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Contents

Constructions of complex analytic spaces

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2. Analytic spectra

Proposition 2.1. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Then the presheaf $F_{\mathcal{A}}$ on $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$ defined by

$$F_{\mathcal{A}}(T \xrightarrow{p} S) = \mathrm{Hom}_{\mathcal{O}_T}(p^*\mathcal{A}, \mathcal{O}_T)$$

is representable.

Proof. By the arguments of [Stacks, Tag 01JJ], the problem is local in S. So we may assume that $\mathcal{A}$ has the following form

$$\mathcal{A} = \mathcal{O}_S[X_1, \ldots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and $\mathcal{I} \subseteq \mathcal{O}_S(S)[X_1, \ldots, X_n]$ an ideal sheaf of finite type.

Step 1. We first handle the case where $\mathcal{A} = \mathcal{O}_S[X_1, \ldots, X_n]$.

In this case, we claim that $F_{\mathcal{A}}$ is represented by $S \times \mathbb{C}^n$. In fact, it suffices to observe that

$$\begin{aligned} F_{\mathcal{A}}(T \xrightarrow{p} S) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \ldots, X_n], \mathcal{O}_T) \xrightarrow{\sim} \mathcal{O}_T(T)^n \\ &= \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}}(T, \mathbb{C}^n) = \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}}(T, S \times \mathbb{C}^n). \end{aligned}$$

From this proof, it is easy to see that the universal morphism is

$$\eta : \mathcal{O}_{S\times\mathbb{C}^n}[X_1, \ldots, X_n] \to \mathcal{O}_{S\times\mathbb{C}^n}$$

sending X_i to z_i, the i-th coordinate of $\mathbb{C}^n$.

Step 2. We handle the general case. We have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S[X_1, \ldots, X_n] \to \mathcal{A} \to 0.$$

For any $p : T \to S$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$, we have an exact sequence

$$p^*\mathcal{I} \to \mathcal{O}_T[X_1, \ldots, X_n] \to p^*\mathcal{A} \to 0.$$

We then have

$$\begin{aligned} F_{\mathcal{A}}(T) &\xrightarrow{\sim} \{h \in \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \ldots, X_n], \mathcal{O}_T) : h|_{p^*\mathcal{I}} = 0\} \\ &\xrightarrow{\sim} \left\{h \in F_{\mathcal{O}_S[X_1,\ldots,X_n]}(T) : h|_{p^*\mathcal{I}} = 0\right\}. \end{aligned}$$

Let $\pi : S \times \mathbb{C}^n \to S$ be the projection. Then $F_{\mathcal{A}}(T)$ is represented by the closed subspace of $S \times \mathbb{C}^n$ defined by the ideal $\eta(\pi^*\mathcal{I})$, which is clearly of finite type. □

Definition 2.2. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Then the complex analytic space representing the functor $F_{\mathcal{A}}$ in Proposition 2.1 is called the *analytic spectrum* of $\mathcal{A}$. We denote it by $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}$. By construction, there is a canonical morphism $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$.

By definition, we have a universal morphism $\xi \in F_{\mathcal{A}}(X) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}_X, \mathcal{O}_X)$ with $X = \mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}$. It defines a morphism of ringed spaces $X \to (|S|, \mathcal{A})$. The pull-back of an $\mathcal{A}$-module $\mathcal{M}$ is denoted by $\tilde{\mathcal{M}}$. The assignment $\mathcal{M} \mapsto \tilde{\mathcal{M}}$ is functorial in M.

It is easy to see that $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}$ is contravaraint in $\mathcal{A}$.

Proposition 2.3. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Consider a morphism $g : S' \to S$ of complex analytic spaces. Then we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}^{\mathrm{an}}_{S'} g^*\mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}^{\mathrm{an}}_S \mathcal{A} \arrow[d] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

PROOF. This is clear at the level of functor of points. $\square$

Corollary 2.4. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Take $s \in S$. Then $\mathrm{Spec}^{\mathrm{an}}_{\{s\}} \mathcal{A}_s \xrightarrow{\sim} (\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A})_s$.

Moreover, the universal morphism $\mathcal{A}_{\mathrm{Spec}^{\mathrm{an}}_{\{s\}} \mathcal{A}_s} \to \mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}_{\{s\}} \mathcal{A}_s}$ is the reduction of the universal morphism $\mathcal{A}_{\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}} \to \mathcal{O}_{\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}}$ modulo $\mathfrak{m}_s$.

PROOF. This follows from Proposition 2.3. $\square$

Proposition 2.5. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Take $s \in S$. Write $X = \mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}$ and $\mathcal{A}_s := \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,s}$. Then the map from X_s to

$$\{\mathfrak{m} \in \mathrm{Spm}_{\mathbb{C}} \mathcal{A}_s : \mathfrak{m} \supseteq \mathfrak{m}_s\}$$

sending $x \in X_s$ to the inverse image of $\mathfrak{m}_x$ with respect to $\mathcal{A}_s \to \mathcal{O}_{X,x}$ is bijective.

If $\mathfrak{m}$ corresponds to $x \in X_s$, then the natural homomorphism $\mathcal{A}_s \to \mathcal{O}_{X,x}$ factorizes through $\mathcal{A}_{s,\mathfrak{m}} \to \mathcal{O}_{X,x}$. The completion of the latter

$$\widehat{\mathcal{A}_{s,\mathfrak{m}}} \to \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism.

PROOF. By Corollary 2.4, we have natural bijections

$$X_s \xrightarrow{\sim} \mathrm{Hom}_{\{s\}}(\{s\}, X_s) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{lg}}(\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s, \mathbb{C}).$$

This gives the desired bijection.

Next we prove the latter part. The problem is local on S, we may assume that

$$\mathcal{A} = \mathcal{O}_S[X_1, \ldots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and some ideal $\mathcal{I}$ of finite type in $\mathcal{O}_S[X_1, \ldots, X_n]$. Recall that the universal morphism

$$\eta : \mathcal{O}_{S\times\mathbb{C}^n}[X_1, \ldots, X_n] \to \mathcal{O}_{S\times\mathbb{C}^n}$$

sends X_i to z_i, the i-th coordinate of $\mathbb{C}^n$.

By construction, we have

$$\mathcal{A}_s \xrightarrow{\sim} \mathcal{O}_{S,s}[X_1, \ldots, X_n]/\mathcal{I}_s$$

and

$$\mathcal{O}_{X,x} = \mathcal{O}_{S\times\mathbb{C}^n,x}/\mathcal{J}_x,$$

where

$$\mathcal{J}_x = \eta_x \left(\mathcal{I}_s \mathcal{O}_{S\times\mathbb{C}^n,x}[X_1,\dots,X_n] \right).$$

We have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_s \arrow[r] \arrow[d] & \mathcal{O}_{S,s}[X_1,\dots,X_n] \arrow[r] \arrow[d] & \mathcal{A}_s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}_x \arrow[r] & \mathcal{O}_{S\times\mathbb{C}^n,x} \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & 0
\end{tikzcd}.$$

The middle vertical map is induced by η_x. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{m}_{S\times\mathbb{C}^n,x}$ under the vertical map in the middle. Then $\mathfrak{p}$ is generated by $\mathfrak{m}_s$ and $X_1 - x_1,\dots,X_n - x_n$, where $x_i \in \mathbb{C}$ is the value of z_i at x for $i=1,\dots,n$. By localization and completion, we find a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widehat{(\mathcal{I}_s)_\mathfrak{p}} \arrow[r] \arrow[d] & (\mathcal{O}_{S,s}[X_1,\dots,X_n])^\wedge_\mathfrak{p} \arrow[r] \arrow[d] & \widehat{(\mathcal{A}_s)_\mathfrak{m}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathcal{J}_x} \arrow[r] & \widehat{\mathcal{O}_{S\times\mathbb{C}^n,x}} \arrow[r] & \widehat{\mathcal{O}_{X,x}} \arrow[r] & 0
\end{tikzcd}.$$

