Affinoid algebras

Contents

1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. We set

$$\begin{aligned} k\{r^{-1}T\} =& k\{r_1^{-1}T_1, \ldots, r_n T_n^{-1}\} \\ &:= \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \ldots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\}. \end{aligned}$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$\|f\|_r = \max_\alpha |a_\alpha| r^\alpha.$$

We call $(k\{r^{-1}T\}, \|\bullet\|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\|\bullet\|_r$ is called the *Gauss norm*.

We omit r from the notation if $r = (1, \ldots, 1)$.

This is a special case of Example 4.15 in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach k-algebra and $\|\bullet\|_r$ is a valuation.

Proof. This is a special case of Proposition 4.16 in the chapter Banach Rings. □

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T\rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T\rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\} \cong k[T_1, \ldots, T_n]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \ldots, T_n]]$ otherwise.

Lemma 2.5. Let A be a Banach k-algebra. For each $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathring{A}$, there is a unique continuous homomorphism $k\{T_1, \ldots, T_n\} \to A$ sending T_i to a_i.

Proof. This is a special case of Proposition 4.17 in the chapter Banach Rings. □

3. Affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

More generally, a Banach k-algebra A is *k_H-affinoid* if there are $n \in \mathbb{N}$, $r \in H^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

A morphism between k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is a bounded k-algebra homomorphism.

The category of k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is denoted by k-$\mathcal{A}$ff$\mathcal{A}$lg (resp. st-k-$\mathcal{A}$ff$\mathcal{A}$lg, resp. k_H-$\mathcal{A}$ff$\mathcal{A}$lg). The opposite categories of these categories are denoted by k-$\mathcal{A}$ff, st-k-$\mathcal{A}$ff and k_H-$\mathcal{A}$ff. For any A in k-$\mathcal{A}$ff$\mathcal{A}$lg (resp. st-k-$\mathcal{A}$ff$\mathcal{A}$lg, resp. k_H-$\mathcal{A}$ff$\mathcal{A}$lg), the corresponding image in the opposite category is denoted by $\operatorname{Sp} A$. We can also identify $\operatorname{Sp} A$ with the topological space defined in Definition 6.1 in the chapter Banach Rings.

For the notion of admissible morphisms, we refer to Definition 2.5 in the chapter Banach rings.

Remark 3.2. Berkovich also introduced the notion of *affinoid k-algebras*: it is a K-affinoid algebra for some complete non-Archimedean field extension K/k. We will not use this notion.

Example 3.3. Let $r \in \mathbb{R}_{>0}$. We let k_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \to 0$ for $i \to \infty$ and $i \to -\infty$. We define a norm $\| \bullet \|_r$ on k_r as follows:

$$\|f\|_r := \max_{i\in\mathbb{Z}} |a_i| r^i.$$

We will show in Proposition 3.4 that k_r is k-affinoid.

Proposition 3.4. Let $r \in \mathbb{R}_{>0}$, then $(k_r, \|\bullet\|_r)$ defined in Example 3.3 is a k-affinoid algebra. Moreover, $\| \bullet \|_r$ is a valuation.

Proof. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \to k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$|a_{i,j}| r^{i-j} \to 0 \tag{3.1}$$

as $i + j \to \infty$. Observe that for each $k \in \mathbb{Z}$, the series

$$c_k := \sum_{i-j=k, i,j\in\mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that $\iota(f) \in k_r$. That is

$$|c_k|r^k \to 0$$

as $k \to \pm\infty$. When $k \geq 0$, we have $|c_k| \leq |a_{k0}|$ by definition of c_k. So $|c_k|r^k \to 0$ as $k \to \infty$ by (3.1). The case $k \to -\infty$ is similar.

We conclude that we have a well-defined map of sets ι. It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g = \sum_{i=-\infty}^{\infty} c_i T^i \in k_r$. We want to show that g lies in the image of ι. As ι is a ring homomorphism, it suffices to treat two cases separately: $g = \sum_{i=0}^{\infty} c_i T^i$ and $g = \sum_{i=-\infty}^{0} c_i T^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$. It is immediate that $\iota(f) = g$.

Next we show that ι is admissible. We first identify the kernel of ι. We claim that the kenrel is the ideal I generated by $T_1T_2 - 1$. It is obvious that $I \subseteq \ker \iota$. Conversely, consider an element

$$f = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kenrel of ι. Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If $f \in \ker \iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by $T_1T_2 - 1$. The condition that $f_k \in \ker \iota$ means

$$\sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find $b_{i,j} \in k$ for $i, j \in \mathbb{N}$, $i - j = k$ such that

$$a_{i,j} = b_{i-1,j-1} - b_{i,j}.$$

Then

$$f_k = (T_1T_2 - 1) \sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$. In particular, the sum of $\sum_{i,j\in\mathbb{N},i-j=k} b_{i,j} T_1^i T_2^j$ for various k converges to some $g \in k\{r^{-1}T_1, rT_2\}$ and hence $f_k = (T_1T_2 - 1)g$. Therefore, we have proved that $\ker \iota$ is generated by $T_1T_2 - 1$.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \to k_r.$$

To see this, take $f = \sum_{k=-\infty}^{\infty} c_k T^k \in k_r$ and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$, then $\iota(g) = f$ and $\|g\|_{(r,r^{-1})} = \|f\|$. So it suffices to show that for any $h = \sum_{(i,j)\in\mathbb{N}^2} d_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$, we have

$$\|f\|_r \leq \|g + h(T_1T_2 - 1)\|_{r,r^{-1}}. \tag{3.2}$$

We compute

$$g+h(T_1T_2-1)=\sum_{k=1}^{\infty}(c_k-d_{k,0})T_1^k+\sum_{k=1}^{\infty}(c_{-k}-d_{0,k})T_2^k+(c_0-d_0)+\sum_{i,j\geq 1}(d_{i-1,j-1}-d_{i,j})T_1^iT_2^j.$$

So

$$\|g+h(T_1T_2-1)\|_{r,r^{-1}}=\max\left\{\max_{k\geq 0}C_{1,k},\max_{k\geq 1}C_{2,k}\right\},$$

where

$$C_{1,k}=\max\left\{|c_k-d_{k,0}|,\left|\sum_{i-j=k,i,j\geq 1}d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k\geq 0$ and

$$C_{2,k}=\max\left\{|c_{-k}-d_{0,k}|,\left|\sum_{i-j=-k,i,j\geq 1}d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k\geq 1$. It follows from the strong triangle inequality that $|c_k|\leq C_{1,k}$ for $k\geq 0$ and $c_{-k}\leq C_{2,k}$ for $k\geq 1$. So (3.2) follows. □

Proposition 3.5. Let $r\in\mathbb{R}_{>0}\setminus\sqrt{|k^\times|}$, then $\|\bullet\|_r$ defined in Example 3.3 is a valuation on k_r.

PROOF. Take $f,g\in k_r$, we need to show that

$$\|fg\|_r\geq\|f\|_r\|g\|_r.$$

Let us expand

$$f=\sum_{i=-\infty}^{\infty}a_iT^i,\quad g=\sum_{i=-\infty}^{\infty}b_iT^i.$$

Take i and j so that

$$|a_i|r^i=\|f\|_r,\quad |b_j|r^j=\|g\|_r. \tag{3.3}$$

By our assumption on r, i,j are unique. Then

$$\|fg\|_r=\max_{k\in\mathbb{Z}}\{|c_k|r^k\},$$

where

$$c_k:=\sum_{u,v\in\mathbb{Z},u+v=k}a_ub_v.$$

It suffices to show that

$$|c_k|r^k=\|f\|_r\|g\|_r. \tag{3.4}$$

for $k=i+j$. Of course, we may assume that $a_i\neq 0$ and $b_j\neq 0$ as otherwise there is nothing to prove. For $u,v\in\mathbb{Z}$, $u+v=i+j$ while $(u,v)\neq(i,j)$, we may assume that $u\neq i$. Then $|a_u|r^u<|a_i|r^i$ and $|b_v|r^v\leq|b_j|r^j$. So $|a_ub_v|<|a_ib_j|$ and we conclude (3.4). □

Remark 3.6. The argument of Proposition 4.16 in the chapter Banch Rings does not work here if $r\in\sqrt{|k^\times|}$, as in general one can not take minimal i,j so that (3.3) is satisfied.

Proposition 3.7. Assume that $r\in\mathbb{R}_{>0}\setminus\sqrt{|k^\times|}$. Then k_r is a valuation field and $\|\bullet\|_r$ is non-trivial.

Proof. We first show that $\operatorname{Sp} k_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet| \in \operatorname{Sp} k_r$. As $\|\bullet\|_r$ is a valuation, we find

$$|\bullet| \leq \|\bullet\|_r. \tag{3.5}$$

In particular, $|\bullet|$ restricted to k is the given valuation on k. It suffices to show that $|T| = r$. This follows from (3.5) applied to T and T^{-1}.

