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The notion of complex analytic spaces

1. Introduction

We introduce the notion of complex analytic spaces in this section.

2. \mathbb{C} -ringed space

Definition 2.1. A \mathbb{C} -ringed space is a pair (X, \mathcal{O}_X) consisting of a topological space X and a sheaf \mathcal{O}_X of \mathbb{C} -algebras on X .

A *morphism of \mathbb{C} -ringed spaces* $f : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ is a pair consisting of a continuous map $f : Y \rightarrow X$ and a morphism of sheaves of \mathbb{C} -algebras $f^\# : f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Y$.

Given two morphisms of \mathbb{C} -ringed spaces $f : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$ and $g : (Z, \mathcal{O}_Z) \rightarrow (Y, \mathcal{O}_Y)$, their *composition* is the morphism $f \circ g : (Z, \mathcal{O}_Z) \rightarrow (X, \mathcal{O}_X)$ consisting of the continuous map $f \circ g : Z \rightarrow X$ and a morphism of sheaves $(f \circ g)^\# = g^\# \circ f^{-1}f^\# : (f \circ g)^{-1}\mathcal{O}_X \xrightarrow{\sim} g^{-1}f^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_Z$.

When there is no risk of confusion, we say X is a \mathbb{C} -ringed space. In this case, we write $|X|$ for the topological space underlying X .

It is straightforward to verify that \mathbb{C} -ringed spaces form a category, which we denote by $\mathbb{C}\text{-}\mathcal{RS}$. Similarly, we denote by \mathcal{RS} the category of ringed spaces defined in [Stacks, Tag 0090].

In fact, by definition a \mathbb{C} -ringed space is nothing but a morphism in the category of ringed spaces $X \rightarrow \mathbb{C}^0$, where \mathbb{C}^0 is a single point $*$ endowed with the sheaf of rings $\mathcal{O}_{\mathbb{C}^0}$ with $\mathcal{O}_{\mathbb{C}^0}(*) = \mathbb{C}$. In terms of slice categories, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{RS} \approx \mathcal{RS}/\mathbb{C}^0.$$

From this identification, most of the basic results above $\mathbb{C}\text{-}\mathcal{RS}$ follows, which we will use freely.

There is an obvious faithful forget functor $\mathbb{C}\text{-}\mathcal{RS} \rightarrow \mathcal{RS}$.

Definition 2.2. A *locally \mathbb{C} -ringed space* is a \mathbb{C} -ringed space (X, \mathcal{O}_X) which when regarded as a ringed space is a locally ringed space.

A *morphism* between two locally \mathbb{C} -ringed spaces is a morphism between the underlying \mathbb{C} -ringed spaces which is a morphism of locally ringed spaces at the same time.

The category of locally \mathbb{C} -ringed spaces is denoted by $\mathbb{C}\text{-}\mathcal{LRS}$.

We refer to [Stacks, Tag 01HA] for the notion of locally ringed spaces. Similar to the case of \mathbb{C} -ringed space, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{LRS} \approx \mathcal{LRS}/\mathbb{C}^0.$$

Example 2.3. Let $n \in \mathbb{N}$, we define a sheaf of \mathbb{C} -algebras $\mathcal{O}_{\mathbb{C}^n}$ on \mathbb{C}^n as follows: for any open subset $U \subseteq \mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^n}(U)$ is the \mathbb{C} -algebra of holomorphic functions on U . It is easy to see that $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf and $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a \mathbb{C} -ringed space. Moreover, it is easy to show that $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a locally \mathbb{C} -ringed space.

Proposition 2.4. Let $n \in \mathbb{N}$, $w \in \mathbb{C}^n$, then there is a natural isomorphism $\mathcal{O}_{\mathbb{C}^n, w} \cong \mathbb{C}\{z_1, \dots, z_n\}$.

The ring on the right-hand side is defined in [Definition 2.1](#) in the Complex Analytic Local Algebras.

PROOF. This is a well-known result from classical complex analysis. [Include details later.](#) \square

3. Complex model spaces and complex analytic spaces

Definition 3.1. Given any domain D in \mathbb{C}^n , we can define a sheaf of \mathbb{C} -algebras \mathcal{O}_D on D as the restriction of $\mathcal{O}_{\mathbb{C}^n}$ defined in [Example 2.3](#) to D . Observe that (D, \mathcal{O}_D) is a locally \mathbb{C} -ringed space.

Definition 3.2. A *complex model space* is a \mathbb{C} -ringed space (X, \mathcal{O}_X) such that there exist

- (1) a domain D in \mathbb{C}^n for some $n \in \mathbb{N}$ and
- (2) an ideal sheaf \mathcal{I} in \mathcal{O}_D of finite type

such that there is an isomorphism

$$(X, \mathcal{O}_X) \cong (\text{Supp } \mathcal{O}_D/\mathcal{I}, i^{-1}(\mathcal{O}_D/\mathcal{I}))$$

in the category of \mathbb{C} -RS, where $i : \text{Supp } \mathcal{O}_D/\mathcal{I} \rightarrow D$ is the inclusion map. Here \mathcal{O}_D is the sheaf of \mathbb{C} -algebras defined in [Definition 3.1](#).

Clearly, (X, \mathcal{O}_X) is a locally \mathbb{C} -ringed space.

Observe that X is always a Hausdorff space.