Observe that

$$(\mathcal{O}_{S,s}[X_1,\dots,X_n])^\wedge_\mathfrak{p} \cong \widehat{\mathcal{O}_{S,s}}[[X_1-x_1,\dots,X_n-x_n]]$$

and

$$\widehat{\mathcal{O}_{S\times\mathbb{C}^n,x}} \cong \widehat{\mathcal{O}_{S,s}} \hat{\otimes}_k \widehat{\mathcal{O}_{\mathbb{C}^n,(x_1,\dots,x_n)}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1-x_1,\dots,X_n-x_n]].$$

It is easy to see that the middle map is an isomorphism. As $\mathcal{J}_x$ is generated by $\mathcal{I}_s$, the first vertical map is also an isomorphism. Our assertion follows. □

Corollary 2.6. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Write $X = \operatorname{Spec}^{\mathrm{an}}_S \mathcal{A}$. Take $s \in S$. Then the fiber X_s is finite and is in bijection with $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_s = \operatorname{Spm} \mathcal{A}_s$. If $\mathfrak{m}$ corresponds to $x \in X_s$, then we have a natural isomorphism

$$\mathcal{A}_{s,\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}.$$

Proof. We first observe that as $\mathcal{A}_s$ is a finite $\mathcal{O}_{S,s}$-algebra, its residue fields at maximal primes are finite extensions of the residue field $\mathbb{C}$ of $\mathcal{O}_{S,s}$. So $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_s = \operatorname{Spm} \mathcal{A}_s$.

As $\mathcal{O}_{S,s} \to \mathcal{A}_s$ is finite, $\mathcal{A}_s$ is semi-local. On the other hand, by Proposition 2.5,

$$\mathcal{A}_{s,\mathfrak{m}} \to \mathcal{O}_{X,x}$$

is injective and $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{S,s}$. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by Theorem 5.4 in Complex analytic local algebras. It follows from Nakayama's lemma that $\mathcal{A}_{s,\mathfrak{m}} \to \mathcal{O}_{X,x}$ is also surjective. □

Corollary 2.7. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Then the image of $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$ is $\operatorname{Supp} \mathcal{A}$.

Proof. This follows from Corollary 2.6 and the fact that $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_s = \operatorname{Spm} \mathcal{A}_s$ for all $s \in S$. □

Proposition 2.8. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Write $f : \operatorname{Spec}^{\mathrm{an}}_S \mathcal{A}$ for the structure map. Then we have the following assertions:

(1) for all $\mathcal{A}$-module $\mathcal{M}$, the natural morphism

$$\mathcal{M} \to f_*\tilde{\mathcal{M}}$$

is an isomorphism,

In particular, $\mathcal{A} \xrightarrow{\sim} f_*\mathcal{O}_X$.

(2) for all $\mathcal{O}_X$-module $\mathcal{F}$, the canonical morphism

$$\widehat{f_*\mathcal{F}} \to \mathcal{F}$$

is an isomorphism.

In particular, the category of $\mathcal{A}$-modules is equivalent to the cateogory of $\mathcal{O}_X$-modules.

Proof. By Corollary 3.8, f is topologically finite. Take $s \in S$. Let $x_1, \dots, x_n$ be the distinct points of $f^{-1}(s)$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ denote the maximal ideals of $\mathcal{A}_s$ corresponding to $x_1, \dots, x_n$.

(1) By Corollary 4.9 in Topology and bornology and Corollary 2.6,

$$(f_*\tilde{\mathcal{M}})_s \cong \prod_{i=1}^n \widehat{M}_{x_i} \cong \prod_{i=1}^n \widehat{\mathcal{M}}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \mathcal{M}_s \otimes_{\mathcal{A}_s} \prod_{i=1}^n \mathcal{A}_{s,\mathfrak{m}_i} \xrightarrow{\sim} \mathcal{M}_s.$$

(2) By Corollary 4.9 in Topology and bornology,

$$f_*\mathcal{F}_s \cong \prod_{i=1}^n \mathcal{F}_{x_i}.$$

It follows that

$$\widetilde{f_*\mathcal{M}}_{x_i} \cong f_*\mathcal{F}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \prod_{j=1}^n \mathcal{F}_{x_j} \otimes_{\mathcal{A}_s} \mathcal{A}_{s,\mathfrak{m}_i}$$

for $i = 1, \dots, n$. But the only non-zero term is when $j = i$, so

$$\widetilde{f_*\mathcal{M}}_{x_i} \cong \mathcal{F}_{x_i}$$

for $i = 1, \dots, n$. □

Corollary 2.9. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Write $f : \mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$ for the structure map. Then for any coherent $\mathcal{O}_X$-module $\mathcal{M}$, $f_*\mathcal{F}$ is coherent.

Moreover, f_* is exact from $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$ to $\mathcal{C}\mathrm{oh}(\mathcal{O}_Y)$.

Proof. The exactness of f_* follows from Proposition 2.8.

We claim that up to shrinking S, we may assume that $\mathcal{M}$ has a global presentation. Fix $s \in S$ and let $x_1, \dots, x_n$ be the distinct points of $f^{-1}(s)$.

For each $j = 1, \dots, n$, we can find an open neighbourhood U_j of x_j in X, pairwise disjoint and an exact sequence

$$\mathcal{O}_{U_j}^{p_j} \to \mathcal{O}_{U_j}^{q_j} \to \mathcal{M}|_{U_j} \to 0$$

for some $p_j, q_j \in \mathbb{Z}_{>0}$. We may assume that $p_1 = \dots = p_n$ and $q_1 = \dots = q_n$. We denote the common values by p and q. Then $U = U_1 \cup \dots \cup U_n$ is a neighbourhood of $f^{-1}(s)$, and we have an exact sequence

$$\mathcal{O}_U^p \to \mathcal{O}_U^q \to \mathcal{M}|_U \to 0.$$

By Lemma 4.2 in Topology and bornology, we may assume that $U = \pi^{-1}(V)$ for some open neighbourhood V of s in S. The induced map $f' : U \to V$ is finite and by Corollary 4.9 in Topology and bornology.

Now let us take a presentation

$$\mathcal{O}^p \to \mathcal{O}^q \to \mathcal{M} \to 0.$$

By Proposition 2.8, we have an exact sequence

$$f_*\mathcal{O}^p \to f_*\mathcal{O}^q \to f_*\mathcal{M} \to 0.$$

By Proposition 2.8 again, this can be written as

$$\mathcal{A}^p \to \mathcal{A}^q \to f_*\mathcal{M} \to 0.$$

It follows that $f_*\mathcal{M}$ is coherent. □

Proposition 2.10. Let S be a complex analytic space and $\mathcal{A}$, $\mathcal{B}$ be $\mathcal{O}_S$-algebras of finite presentation. Assume that $\mathcal{A}$ is finite. Then we have a natural bijection

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}}(\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}, \mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}).$$

PROOF. Let $f : X := \mathrm{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$ be the natural map. We construct the bijection as

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, f_*\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{B}_X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}}(\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}, \mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}).$$

The first map is a bijection by Proposition 2.8 □

Definition 2.11. Let S be a complex analytic space and $\mathcal{E}$ be an $\mathcal{O}_S$-module of finite presentation. We define the *vector bundle* $\mathbf{V}(\mathcal{E})$ generated by $\mathcal{E}$ as

$$\mathbf{V}(\mathcal{E}) = \mathrm{Spec}^{\mathrm{an}}_S \mathrm{Sym}\, \mathcal{E}.$$

We have a natural projection $\mathbf{V}(\mathcal{E}) \to S$.

