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Berkovich analytic spaces

1. Introduction

2. Compact analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 2.1. Let A be a k_H -affinoid algebra. A *compact k_H -analytic domain* V in $\mathrm{Sp} A$ is a finite union of k_H -affinoid domains in $\mathrm{Sp} A$.

Lemma 2.2. Let A be a k_H -affinoid algebra and V be a compact k_H -analytic domain. Write $\mathrm{Sp} A$ as a finite union of k_H -affinoid domains $\mathrm{Sp} A_i$ with $i = 1, \dots, n$ in $\mathrm{Sp} A$. Define $A_{ij} = A_i \hat{\otimes}_A A_j$ and

$$A_V := \ker \left(\prod_{i=1}^n A_i \rightarrow \prod_{i,j=1}^n A_{ij} \right).$$

Then the Banach k -algebra does not depend on the choice of the covering $\{\mathrm{Sp} A_i\}_i$ up to a canonical isomorphism.

The image of the natural continuous map $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ contains V and the map does not depend on the choice of the covering up to the canonical isomorphism between $\mathrm{Sp} A_V$ for different coverings.

PROOF. We first observe that A_V is a Banach k -algebra as it is defined as an equalizer. This follows from ?? in ??.

Let $\{\mathrm{Sp} B_j\}_{j=1, \dots, m}$ be another k_H -affinoid covering of $\mathrm{Sp} A$. We need to show that A_V defined using the two coverings are canonically isomorphic. We write A'_V for

$$\ker \left(\prod_{j=1}^m B_j \rightarrow \prod_{i,j=1}^m B_{ij} \right)$$

to make a distinction. We write $B_{ij} = B_i \hat{\otimes}_A B_j$.

By ?? in ??, the columns in the following commutative diagram are exact:

$$\begin{array}{ccccccc}
& & 0 & & 0 \\
& & \downarrow & & \downarrow \\
0 & \longrightarrow & A_V & \longrightarrow & \prod_{i=1}^n A_i & \longrightarrow & \prod_{i,i'=1}^n A_{ii'} \\
& & \downarrow \text{dotted} & & \downarrow \eta & & \downarrow \\
0 & \longrightarrow & \ker \iota & \longrightarrow & \prod_{i=1}^n \prod_{j=1}^m A_i \hat{\otimes}_A B_j & \xrightarrow{\iota} & \prod_{i,i'=1}^n \prod_{j,j'=1}^m A_{ii'} \hat{\otimes}_A B_{jj'} \\
& & & & \downarrow \tau & & \\
& & & & \prod_{i=1}^n \prod_{j,j'=1}^m A_i \hat{\otimes}_A B_{jj'} & &
\end{array}$$

The rows are exact by definition. By diagram chasing, the dotted arrow is injective. To see it is surjective, it suffices to observe that the factors with $i = i'$ in the lower right corner is exactly the same as the factors of the lower corner, so an element in $\ker \iota$ is necessarily in $\ker \tau$. It follows that the dotted arrow is surjective.

Similarly, we have a natural isomorphism $A'_V \xrightarrow{\sim} \ker \iota$. We conclude the first assertion.

As for the second, observe that $\mathrm{Sp} A_V$ is defined as a colimit in the category of Banach k -algebras, so it follows from general abstract nonsense that there is a natural morphism $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$. It clearly contains V in the image. The compatibility with the isomorphism above follows simply from the fact that the map η is an A -algebra homomorphism. \square

Remark 2.3. This is also a natural continuous map $V \rightarrow \mathrm{Sp} A_V$, given by the natural map $A_V \rightarrow A_i$ for $i = 1, \dots, n$. This map is a section of the continuous map $\mathrm{Sp} A_V \rightarrow A$ we just constructed over V . In [Ber93], Berkovich always uses this map instead of $\mathrm{Sp} A_V \rightarrow A$.

Definition 2.4. Let A be a k -affinoid algebra and V be a compact k -analytic domain in $\mathrm{Sp} A$. We define the Banach k -algebra A_V associated with V as A_V constructed in Lemma 2.2.

The continuous map $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ constructed in Lemma 2.2 is called the *structure map* $\mathrm{ov} V$.

Proposition 2.5. Let A be a k_H -affinoid algebra and V be a compact k_H -analytic domain in $\mathrm{Sp} A$. Then the following are equivalent:

- (1) V is a k_H -affinoid domain.
- (2) A_V is a k_H -affinoid algebra and the image of the structure map $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ is exactly V .

PROOF. (1) \implies (2): By ?? in ??, when V is a k_H -affinoid domain, A_V is a k_H -affinoid algebra and the structure map corresponds to the inclusion of the k_H -affinoid domain. There is nothing to prove.

(2) \implies (1): It suffices to show that the structure map represents the k_H -affinoid domain V . Take a k_H -affinoid algebra D and a morphism $\mathrm{Sp} D \rightarrow \mathrm{Sp} A$ of k_H -affinoid spectra that factorizes through V . We need to construct a morphism

$\mathrm{Sp} D \rightarrow \mathrm{Sp} A_V$ making the following diagram commutative

$$\begin{array}{ccc} \mathrm{Sp} D & & \\ \downarrow \text{dotted} & \searrow & \\ \mathrm{Sp} A_V & \longrightarrow & \mathrm{Sp} A \end{array} \quad .$$

Take k_H -affinoid domains $\mathrm{Sp} B_1, \dots, \mathrm{Sp} B_n$ in $\mathrm{Sp} A$ that cover V . Let $C_i = B_i \hat{\otimes}_A D$ for $i = 1, \dots, n$, then $\mathrm{Sp} C_i$ is a k_H -affinoid domain in $\mathrm{Sp} D$ by ?? in ??. By ?? in ?? and general abstract nonsense, it suffices to construct the dotted arrow after restricting to $\mathrm{Sp} C_i$ for $i = 1, \dots, n$. So we could assume that $\mathrm{Sp} D \rightarrow \mathrm{Sp} A$ factorizes through $\mathrm{Sp} B_1$. From the universal property, we therefore have the dotted morphism making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Sp} D & & \\ \downarrow \text{dotted} & \searrow & \\ \mathrm{Sp} B_1 & \longrightarrow & \mathrm{Sp} A \end{array} \quad .$$

It suffices to show that the natural homomorphism

$$B_1 \rightarrow A_V \hat{\otimes}_A B_1$$

is an isomorphism. But this follows from general abstract nonsense as B_1 is already a Banach A_V -algebra. \square

Remark 2.6. This proposition is not correctly stated in [Ber12, Corollary 2.2.6]. The corresponding statement in [Ber93, Remark 1.2.1] is slightly weaker than our statement.

3. The category of Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. Let X be a locally Hausdorff space and τ be a net of compact subsets. A k_H -affinoid atlas \mathcal{A} on X with the net τ is a map which assigns

- (1) to each $V \in \tau$, a k_H -affinoid algebra A_V and a homeomorphism $\varphi_V : \mathrm{Sp} A_V \rightarrow V$;
- (2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of k_H -affinoid algebras $\alpha_{V/U} : A_V \rightarrow A_U$ representing a k_H -affinoid domain $\mathrm{Sp} A_U$ in $\mathrm{Sp} A_V$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Sp} A_U & \xrightarrow{\mathrm{Sp} \alpha_{V/U}} & \mathrm{Sp} A_V \\ \downarrow \varphi_U & & \downarrow \varphi_V \\ U & \longrightarrow & V \end{array} \quad .$$

The triple (X, \mathcal{A}, τ) as above is called a k_H -analytic space.

A *morphism* between atlases \mathcal{A} and \mathcal{A}' on X with the net τ is an assignment that with each $V \in \tau$, one associates a morphism of k_H -affinoid algebras $\beta_V : A_V \rightarrow A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sp} A'_V & \xrightarrow{\mathrm{Sp} \beta_V} & \mathrm{Sp} A_V \\ \downarrow \varphi'_V & \swarrow \varphi_V & \\ V & & \end{array} ;$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{array}{ccc} A_V & \xrightarrow{\alpha_{V/U}} & A_U \\ \downarrow \beta_V & & \downarrow \beta_U \\ A'_V & \xrightarrow{\alpha'_{V/U}} & A'_U \end{array}$$

Here we have denoted the data associated with \mathcal{A}' with a prime. In this way, the atlases on X with the net τ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), if $W \subseteq U \subseteq V$ are three sets in τ , then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

Remark 3.2. As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps φ_U 's by identifying $\mathrm{Sp} A_U$ with U . We will say U is a k_H -affinoid domain in V .