Definition 3.3. A *complex analytic space* is a locally \mathbb{C} -ringed space (X, \mathcal{O}_X) such that for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is isomorphic to a complex model space in the sense of [Definition 3.2](#) in the category \mathbb{C} -LRS.

When there is no risk of confusion, we also omit \mathcal{O}_X from the notation say X is a complex analytic space.

A morphism between complex analytic spaces is a morphism of the underlying locally \mathbb{C} -ringed spaces. Such a morphism is also known as a *holomorphic map*.

The category of complex analytic spaces is denoted as $\mathbb{C}\text{-An}$.

Remark 3.4. Most authors require X to be Hausdorff as well. We do not follow this convention. We require it whenever necessary.

Remark 3.5. Most of the authors require extra conditions in the definition of a complex analytic space: σ -compactness, paracompactness, having countable basis etc. We will not put these constraints in the definition, instead, we choose to include them into the assumptions of the theorems.

Proposition 3.6. Let X be a complex analytic space, $x \in X$. Then $\mathcal{O}_{X, x}$ is a complex analytic local algebra.

Recall that complex analytic local algebras are defined in [Definition 5.1](#) in the Complex Analytic Local Algebras.

PROOF. The problem is local, so we may assume that X is a complex model space. In this case, the result follows easily from [Proposition 2.4](#). \square

Let X be a complex analytic space. It is a sheaf of \mathbb{C} -algebras. For any sheaf of local \mathbb{C} -algebras \mathcal{A} on X , any open set $U \subseteq X$ and any $s \in \mathcal{A}_X(U)$. We want to construct a function $[s] : U \rightarrow \mathbb{C}$.

Take $x \in U$, there is a canonical splitting

$$(3.1) \quad \mathcal{A}_x \cong \mathbb{C} \oplus \mathfrak{m}_x,$$

where \mathfrak{m}_x is the maximal ideal of \mathcal{A}_x . Then we define $[s](x)$ as the image of s_x in the \mathbb{C} -factor in [\(3.1\)](#). When there is no risk of confusion, we write $s(x)$ instead of s .

Definition 3.7. Let X, \mathcal{A}, U, x, s be as above. The value $[s](x) \in \mathbb{C}$ is called the *value* of s at x . We sometimes denote it by $s(x)$ as well.

Lemma 3.8. Let X be a complex analytic space. We denote by \mathcal{C}_X the sheaf of continuous functions on X . The association $s \mapsto [s]$ in [Definition 3.7](#) defines a homomorphism of sheaves of \mathbb{C} -algebras $\mathcal{O}_X \rightarrow \mathcal{C}_X$.

When there is no risk of confusion, we also write s instead of $[s]$.

PROOF. We need to show that for any open set $U \subseteq X$ and any $s \in \mathcal{O}_X(U)$, $[s]$ is a continuous function on U .

We may clearly assume that $U = X$. The problem is local on X , so we may assume that X is a complex model space in the sense of [Definition 3.2](#) defined by a coherent ideal \mathcal{I} in a domain D in \mathbb{C}^n . By further localizing, we may assume that s can be lifted to a section $f \in \mathcal{O}_D(D)$. Then $[s] = f|_X$ by definition. So the assertion follows from the fact that a holomorphic function on a domain is continuous. \square

4. The spaces \mathbb{C}^n

Fix $n \in \mathbb{N}$. In this section, we will study the complex analytic space \mathbb{C}^n . We will write $z_1, \dots, z_n \in \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$ the coordinate functions.

Lemma 4.1. Given $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, we have

$$a_i = z_i(a) = \epsilon_a(z_i)$$

for $i = 1, \dots, n$. Here $\epsilon_a : \mathcal{O}_{\mathbb{C}^n, a} \rightarrow \mathbb{C}$ is the augmentation map of $\mathcal{O}_{\mathbb{C}^n, a}$.

PROOF. This follows by expanding the definitions. \square

Given a complex analytic space X , we will construct a canonical map

$$(4.1) \quad \varphi(X) : \text{Hom}_{\mathbb{C}\text{-An}}(X, \mathbb{C}^n) \rightarrow \mathcal{O}_X(X)^n$$

as follows: given a morphism $f : X \rightarrow \mathbb{C}^n$, we let $\varphi_i(f) = f^*(z_i) \in \mathcal{O}_X(X)$ for $i = 1, \dots, n$. Then we define $\varphi(X)(f) = (\varphi_1(f), \dots, \varphi_n(f))$.

THEOREM 4.2. Let X be a complex analytic space. Then the map [\(4.1\)](#) is bijective and is functorial in X .

PROOF. The functoriality is clear. It suffices to show that (4.1) is bijective.

We first show that $\varphi(X)$ is injective. Namely, $\varphi(X)(f)$ determines the morphism $f : X \rightarrow \mathbb{C}^n$. Take $x \in X$ and $a = f(x)$. By Lemma 4.1, we have

$$z_i(f(x)) = \epsilon_a(\varphi_i(f))$$

for $i = 1, \dots, n$. So set-theoretically, f is determined by $\varphi(X)(f)$.

