence of (plenty of) Ricci flat metrics is proved by Shing-Tung Yau in [Yau78], answering a conjecture of Calabi. Aubin [Aub76] and Yau [Yau77] established the existence and uniqueness of Kähler–Einstein metrics in the anti-Fano case as well.

The solutions in both cases rest on the the solution to the corresponding PDE. The Kähler–Einstein condition is equivalent to a second-order fully non-linear elliptic PDE, which belongs to the general category of Monge–Ampère equations. Aubin–Yau's method fails in the Fano case, however. In fact, by using similar approaches, Tian [Tia87] showed that this method works only if we impose in addition that an invariant $\alpha(X)$ of X is big. In general, there are examples showing that Kähler–Einstein metrics do not exist on a general Fano manifold.

This raises the question of finding suitable conditions on a Fano manifold guaranteeing the existence of Kähler–Einstein metrics. Since the 80s, Yau, Tian, Donaldson among others formulate a conjecture saying that the existence of Kähler–Einstein metrics should be equivalent to some algebraico-geometric stability condition on the manifold. These stability conditions are called *K-stability*. This conjecture is already answered in different generality. Here we only mention the first solution, due to Chen–Donaldson–Sun [CDS15a; CDS15b; CDS15c].

Finally, let us mention that there is a natural generalization of Kähler–Einstein metrics when $c_1(X)$ is not definite. These metrics are known as constant scalar curvature Käler metrics (cscK metric). A more general version of the Yau–Tian–Donaldson conjecture asserts that the existence of cscK metrics is equivalent to a generalized version of K-stability. Despite of the recent progress [CC21a; CC21b; CC18; Li20], the general Yau–Tian–Donaldson conjecture remains open.

In this thesis, I will apply the techniques of pluripotential theory to attack the problems of cscK metrics. I will explain the details about my work after giving a more technical introduction to Kähler geometry in the next chapter.

CHAPTER 2

Kähler geometry, pluripotential theory and Kähler–Einstein metrics

2.1. Kähler geometry

As a general reference, most results in this section can be found in the textbook [GH14].

In this section, we assume that the readers are familiar with notion of complex manifolds. For us, a complex manifold (X, J) consists of a manifold X together with an integrable almost complex structure J on X. By abuse of language, we usually call X a complex manifold. For $x \in X$, we write $T^{\mathbb{R}}_x X$ for the real tangent space of X at x and $T_x X$ for the holomorphic tangent space at x. Similarly, $T^{\mathbb{R}}X$ and TX denote the real tangent bundle and the holomorphic tangent bundle respectively. Write $T^{\mathbb{C}}X$ for the complexification of $T^{\mathbb{R}}X$.

Definition 2.1.1. Let (X, J) be a complex manifold. We say an Riemannian metric g is *compatible* with J if

$$g(Ju, Jv) = g(u, v), \quad \text{for all } x \in X, u, v \in T^{\mathbb{R}}_x X\,.$$

We call such (X, J, g) a *Hermitian manifold*. We set

$$\omega(u, v) = g(Ju, v), \quad \text{for all } x \in X, u, v \in T^{\mathbb{R}}_x X\,.$$

The 2-form ω is called the *Hermitian form* associated with g.

We say g (or ω) is a *Kähler metric* if ω is closed as a 2-form.

We say (X, J) is a *Kähler manifold* if it admits a Kähler metric.

Example 2.1.2. *Consider $X = \mathbb{P}^n$ with the standard complex structure. In other words, $X = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^n \setminus \{0\}$ by scaling. In this case, $\mathbb{P}^n$ is endowed with the* Fubini–Study metric ω_{FS}*:*

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2}\partial\bar{\partial}\log\left(|X_0|^2 + \cdot + |X_n|^2\right)\,, \tag{2.1.1}$$

where $[X_0 : \cdots : X_n]$ is the homogeneous coordinates on $\mathbb{P}^n$. This equation should be understood as a shorthand notation for the expressions of ω_{FS} in each local chart. For example, in the chart $X_0 \neq 0$, this means

$$\omega_{\mathrm{FS}} = \frac{\mathrm{i}}{2}\partial\bar{\partial}\log\left(1 + |X_1/X_0|^2 + \cdot + |X_n/X_0|^2\right)\,.$$

It is straightforward to verify that ω_{FS} is a Fubini–Study metric. As a consequence, each projective manifold X admits a Kähler metric: just embed X into some $\mathbb{P}^n$ and take the pull-back of the Fubini–Study metric as the Kähler metric on X.

Example 2.1.3. *There are also many non-projective Kähler manifolds. For example, any complex torus is Kähler. Recall that a complex torus is a quotient $\mathbb{C}^n/\Lambda$, where Λ is a lattice of rank $2n$ in $\mathbb{C}^n$. The flat metric is obviously Kähler in this case. However, $\mathbb{C}^n/\Lambda$ is projective (i.e. an Abelian variety) if and only if Λ admits a Riemann bilinear form, which is not always the case. See [MRM74].*

In fact, according to a theorem of Voisin [Voi04], since dimension 4, there exist compact Kähler manifolds which do not even have the homotopy type of a projective manifold.

Example 2.1.4. *There are many non-Kähler surfaces as well. For example, the Hopf surfaces and Inoue surfaces. In fact, it is well-known that a compact complex surface is Kähler if and only if the first Betti number is even. However, higher dimensional examples show that being Kähler or not is not topologically invariant.*

We fix a compact Kähler manifold (X, J, g) of pure dimension $n \geq 1$. Write ω for the corresponding Kähler form. Write ∇ for the Levi-Civita connection of g on $T^{\mathbb{R}}X$. It admits a natural extension as a connection on $T^{\mathbb{C}}X$, which we still denote by ∇. Choose local holomorphic coordinates $z^1, \ldots, z^n$ on X. With respect to the basis ∂_{z^i}, $\partial_{\bar{z}^i}$ of $T^{\mathbb{C}}X$, the non-zero Christoffel symbols are

$$\Gamma^i_{jk} = \frac{\partial g_{k\bar{\ell}}}{\partial z^j} g^{\bar{\ell}i}\,, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}\,. \tag{2.1.2}$$

Correspondingly, the non-zero Riemannian curvature components are

$$R^i_{j\bar{k}\ell} = -\frac{\partial \Gamma^i_{j\ell}}{\partial \bar{z}^k}\,, \quad R^i_{\bar{j}k\ell} = \frac{\partial \Gamma^i_{k\ell}}{\partial \bar{z}^j}\,, \quad R^{\bar{i}}_{\bar{j}k\bar{\ell}} = \overline{R^i_{j\bar{k}\ell}}\,, \quad R^{\bar{i}}_{j\bar{k}\bar{\ell}} = \overline{R^i_{\bar{j}k\ell}}\,. \tag{2.1.3}$$

Using this, we can easily compute the Ricci curvature. The non-zero components are

$$R_{i\bar{j}} = -\frac{\partial^2 \log\det(g_{a\bar{b}})}{\partial z^i \partial \bar{z}^j}\,, \quad R_{\bar{i}j} = \overline{R_{i\bar{j}}}\,. \tag{2.1.4}$$

In particular, the Ricci curvature can be regarded as the complexification of a real $(1,1)$-form $2\pi\,\mathrm{Ric}$:

$$\mathrm{Ric}(u, v) := \frac{1}{2\pi} R(Ju, v)\,, \quad \text{for all } x \in X, u \in T_x X\,, v \in \bar{T}_x X\,. \tag{2.1.5}$$

Now we have our key formula

$$\mathrm{Ric} = -\mathrm{dd^c} \log\det g\,, \tag{2.1.6}$$

where $\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}$ and $\det g$ is short for $\det(g_{i\bar{j}})$.

Next we recall the $\mathrm{dd^c}$-lemma on a compact Kähler manifold.

Theorem 2.1.5. *Let α be a differential form on X. Assume that α is either* d*-closed and* $\mathrm{d^c}$*-exact or* d*-exact and* $\mathrm{d^c}$*-closed. Then α is* $\mathrm{dd^c}$*-exact.*

2.2. Monge–Ampère operators

Let X be a compact Kähler manifold of dimension n.

2.2.1. Plurisubharmonic functions.

Definition 2.2.1. Let E be a holomorphic vector bundle of rank r on X. A *Hermitian metric* on E is a smooth assignment $X \ni x \mapsto h_x$, where h_x is a positive definite Hermitian metric on E_x.

In particular, if we locally trivialize E as $X \times \mathbb{C}^r$, a Hermitian metric can be identified with a smooth family of positive definite Hermitian matrices $h_{\mu\nu}$ parameterized by X.

The pair (E, h) is called a *Hermitian holomorphic vector bundle.*

THEOREM 2.2.2. *Given any Hermitian holomorphic vector bundle (E, h) of rank r on X, there is a unique connection D on E such that*

(1) D is Hermitian: $D_v h(a, b) = h(D_v a, b) + (a, D_v b)$ for any smooth vector fields v and any smooth sections a, b of E.
(2) The $(0, 1)$-component of D is $\bar{\partial}$.

This connection is called the *Chern connection* of (E, h). The curvature form of the Chern connection can be computed as follows: locally trivialize E and identify h with a family of matrices $h = (h_{\mu\nu})$, then

$$\Theta = \bar{\partial}\left(h^{-1}\partial h\right) . \tag{2.2.1}$$

Definition 2.2.3. A metric h on E is *Griffiths positive* if

$$\Theta(\xi \otimes v, \xi \otimes v) > 0$$

for all non-zero decomposable tensor $\xi \otimes v \in TX \otimes E$.

Similarly, a metric h on E is *Nakano positive* if

$$\Theta(\tau, \tau) > 0$$

for all non-zero tensor $\tau \in TX \otimes E$.

Roughly speaking, pluripotential theory is the study of positive metrics on holomorphic vector bundles. For the purpose of this introduction, we will restrict our attention to the case of rank 1, namely, we consider only holomorphic line bundles.

Let L be a holomorphic line bundle on X and h is a Hermitian metric on L. Locally, with respect to a trivialization $L \cong X \times \mathbb{C}$, h can be identified with a smooth function $e^{-\varphi}$, where φ is a real-valued smooth function:

$$h_x(v, v) = |v|^2 e^{-\varphi(x)} . \tag{2.2.2}$$

As a special case of (2.2.1), the curvature of the Chern connection is given by

$$\Theta = -2\pi \mathrm{dd^c} \log h . \tag{2.2.3}$$

We will use the following notations

$$c_1(L, h) = -\mathrm{dd^c} \log h = 2\pi\Theta .$$

We will mostly be interested in the positively curved case, namely, in the case where $c_1(L, h)$ is a non-negative $(1, 1)$-form. In this case, we say h is *plurisubharmonic.*

More generally, one can allow h to have singularities. For a 1-dimensional complex vector space L, the singular metric on L is the assignment $L \to \{0, \infty\}$ which takes value 0 at 0 and takes value ∞ on $L^\times$.

Definition 2.2.4. A *plurisubharmonic metric* (or psh metric for short) on L is an assignment $X \ni x \mapsto h_x$, where h_x is either a positive definite Hermitian metric on L_x or the singular metric, such that if we locally identify h with a function φ taking value in $[-\infty, \infty)$ using (2.2.2), then φ is an upper-semicontinuous function with $\mathrm{dd^c}\varphi \geq 0$ in the sense of currents.

A line bundle L together with a plurisubharmonic metric is called a *Hermitian pseudo-effective line bundle.* A line bundle which admits a plurisubharmonic metric is called a *pseudo-effective line bundle.*

We exclude the constant function $-\infty$ on a connected manifold from upper-semicontinuous functions.

Consider a Hermitian pseudo-effective line bundle (L, ϕ) on X, the curvature current is still well-defined:

$$\Theta := \frac{1}{2\pi}\mathrm{dd^c}\varphi\,,$$

where φ is the local potential of ϕ as in Definition 2.2.4. Observe that Θ is a closed positive $(1,1)$-current representing the cohomology class $(2\pi)^{-1}c_1(L)$. It is more convenient to use

$$c_1(L,h) := 2\pi\Theta$$

instead, as this current represents $c_1(L)$.

Assume that L is a pseudo-effective line bundle on X. Fix a smooth closed $(1,1)$-form $\theta \in c_1(L)$. It is easy to see that there is a (smooth) Hermitian metric h_0 on L with $c_1(L, h_0) = \theta$. Now consider a general plurisubharmonic metric h on L. Then $\varphi := -\log h/h_0$ is a function $X \to [-\infty, \infty)$. It is not hard to see that φ is upper semi-continuous and $\theta + \mathrm{dd^c}\varphi \geq 0$ as currents.

Definition 2.2.5. Let θ be a smooth closed $(1,1)$-form on X. A *θ-plurisubharmonic function* (or θ-psh functions for short) on X is an upper semi-continuous function $\varphi : X \to [-\infty, \infty)$ such that $\theta + \mathrm{dd^c}\varphi \geq 0$ as currents. We write $\mathrm{PSH}(X, \theta)$ for the set of θ-psh functions. Using the $\mathrm{dd^c}$-lemma Theorem 2.1.5, we see that all closed positive $(1,1)$-currents in $c_1(L)$ can be obtained as $\theta + \mathrm{dd^c}\varphi$ for some suitable $\varphi \in \mathrm{PSH}(X, \theta)$.

Observe that in this definition, we do not require that θ come from c_1 of some line bundle.

We sometimes call φ the Kähler potential of $\theta + \mathrm{dd^c}\varphi$ with respect to θ.

Thus, we have shown that any psh metric on L gives rise to a θ-psh function. Conversely, given any θ-psh function φ, we can define a psh metric on L which locally is given by $h_0 e^{-\varphi}$. These operations are inverse to each other, giving

Proposition 2.2.6. *Let L be a pseudo-effective line bundle and h_0 be a smooth Hermitian metric on L. Write $\theta = c_1(L, h_0)$. Then there is a canonical bijection between the set of psh metrics on L and $\mathrm{PSH}(X, \theta)$.*

Moreover, each closed positive $(1,1)$-current in $c_1(L)$ can be written as the first Chern current $c_1(L, h)$ of some psh metric h on L or as $\theta + \mathrm{dd^c}\varphi$ for some $\varphi \in \mathrm{PSH}(X, \theta)$.

In particular, the study of psh metrics on L be be effectively reduced to the study of θ-psh functions. On the other hand, the theory of θ-psh functions include many transcendental examples as well: the cohomology class of θ is not necessarily represented by a line bundle.

2.2.2. Bedford–Taylor theory and non-pluripolar products. In order to study Kähler–Einstein problem using plurisubharmonic function, we need to introduce a product operator called the Monge–Ampère operator. In this section, we recall the classical definition due to Bedford–Taylor and one of its generalizations: the non-pluripolar products.

Let X be a compact Kähler manifold of pure dimension n. Let θ be a closed positive real $(1,1)$-form on X representing a pseudo-effective cohomology class, namely such that $\mathrm{PSH}(X,\theta)$ is not empty. The Monge–Ampère operator sends $\varphi \in \mathrm{PSH}(X,\theta)$ to a measure $(\theta + \mathrm{dd^c}\varphi)^n$. Of course, in general $\theta + \mathrm{dd^c}\varphi$ are just currents, so the product needs to be defined.

When φ is smooth, one can easily make sense of $(\theta + \mathrm{dd^c}\varphi)^n$ because $\theta + \mathrm{dd^c}\varphi$ becomes a differential form and it suffices to interpret this product as the wedge product of forms. In general, it is easier to define the mixed Monge–Ampère operators as well: give $\varphi_1, \dots, \varphi_m \in \mathrm{PSH}(X,\theta)$ $(m \leq n)$, we want to define a closed positive (p,p)-current $\theta_{\varphi_1} \wedge \cdots \wedge \theta_{\varphi_m}$. Here $\theta_{\varphi_1} := \theta + \mathrm{dd^c}\varphi_1$. The theory of Bedford–Taylor handles the case of bounded φ_i's inductively. As the problem is local on X, it will be easier to restrict to a Stein open set U on X. In this case, $\theta = \mathrm{dd^c} f$ for some smooth function f on U, by absorbing f into φ_i, we could reduce to the case where the φ_i's are 0-psh. In this case, we say the φ_i's are *plurisubharmonic* or psh for short. Consider bounded psh functions $\varphi_1, \dots, \varphi_m$ on U. We define

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m := \mathrm{dd^c}\left(\varphi_1 \left(\mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m\right)\right) .$$

It is shown by Bedford–Taylor [BT76] that these products behaves as expected: they are multilinear and symmetric. They are continuous along monotone sequences in each φ_i. We formulate the following case for the purpose of comparison:

THEOREM 2.2.7. *Let $\varphi^j, \varphi \in \mathrm{PSH}(X,\theta)$ $(j \in \mathbb{Z}_{>0})$. Assume that each φ^j and each φ is bounded. Assume that one of the following assumptions holds:*

(1) φ^j is decreasing and converges pointwisely to φ;
(2) φ^j is increasing and converges almost everywhere to φ.

Then $\theta^n_{\varphi^j} \rightharpoonup \theta^n_\varphi$ in the weak sense of currents.

Moreover, the Bedford–Taylor product is local in the pluri-fine topology: the coarsest topology rendering all psh functions on all open subsets continuous. As expected, the Bedford–Taylor theory gives a unique definition to $\theta_{\varphi_1} \wedge \cdots \wedge \theta_{\varphi_m}$, independent of all choices we made.

In particular, the Bedford–Taylor theory allows us to make sense of θ^n_φ when φ is a bounded θ-psh function. It is a positive Radon measure on X.

Unfortunately, in a general pseudo-effective cohomology class, there are usually no bounded θ-psh functions. In order to solve the Monge–Ampère type equations in general, we need to extend the Bedford–Taylor theory to unbounded functions as well.

The non-pluripolar products are defined by Boucksom–Eyssidieux–Guedj–Zeriahi in [BEGZ10]. It is the only extension of the Bedford–Taylor theory that is both local in the plurifine topology and does not put mass on any pluripolar set (a set which is locally contained in the locus $\{\varphi = -\infty\}$ for some psh function φ). Let

$\varphi \in \mathrm{PSH}(X,\theta)$. Locally on a Stein open set U, we absorb θ into φ as before and assume that φ is psh on U. Then the non-pluripolar product θ_φ^n is defined as

$$(\mathrm{dd^c}\varphi)^n := \lim_{C\to\infty} \mathbb{1}_{\{\varphi>-C\}}(\mathrm{dd^c}\max\{\varphi,-C\})^n$$

if the limit exists. Here on the right-hand side, we use the Bedford–Tayloy product and the limit is a weak limit on U.

It turns out that on X, θ_φ^n is always defined and it is always a positive Radon measure on X that puts no mass on any pluripolar set. The non-pluripolar theory is not so well-behaved with respect to monotonely decreasing sequences: the decreasing part of Theorem 2.2.7 fails in the non-pluripolar setting. In this case, one could only expect a lower semi-continuity result [DDL18b]. In other words, non-pluripolar products may lose mass.

Before moving on, let us define the volume of the cohomology class of θ. We set

$$V_\theta := \sup\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \le 0\}\,.$$

It can be shown that $V_\theta \in \mathrm{PSH}(X,\theta)$. In order to understand this definition, we introduce the following ordering on the space $\mathrm{PSH}(X,\theta)$: given $\varphi,\psi \in \mathrm{PSH}(X,\theta)$, we say φ is *more singular* than ψ if $\varphi \le \psi + C$ for some constant C. The thrust is the following theorem of Witt Nyström [Wit19]:

THEOREM 2.2.8. *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)$. Assume that φ is more singular than ψ, then*

$$\int_X \theta_\varphi^n \le \int_X \theta_\psi^n\,.$$

One sees immediately that under this ordering, V_θ is a least singular element in $\mathrm{PSH}(X,\theta)$. In particular,

$$\int_X \theta_\varphi^n \le \int_X \theta_{V_\theta}^n$$

for any $\varphi \in \mathrm{PSH}(X,\theta)$. Based on this, we can simply define the volume of the cohomology class θ as

$$\mathrm{vol}[\theta] := \int_X \theta_{V_\theta}^n\,.$$

It is easy to see that this quantity is intrinsic to $[\theta] \in H^{1,1}(X,\mathbb{R})$, independent of the choice of θ. Moreover, when $[\theta]$ is the first Chern class of some pseudo-effective line bundle L, $\mathrm{vol}[\theta]$ is just the usual Riemann–Roch volume of L. We say the cohomology class $[\theta]$ is *big* if it is pseudo-effective and $\mathrm{vol}[\theta] > 0$.

The usefulness of the non-pluripolar theory is explained in the following theorem proved in [BEGZ10]:

THEOREM 2.2.9. *Assume that $[\theta]$ is a big cohomology class. Let μ be a positive Radon measure on X satisfying:*

(1) μ puts no mass on pluripolar sets.
(2) $\int_X \mu = \mathrm{vol}[\theta]$.

Then there is a unique $\varphi \in \mathrm{PSH}(X,\theta)$ up to addition by a real constant satisfying

$$\theta_\varphi^n = \mu\,.$$

This theorem can be seen as a vast extension of Yau's theorem [Yau78] which handles the case where μ is a smooth volume form. Of course, by allowing more general measures μ, we lose the regularities in the solution φ.

2.3. Kähler–Einstein problems

In this section, we let X be a smooth connected projective Fano manifold of dimension n. Here the Fano condition means that $-K_X$ is ample, where K_X is the canonical line bundle of X.

As mentioned in the introduction, a Kähler–Einstein metric is a Kähler metric satisfying the vacuum Einstein's field equation, which in terms of the associated real $(1,1)$-forms as in (2.1.5), can be reformulated as

$$\mathrm{Ric} = \lambda\omega\,. \tag{2.3.1}$$

Of course, Ric depends on the choice of ω, one may more suggestively write $\mathrm{Ric}\,\omega$ instead. By (2.1.6),

$$\mathrm{Ric}\,\omega = -\mathrm{dd^c}\log\omega^n\,, \tag{2.3.2}$$

where ω^n is understood as follows: take any local holomorphic coordinates $z_1,\dots,z_n$ on X, then ω^n is identified with $\omega^n/\mathrm{d}z_1\wedge\cdots\mathrm{d}z_n$ (the Radon–Nikodym derivative).

Recall that we have assumed that $-K_X$ is ample. But we know that $\mathrm{Ric}\,\omega$ represents $c_1(X)=c_1(-K_X)$. This forces $\lambda>0$ in (2.3.1). By rescaling ω, we may further assume that $\lambda=1$. We finally arrive at the equation:

$$\mathrm{Ric}\,\omega = \omega\,,\quad [\omega]=c_1(X)\,. \tag{2.3.3}$$

Next we reformulate (2.3.3) in terms of pluripotential theory. We fix an arbitrary Kähler form $\omega'\in c_1(X)$. As we have imposed the cohomology constraint $[\omega]=c_1(X)$, we may write $\omega=\omega'+\mathrm{dd^c}\varphi$ for some $\varphi\in\mathrm{PSH}(X,\omega')$ by Proposition 2.2.6. Thus (2.3.3) becomes

$$\mathrm{Ric}(\omega'+\mathrm{dd^c}\varphi)=\omega'+\mathrm{dd^c}\varphi\,. \tag{2.3.4}$$

We need to understand $\mathrm{Ric}(\omega'+\mathrm{dd^c}\varphi)$:

$$\mathrm{Ric}(\omega'+\mathrm{dd^c}\varphi)-\mathrm{Ric}\,\omega' = -\mathrm{dd^c}\log(\omega'+\mathrm{dd^c}\varphi)^n+\mathrm{dd^c}\log\omega'^n = -\mathrm{dd^c}\log\frac{(\omega'+\mathrm{dd^c}\varphi)^n}{\omega'^n}\,.$$

We also introduce the *Ricci potential* $\rho=\rho_{\omega'}$:

$$\omega'-\mathrm{Ric}\,\omega' = -\mathrm{dd^c}\rho\,,\quad \int_X (e^\rho-1)\omega'^n = 0\,.$$

The second condition is just a convenient normalization fixing the additional constant in ρ. Subtracting $\mathrm{Ric}\,\omega'$ from both sides of (2.3.4), we therefore find

$$-\mathrm{dd^c}\log\frac{(\omega'+\mathrm{dd^c}\varphi)^n}{\omega'^n} = -\mathrm{dd^c}\rho+\mathrm{dd^c}\varphi\,.$$

From the elliptic maximum principle, we then conclude that there is a constant $C\in\mathbb{R}$ such that

$$\log\frac{(\omega'+\mathrm{dd^c}\varphi)^n}{\omega'^n} = \rho-\varphi+C\,.$$

We have therefore arrived at our desired equation

$$\omega_\varphi^n = Ce^{\rho-\varphi}\omega^n \tag{2.3.5}$$

Of course, we have renamed ω' as ω and C becomes a different constant. We may further absorb C into ρ and find the general equation:

$$\omega_\varphi^n = fe^{-\varphi}\omega^n\,, \tag{2.3.6}$$

where f is a strictly positive smooth function on X. Although we begin with only smooth φ, it turns out that this equation makes sense for all $\varphi \in \mathrm{PSH}(X,\theta)$. This flexibility turns out to be crucial: the space of smooth θ-psh functions is not *complete* in any reasonable sense. However, by allowing singularities, we can find suitable metric completions and conduct the variational approach to find solutions. We will explain the variational approach in the next section.

Finally let us mention that (2.3.6) also makes sense on a normal varieties, hence giving the notion of Kähler–Einstein metrics on normal varieties.

2.4. The variational approach

In general, finding a solution to the fully non-linear equation (2.3.6) is quite difficult. In this section, we briefly explain the variational approach to solving (2.3.6).

The idea is to find a Lagrangian $D(\varphi)$ of φ so that the associated Euler–Lagrange equation is (2.3.6). Then certain convexity property of $D(\varphi)$, one can easily conclude the existence of a minimizer, namely a solution to (2.3.6).

Let us begin with a finite dimensional example. Consider a smooth convex function f on $\mathbb{R}^n$. When there is a minimizer $x \in \mathbb{R}^n$ of f, then for any geodesic ray ℓ emanating from x, we clearly have

$$\lim_{t\to\infty} \frac{f(\ell_t)}{t} \geq 0\,. \tag{2.4.1}$$

We observe that the condition (2.4.1) is independent of the choice of ℓ_0: it is easy to see that for parallel rays, the left-hand sides of (2.4.1) are equal. Conversely, under some mild assumptions, (2.4.1) also implies the existence of a minimizer.

So we see that the existence of the minimizer of a convex function can be examined along rays. This picture can be generalized to our original problem, which is infinite dimensional in nature. The Lagrangian $D(\varphi)$ in this case is the *Ding functional*. We will review the detailed definition of D in Paper 3.

As explained above, we need to study the behaviour of D along suitable rays. This motivates the study of geodesic rays in the space of Kähler potentials, as we will explain in the next section.

We refer to [BBGZ13; BBEGZ19; BBJ21; BEGZ10] for more details on the variational approach.

2.5. Pluripotential theory and the space of Kähler metrics

The study of Kähler–Einstein problems in the previous section motivates the study of metric completions of the space of smooth Kähler potentials. From an analytic point of view, this leads to the notion of finite energy potentials introduced in [GZ07].

We fix a compact Kähler manifold X of pure dimension n. Assume that ω is a Kähler form on X. Set $V = \int_X \omega^n$. The space of smooth Kähler potentials is denoted by $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \in C^\infty(X), \omega + \mathrm{dd^c}\varphi > 0\}\,.$$

So $\mathcal{H}/\mathbb{R}$ (where $\mathbb{R}$ acts by adding a constant) is the space of Kähler metrics cohomologous to ω.

The space $\mathcal{H}$ has its own geometry. We can naturally regard $\mathcal{H}$ as a infinite dimensional manifold. The tangent space at $\varphi \in \mathcal{H}$ is clearly identified with $C^\infty(X)$:

$$T_\varphi \mathcal{H} = C^\infty(X), \quad \text{for all } \varphi \in \mathcal{H}.$$

Then we can introduce a Riemannian metric on $\mathcal{H}$ as follows: fix $\varphi \in \mathcal{H}$, take $f, g \in C^\infty(X) = T_\varphi \mathcal{H}$, their inner product is then defined as

$$(f, g)_\varphi := \frac{1}{V} \int_X f g \omega_\varphi^n.$$

With respect to this Riemannian metric, the geodesic between $\varphi_0, \varphi_1 \in \mathcal{H}$ is given in the following way. Suppose $(\varphi_t)_{t\in[0,1]}$ is the geodesic, we identify φ_t with a function Φ on $X \times A$, where $A = \{s \in \mathbb{C} : e^{-1} \leq |s| \leq 1\}$. The function Φ on $X \times \{s\}$ is given by $\varphi_{-\log|s|}$. Then the geodesic equation becomes

$$\begin{cases} (\mathrm{dd^c}\Phi)^{n+1} = 0 & \text{on } X \times \mathrm{Int}(A), \\ \Phi|_{X\times\{s\}} = \varphi_1 & \text{when } |s| = 1, \\ \Phi|_{X\times\{s\}} = \varphi_0 & \text{when } |s| = e^{-1}. \end{cases} \tag{2.5.1}$$

The two boundary conditions should be understood as limits. See [Bło12] for the details. The equation (2.5.1) is known as the *homogeneous Monge–Ampère equation*.

It has been studied in detail by Chen [Che00].

Theorem 2.5.1. *Given $\varphi_0, \varphi_1 \in \mathcal{H}$, the equation* (2.5.1) *has a unique solution Φ. Moreover, Φ is a $C^{1,1}$-function.*

This theorem is due to Chen except for the regularity part, proved much later in [CTW17]. In general, Φ is not smooth at the boundary of $X \times A$. It is an open problem if Φ is smooth on $X \times \mathrm{Int}(A)$. Due to the lack of regularity, we usually call $(\varphi_t)_{t\in[0,1]}$ the *weak geodesic* from φ_0 to φ_1.

Having settled the problem of geodesics, we can define the distance between two potentials. Let $\varphi_0, \varphi_1 \in \mathcal{H}$ and consider the weak geodesic $(\varphi_t)_{t\in[0,1]}$ from φ_0 to φ_1. For any $p \geq 1$, we define

$$d_p(\varphi_0, \varphi_1) := \left(\frac{1}{V} \int_X |\dot{\varphi}_0|^p \, \omega_{\varphi_0}^n \right)^{1/p}, \tag{2.5.2}$$

where the dot denotes the derivative in the t-variable. It is shown in [Dar17] that d_p is indeed a metric. Unfortunately $\mathcal{H}$ is not complete with respect to d_p. We introduce the following finite energy spaces to remedy this drawback following [GZ07].

Definition 2.5.2. We say that a potential $\varphi \in \mathrm{PSH}(X, \omega)$ has *full mass* if $\int_X \omega_\varphi^n = V$. The space of full mass potentials is denoted by $\mathcal{E}(X, \omega)$.

Fix $p \geq 1$. We say a full mass potential φ *has finite p-energy* if $\int_X |\varphi|^p \omega_\varphi^n < \infty$. The space of full mass potentials with finite p-energy is denoted by $\mathcal{E}^p(X, \omega)$.

The following theorem is due to Darvas [Dar17].

Theorem 2.5.3. *For any $p \geq 1$, $\mathcal{E}^p(X, \omega)$ admits a natural metric d_p so that the inclusion $\mathcal{H} \hookrightarrow \mathcal{E}^p(X, \omega)$ realizes $\mathcal{E}^p(X, \omega)$ as the metric completion of $\mathcal{H}$ with respect to d_p.*

In all of these spaces $\mathcal{E}^p(X,\omega)$, there is a notion of geodesics, generalizing the weak geodesics. We will not recall the details and simply refer to [Dar17].

The case $p=1$ and $p=2$ are the most interesting ones for our purposes.

In the case $p=1$, there is a simple expression of the metric d_1 on $\mathcal{E}^1(X,\omega)$ in terms of rooftop operators. In the case $p=2$, the space $\mathcal{E}^2(X,\omega)$ satisfies the *CAT(0) property*, a notion of non-positive curvature for metric spaces. These features are crucial to our applications.

So far, we have only been discussing about more or less regular psh functions. The reason is that we are interested in solving some equations and get smooth solutions, so our space of functions should not be too far away from smooth ones. However, this is only one angle of the problem. It turns out that singular psh functions are equally important in the Kähler–Einstein problems, through a construction of Ross–Witt Nyström [RW14]. It is our intention in this thesis to emphasis the importance of this point of view.

This thesis consists of three papers. In the first paper, we establish a generalization of [RW14] and introduce an important class of singular potentials. In this second paper, we will apply the results in the first paper to study K-stability. The third paper belongs to a different lineage: we use the regular psh functions to understand the K-unstable case. In the next few chapters, we will give a more technical introduction to the contents of each paper.

CHAPTER 3

Paper 1

This paper is not directly related to K-stability. This paper sets up a general framework, suitable for various different problems.

We will fix a compact Kähler manifold of pure dimension n in this section. Let ω be a Kähler form on X.

3.1. Legendre transform of geodesic rays

The first result in this paper builds the bridge connecting singular psh functions to regular psh functions. In order to state it, we need a few additional notions.

Definition 3.1.1. A *geodesic ray* ℓ in $\mathcal{E}^1(X,\omega)$ is an assignment $\mathbb{R}_{\geq 0} \ni t \mapsto \ell_t \in \mathcal{E}^1(X,\omega)$ such that for any $0 \leq t_1 \leq t_2$, $\ell|_{[t_1,t_2]}$ is a geodesic in $\mathcal{E}^1(X,\omega)$. We say ℓ is a geodesic ray *emanating* from ℓ_0.

The set of geodesic rays in $\mathcal{E}^1(X,\omega)$ emanating from 0 is denoted by $\mathcal{R}^1(X,\omega)$.

In fact, $\mathcal{R}^1(X,\omega)$ is a metric space: given $\ell,\ell' \in \mathcal{R}^1(X,\omega)$, we define

$$d_1(\ell,\ell') := \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t,\ell'_t)\,.$$

The following result is shown in [DL20].

Theorem 3.1.2. *$(\mathcal{R}^1(X,\omega),d_1)$ is a complete metric space.*

A geodesic ray $\ell \in \mathcal{R}^1(X,\omega)$ is a convex function in the time variable: $\ell_t(x)$ is convex in t for any $x \in X$. Thus, one may consider its Legendre transform:

$$\hat{\ell}_\tau := \inf_{t\geq 0}\left(\ell_t - t\tau\right)\,, \quad \tau \in \mathbb{R}\,.$$

It is a remarkable result that $\hat{\ell}_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$, a result known as *the Kiselman's principle.*

The curve $\hat{\ell}$ turns out to satisfy several obvious conditions, which gives rise to the notion of test curves. We write $\mathrm{PSH}^{\mathrm{Model}}(X,\omega)$ for the set of *model potentials* in $\mathrm{PSH}(X,\omega)$: it is the set of $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\varphi = P[\varphi]$, where $P[\varphi]$ is the supremum of all potentials $\psi \in \mathrm{PSH}(X,\omega)$ satisfying

(1) ψ is less singular than φ;
(2) $\psi \leq 0$;
(3) $\int_X \omega_\psi^n = \int_X \omega_\varphi^n$,

we have $\psi \leq \varphi$. This seemingly technical condition is the translation of the geodesic condition under the Legendre transform, as we will shortly see.

Definition 3.1.3. A *test curve* is a map $\psi = \psi_\bullet : \mathbb{R} \to \mathrm{PSH}^{\mathrm{Model}}(X,\omega) \cup \{-\infty\}$, such that

(1) $\psi_\bullet$ is concave in $\bullet$.

(2) ψ is usc as a function $\mathbb{R} \times X \to [-\infty, \infty)$.
(3) $\lim_{\tau \to -\infty} \psi_\tau = 0$ in L^1.
(4) $\psi_\tau = -\infty$ for τ large enough.

Let $\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$.

The *energy* of a test curve $\psi_\bullet$ is defined as

$$\mathbf{E}(\psi_\bullet) := \tau^+ + \frac{1}{V} \int_{-\infty}^{\tau^+} \left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n \right) \mathrm{d}\tau \,. \tag{3.1.1}$$

A test curve ψ is said to be of *finite energy* if $\mathbf{E}(\psi) > -\infty$. We denote the set of finite energy test curves by $\mathcal{TC}^1(X, \omega)$.

We can now state the first main theorem of Paper 1:

THEOREM 3.1.4. *The Legendre transform is a bijection from $\mathcal{R}^1(X, \omega)$ to $\mathcal{TC}^1(X, \omega)$.*

Various special cases of this theorem are known in the earlier literature, see [RW14] and [DDL18a].

The striking feature of this theorem is that it relates two seemingly unrelated sets: objects in $\mathcal{R}^1(X, \omega)$ are curves of regular (finite energy) ω-psh functions, while objects in $\mathcal{TC}^1(X, \omega)$ are curves of rather singular ω-psh functions. This theorem allows us to effectively translate the techniques dealing with regular and singular potentials back and forth. As we will shortly see, this yields a holomorphic Morse inequality for singular potentials.

The idea of studying geodesic rays in terms of singular potentials dates back to Ross and Witt Nyström [RW14]. The guiding principle that underlies our work is the following: geodesic rays can be considered as the superposition of a bunch of singular potentials, while functionals along geodesic rays can be decomposed as integrals of local functionals of the singularity types of these singular potentials.

3.2. Holomorphic Morse inequalities

Recall the following celebrated theorem of Demailly [Dem85]:

THEOREM 3.2.1. *Consider a holomorphic line bundle L on X together with a smooth Hermitian metric h on L. Then for any $q = 0, \ldots, n$, we have*

$$h^q(X, L^k) \le \frac{k^n}{n!} \int_{X(q,L)} (-1)^q c_1(L, h)^n + o(k^n)$$

as $k \to \infty$, where $X(q, L)$ is the set of $x \in X$ where $c_1(L, h)$ has $n - q$-positive eigenvalues and q-negative eigenvalues.

See [Dem85] for more general statements. Roughly speaking, the holomorphic Morse inequalities are inequalities relating the asymptotic cohomologies of a line bundle L to the curvature integrals of Hermitian metrics on L.

We are also interested in the singular case, allowing singularities in h. For singular h, we require in addition that h is plurisubharmonic. In the thesis of Bonavero [Bon98], Bonavero considered the case where h has algebraic singularities. We recall the meaning:

Definition 3.2.2. We say that $\psi \in \mathrm{PSH}(X,\omega)$ has *algebraic singularities* if there exists $c \in \mathbb{Q}_+$ and around every point of X there exists a Zariski open set $U \ni x$ and $f_j \in \mathcal{O}(U)$ algebraic, such that $\psi|_U - \frac{c}{2}\log\left(\sum_{j=1}^k |f_j|^2\right)$ is smooth.

We say that $\psi \in \mathrm{PSH}(X,\omega)$ has *analytic singularities* if there exists $c \in \mathbb{R}_+$ and around every point of X there exists an open set $U \ni x$ (with respect to the analytic topology) and $f_j \in \mathcal{O}(U)$, such that $\psi|_U - \frac{c}{2}\log\left(\sum_{j=1}^k |f_j|^2\right)$ is locally bounded.

As the singularities are presented, if we want to derive a correct holomorphic Morse inequality, we need to take the singularities into consideration. This leads to Nadel's theory of multiplier ideal sheaves.

Definition 3.2.3. Let φ be a quasi-plurisubharmonic function on a complex manifold Y, that is, there is a closed smooth $(1,1)$-form on Y such that φ is θ-psh. The *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ of φ is the ideal sheaf on Y locally generated by holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable.

The following theorem of Nadel [Nad90] is of fundamental importance:

Theorem 3.2.4. *Let φ be a quasi-plurisubharmonic function on a complex manifold Y, then $\mathcal{I}(\varphi)$ is a coherent ideal sheaf on Y.*

This result is not trivial. It depends on deep techniques involving L^2-estimates. Now we further assume that ω lies in $c_1(L)$ for some ample line bundle L on X. A special case of Bonavero's theorem states:

Theorem 3.2.5. *Assume that $\varphi \in \mathrm{PSH}(X,\omega)$ has algebraic singularities. Then*

$$\lim_{k\to\infty} \frac{h^0(X, L^k \otimes \mathcal{I}(k\varphi))}{k^n} = \frac{1}{n!}\int_X \omega_\varphi^n . \tag{3.2.1}$$

For many purposes, this theorem does not suffice. For example, when X is the toroidal compactification of a mixed Shimura variety and (L,h) is the Lear extension of certain automorphic line bundles equipped with a suitable equivariant metric. In this case, in general, when identifying h with some quasi-plurisubharmonic function φ, φ does not have algebraic singularities in general, as can be shown by simple examples. In this case, the left-hand side of (3.2.1) has natural modular explanations, so it is certainly of interest to extend Theorem 3.2.5 to more general singularities.

In the more general setting, we prove the following result:

Theorem 3.2.6. *Let $\varphi \in \mathrm{PSH}(X,\omega)$, then*

$$\lim_{k\to\infty} \frac{h^0(X, L^k \otimes \mathcal{I}(k\varphi))}{k^n} \geq \frac{1}{n!}\int_X \omega_\varphi^n . \tag{3.2.2}$$

It is a surprising feature that equality can actually fail. This can not be seen from the more familiar situations like analytic singularities or the toric setting. This failure means that we have to treat seriously the pathological potentials. This situation is similar to the fact that not all geodesic rays in $\mathcal{R}^1$ can be recovered from the associated non-Archimedean data. In fact, we will see that these two phenomena are equivalent, given the correspondence Theorem 3.1.4.

It is important to understand the equality case in (3.2.2). For this purpose, we need to introduce the notion of $\mathcal{I}$-model potentials.

Definition 3.2.7. A potential $\varphi \in \mathrm{PSH}(X,\omega)$ is called *$\mathcal{I}$-model* if $\varphi = P[\varphi]_{\mathcal{I}}$, where

$$P[\varphi]_{\mathcal{I}} := \sup{}^{*}\left\{\psi \in \mathrm{PSH}(X,\omega) : \psi \leq 0, \mathcal{I}(k\psi) \supseteq \mathcal{I}(k\varphi) \text{ for all real } k > 0\right\} .$$

We say $\varphi \in \mathrm{PSH}(X,\omega)$ is *$\mathcal{I}$-good* if $\int_X \omega_\varphi^n > 0$ and $P[\varphi]_{\mathcal{I}} = P[\varphi]$.

It can be shown that $P[\bullet]_{\mathcal{I}}$ is a projection operator in $\mathrm{PSH}(X,\omega)$. The reason that we introduce the notion of $\mathcal{I}$-good potential is that this property is intrinsic to the corresponding singular Hermitian metric, in contrast to the properties of being $\mathcal{I}$-model, which depends on the choice of ω.

We deduce that

Theorem 3.2.8. *Let $\varphi \in \mathrm{PSH}(X,\omega)$. Assume that $\int_X \omega_\varphi^n > 0$. Then equality holds in* (3.2.2) *if and only if φ is $\mathcal{I}$-good.*

In the paper, we give several equivalent characterizations of $\mathcal{I}$-good potentials. We will not recall these conditions here.

Let us say a few word about the proof. Observe that using the monotonicity theorem Theorem 2.2.8, Theorem 3.2.8 easily implies the general inequality (3.2.2), so it suffices to handle Theorem 3.2.8. The proof of the inequality

$$\varliminf_{k\to\infty} \frac{h^0(X, L^k \otimes \mathcal{I}(k\varphi))}{k^n} \geq \frac{1}{n!}\int_X \omega_\varphi^n$$

is not super hard. By using a suitable version of Demailly's regularization, one can reduce the general case to the case of algebraic singularities. Hence one can readily apply Bonavero's theorem Theorem 3.2.5.

The other inequality, by contrast, is much harder. Here we rely on a construction generalizing the deformation to the normal cone: given $\varphi \in \mathrm{PSH}(X,\theta)$, we define a test curve

$$\psi_\tau := \begin{cases} 0\,, & \tau \leq -1\,, \\ P[(1+\tau)\varphi]\,, & \tau \in (-1,0]\,, \\ -\infty\,, & \tau > 0\,. \end{cases}$$

Through the correspondence Theorem 3.1.4, $\psi_\bullet$ corresponds to a geodesic ray ℓ. In this case, the desired equality can be seen as a differentiated version of an equality of geodesic rays. The $\mathcal{I}$-goodness of φ can be integrated into a condition of geodesic rays. We prove that this condition is exactly the same as the maximality condition studied in [BBJ21]:

Theorem 3.2.9. *The bijection in Theorem 3.1.4 restricts to a bijection between the set of maximal geodesic rays and the set of $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$ such that each ψ_τ is either $\mathcal{I}$-model of $-\infty$.*

We call a test curve satisfying the conditions in this theorem an *$\mathcal{I}$-model test curve*.

By a careful analysis, we are able to reduce the lower bound in (3.2.2) to a result proved in [BBJ21]: the radial Monge–Ampère energy of a maximal geodesic ray is the same as the Monge–Ampère energy of the associated non-Archimedean potential.

As a corollary of (3.2.9), we deduce that the geodesic rays induced by a filtration is always maximal. We recall the precise meaning.

We use the notation

$$R(X,L) := \bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(X,L^k)\,.$$

Definition 3.2.10. A *filtration* on $R(X,L)$ is a decreasing, left continuous, multiplicative $\mathbb{R}$-filtration $\mathscr{F}^\bullet$ on the ring $R(X,L)$ which is linearly bounded in the sense that there is $C>0$, so that

$$\mathscr{F}^{-k\lambda}H^0(X,L^k) = H^0(X,L^k)\,,\quad \mathscr{F}^{k\lambda}H^0(X,L^k) = 0\,,$$

when $\lambda > C$.

Recall that by [RW14], a filtration induces a test curve in the following manner. Let $\mathscr{F}^\bullet$ be a filtration. For $\tau\in\mathbb{R}$, define

$$\psi_\tau := \sup_{k\in\mathbb{Z}_{>0}}{}^{*}\, k^{-1} \sup{}^{*}\left\{ \log|s|^2_{h^k} : s\in\mathscr{F}^{k\tau}H^0(X,L^k), \sup_X |s|_{h^k}\leq 1 \right\}. \tag{3.2.3}$$

By [DX22, Theorem 3.11], ψ_τ is $\mathscr{I}$-model or $-\infty$ for each $\tau\in\mathbb{R}$. In particular, the geodesic ray induced by a filtration is always maximal. Finally, recall that a finitely generated filtration is induced by a test configuration. So we can regard the space of filtrations as a partial compactification of the space of test configurations.

3.3. Continuity of non-Archimedean envelopes

The third main result of our paper is a proof of the continuity of envelope conjecture for smooth projective varieties over $\mathbb{C}$ endowed with the trivial valuation.

Let X be a smooth projective variety over $\mathbb{C}$. Let X^{an} be the Berkovich analytification of X with respect to the trivial valuation on $\mathbb{C}$. We refer to [Ber12] and [BJ21] for the construction of the analytification. We endow X^{an} with the Berkovich topology. Consider an ample line bundle L on X, through the morphism of locally ringed spaces $X^{\mathrm{an}}\to X$, the line bundle L pulls-back to an invertible sheaf L^{an} on X^{an}.

In this case, Boucksom–Jonsson defined a class of plurisubharmonic functions on L^{an}, denoted by $\mathrm{PSH}(L^{\mathrm{an}})$. Given a continuous metric ϕ on L, one can define $P[\phi]$, the usc regularized supremum of all elements in $\mathrm{PSH}(L^{\mathrm{an}})$ lying below ϕ. The conjecture of continuity of envelopes asserts that $P[\phi]$ is always continuous.

In this case, through the general reduction theory established in [BJ21], it is shown that this conjecture is equivalent to the completeness of $\mathcal{E}^1(L^{\mathrm{an}})$, a metric space analogous to $\mathcal{E}^1$ defined in Definition 2.5.2. By [BBJ21], we know that $\mathcal{E}^1(L^{\mathrm{an}})$ can be identified with the space of maximal geodesic rays. Further, through the bijection in Theorem 3.2.9, one can identify $\mathcal{E}^1(L^{\mathrm{an}})$ with the space of $\mathcal{I}$-model test curves of finite energy.

There is a natural metric d_1 on the space of test curves, corresponding to the d_1-metric through Theorem 3.1.4. In particular, we have reduced the conjecture of continuity of envelopes to the following statement:

Theorem 3.3.1. *The space of $\mathcal{I}$-model test curves of finite energy is complete with respect to d_1. In particular, continuity of envelopes holds for a smooth projective variety over the trivially valued field $\mathbb{C}$.*

In fact, we proved a stronger statement that there is a natural *continuous* contraction from $\mathcal{TC}^1(X,\omega)$ to the space of $\mathcal{I}$-model test curves of finite energy.

CHAPTER 4

Paper 2

In the previous paper, we argued and proved that the study of singular potentials can be viewed as the differentiated version of the study of regular potentials. This paper is a concrete application of this philosophy to the study of K-stability.

Let X be a normal projective variety and L be an ample line bundle on X.

4.1. The notion of K-stability

K-stability is a sort of algebraico-geometric condition imposed on a polarized variety. It has been shown to be the correct condition for the construction of moduli spaces. We recall some basic ideas.

Definition 4.1.1. A *test configuration* of (X, L) consists of a pair $(\mathcal{X}, \mathcal{L})$ consisting of a normal variety $\mathcal{X}$ and a semi-ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$, a morphism $\Pi : \mathcal{X} \to \mathbb{C}$, a $\mathbb{C}^*$-action on $\mathcal{X}, \mathcal{L}$ and an isomorphism $(\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1}) \cong (X, L)$, so that

(1) π is $\mathbb{C}^*$-equivariant.
(2) The fibration π is equivariantly isomorphic to the trivial fibration $(X \times \mathbb{C}^*, p_1^* L)$ through an isomorphism that extends the given one over 1. Here p_1 denotes the projection to the first factor.

A test configuration $(\mathcal{X}, \mathcal{L})$ can be compactified by gluing the trivial fibration over $\mathbb{P}^1 \setminus \{0\}$. We write $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ for the compactified test configuration. We will frequently omit the bars when we talk about compactified test configurations.

Definition 4.1.2 (Donaldson–Futaki invariant)**.** Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (X, L). Take $r \in \mathbb{Z}_{>0}$ so that $\mathcal{L}^r$ is integral. For $k \in \mathbb{Z}_{>0}$, define $w(rk)$ as the weight of the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}^{rk}|_{\mathcal{X}_0})$. By equivariant Riemann–Roch theorem, we can write

$$w(rk) = a(rk)^{n+1} + b(rk)^n + \mathcal{O}(k^{n-1})\,.$$

Define the *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ as

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{n!\bar{S}}{V} a - 2b\frac{n!}{V}\,.$$

Definition 4.1.3. We say (X, L) is

(1) *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ of (X, L);
(2) *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all test configurations $(\mathcal{X}, \mathcal{L})$ and strictly positive if the test configuration is not a product.
(3) *uniformly K-stable* if there is $\epsilon > 0$ such that $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \epsilon \|(\mathcal{X}, \mathcal{L})\|_1$ for all test configurations $(\mathcal{X}, \mathcal{L})$ of (X, L). Here $\|(\mathcal{X}, \mathcal{L})\|_1$ is a suitable norm, see [BHJ17] for the precise definition.

In [BBJ21], the following result is proved:

Theorem 4.1.4. *Assume that X is a Fano variety and $L = -K_X$, then the following are equivalent:*

(1) There is a unique Kähler–Einstein metric on X.
(2) (X, L) is uniformly K-stable.

Although test configurations define good notions of stability, they are not good objects from the metric point of view: they do not form complete metric spaces under any natural metrics. The natural completion of test configurations is given by the non-Archimedean $\mathcal{E}^1(L^{\mathrm{an}})$, considered in the previous paper. We refer the interested readers to [BJ21] for the proof, at least when X is smooth. Through the isomorphism between $\mathcal{E}^1(L^{\mathrm{an}})$ and $\mathcal{I}$-model test curves, one can in principle reduce problems concerning test configurations to problems concerning $\mathcal{I}$-model test curves. In most cases, one can further reduce the problems to problems concerning $\mathcal{I}$-model potentials only. This is the underlying idea of Paper 2.

Next we recall the notion of δ-invariant. We restrict our attention to the case where X is a Fano manifold and $L = -K_X$.

The valuative approach to K-stability is introduced in [Fuj19], [FO18], [BJ20], which we briefly recall. The valuative approach suggest to test the stability through not only test configurations, but also filtrations. As we recalled in the previous paper, filtrations can be regarded as partial compactifications of test configurations. It turns out that a special class of filtrations suffice for detecting the K-stability of Fano varieties. To introduce this class, we just recall that a $\mathbb{Q}$-divisorial valuation on X is a valuation of $\mathbb{C}(X)$ of the following form $v = c\,\mathrm{ord}_E$, where $c \in \mathbb{Q}_{>0}$ and E is a prime divisor lying on a normal birational model $Y \to X$ of X. To each such valuations, one can construct a filtration

$$\mathscr{F}_v^\lambda H^0(X, L^k) = \begin{cases} H^0(X, kL - \lambda c E)\,, & \lambda \geq 0\,, \\ H^0(X, L^k)\,, & \lambda < 0\,. \end{cases}$$

It is shown in [Fuj19] that these valuations suffice for the purpose of test K-stability. More precisely, we can define the following δ-invariant

$$\delta(L) := \inf_E \frac{A_X(E)}{S_L(E)}\,, \tag{4.1.1}$$

where E runs over the set of prime divisors over X, $A_X(E)$ denotes the log discrepancy of E and $S_L(E)$ is the expected order of vanishing of L along E.

Theorem 4.1.5. *X is K-semistable (resp. uniformly K-stable) if and only if $\delta(-K_X) \geq 1$ (resp. $\delta(-K_X) > 1$).*

One can also express $\delta(L)$ in terms of all $\mathcal{E}^1(L^{\mathrm{an}})$ instead of only these special valuations ([BJ18, Section 2.9, Theorem 5.16]):

$$\delta(L) = \inf_{\mu \in \mathcal{M}(X^{\mathrm{an}})} \frac{\mathrm{Ent}^{\mathrm{an}}(\mu)}{E^*(\mu)}\,, \tag{4.1.2}$$

where $\mathcal{M}(X^{\mathrm{an}})$ denotes the set of Radon measures on X^{an} with total mass V,

$$E^*(\mu) := \sup_{\psi \in \mathcal{E}^1(L^{\mathrm{an}})} \left(E(\psi) - \int_{X^{\mathrm{an}}} \psi \,\mathrm{d}\mu \right)\,.$$

Here E denotes the non-Archimedean Monge–Ampère energy and $\mathrm{Ent}^{\mathrm{an}}$ is the non-Archimedean entropy functional: the integral against the log discrepancy functional A_X.

Given the correspondence between $\mathcal{E}^1(L^{\mathrm{an}})$ and the space of $\mathcal{I}$-model test curves, it is natural to wonder if $\delta(L)$ can be expressed purely in terms of quasi-psh functions. It turns out that this is indeed possible, but before getting to that point, we need to introduce a new point of view towards $\mathcal{I}$-good potentials.

4.2. b-divisors associated with $\mathcal{I}$-good potentials

The idea of studying psh singularities using b-divisors come from [BFJ08]. Here we prove a precise version in the global setting. In this section, X will be a general compact Kähler manifold and ω is a Kähler form on X.

A b-divisor is a family of divisors or divisor classes on all birational models $Y \to X$ of X, compatible under pushforward. We will adopt the following definition, which is the most convenient one:

Definition 4.2.1. By a *Weil b-divisor* on X, we mean an element in

$$\mathrm{bWeil}(X) := \varprojlim_{Y} \mathrm{Weil}(Y)\,,$$

where Y runs over all (smooth) birational models of X and $\mathrm{Weil}(Y)$ is the set of numerical classes of $\mathbb{R}$-divisors on Y.

By a *Cartier b-divisor* on X, we mean an element in

$$\mathrm{bCart}(X) := \varinjlim_{Y} \mathrm{Weil}(Y)\,,$$

where Y runs over all (smooth) birational models of X.

Both the limit and the colimit are taken in the category of topological vector spaces.

There is a natural continuous injection $\mathrm{bCart}(X) \hookrightarrow \mathrm{bWeil}(X)$. Given a b-divisor $\mathbb{D}$, we write $\mathbb{D}_Y$ for the projection of $\mathbb{D}$ to the component corresponding to $Y \to X$.

Given $\psi \in \mathrm{PSH}(X, \omega)$, we want to construct a b-divisor out of the singularities of ψ.

Definition 4.2.2. Let $\psi \in \mathrm{PSH}(X, \omega)$. We define the *singularity divisor* of ψ as a Weil b-divisor $\mathrm{div}_{\mathfrak{X}}\,\psi \in \mathrm{bWeil}(X)$:

$$(\mathrm{div}_{\mathfrak{X}}\,\psi)_Y = \mathrm{div}_Y\,\psi\,.$$

Here we have abused the notation by writing $\mathrm{div}_Y\,\psi$ for the numerical class of the corresponding divisor.

We write $\mathbb{D}(L, \psi)_Y := \pi^* L - (\mathrm{div}_{\mathfrak{X}}\,\psi)_Y$.

It turns out that $\mathbb{D}(L, \psi)$ is a nef b-divisor in the sense of [DF20]. In [DF20], an intersection theory of nef b-divisors is established. Using their intersection theory, we can prove the following result:

Theorem 4.2.3. *Let $\psi \in \mathrm{PSH}(X, \omega)$ be a model potential with positive mass. Then ψ is $\mathcal{I}$-good iff*

$$\int_X \omega_\psi^n = (\mathbb{D}(L)^n)\,.$$

So we can say that $\mathcal{I}$-good potentials are exactly the potentials that can be reconstructed from their generic Lelong numbers. This point of view is convenient in dealing with $\mathcal{I}$-good potentials, as b-divisors are more algebraic objects.

4.3. Pluripotential-theoretic delta invariant

We fix a projective manifold X of pure dimension n and an ample line bundle L on X.

Now we can proceed to the main results of this paper. We want to translate (4.1.2) into an expression involving only test curves.

The denominator part is not too hard. We refer to the paper itself for the definition of the I, J-functionals. It is well-known that $E^*(\mu)$ can be expressed as the slope at infinity of the $(I-J)$-functional along geodesic rays, the following theorem suffices:

THEOREM 4.3.1. *Let $\ell \in \mathcal{R}^1(X,\omega)$ be a maximal geodesic ray. Then*

$$\lim_{t\to\infty} \frac{1}{t}(I-J)(\ell_t) = n\mathbf{E}^\omega(\psi_\bullet) - n\mathbf{E}(\psi_\bullet) = \frac{n}{V}\int_{-\infty}^{\infty}\left(\int_X \omega\wedge\omega_{\psi_\tau}^{n-1} - \int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau\,,$$

where $\psi_\bullet$ is the Legendre transform of ℓ.

The proof is based on a technique discovered in [RW14].

The numerator in (4.1.2), namely, the entropy part is much harder. It turns out that this requires the essential use of the Berkovich space and the b-divisor technique.

We introduce the notion of entropy of an $\mathcal{I}$-model test curve:

$$\mathrm{Ent}(\psi_\bullet) := \int_{-\infty}^{\infty} \mathrm{Ent}([\psi_\tau])\,\mathrm{d}\tau\,. \tag{4.3.1}$$

Here

$$\mathrm{Ent}([\psi]) := \frac{n}{V}\lim_{\overleftarrow{Y}}\left(\langle \pi^*L - \mathrm{div}_Y\,\psi\rangle^{n-1}\cdot(K_{Y/X} + \mathrm{red}\,\mathrm{div}_Y\,\psi)\right) \in [0,\infty]\,,$$

where $\pi : Y \to X$ runs over all birational models on X. Here the product $\langle\bullet\rangle$ is the movable intersection in the sense of [BFJ09], [Bou02]. We formally set $\mathrm{Ent}([-\infty]) = 0$.

We prove

THEOREM 4.3.2. *Let $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$ be an $\mathcal{I}$-model test curve. Let ℓ be the geodesic ray defined by $\psi_\bullet$, then*

$$\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \mathrm{Ent}(\psi_\bullet)\,.$$

Equality holds if $\psi_\bullet$ is induced by a test configuration.