Remark 3.3. Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

Lemma 3.4. Let (X, \mathcal{A}, τ) be a k_H -analytic space, $U \in \tau$ and W is a k_H -affinoid domain in U . Then for any $V \in \tau$ containing W , W is a k_H -affinoid domain in V .

PROOF. As $\tau|_{U \cap V}$ is a net and W is compact, we can find $U_1, \dots, U_n \in \tau_{U \cap V}$ with $W \subseteq U_1 \cup \dots \cup U_n$. As W, U_i are k_H -affinoid domains in U , $W_i = W \cap U_i$ is a k_H -affinoid domain in U_i for all $i = 1, \dots, n$ by ?? in ???. It follows from ?? and ?? in ?? that W_i and $W_i \cap W_j$ are both k_H -affinoid domains in V for $i, j = 1, \dots, n$. So W is a compact k_H -analytic domain in V .

By Proposition 2.5,

$$A_W := \ker \left(\prod_{i=1}^n A_{W_i} \rightarrow \prod_{i,j=1}^n A_{W_i \cap W_j} \right)$$

is k_H -affinoid and $\mathrm{Sp} A_W \rightarrow \mathrm{Sp} A$ induces a homeomorphism $\mathrm{Sp} A_W \rightarrow W$ by ?? in ??. By Proposition 2.5 again, W is affinoid in V . \square

Definition 3.5. Let (X, \mathcal{A}, τ) be a k_H -analytic space. We define $\bar{\tau}$ as the set of all $W \subseteq X$ such that there is $U \in \tau$ containing W and W is k_H -affinoid in U .

Lemma 3.6. Let (X, \mathcal{A}, τ) be a k_H -analytic space. Then $\bar{\tau}$ is a net on X and there is a k_H -affinoid atlas $\bar{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending \mathcal{A} . Moreover, the k_H -affinoid atlas $\bar{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending \mathcal{A} is unique up to a canonical isomorphism.

PROOF. **Step 1.** We first show that $\bar{\tau}$ is a net. Let $U, V \in \bar{\tau}$ and $x \in U \cap V$. Take $U', V' \in \tau$ containing U and V respectively. Take $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau$ such that

$$(1) \quad x \in W_1 \cap \dots \cap W_n;$$

(2) $W_1 \cup \dots \cup W_n$ is a neighbourhood of x in $U' \cap V'$.

This is possible because $\tau|_{U' \cap V'}$ is a quasi-net by assumption.

By [Lemma 3.4](#), U (resp. V) and W_1, \dots, W_n are k_H -affinoid domains in U' (resp. V').

By ?? in ??, $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a k_H -affinoid domain in W_i for $i = 1, \dots, n$. By ?? in ?? again, $U_i \cap V_i$ is a k_H -affinoid domain in W_i for $i = 1, \dots, n$. So $U_i \cap V_i \in \bar{\tau}|_{U \cap V}$ for $i = 1, \dots, n$. But

$$\bigcup_{i=1}^n U_i \cap V_i = (U \cap V) \cap \bigcup_{i=1}^n W_i,$$

so $\bigcup_{i=1}^n U_i \cap V_i$ is a neighbourhood of x in $U \cap V$ and $x \in \bigcap_{i=1}^n U_i \cap V_i$. It follows that $\bar{\tau}$ is a net.

Step 2. We extend the k_H -affinoid atlas \mathcal{A} .

For each $V \in \bar{\tau}$, we fix a $V' \in \tau$ containing V .

By [Lemma 3.4](#), V is a k_H -affinoid domain in V' . Let $A_{V'} \rightarrow A_V$ be the morphism of k_H -affinoid algebras representing the k_H -affinoid domain V in $\mathrm{Sp} A_{V'}$. We define the homeomorphism $\varphi_V : \mathrm{Sp} A_V \rightarrow V$ as the morphism induced by $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$.

For $U, V \in \bar{\tau}$ with $U \subseteq V$, we want to define $\alpha_{V/U} : A_V \rightarrow A_U$. We handle two cases. When $V \in \tau$, as $\tau|_{U' \cap V}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \tau|_{U' \cap V}$ such that

$$U = \bigcup_{i=1}^n U_i.$$

By [Lemma 3.4](#), U_1, \dots, U_n are k_H -affinoid domains in U' and in V . By ?? in ??,

$$A_U \xrightarrow{\sim} \ker \left(\prod_{i=1}^n A_{U_i} \rightarrow \prod_{i,j=1}^n A_{U_i \cap U_j} \right).$$

So the morphism $\alpha_{V/U_i} : A_V \rightarrow A_{U_i}$ and $\alpha_{V/U_i \cap U_j} : \alpha_{V/U_i} : A_V \rightarrow A_{U_i \cap U_j}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ induces a morphism $\alpha_{V/U} : A_V \rightarrow A_U$. Observe that $\alpha_{V/U}$ represents the k_H -affinoid domain U in V , so it is independent of the choice of U_1, \dots, U_n .

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \rightarrow A_U$ representing the k_H -affinoid domain U in V' , it follows that U is a k_H -affinoid domain in V , and we therefore get the desired morphism $\alpha_{V/U} : A_V \rightarrow A_U$.

It is easy to verify that the constructions gives a k_H -affinoid atlas with the net $\bar{\tau}$ extending \mathcal{A} . The uniqueness of the extension is immediate. \square

Definition 3.7. Let (X, \mathcal{A}, τ) and $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ is a pair consisting of

- (1) a continuous map $\varphi : X \rightarrow X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
- (2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of k_H -affinoid spectra $\varphi_{V/V'} : V \rightarrow V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq V'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi_{V/V'}} & V' \\ \downarrow & & \downarrow \\ W & \xrightarrow{\varphi_{W/W'}} & W' \end{array}.$$

Recall our convention [Remark 3.2](#), the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \rightarrow A_V$ of k_H -affinoid algebras making the following diagram commutative

$$\begin{array}{ccc} \mathrm{Sp} A_V & \longrightarrow & \mathrm{Sp} A'_{V'} \\ \downarrow \varphi_V & & \downarrow \varphi'_{V'} \\ V & \xrightarrow{\varphi} & V' \end{array}.$$

We will continue our identifications as in [Remark 3.2](#) to simplify our notations.

Proposition 3.8. Let (X, \mathcal{A}, τ) and $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \bar{\mathcal{A}}, \bar{\tau}) \rightarrow (X', \bar{\mathcal{A}}', \bar{\tau}')$.

PROOF. Let $U \in \bar{\tau}$, $U' \in \bar{\tau}'$ with $\varphi(U) \subseteq U'$. Take $V \in \tau$ and $V' \in \tau'$ containing U and U' respectively. By [Lemma 3.4](#), U (resp. V) is a k_H -affinoid domain in V (resp. V'). Take $W \in \tau'$ with $\varphi(V) \subseteq W'$. Then in particular, $\varphi(U) \subseteq W'$. As $\tau'|_{V' \cap W'}$ is a quasi-net and $\varphi(U)$ is compact, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau'|_{V' \cap W'}$ such that

$$\varphi(U) \subseteq W_1 \cup \dots \cup W_n.$$

Now W_i is a k_H -affinoid domain in W' by [Lemma 3.4](#), so $V_i := \varphi_{V/W'}^{-1}(W_i)$ is an affinoid domain in V by ?? in ??, and we have an induced morphism $V_i \rightarrow W_i$ for $i = 1, \dots, n$. This morphism in turn induces a morphism of k_H -affinoid spectra

$$U_i := U \cap V_i \rightarrow U'_i := U' \cap W_i \rightarrow U'$$

for $i = 1, \dots, n$. These morphisms are compatible on their intersections by construction. So by ?? in ??, they glue together to a morphism of k_H -affinoid spectra $\bar{\varphi}_{U/U'} : U \rightarrow U'$. It is easy to see that this construction defines a strong morphism.