Let $a_i = z_i(a) \in \mathbb{C}$ for $i = 1, \dots, n$. Then $z'_i = z_i - a$ for $i = 1, \dots, n$ form a regular system of parameters of $\mathcal{O}_{\mathbb{C}^n, a}$. Consider $f_x^\# : \mathcal{O}_{\mathbb{C}^n, a} \rightarrow \mathcal{O}_{X, x}$, then

$$f_x^\#(z'_i) = \varphi_i(f)_x - a_i$$

for $i = 1, \dots, n$. So $\varphi(X)(f)$ uniquely determines $f_x^\#(z'_i)$. Now f_x induces by completion

$$\widehat{\mathcal{O}_{\mathbb{C}^n, a}} \cong k[[z'_1, \dots, z'_n]] \rightarrow \widehat{\mathcal{O}_{X, x}}.$$

This map is uniquely determined by the $f_x^\#(z'_i)$'s hence by $\varphi(X)(f)$. In particular, $f_x^\# : \mathcal{O}_{\mathbb{C}^n, a} \rightarrow \mathcal{O}_{X, x}$ is uniquely determined by $\varphi(X)(f)$. Hence, the morphism f is uniquely determined.

Next we show that φ is surjective. We first observe that φ induces a morphism from the sheaf

$$(U \subseteq X) \mapsto \text{Hom}_{\mathbb{C}\text{-}\mathcal{A}\text{n}}(U, \mathbb{C}^n)$$

on X to \mathcal{O}_X^n . So the surjectivity of φ is local on X . It suffices to show that for $f_1, \dots, f_n \in \mathcal{O}_X(X)$, there exists a morphism from a neighbourhood U of x in X to \mathbb{C}^n such that $\varphi_i(f)_x = f_{i, x}$ for all $i = 1, \dots, n$. Up to shrinking X , we may assume that X is a closed subspace of an open subset V in \mathbb{C}^m for some $m \in \mathbb{N}$ and f_i lifts to $g_i \in \mathcal{O}_V(U)$ for $i = 1, \dots, n$. Let $g : U \rightarrow \mathbb{C}^n$ be the morphism induced by (g_1, \dots, g_n) . Then $\varphi_i(g) = g_i$ for $i = 1, \dots, n$. Let $f : X \rightarrow \mathbb{C}^n$ be the morphism induced by g . Then $\varphi_i(f) = (\varphi_i(g))|_X = g_i|_X = f_i$. \square

5. Open and closed immersions

Definition 5.1. A morphism $f : X \rightarrow Y$ of complex analytic spaces is an *open immersion* if it is an open immersion of locally ringed spaces.

Recall that an open immersion of locally ringed spaces is defined in [Stacks, Tag 01HE].

Example 5.2. Let X be a complex analytic space and U be an open subset of X . Then U has a structure of complex analytic space induced from X . The inclusion $U \hookrightarrow X$ is an open immersion. We say U is an *open subspace* of X .

Definition 5.3. A morphism $f : X \rightarrow Y$ of complex analytic spaces is a *closed immersion* if it is a closed immersion of locally ringed spaces.

The kernel of the canonical morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is called the *ideal* of X in Y .

Recall that a closed immersion of locally ringed spaces is defined in [Stacks, Tag 01HK]. Note that we have a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X \rightarrow 0.$$

Later on, we will show that \mathcal{I} is coherent after proving Oka's coherence theorem.

Example 5.4. Let X be a complex analytic space and \mathcal{I} be a subsheaf of \mathcal{O}_X locally generated by sections as a sheaf of \mathcal{O}_X -modules in the sense of [Stacks, Tag 01B2]. Set $Z = \text{Supp } \mathcal{O}_X/\mathcal{I}$ and let $i : Z \rightarrow X$ be the inclusion map. Let \mathcal{O}_Z be the unique sheaf of rings on Z such that $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$, whose existence and uniqueness is guaranteed by [Stacks, Tag 01AX]. Then (Z, \mathcal{O}_Z) is a complex analytic space and $i : Z \rightarrow X$ is a closed immersion of complex analytic spaces. We say Z is the *closed subspace* of X defined by \mathcal{I} .

Example 5.5. Let X be a complex analytic space and $x \in X$. Then $\{x\}$ is a closed analytic subspace of X . In fact, it corresponds to the ideal \mathfrak{m}_x .

Definition 5.6. A morphism $f : X \rightarrow Y$ of complex analytic spaces is an *immersion* if it can be factorized as $j \circ i$ where i is a closed immersion and j is an open immersion.

6. Weierstrass map

Definition 6.1. Let $d \in \mathbb{N}$ and B be a domain (non-empty open subset) in \mathbb{C}^d . Let $\omega_j \in \mathcal{O}_B(B)[w_j] \subseteq \mathcal{O}_B(B)[w_1, \dots, w_k]$ be monic polynomials for $j = 1, \dots, k$ for some $k \in \mathbb{N}$. We let A be the closed subspace of $B \times \mathbb{C}^k$ defined by the ideal generated by $\omega_1, \dots, \omega_k$. The projection map $B \times \mathbb{C}^k \rightarrow B$ induces a map $\pi : A \rightarrow B$. The map π is called the *Weierstrass map* defined by $\omega_1, \dots, \omega_k$ over B .

Theorem 6.2. Let $d \in \mathbb{N}$ and B be a domain in \mathbb{C}^d . Let $\omega_j \in \mathcal{O}_B(B)[w_j] \subseteq \mathcal{O}_B(B)[w_1, \dots, w_k]$ be monic polynomials of degree b_j for $j = 1, \dots, k$ for some $k \in \mathbb{N}$. Then the Weierstrass map $\pi : A \rightarrow B$ defined by $\omega_1, \dots, \omega_k$ over B is topologically finite and open.