Having translated both the numerator and the denominator of (4.1.2), we get a complicated expression of the δ-invariant as the infimum over all $\mathcal{I}$-model test curves of finite energy. This expression is not so useful as the latter space is usually very large and hard to control. Fortunately, when $\delta < (n+1)/n$, we can show that it suffices to consider a special type of test curves $\psi_\bullet^+$, associated with ω-psh functions ψ. We will not recall the very technical definition here. Minimizing (4.1.2) on these special test curves gives the following definition:

$$\delta_{\mathrm{pp}} := \inf_{[\psi]} \frac{\int_{-\infty}^{\infty}\mathrm{Ent}([\psi_\tau^+])\,\mathrm{d}\tau}{nV^{-1}\int_{-\infty}^{\infty}\left(\int_X \omega\wedge\omega_{\psi_\tau^+}^{n-1} - \int_X \omega_{\psi_\tau^+}^n\right)\mathrm{d}\tau}\,, \tag{4.3.2}$$

where $[\psi]$ runs over the set of singularity types of quasi-psh functions with some non-zero Lelong numbers on X.

Now we can formulate our main theorem:

THEOREM 4.3.3. *Let (X, L) be a polarized manifold. Then $\delta_{\mathrm{pp}} \geq \delta$. Further, if X is Fano and $L = -K_X$ and $\delta < \frac{n+1}{n}$, then $\delta = \delta_{\mathrm{pp}}$.*

The proof of this theorem relies on the recent progress from the algebraic side [BLZ19] and [LXZ22].

As a corollary,

Corollary 4.3.4. *Assume that X is Fano and $L = -K_X$. Then*

(1) $\delta_{\mathrm{pp}} \geq 1$ iff X is K-semistable.
(2) $\delta_{\mathrm{pp}} > 1$ iff X is uniformly K-stable.

So we conclude that it is possible to detect K-stability using only quasi-psh functions. In particular, taking Theorem 4.1.4 into consideration, we find that if $\delta_{\mathrm{pp}} > 1$, there is a unique Kähler–Einstein metric on X.

The author wants to emphasize that the key innovation of this result is that it shows that singular potentials are useful in concrete geometric problems, even if initially one is only interested in regular potentials.

CHAPTER 5

Paper 3

This paper has a different flavor compared to the previous papers. This paper concerns only regular potentials.

Here we are interested in the case where Kähler–Einstein metrics do not exist. We want to identify the cause of the failure of K-stability.

Let us begin with a more familiar example. Consider a smooth projective curve S. The slope $\mu(E)$ of a vector bundle E on S is the degree of E divided by the rank of E. A vector bundle E on S is stable (resp. semi-stable) if $\mu(E) > \mu(F)$ (resp. $\mu(E) > \mu(F)$) for all proper subbundles F of E. By Donaldson–Uhlenbeck–Yau theorem, the stability of a vector bundle is equivalent to the existence of a Hermitian–Einstein metric on E.

When E is not semi-stable, it is always possible to construct a optimal destabilizing subbundle: a proper subbundle E_1 of E with the maximal slope. Continuing this procedure for E/E_1 etc, we arrive at the so-called *Harder–Narasimhan filtration*: it is a filtration of $E = E_m \supseteq E_{m-1} \supseteq \cdots \supseteq E_0 = 0$ such that for each $i = 1, \dots, m$, E_i/E_{i-1} is non-trivial, semi-stable with slope λ_i and $\lambda_1 > \lambda_2 > \cdots > \lambda_m$.

We find that in this problem, when stability is violated, there is always an optimal destabilizing object. We will prove that the same holds in the setting of K-stability.

5.1. Energy functionals

We will need the notions of Calabi energy.

Definition 5.1.1. Define the Calabi energy $Ca : \mathcal{H} \to \mathbb{R}$ as

$$Ca(\varphi) = \left(\frac{1}{V}\int_X (S(\varphi) - \bar{S})^2\, \omega_\varphi^n\right)^{1/2}, \tag{5.1.1}$$

where $S(\varphi)$ is the scalar curvature of ω_φ and

$$\bar{S} = \frac{1}{V}\int_X S_\varphi\, \omega_\varphi^n$$

is independent of the choice of $\varphi \in \mathcal{H}$.

This functional can be naturally extended to a lsc function on $\mathcal{E}^1(X, \omega)$. Note that in most literature, Calabi energy is defined as $(Ca)^2$. Clearly, the minimizers of Ca is closely related to canonical metrics.

Donaldson ([Don05]) proved the following inequality:

$$\inf_{\varphi \in \mathcal{H}} Ca(\varphi) \geq \max\left(\sup_{(\mathcal{X},\mathcal{L})} \frac{-\mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_{L^2}}, 0\right), \tag{5.1.2}$$

where $(\mathcal{X}, \mathcal{L})$ takes value in the set of non-trivial normal test configurations of (X, L) with reduced central fibre, DF is the Donaldson–Futaki invariant of a test configuration. For the definition of the L^2 norm of a test configuration, see [His16].

Donaldson conjectured in the same paper that equality should hold. Our main theorem is a partial confirmation of Donaldson's theorem, after extending the space of test configurations to the space of geodesic rays. As we explained before, this extension corresponds to metric completion.

THEOREM 5.1.2. *Let X be a compact Kähler manifold. Let ω be a Kähler form on X. We have*

$$\inf_{\phi\in\mathcal{E}^2(X,\omega)} Ca(\phi) = \max_{\ell\in\mathcal{R}^2\setminus\{0\}} \frac{-\mathbf{M}(\ell)}{\|\ell\|}\,. \tag{5.1.3}$$

Moreover, in the K-unstable case, the maximizer ℓ is unique up to rescaling.

Here $\mathbf{M}(\ell)$ is the slope at infinity of the Mabuchi functional along ℓ. A theorem proved in this paper and by Li [Li20] shows that $\mathbf{M}(\ell)$ restricts to the Donaldson–Futaki invariant on the space of test configurations.

The quantity $\|\ell\|$ is a certain norm of the geodesic ray. The readers will find detailed definition in the paper.

The interesting thing here is the uniqueness of the maximizer. It means that in the unstable case, there is always a unique geodesic ray causing the destabilization. Hence, we have found an analogue of the Harder–Narasimhan filtration in the setting of K-stability.

By a theorem of Chi Li [Li20], the maximizer ℓ is always maximal in the sense of [BBJ21]. Hence it can always be approximated by test configurations. In order to answer Donaldson's original conjecture, it remains to show that one can take good enough approximating test configurations so that $\mathbf{M}$ is continuous along this approximation.

In order to prove Theorem 5.1.2, we need to study the properties of the gradient flow of the Mabuchi functional. In fact, we will prove a more general result that works for any gradient flows.

5.2. Weak Calabi flow

The key to the proof of Theorem 5.1.2 is to study the Calabi flow. Recall that by definition, it is just the gradient flow of the Mabuchi functional M. It one writes down the explicit PDE, it becomes a fourth order parabolic equation. The long time existence of the solution is still an open problem. Let us assume anyway that the long time solutions exist and see how to find the maximizer in Theorem 5.1.2.

5.2.1. A finite dimensional example. Let us begin with a much simpler example. Let $G:\mathbb{R}^n\to\mathbb{R}$ be a smooth convex function. We may consider the gradient flow of G, namely

$$\dot{x}_t = -\nabla G(x_t)\,.$$

It is well-known that for any initial value $x_0\in\mathbb{R}^n$, there is always a smooth global solution.

Following the general theory of Hadamard spaces, we define the boundary $\mathbb{R}^n(\infty)$ as the set of equivalence classes of unit speed rays (in the usual sense) in $\mathbb{R}^n$, two rays are considered as equivalent if they are parallel in the sense that they are related by a translation. There is an obvious identification $\mathbb{R}^n(\infty)$ with the unit sphere S^{n-1}.

We can define a radial version of G, namely $\mathbf{G} : \mathbb{R}^n(\infty) \to (-\infty, \infty]$ as follows: let $[\ell] \in \mathbb{R}^n(\infty)$, take $x \in \mathbb{R}^n$, take a representative of ℓ of $[\ell]$ that emanates from x, define

$$\mathbf{G}([\ell]) = \lim_{t\to\infty} \frac{G(\ell_t)}{t}\,.$$

It is easy to show that $\mathbf{G}$ is independent of the choice of x. See the proof of [DL20, Lemma 4.10].

Fix a solution to the flow, say x_t. Set $G(t) = G(x_t)$.

Then we claim that

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} = \max\left\{0, \sup_{[\ell]\in\mathbb{R}^n(\infty)} -\mathbf{G}([\ell])\right\}\,. \tag{5.2.1}$$

Let ℓ be a unit speed ray emanating from $x \in \mathbb{R}^n$. Then by a simple general fact we will prove in the paper,

$$-\mathbf{G}([\ell]) \leq= \left(-\dot{G}(0)\right)^{1/2}\,.$$

Since x is arbitrary, we conclude

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} \geq \max\left\{0, \sup_{[\ell]\in\mathbb{R}^n(\infty)} -\mathbf{G}([\ell])\right\}\,.$$

For the inverse direction, we may assume that

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} > 0\,. \tag{5.2.2}$$

In this case, $|x_0 - x_t| \to \infty$ as $t \to \infty$. Otherwise, let y be a limit point of x_t, it is easy to see that $G(y)$ obtains the minimial value of G. It is a general fact of the gradient flow that the left-hand side of (5.2.2) is independent of the choice of x_0, so we find a contradiction by considering the flow starting at y.

By inspection, we have the following control for $0 \leq t < s$,

$$\left(-\dot{G}(s)\right)^{1/2} \leq \frac{G(t) - G(s)}{|x_t - x_s|} \leq \left(-\dot{G}(t)\right)^{1/2}\,.$$

Now we claim that the sup on right-hand side of (5.2.1) is indeed obtained by a special direction ℓ^∞. The construction is as follows: connect x_0 and x_s by a unit speed segment $\ell^s : [0, |x_0 - x_s|] \to \mathbb{R}^n$. Fix $T > 0$, it easy to see that the images of the maps $\ell^s|_{[0,T]}$ all lie in a fixed compact set when $s \geq T$, so we may take $s_i \to \infty$ so that the corresponding ℓ^{s_i} tends to another segment uniformly. Combining this with a Cantor diagonal argument, we arrive at a subsequence $s_i \to \infty$, so that the corresponding ℓ^{s_i} converge to a ray ℓ^∞ in uniformly on each compact time interval. We then calculate for $0 < A < s$ that

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} \leq \left(-\dot{G}(s)\right)^{1/2} \leq \frac{G(0) - G(s)}{|x_0 - x_s|} \leq \frac{G(0) - G(\ell^s_A)}{A}\,.$$

Let $s \to \infty$ along the subsequence s_i used to define ℓ^∞, we find

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} \leq \frac{G(0) - G(\ell^\infty_A)}{A}\,.$$

Let $A \to \infty$, we conclude

$$\left(-\lim_{t\to\infty} \dot{G}(t)\right)^{1/2} \leq -\mathbf{G}([\ell^\infty])\,.$$

Hence equality in (5.2.1) indeed holds.

5.2.2. Hadamard spaces and the weak Calabi flow. In the case of Calabi flows, we want to have a space of potentials having similar properties as $\mathbb{R}^n$ in the previous example. It turns out that the correct notion is that of a complete CAT(0) metric space, also known as an *Hadamard space*. Roughly speaking, the CAT(0) condition is a metric space version notion corresponding to the intuition of non-positive curvature. The following theorem of Darvas [Dar17] is the key:

THEOREM 5.2.1. *The metric space $(\mathcal{E}^2(X,\omega), d_2)$ is an Hadamard space.*

Hence, assuming the long time existence of solutions to the Calabi flow, we can perform similar constructions as in the previous example and prove Theorem 5.1.2. Unfortunately, as the long time existence of solutions to the Calabi flow is still open, we need some innovations. In fact, for people working on metric geometry, the solution is already obvious. On a general Hadamard space, there is the notion of gradient flows. In fact, given any lower semi-continuous function M on any Hadamard space, one can always construct the *weak gradient flow* of M. This weak gradient flow exists for all time by construction. The book [Bač14] contains a self-contained introduction to this subject. In the paper, we will also recall the basic facts about the weak gradient flows.

In the case of the Mabuchi functional M, it is known that M extends to a convex lower-semicontinuous function on $\mathcal{E}^2$ [BDL17]. In particular, it admits a weak gradient flow, which we call the *weak Calabi flow*. The weak Calabi flow coincides with the Calabi flow as long as the latter exists. Now using the weak Calabi flow, it is not hard to construct a maximizer in Theorem 5.1.2 as in the previous example.

We remark that the existence of the maximizer and an equality like (5.1.3) hold for general weak gradient flows under mild assumption. However, the uniqueness of the maximizer in Theorem 5.1.2 is a special feature of the weak Calabi flow. It relies on the non-trivial fact that $\mathcal{R}^2$ is also an Hadamard space.

In the paper, we will also establish another version of (5.1.3), replacing the Mabuchi functional by the Ding functional.

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Part 2

Appended papers

PAPER 1

The closures of test configurations and algebraic singularity types

Tamás Darvas and Mingchen Xia

Advances in Mathematics (2022)

Paper 1. The closures of test configurations and algebraic singularity types

Tamás Darvas and Mingchen Xia

Abstract

Given a Kähler manifold X with an ample line bundle L, we consider the metric space of finite energy geodesic rays associated to the Chern class $c_1(L)$. We characterize rays that can be approximated by ample test configurations. At the same time, we also characterize the closure of algebraic singularity types among all singularity types of quasi-plurisubharmonic functions, pointing out the very close relationship between these two seemingly unrelated problems.

By Bonavero's holomorphic Morse inequalities, the arithmetic and non-pluripolar volumes of algebraic singularity types coincide. We show that in general the arithmetic volume dominates the non-pluripolar one, and equality holds exactly on the closure of algebraic singularity types. Analogously, we give an estimate for the Monge–Ampère energy of a general finite energy ray in terms of the arithmetic volumes along its Legendre transform. Equality holds exactly for rays approximable by test configurations.

Various other cohomological and potential theoretic characterizations are given in both settings. As applications, we give a concrete formula for the non-Archimedean Monge–Ampère energy in terms of asymptotic expansion, and show the continuity of the projection map from L^1 rays to non-Archimedean rays.

1. Introduction

We fix notation and terminology for the entire paper. We consider X a compact Kähler manifold of dimension n with an ample line bundle L. We pick a positive smooth Hermitian metric h on L and let $\omega := \frac{\mathrm{i}}{2\pi}\Theta(h) > 0$ be the background Kähler form of X, where $\Theta(h)$ denotes the curvature form of the Hermitian metric h.

By an ω-plurisubharmonic (ω-psh) function $u \in \mathrm{PSH}(X,\omega)$, we understand a quasi-plurisubharmonic function on X, such that $\omega + \mathrm{dd^c}u := \omega + \frac{\mathrm{i}}{2\pi}\partial\bar\partial u \geq 0$, where $\mathrm{d} = \partial + \bar\partial$ and $\mathrm{d^c} = \frac{\mathrm{i}}{4\pi}(-\partial + \bar\partial)$. We will often use ω_u shorthand for $\omega + \mathrm{dd^c}u$.

Two potentials $u, v \in \mathrm{PSH}(X,\omega)$ have the same singularity type if $u - C \leq v \leq u + C$ for some $C > 0$, inducing an equivalence relation $u \simeq v$, with equivalence classes $[u] = [v] \in \mathcal{S}$, called *singularity types*. It turns out that this latter space has a natural pseudometric structure $(\mathcal{S}, d_{\mathcal{S}})$, introduced in [DDL21] (recalled in Section 2.1).

For $u \in \mathrm{PSH}(X,\omega)$, by $H^0(X, L^k \otimes \mathcal{I}(ku)) \subseteq H^0(X, L^k)$ we denote the space of sections s satisfying the L^2 integrability condition $\int_X h^k(s,s)e^{-ku}\,\omega^n < \infty$. We also denote

$$h^0(X, L^k \otimes \mathcal{I}(ku)) := \dim_{\mathbb{C}} H^0(X, L^k \otimes \mathcal{I}(ku))\,.$$

A major theme in Kähler geometry is to relate algebraic objects to analytic ones. In this work we address two such problems. First, we give cohomological and potential theoretic characterizations for L^1 geodesic rays in the space of Kähler metrics that lie in the closure of test configurations. Second, we characterize the closure of algebraic singularity types in $\mathcal{S}$, with respect to the complete pseudometric $d_{\mathcal{S}}$. Potentials with algebraic singularity types are among the nicest ones one could hope for in practice (see Definition 1.3).

According to our work, it is most natural to treat both of these seemingly different problems at the same time, with our final answers also paralleling each other on many different levels. We now present one facet of this, with slight abuse of precision.

Given $\varphi \in \mathrm{PSH}(X,\omega)$ with algebraic singularity type $[\varphi] \in \mathcal{S}$, the arithmetic and non-pluripolar volumes coincide, according to the singular holomorphic Morse inequalities of Bonavero [Bon98] (see Theorem 2.26):

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \frac{1}{n!}\int_X \omega_\varphi^n. \tag{1.1}$$

All volumes in this work, in particular the one on the right hand side above, are interpreted in the non-pluripolar sense of Guedj–Zeriahi [GZ07], [BEGZ10] (see (2.1) below). Both the left and right hand sides only depend on the singularity type $[\varphi]$ ([Wit19, Theorem 1.1]).

In Theorem 1.4 we show that in (1.1) the limit on the left exists for *all* $\varphi \in \mathrm{PSH}(X,\omega)$ and dominates the right hand side in general. Moreover, $[\varphi] \in \mathcal{S}$ lies in the $d_{\mathcal{S}}$-closure of algebraic singularity types if and only if Bonavero's identity (1.1) holds for φ.

Paralleling the above, in [DL20] the authors introduced a metric d_1^c on the space of L^1 geodesic rays $\mathcal{R}^1$ (recalled in Section 2.1), making $(\mathcal{R}^1, d_1^c)$ a complete geodesic metric space. As is well known, to any ample test configuration one can associate a geodesic ray, a construction going back to Phong–Sturm [PS07]. We show that a geodesic ray $\{r_t\}_t \in \mathcal{R}^1$ is in the d_1^c-completion of the space of rays induced by test configurations if and only if $\hat{r}_\tau \in \mathrm{PSH}(X,\omega)$ satisfies Bonavero's identity (1.1) for all $\tau \in \mathbb{R}$, where $\hat{r}_\tau := \inf_{t>0}(r_t - t\tau)$ is the Legendre transform of the ray $\{r_t\}_t$. In particular, the ray $\{r_t\}_t \in \mathcal{R}^1$ is approximable by test configurations if and only if the singularity types $[\hat{r}_\tau] \in \mathcal{S}$ are approximable by algebraic singularity types! This parallels previous characterizations of approximable rays in the non-Archimedean context from [BBJ21].

We refer to Theorem 1.1 and Theorem 1.4 for additional cohomological and potential theoretic characterizations complementing each other in both settings.

In addition, in Theorem 1.2 we express the non-Archimedean Monge–Ampère energy of a ray as the first order term in the expansion of the non-Archimedean Donaldson's $\mathcal{L}$-functional, a result paralleling [BB10, Theorem A].

The closure of the rays induced by test configurations. Let d_1 be the metric on the space of smooth Kähler potentials $\mathcal{H}_\omega := \{u \in C^\infty(X) : \omega + dd^c u > 0\}$ associated with the L^1 Finsler metric [Dar15]:

$$\|\psi\|_1 := \frac{1}{V}\int_X |\psi|\,\omega_u^n\,, \quad u \in \mathcal{H}_\omega \text{ and } \psi \in T_u\mathcal{H}_\omega\,,$$

where $V = \int_X \omega^n$ is the total volume. By $\mathcal{E}^1$ we denote the d_1-completion of $\mathcal{H}_\omega$, that can be identified with a finite energy space studied by Guedj–Zeriahi [GZ07]. Then

$(\mathcal{E}^1, d_1)$ is a complete geodesic metric space. Its geodesics are limits of solutions to a degenerate complex Monge–Ampère equation [Dar15, Theorem 2].

By $\mathcal{R}^1$ we denote the space of d_1-geodesic rays $\{r_t\}_{t\geq 0}$ in $\mathcal{E}^1$ emanating from $r_0 = 0 \in \mathcal{H}_\omega$. As shown in [DL20, Theorem 1.3, Theorem 1.4], $\mathcal{R}^1$ has a a natural chordal metric d_1^c (see (2.3)), compatible with d_1, making $(\mathcal{R}^1, d_1^c)$ a complete geodesic metric space.

Of special importance is the subspace $\mathcal{T} \subseteq \mathcal{R}^1$, composed of the rays induced by ample test configurations [PS07], [PS10], [CT08]. Similarly, one can consider the bigger subspace $\mathcal{F} \subseteq \mathcal{R}^1$, the space of rays induced by filtrations [RW14]. In this work it is advantageous to think of ample test configurations as special kind of filtrations on the ring of sections of (X, L), and we refer to Section 2.5 for details on this. Understanding the closures $\overline{\mathcal{T}}$ and $\overline{\mathcal{F}}$ is one of the main problems we take up in this work.

The well-known Monge–Ampère energy $\mathrm{I}(\cdot) : \mathcal{E}^1 \to \mathbb{R}$, whose Euler–Lagrange equation is simply the Monge–Ampère equation (see (2.2)), is linear along d_1-geodesics. One can simply consider its radial version $\mathrm{I}\{\cdot\} : \mathcal{R}^1 \to \mathbb{R}$ defined by the following slope

$$\mathrm{I}\{r_t\} = \mathrm{I}(r_1) = \lim_{t\to\infty} \frac{\mathrm{I}(r_t)}{t}\,. \tag{1.2}$$

Another quantity that is linear along a ray $\{r_t\}_t$ is the supremum of potentials (Lemma 3.2). For simplicity, we will often assume that $\sup_X r_t = 0$ for $t \geq 0$, and such rays will be called *sup-normalized*. Note that all our results hold in an appropriate form without normalization, even when these are not specifically mentioned.

First, in Theorem 3.7, we develop ideas from [RW14] further, and show a precise formula for the radial Monge–Ampère energy of sup-normalized rays $\{r_t\}_t \in \mathcal{R}^1$:

$$\mathrm{I}\{r_t\} = \int_{-\infty}^{0} \left(\frac{\int_X \omega^n_{\hat{r}_\tau}}{\int_X \omega^n} - 1 \right) \mathrm{d}\tau\,, \quad t > 0\,, \tag{1.3}$$

where $\hat{r}_\tau \in \mathrm{PSH}(X, \omega)$ for $\tau < 0$ is the Legendre transform of the ray:

$$\hat{r}_\tau := \inf_{t\geq 0}(r_t - t\tau)\,.$$

We attempt to approximate the non-pluripolar volumes in the integrand of (1.3) using arithmetic volumes (in the spirit of Bonavero's identity (1.1)). In our first main result we show that this fails in general, and it works exactly for rays in the d_1^c-closure of $\mathcal{T}$:

Theorem 1.1 (Theorem 4.7, Corollary 5.6). *For $\{r_t\}_t \in \mathcal{R}^1$ with $\sup_X r_t = 0$ we have*

$$\lim_{k\to\infty} \int_{-\infty}^{0} \left(\frac{h^0(X, L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, L^k)} - 1 \right) \mathrm{d}\tau \geq \mathrm{I}\{r_t\}\,. \tag{1.4}$$

Moreover, equality holds in (1.4) if and only if the following equivalent conditions hold:

(i) $\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\hat{r}_\tau)) = \frac{1}{n!}\int_X \omega^n_{\hat{r}_\tau}$, $\tau < 0$.

(ii) $P[\hat{r}_\tau]_{\mathcal{I}} = \hat{r}_\tau$, $\tau \leq 0$.

(iii) $\{r_t\}_t \in \overline{\mathcal{T}}$.

(iv) $\{r_t\}_t \in \overline{\mathcal{F}}$.

A ray satisfying condition (iii) is called a *maximal* geodesic ray in the work [BBJ21], giving non-Archimedean characterizations of $\overline{\mathcal{T}}$ recalled below. Here we do not use this terminology, to avoid potential confusion with other notions of maximality.

In condition (ii) the operator $P[u]_{\mathcal{I}}$ is the $\mathcal{I}$-envelope of the singularity type $[u] \in \mathcal{S}$, used explicitly and implicitly in [KS20] and [Cao14] respectively:

$$P[u]_{\mathcal{I}} = \sup \{ \chi \in \mathrm{PSH}(X,\omega) : \chi \leq 0 \text{ and } \mathcal{I}(c\chi) = \mathcal{I}(cu) \text{ for all } c > 0 \} \,, \tag{1.5}$$

where $\mathcal{I}(u)$ is simply the multiplier ideal sheaf of a quasi-psh function u on X.

As part of showing (1.4), we will argue that the limit on the left hand side exists. We think of (i) as the *cohomological characterization* of the closure of $\mathcal{T}$. On the other hand, we think of (ii) as the *potential theoretic* characterization.

The equivalence between (iii) and (iv) indicates that uniform notions of K-stability with respect to test configurations and filtrations are very likely the same (c.f. [CC18, Question 1.12]). To fully confirm this, one needs to show that elements of $\mathcal{F}$ can be approximated by $\mathcal{T}$ while also preserving the slope of the K-energy functional, as predicted by [Li20, Conjecture 1.5]. Of course, due to the examples of [ACGT08], one might still expect that relative K-stability with respect to $\mathcal{F}$ and $\mathcal{T}$ are different notions.

It is not hard to see that for many of the rays constructed in [Dar17] there is strict inequality in (1.4), implying $\overline{\mathcal{T}} \subsetneq \mathcal{R}^1$. This strict containment was noticed in [BBJ21, Remark 5.9] using non-Archimedean methods. However, as a result of condition (i) above and our Theorem 3.7 (iv) (that allows for flexible construction of L^1 rays using test curves) the containment $\overline{\mathcal{T}} \subsetneq \mathcal{R}^1$ is seen to be nowhere d_1^c-dense (c.f. [CC18, Question 1.10]).

Finally, let us put our Theorem 1.1 in historical context, and discuss the possible connection with the general version of the Yau–Tian–Donaldson conjecture, seeking to characterize existence of constant scalar curvature Kähler (cscK) metrics cohomolgous to ω in terms of algebro-geometric properties of the bundle (X, L). Despite the difficulties arising due to infinite dimensionality, and the underlying fourth order PDE, by now we have a comprehensive understanding on the analytic side (see [BDL17], [CC21a], [CC21b], [CC18], [DL20]), allowing to characterize existence of cscK metrics in terms of uniform geodesic stability along $C^{1,\bar{1}}$-rays of the space of Kähler metrics, yielding the essentially optimal version of what was conjectured by Donaldson [Don99].

Similarly, with the development of the non-Archimedean toolbox, we have a very good understanding of the algebraic side as well (see [BBJ21], [BHJ17], [BJ18], [BHJ19], [Der16]), allowing not only to embed test configurations into $\mathcal{R}^1$ (along with their invariants), but to also keeping track of algebraic invariants using non-Archimedean metrics, an intermediate notion lying between the algebraic and analytic data.

The remaining step in the variational program for the Yau–Tian–Donaldson conjecture is to understand what L^1 rays are approximable by ample test configurations, while also preserving the slope of the radial K-energy in the limit. This is the connection point with our characterization theorem above, though here we completely ignored the behavior of the K-energy in the approximation process.

During the final stages of writing up our work we learned of the preprint of C. Li [Li20], who proved that L^1 rays with bounded radial K-energy are in $\overline{\mathcal{T}}$. Though not a characterization of $\overline{\mathcal{T}}$, this result is more closely lined up with the variational program, and it is an intriguing prospect to examine the relationship between our results and the ones in [Li20].

Non-Archimedean interpretation. The non-Archimedean approach to singularities in pluripotential theory developed in [BFJ08], [BBJ21] will play a crucial role in our discussion (especially in the form of valuative criteria), and we mention here how Theorem 1.1 can be interpreted in this context.

In this approach $\mathcal{T}$ can be identified with $\mathcal{H}^{\mathrm{an}}$, the space of Fubini–Study metrics on the Berkovich analytification $(X^{\mathrm{an}}, L^{\mathrm{an}})$ with respect to the trivial valuation on $\mathbb{C}$. On the other hand, the closure $\overline{\mathcal{T}}$ can naturally be identified with the space of finite energy metrics on $(X^{\mathrm{an}}, L^{\mathrm{an}})$, leading to a characterization of $\overline{\mathcal{T}} = \mathcal{E}^{1,\mathrm{an}}$ in the non-Archimedean context [BBJ21], in addition to the ones given in Theorem 1.1.

Given an arbitrary ray $\{r_t\}_t \in \mathcal{R}^1$, in [BBJ21] the authors introduce a natural projection

$$\Pi : \mathcal{R}^1 \to \overline{\mathcal{T}} = \mathcal{E}^{1,\mathrm{an}} \subset \mathcal{R}^1 ,$$

satisfying $r_t \leq \Pi(r)_t$ and one can think of $\{\Pi(r)_t\}_t$ as the closest ray to $\{r_t\}_t$ that is approximable by test configurations (see Section 3.2 for more details). Using Π, one can conveniently introduce the non-Archimedean Monge–Ampère energy as follows:

$$\mathrm{I}^{\mathrm{an}}\{r_t\} := \mathrm{I}\{\Pi(r)_t\} .$$

The original definition is given by means of the non-Archimedean Monge–Ampère measures introduced in [Cha06], [CLD12] that only depend on the non-Archimedean data r^{an} (see [BJ18] and references therein). Here we show that Π is d_1^c-continuous, and give the following expansion interpretation for I^{an}:

Theorem 1.2 (Theorem 3.18, Corollary 4.9). *The map $\Pi : \mathcal{R}^1 \to \overline{\mathcal{T}}$ is d_1^c-continuous. Moreover, for any sup-normalized $\{r_t\}_t \in \mathcal{R}^1$ we have*

$$\mathrm{I}^{\mathrm{an}}\{r_t\} = \mathrm{I}\{\Pi(r)_t\} = \lim_{k\to\infty} \frac{n!}{Vk^n} \int_{-\infty}^{0} \Big(h^0(X, L^k \otimes \mathcal{I}(k\hat{r}_\tau)) - h^0(X, L^k)\Big)\, \mathrm{d}\tau . \tag{1.6}$$

The integral in (1.6) can be interpreted as $\mathcal{L}_k^{\mathrm{an}}\{r_t\}$, the non-Archimedean analogue of Donaldson's $\mathcal{L}$-functional (see (4.1) and Proposition 4.4). Theorem 1.2 says that the leading order term in the expansion of $\mathcal{L}_k^{\mathrm{an}}\{r_t\}$ is given by the non-Archimedean Monge–Ampère energy. This is the non-Archimedean analogue of [BF14, Theorem 3.5], where based on [Don05] and [BB10], it is proved that Donaldson's $\mathcal{L}$-functional from [Don05] admits an expansion whose leading order term is given by the usual Monge–Ampère energy of $\mathcal{E}^1$.

Similar flavour results in the non-Archimedean setting were obtained in [BE21, Theorem A] and [BGJKM20, Theorem A] under different assumptions on the ground field and for continuous metrics. It would be interesting to see if one could extend their results to finite energy metrics in case of trivially valued base fields of characteristic 0, using our Theorem 1.2.

The closure of the space of algebraic singularity types. Finally we discuss approximation in the space of singularity types $\mathcal{S}$. We start with precisely defining algebraic/analytic singularity types.

Definition 1.3. We say that $[\psi]$ is an *algebraic singularity type* (notation: $[\psi] \in \mathcal{Z} \subseteq \mathcal{S}$), if there exists $c \in \mathbb{Q}_+$ and around every point of X there exists a Zariski open set $U \ni x$ and $f_j \in \mathcal{O}(U)$ algebraic, such that $\psi|_U - \frac{c}{2}\log\left(\sum_{j=1}^k |f_j|^2\right)$ is smooth.

We say that $[\psi]$ is an *analytic singularity type* (notation: $[\psi] \in \mathcal{A} \subseteq \mathcal{S}$), if there exists $c \in \mathbb{R}_+$ and around every point of X there exists an open set $U \ni x$ (with respect to the analytic topology) and $f_j \in \mathcal{O}(U)$, such that $\psi|_U - \frac{c}{2}\log\left(\sum_{j=1}^k |f_j|^2\right)$ is locally bounded.

Many different conventions are in place regarding the definition of analytic/algebraic singularity types in the literature (see [Dem12, Definition 1.10], [MM07, Definition 2.3.9] or [RN17, (4)]). Out of all possible definitions, our choice of $\mathcal{Z}$ is the smallest family one can consider, and $\mathcal{A}$ is perhaps the biggest. As we will show below, for purposes of approximation, using $\mathcal{A}$ or $\mathcal{Z}$ does not make a difference.

The study of partial Bergman kernels in connection with approximation in Kähler geometry has a long history, having both potential theoretic and spectral theoretic applications (see [DPS01], [Bou02], [DP04], [Ras13], [Cao14], [Dem15], [RN17], [RS17], [ZZ19] in a very fast expanding literature). Even in case of smooth potentials $u \in \mathcal{H}_\omega$ one can use approximation by Bergman kernels for various geometric purposes, going back to questions of Yau [Yau87], early work of Tian [Tia88; Tia90], and many others [Cat99], [Zel98], [Lu00].

Since many of the important invariants involved only depend on the singularity type of the potentials, in [DDL21] the authors introduced a pseudometric $d_\mathcal{S}$ on $\mathcal{S}$, to study the effectiveness of the approximation methods in the literature. $d_\mathcal{S}$-convergence implies convergence of all mixed complex Monge–Ampere masses [DDL21, Lemma 3.7], together with semicontinuity of multiplier ideal sheaves [DDL21, Theorem 1.3]. Being also complete in the presence of positive mass [DDL21, Theorem 1.1], $d_\mathcal{S}$ seems well suited for this purpose.

We refer to Section 2.1 for a detailed discussion on the $d_\mathcal{S}$ metric, as well as the paper [DDL21]. We only mention the following double inequality of [DDL21, Lemma 3.4 and Proposition 3.5], giving intuition about what $d_\mathcal{S}$-convergence means:

$$d_\mathcal{S}([u],[v]) \leq \sum_{j=0}^n \left(2\int_X \omega^j \wedge \omega^{n-j}_{\max(u,v)} - \int_X \omega^j \wedge \omega_u^{n-j} - \int_X \omega^j \wedge \omega_v^{n-j}\right) \leq C d_\mathcal{S}([u],[v])\,,$$

where $C > 1$ only depends on $\dim X$.

The pseudometric $d_\mathcal{S}$ is slightly degenerate, however $d_\mathcal{S}([\phi],[\psi]) = 0$ implies that the singularities of ϕ, ψ are essentially indistinguishable (for example all Lelong numbers, multiplier ideal sheaves, and mixed complex Monge–Ampère masses need to agree), so in many ways this is a blessing in disguise.

In our last main result we prove the inequality between arithmetic and non-pluripolar volumes for general ω-psh functions, complementing (1.1) (c.f. [Bou02, Theorem 1.2], [Cao14, Proposition 1.1]). Moreover, in the presence of positive mass, we show that Bonavero's formula holds exactly for the $d_\mathcal{S}$-closure $\overline{\mathcal{Z}}$.

We also give a potential theoretic characterization for elements of $\overline{\mathcal{Z}}$ in terms of the coincidence locus of $P[\cdot]_{\mathcal{I}}$ (defined in (1.5)) and its analytic counterpart $P[\cdot]$:

$$P[u] := \operatorname{usc}\left(\sup\{\, v \in \operatorname{PSH}(X,\omega) : [v] = [u] \text{ and } v \leq 0 \,\}\right) .$$

THEOREM 1.4 (Theorem 5.5). *For $u \in \operatorname{PSH}(X,\omega)$ we have*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(ku)) = \frac{1}{n!}\int_X \omega^n_{P[u]_{\mathcal{I}}} \geq \frac{1}{n!}\int_X \omega_u^n . \tag{1.7}$$

Assume that $\int_X \omega_u^n > 0$. Then equaility holds in (1.7) if and only if one the following equivalent conditions hold:

(i) $\lim_{k\to\infty} \frac{1}{k^n} h^0(X, L^k \otimes \mathcal{I}(ku)) = \frac{1}{n!}\int_X \omega_u^n$.

(ii) $P[u] = P[u]_{\mathcal{I}}$.

(iii) $[u] \in \overline{\mathcal{Z}}$.

(iv) $[u] \in \overline{\mathcal{A}}$.

It is part of showing (1.7) that the limit on the left hand side exists. The equality part of (1.7) can be interpreted as singular version of the Riemann–Roch theorem. There are many known examples of potentials $u \in \operatorname{PSH}(X,\omega)$ for which the inequality (1.7) is strict. One can even construct potentials u that have zero Lelong numbers but don't have full mass, i.e., $u \notin \mathcal{E}$ [GZ07]. In particular, $\overline{\mathcal{Z}} \subsetneq \mathcal{S}$. What is more, by taking convex combinations of this u with a potential of $\overline{\mathcal{Z}}$ (and checking failure of condition (i) above), one can see that the containment $\overline{\mathcal{Z}} \subsetneq \mathcal{S}$ is nowhere $d_{\mathcal{S}}$-dense.

That the equivalences of Theorem 1.4 are only proved in the presence of positive mass is perhaps not surprising, in light of [DDL21, Theorem 1.1, Section 4.3], where it was shown that $d_{\mathcal{S}}$ is complete *only* in the presence of such condition. Still, it remains to be seen if this condition is essential in Theorem 1.4.

With different motivation, Rashkovskii studied the approximability of local isolated psh singularties using isolated analytic singularities in [Ras13]. It is an interesting prospect to find the local analog of the $d_{\mathcal{S}}$ metric, and to relate our findings to the ones in [Ras13].

As we will see, in all of our main theorems one can allow an additional twisting Hermitian line bundle (T, h_T) as well (see Theorem 4.7, Theorem 4.8, Theorem 5.5 and Corollary 4.9).

Organization. In Section 1.2 we recall previous results and adapt them to our context. In Section 1.3 we extend the Ross–Witt Nyström correspondence to finite energy L^1 geodesic rays. In Section 1.4 we prove Theorem 1.1 and Theorem 1.2. In Section 1.5 we prove Theorem 1.4.

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2. Preliminaries

2.1. The metric space of L^1 geodesic rays and singularity types.

The L^1 metric on $\mathcal{H}_\omega$ and its completion. We recall the basics of the L^1 metric structure of $\mathcal{H}_\omega$, introduced in [Dar15]. For a survey we refer to [Dar19, Chapter 3], and perhaps [DR17, Section 4] is a convenient quick summary. For historical context, we refer to [Rub20].

The d_1 metric on $\mathcal{H}_\omega$ is simply the path length pseudometric associated with the following L^1 Finsler metric:

$$\|\psi\|_1 := \frac{1}{V}\int_X |\psi|\,\omega_u^n\,, \quad u \in \mathcal{H}_\omega \text{ and } \psi \in T_u\mathcal{H}_\omega\,,$$

where $V = \int_X \omega^n$ is the total volume. One then shows that d_1 is non-degenerate, making $(\mathcal{H}_\omega, d_1)$ a *bona fide* metric space [Dar15, Theorem 1].

When trying to find the d_1-completion of $\mathcal{H}_\omega$, one encounters the space $\mathcal{E}^1 \subseteq \mathrm{PSH}(X,\omega)$ that is defined in the following manner. One first defines the space of full mass potentials $\mathcal{E} \subseteq \mathrm{PSH}(X,\omega)$. Potentials in this space are characterized by the property $\int_X \omega_u^n = \int_X \omega^n$. Here ω_u^n is the following limit of measures

$$\omega_u^n := \lim_{k\to\infty} \mathbb{1}_{\{u>-k\}}\,\omega^n_{\max(u,-k)}\,, \tag{2.1}$$

where $\omega^n_{\max(u,-k)}$ can be made sense of using Bedford–Taylor theory, since $\max(u,-k)$ is bounded [BT76]. For a general ω-psh potential u we have $\int_X \omega_u^n \in [0, \int_X \omega^n]$, with all values taken up. For more on this we refer to the original papers [GZ07] and [BEGZ10] (for a minimalist survey see [Dar19, Chapter 2]).

Then, $\mathcal{E}^1 \subseteq \mathcal{E}$ is the class of full mass potentials satisfying $\int_X |u|\,\omega_u^n < \infty$. By [Dar15, Theorem 2], one can extend the metric d_1 to $\mathcal{E}^1$. In addition, $(\mathcal{E}^1, d_1)$ is a complete geodesic metric space whose geodesics are decreasing limits of $C^{1,1}$-solutions to a degenerate complex Monge–Ampère equation ([Che00; CTW18; Dar15]). Unfortunately, such limits are not the only d_1-geodesics connecting points of $\mathcal{E}^1$ (see the comments after [Dar15, Theorem 4.17]). However, when talking about d_1-geodesics, we will only consider this distinguished class of length minimizing segments.

We recall that the definition of the Monge–Ampère energy $\mathrm{I} : \mathcal{E}^1 \to \mathbb{R}$ (sometimes denoted Aubin–Yau, or Aubin–Mabuchi energy):

$$\mathrm{I}(u) = \frac{1}{V(n+1)}\sum_{j=0}^{n}\int_X u\,\omega^j \wedge \omega_u^{n-j}\,. \tag{2.2}$$

Using the Monge–Ampère energy one can give the following potential theoretic description of d_1 [Dar15, Corollary 4.14]:

$$d_1(u,v) = \mathrm{I}(u) + \mathrm{I}(v) - 2\mathrm{I}(P(u,v))\,, \quad u,v \in \mathcal{E}^1\,,$$

where $P(u,v) \in \mathcal{E}^1$ is the following *rooftop* envelope:

$$P(u,v) = \sup\{\, h \in \mathrm{PSH}(X,\omega) : h \le u \text{ and } h \le v\,\}\,.$$

To understand d_1-convergence from a purely analytical point of view, the following double estimate is often very useful [Dar15, Theorem 3]:

$$\frac{1}{C}d_1(u,v) \le \int_X |u-v|\,\omega_u^n + \int_X |u-v|\,\omega_v^n \le C d_1(u,v)\,,$$

where C is a constant only dependent on $n = \dim X$.

The complete metric space of L^1 rays. Building on the previous paragraph, we recall the basics of the L^1 metric structure of $\mathcal{R}^1$, the space of d_1-geodesic rays in $\mathcal{E}^1$ emanating from $0 \in \mathcal{H}_\omega$, explored in detail in [DL20].

To fix notation, a d_1-geodesic ray $[0,\infty) \ni t \mapsto u_t \in \mathcal{E}^1$ with $u_0 = 0$ will simply be denoted $\{u_t\}_t \in \mathcal{R}^1$. The chordal metric d_1^c on $\mathcal{R}^1$ is introduced in the following manner:

$$d_1^c(\{u_t\}_t, \{v_t\}_t) = \lim_{t\to\infty} \frac{d_1(u_t, v_t)}{t} \,. \tag{2.3}$$

Since $t \mapsto d_1(u_t, v_t)$ is convex [BDL17, Proposition 5.1], the limit on the right hand side exists, and one can show that d_1^c is non-degenerate, satisfies the triangle inequality, moreover $(\mathcal{R}^1, d_1^c)$ is a complete geodesic metric space [DL20, Theorem 1.3, Theorem 1.4].

The subspace $\mathcal{R}^\infty \subseteq \mathcal{R}^1$ is the space of *bounded* geodesic rays $\{u_t\}_t$, satisfying the property $u_t \in L^\infty \cap \mathcal{E}^1,\ t \geq 0$. Such rays allow for an important approximation property [DL20, Theorem 1.4] that will be used in this work, as well as its proof:

Theorem 2.1. *For any $\{u_t\}_t \in \mathcal{R}^1$ there exists $\{u_t^j\}_t \in \mathcal{R}^\infty$ such that $u_t^j \searrow u_t$, for $t \geq 0$ and $d_1^c(\{u_t\}_t, \{u_t^j\}_t) \to 0$.*

The pseudo-metric space of singularity types. We recall the basics of the pseudo-metric structure on $\mathcal{S}$, the space of singularity types, first explored in [DDL21]. First one needs to construct a map from $\mathcal{S}$ to $\mathcal{R}^\infty \subseteq \mathcal{R}^1$, using ideas going back to [Dar17]. Starting with $[u] \in \mathcal{S}$, one constructs d_1-geodesic segments $[0,l] \ni t \mapsto s(u)_t^l \in \mathcal{E}^1 \cap L^\infty$ connecting $s(u)_0^l = 0$ and $s(u)_l^l = \max(u, -l)$. Moreover, using the maximum principle one can show that $\{s(u)_t^l\}_{l \geq t}$ is an l-increasing sequence converging to $r[u]_t \in \mathcal{E}^1 \cap L^\infty$, yielding a geodesic ray $\{r[u]_t\}_t \in \mathcal{R}^\infty$ [Dar17, Lemma 4.2].

Using the map $[u] \mapsto \{r[u]_t\}_t$ we define the following pseudometric [DDL21, Section 3]:

$$d_{\mathcal{S}}([u],[v]) = d_1^c(\{r[u]_t\}, \{r[v]_t\}) \,, \quad [u],[v] \in \mathcal{S} \,.$$

Due to non-degeneracy of d_1^c, one immediately sees that $d_{\mathcal{S}}([u],[v]) = 0$ if and only if $r[u]_t = r[v]_t$ for $t \geq 0$. As shown in [DDL21, Theorem 3.3], in the presence of non-vanishing mass ($\int_X \omega_u^n > 0$ and $\int_X \omega_v^n > 0$), this condition is equivalent to $P[u] = P[v]$, where $P[\chi]$ is the envelope of the singularity type $[\chi]$, first considered in [RW14] in the Kähler context:

$$P[\chi] = \lim_{C\to\infty} P(0, \chi + C) = \operatorname{usc}\left(\sup\{\psi \in \mathrm{PSH}(X,\omega), \psi \leq 0 \text{ and } [\psi] = [\chi]\}\right) .$$

As pointed out in [DDL21], if $P[u] = P[v]$ holds, then the singularity types $[u]$ and $[v]$ are indistinguishable using Lelong numbers, multiplier ideal sheaves and mixed masses.

Unfortunately the pseudomertic space $(\mathcal{S}, d_{\mathcal{S}})$ is incomplete [DDL21, Section 4.2]. However when restricting to the subspaces $\mathcal{S}_\delta := \{\, [u] \in \mathcal{S} : \int_X \omega_u \geq \delta > 0 \,\}$, one obtains complete metric spaces $(\mathcal{S}_\delta, d_{\mathcal{S}})$ [DDL21, Theorem 4.9] that are able to govern the variation of singularity types in equations of complex Monge–Ampère type [DDL21, Theorem 1.4].

Lastly, we mention the following double inequality that often comes handy when discussing $d_{\mathcal{S}}$-convergence in practice [DDL21, Lemma 3.4 and Proposition 3.5]:

$$d_{\mathcal{S}}([u],[v]) \leq \sum_{j=0}^{n}\Big(2\int_X \omega^j\wedge\omega^{n-j}_{\max(u,v)} - \int_X \omega^j\wedge\omega_u^{n-j} - \int_X \omega^j\wedge\omega_v^{n-j}\Big) \leq C d_{\mathcal{S}}([u],[v])\,,$$

where $C > 1$ only depends on n. That the expression in the middle is non-negative and only dependent on $[u]$ and $[v]$ is a consequence of [Wit19, Theorem 1.1].

2.2. Exponents and filtrations of a family of Hermitian metrics. In this section we relate the log-slope of the volume of a one dimensional family of Hermitian metrics with the associated filtration. In many ways we simply tailor the arguments of [Ber20] to our needs, and for more thorough treatment of related results we refer to [BE21, Part 1].

Let V be a finite dimensional complex vector space of dimension N. By $\mathrm{Herm}(V)$ we denote the set of positive Hermitian inner products on V. Throughout this section, $H_s \in \mathrm{Herm}(V)$ $(s \geq 0)$ will denote a continuous family of Hermitian inner products, simply referred to as $s \mapsto H_s$.

We denote by V^* the dual vector space of V. Recall that given any $H \in \mathrm{Herm}(V)$, it naturally induces a dual inner product $H^* \in \mathrm{Herm}(V^*)$.

Definition 2.2. Let $I \subseteq \mathbb{R}$ be an interval. We say that a family $H_s \in \mathrm{Herm}(V)$ $(s \in I)$ is *negative* if its trivial complexification $z \mapsto H_{\mathrm{Re}\, z}$ is a Griffiths negative vector bundle on V with base $\{\mathrm{Re}\, z \in I\}$. This is equivalent to $s \mapsto \log H_s(v,v)$ being convex on I for any $v \in V\setminus\{0\}$ ([Dem12, Section VII.6]). Analogously, $s \mapsto H_s$ is *positive* if its dual bundle $(H_s^*)_{s\in I}$ is negative.

Let $I = [0,\infty)$. We do not assume that $s \mapsto H_s$ is positive or negative for the moment. The *exponent* $\lambda_H : V \to [-\infty,\infty]$ of $s \mapsto H_s$ is defined by

$$\lambda_H(v) := \overline{\lim_{s\to\infty}}\, \frac{1}{s}\log H_s(v,v)\,, \quad v \in V\,. \tag{2.4}$$

Note that $\lambda_H(0) = -\infty$. Moreover, one sees that $\lambda_H(cv) = \lambda_H(v)$ for any $c \in \mathbb{C}^*$, and $\lambda_H(u+v) \leq \max(\lambda_H(u), \lambda_H(v))$. Thus for any $c \in [-\infty,\infty]$, the set $\{\lambda_H \leq c\} \subseteq V$ is a sub- vector space. Hence λ_H takes up only a finite number of values. If ∞ is not one of them, then λ_H is the exponent of the non-Archimedean pseudometric e^{λ_H}, motivating our terminology.

The above properties of the exponent λ_H also allow to introduce the associated *filtration* of $s \mapsto H_s$:

$$\mathcal{F}^H_\lambda := \{v \in V : \lambda_H(v) \leq \lambda\}\,. \tag{2.5}$$

Notice that $\mathcal{F}^H_\lambda$ defines an increasing right-continuous filtration of V by linear subspaces. This filtration is bounded from above (in the sense that $\mathcal{F}^H_\lambda = V$ for some $\lambda \in \mathbb{R}$) if and only if $\lambda_H < \infty$. We call a number $\lambda \in \mathbb{R}$ a *jumping number* of the filtration $\mathcal{F}^H$ if $\mathcal{F}^H_\lambda \neq \mathcal{F}^H_{\lambda-\epsilon}$ for any $\epsilon > 0$.

Given $U_0, U_1 \in \mathrm{Herm}(V)$, one can diagonalize U_1 with respect to U_0 to find eigenvalues $e^{\lambda_1}, \dots, e^{\lambda_N}$ counting multiplicity. Then one can introduce the following metric:

$$d_1^V(U_0,U_1) := \frac{1}{\dim V}\sum_{j=1}^{N}|\lambda_j|\,. \tag{2.6}$$

This metric, along with its L^p-counterparts, was studied extensively in [DLR20], where it was shown that d_1^V quantizes d_1 in the appropriate context.

In particular, (in the appropriate diagonalizing basis) the curve $[0,1] \ni t \mapsto U_t := \operatorname{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_N}) \in \operatorname{Herm}(V)$ provides a d_1^V-geodesic joining U_0 and U_1 ([DLR20, Theorem 1.1], [BE21, Theorem 3.7]). There are other d_1^V-geodesics joining U_0, U_1, but we will only consider the above type of length minimizing curves.

We emphasize the following formula, pointing out that the dualization map $U \mapsto U^*$ between $\operatorname{Herm}(V)$ and $\operatorname{Herm}(V^*)$ is an isometry:

$$d_1^V(U_0, U_1) = d_1^{V^*}(U_0^*, U_1^*)\,, \quad U_0, U_1 \in \operatorname{Herm}(V)\,. \tag{2.7}$$

This can be verified by picking an appropriate diagonalizing basis of V.

In studying the growth of the volume of the unit ball with respect to H_s as $s \to \infty$, we start with the following lemma that one can justify simply by diagonalizing:

Lemma 2.3. *Suppose $s \mapsto H_s$ is a d_1^V-geodesic ray and let $\lambda_1 \leq \ldots \leq \lambda_m$ be the jumping numbers of $\mathcal{F}_\lambda^H$. Then*

$$\lim_{s\to\infty} \frac{1}{s} \log\left(\frac{\det H_s}{\det H_0}\right) = \sum_{j=0}^{m-1} \lambda_{j+1}(\dim \mathcal{F}^H_{\lambda_{j+1}} - \dim \mathcal{F}^H_{\lambda_j}) = \int_{-\infty}^{\infty} \lambda \, d(\dim \mathcal{F}_\lambda A)\,, \tag{2.8}$$

where the integral on the right is interpreted in the Stieltjes sense. Moreover, $\dim \mathcal{F}^H_{\lambda_0} = \dim \bigcap_{\lambda\in\mathbb{R}} \mathcal{F}^H_\lambda = 0$ in the middle sum, by convention.

By $\det H$ we mean the determinant of a matrix representative of the sesquilinear form $H \in \operatorname{Herm}(V)$ with respect to a fixed basis, making $\det H_s / \det H_0$ in (2.8) well-defined. Note that our convention is different from that in [BE21] by a square.

Using Hadamard's inequality, for $s \mapsto H_s$ only satisfying $\lambda_H < \infty$, one can show that in general the left hand side is dominated by the right hand side in (2.8).

As we will see, equality holds in (2.8) when $s \mapsto H_s$ is only positive, satisfying a mild decay condition. Before we prove this, we will construct a geodesic ray $s \mapsto \tilde{H}_s$ asymptotic to any $s \mapsto H_s$, closely following [Ber20, Proposition 2.2].

Lemma 2.4. *Assume that $[0,\infty) \ni s \mapsto H_s$ is positive and $\lambda_{H^*} < \infty$. Then there exists a d_1^V-geodesic ray $s \mapsto \tilde{H}_s$ such that*

(i) $H_0 = \tilde{H}_0$.

(ii) $H_s \geq \tilde{H}_s$.

(iii) $\lambda_H = \lambda_{\tilde{H}}$. *In particular,* $\lambda_H < \infty$.

(iv) $\displaystyle \lim_{s\to\infty} \frac{1}{s} \log\left(\frac{\det H_s}{\det H_0}\right) = \lim_{s\to\infty} \frac{1}{s} \log\left(\frac{\det \tilde{H}_s}{\det \tilde{H}_0}\right).$

Recall the following comparison principle that will be used multiple times in the argument below: if $[a,b] \ni s \mapsto U_s, W_s \in \operatorname{Herm}(V)$ are such that $s \mapsto U_s$ is positive and $s \mapsto W_s$ is a geodesic with $W_a \leq U_a$ and $W_b \leq U_b$ then $W_s \leq U_s$ for $s \in [a,b]$ [BK12, Lemma 8.11]. Note that $s \mapsto \log\det W_s$ is linear and also $\log\det W_s \leq \log\det U_s$. Varying the endpoints a, b we obtain that $s \mapsto \log\det H_s$ is concave, whenever $s \mapsto H_s$ is positive. As a result, the limit on the left of (iv) exists.

Proof. First we interpret the condition $\lambda_{H^*} < \infty$. Since $s \mapsto H_s^*$ is negative, $s \mapsto \log H_s^*(v,v)$ is convex for any $v \in V^* \setminus \{0\}$, hence $H_s^*(v,v) \leq e^{s\lambda_{H^*}(v)} H_0^*(v,v)$, for $s \geq 0$. Dualizing, we arrive at

$$H_s \geq e^{-s\lambda_{H^*}} H_0\,, \quad s \geq 0\,. \tag{2.9}$$

Now we construct $s \mapsto \tilde{H}_s$. For each $t \geq 0$, we define $[0,\infty) \ni s \mapsto H^t_s \in \mathrm{Herm}(A)$ as follows: for $[0,t] \ni s \mapsto H^t_s$ is the geodesic connecting H_0 and H_t and $H^t_s = H_s$ for $s > t$.

By the comparison principle, we get that H^t_s is t-decreasing for any $s \geq 0$ (in fact $s \mapsto H^t_s$ is positive for any t, but this will not be needed). Due to (2.9) we can take the decreasing t-limit to obtain

$$\tilde{H}_s(v,v) := \lim_{t\to\infty} H^t_s(v,v)\,, \quad v \in V\,.$$

It is immediate that $s \mapsto H_s$ is a d_1^V-geodesic ray satisfying (i) and (ii) .

Recall that $s \mapsto \log\det H_s$ is concave (due to positivity) and of course $s \mapsto \log\det \tilde{H}_s$ is linear (since $s \mapsto \tilde{H}_s$ is a geodesic). Using this, due to the construction of $s \mapsto \tilde{H}_s$, one immediately sees that $\lim_{s\to\infty} s^{-1}(\log\det H_s - \log\det \tilde{H}_s) = 0$, proving (iv).

Since $H_s \geq \tilde{H}_s$, comparing with (2.6) we arrive at

$$\lim_{s\to\infty} \frac{d_1^V(\tilde{H}_s, H_s)}{s} = \lim_{s\to\infty} \frac{\log\det H_s - \log\det \tilde{H}_s}{s} = 0\,.$$

Because of this, by Lemma 2.5 below, for any $\epsilon > 0$ there exists s_0 such that $e^{-\epsilon s}\tilde{H}_s \leq H_s \leq e^{\epsilon s}\tilde{H}_s$, for $s \geq s_0$. This is immediately seen to imply (iii) . □

Lemma 2.5. *Let $U_1, U_2 \in \mathrm{Herm}(V)$. Assume that $d_1^V(U_1,U_2) \leq \epsilon$ for some $\epsilon > 0$. Then*

$$e^{-\epsilon \dim V} U_2 \leq U_1 \leq e^{\epsilon \dim V} U_2\,.$$

Proof. We fix a basis $(e_1, \ldots, e_{\dim V})$ that is orthonormal with respect to U_1 and orthogonal with respect to U_2, with eigenvalues $e^{\lambda_1}, \ldots, e^{\lambda_{\dim V}}$. Then by definition, $d_1(U_1,U_2) = \frac{1}{\dim V}\sum_{j=1}^{\dim V} |\lambda_j|$. Hence, $|\lambda_j| \leq \epsilon \dim V$, so $e^{-\epsilon\dim V}U_2 \leq U_1 \leq e^{\epsilon \dim V} U_2$. □

Theorem 2.6. *Assume that $[0,\infty) \ni s \mapsto H_s$ is positive with $\lambda_{H^*} < \infty$, and $\lambda_1 \leq \ldots \leq \lambda_m$ are the jumping numbers of the filtration $\mathcal{F}^H_\lambda$. Then*

$$\lim_{s\to\infty} \frac{1}{s}\log\left(\frac{\det H_s}{\det H_0}\right) = \sum_{j=0}^{m-1} \lambda_{j+1}(\dim \mathcal{F}^H_{\lambda_{j+1}} - \dim \mathcal{F}^H_{\lambda_j}) = \int_{-\infty}^{\infty} \lambda\, \mathrm{d}\left(\dim \mathcal{F}^H_\lambda\right)\,, \tag{2.10}$$

where $\dim \mathcal{F}^H_{\lambda_0} = 0$ by convention.

Proof. As discussed below the statement of Lemma 2.4, the limit on the left hand side of (2.10) exists and is finite. In fact, for the ray $s \mapsto \tilde{H}_s$ constructed in Lemma 2.4 we have that

$$\lim_{s\to\infty} \frac{1}{s}\log\left(\frac{\det H_s}{\det H_0}\right) = \lim_{s\to\infty} \frac{1}{s}\log\left(\frac{\det \tilde{H}_s}{\det \tilde{H}_0}\right)\,.$$

Since $\lambda_H = \lambda_{\tilde{H}}$ implies $\mathcal{F}^H_\lambda = \mathcal{F}^{\tilde{H}}_\lambda$, the conclusion follows from Lemma 2.3. □

2.3. Quantization of the Monge–Ampère energy. Recall that we have a positive Hermitian line bundle (L,h) inducing a background Kähler metric $\omega = \frac{\mathrm{i}}{2\pi}\Theta(h) > 0$ with class $[\omega] \in c_1(L)$. For any $k \geq 1$, the metric h induces a Hermitian metric h^k on L^k.

For the rest of the paper we also fix a holomorphic (twisting) line bundle T on X together with a smooth Hermitian metric h_T. By slight abuse of notation, we also denote the induced metric on $T \otimes L^k$ by h^k.

Given $\varphi \in \mathrm{PSH}(X,\omega)$, by $H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, T \otimes L^k)$, we denote the space of holomorphic sections of L^k over X that are L^2-integrable with respect to the weight $e^{-k\varphi}$, i.e., $\int_X h^k(s,s)e^{-k\varphi}\,\omega^n < \infty$. Also $h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) = \dim H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$.