It follows that k_r does not have any non-zero proper closed ideals: if I is such an ideal, k_r/I is a Banach k-algebra. By Proposition 6.10 in the chapter Banach rings, $\operatorname{Sp} k_r$ is non-empty. So k_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.7 in the chapter Banach rings, the only maximal ideal of k_r is 0. It follows that k_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r = r$. □

Definition 3.8. An element $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$ for some $n \in \mathbb{N}$ is called a *k-free polyray* if $r_1, \dots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_{>0}/\sqrt{|k^\times|}$.

Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Assume that r is a k-free polyray. We define

$$k_r = k_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{r_n}.$$

By an interated application of Proposition 3.7, k_r is a complete valuation field.

As a general explanation of why k_r is useful, we prove the following proposition:

Proposition 3.9. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k-free polyray.

(1) For any k-Banach space X, the natural map

$$X \to X \hat{\otimes}_k k_r$$

is an isometric embedding.

(2) Consider a sequence of bounded homomorphisms of k-Banch spaces $X \to Y \to Z$. Then the sequence is admissible and exact (resp. coexact) if and only if $X \hat{\otimes}_k k_r \to Y \hat{\otimes}_k k_r \to Z \hat{\otimes}_k k_r$ is admissible and exact (resp. coexact).

Proof. We may assume that $n = 1$.

(1) We have a more explicit description of $X \hat{\otimes}_k k_r$: as a vector space, it is the space of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ with $a_i \in X$ and $\|a_i\| r^i \to 0$ when $|i| \to \infty$. The norm is given by $\max_i \|a_i\| r^i$. From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1). □

When X is a Banach k-algebra, $X \hat{\otimes}_k k_r$ is a Banach k_r-algebra.

Example 3.10. For any $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$, not necessarily k-free. We define k_r as the completed fraction field of $k\{r^{-1}T\}$ provided with the extended valuation $|\bullet|_r$. Then k_r is still a valuation field extending k.

When r is a k-free polyray, we claim that k_r coincides with k_r defined in Definition 3.8. To see this, let us temporarily denote the k_r defined in this example as k'_r consider the extension of field:

$$\operatorname{Frac} k\{r^{-1}T\} \to k_r = k\{r^{-1}T, rS\}/(T_1 S_1 - 1, \dots, T_n S_n - 1)$$

sending T_i to T_i for $i = 1, \dots, n$. Observe that this is an extension of valuation field as well by the same arguments as in Proposition 3.4. In particular, it induces an extension of complete valuation fields $k'_r \to k_r$. But the image clearly contains the classes of all polynomials in $k[T, S]$, so $k'_r \to k_r$ is an isometric isomorphism.

Proposition 3.11. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite bounded homomorphism into a k-Banach algebra A. Then A is also strictly k-affinoid.

PROOF. We may assume that $B = k\{T_1, \dots, T_n\}$ for some $n \in \mathbb{N}$. By assumption, we can find finitely many $a_1, \dots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(B)a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.17 in the chapter Banach Rings, φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{T_1, \dots, T_n, S_1, \dots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is strictly k-affinoid. $\square$

Lemma 3.12. Assume that k is non-trivially valued. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. The algebra $k\{r^{-1}T\}$ is strictly k-affinoid if $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \dots, n$.

Remark 3.13. The converse is also true.

PROOF. Assume that $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \dots, n$. Take $s_i \in \mathbb{N}$ and $c_i \in k^\times$ such that

$$r_i^{s_i} = |c_i^{-1}|$$

for $i = 1, \dots, n$. We deifne a bounded k-algebra homomorphism $\varphi : k\{T_1, \dots, T_n\} \to k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ by sending T_i to $c_i T_i^{s_i}$. This is possible by Proposition 4.17 in the chapter Banach Rings.

We claim that φ is finite. To see this, it suffices to observe that if we expand $f \in k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (cT^s)^\gamma,$$

where the product γs is taken component-wise. For each $\beta \in \mathbb{N}^n, \beta_i < s_i$, we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \dots, T_n\}.$$

While $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$. So We have shown that φ is finite. Hence, $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is k-affinoid by Proposition 3.11. $\square$

Proposition 3.14. Let A be a k-affinoid algebra, then there is $n \in \mathbb{N}$ and a k-free polyray $r = (r_1, \dots, r_n)$ such that $A \hat{\otimes}_k k_r$ is strictly k_r-affinoid. Moreover, we can guarantee that k_r is non-trivially valued.

PROOF. By Proposition 3.9, we may assume that $A = k\{t^{-1}T\}$ for some $t \in \mathbb{R}^m_{>0}$. By Lemma 3.12, it suffices to take r so that the linear subspace of $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ generated by $r_1, \dots, r_n$ contains all components of t. By taking $n \geq 1$, we can guarantee that k_r is non-trivially valued. $\square$

Proposition 3.15. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a monomorphism in k_H-$\mathcal{A}\mathrm{ff}$. Then for any $y \in \operatorname{Sp} B$ with $x = \varphi(y)$, one has $\varphi^{-1}(x) = \{y\}$ and the natural map $\mathscr{H}(x) \to \mathscr{H}(y)$ is an isomorphism of complete valuation rings.

Proof. It suffices to show that $\mathscr{H}(x) \to B\hat{\otimes}_A \mathscr{H}(y)$ is an isomorphism as Banach k-algebras. Include details about cofiber products in affalg. By assumption, the codiagonal map $B\hat{\otimes}_A B \to B$ is an isomorphism. It follows that the base change with respect to $A \to \mathscr{H}(x)$ is also an isomorphism: $B'\hat{\otimes}_{\mathscr{H}(x)} B' \to B'$, where $B' = B\hat{\otimes}_A \mathscr{H}(x)$.

Include the fact that the first map is injective. It follows that the composition $B' \otimes_{\mathscr{H}(x)} B \to B'\hat{\otimes}_{\mathscr{H}(x)} B' \to B'$ is injective. Therefore, $\mathscr{H}(x) \to B'$ is an isomorphism of rings. We also know that this map is bounded. But we already know that $\mathscr{H}(x)$ is a complete valuation ring, so the map $\mathscr{H}(x) \to B'$ is an isomorphism of complete valuation rings. $\square$

4. Weierstrass theory

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 4.1. We have canonical identifications

$$(k\{T_1, \dots, T_n\})^\circ \cong \mathring{k}\{T_1, \dots, T_n\},$$
$$(k\{T_1, \dots, T_n\})^{\check{}} \cong \check{k}\{T_1, \dots, T_n\},$$
$$\widetilde{k\{T_1, \dots, T_n\}} \cong \tilde{k}[T_1, \dots, T_n].$$

The last identification extends $\mathring{k} \to \tilde{k}$ and T_i is mapped to T_i.

Proof. This follows from Corollary 4.19 from the chapter Banach rings. $\square$

We will denote the reduction map $\mathring{k}\{T_1, \dots, T_n\} \to \tilde{k}[T_1, \dots, T_n]$ by $\tilde{\bullet}$.

Definition 4.2. Let $n \in \mathbb{N}$. A system $f_1, \dots, f_n \in k\{T_1, \dots, T_n\}$ is called an *affinoid chart* of $k\{T_1, \dots, T_n\}$ if $f_i \in \mathring{k}\{T_1, \dots, T_n\}$ for each $i = 1, \dots, n$ and the continuous k-algebra homomorphism $k\{T_1, \dots, T_n\} \to k\{T_1, \dots, T_n\}$ sending T_i to f_i is an isomorphism.

The map $k\{T_1, \dots, T_n\} \to k\{T_1, \dots, T_n\}$ is well-defined by Proposition 4.1 and Lemma 2.5.

Lemma 4.3. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \dots, T_n\}$. Assume that $\|f\|_1 = 1$. Then the following are equivalent:

(1) f is a unit $k\{T_1, \dots, T_n\}$.
(2) $\tilde{f}$ is a unit in $\tilde{k}[T_1, \dots, T_n]$.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 3.4, f is a unit in $k\{T_1, \dots, T_n\}$ if and only if it is a unit in $(k\{T_1, \dots, T_n\})^\circ$, which is identified with $\mathring{k}\{T_1, \dots, T_n\}$ by Proposition 4.1. This result then follows from Corollary 4.20 in the chapter Banach Rings. $\square$

Definition 4.4. Let $n \in \mathbb{N}$. Consider $g \in k\{T_1, \dots, T_n\}$. We expand g as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

For $s \in \mathbb{N}$, we say g is *X_n-distinguished of degree s* if g_s is a unit in $k\{T_1, \ldots, T_{n-1}\}$, $\|g_s\|_1 = \|g\|_1$ and $\|g_s\|_1 > \|g_t\|_1$ for all $t > s$.