PROOF. We first prove that $\pi : A \rightarrow B$ is topologically finite. The only non-trivial point is to show that π is closed. Let M be a closed subset in A and y be a point in the closure of $\pi(M)$ in B . Then we can find a sequence $(y_i, c_{1i}, \dots, c_{ki}) \in M$ with $y_i \in B$ and $(c_{1i}, \dots, c_{ki}) \in \mathbb{C}^k$ for $i \in \mathbb{N}$ such that $y_i \rightarrow y$ in B as $i \rightarrow \infty$. Then for each $i \in \mathbb{N}$ and $j = 1, \dots, k$, (y_i, c_{ji}) is a solution to $\omega_j(y_i, \bullet)$. By continuity of roots, up to extracting a subsequence, we can find $c_j \in \mathbb{C}$ such that $c_{ji} \rightarrow c_j$ when $i \rightarrow \infty$ for $j = 1, \dots, k$. It follows that $y = \pi(y, c_1, \dots, c_k) \in \pi(M)$.

It remains to show that π is open. Take $p = (q, c) \in A$ with $q \in B$ and $c = (c_1, \dots, c_k) \in \mathbb{C}^k$. Let U be an open neighbourhood of p in A . We need to show that $\pi(U)$ contains an open neighbourhood of $\pi(p)$. We may assume that $U = A \cap (D \times W)$, where D is an open neighbourhood of q in B and W is an open neighbourhood of c in \mathbb{C}^k .

By the proof of Theorem 2.11 in Complex analytic local algebras, choosing D small enough, we may guarantee that there are monic polynomials $\omega'_j, \omega''_j \in \mathcal{O}(B)[w_j]$ and $n_j \in \mathbb{Z}_{>0}$ such that

$$\omega_j|_{D \times \mathbb{C}^k} = \omega'_j \omega''_j, \quad \omega'_j(q, w) = (w - c_j)^{n_j}$$

for $j = 1, \dots, k$. Let A' be the closed subspace of $D \times \mathbb{C}^k$ defined by $\omega'_1, \dots, \omega'_k$. Then $A' \subseteq A$ by construction. Let $\pi' : A' \rightarrow D$ be the natural projection. Then p is the only point on $\pi'^{-1}(q)$. We claim that there is an open neighbourhood $V \subseteq D$ of q such that

$$A' \cap (V \times \mathbb{C}^k) \subseteq V \times W.$$

In fact, by [Lemma 4.2](#) in [Topology and bornology](#), we can find an open neighbourhood $V \subseteq D$ of q such that

$$\pi'^{-1}(V) \subseteq A' \cap (D \times W).$$

But $\pi'^{-1}(V) = A' \cap (V \times \mathbb{C}^k)$. So our claim follows.

It follows that $V \subseteq \pi(U)$ and our assertion follows. \square

Lemma 6.3. Let $d \in \mathbb{N}$, B be a domain in \mathbb{C}^d and

$$\omega = w^b + a_1 w^{b-1} + \cdots + a_b \in \mathcal{O}_B(B)[w]$$

be a monic polynomial. Let $\pi : A \rightarrow B$ be the Weierstrass map defined by ω over B . For any $y \in B$ and x_1, \dots, x_n be the distinct points in the fiber $\pi^{-1}(y)$. Then for any $f_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$, there exist germs $q_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$ and a polynomial $r \in \mathcal{O}_y[w]$ with $\deg r < b$ such that

$$f_j = \omega_{x_j} q_j + r_{x_j}$$

for $j = 1, \dots, n$. The polynomial r and the germs q_1, \dots, q_n are uniquely determined.

PROOF. We write

$$\omega(y, w) = (w - c_1)^{b_1} \cdots (w - c_n)^{b_n}$$

with c_1, \dots, c_n with $x_i = (y, c_i)$ for $i = 1, \dots, n$ and $b_1, \dots, b_n \in \mathbb{Z}_{>0}$.

By [Theorem 2.11](#) in [Complex analytic local algebras](#), we can find $\omega_1, \dots, \omega_n \in \mathcal{O}_y[w]$ such that

$$\omega_{x_j} = \omega_{1x_j} \cdots \omega_{nx_j}, \quad \omega_j(y, w) = (w - c_j)^{b_j}$$

for $j = 1, \dots, n$. We define

$$e_j := \prod_{i \neq j} \omega_i \in \mathcal{O}_y[w]$$

for $j = 1, \dots, n$. Then e_{jx_j} is a unit in \mathcal{O}_{x_j} for $j = 1, \dots, n$ as $e_j(x_j) = \prod_{i \neq j} (c_j - c_i)^{b_i} \neq 0$.

By [Theorem 3.2](#) in [Complex analytic local algebras](#), each germ $f_j e_{jx_j}^{-1} \in \mathcal{O}_{x_j}$ can be written as

$$f_j e_{jx_j}^{-1} = \omega_{jx_j} q'_j + r_{jx_j},$$

where $q'_j \in \mathcal{O}_{x_j}$ and $r_j \in \mathcal{O}_y[w - c_j]$ with $\deg r_j < b_j$ for $j = 1, \dots, n$. Set $e_{ij} := \prod_{k \neq i, j} \omega_k \in \mathcal{O}_y[w]$ for any $i, j = 1, \dots, n$ with $i \neq j$. For $j = 1, \dots, n$, we define

$$q_j := q'_j - \sum_{i \neq j} r_{ix_j} e_{ijx_j}$$

and

$$r = r_1 e_1 + \cdots + r_n e_n \in \mathcal{O}_y[w].$$

Then $f_j = \omega_{x_j} q_j + r_{x_j}$ for $j = 1, \dots, n$. This proves the uniqueness part.

