# Banach algebras

# Contents

## 1. Introduction

This section conerns the theory of Banach algebras. Our references are [Ber12] and [BGR84].

In this chapter, all rings are assumed to be commutative.

## 2. Semi-normed Abelian groups

**Definition 2.1.** Let $A$ be an Abelian group. A *semi-norm* on $A$ is a function $\|\bullet\| : A \to [0, \infty]$ satisfying

(1) $\|0\| = 0$;
(2) $\|f - g\| \leq \|f\| + \|g\|$ for all $f, g \in A$.

A semi-norm $\|\bullet\|$ on $A$ is a *norm* if moreover the following conditions is satisfied:

(0) if $\|f\| = 0$ for some $f \in A$, then $f = 0$.

A semi-norm $\|\bullet\|$ on $A$ is *non-Archimedean* or *ultra-metric* if Condition (2) can be replaced by

(2') $\|f - g\| \leq \max\{\|f\|, \|g\|\}$ for all $f, g \in A$.

**Definition 2.2.** A *semi-normed Abelian group* (resp. *normed Abelian group*) is a pair $(A, \|\bullet\|)$ consisting of an Abelian group $A$ and a semi-norm (resp. norm) $\|\bullet\|$ on $A$. When $\|\bullet\|$ is clear from the context, we also say $A$ is a semi-normed Abelian group (resp. normed Abelian group).

**Definition 2.3.** Let $(A, \|\bullet\|_A)$ be a semi-normed Abelian group and $B \subseteq A$ be a subgroup. Then we define the *quotient semi-norm* $\|\bullet\|_{A/B}$ on $A/B$ as follows:

$$\|a + B\|_{A/B} := \inf\{\|a + b\|_A : b \in B\}$$

for all $a + B \in A/B$.

We define the *subgroup semi-norm* on $B$ as follows:

$$\|b\|_B = \|b\|_A$$

for all $b \in B$.

**Definition 2.4.** Let $A$ be an Abelian group and $\|\bullet\|$, $\|\bullet\|'$ be two seminorms on $A$. We say $\|\bullet\|$ and $\|\bullet\|'$ are *equivalent* if there is a constant $C > 0$ such that

$$C^{-1}\|f\| \leq \|f\|' \leq C\|f\|$$

for all $f \in A$.

**Definition 2.5.** Let $(A, \|\bullet\|_A)$, $(B, \|\bullet\|_B)$ be semi-normed Abelian groups. A homomorphism $\varphi : A \to B$ is said to be

(1) *bounded* if there is a constant $C > 0$ such that $\|\varphi(f)\|_B \leq C\|f\|_A$ for any $f \in A$;
(2) *admissible* if the quotient semi-norm on $A/\ker\varphi$ is equivalent to the subspace semi-norm on $\operatorname{Im}\varphi$.

Observe that an admissible homomorphism is always bounded.

Next we study the topology defined by a semi-norm.

**Lemma 2.6.** Let $(A, \|\bullet\|)$ be a semi-normed Abelian group. Define

$$d(a, b) = \|a - b\|$$

for $a, b \in A$. Then $\|\bullet\|$ is a pseudo-metric on $A$. This psuedo-metric is a metric if and only if $\|\bullet\|$ is a norm.

PROOF. This is clear from the definitions. □

We always endow $A$ with the topology induced by the psuedo-metric $d$.

## 3. Semi-normed rings

**Definition 3.1.** Let $A$ be a ring. A *semi-norm* $\|\bullet\|$ on $A$ is a semi-norm $\|\bullet\|$ on the underlying additive group satisfying the following extra properties:

(3) $\|1\| = 1$;
(4) for any $f, g \in A$, $\|fg\| \leq \|f\|\|g\|$.

A semi-norm $\|\bullet\|$ on $A$ is called *power-multiplicative* if $\|f\|^n = \|f^n\|$ for all $f \in A$ and $n \in \mathbb{N}$.

A semi-norm $\|\bullet\|$ on $A$ is called *multiplicative* if $\|fg\| = \|f\|\|g\|$ for all $f, g \in A$.

**Definition 3.2.** A *semi-normed ring* (resp. *normed ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring $A$ and a semi-norm (resp. norm) $\|\bullet\|$ on $A$. When $\|\bullet\|$ is clear from the context, we also say $A$ is a semi-normed ring (resp. normed ring).

**Definition 3.3.** Let $A$ be a ring. A *semi-valuation* on $A$ is a multiplicative semi-norm on $A$. A semi-valuation on $A$ is a *valuation* on $A$ if its underlying semi-norm of Abelian groups is a norm.

**Definition 3.4.** A *semi-valued ring* (resp. *valued ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring $A$ and a semi-valuation (resp. valuation) $\|\bullet\|$ on $A$. When $\|\bullet\|$ is clear from the context, we also say $A$ is a semi-valued ring (resp. valued ring).