For each $k \geq 1$, define the Hilbert map $\mathrm{Hilb}_k : \mathcal{E}^1 \to \mathrm{Herm}(H^0(X, T \otimes L^k))$ as follows:

$$\mathrm{Hilb}_k(\varphi)(f,g) := \int_X h^k(f,g)e^{-k\varphi}\,\omega^n\,, \quad \varphi \in \mathcal{E}^1 \text{ and } f,g \in H^0(X, T \otimes L^k)\,. \tag{2.11}$$

Define the quantum Monge–Ampère energy $\mathrm{I}_k : \mathrm{Herm}(H^0(X, T \otimes L^k)) \to \mathbb{R}$ by the formula

$$\mathrm{I}_k(U) := -\frac{1}{kV}\log\left(\frac{\det U}{\det \mathrm{Hilb}_k(0)}\right)\,. \tag{2.12}$$

The expression $\mathrm{I}_k(U) - \mathrm{I}_k(V)$ is nothing but Donaldson's original $\mathcal{L}$-functional from [Don05]. As we will see, the I_k quantizes the usual Monge–Ampère energy I, motivating our notation.

Now we define $\mathcal{L}_k : \mathcal{E}^1 \to \mathbb{R}$ by

$$\mathcal{L}_k(\varphi) := \mathrm{I}_k \circ \mathrm{Hilb}_k(\varphi)\,. \tag{2.13}$$

Remark 2.7. When (T,h_T) is trivial and φ is equal to $P(\phi)$ for some continuous function ϕ on X, the functional $\mathcal{L}_k(\varphi)$ is defined and studied in [BB10]. Note that our $\mathcal{L}_k(\varphi)$ corresponds to $h^0(X,L^k)\mathcal{L}_k(X,\phi/2)$ in their paper, with the extra $1/2$ due to the difference in conventions.

Let $V_k := H^0(X, T \otimes L^k)$. As recalled in Section 2.2, there is a natural metric $d_1^{V_k}$ on $\mathrm{Herm}(V_k)$. With the focus of this section on quantization, we define a scaled version of this metric:

$$d_1^k(H_1,H_2) := \frac{1}{k}d_1^{V_k}(H_1,H_2)\,, \quad H_1,H_2 \in V_k\,.$$

This convention coincides with the one used in [DLR20].

Let $S = \{0 < \mathrm{Re}\, z < 1\} \subset \mathbb{C}$ be the unit strip and $\pi : S \times X \to X$ be the natural projection. We say that $(0,1) \ni t \mapsto \varphi_t \in \mathcal{E}^1$ is a *subgeodesic* if its complexification Φ satisfies $\pi^*\omega + \mathrm{dd^c}\Phi \geq 0$ on $S \times X$ in the sense of currents. Let us recall the following version of Berndtsson's convexity theorem [Ber09, Theorem 1.2].

Theorem 2.8. *Let $[0,1] \ni t \mapsto \varphi_t \in \mathcal{E}^1$ be a subgeodesic connecting $\varphi_0, \varphi_1 \in \mathcal{E}^1$. Let Φ be the complexification of $t \mapsto \varphi_t$ and assume that*

$$\pi^*\omega + \mathrm{dd^c}\Phi \geq \epsilon\pi^*\omega$$

for some $\epsilon > 0$. Let $[0,1] \ni t \mapsto H_t \in \mathrm{Herm}(H^0(X, T \otimes L^k))$ be the geodesic connecting $\mathrm{Hilb}_k(\varphi_0)$ with $\mathrm{Hilb}_k(\varphi_1)$. Then there exists $k_0(\varepsilon) > 0$, so that for any $k \geq k_0$ we have

$$H_t \leq \mathrm{Hilb}_k(\varphi_t)\,, \quad t \in [0,1]\,.$$

Moreover, $t \mapsto \mathrm{Hilb}_k(\varphi_t)$ is positive (see Definition 2.2).

If additionally $t \mapsto \varphi_t$ is t-increasing, then for any $s \in H^0(X, T \otimes L^k)$,

$$-k \int_X \dot{\varphi}_1 h^k(s,s) e^{-k\varphi_1}\, \omega^n \leq \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=1} H_t(s,s)\,. \tag{2.14}$$

The proof follows line by line from that of [DLR20, Corollary 2.13, Lemma 2.14].

Lemma 2.9. *For $\varphi_0, \varphi_1 \in \mathcal{E}^1$ we have*

$$|\mathcal{L}_k(\varphi_0) - \mathcal{L}_k(\varphi_1)| \leq C k^n d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_1))\,,$$

where C depends only on X.

Proof. Notice that

$$\mathcal{L}_k(\varphi_1) - \mathcal{L}_k(\varphi_0) = -\frac{1}{kV} \log \frac{\det \mathrm{Hilb}_k(\varphi_1)}{\det \mathrm{Hilb}_k(\varphi_0)}\,.$$

Take a basis $(e_1, \ldots, e_{N_k})$ of $H^0(X, T \otimes L^k)$ which is orthonormal with respect to $\mathrm{Hilb}_k(\varphi_0)$ and is orthogonal with respect to $\mathrm{Hilb}_k(\varphi_1)$. Let $\lambda_j := \log \mathrm{Hilb}_k(\varphi_1)(e_j, e_j)$ for $j = 1, \ldots, N_k$. Then

$$\mathcal{L}_k(\varphi_1) - \mathcal{L}_k(\varphi_0) = -\frac{1}{kV} \sum_{j=1}^{N_k} \lambda_j \leq \frac{N_k}{V} d_1^k\left(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_1)\right)\,.$$

By the Riemann–Roch theorem, N_k is dominated by Vk^n, and the result follows. ☐

Lemma 2.10. *For $\varphi_0, \varphi_1 \in \mathcal{E}^1$ we have*

$$\lim_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_1)) = d_1(\varphi_0, \varphi_1)\,.$$

This result is the twisted version of [DLR20, Theorem 1.2(ii)]. We reproduce the proof for convenience of the reader.

Proof. Without loss of generality, let us assume that $\varphi_0, \varphi_1 \leq -1$. Due to [Ber18, Theorem 3.3] the results is known for $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$.

By [BK07] we can find φ_0^j, $\varphi_1^j \in \mathcal{H}_\omega$, sequences decreasing to φ_0, φ_1, respectively. We may assume without loss of generality that $\varphi_0^1, \varphi_1^1 \leq 0$. By our assumption, for any $j \geq 1$ we have,

$$\lim_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_0^j), \mathrm{Hilb}_k(\varphi_1^j)) = d_1(\varphi_0^j, \varphi_1^j)\,.$$

Hence it is enough to show that for any $\epsilon > 0$, we can find $j_0 > 0$ such that

$$\varlimsup_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_0^j), \mathrm{Hilb}_k(\varphi_0)) < \epsilon\,, \quad \varlimsup_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_1^j), \mathrm{Hilb}_k(\varphi_1)) < \epsilon\,,$$

for any $j \geq j_0$. By symmetry, we only prove the former. We fix some real number $\delta > 1$ for now. Let $[0,1] \ni t \mapsto \ell_t \in \mathcal{E}^1$ be the geodesic from $P(\delta\varphi_0)$ to φ_0^j. For $t \in [0,1]$, let

$$\ell_t' := \frac{1}{\delta}\ell_t + \left(1 - \frac{1}{\delta}\right)\varphi_0^j\,.$$

Notice that $\ell'_0 \leq \varphi_0 \leq \varphi_0^j = \ell'_1$. As a result, $\mathrm{Hilb}_k(\ell'_1) \leq \mathrm{Hilb}_k(\varphi_0) \leq \mathrm{Hilb}_k(\ell'_0)$. Hence, by comparison of the tangent vectors at ℓ'_1, we conclude

$$d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_0^j)) \leq d_1^k(\mathrm{Hilb}_k(\ell'_0), \mathrm{Hilb}_k(\ell'_1))\,. \tag{2.15}$$

Let $t \to G_t^k \in \mathrm{Herm}(H^0(X, T \otimes L^k))$ ($t \in [0,1]$) be the geodesic from $\mathrm{Hilb}_k(\ell'_0)$ to $\mathrm{Hilb}_k(\ell'_1)$. Observe that the conditions of Theorem 2.8 are satisfied by $t \to \ell'_t$. Hence for $k \geq k_0(\delta)$,

$$G_t^k \leq \mathrm{Hilb}_k(\ell'_t)$$

for all $t \in [0,1]$. By (2.14), for any $f \in H^0(X, T \otimes L^k)$, we have

$$-\frac{1}{\delta}\int_X \dot{\ell}_1 h^k(f,f) e^{-k\varphi_0^j}\,\omega^n \leq \frac{1}{k}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} G_t^k(f,f) \leq 0\,. \tag{2.16}$$

By [DLR20, Lemma 4.5], the left hand side is finite. Now we find a basis $e_1, \ldots, e_N$ of $H^0(X, T \otimes L^k)$ that is orthonormal with respect to $\mathrm{Hilb}_k(\ell'_1)$ and such that the quadratic form

$$f \mapsto -\frac{1}{\delta}\int_X \dot{\ell}_1 h^k(f,f) e^{-k\varphi_0^j}\,\omega^n$$

is orthogonal with eigenvalues $\lambda_1, \ldots, \lambda_N$. Then, using (2.15) and (2.16), we get

$$d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_0^j)) \leq d_1^k(\mathrm{Hilb}_k(\ell'_0), \mathrm{Hilb}_k(\ell'_1)) \leq \frac{1}{N}\sum_{a=1}^N |\lambda_a| \leq \frac{1}{N}\sum_{a=1}^N \int_X |\dot{\ell}_1| h^k(e_a, e_a) e^{-k\varphi_0^j}\,\omega^n\,.$$

Letting $k \to \infty$, it follows from the classical Bergman kernel expansion that (see the elementary calculations following [DLR20, (35)])

$$\varlimsup_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_0^j)) \leq \frac{1}{V\delta}\int_X |\dot{\ell}_1|\,\omega^n_{\varphi_0^j} = \frac{1}{\delta} d_1(P(\delta\varphi_0), \varphi_0^j)\,,$$

where the last equality follows from [DLR20, Lemma 4.5]. Letting $\delta \searrow 1$, we find

$$\varlimsup_{k\to\infty} d_1^k(\mathrm{Hilb}_k(\varphi_0), \mathrm{Hilb}_k(\varphi_0^j)) \leq d_1(P(\varphi_0, \varphi_0^j)\,,$$

finishing the proof of the claim, and the argument. □

Next we quantize the Monge–Ampère energy (see (2.2)) on the space $\mathcal{E}^1$, extending the corresponding result for smooth metrics [Don05], continuous metrics [BB10, Theorem A], and the case of K_X-twisting [BF14, Theorem 3.5]:

THEOREM 2.11. *For any $\varphi \in \mathcal{E}^1$, we have*

$$\lim_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k(\varphi) = \mathrm{I}(\varphi)\,. \tag{2.17}$$

PROOF. Assume that this result is true for $\varphi \in \mathcal{H}_\omega$. For a general $\varphi \in \mathcal{E}^1$, take a decreasing sequence $\varphi_j \in \mathcal{H}_\omega$ that converges to φ. Then by Lemma 2.9 and Lemma 2.10, for any $j \geq 1$,

$$\varlimsup_{k\to\infty} \left|\frac{n!}{k^n}\mathcal{L}_k(\varphi_j) - \frac{n!}{k^n}\mathcal{L}_k(\varphi)\right| \leq C \varlimsup_{k\to\infty} d_1^k(\varphi_j, \varphi) = C d_1(\varphi_j, \varphi)\,.$$

By our assumption, $\lim_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k(\varphi_j) = \mathrm{I}(\varphi_j)$. This implies that

$$\varlimsup_{k\to\infty} \left|\mathrm{I}(\varphi_j) - \frac{n!}{k^n}\mathcal{L}_k(\varphi)\right| \leq C d_1(\varphi_j, \varphi)\,.$$

Letting $j \to \infty$, we conclude.

It remains to prove (2.17) when $\varphi \in \mathcal{H}_\omega$. When (T, h_T) is trivial, this was carried out in [Don05]. Indeed, it suffices to observe that

$$\frac{n!}{k^n} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_k(t\varphi) = \int_X \varphi B_k(t\varphi)\,\omega^n\,, \quad t \in [0,1]\,,$$

where $B_k(t\varphi)$ denotes the k-th T-twisted Bergman kernel at $t\varphi \in \mathrm{PSH}(X,\omega)$. The well-known Bergman kernel expansion [MM07, Theorem 4.1.1] implies that $B_k(t\varphi)\,\omega^n \to V^{-1}\omega_{t\varphi}^n$ uniformly. Consequently,

$$\frac{n!}{k^n} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_k(t\varphi) \to \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{I}(t\varphi)$$

uniformly. Taking integral with respect to $t \in [0,1]$, we conclude (2.17) in this case. □

2.4. An algebraic notion of singularity type.

Detecting singularities using algebraic tools. In this section, let (X,ω) be a compact Kähler manifold of dimension n. Let θ be a smooth real $(1,1)$-form on X representing a pseudo-effective cohomology class.

Given $u \in \mathrm{PSH}(X,\theta)$, as pointed out in the literature (see for example [BFJ08], [Kim15]), one can not characterize the singularity type $[u]$ using "mainstream" algebraic data, like multiplier ideal sheaves $\mathcal{I}(cu), c > 0$ or Lelong numbers. Instead, one can introduce an algebraic notion that is coarser than equivalence up to singularity types, considered in [KS20, Section 2.1]:

Definition 2.12. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. We put $\varphi \preceq_{\mathcal{I}} \psi$ in case $\mathcal{I}(a\varphi) \subseteq \mathcal{I}(a\psi)$ for all $a > 0$. Then $\preceq_{\mathcal{I}}$ is a preorder, with equivalence relation $\varphi \simeq_{\mathcal{I}} \psi$ characterized by $\mathcal{I}(a\varphi) = \mathcal{I}(a\psi)$ for all $a > 0$. The corresponding classes are called *$\mathcal{I}$-singularity types*, and are denoted by $[\chi]_{\mathcal{I}}$, where $\chi \in \mathrm{PSH}(X,\theta)$ is a representative of the class.

In the language of [KS20, Section 2.1] the relation $\simeq_{\mathcal{I}}$ is called *v-equivalence*. Obviously $[\varphi] = [\psi]$ implies $[\varphi]_{\mathcal{I}} = [\psi]_{\mathcal{I}}$. However the reverse direction does not hold in general, and this phenomenon is at the center of the discussion in this subsection. Before we dive deeper into this, let us recall the following characterization of $\mathcal{I}$-equivalence via Lelong numbers, a direct consequence of [BFJ08, Theorem A]:

Theorem 2.13. *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(i) $\varphi \simeq_{\mathcal{I}} \psi$.

(ii) $\nu(\varphi, y) = \nu(\psi, y)$ *for any projective modification $\pi : Y \to X$, with Y smooth, and $y \in Y$.*

In the above statement $\nu(\varphi, y)$ is the Lelong number of $\varphi \circ \pi$ at y, in local coordinates defined by

$$\nu(\varphi, y) = \nu^\pi(\varphi, y) := \sup\{\, c \geq 0 : \varphi \circ \pi(z) \leq c \log \|z - y\| + O(1) \text{ near } y \,\}\,.$$

Given a prime divisor Z of Y, the *generic Lelong number* of φ along Z is defined as:

$$\nu(\varphi, Z) = \inf_{z \in Z} \nu(\varphi, z)\,.$$

Due to Siu's semicontinuity theorem, for a set $S \subseteq Z$ of measure zero, we have that $\nu(\varphi, z) = \nu(\varphi, Z)$ for $z \in Z \setminus S$, motivating the terminology.

Since we work with smooth models Y, for a coherent ideal $\mathcal{J} \subseteq \mathcal{O}_X$ one can talk about $\nu(\mathcal{J}, y)$ ($\nu(\mathcal{J}, Z)$) as the minimum vanishing order of $f_j \circ \pi$ at y (along Z) for a finite set of generators $\{f_j\}_j$ of $\mathcal{J}_y$ ($\mathcal{J}_z$ for some $z \in Z$). Moreover, one can see that $\nu(\mathcal{J}, y) := \nu(\mathcal{J}, E_y)$, where E_y is the exceptional divisor of $p_y : \mathrm{Bl}_{\{y\}} Y \to Y$, the blowing up of Y at y.

The following result is implicit in [BFJ08], and clarifies the relationship between multiplier ideal sheaves and Lelong numbers in Theorem 2.13. We give a detailed sketch of the argument for the convenience of the reader:

Proposition 2.14. *Let $\pi : Y \to X$ be a projective modification with Y smooth, and $y \in Y$. For $\varphi \in \mathrm{PSH}(X, \theta)$ we have*

$$\nu(\varphi, y) = \lim_{k\to\infty} \frac{1}{k} \nu(\mathcal{I}(k\varphi), y)\,. \tag{2.18}$$

Proof. That $\nu(\varphi, y) \geq \frac{1}{k}\nu(\mathcal{I}(k\varphi), y)$ follows from the fact that the local potential with singularity governed by $\frac{1}{k}\mathcal{I}(k\varphi)$ is always less singular than φ (by the Ohsawa–Takegoshi theorem). Taking $\overline{\lim}$, we get that the left hand side is greater than $\nu(\varphi, y) \geq \overline{\lim}_{k\to\infty} \frac{1}{k}\nu(\mathcal{I}(k\varphi))$.

For the reverse inequality, we start with noticing that $\nu(\varphi \circ p_y, z) \geq \nu(\varphi, y)$ for any $z \in E_y$. Indeed, Lelong numbers can only increase under pullbacks. In particular, $\nu(\varphi, E_y) \geq \nu(\varphi, y)$. That $\underline{\lim}_{k\to\infty} \frac{1}{k}\nu(\mathcal{I}(k\varphi), E_y) \geq \nu(\varphi, E_y)$, follows from an application of Fubini–Tonelli's theorem, as elaborated in the proof of [BFJ08, (5.3)]. ☐

Remark 2.15. That (i) implies (ii) in Theorem 2.13 is seen to follow from (2.18). The reverse direction now follows from the local result [BFJ08, Theorem A]. More broadly, the reverse direction is the consequence of the valuative criteria for integrability (see [Bou17, Theorem 10.12] and its proof).

Corollary 2.16. *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Then the following are equivalent:*

(i) $\varphi \preceq_{\mathcal{I}} \psi$.

(ii) $\nu(\varphi, y) \geq \nu(\psi, y)$ *for any projective modification $\pi : Y \to X$, with Y smooth, and $y \in Y$.*

Proof. Assume that (i) holds. Then (ii) holds by Proposition 2.14. Conversely, assume that (ii) holds. Then $\max\{\varphi, \psi\} \simeq_{\mathcal{I}} \psi$ by Theorem 2.13, and the fact that $\nu(\max(\varphi, \psi), y) = \min(\nu(\psi, y), \nu(\varphi, y))$ [Bou17, Corollary 2.10]. Hence $\varphi \preceq_{\mathcal{I}} \max(\varphi, \psi) \simeq_{\mathcal{I}} \psi$ as desired. ☐

As we saw in the above argument, the class of potentials χ satisfying $\chi \preceq_{\mathcal{I}} \varphi$ are stable under taking max, hence we can introduce the notion of an envelope with respect to $\mathcal{I}$-singularity:

$$\begin{aligned} P[\varphi]_{\mathcal{I}} &:= \mathrm{usc}\,(\sup\{\,\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0\,, \psi \preceq_{\mathcal{I}} \varphi\,\}) \\ &= \mathrm{usc}\left(\sup\left\{\,\max\{\psi, \varphi - \sup_X \varphi\} : \psi \in \mathrm{PSH}(X, \theta)\,, \psi \leq 0\,, \psi \preceq_{\mathcal{I}} \varphi\,\right\}\right) \\ &= \mathrm{usc}\,(\sup\{\,\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0\,, \psi \simeq_{\mathcal{I}} \varphi\,\})\,. \end{aligned} \tag{2.19}$$

The above envelope should be compared with the well-known envelope with respect to singularity type (going back to [RW14] and [RS05] in the local case):

$$P[\varphi] := \operatorname{usc}\left(\sup\{\, v \in \mathrm{PSH}(X,\theta) : [v] = [\varphi] \text{ and } v \leq 0 \,\}\right) .$$

We refer the reader to [RW14], [DDL18b], [DDL21] for basic properties of $P[u]$.

Definition 2.17. Recall that $u \in \mathrm{PSH}(X,\theta)$ is a *model potential* if $u = P[u]$. Also, the singularity $[u]$ of a model potential u is called a *model singularity type*. Analogously, a potential $\varphi \in \mathrm{PSH}(X,\theta)$ is called $\mathcal{I}$*-model* if $\varphi = P[\varphi]_{\mathcal{I}}$. The singularity type $[\varphi]$ of an $\mathcal{I}$-model potential φ is called an $\mathcal{I}$*-model singularity type*.

We begin to discuss the parallel between the above notions:

Proposition 2.18. *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Then*

(i) *$P[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$ is a model potential ($P[P[\varphi]_{\mathcal{I}}] = P[\varphi]_{\mathcal{I}}$), moreover $P[\varphi]_{\mathcal{I}} \geq P[\varphi]$. In particular, all $\mathcal{I}$-model potentials are also model potentials.*

(ii) *$\varphi \simeq_{\mathcal{I}} P[\varphi]_{\mathcal{I}}$. In particular, $P[P[\varphi]_{\mathcal{I}}]_{\mathcal{I}} = P[\varphi]_{\mathcal{I}}$, and the* usc *is unnecessary in* (2.19).

According to the above result $P[u]_{\mathcal{I}} = u$ implies $P[u] = u$. As a result, if u is $\mathcal{I}$-model then it is automatically model, but not vice versa.

Proof. (i) $P[\varphi]_{\mathcal{I}} = P[P[\varphi]_{\mathcal{I}}]$ because $aP[\varphi]_{\mathcal{I}}$ and $aP[P[u]_{\mathcal{I}}] = a\lim_{C\to\infty} P(P[\varphi]_{\mathcal{I}}+C, 0)$ have that same multiplier ideal sheaves for any $a \geq 0$. Indeed, multiplier ideal sheaves are stable under taking increasing limits [GZ15]. Since $\varphi \simeq_{\mathcal{I}} P[\varphi]$, we get $P[\varphi]_{\mathcal{I}} \geq P[\varphi]$.

(ii) By Choquet's lemma we can take $\psi_j \in \mathrm{PSH}(X,\theta)$ ($j \geq 0$), such that $\psi_j \leq 0$, $\psi_j \sim_{\mathcal{I}} \varphi$ and that ψ_j increases to $P[\varphi]_{\mathcal{I}}$ a.e.. It follows from Guan–Zhou's strong openness theorem [GZ15] that $\varphi \sim_{\mathcal{I}} P[\varphi]_{\mathcal{I}}$. □

Example 2.19. *Following [BBJ21, Example 6.10], we give an example showing that not all model potentials are $\mathcal{I}$-model. Consider $X = \mathbb{P}^1$ and let ω be the Fubini–Study form on X. Let $K \subseteq \mathbb{P}^1$ be a Cantor set. Then K carries an atom-free probability measure, whose potential v has zero Lelong numbers. Then the pull-back of v to any proper modification of X has zero Lelong numbers as well [Kim15, Corollary 2.4]. Hence $P[v]_{\mathcal{I}} = 0$. But v does not have full mass. Hence $P[v] \neq 0$, i.e., $P[v]$ is model but not $\mathcal{I}$-model.*

Proposition 2.20. *Assume that $\psi \in \mathrm{PSH}(X,\theta) \in \mathcal{A}$ (See Definition 1.3). Then*

$$P[\psi]_{\mathcal{I}} = P[\psi] .$$

In particular, $P[\psi]$ is $\mathcal{I}$-model, and has the same singularity type as ψ.

Proof. First one notices that $[P[\psi]_{\mathcal{I}}] = [\psi]$ for $[\psi]$ analytic. This is a consequence of [Kim15, Theorem 4.3]. It can also be seen after an analysis on the pullback $\pi : Y \to X$, where π is the normalized blowup of the ideal of ψ, precomposed with a log resolution.

Since $[P[\psi]_{\mathcal{I}}] = [\psi]$, we get $P[\psi]_{\mathcal{I}} \leq P[\psi]$, with the reverse being true by Proposition 2.18(i). □

Lemma 2.21. *Suppose that* $\{\varphi_j\}_j \in \mathrm{PSH}(X,\theta)$ *and* $\varphi \in \mathrm{PSH}(X,\theta)$ *are model potentials.*

(i) *If* $\varphi_j \searrow \varphi$ *and* φ_j *are* $\mathcal{I}$*-model, then* φ *is* $\mathcal{I}$*-model as well.*

(ii) *If* $\varphi_j \searrow \varphi$ *and* $\int_X \theta_\varphi^n > 0$*, then* $P[\varphi_j]_\mathcal{I} \searrow P[\varphi]_\mathcal{I}$.

(iii) *If* $\varphi_j \nearrow \varphi$ *a.e. and* $\int_X \theta_\varphi^n > 0$*, then* $P[\varphi_j]_\mathcal{I} \nearrow P[\varphi]_\mathcal{I}$ *a.e. as well. In particular, if the* φ_j *are additionally* $\mathcal{I}$*-model, then* φ *is* $\mathcal{I}$*-model as well.*

Proof. First we prove (i). Note that $P[\varphi]_\mathcal{I} \simeq_\mathcal{I} \varphi \preceq_\mathcal{I} \varphi_j$ for any $j \geq 1$. Hence by Proposition 2.18, $P[\varphi]_\mathcal{I} \leq P[\varphi_j]_\mathcal{I} = \varphi_j$. Letting $j \to \infty$, we obtain $P[\varphi]_\mathcal{I} \leq \varphi$. Since $\varphi \leq \varphi_j \leq 0$, we know that $\varphi \leq P[\varphi]_\mathcal{I}$, hence φ is $\mathcal{I}$-model.

We deal with (ii). Since $\int_X \theta_{\varphi_j}^n \searrow \int_X \theta_\varphi^n > 0$ [DDL21, Proposition 4.8], by [DDL21, Lemma 4.3] there exists $\alpha_j \searrow 0$ and $v_j := P(\frac{1}{\alpha_j}\varphi + (1-\frac{1}{\alpha_j})\varphi_j) \in \mathrm{PSH}(X,\theta)$ satisfying $(1-\alpha_j)\varphi_j + \alpha_j v_j \leq \varphi$. Taking $P[\cdot]_\mathcal{I}$ we arrive at

$$(1-\alpha_j)P[\varphi_j]_\mathcal{I} + \alpha_j P[v_j]_\mathcal{I} \leq P[(1-\alpha_j)\varphi_j + \alpha_j v_j]_\mathcal{I} \leq P[\varphi]_\mathcal{I}\,,$$

where in the first inequality we have used $P[\psi]_\mathcal{I} \sim_\mathcal{I} \psi$, Theorem 2.13, and additivity of Lelong numbers. Since $\{\varphi_j\}_j$ is decreasing, so is $\{P[\varphi_j]_\mathcal{I}\}_j$, hence $w := \lim_j P[\varphi_j]_\mathcal{I} \geq P[\varphi]_\mathcal{I}$ exists. Since $\alpha_j \to 0$ and $\sup_X P[v_j]_\mathcal{I} = 0$, comparison with the above gives $w = P[\varphi]_\mathcal{I}$.

Dealing with (iii) is similar. Since $\int_X \theta_{\varphi_j}^n \nearrow \int_X \theta_\varphi^n > 0$ [DDL18b, Theorem 2.3], by [DDL21, Lemma 4.3] there exists $\alpha_j \searrow 0$ and $v_j := P(\frac{1}{\alpha_j}\varphi_j + (1-\frac{1}{\alpha_j})\varphi) \in \mathrm{PSH}(X,\theta)$ satisfying $(1-\alpha_j)\varphi + \alpha_j v_j \leq \varphi_j$. Taking $P[\cdot]_\mathcal{I}$ we arrive at

$$(1-\alpha_j)P[\varphi]_\mathcal{I} + \alpha_j P[v_j]_\mathcal{I} \leq P[(1-\alpha_j)\varphi + \alpha_j v_j]_\mathcal{I} \leq P[\varphi_j]_\mathcal{I}\,,$$

where in the first inequality we have used that $P[\psi]_\mathcal{I} \sim_\mathcal{I} \psi$, Theorem 2.13, and additivity of Lelong numbers. Since $\{\varphi_j\}_j$ is increasing, so is $\{P[\varphi_j]_\mathcal{I}\}_j$, hence $w := \lim_j P[\varphi_j]_\mathcal{I} \leq P[\varphi]_\mathcal{I}$ exists. Since $\alpha_j \to 0$ and $\sup_X P[v_j]_\mathcal{I} = 0$, comparison with the above yields $w = P[\varphi]_\mathcal{I}$. □

Remark 2.22. The condition $\varphi \simeq_\mathcal{I} \psi$ is strictly stronger than requiring φ and ψ have the same Lelong number everywhere on X (See [Kim15, Example 2.5]). As we will see in the next section, in terms of valuations, $\varphi \preceq_\mathcal{I} \psi$ means exactly that the induced non-Archimedean functions on the space of divisorial valuations $X_\mathbb{Q}^{\mathrm{div}}$ satisfy $\varphi^{\mathrm{an}} \leq \psi^{\mathrm{an}}$. In particular, $\varphi^{\mathrm{an}} = \psi^{\mathrm{an}}$ is equivalent to $\varphi \simeq_\mathcal{I} \psi$. See [BFJ08] for further details.

Algebraic approximation of $\mathcal{I}$-model potentials. For the remained of this subsection, we return to the context of an ample line bundle $L \to X$, with hermitian metric h, whose first Chern form is equal to the Kähler form ω. Let us recall the following well known result, originated from [DPS01, Theorem 2.2.1]:

Theorem 2.23. *Let* $u \in \mathrm{PSH}(X,\omega)$*. Let* $u_k \in \mathrm{PSH}(X,\omega)$ *be the partial Bergman kernel of* $V_k^u := H^0(X, L^{2^k+k_0} \otimes \mathcal{I}(2^k u))$*:*

$$u_k = \frac{1}{2^k+k_0} \sup_{\substack{s \in V_k^u,\\ \mathrm{Hilb}_{2^k}(u)(s,s) \leq 1}} \log h^{2^k+k_0}(s,s)\,, \tag{2.20}$$

where $\mathrm{Hilb}_{2^k}(u)$ *is the Hilbert map of* L^{2^k} *with twisting* $T = L^{k_0}$ *(see Section 2.3). Then* u_k *has algebraic singularity type (See Definition 1.3), and for some* $k_0 = k_0(X, L, \omega)$ *the following hold:*
(i) u_k *converges to* u *in* L^1 *as* $k \to \infty$.
(ii) $[u_{k+1}] \preceq [u_k]$.
(iii) *for all* $m > 0$ *and* $k > m$ *one can find* $\delta_{k,m} > 1$ *such that* $\mathcal{I}(m\delta_{k,m}u_k) \subseteq \mathcal{I}(mu)$ *and* $\delta_{k,m} \searrow 1$ *as* $k \to \infty$.

Sketch of proof. (i) follows from Step 1 in the proof of [GZ05, Theorem 7.1]. By Step 2 in the proof of [GZ05, Theorem 7.1] we get that for sufficiently high k_0 the sequence $[u_k]$ is decreasing, satisfying (ii). Condition (iii) is a consequence of the *comparison of integrals method* of [DPS01, Theorem 2.2.1] (see [Cao14, Lemma 3.2] where this is written out explicitly). □

Following terminology of [Cao14], approximations $\{u_j\}_j$ of $u \in \mathrm{PSH}(X, \omega)$ of the type (2.20), satisfying all three conditions in Theorem 2.23 will be referred to as *quasi-equisingular approximations* of u.

We arrive at the following result, characterizing the difference between model and $\mathcal{I}$-model potentials in terms of $d_{\mathcal{S}}$-approximability via quasi-equisingular sequences:

Theorem 2.24. *Let* $\varphi \in \mathrm{PSH}(X, \omega)$ *be a model potential* ($P[\varphi] = \varphi$) *with* $\int_X \omega_\varphi^n > 0$. *Then* φ *is* $\mathcal{I}$*-model* ($P[\varphi]_{\mathcal{I}} = \varphi$) *if and only if* $[\varphi]$ *is the* $d_{\mathcal{S}}$*-limit of a quasi-equisingular approximation* $[\varphi_j] \in \mathcal{Z}$.

Remark 2.25. Due to this theorem and Proposition 2.20, the class of analytic singularity types $\mathcal{A}$ are $d_{\mathcal{S}}$-approximable by algebraic singularity types of $\mathcal{Z}$ (in the presence of positive mass), already proving the (easy) equivalences between (iv) and (v) in Theorem 1.4.

Proof. Let $\varphi \in \mathrm{PSH}(X, \omega)$ be an $\mathcal{I}$-model potential with $\int_X \omega_\varphi^n > 0$. Let φ_k be the corresponding quasi-equisingular approximation of φ. By [DDL18b, Theorem 1.1] the sum $\sum_{j=0}^n \int_X \omega^{n-j} \wedge \omega_{\varphi_k}^j$ is decreasing in k, hence converges. By [DDL21, Lemma 3.4], $\{[\varphi_k]\}_k \subseteq \mathcal{S}$ forms a $d_{\mathcal{S}}$-Cauchy sequence with $\int_X \omega_{\varphi_j}^n \geq \int_X \omega_\varphi^n > 0$. Hence by [DDL21, Theorem 1], the sequence $d_{\mathcal{S}}$-converges to $[\varphi'] \in \mathcal{S}$, where $\varphi' = P[\varphi]' = \lim_j P[\varphi_j] \geq P[\varphi] = \varphi$ [DDL21, Corollary 4.7]. We claim that

$$\varphi = \varphi' . \tag{2.21}$$

By Proposition 2.20 and Lemma 2.21(i), both φ and φ' are $\mathcal{I}$-model, so it suffices to show that $\mathcal{I}(m\varphi) = \mathcal{I}(m\varphi')$ for any $m > 0$. Since $\varphi \leq \varphi'$, the non-trivial inclusion is $\mathcal{I}(m\varphi) \supseteq \mathcal{I}(m\varphi')$. To prove this, by the last statement of the above theorem, we notice that

$$\mathcal{I}(m\varphi) \supseteq \mathcal{I}(m\delta_{k,m}\varphi_k) = \mathcal{I}(m\delta_{k,m}P[\varphi_k]) \supseteq \mathcal{I}(m\delta_{k,m}\varphi') .$$

By the strong openness theorem [GZ15], we can let $k \to \infty$ to arrive at $\mathcal{I}(m\varphi) \supseteq \mathcal{I}(m\varphi')$, as desired.

Conversely, let φ_j be the quasi-equisingular approximation of φ such that $d_{\mathcal{S}}([\varphi_j], [\varphi]) \to 0$. Taking envelopes, by Proposition 2.20 we conclude that $\varphi_j' := P[\varphi_j] = P[\varphi_j]_{\mathcal{I}}$ is pointwise decreasing, and $d_{\mathcal{S}}([\varphi_j], [\varphi]) = d_{\mathcal{S}}([\varphi_j'], [\varphi]) \to 0$. [DDL21, Lemma 3.6] now gives that $\int_X \omega_{\varphi'}^n \searrow \int_X \omega_\varphi^n$. Since $\varphi = P[\varphi]$, $\lim_j \varphi_j' \geq \varphi$, and

$\int_X \omega_\varphi^n > 0$, by [DDL18b, Theorem 3.12] we obtain that $\lim_j \varphi'_j = \varphi$. Finally, Lemma 2.21(i) implies that φ is $\mathcal{I}$-model. □

Before we proceed further, we recall that the conventions set at the beginning of Section 1.2 guarantee that the leading order Riemann–Roch expansion takes the form $h^0(X, T \otimes L^k) = \frac{V}{n!}k^n + \mathcal{O}(k^{n-1})$, where T is an arbitrary line bundle. We recall the following result of Bonavero:

THEOREM 2.26. *Assume that $\varphi \in \mathrm{PSH}(X,\omega)$ has algebraic singularity type ($[\varphi] \in \mathcal{Z}$). Then*

$$\lim_{k\to\infty} \frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))}{k^n} = \frac{1}{n!}\int_X \omega_\varphi^n .$$

This is indeed a special case of the singular holomorphic Morse inequalities proved by Bonavero [Bon98], surveyed in [MM07, Theorem 2.3.18].

PROOF. According to [Bon98, Théorème 1.1] for $q = 0$, we have $k^{-n}h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \frac{1}{n!}\int_X \omega_\varphi^n + o(1)$, hence

$$\overline{\lim_{k\to\infty}} \frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))}{k^n} \leq \frac{1}{n!}\int_X \omega_\varphi^n .$$

Applying [Bon98, Théorème 1.1] with $q = 1$, we get

$$-\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))}{k^n} + \frac{h^1(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))}{k^n} \leq -\frac{1}{n!}\int_X \omega_\varphi^n + o(1) .$$

But according to [Bon98, Corollaire 2.2], $h^1(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) = o(k^n)$, hence

$$\underline{\lim_{k\to\infty}} \frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))}{k^n} \geq \frac{1}{n!}\int_X \omega_\varphi^n .$$

Hence equality indeed holds. □

Remark 2.27. Note that our convention for the multiplier ideal sheaves is different from that of Bonavero's. In fact, Bonavero's definition of $\mathcal{I}(\varphi/2)$ corresponds to our $\mathcal{I}(\varphi)$. But the volume of $\varphi/2$ in the sense of Bonavero is exactly the same as $\int_X \omega_\varphi^n$ in our sense, hence the holomorphic Morse inequalities take exactly the same form, despite the difference in conventions.

We additionally note that Bonavero proved the above result for potentials with *analytic* singularity type, however his definition of this notion is less general than ours in Definition 1.3, this being the reason for our more conservative statement above.

THEOREM 2.28. *Let $\varphi \in \mathrm{PSH}(X,\omega)$ be an $\mathcal{I}$-model potential. Then*

$$\overline{\lim_{k\to\infty}} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \frac{1}{n!}\int_X \omega_\varphi^n .$$

PROOF. If $\int_X \omega_\varphi^n > 0$, the estimate follows directly from Theorem 2.26, Theorem 2.24 and [DDL21, Lemma 3.6].

Now let $\int_X \omega_\varphi^n = 0$ and φ_j be a quasi-equisingular approximation of φ. Let $\epsilon \in (0,1) \cap \mathbb{Q}$. Let $P_\epsilon[\cdot]$ denote the envelope with respect to singularity type with respect to $\omega + \epsilon\omega$. Note that the potentials $P_\epsilon[\varphi_j] \in \mathrm{PSH}(X, \omega + \epsilon\omega)$ have positive masses bounded away from zero, for each $\epsilon > 0$ fixed. Moreover, $P_\epsilon[\varphi_j]$ has the same singularity type as φ_j. Let ψ^ϵ be the decreasing limit of $P_\epsilon[\varphi_j]$. By Lemma 2.21(i), ψ^ϵ is an $\mathcal{I}$-model potential and

$$\int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n = \lim_{j\to\infty} \int_X (\omega + \epsilon\omega + \mathrm{dd^c}\varphi_j)^n \,.$$

We have $\psi_\epsilon \searrow \psi \in \mathrm{PSH}(X,\omega)$ as $\epsilon \searrow 0$. From the condition $\mathcal{I}(m\delta_{k,m}\varphi_k) \subseteq \mathcal{I}(m\varphi)$, we see that $\mathcal{I}(m\psi_\epsilon) = \lim_k \mathcal{I}(m\delta_{k,m}\psi_\epsilon) \subseteq \lim_k \mathcal{I}(m\delta_{k,m}\varphi_k) \subseteq \mathcal{I}(m\varphi)$, for any $m \geq 0$. Hence it follows that $\mathcal{I}(m\psi) \subseteq \mathcal{I}(m\varphi)$. Hence $P[\psi]_{\mathcal{I}} \leq \varphi$. But as $\psi \leq 0$, we know that $P[\psi]_{\mathcal{I}} \geq \psi \geq \varphi$, hence we conclude that $\varphi = \psi$.

Now we claim that

$$\lim_{\epsilon\to 0+} \int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n = 0 \,. \tag{2.22}$$

Indeed, for $c > 0$ let $\psi^\epsilon_c := \max(\psi^\epsilon, -c)$ and $\varphi_c := \max(\varphi, -c)$. For any $b > 0$ We have that

$$\varlimsup_{\epsilon\to 0+} \int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n \leq \varlimsup_{\epsilon\to 0+} \int_X e^{b\psi^\epsilon} (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon_c)^n = \int_X e^{b\varphi} (\omega + \mathrm{dd^c}\varphi_c)^n,$$

where in the first estimate we used that $(\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n$ is supported on $\{\psi^\epsilon = 0\}$ [DDL18b, Theorem 3.8], and in the last equality we used that $e^{b\psi^\epsilon}$ is bounded and quasi-continuous, converging to $e^{b\varphi}$ in capacity. Similarly, $(\omega + \mathrm{dd^c}\varphi)^n$ is supported on $\{\varphi = 0\}$, hence letting $b \to \infty$ we arrive at our claim

$$\varlimsup_{\epsilon\to 0+} \int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n \leq \int_X \omega_\varphi^n = 0 \,.$$

Let $\delta > 0$ be arbitrary, and take $\epsilon = p/q \in \mathbb{Q}_+$ such that $\frac{1}{n!}\int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n < \delta$. By the positive mass case of this theorem,

$$\begin{aligned} \varlimsup_{k\to\infty, q|k} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) &\leq \varlimsup_{k\to\infty, q|k} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes L^{k\epsilon} \otimes \mathcal{I}(k\psi^\epsilon)) \\ &\leq \frac{1}{n!}\int_X (\omega + \epsilon\omega + \mathrm{dd^c}\psi^\epsilon)^n < \delta \,. \end{aligned} \tag{2.23}$$

For a general k (possibly not divisible by q) write $k = dq + r$ with $d \in \mathbb{Z}_{\geq 0}$, $r = 0, \ldots, q-1$. Then

$$\frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \frac{1}{(dq)^n} h^0(T \otimes L^r \otimes L^{dq} \otimes \mathcal{I}(dq\varphi))$$

for q large enough. Thus, replacing T with $T \otimes L^r$ as the twisting line bundle, we are reduced to the case $r = 0$, dealt with in (2.23). Letting $\delta \to 0$, the proof is finished. □

2.5. Filtrations, flag ideals and the non-Archimedean formalism.

Filtrations of the ring of sections. Let us recall the basics of filtrations in the context of canonical Kähler metrics, going back to work of Székelyhidi [Szé15]. We refer to [BHJ17, Section 1, Section 5] and [BJ18, Section 3] for a much more detailed description. In the sequel, we will focus on the point of view advocated by Ross–Witt Nyström [RW14]. For $r > 0$ we will consider

$$R(X, L^r) := \bigoplus_{k=0}^{\infty} H^0(X, L^{kr})$$

the graded ring associated to (X, L^r). When dealing with filtrations, we always assume that r is big enough so that $R(X, L^r)$ is generated in degree 1.

A *filtration* $\big(\{\mathcal{F}_k^\lambda\}_{\lambda\in\mathbb{R},k\in\mathbb{N}}, r\big)$ of $R(X, L^r)$ is a collection of decreasing left-continuous filtrations $\{\mathcal{F}_k^\lambda\}_\lambda$ on each vector space $H^0(X, L^{kr})$, that is *multiplicative* ($\mathcal{F}_k^\lambda \cdot \mathcal{F}_{k'}^{\lambda'} \subseteq \mathcal{F}_{k+k'}^{\lambda+\lambda'}$ for any $k, k' \in \mathbb{N}$, $\lambda, \lambda' \in \mathbb{R}$) and *linearly bounded* (there exists $\tilde{\lambda} > 0$ big enough such that $\mathcal{F}_k^{-\tilde{\lambda}k} = H^0(X, L^{kr})$ and $\mathcal{F}_k^{\tilde{\lambda}k} = \{0\}$, for $k \geq 0$.)

Let $(\mathbb{C}, \|\cdot\|)$ be the trivially normed complex line. A *non-Archimedean graded norm* on $R(X, L^r)$ is a norm on $R(X, L^r)$ considered as a $(\mathbb{C}, \|\cdot\|)$-algebra satisfying exponential boundedness (there exists $\tilde{\lambda} > 0$, such that for any $k \in \mathbb{N}$ and any non-zero $s \in H^0(X, L^{kr})$, $e^{-\tilde{\lambda}k} \leq \|s\|_k \leq e^{\tilde{\lambda}k}$) and sub-multiplicativity ($\|s \cdot s'\|_{k+k'} \leq \|s\|_k \|s'\|_{k'}$ for any $s \in H^0(X, L^{kr})$ and $s' \in H^0(X, L^{k'r})$).

It is elementary to verify that there is a bijection between filtrations $\big(\{\mathcal{F}_k^\lambda\}, r\big)$ and non-Archimedean graded norms $\{\|\cdot\|_k\}_{k\in\mathbb{N}}$ on $R(X, L^r)$ given by

$$\|s\|_k \leq e^{-\lambda} \Leftrightarrow s \in \mathcal{F}_k^\lambda\,, \quad k \in \mathbb{N}\,, \lambda \in \mathbb{R}\,, s \in H^0(X, L^{kr})\,.$$

Due to this, we will use the terms filtrations and non-Archimedean norms interchangeably.

Filtrations induced by test configurations and flag ideals. A filtration $\big(\{\mathcal{F}_k^\lambda\}, r\big)$ is a $\mathbb{Z}$-*filtration* if the jumping numbers/points of discontinuity of $\lambda \mapsto \mathcal{F}_k^\lambda$ are integers for all $k \geq 0$.

Due to the fact that $R(X, L^r)$ is generated in degree 1, we have a surjective map

$$\big(H^0(X, L^r)\big)^{\otimes k} \to H^0(X, L^{kr})\,. \tag{2.24}$$

Naturally, $\|\cdot\|_1$ induces a non-Archimedean norm on $(H^0(X, L^r))^{\otimes k}$, as well as on any quotient $(H^0(X, L^r))^{\otimes k}/W$, where $W \subseteq (H^0(X, L^r))^{\otimes k}$ is a subspace.

As a result, given a filtration $\big(\{\mathcal{F}_k^\lambda\}, r\big)$, it is possible to define a non-Archimedean graded norm $\|\cdot\|_k^T$ on each $H^0(X, L^{kr})$ only using $\|\cdot\|_1$ and the maps (2.24).

We say that $\big(\{\mathcal{F}_k^\lambda\}, r\big)$ is *induced by an (ample) test configuration* if it is a $\mathbb{Z}$-filtration, and the map (2.24) induces an isometry between the graded non-Archimedean norms $\|\cdot\|_k^T$ and $\|\cdot\|_k$ for any $k \geq 0$.

This of course is not the usual definition of (ample) test configurations. However, as pointed out in [BHJ17, Proposition 2.15], this construction is in a one-to-one correspondence with the usual one going back to Tian [Tia97] and Donaldson [Don01].

Flag ideals yield an important (and in many ways exhaustive) class of filtrations induced by test configurations, going back to Odaka [Oda13]. A *flag ideal* $\mathfrak{a}$ is a $\mathbb{C}^*$-invariant coherent ideal of $\mathcal{O}_{X\times\mathbb{C}}$, cosupported in $X \times \{0\}$. Such an ideal is always of the form

$$\mathfrak{a} = \sum_{j=0}^{d-1} \tau^j \mathfrak{a}_j + \tau^d \mathcal{O}_X\,,$$

where $\mathfrak{a}_j$ is an increasing sequence of coherent ideals of $\mathcal{O}_X$ and τ is the variable in $\mathbb{C}$. As a convention, we write $\mathfrak{a}_j = \mathcal{O}_X$, when $j \geq d$ and $\mathfrak{a}_j = 0$, when $j < 0$.

If for some $r > 0$, the sheaves $L^r \otimes \mathfrak{a}_i$ are globally generated for every $i \geq 0$, then we associate a filtration to $\mathfrak{a}$ in the following way. First we define $\mathcal{F}_0^\lambda := H^0(X, L^r \otimes \mathfrak{a}_{\lfloor -\lambda \rfloor})$. As $\{\mathcal{F}_0^\lambda\}_\lambda$ is decreasing and left-continuous, it induces a non-Archimedean norm $\|\cdot\|_1$

on $H^0(X, L^r)$, which further introduces a non-Archimedean graded norm $\|\cdot\|$ on $R(X, L^r)$ via the surjections (2.24).

By construction, the underlying $\mathbb{Z}$-filtration is clearly induced by a test configuration, with the jumping numbers of $\{\mathcal{F}_0^\lambda\}_\lambda$ being exactly the integers j such that $\mathfrak{a}_j \subsetneq \mathfrak{a}_{j+1}$. As pointed out by Odaka [Oda13], essentially all test configurations arise via this construction.

The non-Archimedean formalism. Here we recall some of the formalism developed in [BHJ17; BHJ19; BBJ21], and later tailor some of their results to our context.

By $X^{\mathrm{div}}_{\mathbb{Q}}$ we denote the set of rational divisorial valuations on X, i.e., valuations $v : \mathbb{C}(X) \to \mathbb{Q}$ of the form $v = c\,\mathrm{ord}_D$, with D being a prime divisor on some smooth variety Y, mapping to X via a projective modification, and $c \in \mathbb{Q}_+$. By convention, we also take the trivial valuation v_{triv} to be part of $X^{\mathrm{div}}_{\mathbb{Q}}$.

To any $v \in X^{\mathrm{div}}_{\mathbb{Q}}$ one associates $\sigma(v) \in (X \times \mathbb{C})^{\mathrm{div}}_{\mathbb{Q}}$, the *Gauss extension* of v. The construction is described in detail in [BHJ17, Section 4.1].

The Gauss extension is defined as $\sigma(v)(\sum_j f_j \tau^j) := \min_j(v(f_j) + j)$, where $f_j \in \mathbb{C}(X)$ and τ is the coordinate of $\mathbb{C}$. It can be immediately verified that $\sigma(v)$ thus defined is a valuation, moreover as shown in [BHJ17, Lemma 4.5, Theorem 4.6], $\sigma(v)$ is also divisorial on $X \times \mathbb{C}$.

Remark 2.29. In [BBJ21, Section 3] the authors define $X^{\mathrm{div}}_{\mathbb{Q}}$ as divisorial valuations on normal models of X. However, due to Hironaka's theorem, one can always take log-resolutions of normal models, hence no information is lost if one considers only prime divisors of smooth models, as we do in this work.

The non-Archimedean data of potentials, rays and flag ideals. In the non-Archimedean approach to canonical Kähler metrics one converts both analytic and algebraic data into *non-Archimedean data*, i.e., various functions on $X^{\mathrm{div}}_{\mathbb{Q}}$. We describe how this is carried out with ω-psh functions, geodesic rays and flag ideals.

Given $u \in \mathrm{PSH}(X, \omega)$, one defines $u^{\mathrm{an}} : X^{\mathrm{div}}_{\mathbb{Q}} \to \mathbb{R}$ by

$$u^{\mathrm{an}}(V) := -c\nu(u, D)\,, \quad V = c\,\mathrm{ord}_D \in X^{\mathrm{div}}_{\mathbb{Q}}\,. \tag{2.25}$$

Recall that $\nu(u, D)$ is the generic Lelong number of u along D (see Section 2.4). In accordance with the literature, sometimes we will also write $V(u) := c\nu(u, D)$.

Before proceeding, we note the following result, which corresponds to Theorem 2.13 in the non-Archimedean dictionary. Indeed, for any projective modification $\pi' : Y \to X$ with Y smooth, the Lelong number for any $u \in \mathrm{PSH}(X, \omega)$ at $y \in Y$ is the same as the generic Lelong number along the exceptional divisor of the blowup $\mathrm{Bl}_{\{y\}} Y$.

Proposition 2.30. *Suppose that $u, w \in \mathrm{PSH}(X, \omega)$. Then the following are equivalent:*

(i) $u \simeq_{\mathcal{I}} w$.

(ii) $u^{\mathrm{an}} = w^{\mathrm{an}}$.

Given a ray $\{r_t\}_t \in \mathcal{R}^1$, it is known that $t \mapsto \sup_X r_t$ is linear, in particular, there exists $l \in \mathbb{R}$ such that $\Phi(s, x) = r_{-\log|s|} + l \log|s| \in \mathrm{PSH}(X \times \mathbb{D}, \pi^*\omega)$, where $\mathbb{D}$ is the unit disk and $\pi : X \times \mathbb{D} \to X$ is the usual projection.

We define $r^{\mathrm{an}} : X^{\mathrm{div}}_{\mathbb{Q}} \to \mathbb{R}$ using the Gauss extension in the following manner:

$$r^{\mathrm{an}}(v) := -\sigma(v)(\Phi) + l\,, \tag{2.26}$$

where $\sigma(v) \in (X \times \mathbb{C})^{\mathrm{div}}_{\mathbb{Q}}$ is the Gauss extension of $v \in X^{\mathrm{div}}_{\mathbb{Q}}$ and $\sigma(v)(\Phi)$ is to be interpreted as a suitable multiple of the generic Lelong number along the center divisor E' of $\sigma(v)$. It can be seen that this definition does not depend on $l \in \mathbb{R}$, nor on the choice of smooth model hosting E'.

Lastly, by the same construction u^{an} can be defined for sublinear subgeodesic rays $\{u_t\}_t$ (as defined in Section 3.1 below).

Given a *flag ideal* $\mathfrak{a} = \sum_{j=0}^{d-1} \tau^j \mathfrak{a}_j + \tau^d \mathcal{O}_X$, such that $L^m \otimes \mathfrak{a}_j$ are globally generated for all j, we define the corresponding function non-Archimedean function $\varphi^{\mathrm{an}}_{\mathfrak{a}} : X^{\mathrm{div}}_{\mathbb{Q}} \to \mathbb{R}$ as follows:

$$\varphi^{\mathrm{an}}_{\mathfrak{a}}(v) := -\min_j (v(\mathfrak{a}_j) + j)\,,$$

where $v \in X^{\mathrm{div}}_{\mathbb{Q}}$, and $v(\mathfrak{a}_j)$ is the valuation of $\mathfrak{a}_j$ given by $v(\mathfrak{a}_j) := \inf\{\, v(a) : a \in \mathfrak{a}_j \,\}$.

Approximation by flag ideals. Given a ray $\{r_t\}_t \in \mathcal{R}^1$, in [BBJ21] the authors define an approximation scheme by flag ideals $\mathfrak{a}^m$ such that $\varphi^{\mathrm{an}}_{\mathfrak{a}^m} \searrow r^{\mathrm{an}}$. We describe the main point of this procedure, as it will be important in the sequel.

For simplicity, let us assume that $\{r_t\}_t \in \mathcal{R}^1$ satisfies $\sup_X r_t \leq 0$ for $t \geq 0$. The general case, can easily be reduced to this case, but one needs to slightly extend the definition of flag ideals (to allow for fractional ideals). We have $\Phi(s,x) = r_{-\log|s|} \in \mathrm{PSH}(X \times \mathbb{D}, \pi^*\omega)$, and we simply define [BBJ21, Section 5.3]:

$$\mathfrak{a}^m := \mathcal{I}(2^m \Phi)\,.$$

As pointed out in [BBJ21, Lemma 5.6] $L^{2^m + m_0} \otimes \mathfrak{a}^m_j$ is globally generated for some $m_0 > 0$ and all m, j. In addition, by the proof of [BBJ21, Theorem 6.2], the subbaditivity of multiplier ideals implies that $\varphi^{\mathrm{an}}_{\mathfrak{a}^m}$ is m-decreasing, moreover $\varphi^{\mathrm{an}}_{\mathfrak{a}^m}(v) \searrow r^{\mathrm{an}}(v)$ for $v \in X^{\mathrm{div}}_{\mathbb{Q}}$.

3. The structure of $\mathcal{R}^1$ and approximability

3.1. The extended Ross–Witt Nyström correspondence. The results of this subsection hold for an arbitrary Kähler manifold (X, ω). The goal is to give a duality between the finite energy geodesic rays of $\mathcal{R}^1_\omega$ and certain maximal test curves, reminiscent of [RW14] and [DDL18a], but to also give a formula the Monge–Ampère slope of L^1 rays in terms of their Legendre transforms. To do this we consider a wider context and generalize the discussion going back to [RW14], revisited in [DDL18a].

A *sublinear subgeodesic ray* is a subgeodesic ray $(0, +\infty) \ni t \mapsto u_t \in \mathrm{PSH}(X, \omega)$ (notation $\{u_t\}_{t>0}$) such that $u_t \to_{L^1} u_0 := 0$ as $t \to 0$, and there exists $C \in \mathbb{R}$ such that $u_t(x) \leq Ct$ for all $t \geq 0$, $x \in X$.

Due to t-convexity, we obtain some immediate properties of sublinear subgeodesic rays:

Lemma 3.1. *Suppose that $\{u_t\}_t$ is a sublinear subgeodesic ray. Then the set $\{u_t > -\infty\}$ is the same for any $t > 0$. In particular, for any $x \in X$ the curve $t \mapsto u_t(x)$ is either finite and convex on $(0, \infty)$, or equal to $-\infty$ on this interval.*

A *psh geodesic ray* is a sublinear subgeodesic ray that additionally satisfies the following maximality property: for any $0 < a < b$, the subgeodesic $(0,1) \ni t \mapsto v_t^{a,b} := u_{a(1-t)+bt} \in \mathrm{PSH}(X,\omega)$ can be recovered in the following manner:

$$v_t^{a,b} := \sup_{h \in \mathcal{S}} h_t\,, \quad t \in [0,1]\,, \tag{3.1}$$

where $\mathcal{S}$ is the set of subgeodesics $(0,1) \ni t \to h_t \in \mathrm{PSH}(X,\omega)$ such that

$$\lim_{t \searrow 0} h_t \leq u_a\,, \quad \lim_{t \nearrow 1} h_t \leq u_b\,.$$

We note the following properties of the map $v \mapsto \sup_X v$ along rays:

Lemma 3.2. *For any psh geodesic ray $\{u_t\}_t$, the map $t \mapsto \sup_X u_t$ is linear. For sublinear subgeodesics, the map $t \mapsto \sup_X u_t$ is only convex.*

The statement for subgeodesics is a consequence of t-convexity. To argue the statement for psh geodesic rays, one can simply use [Dar17, Theorem 1] together with approximation by bounded geodesics, and the continuity of $u \mapsto \sup_X u$ in the weak L^1-topology of $\mathrm{PSH}(X,\omega)$.

Making small tweaks to [RW14, Definition 5.1], we are ready to give the definition of test curves:

Definition 3.3. A map $\mathbb{R} \ni \tau \mapsto \psi_\tau \in \mathrm{PSH}(X,\omega)$ is a *psh test curve*, denoted $\{\psi_\tau\}_{\tau\in\mathbb{R}}$, if

(i) $\tau \mapsto \psi_\tau(x)$ is concave, decreasing and usc for any $x \in X$.

(ii) $\psi_\tau \equiv -\infty$ for all τ big enough, and ψ_τ increases a.e. to 0 as $\tau \to -\infty$.

Note that this definition is more general than the one in [RW14] (where the authors only considered potentials with small unbounded locus), even more general than the one in [DDL18a] (where the authors considered only bounded test curves). Moreover, condition (ii) allows for the introduction of the following constant:

$$\tau_\psi^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau \equiv -\infty\}\,. \tag{3.2}$$

Remark 3.4. We adopt the following notational convention: psh test curves will always be parametrized by τ, whereas rays will be parametrized by t. Hence $\{\psi_t\}_t$ will always refer to some kind of ray, whereas $\{\phi_\tau\}_\tau$ will refer to some type of test curve. As we prove below, rays and test curves are dual to each other, so one should think of the parameters t and τ to be dual to each other as well.

Definition 3.5. A psh test curve $\{\psi_\tau\}_\tau$ can have the following properties:

(i) $\{\psi_\tau\}_\tau$ is *maximal* if $P[\psi_\tau] = \psi_\tau$ for any $\tau \in \mathbb{R}$.

(ii) $\{\psi_\tau\}_\tau$ is *$\mathcal{I}$-maximal* if $P[\psi_\tau]_\mathcal{I} = \psi_\tau$ for any $\tau \in \mathbb{R}$.