Theorem 4.5 (Weierstrass division theorem)**.** Let $n, s \in \mathbb{N}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished of degree s. Then for each $f \in k\{T_1, \ldots, T_n\}$, there exist $q \in k\{T_1, \ldots, T_n\}$ and $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$ such that

$$f = qg + r.$$

Moreover, q and r are uniquely determined. We have the following estimates

$$\|q\|_1 \leq \|g\|_1^{-1}\|f\|_1, \quad \|r\|_1 \leq \|f\|_1. \tag{4.1}$$

If in addition, $f, g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, then $q \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ as well.

Proof. We may assume that $\|g\|_1 = 1$.

Step 1. Assuming the existence of the division. Let us prove (4.1). We may assume that $f \neq 0$, so that one of q, r is non-zero. Up to replacing q, r by a scalar multiple, we may assume that $\max\{\|q\|_1, \|r\|_1\} = 1$. So $\|f\|_1 \leq 1$ as well. We need to show that $\|f\|_1 = 1$. Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here $\tilde{\bullet}$ denotes the reduction map. By our assumption, $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$. From Proposition 4.1, the equality is in $\tilde{k}[T_1, \ldots, T_n]$. From the usual Euclidean division, we have $\tilde{q} = \tilde{r} = 0$. This is a contradiction to our assumption.

Step 2. Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with q and r as in the theorem. The estimate in Step 1 shows that $q = r = 0$.

Step 3. We prove the existence of the division.

We define

$$B := \left\{qg + r : r \in k\{T_1, \ldots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \ldots, T_n\}\right\}.$$

From Step 1, B is a closed subgroup of $k\{T_1, \ldots, T_n\}$. In fact, suppose $f_i \in B$ is a sequence converging to $f \in k\{T_1, \ldots, T_n\}$. From Step 1, we can represent $f_i = q_i g + r_i$, then from Step 1, q_i and r_i are both Cauchy sequences, we may assume that $q_i \to q \in k\{T_1, \ldots, T_n\}$ and $r_i \to r$. As $\deg_{T_n} r_i < s$, it follows that $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ and $\deg_{T_n} r < s$. So $f = qg + r$ and hence B is closed.

It suffices to show that B is dense $k\{T_1, \ldots, T_n\}$. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \ldots, T_{n-1}\}.$$

We may assume that $\|g\|_1 = 1$. Define $\epsilon := \max_{j \geq s} \|g_j\|$. Then $\epsilon < 1$ by our assumption. Let $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$ for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \ldots, T_n\})^\circ \to (\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$$

with kernel $\{f \in k\{T_1, \ldots, T_n\} : \|f\|_1 \leq \epsilon\}$. We now apply Euclidean division in the ring $(\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$ to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some $q \in (k\{T_1, \ldots, T_n\})^\circ$ and $r \in (k\{T_1, \ldots, T_{n-1}\})^\circ[T_n]$ with $\deg_{T_n} r < s$. So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that B is dense in $k\{T_1,\ldots,T_n\}$ by Proposition 2.8 in the chapter Banach rings.

Step 4. It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring $k\{T_1,\ldots,T_{n-1}\}[T_n]$ and the uniqueness proved in Step 2. □

Lemma 4.6. Let $\omega\in k\{T_1,\ldots,T_{n-1}\}[T_n]$ be a Weierstrass polynomial and $g\in k\{T_1,\ldots,T_n\}$. Assume that $\omega g\in k\{T_1,\ldots,T_{n-1}\}[T_n]$, then $g\in k\{T_1,\ldots,T_{n-1}\}[T_n]$.

Proof. By the division theorem of polynomial rings, we can write

$$\omega g=q\omega+r$$

for some $q,r\in k\{T_1,\ldots,T_{n-1}\}[T_n]$, $\deg_{T_n} r<\deg_{T_n}\omega g$. But we can write $\omega g=\omega\cdot g$. From the uniqueness part of Theorem 4.5, we know that $q=g$, so g is a polynomial in T_n. □

As a consequence, we deduce Weierstrass preparation theorem.

Definition 4.7. Let $n\in\mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n-variables is a monic polynomial $\omega\in k\{T_1,\ldots,T_{n-1}\}[T_n]$ with $\|\omega\|_1=1$.

Lemma 4.8. Let $n\in\mathbb{Z}_{>0}$ and $\omega_1,\omega\in k\{T_1,\ldots,T_{n-1}\}[T_n]$ be two monic polynomials. If $\omega_1\omega_2$ is a Weierstrass polynomial then so are ω_1 and ω_2.

Proof. As ω_1 and ω_2 are monic, $\|\omega_i\|_1\geq 1$ for $i=1,2$. On the other hand, $\|\omega_1\|_1\cdot\|\omega_2\|_1=\|\omega_1\omega_2\|_1=1$, so $\|\omega_i\|_1=1$ for $i=1,2$. □

Theorem 4.9 (Weierstrass preparation theorem)**.** Let $n\in\mathbb{Z}_{>0}$ and $g\in k\{T_1,\ldots,T_n\}$ be X_n-distinguished of degree s. Then there are a Weierstrass polynomial $\omega\in k\{T_1,\ldots,T_{n-1}\}[T_n]$ of degree s and a unit $e\in k\{T_1,\ldots,T_n\}$ such that

$$g=e\omega.$$

Moreover, e and ω are unique. If $g\in k\{T_1,\ldots,T_{n-1}\}[T_n]$, then so is e.

Proof. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let $r=T_n^s-\omega$. Then $T_n^s=e^{-1}g+r$. The uniqueness part of Theorem 4.5 implies that e and r are uniquely determined, hence so is ω.

Next we prove the existence. By Weierstrass division theorem Theorem 4.5, we can write

$$T_n^s=qg+r$$

for some $q\in k\{T_1,\ldots,T_n\}$ and $r\in k\{T_1,\ldots,T_{n-1}\}[T_n]$ with $\deg_{T_n} r<s$. Let $\omega=T_n^s-r$. From the estimates in Theorem 4.5, $\|r\|_1\leq 1$. So $\|\omega\|_1=1$. Then ω is a Weierstrass polynomial of degree s and $\omega=qg$. It suffices to argue that q is a unit.

We may assume that $\|g\|_1=1$. By taking reductions, we find

$$\tilde{\omega}=\tilde{q}\tilde{g}.$$

As $\deg_{T_n}\tilde{g}=\deg_{T_n}\tilde{\omega}$ and the leading coefficients of both polynomials are units in $\tilde{k}[T_1,\ldots,T_{n-1}]$, it follows that $\tilde{q}$ is a unit in $\tilde{k}[T_1,\ldots,T_{n-1}]$. It follows that $\tilde{q}$ is also a unit in $\tilde{k}[T_1,\ldots,T_n]$. By Lemma 4.3, q is a unit in $k\{T_1,\ldots,T_n\}$.

The lsat assertion is already proved in Theorem 4.5.

□

Definition 4.10. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished. Then the Weierstrass polynomial ω constructed in Theorem 4.9 is called the *Weierstrass polynomial* defined by g.

Corollary 4.11. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished. Let ω be the Weierstrass polynomial of g. Then the injection

$$k\{T_1, \ldots, T_{n-1}\}[T_n] \to k\{T_1, \ldots, T_n\}$$

induces an isomorphism of k-algebras

$$k\{T_1, \ldots, T_{n-1}\}[T_n]/(\omega) \to k\{T_1, \ldots, T_n\}/(g).$$

Proof. The surjectivity follows from Theorem 4.5 and the injectivity follows from Lemma 4.6. □

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

Lemma 4.12. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ is non-zero. Then there is a k-algebra automorphism σ of $k\{T_1, \ldots, T_n\}$ so that $\sigma(g)$ is T_n-distinguished.

Proof. We may assume that $\|g\|_1 = 1$. We expand g as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow $\mathbb{N}^n$ with the lexicographic order. Take the maximal $\beta \in \mathbb{N}^n$ so that $|a_\beta| = 1$. Take $t \in \mathbb{Z}_{>0}$ so that $t \geq \max_{i=1,\ldots,n} \alpha_i$ for all $\alpha \in \mathbb{N}^n$ with $\tilde{a}_\alpha \neq 0$.

We will define σ by sending T_i to $T_i + T_n^{c_i}$ for all $i = 1, \ldots, n-1$. The c_i's are to be defined. We begin with $c_n = 1$ and define the other c_i's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for $j = 1, \ldots, n-1$. We claim that $\sigma(f)$ is T_n-distinguished of order $s = \sum_{i=1}^n c_i \beta_i$.

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^{s} p_i T_n^i$$

for some $p_i \in \tilde{k}[T_1, \ldots, T_{n-1}]$ and $p_s = \tilde{a_\beta}$. Our claim follows. □

Proposition 4.13. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Noetherian.