We remind the readers that we are following Grothendieck's convention for $\mathbf{V}(\mathcal{E})$, which is different from Fulton's.

3. Analytic germs

Definition 3.1. A *pointed complex analytic space* is a pair (X, x) consisting of a complex analytic space X and a point $x \in X$. A morphism between pointed complex analytic spaces (X, x) and (Y, y) is a morphism $f : X \to Y$ of complex analytic spaces such that $f(x) = y$. The category of pointed complex analytic spaces is denoted by $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_*$.

The category of *complex analytic germs* $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is the right category of fractions of $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$ with respect to the system of morphisms $f : (X, x) \to (Y, y)$ such that $f : X \to Y$ is an open immersion. An element in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is called a *complex analytic germ*. A complex analytic germ represented by (X, x) is denoted by X_x.

Given a complex analytic germ X_x, we write $\mathcal{O}_{X,x}$ for the local ring of X at x. Clearly, it does not depend on the choice of (X, x). Given any morphism $f : X_x \to Y_y$ of complex analytic germs, we have an obvious local homomorphism $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Definition 3.2. Given a complex analytic germ X_x, a *closed subgerm* of X_x is an isomorphism class in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}_{/X_x}$ of Y_x represented by a closed analytic subspace of X containing x for any representation (X,x) of X_x.

In particular, X_x is a closed subgerm of X_x. A closed subgerm Y_y of X_x is *proper* if Y_y is different from X_x as subgerms.

Given a closed subgerm Y_x of X_x, we have an induced surjective homomorphism $\mathcal{O}_{X,x}\to\mathcal{O}_{Y,y}$. The kernel is denoted by $I(Y,x)$ or $I_X(Y,x)$.

Theorem 3.3. The functor $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}^{\mathrm{op}}\to\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ defined in Definition 3.1 is an equivalence.

Proof. **Step 1**. We show that the functor is faithfully.

In order words, let (X,x) and (Y,y) be two pointed complex analytic spaces and $f,g:(X,x)\to(Y,y)$ be two morphsims inducing the same map $\mathcal{O}_{Y,y}\to\mathcal{O}_{X,x}$, then f and g coincide on a neighbourhood of x in X.

The question is open on Y, so we may reduce to the case where Y is a complex model space. We then further reduce to the case where Y is a domain in $\mathbb{C}^n$ for some $n\in\mathbb{N}$ and then to $Y=\mathbb{C}^n$.

By Theorem 4.2 in The notion of complex analytic spaces, f and g can be identified with systems $(f_1,\ldots,f_n)\in\mathcal{O}_X(X)^n$ and $(g_1,\ldots,g_n)\in\mathcal{O}_X(X)^n$. The assumption $f_x^\#=g_x^\#$ menas $f_{i,x}=g_{i,x}$ for $i=1,\ldots,n$. So $f_i=g_i$ after shrinking X. We conclude by Theorem 4.2 in The notion of complex analytic spaces again.

Step 2. We show that the functor is fully faithful.

In other words, let (X,x) and (Y,y) be two pointed complex analytic spaces and $\varphi:\mathcal{O}_{Y,y}\to\mathcal{O}_{X,x}$ be a morphism in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$. Then we can find an open neighbourhood U of x in X and a morphism $(U,x)\to(Y,y)$ inducing φ.

The problem is local on Y, so we may assuem that Y is a complex model space, say Y is a closed subspace of a domain V in $\mathbb{C}^n$ defined by a coherent ideal $\mathcal{I}$. We write $\psi:\mathcal{O}_{V,y}\to\mathcal{O}_{X,x}$ the homomorphism induced by φ, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{V,y} \arrow[r, "\psi"] \arrow[d, two heads] & \mathcal{O}_{Y,y} \\
\mathcal{O}_{X,x} \arrow[ur, "\varphi"] &
\end{tikzcd}.$$

Let $z_1,\ldots,z_n$ be the coordinates on V. Let $f_{i,x}$ be the image of $z_{i,x}$ under ψ for $i=1,\ldots,n$. Take an open neighbourhood U of x in X so that $f_{i,x}$ lifts to $f_i\in\mathcal{O}_X(U)$ for $i=1,\ldots,n$. By Theorem 4.2 in The notion of complex analytic spaces, $f_1,\ldots,f_n$ then defines a morphism $g:U\to\mathbb{C}^n$. Clearly $g(x)=y$. But $g_x^\#$ and ψ coincide on $z_{i,y}$ so $g_x^\#=\psi$ as $\mathcal{O}_{V,y}=\mathbb{C}\{z_{1,y}-a_1,\ldots,z_{n,y}-a_n\}$ with $a_i=\epsilon(z_{i,y})$ for $i=1,\ldots,n$. Therefore, $g_x^\#(\mathcal{I}_y)=0$. Up to shrinking U, we may guarantee that $g(U)\subseteq V$ and $g^*(\mathcal{I})=0$ on U. Namely, g factorizes through $f:U\to Y$ and $f_x^*=\varphi$.

Step 3. We show that the functor is essentially surjective.

In other words, let A be a complex analytic local algebra, then there is a pointed complex analytic space (X,x) with $\mathcal{O}_{X,x}\cong A$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$.

We may assume that $A=\mathbb{C}\{z_1,\ldots,z_n\}/I$ for some $n\in\mathbb{N}$ and ideal I in $\mathbb{C}\{z_1,\ldots,z_n\}$. Then I is finitely generated as $\mathbb{C}\{z_1,\ldots,z_n\}$ is noetherian. Take finitely many generators $f_1,\ldots,f_m\in I$. We extend $f_1,\ldots,f_m$ to $g_1,\ldots,g_m\in$

$\mathcal{O}_{\mathbb{C}^n}(U)$ for some open neighbourhood U of 0 in $\mathbb{C}^n$. Then the closed subspace X of U defined by $f_1, \ldots, f_m$ satisfies the required conditions. $\square$

Corollary 3.4. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is a local isomorphism;
(2) $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an isomorphism;
(3) $\widehat{f^\#_x} : \hat{\mathcal{O}}_{Y,f(x)} \to \hat{\mathcal{O}}_{X,x}$ is an isomorphism.

Later on, we will see that Condition (3) means f is étale at x.

Proof. (1) $\Leftrightarrow$ (2): This follows from Theorem 3.3.
(2) $\implies$ (3): This is clear.
(3) $\implies$ (2): As $f^\#_x$ is quasi-finite, the $\mathfrak{m}_x$-adic topology on $\mathcal{O}_{X,x}$ coincides with the $\mathfrak{m}_{f(x)}$-adic topology on it regarded as an $\mathcal{O}_{Y,f(x)}$-module. By Theorem 5.4 in Complex analytic local algebras, $f^\#_x$ is finite. So

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \hat{\mathcal{O}}_{Y,f(x)}.$$

So (2) follows from the fact that $\hat{\mathcal{O}}_{Y,f(x)}$ is faithfully flat over $\mathcal{O}_{Y,f(x)}$, see [Stacks, Tag 00MC]. $\square$

Corollary 3.5. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is a local immersion at x;
(2) $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective;
(3) $\widehat{f^\#_x} : \hat{\mathcal{O}}_{Y,f(x)} \to \hat{\mathcal{O}}_{X,x}$ is surjective;
(4) $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} \xrightarrow{\sim} \mathbb{C}$.