As for the uniqueness, it suffices to show that the morphism $U_i \rightarrow U'_i$ is uniquely determined for $i = 1, \dots, n$. In other words, we need to show that the dotted arrow that makes the following diagram commutes is unique:

$$\begin{array}{ccc} U_i & \cdots \cdots \cdots & U'_i \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi_{V/W'}} & W' \end{array}$$

for $i = 1, \dots, n$. It suffices to apply the universal property of the k_H -affinoid domain $U'_i \rightarrow W'$. \square

Definition 3.9. Let (X, \mathcal{A}, τ) , $(X', \mathcal{A}', \tau')$, $(X'', \mathcal{A}'', \tau'')$ be k_H -analytic spaces. Let

$$\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau'), \quad \psi : (X', \mathcal{A}', \tau') \rightarrow (X'', \mathcal{A}'', \tau'')$$

be strong morphisms. We will define their *composition* $\chi = \psi \circ \varphi$ as follows. The underlying map of topological spaces is just the composition of the underlying maps of topological spaces corresponding to ψ and φ .

Let $\bar{\varphi}$ and $\bar{\psi}$ be the extensions of φ and ψ to $\bar{\tau}$ and $\bar{\tau}'$ as in [Proposition 3.8](#).

Given $V \in \tau$ and $V'' \in \tau''$ with $\chi(V) \subseteq V''$, we need to define a morphism of k_H -affinoid spectra $\chi_{V/V''} : V \rightarrow V''$. Take $V' \in \tau'$ and $U'' \in \tau''$ such that $\varphi(V) \subseteq V'$ and $\psi(V') \subseteq U''$. Since $\chi(V) \subseteq U'' \cap V''$ and V is compact, we can take $n \in \mathbb{Z}_{>0}$ and $V_1'', \dots, V_n'' \in \tau''|_{U'' \cap V''}$ with $\chi(V) \subseteq V_1'' \cup \dots \cup V_n''$. Then $V_i' := \psi_{V'/U''}^{-1}(V_i'')$ and $V_i := \varphi_{V/V'}^{-1}(V_i')$ are k_H -affinoid domains in V' and V respectively for $i = 1, \dots, n$ and $V = V_1 \cup \dots \cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ then induce a morphism $V_i \rightarrow V_i'' \rightarrow V$ of k_H -affinoid spectra. These morphisms are clearly compatible on the intersections and hence induce a morphism $V \rightarrow V''$ of k_H -affinoid spectra by ?? in ??.

It is easy to verify that $\psi \circ \varphi$ is a strong morphism.

In this way, we get a category $k_H\text{-}\widetilde{\mathcal{A}n}$ of k_H -analytic spaces.

Definition 3.10. Let (X, \mathcal{A}, τ) and $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if

- (1) φ is a homeomorphism between X and X' ;
- (2) for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$, $\text{Sp } \varphi_{V/V'}$ identifies V with an affinoid domain in V' .

Lemma 3.11. Let (X, \mathcal{A}, τ) and $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces and $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a strong morphism. Then for any $V \in \tau$ and $V' \in \tau'$, the intersection $V \cap \varphi^{-1}(V')$ is a compact k_H -analytic domain in V .

PROOF. Take $U' \in \bar{\tau}'$ with $\varphi(V) \subseteq U'$. As $\tau|_{U' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1', \dots, U_n' \in \tau|_{U' \cap V'}$ with $\varphi(V) \subseteq U_1' \cup \dots \cup U_n'$ and

$$V \cap \varphi^{-1}(V') = \bigcup_{i=1}^n \varphi_{V/U}^{-1}(U_i').$$

□

Lemma 3.12. The system of quasi-isomorphisms in $k_H\text{-}\widetilde{\mathcal{A}n}$ is a right multiplicative system.

For the notion of right multiplicative system, we refer to [\[Stacks, Tag 04VC\]](#).

PROOF. We verify the three axioms as in [\[Stacks, Tag 04VC\]](#).

RMS1. The identity is clear a quasi-isomorphism. It remains to verify that the composition of quasi-isomorphisms is still a quasi-isomorphism.

We take φ, ψ as in [Definition 3.9](#). We will use the same notations as in [Definition 3.9](#). We need to show that $V \rightarrow V''$ identifies V with a k_H -affinoid domain in V'' . From the construction, we know that φ identifies V_i with a k_H -affinoid domain in V_i' and ψ identifies V_i' with a k_H -affinoid domain in V_i'' for $i = 1, \dots, n$. In particular, $\chi(V)$ is a compact k_H -analytic domain in V'' . It follows from [Proposition 2.5](#) that $\chi(V)$ is a k_H -affinoid domain in V'' .

RMS2. If $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ and $f : (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}') \rightarrow (X', \mathcal{A}', \tau')$ are given strong morphisms of k_H -analytic spaces and g is a quasi-isomorphism, then there are k_H -analytic space $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau})$ and strong morphisms $\bar{\varphi} : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \rightarrow$

$(\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$ and $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \rightarrow (X, \mathcal{A}, \tau)$ such that f is a quasi-isomorphism and the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) & \xrightarrow{\widetilde{\varphi}} & (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}') \\ \downarrow f & & \downarrow g \\ (X, \mathcal{A}, \tau) & \xrightarrow{\varphi} & (X', \mathcal{A}', \tau') \end{array}.$$

We may assume that $\widetilde{X}' = X'$. Then $\widetilde{\tau}' \subseteq \overline{\tau}'$. We let $\widetilde{X} = X$. Let $\widetilde{\tau}$ be the family of all $V \in \tau$ for which there is $\widetilde{V}' \in \widetilde{\tau}'$ with $\varphi(V) \subseteq \widetilde{V}'$. By [Lemma 3.11](#), $\widetilde{\tau}$ is a net on \widetilde{X} . The k_H -atlas $\widetilde{\mathcal{A}}$ defines a k_H -affinoid atlas $\widetilde{\mathcal{A}}$ with the net $\widetilde{\tau}$. The strong morphism $\widetilde{\varphi}$ induces $\widetilde{\varphi}$. The morphism f is the canonical quasi-isomorphism. It is immediate that these constructions satisfy the desired conditions.

RMS3. If $\varphi, \psi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ are strong morphisms of k_H -analytic spaces and there is a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \rightarrow (\widetilde{X}', \widetilde{\mathcal{A}}', \widetilde{\tau}')$ of k_H -analytic spaces such that $g \circ \varphi = g \circ \psi$, then there is a quasi-isomorphism $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \rightarrow (X, \mathcal{A}, \tau)$ with $\varphi \circ f = \psi \circ f$.

We will in fact show that $\varphi = \psi$. It is clear that they coincide as maps of topological spaces. Let $V \in \tau$, $V' \in \tau'$ such that $\varphi(V) \subseteq V'$. Take $\widetilde{V}' \in \widetilde{\tau}'$ with $g(V') \subseteq \widetilde{V}'$. Then we have two morphisms of k -affinoid spectra $\varphi_{V/V'}, \psi_{V/V'} : V \rightarrow V'$ such that their compositions with $g_{V'/\widetilde{V}'}$ coincide. As V' is an affinoid domain in \widetilde{V}' , it follows that $\varphi_{V/V'} = \psi_{V/V'}$ by the universal property. \square

Definition 3.13. The category $k_H\text{-}\widetilde{\mathcal{A}\text{n}}$ is the right category of fractions of $k_H\text{-}\widetilde{\mathcal{A}\text{n}}$ with respect to the system of quasi-isomorphisms. A morphism in $k_H\text{-}\widetilde{\mathcal{A}\text{n}}$ is called a *morphism* between k_H -analytic spaces.