Proof. We make induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. It suffices to show that for any non-zero $g \in k\{T_1, \ldots, T_n\}$, $k\{T_1, \ldots, T_n\}/(g)$ is Noetherian. By Lemma 4.12, we may assume that g is T_n-distinguished. By Theorem 4.5, $k\{T_1, \ldots, T_n\}/(g)$ is a finite free $k\{T_1, \ldots, T_{n-1}\}$-module. By the inductive hypothesis and Hilbert basis theorem, $k\{T_1, \ldots, T_n\}/(g)$ is indeed Noetherian. □

Proposition 4.14. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Jacobson.

Proof. When $n = 0$, there is nothing to prove. We make induction on n and assume that $n > 0$. Let $\mathfrak{p}$ be a prime ideal in $k\{T_1, \ldots, T_n\}$, we want to show that the Jacobson radical of $\mathfrak{p}$ is equal to $\mathfrak{p}$.

We distinguish two cases. First we assume that $\mathfrak{p} \neq 0$. Let $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \dots, T_{n-1}\}$. By Lemma 4.12, we may assume that $\mathfrak{p}$ contains a Weierstrass polynomial ω. Observe that

$$k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' \to k\{T_1, \dots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any $b \in J(k\{T_1, \dots, T_n\}/\mathfrak{p})$ (where J denotes the Jacobson radical), we consider a monic integral equation of minimal degree over $k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'$:

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0, \quad a_i \in k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \dots, T_n\}/\mathfrak{p}) \cap k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that $n = 1$ and so $b = 0$. This proves $J(k\{T_1, \dots, T_n\}/\mathfrak{p}) = 0$.

On the other hand, let us consider the case $\mathfrak{p} = 0$. As $k\{T_1, \dots, T_n\}$ is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \dots, T_n\}) = 0.$$

Assume that there is a non-zero element f in $J(k\{T_1, \dots, T_n\})$. We may assume that $\|f\|_1 = 1$.

We claim that there is $c \in k$ with $|c| = 1$ such that $c + f$ is not a unit in $k\{T_1, \dots, T_n\}$. Assuming this claim for the moment, we can find a maximal ideal $\mathfrak{m}$ of $k\{T_1, \dots, T_n\}$ such that $c + f \in \mathfrak{m}$. But $f \in \mathfrak{m}$ by our assumption, so $c \in \mathfrak{m}$ as well. This contradicts the fact that $c \in k^\times$.

It remains to prove the claim. We treat two cases separately. When $|f(0)| < 1$, we simply take $c = 1$, which works thanks to Lemma 4.3. If $|f(0)| = 1$, we just take $c = -f(0)$. □

Proposition 4.15. Let $n \in \mathbb{N}$. Then $k\{T_1, \dots, T_n\}$ is UFD. In particular, $k\{T_1, \dots, T_n\}$ is normal.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 2.2, $k\{T_1, \dots, T_n\}$ is an integral domain. In order to see that $k\{T_1, \dots, T_n\}$ has the unique factorization property, we make induction on $n \geq 0$. When $n = 0$, there is nothing to prove. Assume that $n > 0$. Take a non-unit element $f \in k\{T_1, \dots, T_n\}$. By Theorem 4.9 and Lemma 4.12, we may assume that f is a Weierstrass polynomial. By inductive hypothesis, $k\{T_1, \dots, T_{n-1}\}$ is a UFD, hence so is $k\{T_1, \dots, T_{n-1}\}[T_n]$ by [Stacks, Tag 0BC1]. It follows that f can be decomposed into the products of monic prime elements $f_1, \dots, f_r \in k\{T_1, \dots, T_{n-1}\}[T_n]$, which are all Weierstrass polynomials by Lemma 4.8. Then by Corollary 4.11, we see that each f_i is prime in $k\{T_1, \dots, T_n\}$.

Any UFD is normal by [Stacks, Tag 0AFV]. □

5. Noetherian normalization and maximal modulus principle

Let $(k, |\bullet|)$ be a complete non-trivially valued non-Archimedean valued-field.

Theorem 5.1. Let A be a non-zero strictly k-affinoid algebra, $n \in \mathbb{N}$ and $\alpha : k\{T_1, \dots, T_n\} \to A$ be a finite (resp. integral) k-algebra homomorphism. Then up to replacing $T_1, \dots, T_n$ by an affinoid chart, we can guarantee that there exists $d \in \mathbb{N}$, $d \leq n$ such that α when restricted to $k\{T_1, \dots, T_d\}$ is finite (resp. integral) and injective.

PROOF. We make an induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. If $\ker \alpha = 0$, there is nothing to prove, so we may assume that $\ker \alpha \neq 0$. By Lemma 4.12 and Theorem 4.9, we may assume that there is a Weierstrass polynomial $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ in $\ker \alpha$. Then α induces a finite (resp. integral) homomorphism $\beta : k\{T_1, \ldots, T_n\}/(\omega) \to A$. By Theorem 4.5, $k\{T_1, \ldots, T_{n-1}\} \to k\{T_1, \ldots, T_n\}/(\omega)$ is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism $k\{T_1, \ldots, T_{n-1}\} \to A$. We can apply the inductive hypothesis to conclude. $\square$

Corollary 5.2. Let A be a non-zero strictly k-affinoid algebra, then there is $d \in \mathbb{N}$ and a finite injective k-algebra homomorphism: $k\{T_1, \ldots, T_d\} \to A$.

PROOF. Take some $n \in \mathbb{N}$ and a surjective k-algebra homomorphism $k\{T_1, \ldots, T_n\} \to A$ and apply Theorem 5.1, we conclude. $\square$

Corollary 5.3. Let A be a strictly k-affinoid algebra and I be an ideal in A such that $\sqrt{I}$ is a maximal ideal in A, then A/I is finite-dimensional over k.

In particular, $\operatorname{Spm} A = \operatorname{Spm}_k A$.

PROOF. By Corollary 5.2, there is $d \in \mathbb{N}$ and a finite monomorphism $f : k\{T_1, \ldots, T_d\} \to A/I$. It suffices to show that $d = 0$. Observe that the composition

$$k\{T_1, \ldots, T_d\} \xrightarrow{f} A/I \to A/\sqrt{I}$$

is finite and injective as $k\{T_1, \ldots, T_d\}$ is an integral domain, so $k\{T_1, \ldots, T_d\}$ is a field. This is possible only when $d = 0$. $\square$

Definition 5.4. For any non-Archimedean valuation field $(K, |\bullet|)$ and $n \in \mathbb{N}$, we define the *n-dimensional polydisk* with value in K:

$$B^n(K) := \left\{ (x_1, \ldots, x_n) \in K^n : \max_{i=1,\ldots,n} |x_i| \leq 1 \right\}.$$

Definition 5.5. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ as sending $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

Lemma 5.6. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, then $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ is continuous and for any $x \in B^n(k^{\mathrm{alg}})$,

$$|f(x)| \leq \|f\|_1.$$

There is $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ such that $|f(x)| = \|f\|_1$.

PROOF. To see that f is continuous, it suffices to observe that f is a uniform limit of polynomials. For any $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$, we have

$$|f(x)| = \left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \right| \leq \max_{\alpha \in \mathbb{N}^n} |a_\alpha x^\alpha| \leq \|f\|_1.$$

To prove the last assertion, we may assume that $\|f\|_1 = 1$. As the residue field of k^{alg} is equal to $\tilde{k}^{\mathrm{alg}}$, it has infinitely many elements, so there is a point $x \in B^n(k^{\mathrm{alg}})$ such that $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$. In other words, $\|f(x)\|_1 = 1$. $\square$

Proposition 5.7. Let $n \in \mathbb{N}$, then the maximal modulus principle holds for $k\{T_1, \ldots, T_n\}$. Moreover, for any $f \in k\{T_1, \ldots, T_n\}$, $\|f\|_1 = |f|_{\sup}$.

PROOF. By Lemma 6.3 in the chapter Banach Rings, we have

$$\|f\|_1 \geq |f|_{\sup}$$

for any $f \in A$. We only have to show that for any $f \in k\{T_1, \ldots, T_n\}$ there is a maximal ideal $\mathfrak{m} \subseteq k\{T_1, \ldots, T_n\}$ such that $|f(\mathfrak{m})| = \|f\|_1$.

By Lemma 5.6 we can take $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ such that $|f(x)| = \|f\|_1$. Let L be the field extension of k generated by $x_1, \ldots, x_n$, then L/k is finite. Then we can define a homomorphism

$$\mathrm{ev}_x : k\{T_1, \ldots, T_n\} \to L$$

sending $g \in k\{T_1, \ldots, T_n\}$ to $g(x)$. Observe that the image is indeed in L. Clearly ev_x is surjective. So $\mathfrak{m}_x := \ker \mathrm{ev}_x$ is a k-algebraic maximal ideal in $k\{T_1, \ldots, T_n\}$. Then

$$|f(\mathfrak{m}_x)| = |f(x)| = \|f\|_1.$$

$\square$

Corollary 5.8. Let A be a strictly k-affinoid algebra. Then for any $f \in A$,

$$|f|_{\sup} \subseteq \sqrt{|k^\times|} \cup \{0\}.$$

PROOF. We may assume that $A \neq 0$. By Corollary 5.2 and Proposition 8.11 in the chapter Banach Rings, we may assume that $A = k\{T_1, \ldots, T_n\}$ for some $n \in \mathbb{N}$. The result then follows from Proposition 5.7. $\square$

Corollary 5.9. Maximal modulus principle holds for any strictly k-affinoid algebras.