(iii) $\{\psi_\tau\}_\tau$ is a *finite energy test curve* if

$$\int_{-\infty}^{\tau_\psi^+} \left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n \right) d\tau > -\infty\,. \tag{3.3}$$

(iv) We say (ψ_τ) is *bounded* if $\psi_\tau = 0$ for all τ small enough. In this case, one can introduce the following constant, complementing (3.2):

$$\tau_\psi^- := \sup\{\, \tau \in \mathbb{R} : \psi_\tau \equiv 0 \,\}\,. \tag{3.4}$$

In the above definition, we followed the convention $P[-\infty] = P[-\infty]_{\mathcal{I}} = -\infty$. Note that bounded test curves are clearly of finite energy.

We recall the *Legendre transform*, that will help establish the duality between various types of maximal test curves and geodesic rays. Given a convex function $f : [0, +\infty) \to \mathbb{R}$, its Legendre transform is defined as

$$\hat{f}(\tau) := \inf_{t \geq 0}(f(t) - t\tau) = \inf_{t>0}(f(t) - t\tau)\,, \quad \tau \in \mathbb{R}\,. \tag{3.5}$$

The *(inverse) Legendre transform* of a decreasing concave function $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is

$$\check{g}(t) := \sup_{\tau \in \mathbb{R}}(g(\tau) + t\tau)\,, \quad t \geq 0\,. \tag{3.6}$$

We point out that there is a sign difference in our choice of Legendre transform compared to the convex analysis literature, however this choice will be more suitable for us.

As it is well known, for every $\tau \in \mathbb{R}$ we have that $\hat{\check{g}}(\tau) \geq g(\tau)$ with equality if and only if g is additionally τ-usc. Similarly, $\check{\hat{f}}(t) \leq f(t)$ for all $t \geq 0$ with equality if and only if f is t-lsc. In general, $\hat{\check{g}}$ is the τ-usc envelope of g, and $\check{\hat{f}}$ is the t-lsc envelope of f. We will refer to these facts commonly as the involution property of the Legendre transform.

Starting with a psh test curve $\{\psi_\tau\}_\tau$, our goal will be to construct a geodesic/subgeodesic ray by taking the τ-inverse Legendre transform. The first step is the next proposition which was essentially proved in [Dar17]:

Proposition 3.6. *Suppose $\{\psi_\tau\}_\tau$ is a psh test curve. Then $\sup_\tau(\psi_\tau(x) + t\tau)$ is usc with respect to $(t, x) \in (0, \infty) \times X$.*

Since $\tau^+_\psi < \infty$ and $\psi_\tau \leq 0, \tau \in \mathbb{R}$, we note that $\sup_\tau(\psi_\tau + t\tau) \leq t\tau^+_\psi$ for $t \geq 0$. Also, for $t = 0$ the above proposition may fail.

Proof. Let $S = \{\operatorname{Re} s > 0\}$. In the proof, $\operatorname{usc}(\cdot)$ will denote the usc regularization on $S \times X$. We consider the usual complexification of the inverse Legendre transform:

$$u(s, z) := \sup_\tau(\psi_\tau(z) + \tau \operatorname{Re} s)\,, \quad (s, z) \in S \times X\,.$$

Also, $u_t(x) := u(t, x) \leq t\tau^+_\psi, t > 0$. Clearly, $\operatorname{usc} u \in \operatorname{PSH}(S \times X, \pi^*\omega)$, where $\operatorname{usc} u$ is the usc regularization of u on $S \times X$. Let $\pi : S \times X \to X$ be the natural projection. It will be enough to show that $\operatorname{usc} u = u$.

We introduce $E = \{u < \operatorname{usc} u\} \subseteq S \times X$. As both u and $\operatorname{usc} u$ are $\mathbb{R}$-invariant in the imaginary direction of S, it follows that E is also $\mathbb{R}$-invariant, i.e., there exists $B \subseteq (0, \infty) \times X$ such that $E = B \times \mathbb{R}$.

As E has Monge–Ampère capacity zero, it follows that E has Lebesgue measure zero. By Fubini's theorem $B \subseteq (0, \infty) \times X$ has Lebesgue measure zero as well. For $z \in X$, we introduce the slices:

$$B_z = B \cap ((0, \infty) \times \{z\})\,.$$

By Fubini's theorem again, we have that B_z has Lebesgue measure 0 for all $z \in X \setminus F$, where $F \subseteq X$ is some set of Lebesgue measure 0.

By slightly increasing F, but not its zero Lebesgue measure(!), we can additionally assume that $u_t(z) > -\infty$ for all $t > 0$ and $z \in X \setminus F$ (indeed, at least one potential ψ_τ is not identically equal to $-\infty$).

Let $z \in X \setminus F$. We argue that B_z is in fact empty. By our assumptions on F, both maps $t \mapsto u_t(z)$ and $t \mapsto (\operatorname{usc} u)(t,z)$ are locally bounded and convex (hence continuous) on $(0,\infty)$. As they agree on the dense set $(0,\infty) \setminus B_z$, it follows that they have to be the same, hence $B_z = \emptyset$. This allows to conclude that

$$\inf_{t>0}[u_t(x) - \tau t] = \chi_\tau := \inf_{t>0}\left[(\operatorname{usc} u)(t,x) - \tau t\right], \quad \tau \in \mathbb{R} \text{ and } z \in X \setminus F. \tag{3.7}$$

By duality of the Legendre transform $\psi_\tau(x) = \inf_{t>0}[u_t(x) - t\tau]$ for all $x \in X$ and $\tau \in \mathbb{R}$ (here is where the τ-usc property of $\tau \mapsto \psi_\tau$ is used). From this and (3.7) it follows that $\psi_\tau = \chi_\tau$ a.e. on X, for all $\tau \in \mathbb{R}$. Since both ψ_τ and χ_τ are ω-psh (the former by definition, the latter by Kiselman's minimum principle [Dem12, Theorem I.7.5]), it follows that in fact $\psi_\tau \equiv \chi_\tau$ for all $\tau \in \mathbb{R}$.

Consequently, applying the τ-Legendre transform to the τ-usc and τ-concave curves $\tau \mapsto \psi_\tau$ and $\tau \mapsto \chi_\tau$, we obtain that $u_t(x) = \operatorname{usc} u(t,x)$ for all $(t,x) \in (0,\infty) \times X$. □

Given a sublinear subgeodesic ray $\{\phi_t\}_t$ (psh test curve $\{\psi_\tau\}_\tau$), we can associate its (inverse) Legendre transform at $x \in X$ as

$$\begin{aligned} \hat{\phi}_\tau(x) &:= \inf_{t>0}(\phi_t(x) - t\tau), \quad \tau \in \mathbb{R}, \\ \check{\psi}_t(x) &:= \sup_{\tau\in\mathbb{R}}(\psi_\tau(x) + t\tau), \quad t > 0. \end{aligned} \tag{3.8}$$

Our main theorem describes a duality between various types of rays and maximal test curves, extending various particular cases from [RW14], [DDL18a]:

Theorem 3.7. *The Legendre transform $\{\psi_\tau\}_\tau \mapsto \{\check{\psi}_t\}_t$ gives a bijective map with inverse $\{\phi_t\}_t \mapsto \{\hat{\phi}_\tau\}_\tau$ between:*

(i) *psh test curves and sublinear subgeodesic rays,*

(ii) *maximal psh test curves and psh geodesic rays,*

(iii) *[RW14], [DDL18a] maximal bounded test curves and bounded geodesic rays. In this case, we additionally have that*

$$\tau_\psi^- t \le \check{\psi}_t \le \tau_\psi^+ t, \quad t \ge 0.$$

(iv) *maximal finite energy test curves and finite energy geodesic rays. In this case, we additionally have that*

$$\mathrm{I}\{\check{\psi}\} = \frac{1}{V}\int_{-\infty}^{\tau_\psi^+}\left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n\right) d\tau + \tau_\psi^+. \tag{3.9}$$

Recall that the functional I is defined in (1.2).

Proof. We prove (i). This is essentially [DDL18a, Proposition 4.4], where an important particular case was addressed. Let $\{\psi_\tau\}_\tau$ be a psh test curve. Then $\check{\psi}_t \in \mathrm{PSH}(X,\omega)$ for all $t > 0$ due to Proposition 3.6. We also see that $\sup_X \check{\psi}_t \le t\tau_\psi^+$, and $\check{\psi}_t \to_{L^1} 0$ as $t \to 0$, proving that $\{\check{\psi}_t\}_t$ is a subgeodesic.

For the reverse direction, let $\{\phi_t\}_t$ be a sublinear subgeodesic ray. Then $\hat{\phi}_\tau \in \mathrm{PSH}(X,\omega)$ or $\hat{\phi}_\tau \equiv \infty$ for any $\tau \in \mathbb{R}$ due to Kiselman's minimum principle. By properties of Legendre transforms and Lemma 3.1, we get that $\tau \mapsto \hat{\phi}_\tau(x)$ is τ-usc, τ-concave and decreasing. Due to sublinearity of $\{\phi_t\}_t$ we get that $\hat{\phi}_\tau \equiv -\infty$ for τ big enough. Lastly $\psi_\tau \nearrow 0$ a.e. as $\tau \to -\infty$, since $\phi_t \to_{L^1} 0$ as $t \to 0$.

We prove (ii). From [Dar17, Propisition 5.1] (that only uses the maximum principle (3.1)) we obtain that for any psh geodesic ray $\{u_t\}_t$, the curve $\{\hat{u}_\tau\}_\tau$ is a maximal psh test curve.

Let $\{\psi_\tau\}_\tau$ be a maximal psh test curve. We will show that the sublinear subgeodesic $\{\check{\psi}_t\}_t$ is a psh geodesic ray. By elementary properties of the Legendre transform we can assume that $\tau^+_\psi = 0$, in particular $\{\check{\psi}_t\}_t$ is t-decreasing.

Now assume by contradiction that $\{\check{\psi}_t\}_t$ is not a psh geodesic ray. Comparing with (3.1), there exists $0 < a < b$ such that

$$\check{\psi}_{(1-t)a+tb} \lneq \chi_t := \sup_{h \in \mathcal{S}} h_t\,, \quad t \in [0,1]\,,$$

where $\mathcal{S}$ is the set of subgeodesics $(a,b) \ni t \mapsto h_t \in \mathrm{PSH}(X,\omega)$ satisfying $\lim_{t\to a+} h_t \leq \check{\psi}_a$ and $\lim_{t \to b-} h_t \leq \check{\psi}_b$. Now let $\{\phi_t\}_t$ be the sublinear subgeodesic such that $\phi_t := \check{\psi}_t$ for $t \notin (a,b)$ and $\phi_{a(1-t)+bt} := \chi_t$ otherwise.

Trivially, $\check{\psi}_t \leq \phi_t \leq 0$, hence by duality, $\psi_\tau \leq \hat{\phi}_\tau$ and $\tau^+_\psi = \tau^+_{\hat{\phi}} = 0$. However, comparing with (3.8), we claim that $\hat{\phi}_\tau \leq \psi_\tau + \tau(a-b)$ for any $\tau \in \mathbb{R}$. Since $\tau^+_\psi = \tau^+_{\hat{\phi}} = 0$, we only need to show this for $\tau \leq 0$. For such τ we indeed have

$$\inf_{t\in[a,b]}(\phi_t - t\tau) \leq \phi_b - b\tau = \check{\psi}_b - b\tau \leq \inf_{t \in [a,b]}(\check{\psi}_t - t\tau) + (a-b)\tau\,,$$

where in the last inequality we used that $t \mapsto \check{\psi}_t$ is decreasing.

By the maximality of $\{\psi_\tau\}_\tau$, we obtain that $\psi_\tau = \hat{\phi}_\tau$. An application of the Legendre transform now gives that $\check{\psi}_t = \phi_t$, a contradiction. Hence $\{\psi_t\}_t$ is a psh geodesic ray.

The duality of (iii) is simply [DDL18a, Theorem 1.3], closely following [RW14].

We deal with (iv). As before, we may assume that $\tau^+_\psi = 0$. As a preliminary result, in Proposition 3.8 below we prove (3.9) for bounded maximal test curves.

Given a finite energy maximal test curve $\{\psi_\tau\}_\tau$, we know that $\{\check{\psi}\}_t$ is a psh geodesic ray. By [DL20, Theorem 4.5] and its proof there exists bounded geodesic rays $\{\check{\psi}^k_t\}_t$ such that $\check{\psi}^k_t \searrow \check{\psi}_t$ for any $t \geq 0$, and $\int_X \omega^n_{\psi^k_\tau} \searrow \int_X \omega^n_{\psi_\tau}$ for any $\tau < \tau^+_\psi = \tau^+_{\psi^k} = 0$ (see especially the last displayed equation of [DL20, pp. 17]). Indeed, the arguments of [DL20, Theorem 4.5, Lemma 4.6] work for general psh rays, without change.

By Proposition 3.8 below

$$\mathrm{I}\{\check{\psi}^k_t\} = \frac{1}{V}\int_{-\infty}^0 \left(\int_X \omega^n_{\psi^k_\tau} - \int_X \omega^n\right) \mathrm{d}\tau\,.$$

The right hand side is bounded from below, since $\{\psi_\tau\}_\tau$ is a finite energy test curve. Since $\int_X \omega^n_{\psi^k_\tau} \searrow \int_X \omega^n_{\psi_\tau}$, we can take the limit on both the left and right hand side, to arrive at (3.9), also implying that $\{\check{\psi}_t\}_t$ is a finite energy geodesic ray.

Conversely, assume that $\{\phi_t\}_t$ is a finite energy geodesic ray, with decreasing approximating sequence of bounded rays $\{\phi^k_t\}_t$, as detailed above. For similar reasons we have $\mathrm{I}\{\phi_t^k\} = \frac{1}{V}\int_{-\infty}^0 \left(\int_X \omega^n_{\hat\phi^k_\tau} - \int_X \omega^n\right) d\tau$. Since $\mathbf{I}\{\phi_t^k\} \searrow \mathrm{I}\{\phi_t\}$, the monotone convergence theorem gives that (3.9) holds for $\{\hat\phi_\tau\}_\tau$, finishing the proof. □

As promised, to complete the argument of Theorem 3.7, we prove the next proposition, whose argument can be extracted from [RW14, Section 6] with additional references to [DDL18b]. We recall the precise details here as the results of [RW14] were proved in the context of potentials with small unbounded locus.

Proposition 3.8. *Suppose that $\{\psi_\tau\}_\tau$ is a bounded maximal test curve with $\tau^+_\psi = 0$. Then*

$$\frac{\mathrm{I}(\check\psi_t)}{t} = \mathrm{I}\{\check\psi_t\} = \frac{1}{V}\int_{-\infty}^0 \left(\int_X \omega^n_{\psi_\tau} - \int_X \omega^n\right) d\tau\,, \quad t>0\,. \tag{3.10}$$

Proof. Without loss of generality we assume that $V=1$. For $N\in\mathbb{Z}_+, M\in\mathbb{Z}$ and $t>0$, we introduce the following:

$$\check\psi_t^{N,M} := \max_{\substack{k\in\mathbb{Z}\\ k\le M}}\left(\psi_{k/2^N} + tk/2^N\right)\,.$$

It is clear that $\check\psi_t^{N,M}\in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, since it is a maximum of a finite number of ω-psh potentials (here we also used that $\{\psi_\tau\}_\tau$ is a bounded test curve). Moreover, we now argue that

$$\frac{t}{2^N}\int_X \omega^n_{\psi_{(M+1)/2^N}} \le \mathrm{I}(\check\psi_t^{N,M+1}) - \mathrm{I}(\check\psi_t^{N,M}) \le \frac{t}{2^N}\int_X \omega^n_{\psi_{M/2^N}}\,. \tag{3.11}$$

Indeed, for elementary reasons:

$$\int_X\left(\check\psi_t^{N,M+1}-\check\psi_t^{N,M}\right)\omega^n_{\check\psi_t^{N,M+1}} \le \mathrm{I}(\check\psi_t^{N,M+1})-\mathrm{I}(\check\psi_t^{N,M}) \le \int_X\left(\check\psi_t^{N,M+1}-\check\psi_t^{N,M}\right)\omega^n_{\check\psi_t^{N,M}}\,. \tag{3.12}$$

Clearly $\check\psi_t^{N,M+1}\ge\check\psi_t^{N,M}$, and using τ-concavity we notice that

$$U_t := \left\{\check\psi_t^{N,M+1}-\check\psi_t^{N,M}>0\right\} = \left\{\psi_{(M+1)/2^N}+2^{-N}t-\psi_{M/2^N}>0\right\}\,.$$

Moreover, on U_t we have

$$\check\psi_t^{N,M+1} = \psi_{(M+1)/2^N}+t(M+1)/2^N\,,\quad \check\psi_t^{N,M} = \psi_{M/2^N}+tM/2^N\,.$$

We also note that U_t is an open set in the plurifine topology, implying that $\omega^n_{\psi_{(M+1)/2^N}}\big|_{U_t} = \omega^n_{\check\psi_t^{N,M+1}}\big|_{U_t}$ and $\omega^n_{\psi_{M/2^N}}\big|_{U_t} = \omega^n_{\check\psi_t^{N,M}}\big|_{U_t}$. Recall that $\omega^n_{\psi_{M/2^N}}$ and $\omega^n_{\psi_{(M+1)/2^N}}$ are supported on the sets $\{\psi_{M/2^N}=0\}$ and $\{\psi_{(M+1)/2^N}=0\}$ respectively [DDL18b, Theorem 3.8]. Since $\{\psi_{(M+1)/2^N}=0\}\subseteq U_t$ and $\{\psi_{(M+1)/2^N}=0\}\subset\{\psi_{M/2^N}=0\}$, applying the above to (3.12), we arrive at (3.11).

Fixing N, let M be the biggest integer to the left of $2^N\tau^-_\psi$. Then repeated application of (3.11) yields

$$\sum_{M+1\le j\le 0}\frac{t}{2^N}\int_X\omega^n_{\psi_{j/2^N}} \le \mathrm{I}(\check\psi_t^{N,0})-\mathrm{I}(\check\psi_t^{N,M}) \le \sum_{M\le j\le -1}\frac{t}{2^N}\int_X\omega^n_{\psi_{j/2^N}}\,.$$

Since $M\le 2^N\tau^-_\psi$ we have that $\check\psi_t^{N,M} = \psi_{M/2^N}+tM/2^N = tM/2^N$, we can continue to write

$$\sum_{j=M+1}^{0}\frac{t}{2^N}\left(\int_X\omega^n_{\psi_{j/2^N}}-\int_X\omega^n\right) \le \mathrm{I}(\check\psi_t^{N,0}) \le \sum_{j=M}^{-1}\frac{t}{2^N}\left(\int_X\omega^n_{\psi_{j/2^N}}-\int_X\omega^n\right)\,.$$

We now notice that we have Riemann sums on both the left and right of the above inequality. Using Lemma 3.9 below, it is possible to let $N \to \infty$ and obtain (3.10), as desired. □

Lemma 3.9. *Suppose that $\{\psi_\tau\}_\tau$ is a psh test curve. Then $\tau \mapsto \int_X \omega^n_{\psi_\tau} > 0$ is a continuous function for $\tau \in (-\infty, \tau^+_\psi)$.*

By working harder, using [DDL21, Theorem B], one can show that $\tau \mapsto \left(\int_X \omega^n_{\psi_\tau}\right)^{1/n}$ is concave, however we will not need this in the sequel.

PROOF. First we argue positivity. Since $\psi_\tau \nearrow 0$ a.e, as $\tau \to -\infty$, [DDL18b, Theorem 2.3] gives $\int_X \omega^n_{\psi_\tau} \nearrow \int_X \omega^n > 0$ as $\tau \to -\infty$. Let $\tau \in (-\infty, \tau^+_\psi)$ be arbitary. Pick $\tau_1 \in (\tau, \tau^+_\psi)$ and $\tau_0 < \tau$ such that $\int_X \omega_{\psi^n_{\tau_0}} > 0$. Let $\alpha \in (0,1)$ such that $\tau = \alpha\tau_0 + (1-\alpha)\tau_1$. By τ-concavity and [Wit19, Theorem 1.1] we obtain that $\int_X \omega^n_{\psi_\tau} \geq \int_X \omega^n_{\psi_{\alpha\tau_0+(1-\alpha)\tau_1}} \geq \alpha^n \int_X \omega^n_{\psi_{\tau_0}} > 0$, as desired.

Next, we argue continuity. We know that $\tau \mapsto \psi_\tau$ is τ-decreasing. Fix $\tau_0 \in (-\infty, \tau^+_\psi)$ then $\int_X \omega^n_{\psi_\tau} \nearrow \int_X \omega^n_{\psi_{\tau_0}}$ as $\tau \searrow \tau_0$ by [DDL18b, Theorem 2.3]. Now we argue that $\int_X \omega^n_{\psi_\tau} \searrow \int_X \omega^n_{\psi_{\tau_0}}$ as $\tau \nearrow \tau_0$. For $\epsilon > 0$ small, using the τ-convexity of $\tau \mapsto \psi_\tau$ together with monotonicity and multi-linearity of the non-pluripolar measure, we have

$$\frac{1}{2^n}\int_X \omega^n_{\psi_{\tau_0-\epsilon}} + \frac{2^n-1}{2^n}\int_X \omega^n_{\psi_{\tau_0+\epsilon}} \leq \int_X \omega^n_{\frac{1}{2}\psi_{\tau_0-\epsilon}+\frac{1}{2}\psi_{\tau_0+\epsilon}} \leq \int_X \omega^n_{\psi_{\tau_0}}\,.$$

Letting $\epsilon \searrow 0$, we know that $\int_X \omega^n_{\psi_{\tau_0+\epsilon}} \to \int_X \omega^n_{\psi_{\tau_0}}$, hence $\int_X \omega^n_{\psi_{\tau_0-\epsilon}} \searrow \int_X \omega^n_{\psi_{\tau_0}}$. □

The technique in the proof of Proposition 3.8 can also be applied to other energy functionals. We refer to [Xia20] for details.

3.2. Rays induced by filtrations and approximability. We fix a filtration $(\{\mathcal{F}^\lambda_k\}, r)$ on $R(X, L^r)$. Following [RW14, Section 7], one can associate a maximal test curve to this filtration in the following manner. The corresponding construction for test configurations is due to Phong–Sturm [PS07], [PS10]. For $\tau \in \mathbb{R}$, let

$$\hat{u}^k_\tau := \sup_{s \in \mathcal{F}^{\tau k}_k, h^k(s,s)\leq 1} \frac{1}{kr}\log h^{kr}(s,s) \leq 0\,. \tag{3.13}$$

Since each $\mathcal{F}^{\tau k}_k$ is finite dimensional, one notices that $\hat{u}^k_\tau \in \mathrm{PSH}(X,\omega)$ has analytic singularity type. Moreover, by the multiplicativity of the filtration we have that

$$k\hat{u}^k_\tau + k'\hat{u}^{k'}_\tau \leq (k+k')\hat{u}^{k+k'}_\tau \leq 0\,. \tag{3.14}$$

As a result, Fekete's lemma implies that $\hat{u}_\tau := \lim_k \hat{u}^k_\tau = \sup_k \hat{u}^k_\tau \in \mathrm{PSH}(X,\omega)$ exists, and the curve $\tau \mapsto \hat{u}_\tau$ has a number of special properties.

THEOREM 3.10. *[RW14, Proposition 7.7, Proposition 7.11] For any filtration $(\{\mathcal{F}^\lambda_k\}, r)$ the potentials $\{\hat{u}_\tau\}_\tau$ form a maximal bounded test curve. In particular, by Theorem 3.7 they induce a ray of bounded potentials $\{u_t\}_t \in \mathcal{R}^\infty$.*

We give a very brief sketch of the argument. As elaborated in [RW14, Section 7], that $\{\hat{u}_\tau\}_\tau$ is τ-concave and τ-decreasing is a consequence of the multiplicativity of the filtration. Boundedness follows from linear boundedness of the filtration. To make sure that $\{\hat{u}_\tau\}_\tau$ is τ-usc we take $\hat{u}_{\tau^+_u} := \lim_{\tau \searrow \tau^+_u} \hat{u}_\tau$ [DDL18a, Lemma 4.3]. Regarding maximality, $P[\hat{u}_\tau] = \hat{u}_\tau$, for $\tau < \tau_u^+$ is a consequence of the Skoda division theorem. That $P[\hat{u}_{\tau_u^+}] = \hat{u}_{\tau_u^+}$ follows since $\hat{u}_\tau \searrow \hat{u}_{\tau_u^+}$ as $\tau \searrow \tau_u^+$ [DDL21, Corollary 4.7].

Recall the notion of $\mathcal{I}$-maximal test curves from Definition 3.5. As $\mathcal{I}$-maximal test curves are maximal (Proposition 2.18), we give the following improvement to the above result:

THEOREM 3.11. *For any filtration $(\{\mathcal{F}_k^\lambda\}, r)$ the curve $\{\hat{u}_\tau\}_\tau$ is a bounded $\mathcal{I}$-maximal test curve.*

PROOF. Using the previous result, we only have to show that $P[\hat{u}_\tau]_\mathcal{I} = \hat{u}_\tau$ for $\tau \leq \tau_u^+$. Due to (3.14) we have that $\hat{u}_\tau^{2^j} \leq \hat{u}_\tau^{2^{j+1}}$, moreover $\hat{u}_\tau = \lim_j \hat{u}_\tau^{2^j}$. By maximality of $\hat{u}_\tau$, we have that $\hat{u}_\tau^{2^j} \leq P[\hat{u}_\tau^{2^j}] \leq P[\hat{u}_\tau] = \hat{u}_\tau$, in particular, $P[\hat{u}_\tau^{2^j}] \nearrow \hat{u}_\tau$.

Let us assume momentarily that $\tau < \tau_u^+$. Then $\int_X \omega_{\hat{u}_\tau} > 0$ by Lemma 3.9. By Lemma 2.21(iii), to conclude that $P[\hat{u}_\tau]_\mathcal{I} = \hat{u}_\tau$, we only need to argue that $P[\hat{u}_\tau^{2^j}]$ is $\mathcal{I}$-model, which follows from Proposition 2.20.

In case when $\tau = \tau_u^+$, notice that $P[\hat{u}_{\tau-\epsilon}]_\mathcal{I} = \hat{u}_{\tau-\epsilon} \searrow \hat{u}_\tau$ as $\epsilon \searrow 0$. Hence by Lemma 2.21(i), and what we just proved, we get that $P[\hat{u}_\tau]_\mathcal{I} = \hat{u}_\tau$, as desired. □

Test configurations and approximable rays. We introduce some preliminary terminology, aiding our discussion in this paragraph.

Definition 3.12. We say that a ray $\{r_t\}_t \in \mathcal{R}^1$ is *approximable* if there exists rays $\{r_t^j\}_t \in \mathcal{R}^1$ induced by test configurations such that $r_t^j \searrow r_t$ for $t \geq 0$.

In the terminology of [BBJ21], approximable rays $\{r_t\}_t$ are called *maximal*. By [DL20, Lemma 4.3] we obtain that $d_1^c(\{r_t^j\}_t, \{r_t\}_t) \to 0$, in particular $\{r_t\}_t \in \overline{\mathcal{T}}$, where $\mathcal{T}$ is the set of rays induced by ample test configurations.

Due to the completeness of $\mathcal{E}^{1,\mathrm{an}}$ proved by Boucksom–Jonsson (see [Xia21, Example 3.3], or Theorem 3.18 below), $\{v_t\}_t \in \overline{\mathcal{T}}$ if and only if it is approximable, but we will not rely on this property in the present section.

We recall the following potential theoretic interpretation of r^{an} from [BBJ21, Section 4.3] in terms of the Legendre transform:

Proposition 3.13. *Let $\{r_t\}_t \in \mathcal{R}^1$ such that $\tau_{\hat{r}}^+ = \sup_X r_t \leq 0$, for $t \geq 0$. Let $v \in X_\mathbb{Q}^{\mathrm{div}}$. Then*

$$
\begin{aligned}
r^{\mathrm{an}}(v) &= -\inf_{\tau<\tau_{\hat{r}}^+}(v(\hat{r}_\tau)-\tau) = -\inf_{\tau\in\mathbb{R}}(v(\hat{r}_\tau)-\tau) \\
&= \sup_{\tau<\tau_{\hat{r}}^+}(\hat{r}_\tau^{\mathrm{an}}(v)+\tau) = \sup_{\tau\in\mathbb{R}}(\hat{r}_\tau^{\mathrm{an}}(v)+\tau)\,.
\end{aligned} \tag{3.15}
$$

where $\{\hat{r}_\tau\}_\tau$ is the maximal finite energy test curve of $\{r_t\}_t$. By convention, $v(-\infty) = \infty$.

PROOF. We only need to prove the very first equality. Recall from Theorem 3.7 the following duality between $\{r_t\}_t$ and its maximal test curve:

$$
r_t = \sup_{\tau<\tau_{\hat{r}}^+}(\hat{r}_\tau + t\tau)\,, \quad t \geq 0\,. \tag{3.16}
$$

Let $v \in X_\mathbb{Q}^{\mathrm{div}}$, and $\sigma(v) \in (X \times \mathbb{C})_\mathbb{Q}^{\mathrm{div}}$ be the corresponding Gauss extension (see Section 2.5). Since $\tau_{\hat{r}}^+ \leq 0$, by Lemma 3.14 below we conclude that $-r^{\mathrm{an}}(v) = \sigma(v)(\Phi) = \inf_{\tau<\tau_{\hat{r}}^+}(v(\hat{r}_\tau)-\tau)$. □

Lemma 3.14. *Let Ω be a complex manifold. Let $\mathcal{F}$ be a non-empty family of non-positive psh functions on Ω and $\psi := \operatorname{usc}\left(\sup_{\varphi\in\mathcal{F}} \varphi\right)$. Then for any $x \in \Omega$,*

$$\nu(\psi, x) = \inf_{\varphi\in\mathcal{F}} \nu(\varphi, x)\,. \tag{3.17}$$

Proof. By Choquet's lemma, we may assume that $\mathcal{F}$ consists of only countably many functions φ_j ($j \in \mathbb{N}$) and $\psi = \operatorname{usc}\left(\sup_{j\in\mathbb{N}} \varphi_j\right)$.

By upper semicontinuity of Lelong numbers, $\nu(\psi, x) \geq \overline{\lim}_{j\in\mathbb{N}} \nu(\max\{\varphi_0, \dots, \varphi_j\}, x)$. In addition, by monotonicity of Lelong numbers, $\nu(\psi, x) = \lim_{j\in\mathbb{N}} \nu(\max\{\varphi_0, \dots, \varphi_j\}, x) = \inf_j \nu(\varphi_j, x)$, where in the last step we used [Bou17, Corollary 2.10]. $\square$

Given a ray $\{r_t\}_t \in \mathcal{R}^1$, we define two associated envelopes, based on the ideas of [BBJ21]. For $t \geq 0$ let

$$\Pi(r_t) := \inf\left\{ r'_t : \{r'_t\}_t \in \mathcal{R}^1 \text{ is induced by a test configuration and } r'_t \geq r_t \right\},$$

A priori, it is not even clear that $\{\Pi(r_t)\}_t$ is a geodesic ray. On the other hand, following the argument of [BBJ21, Theorem 6.6], for $t \geq 0$ we can also consider

$$\pi(r_t) := \sup\left\{ \{r''_t\}_t \in \mathcal{R}^1 : r''^{\mathrm{an}} = r^{\mathrm{an}} \right\}.$$

As we prove now, these two projections coincide to give a ray, whose maximal test curve can be described concretely:

Theorem 3.15. *Let $\{r_t\}_t \in \mathcal{R}^1$. Then $\{\Pi(r)_t\}_t$ is an approximable geodesic ray. Moreover the following hold:*

(i) $\widehat{\Pi(r)}_\tau = \widehat{\pi(r)}_\tau = P[\hat{r}_\tau]_{\mathcal{I}}$, $\tau \neq \tau^+_{\hat{r}}$.

(ii) $\Pi(r)_t = \pi(r)_t$ *for* $t \geq 0$. *Moreover* $\widehat{\Pi(r)}_\tau \simeq_{\mathcal{I}} \hat{r}_\tau$, $\tau \neq \tau^+_{\hat{r}}$.

(iii) $\Pi(r)^{\mathrm{an}} = \pi(r)^{\mathrm{an}} = r^{\mathrm{an}}$.

Since $\{\Pi(r)_t\}_t$ is always approximable, we get that $\Pi \circ \Pi = \Pi$, i.e., Π is a projection. It is not clear if (i) and (ii) hold for $\tau = \tau^+_{\hat{r}}$, though this is not essential to our discussion.

Proof. First we observe that no generality is lost if we assume the condition $\sup_X r_t \leq 0$ for $t \geq 0$, after possibly replacing r_t with $r_t - mt$ for some $m \in \mathbb{N}$ big enough.

We note that (i) immediately implies (ii), as $P[\hat{r}_\tau]_{\mathcal{I}} \simeq_{\mathcal{I}} \hat{r}_\tau$ (Proposition 2.18(ii)). On the other hand, due to Theorem 2.13 and (3.15), we have that (ii) implies (iii) as well. Lastly, (i) together with Lemma 3.16 below imply that $\{\Pi(r)_t\}_t$ is approximable.

Hence we only need to argue (i). In fact, from Lemma 3.16 below we know that $\{\pi(r_t)\}_t$ is an approximable ray, so $\Pi(r)_t \leq \pi(r)_t$, since $r_t \leq \pi(r)_t$. To show that $\Pi(r)_t \geq \pi(r)_t$, it suffices to show that for any Phong–Sturm ray $\{w_t\}_t$ satisfying $w_t \geq r_t$, we have $w_t \geq \pi(r)_t$. We have that $w^{\mathrm{an}} \geq r^{\mathrm{an}} = r''^{\mathrm{an}}$ for any candidate $\{r''_t\}_t$ of $\{\pi(r_t)\}_t$. Hence, [BBJ21, Lemma 4.6] implies that $w_t \geq r''_t$. Taking supremum over $\{r''_t\}_t$ we obtain $w_t \geq \pi(r)_t$, finishing the proof of $\Pi(r)_t = \pi(r)_t$.

Due to how $\Pi(r_t)$ is defined, we immediately obtain $\tau^+_{\widehat{\Pi(r)}} = \tau^+_{\widehat{\pi(r)}} = \tau^+_{\hat{r}}$, giving

$$\widehat{\Pi(r)}_\tau = \widehat{\pi(r)}_\tau = P[\hat{r}_\tau]_{\mathcal{I}} = -\infty\,, \quad \tau > \tau^+_{\hat{r}}\,.$$

To finish the proof of (i), we need to show that $\widehat{\pi(r)}_\tau = P[\hat{r}_\tau]_{\mathcal{I}}$, for $\tau < \tau^+_{\hat{r}}$. Due to the lemma below, $\{\pi[r]_t\}_t$ is approximable. Hence, by Theorem 3.11 and Lemma 2.21(i), $\widehat{\pi(r)}_\tau$ is $\mathcal{I}$-maximal for any $\tau \in \mathbb{R}$. In particular, to show that $\widehat{\pi(r)}_\tau = P[\hat{r}_\tau]_{\mathcal{I}}$, it is enough to argue that $\widehat{\pi(r)}_\tau \simeq_{\mathcal{I}} \hat{r}_\tau$ for $\tau < \tau^+_{\hat{r}}$. To show this, due to Proposition 2.30 it is enough to argue that

$$\widehat{\pi(r)}_\tau^{\mathrm{an}}(v) = \hat{r}_\tau^{\mathrm{an}}(v)\,, \tag{3.18}$$

for all $v = c\,\mathrm{ord}_E \in X^{\mathrm{div}}_{\mathbb{Q}}$ and $\tau < \tau^+_{\hat{r}}$.

Since test curves are τ-concave, both sides in (3.18) are τ-concave on $\mathbb{R}$, with the τ-usc property failing at most at $\tau = \tau^+_{\hat{r}}$, the point of discontinuity.

Due to the comments following (3.6), to argue (3.18) it is enough to show that both sides have the same Legendre transform on positive rational values, namely

$$\sup_{\tau < \tau^+_{\hat{r}}} \left(\widehat{\pi(r)}_\tau^{\mathrm{an}}(v) + t\tau\right) = \sup_{\tau < \tau^+_{\hat{r}}} \left(\hat{r}_\tau^{\mathrm{an}}(v) + t\tau\right),$$

for any $t \in \mathbb{Q}_{>0}$. We may assume $t = 1$ by considering the valuation $t^{-1}v$ instead. From (3.15), this is equivalent to $r^{\mathrm{an}}(v) = \pi(r)^{\mathrm{an}}(v)$, which is known to hold by the lemma below. $\square$

Lemma 3.16 ([BBJ21]). *For any $\{r_t\}_t \in \mathcal{R}^1$ the ray $\{\pi(r)_t\}_t \in \mathcal{R}^1$ is approximable and $\pi(r)^{\mathrm{an}} = r^{\mathrm{an}}$.*

Proof. As before, we can assume that $\sup_X r_t \leq 0$, for $t \geq 0$, after possibly replacing r_t with $r_t - mt$ for some $m \in \mathbb{N}$ big. Recall that in the proof of [BBJ21, Theorem 6.2], one constructs a sequence of globally generated flag ideals $\mathfrak{a}^k$, such that $\varphi_{\mathfrak{a}^k} \in \mathcal{H}^{\mathrm{an}}$ and $\varphi_{\mathfrak{a}^k} \searrow r^{\mathrm{an}}$. Let $\{r^k_t\}_t$ be the Phong–Sturm geodesic ray induced by the fractional ideals $\mathfrak{a}^k$.

Due to [BBJ21, Lemma 4.6], $r^k_t \geq \rho_t$ for any ray $\{\rho_t\}_t \in \mathcal{R}^1$ that is a candidate for $\{\pi(r)_t\}_t \in \mathcal{R}^1$. Let $\{r'_t\}_t$ be the decreasing limit of $\{r^k_t\}_t$. Since $r^k_t \geq r'_t \geq r_t$, and ${r^k}^{\mathrm{an}} = \varphi_{\mathfrak{a}^k} \to r^{\mathrm{an}}$, we get that $r'^{\mathrm{an}} = r^{\mathrm{an}}$. This implies that $\{r'_t\}_t$ is a candidate for $\{\pi(r)_t\}_t$ implying that $\{r'_t\}_t = \{\pi(r)_t\}_t$. $\square$

Finally, we arrive at one of the main results of this section:

Theorem 3.17. *There is a bijective correspondence between $\mathcal{I}$-maximal finite energy test curves and the approximable geodesic rays of $\mathcal{R}^1$.*

Proof. Let $\{r_t\}_t \in \mathcal{R}^1$ be an approximable geodesic ray. Let $\{r^k_t\}_t$ be a sequence of Phong–Sturm geodesic rays decreasing to $\{r_t\}_t$. Since test configurations induce filtrations, that in turn induce geodesic rays (see Section 2.5), we can use Theorem 3.11 to conclude that $\{\hat{r}^k_\tau\}_\tau$ is $\mathcal{I}$-maximal. So $\{\hat{r}_\tau\}_\tau$ is $\mathcal{I}$-maximal by Lemma 2.21(i).

Assume now that $\{\psi_\tau\}_\tau$ is an $\mathcal{I}$-maximal finite energy test curve. Due to $\mathcal{I}$-maximality, by Theorem 3.15(i) we have that

$$\widehat{\Pi(\check{\psi})}_\tau = P[\psi_\tau]_{\mathcal{I}} = \psi_\tau\,, \quad \tau \neq \tau^+_\psi\,.$$

By duality, $\{\Pi(\check{\psi})_t\}_t = \{\psi_t\}_t \in \mathcal{R}^1$. Finally, by Theorem 3.15 the ray $\{\Pi(\check{\psi})_t\}_t$ is approximable, finishing the proof. $\square$

In addition to the above characterization, we show below that the projection Π is continuous. First, we recall radial analogs of some known properties of $(\mathcal{E}^1, d_1)$.

Given $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}^1$, it is possible to construct $\{\max_{\mathcal{R}}(u,v)_t\}_t, \{P_{\mathcal{R}}(u,v)_t\}_t \in \mathcal{R}^1$ the smallest/biggest ray that is above/below $\{u_t\}$ and $\{v_t\}$ respectively. The ray $\{P_{\mathcal{R}}(u,v)_t\}_t$ was constructed in [Xia21, Example 3.2], and $\{\max_R(u,v)_t\}_t$ was constructed above [DDL21, Proposition 2.15]. These two rays satisfy the following metric estimates/identities for some $C(n) > 1$, as argued in [DDL21, Proposition 2.15] and [Xia21, Example 3.2]:

$$d_1^c(\{u_t\}_t, \{v_t\}_t) \leq d_1^c(\{u_t\}_t, \{\max_{\mathcal{R}}(u,v)_t\}_t)) + d_1^c(\{\max_{\mathcal{R}}(u,v)_t\}_t, \{v_t\}_t) \leq C d_1^c(\{u_t\}_t, \{v_t\}_t),$$

$$d_1^c(\{u_t\}_t, \{v_t\}_t) = d_1^c(\{u_t\}_t, \{P_{\mathcal{R}}(u,v)_t\}_t)) + d_1^c(\{P_{\mathcal{R}}(u,v)_t\}_t, \{v_t\}_t)\,. \tag{3.19}$$

THEOREM 3.18. *The projection map $\Pi : \mathcal{R}^1 \to \overline{\mathcal{T}}$ is d_1^c-continuous. In particular, the set of approximable rays is d_1^c-closed.*

The last sentence also follows from the completeness of $\mathcal{E}^{1,\mathrm{an}}$ proved by Boucksom–Jonsson ([BJ21, Theorem 9.8]). This theorem proves the first part of Theorem 1.2.

PROOF. Let $\{u_t^j\}_t, \{u_t\}_t \in \mathcal{R}^1$ with $d_1^c(\{u_t^j\}, \{u_t\}) \to 0$. To derive a contradiction, we can suppose that $d_1^c(\Pi\{u_t^j\}, \Pi\{u_t\}) \geq \delta > 0$.

After possibly taking a subsequence of $\{u_t^j\}_t$, the radial version of [BDL17, Proposition 2.6] (whose proof is the same, and only depends on the estimates (3.19)) gives existence of two sequences $\{v_t^j\}_t, \{w_t^j\}_t \in \mathcal{R}^1$ that are decreasing and increasing respectively, satisfying $w_t^j \leq u_t^j \leq v_t^j$, $w_t^j \leq u_t \leq v_t^j$ together with $d_1^c(\{v_t^j\}, \{u_t\}) \to 0$, $d_1^c(\{w_t^j\}, \{u_t\}) \to 0$. For closely related arguments, see [Xia21, Proposition 3.1] and [DDL21, Proposition 4.2].

Naturally we also get $\Pi\{w^j\}_t \leq \Pi\{u_t^j\} \leq \Pi\{v_t^j\}$, $\Pi\{w^j\}_t \leq \Pi\{u_t\} \leq \Pi\{v_t^j\}$, hence to conclude it is enough to show that $\Pi\{w_t^j\} \nearrow \Pi\{u_t\}$ a.e., and $\Pi\{v_t^j\} \searrow \Pi\{u_t\}$ for all $t \geq 0$ [DL20, Lemma 4.3].

Note that we have $\sup_X v_1^j = \tau^+_{\hat{v}^j} \searrow \tau^+_{\hat{u}} = \sup_X u_1$ and $\sup_X w_1^j = \tau^+_{\hat{w}^j} \nearrow \tau^+_{\hat{u}} = \sup_X u_1$ by the Hartogs lemma for L^1-convergence of quasi-psh functions. Because of this, by the duality of Theorem 3.17, we only need to show that $P[\hat{w}_\tau^j]_{\mathcal{I}} \nearrow P[\hat{u}_\tau]_{\mathcal{I}}$ a.e. and $P[\hat{v}_\tau^j]_{\mathcal{I}} \searrow P[\hat{u}_\tau]_{\mathcal{I}}$ for $\tau < \tau^+_{\hat{u}}$. But this follows from Lemma 2.21, since $\hat{w}_\tau^j \nearrow \hat{u}_\tau$ a.e. and $\hat{v}_\tau^j \searrow \hat{u}_\tau$ for $\tau < \tau^+_{\hat{u}}$. □

3.3. Approximation of rays from below via subgeodesics of Kähler currents. We prove the following result, which is the radial version of [DLR20, Proposition 2.15]:

THEOREM 3.19. *Let $\{u_t\}_t \in \mathcal{R}^1$. Then for any $\epsilon > 0$ there exists subgeodesics $[0,\infty) \ni t \mapsto u_t^\epsilon \in \mathcal{E}^1$ such that, $u_0^\epsilon = 0$, $\omega_{u_t} \geq \epsilon\omega$, $u_t^\epsilon \leq u_t$ and*

$$\mathrm{I}\{u_t\} - \mathrm{I}\{u_t^\epsilon\} \leq \epsilon\, |\mathrm{I}\{u\}|\,.$$

In addition, $u_t^\epsilon \nearrow u_t$ a.e, as $\epsilon \to 0$ for any $t \geq 0$.

PROOF. We can assume without loss of generality that $\sup_X u_t = 0$. By [DLR20, Proposition 2.15] we have that $P((1+\epsilon)u_t) \in \mathcal{E}^1$ for any $\epsilon > 0$.

We fix $\epsilon > 0$ and $t \geq 0$ momentarily. Let $[0,t] \ni l \mapsto v_l^{\epsilon,t} \in \mathcal{E}^1$ be the geodesic connecting 0 and $P((1+\epsilon)u_t)$. Then $l \mapsto \frac{1}{1+\epsilon} v_l^{\epsilon,t}$ is a subgeodesic connecting 0 and $\frac{1}{1+\epsilon}P((1+\epsilon)u_t) \leq u_t$. Hence by the maximum principle, $\frac{1}{1+\epsilon}v_l^{\epsilon,t} \leq u_l$, i.e.,

$v_l^{\epsilon,t} \leq (1+\epsilon)u_l$, i.e., $v_l^{\epsilon,t} \leq P((1+\epsilon)u_l)$ for all $l \in [0,t]$. In particular, another application of the maximum principle gives that $\{v_l^{\epsilon,t}\}_{t\geq l}$ is a decreasing sequence for any $\epsilon > 0$ and $l \geq 0$ fixed.

Next we notice the following: for any $t \geq 0$ and $l \in [0,t]$,

$$\begin{aligned}\frac{t}{l}(\mathrm{I}(u_l) - \mathrm{I}(v_l^{\epsilon,t})) = \mathrm{I}(u_t) - \mathrm{I}(v_t^{\epsilon,t}) &= \mathrm{I}(u_t) - \mathrm{I}(P((1+\epsilon)u_t)) \\ &\leq \frac{1}{V}\int_X (u_t - P((1+\epsilon)u_t))\,\omega^n_{P((1+\epsilon)u_t)} \\ &\leq -\frac{\epsilon(1+\epsilon)^n}{V}\int_X u_t\,\omega^n_{u_t} \leq \epsilon(1+\epsilon)^n(n+1)|\mathrm{I}(u_t)|\,,\end{aligned}$$

where in the third inequality we have used [DDL21, Lemma 4.4], and in the very last inequality we have used that $\sup_X u_t = 0$.

Now linearity of I along geodesic segments gives that $v_l^\epsilon := \lim_{t\to\infty} v_l^{\epsilon,t} \in \mathcal{E}^1$. Moreover, endpoint stability of geodesics gives that $\{v_l^\epsilon\}_l \in \mathcal{R}^1$ [BDL17, Proposition 4.3]. Lastly, the sequence of rays $\{v_t^\epsilon\}_t$ is increasing to $\{u_t\}_t$. In addition, by the maximum principle, $v_l^\epsilon \leq P((1+\epsilon)u_l)$.

Finally, we introduce the subgeodesics $u_l^\epsilon := \frac{1}{1+\epsilon}v_l^\epsilon \geq v_l^\epsilon$, for $l \geq 0$. We immediately obtain that $\omega_{u_l^\epsilon} \geq \frac{\epsilon}{1+\epsilon}\omega$. Since $v_l^\epsilon \leq P((1+\epsilon)u_l)$, we get that $v_l^\epsilon \leq u_l^\epsilon \leq u_l$. Lastly,

$$\mathrm{I}\{u_t\} - \mathrm{I}\{u_t^\epsilon\} \leq \mathrm{I}\{u_t\} - \mathrm{I}\{v_t^\epsilon\} \leq \epsilon(1+\epsilon)^n(n+1)|\mathrm{I}\{u_t\}|\,.$$

After re-scaling $\epsilon > 0$, the result follows. □

4. The closure of rays induced by test configurations

For this section, let (T, h_T) be a fixed Hermitian line bundle on X with smooth metric h_T.

To start, we notice that a sublinear subgeodesic ray $\{r_t\}_t$ satisfies

$$\lim_{t\to\infty}\frac{1}{t}\sup_X r_t = \tau^+_{\hat{r}} = \sup_{v\in X^{\mathrm{div}}_{\mathbb{Q}}} r^{\mathrm{an}}(v)\,.$$

The first equality already follows from Lemma 3.2 and the correspondence in Theorem 3.7(i). The last equality is pointed out in [BBJ21, Lemma 4.3]. In particular, the above constant(s) can be recovered using only the non-Archimedean data r^{an}.

Now we introduce the non-Archimedean analogue of Donaldson's $\mathcal{L}$-functionals. For each $k \geq 1$ and $\{r_t\}_t$ sublinear subgeodesic ray we define

$$\begin{aligned}\mathcal{L}_k^{\mathrm{an}}\{r_t\} :=& \frac{1}{V}h^0(X, T\otimes L^k)\cdot\tau^+_{\hat{r}} + \frac{1}{V}\int_{-\infty}^{\tau^+_{\hat{r}}}\Big(h^0(X, T\otimes L^k\otimes\mathcal{I}(k\hat{r}_\tau)) - h^0(X, T\otimes L^k)\Big)\,\mathrm{d}\tau \\ =& -\frac{1}{V}\int_{-\infty}^{\infty}\tau\,\mathrm{d}h^0(X, T\otimes L^k\otimes\mathcal{I}(k\hat{r}_\tau))\,,\end{aligned} \tag{4.1}$$

where we integrated by parts for Riemann–Stieltjes integrals on some interval $[\tau_0, \tau^+_{\hat{r}} + \epsilon]$, with $h^0(X, T\otimes L^k\otimes\mathcal{I}(k\hat{r}_{\tau_0})) = h^0(X, T\otimes L^k)$ and $\epsilon \searrow 0$ [Apo74, Theorem 7.6]. Indeed, such $\tau_0 \in (-\infty, \tau^+_{\hat{r}})$ exists due to the openness theorem of Guan–Zhou [GZ15].

Lemma 4.1. (i) *Let $\{r_t\}_t$ be a sublinear subgeodesic ray and $r'_t := r_t + tc$ for some $c \in \mathbb{R}$. Then*

$$\mathcal{L}_k^{\mathrm{an}}\{r'_t\} = \mathcal{L}_k^{\mathrm{an}}\{r_t\} + \frac{1}{V} h^0(X, T \otimes L^k) \cdot c\,. \tag{4.2}$$

(ii) *If $\{u_t\}_t$ and $\{v_t\}_t$ are sublinear subgeodesics such that $u_t \le v_t$, then $\mathcal{L}_k^{\mathrm{an}}\{u_t\} \le \mathcal{L}_k^{\mathrm{an}}\{v_t\}$.*

Proof. (i) is obvious. Let us argue (ii). One can see that

$$\mathcal{L}_k^{\mathrm{an}}\{v_t\} - \mathcal{L}_k^{\mathrm{an}}\{u_t\} = \frac{1}{V}\int_{-\infty}^{\tau_{\hat{u}}^+} \Big(h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{v}_\tau)) - h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{u}_\tau)) \Big)\, \mathrm{d}\tau$$

$$+ \frac{1}{V} h^0(X, T \otimes L^k) \cdot (\tau_{\hat{v}}^+ - \tau_{\hat{u}}^+) + \frac{1}{V}\int_{\tau_{\hat{u}}^+}^{\tau_{\hat{v}}^+} \Big(h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{v}_\tau)) - h^0(X, T \otimes L^k) \Big)\, \mathrm{d}\tau\,.$$

To conclude, one observes that both the first and second lines are positive quantities. □

Next we provide an important estimate for the radial Monge–Ampère energy of approximable rays, in terms of $\mathcal{L}_k^{\mathrm{an}}$.

Proposition 4.2. *Let $\{r_t\}_t \in \mathcal{R}^1$ be an approximable ray, i.e., $\{\hat{r}_\tau\}_\tau$ is $\mathcal{I}$-maximal. Then*

$$\mathrm{I}\{r_t\} \ge \varlimsup_{k\to\infty} \frac{n!}{k^n} \mathcal{L}_k^{\mathrm{an}}\{r_t\}\,. \tag{4.3}$$

Proof. By Lemma 4.1, we may assume that $\tau_{\hat{r}}^+ = 0$. By Lemma 3.9, we have $\int_X \omega_{\hat{r}_\tau}^n > 0$ for $\tau < 0$. Moreover, $\hat{r}_\tau$ is $\mathcal{I}$-model for all $\tau \in \mathbb{R}$ by Theorem 3.17. We can calculate

$$\begin{aligned}
\mathrm{I}\{r_t\} &= \frac{1}{V}\int_{-\infty}^0 \left(\int_X \omega_{\hat{r}_\tau}^n - V \right) \mathrm{d}\tau = \int_{-\infty}^0 \left(\frac{\int_X \omega_{\hat{r}_\tau}^n}{\int_X \omega^n} - 1 \right) \mathrm{d}\tau \\
&\ge \int_{-\infty}^0 \varlimsup_{k\to\infty} \left(\frac{h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T \otimes L^k)} - 1 \right) \mathrm{d}\tau \\
&\ge \varlimsup_{k\to\infty} \int_{-\infty}^0 \left(\frac{h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T \otimes L^k)} - 1 \right) \mathrm{d}\tau = \varlimsup_{k\to\infty} \frac{n!}{k^n} \mathcal{L}_k^{\mathrm{an}}\{r_t\}\,,
\end{aligned} \tag{4.4}$$

where the first line we used (3.9), in the second line we used the Riemann–Roch theorem together with Theorem 2.28, and in the third line we used Fatou's lemma. □

Using the results of Section 2.2, in the next lemma we provide a formula that will be an important technical ingredient (closely related to [Ber20, Theorem 1.1]). Recall the definition of the Hilbert map from (2.11):

Lemma 4.3. *Let $\{r_t\}_t$ be a sublinear subgeodesic ray with such that $r_t \le 0$ for all $t \ge 0$. Let*

$$\lambda_H(s) := \varlimsup_{t\to\infty} t^{-1} \log \mathrm{Hilb}_k(r_t)(s, s)\,, \quad s \in H^0(X, T \otimes L^k)\,.$$

Then for any $s \in H^0(X, T \otimes L^k)$,

$$\lambda_H(s) = -k \sup\Big\{ \lambda < 0 : s \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\lambda)) \Big\} < \infty\,. \tag{4.5}$$

Proof. Let $\lambda < \sup\{\tau < 0 : s \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau)\}$. Let $C := \int_X h^k(s,s) e^{-k\hat{r}_\lambda}\,\omega^n < \infty$. By definition, for any $t \geq 0$ we have $\hat{r}_\lambda \leq r_t - t\lambda$, so $C \geq \int_X h^k(s,s) e^{-k(r_t - t\lambda)}\,\omega^n$. As a result, $\lambda_H(s) \leq -k\lambda$, hence

$$\lambda_H(s) \leq -k \sup\Big\{\, \lambda < 0 : s \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\lambda)) \,\Big\}\,.$$

Now we prove the reverse inequality. We fix $p > \lambda_H(s)$ and $\epsilon > 0$ satisfying $p - \epsilon > \lambda_H(s)$. We can find $t_0 > 0$ such that

$$\int_X h^k(s,s) e^{-kr_t}\,\omega^n < e^{(p-\epsilon)t}\,, \quad t \geq t_0\,.$$

Hence $\int_0^\infty e^{-pt} \int_X h^k(s,s) e^{-kr_t}\,\omega^n\,\mathrm{d}t < \infty$. By Tonelli's theorem, this is equivalent to

$$\int_X h^k(s,s) \left(\int_0^\infty e^{-pt} e^{-kr_t}\,\mathrm{d}t \right) \omega^n < \infty\,. \tag{4.6}$$

Before proceeding further, we show that $\lambda_H(s) \geq -k\tau_r^+ = -k \lim_t \frac{\sup_X r_t}{t}$. Indeed, we get this after letting $t \to \infty$ in the following inequality:

$$\frac{1}{t}\log \mathrm{Hilb}_k(r_t)(s,s) = -k\frac{\sup_X r_t}{t} + \frac{1}{t}\log \mathrm{Hilb}_k(r_t - \sup_X r_t)(s,s) \geq -k\frac{\sup_X r_t}{t} + \frac{1}{t}\log \mathrm{Hilb}_k(0)(s,s)\,.$$

As a result, $-p/k < -k^{-1}\lambda_H(s) \leq \lim_{t\to\infty} t^{-1} \sup_X r_t$, giving that $\hat{r}_{-p/k}$ is not identically equal to $-\infty$.

Next, for any $x \in X$ such that $\hat{r}_{-p/k}(x)$ is finite, we claim that

$$\int_0^\infty e^{-pt} e^{-kr_t(x)}\,\mathrm{d}t \geq e^{-p-k} e^{-k\hat{r}_{-p/k}(x)}\,. \tag{4.7}$$

By definition of $\hat{r}_\tau$, we can find $t_0 > 0$ so that $\hat{r}_{-p/k}(x) + 1 \geq r_{t_0}(x) + pk^{-1}t_0$. Since $t \mapsto r_t(x)$ is decreasing, we have $\hat{r}_{-p/k}(x) + pk^{-1} + 1 \geq r_t(x) + pk^{-1}t$ for $t \in [t_0, t_0+1]$. Hence

$$\int_0^\infty e^{-pt} e^{-kr_t(x)}\,\mathrm{d}t \geq \int_{t_0}^{t_0+1} e^{-pt} e^{-kr_t(x)}\,\mathrm{d}t \geq \int_{t_0}^{t_0+1} e^{-k\hat{r}_{-p/k}(x)} e^{-p-k}\,\mathrm{d}t \geq e^{-p-k} e^{-k\hat{r}_{-p/k}(x)}\,.$$

This proves the claim (4.7). So by (4.6) and the claim, $\int_X h^k(s,s) e^{-k\hat{r}_{-p/k}}\,\omega^n < \infty$, hence $p \geq -k \sup\{\, \lambda < 0 : s \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\lambda)) \,\}$, concluding the proof. $\square$

Next we link the non-Archimedean functional $\mathcal{L}_k^{\mathrm{an}}$ to the classical functional $\mathcal{L}_k$ for sufficiently positive subgeodesic rays:

Proposition 4.4. *Let $\{r_t\}_t$ be a sublinear subgeodesic ray and $\delta > 0$ such that $r_t \in \mathcal{E}^1$, and $\omega_{r_t} \geq \delta\omega$ for all $t \geq 0$. Then there exists $k_0(\delta) > 0$ such that $t \to \mathcal{L}_k(r_t)$ is convex, moreover*

$$\mathcal{L}_k^{\mathrm{an}}\{r_t\} = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}_k(r_t)\,, \quad k \geq k_0\,. \tag{4.8}$$

As it will be clear from the proof below, in case $T = K_X$ and h_T is dual to ω^n, one can omit the condition $\omega_{r_t} \geq \delta\omega$ from the assumptions.

Proof. By Lemma 4.1, we may assume that $\sup_X r_t = 0$ for any $t \geq 0$.

By Lemma 4.3 for $f \in H^0(X, T \otimes L^k)$ we have that

$$\lambda_{\mathrm{Hilb}_k}(f) = \overline{\lim_{t\to\infty}}\, t^{-1} \log \mathrm{Hilb}_k(r_t)(f,f) = -k \sup\Big\{\, \lambda < 0 : f \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\lambda)) \,\Big\}\,. \tag{4.9}$$

In particular, for $\lambda \geq 0$,

$$\mathcal{F}_\lambda^{\mathrm{Hilb}_k} := \left\{ f \in H^0(X, T \otimes L^k) : \lambda_{\mathrm{Hilb}_k}(f) \leq \lambda \right\} = H^0(X, T \otimes L^k \otimes \mathcal{I}_-(k\hat{r}_{-\lambda/k})) \tag{4.10}$$

where $\mathcal{I}_-(k\hat{r}_\tau) := \bigcap_{\lambda<\tau} \mathcal{I}(k\hat{r}_\lambda)$, and $\mathcal{F}_\lambda^{\mathrm{Hilb}_k}$ is the filtration associated to $\lambda_{\mathrm{Hilb}_k}$, defined in (2.5).