We refer to [\[Stacks, Tag 04VB\]](#) for the definition of right category of fractions. For later references, we explicitly write down the morphisms in $k_H\text{-}\widetilde{\mathcal{A}\text{n}}$.

Lemma 3.14. Let $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a morphism of k_H -analytic spaces. We define a partial order on the set of nets on X : $\tau_1 \preceq \tau_0$ if $\tau_1 \subseteq \overline{\tau_0}$. Then the set of nets is a directed set and

$$\text{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\text{n}}}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')) = \lim_{\substack{\longrightarrow \\ \sigma \preceq \tau}} \text{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\text{n}}}((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau'))$$

in the category of sets, where \mathcal{A}_σ is induced by $\overline{\mathcal{A}}$. The transition maps are all injective.

PROOF. This follows immediately from the definition. \square

Definition 3.15. Let (X, \mathcal{A}, τ) be a k_H -analytic space. We say a subset $W \subseteq X$ is τ -special if it is compact and there exist $n \in \mathbb{Z}_{>0}$ and a covering $W = W_1 \cup \dots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \dots, n$ and the natural map

$$A_{W_i} \hat{\otimes}_k A_{W_j} \rightarrow A_{W_i \cap W_j}$$

is an admissible epimorphism.

The covering W_1, \dots, W_n is called a τ -special covering of W .

Under our convention, the assumption means that $W_i \cap W_j \rightarrow W_i \times W_j$ is a closed immersion of k_H -affinoid spectra.

Example 3.16. Let (X, \mathcal{A}, τ) be a k_H -analytic space. Suppose that $V \in \tau$ and W is a compact k_H -analytic domain in V . Let $n \in \mathbb{Z}_{>0}$ and $W = W_1 \cup \dots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \dots, n$. Then $\{W_i\}_i$ is a τ -special covering of W . This follows from ?? in ??.

Lemma 3.17. Let (X, \mathcal{A}, τ) be a k_H -analytic space and W be a τ -special subset of X . If $U, V \in \tau|_W$, then $U \cap V \in \bar{\tau}$ and the natural map

$$A_U \hat{\otimes}_k A_V \rightarrow A_{U \cap V}$$

is an admissible epimorphism.

PROOF. Let $n \in \mathbb{Z}_{>0}$ and W_1, \dots, W_n be a τ -special covering of W . As $U \cap W_i$ and $V \cap W_i$ are compact for $i = 1, \dots, n$, we can find $m_i \in \mathbb{Z}_{>0}$ (resp. $k_i \in \mathbb{Z}_{>0}$) and finite coverings $U_{i1}, \dots, U_{im_i} \in \tau$ of $U \cap W_i$ (resp. $V_{i1}, \dots, V_{ik_i} \in \tau$ of $V \cap W_i$).

Observe that $U_{ik} \cap V_{jl}$ is a k_H -affinoid domain in $U \cap V$, hence $U_{ik} \cap V_{jl} \in \bar{\tau}$ for any $i, j = 1, \dots, n$, $k = 1, \dots, m_i$ and $l = 1, \dots, k_l$. By ?? in ??, $U_{ik} \cap V_{jl} \rightarrow U_{ik} \times V_{jl}$ is a closed immersion since $W_i \cap W_j \rightarrow W_i \times W_j$ is by our assumption.

Consider the finite covering

$$\mathcal{U} := \{U_{ik} \times V_{jl} : i, j = 1, \dots, n; k = 1, \dots, m_i; l = 1, \dots, k_l\}$$

of $U \times V$. For each tuple (i, j, k, l) , $A_{U_{ik} \cap V_{jl}}$ is a finite $A_{U_{ik} \times V_{jl}}$ -algebra. By ?? in ??, we can construct a finite $A_{U \times V}$ -algebra $A_{U \cap V}$ inducing all of these $A_{U_{ik} \cap V_{jl}}$'s. By ?? in ??, $A_{U \cap V}$ is k_H -affinoid.

As \mathcal{U} is a finite k_H -affinoid covering of $U \times V$, $\{A_{U_{ik} \cap V_{jl}}\}_{i,k,j,l}$ is a finite k_H -affinoid covering of $U \cap V$ by ?? in ?? . In particular, we have a natural homeomorphism

$$\mathrm{Sp} A_{U \cap V} \xrightarrow{\sim} U \cap V.$$

Observe that $A_U \hat{\otimes}_k A_V \rightarrow A_{U \cap V}$ is surjective. We endow $A_{U \cap V}$ with the structure of finite $A_U \hat{\otimes}_k A_V$ -Banach algebras by ?? in ?? . Then $A_U \hat{\otimes}_k A_V \rightarrow A_{U \cap V}$ is an admissible epimorphism by ?? in ?? .

On the other hand $U \cap V$ is a compact k_H -analytic domain in U , so by [Proposition 2.5](#), $U \cap V$ is a k_H -affinoid in U . In particular, $U \cap V \in \bar{\tau}$. \square

Lemma 3.18. Let (X, \mathcal{A}, τ) be a k_H -analytic space and $W \subseteq X$ be a τ -special set. Then for any finite covering $\{W_i\}_{i \in I}$ of W with $W_i \in \tau$ for $i \in I$, the Banach k -algebra

$$A_W := \ker \left(\prod_{i \in I} A_{W_i} \rightarrow A_{W_i \cap W_j} \right)$$

does not depend on the choice of $\{W_i\}_{i \in I}$ up to canonical isomorphisms.

Moreover, we have a canonical map $W \rightarrow \mathrm{Sp} A_W$, which does not depend on the choice of the covering modulo the canonical isomorphism between A_W .

PROOF. It follows from [Lemma 3.17](#) that the covering $\{W_i\}_{i \in I}$ is τ -special. It suffices to apply the same argument of [Lemma 2.2](#). \square

Definition 3.19. Let (X, \mathcal{A}, τ) be a k_H -analytic space. Let $\hat{\tau}$ denote the collection of $\bar{\tau}$ -special subsets $W \subseteq X$ such that

- (1) A_W is k -affinoid;
- (2) the natural map $W \rightarrow \mathrm{Sp} A_W$ is bijective;
- (3) there is a $\bar{\tau}$ -special covering $\{W_i\}_{i \in I}$ of W such that W_i is a k -affinoid domain in W for $i \in I$.

The sets from $\hat{\tau}$ are called *k_H -affinoid domains in (X, \mathcal{A}, τ)* .

Observe that W is k_H -affinoid and W_i is a k_H -affinoid domain in W by ?? in ?. Condition (3) holds for any $\bar{\tau}$ -special covering.

Proposition 3.20. Let (X, \mathcal{A}, τ) be a k_H -analytic space. Then $\hat{\tau}$ is a net. For any net σ on X contained in $\bar{\tau}$, we have $\hat{\sigma} = \hat{\tau}$.

Moreover, $\hat{\hat{\tau}} = \hat{\tau}$.

PROOF. Let $U, V \in \hat{\tau}$. Take $\bar{\tau}$ -special coverings $\{U_i\}_{i \in I}$, $\{V_j\}_{j \in J}$ of U and V respectively. In order to show that $\hat{\tau}|_{U \cap V}$ is a quasi-net, it suffices to show that $\hat{\tau}|_{U_i \cap V_j}$ is for any $i \in I$ and $j \in J$. This follows simply from the fact that $\bar{\tau}|_{U_i \cap V_j}$ is a quasi-net. Similarly, as $\hat{\tau}$ is a quasi-net as $\bar{\tau}$ is. So $\hat{\tau}$ is a net.

Let σ be a net on X contained in $\bar{\tau}$. By Lemma 3.17, it suffices to verify that for any $V \in \bar{\tau}$, there are $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \sigma$ with $V = U_1 \cup \dots \cup U_n$. As σ is a net on X , we can find $m \in \mathbb{Z}_{>0}$, $W_1, \dots, W_m \in \sigma$ such that

$$V \subseteq W_1 \cup \dots \cup W_m.$$

As $V, W_j \in \bar{\tau}$ for $j = 1, \dots, m$, by ?? in ??, we can find $U_1, \dots, U_n \in \bar{\tau}$ such that $V = U_1 \cup \dots \cup U_n$ and each U_i is contained in some W_j . As $W_j \in \sigma$ for $j = 1, \dots, m$, it follows that $U_i \in \sigma$ for $i = 1, \dots, n$.