PROOF. This follows from Corollary 5.2, Proposition 8.11 in the chapter Banach Rings and Proposition 5.7. $\square$

6. Properties of affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $R_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Proposition 6.1. Assume that k is non-trivially valued. Let A be a strictly k-afifnoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\sup} \leq 1\}.$$

PROOF. By Lemma 6.3, we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\sup} \leq 1\}.$$

Conversely, let $f \in A$, $|f|_{\sup} \leq 1$. Choose $d \in \mathbb{N}$ and a surjective k-algebra homomorphism

$$\varphi : k\{T_1, \ldots, T_d\} \to A.$$

Let $f^n+t_1f^{n-1}+\cdots+t_n=0$ be the minimal equation of f over $k\{T_1,\ldots,T_d\}$. Then $t_i\in(k\{T_1,\ldots,T_d\})^\circ$ by Proposition 8.11 in the chapter Banach Rings. An induction on $i\geq 0$ shows that

$$f^{n+i}\in\sum_{j=0}^{n-1}\varphi\left((k\{T_1,\ldots,T_d\})^\circ\right)f^j.$$

The right-hand side is clearly bounded. □

Corollary 6.2. Assume that k is non-trivially valued. Let $(A,\|\bullet\|)$ be a strictly k-affinoid algebra. For any $f\in A$,

$$\rho(f)=|f|_{\sup}.$$

Proof. We have shown that $\rho(f)\geq|f|_{\sup}$ in Lemma 6.3 from the chapter Banach Rings. Assume tha tthe inverse inequality fails: for some $f\in A$,

$$\rho(f)>|f|_{\sup}.$$

If $|f|_{\sup}=0$, then f lies in the Jacobson radical of A, which is equal to the nilradial of A by Proposition 4.14. But then $\rho(f)=0$ as well. We may therefore assume that $|f|_{\sup}\neq 0$. By Corollary 5.8, we may assume that $|f|_{\sup}=1$ as ρ is power-multiplicative. Then $\rho(f)>1$. This contradicts Proposition 6.1. □

Theorem 6.3. A k-affinoid algebra A is Noetherian and all ideals of A are closed.

Proof. Let I be an ideal in A. By Proposition 3.14, we can take a suitable $r\in\mathbb{R}^m_{>0}$ so that $A\hat{\otimes}k_r$ is strictly k_r-affinoid. Then $I(A\hat{\otimes}k_r)$ is an ideal in $A\hat{\otimes}k_r$. By Proposition 4.13, the latter ring is Noetherian. So we may take finitely many generators $f_1,\ldots,f_k\in I$. Each $f\in I$ can be written as

$$f=\sum_{i=1}^k f_ig_i$$

with $g_i=\sum_{j=-\infty}^{\infty}g_{i,j}T^j\in A\hat{\otimes}k_r$. But then

$$f=\sum_{i=1}^k f_ig_{i,0}.$$

So I is finitely generated.

As $I=A\cap(I(A\hat{\otimes}k_r))$, by Corollary 7.4 in the chapter Banach Rings, we see that I is closed in $A\hat{\otimes}k_r$ and hence closed in A. □

Proposition 6.4. Let $(A,\|\bullet\|)$ be a k-affinoid algebra and $f\in A$. Then there is $C>0$ and $N\geq 1$ such that for any $n\geq N$, we have

$$\|f^n\|\leq C\rho(f)^n.$$

Recall that ρ is the spectral radius map defined in Definition 4.9 in the chatper Banach Rings.

Proof. By Proposition 3.9, we may assume that k is non-trivially valued and k is non-trivially valued.

If $\rho(f)=0$, then f lies in each maximal ideal of A. To see this, we may assume that A is a field, then by Proposition 6.10 in the chapter Banach Rings, there is a bounded valuation $\|\bullet\|'$ on A. But then $\rho(f)=0$ implies that $\|f\|'=0$ and hence $f=0$.

It follows that if $\rho(f) = 0$ then f lies in $J(A)$, the Jacobson radical of A. By Proposition 4.14, A is a Jacobson ring. So f is nilpotent. The assertion follows.

So we can assume that $\rho(f) > 0$. In this case, by Corollary 5.2 and Proposition 8.11 in the chapter Banach Rings, we have $\rho(f) \in \sqrt{|k^\times|}$. Take $a \in k^\times$ and $d \in \mathbb{Z}_{>0}$ so that $\rho(f)^d = |a|$. Then $\rho(f^d/a) = 1$ and hence it is powerly-bounded by Proposition 6.1. It follows that there is $C > 0$ so that for $n \geq 1$,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that $\|f^n\| \leq C\rho(f)$ for $n \geq d$ as long as we enlarge C. □

Corollary 6.5. Let $\varphi : A \to B$ be a bounded homomorphism of k-affinoid algebras. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n \in B$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$ for $i = 1, \dots, n$. Write $r = (r_1, \dots, r_n)$, then there is a unique bounded homomorphism $\Phi : A\{r^{-1}T\} \to B$ extending φ and sending T_i to f_i.

PROOF. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from Proposition 6.4 that the right-hand side the series converges. The boundedness of Φ is obvious. □

Definition 6.6. Let A be an affinoid algebra, $f \in A$ is a non-zero element and $r \in \mathbb{R}_{>0}$, we define the *localization* $A\{rf^{-1}\}$ of A at $r^{-1}f$ as follows:

$$A\{rf^{-1}\} := A\{rT\}/(Tf - 1).$$

Observe that $A\{rf^{-1}\}$ is k-affinoid by Theorem 6.3.

Proposition 6.7. Let A be an affinoid algebra, $f \in A$ is a non-zero element and $r \in \mathbb{R}_{>0}$. Consider the natural map $\iota : A \to A\{rf^{-1}\}$, then $\operatorname{Sp}\iota : \operatorname{Sp} A\{rf^{-1}\} \to \operatorname{Sp} A$ is injective. We will identify $\operatorname{Sp} A\{rf^{-1}\}$ with a subset of $\operatorname{Sp} A$. Then

$$\operatorname{Sp} A\{rf^{-1}\} = \{x \in \operatorname{Sp} A : |f(x)| \geq r\}.$$

For any $x \in \operatorname{Sp} A\{rf^{-1}\}$, we have

$$|f(x)| \geq r.$$

PROOF. The first assertion means that each bounded semi-valuation on A admits at most one bounded extension to $A\{r^{-1}T\}$. This is obvious as the image of A in $A\{r^{-1}T\}$ is dense.

For the second statement, let $\|\bullet\|_x$ be the bounded semi-norm on $A\{r^{-1}T\}$ corresponding to x. We need to show that

$$\|f\|_x \geq r.$$

We know that

$$\|T\|_{r^{-1}} = r^{-1}$$

so

$$\|T\|_x \leq r^{-1}.$$

From $Tf = 1$, we find

$$1 \leq \|f\|_x \cdot \|T\|_x \leq r^{-1}\|f\|_x.$$

Conversely, let $x \in \operatorname{Sp} A$ with $|f(x)| \geq r$. Let $\|\bullet\|_x$ be the bounded semi-valuation on A corresponding to x. We can extend $\|\bullet\|_x$ to a semi-valuation $\|\bullet\|'_x$ on by **??** in the chapter Banach Rings. The assumption $|f(x)| \geq r$ guarantees exactly that $\|\bullet\|'_x$ is bounded. □

Proposition 6.8. Let $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$ be k-affinoid algebras, $r \in \mathbb{R}^n_{>0}$ and $\varphi : A\{r^{-1}T\} \to B$ be an admissible epimorphism. Write $f_i = \varphi(T_i)$ for $i = 1, \dots, n$. Then there is $\epsilon > 0$ such that for any $g = (g_1, \dots, g_n) \in B^n$ with $\|f_i - g_i\|_B < \epsilon$ for all $i = 1, \dots, n$, there exists a unique bounded k-algebra homomorphism $\psi : A\{r^{-1}T\} \to B$ that coincides with φ on A and sends T_i to g_i. Moreover, ψ is also an admissible epimorphism.

Proof. The uniqueness of ψ is obvious. We prove the remaining assertions. Taking $\epsilon > 0$ small enough, we could further guarantee that $\rho(g_i) \leq r_i$. It follows from Corollary 6.5 that there exists a bounded homomorphism ψ as in the statement of the proposition.

As φ is an admissible epimorphism, we may assume that $\|\bullet\|_B$ is the residue induced by $\|\bullet\|_r$ on $A\{r^{-1}T\}$.