A semi-valued ring (resp. valued ring) $(A, \|\bullet\|)$ is called a *semi-valued field* (resp. *valued field*) if $A$ is a field.

## 4. Banach rings

**Definition 4.1.** A *Banach ring* is a normed ring that is complete with respect to the metric defined in Lemma 2.6.

**Proposition 4.2.** Let $(A, \|\bullet\|)$ be a Banach ring and $f \in A$. Assume that $\|f\| < 1$, then $1 - f$ is invertible.

PROOF. Define
$$g = \sum_{i=0}^{\infty} f^i.$$
From our assumption, the series converges and $g \in A$. It is elementary to check that $g$ is the inverse of $1 - f$. □

**Example 4.3.** The ring $\mathbb{C}$ with its usual norm $|\bullet|$ is a Banach ring. In fact, $(\mathbb{C}, |\bullet|)$ is a complete valued field.

**Example 4.4.** For any Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, we define $A\langle r^{-1}z\rangle = A\langle r_1^{-1}z_1, \ldots, r_n^{-1}z_n\rangle_{r_1,\ldots,r_n}$ as the subring of $A[[z_1, \ldots, z_n]]$ consisting of formal power series
$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in A$$
such that
$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha < \infty.$$

We will verify in Proposition 4.5 that $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

When $r = (1, \dots, 1)$, we omit $r^{-1}$ from our notations.

**Proposition 4.5.** In the setting of Example 4.4, $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

Proof. By induction, we may assume that $n = 1$.

It is obvious that $\|\bullet\|_r$ is a norm on the undelrying Abelian group. To see that $\|\bullet\|_r$ is a norm on the ring $A\langle r^{-1}z\rangle$, we need to verify the condition in Definition 3.1. Condition (3) in Definition 3.1 is obvious. Let us consider Condition (4). Let

$$f = \sum_{i=0}^{\infty} a_i z^i, \quad g = \sum_{j=0}^{\infty} b_j z^j$$

be two elements in $A\langle r^{-1}z\rangle$. Then

$$fg = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} a_i b_j \right) z^k.$$

We compute

$$\|fg\|_r = \sum_{k=0}^{\infty} \left\| \sum_{i+j=k} a_i b_j \right\| r^k \leq \sum_{k=0}^{\infty} \left( \sum_{i+j=k} \|a_i\| \cdot \|b_j\| \right) r^k = \|f\|_r \cdot \|g\|_r.$$

It remains to verify that $A\langle r^{-1}z\rangle$ is complete.

For this purpose, take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b z^i \in A\langle r^{-1}z\rangle$$

for $b \in \mathbb{N}$. Then for each $i$, the coefficients $(a_i^b)_b$ is a Cauchy sequence in $A$. Let $a_i$ be the limit of $a_i^b$ as $b \to \infty$ and set

$$f = \sum_{i=0}^{\infty} a_i z^i \in A[[z]].$$

We need to show that $f \in A\langle r^{-1}z\rangle$ and $f^b \to f$.

Fix a constant $\epsilon > 0$. There is $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$, we have

$$\sum_{i=0}^{\infty} \|a_i^{j+k} - a_i^j\| r^i < \epsilon/2.$$

In particular, for any $s > 0$, we have

$$\sum_{i=0}^{s} \|a_i - a_i^j\| r^i \leq \sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i + \sum_{i=0}^{s} \|a_i^j - a_i^{j+k}\| r^i \leq \sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i + \epsilon/2.$$

When $k$ is large enough, we can guarantee that

$$\sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i < \epsilon/2.$$

So

$$\sum_{i=0}^{s} \|a_i - a_i^j\| r^i \leq \epsilon.$$

Let $s \to \infty$, we find

$$\|f - f^j\|_r \leq \sum_{i=0}^{\infty} \|a_i - a_i^j\| r^i \leq \epsilon.$$

In particular, $\|f\|_r < \infty$ and $f^j \to f$ as $j \to \infty$. □

## 5. Semi-normed modules

**Definition 5.1.** Let $(A, \|\bullet\|_A)$ be a normed ring. A *semi-normed $A$-module* (resp. *normed $A$-module*) is a pair $(M, \|\bullet\|_M)$ consisting of a $A$-module $M$ and a semi-norm (resp. norm) on the underlying Abelian group of $M$ such that there is a constant $C > 0$ such that

$$\|fm\|_M \leq C\|f\|_A \|m\|_M$$

for all $f \in A$ and $m \in M$. When $\|\bullet\|_M$ is clear from the context, we say $M$ is a semi-normed $A$-module (resp. normed $A$-module).

A *Banach $\mathcal{A}$-module* is a normed $A$-module which is complete with respect to the metric Lemma 2.6.

# Bibliography


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