Proof. (1) $\implies$ (2): This is clear.
(2) $\implies$ (1): Let I be the kernel of $f^\#_x$. Up to shrinking X, we may assume that I spreads to a coherent ideal sheaf $\mathcal{I}$ on Y. Let Y' be the closed analytic subspace of Y defined by $\mathcal{I}$. Up to shrinking X, we may assume that f factorizes through $f' : X \to Y'$ by Theorem 3.3. But $f'^\#_x$ is an isomorphism, so f' is a local isomorphism by Corollary 3.4.
(2) $\Leftrightarrow$ (3): This follows from the same arguments as in Corollary 3.4.
(2) $\Leftrightarrow$ (4): This follows from Nakayama's lemma. $\square$

Corollary 3.6. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then the following are equivalent:

(1) f is an immersion;
(2) $|f|$ induces a homeomorphism of $|X|$ with a locally closed subset of $|Y|$ and for all $x \in X$, the homomorphism $f^\#_c : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

The condition in (2) is the usual definition of an immersion of ringed spaces. Our notion of immersion is usually called a locally closed immersion.

Proof. (1) $\implies$ (2): This is clear by definition.
(2) $\implies$ (1): We may clearly assume that $f(X)$ is closed in Y. We need to show that the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is of finite type. This follows from Corollary 3.5. $\square$

Lemma 3.7. Let S be a complex analytic space and $s \in S$. For any finite $\mathcal{O}_{S,s}$-algebra A, there is an open neighbourhood U of s in S and a finite $\mathcal{O}_U$-algebra such that $\mathcal{A}_s \cong A$.

PROOF. Let $s \in S$, as $\mathcal{A}_s$ is a finite $\mathcal{O}_{S,s}$-algebra, we can find finitely many generators $\sigma_{1,s}, \ldots, \sigma_{n,s}$. As $\mathcal{A}_s$ is integral over $\mathcal{O}_{S,s}$, we can find unitary polynomials $F_{i,s} \in \mathcal{O}_{S,s}[X_i]$ such that $F_{i,s}(\sigma_{i,s}) = 0$ for $i = 1, \ldots, n$. Take a sufficient small neighbourhood U of s so that $\sigma_{i,s}$ lifts to $\sigma_i \in \mathcal{O}_S(U)$ and $F_{i,s}$ lifts to a unitary polynomial $F_i \in H^0(U, \mathcal{O}_S[X_i])$ for $i = 1, \ldots, n$. Up to shrinking U, we may guarantee that $\sigma_1, \ldots, \sigma_n$ generate $\mathcal{A}|_U$ at all points and $F_i(\sigma_i) = 0$ for $i = 1, \ldots, n$. Then $\mathcal{B} := \mathcal{O}_U[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$ is coherent and we have a surjective homomorphism $\mathcal{B} \to \mathcal{A}|_U$ sneding X_i to σ_i for $i = 1, \ldots, n$. As the kenrel of this homomorphism is of finite ytpe, up to shrinking U, we may take finitely many $G_1, \ldots, G_m \in \mathcal{B}(U)$ that generate the kernel. Lift $G_1, \ldots, G_m$ to $H_1, \ldots, H_m \in H^0(U, \mathcal{O}_S[X_1, \ldots, X_m])$, then

$$\mathcal{A}|_U \cong \mathcal{O}_U[X_1, \ldots, X_n]/(F_1, \ldots, F_n, G_1, \ldots, G_m).$$

This follows from the same arguments of the proof of Theorem 3.3 Step 3. $\square$

Corollary 3.8. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra, then the map $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$ is topologically finite.

PROOF. By Corollary 2.6, the fibers of $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$ is finite. The map $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A} \to S$ is separated by construction. It remains to show that the map is closed.

The problem is local on S. By the proof of Lemma 3.7, we can find a closed immersion over S: $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A} \to \operatorname{Spec}^{\mathrm{an}}_S \mathcal{B}$, where $\mathcal{B} = \mathcal{O}_S[X_1, \ldots, X_n]/(F_1, \ldots, F_n)$ for some $n \in \mathbb{N}$, where F_i is a unitary polynomial in $\mathcal{O}_S(S)[X_i]$ for $i = 1, \ldots, n$. It suffices to show that $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{B} \to S$ is closed.

Observe that

$$\operatorname{Spec}^{\mathrm{an}}_S \mathcal{B} \cong \operatorname{Spec}^{\mathrm{an}}_S \prod_{j=1}^{n} \mathcal{O}_S[X_j]/(F_j)$$

in $\mathcal{A}\mathrm{n}_{/S}$ as can be seen from the functor of points. So the problem reduces to showing that

$$\operatorname{Spec}^{\mathrm{an}}_S \mathcal{O}_S[X]/(F) \to S$$

for a unitary polynomial is closed. This is the classical continuity of roots. $\square$

Next we describe the local structure of a complex analytic germ.

Theorem 3.9. Let X_x be a complex analytic germ, $n \in \mathbb{Z}_{>0}$ and $f_1, \ldots, f_n \in \mathcal{O}_{X,x}$ be a system of parameters. We have a morphism $X_x \to \mathbb{C}^n_0$ induced by $f_1, \ldots, f_n$. Then there is an open neighbourhood U of 0 in $\mathbb{C}^n$ and a finite $\mathcal{O}_U$-algebra $\mathcal{A}$ such that $\mathcal{A}_0 \cong \mathcal{O}_{X,x}$. The space $\operatorname{Spec}^{\mathrm{an}}_U(\mathcal{A})$ admits a unique point x' over 0 and X_x is isomorphic to $\operatorname{Spec}^{\mathrm{an}}_U(\mathcal{A})_{x'}$ in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}_{/\mathbb{C}^n_0}$.

PROOF. As $f_1, \ldots, f_n$ is a system of parameters, $\mathcal{O}_{X,x} \to \mathcal{O}_{\mathbb{C}^n,0}$ is finite. By Lemma 3.7, we can spread $\mathcal{O}_{X,x}$ to a finite $\mathcal{O}_U$-algebra on an open neighbourhood U of 0 in $\mathbb{C}^n$. Let $Y = \operatorname{Spec}^{\mathrm{an}}_U(\mathcal{A})$. It follows from Corollary 2.6 that Y has a unique point x' over 0. By Theorem 3.3, up to shrinking U, we may guarantee that X_x and $Y_{x'}$ are isomorphic over $\mathbb{C}^n_0$. $\square$

Proposition 3.10. Let X_x be a complex analytic germ. The map $Y_x \mapsto I_X(Y,x)$ defines a bijection between the set of closed subgerms of X_x and the set of ideals of $\mathcal{O}_{X,x}$.

In particular, we can view a germ Y_x as a closed subscheme $\operatorname{Spec} \mathcal{O}_{X,x}/I_X(Y,x)$ of $\operatorname{Spec} \mathcal{O}_{X,x}$.

Proof. We construct a reverse map. Given an ideal I of $\mathcal{O}_{X,x}$, as $\mathcal{O}_{X,x}$ is noetherian, I is finitely generated. We can find an open neighbourhood U of x in X and an ideal sheaf of finite type $\mathcal{I}$ of U with $\mathcal{I}_x = I$. Let Y be the closed analytic subspace of X defined by $\mathcal{I}$. We associated Y_x with I.