By definition of the residue norm, for any $\delta > 0$ and any $h \in B$, we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1+\delta)\|h\|_B$$

for any $\alpha \in \mathbb{N}^n$. Choose $\epsilon \in (0, (1+\delta)^{-1})$. Now for $g_1, \dots, g_n$ as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1+\delta)\epsilon \|h\|_B.$$

Repeating this procedure, we can construct $k_i \in A\{r^{-1}T\}$ for $i \in \mathbb{N}$ and $h_j \in B$ for $j \in \mathbb{Z}_{>0}$ such that for any $i \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} h =& \psi(k_0 + \dots + k_{i-1}) + h_i, \\ \|k_i\|_r \leq& ((1+\delta)\epsilon)^i (1+\delta)\|h\|_B, \\ \|h_i\|_B \leq& ((1+\delta)\epsilon)^i \|h\|_B. \end{aligned}$$

In particular, $k := \sum_{i=0}^\infty k_i$ converges in $A\{r^{-1}T\}$ and

$$\|k\|_r \leq (1+\delta)\|h\|_B.$$

It follows that ψ is an admissible epimorphism. □

Corollary 6.9. Let A be a Banach k-algebra, $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k-free polyray. Assume that $A\hat{\otimes}_k k_r$ is k_r-affinoid, then A is k-affinoid.

If $A\hat{\otimes}_k k_r$ is k_H-affinoid and $r \in H$, then A is also k_H-affinoid.

Proof. We may assume that r has only one component.

Take $m \in \mathbb{N}$, $p_1, \dots, p_m \in \mathbb{R}_{>0}$ and an admissible epimorphism

$$\pi : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \to A\hat{\otimes}_k k_r.$$

Let

$$\pi(S_i) = \sum_{j=-\infty}^{\infty} a_{i,j}T^j, \quad a_{i,j} \in A$$

for $i = 1, \dots, m$. By Proposition 6.8, we may assume that there is a large integer l such that $a_{i,j} = 0$ for $|j| > l$ and for any $i = 1, \dots, m$. We define $B = k\{p_i^{-1}r^jT_{i,j}\}$, $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$. Let $\varphi : B \to A$ be the bounded k-algebra homomorphism sending $T_{i,j}$ to $a_{i,j}$. The existence of φ is guaranteed by Corollary 6.5.

We claim that φ is an admissible epimorphism. It is clearly an epimorphism. Let us show that φ is admissible. Let $\eta : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \to B\hat{\otimes}_k k_r$ be the bounded homomorphism sending S_i to $\sum_{j=-l}^{l} T_{i,j}T^j$, then we have the following commutative diagram

$$\begin{tikzcd} k_r\{p^{-1}S\} \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\ B\hat{\otimes}_k k_r \arrow[r, "\varphi\hat{\otimes}_k k_r"] & A\hat{\otimes}_k k_r \end{tikzcd}$$

It follows that $\varphi\hat{\otimes}_k k_r$ is also an admissible epimorphism. By Proposition 3.9, φ is also admissible. □

7. H-strict affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $R_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

We next give a non-strict extension of Proposition 3.11.

Proposition 7.1. Let B be a k_H-affinoid algebra and $\varphi : B \to A$ be a finite bounded homomorphism into a k-Banach algebra A. Then A is also k_H-affinoid.

Proof. We first assume that k is non-trivially valued.

We may assume that $B = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ for some $n \in \mathbb{N}$ and $r_1, \dots, r_n \in H$. By assumption, we can find finitely many $a_1, \dots, a_m \in A$ such that $A = \sum_{i=1}^{m} \varphi(B)a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.17 in the chapter Banach Rings, φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S_1, \dots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is k_H-affinoid.

If k is trivially valued, then H is non-trivial. Take $s \in H \setminus \{1\}$. It follows from the previous case applied to $\varphi\hat{\otimes}k_s : B\hat{\otimes}k_s \to A\hat{\otimes}k_s$ that $A\hat{\otimes}k_s$ is k_H-affinoid. By Corollary 6.9, A is also k_H-affinoid. □

Proposition 7.2. Let A be a Banach k-algebra. Then the following are equivalent:

(1) A is k_H-affinoid;

(2) there are $n \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

Proof. The non-trivial direction is (2). Assume (2). Take $s_1, \dots, s_n \in \mathbb{Z}_{>0}$, $c_1, \dots, c_n \in k^\times$ and $h_1, \dots, h_n \in H$ such that

$$r_i^{s_i} = |c_i^{-1}| h_i$$

for $i = 1, \dots, n$. We define a bounded k-algebra homomorphism

$$\varphi : k\{h_1^{-1}T_1, \dots, h_n^{-1}T_n\} \to k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

by sending T_i to $c_i T_i^{s_i}$. The existence of such a homomorphism is guaranteed by Corollary 6.5. The same proof of Lemma 3.12 shows that φ is finite. By Proposition 7.1, $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is k_H-affinoid. □

Lemma 7.3. Assume that k is non-trivially valued. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is strictly k-affinoid;
(2) for any $a \in A$, $\rho(a) \in \sqrt{|k^\times|} \cup \{0\}$.

Proof. (1) $\implies$ (2) by Corollary 5.8 and Corollary 6.2.

(2) $\implies$ (1): Take $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \to A.$$

Let $f_i = \varphi(T_i)$ for $i = 1, \dots, n$. Suppose $r_1, \dots, r_m \notin \sqrt{|k^\times|}$ and $r_{m+1}, \dots, r_n \in \sqrt{|k^\times|}$. Then $\rho(f_i) < r_i$ for $i = 1, \dots, m$ and we can choose $r'_1, \dots, r'_m \in \sqrt{|k^\times|}$ such that

$$\rho(f_i) \leq r'_i < r_i$$

for $i = 1, \dots, m$. Set $r'_i = r_i$ when $i = m+1, \dots, n$. We can then define a bounded k-algebra homomorphism $\psi : k\{r'^{-1}T\} \to A$ sending T_i to f_i for $i = 1, \dots, n$. The existence of ψ is guaranteed by Corollary 6.5. Observe that ψ is surjective and admissible. It follows that A is strictly k-affinoid. □

Theorem 7.4. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is k_H-affinoid;
(2) A is $k_{\sqrt{|k^\times| \cdot H}}$-affinoid;
(3) For any non-zero $a \in A$, $\rho(a) \in \sqrt{|k^\times| \cdot H} \cup \{0\}$.

Proof. The equivalence between (1) and (2) follows from Proposition 7.2.

(1) $\implies$ (3): we may assume that $H \supseteq |k^\times|$. Take $n \in \mathbb{N}$, $r = (r_1, \dots, r_n) \in H^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \to A.$$

Take a k-free polyray s with at least one component so that $|k_s| \supseteq \{r_1, \dots, r_n\}$. We can apply Lemma 7.3 to $\varphi \hat{\otimes}_k k_s$, it follows that $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$.

(3) $\implies$ (2): we may assume that $H \supseteq |k^\times|$. It suffices to apply the same argument as (2) $\implies$ (1) in the proof of Lemma 7.3.

□

8. Finite modules over affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

For any k-affinoid algebra A, we have defined the category $\mathcal{B}\mathrm{an}^f_A$ of finite Banach A-modules in Definition 5.3 in the chapter Banach Rings. We write $\mathcal{M}\mathrm{od}^f_A$ for the category of finite A-modules.

Lemma 8.1. Let A be a k-affinoid algebra, $(M, \|\bullet\|_M)$ be a finite Banach A-module and $(N, \|\bullet\|_N)$ be a Banach A-module N. Let $\varphi : M \to N$ be an A-linear homomorphism. Then φ is bounded.

Proof. Take $n \in \mathbb{N}$ such that there is an admissible epimorphism

$$\pi : A^n \to M.$$

It suffices to show that $\varphi \circ \pi$ is bounded. So we may assume that $M = A^n$. For $i = 1, \dots, n$, let e_i be the vector with $(0, \dots, 0, 1, 0, \dots, 0)$ of A^n with 1 placed at the i-th place. Set $C = \max_{i=1,\dots,n} \|\varphi(e_i)\|_N$. For a general $f = \sum_{i=1}^n a_i e_i$ with $a_i \in A$, we have

$$\|\varphi(f)\|_N \leq C\|f\|_M.$$

So φ is bounded. □

Proposition 8.2. Let A be a k-affinoid algebra. The forgetful functor $\mathcal{B}\mathrm{an}^f_A \to \mathcal{M}\mathrm{od}^f_A$ is an equivalence of categories.

Proof. It suffices to construct the inverse functor. Let M be a finite A-module. Choose $n \in \mathbb{N}$ and an A-linear epimorphism $\pi : A^n \to M$. By Theorem 6.3, $\ker \pi$ is closed in A^n. We can endow M with the residue norm. By Lemma 8.1, the equivalence class of the norm does not depend on the choice of π.