By Lemma 3.17,

$$\bar{\hat{\tau}} = \hat{\tau}.$$

Let $V \in \hat{\hat{\tau}}$. Let $\{V_i\}_{i \in I}$ be a $\hat{\tau}$ -special covering of V . For each $i \in I$, take a $\bar{\tau}$ -special covering $\{V_{ij}\}_{j \in J_i}$ of V_i . Then $\{V_{ij}\}_{i,j}$ is a $\bar{\tau}$ -special covering of V . It follows that $V \in \hat{\tau}$. \square

Proposition 3.21. Let (X, \mathcal{A}, τ) be a k_H -analytic space. There is a k_H -analytic atlas $\hat{\mathcal{A}}$ on X with the net $\hat{\tau}$ extending \mathcal{A} . Moreover, $\hat{\mathcal{A}}$ is unique up to a canonical isomorphism.

PROOF. For each $V \in \hat{\tau}$, Fix a $\bar{\tau}$ -special covering $\{V_i\}_{i \in I_V}$.

We define A_V using this covering as in Lemma 3.18. By definition, the canonical map $V \rightarrow \text{Sp } A_V$ is a homeomorphism.

Next take $U, V \in \hat{\tau}$ with $U \subseteq V$. We want to identify U with a k_H -affinoid domain in V . First assume that $U \in \tau$, then $U \cap V_i$ is a k_H -affinoid domain in V_i for $i \in I_V$ by Lemma 3.17. Hence, U is a k_H -affinoid domain in V . If we only know $U \in \hat{\tau}$, we know that U_i is a k_H -affinoid domain in V for any $i \in I_U$. It follows that U is a k_H -affinoid domain in V by Proposition 2.5.

The uniqueness is immediate. \square

Definition 3.22. Let (X, \mathcal{A}, τ) be a k_H -analytic space. A $\hat{\tau}$ -special set is called a *k_H -special domain in X* .

Observe that a k_H -special domain inherits a structure of k_H -analytic space from (X, \mathcal{A}, τ) .

Proposition 3.23. Let $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a morphism of k_H -analytic spaces. Then for any k_H -affinoid domains $V \subseteq X$ and $V' \subseteq X'$, the intersection $V \cap \varphi^{-1}(V')$ is a k_H -special domain in X .

PROOF. By Proposition 3.20, we may assume that φ is a strong morphism. In this case, it suffices to apply Lemma 3.11. \square

Lemma 3.24. Let (X, \mathcal{A}, τ) and $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \widehat{\mathcal{A}}, \widehat{\tau}) \rightarrow (X', \widehat{\mathcal{A}'}, \widehat{\tau}')$.

PROOF. Let $V \in \widehat{\tau}$ and $V' \in \widehat{\tau}'$ with $\varphi(V) \subseteq V'$. We want to define $\varphi_{V/V'} : V \rightarrow V'$ of k_H -affinoid spectra. By [Proposition 3.8](#), we may extend φ uniquely to $\bar{\tau}$. Take a $\bar{\tau}$ -special covering of V , we may reduce to the case where $V \in \bar{\tau}$. Take $W' \in \tau'$ such that $\varphi(V) \subseteq W'$. As $\tau|_{W' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau'|_{V' \cap W}$ such that $\varphi(V) \subseteq W_1 \cup \dots \cup W_n$. Considering the inverse images of W_i 's and $W_i \cap W_j$'s using [Lemma 3.17](#), we are reduced to the case where $V' \in \bar{\tau}'$. This is already handled in [Proposition 3.8](#). The uniqueness of the extension is clear. \square

Proposition 3.25. Let (X, \mathcal{A}, τ) , $(X', \mathcal{A}', \tau')$ be k_H -analytic spaces.

- (1) There is a canonical bijection between

$$\text{Hom}_{k_H\text{-An}}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau'))$$

and the set of pairs consisting of

- (a) a continuous map $\varphi : X \rightarrow X'$ such that for all $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, neighbourhoods $V_1 \cup \dots \cup V_n$ of x and $V'_1 \cup \dots \cup V'_n$ of $\varphi(x)$ with $x \in V_1 \cap \dots \cap V_n$ and $\varphi(V_i) \subseteq V'_i$ for $i = 1, \dots, n$, where $V_i \subseteq X$ and $V'_i \subseteq X'$ are k_H -affinoid domains;
- (b) for each pair of k_H -affinoid domains $V \subseteq X$, $V' \subseteq X'$ with $\varphi(V) \subseteq V'$, a morphism of k_H -affinoid spectra $\varphi_{V/V'} : V \rightarrow V'$ such that if $V, W \subseteq X$ and $V', W' \subseteq X'$ are k_H -affinoid domains with $\varphi(V) \subseteq V'$, $\varphi(W) \subseteq W'$, the diagram below commutes

$$\begin{array}{ccc} V & \xrightarrow{\varphi_{V/V'}} & V' \\ \downarrow & & \downarrow \\ W & \xrightarrow{\varphi_{W/W'}} & W' \end{array}.$$

- (2) Under the bijection in (1), an isomorphism corresponds to the pair where φ is a homeomorphism such that $\varphi(\widehat{\tau}) = \widehat{\tau}'$ and for any $V \in \widehat{\tau}$, $\varphi_{V/\varphi(V)}$ is an isomorphism of k_H -affinoid spectra.

PROOF. (2) follows immediately from (1). So it suffices to prove (1).

We construct the forward map. Let $\varphi : (X, \mathcal{A}, \tau) \rightarrow (X', \mathcal{A}', \tau')$ be a morphism. Take a subnet σ of $\bar{\tau}$ such that φ is represented by a strong morphism

$$\varphi : (X, \mathcal{A}_\sigma, \sigma) \rightarrow (X', \mathcal{A}', \tau').$$

By [Lemma 3.24](#), this extends to a strong morphism

$$\varphi : (X, \widehat{\mathcal{A}}_\sigma, \widehat{\sigma}) \rightarrow (X', \widehat{\mathcal{A}'}, \widehat{\tau}').$$

We get an injective map from the first set into the second set.

Conversely, we need to show that any given map from the second map comes from the first set. It suffices to show that

$$\sigma := \left\{ V \in \widehat{\tau} : \varphi(V) \subseteq V' \text{ for some } V' \in \widehat{\tau}' \right\}$$

is a net. Take $x \in X$ and neighbourhoods $V_1 \cup \dots \cup V_n$ of x and $V'_1 \cup \dots \cup V'_n$ of $\varphi(x)$ as in the statement of (1). Then $V_i \in \sigma$, so we conclude. \square

In practice, we do not distinguish a k_H -analytic space from the isomorphic k_H -analytic spaces. In particular, we will write (X, \mathcal{A}, τ) as X and always endow it with the structure $(X, \hat{\mathcal{A}}, \hat{\tau})$ of k_H -analytic space. If necessarily, we will write $|X|$ for the underlying topological space.

Corollary 3.26. The natural functor $k_H\text{-Aff} \rightarrow k_H\text{-An}$ is fully faithful.

PROOF. Let $X = \text{Sp } A$ be a k_H -affinoid spectrum. We endow it with the net $\tau = \{X\}$. The k_H -atlas with the net τ assigns $X \in \tau$ with A . It is easily verified that this is a functor. By [Proposition 3.25](#), the functor is fully faithful. \square

Definition 3.27. A k_H -affinoid space is an object of $k_H\text{-An}$ lying in the essential image of the functor $k_H\text{-Aff} \rightarrow k_H\text{-An}$.

The category of k_H -affinoid spaces is denoted by $k_H\text{-Aff}$.

The notation for the category of k_H -affinoid spaces is the same as the notation for the category of k_H -affinoid spectra, as the two categories are canonically equivalent.

Definition 3.28. A k_H -analytic space X is *good* if any point $x \in X$ admits a k_H -affinoid neighbourhood.

