# The notion of complex analytic spaces

# Contents

## 1. Introduction

## 2. $\mathbb{C}$-ringed space

**Definition 2.1.** A $\mathbb{C}$-*ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras on $X$.

A *morphism of* $\mathbb{C}$-*ringed spaces* $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair consisting of a continuous map $f : Y \to X$ and a morphism of sheaves of $\mathbb{C}$-algebras $f^\# : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$.

Given two morphisms of $\mathbb{C}$-ringed spaces $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $g : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$, their *composition* is the morphism $f \circ g : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ consisting of the continuous map $f \circ g : Z \to X$ and a morphism of sheaves $(f \circ g)^\# = g^\# \circ g^{-1} f^\# : (f \circ g)^{-1}\mathcal{O}_X \xrightarrow{\sim} g^{-1}f^{-1}\mathcal{O}_X \to \mathcal{O}_Z$.

It is straightforward to verify that $\mathbb{C}$-ringed spaces form a category, which we denote by $\mathbb{C}$-$\mathcal{R}$S. Similarly, we denote by $\mathcal{R}$S the category of ringed spaces defined in [Stacks, Tag 0090].

There is an obvious faithful forget functor $\mathbb{C}$-$\mathcal{R}$S $\to$ $\mathcal{R}$S.

**Definition 2.2.** A *locally* $\mathbb{C}$-*ringed space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ which when regarded as a ringed space is a locally ringed space.

A *morphism* between two locally $\mathbb{C}$-ringed spaces is a morphism between the underlying $\mathbb{C}$-ringed spaces which is a morphism of locally ringed spaces at the same time.

We refer to [Stacks, Tag 01HA] for the notion of locally ringed spaces.

**Example 2.3.** *Let $n \in \mathbb{N}$, we define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{\mathbb{C}^n}$ on $\mathbb{C}^n$ as follows: for any open subset $U \subseteq \mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^n}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on $U$. It is easy to see that $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf and $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a $\mathbb{C}$-ringed space. Moreover, it is easy to show that $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a locally $\mathbb{C}$-ringed space.*

## 3. Complex model spaces and complex analytic spaces

**Definition 3.1.** Given any domain $D$ in $\mathbb{C}^n$, we can define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_D$ on $D$ as the restriction of $\mathcal{O}_{\mathbb{C}^n}$ defined in Example 2.3 to $D$. Observe that $(D, \mathcal{O}_D)$ is a locally $\mathbb{C}$-ringed space.

**Definition 3.2.** A *complex model space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that there exist

(1) a domain $D$ in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and
(2) an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_D$ of finite type

such that thre is an isomorphism

$$(X, \mathcal{O}_X) \cong (\operatorname{Supp} \mathcal{O}_D/\mathcal{I}, i^{-1}(\mathcal{O}_D/\mathcal{I}))$$

in the category of $\mathbb{C}$-$\mathcal{R}$S, where $i : \operatorname{Supp} \mathcal{O}_D/\mathcal{I} \to D$ is the inclusion map. Here $\mathcal{O}_D$ is the sheaf of $\mathbb{C}$-algebras defined in Definition 3.1.

Clearly, $(X, \mathcal{O}_X)$ is a locally $\mathbb{C}$-ringed space.

Observe that $X$ is always a Hausdorff space.

**Definition 3.3.** A *complex analytic space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that

(1) $X$ is a Hausdorff space.

(2) For any $x \in X$, there is an open neighbourhood $U \subseteq X$ of $x$ such that $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is isomorphic to a complex model space in the sense of Definition 3.2 in the category $\mathbb{C}$-$\mathcal{R}$S.

When there is no risk of confusion, we also omit $\mathcal{O}_X$ from the notation say $X$ is a $\mathbb{C}$-ringed space.

A morphism between complex analytic spaces is a morphism of the underlying locally $\mathbb{C}$-ringed spaces. Such a morphism is also known as a *holomorphic map*.

**Remark 3.4.** It seems that all authors on this subject requires that complex analytic spaces be Hausdorff, which may seem unnatural from the eyes of an algebro-geometrist. Morally, Hausdorffness corresponds to separatedness in the scheme world. However, non-Hausdorff analytic spaces do not seem to play a major role, in contrast to non-separated schemes, so we stick to the current definition.

**Remark 3.5.** Most of the authors require extra conditions in the definition of a complex analytic space: $\sigma$-compactness, paracompactness, having countable basis etc. We will not put these constraints in the definition, instead, we choose to include them into the assumptions of the theorems.

Observe that a complex analytic space is always a locally $\mathbb{C}$-ringed space.