For any A-linear homomorphism $f : M \to N$ of finite A-modules, we endow M and N with the Banach structures as above. It follows from Lemma 8.1 that f is bounded. We have defined the inverse functor of the forgetful functor $\mathcal{B}\mathrm{an}^f_A \to \mathcal{M}\mathrm{od}^f_A$. □

Remark 8.3. Let A be a k-affinoid algebra. It is not true that a Banach A-module which is finite as A-module is finite as Banach A-module.

As an example, take $0 < p < q < 1$ and $A = k\{q^{-1}T\}$, $B = k\{p^{-1}T\}$. Then B is a Banach A-module. By Example 2.4, the underlying rings of A and B are both $k[[T]]$. So the canonical map $A \to B$ is bijective. But B is not a finite A-module. As otherwise, the inverse map $B \to A$ is bounded by Lemma 8.1, which is not the case.

The correct statement is the following: consider a Banach A-module $(M, \|\bullet\|_M)$ which is finite as A-module, then there is a norm on M such that M becomes a finite Banach A-module. The new norm is not necessarily equivalent to the given norm $\|\bullet\|_M$.

Proposition 8.4. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Then the natural map

$$M \otimes_A N \to M \hat{\otimes}_A N$$

is an isomorphism of Banach A-modules and $M \hat{\otimes}_A N$ is a finite Banach A-module.

Here the Banach A-module structure on $M \otimes_A N$ is given by Proposition 8.2.

Proof. Choose $m, m' \in \mathbb{N}$ an admissibly coexact sequence

$$A^{m'} \to A^m \to M \to 0$$

of Banach A-modules. Then we have a commutative diagram of A-modules:

$$\begin{array}{ccccccc} A^{m'} \otimes_A N & \longrightarrow & A^m \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{m'} \hat{\otimes}_A N & \longrightarrow & A^m \hat{\otimes}_A N & \longrightarrow & M \hat{\otimes}_A N & \longrightarrow & 0 \end{array}$$

with exact rows. By 5-lemma, in order to prove $M \otimes_A N \xrightarrow{\sim} M\hat{\otimes}_A N$ and $M\hat{\otimes}_A N$ is a finite Banach A-module, we may assume that $M = A^m$ for some $m \in \mathbb{N}$. Similarly, we can assume $N = A^n$ for some $n \in \mathbb{N}$. In this case, the isomorphism is immediate and $M\hat{\otimes}_A N$ is clearly a finite Banach A-module. By Lemma 8.1, the Banach A-module structure on $M\hat{\otimes}_A N$ coincides with the Banach A-module strucutre on $M \otimes_A N$ induced by Proposition 8.2. □

Proposition 8.5. Let A, B be a k-affinoid algebra and $A \to B$ be a bounded k-algebra homomorphism. Let M be a finite Banach A-module, then the natural map

$$M \otimes_A B \to M\hat{\otimes}_A B$$

is an isomorphism of Banach B-modules and $M\hat{\otimes}_A B$ is a finite Banach B-module.

Proof. By the same argument as Proposition 8.4, we may assume that $M = A^n$ for some $n \in \mathbb{N}$. In this case, the assertions are trivial. □

Proposition 8.6. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Let $\varphi : M \to N$ be an A-linear map. Then φ is admissible.

Proof. By Lemma 8.1, φ is always bounded. By Proposition 8.5 and Proposition 3.9, we may assume that k is non-trivially valued. By Theorem 6.3, N is a Noetherian A-module. It follows from Corollary 7.4 in the chapter Banach Rings that $\operatorname{Im}\varphi$ is closed in N and is finite as an A module. In particular, the norm induced from N and from M are equivalent by Lemma 8.1. It follows that φ is admissible. □

Proposition 8.7. Let A be a k-affinoid algebra. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k-free polyray. Then M is a finite Banach A-module if and only if $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module.

Proof. We may assume that r has only one component and write $r_1 = r$. The direct implication is trivial. Let us assume that $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module. Take $n \in \mathbb{N}$ and an admissible epimorphism of $A\hat{\otimes}_k k_r$-modules

$$\varphi : (A\hat{\otimes}_k k_r)^n \to M\hat{\otimes}_k k_r.$$

Let $e_1, \dots, e_n$ denotes the standard basis of $(A\hat{\otimes}_k k_r)^n$. We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By Proposition 6.8, we can assume that there is $l > 0$ such that $m_{i,j} = 0$ for all $i = 1, \dots, n$ and $|j| > l$. It follows that

$$A^{n(2l+1)} \to M$$

sending the standard basis to $m_{i,j}$ with $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$ is an admissible epimorphism. $\square$

For any ring A, $\mathcal{A}\mathrm{lg}^f_A$ denotes the category of finitely generated A-algebras.

Proposition 8.8. Let A be a k-affinoid algebra. Then the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A \to \mathcal{A}\mathrm{lg}^f_A$ is an equivalence of categories.

Recall that $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A$ is defined in Definition 5.9 in the chapter Banach Rings.

Proof. It suffices to construct an inverse functor. Let B be a finite A-algebra. We endow B with the norm $\| \bullet \|_B$ as in Proposition 8.2. We claim that B is a Banach A-algebra.

Let us recall the definition of the norm. Take $n \in \mathbb{N}$ an epimorphism $\varphi : A^n \to B$ of A-modules. Then $\| \bullet \|_B$ is the residue norm induced by φ.

Consider the A-linear epimorphism $\psi : A^n \otimes_A A^n \to B \otimes_A B$. By Proposition 8.6, when both sides are endowed with the norms $\| \bullet \|_{A^n \otimes_A A^n}$ and $\| \bullet \|_{B \otimes_A B}$ as in Proposition 8.2, ψ is admissible. It follows that there is $C > 0$ such that for any $f, g \in B$,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by Proposition 8.2, the natural map $B \otimes_A B \to B$ is bounded. It follows that there is a constant $C' > 0$ such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in B is bounded and hence B is a finite Banach algebra. Given any morphism $B \to B'$ in $\mathcal{A}\mathrm{lg}^f_A$, we endow B and B' with the norms given by Proposition 8.2. It follows from Lemma 8.1 that $B \to B'$ is a bounded homomorphism of finite Banach A-algebras. So we have defined an inverse functor to the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A \to \mathcal{A}\mathrm{lg}^f_A$. $\square$

Remark 8.9. It is not true that any homomorphism of k-affinoid algebras is bounded. For example, if the valuation on k is trivial. Take $0 < p < q < 1$ and consider the natural homomorphism $k_p \to k_q$. This homomorphism is bijective but not bounded.

9. Graded reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 9.1. Let A be a k_H-affinoid algebra. We define the *k_H-graded reduction* of A as the $\sqrt{|k^\times| \cdot H}$-graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times| \cdot H}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define $\tilde{f}$ as the image of f in the $\rho(f)$-graded piece of $\tilde{A}^H$.

For any morphism $f : A \to B$ of k_H-affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \to \tilde{B}^H$$

as the map induced by sending the class of $x \in A$ with $\rho(x) \leq h$ for any $h \in \sqrt{|k^\times| \cdot H}$ to the class of $f(x) \in B$.

Recall that $\rho(A) = \sqrt{|k^\times| \cdot H} \cup \{0\}$ by Theorem 7.4, so $\tilde{f}$ is well-defined.

Example 9.2. If K is a k_H-affinoid algebra which is a field as well, then $\tilde{K}^H$ is a $\sqrt{|k^\times| \cdot H}$-graded field. This is immediate from the definition.

Lemma 9.3. Let $(A, \|\bullet\|)$ be a k-affinoid algebra, $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Let $f \in k\{r^{-1}T\}$. Expand f as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha.$$

Proof. By induction, we may assume that $n = 1$ and write $r = r_1$. As ρ is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j)\rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j.$$

Observe that $\rho(a_j)$ is not ambiguous: when intepreted as in A and in $A\{r^{-1}T\}$, it has the same value.

Conversely, we need to show that for any $j \in \mathbb{N}$,

$$\rho(f) \geq \rho(a_j) r^j.$$

Equivalently, this means for any $k \in \mathbb{Z}_{>0}$ and any $j \in \mathbb{N}$, we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk}.$$

Fix j and k as above. We compute the left-hand side:

$$f^k = \sum_{\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^k a_{\beta_l}.$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|}.$$

Take $\beta = (j, j, \ldots, j)$, we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk}.$$

□

Lemma 9.4. Assume that k is non-trivially valued. Let A be a strictly k-affinoid algebra. Then for any $a, f \in A$, the set of non-zero values $\rho(f^n a)$ for $n \in \mathbb{N}$ is a discrete subset of $\mathbb{R}_{>0}$.