Example 3.29. Fix $n \in \mathbb{N}$. Let \mathbb{A}_k^n denote the set of all semi-valuations on $k[T_1, \dots, T_n]$ whose restriction to k coincides with the given valuation on k . We provide \mathbb{A}_k^n with the weakest topology such that for any $f \in k[T_1, \dots, T_n]$, the map $|\bullet| \mapsto |f|$ is continuous.

Observe that as a topological space,

$$(3.1) \quad \mathbb{A}_k^n \xrightarrow{\sim} \varinjlim_{r \in \mathbb{R}_{>0}^n} \text{Sp } k\{r^{-1}T\}.$$

As a set, this is clear: if $|\bullet| \in \mathbb{A}_k^n$, we take $r = (|T_1|, \dots, |T_n|)$, then $|\bullet| \leq \|\bullet\|_r$, so $|\bullet| \in \text{Sp } k\{r^{-1}T\}$. As

$$\bigcap_{r \in \mathbb{R}_{>0}^n} k\{r^{-1}T\} = k[T_1, \dots, T_n],$$

so the topology on the right-hand side of (3.1) is the weakest topology making $|\bullet| \mapsto |f|$ continuous for any $f \in k[T_1, \dots, T_n]$. It follows immediately that (3.1) is an identification of topological spaces.

It is clear that \mathbb{A}_k^n has a structure of good k_H -analytic space.

Proposition 3.30. Let X be a k_H -analytic space, $x \in X$ and U be a neighbourhood of x in X . Then there is a neighbourhood V of x in X contained in U such that V is open connected locally compact paracompact and Hausdorff. Moreover, we can guarantee that $\bar{V} \subseteq U$ and V is a countable union of k_H -affinoid domains.

PROOF. Take $n \in \mathbb{Z}_{>0}$ and k_H -affinoid spaces V_1, \dots, V_n containing x and $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X . If we have proved the proposition for V_i in place of X and $U \cap V_i$ in place of U for $i = 1, \dots, n$, namely, if we have found open connected locally compact paracompact and Hausdorff sets W_i containing x and contained in $U \cap V_i$ whose closure in V_i is contained in $U \cap V_i$, then we can take $V = W_1 \cup \dots \cup W_n$.

So we may assume that X is a k_H -affinoid space, say $X = \text{Sp } A$. Choose a k_H -rational neighbourhood

$$W = \text{Sp } A\{r^{-1} \frac{f}{g}\}$$

of x in U , where $n \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in A^n$, $r \in \sqrt{|k^\times| \cdot H^n}$, $g \in A$ and f_1, \dots, f_n, g generate the unit ideal in A . This is possible by ?? and ?? in ?. Take $\delta > 0$ so that $x \in \text{Sp } A\{((1 - \delta)r)^{-1} \frac{f}{g}\}$. Choose a strictly increasing sequence $\epsilon_i \in (0, 1) \cap \sqrt{|k^\times| \cdot H}$ converging to $1 - \delta/2$ for $i \in \mathbb{Z}_{>0}$. Let

$$W_i = \text{Sp } A \left\{ (\epsilon_i r)^{-1} \frac{f}{g} \right\}$$

for $i \in \mathbb{Z}_{>0}$. Then W_i lies in the interior of W_{i+1} for $i \in \mathbb{Z}_{>0}$. Choose a connected component V_i of W_i so that $V_1 \subseteq V_2 \subseteq \dots$ and $x \in V := \bigcup_{i=1}^{\infty} V_i$. If $x \in V_i$ for some $i \in \mathbb{Z}_{>0}$, then x lies in the topological interior of V_{i+1} . Hence, x lies in the interior of V . By construction, V is open connected paracompact locally compact and Hausdorff. Moreover, $V \subseteq U$ by our construction. \square

Proposition 3.31. Let $\{X_i\}_{i \in I}$ be a family of k_H -analytic spaces. Suppose that for $i, j \in I$, we are given a k_H -analytic domain $X_{ij} \subseteq X_i$ and an isomorphism $\nu_{ij} : X_{ij} \rightarrow X_{ji}$ satisfying the *cocycle condition*: $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $X_{ij} \cap X_{il}$ for $i, j, l \in I$.

Assume that either of the following conditions holds:

- (1) X_{ij} is open in X_i for all $i, j \in I$;
- (2) for any $i \in I$, all X_{ij} 's are closed in X_i and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite.

Then there is a k_H -analytic space X and morphisms $\mu_i : X_i \rightarrow X$ for $i \in I$ such that

- (1) μ_i is an isomorphism of X_i with a k_H -analytic domain in X ;
- (2) $X = \bigcup_{i \in I} \mu_i(X_i)$;
- (3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$ for $i, j \in I$;
- (4) $\mu_i = \mu_j \circ \nu_{ij}$ on X_{ij} for $i, j \in I$.

The space X is unique up to a canonical isomorphism. Moreover, under Condition (1), $\mu_i(X_i)$ is open in X for $i \in I$; under Condition (2), $\mu_i(X_i)$ is closed in X for $i \in I$.

Under both conditions, if all X_i 's are Hausdorff (resp. paracompact), then so is X .

We will call X the *gluing* of the X_i 's along the X_{ij} 's.

PROOF. By Proposition 4.10, the uniqueness of X is clear.

Let

$$\tilde{X} = \coprod_{i \in I} X_i$$

in $k_H\text{-An}$. Observe that

$$|\tilde{X}| = \coprod_{i \in I} |X_i|$$

in the category $\mathcal{T}\text{op}$. The system ν_{ij} 's defines an equivalence relation R on $|\tilde{X}|$. Let $|X| = |\tilde{X}|/R$ and $\mu_i : |X_i| \rightarrow |X|$ be the induced map for $i \in I$.

Under Condition (1), $\mu_i(|X_i|)$ is open in $|X|$ for $i \in I$. Under Condition (2), $\mu_i(|X_i|)$ is closed in $|X|$ for $i \in I$.

Under both conditions, the map μ_i induces a homeomorphism $|X_i| \rightarrow \mu_i(|X_i|)$ for $i \in I$. If all $|X_i|$'s are Hausdorff (resp. paracompact), so is $|X|$.

All these claims follow from well-known results in general topology.

We will endow $|X|$ with a structure of k_H -analytic space. Let τ be the set of $V \subseteq |X|$ for which there is $i \in I$ such that $V \subseteq \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is a k_H -affinoid domain in X_i . Then τ is a net on X . There is an obvious k -affinoid atlas on X with the net τ . All properties in the proposition are satisfied by $X = (|X|, \mathcal{A}, \tau)$. \square

Proposition 3.32. Let (X, \mathcal{A}, τ) be a paracompact Hausdorff k_H -analytic space. Then the locally finite nets are cofinal in the directed set of nets σ on X contained in $\hat{\tau}$.

PROOF. Let σ be a net on X contained in $\bar{\tau}$. We need to find a refinement η of σ contained in $\bar{\tau}$ that is locally finite.

We first choose an open covering \mathcal{U} of X by open σ -compact Hausdorff and locally compact subsets. This is possible by [Proposition 3.30](#). *Is this possible?* Up to replacing \mathcal{U} by a refinement, we may assume that \mathcal{U} is locally finite.

Now we can choose a locally finite covering \mathcal{V}_U of each $U \in \mathcal{U}$ by k_H -affinoid domains in X which are contained in some element in σ . Let $\mathcal{V} = \bigcup_{U \in \mathcal{U}} \mathcal{V}_U$. Then \mathcal{V} is a locally finite covering of X .

Refining \mathcal{V} , we may assume that each pair of elements in \mathcal{V} are either disjoint or both contained in a common k_H -affinoid domain in X . Then the collection of all finite intersections in \mathcal{V} is a net. \square

Proposition 3.33. The category $k_H\text{-An}$ admits finite limits.

Many details need to be included in this proof!

PROOF. By general abstract nonsense, it suffices to show that $k_H\text{-An}$ admits finite fiber products.