As $\mathrm{Hilb}_k(r_s)$ is increasing in s, $\mathrm{Hilb}_k(r_s)^*$ is decreasing in s, hence the exponent $\lambda_{\mathrm{Hilb}_k^*}$ of $\mathrm{Hilb}_k(r_s)^*$ on $H^0(X, T \otimes L^k)^*$ is bounded above. Moreover, the family $(\mathrm{Hilb}_k(r_t))_{t\geq 0}$ is positive when $k \geq k_0(\delta)$ by Theorem 2.8. As a result, $t \to \mathcal{L}_k(r_t)$ is convex (see the comments after Lemma 2.4) and the conditions of Theorem 2.6 are satisfied to imply that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log\left(\frac{\det \mathrm{Hilb}_k(r_t)}{\det \mathrm{Hilb}_k(r_0)}\right) &= \int_0^\infty \lambda \, \mathrm{d}h^0(X, T \otimes L^k \otimes \mathcal{I}_-(k\hat{r}_{-\lambda/k})) \\ &= k \int_0^\infty \lambda \, \mathrm{d}h^0(X, T \otimes L^k \otimes \mathcal{I}_-(k\hat{r}_{-\lambda}))\end{aligned}$$

for $k \geq k_0(\delta)$, where in the first line we also used (4.10). As $\mathcal{I}(k\hat{r}_\tau) \subseteq \mathcal{I}_-(k\hat{r}_\tau) \subseteq \mathcal{I}(k\hat{r}_{\tau-\epsilon})$ for any $\epsilon > 0$, and $\mathcal{F}_\lambda^{\mathrm{Hilb}_k}$ can only have finitely many jumping numbers, we get

$$\lim_{t\to\infty} \frac{1}{t}\mathcal{L}_k(r_t) = -\frac{1}{kV} \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{\det \mathrm{Hilb}_k(r_t)}{\det \mathrm{Hilb}_k(r_0)}\right) = -\frac{1}{V}\int_{-\infty}^0 \lambda \, \mathrm{d}h^0(X, T \otimes L^k \otimes \mathcal{I}(k\hat{r}_\lambda)) \,.$$

Comparing with (4.1), the proof is finished. □

Before proceeding, we recall the following basic lemma:

Lemma 4.5. *Let $I \subseteq \mathbb{R}$ be an open interval. Let $f_j, f : I \to \mathbb{R}$ ($j \geq 1$) be convex functions such that $f_j \to f$ pointwise. Then for all $x \in I$, we have that*

$$f'_-(x) \leq \varliminf_{j\to\infty} f'_{j-}(x) \leq \varlimsup_{j\to\infty} f'_{j+}(x) \leq f'_+(x) \,.$$

Proof. Due to convexity, for $h > 0$ small enough, we have that $(f_j(x-h) - f_j(x))/(-h) \leq f'_{j-}(x)$. Letting $j \to \infty$, we arrive at $(f(x-h) - f(x))/(-h) \leq \varliminf_{j\to\infty} f'_{j-}(x)$. Now $h \to 0$ gives the first inequality. The other inequality follows similarly. □

Theorem 4.6. *Let $\{r_t\}_t \in \mathcal{R}^1$. Then*

$$\varliminf_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{r_t\} \geq \mathrm{I}\{r_t\} \,. \tag{4.11}$$

Proof. First, we consider $\{v_t\}_t$ a sublinear subgeodesic such that $v_t \in \mathcal{E}^1$, $v_t \leq 0$, and $\omega_{v_t} \geq \delta\omega$ for all $t \geq 0$ and some $\delta > 0$.

By Theorem 2.11, we have $\lim_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k(v_t) = \mathrm{I}(v_t)$ for $t \geq 0$. So by Lemma 4.5 above and Proposition 4.4,

$$\mathrm{I}\{v_t\} = \lim_{t\to\infty} \frac{\mathrm{I}(v_t)}{t} \leq \varliminf_{k\to\infty} \frac{n!}{k^n}\left(\lim_{t\to\infty} \frac{1}{t}\mathcal{L}_k(v_t)\right) = \varliminf_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{v_t\} \,. \tag{4.12}$$

By Lemma 4.1 it is enough to prove the theorem for $\{r_t\}_t \in \mathcal{R}^1$ with $\sup_X r_t = 0$. By Theorem 3.19, we can find sublinear subgeodesics $\{v_t^k\}_t$ such that $v_t^k \in \mathcal{E}^1$, $v_t^k \nearrow r_t$

and $\omega_{v_t^k} \geq \delta_k \omega$ for all $t \geq 0$ and some $\delta_k \searrow 0$. Moreover, $\mathrm{I}\left\{v_t^k\right\} \to \mathrm{I}\{r_t\}$. By monotonicity of $\mathcal{L}_k^{\mathrm{an}}\{\cdot\}$ (Lemma 4.1) we have

$$\varliminf_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{r_t\} \geq \varliminf_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{v_t^k\} \geq \mathrm{I}\left\{v_t^k\right\}.$$

Letting $k \to \infty$, we conclude (4.11). $\square$

THEOREM 4.7. *Let $\{r_t\}_t \in \mathcal{R}^1$ with $\sup_X r_t = 0$. Then the following are equivalent:*

(i) $\{r_t\}_t \in \overline{\mathcal{T}}$.

(ii) $\{r_t\}_t \in \overline{\mathcal{F}}$.

(iii) $P[\hat{r}_\tau]_{\mathcal{I}} = \hat{r}_\tau$, *for all* $\tau \leq 0$.

(iv) $\displaystyle\lim_{k\to\infty} \int_{-\infty}^0 \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) \mathrm{d}\tau = \mathrm{I}\{r_t\}$.

This theorem proves most of Theorem 1.1. The remaining point will be completed in Corollary 5.6.

PROOF. First we show (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). Then we show (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Due to [Xia21, Example 3.3] or (Theorem 3.18), (i) implies that $\{r_t\}_t$ is approximable. This in turn is equivalent with (iii) due to Theorem 3.17. However (iii) implies (iv) due to Proposition 4.2 and Theorem 4.6.

Now we show that (iv) implies (i) . For this, let us consider the approximable ray $\{\Pi(r)_t\}_t \in \mathcal{R}^1$ (Theorem 3.15). From the same result we know that $\widehat{\Pi(r)}_\tau = P[\hat{r}_\tau]$ for $\tau < 0$. In particular,

$$h^0(X, T\otimes L^k \otimes \mathcal{I}(k\widehat{\Pi(r)}_\tau)) = h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau)).$$

From the direction (iii) implies (iv) already proved, we obtain that (iv) holds for $\{\Pi(r)_t\}_t$ in the following manner:

$$\mathrm{I}\{\Pi(r)_t\} = \lim_{k\to\infty} \int_{-\infty}^0 \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) \mathrm{d}\tau. \tag{4.13}$$

Condition (iv) now gives $\mathrm{I}\{\Pi(r)_t\} = \mathrm{I}\{r_t\}$. Since $r_t \leq \Pi(r)_t$, this gives $r_t = \Pi(r)_t$ for $t \geq 0$. Since $\{\Pi(r)_t\}_t$ is approximable due to Theorem 3.15, so is $\{r_t\}_t$, concluding (i) .

Finally, since $\mathcal{T} \subseteq \mathcal{F}$, we obtain that (i) implies (ii) . For the other direction, it is enough to show that elements of $\mathcal{F}$ are approximable. However the rays of $\mathcal{F}$ are all $\mathcal{I}$-maximal, due to Theorem 3.11, so they are approximable due to Theorem 3.17, proving (i) . $\square$

THEOREM 4.8. *Let $\{r_t\}_t \in \mathcal{R}^1$ with $\sup_X r_t = 0$ for any $t \geq 0$. Then $\lim_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{r_t\}$ exists and can be estimated the following way*

$$\lim_{k\to\infty} \frac{n!}{k^n}\mathcal{L}_k^{\mathrm{an}}\{r_t\} = \lim_{k\to\infty} \int_{-\infty}^0 \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) \mathrm{d}\tau \geq \mathrm{I}\{r_t\}. \tag{4.14}$$

Proof. Consider the approximable ray $\{\Pi(r)_t\}_t \in \mathcal{R}^1$. In the argument (iii) implies (iv) of the previous theorem, we actually showed that the limit on the left hand side of (4.14) exists and is equal to $\mathrm{I}\{\Pi(r)_t\}$. The inequality now readily follows from the fact that $r_t \le \Pi(r)_t$, implying $\mathrm{I}\{r_t\} \le \mathrm{I}\{\Pi(r)_t\}$. $\square$

For $\{r_t\}_t \in \mathcal{R}^1$, it is possible to introduce the non-Archimedean Monge–Ampère energy in the following manner:

$$\mathrm{I}^{\mathrm{an}}\{r_t\} := \mathrm{I}\{\Pi(r)_t\}\,. \tag{4.15}$$

In particular, when $\{r_t\}_t \in \overline{\mathcal{T}}$ we have $\mathrm{I}^{\mathrm{an}}\{r_t\} = \mathrm{I}\{r_t\}$. Comparing with (4.13), we obtain a new interpretation for the non-Archimedean Monge–Ampère energy:

Corollary 4.9. *For $\{r_t\}_t \in \mathcal{R}^1$ we have*

$$\mathrm{I}^{\mathrm{an}}\{r_t\} = \lim_{k\to\infty} \frac{n!}{k^n} \mathcal{L}_k^{\mathrm{an}}\{r_t\}\,. \tag{4.16}$$

In particular, if $\sup_X r_t = 0$ for any $t \ge 0$, then

$$\mathrm{I}^{\mathrm{an}}\{r_t\} = \lim_{k\to\infty} \frac{n!}{Vk^n} \int_{-\infty}^0 \Big(h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau)) - h^0(X, T\otimes L^k)\Big)\, d\tau\,.$$

This proves the second part of Theorem 1.2.

5. The closure of algebraic singularity types

We start with the following result about approximable bounded geodesic rays.

Proposition 5.1. *Let $\{r_t\}_t \in \overline{\mathcal{T}} \subseteq \mathcal{R}^1$ be a ray of bounded potentials. Then for all $\tau \in (\tau^-_{\hat{r}}, \tau^+_{\hat{r}})$ we have*

$$\int_X \omega^n_{\hat{r}_\tau} = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))\,. \tag{5.1}$$

In particular the limit on the right hand side exists.

Proof. Without loss of generality, we may assume that $\{r_t\}_t$ is sup-normalized, i.e., $\tau^+_{\hat{r}} = 0$.

Using (3.9), Theorem 4.7(iv), and Fatou's lemma, we have the following estimate

$$\begin{aligned}\int_{\tau^-_{\hat{r}}}^0 \left(\frac{\int_X \omega^n_{\hat{r}_\tau}}{V} - 1\right) d\tau = \mathrm{I}\{r_t\} &= \lim_{k\to\infty} \int_{\tau^-_{\hat{r}}}^0 \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) d\tau \\ &\le \int_{\tau^-_{\hat{r}}}^0 \varlimsup_{k\to\infty} \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) d\tau\,.\end{aligned} \tag{5.2}$$

Comparing with (4.4) we arrive at

$$\int_{\tau^-_{\hat{r}}}^0 \left(\frac{\int_X \omega^n_{\hat{r}_\tau}}{V} - 1\right) d\tau = \int_{\tau^-_{\hat{r}}}^0 \varlimsup_{k\to\infty} \left(\frac{h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))}{h^0(X, T\otimes L^k)} - 1\right) d\tau\,. \tag{5.3}$$

Since each $\hat{r}_\tau$ is $\mathcal{I}$-model, by Theorem 2.28 the integrand of the left hand side is greater or equal to the integrand on the right hand side, so for almost every $\tau \in (\tau^-_{\hat{r}}, 0)$ we have

$$\frac{1}{n!}\int_X \omega^n_{\hat{r}_\tau} = \varlimsup_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\hat{r}_\tau))\,. \tag{5.4}$$

Due to (5.2) and (5.3), the conditions of Lemma 5.2 below are satisfied for $I = (\tau_{\hat{r}}^-, 0)$, $I' = (-2, 1)$ and f_k being the integrand on the right hand side of (5.2). Due to Lemma 3.9, the function $(\tau_{\hat{r}}^-, 0) \ni \tau \mapsto \int_X \omega_{\hat{r}_\tau}^n$ is continuous and decreasing. We conclude that the limsup in (5.2) is a limit for all $\tau \in (\tau_{\hat{r}}^-, 0)$ and (5.1) holds as desired. □

Lemma 5.2. *Let $I, I' \subseteq \mathbb{R}$ be two bounded open intervals, and $f_k : I \to I'$ for $k \in \mathbb{N}$ be a sequence of decreasing functions. Suppose that*

$$\lim_{k\to\infty} \int_I f_k \,\mathrm{d}\lambda = \int_I \varlimsup_{k\to\infty} f_k \,\mathrm{d}\lambda \,, \tag{5.5}$$

where $\mathrm{d}\lambda$ is the Lebesgue measure. Denote $f := \varlimsup_{k\to\infty} f_k$. Then the limit $\lim_{k\to\infty} f_k$ exists at each point of right continuity of f. In particular, $f(x) = \lim_{k\to\infty} f_k(x)$ for a.e. $x \in I$.

The proof of this lemma is due to Fan Zheng.

Proof. Without loss of generality, we may assume that $I = (0, 1)$, $I' = (0, 1)$. Let $x \in (0, 1)$, such that

$$a := f(x) - \varliminf_{k\to\infty} f_k(x) > 0 \,.$$

We assume that f is right continuous at x, to obtain a contradiction. There exists $\delta > 0$, so that on $[x, x + \delta]$, $f > f(x) - a/2$.

We may take a subsequence g_k of f_k so that $g_k(x) \to f(x) - a$. We automatically have

$$\lim_{k\to\infty} \int_0^1 g_k \,\mathrm{d}\lambda = \int_0^1 f \,\mathrm{d}\lambda \,, \quad \varlimsup_{k\to\infty} g_k \leq f \,. \tag{5.6}$$

We deduce the estimates

$$\varlimsup_{k\to\infty} \int_x^{x+\delta} g_k \,\mathrm{d}\lambda \leq \varlimsup_{k\to\infty} \delta g_k(x) \leq \delta f(x) - \delta a \leq \int_x^{x+\delta} f \,\mathrm{d}\lambda - \frac{\delta a}{2} \,.$$

By Fatou's lemma, on the complement $S := (0, 1) \setminus [x, x + \delta]$ we have

$$\varlimsup_{k\to\infty} \int_S g_k \,\mathrm{d}\lambda \leq \int_S \varlimsup_{k\to\infty} g_k \,\mathrm{d}\lambda \leq \int_S f \,\mathrm{d}\lambda \,.$$

Adding these estimates, we get $\varlimsup_{k\to\infty} \int_0^1 g_k \,\mathrm{d}\lambda \leq \int_0^1 f \,\mathrm{d}\lambda - \frac{\delta a}{2}$, contradicting (5.6). □

Next compare the arithmetic and non-pluripolar volumes of arbitrary ω-psh functions:

Proposition 5.3. *For $\varphi \in \mathrm{PSH}(X, \omega)$, the limit $\lim_{k\to\infty} k^{-n} h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ always exists. Moreover,*

$$\frac{1}{n!} \int_X \omega_\varphi^n \leq \frac{1}{n!} \int_X \omega_{P[\varphi]_{\mathcal{I}}}^n = \lim_{k\to\infty} \frac{1}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \,. \tag{5.7}$$

Proof. We note that $h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) = h^0(X, T \otimes L^k \otimes \mathcal{I}(kP[\varphi]_{\mathcal{I}}))$, hence we can assume that φ is $\mathcal{I}$-model by replacing φ with $P[\varphi]_{\mathcal{I}}$. Further, due to Theorem 2.28, we can also assume that $\int_X \omega_\varphi^n = \int_X \omega_{P[\varphi]_{\mathcal{I}}}^n > 0$.

By [DDL21, Lemma 4.3], $\tau \mapsto P(\tau\varphi)$ is well defined for $\tau \in [0, 1+\delta)$, where $\delta > 0$ is a small constant depending on $\int_X \omega_\varphi^n > 0$. We consider the bounded $\mathcal{I}$-maximal test curve $\{\psi_\tau\}_\tau$ with $\tau^-_\psi = -1-\delta$ and $\tau^+_\psi = 0$ (for notations, see Theorem 3.7(iii) and Definition 3.5) such that

$$\psi_\tau := P[P((1+\delta+\tau)\varphi)]_{\mathcal{I}}\ , \quad \tau \in [-1-\delta, 0)$$

and $\psi_0 := \lim_{\tau\nearrow 0}\psi_\tau$. Since φ is $\mathcal{I}$-model, we conclude that $\psi_{-\tau} = \varphi$. By Proposition 5.1, for $\tau \in [-1-\delta, 0)$, we have

$$\frac{1}{n!}\int_X \omega^n_{\psi_\tau} = \lim_{k\to\infty} k^{-n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\psi_\tau))\,. \tag{5.8}$$

Since $\psi_{-\delta} = \varphi$, plugging $\tau = -\delta$ in the above formula, we conclude that the limit on the right hand side of (5.7) exists. Moreover this limit is equal to $\int_X \omega^n_{P[\varphi]_{\mathcal{I}}}$. □

Next we characterize equality in (5.7), in the presence of positive mass.

Proposition 5.4. *Let $\varphi \in \mathrm{PSH}(X,\omega)$ with $\int_X \omega^n_\varphi > 0$. Then $P[\varphi] = P[\varphi]_{\mathcal{I}}$ if and only if*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi)) = \frac{1}{n!}\int_X \omega^n_\varphi. \tag{5.9}$$

Proof. Since $\int_X \omega^n_{P[\varphi]} = \int_X \omega^n_\varphi > 0$ and $\mathcal{I}(k\varphi) = \mathcal{I}(kP[\varphi])$, we can assume that φ is model, i.e., $\varphi = P[\varphi]$. If φ is $\mathcal{I}$-model, i.e., $P[\varphi]_{\mathcal{I}} = \varphi$, then (5.9) follows from (5.7).

If (5.9) holds, then (5.7) implies that $\int_X \omega^n_\varphi = \int_X \omega^n_{P[\varphi]_{\mathcal{I}}} > 0$. Since $\varphi \le P[\varphi]_{\mathcal{I}}$ and φ is model, [DDL18b, Theorem 3.12] gives $P[\varphi]_{\mathcal{I}} = \varphi$, as desired. □

Finally, we state our last main result, collecting many of our previous findings:

Theorem 5.5. *For $u \in \mathrm{PSH}(X,\omega)$ we have*

$$\lim_{k\to\infty} \frac{h^0(X, T\otimes L^k\otimes\mathcal{I}(ku))}{k^n} = \frac{1}{n!}\int_X \omega^n_{P[u]_{\mathcal{I}}} \ge \frac{1}{n!}\int_X \omega^n_u\,. \tag{5.10}$$

Moreover, when $\int_X \omega^n_u > 0$, we have equality in the above estimate if and only if one of the the following equivalent conditions hold:

(i) $\lim_{k\to\infty} \frac{h^0(X, T\otimes L^k\otimes\mathcal{I}(ku))}{k^n} = \frac{1}{n!}\int_X \omega^n_u$.

(ii) $P[u] = P[u]_{\mathcal{I}}$.

(iii) *$[u]$ is $d_{\mathcal{S}}$-approximable by the quasi-equisingular sequence $[u_j]$ (see (2.20)).*

(iv) $[u] \in \overline{\mathcal{Z}}$.

(v) $[u] \in \overline{\mathcal{A}}$.

In particular, when $\int_X \omega^n_{P[u]_{\mathcal{I}}} > 0$ (i.e. $\mathrm{nd}(L,u) = n$ in the terminology of [Cao14]), we have $\lim_{k\to\infty} k^{-n}h^0(X, T\otimes L^k\otimes\mathcal{I}(ku)) > 0$. This reproves [Cao14, Proposition 3.6] in the case of ample line bundles.

This theorem corresponds to Theorem 1.4 in the introduction. Moreover, we can now complete the proof of Theorem 1.1 as well.

Corollary 5.6. *Let $\{r_t\}_t \in \mathcal{R}^1$ with $\sup_X r_t = 0$. Then $\hat{r}_\tau = P[\hat{r}_\tau]$ and $\int_X \omega^n_{\hat{r}_\tau} > 0$ for any $\tau \in (-\infty, 0)$. Moreover, condition* (iii) *of Theorem 4.7 is equivalent to*

$$\text{(v)}\ \lim_{k\to\infty} \frac{h^0(X, T\otimes L^k\otimes\mathcal{I}(k\hat{r}_\tau))}{k^n} = \frac{1}{n!}\int_X \omega^n_{\hat{r}_\tau}\,, \quad \tau\in(-\infty,0)\,.$$

Proof of Theorem 5.5. The inequality (5.10) was proved in Proposition 5.3. The equivalence of (i) and (ii) was proved in Proposition 5.4.

That $[u]$ is $d_{\mathcal{S}}$-approximable by its quasi-equisingular sequence is equivalent to $[P[u]]$ being $d_{\mathcal{S}}$-approximable by its quasi-equisingular sequence (Indeed, $d_{\mathcal{S}}(u, P[u]) = 0$. Also, we have $V_u^k = V_{P[u]}^k$ in the language of Theorem 2.23, hence $[u_k] = [P[u]_k]$ for the corresponding quasi-equisingular approximations). As a result, Theorem 2.24 immediately gives the equivalence between (ii) and (iii) .

Trivially, (iii) $\implies$ (iv) $\implies$ (v). To finish, it is enough to argue that (v) implies (ii) . As before, we can assume that u is model, i.e., $P[u] = u$.

Let $[u_j] \in \mathcal{A}$ be such that $d_{\mathcal{S}}([u], [u_j]) \to 0$. Since each $[u_j]$ is analytic, (ii) holds for each u_j (Proposition 2.20). Since (ii) is equivalent to (iii), we can replace each u_j with a potential of the type (2.20), that is algebraic.

Further, after passing to a subsequence, we can assume that $d_{\mathcal{S}}([u_j], [u_{j+1}]) \leq C^{-2j}$, where $C > 1$ is the constant of [DDL21, Proposition 3.5]. Let $v_j^l := \max(u_j, u_{j_1}, \dots u_{j+l})$. Using the triangle inequality and [DDL21, Proposition 3.5] we have

$$\begin{aligned} d_{\mathcal{S}}([u_j], [v_j^l]) &= d_{\mathcal{S}}([u_j], [\max(u_j, v_{j+1}^{l-1})]) \leq C d_{\mathcal{S}}([u_j], [v_{j+1}^{l-1}]) \\ &\leq C \left(d_{\mathcal{S}}([u_j], [u_{j+1}]) + d_{\mathcal{S}}([u_{j+1}], [v_{j+1}^{l-1}]) \right) . \end{aligned}$$

After iterating the above inequality l times and observing that $d_{\mathcal{S}}([u_{j+l}], [v_{j+l}^0]) = 0$, we conclude that

$$d_{\mathcal{S}}([u_j], [v_j^l]) \leq \sum_{k=j}^{j+l-1} C^{k+1-j} d_{\mathcal{S}}([u_k], [u_{k+1}]) = \sum_{k=j}^{j+l-1} \frac{C^{k+1-j}}{C^{2k}} \leq \frac{1}{C^{j-2}(C-1)} . \tag{5.11}$$

Now let $w_j^l := P\left[v_j^l\right]_{\mathcal{I}}$ and $w_j := \lim_{l\to\infty} w_j^l$. By Lemma 5.7 below and Proposition 2.20 we get that w_j^l is $\mathcal{I}$-model and has the same singularity type as v_j^l. Moreover, by Lemma 2.21 (iii), we get that w_j is $\mathcal{I}$-model.

Comparing with (5.11), we obtain that $d_{\mathcal{S}}([u_j], [w_j^l]) = d_{\mathcal{S}}([u_j], [v_j^l]) \leq C^{2-j}(C-1)^{-1}$. Letting $l \to \infty$, and using [DDL21, Lemma 4.1], we arrive at $d_{\mathcal{S}}([u], [w_j]) \leq C^{2-j}(C-1)^{-1}$, i.e., $d_{\mathcal{S}}([u], [w_j]) \to 0$ as $j \to \infty$. Since $\int_X \omega_u^n > 0$, each w_j and u is model, we obtain that $u = \lim_{j\to\infty} w_j$ ([DDL18b, Theorem 3.12]).

Since $\{w_j\}_j$ is decreasing, by Lemma 2.21 (i) we obtain that $u = P[u]_{\mathcal{I}}$. Since $u = P[u]$ by assumption, (ii) follows. ☐

Lemma 5.7. *Let* $u \in \mathrm{PSH}(X, \omega)$. *Suppose that* $u_1, \dots, u_l \in \mathrm{PSH}(X, \omega)$ *are the potentials arising from the construction in* (2.20). *Then* $\max(u_1, \dots, u_l) \in \mathrm{PSH}(X, \omega)$ *has analytic singularity type.*

Proof. Examining the expression (2.20) one notices that for each $x \in X$ we can find (a common denominator) $m \in \mathbb{N}$ and an open neighborhood $x \in U_x \subseteq X$ such that $u_k - \frac{1}{m}\max_j \log|f_j^k|^2$ is locally bounded on U_x for a finite number of holomorphic functions $f_j^k \in \mathcal{O}(U_x)$. Then $\max(u_1, \dots, u_l) - \frac{1}{m}\max_{j,k} \log|f_j^k|^2$ is locally bounded on U_x as well. ☐

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PAPER 2

Pluripotential-theoretic stability thresholds

Mingchen Xia

IMRN (2022)

Paper 2. Pluripotential-theoretic stability thresholds

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Abstract

Given a compact polarized manifold (X, L), we introduce two new stability thresholds in terms of singularity types of global quasi-plurisubharmonic functions on X. We prove that in the Fano setting, the new invariants can effectively detect K-stability of X. We study some functionals of geodesic rays in the space of Kähler potentials by means of the corresponding test curves. In particular, we introduce a new entropy functional of quasi-plurisubharmonic functions and relate the radial entropy functional to this new entropy functional.

1. Introduction

Let X be a complex projective manifold of dimension n. Let L be an ample line bundle on X. Fix a Kähler form $\omega \in c_1(L)$. Let $V = (L^n)$.

A central problem in Kähler geometry is to give conditions for the existence of canonical metrics in the Kähler class $[\omega]$. The celebrated Yau–Tian–Donaldson conjecture asserts that the existence of Kähler–Einstein metrics (or more generally cscK metrics) is equivalent to certain algebro-geometric stability conditions. Classically, dating back to the work of Ding–Tian ([DT92]), Tian ([Tia97]) and Donaldson ([Don02]), the algebro-geometric stability notion, known as *K-stability* has been defined in terms of test configurations. Later on, a stronger condition known as *uniform K-stability* is also introduced and studied in [Der16] and in [BHJ16], [BHJ17]. It is known that the equivariant version of uniform K-stability gives a characterization of the existence of Kähler–Einstein metrics, which even generalizes to the log Fano setting, see [Li22] and references therein.

On the other hand, more recently a valuative approach to K-stability is introduced in [Fuj19], [FO18], [BJ20], which we briefly recall. The δ-invariant of L is defined as

$$\delta(L) := \inf_{E} \frac{A_X(E)}{S_L(E)}\,, \tag{1.1}$$

where E runs over the set of prime divisors over X, $A_X(E)$ denotes the log discrepancy of E and $S_L(E)$ is the expected order of vanishing of L along E. It is well-known that uniform twisted K-stability (resp. twisted K-semistability) is equivalent to $\delta(L) > 1$ (resp. $\delta(L) \geq 1$). See [Fuj19], [Li17], [FO18], [BJ18] for details.

In this paper, we introduce a different stability threshold in terms of the singularity types of quasi-plurisubharmonic functions on X:

$$\delta_{\mathrm{pp}} := \inf_{[\psi]} \frac{\int_{-\infty}^{\infty} \mathrm{Ent}([\psi^+_\tau])\,\mathrm{d}\tau}{nV^{-1}\int_{-\infty}^{\infty}\left(\int_X \omega\wedge\omega^{n-1}_{\psi^+_\tau} - \int_X \omega^n_{\psi^+_\tau}\right)\mathrm{d}\tau}\,, \tag{1.2}$$

where $[\psi]$ runs over the set of singularity types of quasi-psh functions with some non-zero Lelong numbers on X, $\psi^+_\bullet$ is a test curve associated to ψ. Recall that a test curve is the Legendre transform of a geodesic ray as in [RW14]. The test curve $\psi^+_\bullet$ is the maximal extension of the test curve corresponding to deformation to the normal cone (see Section 2.6). The quantity $\mathrm{Ent}[\bullet]$ is an invariant of the singularity types of quasi-psh functions (see Definition 2.21). To the best of the author's knowledge, this invariant has never been studied in the literature. Observe that the quotient in (1.2) depends only on the singularity type of ψ. Moreover, it is easy to see that the quotient in (1.2) does not change under the rescaling $\psi \to c\psi$ for $c \in \mathbb{R}_{>0}$, hence one could restrict ψ to run only in the set of ω-psh functions.

Now we state the main theorem.

THEOREM 1.1. *Let (X, L) be a polarized manifold. Then $\delta_{\mathrm{pp}} \geq \delta$. Further, if X is Fano and $L = -K_X$ and $\delta < \frac{n+1}{n}$, then $\delta = \delta_{\mathrm{pp}}$.*

We recall that in the Fano setting, when $\delta \leq 1$, δ is also equal to the greatest Ricci lower bound, see Section 9.1 for details. We remark that although our theorem concerns only Kähler geometry, our proof relies essentially on the non-Archimedean tools developed by Boucksom–Jonsson, as we recall later.

As a corollary,

Corollary 1.2. *Assume that X is Fano and $L = -K_X$. Then*

(1) $\delta_{\mathrm{pp}} \geq 1$ iff X is K-semistable.
(2) $\delta_{\mathrm{pp}} > 1$ iff X is uniformly K-stable.

Corollary 1.2 integrates into the program of characterizing K-stability in terms of some more explicit data dating back to [RT07]. In [RT07], Ross–Thomas introduced the notion of slope stability in terms of test configurations associated to deformation to the normal cone, which gives a necessary condition for K-stability. Later on, this theory was extended in [Oda13] using flag ideals, in [Wit12], [Szé15] using filtrations. In [Fuj19] and [Li17], Fujita and Li give a characterization of K-stability in terms of all divisorial valuations. Our result gives a different characterization in terms of psh singularity types. Our approach can also be seen as a generalization of those of [RT07] in the sense that our definition of δ_{pp} is based on a generalization of deformation to the normal cone. We also notice that very recently in [DL22], Dervan–Legendre have partially extended Fujita's work to general polarizations and studied valuative stability.

When the stability threshold is less than 1, it is interesting to understand its minimizers. We propose the following conjecture:

Conjecture 1.3. *When $\delta_{\mathrm{pp}} \leq 1$, there is always a minimizer of δ_{pp}.*

When X is a Fano manifold, $L = -K_X$ and when $\delta < 1$, the pluricomplex Green function G in the sense of [MT19] is a minimizer of δ_{pp}.

The first part in the Fano case follows from our proof of the main theorem. For general polarization, it seems difficult.

There are some similar results for δ in the more general log Fano variety setting: there is always a quasi-monomial valuation that computes δ ([BLX19]). In the smooth Fano setting, there is a divisor computing δ ([DS20], see [BLZ19, Theorem 6.7] for details). The same holds in the log Fano setting by the recent breakthrough [LXZ22].

The δ_{pp}-invariant is closely related to δ in the following manner: take an extractable (Definition 2.11) divisor E. One can prove that in this case, there is always an ω-psh function ψ with analytic singularities such that on a suitable birational model $\pi : Y \to X$, the singularities of $\pi^*\psi$ are just the hyperplane singularity along E. Recall that E induces a test configuration $(\mathcal{X}, \mathcal{L})$. Now one can make explicit computations to express various functionals of $(\mathcal{X}, \mathcal{L})$ in terms of $\psi_\bullet$, the result turns out to be of the form of (1.2). More precisely, we prove that for a test curve induced by a general (semi-ample) test configuration, the non-Archimedean entropy and $\tilde{J}^{\mathrm{an}}$-functionals (the latter is more frequently denoted by $I^{\mathrm{an}} - J^{\mathrm{an}}$ in the literature) are both integrals of some corresponding functionals of psh singularities along the test curve (see Theorem 7.5, Corollary 7.6, Corollary 6.11). Conversely, given an ω-psh function ψ with analytic singularities, we can always take a log resolution so that ψ has singularities along a snc (strictly normal crossing) $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$. This divisor then induces a higher rank valuation $(a_i^{-1} \operatorname{ord}_{D_i})_i$ of $\mathbb{C}(X)$.

As a byproduct of our work, we could also define a slightly different stability threshold:

$$\delta' := \inf_{\psi} \frac{(K_{Y/X} \cdot (-\operatorname{div}_Y \psi)^{n-1}) + n\left(G_{n-1}(L, \operatorname{div}_Y \psi) \cdot \operatorname{red} \operatorname{div}_Y \psi\right)}{n \int_0^1 \left(\int_X \omega \wedge \omega_{\tau\psi}^{n-1} - \int_X \omega_{\tau\psi}^n\right) \mathrm{d}\tau}, \tag{1.3}$$

where $[\psi]$ runs over the set of singularity types of unbounded ω-psh functions with analytic singularities, $\pi : Y \to X$ is a log resolution of ψ, G_{n-1} is a polynomial defined by (2.6), red of a divisor D is the divisor with the same support as D but with all non-zero coefficients of D set to 1. Note that the quotient in (1.3) is *not* invariant under the rescaling $\psi \mapsto c\psi$ ($c \in \mathbb{Q}_{>0}$), hence δ' is an invariant of ω-psh functions on X, not an invariant of all quasi-psh functions on X as δ_{pp} is. We also prove that

THEOREM 1.4. *We always have $\delta' \geq \delta$.*

In general, we do not expect δ and δ' to be equal even if $\delta \leq 1$. The invariant δ' restricts the possible singularities of an ω-psh function. Although δ' does not seem to have direct applications in K-stability, it may play some interesting roles in pluripotential theory.

Philosophy behind the theorems

Before discussing the proofs, let us explain the philosophy behind these theorems.

We regard the global pluripotential theory of singular metrics on a compact Kähler manifold as a differential version of the theory of geodesic rays in the space of Kähler potentials. In fancier terms, we could roughly regard the space of quasi-psh singularity types as the boundary at infinity of the space of geodesic rays: on one hand, each quasi-psh singularity type induces a geodesic ray; on the other hand, each geodesic ray degenerates to a quasi-psh singularity type at infinity. As in the finite dimensional picture between the space of rays in $\mathbb{R}^n$ and the sphere at infinity S^{n-1}, it is natural to expect a closer relation between these spaces. One of the justifications is given by Theorem 6.8 (namely, [DX22, Theorem 1.1]). Similarly, by the computations in this paper and in [DX22], a number of radial invariants of geodesic rays are in fact an integral along the corresponding test curves of some corresponding invariants defined by quasi-psh functions. See Table 1 for more examples.

Classically, K-stability of a polarized manifold is detected by test configurations, valuations, filtrations and non-Archimedean potentials, which can all be embedded in the space of geodesic rays. By our philosophy, there should be a pluripotential-theoretic counterpart, which leads to the present paper. Note that our previous work [DX22] already followed this philosophy.

We do not expect Theorem 1.1, Theorem 1.4 to be useful when trying to find new examples of K-stable varieties. However, from the pluripotential-theoretic point of view, these results provide strong restrictions on the possible singularity types of quasi-psh functions using global geometric conditions. To the best of my knowledge, this kind of results has never been studied before.

Strategy of the proof

In the discussion, we fix a maximal geodesic ray in the sense of [BBJ21] and its Legendre transform $\psi = \hat{\ell}$.

As discussed above, we need to express various energy functionals of ℓ in terms of $\psi_\bullet$.

The part for $\tilde{\mathbf{J}}$ follows from the strategy introduced in [RW14] and further developed in [DX22]. See the proof of Theorem 6.10 for details.

The corresponding result for the entropy functional is the main new feature in this paper. As the variation of the entropy functional is not easily controlled, we try to tackle the non-Archimedean counterpart of the entropy at first, namely the non-Archimedean entropy:

$$\mathrm{Ent}^{\mathrm{an}}(\phi) := \frac{1}{V}\int_{X^{\mathrm{an}}} A_X \,\mathrm{MA}(\phi)\,, \quad \phi \in \mathcal{E}^{1,\mathrm{an}}\,.$$

In [Li20], Li showed that $\mathrm{Ent}^{\mathrm{an}}(\phi)$ is dominated by the slope at infinity of the usual entropy functional Ent along ℓ, where ϕ is the non-Archimedean potential induced by ℓ. In [DX22], we have expressed the non-Archimedean Monge–Ampère energy in terms of the test curves. Since the non-Archimedean Monge–Ampère energy is nothing but the primitive function of the Chambert-Loir measure $\mathrm{MA}(\phi)$, we get *a fortiori* a good understanding of $\mathrm{MA}(\phi)$. We make use of this description to compute the non-Archimedean entropy functional. The result turns out to be an integral of the variation of volumes along $\psi_\bullet$.

Recall the potentials ψ_τ are all $\mathcal{I}$-model in the sense of [DX22]. In order to compute the variation of volumes of an $\mathcal{I}$-model potential, we need to express this volume algebraically. We prove that the volume of an $\mathcal{I}$-model potential can be realized as certain (movable) intersection number of a b-divisor (in the sense of Shokurov) associated to the singularities of the potential, if the intersection number is properly defined (see Theorem 5.4). Now we can carry out a purely algebraic computation to get a formula for the non-Archimedean entropy (See Theorem 7.5, Corollary 7.6).

Now it comes to Theorem 1.1 and Theorem 1.4. That these new invariants dominate δ is an easy consequence of the formulae of $\tilde{\mathbf{J}}$ and $\mathrm{Ent}^{\mathrm{an}}$. We simply embed the set of ω-psh functions into the set of maximal geodesic rays using the deformation to the normal cone like construction. This kind of embedding was already studied in [DDL21b]. For the equality $\delta = \delta_{\mathrm{pp}}$ when $\delta \leq 1$. We make use of the results from [BLZ19], which says that when $\delta \leq 1$, δ can be computed by a divisor E. Then [BCHM10] allows us to extract the divisor. We can therefore construct an ω-psh ψ whose singularities are exactly given by E. Then ψ minimizes δ_{pp} as well and we conclude that $\delta = \delta_{\mathrm{pp}}$.

We remark that although we have carried out our computations only on smooth complex varieties, it is easy to generalize most results to normal Kähler varieties. However we decide to limit ourselves to the smooth setting in order to keep the present paper at a readable length.

Organization of the paper

In Section 2.2, we present a few results necessary for understanding the definition of δ_{pp}.

In Section 2.3 and Section 2.4, we recall some basic notions in Kähler geometry and pluripotential theory.

In Section 2.5, we recall the notion of Shokurov's b-divisors and apply it to define the entropy of qpsh singularities.

In Section 2.6 and Section 2.7, we express several functionals on the space of geodesic rays in terms of the corresponding test curves.

In Section 2.8, we relate the new δ-invariants to the classical δ-invariant.

In Section 2.9, we propose several further problems.

Conventions

In this paper, all Monge–Ampère type operators are taken in the non-pluripolar sense (see [BEGZ10]). The functional $\tilde{J}$ defined in (3.1) is usually written as $I-J$ in the literature. Our definition of *test curves* in Definition 4.1 corresponds to *maximal test curves* in the literature (see [RW14] for example). The $\mathrm{d^c}$ operator is normalized so that $\mathrm{dd^c} = \frac{i}{2\pi}\partial\bar{\partial}$. The definition of a birational model in Definition 2.3 requires that the model be smooth, hence stronger than the usual definition. When ω is a Kähler form, we adopt the convention that $\omega_{-\infty} = \omega + \mathrm{dd^c}(-\infty) = 0$. A snc divisor is always assumed to be effective. By a valuation of a field, we refer to real valuations unless otherwise specified. We adopt the additive convention for valuations. We allow ($\mathbb{Q}$-)Weil divisors to have countably many components.

We do not distinguish a holomorphic line bundle and the corresponding invertible sheaf in the *analytic* category. We use interchangeably additive and multiplicative notations for tensor products of invertible sheaves.

We make use of the results of [BHJ19] in an essential way. We only refer to the latest version on arXiv [BHJ16] with errata instead of the journal version.

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2. Setup

In this section, we present a minimal amount of preliminaries necessary to understand the definition of the new delta invariant (1.2).

2.1. Log resolution of analytic singularities. Let X be a projective manifold of dimension n. Let L be a big and semi-ample line bundle on X. Let h be a smooth non-negatively curved Hermitian metric on L. Let $\omega = c_1(L,h)$. Let $\mathrm{PSH}(X,\omega)$ denote the set of all ω-psh functions on X, namely the set of usc functions $\varphi : X \to [-\infty,\infty)$ such that $\omega + \mathrm{dd^c}\varphi \geq 0$ as currents. See [GZ17] for more details.

Definition 2.1. A potential $\varphi \in \mathrm{PSH}(X,\omega)$ is said to have *analytic singularities* if for each $x \in X$, there is a neighbourhood $U_x \subseteq X$ of x in the Euclidean topology, such that on U_x,

$$\varphi = c \log \left(\sum_{j=1}^{N_x} |f_j|^2 \right) + \psi \,,$$

where $c \in \mathbb{Q}_{\geq 0}$, f_j are analytic functions on U_x, $N_x \in \mathbb{Z}_{>0}$ is an integer depending on x, $\psi \in C^\infty(U_x)$.

Definition 2.2. Let D be an effective snc $\mathbb{R}$-divisor on X. Let $D = \sum_i a_i D_i$ with D_i being prime divisors and $a_i \in \mathbb{R}_{>0}$. We say that $\varphi \in \mathrm{PSH}(X,\omega)$ has *analytic singularities along D* if locally (in the Euclidean topology),

$$\varphi = \sum_i a_i \log |s_i|_h^2 + \psi \,,$$

where s_i is a local section of L that defines D_i, ψ is a smooth function.

Note that a potential with analytic singularities along a snc $\mathbb{Q}$-divisor has analytic singularities in the sense of Definition 2.1.

Definition 2.3. A *birational model* of X is a projective birational morphism $\pi : Y \to X$ from a *smooth* projective variety Y to X.

Definition 2.4. Let $\varphi \in \mathrm{PSH}(X,\omega)$ be a potential with analytic singularities. Then there is a birational model $\pi : Y \to X$ of X, such that $\pi^*\varphi$ has analytic singularities along a snc $\mathbb{Q}$-divisor (see [MM07, Lemma 2.3.19]). We call any such π a *log resolution* of φ.

2.2. $\mathcal{I}$-model potentials. Let X be a compact Kähler manifold of dimension n. Let L be an ample line bundle. Let $\omega \in c_1(L)$ be a Kähler form. For any quasi-psh function φ on X, let $\mathcal{I}(\varphi)$ denote Nadel's multiplier ideal sheaf of φ, namely, the coherent ideal sheaf on X locally generated by holomorphic functions f such that $\int |f|^2 e^{-\varphi}\omega^n < \infty$.

The concept of $\mathcal{I}$-model potential is developed in [DX22].

Definition 2.5. Let $\varphi \in \mathrm{PSH}(X,\omega)$. A *quasi-equisingular approximation* of φ is a sequence φ^j of potentials in $\mathrm{PSH}(X,\omega)$ with analytic singularities, such that

(1) φ^j converges to φ in L^1.
(2) The singularity types of φ^j are decreasing.
(3) For any $\delta > 0$, $k > 0$, we can find $j_0 = j_0(\delta,k) > 0$, so that for $j \geq j_0$,

$$\mathcal{I}((1+\delta)k\varphi^j) \subseteq \mathcal{I}(k\varphi) \,.$$

Recall that quasi-equisingular approximations always exist ([Cao14, Lemma 3.2], [DPS01, Theorem 2.2.1]).

Recall that a potential $\varphi \in \mathrm{PSH}(X,\omega)$ is said to be $\mathcal{I}$-*model* if

$$\varphi = P[\varphi]_{\mathcal{I}} := \sup{}^* \{ \psi \in \mathrm{PSH}(X,\omega) : \psi \le 0, \psi^{\mathrm{an}} \le \mathcal{I}(k\psi) \subseteq \mathcal{I}(k\varphi) \text{ for all } k \in \mathbb{N} \} .$$

We use the notation $\mathrm{PSH}^{\mathrm{Model}}_{\mathcal{I}}(X,\omega)$ to denote the set of $\mathcal{I}$-model potentials in $\mathrm{PSH}(X,\omega)$.

Theorem 2.6 ([DX22, Theorem 1.4]). *Let $\varphi \in \mathrm{PSH}^{\mathrm{Model}}(X,\omega)$, $\int_X \omega_\varphi^n > 0$. Then the following are equivalent:*

(1) $\varphi \in \mathrm{PSH}^{\mathrm{Model}}_{\mathcal{I}}(X,\omega)$.

(2)

$$\varphi = \sup{}^* \{ \psi \in \mathrm{PSH}(X,\omega) : \psi \le 0, \mathcal{I}(k\psi) \subseteq \mathcal{I}(k\varphi) \textit{ for any } k \in \mathbb{R}_{>0} \} .$$

(3)

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, K_X \otimes L^k \otimes \mathcal{I}(k\varphi)) = \int_X \omega_\varphi^n .$$

(4) For one (or equivalently any) quasi-equisingular approximation φ^j of φ,

$$\lim_{j\to\infty} \int_X \omega_{\varphi^j}^n = \int_X \omega_\varphi^n .$$

In terms of the function φ^{an} introduced below, these conditions are also equivalent to

$$\varphi = \sup{}^* \{ \psi \in \mathrm{PSH}(X,\omega) : \psi \le 0, \psi^{\mathrm{an}} \le \varphi^{\mathrm{an}} \} .$$

Here and in the whole paper, products like ω_φ^n are taken in the non-pluripolar sense, see [BEGZ10].

For the definition of model potentials, we refer to [DDL18b]. The set of model potentials in $\mathrm{PSH}(X,\omega)$ is denoted by $\mathrm{PSH}^{\mathrm{Model}}(X,\omega)$. Recall that a model potential with analytic singularities is $\mathcal{I}$-model ([Bon98]).

Let $X^{\mathrm{div}}_{\mathbb{Q}}$ denote the set of all $\mathbb{Q}$-divisorial geometric valuation on X. Namely, elements of $X^{\mathrm{div}}_{\mathbb{Q}}$ are $c\,\mathrm{ord}_E$, where $c \in \mathbb{Q}_{>0}$, E is a prime divisor over X (i.e. a prime divisor on a birational model of X). Let $\psi \in \mathrm{PSH}(X,\omega)$, recall that ψ^{an} is a function on $X^{\mathrm{div}}_{\mathbb{Q}}$ defined as follows: let $v \in X^{\mathrm{div}}_{\mathbb{Q}}$, then set

$$-v(\psi) = \psi^{\mathrm{an}}(v) := -\lim_{k\to\infty} \frac{1}{k} v\left(\mathcal{I}(k\psi)\right) . \tag{2.1}$$

Lemma 2.7. *Let $\varphi \in \mathrm{PSH}(X,\omega)$. Let φ^j be a quasi-equisingular approximation of φ^j. Then $\varphi^{j,\mathrm{an}} \to \varphi^{\mathrm{an}}$ pointwisely on $X^{\mathrm{div}}_{\mathbb{Q}}$ as $j \to \infty$.*

Proof. It suffices to prove that for any prime divisor E over X, $\varphi^{j,\mathrm{an}}(\mathrm{ord}_E) \to \varphi^{\mathrm{an}}(\mathrm{ord}_E)$. Fix $k \in \mathbb{Z}_{>0}$, $\delta \in \mathbb{Q}_{>0}$, take $j_0 > 0$, so that when $j > j_0$, $\mathcal{I}((1+\delta)k\varphi^j) \subseteq \mathcal{I}(k\varphi)$. When $j > j_0$, we get

$$\frac{1}{k} \mathrm{ord}_E(\mathcal{I}(k\varphi)) \le \frac{1}{k} \mathrm{ord}_E(\mathcal{I}((1+\delta)k\varphi^j)) .$$

By Fekete's lemma,

$$-\varphi^{j,\mathrm{an}}(\mathrm{ord}_E) = \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k} \mathrm{ord}_E(\mathcal{I}(k\varphi^j)) .$$

So

$$\frac{1}{k} \mathrm{ord}_E(\mathcal{I}(k\varphi)) \le (1+\delta)(-\varphi^{j,\mathrm{an}}(\mathrm{ord}_E)) .$$

Take sup with respect to $k \in \mathbb{Z}_{>0}$, we get

$$-\varphi^{\mathrm{an}}(\mathrm{ord}_E) \leq (1+\delta)(-\varphi^{j,\mathrm{an}}(\mathrm{ord}_E)) .$$

Let $\delta \to 0+$, we get

$$\varphi^{\mathrm{an}}(\mathrm{ord}_E) \geq \lim_{j\to\infty} \varphi^{j,\mathrm{an}}(\mathrm{ord}_E) .$$

The converse is trivial. □

Remark 2.8. For readers familiar with the non-Archimedean language of [BJ21], our proof in fact implies the following stronger result: φ^{an} extends uniquely to a function in $\mathrm{PSH}^{\mathrm{an}}(L)$ and $\varphi^{j,\mathrm{an}} \to \varphi^{\mathrm{an}}$ in $\mathrm{PSH}^{\mathrm{an}}(L)$. See [BJ21, Theorem 4.28, Corollary 4.58].

2.3. Singularity divisors. Let X be a projective manifold of dimension n. Let L be a semi-ample line bundle with a smooth non-negatively curved Hermitian metric h. Let $\omega = c_1(L,h)$.

Definition 2.9. Let $\psi \in \mathrm{PSH}(X,\omega)$. Let $\pi : Y \to X$ be a birational morphism from a normal $\mathbb{Q}$-factorial projective variety. Define the *singularity divisor* of ψ on Y as

$$\mathrm{div}_Y\,\psi := \sum_E \nu_E(\psi)\,E ,$$

where E runs over the set of prime divisors on Y, $\nu_E(\psi)$ is the generic Lelong number of $\pi^*\psi$ along E. Note that this is a countable sum by Siu's semi-continuity theorem.

Let D be an effective $\mathbb{R}$-divisor on Y. We say that the singularities of ψ are *determined* on Y by D if for any birational model $\Pi : Z \to Y$, $\mathrm{div}_Z\,\psi = \Pi^*D$.

We can regard $\mathrm{div}_Y\,\psi$ as the divisorial part of Siu's decomposition of $\mathrm{dd^c}\pi^*\psi$.

Remark 2.10. In general, a divisor with countably many components does not define a class in the Néron–Severi group, but in the case of $\mathrm{div}_Y\,\psi$, this can be easily defined. In fact, write $\mathrm{div}_Y\,\psi = \sum_{i=1}^{\infty} a_i D_i$. Here we allow a_i to be 0. Clearly, $\pi^*L - \sum_{i=1}^{r} a_i D_i$ is pseudo-effective. It follows that $\sum_{i=1}^{\infty} a_i D_i$ converges as a sum in the Néron–Severi group $\mathrm{NS}^1(Y)\otimes\mathbb{R}$, see [BFJ09, Proposition 1.3]. In particular, we can talk about the intersection between $\mathrm{div}_Y\,\psi$ and divisors.

As a consequence of resolution of singularities, any potential with analytic singularity admits a model where its singularities are determined ([MM07, Lemma 2.3.19]).

Definition 2.11. Let E be a prime divisor over X. An *extraction* of E is a proper birational morphism $\pi : Y \to X$ from a *normal* $\mathbb{Q}$*-factorial variety* Y, such that E is a prime divisor on Y and that $-E$ is π-ample.

If there is an extraction of E, we call E an *extractable* divisor.

Observe that when X is Fano, an extractable divisor E is dreamy in the sense that the doubly graded algebra

$$\bigoplus_{m\in\mathbb{Z}_{\geq 0}} \bigoplus_{p\in\mathbb{Z}} H^0(Y, -m\pi^*K_X - pE) \tag{2.2}$$

is finitely generated.

In general, when the log discrepancy of E is well-behaved, one can run a suitable MMP to extract E. See [BCHM10, Corollary 1.4.3], [Kol13, Section 1.4] for details.

Assume that L is ample. Let F be an extractable divisor. Let $\pi : Y \to X$ be an extraction of F. We can take $A \in \mathbb{Q}_{>0}$ large enough, so that $A\pi^*L - F$ is semi-ample.

In particular, take B large enough, so that $B(A\pi^*L - F)$ is base-point free. Take a basis $s_1, \dots, s_N$ of $H^0(X, B(A\pi^*L - F))$. Let

$$\psi = \frac{1}{AB} \log \max_{i=1,\dots,N} |s_j|^2_{h^{AB}} .$$

Then the singularities of ψ are determined on Y by $A^{-1}F$ (see Definition 2.9).

2.4. Quasi-analytic singularities. Let X be a projective manifold of dimension n. Let L be a big and semi-ample line bundle on X. Let h be a smooth non-negatively curved Hermitian metric on L. Let $\omega = c_1(L, h)$.

Definition 2.12. We say a potential $\varphi \in \mathrm{PSH}(X, \omega)$ has *quasi-analytic singularities* if there is a birational model $\pi : Y \to X$, a snc $\mathbb{R}$-divisor D on Y, such that the singularities of ψ are determined on Y by D (see Definition 2.9). In this case, we also say that φ has quasi-analytic singularities along D.

Lemma 2.13. *Let $\varphi \in \mathrm{PSH}(X, \omega)$ be a potential with quasi-analytic singularities along a snc $\mathbb{Q}$-divisor D on a birational model $\pi : Y \to X$, then*

$$\mathcal{I}(k\pi^*\varphi) = \mathcal{O}_Y(-\lfloor kD \rfloor)$$

for any $k \in \mathbb{Q}_{>0}$.

Proof. Without loss of generality, we take $k = 1$. Recall that we have assumed that the model is projective. Take a sufficiently ample line bundle H on Y, so that $H - D$ is semi-ample and $H - \pi^*L$ is ample. Take a $m \in \mathbb{Z}_{>0}$ so that $m(H - D)$ is globally generated. Fix a smooth positively curved metric h on H. Let $\omega' := c_1(H, h)$, we may assume that $\omega' > \pi^*\omega$. Take a basis $s_1, \dots, s_N$ of $H^0(Y, m(H - D))$. Let

$$\psi = \frac{1}{m} \log \max_{i=1,\dots,N} |s_i|^2_{h^m} .$$

Then we know that

$$\mathcal{I}(\psi) = \mathcal{O}_Y(-\lfloor D \rfloor) .$$

But we know that $\pi^*\varphi \sim_{\mathcal{I}} \psi$ as ω'-psh functions, so we conclude. ☐

Remark 2.14. We rephrase the proof of Lemma 2.13 in fancier terms: Let

$$\mathrm{PSH}^{\mathrm{Model}}(X) := \varinjlim_{\omega} \mathrm{PSH}^{\mathrm{Model}}(X, \omega) ,$$

where ω runs over all Kähler forms on X, when $\omega \leq \omega'$, the map $\mathrm{PSH}^{\mathrm{Model}}(X, \omega) \to \mathrm{PSH}^{\mathrm{Model}}(X, \omega')$ is given by the $P_{\omega'}[\bullet]$. We take the filtered colimit in the category of sets. We define a class $[\varphi] \in \mathrm{PSH}^{\mathrm{Model}}(X)$ to be analytic if some representative is analytic. Now the proof of Lemma 2.13 says that when $\varphi \in \mathrm{PSH}^{\mathrm{Model}}(X, \omega)$ is quasi-analytic, the class $[\varphi] \in \mathrm{PSH}^{\mathrm{Model}}(X)$ is analytic.

Lemma 2.15. *Let $\varphi \in \mathrm{PSH}(X, \omega)$ be a potential with quasi-analytic singularities along a snc $\mathbb{R}$-divisor D on X, then $L - D$ is nef. If moreover $\int_X \omega_\varphi^n > 0$, then $L - D$ is big and nef.*

Proof. Consider the positive current $\omega_\varphi - [D]$ in $c_1(L - D)$. Take a quasi-equisingular approximation h_j of $\omega_\varphi - [D]$ ([Cao14]). The Lelong number condition and the fact that h_j has analytic singularities show that its local potential is in fact bounded. Hence $L - D$ is nef. Now the assumption $\int_X \omega_\varphi^n > 0$ implies that $(L - D)^n > 0$, hence $L - D$ is big ([DP04]). ☐

2.5. Non-archimedean envelopes. Let X be a compact Kähler manifold of dimension n. Let L be an ample line bundle with a smooth strictly positively curved metric h. Let $\omega = c_1(L,h)$.

Let $\mathbf{v} = (v_1, \ldots, v_m)$ be a valuation of $\mathbb{C}(X)$ with value in $\mathbb{R}^m$. We assume for simplicity that each v_i is divisorial.

Definition 2.16. Let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m_{\geq 0}$. Define a potential in $\mathrm{PSH}(X,\omega) \cup \{-\infty\}$:

$$\psi_{\mathbf{v}\geq\mathbf{a}} := \sup{}^* \{ \psi \in \mathrm{PSH}(X,\omega) : \psi \leq 0, v_i(\psi) \geq a_i \text{ for } i = 1, \ldots, m \} . \tag{2.3}$$

We also define

$$\psi'_{\mathbf{v}\geq\mathbf{a}} := \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k} \sup{}^* \left\{ \log |s|^2_{h^k} : s \in H^0(X, L^k), \sup_X |s|_{h^k} \leq 1, v_i(s) \geq ka_i \text{ for } i = 1, \ldots, m \right\} . \tag{2.4}$$

Observe that $\psi_{\mathbf{v}\geq\mathbf{a}}$ itself is a candidate in the sup in (2.3), provided that $\psi_{\mathbf{v}\geq\mathbf{a}} \neq -\infty$. Obviously, $\psi_{\mathbf{v}\geq\mathbf{a}}$ is either $\mathcal{I}$-model or $-\infty$.

Lemma 2.17. *Assume that $\psi_{\mathbf{v}\geq\mathbf{a}}$ has positive mass, then $P[\psi'_{\mathbf{v}\geq\mathbf{a}}]_{\mathcal{I}} = \psi_{\mathbf{v}\geq\mathbf{a}}$.*

Proof. That $P[\psi'_{\mathbf{v}\geq\mathbf{a}}]_{\mathcal{I}} \leq \psi_{\mathbf{v}\geq\mathbf{a}}$ is trivial, we prove the converse. Write $\psi = \psi_{\mathbf{v}\geq\mathbf{a}}$. It suffices to show that for any small enough $\epsilon > 0$, any fixed divisorial valuation v of $\mathbb{C}(X)$, we can construct a section $s \in H^0(X, L^k)$ for some large k, so that $k^{-1}v_i(s) \geq v_i(\psi)$ for all i and $k^{-1}v(s) \leq v(\psi) + \epsilon$. We may assume that $a_i > 0$ for all i. Let $b_i = v_i(\psi)$. Then $b_i \geq a_i$. Let $b = v(\psi)$.