## 4. Oka's coherence theorem

This lemma needs to be placed elsewhere. Proof at CAS p58 needs to be included

**Lemma 4.1.** *Let $X$ be a topological space and $\mathcal{A}$ be a Hausdorff sheaf of rings on $X$ (in the sense that the espace étalé of $\mathcal{A}$ is Hausdorff) such that all stalks of $\mathcal{A}$ are integral domains. Then $\mathcal{A}$ is coherent if and only if for any open set $V \subseteq X$ and any section $s \in \mathcal{A}(X)$, $\mathcal{A}_V/s\mathcal{A}_V$ is coherent at every $x \in V$ where $s_x \neq 0$.*

**Lemma 4.2** (Oka)**.** *For any $n \in \mathbb{N}$, $\mathcal{O}_{\mathcal{C}^n}$ is coherent.*

Proof. As a preparation, observe that $\mathcal{O}_{\mathbb{C}^n}$ is a Hausdorff sheaf.

For any two germs $s_i \in \mathcal{O}_{\mathbb{C}^n, a_i}$ $(i = 1, 2)$, we need to construct disjoint open neighbourhoods $U_i$ in the espace étalé of $\mathcal{O}_{\mathbb{C}^n}$ of $s_i$. If $a_1 \neq a_2$, the assertion is clear. So assume that $a_1 = a_2 = 0$. We extend $s_i$ to $f_i \in \mathcal{O}_{\mathbb{C}^n}(U)$ for a connected open neighbourhood $U \subseteq \mathbb{C}^n$ of 0. Then $\{f_x : x \in U\}$ and $\{g_x : x \in U\}$ are disjoint: if for some $z \in U$, $f_z = g_z$, then the same holds in a neighbourhood of $z$ and so $f = g$ on $U$ by Identitätssatz. Include the proof

We will prove the coherence of $\mathcal{O}_{\mathcal{C}^n}$ by induction on $n$. The case $n = 0$ is trivial. Assume that $n > 0$ and the theorem has been proved for all smaller $n$. We will apply Lemma 4.1. Take an open set $U \subseteq \mathbb{C}^n$ and $g \in \mathcal{O}_{\mathbb{C}^n}(U)$. We need to show that $\mathcal{O}_U/g\mathcal{O}_U$ is coherent at all $x \in U$ with $g_x \neq 0$.

Fix such a point $x$, which may be assumed to be 0. We may assume that $g(0) = 0$ as otherwise, the stalk of $\mathcal{O}_U/g\mathcal{O}_U$ at 0 is trivial. By perturbing the coordinates, we may guarantee that $g_0(0, w)$ is not identically 0 for $w \in \mathbb{C}$. By Weierstrass preparation theorem Include a proof, there is a monic polynomial $\omega_0 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ such that $g_0\mathcal{O}_{\mathbb{C}^n,0} = \omega_0\mathcal{O}_{\mathbb{C}^n,0}$. Lift $\omega_0$ to $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)$ for some neighbourhood $B \subseteq \mathbb{C}^{n-1}$ of 0. $\square$

## 5. Implicit function theorem

## 6. Rückert Nullstellensatz

Let $X$ be a complex analytic space. It is a sheaf of $\mathbb{C}$-algebras. For any sheaf of local $\mathbb{C}$-algebras $\mathcal{A}$ on $X$, any open set $U \subseteq X$ and any $s \in \mathcal{A}_X(U)$. We want to construct a function $[s] : U \to \mathbb{C}$.

Take $x \in U$, there is a canonical splitting

$$\mathcal{A}_x \cong \mathbb{C} \oplus \mathfrak{m}_x, \tag{6.1}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{A}_x$. Then we define $[s](x)$ as the image of $s_x$ in the $\mathbb{C}$-factor in (6.1).

**Definition 6.1.** Let $X, \mathcal{A}, U, x, s$ be as above. The value $[s](x) \in \mathbb{C}$ is called the *value* of $s$ at $x$. We sometimes denote it by $s(x)$ as well.

**Lemma 6.2.** *Let $X$ be a complex analytic space. We denote by $\mathcal{C}_X$ the sheaf of continuous functions on $X$. The association $s \mapsto [s]$ in Definition 6.1 defines a homomorphism of sheaves of $\mathbb{C}$-algebras $\mathcal{O}_X \to \mathcal{C}_X$.*

When there is no risk of confusion, we also write $s$ instead of $[s]$.

Proof. We need to show that for any open set $U \subseteq X$ and any $s \in \mathcal{O}_X(U)$, $[s]$ is a continuous function on $U$.