It is easy to verify that this map is the inverse of the given map. $\square$

Definition 3.11. Let X_x be a complex analytic germ and Y_x, Z_x be two closed subgerms of X_x. We say Y_x is contained in Z_x and write $Y_x \subseteq Z_x$ if $I(Y,x) \supseteq I_X(Z,x)$. This defines a partial order on the set of closed subgerms of X_x.

Definition 3.12. A complex analytic germ X_x is *integral* if $\mathcal{O}_{X,x}$ is integral.

We also say (X,x) is *integral*.

Theorem 3.13 (Nullstellensatz)**.** Let X_x be an integral complex analytic germ and Y_y be a closed subgerm of X_x. Then the following are equivalent:

(1) Y_x is a proper closed subgerm of X_x;
(2) $|Y|_x$ is a proper closed subgerm of $|X|_x$.

Proof. (2) $\implies$ (1): This is obvious.

(1) $\implies$ (2): Consider a proper closed subgerm Y_x of X_x. By Proposition 3.10, $I(Y,x) \neq 0$.

Step 1. We reduce to the case $I(Y,x) = (f)$ for some non-zero element $f \in \mathcal{O}_{X,x}$.

Take a non-zero element $f \in I(Y,x)$. Let Y'_x be the subgerm of X_x corresponding to the ideal (f) of $\mathcal{O}_{X,x}$. Then $Y_x \subseteq Y'_x$. It suffices to show that $|Y'|_x \neq |X|_x$. We may replace Y by Y'.

Step 2. We prove that $|Y|_x \neq |X|_x$.

Note that f is not a zero-divisor as $\mathcal{O}_{X,x}$ is integral. Write $n = \dim \mathcal{O}_{X,x}$. By Krulls Hauptidealsatz, $\dim \mathcal{O}_{X,x}/(f) = n-1$. Let $\overline{f_1}, \dots, \overline{f_{n-1}}$ be a system of parameters ([Stacks, Tag 00KU]) of $\mathcal{O}_{X,x}/(f)$. Lift them to $f_1, \dots, f_{n-1} \in \mathcal{O}_{X,x}$. Then $(f_1, \dots, f_{n-1}, f)$ is a system of parameters of $\mathcal{O}_{X,x}$. Let $\varphi : X_x \to \mathbb{C}^n_0$ and $\psi : Y_x \to \mathbb{C}^{n-1}_0$ be the morphisms defined by these systems of parameters. We then have a commutative diagram in $\mathbb{C}$-$\mathcal{G}$er:

$$\begin{tikzcd} Y_x \arrow[r, hook] \arrow[d, "\psi"] & X_x \arrow[d, "\varphi"] \\ \mathbb{C}^{n-1}_0 \arrow[r, hook] & \mathbb{C}^n_0 \end{tikzcd}$$

It induces a commutative diagram of topological germs:

$$\begin{tikzcd} {|Y|_x} \arrow[r, hook] \arrow[d, "|\psi|"] & {|X|_x} \arrow[d, "|\varphi|"] \\ \mathbb{C}^{n-1}_0 \arrow[r, hook] & \mathbb{C}^n_0 \end{tikzcd}$$

The morphism of topological germs of $\mathbb{C}^{n-1}_0 \to \mathbb{C}^n_0$ is clearly not an isomorphism, so it suffices to show that $|\varphi| : |X|_x \to \mathbb{C}^n_0$ is surjective, in the sense that if we represent $|\varphi|$ by a morphism $(U, x) \to (\mathbb{C}^n, 0)$ from an open neighbourhood U of x in X to $\mathbb{C}^n$, then its image contains an open neighbourhood of 0 in $\mathbb{C}^n$.

By Theorem 3.9, we may assume that $X = \operatorname{Spec}^{\mathrm{an}}_X \mathcal{A}$ for some finite $\mathcal{O}_X$-algebra $\mathcal{A}$ and X has a unique point over 0. Then by Corollary 2.6, we have $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{O}_{X,x}$. By Corollary 5.5 in Complex analytic local algebras, the natural homomorphism

$$\varphi^{\#} : \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{X_1, \ldots, X_n\} \to \mathcal{A}_0$$

is injective.

By Corollary 2.7, it remains to show that $\operatorname{Supp} \mathcal{A}$ is a neighbourhood of s in S. But the kernel of $\mathcal{O}_S \to \mathcal{A}$ is 0 at s hence 0 in a neighbourhood of s since both $\mathcal{O}_S$ and $\mathcal{A}$ are coherent by Corollary 7.4 in The notion of complex analytic spaces. $\square$

Corollary 3.14. Let X_x be a complex analytic germ and I, J be two ideals in $\mathcal{O}_{X,x}$. We let $W(I)$, $W(J)$ denote the topological germs of the closed analytic subgerms of X_x defined by I and J respectively. Then the following are equivalent:

(1) $W(I) \subseteq W(J)$;
(2) $J \subseteq \sqrt{I}$.

Proof. If (2) is true, as $\mathcal{O}_{X,x}$ is noetherian, we can find $n \in \mathbb{Z}_{>0}$ such that $J^n \subseteq I$. Extend I, J to coherent ideals $\mathcal{I}, \mathcal{J}$ on X up to shrinking X. Then $\operatorname{Supp} \mathcal{O}_X/\mathcal{J} \subseteq \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$. Hence, (1) holds.

Suppose that (1) holds. In order to prove (2), we may assume that I is prime. Then the closed analytic subgerm Y_x of X_x defined by I is integral. Let Z_x denote the closed analytic subgerm of X_x defined by J. The intersection $Y_x \cap Z_x$ of the germs Y_x and Z_x is by definition the closed analytic subgerm of X_x defined by $I + J$. Then

$$|Y_x \cap Z_x| = |Y|_x \cap |Z|_x = W(I).$$

By Theorem 3.13, $Y_x \subseteq Z_x$. Namely, (2) holds. $\square$

Corollary 3.15. Let X_x be a complex analytic germ and Y_x be a closed analytic subgerm. Then the following are equivalent:

(1) $|X|_x = |Y|_x$;
(2) $I_X(Y, x)$ is nilpotent.

In particular, if these conditions hold, $\dim \mathcal{O}_{Y,x} = \dim \mathcal{O}_{X,x}$.

Proof. This follows immediately from Corollary 3.14. $\square$

Corollary 3.16. Let X be a complex analytic space and $x \in X$. Then the following are equvalent:

(1) x is isolated in X;
(2) $\mathcal{O}_{X,x}$ is artinian.

Proof. (1) simply means that $X_x = \{x\}_x$. By Corollary 3.15, this holds if and only if $\mathfrak{m}_x$ is nilpotent. As $\mathcal{O}_{X,x}$ is noetherian, the latter is equivalent to that $\mathcal{O}_{X,x}$ is artinian. $\square$

Corollary 3.17. Let X be a complex analytic space and Y be a closed analytic subspace defined by a coherent ideal $\mathcal{I}$. Then the following are equivalent:

(1) $|X| = |Y|$;

(2) $\mathcal{I}$ is locally nilpotent.

PROOF. This follows immediately from Corollary 3.15. □

Corollary 3.18. Let X be a complex analytic space and $f \in \mathcal{O}_X(X)$. Then the following are equivalent:

(1) $f(x) = 0$ for all $x \in X$;
(2) f is locally nilpotent.