By [DDL21b, Lemma 4.4], we can construct a potential ψ', more singular than ψ, such that

$$v_i(\psi') > b_i , \quad v(\psi') \leq v(\psi) + \epsilon .$$

Take a small enough $\delta \in \mathbb{Q}_{>0}$, such that

$$(1+\delta)^{-1} v_i(\psi') > b_i$$

for all i. Regarding ψ' as a metric on $(1+\delta)L$ and applying [Dem12, Corollary 13.23] and its proof, we find a sequence of sections $s_k \in H^0(X, L^k)$ for some sequence k increasing to ∞, such that

$$\frac{1+\delta}{k}[\operatorname{div} s_k] \to \delta\omega + \omega_{\psi'} , \quad \frac{1+\delta}{k} v_i(s_k) \to v_i(\psi') .$$

Thus for k large enough,

$$\frac{1}{k} v(s_k) \leq b + \epsilon , \quad \frac{1}{k} v_i(s_k) > b_i .$$

□

As a particular case, let $v_i = c_i \operatorname{ord}_{F_i}$ be a $\mathbb{Q}$-divisorial valuation, $a_i \in \mathbb{R}$ ($i = 1, \ldots, m$). We have

$$\psi'_{\mathbf{v}\geq\mathbf{a}} = \sup_{k\in\mathbb{Z}_{>0} \text{ sufficiently divisible}}{}^* \frac{1}{k} \sup{}^* \left\{ \log |s|^2_{h^k} : s \in H^0(X, kL - \sum_{i=1}^m ka_i c_i^{-1} F_i), \sup_X |s|_{h^k} \leq 1 \right\} .$$

Let D be an effective $\mathbb{Q}$-divisor on X. In this paper, $\mathbb{Q}$-divisor are allowed to have countably many components. If D has finitely many irreducible components, say $D = \sum_{i=1}^{r} a_i D_i$, we define

$$\psi_{\geq D} := \psi_{(\mathrm{ord}_{D_i}) \geq (a_i)} . \tag{2.5}$$

In general, if D has countably many components, say $D = \sum_{i=1}^{\infty} a_i D_i$, we just let

$$\psi_{\geq D} := \inf_{j=1,\dots,\infty} \psi_{\geq \sum_{i=1}^{j} a_i D_i} .$$

2.6. Extended deformation to the normal cone. Let $\psi \in \mathrm{PSH}(X, \omega)$ be a potential with analytic singularities. Let $\pi : Y \to X$ be a log resolution of ψ. Let $\mathrm{Psef}(\psi)$ be the pseudo-effective threshold of ψ. Namely,

$$\mathrm{Psef}(\psi) = \sup \left\{ t \geq 0 : \pi^* L - t \operatorname{div}_Y \psi \text{ is pseudo-effective} \right\} .$$

We define a test curve $\psi^+_\bullet$ as follows (see Section 2.4 for the general theory of test curves):

$$\psi^+_\tau := \begin{cases} 0 , & \tau \leq 0 , \\ \psi_{\geq \tau \operatorname{div}_Y \psi} , & \tau \in 0 < \tau \leq \mathrm{Psef}(\psi) , \\ -\infty , & \tau > \mathrm{Psef}(\psi) . \end{cases}$$

We call this construction the *extended deformation to the normal cone* with respect to ψ. See Lemma 2.19 for an explanation of this terminology.

This construction can be realized geometrically.

Definition 2.18. Let $\psi \in \mathrm{PSH}(X, C\omega)$ ($C \in \mathbb{Z}_{>0}$) be a potential with analytic singularities. Let $\pi : Y \to X$ be a log resolution. Let $A > 0$ be an integer so that $A \operatorname{div}_Y \psi$ is integral. We say ψ is *dreamy* if the double-graded ring

$$R(X, L, \psi) := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{s \in \mathbb{Z}_{\geq 0}} H^0(Y, kACL - sA \operatorname{div}_Y \psi)$$

is finitely generated.

Note that whether or not ψ is dreamy does not depend on the choice of A.

Assume that $\psi \in \mathrm{PSH}(X, \omega)$ is dreamy and L is ample. Let $(\mathcal{X}, \mathcal{L})$ be the relative proj of

$$\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigoplus_{s \in \mathbb{Z}_{\geq 0}} t^{-s} H^0(Y, kAL - sA \operatorname{div}_Y \psi)$$

over $\mathbb{C}$. Then $(\mathcal{X}, \mathcal{L})$ is a test configuration of (X, L^A). It follows from Lemma 2.17 that the corresponding test curve is just ψ^+.

More generally, let $\psi \in \mathrm{PSH}(X, \omega)$. We define $\mathrm{Psef}(\psi)$ as the sup of $t \geq 0$, such that on each birational model $\pi : Y \to X$, $\pi^* L - t \operatorname{div}_Y \psi$ is pseudo-effective. This definition coincides with the previous one when ψ has analytic singularities.

We define the corresponding test curve $\psi^+_\bullet$ as follows: When $\tau \leq 0$, set $\psi^+_\tau = 0$. When $0 < \tau < \mathrm{Psef}(\psi)$, we define

$$\psi^+_\tau = \lim_{Y} \psi_{\geq \tau \operatorname{div}_Y \psi} ,$$

where the limit is a limit of decreasing net taken over all birational models $\pi : Y \to X$. Note that the limit is $\mathcal{I}$-model by [DX22, Lemma 2.20] (Strictly speaking, [DX22, Lemma 2.20] only deals with decreasing sequences, but the proof works for decreasing nets as well). Define

$$\psi^+_{\mathrm{Psef}(\psi)} = \lim_{\tau \to \mathrm{Psef}(\psi)-} \psi^+_\tau$$

and

$$\psi^+_\tau = -\infty$$

if $\tau > \mathrm{Psef}(\psi)$.

2.7. Generalized deformation to the normal cone. Let $\psi \in \mathrm{PSH}(X,\omega)$ be a model potential with analytic singularities. We define a test curve (see Section 2.4 for the precise definition) $\psi_\bullet$ by

$$\psi_\tau := \begin{cases} 0\,, & \tau \leq -1\,, \\ P[(1+\tau)\psi]\,, & \tau \in (-1,0]\,, \\ -\infty\,, & \tau > 0\,. \end{cases}$$

The test curve $\psi_\bullet$ is a truncated version of $\psi^+_\bullet$:

Lemma 2.19. *When $\tau \in [0,1]$, $\psi^+_\tau = \psi_{\tau-1}$.*

The test curve $\psi_\bullet$ and its associated geodesic ray were studied in [Dar17a] and [DDL21b].

The following result due to Darvas ([Dar17b]) characterizes the geodesic ray induced by $\psi_\bullet$.

Proposition 2.20. *Let $\psi \in \mathrm{PSH}^{\mathrm{Model}}(X,\omega)$, then $\check{\psi}_t$ $(t \geq 0)$ is the increasing limit of ℓ^k_t, where $(\ell^k_t)_{t\in[0,-E(\max\{-k,\psi\})]}$ is the geodesic from 0 to $\max\{-k,\psi\}$.*

Assume that ψ has analytic singularities along a $\mathbb{Z}$-divisor $\mathrm{div}_X\,\psi$ on X and that $L-\mathrm{div}_X\,\psi$ is semi-ample. In this case, let $\mathcal{X} = \mathrm{Bl}_{\mathrm{div}_X\,\psi\times\{0\}}\, X\times\mathbb{C}$ be the deformation to the normal cone. Let $\mathcal{E}$ be the exceptional divisor. Let $\Pi : \mathcal{X} \to X\times\mathbb{C}$ be the natural map and let $p_1 : X\times\mathbb{C} \to X$ be the natural projection. Let $\mathcal{L} = \Pi^* p_1^* L \otimes \mathcal{O}_{\mathcal{X}}(-\mathcal{E})$. Then we have a test configuration $(\mathcal{X},\mathcal{L})$ of (X,L). By Example 4.13, the test curve induced by the filtration of this test configuration is exactly $\psi_\bullet$. Note that $\psi_\bullet$ is induced by the filtration in Example 4.15.

2.8. Entropy and delta invariant. Let X be a compact Kähler manifold of dimension n. Let L be an ample line bundle. Let $\omega \in c_1(L)$ be a Kähler form.

We recall that for an $\mathbb{R}$-Weil divisor $D = \sum_i a_i D_i$ with $a_i \neq 0$, D_i prime and pairwise distinct, $\mathrm{red}\, D := \sum_i D_i$.

Definition 2.21. Let $\psi \in \mathrm{PSH}(X,\omega)$. We define the *entropy* of $[\psi]$ as

$$\mathrm{Ent}([\psi]) := \frac{n}{V}\varliminf_{Y}\left(\langle \pi^*L - \mathrm{div}_Y\,\psi\rangle^{n-1}\cdot\left(K_{Y/X} + \mathrm{red}\,\mathrm{div}_Y\,\psi\right)\right) \in [0,\infty]\,,$$

where $\pi : Y \to X$ runs over all birational models on X. Here the product $\langle\bullet\rangle$ is the movable intersection in the sense of [BFJ09], [Bou02]. We formally set $\mathrm{Ent}([-\infty]) = 0$.

We observe that $\mathrm{Ent}([\psi])$ depends only on the $\mathcal{I}$-singularity type of ψ. To the best of the author's knowledge, this invariant has never been defined in the literature.

Remark 2.22. The condition $\psi \in \mathrm{PSH}(X,\omega)$ is not essential. We can define the same quantity for any quasi-psh function.

We can now define our new delta invariant:

Definition 2.23. We define the *pluripotential-theoretic δ-invariant* as

$$\delta_{\mathrm{pp}} = \inf_{\psi} \frac{\int_{-\infty}^{\infty} \mathrm{Ent}([\psi_\tau^+])\,\mathrm{d}\tau}{nV^{-1}\int_{-\infty}^{\infty}\left(\int_X \omega\wedge\omega_{\psi_\tau^+}^{n-1} - \int_X \omega_{\psi_\tau^+}^n\right)\mathrm{d}\tau}\,,$$

where $V=(L^n)$, ψ runs over the set of ω-psh functions with some non-zero Lelong number on X. The quotient depends only on the $\mathcal{I}$-singularity type of ψ.

Remark 2.24. We remark that $\psi_\bullet^+$ depends only on the $\mathcal{I}$-singularity type of ψ, hence in the definition above, it suffices to take $\mathcal{I}$-model potentials ψ.

For the next definition, we need to introduce some notations. We define a polynomial
(2.6)

$$G_{n-1}(A,B) = \sum_{j=0}^{n-1}\frac{1}{j+1}\binom{n-1}{j}(-1)^j\left(A^{n-1-j}\cdot B^j\right) = \frac{1}{nB}\left(A^n-(A-B)^n\right).$$

When A, B are divisors on X, $G_{n-1}(A,B)$ is considered as an element in the Chow ring of X. We observe that when $A,B_1,B_0\in\mathbb{R}$ and if we set $B_t = tB_1+(1-t)B_0$ ($t\in[0,1]$), then

$$\int_0^1 (A-B_t)^{n-1}\,\mathrm{d}t = G_{n-1}(A-B_0,B_1-B_0)\,. \tag{2.7}$$

Definition 2.25. We define the *δ'-invariant* of (X,L) as

$$\delta' := \inf_{\psi}\frac{(K_{Y/X}\cdot(-\operatorname{div}_Y\psi)^{n-1}) + n\left(G_{n-1}(L,\operatorname{div}_Y\psi)\cdot\operatorname{red}\operatorname{div}_Y\psi\right)}{n\int_0^1\left(\int_X\omega\wedge\omega_{\tau\psi}^{n-1} - \int_X\omega_{\tau\psi}^n\right)\mathrm{d}\tau}\,,$$

where $\pi: Y\to X$ is a log resolution of ψ. Here ψ runs over the set of unbounded ω-psh functions with analytic singularities. The quotient depends only on the singularity type of ψ.

3. Preliminaries

Let X be a compact Kähler manifold of dimension n. Let ω be a Kähler form on X. We introduce a number of functionals on the space of Kähler potentials and on the space of geodesic rays.

3.1. Archimedean functionals.

In this section, we recall the definitions of several functionals in Kähler geometry. For the definition of $\mathcal{E}^1=\mathcal{E}^1(X,\omega)$, we refer to [Dar19] and references there in. We write $\mathcal{E}^\infty(X,\omega)$ for the set of bounded potentials in $\mathrm{PSH}(X,\omega)$.

Define $V=V_\omega := \int_X \omega^n$. Let $E:\mathcal{E}^1\to\mathbb{R}$ denote the Monge–Ampère energy functional:

$$E(\varphi) = \frac{1}{V}\sum_{j=0}^{n}\int_X \varphi\,\omega_\varphi^j\wedge\omega^{n-j}\,.$$

For $\varphi \in \mathcal{E}^1(X,\omega)$, define

$$\mathrm{Ent}(\varphi) := \begin{cases} \dfrac{1}{V}\displaystyle\int_X \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\omega_\varphi^n\,, & \text{if } \omega_{\varphi^n} \text{ is absolutely continuous with respect to } \omega^n\,, \\ \infty\,, & \text{otherwise}\,. \end{cases}$$

Let α be a smooth real $(1,1)$-form on X. We define the functional $E^\alpha : \mathcal{E}^1 \to \mathbb{R}$ by

$$E^\alpha(\varphi) := \frac{1}{nV}\sum_{j=0}^{n-1}\int_X \varphi\,\alpha \wedge \omega_\varphi^j \wedge \omega^{n-1-j}\,.$$

In particular, the Ricci energy is defined as

$$E_R := E^{-n\,\mathrm{Ric}\,\omega} = -\frac{1}{V}\sum_{j=0}^{n-1}\int_X \varphi\ \mathrm{Ric}\,\omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j}\,.$$

Define the $\tilde{J}$ functional as $I - J$, namely

$$\tilde{J}(\varphi) = E(\varphi) - \frac{1}{V}\int_X \varphi\,\omega_\varphi^n\,. \tag{3.1}$$

Note that

$$E^\omega = E + \frac{1}{n}\tilde{J}\,. \tag{3.2}$$

Let $M : \mathcal{E}^1 \to (-\infty,\infty]$ denote the Mabuchi functional:

$$M(\varphi) = \bar{S}E(\varphi) + \mathrm{Ent}(\varphi) + E_R(\varphi)\,,$$

where $\bar{S}$ is the average scalar curvature. In this paper, it is convenient to use a different normalization of the Mabuchi functional, so we define the *twisted Mabuchi functional* $\tilde{M} : \mathcal{E}^1(X,\omega) \to (-\infty,\infty]$ as

$$\tilde{M} := M - \bar{S}E = \mathrm{Ent} + E_R\,.$$

Now assume that $[\omega] = c_1(L)$ for some ample line bundle L on X. Fix a smooth Hermitian metric h on L with $c_1(L,h) = \omega$.

For any $k \in \mathbb{Z}_{>0}$, the Donaldson's $\mathscr{L}_k$-functional ([Don05]) is defined as

$$\mathscr{L}_k(\varphi) := -\frac{2}{kV}\log\frac{\det \|\cdot\|_{\mathrm{Hilb}_k(\varphi)}}{\det \|\cdot\|_{\mathrm{Hilb}_k(0)}}\,.$$

Here $\mathrm{Hilb}_k(\varphi)$ is the norm on $H^0(X, K_X \otimes L^k)$ defined by

$$\|s\|^2_{\mathrm{Hilb}_k(\varphi)} = \int_X (s,\bar{s})_{h^k}\, e^{-k\varphi}\,.$$

Definition 3.1. Here our convention of the determinant follows that in [BE21], which differs from the convention of [DX22] by a factor of 2.

Theorem 3.2. *For each $k \geq 1$, the functional $\mathscr{L}_k$ is convex along finite energy geodesics in $\mathcal{E}^1$.*

This result is essentially Berndtsson's convexity theorem ([Ber09b; Ber09a])). See [DLR20, Proposition 2.12] for details.

3.2. Radial functionals. In this section, we assume that the Kähler class $[\omega]$ is in the integral Néron–Severi group. Take an ample line bundle L on X so that $[\omega] = c_1(L)$. Fix a smooth positive metric h on L with $c_1(L, h) = \omega$.

Let $\mathcal{R}^1(X, \omega)$ be the space of $\mathcal{E}^1(X, \omega)$ geodesic rays emanating from 0. That is, a general element $\ell \in \mathcal{R}^1$ is a map $[0, \infty) \to \mathcal{E}^1$, such that $\ell_0 = 0$ and such that $\ell|_{[0,A]}$ is a (finite energy) geodesic in $\mathcal{E}^1$ for any $A > 0$. See [DL20] for details.

We also write $\mathcal{R}^\infty(X, \omega)$ for the set of locally bounded geodesic rays emanating from 0.

For $F = \mathrm{Ent}, E^\alpha, M, \tilde{M}, E, \tilde{J}$, we define a corresponding radial functional $\mathbf{F}$ on $\mathcal{R}^1$ by

$$\mathbf{F}(\ell) := \lim_{t\to\infty} \frac{1}{t} F(\ell_t)\,.$$

For each of them, the limit is well-defined by [BDL17, Proposition 4.5, Theorem 4.7].

3.3. Non-Archimedean functionals. We write X^{an} for the Berkovich analytification of X with respect to the trivial valuation on $\mathbb{C}$. As a set, X^{an} consists of all real semi-valuations (up to equivalence) extending the trivial valuation on $\mathbb{C}$. There is a natural topology known as the Berkovich topology on X^{an}. We always endow X^{an} with this topology. There is a continuous morphism of locally ringed spaces from X^{an} to X with the Zariski topology. Let L^{an} be the pull-back of L to X^{an}. See [Ber12, Section 3.5]. We refer to [BJ21] for the definition of $\mathcal{E}^{1,\mathrm{an}} = \mathcal{E}^1(X^{\mathrm{an}}, L^{\mathrm{an}})$. For $\ell \in \mathcal{R}^1$, we write ℓ^{an} for the corresponding potential in $\mathcal{E}^{1,\mathrm{an}}$ in the sense of [BBJ21], namely

$$\ell^{\mathrm{an}}(v) := -G(v)(\Phi)\,,$$

where $G(v)$ is the Gauss extension of v and Φ is the potential on $X \times \Delta$ corresponding to ℓ. Recall that there is a natural embedding $\mathcal{E}^{1,\mathrm{an}} \hookrightarrow \mathcal{R}^1$. We will often use this embedding implicitly. Geodesic rays in the image of this embedding are known as *maximal* geodesic rays.

For $F = E, E^\alpha, \tilde{J}$, we write

$$F^{\mathrm{an}}(\ell^{\mathrm{an}}) := \mathbf{F}(\ell)\,,$$

when ℓ is a maximal geodesic ray. For explanation of this terminology, see [BHJ16] and [Li20, Proposition 2.38]. We also remark that $\tilde{J}^{\mathrm{an}}$-functional appeared already in [Der16] under the name of the minimum norm.

Let $\psi \in \mathcal{E}^{1,\mathrm{an}}$, we write

$$\mathrm{Ent}^{\mathrm{an}}(\psi) = \mathrm{Ent}^{\mathrm{an}}\left(\mathrm{MA}(\psi)\right) := \frac{1}{V}\int_{X^{\mathrm{an}}} A_X\, \mathrm{MA}(\psi)\,,$$

where $A_X : X^{\mathrm{an}} \to [0, \infty]$ denotes the log discrepancy functional (see [JM12]) and $\mathrm{MA}(\psi)$ denotes the Chambert-Loir measure (see [CLD12], [Cha06], [BJ21]). We also write

$$\tilde{M}^{\mathrm{an}} = E_R^{\mathrm{an}} + \mathrm{Ent}^{\mathrm{an}}\,.$$

In the case of $\mathscr{L}_k$ and ℓ is maximal, we write

$$\mathscr{L}_k^{\mathrm{an}}(\ell^{\mathrm{an}}) = \lim_{t\to\infty} \frac{1}{t}\mathscr{L}_k(\ell_t)\,. \tag{3.3}$$

Recall the definition of δ-invariant:

$$\delta = \delta([\omega]) := \inf_{v \in \mathrm{Val}_X^*} \frac{A_X(v)}{S_L(v)}\,, \tag{3.4}$$

where Val_X^* denotes the space of non-trivial real valuations of $\mathbb{C}(X)$ ([JM12]) and

$$S_L(v) := \int_0^\infty \mathrm{vol}(L - tv)\,\mathrm{d}t$$

and

$$\mathrm{vol}(L - tv) = \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathfrak{a}_{tk})$$

with $\mathfrak{a}_t$ being the ideal sheaf defined by the condition that $v \geq t$. Recall that in the Fano setting, there is always a quasi-monomial valuation that achieves the minimum in (3.4) (see [Xu21, Theorem 4.20], [BLZ19])). Recall that ([BJ18, Section 2.9, Theorem 5.16])

$$\delta([\omega]) = \inf_{\mu\in\mathcal{M}(X^{\mathrm{an}})} \frac{\mathrm{Ent}^{\mathrm{an}}(\mu)}{E^*(\mu)}\,, \tag{3.5}$$

where $\mathcal{M}(X^{\mathrm{an}})$ denotes the set of Radon measures on X^{an} with total mass V,

$$E^*(\mu) := \sup_{\psi\in\mathcal{E}^1(X^{\mathrm{an}},L^{\mathrm{an}})} \left(E(\psi) - \int_{X^{\mathrm{an}}} \psi\,\mathrm{d}\mu \right).$$

It is easy to see that for $\varphi \in \mathcal{E}^{1,\mathrm{an}}$,

$$E^*(\mathrm{MA}(\varphi)) = \tilde{J}^{\mathrm{an}}(\varphi)\,. \tag{3.6}$$

3.4. Flag ideals and test configurations. Let X be a compact Kähler manifold of dimension n. Let L be a big and semi-ample line bundle on X. Let h be a smooth, non-negatively curved metric on L. Let $\omega = c_1(L, h)$.

Definition 3.3. A *flag ideal* on $X \times \mathbb{C}$ is a $\mathbb{C}^*$-invariant coherent ideal sheaf of $X \times \mathbb{C}$ that is cosupported on the central fibre. Equivalently, a flag ideal is an ideal of the form

$$\mathcal{I} = I_0 + I_1 t + \cdots + I_{N-1} t^{N-1} + (t^N)\,, \tag{3.7}$$

where $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{N-1} \subseteq I_N = \mathcal{O}_X$ are coherent ideal sheaves on X, t is the variable on $\mathbb{C}$.

Definition 3.4. A *test configuration* of (X, L) consists of a pair $(\mathcal{X}, \mathcal{L})$ consisting of a variety $\mathcal{X}$ and a semi-ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$, a morphism $\Pi : \mathcal{X} \to \mathbb{C}$, a $\mathbb{C}^*$-action on $\mathcal{X}, \mathcal{L}$ and an isomorphism $(\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1}) \cong (X, L)$, so that

(1) π is $\mathbb{C}^*$-equivariant.
(2) The fibration π is equivariantly isomorphic to the trivial fibration $(X \times \mathbb{C}^*, p_1^* L)$ through an isomorphism that extends the given one over 1. Here p_1 denotes the projection to the first factor.

A test configuration $(\mathcal{X}, \mathcal{L})$ can be compactified by gluing the trivial fibration over $\mathbb{P}^1 \setminus \{0\}$. We write $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ for the compactified test configuration. We will frequently omit the bars when we talk about compactified test configurations.

Definition 3.5 (Donaldson–Futaki invariant)**.** Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (X, L). Take $r \in \mathbb{Z}_{>0}$ so that $\mathcal{L}^r$ is integral. For $k \in \mathbb{Z}_{>0}$, define $w(rk)$ as the weight of the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}^{rk}|_{\mathcal{X}_0})$. By equivariant Riemann–Roch theorem, we can write

$$w(rk) = a(rk)^{n+1} + b(rk)^n + \mathcal{O}(k^{n-1})\,.$$

Define the *twisted Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$ as

$$\widetilde{\mathrm{DF}}(\mathcal{X}, \mathcal{L}) = -2b\frac{n!}{V}\,.$$

Let ℓ be the Phong–Sturm geodesic ray associated to $(\mathcal{X}, \mathcal{L})$ and let $\phi = \ell^{\mathrm{an}} \in \mathcal{H}^{\mathrm{an}}$ be the non-Archimedean potential defined by $(\mathcal{X}, \mathcal{L})$.

Proposition 3.6 ([BHJ16, Proposition 2.8],[Li20, Theorem 5.3])**.** *Assume that L is ample.*

(1) *Let $\ell \in \mathcal{E}^{1,\mathrm{an}}$, then*

$$\tilde{M}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \tilde{\mathbf{M}}(\ell)\,, \quad \mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \mathbf{Ent}(\ell)\,.$$

Equality holds if ℓ is the Phong–Sturm geodesic ray of some test configuration.

(2) *Let $(\mathcal{X}, \mathcal{L})$ be a (not necessarily normal) test configuration of (X, L). Let $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be the normalization. Let $\tilde{L} = p^*\mathcal{L}$. Then*

$$\tilde{\mathbf{M}}(\ell) = \tilde{M}^{\mathrm{an}}(\phi) = \widetilde{\mathrm{DF}}(\mathcal{X}, \mathcal{L}) - \frac{1}{V}\left(\left(\tilde{X}_0 - \tilde{X}_0^{\mathrm{red}}\right) \cdot \tilde{\mathcal{L}}^n\right)\,. \tag{3.8}$$

The intersection-theoretic formulae of the Donaldson–Futaki invariant were obtained first in [Oda13] and [Wan12].

4. The theory of test curves

In this section, we review and extend the theory of test curves.

4.1. Ross–Witt Nyström correspondence. Results in this section are contained in [RW14], [DDL18a] and [DX22]. The references work with ample line bundles and Kähler forms, but the readers can readily check that all arguments work for semi-ample line bundles and real semi-positive forms.

Let X be a compact Kähler manifold of dimension n. Let ω be a real semi-positive form on X. Assume that $\int_X \omega^n > 0$. Let $\mathrm{PSH}^{\mathrm{Model}}(X, \omega)$ denote the set of model potentials in $\mathrm{PSH}(X, \omega)$.

Definition 4.1. A *test curve* is a map $\psi = \psi_\bullet : \mathbb{R} \to \mathrm{PSH}^{\mathrm{Model}}(X, \omega) \cup \{-\infty\}$, such that

(1) $\psi_\bullet$ is concave in $\bullet$.
(2) ψ is usc as a function $\mathbb{R} \times X \to [-\infty, \infty)$.
(3) $\lim_{\tau \to -\infty} \psi_\tau = 0$ in L^1.
(4) $\psi_\tau = -\infty$ for τ large enough.

Let $\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$. We say ψ is *normalized* if $\tau^+ = 0$. The test curve is called *bounded* if $\psi_\tau = 0$ for τ small enough. Let $\tau^- := \sup\{\tau \in \mathbb{R} : \psi_\tau = 0\}$ in this case.

The set of bounded test curves is denoted by $\mathcal{TC}^\infty(X, \omega)$.

Remark 4.2. We remind the readers that our test curves correspond to *maximal* test curves in the literature.

Remark 4.3. In fact, it is more natural to define a test curve only on the interval $(-\infty, \tau^+)$. But we adopt the traditional definition here to facilitate the comparison with the literature.

Definition 4.4. The *energy* of a test curve $\psi_\bullet$ is defined as

$$\mathbf{E}(\psi_\bullet) := \tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \omega_{\psi_\tau}^n - \int_X \omega^n\right) \mathrm{d}\tau\,. \tag{4.1}$$

A test curve ψ is said to be of *finite energy* if $\mathbf{E}(\psi) > -\infty$. We denote the set of finite energy test curves by $\mathcal{TC}^1(X,\omega)$.

Proposition 4.5. *Let $\psi_\bullet$ be a test curve. Then*
(1) $\tau \mapsto \int_X \omega_{\psi_\tau}^n$ is a continuous function for $\tau \in (-\infty, \tau^+)$.
(2) For any $\tau < \tau^+$, $\int_X \omega_{\psi_\tau}^n > 0$.
(3) The function $\tau \mapsto \log \int_X \omega_{\psi_\tau}^n$ is concave for $\tau \in (-\infty, \tau^+)$.

Proof. Part (1) and Part (2) follow from [DX22, Lemma 3.9]. Part (3) is a consequence of [DDL21a, Theorem 6.1] and the monotonicity theorem [Wit19]. ☐

Definition 4.6. Let $\ell \in \mathcal{R}^1(X,\omega)$. The *Legendre transform* of ℓ is defined as

$$\hat{\ell}_\tau := \inf_{t\geq 0}\left(\ell_t - t\tau\right), \quad \tau \in \mathbb{R}\,.$$

Let $\psi \in \mathcal{TC}^1(X,\omega)$, the *inverse Legendre transform* of ψ is defined as

$$\check{\psi}_t := \sup_{\tau\in\mathbb{R}}\left(\psi_\tau + t\tau\right), \quad t \geq 0\,.$$

Theorem 4.7 ([DX22, Theorem 3.7]). *The Legendre transform and inverse Legendre transform establish a bijection from $\mathcal{R}^1(X,\omega)$ to $\mathcal{TC}^1(X,\omega)$. For $\ell \in \mathcal{R}^1(X,\omega)$, We have $\sup_X \ell_1 = \tau^+$ and $\mathbf{E}(\ell) = \mathbf{E}(\hat{\ell})$.*

Moreover, under this correspondence, $\mathcal{R}^\infty$ corresponds to the set of bounded test curves. When $\ell \in \mathcal{R}^\infty$,$\inf_X \ell_1 = \tau^-$.

Now assume that $\omega = c_1(L,h)$ for some ample line bundle L and a strictly positively curved smooth Hermitian metric h on L.

Definition 4.8. An *$\mathcal{I}$-model test curve* is a test curve $\psi_\bullet$ such that for every $\tau < \tau^+$, ψ_τ is $\mathcal{I}$-model. The set of $\mathcal{I}$-model test curves of finite energy is denoted by $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$.

Theorem 4.9 ([DX22, Theorem 3.7]). *The Legendre transform and inverse Legendre transform establish a bijection between $\mathcal{E}^{1,\mathrm{an}}$ and $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$.*

4.2. Test curves induced by filtrations. Let X be a compact Kähler manifold of dimension n. Let L be a big and semi-ample line bundle on X. Let h be a smooth, non-negatively curved metric on L. Let $\omega = c_1(L,h)$. We use the notation

$$R(X,L) := \bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(X,L^k)\,.$$

Definition 4.10. A *filtration* on $R(X,L)$ is a decreasing, left continuous, multiplicative $\mathbb{R}$-filtration $\mathscr{F}^\bullet$ on the ring $R(X,L)$ which is linearly bounded in the sense that there is $C > 0$, so that

$$\mathscr{F}^{-k\lambda}H^0(X,L^k) = H^0(X,L^k)\,, \quad \mathscr{F}^{k\lambda}H^0(X,L^k) = 0\,,$$

when $\lambda > C$.

A filtration $\mathscr{F}$ is called a *$\mathbb{Z}$-filtration* if $\mathscr{F}^\lambda = \mathscr{F}^{\lfloor\lambda\rfloor}$ for any $\lambda \in \mathbb{R}$.

A $\mathbb{Z}$-filtration $\mathscr{F}$ is called *finitely generated* if the bigraded algebra

$$\bigoplus_{\lambda\in\mathbb{Z},k\in\mathbb{Z}_{\geq 0}} \mathscr{F}^\lambda H^0(X,L^k)$$

is finitely generated over $\mathbb{C}$.

Recall that by [RW14], a filtration induces a test curve in the following manner. Let $\mathscr{F}^\bullet$ be a filtration. For $\tau\in\mathbb{R}$, define

$$\psi_\tau := \sup_{k\in\mathbb{Z}_{>0}}{}^* k^{-1}\sup{}^*\left\{\log|s|^2_{h^k} : s\in\mathscr{F}^{k\tau}H^0(X,L^k), \sup_X |s|_{h^k}\leq 1\right\}. \tag{4.2}$$

By [DX22, Theorem 3.11], ψ_τ is $\mathcal{I}$-model or $-\infty$ for each $\tau\in\mathbb{R}$.

Lemma 4.11. *Let $\psi_\bullet$ be the test curve induced by a filtration $\mathscr{F}^\bullet$ on $R(X,L)$. Let v be a real valuation of $\mathbb{C}(X)$. Then*

$$v(\psi_\tau) = \inf_{k\in\mathbb{Z}_{>0}} k^{-1}\inf\left\{v(s) : s\in\mathscr{F}^{k\tau}H^0(X,L^k)\right\}.$$

Proof. For $k\in\mathbb{Z}_{>0}$, let

$$F_k := \sup{}^*\left\{\log|s|^2_{h^k} : s\in\mathscr{F}^{k\tau}H^0(X,L^k), \sup_X |s|_{h^k}\leq 1\right\}.$$

For $k,m\in\mathbb{Z}_{>0}$,

$$F_{k+m}\geq F_k+F_m.$$

So by Fekete's lemma, ψ_τ is the usc regularization of the increasing limit $2^{-k}F_{2^k}$. We conclude by the monotonicity and the upper semi-continuity of Lelong numbers. □

Lemma 4.12. *Let $\psi_\bullet$ be the test curve induced by a $\mathbb{Z}$-filtration $\mathscr{F}^\bullet$ on $R(X,L)$. Then*

$$\int_X \omega^n_{\psi_\tau} \geq \lim_{k\to\infty}\frac{n!}{k^n}\dim\mathscr{F}^{k\tau}H^0(X,L^k). \tag{4.3}$$

Equality holds if $\mathscr{F}^\bullet$ is finitely generated, $\tau<\tau^+$.

Note that the limit on the right-hand side exists by [LM09].

Proof. By [DX22, Theorem 1.1],

$$\int_X \omega^n_{\psi_\tau} = \lim_{k\to\infty}\frac{n!}{k^n}h^0(X,L^k\otimes\mathcal{I}(k\psi_\tau)).$$

Each element in $\mathscr{F}^{k\tau}H^0(X,L^k)$ is obviously square integrable with respect to $k\psi_\tau$, (4.3) follows.

Now assume that $\mathscr{F}^\bullet$ is finitely generated. Then it is the filtration induced by some test configuration $(\mathcal{X},\mathcal{L})$ of (X,L) by [BHJ17, Proposition 2.15]. Without loss of generality, we may assume that $\tau^+=0$ for the test curve $\psi_\bullet$. Then by [BHJ17, Section 5], the Duistermaat–Heckman measure of $(\mathcal{X},\mathcal{L})$ is given by

$$\nu = -\frac{1}{V}\frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{vol}(R^{(\tau)}),$$

where

$$\operatorname{vol}(R^{(\tau)}) := \lim_{k\to\infty}\frac{n!}{k^n}\dim\mathscr{F}^{k\tau}H^0(X,L^k).$$

By [BHJ17, Lemma 7.3], the non-Archimedean Monge–Ampère energy of $(\mathcal{X},\mathcal{L})$ is given by

$$E^{\mathrm{an}}(\mathcal{X},\mathcal{L})=\int_{-\infty}^{\infty}\tau\,\mathrm{d}\nu(\tau)=\int_{-\infty}^{0}\left(\frac{1}{V}\operatorname{vol}R^{(\tau)}-1\right)\mathrm{d}\tau\,.$$

On the other hand, by [DX22, Theorem 1.1],

$$E^{\mathrm{an}}(\mathcal{X},\mathcal{L})=\int_{-\infty}^{0}\left(\frac{1}{V}\int_X\omega_{\psi_\tau}^n-1\right)\mathrm{d}\tau\,.$$

Now by (4.3), Proposition 4.5 and [BHJ17, Theorem 5.3], we conclude that equality holds in (4.3) when $\tau<\tau^+$. □

Let $(\mathcal{X},\mathcal{L})$ be a test configuration of (X,L). It induces a filtration as follows: Take $r\in\mathbb{Z}_{>0}$ so that $\mathcal{L}^r$ is integral. Then $(\mathcal{X},\mathcal{L})$ induces a $\mathbb{Z}$-filtration of $R(X,rL)$ as follows: let $s\in H^0(X,rkL)$, then $s\in\mathscr{F}^\lambda H^0(X,rkL)$ iff $t^{-\lambda}s\in H^0(\mathcal{X},\mathcal{L}^{rk})$. Here we have abused the notation by writing s for the equivariant extension of s as well. See [BHJ17]. The weight of the $\mathbb{C}^*$-action on the central fibre of $\mathcal{L}^{rk}$ is given by

$$w(rk)=-\int_{-\infty}^{\infty}\lambda\,\mathrm{d}\dim\mathscr{F}^\lambda H^0(X,L^{rk})\,.$$

Example 4.13. *Let $I=I_0+I_1t+\cdots+I_{N-1}t^{N-1}+(t^N)$ be a flag ideal on $X\times\mathbb{P}^1$. Let $\mathcal{X}=\mathrm{Bl}_I X\times\mathbb{P}^1$. Denote by $\Pi:\mathcal{X}\to X\times\mathbb{P}^1$ the natural morphism. Let E be the exceptional divisor. Let $p_1:X\times\mathbb{P}^1\to X$ be the natural projection. Assume that $\mathcal{L}:=\Pi^*p_1^*L\otimes\mathcal{O}_{\mathcal{X}}(-E)$ is π-semiample.*

Write

$$I^k=\sum_{j=0}^{Nk-1}J_{k,j}t^j+(t^{Nk})\,.$$

Then

$$J_{k,j}=\sum_{\alpha\in\mathbb{N}^N,|\alpha|=k,|\alpha|'=j}I_\bullet^\alpha\,.$$

Here $|\alpha|':=\sum_i i\alpha_i$. Set $J_{k,kN}=\mathcal{O}_X$.

Let $\mathscr{F}^\bullet$ be the filtration on $R(X,L)$ induced by $(\mathcal{X},\mathcal{L})$. Let $\psi_\bullet$ be the corresponding test curve.

We claim that

$$\psi_\tau^{\mathrm{an}}(v)=-\min_{\alpha\in\mathbb{Q}^N_{\geq0},|\alpha|=1,|\alpha|'=-\tau}\sum_i\alpha_i v(I_i)\,.$$

In particular,

$$\psi_0^{\mathrm{an}}(v)=-v(I_0)\,.$$

Proof. Let $\lambda\in\mathbb{Z}$ and $s\in H^0(X,L^k)$, then $s\in\mathscr{F}^\lambda H^0(X,L^k)$ iff $t^{-\lambda}s$ extends to a section of $\mathcal{L}^k$ iff

$$t^{-\lambda}s\in H^0(X\times\mathbb{C},L^k\otimes I^{\otimes k})$$

iff $s\in J_{k,-\lambda}$. Hence we have

$$v(\psi_\tau)=\inf_k\frac{1}{k}\inf\{v(s):s\in J_{k,-\lceil k\tau\rceil}\}\,.$$

Observe that

$$\inf\{v(s):s\in J_{k,-\lceil k\tau\rceil}\}=\min_{\alpha\in\mathbb{N}^N,|\alpha|=k,|\alpha|'=-\lceil k\tau\rceil}\sum_i\alpha_i v(I_i)\,.$$

So

$$v(\psi_\tau) \geq \min_{\alpha\in\mathbb{Q}^N_{\geq 0},|\alpha|=1,|\alpha|'=-\tau} \sum_i \alpha_i v(I_i)\,.$$

On the other hand, observe that the minimizer is indeed rational when τ is rational, so the reverse inequality also holds. □

Observe that when $\tau < 0$, ψ_τ has quasi-analytic singularities (Definition 2.12).

Example 4.14. *Let $v = c\operatorname{ord}_F$ be a divisorial valuation of $\mathbb{C}(X)$, where $c \in \mathbb{Q}_{>0}$, F is a prime divisor over X. Then v induces a filtration $\mathscr{F}_v^\bullet$ on $R(X,L)$:*

$$\mathscr{F}_v^\lambda H^0(X,L^k) = \begin{cases} H^0(X,kL-\lambda cF)\,, & \lambda \geq 0\,,\\ H^0(X,L^k)\,, & \lambda < 0\,.\end{cases}$$

Here we have omitted the pull-back of L to a model.

Example 4.15. *Let $\psi \in \mathrm{PSH}(X,\omega)$ be a potential with analytic singularities. Let $\pi : Y \to X$ be a log resolution of the singularities of ψ. Assume that $\pi^*L - \operatorname{div}_Y\psi$ is semi-ample. Then ψ induces a test configuration of (Y,π^*L) by deformation to the normal cone with respect to $\operatorname{div}_Y\psi$. Then a section $s \in H^0(X,L^k) = H^0(Y,\pi^*L^k)$ is in $\mathscr{F}^\lambda$ iff $t^{-\lambda}s$ extends to the central fibre, that is, $s \in \mathcal{O}_Y(-(k+\lambda)\operatorname{div}_Y\psi)$. Hence*

$$\mathscr{F}_\psi^\lambda H^0(X,L^k) = \begin{cases} H^0(Y,k\pi^*L-(\lambda+k)\operatorname{div}_Y\psi)\,, & \lambda \leq 0\,,\\ H^0(X,L^k)\,, & \lambda > 0\,.\end{cases}$$

The test curve $\psi_\bullet$ defined in Section 2.7 is induced by this filtration.

Example 4.16. *Let $\psi \in \mathrm{PSH}(X,\omega)$ be a potential with analytic singularities. Let $\pi : Y \to X$ be a log resolution of the singularities of ψ. The deformation to the normal cone defined in Example 4.15 can be extended as follows:*

$$\mathscr{F}_{\psi^+}^\lambda H^0(X,L^k) = \begin{cases} H^0(X,k\pi^*L-\lambda\operatorname{div}_Y\psi)\,, & \lambda \geq 0\,,\\ H^0(X,L^k)\,, & \lambda < 0\,.\end{cases}$$

The test curve $\psi_\bullet^+$ defined in Section 2.6 is induced by this filtration.

4.3. The Phong–Sturm geodesic ray. Let X be a compact Kähler manifold of dimension n. Let ω be a real smooth *semi-positive* $(1,1)$-form on X. Let $(\mathcal{X},\mathcal{L})$ be a semi-ample test configuration of (X,L). Fix a S^1-invariant smooth metric Φ on $\mathcal{L}$ with $c_1(\mathcal{L},\Phi) = \Omega$, we may assume that $\Omega|_{X\times S^1}$ is the pull-back of ω. Let $\pi : \mathcal{X} \to \mathbb{C}$ be the natural map. Let $\mathcal{X}^\circ := \pi^{-1}(\Delta)$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Consider the homogeneous Monge–Ampère equation

$$\begin{cases} (\Omega + \mathrm{dd^c}\Psi)^{n+1} = 0 & \text{on } \mathcal{X}^\circ\,,\\ \Psi|_{X\times S^1} = 0\,. \end{cases} \tag{4.4}$$

By [CTW18], there is a unique bounded solution to (4.4) and the solution is $C^{1,1}$ outside the central fibre.

Let ℓ be the geodesic ray in $\mathcal{E}^1(X,\omega)$ corresponding to Ψ, then ℓ is known as the *Phong–Sturm geodesic ray* induced by $(\mathcal{X},\mathcal{L})$. This construction was first studied in [PS07] and [PS10].

THEOREM 4.17 ([RW14, Theorem 9.2]). *Let $(\mathcal{X},\mathcal{L})$ be a semi-ample test configuration of (X,L). Let ℓ be the Phong–Sturm geodesic ray induced by $(\mathcal{X},\mathcal{L})$. Let $\mathscr{F}^\bullet$ be the filtration induced by $(\mathcal{X},\mathcal{L})$. Let $\psi_\bullet$ be the test curve induced by $\mathscr{F}^\bullet$ as in (4.2). Then $\check{\psi}=\ell$.*

4.4. Non-Archimedean analogue of Ross–Witt Nyström correspondence. Assume that L is ample.

Definition 4.18. A function $\psi : X^{\mathrm{an}} \to [-\infty,\infty)$ is called a *good potential* if there exists $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\psi=\varphi^{\mathrm{an}}$.

The set of good potential is denoted as $\mathrm{PSH}^{\mathrm{an}}_{\mathrm{g}}(X,\omega)$.

See (2.1) for the definition of φ^{an}.

Proposition 4.19. *The map $\psi \mapsto \psi^{\mathrm{an}}$ is a bijection from $\mathrm{PSH}^{\mathrm{Model}}_{\mathcal{I}}(X,\omega)$ to $\mathrm{PSH}^{\mathrm{an}}_{\mathrm{g}}(X,\omega)$.*

This is obvious by definition.

Definition 4.20. A test curve $\psi \in \mathcal{TC}^\infty(X,\omega)$ is *piecewise linear* if ψ^{an} is piecewise linear with finitely many breaking points (i.e. non-differentiable points).

Definition 4.21. A *non-Archimedean test curve* is a map $\psi : (-\infty,\tau^+) \to \mathrm{PSH}^{\mathrm{an}}_{\mathrm{g}}(X,\omega)$ for some $\tau^+\in\mathbb{R}$, such that

(1) ψ is concave.
(2) $\lim_{\tau\to-\infty}\psi_\tau=0$ in L^1.

We define τ^- as in the Archimedean case.

The non-Archimedean test curve $\psi_\bullet$ is *of finite energy* if

$$\mathbf{E}(\psi_\bullet) := \tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \omega_{\varphi_\tau}^n - \int_X \omega^n\right)\mathrm{d}\tau > -\infty\,, \tag{4.5}$$

where φ_τ is the $\mathcal{I}$-model potential in $\mathrm{PSH}(X,\omega)$ with $\varphi_\tau^{\mathrm{an}}=\psi_\tau$.

The set of non-Archimedean test-curves of finite energy is denoted by $\mathcal{TC}^{1,\mathrm{an}}(X,\omega)$.

Proposition 4.22. *The map $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)\to\mathcal{TC}^{1,\mathrm{an}}(X,\omega)$ defined by $\psi_\bullet\mapsto(\psi_\tau^{\mathrm{an}})_{\tau<\tau^+}$ is a bijection.*

This again is immediate by definition.

Remark 4.23. In Definition 4.21, we deliberately define ψ_τ only for $\tau<\tau^+$. This is because it is *not* always true that for an Archimedean test curve $\psi_\bullet$,

$$\psi^{\mathrm{an}}_{\tau^+} = \lim_{\tau\to\tau^+-}\psi^{\mathrm{an}}_\tau\,.$$

THEOREM 4.24 ([DX22, Proposition 3.13]). *The map $\check{}: \mathcal{TC}^{1,\mathrm{an}}(X,\omega)\to\mathcal{E}^{1,\mathrm{an}}$ given by*

$$\psi^{\mathrm{an}}_\bullet \mapsto \sup_{\tau<\tau^+}\left(\psi^{\mathrm{an}}_\tau+\tau\right)$$

is a bijection. Moreover, when $\psi_\bullet\in\mathcal{TC}^1(X,\omega)$,

$$\left(\psi^{\mathrm{an}}_\bullet\right)\check{} = \left(\check{\psi}_\bullet\right)^{\mathrm{an}}\,,$$

namely, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{TC}^1 \arrow[r, "\mathrm{an}"] \arrow[d, "\check{}"] & \mathcal{TC}^{1,\mathrm{an}} \arrow[d, "\check{}"] \\
\mathcal{R}^1 \arrow[r, "\mathrm{an}"] & \mathcal{E}^{1,\mathrm{an}}
\end{tikzcd}$$

Remark 4.25. Ideally when we are considering only maximal geodesic rays, it should be possible to carry out the computations in Section 2.6 purely in terms of non-Archimedean test curves, without referring to the machinery of test curves, filtrations and test configurations. However, the difficulty is that we do not have a good understanding of the following non-Archimedean Monge–Ampère measure

$$\mathrm{MA}\left(\sup_{\tau<\tau^+}(\psi_\tau^{\mathrm{an}}+\tau)\right).$$

It is highly desirable to have a description of this measure in terms of certain real Monge–Ampère measures on some dual complexes. In the non-trivially valued case, a partial result is derived by Vilsmeier ([Vil21]).

5. Intersection theory of b-divisors

In this section, we apply the intersection theory of Shokurov's b-divisors to the study of singularities of psh functions. Due to the technical assumptions in [DF20a] and [DF20b], we can not apply Dang–Favre's intersection theory directly. Although it seems possible to remove the technical assumptions in Dang–Favre's theory, we do not pursue this most general theory here.*

References to this section are [DF20a], [DF20b], [BFJ09], [BDPP13], [KK14].

Let X be a projective manifold of dimension n.

5.1. b-divisors. Recall that the Riemann–Zariski space of X is the locally ringed space defined by

$$\mathfrak{X} := \varprojlim_Y Y,$$

where Y runs over all birational models of X. Here the projective limit is taken in the category of locally ringed spaces. For valuative interpretation of $\mathfrak{X}$, see [Tem11]. We do not make use of the theory of Riemann–Zariski spaces in an essential way in this paper. Instead, we give an *ad hoc* treatment of divisors on $\mathfrak{X}$.

Definition 5.1. By a *Weil divisor on* $\mathfrak{X}$ or a *Weil b-divisor* on X, we mean an element in

$$\mathrm{bWeil}(X) := \varprojlim_Y \mathrm{Weil}(Y),$$

where Y runs over all (smooth) birational models of X and $\mathrm{Weil}(Y)$ is the set of numerical classes of $\mathbb{R}$-divisors on Y.

By a *Cartier divisor on* $\mathfrak{X}$ or a *Cartier b-divisor* on X, we mean an element in

$$\mathrm{bCart}(X) := \varinjlim_Y \mathrm{Weil}(Y),$$

*When the current paper was written, the second version of [DF20a] was not available yet, where Dang–Favre developed the general intersection theory of nef b-divisors. Our definition of volumes is essentially the same as the intersection number defined using [DF20a].

where Y runs over all (smooth) birational models of X.

Both the limit and the colimit are taken in the category of topological vector spaces.

There is a natural continuous injection $\mathrm{bCart}(X) \hookrightarrow \mathrm{bWeil}(X)$.

5.2. Differentiability of the volume. General references of results in this section are [BFJ09], [DP04].

Let X be a compact Kähler manifold of dimension n. Let L be a big line bundle on X. Recall that the volume of L is defined as

$$\mathrm{vol}(L) := \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k)\,.$$

More generally, by requiring

$$\mathrm{vol}(L^k) = k^n \,\mathrm{vol}(L)\,,$$

we extend the definition of volume to all big $\mathbb{Q}$-line bundles. By continuity, this definition further extends to all pseudo-effective $\mathbb{R}$-line bundles.

When L is a nef $\mathbb{R}$-line bundle, we have

$$\mathrm{vol}(L) = (L^n)\,. \tag{5.1}$$

Recall the following basic fact,

Theorem 5.2 ([BFJ09]). *The volume function* vol *is continuously differentiable in the big cone. Moreover, let L be a big and nef $\mathbb{R}$-line bundle, let L' be a line bundle, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{vol}(L+\epsilon L') = n\left(L^{n-1}\cdot L'\right)\,. \tag{5.2}$$

Now assume that L is big and semi-ample. Fix a smooth semi-positive real $(1,1)$-form $\omega \in c_1(L)$. Let $\psi \in \mathrm{PSH}(X,\omega)$ be a potential with quasi-analytic singularities along a snc $\mathbb{R}$-divisor $\mathrm{div}_X \psi$. Assume that ψ has positive mass. Recall that by Lemma 2.15, $L - \mathrm{div}_X \psi$ is nef and big. Let L' be an $\mathbb{R}$-line bundle on X. Now we define

$$D_L(\psi, L') = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \mathrm{vol}(L - \mathrm{div}_X \psi + \epsilon L') = n\left((L - \mathrm{div}_X \psi)^{n-1}\cdot L'\right)\,. \tag{5.3}$$

When $\psi \in \mathrm{PSH}(X,\omega)$ has positive mass and there exists a birational model $\pi : Y \to X$, ψ has quasi-analytic singularities along a snc $\mathbb{R}$-divisor $\mathrm{div}_Y \psi$, let L' be an $\mathbb{R}$-line bundle on Y, we define

$$D_L(\psi, L') := D_{\pi^*L}(\pi^*\psi, L')\,. \tag{5.4}$$

We formally set $D_L(-\infty, L') = 0$.

5.3. Singularity divisors. Let L be a semi-ample line bundle on X. Let h be a non-negatively curved metric on L. Let $\omega = c_1(L,h)$.

Definition 5.3. Let $\psi \in \mathrm{PSH}(X,\omega)$. We define the *singularity divisor* of ψ as a Weil b-divisor $\mathrm{div}_{\mathfrak{X}}\,\psi \in \mathrm{bWeil}(X)$:

$$(\mathrm{div}_{\mathfrak{X}}\,\psi)_Y = \mathrm{div}_Y\,\psi\,.$$

Here we have abused the notation by writing $\mathrm{div}_Y\,\psi$ for the numerical class of the corresponding divisor, which makes sense as explained in Remark 2.10.

We set

$$\mathrm{vol}(L - \mathrm{div}_{\mathfrak{X}}\,\psi) := \lim_{Y} \mathrm{vol}(\pi^* L - \mathrm{div}_Y\,\psi)\,,$$

where $\pi : Y \to X$ runs over all birational model of X. The net is decreasing, hence the limit is well-defined.

THEOREM 5.4. *Assume that ψ is $\mathcal{I}$-model and of positive mass, then*

$$\int_X \omega_\psi^n = \mathrm{vol}\left(L - \mathrm{div}_{\mathfrak{X}}\,\psi\right)\,. \tag{5.5}$$

PROOF. Let ψ^j be a quasi-equisingular approximation to ψ. By [DX22, Theorem 1.4], $\int_X \omega_{\psi^j}^n \to \int_X \omega_\psi^n$. Similarly, the right-hand side converges along ψ^j as follows from [DF20a, Proof of Theorem 6(3)]. To be more precise, it suffices to prove that for any $\epsilon > 0$, any model $\pi : Y \to X$, we can find $j_0 > 0$, such that for $j \geq j_0$,

$$\mathrm{vol}\left(L - \mathrm{div}_{\mathfrak{X}}\,\psi\right) \leq \mathrm{vol}\left(L - \mathrm{div}_{\mathfrak{X}}\,\psi^j\right) \leq \mathrm{vol}\left(\pi^*L - \mathrm{div}_Y\,\psi\right) + \epsilon\,.$$

The first inequality is trivial. For the second inequality, observe that by Lemma 2.7, $\mathrm{div}_Y\,\psi^j \to \mathrm{div}_Y\,\psi$. Fix some $C > 0$, depending on π, we may take j_0 large enough, so that when $j \geq j_0$,

$$\pi^*L - \mathrm{div}_Y\,\psi^j \leq \pi^*L - \mathrm{div}_Y\,\psi + C^{-1}\epsilon\pi^*\omega\,.$$

Then it follows that

$$\mathrm{vol}\left(\pi^*L - \mathrm{div}_Y\,\psi^j\right) \leq \mathrm{vol}\left(\pi^*L - \mathrm{div}_Y\,\psi\right) + \epsilon\,.$$

Hence

$$\mathrm{vol}\left(L - \mathrm{div}_{\mathfrak{X}}\,\psi^j\right) \leq \mathrm{vol}\left(\pi^*L - \mathrm{div}_Y\,\psi\right) + \epsilon\,.$$

□

In particular, this gives an additional characterization of $\mathcal{I}$-model potentials.

Corollary 5.5. *Let $\psi \in \mathrm{PSH}^{\mathrm{Model}}(X,\omega)$ be a model potential with positive mass. Then ψ is $\mathcal{I}$-model iff*

$$\int_X \omega_\psi^n = \mathrm{vol}\left(L - \mathrm{div}_{\mathfrak{X}}\,\psi\right)\,.$$

Remark 5.6. As the techniques of [DX22] have been extended to pseudo-effective line bundles in [DX21], this corollary and its proof actually work in the setting of big line bundles. In terms of [DF20a], our proof also shows that $L - \mathrm{div}_{\mathfrak{X}}\,\psi$ is nef. A special case of this result is also discovered in [BBGHdJ21].

6. Radial functionals in terms of Legendre transforms

In this section, let X be a compact Kähler manifold of dimension n. Let L be a big and semi-ample line bundle on X. Let h be a smooth non-negatively curved metric on L. Let $\omega = c_1(L,h)$.

From Section 6.2 on, we assume that L is an ample line bundle and h is strictly positively curved.

In this section, we study several functionals on the space of geodesic rays and express them in terms of test curves.

6.1. Functionals on the space of test curves. Let $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$. Recall that $\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$.

We have already defined the Monge–Ampère energy $\mathbf{E}(\psi_\bullet)$ in (4.1). For any real smooth $(1,1)$-form α on X, define the α-energy of $\psi_\bullet$ as

$$\mathbf{E}^\alpha(\psi_\bullet) := \tau^+ \frac{1}{V}\int_X \alpha\wedge\omega^{n-1} + \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \alpha\wedge\omega_{\psi_\tau}^{n-1} - \int_X \alpha\wedge\omega^{n-1}\right)\mathrm{d}\tau\,. \tag{6.1}$$

The *Ricci energy* of $\psi_\bullet$ is defined as

(6.2)

$$\mathbf{E}_R(\psi_\bullet) := -n\tau^+\frac{1}{V}\int_X \operatorname{Ric}\omega'\wedge\omega^{n-1} - \frac{n}{V}\int_{-\infty}^{\tau^+}\left(\int_X \operatorname{Ric}\omega'\wedge\omega_{\psi_\tau}^{n-1} - \int_X \operatorname{Ric}\omega'\wedge\omega^{n-1}\right)\mathrm{d}\tau\,,$$

where ω' denotes a Kähler form on X.

The $\tilde{J}$-*functional* of $\psi_\bullet$ is defined as

$$\tilde{\mathbf{J}}(\psi_\bullet) = n\mathbf{E}^\omega(\psi_\bullet) - n\mathbf{E}(\psi_\bullet) = \frac{n}{V}\int_{-\infty}^{\infty}\left(\int_X \omega\wedge\omega_{\psi_\tau}^{n-1} - \int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau\,. \tag{6.3}$$

Remark 6.1. It is interesting to observe that $\mathbf{E}^\alpha(\psi_\bullet)$ depends only on the cohomology class of α.

Assume that $\psi_\bullet$ is $\mathcal{I}$-model. The *non-Archimedean* $\mathscr{L}_k$-*functional* of $\psi_\bullet$ is defined as

$$\mathscr{L}_k^{\mathrm{an}}(\psi_\bullet) := \frac{1}{V}\int_{-\infty}^{\infty} \tau\,\mathrm{d}h^0(X, K_X\otimes L^k\otimes\mathcal{I}(k\psi_\tau))\,. \tag{6.4}$$

Assume that $\psi_\bullet$ is $\mathcal{I}$-model, the *entropy* of $\psi_\bullet$ is defined as

$$\operatorname{Ent}(\psi_\bullet) := \int_{-\infty}^{\infty}\operatorname{Ent}([\psi_\tau])\,\mathrm{d}\tau\,. \tag{6.5}$$

Recall that $\operatorname{Ent}[\bullet]$ is defined in Definition 2.21.

Definition 6.2. Let $\psi_\bullet \in \mathcal{TC}^\infty(X,\omega)$. We say $\psi_\bullet$ is *analytic* if ψ_τ has quasi-analytic singularities for any $\tau < \tau^+$.

We say $\psi_\bullet$ is *piecewise linear* if $\psi_\bullet^{\mathrm{an}}$ is piecewise linear with finitely many breaking points (non-differentiable points).

We need the following observation.

Lemma 6.3. *The test curves in Example 4.13 are analytic and piecewise linear.*

Corollary 6.4. *The test curve induced by a test configuration is analytic and piecewise linear.*

Proof. This follows from [Oda13, Proposition 3.10] and Lemma 6.3. ∎

Remark 6.5. The statement of [Oda13, Proposition 3.10] needs to be corrected as follows: $\mathcal{L}^r(-E) = f^*\mathcal{M} + c\mathcal{B}_0$ for some constant $c \in \mathbb{Q}$. The mistake in the proof is on the fourth line, where we need to make sure that the isomorphism between $h^*\mathcal{M}^s$ and $\mathcal{L}^r$ extends to the generic point of the central fibre.

We observe the following obvious lemma.

Lemma 6.6. *Let $\psi_\bullet$ be an analytic test curve. Then*

$$\operatorname{Ent}(\psi_\bullet) = \frac{1}{V}\int_{-\infty}^{\infty} D_L(\psi_\tau, K_{Y_\tau/X})\,\mathrm{d}\tau + \frac{1}{V}\int_{-\infty}^{\infty} D_L(\psi_\tau, \operatorname{red}\operatorname{div}_Y \psi_\tau)\,\mathrm{d}\tau\,, \tag{6.6}$$

where $\pi_\tau : Y_\tau \to X$ is a log resolution of ψ_τ.

See Section 5.2 for the definition of D_L.

6.2. Monge–Ampère energy. From this section on, we assume that L is ample and h is strictly positively curved, so that ω is a Kähler form.

THEOREM 6.7 ([DX22, Theorem 3.7]). *Let $\ell \in \mathcal{R}^1$. Then*

$$\mathbf{E}(\ell) = \mathbf{E}(\hat{\ell})\,. \tag{6.7}$$

Recall that the right-hand side is defined in (4.1).

6.3. Non-archimedean $\mathcal{L}$-functionals.

THEOREM 6.8 ([DX22, Theorem 1.1]). *Let $\ell \in \mathcal{E}^{1,\mathrm{an}}$. For each $k \in \mathbb{Z}_{>0}$,*

$$\mathscr{L}_k^{\mathrm{an}}(\ell) = \mathscr{L}_k^{\mathrm{an}}(\hat{\ell})\,. \tag{6.8}$$

The right-hand side is defined in (6.4) and the left-hand side is defined in (3.3).

6.4. α-energy. Let α be a smooth real $(1,1)$-form on X.