Next we show the uniqueness. Assume that

$$0 = \omega_{x_j} q_j + r_{x_j}$$

with $q_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$ and $r \in \mathcal{O}_y[w]$ with degree less than b . We need to show that $r = 0$. Assume by contrary that $r \neq 0$, then $p_j := r(\omega_1 \cdots \omega_j)^{-1} \neq 0$ for $j = 1, \dots, n$. Now $-r_{x_j} = (\omega_1 \cdots \omega_n)_{x_j} q_j$ implies that $p_{jx_j} = -q_j(\omega_{j+1} \cdots \omega_n)_{x_j} \in$

\mathcal{O}_{x_j} for $j = 1, \dots, n$. Since $\omega_{jx_j} \in \mathcal{O}_y[w - c_j]$ is a Weierstrass polynomial, it follows from [Lemma 4.2 in Complex analytic local algebras](#) and

$$r = p_1\omega_1, \quad p_{j-1} = p_j\omega_j \text{ for } j = 2, \dots, n$$

that p_1, \dots, p_n are all polynomials in w . As $r = p_n\omega$, we have a contradiction as $\deg r < b$. \square

Theorem 6.4. Let $d \in \mathbb{N}$, B be a domain in \mathbb{C}^d and

$$\omega = w^b + a_1w^{b-1} + \dots + a_b \in \mathcal{O}_B(B)[w]$$

be a monic polynomial. Let $\pi : A \rightarrow B$ be the Weierstrass map defined by ω over B . Then we have a natural isomorphism of \mathcal{O}_B -modules

$$\mathcal{O}_B^b \xrightarrow{\sim} \pi_*\mathcal{O}_A.$$

PROOF. We first define the map. Let $V \subseteq B$ be an open subset and $s = (s_0, \dots, s_{b-1}) \in \mathcal{O}_B(V)^b$. The polynomial $\sum_{j=0}^{b-1} s_j w^j$ determines a section $s' \in \mathcal{O}_A(\pi^{-1}(V)) = \pi_*\mathcal{O}_A(V)$. The map $s \mapsto s'$ clearly defines a map of \mathcal{O}_B -modules $\mathcal{O}_B^b \xrightarrow{\sim} \pi_*\mathcal{O}_A$. In order to prove that this map is an isomorphism, it suffices to do so for each germ. Let $y \in B$ and x_1, \dots, x_n denote the points in the fiber $\pi^{-1}(y)$. By [Theorem 6.2](#) and [Corollary 4.8 in Topology and bornology](#), we have a natural identification

$$(\pi_*\mathcal{O}_A)_y \xrightarrow{\sim} \prod_{j=1}^n \mathcal{O}_{A,x_j}.$$

A germ $g \in (\pi_*\mathcal{O}_A)_y$ corresponds to $(g_1, \dots, g_n) \in \prod_{j=1}^n \mathcal{O}_{A,x_j}$. By [Lemma 6.3](#), the latter can be uniquely lifted to $f_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$ such that if we define

$$r := \sum_{j=0}^{b-1} r_j w^j \in \mathcal{O}_y[w],$$

then r_{x_j} restricts to g_j for $j = 1, \dots, n$. This shows that the map of germs

$$\mathcal{O}_y^b \rightarrow (\pi_*\mathcal{O}_A)_y$$

is bijective. \square

7. Oka's coherence theorem

This lemma needs to be placed elsewhere. Proof at CAS p58 needs to be included

Lemma 7.1. Let X be a topological space and \mathcal{A} be a Hausdorff sheaf of rings on X (in the sense that the espace étalé of \mathcal{A} is Hausdorff) such that all stalks of \mathcal{A} are integral domains. Then \mathcal{A} is coherent if and only if for any open set $V \subseteq X$ and any section $s \in \mathcal{A}(V)$, $\mathcal{A}_V/s\mathcal{A}_V$ is coherent at every $x \in V$ where $s_x \neq 0$.

Lemma 7.2 (Oka). For any $n \in \mathbb{N}$, $\mathcal{O}_{\mathbb{C}^n}$ is coherent.

PROOF. As a preparation, observe that $\mathcal{O}_{\mathbb{C}^n}$ is a Hausdorff sheaf.

For any two germs $s_i \in \mathcal{O}_{\mathbb{C}^n, a_i}$ ($i = 1, 2$), we need to construct disjoint open neighbourhoods U_i in the espace étalé of $\mathcal{O}_{\mathbb{C}^n}$ of s_i . If $a_1 \neq a_2$, the assertion is clear. So assume that $a_1 = a_2 = 0$. We extend s_i to $f_i \in \mathcal{O}_{\mathbb{C}^n}(U)$ for a connected open neighbourhood $U \subseteq \mathbb{C}^n$ of 0. Then $\{f_x : x \in U\}$ and $\{g_x : x \in U\}$ are disjoint: if

for some $z \in U$, $f_z = g_z$, then the same holds in a neighbourhood of z and so $f = g$ on U by Identitätssatz. **Include the proof**

We will prove the coherence of $\mathcal{O}_{\mathbb{C}^n}$ by induction on n . The case $n = 0$ is trivial. Assume that $n > 0$ and the theorem has been proved for all smaller n . We will apply **Lemma 7.1**. Take an open set $U \subseteq \mathbb{C}^n$ and $g \in \mathcal{O}_{\mathbb{C}^n}(U)$. We need to show that $\mathcal{O}_U/g\mathcal{O}_U$ is coherent at all $x \in U$ with $g_x \neq 0$.