Proof. As A is noetherian Theorem 6.3, it has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. It follows that

$$\operatorname{Sp} A = \bigcup_{i=1}^m \operatorname{Sp} A/\mathfrak{p}_i.$$

Here we make the obvious identification by identifying $\operatorname{Sp} A/\mathfrak{p}_i$ with a subset of $\operatorname{Sp} A$.

By Corollary 6.12 in the chapter Banach Rings, it suffices to consider each of $\operatorname{Sp} A/\mathfrak{p}_i$ separately, so we may assume that A is an integral domain.

By Theorem 5.1, we can take $d \in \mathbb{N}$ and a finite injective homomorphism of k-algebras $\iota : k\{T_1, \ldots, T_d\} \to A$. According to Proposition 8.11 in the chapter

Banach Rings, ρ_A is the restriction of the norm $\|\bullet\|_{\operatorname{Frac} A}$ on $\operatorname{Frac} A$ induced by the finite extension $\operatorname{Frac} A/\operatorname{Frac} k\{T_1,\dots,T_d\}$ from the Gauss valuation. But it is well-known that $\|\bullet\|_{\operatorname{Frac} A}$ is the maximum of finitely many valuations on $\operatorname{Frac} A$. Reproduce BGR3.3.3.1 somewhere. The assertion is by now obvious. □

Lemma 9.5. Let $(A,\|\bullet\|)$ be a k-affinoid algebra, $f\in A$ with $r=\rho(f)>0$. Let $B=A\{r^{-1}f\}$. Then for any $a\in A$, we have

$$\rho_B(a)=\lim_{n\to\infty} r^{-n}\rho_A(f^n a).$$

If moreover, $\rho_B(a)>0$, then there is $n_0>0$ such that for $n\geq n_0$,

$$\rho_B(a)=r^{-n}\rho_A(f^n a),\quad \rho_B(f^n a)=r^{-n}\rho_A(a).$$

Proof. We observe that for any $a\in A$, $n\in\mathbb{Z}_{>0}$, we have

$$\rho_B(f^n a)=r^n\rho_B(a).$$

So the last two assertions are equivalent.

Take a k-free polyradius s such that $A\hat{\otimes}_k k_s$ and $B\hat{\otimes}_k k_s$ are both strictly k_s-affinoid. By Proposition 3.9, $A\hat{\otimes}_k k_s\{r^{-1}f\}\xrightarrow{\sim} B\hat{\otimes}_k k_s$. Moreover, ρ_A and ρ_B are both preserved after base change to k_s. So we may assume that k is non-trivially valued and A and B are strictly k-affinoid.

Observe that for $n\in\mathbb{Z}_{>0}$,

$$\rho_A(f^{n+1}a)\leq\rho_A(f)\rho_A(f^n a)=r\rho_A(f^n a).$$

So $r^{-n}\rho_A(f^n a)$ is decreasing in n. Moreover, for any $x\in\operatorname{Sp} A\{r^{-1}f\}$, by Proposition 6.7, we have

$$|f(x)|\geq r.$$

By Corollary 6.12 in the chapter Banach Rings, we have

$$|f(x)|=r$$

for any $x\in\operatorname{Sp} A\{r^{-1}f\}$. It follows from Corollary 6.12 in the chapter Banach Rings that for any $n\in\mathbb{Z}_{>0}$,

$$\rho_A(f^n a)=\sup_{x\in\operatorname{Sp} A}|f^n a(x)|\geq r^n\sup_{x\in\operatorname{Sp} A\{rf^{-1}\}}|a(x)|=r^n\rho_B(a).$$

By Lemma 9.4, the decreasing sequence $\{r^{-n}\rho_A(f^n a)\}_n$ either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is $\alpha\in\mathbb{R}_{>0}$ and $n_0>0$ such that for $n\geq n_0$, we have

$$r^{-n}\rho_A(f^n a)=\alpha.$$

We have to show that $\alpha\leq\rho_B(a)$. Assume the contrary $\alpha>\rho_B(a)$. Then for all $x\in\operatorname{Sp} A$, we have

$$|f^n a(x)|\leq r^n|a(x)|.$$

So $f^n a$ must obtain its maximum on $U:=\{x\in\operatorname{Sp} A:|a(x)|\geq\alpha\}$. But U is disjoint from $\operatorname{Sp} A\{r^{-1}f\}$ as

$$\alpha>\rho_B(a).$$

It follows from Proposition 6.7 that

$$\beta:=\sup_{x\in U}|f(x)|=\max_{x\in U}|f(x)|<r.$$

So

$$\rho(f^n a)=\sup_{x\in\operatorname{Sp} A}|f^n a(x)|=\sup_{x\in U}|f^n a(x)|\leq\beta^n\sup_{x\in U}|a(x)|.$$

This contradicts the fact that $\alpha > 0$. $\square$

Proposition 9.6. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that k_r is defined in Example 3.10.

Proof. By Lemma 9.3, we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}^H_s \xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{N}^n} \tilde{A}^H_{sr^{-\alpha}}$$

for any $s \in \sqrt{|k^\times| \cdot H}$. This establishes the desired isomorphism. $\square$

Proposition 9.7. Let A be a k_H-affinoid algebra and $f \in A$ with $r = \rho(f) > 0$. Then there is a natural isomorphism

$$\tilde{A}^H_{\tilde{f}} \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that $A\{rf^{-1}\}$ is defined in Definition 6.6, by Theorem 7.4, it is k_H-affinoid.

Proof. $\square$

Corollary 9.8. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H \cong \widetilde{A\hat{\otimes}_k k_r}^H .$$

10. Affinoid domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H \supseteq |k^\times|$ be a subgroup of $R_{>0}$.

Definition 10.1. Let A be a k_H-affinoid algebra. A subset $V \subseteq \operatorname{Sp} A$ is said to be a k_H*-affinoid domain* in X if there is a bounded homomorphism of k_H-affinoid algebras $\varphi : A \to A_V$ satisfying

(1) $\operatorname{Im} \operatorname{Sp} \varphi = V$;
(2) given a bounded homomorphism of k_H-affinoid algebras $\psi : A \to B$ such that $\operatorname{Sp} \psi : \operatorname{Sp} B \to \operatorname{Sp} A$ factorizes through V, there is a unique bounded homomorphism $A_V \to B$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\psi"] & A_V \arrow[dl] \\ B & \end{tikzcd} .$$

We say V is *represented by* the morphism φ.

When $k_H = \mathbb{R}_{>0}$, we say V is a k*-affinoid domain* in X. When $k_H = |k^\times|$, we say V is a *strict* k*-affinoid domain* in X.

Remark 10.2. This definition differs from the original definition of [Ber12], we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the orignal definition of Berkovich when $H = \mathbb{R}_{>0}$.

Proposition 10.3. Let A be a k_H-affinoid algebra and $V \subseteq \operatorname{Sp} A$ be a k_H-affinoid domain represented by $\varphi : A \to A_V$. Then $\operatorname{Sp}\varphi$ induces a bijection $\operatorname{Sp} A_V \to \operatorname{Sp} A$.

Proof. We observe that $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a monomorphism in the category k_H-$\mathcal{A}$ff. In other words, $A \to A_V$ is an epimorhism in the category k_H-$\mathcal{A}$ff$\mathcal{A}$lg. To see this, let $\eta_1, \eta_2 : A_V \to B$ be two arrows in k_H-$\mathcal{A}$ff$\mathcal{A}$lg such that $\eta_1 \circ \varphi = \eta_2 \circ \varphi$. It follows from the universal property in Definition 10.1 that $\eta_1 = \eta_2$. We claim that $\operatorname{Sp} A_V \to V$ is a bijection. □

It is not immediately clear that A_V is canonically assocaited with V. We will prove this now.

Proposition 10.4. Let A be a k_H-affinoid algebra and V be an affinoid domain in X represented by $\varphi : A \to A_V$. Then $\operatorname{Sp}\varphi : \operatorname{Sp} A_V \to \operatorname{Sp} A$ induces a homeomorphism $\operatorname{Sp} A_V \to V$.

In particular, A_V is uniquely determined by V up to isomorphisms of Banach k-algebras.

Proof. Let us reduce the problem to the case where k is non-trivially valued and A and A_V are both strictly k-affinoid.

By Proposition 3.14, taking a suitable $r = r(r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ such that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$, we may guarantee that $A \hat{\otimes}_k k_r$ and $A_V \hat{\otimes}_k k_r$ are both strictly k_r-affinoid.

Let V' be the inverse image of V in $\operatorname{Sp} A \hat{\otimes}_k k_r$. We claim that V' is a strict k_r-affinoid domain in $\operatorname{Sp} A \hat{\otimes}_k k_r$ represented by $A \hat{\otimes}_k k_r \to A_V \hat{\otimes}_k k_r$. □

Bibliography

[Ber12] V. G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. 33. American Mathematical Soc., 2012.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. Non-Archimedean analysis. Vol. 261. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436. URL: https://doi.org/10.1007/978-3-642-52229-1.

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