Lemma 6.9. *Let $\varphi, \psi \in \mathcal{E}^\infty$, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E^\alpha(s\psi + (1-s)\varphi) = \frac{1}{V}\int_X (\psi - \varphi)\,\alpha \wedge \omega_\varphi^{n-1}\,.$$

PROOF. This result is well-known when ψ and φ are smooth. In general, it follows from a direct computation using integration by parts ([Xia19], [Lu21]). $\square$

THEOREM 6.10. *Let $\ell \in \mathcal{E}^{1,\mathrm{an}}$ or $\ell \in \mathcal{R}^\infty$. Then*

$$\mathbf{E}^\alpha(\ell) = \mathbf{E}^\alpha(\hat{\ell})\,. \tag{6.9}$$

The strategy of the proof first appeared in [RW14].

PROOF. Without loss of generality, we may assume that α is a Kähler form and $\sup_X \ell_1 = 0$.

We first assume that $\ell \in \mathcal{R}^\infty$. We fix a few notations. Let $\psi_\bullet$ be the Legendre transform of ℓ. Now for each $N \in \mathbb{N}$, $M \in \mathbb{Z}$, $t \geq 0$, we introduce

$$\check{\psi}_t^{N,M} := \max_{\substack{k\in\mathbb{Z}\\ k\leq M}} (\psi_{k2^{-N}} + tk2^{-N})\,.$$

Let

$$U_t^{N,M} := \left\{ x \in X : \check{\psi}_t^{N,M+1}(x) > \check{\psi}_t^{N,M}(x) \right\}\,.$$

Observe that on $U_t^{N,M}$,

$$\check{\psi}_t^{N,M+1} = \psi_{(M+1)2^{-N}} + t(M+1)2^{-N}\,, \quad \check{\psi}_t^{N,M} = \psi_{M2^{-N}} + tM2^{-N}\,. \tag{6.10}$$

By Lemma 6.9,

$$E^\alpha(\check{\psi}_t^{N,M+1}) - E^\alpha(\check{\psi}_t^{N,M}) = \frac{1}{V}\int_0^1\int_X \left(\check{\psi}_t^{N,M+1} - \check{\psi}_t^{N,M}\right)\alpha\wedge\omega^{n-1}_{s\check{\psi}_t^{N,M+1}+(1-s)\check{\psi}_t^{N,M}}\,\mathrm{d}s\,.$$

By the comparison principle ([DDL18b, Proposition 3.5]),

$$\begin{aligned}\int_{U_t^{N,M}}(\check{\psi}_t^{N,M+1} - \check{\psi}_t^{N,M})\alpha\wedge\omega^{n-1}_{\check{\psi}_t^{N,M+1}} \leq& \int_X(\check{\psi}_t^{N,M+1} - \check{\psi}_t^{N,M})\alpha\wedge\omega^{n-1}_{s\check{\psi}_t^{N,M+1}+(1-s)\check{\psi}_t^{N,M}}\\ \leq& \int_{U_t^{N,M}}(\check{\psi}_t^{N,M+1} - \check{\psi}_t^{N,M})\alpha\wedge\omega^{n-1}_{\check{\psi}_t^{N,M}}\,.\end{aligned}$$

We first deal with the upper bound,

$$E^\alpha(\check{\psi}_t^{N,M+1}) - E^\alpha(\check{\psi}_t^{N,M}) \leq 2^{-N}V^{-1}t\int_{U_t^{N,M}}\alpha\wedge\omega^{n-1}_{\psi_{M2^{-N}}}\,.$$

Set $\tau^- := \inf_X \ell_1$. Take the sum with respect to M from $[\tau^-]2^N$ to -1, we get

$$\begin{aligned}-t[\tau^-]V^{-1}\int_X\omega^{n-1}\wedge\alpha + E^\alpha(\check{\psi}_t^{N,0}) \leq& 2^{-N}V^{-1}t\sum_{M=[\tau^-]2^N}^{-1}\int_{U_t^{N,M}}\alpha\wedge\omega^{n-1}_{\psi_{M2^{-N}}}\\ \leq& 2^{-N}V^{-1}t\sum_{M=[\tau^-]2^N}^{-1}\int_X\alpha\wedge\omega^{n-1}_{\psi_{M2^{-N}}}\,.\end{aligned}$$

Let $N\to\infty$ and then $t\to\infty$, we get

$$\mathbf{E}^\alpha(\ell) \leq \frac{1}{V}\int_{[\tau^-]}^0\left(\int_X\alpha\wedge\omega^{n-1}_{\psi_\tau} - \int_X\alpha\wedge\omega^{n-1}\right)\mathrm{d}\tau\,.$$

Now we deal with the lower bound part. We have

$$\begin{aligned}E^\alpha(\check{\psi}_t^{N,M+1}) - E^\alpha(\check{\psi}_t^{N,M}) \geq& 2^{-N}V^{-1}t\int_{U_t^{N,M}}\alpha\wedge\omega^{n-1}_{\psi_{(M+1)2^{-N}}}\\ &+ V^{-1}\int_{U_t^{N,M}}(\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}})\alpha\wedge\omega^{n-1}_{\psi_{(M+1)2^{-N}}}\,.\end{aligned}$$

Taking summation with respect to M from $[\tau^-]2^N$ to -1, we get

$$\begin{aligned}\frac{1}{t}E^\alpha(\check{\psi}_t^{N,0}) \geq& 2^{-N}V^{-1}\sum_{M=[\tau^-]2^N}^{-1}\int_{U_t^{N,M}}\alpha\wedge\omega^{n-1}_{\psi_{(M+1)2^{-N}}}\\ &+ V^{-1}t^{-1}\sum_{M=[\tau^-]2^N}^{-1}\int_{U_t^{N,M}}(\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}})\alpha\wedge\omega^{n-1}_{\psi_{(M+1)2^{-N}}}\\ &+ [\tau^-]V^{-1}\int_X\omega^{n-1}\wedge\alpha\,.\end{aligned}$$

Note that as $t \to \infty$, $\mathbb{1}_{U_t^{N,M}} \to 1$ outside a pluripolar set if $M < -1$. Hence

$$\begin{aligned}\varliminf_{t\to\infty} \frac{1}{t} E^\alpha(\check{\psi}_t^{N,0}) \geq & 2^{-N} V^{-1} \sum_{M=[\tau^-]2^N}^{-2} \int_X \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \\ & + \varliminf_{t\to\infty} (Vt)^{-1} \sum_{M=[\tau^-]2^N}^{-1} \int_{U_t^{N,M}} (\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \\ & + [\tau^-] V^{-1} \int_X \alpha \wedge \omega^{n-1} .\end{aligned}$$

Observe that $\check{\psi}_t^{N,0} \leq \ell_t$, so

$$\varliminf_{t\to\infty} \frac{1}{t} E^\alpha(\check{\psi}_t^{N,0}) \leq \lim_{t\to\infty} \frac{1}{t} E^\alpha(\ell_t) = \mathbf{E}^\alpha(\ell) .$$

Observe that

$$U_t^{N,M} \setminus S \subseteq \left\{ 2^N (\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) > -t \right\} \setminus S ,$$

where S is the pluripolar set $\{\psi_{M2^{-N}} = -\infty\}$.

Let

$$F^N(t) := 2^{-N} \sum_{M=[\tau^-]2^N}^{-1} \int_{\left\{2^N(\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) > -t\right\}} \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} .$$

Then

$$\begin{aligned} & \sum_{M=[\tau^-]2^N}^{-1} \int_{U_t^{N,M}} (\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \\ \geq & 2^{-N} \sum_{M=[\tau^-]2^N}^{-1} \int_{\left\{2^N(\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) > -t\right\}} 2^N (\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) \, \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \\ = & -2^{-N} \sum_{M=[\tau^-]2^N}^{-1} \int_0^t \mathrm{d}a \int_{\left\{-a \geq 2^N(\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) > -t\right\}} \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \\ = & - \int_0^t \left(F^N(t) - F^N(a) \right) \mathrm{d}a .\end{aligned}$$

Observe that F^N is bounded and increasing, so we conclude

$$\varliminf_{t\to\infty} t^{-1} \sum_{M=[\tau^-]2^N}^{-1} \int_{U_t^{N,M}} (\psi_{(M+1)2^{-N}} - \psi_{M2^{-N}}) \alpha \wedge \omega_{\psi_{(M+1)2^{-N}}}^{n-1} \geq 0 .$$

We conclude

$$\mathbf{E}^\alpha(\ell) \geq \frac{1}{V} \int_{[\tau^-]}^0 \left(\int_X \alpha \wedge \omega_{\psi_\tau}^{n-1} - \int_X \alpha \wedge \omega^{n-1} \right) \mathrm{d}\tau .$$

Now we deal with the case where $\ell \in \mathcal{E}^{1,\mathrm{an}}$. It suffices to write ℓ as a decreasing limit of a sequence of Phong–Sturm geodesic rays $\ell^j \in \mathcal{R}^\infty$ as in [BBJ21] and apply the monotone convergence theorem and [Li20, (121)]. □

Corollary 6.11. *Let $\ell \in \mathcal{E}^{1,\mathrm{an}}$, let $\psi = \hat{\ell}$, then*

$$\tilde{\mathbf{J}}(\ell) = \tilde{\mathbf{J}}(\psi_\bullet) .$$

Proof. This follows from Theorem 6.10, Theorem 6.7 and (3.2). ☐

Corollary 6.12. *Let $\ell^m \in \mathcal{E}^{1,\mathrm{an}}$ ($m \in \mathbb{Z}_{>0}$) be a decreasing sequence of maximal geodesic rays. Let $\ell \in \mathcal{R}^1$ be its limit. Then $\mathbf{E}^\alpha(\ell^m) \to \mathbf{E}^\alpha(\ell)$ as $m \to \infty$.*

This generalizes [Li20, (121)]. From our proof, it is easy to drop the condition that ℓ^m be decreasing when $\ell \in \mathcal{R}^\infty$.

Proof. Without loss of generality, we may assume that α is a Kähler form. We may and do assume that $\ell_0^m = 0$, $\sup_X \ell_1^m = 0$.

Observe that ℓ is maximal by the completeness of $\mathcal{E}^{1,\mathrm{an}}$ (see for example [DX22, Theorem 1.2], [Xia19, Example 3.3]). By Theorem 6.10, it suffices to prove that

$$\int_{-\infty}^0 \left(\int_X \alpha \wedge \omega_{\hat{\ell}_\tau^j}^{n-1} - \int_X \alpha \wedge \omega^{n-1}\right) \mathrm{d}\tau \to \int_{-\infty}^0 \left(\int_X \alpha \wedge \omega_{\hat{\ell}_\tau}^{n-1} - \int_X \alpha \wedge \omega^{n-1}\right) \mathrm{d}\tau\,.$$

By monotone convergence theorem, it suffices to prove that for almost all $\tau < 0$,

$$\int_X \alpha \wedge \omega_{\hat{\ell}_\tau^j}^{n-1} \to \int_X \alpha \wedge \omega_{\hat{\ell}_\tau}^{n-1}\,.$$

It suffices to show that $\hat{\ell}_\tau$ is the $d_{\mathcal{S}}$-limit ([DDL21b, Theorem 1.1]) of $\hat{\ell}_\tau^j$ for almost all $\tau < 0$. In turn, it suffices to show that $\int_X \omega_{\hat{\ell}_\tau^j}^n \to \int_X \omega_{\hat{\ell}_\tau}^n$ for almost all $\tau < 0$. This follows from the continuity of E^{an} along decreasing sequences and [DX22, Theorem 1.1]. ☐

6.5. Entropy, dreamy quasi-psh function. Results in this section are special cases of the results of Section 2.7, we will be sketchy here.

Let $\psi \in \mathrm{PSH}(X,\omega)$ be a dreamy (Definition 2.18) potential with analytic singularities. Let $\pi : Y \to X$ be a log resolution of ψ, which is a composition of blowing-ups with smooth centers. Assume that $\mathrm{div}_Y \psi$ is integral. Let $(\mathcal{X},\mathcal{L})$ be the test configuration induced by ψ, namely

$$\mathcal{X} = \mathcal{P}\mathrm{roj}_{\mathbb{C}} \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \bigoplus_{j\in\mathbb{Z}_{\geq 0}} t^{-j} H^0(Y, kAL - j\,\mathrm{div}_Y \psi)\,,$$

where $A \in \mathbb{Z}_{>0}$ is a sufficiently divisible integer such that the algebra is generated in degree $k = 1$. Take $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1)$. Recall that the test curve induced by $(\mathcal{X},\mathcal{L})$ is $\psi_\bullet^+$. By taking further blowing-ups, we may assume that π also resolves the singularities of all ψ_τ, for $\tau \in \mathbb{Q}$, $\tau < \mathrm{Psef}(\psi)$. The filtration induced by $(\mathcal{X},\mathcal{L})$ is

$$\mathscr{F}^\lambda H^0(X, L^k) = H^0(Y, k\pi^*L - \lambda\,\mathrm{div}_Y \psi)$$

for $\lambda \geq 0$.

We slightly reformulate Lemma 4.12 in our setting.

Lemma 6.13. *For any $\tau < \mathrm{Psef}(\psi)$, we have*

$$\int_X \omega_{\psi_\tau^+}^n = \mathrm{vol}(\pi^*L - \tau\,\mathrm{div}_Y \psi)\,. \tag{6.11}$$

Corollary 6.14. *For any $0 \leq \tau < \mathrm{Psef}(\psi)$, the decomposition*

$$\pi^*L - \tau\,\mathrm{div}_Y \psi = (\pi^*L - \mathrm{div}_Y \psi_\tau^+) + N_\tau$$

is the divisorial Zariski decomposition ([Bou04], [Nak04]). More precisely, $(\pi^*L - \mathrm{div}_Y \psi_\tau^+)$ is the movable part of $\pi^*L - \tau\,\mathrm{div}_Y \psi$.

Proof. Recall that by our definition of ψ^+_τ, N_τ is effective. So the result follows from Lemma 6.13 and [FKL16]. □

Remark 6.15. Using Lemma 4.12, one can easily generalize Corollary 6.14 to general test configurations.

We leave the details to the readers.

In particular,

$$D_L(\psi^+_\tau, \operatorname{red} \operatorname{div}_Y \psi) = D_L(\psi^+_\tau, \operatorname{red} \operatorname{div}_Y \psi^+_\tau)\,. \tag{6.12}$$

See Section 5.2 for the definition of D_L.

Theorem 6.16. *Let $(\mathcal{X}, \mathcal{L})$ be as above, then*

$$\widetilde{\mathrm{DF}}(\mathcal{X}, \mathcal{L}) - \mathbf{E}_R(\psi^+) = \frac{1}{V}\int_{-\infty}^{\infty} D_L(\psi^+_\tau, K_{Y/X})\,\mathrm{d}\tau + \frac{1}{V}\int_{-\infty}^{\infty} D_L(\psi^+_\tau, \operatorname{div}_Y \psi)\,\mathrm{d}\tau\,.$$

When $\operatorname{div}_Y \psi$ has a single irreducible component with coefficient 1,

$$\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) = \mathbf{Ent}(\ell) = \mathrm{Ent}(\psi^+_\bullet)\,.$$

Proof. For the first part, the computation is a generalization of those in [Fuj19]. As this method has been elaborated in [DL22, Section 3], we omit the proof.

As for the second part, assume that $\operatorname{div}_Y \psi = F$ for some prime divisor F over X. Then $\mathcal{X}_0$ is clearly irreducible and reduced and $\mathcal{X}$ is normal (see [DL22, Lemma 3.2] for example). Hence we conclude by Proposition 3.6. □

7. Variational approach on Berkovich spaces

Let X be a compact Kähler manifold of dimension n. Let L be an ample line bundle on X. Fix a smooth strictly positively-curved metric h on L. Let $\omega = c_1(L, h)$. In this section, we study the variation of the energy functional on the Berkovich space. As a consequence, we prove a comparison theorem of two entropy functionals Theorem 7.5, which is the key inequality used in comparing δ-invariants.

Proposition 7.1. *Let $\phi \in \mathcal{E}^{1,\mathrm{an}}$. We have*

$$\varliminf_{\epsilon\to 0+} \frac{1}{\epsilon}\left(E^{\mathrm{an}}(P[\phi + \epsilon A_X]) - E^{\mathrm{an}}(\phi)\right) \geq \frac{1}{V}\int_{X^{\mathrm{an}}} A_X \,\mathrm{MA}(\phi)\,. \tag{7.1}$$

Here $P[\bullet]$ is usc-regularized supremum of all elements in $\mathcal{E}^{1,\mathrm{an}}$ lying below $\bullet$. When $\phi \in \mathcal{H}^{\mathrm{an}}$, the limit exists and equality holds.

Remark 7.2. We expect that equality holds.

Proof. Let $\mathcal{Y}$ run over the set of snc models of $X \times_{\mathbb{C}} \mathbb{C}((T))$ (see [BJ21, Section 1.3] for the precise definition). Let $r_{\mathcal{Y}} : X^{\mathrm{an}} \to \Delta_{\mathcal{Y}}$ be the natural retraction. Let $f^{\mathcal{Y}} = A_X \circ r_{\mathcal{Y}}$. Then $(f^{\mathcal{Y}})_{\mathcal{Y}}$ is an increasing net of non-negative continuous functions converging to A_X pointwisely. See [JM12] for details.

Then for any $\epsilon > 0$, $\phi + \epsilon f^{\mathcal{Y}} \leq \phi + \epsilon A_X$, so

$$E^{\mathrm{an}}\left(P[\phi + \epsilon f^{\mathcal{Y}}]\right) \leq E^{\mathrm{an}}\left(P[\phi + \epsilon A_X]\right)\,.$$

Hence

$$\varliminf_{\epsilon\to 0+} \frac{1}{\epsilon}\left(E^{\mathrm{an}}(P[\phi + \epsilon A_X]) - E^{\mathrm{an}}(\phi)\right) \geq \frac{1}{V}\int_{X^{\mathrm{an}}} f^{\mathcal{Y}} \,\mathrm{MA}(\phi)$$

by [BJ21, Corollary 6.32]. By monotone convergence theorem ([Fol99, Proposition 7.12]), we conclude (7.1).

Finally, let us deal with the case where ϕ is associated to some test configuration $(\mathcal{X},\mathcal{L})$. We may assume that $\mathcal{X}_0$ is snc. We claim that for any $\epsilon>0$,

$$P[\phi+\epsilon f^{\mathcal{X}}]=P[\phi+\epsilon A_X]\,. \tag{7.2}$$

We only have to prove

$$P[\phi+\epsilon f^{\mathcal{X}}]\geq P[\phi+\epsilon A_X]\,. \tag{7.3}$$

By [BJ21, Theorem 5.29] (here we refer to the first version, this theorem does not appear in the final version),

$$P[\phi+\epsilon A_X]\leq P[\phi+\epsilon A_X]\circ r_{\mathcal{X}}\,.$$

But observe that

$$P[\phi+\epsilon A_X]\circ r_{\mathcal{X}}\leq \phi+\epsilon f^{\mathcal{X}}\,.$$

In fact, it suffices to check this on $\Delta_{\mathcal{X}}$, where the inequality follows from the definition of P. Hence (7.3) follows and *a fortiori* equality holds in (7.1). □

For $\psi\in\mathrm{PSH}(X,\omega)$, each $\epsilon>0$, we define

$$\psi^{\epsilon}:=\sup{}^*\left\{\varphi\in\mathrm{PSH}(X,\omega):\varphi\leq 0,\varphi^{\mathrm{an}}\leq\psi^{\mathrm{an}}+\epsilon A_X\text{ on }X^{\mathrm{div}}_{\mathbb{Q}}\right\}\,. \tag{7.4}$$

Note that ψ^ϵ is increasing and concave in $\epsilon>0$. By [DDL21a, Theorem 6.1], [Wit19], $\log\int_X\omega_{\psi^\epsilon}^n$ is concave in ϵ. When ψ is $\mathcal{I}$-model, the mass $\int_X\omega_{\psi^\epsilon}^n$ is right-continuous at $\epsilon=0$.

Lemma 7.3. *Let $\psi\in\mathrm{PSH}(X,\omega)$. Then for any birational model $\pi:Y\to X$,*

$$-\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\mathrm{div}_Y\,\psi^\epsilon\leq\sum_E A_X(E)E\leq\mathrm{red}\,\mathrm{div}_Y\,\psi+K_{Y/X}\,,$$

where E runs over all irreducible divisors in $\mathrm{div}_Y\,\psi$.

Proof. It follows from (7.4) that the only possible components of $-\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\mathrm{div}_Y\,\psi^\epsilon$ are components of $\mathrm{div}_Y\,\psi$. Also by (7.4), the multiplicity of each component E is bounded by $A_X(E)$, hence

$$-\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\mathrm{div}_Y\,\psi^\epsilon\leq\sum_E A_X(E)E\,.$$

The second inequality is trivial. □

Lemma 7.4. *Assume that $\psi\in\mathrm{PSH}(X,\omega)$ is $\mathcal{I}$-model and has positive mass. Then*

$$\frac{1}{V}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_X\omega_{\psi^\epsilon}^n\leq\mathrm{Ent}([\psi])\,.$$

PROOF. By Theorem 5.4,

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} \log\int_X \omega_{\psi^\epsilon}^n &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} \log \operatorname{vol}\left(L - \operatorname{div}_{\mathfrak{X}} \psi^\epsilon\right) \\
&\leq \varliminf_Y \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} \log\operatorname{vol}\left(L-\operatorname{div}_Y \psi^\epsilon\right) \\
&= \frac{n}{\int_X \omega_\psi^n} \varliminf_Y \left(\left\langle \pi^* L - \operatorname{div}_Y \psi)^{n-1}\right\rangle \cdot \left(-\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} \operatorname{div}_Y \psi^\epsilon\right)\right) \\
&\leq \frac{n}{\int_X \omega_\psi^n} \varliminf_Y \left(\left\langle \pi^* L - \operatorname{div}_Y \psi)^{n-1}\right\rangle \cdot \left(\operatorname{red}\operatorname{div}_Y \psi + K_{Y/X}\right)\right) \\
&= \frac{V}{\int_X \omega_\psi^n} \operatorname{Ent}([\psi])\,.
\end{aligned}
\tag{7.5}
$$

Here $\pi : Y \to X$ runs over all birational models of X, the second line follows from the log concavity of the masses of ψ^ϵ in ϵ, the third line follows from [BFJ09, Theorem A], the fourth line follows from Lemma 7.3. □

THEOREM 7.5. *Let $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$ be an $\mathcal{I}$-model test curve. Let ℓ be the geodesic ray defined by $\psi_\bullet$, then*

$$\operatorname{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \operatorname{Ent}(\psi_\bullet)\,.$$

PROOF. We may assume that $\operatorname{Ent}(\psi_\bullet) < \infty$. We first assume that ψ_{τ^+} has positive mass.

By Fatou's lemma, we always have

$$\frac{1}{V}\int_{-\infty}^{\infty} \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} \int_X \omega_{\psi_\tau^\epsilon}^n \,\mathrm{d}\tau \leq \varlimsup_{\epsilon\to 0+} \epsilon^{-1}\int_{-\infty}^{\infty}\left(\frac{1}{V}\int_X \omega_{\psi_\tau^\epsilon}^n - \frac{1}{V}\int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau\,.$$

On the other hand, since $\log\int_X \omega_{\psi_\tau^\epsilon}^n$ is concave in $\epsilon \geq 0$, fix $\epsilon_0 > 0$, for $\epsilon \in (0,\epsilon_0)$, we have

$$V^{-1}\int_{-\infty}^{\infty} \epsilon^{-1}\left(\int_X \omega_{\psi_\tau^\epsilon}^n - \int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau \leq V^{-1}\int_{-\infty}^{\tau^+}\epsilon^{-1}\left(\int_X \omega_{\psi_\tau^{\epsilon_0}}^n\right)\left(\log\int_X \omega_{\psi_\tau^\epsilon}^n - \log\int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau\,.$$

Thus by monotone convergence theorem,

$$\varlimsup_{\epsilon\to 0+} \epsilon^{-1}\int_{-\infty}^{\infty}\left(\frac{1}{V}\int_X \omega_{\psi_\tau^\epsilon}^n - \frac{1}{V}\int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau \leq \frac{1}{V}\int_{-\infty}^{\tau^+}\left(\int_X \omega_{\psi_\tau^{\epsilon_0}}^n\right)\left(\int_X \omega_{\psi_\tau}^n\right)^{-1}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_X \omega_{\psi_\tau^\epsilon}^n\,\mathrm{d}\tau\,.$$

Let $\epsilon_0 \to 0+$, by dominated convergence theorem, we get

$$\varlimsup_{\epsilon\to 0+} \epsilon^{-1}\int_{-\infty}^{\infty}\left(\frac{1}{V}\int_X \omega_{\psi_\tau^\epsilon}^n - \frac{1}{V}\int_X \omega_{\psi_\tau}^n\right)\mathrm{d}\tau \leq \frac{1}{V}\int_{-\infty}^{\infty}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_X \omega_{\psi_\tau^\epsilon}^n\,\mathrm{d}\tau\,.$$

Thus

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+} E^{\mathrm{an}}(\psi_\bullet^\epsilon) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_{-\infty}^{\infty}\left(\frac{1}{V}\int_X \omega_{\psi_\tau^\epsilon}^n - 1\right)\mathrm{d}\tau \\
&= \frac{1}{V}\int_{-\infty}^{\infty}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_X \omega_{\psi_\tau^\epsilon}^n\,\mathrm{d}\tau \\
&\leq \int_{-\infty}^{\infty}\operatorname{Ent}([\psi_\tau])\,\mathrm{d}\tau\,,
\end{aligned}
$$

where the first equality follows from Theorem 6.7. By Theorem 4.24, the non-Archimedean potential associated to $\psi^\epsilon_\bullet$ is just $P[\ell^{\mathrm{an}}+\epsilon A_X]$, hence we can apply Proposition 7.1 to conclude.

For a general $\psi_\bullet$, for each $\delta>0$, we define a new test curve ψ^τ that agrees with ψ_τ when $\tau\leq\tau^+-\delta$ and equals $-\infty$ otherwise. We apply the previous step and the fact that $\mathrm{Ent}^{\mathrm{an}}(\bullet)$ is lsc. $\square$

The same proof actually yields equality in the case of test configurations.

Corollary 7.6. *Let $(\mathcal{X},\mathcal{L})$ be a test configuration of (X,L). Let ℓ be the induced Phong–Sturm geodesic ray, let $\psi=\hat{\ell}$. Then*

$$\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}})=\mathbf{Ent}(\ell)=\mathrm{Ent}(\psi_\bullet)\,.$$

Proof. The first equality follows from Proposition 3.6. The inequality $\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}})\leq\mathrm{Ent}(\psi_\bullet)$ follows from Theorem 7.5.

Now we prove the converse. Replacing $\mathcal{L}$ by $(1+\delta)\mathcal{L}$ for a small $\delta\in\mathbb{Q}_{>0}$, we may assume that ψ_{τ^+} has positive mass. We may assume that $\mathcal{X}_0=\sum b_E E$ is snc and $\mathcal{X}$ dominates $X\times\mathbb{C}$ by a map $\Pi:\mathcal{X}\to X\times\mathbb{C}$. Let D be a divisor supported on the central fibre and $\mathcal{O}(D)=\mathcal{L}-p_1^*L$, where $p_1:X\times\mathbb{C}\to X$ is the natural map.

Observe that $\psi^\epsilon_\bullet$ is the test curve defined by the (not necessarily finitely generated) $\mathbb{Z}$-filtration $\mathscr{F}_\epsilon$ associated to the model $(\mathcal{X},\mathcal{L}+\epsilon K^{\log}_{\mathcal{X}/X\times\mathbb{C}})$. In fact, this follows from [BHJ17, Corollary 4.12], [BFJ16, Theorem 8.5] and (7.2) (see also discussions in [Li20] after Definition 2.7). By [BHJ17, Corollary 4.12], [BHJ17, Proof of Lemma 5.17], $\mathscr{F}_\epsilon$ is given by

$$\mathscr{F}^\lambda_\epsilon H^0(X,L^k)=\left\{s\in H^0(X,L^k):r(\mathrm{ord}_E)(s)+k\,\mathrm{ord}_E D+k\epsilon A_X(r(\mathrm{ord}_E))\geq b_E\lambda\,,\forall E\right\}\,, \tag{7.6}$$

where E runs over all components of $\mathcal{X}_0$, $r(\mathrm{ord}_E)$ is the restriction of ord_E to $\mathbb{C}(X)$. Recall that $r(\mathrm{ord}_E)$ is a divisorial valuation ([BHJ17, Section 4.2]). Let $\pi:Y\to X$ be a birational model on which the divisors corresponding to all $r(\mathrm{ord}_E)$ lie and which resolves the singularities of all ψ_τ, which is possible by Corollary 6.4. Then $\pi^*L-\mathrm{div}_Y\psi_\tau$ is nef by Lemma 2.15. Now by Lemma 4.12 and (7.6),

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\int_X\omega^n_{\psi^\epsilon_\tau}\geq{}&\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0+}\mathrm{vol}\left(\pi^*L-\mathrm{div}_Y\psi_\tau+\epsilon\sum_F A_X(F)F\right)\\ ={}&n\,(\pi^*L-\mathrm{div}_Y\psi_\tau)^{n-1}\cdot\sum_F A_X(F)F\\ ={}&n\,(\pi^*L-\mathrm{div}_Y\psi_\tau)^{n-1}\cdot\left(K_{Y/X}+\mathrm{red}\,\mathrm{div}_Y\psi_\tau\right)\,,\end{aligned}$$

where F runs over all irreducible components of $\mathrm{div}_Y\psi_\tau$, the second step follows from Theorem 5.2. In the first and the last step, we applied the negativity lemma ([KM08, Lemma 3.39]). We conclude by the same argument as above. $\square$

8. Stability thresholds

Let X be a compact Kähler manifold of dimension n. Let L be an ample line bundle on X. Fix a smooth strictly positively-curved Hermitian metric h on X and let $\omega=c_1(L,h)$.

We will compare various δ invariants and prove the main theorem of the paper Theorem 8.3.

We refer to Definition 2.23 and Definition 2.25 for the definitions of δ_{pp} and δ'.

Proposition 8.1. *We always have $\delta' \geq \delta$.*

PROOF. Let $\psi \in \mathrm{PSH}(X,\omega)$ be an unbounded potential with analytic singularities. Let ℓ be the geodesic ray induced by the generalized deformation to the normal cone with respect to ψ (see Section 2.7). By Theorem 7.5 and (2.7),

$$\operatorname{Ent}^{\mathrm{an}}(\mathrm{MA}(\ell^{\mathrm{an}})) \leq \operatorname{Ent}(\psi_\bullet) = \frac{1}{V}\left(K_{Y/X}\cdot(-\operatorname{div}_Y\psi)^{n-1}\right) + \frac{n}{V}\left(G_{n-1}(L,\operatorname{div}_Y\psi)\cdot\operatorname{red}\operatorname{div}_Y\psi\right). \tag{8.1}$$

By Corollary 6.11, we have

$$\tilde{\mathbf{J}}(\ell) = \frac{n}{V}\int_0^1 \left(\int_X \omega\wedge\omega_{\tau\psi}^{n-1} - \int_X \omega_{\tau\psi}^n\right)\mathrm{d}\tau\,.$$

By our definition (see also [Li20, Proposition 2.38]),

$$\tilde{J}^{\mathrm{an}}(\ell^{\mathrm{an}}) = \tilde{\mathbf{J}}(\ell)\,.$$

Hence by (3.5) and (3.6),

$$\delta \leq \frac{\operatorname{Ent}^{\mathrm{an}}(\mathrm{MA}(\ell^{\mathrm{an}}))}{E^*(\mathrm{MA}(\ell^{\mathrm{an}}))} \leq \frac{\left(K_{Y/X}\cdot(-\operatorname{div}_Y\psi)^{n-1}\right) + n\left(G_{n-1}(L,\operatorname{div}_Y\psi)\cdot\operatorname{red}\operatorname{div}_Y\psi\right)}{n\int_0^1\left(\int_X\omega\wedge\omega_{\tau\psi}^{n-1} - \int_X\omega_{\tau\psi}^n\right)\mathrm{d}\tau}\,.$$

□

Similarly, we have

Proposition 8.2. *We always have $\delta_{\mathrm{pp}} \geq \delta$.*

PROOF. Let $\psi \in \mathrm{PSH}(X,\omega)$ be an $\mathcal{I}$-model potential. Let ℓ be the geodesic ray induced by $\psi^+_\bullet$. Then by Theorem 7.5,

$$\operatorname{Ent}^{\mathrm{an}}(\mathrm{MA}(\ell^{\mathrm{an}})) \leq \operatorname{Ent}(\psi^+_\bullet)\,.$$

While $\tilde{J}^{\mathrm{an}}(\ell) = \tilde{\mathbf{J}}(\psi^+_\bullet)$ as in the previous proof. Hence

$$\delta \leq \frac{\operatorname{Ent}^{\mathrm{an}}(\mathrm{MA}(\ell^{\mathrm{an}}))}{E^*(\mathrm{MA}(\ell^{\mathrm{an}}))} \leq \frac{\operatorname{Ent}(\psi^+_\bullet)}{\tilde{\mathbf{J}}(\psi^+_\bullet)}\,.$$

We conclude (c.f. Remark 2.24). □

THEOREM 8.3. *Assume that X is Fano, $L = -K_X$. If $\delta < \frac{n+1}{n}$, then $\delta \geq \delta_{\mathrm{pp}}$.*

Proposition 8.2 and Theorem 8.3 together are just Theorem 1.1 in the introduction.

PROOF. Assume that $\delta < 1$, by [BLZ19, Proof of Theorem 4.1], δ can be computed by a sequence of extractable divisors, say E_k in the sense that

$$\delta = \lim_{k\to\infty}\frac{A_X(E_k)}{S_L(E_k)}\,.$$

Let $\pi_k : (Y_k,\Delta_k) \to X$ be a dlt (divisorially log terminal) extractions of E_k so that E_k is the only exceptional divisor and $A_X(E_k) \in (0,1)$. See [Kol13, Corollary 1.38] for the notion and existence of dlt extraction. Choose $\epsilon' \in \mathbb{Q}_{>0}$, so that $\pi^*L - \epsilon' E_k$ is semi-ample. Take $m \in \mathbb{Z}_{>0}$, so that $m(\pi^*L - \epsilon' E_k)$ is base-point free. Take a basis $s_1,\ldots,s_M$ of $H^0(Y_k, m(\pi^*L - \epsilon' E_k))$, regarded as a subspace of $H^0(X,L^m)$. Let

$$\psi_k := \frac{1}{m}\log\max_{i=1,\ldots,M}|s_i|^2_{h^m}\,.$$

The filtration induced by ψ_k on $R(X, L^m)$ in the sense of Example 4.16 is the same as that defined by ord_{E_k}. So the geodesic ray ℓ_k induced by ψ_k through the extended deformation to the normal cone construction is the same as the geodesic ray induced by the filtration of ord_{E_k}. By Theorem 6.16 and Corollary 6.11,

$$\frac{A_X(E_k)}{S_L(E_k)} \geq \delta_{\mathrm{pp}} .$$

Let $k \to \infty$, we conclude.

Now assume that $\delta = 1$. By [BLZ19, Theorem 6.7], there is a prime divisor E over X computing $\delta(X)$. By [BLZ19, Proof of Theorem 4.5], E is extractable. So we can proceed as in the case $\delta < 1$.

In general, if $\delta < \frac{n+1}{n}$, it suffices to apply [LXZ22] instead of [BLZ19] and run the same arguments. □

Remark 8.4. By slightly refining the argument, one finds that when $\delta < \frac{n+1}{n}$, there is always a qpsh function with analytic singularities that computes δ_{pp}.

Corollary 8.5. *Assume that X is Fano and $L = -K_X$. Then $\delta_{\mathrm{pp}} \geq 1$ (resp. $\delta_{\mathrm{pp}} > 1$) iff X is K-semistable (resp. uniformly K-stable).*

9. Further problems

9.1. Minimizers. Let X be a Fano manifold and $L = -K_X$. We assume that $\delta < 1$. Fix a smooth strictly positively-curved Hermitian metric h on L. Let $\omega = c_1(L, h)$. In this case, it is well-known that δ is equal to the greatest Ricci lower bound $R(X)$:

$$R(X) := \sup \left\{ t \in [0,1] : \exists \omega \in c_1(X) \text{ s.t. } \operatorname{Ric}\omega > t\omega \right\} .$$

This quantity was first explicitly introduced by Rubinstein in [Rub08; Rub09]. See [Szé11] for further results. This invariant also appears in an implicit form in [Tia92]. One could always solve Aubin's continuity path ([Aub84]) for $t < R(X)$:

$$\omega_{\varphi_t}^n = e^{F - t\varphi_t}\omega^n ,$$

where F is the Ricci potential of ω: $\operatorname{Ric}\omega - \omega = \mathrm{dd^c}F$, $\int_X (\exp(F) - 1)\omega^n = 0$.

The following are known about φ_t:

(1) Blowing-up at the limit time:

$$\lim_{t \to R(X)-} \sup_X \varphi_t = \infty .$$

See [Siu88], [Tia87].

(2) There is a proper closed subvariety $V \subseteq X$, such that on each compact subset of $X \setminus V$, for any increasing sequence $t_i \to R(X)$ and $t_i < R(X)$, up to passing to a subsequence, $\omega_{\varphi_{t_i}}^n$ converges to 0 uniformly ([Tos13]).

(3) Tian's partial C^0-estimate: let $\beta_{m,t}$ be the m-th Bergman kernel defined by ω_{φ_t}. Then there exists $m \in \mathbb{Z}_{>0}$ and $C > 0$, such that

$$\inf_X \rho_{m,t} \geq C^{-1}$$

for any $t \in [0, R(X))$. See [Szé16], [LS21],[Zha21c], [CW20], [Bam18].

It follows from the partial C^0-estimate that for any increasing sequence $t_i \to R(X)$, $t_i < R(X)$, up to subtracting a subsequence, there is $G \in \mathrm{PSH}(X, \omega)$, such that

$$\varphi_{t_i} - \sup_X \varphi_{t_i} \to G$$

in L^1. Moreover, G has the following type of singularities:

$$\frac{1}{m} \log \sum_{j=1}^{N} \lambda_j^2 |S_j|^2_{h^m} ,$$

where $m \in \mathbb{Z}_{>0}$, $\lambda_j \in (0, 1]$, $S_j \in H^0(X, K_X^{-m})$. Moreover, $\omega_G^n = 0$. See [MT19] for details. The function G is known as a *pluricomplex Green function* of X.

We make the following conjecture:

Conjecture 9.1. *When $\delta < 1$, the pluricomplex Green function G is a minimizer of δ_{pp}.*

9.2. Moser–Trudinger type inequalities. Let X be a compact Kähler manifold of dimension n. Let $[\omega]$ be a Kähler class on X with a representative Kähler form ω.

Definition 9.2 ([Zha21a, Definition 3.1, Proposition 3.5])**.** We define the *analytic δ-invariant* δ^A of $[\omega]$ as

$$\begin{aligned} \delta^A([\omega]) :=& \sup \left\{ \lambda > 0 : \int_X e^{-\lambda(\varphi - E(\varphi))} = \mathcal{O}_\lambda(1) \text{ for any } \varphi \in \mathcal{H}(X, \omega) \right\} \\ =& \sup \left\{ \lambda > 0 : \mathrm{Ent}(\varphi) \geq \lambda \tilde{J}(\varphi) - \mathcal{O}_\lambda(1) \text{ for any } \varphi \in \mathcal{H}(X, \omega) \right\} . \end{aligned} \tag{9.1}$$

Inequalities as in the first line of (9.1) are known as *Moser–Trudinger type inequalities*, they were first studied in [BB11]. See [DGL21] for recent progress in Moser–Trudinger type inequalities. The equality of two lines in (9.1) follows essentially from [BBEGZ19, Proposition 4.11], as explained in [Zha21a, Proposition 3.5].

In [Zha21b], Zhang proved that $\delta^A([\omega]) = \delta(L)$, improving previous partial results in [Zha21a, Proposition 3.11], [RTZ21, Proposition 5.3]. Hence in this case, $\delta^A \leq \delta_{\mathrm{pp}}$ by Proposition 8.2. Moreover, both δ^A and our δ_{pp} make sense for a transcendental Kähler class. It is interesting to understand the exact relation between δ^A and δ_{pp}.

9.3. Non-Archimedean entropy in terms of test curves. Let X be a compact Kähler manifold of dimension n. Let ω be a Kähler form on X.

When ω is in the first Chern class of an ample $\mathbb{Q}$-line bundle, $\mathcal{E}^{1,\mathrm{an}}(L)$ makes sense as in [BJ21]. In general, we define $\mathcal{E}^{1,\mathrm{an}}([\omega])$ as the subspace of $\mathcal{R}^1$ consisting of $\ell \in \mathcal{R}^1$, such that $\hat{\ell}_\tau$ is either $-\infty$ or $\mathcal{I}$-model for all τ.

Conjecture 9.3. *Let $\ell \in \mathcal{R}^1$. Assume that* $\mathbf{Ent}(\ell) < \infty$*, then $\ell \in \mathcal{E}^{1,\mathrm{an}}(L)$.*

When $[\omega]$ is integral, this follows from [Li20].

Conjecture 9.4. *Let $\ell \in \mathcal{E}^{1,\mathrm{an}}$, let $\psi = \hat{\ell}$, then*

$$\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) = \mathbf{Ent}(\ell) = \mathrm{Ent}(\psi_\bullet) .$$

Many special cases are known: when $[\omega]$ is integral, we know that $\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \mathbf{Ent}(\ell)$ (Proposition 3.6), $\mathrm{Ent}^{\mathrm{an}}(\ell^{\mathrm{an}}) \leq \mathrm{Ent}(\psi_\bullet)$ (Theorem 7.5). When ℓ is the Phong–Sturm geodesic ray of some test configuration, both equalities hold (Corollary 7.6).

When $[\omega]$ is not integral, all three terms are still defined, but very little information is known.

We summarize the information we know so far about various functionals in Table 1.

TABLE 1. Comparison of functionals

Maximal geodesic rays	NA potentials	Test curves	Known facts
$\mathbf{E}$	E^{an}	$\mathbf{E}$	All equal
$\mathbf{E}_R$	E^{an}_R	$\mathbf{E}_R$	All equal
$\mathcal{L}^{\mathrm{an}}_k$	?	$\mathcal{L}^{\mathrm{an}}_k$	First=Third
$\mathbf{Ent}$	$\mathrm{Ent}^{\mathrm{an}}$	Ent	Second $\leq$ First Second$\leq$ Third

This missing term in Table 1 is given by a construction similar to the relative volume in [BE21, (0.1)] up to an error term. One evidence of this is given by the analogy between [BGM21, Theorem 1.1] and [DX22, Theorem 1.2]. Note that every term on the third column is defined as an integral of some functional of psh singularities along the test curve.

Finally, let us explain the relation between Conjecture 9.4 and the celebrated Yau–Tian–Donaldson (YTD) conjecture. Up to now, it is well-understood that in order to achieve the variational approach of the YTD conjecture, it suffices to show that for a maximal geodesic ray, $\mathbf{Ent}(\ell)$ is continuous along the approximation of Berman–Boucksom–Jonsson ([BBJ21], [Li20], [CC21a], [CC21b], [CC18]). If Conjecture 9.4 holds, up to some technical subtleties, the problem can be reduced to showing that $\mathrm{Ent}([\bullet])$ of a psh singularity is continuous along a suitable quasi-equisingular approximation.

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PAPER 3

On sharp lower bounds for Calabi-type functionals and destabilizing properties of gradient flows

Mingchen Xia

Analysis & PDE (2021)

Paper 3. On sharp lower bounds for Calabi-type functionals and destabilizing properties of gradient flows

Mingchen Xia

Abstract

Let X be a compact Kähler manifold with a given ample line bundle L. Donaldson proved one inequality between the Calabi energy of a Kähler metric in $c_1(L)$ and the negative of normalized Donaldson–Futaki invariants of test configurations of (X, L). He also conjectured that the bound is sharp.

In this paper, we prove a metric analogue of Donaldson's conjecture, we show that if we enlarge the space of test configurations to the space of geodesic rays in $\mathcal{E}^2$ and replace the Donaldson–Futaki invariant by the radial Mabuchi K-energy $\mathbf{M}$, then a similar bound holds and the bound is indeed sharp. Moreover, we construct explicitly a minimizer of $\mathbf{M}$. On a Fano manifold, a similar sharp bound for the Ricci–Calabi energy is also derived.

1. Introduction

Motivation. Let (X, L) be a polarized manifold of dimension n, namely, X is a compact complex manifold of dimension n and L is an ample line bundle on X. We fix a Kähler metric on X in the class $c_1(L)$. Let $\mathcal{H}$ be the space of smooth strictly ω-psh functions on X. It is well-known that $\mathcal{H}$ is a Fréchet–Riemann manifold of constant non-positive curvature with respect to the standard Mabuchi–Donaldson–Semmes L^2 metric structure. See [Bło12] for details.

Donaldson [Don05] proved the following inequality:

$$\inf_{\varphi \in \mathcal{H}} Ca(\varphi) \geq \max \left(\sup_{(\mathcal{X},\mathcal{L})} \frac{-\mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|_{L^2}}, 0 \right), \tag{1.1}$$

where Ca is the Calabi functional, $(\mathcal{X}, \mathcal{L})$ takes value in the set of non-trivial normal test configurations of (X, L) with reduced central fibre, DF is the Donaldson–Futaki invariant of a test configuration. For the definition of the L^2 norm of a test configuration, see [His16]. Donaldson conjectured in the same paper that equality should hold.

To appreciate (1.1), we recall that $Ca(\varphi) = 0$ iff φ is a cscK metric, on the other hand the right-hand side of (1.1) is zero iff (X, L) is K-semistable. So (1.1) establishes a connection between the canonical metrics and the GIT stability.

In terms of non-Archimedean metrics introduced by Boucksom, Hisamoto, Jonsson [BHJ19; BHJ17], (1.1) can be reformulated as (see Section 5.1)

$$\inf_{\varphi \in \mathcal{H}} Ca(\varphi) \geq \max \left(0, \sup_{\psi \in \mathcal{H}^{\mathrm{an}} \setminus \{\phi_{\mathrm{triv}}\}} \frac{-M^{\mathrm{an}}(\psi)}{\|\psi\|_{L^2}} \right), \tag{1.2}$$

where $\mathcal{H}^{\mathrm{an}}$ is the space of non-Archimedean FS metrics on (X, L) (i.e. a FS metric on the Berkovich analytification of (X, L) with respect to the trivial norm on $\mathbb{C}$), ϕ_{triv} denotes the trivial metric, M is the Mabuchi K-energy, the super-index an denotes the non-Archimedean version of a functional.

In the present paper, we will prove a metric analogue of Donaldson's conjecture. That is, we prove that equality holds in (1.2) if we enlarge $\mathcal{H}$ to $\mathcal{E}^2$ and $\mathcal{H}^{\mathrm{an}}$ to $\mathcal{R}^2$ (the space of $\mathcal{E}^2$ geodesic rays) and if we replace the non-Archimedean functional M^{an} by the corresponding radial functional $\mathbf{M}$. We also prove an analogous result for the radial Ding functional $\mathbf{D}$ and the Ricci–Calabi energy R. See Section 3.2 for the definitions of various functionals.

Recall that the space $\mathcal{E}^2$ is the metric completion of $\mathcal{H}$ with respect to the L^2 metric. It is a deep theorem of Darvas (previously conjectured by Guedj) that the space $\mathcal{E}^2$ can be concretely realized as a subset of $\mathrm{PSH}(X, \omega)$ consisting of ω-psh functions with finite energy. See [Gue14] for a survey of these facts.

Statement of the main result. Our proof of the main result will rely on the gradient flows of M and D, which we recall now. The definitions of various functionals will be recalled in Section 3.2.

The gradient flow of M is known as the Calabi flow:

$$\begin{cases} \partial_t \varphi_t = S(\varphi_t) - \bar{S}\,, \\ \varphi_t|_{t=0} = \varphi_0\,, \end{cases} \tag{1.3}$$

where S denotes the scalar curvature of a metric, $\varphi_0 \in \mathcal{H}$ and

$$\bar{S} = \frac{1}{V}\int_X S(\varphi)\omega_\varphi^n$$

is independent of the choice of $\varphi \in \mathcal{H}$.

The main difficulty is that the equation is of 4-th order. The short time existence of the solution is proved in [CH08] using a general method of 4-th order quasi-linear parabolic equations. However, the long time existence is still widely open. Chen, Cheng [CC21] proved the existence of long-time solution under the assumption of the existence of *a priori* bounds of the scalar curvature.

In contrast, if we enlarge the space $\mathcal{H}$ to the finite energy space $\mathcal{E}^2$, it is shown in [BDL17] that the long time solution does exist and coincides with the smooth solution on the time interval where the latter exists. We refer to such a flow as the *weak Calabi flow*. The study of the weak Calabi flow dates back to [Str14] and [Str16].

In the Fano setting, namely, when X is a Fano manifold and $L = -K_X$, the gradient flow of D is known as the inverse Monge–Ampère flow:

$$\begin{cases} \partial_t \varphi_t = 1 - e^{\rho_t}\,, \\ \varphi_t|_{t=0} = \varphi_0\,, \end{cases} \tag{1.4}$$

where $\varphi_0 \in \mathcal{H}$, ρ denotes the Ricci potential, $\rho_t = \rho_{\varphi_t}$. See Section 3.2 for the precise definition.

The study of this flow is initiated recently by Collins, Hisamoto and Takahashi [CHT22]. A crucial advantage of this flow is that the flow equation is a second order parabolic equation, hence the short-time existence follows from the general theory. For the long time behaviour, the standard theory of Monge–Ampère equations reduces the long time existence to derive *a priori* C^0 bound of φ_t. This is done by a compactness argument in [CHT22].

A key feature of the (weak) Calabi flow is that M is convex along the flow. Hence, Ca is decreasing along the flow and it makes sense to consider the limit value of Ca along the flow. It is easy to prove that the limit value of Ca does not depend on the initial value (see Proposition 3.4).

These remarks apply equally to the inverse Monge–Ampère flow with D in place of M.

The main result of this paper is the following metric analogue of Donaldson's conjecture (1.2).

THEOREM 1.1. *Let X be a compact Kähler manifold. Let ω be a Kähler form on X. Let $\mathcal{E}^2 = \mathcal{E}^2(X, \omega)$, $\mathcal{H} = \mathcal{H}(X, \omega)$.*

1. We have

$$\inf_{\phi \in \mathcal{E}^2} Ca(\phi) = \max_{\ell \in \mathcal{R}^2 \setminus \{0\}} \frac{-\mathbf{M}(\ell)}{\|\ell\|}.$$

2. In the Fano case,

$$\inf_{\varphi \in \mathcal{H}} R(\varphi) = \max_{\ell \in \mathcal{R}^2 \setminus \{0\}} \frac{-\mathbf{D}(\ell)}{\|\ell\|}.$$

Moreover, the inf in 1. (resp 2.) can be obtained as follows: let $\phi_0 \in \mathcal{E}^2$ with $M(\phi_0) < \infty$ (resp. $\varphi_0 \in \mathcal{H}$), let ϕ_t (resp. φ_t) be the weak Calabi flow (resp. inverse Monge–Ampère flow) with initial value ϕ_0 (resp. φ_0), then

$$\inf_{\phi \in \mathcal{E}^2} Ca(\phi) = \lim_{t \to \infty} Ca(\phi_t), \quad \inf_{\varphi \in \mathcal{H}} R(\varphi) = \lim_{t \to \infty} R(\varphi_t).$$

Notice that in our theorem, we do not require that the polarization of X be integral anymore.

Here $\mathcal{R}^2$ is the space of geodesic rays in $\mathcal{E}^2$ emanating from a point $\varphi \in \mathcal{H}$. The norm $\|\ell\|$ of $\ell \in \mathcal{R}^2$ is defined as the d_2 distance between ℓ_0 and ℓ_1. The notation 0 is used for the constant geodesic. According to the recent work of Darvas–Lu [DL20], the max terms of both statements do not depend on the choice of φ. In the general context of Hadamard spaces, $\mathcal{R}^2$ is also known as the cone at infinity of $\mathcal{H}$ [Bal95]. For the definition of Ca on $\mathcal{E}^2$, see Section 3.4. We also notice that by considering the following geodesic ray $(\varphi + t)_t \in \mathcal{R}^2$, both max terms in Theorem 1.1 are non-negative.

An abstract version of this result, which applies to general gradient flows in Hadamard spaces is also included, see Theorem 4.1.

In Section 5.1, we explain the relation between Donaldson's conjecture and Theorem 1.1.

Our proof is constructive. We construct a geodesic ray (called the *Darvas–He geodesic ray*) following the method in [DH17], which was designed originally for the Kähler–Ricci flow. We calculate the radial M or D functional along this ray and show that this ray is indeed a maximizer.

In the unstable case, the situation is rather simple. We prove

Corollary 1.2. *1. Assume that (X, ω) is geodesically unstable (Definition 4.9), then there is a unique maximizer of $-\mathbf{M}$ on the unit sphere in $\mathcal{R}^2$.*

2. In the Fano case, assume that X is K-unstable, then there is a unique maximizer of $-\mathbf{D}$ on the unit sphere in $\mathcal{R}^2$.

Relations to other results. In the toric setting, various special cases are already known.

Part 2 of Theorem 1.1 is proved in the toric setting in [CHT22, Theorem 1.4], see also [Yao17].

As for Part 1 of Theorem 1.1, in the toric setting, it is proved in [Szé08] (1). Moreover, assuming the long time existence of smooth solutions to the Calabi flow, the original version of Donaldson's conjecture is also proved in the toric setting in the same paper.

A similar result for the H functional on Fano manifolds is proved in [DS20].

After finishing this paper, the author was informed that T. Hisamoto [His19] has independently proved the Fano case of the main theorem. Moreover, in the Fano case, Hisamoto also proved that the max in Theorem 1.1 can be obtained by a sequence of test configurations.

After the first version of this paper on arXiv, there have been a number of related papers about optimal distabilizing properties in various settings. See [BLZ19; Der20; Tak20; Sjö20].

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2. Preliminaries on Kähler geometry, pluripotential theory and Mabuchi geometry

Let X be a compact polarized manifold of dimension n. Let ω be a Kähler form on X. We will frequently consider the special case where X is Fano and $\omega \in c_1(X)$, which we refer to as the Fano case.

Set $V = \int_X \omega^n$. Let $\mathcal{H}$ be the space of smooth strictly ω-psh functions with the usual Mabuchi–Semmes–Donaldson L^2-metric: take $f, g \in C^\infty(X) = T_\varphi\mathcal{H}$ for some $\varphi \in \mathcal{H}$, define

$$\langle f, g\rangle_\varphi = \frac{1}{V}\int_X f\,g\,\omega_\varphi^n\,.$$

It is well-known that $\mathcal{H}$ is a Fréchet–Riemann manifold of constant non-positive curvature. See [Bło12] for details.

Given $\varphi \in \mathcal{H}$, write $\omega_\varphi = \omega + \mathrm{dd}^{\mathrm{c}}\varphi$, where we use the convention

$$\mathrm{dd}^{\mathrm{c}} := \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\,.$$

2.1. Finite energy class. It is proved by Darvas [Dar15] that the metric completion of $\mathcal{H}$ with respect to the L^2 metric can be realized by the set $\mathcal{E}^2$ of finite energy ω-psh functions. We briefly recall the related definitions.

We define

$$\mathcal{E}(X,\omega) = \left\{ \varphi \in \mathrm{PSH}(X,\omega) : \int_X \omega_\varphi^n = V \right\}.$$

Here and in the sequel, the product ω_φ^n is always interpreted in the non-pluripolar sense of [BEGZ10].

Define the following classes for $1 \leq p < \infty$

$$\mathcal{E}^p := \left\{ \varphi \in \mathcal{E}(X, \omega) : \int_X |\varphi|^p \, \omega_\varphi^n < \infty \right\} .$$

We also define $\mathcal{E}^\infty$ to be the set of bounded ω-psh functions on X.

According to Chen [Che00], for any $\varphi_0, \varphi_1 \in \mathcal{H}$, there is a unique weak geodesic connecting φ_t connecting them. According to a recent regularity result [CTW17], this weak geodesic has $C^{1,1}$-regularity. One can define a distance d_p on $\mathcal{H}$ for each $p \in [1, \infty)$ by

$$d_p(\varphi_0, \varphi_1) = \left(\frac{1}{V} \int_X |\dot{\varphi}_0|^p \, \omega_{\varphi_0}^n \right)^{1/p} . \tag{2.1}$$

It is shown in [Dar15, Theorem 3.5] that d_p is indeed a metric on $\mathcal{H}$. However, this metric is not complete. It is natural to look for the metric completion of d_p. In the same paper [Dar15], Darvas proved that the metric completion of $\mathcal{H}$ with respect to d_p can be realized as $\mathcal{E}^p$. For the definition of d_p on $\mathcal{E}^p$, we refer to [Dar15] (5). Moreover, $\mathcal{E}^p$ is indeed a geodesic metric space [Dar15, Theorem 4.17]. We will recall some related definitions below in Section 2.3 and Section 3.1.

Recall for $\varphi, \psi \in \mathcal{E}^2$, we have

$$C^{-1} I_p(\varphi, \psi) \leq d_p(\varphi, \psi) \leq C I_p(\varphi, \psi) , \tag{2.2}$$

where $C > 0$ is a universal constant and

$$I_p(\varphi, \psi) = \left(\int_X |\varphi - \psi|^p \, \omega_\psi^n \right)^{1/p} + \left(\int_X |\varphi - \psi|^p \, \omega_\varphi^n \right)^{1/p} .$$

For a proof, see [Dar15, Theorem 3].

The metric topology on $\mathcal{E}^1$ is also known as the strong topology. It is studied in detail in [BBEGZ19]. In this case, the topology admits a very explicit description.

Recall that the usual Monge–Ampère energy $E : \mathcal{H} \to \mathbb{R}$ (See (2.3)) extends to $E : \mathcal{E}^1 \to \mathbb{R}$. The functional is concave, increasing. See [BB10, Section 3] for example. The strong topology on $\mathcal{E}^1$ is then the coarsest refinement of the L^1-topology that makes E continuous. For the proof of this fact, see [Dar15, Proposition 5.9].

We refer to [Dar19] for a systematic introduction to this material.

2.2. Functionals. Let $E : \mathcal{H} \to \mathbb{R}$ be the Monge–Ampère energy functional:

$$E(\varphi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi \, \omega^j \wedge \omega_\varphi^{n-j} . \tag{2.3}$$

This functional extends to a concave, increasing functional on $\mathcal{E}^1$ in a natural way. See [BB10, Section 3].

Define the Calabi energy $Ca : \mathcal{H} \to \mathbb{R}$ as

$$Ca(\varphi) = \left(\frac{1}{V} \int_X (S(\varphi) - \bar{S})^2 \, \omega_\varphi^n \right)^{1/2} , \tag{2.4}$$

where $S(\varphi)$ is the scalar curvature of φ and

$$\bar{S} = \frac{1}{V} \int_X S_\varphi \, \omega_\varphi^n$$

is independent of the choice of $\varphi \in \mathcal{H}$. Note that in most literature, Calabi energy is defined as $(Ca)^2$.

We will show in Section 3.4 that Ca has a natural lsc extension to $\mathcal{E}^2 \to (-\infty, \infty]$.

Recall the definition of $E_R : \mathcal{H} \to \mathbb{R}$:

$$E_R(\varphi) = \frac{1}{nV} \sum_{j=0}^{n-1} \int_X \varphi \operatorname{Ric} \omega \wedge \omega_\varphi^j \wedge \omega^{n-1-j} . \tag{2.5}$$

As in [BDL17, Section 4.2], this functional extends naturally to a continuous functional $E_R : \mathcal{E}^1 \to \mathbb{R}$.

Recall the definition of the entropy $H : \mathcal{H} \to \mathbb{R}$:

$$H(\varphi) = \frac{1}{V} \int_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n . \tag{2.6}$$

This functional extends naturally to $H : \mathcal{E}^1 \to [0, \infty]$.

Let us also recall the definition of the Mabuchi functional $M : \mathcal{H} \to \mathbb{R}$:

$$M(\varphi) = H(\varphi) + \bar{S} E(\varphi) - n E_R(\varphi) . \tag{2.7}$$

We have extended every term, hence we get $M : \mathcal{E}^1 \to (-\infty, \infty]$. The extension is lsc and convex along finite energy geodesics. See [BDL17, Theorem 4.7], [BB17], [CLP14] for details.