We may clearly assume that $U = X$. The problem is local on $X$, so we may assume that $X$ is a complex model space in the sense of Definition 3.2 defined by a coherent ideal $\mathcal{I}$ in a domain $D$ in $\mathbb{C}^n$. By further localizing, we may assume that $s$ can be lifted to a section $f \in \mathcal{O}_D(D)$. Then $[s] = f|_X$ by definition. So the assertion follows from the fact that a holomorphic function on a domain is continuous. □

Theorem 6.3 (Rückert Nullstellensatz). *Let $X$ be a complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\operatorname{Supp}\mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of $x$ and $m \in \mathbb{Z}_{>0}$ such that $f^m\mathcal{F}|_U = 0$.*

Proof. We may assume that $x \in \operatorname{Supp}\mathcal{F}$ as otherwise there is nothing to prove. In particular, $f(x) = 0$.

**Step 1**. We first reduce the problem to a relatively simple situation.

The problem is local on $X$, so we may assume that there is a domain $D$ containing 0 in $\mathbb{C}^n$ and a closed immersion $\iota : X \to D$ sending $x$ to 0. Consider the closed immersion $g : V \to D \times \mathbb{C}$ induced by $\iota$ and $f$. Assume that this theorem has been proved for $w$, $B \times \mathbb{C}$, $g_*\mathcal{F}$ in place of $f$, $X$, $\mathcal{F}$ respectively, then we would find an integer $m \in \mathbb{Z}_{>0}$ such that $w^m(g_*\mathcal{F})_0 = 0$. In particular, $f^m\mathcal{F}_x = 0$. As $\mathcal{F}$ is coherent, there is an open neighbourhood $U \subseteq X$ of $x$ such that $f^t\mathcal{F}|_U = 0$.

**Step 2**. We are reduced to prove the following special case: let $D$ be a domain in $\mathbb{C}^n$ containing 0, $\mathcal{F}$ is a coherent sheaf on $D$ whose support is contained in $\{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : (z, w) \in D, w = 0\}$. Then there is $m \in \mathbb{Z}_{>0}$ such that $w^m\mathcal{F}_0 = 0$.

Let $\mathcal{G}$ be the annihilator sheaf of $\mathcal{F}$:

$$\mathcal{G} := \ker\left(\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F})\right),$$

where the map $\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F})$ sends a local section $f$ of $\mathcal{O}_D$ to the endohomomorphism of multiplying by $f$ of $\mathcal{F}$. Then $\mathcal{G}$ is a coherent sheaf We need to prove a weak version of Oka's property first. So it has closed supports. But by our assumtion, the support of $\mathcal{G}$ contains all $w \neq 0$, so $\operatorname{Supp} \mathcal{G} = D$.

Let $f \in \mathcal{G}_0$ be a non-zero element. We write The structure of the local ring needs to be presented earlier

$$f = \sum_{j=b}^{\infty} a_j w^j, \quad a_j \in \mathcal{O}_{\mathbb{C}^{n-1},0}, a_b \neq 0$$

for some $b \in \mathbb{N}$. We may assume that $b = 0$ by replacing $f$ and $\mathcal{F}$ with $w^{-b} f$ and $w^b \mathcal{F}$ respectively. We want to show that $w^m \mathcal{F}_0 = 0$ for some positive integer $m$.

When $a_0$ is a unit, namely when $a_0(0) \neq 0$, then $f$ is a unit, so $\mathcal{F}_0 = 0$. We make an induction on $n$. The case $n = 1$ is trivial, as $a_0$ is always a unit. So we may assume that $a_0(0) = 0$ and $n > 1$. By perturbing the coordinates in $\mathbb{C}^{n-1}$, we may assume that $a_0$ is not identically zero in the variable $z_1$. We need to finish the Weierstrass theory first.

Shrinking $D$, we may assume that $f$ can be lifted to a holomorphic function $g \in \mathcal{O}_D(D)$ with $g\mathcal{F} = 0$. By our assumption on $a_0$, we may assume that $Z(g) \cap \{(z_1, 0, \ldots, 0) \in D\} = \{0\}$. Hence, $D \cap \operatorname{Supp} \mathcal{F}$, which is a subset of $Z(g)$ also intersects the $z_1$-axis only at the origin.

By To be included, we can find a product domain $B \times W \subseteq D$ with $B \subseteq \mathbb{C}$ and $W \subseteq \mathbb{C}^{n-1}$ containing 0 such that the projection $h : (B \times W) \cap \operatorname{Supp} \mathcal{F} \to B$ is finite and $\mathcal{F}' := h_*(\mathcal{F}|_{B \times W})$ is a coherent sheaf of $\mathcal{O}_B$-modules. Observe that $\operatorname{Supp} \mathcal{F}' \subseteq \{(z_2, \ldots, z_{n-1}, w) \in B : w = 0\}$, we can apply the induction hypothesis to obtain $m \in \mathbb{Z}_{>0}$ such that $w^m \mathcal{F}'_0 = 0$. It follows that $w^m \mathcal{F}_0 = 0$. $\square$

## 7. Fiber products

# Bibliography

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.