PROOF. This follows from Corollary 3.17, where we take $\mathcal{I}$ as the coherent ideal generated by f. □

Corollary 3.19 (Rückert Nullstellensatz)**.** Let X be a complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\operatorname{Supp}\mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m\mathcal{F}|_U = 0$.

PROOF. Let $\mathcal{G}$ be the annihilator sheaf of $\mathcal{F}$:

$$\mathcal{G} := \ker\left(\mathcal{O}_X \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})\right),$$

where the map $\mathcal{O}_X \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ sends a local section f of $\mathcal{O}_X$ to the endo-homomorphism of multiplying by f of $\mathcal{F}$. Then $\mathcal{G}$ is a coherent sheaf by Oka's coherence theorem Theorem 7.3 in The notion of complex analytic spaces. Let Y be the closed analytic subspace defined by $\mathcal{G}$. By our assumption, f is everywhere zero on Y, so f is locally nilpotent in $\mathcal{O}_X/\mathcal{G} \subseteq \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. □

Corollary 3.20. Let X be a complex analytic space and $\mathcal{I}$ and $\mathcal{J}$ be coherent ideal sheaves on X. Then the following are equivalent:

(1) $\operatorname{Supp}\mathcal{O}_X/\mathcal{I} \subseteq \operatorname{Supp}\mathcal{O}_X/\mathcal{J}$;
(2) For any $x \in X$, there is an open neighbourhood U of x in X and $n \in \mathbb{Z}_{>0}$ such that
$$J^n|_U \subseteq I|_U.$$

PROOF. This follows immediately from Corollary 3.14. □

4. Analytic subsets

Definition 4.1. Let X be a complex analytic space. A subset $A \subseteq X$ is *analytic at* $x \in X$ if there is an open neighbourhood U of x in X and finitely many $f_1, \dots, f_m \in \mathcal{O}_X(U)$ such that

$$A \cap U = \{x \in U : f_1(x) = \cdots = f_m(x) = 0\}.$$

We will denote the set on the right-hand side as $N_U(f_1, \dots, f_m)$. A subset $A \subseteq X$ is *analytic* in X if it is analytic at all $x \in X$.

A subset $B \subseteq X$ is *co-analytic* in X if $X \setminus B$ is analytic in X.

We observe that given $A \subseteq X$, the set of points $x \in X$ such that A is analytic at x is open. Also observe that an analytic set is necessarily closed. Analytic sets are clearly closed under finite intersection and finite unions.

Example 4.2. Let X be a complex analytic space. The underlying set of a closed analytic subspace of X is an analytic set in X.

In particular, the support of a coherent sheaf of $\mathcal{O}_X$-modules is an analytic set in X.

Proposition 4.3. Let X be a complex analytic space and Y be a closed analytic subspace of X. Then each analytic set A in Y is also an analytic set in X.

Conversely, if A is an analytic subset of X, then $A \cap Y$ is an analytic set in Y.

Proof. We prove the first part. Let A be an analytic set in Y. Then A is closed in Y. It follows that A is closed in X. Let $a \in A$, we can find an open neighbourhood V of a in Y and finitely many $g_1, \ldots, g_k \in \mathcal{O}_Y(V)$ such that

$$A \cap V = N_V(g_1, \ldots, g_k).$$

Up to shrinking V, we may find a neighbourhood U of a in X with $V = Y \cap U$ and $f_1, \ldots, f_k \in \mathcal{O}_X(U)$ lifting $g_1, \ldots, g_k$. Then

$$A \cap U = N_U(f_1, \ldots, f_k) \cap Y.$$

So by Example 4.2, $A \cap U$ is analytic at a as a subset of X.

The second part is obvious. □

Definition 4.4. Let X be a complex analytic space and $A \subseteq X$ be an analytic set. We define the *sheaf of ideals* $\mathcal{J}_A$ of A as the sheafification of the presheaf of ideals on X defined by

$$U \mapsto \{f \in \mathcal{O}_X(U) : N_U(f) \supseteq M \cap U\}$$

for any open subset $U \subseteq X$.

Observe that $\mathcal{J}_A$ is reduced.

Lemma 4.5. Let X be a complex analytic space and $A, B \subseteq X$ be analytic sets. Take $x \in X$. Then the following are equivalent:

(1) $\mathcal{J}_{A,x} \subseteq \mathcal{J}_{B,x}$;
(2) $A \cap U \supseteq B \cap U$ for some neighbourhood U of x in X.

Proof. (2) $\implies$ (1): This is trivial.

(1) $\implies$ (2): Choose a neighbourhood U of x and finitely many $f_1, \ldots, f_k \in \mathcal{O}_X(U)$ such that $A \cap U = N_U(f_1, \ldots, f_k)$. Then $f_{1,x}, \ldots, f_{k,x} \in \mathcal{J}_{A,x} \subseteq \mathcal{J}_{B,x}$. Up to shrinking U, we may assume that $f_1, \ldots, f_k \in \mathcal{J}_B(U)$. It follows that $A \cap U \supseteq B \cap U$. □

Lemma 4.6. Let X be a complex analytic space and A be an analytic set in X. Take $a \in A$. Let $\mathcal{I}$ be a coherent ideal sheaf on X with $\mathcal{I}_a = \mathcal{J}_{A,a}$. Then there is an open neighbourhood U of a in X such that

$$W(\mathcal{I}|_U) = A \cap U.$$

The lemma tells that an analytic set can always be locally wrtten in the form $W(\mathcal{I})$ for some open set $U \subseteq X$ and a coherent ideal $\mathcal{I}$ on U.

Proof. Choose an open neighbourhood U of x in X and finitely many sections $f_1, \ldots, f_k \in \mathcal{J}_A(U)$ such that

$$\mathcal{I}|_U = \mathcal{O}_U f_1 + \cdots + \mathcal{O}_U f_k.$$

After shrinking U, we may assume that

$$A \cap U = N_U(g_1, \ldots, g_l)$$

for finitely many $g_1, \dots, g_l \in \mathcal{J}_A(U)$. Then $g_{1,a}, \dots, g_{l,a} \in \mathcal{J}_{A,a} = \mathcal{I}_a$. So up to shrinking U, we can find equations for all $j = 1, \dots, l$:

$$g_j = \sum_{i=1}^{k} a_{ij} f_i$$

for some $a_{ij} \in \mathcal{O}_X(U)$ with $i = 1, \dots, k$, $j = 1, \dots, l$. This implies that $W(\mathcal{I}|_U) \subseteq A \cap U$. The reverse inclusion is clear. $\square$

5. Lasker–Noether decomposition

Definition 5.1. Let X be a complex analytic space. An analytic set A in X is *irreducible* at $a \in A$ if $\mathcal{J}_{A,a}$ is a prime ideal in $\mathcal{O}_{X,a}$.

Definition 5.2. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. A *local decomposition* of A at a consists of an open neighbourhood U of a in X and finitely many analytic sets $A_1, \dots, A_s$ in U such that

(1)
$$A \cap U = A_1 \cup \cdots \cup A_s;$$
(2) A_i is irreducible at a for $i = 1, \dots, s$;
(3) for any open neighbourhood V of a in U, $A_j \cap V \not\subseteq A_k \cap V$ for $j, k = 1, \dots, s$, $j \neq k$.

We also say $A_1 \cup \cdots \cup A_s$ is a *local decomposition* of $A \cap U$.

Proposition 5.3. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let

$$\mathcal{J}_{A,a} = \bigcap_{j=1}^{s} \mathfrak{p}_j$$

be the Lasker–Noether decomposition. Then there is a local decompose of A at a:

$$A \cap U = A_1 \cup \cdots \cup A_s$$

with $\mathcal{J}_{A_j,a} = \mathfrak{p}_j$ for $j = 1, \dots, s$.