In the Fano setting, we have two more functionals R and D.

Let $D : \mathcal{H} \to \mathbb{R}$ be the Ding functional. Recall that by definition, this means

$$\begin{cases} \delta D(\varphi) = \dfrac{1}{V}(e^{\rho_\varphi} - 1)\omega_\varphi^n , \\ D(\omega) = 0 . \end{cases} \tag{2.8}$$

where ρ_φ is the Ricci potential of φ:

$$\begin{cases} \operatorname{Ric} \omega_\varphi - \omega_\varphi = \mathrm{dd^c} \rho_\varphi , \\ \displaystyle\int_X (e^{\rho_\varphi} - 1)\, \omega_\varphi^n = 0 . \end{cases} \tag{2.9}$$

More explicitly, this means

$$D(\varphi) = -E(\varphi) - \log \int_X e^{-\varphi + \rho} \omega^n , \tag{2.10}$$

where ρ is the Ricci potential of ω.

This formula then extends directly to $\mathcal{E}^1 \to \mathbb{R}$. The extension is continuous and convex along finite energy geodesics. We refer to [Ber09], [Ber15], [Dar17a, Chapter 4] for details.

Define the Ricci–Calabi energy $R : \mathcal{H} \to \mathbb{R}$ as

$$R(\varphi) = \left(\frac{1}{V} \int_X (e^{\rho_\varphi} - 1)^2 \omega_\varphi^n \right)^{1/2} .$$

2.3. The space of weak geodesic rays. In this section, we recall some notions from the very recent work of Darvas–Lu [DL20].

We first recall the definition of (weak) geodesics.

Let $\Delta(r) \subset \mathbb{C}$ be the open disc of radius r centered at 0. Let $\Delta = \Delta(1)$. Let $\Delta^* = \Delta \setminus \{0\}$. Let $\pi : X \times \Delta^* \to X$ be the natural projection.

Let ℓ_t ($t \in [0, a]$, $a \in (0, \infty]$) be a ray or segment in $\mathcal{E}^\infty(X, \omega)$. Define $D = \bar{\Delta} \setminus \Delta(e^{-a})$. The complexification Φ of ℓ_t is by definition a function on $X \times D$, such that $\Phi_s = \ell_{-\log|s|}$, $s \in D$. When Φ is $\pi^*\omega$-psh and solves the homogeneous Monge–Ampère equation

$$(\pi^*\omega + \mathrm{dd^c}\Phi)^{n+1} = 0 \quad \text{on } X \times \operatorname{Int} D\,,$$

we call ℓ a weak geodesic. Similarly, ℓ_t is called a subgeodesic, if Φ is just $\pi^*\omega$-psh.

For two points $\varphi, \psi \in \mathcal{H}$, there is a unique (up to normalization) weak geodesic segment connecting φ and ψ, the geodesic segment has $C^{1,1}$ regularity [CTW17].

In general, for any two points $\varphi, \psi \in \mathcal{E}^p$ ($p \in [1, \infty]$), we may take a Demailly approximation, namely, decreasing sequences φ_j, ψ_j in $\mathcal{H}$, converging to φ and ψ respectively. Then the geodesic segment connecting φ_j and ψ_j converge to a unique segment in $\mathcal{E}^p$, which does not depend on the choice of φ_j and ψ_j. The limit is known as the finite energy geodesic segment in $\mathcal{E}^p$ connecting φ and ψ. The finite energy geodesic is indeed a d_p-metric geodesic. Moreover, $\mathcal{E}^p$ is a geodesic metric space. The definitions of a metric geodesic and a geodesic metric space are recalled in Section 3.1. It is known that the d_p-metric geodesic between points in $\mathcal{E}^p$ when $p > 1$ is unique, so in these cases [DL20], we use the term *geodesic* instead of *finite energy geodesic*. Note however that, the d_1-geodesics are not unique in general.

Now a ray ℓ_t ($t \geq 0$) in $\mathcal{E}^p$ is called a *finite energy geodesic ray* in $\mathcal{E}^p$ emanating from ℓ_0 if for any $s_2 > s_1 \geq 0$, the restriction of ℓ to $[s_1, s_2]$ is a finite energy geodesic segment in $\mathcal{E}^p$.

Let $\varphi \in \mathcal{H}$. Let $\mathcal{R}^p_\varphi$ be the set of finite energy geodesic rays in $\mathcal{E}^p$ emanating from φ. There is a special ray, namely the constant geodesic. This ray will be referred to as *the origin*. We sometimes use the notation 0 for the origin.

Define the chordal metric on $\mathcal{R}^p_\varphi$ as follows: let ℓ^1 and ℓ^2 be two elements in $\mathcal{R}^p_\varphi$, the distance is defined by

$$d_p^c(\ell^1, \ell^2) := \lim_{t\to\infty} \frac{d_p(\ell^1_t, \ell^2_t)}{t}\,. \tag{2.11}$$

Now assume that $1 \leq p < \infty$, then $(\mathcal{R}^p_\varphi, d^c_p)$ is a complete geodesic metric space [DL20, Theorem 4.7, Theorem 4.9] .

For any $\varphi, \psi \in \mathcal{E}^p$, there is a canonical isometry

$$P_{\varphi,\psi} : \mathcal{R}^p_\varphi \to \mathcal{R}^p_\psi$$

mapping each finite energy geodesic ray ℓ emanating from φ to the unique parallel finite energy geodesic ray ℓ' emanating from ψ [DL20, Theorem 1.3]. Here parallel means that $d_p(\ell_t, \ell'_t)$ is bounded. Moreover, if $\ell^0 \in \mathcal{R}^p_\varphi$ and $\ell^1 \in \mathcal{R}^p_\psi$ are parallel, the radial functional $\mathbf{M}$ (resp. $\mathbf{D}$) to be defined in Section 2.4 takes same value on ℓ^1 and ℓ^2 if $M(\varphi), M(\psi) < \infty$ (resp. no restriction for D). See [DL20, Lemma 4.10].

Hence, for our purpose, we simply identify $\mathcal{R}^p_\varphi$ for various φ and write $\mathcal{R}^p$ when $p < \infty$.

Now $\mathcal{R}^p_\varphi$ forms a decreasing chain indexed by p. We know that $\mathcal{R}^\infty_\varphi$ is dense in arbitrary $\mathcal{R}^p_\varphi$ [DL20, Theorem 1.5].

2.4. Radial functionals. As M and D are both convex along finite energy geodesics, it is natural to define the radial version of these functionals. Fix $\varphi \in \mathcal{E}^1$.

Define $\mathbf{M} : \mathcal{R}^1_\varphi \to (-\infty, \infty]$ by

$$\mathbf{M}(\ell) := \lim_{t\to\infty} \frac{M(\ell_t)}{t}\,. \tag{2.12}$$

Similarly, in the Fano case, define $\mathbf{D} : \mathcal{R}^1_\varphi \to (-\infty, \infty]$ by

$$\mathbf{D}(\ell) := \lim_{t\to\infty} \frac{D(\ell_t)}{t}\,. \tag{2.13}$$

We also define the p-energy of $\ell \in \mathcal{R}^p$ as follows:

$$\|\ell\|_p := E_p(\ell) := d_p^c(\ell, 0)\,. \tag{2.14}$$

Here 0 denotes the constant geodesic. When $p = 2$, we omit the subindex 2.

Let ℓ_t ($t \in [0, s]$, $s > 0$) be a weak geodesic segment between $\ell_0, \ell_s \in \mathcal{H}$. We define

$$\|\ell\| = E_2(\ell) := \left(\frac{1}{V}\int_X |\dot{\ell}_t|^2 \omega^n_{\ell_t}\right)^{1/2} \tag{2.15}$$

for any $t \in [0, s]$. It is well-known that this definition does not depend on the choice of t and is equal to $s^{-1}d_2(\ell_0, \ell_s)$. See [Dar15, Lemma 4.11].

3. Preliminaries on metric geometry and gradient flows

In this section, we review some basic facts about weak gradient flows on Hadamard spaces. We refer to [Bač14; AGS08; Bač18] for details.

3.1. Metric geometry.

We review several basic definitions from metric geometry.

Let (M, d) be a metric space. A *path* in M is an element in $C^0([0, 1], M)$. Let γ be a path in M, the *length* of γ is defined as

$$\ell(\gamma) := \sup \sum_{i=1}^n d(\gamma_{t_{i-1}}, \gamma_{t_i})\,,$$

where the sup is taken over the set of partitions $0 = t_0 < t_1 < \cdots < t_n = 1$ for various $n \in \mathbb{Z}_{>0}$.

The metric space (M, d) is a *length space* if for any $x, y \in M$, for any $\epsilon > 0$, there is a path γ in M with $\gamma_0 = x$, $\gamma_1 = y$ and

$$\ell(\gamma) \leq d(x, y) + \epsilon\,.$$

A path γ in M is called a *geodesic* if

$$d(\gamma_s, \gamma_t) = d(\gamma_0, \gamma_1)|s - t|$$

for any $s, t \in [0, 1]$.

The metric space (M, d) is a *geodesic space* if for any $x, y \in M$, there is a geodesic γ with $\gamma_0 = x$, $\gamma_1 = y$.

From now on, we always assume that (M, d) is a geodesic space. A *geodesic triangle* with vertices $x, y, z \in M$ consists of three geodesics g_{xy}, g_{yz}, g_{zx}, joining x to y, y to z, z to x respectively. The triangle will be denoted as $\Delta(x, y, z)$ although it is not uniquely determined by x, y, z. A *companion triangle* $\Delta(\bar{x}, \bar{y}, \bar{z})$ of $\Delta(x, y, z)$ is a triangle in $\mathbb{R}^2$, whose vertices are denoted as $\bar{x}, \bar{y}, \bar{z}$, such that

$$|\bar{x} - \bar{y}| = d(x, y)\,, \quad |\bar{y} - \bar{z}| = d(y, z)\,, \quad |\bar{z} - \bar{x}| = d(x, y)\,.$$

Let w be a point on the geodesic g_{xy}. The *companion point* of w is a point $\bar{w}$ on the line segment from $\bar{x}$ to $\bar{y}$, such that

$$d(w, y) = |\bar{w} - \bar{y}|\,.$$

Similarly one can define the companion point of a point on g_{yz} and g_{zx}.

The geodesic metric space (M, d) is a *CAT(0) space* if for any geodesic triangle $\Delta(x, y, z)$ in M with companion triangle $\Delta(\bar{x}, \bar{y}, \bar{z})$, for any a on g_{xy}, b on g_{xz} with companion points $\bar{a}$, $\bar{b}$, we have

$$d(a, b) \leq |\bar{a} - \bar{b}| .$$

Geometrically, the CAT(0) condition means that (M, d) has non-positive curvature. See [Bač14] for a detailed explanation.

The geodesic metric space (M, d) is a *Hadamard space* if it is complete and is a CAT(0) space.

Examples of Hadamard spaces include complete Riemannian manifolds of non-positive curvature, the space $\mathcal{E}^2$, Hilbert spaces, e.t.c..

We recall the concept of weak convergence (also called Δ-convergence) in a Hadamard space. See [KP08] for a thorough treatment. Let (M, d) be a Hadamard space. Let $x_n \in M$ be a bounded sequence. For $x \in M$, define

$$r(x) := \varlimsup_{n\to\infty} d(x, x_n) .$$

The *asymptotic radius* of (x_n) is defined as $\inf_{x\in M} r(x)$. The *asymptotic center* of (x_n) is defined as the set

$$\left\{ x \in M : r(x) = \inf_{y\in M} r(y) \right\} .$$

According to [DKS06, Proposition 7], the set consists of a single element. By abuse of language, we also call this element the asymptotic center of (x_n). If $x \in M$ is the asymptotic center of every subsequence of (x_n), we say that (x_n) *converges weakly* (or Δ-*converges*) to x.

Proposition 3.1. *Let (M, d) be a Hadamard space. Assume that $x_n \in M$ is a sequence that converges weakly to $x \in M$. Let $y \in M$, then*

$$d(y, x) \leq \varliminf_{n\to\infty} d(y, x_n) . \tag{3.1}$$

This proposition is a special case of [Bač13, Lemma 3.1], which says that a convex lsc function on a Hadamard space is weakly lsc.

3.2. Weak gradient flows on Hadamard spaces. In this subsection, following [Bač14, Chapter 5], we explore the general theory of weak gradient flows on Hadamard spaces.

Let (M, d) be a Hadamard space. Let $G : X \to (-\infty, \infty]$ be a convex lsc function. We will use the notation

$$\operatorname{Dom} G = G^{-1}(\mathbb{R}) .$$

The *slope* of G is a function $|\partial G| : M \to [0, \infty]$:

$$|\partial G|(y) = \begin{cases} \varlimsup_{z\to y} \dfrac{\max\{G(y) - G(z), 0\}}{d(y, z)} , & y \in \operatorname{Dom}(G) , \\ \infty , & y \in G^{-1}(\infty) . \end{cases}$$

It is a general fact that $|\partial G|$ is always lsc. Moreover

$$|\partial G|(y) = \sup_{z\in M-\{y\}} \frac{\max\{G(y) - G(z), 0\}}{d(y, z)} , \quad y \in \operatorname{Dom}(G) . \tag{3.2}$$

See [Bač14, Lemma 5.1.2] for a proof.

Inspired by the gradient flow on Hilbert spaces, we look for a gradient flow on a general Hadamard space as follows: given $c_0 \in \mathrm{Dom}(G)$, we want to define a curve c_t so that

$$|\dot{c}_t| := \lim_{s\to t+} \frac{d(c_t, c_s)}{s-t}$$

is as large as possible. That is, we hope that

$$|\dot{c}_t| = |\partial G(c_t)|\,, \quad t > 0\,.$$

This is indeed possible, we recall the construction.

We define $c^{m,j} : [0,\infty) \to M$ $(m, j \in \mathbb{Z}_{\geq 0})$ by iteration:

1. $c_t^{m,0} = c_0$.
2. $c_t^{m,j+1}$ is the minimizer of

$$v \mapsto \frac{1}{2} d(v, c_t^{m,j})^2 + \frac{t}{m} G(v)\,.$$

Set $c_t^m = c_t^{m,m}$. Set

$$c_t = \lim_{m\to\infty} c_t^m\,.$$

It is shown by Mayer [May98] that the above procedure is well-defined, $c_t \in \mathrm{Dom}(G)$. The curve c_t is called the *weak gradient flow* of G starting from c_0. See also [Bač14, Theorem 5.1.6].

The curve c_t has the following property:

$$-\frac{\mathrm{d}}{\mathrm{d}t} G(c_t) = |\partial G(c_t)|^2 = |\dot{c}_t|^2 < \infty\,, \quad t > 0\,. \tag{3.3}$$

Here the derivative on the left-hand side is understood as the right derivative. In particular, $G(c_t)$ is right differentiable at $t > 0$. See [Bač14, Theorem 5.1.13], [AGS08, Theorem 2.4.15]. By [Bač14, Proposition 5.1.14], $|\partial G(c_t)|$ is decreasing in $t \geq 0$, so $G(c_t)$ is convex in $t \geq 0$.

Moreover, the following evolution variation inequality holds [Bač14, Theorem 5.1.11]:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} d(c_t, v)^2 \leq G(v) - G(c_t)\,, \tag{3.4}$$

where $v \in \mathrm{Dom}(G)$. Here the left-hand side is understood as the right upper derivative (Dini derivative), namely

$$\frac{\mathrm{d}}{\mathrm{d}t} d(c_t, v)^2 := \varlimsup_{s\to t+} \frac{d(c_s, v)^2 - d(c_t, v)^2}{s-t}\,.$$

Remark 3.2. In [Bač14], this theorem is stated for usual derivative and for almost all t. Moreover, it is shown that $d^2(c_t, v)$ is absolutely continuous. Our formulation follows easily from taking Dini derivative of the integral version of the theorem in [Bač14].

Now fix a weak gradient flow c_t with $c_0 \in \mathrm{Dom}(G)$.

Proposition 3.3. *Let $0 < t < s$, then*

$$|\partial G|(c_t)d(c_t, c_s) \geq G(c_t) - G(c_s) \geq |\partial G|(c_s)d(c_t, c_s)\,. \tag{3.5}$$

Moreover, for $t = 0$, the left-hand part of (3.5) is still true, namely

$$|\partial G|(c_0)d(c_0, c_s) \geq G(c_0) - G(c_s)\,.$$

Proof. The left-hand part of (3.5) (including the case $t = 0$) follows directly from (3.2).

We prove the right-hand part. To prove (3.5), without loss of generality, assume that $t = 0$, that (3.3) holds also at $t = 0$ and that c_t is Lipschitz on $[0, \infty)$ [Bač14, Proposition 5.1.10].

Define two functions

$$F(r) = (G(c_0) - G(c_r))^2\,,\; L(r) = d(c_0, c_r)^2\,, \quad r \geq 0\,,$$

We may assume that $L(s) > 0$, since otherwise, by [Bač14, Proposition 5.1.14], $|\partial G|(c_t)$ is constant for $t \in [0, s]$, hence by (3.3) and the fact that $c_0 = c_s$, this constant is indeed 0. So the flow c_t is just the constant at c_0, the result is obvious.

Define a function $H : [0, s] \to \mathbb{R}$ as follows:

$$H(a) = F(a) - \frac{F(s)}{L(s)}L(a)\,.$$

Obviously, $H(0) = H(s) = 0$, H is a usc function. Let $x \in [0, s)$ be a maximizer of H. Then the right upper derivative of H at x must be non-positive, namely

$$0 \geq \varlimsup_{y \to x+} \frac{H(y) - H(x)}{y - x} = F'(x) - \frac{F(s)}{L(s)} \varliminf_{y \to x+} \frac{L(y) - L(x)}{y - x} \geq F'(x) - \frac{F(s)}{L(s)}L'(x)\,,$$

where in the second step, we made use of the fact that F is right differentiable since G is also right differentiable, as recalled after (3.3). Here each derivative denotes the right upper derivative.

Since G is right differentiable, by (3.3), we have

$$F'(x) = 2(G(c_0) - G(c_x))|\partial G(c_x)|^2 \geq 0\,.$$

By (3.4), we also have

$$0 \leq L'(s) \leq 2\,(G(c_0) - G(c_s))\,.$$

When $L'(x) = 0$, we conclude $F'(x) = 0$ as well. Hence either $F(x) = 0$ or $|\partial G(c_x)| = 0$. In both cases, (3.5) is obvious. When $L'(x) > 0$,

$$\frac{F(s)}{L(s)}L'(x) \geq F'(x) \geq L'(x)|\partial G(c_x)|^2\,.$$

This concludes the proof of (3.5) since $|\partial G(c_x)|$ is decreasing in x [Bač14, Proposition 5.1.14]. □

Proposition 3.4. *Let $\phi_0, \psi_0 \in \mathrm{Dom}(G)$. Let ϕ_t (resp. ψ_t) be the weak gradient flow of G with initial value ϕ_0 (resp. ψ_0). Then*

$$\lim_{t \to \infty} |\partial G|(\phi_t) = \lim_{t \to \infty} |\partial G|(\psi_t)\,.$$

This is proved in [He15, Corollary 2.2].

Proof. We may assume that the curves ϕ_t and ψ_t do not intersect. Moreover, we may assume that (3.3) holds up to $t=0$. Assume that the conclusion is not true, we may assume that there is a constant $\delta>0$, so that for all $t\geq 0$

$$|\partial G|^2(\phi_t)\leq |\partial G|^2(\psi_t)-\delta\,.$$

Now by (3.4),

$$2(G(\psi_t)-G(\phi_{t+1}))\geq d(\psi_t,\phi_{t+1})^2-d(\psi_t,\phi_t)^2\geq -d(\psi_0,\phi_0)^2\,,$$

where we have used the fact that $d(\phi_t,\psi_t)\leq d(\phi_0,\psi_0)$ in the second inequality [Bač14, Theorem 5.1.6].

Now by (3.3),

$$G(\phi_t)-G(\phi_{t+1})\leq |\partial G|^2(\phi_0)\,.$$

By (3.3),

$$(G(\psi_t)-G(\phi_t))-(G(\psi_0)-G(\phi_0))=\int_0^t \left(|\partial G(\phi_s)|^2-|\partial G(\psi_s)|^2\right)\,\mathrm{d}s\leq -\delta t\,.$$

In all, we get

$$-d(\psi_0,\phi_0)\leq -2\delta t+C$$

for some constant C. This is a contradiction. □

3.3. Moment-weight inequality. Let (M,d) be a Hadamard space. Let $G:M\to(-\infty,\infty]$ be a convex lsc function. Let $\mathcal{R}$ be the space of geodesic rays in M emanating from a fixed point $x_0\in M$. Define $\mathbf{G}:\mathcal{R}\to(-\infty,\infty]$ by

$$\mathbf{G}(\ell):=\lim_{t\to\infty}\frac{G(\ell_t)}{t}\,. \tag{3.6}$$

As before, we may identify $\mathcal{R}$ for different x_0, the $\mathbf{G}$ functionals for different x_0 correspond to each other.

For $\ell\in\mathcal{R}$, let

$$\|\ell\|:=d(\ell_0,\ell_1)\,. \tag{3.7}$$

This agrees with the definition in (2.14) for the Hadamard space $\mathcal{E}^2$.

We denote the trivial ray in $\mathcal{R}$ by 0.

Proposition 3.5.

$$\inf_{x\in M}|\partial G|(x)\geq \sup_{\ell\in\mathcal{R}\setminus\{0\}}\frac{-\mathbf{G}(\ell)}{\|\ell\|}\,. \tag{3.8}$$

Proof. Take $\ell\in\mathcal{R}\setminus\{0\}$. Fix $x_0\in M$. Then

$$-\mathbf{G}(\ell)\leq -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0+}G(\ell_t)\leq |\partial G|(x_0)\|\ell\|\,,$$

where the first inequality follows from the convexity of G, the second inequality follows from (3.2). Since x_0 is arbitrary, the inequality follows. □

This is also known as the moment-weight inequality in the general GIT setting.

3.4. Weak Calabi flow. In this subsection, we explore the weak Calabi flow following [BDL20].

Fix a compact Kähler manifold X and a Kähler form ω as before.

The following theorem is the basis of this part.

THEOREM 3.6. *The space $\mathcal{E}^2(X,\omega)$ is a Hadamard space.*

This result is proved by Darvas in [Dar17a]. See also [Gue14, Theorem 3.11, Theorem 3.6].

The weak Calabi flow is an analogue of the Calabi flow recalled in the introduction. By definition, the weak Calabi flow is the weak gradient flow of the functional M on $\mathcal{E}^2$. See [BDL20, Section 6] for a thorough treatment.

We recall that for an initial value $\phi_0 \in \mathcal{H}$, the weak Calabi flow coincides with the Calabi flow on the maximal existence time interval of the latter [BDL20, Proposition 6.1].

Now we define a functional $\overline{Ca} : \mathcal{E}^2 \to [0,\infty]$ as $|\partial M|$. As recalled above, $\overline{Ca}$ is lsc.

Proposition 3.7. *For $\phi \in \mathcal{H}$,*

$$\overline{Ca}(\phi) = Ca(\phi)\,.$$

PROOF. Recall that the evolution variation inequality also holds for the Calabi flow with smooth initial value (See [He15] the equation below (2.4)). So (3.5) also holds on the time interval where the Calabi flow is defined. Moreover, (3.5) extends to $t = 0$.

Now fix ϕ_t be a solution to the weak Calabi flow with $\phi_0 \in \mathcal{H}$, since the flow coincides with the Calabi flow on a short time interval, we conclude that $M(\phi_t)$ is smooth in t for small t, so by (3.3) and the fact that $\overline{Ca}$ is lsc,

$$Ca(\phi_0)^2 = -\dot{M}(\phi_0) \geq \overline{Ca}(\phi_0)^2\,.$$

For the other inequality, by Proposition 3.3,

$$\overline{Ca}(\phi_0) \geq \frac{M(\phi_0) - M(\phi_t)}{d_2(\phi_t,\phi_0)} \geq Ca(\phi_t)\,.$$

for $t > 0$ small. Let $t \to 0+$, we conclude. □

From now on, we will no longer use the notation $\overline{Ca}$, we denote it simply as Ca.

Let ϕ_t be a solution to the weak Calabi flow with $M(\phi_0) < \infty$. As we have recalled above, $Ca(\phi_t)$ is decreasing in t, so one can define

$$B := \lim_{t\to\infty} Ca(\phi_t)\,. \tag{3.9}$$

According to Proposition 3.4, the value of B is independent of the choice of ϕ_0.

3.5. Inverse Monge–Ampère flow. Now assume that we are in the Fano case, we recall the basic theory of the inverse Monge–Ampère flow following [CHT22].

The inverse Monge–Ampère flow is the gradient flow of D on $\mathcal{H}$, namely,

$$\begin{cases} \partial_t \varphi_t = 1 - e^{\rho_t}\,, \\ \varphi_t|_{t=0} = \varphi_0\,, \end{cases} \tag{3.10}$$

where ρ_t is short for ρ_{φ_t}. In the same spirit, we write $\omega_t = \omega_{\varphi_t}$. We assume that $\varphi_0 \in \mathcal{H}$.

Theorem 3.8 ([CHT22]). *The solution to* (3.10) *exists for* $t \in [0, \infty)$ *and is smooth.*

One could of course define the weak gradient flow of D as we did for M. But due to this theorem and a similar argument as [BDL20, Proposition 6.1], the weak flow and the inverse Monge–Ampère flow are exactly the same when the initial value lies in $\mathcal{H}$. As we will see, this is enough for our purpose.

Fix a smooth solution φ_t to (3.10). Note the following

$$-\frac{\mathrm{d}}{\mathrm{d}t} D(\varphi_t) = R(\varphi_t)^2 \,.$$

Proposition 3.9. *(1) E is constant along* (3.10).

(2) R is decreasing along (3.10).

(3) M is decreasing along (3.10).

See [CHT22] for a proof.

According to Proposition 3.9, D is convex and decreasing along the flow. Define

$$B := \lim_{t\to\infty} R(\varphi_t) \in [0, \infty) \,. \tag{3.11}$$

Again, B is independent of the choice of φ_0.

Remark 3.10. When $B > 0$ (B is defined in (3.11)), X does not admit Kähler–Einstein metrics. Otherwise, as is well-known, the Kähler–Einstein metric is a global minimizer of D, and as D is convex and decreasing along φ_t, we infer that $B = 0$, this is a contradiction.

The same remark applies to the weak Calabi flow setting. Hence if $B > 0$ (B is defined in (3.9)), there is no cscK metric.

4. Proof of the main theorem

4.1. Analogue in finite dimensions. Let us explain the idea of the proof in the finite dimensional setting.

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a smooth convex function. We may consider the gradient flow of G, namely

$$\dot{x}_t = -\nabla G(x_t) \,.$$

It is well-known that for any initial value $x_0 \in \mathbb{R}^n$, there is always a smooth global solution.

Following the general theory of Hadamard spaces, we define the boundary $\mathbb{R}^n(\infty)$ as the set of equivalence classes of unit speed rays (in the usual sense) in $\mathbb{R}^n$, two rays are considered as equivalent if they are parallel in the sense that they are related by a translation. There is an obvious identification $\mathbb{R}^n(\infty)$ with the unit sphere S^{n-1}.

We can define a radial version of G, namely $\mathbf{G} : \mathbb{R}^n(\infty) \to (-\infty, \infty]$ as follows: let $[\ell] \in \mathbb{R}^n(\infty)$, take $x \in \mathbb{R}^n$, take a representative of ℓ of $[\ell]$ that emanates from x, define

$$\mathbf{G}([\ell]) = \lim_{t\to\infty} \frac{G(\ell_t)}{t} \,. \tag{4.1}$$

It is easy to show that $\mathbf{G}$ is independent of the choice of x. See the proof of [DL20, Lemma 4.10].

Fix a solution to the flow, say x_t. Set $G(t) = G(x_t)$.

Then we claim that

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}=\max\left\{0,\sup_{[\ell]\in\mathbb{R}^n(\infty)}-\mathbf{G}([\ell])\right\}. \tag{4.2}$$

Let ℓ be a unit speed ray emanating from $x\in\mathbb{R}^n$. Then by Proposition 3.5, we have

$$-\mathbf{G}([\ell])\leq=\left(-\dot{G}(0)\right)^{1/2}.$$

Since x is arbitrary, we conclude

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}\geq\max\left\{0,\sup_{[\ell]\in\mathbb{R}^n(\infty)}-\mathbf{G}([\ell])\right\}.$$

For the inverse direction, we may assume that

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}>0\,. \tag{4.3}$$

In this case, $|x_0-x_t|\to\infty$ as $t\to\infty$. Otherwise, let y be a limit point of x_t, it is easy to see that $G(y)$ obtains the minimial value of G. It is a general fact of the gradient flow that the left-hand side of (4.3) is independent of the choice of x_0 (Proposition 3.4), so we find a contradiction by considering the flow starting at y.

By Proposition 3.3, we have the following control for $0\leq t<s$,

$$\left(-\dot{G}(s)\right)^{1/2}\leq\frac{G(t)-G(s)}{|x_t-x_s|}\leq\left(-\dot{G}(t)\right)^{1/2}.$$

Now we claim that the sup on right-hand side of (4.2) is indeed obtained by a special direction ℓ^∞. The construction is as follows: connect x_0 and x_s by a unit speed segment $\ell^s:[0,|x_0-x_s|]\to\mathbb{R}^n$. Fix $T>0$, it easy to see that the images of the maps $\ell^s|_{[0,T]}$ all lie in a fixed compact set when $s\geq T$, so we may take $s_i\to\infty$ so that the corresponding ℓ^{s_i} tends to another segment uniformly. Combining this with a Cantor diagonal argument, we arrive at a subsequence $s_i\to\infty$, so that the corresponding ℓ^{s_i} converge to a ray ℓ^∞ in uniformly on each compact time interval. We then calculate for $0<A<s$ that

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}\leq\left(-\dot{G}(s)\right)^{1/2}\leq\frac{G(0)-G(s)}{|x_0-x_s|}\leq\frac{G(0)-G(\ell^s_A)}{A}.$$

Let $s\to\infty$ along the subsequence s_i used to define ℓ^∞, we find

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}\leq\frac{G(0)-G(\ell^\infty_A)}{A}.$$

Let $A\to\infty$, we conclude

$$\left(-\lim_{t\to\infty}\dot{G}(t)\right)^{1/2}\leq-\mathbf{G}([\ell^\infty])\,.$$

Hence equality in (4.2) indeed holds.

It is not hard to generalize the proof to a general locally compact Hadamard space and to lsc and convex G. But in the situation we are interested in, the underlying space is $\mathcal{E}^2$, which is not locally compact. So one need some additional compactness theorem. In $\mathcal{E}^2$, the compactness is usually lacking, so we instead apply the compactness theorem for the level set of H in $\mathcal{E}^1$ proved in [BBEGZ19]. The details will be treated in the subsequent subsections.

4.2. An abstract version. Let (M,d) be a Hadamard space. Let σ be a topology on M. We say σ is *compatible* with (M,d) if the followings hold:

(1) σ is a Hausdorff topology.
(2) σ is weaker that the d-topology. Moreover, let x_j be a bounded sequence in (M,d), such that $x_j \to x \in M$ with respect to the σ-topology. Then $x_j \to x$ with respect to the weak topology.
(3) For any bounded σ-converging sequences $x_j \to x$, $y_j \to y$ in M,
$$d(x,y) \leq \varliminf_{j\to\infty} d(x_j,y_j)\,.$$
(4) Let $(x_t^j)_{t\in[0,1]}$ be geodesics in M for any $j \geq 1$. Assume that there are $x_0, x_1 \in M$, such that $x_0^j \to x_0$, $x_1^j \to x_1$ in σ-topology. Let $(x_t)_{t\in[0,1]}$ be the geodesic from x_0 to x_1. Then for any $t \in [0,1]$, $x_t^j \to x_t$ in σ-topology.

Theorem 4.1. *Let (M,d) be a Hadamard space. Let σ be a topology on M compatible with (M,d). Let $F, G : M \to (-\infty,\infty]$ be two convex lsc functions such that $F \leq G$ and such that G is decreasing along the gradient flow of F. Fix an arbitrary point $x_0 \in \operatorname{Dom} G$. Assume that for any constant $C > 0$, the following set*
$$\mathcal{K}_C := \{\, x \in M : d(x,x_0) \leq C, G(x) \leq C \,\} \subseteq M\,.$$
is σ-sequentially compact. Then
$$\inf_{x\in M} |\partial F|(x) = \max\left(0, \max_{\ell\in\mathcal{R}\setminus\{0\}} \frac{-\mathbf{F}(\ell)}{\|\ell\|}\right)\,. \tag{4.4}$$

Here $\mathcal{R}$ denotes the space of all geodesic rays emanating from x_0 and 0 denotes the trivial ray in $\mathcal{R}$. The functional $\mathbf{F} : \mathcal{R} \to (-\infty,\infty]$ is defined by
$$\mathbf{F}(\ell) := \lim_{t\to\infty} \frac{F(\ell_t)}{t}\,. \tag{4.5}$$
The norm of a geodesic ray ℓ is defined as
$$\|\ell\| := d(\ell_0,\ell_1)\,.$$
As before, we identify $\mathcal{R}$ with respect to different x_0. The functional $\mathbf{F}$ does not depend on the choice of x_0.

Proof. Let $(x_t)_{t\geq 0}$ be the gradient flow of F with starting point x_0.

Case 1. Assume that $d(x_0,x_t)$ is bounded.

In this case, by our assumption, the set $\{x_t : t \in [0,\infty)\}$ is weakly relatively compact. In particular, we can take $t_j \to \infty$ ($j \geq 1$), such that x_{t_j} converges weakly to $x_\infty \in M$ as $j \to \infty$. By [Bač13, Lemma 3.1], F is weakly lsc, so
$$F(x_\infty) \leq \varliminf_{j\to\infty} F(x_\infty)\,.$$
By [Bač14, Proposition 5.1.12], we conclude that x_∞ is indeed a minimizer of F. Also observe that by the same argument, $G(x_\infty) < \infty$. In particular, we can replace x_0 by x_∞. In this case, both sides of (4.4) are 0.

Case 2. Assume that $d(x_0,x_t)$ is not bounded. Then we can take $t_i \to \infty$ ($i \geq 1$) so that $d(x_0,x_{t_i}) \to \infty$. Replacing x_0 with x_ϵ for a small $\epsilon > 0$, we may assume that Proposition 3.3 holds up to $t = 0$.

For each $t \geq 0$, let $(\ell_s^t)_{s\in[0,d(x_0,x_t)]}$ be the unit-speed geodesic segment from x_0 to x_t. By the convexity of G, we get

$$G(\ell_s^t) \leq \frac{d(x_0,x_t)-s}{d(x_0,x_t)}G(x_0) + \frac{s}{d(x_0,x_t)}G(x_t)\,.$$

By our assumption, $G(x_t) \leq G(x_0)$. So

$$G(\ell_s^t) \leq G(x_0) < \infty\,.$$

For a fixed s_0, we can take large enough i so that $d(x_0,x_{t_j}) > s_0$ for any $j \geq i$. Then there is a constant $C > 0$ so that $\ell_s^{t_j} \in \mathcal{K}_C$ for any $j \geq i$, $s \in [0,s_0]$. By the compactness assumption, the Ascoli–Arzelà theorem [AGS08, Proposition 3.3.1] and the diagonal argument, after possibly replacing t_j by a subsequence, we may assume that there is a geodesic ray $\ell^\infty \in \mathcal{R}$, such that $\ell_s^{t_j}$ σ-converges to ℓ_s^∞ as $j \to \infty$ for all $s \geq 0$.

Fix $s \geq 0$, when $t_j \geq s$,

$$\inf_{x\in M} |\partial F|(x) \leq |\partial F|(x_{t_j}) \leq \frac{F(x_0)-F(x_{t_j})}{d(x_0,x_{t_j})} \leq \frac{F(x_0)-F(\ell_s^{t_j})}{s}\,,$$

where the second inequality follows from Proposition 3.3, the third follows from the convexity of F. Let $j \to \infty$, since F is weakly lsc, we get

$$\inf_{x\in M} |\partial F|(x) \leq \frac{F(\ell_0^\infty)-F(\ell_s^\infty)}{s}\,,$$

Let $s \to \infty$, we conclude that

$$\inf_{x\in M} |\partial F|(x) \leq -\mathbf{F}(\ell^\infty)\,.$$

When ℓ^∞ is trivial, we conclude immediately. Now assume that ℓ^∞ is not trivial. By Proposition 3.1, $\|\ell^\infty\| \leq 1$. So

$$\inf_{x\in M} |\partial F|(x) \leq -\mathbf{F}(\ell^\infty) \leq \frac{-\mathbf{F}(\ell^\infty)}{\|\ell^\infty\|} \leq \inf_{x\in M} |\partial F|(x)\,,$$

where the last inequality follows from Proposition 3.5. Now (4.4) follows. □

As a by-product of the proof, we find that if

$$\inf_{x\in M} |\partial F|(x) > 0\,,$$

then

$$\|\ell^\infty\| = 1\,. \tag{4.6}$$

We call the geodesic rays that minimizes $\mathbf{F}(\ell)/\|\ell\|$ the *Darvas–He geodesic rays*.

Corollary 4.2. *Assume that*

$$\max_{\ell\in\mathcal{R}\setminus\{0\}} \frac{-\mathbf{F}(\ell)}{\|\ell\|} > 0 \tag{4.7}$$

and that the maximizer is unique. Then for any $s \geq 0$, ℓ_s^t constructed in the previous proof starting from x_ϵ for any $\epsilon > 0$ converges to ℓ_s^∞ in M as $t \to \infty$, where ℓ^∞ is moved parallelly so that $\ell_0^\infty = x_\epsilon$.

PROOF. We use the same notations as in the proof of Theorem 4.1. By (4.7), we are in Case 2. By replacing x_0 by x_ϵ, we may set $\epsilon = 0$.

By [Bač14, Proposition 3.1.6], Theorem 4.1 and (4.6), it suffices to prove that for any $s \geq 0$, ℓ_s^t converges weakly to ℓ_s^∞ as $t \to \infty$. For this purpose, it suffices to prove that for any sequence $t_i \to \infty$, we can find a subsequence $t_{n_i} \to \infty$ such that $\ell_s^{t_{n_i}}$ converges weakly to ℓ_s^∞.

Due to (4.7), we have

$$\lim_{i \to \infty} d(x_0, x_{t_i}) = \infty\,.$$

So we can construct a Darvas–He geodesic ℓ from a subsequence t_{n_i}. We know that $\ell_s^{t_{n_i}}$ converges weakly to $\ell_s \in M$. By the uniqueness of the maximizer, we conclude that $\ell_s = \ell_s^\infty$. The result follows. □

4.3. Proof of Theorem 1.1. Now to get Theorem 1.1, one takes (M, d) to be $(\mathcal{E}^2, d_2)$, $G = M$ and F is M for the weak Calabi flow, D for the inverse Monge–Ampère flow. It remains to check the compactness properties of $\mathcal{K}_C$.

Lemma 4.3. *Let φ_j $(j \in \mathbb{N})$ be a bounded sequence in $\mathcal{E}^2$. Let $\varphi \in \mathcal{E}^1$. Assume that $\varphi_j \to \varphi$ in $\mathcal{E}^1$. Then $\varphi \in \mathcal{E}^2$. Moreover, for any $\psi \in \mathcal{E}^2$,*

$$d_2(\psi, \varphi) \leq \varliminf_{j \to \infty} d_2(\psi, \varphi_j)\,.$$

PROOF. Since $\varphi_j \to \varphi$ in $\mathcal{E}^1$, we know that

$$\varphi_j \to \varphi \text{ a.e.}\,, \qquad \left|\sup_X \varphi_j\right| \leq C\,. \tag{4.8}$$

Define

$$w_j = \sup_{i \geq j}{}^* \varphi_i\,.$$

Then (4.8) together with the Choquet lemma implies that w_j decreases and converges to φ a.e..

According to [Dar15, Lemma 4.16], in order to prove that $\varphi \in \mathcal{E}^2$, it suffices to prove that $d_2(0, w_j)$ is bounded. According to (3.5), this is equivalent to prove

$$\int_X |w_j|^2\, \omega^n \leq C\,, \qquad \int_X |w_j|^2\, \omega_{w_j}^n \leq C\,.$$

For the former, it suffices to consider the negative part of w_t, which is bounded from below by φ_j, so it suffices to prove

$$\int_X |\varphi_j|^2\, \omega^n \leq C\,.$$

This follows again from (3.5) and the assumption that φ_j is bounded in $\mathcal{E}^2$.

For the latter, according to [GZ07] and (3.5), we have

$$\int_X |w_j|^2\, \omega_{w_j}^n \leq C \int_X |\varphi_j|^2\, \omega_{\varphi_j}^n + C \leq C\,.$$

So we conclude that $\varphi \in \mathcal{E}^2$.

According to [BDL17, Theorem 5.3]. φ is the weak limit of φ_j. So we conclude by Proposition 3.1. □

Recall the following version of the compactness theorem of [BBEGZ19].

THEOREM 4.4. *For any $C > 0$, $\varphi_0 \in \mathcal{E}^1$, the set*

$$K_C := \{\varphi \in \mathcal{E}^1 : M(\varphi) \leq C, d_1(\varphi, \varphi_0) \leq C\} \subseteq \mathcal{E}^1$$

is compact with respect to the strong topology.

PROOF. Let $\varphi \in \mathcal{E}^1$ be a potential such that $M(\varphi) \leq C$, $d_1(\varphi, \varphi_0) \leq C$.
By [DH17, Proposition 2.5]*, $H(\varphi) \leq C$ for a constant C_1. Moreover, according to [DDL18, Lemma 3.9],

$$|\sup \varphi| \leq C_2 \,.$$

So according to [BBEGZ19, Theorem 2.17, Proposition 2.6], for any sequence $\varphi_j \in K_C$, up to selecting a subsequence, we may assume that φ_j converges to $\varphi \in \mathcal{E}^1$ in the strong topology. Now as M is lsc, we conclude that

$$M(\varphi) \leq C \,,$$

so $\varphi \in K_C$. This concludes the proof. □

Corollary 4.5. *For any $C > 0$, $\varphi_0 \in \mathcal{E}^2$, the set*

$$\mathcal{K}_C := \left\{\varphi \in \mathcal{E}^2 : M(\varphi) \leq C \,,\ d_2(\varphi, \varphi_0) \leq C\right\} \subseteq \mathcal{E}^2$$

is compact with respect to d_1-topology.

PROOF. Let $\varphi_j \in \mathcal{K}_C$. By Theorem 4.4, up to selecting a subsequence, we may assume that φ_j converges to $\varphi \in \mathcal{E}^1$ in the d_1-topology. Moreover, $M(\varphi) \leq C$. Then according to Lemma 4.3, we have $\varphi \in \mathcal{E}^2$ and $d_2(\varphi, \varphi_0) \leq C$. □

Proposition 4.6. *The d_1-topology on $\mathcal{E}^2$ is compatible with $(\mathcal{E}^2, d_2)$.*

For the definition of compatibility, see Section 4.2.

PROOF. Condition (1) is obvious. For Condition (2), recall that for a bounded sequence in $\mathcal{E}^2$, convergence in $\mathcal{E}^1$ implies convergence in the weak topology [BDL17, Theorem 1.6]. Condition (3) follows from [Bač13, Lemma 3.1] and Condition (2). Finally, Condition (4) follows from [BBJ21, Proposition 1.11]. □

PROOF OF THEOREM 1.1. Let $(M, d) = (\mathcal{E}^2, d_2)$. Let σ be the d_1-topology on $\mathcal{E}^1$. By Proposition 4.6, σ is compatible with (M, d).

(1) We apply Theorem 4.1 with $F = G = M$. The compactness condition is guaranteed by Corollary 4.5.

(2) Recall that M is decreasing along the inverse Monge–Ampère flow according to [CHT22, Lemma 4.6]. We apply Theorem 4.1 with $F = D$, $G = M$. The compactness condition is guaranteed by Corollary 4.5. As the inverse Monge–Ampère flow admits global smooth solutions, by Proposition 3.4, we have

$$\inf_{\varphi \in \mathcal{H}} R(\varphi) = \inf_{\varphi \in \mathcal{E}^2} R(\varphi) \,.$$

Finally observe that in the Fano case,

$$\max_{\ell \in \mathcal{R}^2 \setminus \{0\}} \frac{-\mathbf{D}(\ell)}{\|\ell\|} = 0$$

implies that X is K-semistable [Ber16]. □

*It was only stated for $\varphi \in \mathcal{H}$, but since E_R is continuous on $\mathcal{E}^1$, it also holds for $\varphi \in \mathcal{E}^1$.

Remark 4.7. In contrast to general Hadamard spaces, in $\mathcal{E}^2$ we have geodesic rays of the form $(Ct)_{t\geq 0}$. These rays have vanishing $\mathbf{M}$. So

$$\max_{\ell\in\mathcal{R}^2\setminus\{0\}}\frac{-\mathbf{M}(\ell)}{\|\ell\|}$$

is always non-negative. Similar remark holds for $\mathbf{D}$.

Remark 4.8. If the Calabi flow admits a global smooth solution, it will follow from the same proof that

$$\inf_{\phi\in\mathcal{H}} Ca(\phi)=\max_{\ell\in\mathcal{R}^2\setminus\{0\}}\frac{-\mathbf{M}(\ell)}{\|\ell\|}\,.$$

4.4. Uniqueness of the maximizer.

Definition 4.9. We say (X,ω) is *geodesically unstable* if

$$\max_{\ell\in\mathcal{R}^2\setminus\{0\}}\frac{-\mathbf{M}(\ell)}{\|\ell\|}>0\,.$$

Otherwise, we say (X,L) is *geodesically semistable.*

According to [DL20, Theorem 1.5], (X,ω) is geodesically unstable iff there is a $C^{1,\bar{1}}$ geodesic ray ℓ, such that $\mathbf{M}(\ell)<0$.

THEOREM 4.10. *$\mathcal{R}^2$ is a Hadamard space.*

PROOF. It is known that $\mathcal{R}^2$ is a complete geodesic metric space [DL20, Theorem 1.3, Theorem 1.4]. So it suffices to prove that $\mathcal{R}^2$ satisfies the CAT(0)-inequality. More concretely, we need to show: if $\ell,\ell^s\in\mathcal{R}^2$ $(s\in[0,1])$, ℓ^s is a geodesic segment in $\mathcal{R}^2$, then for any $s\in[0,1]$, we have

$$d_2^c(\ell,\ell^s)^2\leq(1-s)d_2^c(\ell,\ell^0)^2+sd_2^c(\ell,\ell^1)^2-s(1-s)d_2^c(\ell^0,\ell^1)^2\,.$$

Without loss of generality, we may assume that the starting point of geodesic rays in $\mathcal{R}^2$ are 0. We recall the construction of ℓ^s from ℓ^0 and ℓ^1. For each $t\geq 0$, let $(\ell_s^{;t})_{s\in[0,1]}$ be the geodesic segment from ℓ_t^0 to ℓ_t^1. Let $(L_T^{t,s})_{T\in[0,t]}$ be the geodesic segment from 0 to $\ell_s^{;t}$. Then for any fixed $T\geq 0$, $L_T^{t,s}$ for $t\to\infty$ has a unique limit, the limit is defined to be ℓ_T^s.

Now for any $T\geq 0$,

$$\frac{1}{T^2}d_2(\ell_T,\ell_T^s)^2=\lim_{t\to\infty}\frac{1}{T^2}d_2(\ell_T,L_T^{t,s})^2\leq\varlimsup_{t\to\infty}\frac{1}{t^2}d_2(\ell_t,\ell_s^{;t})\,,$$

where the last inequality follows from [DL20] (1).

Now since $\mathcal{E}^2$ is a Hadamard space, we find for any $t\geq 0$

$$d_2(\ell_t,\ell_s^{;t})^2\leq(1-s)d_2(\ell_t,\ell_t^0)^2+sd_2(\ell_t,\ell_t^1)^2-s(1-s)d_2(\ell_t^0,\ell_t^1)\,.$$

Hence

$$\begin{aligned}\frac{1}{T^2}d_2(\ell_T,\ell_T^s)^2&\leq\varlimsup_{t\to\infty}\frac{1}{t^2}\Big((1-s)d_2(\ell_t,\ell_t^0)^2+sd_2(\ell_t,\ell_t^1)^2-s(1-s)d_2(\ell_t^0,\ell_t^1)\Big)\\&=(1-s)d_2^c(\ell,\ell^0)^2+sd_2^c(\ell,\ell^1)^2-s(1-s)d_2^c(\ell^0,\ell^1)^2\,.\end{aligned}$$

Let $T\to\infty$, we conclude. ∎

Proof of Corollary 1.2. We only prove part 1, since part 2 is similar.

Assume that (X,ω) is geodesically unstable. Let $\varphi \in \mathcal{E}^2$ with $M(\varphi) < \infty$. Let ℓ^0, ℓ^1 be two *different* minimizers of $\mathbf{M}$ on the unit sphere. Let $(\ell^s)_{s\in[0,1]}$ be the unique d_2^c-geodesic between them. Since $\mathbf{M}$ is convex in $\mathcal{R}^2$ [DL20, Theorem 4.11], we have

$$-\mathbf{M}(\ell^s) \geq \inf_{\phi\in\mathcal{E}^2} Ca(\phi). \tag{4.9}$$

By the CAT(0)-inequality of $\mathcal{R}^2$,

$$\|\ell^s\| < 1\,, \quad s \in (0,1)\,.$$

Hence

$$\frac{-\mathbf{M}(\ell^s)}{\|\ell^s\|} > \inf_{\phi\in\mathcal{E}^2} Ca(\phi)\,.$$

This is a contradiction. □

In particular, the conditions of Corollary 4.2 are satisfied.

5. Further remarks and conjectures

5.1. Relations between Theorem 1.1 and Donaldson's conjecture. In this section, we assume that the polarization of X is integral, namely, coming from an ample line bundle L on X. This assumption is not essential, but makes notations simpler.

Let $\mathcal{H}^{\mathrm{an}}$ be the space of non-Archimedean metrics defined in [BHJ19; BHJ17]. Recall that there is a natural map $\iota : \mathcal{H}^{\mathrm{an}} \to \mathcal{R}^p$ for $p \geq 1$. Moreover, the geodesic rays in the image of ι have $C^{1,1}$-regularity ([CTW18]). This construction dates back to [PS07]. See also [RW14; DDL21].

The map admits a natural extension to an embedding $\iota : \mathcal{E}^{1,\mathrm{an}} \to \mathcal{R}^1$. See Theorem 6.6 in [BBJ21]. Here $\mathcal{E}^{1,\mathrm{an}}$ is the non-Archimedean analogue of the usual $\mathcal{E}^1$ space. For the precise definition, we refer to [BBJ21; BJ18; Bou18] and references therein.

Now let us explain the relation between Donaldson's conjecture (i.e. equality in (1.2), (1.1)) and Theorem 1.1.

Let ℓ be the image of a non-Archimedean metric $\psi \in \mathcal{H}^{\mathrm{an}}$ under the map ι. According to [His16, Theorem 1.2],

$$\|\psi\|_{L^2}^2 = \frac{1}{V}\int_X |\dot{\ell}_0|^2\,\omega_{\ell_0}^n\,.$$

Since we already know that ℓ has $C^{1,1}$ regularity, it follows from [Dar15, Lemma 4.11] that

$$\frac{1}{V}\int_X |\dot{\ell}_0|^2\,\omega_{\ell_0}^n = \|\ell\|^2\,.$$

According to [BHJ17, Proposition 2.8],

$$\mathrm{DF}(\mathcal{X},\mathcal{L}) = M^{\mathrm{an}}(\psi)\,,$$

where $(\mathcal{X},\mathcal{L})$ is a normal representative of ψ with reduced central fibre. This shows the equivalence between (1.1) and (1.2).

Proposition 5.1. *Notations as above, then*

$$\mathbf{M}(\ell) \leq M^{\mathrm{an}}(\psi)\,.$$

Proof. According to [BDL20] (4.2) and (4.3), we have a subgeodesic ray $\tilde{\ell}_t$, so that

$$M(\tilde{\ell}_t) = \mathrm{DF}(\mathcal{X}, \mathcal{L})t + \mathcal{O}(1)\,, \quad d_2(\ell_t, \tilde{\ell}_t) \leq C\,.$$

For each $t > 0$, let $[0, t] \ni a \mapsto v_a^t$ be the d_2-geodesic connecting ℓ_0 to $\tilde{\ell}_t$. Let ℓ' be the geodesic ray with $\ell'_0 = \tilde{\ell}_0$, which is parallel to ℓ. The existence and uniqueness of ℓ' is guaranteed by [DL20, Proposition 4.1]. Let $[0, t] \ni a \mapsto u_a^t$ be the d_2-geodesic connecting ℓ_0 to ℓ'_t. As in the proof of [DL20, Proposition 4.1], for fixed $a \geq 0$, $u_a^t \to \ell_a$ as $t \to \infty$. Now by [DL20] (1),

$$d_2(u_a^t, v_a^t) = \mathcal{O}(1/t)\,.$$

Hence we conclude

$$v_a^t \to \ell_a\,, \quad t \to \infty\,.$$

By the convexity of M, we find

$$M(v_a^t) \leq \left(1 - \frac{a}{t}\right) M(\ell_0) + \frac{a}{t} M(\tilde{\ell}_t)\,.$$

Let $t \to \infty$ and use the fact that M is lsc, we find

$$\frac{M(\ell_a)}{a} \leq \frac{M(\ell_0)}{a} + \mathrm{DF}(\mathcal{X}, \mathcal{L})\,.$$

Finally, let $a \to \infty$, we conclude

$$\mathbf{M}(\ell) \leq \mathrm{DF}(\mathcal{X}, \mathcal{L})\,.$$

□

Remark 5.2. The reverse inequality is recently proved by Chi Li in [Li20].

Conjecture 5.3. [†] *The Darvas–He geodesic lies in $\iota(\mathcal{E}^{1,\mathrm{an}})$.*

In terms of the terminology of [BBJ21], we conjecture that the Darvas–He geodesic is *maximal*.

Observe that Donaldson's conjecture (equality in (1.1) and (1.2)) will follow from our result if the followings are true:

(1) Conjecture 5.3 is true and we have the following recovery property: for each $\ell \in \mathcal{R}^2 \cap \iota(\mathcal{E}^{1,\mathrm{an}})$, one could find a sequence ℓ^j in $\iota(\mathcal{H}^{\mathrm{an}})$ such that $d_2^c(\ell, \ell^j) \to 0$, and such that

$$\mathbf{M}(\ell^j) \to \mathbf{M}(\ell)\,.$$

(2) Chen's conjecture is true: the Calabi flow admits long time smooth solution for an arbitrary smooth initial value (See Remark 4.8).

A positive result in this direction is recently proved by Darvas and Lu [DL20, Theorem 1.5]. They showed that $\mathcal{R}^{1,\bar{1}}$ (the space of $C^{1,\bar{1}}$ geodesics) is dense in $\mathcal{R}^p$ for any $p \in [1, \infty)$. Moreover, a recovery property holds in this case.

Due to Theorem 4.10, one can study the gradient flow of $\mathbf{M}$ on $\mathcal{R}^2$. This flow can be properly called the *radial Calabi flow*. The behaviour of this flow will be closely related to our conjecture.

[†] The conjecture is true by the recent work [Li20].

5.2. Harnack estimate. We restrict our discussion to the inverse Monge–Ampère flow here.

It is natural to guess that the Darvas–He geodesic rays that we construct should be locally bounded. By using Theorem 3.4 in [Dar17b], this will follow from a lower bound

$$\inf_X \varphi_t \geq -Ct - C$$

for a solution φ_t to (3.10).

The proof of *a priori* bound of $\inf_X \varphi_t$ on finite time intervals in [CHT22] is by means of contradiction, and it seems impossible to get qualitative bounds using their methods.

A similar situation exists for Kähler–Ricci flows. However, in that case, the Sobolev constant along the flow is uniformly bounded, as a consequence of the monotonicity of the Perelman's W-entropy (See [Ye07] for details). Then applying the usual Moser iteration, we arrive at a Harnack inequality (See [Rub09], for example).

The problem for the inverse Monge–Ampère flow is that, the Perelman entropy, in its original form, is not monotone. And there does not seem to be any method to control the Sobolev constant in this case.

We also notice that it is easy to deduce a lower bound exponential in t using the Moser–Trudinger inequality [BB11] and Kołodziej's L^∞-estimate. See [BEGZ10] for an explicit version of Kołodziej's estimate.

If the Harnack estimate does hold, we conclude immediately that the Darvas–He geodesic $\ell^{(t)}$ is non-trivial. So we get plenty of criteria for the existence of Kähler–Einstein metrics.

Similar remarks hold also in the weak Calabi flow setting. Note that we do not require that the Calabi flow has a global smooth solution.

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APPENDIX A

Recent developments

In this chapter, we collect include some later developments related to the papers in this thesis.

1. Paper 1

This paper handles $\mathcal{I}$-model potentials in Kähler cohomology classes. This does not suffice for several applications. For example, the automorphic line bundles on Shimura varieties are usually just semi-ample.

The general case of a pseudo-effective class is handled in [DX21]. More precisely, we prove that the inequality (1.4) in Theorem 4.7 remains true when L is just pseudo-effective and the characterizations of the equality case in Theorem 4.7 remain true after suitable modifications.

There are several subtleties in the study of singular potentials on pseudo-effective line bundles. We developed a general approach that works for more general problems as well.

A particular case of the singular holomorphic Morse inequality is discovered independently by Botero–Burgos Gil–Holmes–de Jong [BBGHdJ21]. They introduce the notions of almost asymptotically algebraic singularities and toroidal singularities of a psh metric. They prove that under these conditions, (1.4) holds. Their result is less general than [DX21], but suffices for their applications on the Siegel–Jacobi modular varieties.

Toroidal singularities are special cases of $\mathcal{I}$-good singularities. The original definition of $\mathcal{I}$-good singularities is global in nature and therefore hard to verify in practice. One interesting thing about the toroidal singularities is that it is a local condition on the psh metric. In practice, it is much easier to verify a concrete psh metric have toroidal singularities.

In another direction, the papers [DX21; DX22; BBGHdJ21] only handle the case of line bundles. The case of vector bundles is treated in [Xia22]. In that paper, we introduce the notion of $\mathcal{I}$-good metrics on vector bundles. It seems that $\mathcal{I}$-good metrics are the correct replacement in the setting of mixed Shimura varieties of Mumford's notion of good metrics in the setting of Shimura varieties.

2. Paper 2

This paper handles the b-divisors associated with psh metrics. This result was stated in a restricted form because we rely on Dang–Favre's intersection theory of b-divisors [DF20]. The general case of Dang–Favre's theory in the second version of their paper was not available when the paper was written.

Now in the second version of their paper, Dang–Favre defined and studied the intersection number of nef b-divisors on a smooth projective variety over an algebraically closed field of characteristic 0, we can say more about our results as well. In [Xia22], we proved that Theorem 5.4 holds in the pseudo-effective setting as well.

A special case of this result is found in [BBGHdJ21] under the name of *Chern–Weil formula*.

Another generalization is about Dang–Favre's intersection. In [Xia22], we establish a general intersection theory of b-divisors for smooth projective varieties over a perfect field. This will be useful when we consider the canonical models the mixed Shimura varieties: we need to consider both the reflex fields and the finite residue fields. In an ongoing project, the author is trying to develop a more general intersection on Riemann–Zariski spaces based on their K-theory.

3. Paper 3

This paper handles the optimal destabilizing properties of the Calabi flow. Since the appearance of this paper, there are several related papers studying optimal destabilizing properties in various related settings: [BLZ19; Der20; Tak20; Sjö20].

A closely related problem is the radial/non-Archimedean version of the Calabi flow introduced in [Xia20]. Recall that the space $\mathcal{R}^2$ of $\mathcal{E}^2$-geodesic rays is an Hadamard space, as proved in the paper. It follows from an argument like Theorem 3.18 in Paper 1 that non-Archimedean space $\mathcal{E}^2(L^{\mathrm{an}})$ is also an Hadamard space. Moreover, it is easy to see that the radial Mabuchi functional $\mathbf{M}$ is a convex lower semi-continuous functional on both spaces. So the general framework of gradient flows makes sense. We call this flow the *radial Calabi flow*. See [Xia20] for the conjectures concerning this flow.

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