Let $\varphi : Y \rightarrow X$ and $f : X' \rightarrow X$ be morphisms of k_H -affinoid spaces.

Step 1. We show assume that $H = \mathbb{R}_{>0}$. We show that $Y \times_X X'$ exists in $k\text{-An}$.

Step 1.1. Assume that Y, X, X' are all paracompact and Hausdorff.

We first claim that we can represent φ and f by strong morphisms $(Y, \mathcal{B}, \sigma) \rightarrow (X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau') \rightarrow (X, \mathcal{A}, \tau)$ with σ, τ', τ locally finite.

We first choose a locally finite net τ consisting of k -affinoid domains on X by [Proposition 3.32](#). Consider the net σ' consisting of all k -affinoid domains on Y that are mapped into some element in τ . Then φ can be represented by a strong morphism $(Y, \mathcal{B}, \sigma') \rightarrow (X, \mathcal{A}, \tau)$. We now refine σ' using [Proposition 3.32](#) again to construct σ . The locally finite net τ' is constructed in the same way.

Let

$$\mathcal{S} := \{(V, U, U') \in \sigma \times \tau \times \tau' : \varphi(V) \subseteq U, f(U') \subseteq U\}.$$

For any $\alpha = (V, U, U') \in \mathcal{S}$, we write

$$W_\alpha := V \times_U U', \quad \Sigma_\alpha = |V| \times_{|U|} |U'|,$$

and we have an obvious continuous map of topological spaces $W_\alpha \rightarrow \Sigma_\alpha$. Let

$$\Sigma = |Y| \times_{|X|} |X'|.$$

Then we have a canonical map $\pi_\alpha : W_\alpha \rightarrow \Sigma$ for any $\alpha \in \mathcal{S}$.

We claim that for any $\alpha, \beta \in \mathcal{S}$,

$$W_{\alpha\beta} := \pi_\alpha^{-1}(\Sigma_\alpha \cap \Sigma_\beta)$$

is a k -special domain in W_α and there is a canonical isomorphism $\nu_{\alpha\beta} : W_{\alpha\beta} \rightarrow W_{\beta\alpha}$ satisfying the cocycle condition.

In fact, if $\alpha = (V, U, U')$ and $\beta = (\bar{U}, \bar{U}, \bar{U}')$, we can write

$$U \cap \bar{U} = U_1 \cup \dots \cup U_n$$

for some $n \in \mathbb{N}$ and $U_1, \dots, U_n \in \tau$. For each $i = 1, \dots, n$, we write

$$\varphi_{V/U}^{-1}(U_i) \cap \varphi_{\bar{V}/\bar{U}}^{-1}(U_i) = \bigcup_{j=1}^{p_i} V_{ij}, \quad f_{U'/U}^{-1}(U_i) \cap f_{\bar{U}'/\bar{U}}^{-1}(U_i) = \bigcup_{l=1}^{q_i} U'_{il},$$

where $p_i, q_i \in \mathbb{N}$, $V_{ij} \in \sigma$ and $U'_{il} \in \tau'$ for $j = 1, \dots, p_i$ and $l = 1, \dots, q_i$. Then

$$W_{\alpha\beta} = \bigcup_{i,j,l} V_{ij} \times_{U_i} U'_{il}.$$

So $W_{\alpha\beta}$ is a k -special domain in W_α . **We need to add a few lemmas** We have an obvious isomorphism $\nu_{\alpha\beta} : W_{\alpha\beta} \rightarrow W_{\beta\alpha}$ satisfying the cocycle condition. We can then glue the W_α 's along the $W_{\alpha\beta}$'s by ?? to obtain a k -analytic space W . By **Proposition 4.10**, W represents the fiber product $Y \times_X X'$.

Step 1.2. Now we only assume X to be paracompact and Hausdorff.

Take open paracompact coverings $\{Y_i\}_{i \in I}$ of Y and $\{X'_j\}_{j \in J}$ of X' . The existence of these coverings follows from **Proposition 3.30**. Similar to Step 1.1, we glue the $Y_i \times_X X'_j$'s along the open subsets $(Y_i \cap Y_k) \times_X (X'_j \cap X'_l)$'s, we get a locally Hausdorff k_H -analytic space Y' . Then by **Proposition 4.10** again, Y' represents the fiber product $Y \times_X X'$.

Step 1.3. We do not assume that X is paracompact. Take a covering $\{X_i\}_{i \in I}$ by open paracompact subsets. Let Y' be the gluing of $\varphi^{-1}(X_i) \times_{X_i} f^{-1}(X_i)$'s along $\varphi^{-1}(X_i \cap X_j) \times_{X_i \cap X_j} f^{-1}(X_i \cap X_j)$'s. Then by **Proposition 4.10** again, Y' represents the fiber product $Y \times_X X'$.

Step 2. We handle the general case. Let σ' be the collection of all k_H -affinoid domains of the form $V \times_U U'$, where $V \subseteq Y$, $U \subseteq X$, $U' \subseteq X'$ are k_H -affinoid domains with $\varphi(V) \subseteq U$, $f(U') \subseteq U$. Then σ' is a net on Y' and there is an obvious k_H -affinoid atlas on Y' with the net σ' . This defines a structure a k_H -affinoid spaces on Y' . It is clear that Y' represents the fiber product $Y \times_X X'$ in the category of k_H -An. \square

4. Analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 4.1. Let X be a k_H -analytic space. A subset $Y \subseteq X$ is called a k_H -analytic domain if for any $y \in Y$, there exist $n \in \mathbb{Z}_{>0}$, k_H -affinoid domains V_1, \dots, V_n contained in Y such that

- (1) $y \in V_1 \cap \dots \cap V_n$;
- (2) $V_1 \cup \dots \cup V_n$ is a neighbourhood of y in Y .

Observe that the net of k_H -affinoid domains in X that are contained in Y form a net on Y . In particular, Y inherits a k_H -analytic space structure from X , and we have a canonical morphism $Y \rightarrow X$ in k_H -An.

Example 4.2. Let X be a k_H -analytic space. Then any open subset U of X is a k_H -analytic domain.

In fact, for $x \in U$, take V_1, \dots, V_n as in [Definition 4.1](#). By ?? in ??, up to replacing V_i 's by k_H -Laurent domains in them, we may guarantee that $V_i \subseteq U$ for all $i = 1, \dots, n$.

Proposition 4.3. Let X, X' be k_H -analytic spaces and $\varphi : X' \rightarrow X$ a morphism of k_H -analytic spaces.

- (1) Let Y, Z be k_H -analytic domains in X , then so is $Y \cap Z$.
- (2) Let Y be a k_H -analytic domain in X , then $\varphi^{-1}(Y)$ is a k_H -analytic domain in X' .

PROOF. (1) Let $x \in Y \cap Z$. Take k_H -affinoid domains V_1, \dots, V_n contained in Y and k_H -affinoid domains W_1, \dots, W_m contained in Z such that

$$x \in V_1 \cap \dots \cap V_n, \quad x \in W_1 \cap \dots \cap W_m$$

and $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in Y , $W_1 \cup \dots \cup W_m$ is a neighbourhood of x in Z . For each $i = 1, \dots, n$ and $j = 1, \dots, m$, $\hat{\tau}|_{V_i \cap W_j}$ is a quasi-net, so we can find a neighbourhood of x in $V_i \cap W_j$ of the form $U_1^{ij} \cup \dots \cup U_{m_{ij}}^{ij}$ with $U_1^{ij}, \dots, U_{m_{ij}}^{ij}$ being k_H -affinoid domains in X containing x . Then each element in the collection $\{U_k^{ij}\}$ contains x and the union is a neighbourhood of x in $Y \cap Z$.