Fix such a point x , which may be assumed to be 0. We may assume that $g(0) = 0$ as otherwise, the stalk of $\mathcal{O}_U/g\mathcal{O}_U$ at 0 is trivial. By perturbing the coordinates, we may guarantee that $g_0(0, w)$ is not identically 0 for $w \in \mathbb{C}$. By Weierstrass preparation theorem ??, there is a Weierstrass polynomial $\omega_0 \in \mathcal{O}_{\mathbb{C}^{n-1}, 0}[w]$ such that $g_0\mathcal{O}_{\mathbb{C}^n, 0} = \omega_0\mathcal{O}_{\mathbb{C}^n, 0}$. Lift ω_0 to $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)$ for some neighbourhood $B \subseteq \mathbb{C}^{n-1}$ of 0. In order to show the coherence of $\mathcal{O}_U/g\mathcal{O}_U$ near 0, it suffices to show that $\mathcal{O}_{B \times \mathbb{C}}/\omega\mathcal{O}_{B \times \mathbb{C}}$ near 0. Let $A \subseteq B \times \mathbb{C}$ be the closed subspace defined by ω and $\pi : A \rightarrow B$ be the Weierstrass map, then it suffices to show that \mathcal{O}_A is coherent near 0. By our inductive hypothesis, \mathcal{O}_B is coherent. We claim that \mathcal{O}_A is also coherent. Let b be the degree of ω . We recall that π is topologically finite by **Theorem 6.2**.

We first prove a special case: let $p \in \mathbb{N}$ and $\varphi : \mathcal{O}_A^p \rightarrow \mathcal{O}_A$ be an \mathcal{O}_A -homomorphism. We show that $\ker \varphi$ is of finite type. By **Theorem 6.4**, $\pi_*\mathcal{O}_A^p$ is coherent. So $\pi_*\ker \varphi$ is coherent by **Corollary 4.8** in **Topology and bornology**. It follows that $\ker \varphi$ is of finite type.

Next let $U \subseteq A$ be an open subset and $s_1, \dots, s_p \in \mathcal{O}_A(U)$. We need to show that the kernel of the associated map

$$\mathcal{O}_U^p \rightarrow \mathcal{O}_A|_U$$

is of finite type. By **Theorem 4.6** in **Topology and bornology**, for each $x \in U$, we can find an open neighbourhood V of x in U such that $\pi^{-1}(V)$ is the disjoint union of open neighbourhoods U_1, \dots, U_n of the points in $\pi^{-1}(\pi(x))$. We may assume that $x \in U_1$. Extend $s_j|_{U_1}$ to $s'_j \in \mathcal{O}_A(\pi^{-1}(V))$ by setting its values to be 0 on U_2, \dots, U_n for $j = 1, \dots, p$. Then φ extends to $\varphi' : \mathcal{O}_{\pi^{-1}(V)}^p \rightarrow \mathcal{O}_{\pi^{-1}(V)}$ with the same kernel over U_1 . By what we have proved, $\ker \varphi'$ is of finite type. Hence, so is $\ker \varphi$. \square

As a corollary, we have the important Oka's coherence theorem.

Theorem 7.3. Let X be a complex analytic space, then \mathcal{O}_X is coherent.

PROOF. The problem is local on X , so we may assume that X is a complex model space, say there is a closed immersion into a domain D in \mathbb{C}^n defined by an ideal of finite type \mathcal{I} . By **Lemma 7.2**, \mathcal{O}_D is coherent and hence \mathcal{I} is coherent. It follows that $\mathcal{O}_D/\mathcal{I}$ is coherent and hence \mathcal{O}_X is coherent. \square

Corollary 7.4. Let X be a complex analytic space and \mathcal{M} be a sheaf of \mathcal{O}_X -modules. Then the following are equivalent:

- (1) \mathcal{M} is coherent;
- (2) \mathcal{M} is of finite presentation.

If \mathcal{M} is a subsheaf of \mathcal{O}_X , then these conditions are further equivalent to:

- (3) \mathcal{M} is of finite type.

PROOF. The equivalence between (1) and (2) follows from **[Stacks, Tag 01BZ]** and **Theorem 7.3**.

The equivalence of (1) and (3) follows from **[Stacks, Tag 01BY]**. \square

Definition 7.5. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces such that $f_*\mathcal{O}_X$ is coherent. We define the *complex image space* $f(X)$ of f as the closed subspace of Y defined by $\text{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$.

Here we need [Theorem 7.3](#) to guarantee that $\text{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$ is coherent. Observe that the support of $\mathcal{O}_Y/\text{Ann}_{\mathcal{O}_Y}(f_*\mathcal{O}_X)$ is exactly $f(X)$ in the set-theoretic sense. Moreover, we have f factorizes canonically through a morphism $X \rightarrow Z$.

Corollary 7.6. Let $i : X \rightarrow Y$ be a closed immersion. Then the ideal \mathcal{I} of i is coherent.