Let $A \cap U' = A'_1 \cup \cdots \cup A'_r$ be another local decomposition of A at a. Then $r = s$ and we can find an open neighbourhood $W \subseteq U \cap U'$ and a bijection $\sigma : \{1, \dots, s\} \to \{1, \dots, s\}$ such that

$$A'_j \cap W = A_{\sigma(j)} \cap W$$

for $j = 1, \dots, s$.

PROOF. We first prove the existence part. Take an open neighbourhood U of a in X and coherent ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_s$ on U such that

$$\mathcal{I}_{j,a} = \mathfrak{p}_j$$

for $j = 1, \dots, s$. Define

$$\mathcal{I} = \bigcap_{j=1}^{s} \mathcal{I}_j.$$

Then $\mathcal{I}_a = \mathcal{J}_{A,a}$. By Lemma 4.6, up to shrinking U, we may guarantee that

$$W(\mathcal{I}) = A \cap U.$$

We set $A_j = W(\mathcal{I}_j)$ for $j = 1, \dots, s$. Then A_j is an analytic set in U and

$$A \cap U = W(\mathcal{I}) = \bigcup_{j=1}^{s} W(\mathcal{I}_j) = A_1 \cup \cdots \cup A_s.$$

Observe that $\mathfrak{p}_j = \mathcal{I}_{j,a} \subseteq \mathcal{J}_{A_j,a}$ for all $j = 1, \dots, s$. We need to prove the reverse inclusion. Assume that this is not true, say it fails for $j = 1$. Then there is $g_1 \in \mathcal{J}_{A_1,a} \setminus \mathfrak{p}_1$. As $\mathfrak{p}_j \not\subset \mathfrak{p}_1$ for $j = 2, \dots, s$, we can find $g_j \in \mathfrak{p}_j \setminus \mathfrak{p}_1$ for $j = 2, \dots, s$. Then

$$g_1 \cdots g_s \in \mathcal{J}_{A_1,a} \cap \cdots \cap \mathcal{J}_{A_s,a} = \mathcal{J}_{A,a} \subseteq \mathfrak{p}_1.$$

This is a contradiction. So $\mathcal{J}_{A_j,a} = \mathfrak{p}_j$ for $j = 1, \dots, s$. We conclude that $A \cap U = A_1 \cup \cdots \cup A_s$ is a local decomposition by Lemma 4.5.

Next we prove the uniqueness statement. We take U' and $A'_1, \dots, A'_r$ as in the statement of the theorem. Then

$$\mathcal{J}_{A,a} = \mathcal{J}_{A'_1,a} \cap \cdots \cap \mathcal{J}_{A'_r,a}.$$

By Lemma 4.5, we find that this is the Lasker–Noether decomposition of $\mathcal{J}_{A,a}$. The uniqueness follows from the uniqueness of Lasker–Noether decomposition and Lemma 4.5. □

Definition 5.4. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let

$$A \cap U = A_1 \cup \cdots \cup A_s$$

be a local decomposition of A at a. We call $A_{1,a}, \dots, A_{s,a}$ the *prime components* of A at a.

By Proposition 5.3, the prime compoments are uniquely determined by the germ of X at x.

Lemma 5.5. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. Let $A_1, \dots, A_s$ be the prime components of A at a. Then A_1 is not contained in $A_2 \cup \cdots \cup A_s$.

Proof. If not, we have

$$\mathcal{J}_{A_1,a} \supseteq \bigcap_{j=2}^{s} \mathcal{J}_{A_j,a}.$$

So

$$\mathcal{J}_{A,a} = \bigcap_{j=2}^{s} \mathcal{J}_{A_j,a}.$$

This contradicts the uniqueness of the Lasker–Noether decomposition. □

Proposition 5.6. Let X be a complex analytic space, A be an analytic set in X and $a \in A$. The following are equivalent:

(1) A is not irreducible at a;

(2) there is an open neighbourhood U of a in X and a decomposition

$$A \cap U = A' \cup A'',$$

where A' and A'' are analytic sets in U such that $A'_a \neq A_a$ and $A''_a \neq A_a$.

Proof. (1) $\implies$ (2): Let $A_{1,x},\dots,A_{s,x}$ be the prime components of A at a. Then $s\geq 2$. Take an open neighbourhood U of a in X such that $A_{1,x},\dots,A_{s,x}$ lifts to analytic subsets $A_1,\dots,A_s$ of U. It suffices to let $A'=A_1$ and $A''=A_2\cup\cdots\cup A_s$. By Lemma 5.5, A' and A'' satisfies the conditions in (2).

(2) $\implies$ (1): We have $\mathcal{J}_{A,a}\neq\mathcal{J}_{A',a}$ and $\mathcal{J}_{A,a}\neq\mathcal{J}_{A'',a}$. Take $f\in\mathcal{J}_{A',a}\setminus\mathcal{J}_{A,a}$ and $g\in\mathcal{J}_{A'',a}\setminus\mathcal{J}_{A,a}$. Then $fg\in\mathcal{J}_{A',a}\cap\mathcal{J}_{A'',a}=\mathcal{J}_{A,a}$. So $\mathcal{J}_{A,a}$ is not a prime ideal. □

6. Diagonal morphism

Definition 6.1. Let $f:X\to Y$ be a morphism of complex analytic space. The *diagonal* of f is by definition the morphism:

$$\Delta_f=\Delta_{X/Y}:X\to X\times_Y X$$

induced by the identity maps $X\to X$ and $X\to X$.

When $Y=\mathbb{C}^0$, we write Δ_X instead of $\Delta_{X/\mathbb{C}^0}$.

Example 6.2. Let $n\in\mathbb{N}$. The diagonal morphism $\mathbb{C}^n\to\mathbb{C}^n\times\mathbb{C}^n$ is a closed immersion corresponding to the ideal generated by $p_1^*z_1-p_2^*z_1,\dots,p_1^*z_n-p_2^*z_n$, where $p_1,p_2:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$ are the two projections and $z_1,\dots,z_n$ are the coordinates on $\mathbb{C}^n$.

This can be seen through the functor of points by Theorem 4.2 in The notion of complex analytic spaces.

Proposition 6.3. Let $f:X\to Y$ be a morphism of complex analytic space. Then $\Delta_{X/Y}$ is an immersion.

Proof. **Step 1**. We first reduce to the case $Y=\mathbb{C}^0$.

By general abstract nonsense, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta_{X/Y}"'] \arrow[rr, bend left, "\Delta_X"] & X\times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X\times X \arrow[d] \\
 & Y \arrow[r, "\Delta_Y"] & Y\times Y
\end{tikzcd}$$

So in order to show that $\Delta_{X/Y}$ is an immersion, it suffices to show that X and Y are.

Step 2. We reduce to the case $X=\mathbb{C}^n$ for some $n\in\mathbb{N}$.

We want to show that $\Delta_X:X\to X\times X$ is an immersion.

The problem is local on X, so we may assume that X is a complex model space, say X is a closed analytic subspace of an open set U in $\mathbb{C}^n$ for some $n\in\mathbb{N}$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d] & X\times X \arrow[d] \\
U \arrow[r, "\Delta_U"] & U\times U
\end{tikzcd}.$$

It suffices to show that Δ_U is an immersion. As the problem is local, it suffices to show that $\Delta_{\mathbb{C}^n}$ is an immersion.