PROOF. Recall that we have a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_Y \rightarrow i_*\mathcal{O}_X \rightarrow 0.$$

Then \mathcal{O}_Y and \mathcal{O}_X are coherent by [Theorem 7.3](#). Then $i_*\mathcal{O}_X$ is coherent as the zero-extension of a coherent sheaf. It follows that \mathcal{I} is also coherent. \square

Corollary 7.7. Let X be a complex analytic space. Then there are natural bijections between the sets of

- (1) closed subspaces of X ;
- (2) coherent ideal sheaves on X .

PROOF. This follows from [Corollary 7.6](#) and [Example 5.4](#). \square

8. Finite limits in the category of complex analytic spaces

The goal of this section is to show that the category of complex analytic spaces admits finite limits.

As the category $\mathbb{C}\text{-An}$ admits a final object, namely \mathbb{C}^0 , the existence of finite limits is the same as the existence of fiber products by general abstract nonsense [[Stacks](#), [Tag 002O](#)].

We begin by considering direct products, namely fiber products over \mathbb{C}^0 .

Lemma 8.1. Let $m, n \in \mathbb{N}$. Then

$$\mathbb{C}^m \times \mathbb{C}^n \cong \mathbb{C}^{m+n}.$$

Here \times denotes the product in $\mathbb{C}\text{-An}$.

PROOF. By Yoneda lemma [[Stacks](#), [Tag 001P](#)], it suffices to establish

$$h_{\mathbb{C}^m \times \mathbb{C}^n} \cong h_{\mathbb{C}^{m+n}},$$

where h_\bullet denotes the functor of points [[Stacks](#), [Tag 001O](#)]. Take $T \in \mathbb{C}\text{-An}$, then there are isomorphisms

$$h_{\mathbb{C}^m \times \mathbb{C}^n}(T) \xrightarrow{\sim} h_{\mathbb{C}^m}(T) \times h_{\mathbb{C}^n}(T) \xrightarrow{\sim} (\mathcal{O}_T(T))^{m+n} \xrightarrow{\sim} h_{\mathbb{C}^{m+n}}(T),$$

which are all functorial in T . We conclude. \square

Lemma 8.2. Let $f : X \rightarrow Y$ be a morphism in $\mathbb{C}\text{-An}$. Let $i : Z \rightarrow Y$ be a closed (resp. an open) immersion. Then the fiber product $X \times_Y Z$ exists. Moreover, $X \times_Y Z \rightarrow X$ is a closed (resp. an open) immersion and there is a natural identification $|X \times_Y Z| \cong |X| \times_{|Y|} |Z|$.

We can draw a Cartesian diagram

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & \square & \downarrow f \\ Z & \xrightarrow{i} & Y \end{array}$$

PROOF. When i is an open immersion, it suffices to take $X \times_Y Z$ as the open subspace of X defined by $f^{-1}(i(Z))$.

Let us consider the case where i is a closed immersion defined by a coherent ideal sheaf \mathcal{I} . It is a general result that $X \times_Y Z$ in the category \mathcal{LRS} exists [Stacks, Tag 01HQ]. Let us show that $X \times_Y Z$ is a closed complex analytic subspace of X and conclude. To do so, recall that $X \times_Y Z$ is by construction a closed subspace of X defined by $\mathcal{J} := \text{Im}(f^*\mathcal{I} \rightarrow f^*\mathcal{O}_Y = \mathcal{O}_X)$. It suffices to show that \mathcal{J} is of finite type. By this is clear as \mathcal{I} is of finite type.

The identification of the underlying topological space is obvious. \square

Lemma 8.3. Let X, Y be complex analytic spaces. Consider open (resp. closed) immersions $X' \rightarrow X$ and $Y' \rightarrow Y$. If $X \times Y$ exists, then so is $X' \times Y'$ and the natural morphism $X' \times Y' \rightarrow X \times Y$ is an open (resp. a closed) immersion.

PROOF. We form the following large Cartesian diagram

$$\begin{array}{ccccc} Z & \longrightarrow & X'' & \longrightarrow & X' \\ \downarrow & \square & \downarrow & \square & \downarrow \\ Y'' & \longrightarrow & X \times Y & \longrightarrow & X \\ \downarrow & \square & \downarrow & \square & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & \mathbb{C}^0 \end{array}$$

The existences of all but the lower right square are guaranteed by Lemma 8.2. More precisely, we first define the upper right square and the lower left square by Lemma 8.2. It follows from Lemma 8.2 that $X'' \rightarrow X \times Y$ is an open (resp. a closed) immersion. So we can apply Lemma 8.2 again to construct the upper left square.

It follows from general abstract nonsense that the big square is also Cartesian. Moreover, by Lemma 8.2 again, $Z \rightarrow Y''$ and $Y'' \rightarrow X \times Y$ are both open (resp. closed) immersions. It follows that $Z \rightarrow X \times Y$ is also an open (resp. a closed) immersion. \square

Corollary 8.4. Let X, Y be complex model spaces. Then $X \times Y$ exists.

PROOF. By Lemma 8.3, we may assume that X and Y are both domains in some \mathbb{C}^m and \mathbb{C}^n respectively. Then applying Lemma 8.3 again, we reduce to the case where $X = \mathbb{C}^m$ and $Y = \mathbb{C}^n$. This case is handled in Lemma 8.1. \square

Corollary 8.5. Let X, Y be complex analytic spaces. Then $X \times Y$ exists in $\mathbb{C}\text{-An}$. Moreover, there is a natural identification $|X \times Y| \cong |X| \times |Y|$.

PROOF. Let

$$X = \bigcup_{i \in I_X} X_i, \quad Y = \bigcup_{j \in I_Y} Y_j$$

be open coverings of X by complex model spaces. Let $K = I_X \times I_Y$. For each $k = (i, j) \in K$, we let $Z_k = X_i \times Y_j$, whose existence is guaranteed by [Corollary 8.4](#). Take another $k' = (i', j') \in K$, then

$$Z_{kk'} := Z_k \cap Z_{k'} = (X_i \times X_{i'}) \cap (Y_j \times Y_{j'})$$

is an open subspace of Z_k . It is clear that $Z_{kk'}$ forms a glueing data. From the general result [[Stacks](#), [Tag 01JB](#)], we can glue Z_k 's into a locally ringed space Z . From the construction, $|Z| = |X| \times |Y|$ in the category of topological spaces, so $|Z|$ is Hausdorff. On the other hand, from the construction, locally Z is isomorphic to some Z_k , so Z is a complex analytic space. As Z is clearly the product in the category of locally \mathbb{C} -ringed spaces, we conclude that $Z = X \times Y$ in $\mathbb{C}\text{-An}$. \square

Corollary 8.6. The category $\mathbb{C}\text{-An}$ admits all finite limits. Moreover, finite limits commute with the forgetful functor $\mathbb{C}\text{-An} \rightarrow \mathcal{T}\text{op}$.

PROOF. By [[Stacks](#), [Tag 002O](#)], [Corollary 8.5](#) and the existence of a final object in $\mathbb{C}\text{-An}$ (namely, \mathbb{C}^0), it suffices to show the existence of fiber products. In other words, suppose that we are given three complex analytic spaces Z, X, Y and morphisms $X \rightarrow Z$ and $Y \rightarrow Z$ in $\mathbb{C}\text{-An}$, we need to prove the existence of $X \times_Z Y$. From the general abstract nonsense, we can define $X \times_Z Y = (X \times Z)_{Y \times Y, \Delta_Y}$:

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \times Z \\ \downarrow & \square & \downarrow \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array},$$

where $\Delta_Y : Y \rightarrow Y \times Y$ is the diagonal morphism, which is a closed immersion, the existence of $X \times Z$ is guaranteed by [Corollary 8.5](#) and the existence of the fiber product is guaranteed by [Lemma 8.2](#).

In order to verify that finite limits commute with the forgetful functor $\mathbb{C}\text{-An} \rightarrow \mathcal{T}\text{op}$, it suffices to consider fiber products. By [Lemma 8.2](#), we reduced to the case of finite products. In this case, the result is proved in [Corollary 8.5](#). \square

Remark 8.7. It is important to remember that the forgetful functor $\mathbb{C}\text{-An} \rightarrow \mathbb{C}\text{-LRS}$ does *not* commute with finite limits, in contrast to the case of schemes [[Stacks](#), [Tag 01JN](#)]. While the forgetful functor from the category of schemes $\mathcal{S}\text{ch}$ to $\mathcal{T}\text{op}$ does not commute with finite limits.

These facts indicate that there are essential differences between the theory of analytic spaces and the theory of schemes.

Definition 8.8. Let $f : X \rightarrow S$ be a morphism of complex analytic spaces and $s \in S$. Then the fiber X_s or $f^{-1}\{s\}$ is the fiber product $X \times_S \{s\}$. It is a closed analytic subspace of X by [Example 5.5](#) and ...[Include a lemma saying the base change of closed immersion is closed immersion](#).

Next we study the local rings of fiber products.

Theorem 8.9. Let Y be an object in $\mathbb{C}\text{-An}$ and $X_1, X_2 \in \mathbb{C}\text{-An}/_Y$. Let (x_1, x_2) be a point of $X_1 \times_Y X_2$, namely, $x_i \in X_i$ for $i = 1, 2$ and the images of x_1 and x_2 in Y coincide, say $y \in Y$. Then there is a canonical isomorphism

$$\mathcal{O}_{X_1 \times_Y X_2, (x_1, x_2)} \cong \mathcal{O}_{X_1, x_1} \bar{\otimes}_{\mathcal{O}_{Y, y}} \mathcal{O}_{X_2, x_2}.$$

The analytic tensor product here is defined [Definition 5.6](#) in the Complex Analytic Local Algebras. We have shown its existence in [Theorem 5.11](#) in the same chapter.

PROOF. Comparing the constructions of both sides, we see that it suffices to prove the theorem in two special cases: when $Y = \mathbb{C}^0$ and when $X_2 \rightarrow Y$ is a closed immersion.

We first consider the case where $Y = \mathbb{C}^0$. As our problem is local, we may assume that X_1 and X_2 are both complex model spaces. From the constructions, we easily reduce to the case where X_1 and X_2 are both domains in \mathbb{C}^m and \mathbb{C}^n respectively. In this case, the result is proved in [Lemma 5.7](#) in the Complex Analytic Local Algebras and [Proposition 2.4](#).

Next we handle the case where $X_2 \rightarrow Y$ is a closed immersion. This case is immediately clear from the constructions of both sides. \square

Bibliography

- [Stacks] T. Stacks Project Authors. Stacks Project. <http://stacks.math.columbia.edu>. 2020.