Step 3. We show that $\Delta_{\mathbb{C}^n}$ is a closed immersion.

This is exactly Example 6.2. □

7. Conormal sheaf

Definition 7.1. Let $i : X \to Y$ be an immersion of complex analytic spaces. The *conormal sheaf* of f is a sheaf of $\mathcal{O}_X$-modules $\mathcal{C}_f = \mathcal{C}_{X/Y}$ with $i_*\mathcal{C}_{X/Y} \cong \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the kernel of $i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

The conormal sheaf is defined up to a unique isomorphism. A choice of a factorization of i into a closed immersion $i' : X \to Z$ followed by an open immersion $j : Z \to Y$ determines a realization of $\mathcal{C}_{X/Y}$. Namely, if $\mathcal{J}$ is the ideal sheaf of i', then $\mathcal{C}_{X/Y}$ is (isomorphic to) $i'^*\mathcal{J}$.

Proposition 7.2. Let $i : X \to Y$ be an immersion of complex analytic spaces. Then $\mathcal{C}_{X/Y}$ is coherent.

Proof. We may assume that i is a closed immersion defined by a coherent ideal $\mathcal{J}$. Then $\mathcal{C}_{X/Y} \cong i^*\mathcal{J}$ is coherent by Corollary 7.5 in The notion of complex analytic spaces. $\square$

8. Kähler differentials

We will make free use of results and notations in [Stacks, Tag 08RL]. In particular, for a morphism $f : X \to S$ of complex analytic spaces, $\Omega_{X/S}$ denotes the sheaf of Kähler differentials and $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$ denotes the universal S-derivation.

Include principal parts etc. here

Proposition 8.1. Let $f : X \to S$ be a morphism of complex analytic spaces. Then there is a canonical isomorphism

$$\Omega_{X/S} \xrightarrow{\sim} \mathcal{C}_{\Delta_{X/S}}.$$

Proof. We first define the map in question. Factorize $\Delta_{X/S}$ as $X \to W \to X \times_S X$, where $X \to W$ is a closed immersion define by a coherent ideal $\mathcal{I}$ and $W \to X \times_S X$ is an open immersion. We have a short exact sequence

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta_{X/S}^{-1}(\mathcal{O}_W/\mathcal{I}^2) \to \mathcal{O}_X \to 0.$$

Let $p_1, p_2 : X \times_S X \to X$ be the two projection maps. Then the natural maps $p_i^\# : p_i^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$ induce $p_i^{-1}\mathcal{O}_X \to \mathcal{O}_W/\mathcal{I}^2$ for $i = 1, 2$. Take Δ^{-1}, we find natural maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

The difference $\mathrm{d} = s_2 - s_1$ is clearly an S-derivation. By the universal property of $\Omega_{X/S}$, we get a unique $\mathcal{O}_X$-linear map $\Omega_{X/S} \to \mathcal{C}_{X/X\times_S X}$.

Now in order to verify

$$\Omega_{X/S} \xrightarrow{\sim} \mathcal{C}_{\Delta_{X/S}}$$

is an isomorphism, it suffices to work on each stalk. This reduces the problem to the corresponding problem of local rings, which is handled in [Stacks, Tag 08S2]. $\square$

We will write $\mathcal{P}^{(1)}_{X/S}$ for $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2)$ introduced in the proof.

Corollary 8.2. Let $f : X \to S$ be a morphism of complex analytic spaces. Then $\Omega_{X/S}$ is coherent.

Proof. This follows from Proposition 8.1 and Proposition 7.2. $\square$

Proposition 8.3. Let $f : X \to Y$, $g : Y \to S$ be morphisms of complex analytic spaces. Then there is a canonical exact sequence

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.$$

PROOF. The existence of the morphisms is obvious. To prove that the sequence is exact, it suffices to localize along each $x \in X$. The result then follows from the algebraic case [Stacks, Tag 01UX]. □

Proposition 8.4. Let $X \to S$ be a morphism of complex analytic spaces and $i : Z \to X$ be an immersion. Then we have a canonical exact sequence

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

PROOF. The existence of the morphisms is obvious. To prove that the sequence is exact, it suffices to localize along each $x \in X$. The result then follows from the algebraic case [Stacks, Tag 01UZ]. □

Proposition 8.5. Let $f : X \to S$, $g : S' \to S$ be morphisms of complex analytic spaces. Consider the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "f"] & S \end{tikzcd}.$$

Then we have a canonical isomorphism

$$g'^*\Omega_{X/S} \to \Omega_{X'/S'}.$$

PROOF. It suffices to show that the canonical morphism $g'^*\mathcal{P}^{(1)}_{X/S} \to \mathcal{P}^{(1)}_{X'/S'}$ is an isomorphism. For this purpose, it suffices to prove it after localizing around $x' \in X'$. But observe that the local rings of $\mathcal{P}^{(1)}_{X/S}$ are finite over the corresponding local rings of X, so the analytic tensor products reduce to usual tensor products. The result then follows from the corresponding algebraic results. □

Corollary 8.6. Let $f : X \to S$, $g : X \to S$ be morphisms of complex analytic spaces. Consider the Cartesian diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd}.$$

Then we have a canonical isomorphism

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \to \Omega_{X \times_S Y/S}.$$

PROOF. The existence of the morphism follows from [Stacks, Tag 08RU]. By Proposition 8.5, the composition

$$p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to \Omega_{X\times_S Y/Y}$$

is an isomorphism. In particular, $p^*\Omega_{X/S} \to \Omega_{X\times_S Y/Y}$ is injective. Similarly, we have a natural isomorphism

$$q^*\Omega_{Y/S} \xrightarrow{\sim} \Omega_{X\times_S Y/X}$$

By Proposition 8.3, we have a short exact sequence

$$0 \to p^*\Omega_{X/S} \to \Omega_{X\times_S Y/S} \to q^*\Omega_{Y/S} \to 0,$$

which clearly splits. □

Example 8.7. Let $n \in \mathbb{N}$. We claim that $\Omega_{\mathbb{C}^n}$ is the free $\mathcal{O}_{\mathbb{C}^n}$-module generated by $\mathrm{d}z_1, \dots, \mathrm{d}z_n$, where $z_1, \dots, z_n \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ are the coordinates on $\mathbb{C}^n$.

By Example 6.2, we know that $\Omega_{\mathbb{C}^n}$ is generated by $\mathrm{d}z_1, \dots, \mathrm{d}z_n$ as an $\mathcal{O}_{\mathbb{C}^n}$-module. Assume that there is $x \in \mathbb{C}^n$, $f_{1,x}, \dots, f_{n,x} \in \mathcal{O}_{X,x}$ such that

$$\sum_{i=1}^{n} f_{i,x}\,\mathrm{d}z_i = 0.$$

We need to show that $f_{i,x} = 0$ for all $i = 1, \dots, n$. We may assume that $x = 0$. Observe that

$$\Omega^1_{\mathbb{C}^n,0} \otimes_{\mathcal{O}_{\mathbb{C}^n,0}} \mathbb{C} \xrightarrow{\sim} \mathfrak{m}_0/\mathfrak{m}_0^2$$

by the algebraic results. Taking the residue of our linear relation at 0, we find

$$\sum_{i=1}^{n} f_{i,0} z_{i,0} \in \mathfrak{m}_0^2.$$

As $z_{i,0}, \dots, z_{n,0}$ form a basis of $\mathfrak{m}_0/\mathfrak{m}_0^2$, we have $f_{i,0} = 0$ for $i = 1, \dots, n$.

Bibliography

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