(2) Let $x' \in \varphi^{-1}(Y)$ and $x = \varphi(x')$. By [Proposition 3.25](#), we can find $n \in \mathbb{Z}_{>0}$, k_H -affinoid domains V'_1, \dots, V'_n on X' and k_H -affinoid domains V_1, \dots, V_n on X such that

$$x' \in V'_1 \cap \dots \cap V'_n, \quad x \in V_1 \cap \dots \cap V_n, \\ \varphi(V'_i) \subseteq V_i \text{ for } i = 1, \dots, n,$$

and $V'_1 \cup \dots \cup V'_n$ (resp. $V_1 \cup \dots \cup V_n$) is a neighbourhood of x' (resp. x) in X' (resp. X). Take k_H -affinoid domains W_1, \dots, W_m in X contained in Y , each containing x such that $W_1 \cup \dots \cup W_m$ is a neighbourhood of x in Y .

Then for each $i = 1, \dots, n$, $j = 1, \dots, m$, we can find k_H -affinoid domains W_{ij}^k for $k = 1, \dots, r_{ij}$ contained in $W_j \cap V_i$ and containing x such that $\cup_k W_{ij}^k$ is a neighbourhood of x in $W_j \cap V_i$. Thus, $\cup_{j,k} W_{ij}^k$ is a neighbourhood of x in $V_i \cap Y$. Then $U_{ij}^k := \varphi^{-1}(W_{ij}^k) \cap V'_i$ is a k_H -affinoid domain in V'_i by ?? in ??. Moreover, $\cup_{j,k} U_{ij}^k$ is a neighbourhood of x' in $V'_i \cap Y'$. So $\cup_{i,j,k} U_{ij}^k$ is a neighbourhood of x' in Y' . \square

Proposition 4.4. Let X be a k_H -analytic space and Y be a k_H -analytic domain in X . Then for any k_H -analytic space Z and any morphism $\varphi : Z \rightarrow X$ whose image is contained in Y , there is a unique morphism $\psi : Z \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} Z & & \\ \downarrow \psi & \searrow \varphi & \\ Y & \longrightarrow & X \end{array}.$$

PROOF. The uniqueness of ψ is obvious. We only need to prove the existence. This is an immediate consequence of [Proposition 3.25](#) and [Proposition 4.3](#).

To be more precise, assume that φ is given by a data as in [Proposition 3.25](#), we only have to show that each k_H -affinoid domain V in X , $V \cap Y$ is a k_H -affinoid domain in Y . This follows from [Proposition 4.3](#). \square

Corollary 4.5. Let $\varphi : X' \rightarrow X$ be a morphism of k_H -analytic spaces and Y be a k_H -analytic domain in X . Then $X' \times_Y X$ in the category $k_H\text{-An}$ exists and $\varphi^{-1}(Y)$ represents $X' \times_Y X$.

PROOF. This follows from [Proposition 4.4](#) and [Proposition 4.3](#). \square

Corollary 4.6. Let $\text{Sp } B$ be a k_H -affinoid space, then we have a functorial isomorphism

$$\text{Hom}_{k_H\text{-An}}(\text{Sp } B, \mathbb{A}_k^1) \xrightarrow{\sim} B.$$

PROOF. As $\text{Sp } B$ is compact as a topological space, its image in \mathbb{A}_k^1 is contained in $\text{Sp } k\{r^{-1}T\}$ for some $r > 0$. By [Proposition 4.4](#), we have natural bijections

$$\text{Hom}_{k_H\text{-An}}(\text{Sp } B, \mathbb{A}_k^1) \xrightarrow{\sim} \varinjlim_{r>0} \text{Hom}_{k_H\text{-An}}(\text{Sp } B, \text{Sp } k\{r^{-1}T\}) \xrightarrow{\sim} \varinjlim_{r>0} \text{Hom}_{k\text{-AffAlg}}(k\{r^{-1}T\}, B).$$

By ?? in ??, the right-hand side is identified with B . \square

Proposition 4.7. Let X be a k_H -analytic space, Y be a k_H -analytic domain in X . For a subset $Z \subseteq Y$, the following are equivalent:

- (1) Z be a k_H -analytic domain in X ;
- (2) Z is a k_H -analytic domain in Y .

PROOF. (1) \implies (2): Let $z \in Z$, we can find $n \in \mathbb{Z}_{>0}$ and k_H -affinoid domains V_1, \dots, V_n in X containing z and contained in Z such that $V_1 \cup \dots \cup V_n$ is a neighbourhood of z in Z . But observe that V_1, \dots, V_n are k_H -affinoid domains in Y as well, so we conclude.

(2) \implies (1): This follows from the same argument. It suffices to observe that a k_H -affinoid domain in Y is necessarily k_H -affinoid in X , as can be seen from [Definition 3.19](#). \square

Definition 4.8. Let X, Y be k_H -analytic spaces and $\varphi : Y \rightarrow X$ be a morphism. We say φ is an *open immersion* if $\varphi(Y)$ is open in X and φ induces an isomorphism between Y and $\varphi(Y)$ as k_H -analytic spaces.

By [Example 4.2](#), $\varphi(Y)$ is a k_H -analytic domain in X and by [Proposition 4.4](#), we have a morphism of k_H -analytic spaces $Y \rightarrow \varphi(Y)$.

Proposition 4.9. Let X be a k_H -analytic space and Y be a k_H -analytic domain in X . Assume that Y is a k_H -affinoid space, then Y is a k_H -affinoid domain in X .

PROOF. As Y is a k_H -affinoid space, we know that $|Y|$ is compact. Take finitely many k_H -affinoid domains V_1, \dots, V_n in X such that

$$Y = V_1 \cup \dots \cup V_n.$$

Then V_1, \dots, V_n are k_H -affinoid domains in Y : let $\text{Sp } D \rightarrow Y$ be a morphism of k_H -affinoid spectra, whose image lies in V_i for some $i = 1, \dots, n$. Consider the following commutative diagram

$$\begin{array}{ccccc} & & \text{Sp } D & & \\ & \swarrow & \downarrow & \searrow & \\ V_i & \longrightarrow & Y & \longrightarrow & X \end{array}$$

By [Proposition 4.4](#), there is a unique dotted morphism making the outer triangle commutative, hence making the whole diagram commutative. We have therefore shown that V_i is a k_H -affinoid domain in Y .

So the covering $\{V_1, \dots, V_n\}$ of Y satisfies the assumptions in [Definition 3.19](#) and Y is k_H -affinoid. \square

Proposition 4.10. Let X be a k_H -analytic space and $\{Y_i\}_{i \in I}$ be a family of k_H -analytic domains in X which forms a quasi-net on X . Then for any k_H -analytic space X' , the following sequence is exact

$$\mathrm{Hom}_{k_H\text{-An}}(X, X') \rightarrow \prod_{i \in I} \mathrm{Hom}_{k_H\text{-An}}(Y_i, X') \rightrightarrows \prod_{i, j \in I} \mathrm{Hom}_{k_H\text{-An}}(Y_i \cap Y_j, X').$$

PROOF. Let $\{\varphi_i : Y_i \rightarrow X'\}_{i \in I}$ be a family of morphisms such that φ_i, φ_j coincides on $Y_i \cap Y_j$ for $i, j \in I$. We need to glue the φ_i 's into a single morphism $\varphi : X \rightarrow X'$. Clearly, the underlying maps glue together to a continuous map $\varphi : X \rightarrow X'$ by ?? in ??.

Let τ be the collection of k_H -affinoid domains V in X such that there is $i \in I$ and a k_H -affinoid domain $V' \subseteq X'$ with $V \subseteq Y_i$ and $\varphi_i(V) \subseteq V'$. Then τ is a net on X , and we have a morphism $X \rightarrow X'$. \square

5. Berkovich site

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Lemma 5.1. Let X be a k_H -analytic space. Consider the category \mathcal{C} of k_H -analytic domains in X , where the morphisms are inclusions of k_H -analytic domains. For each $Y \in \mathcal{C}$, consider the set of coverings $\mathrm{Cov}(Y)$ consisting of all $\{Y_i \rightarrow Y\}_{i \in I}$ such that Y_i is a k_H -analytic domain in Y and $\{Y_i\}_{i \in I}$ is a quasi-net on Y . The class of coverings $\{\mathrm{Cov}(Y)\}_Y$ defines a Grothendieck pretopology.

PROOF. It suffices to verify the axioms in [\[Stacks, Tag 03NH\]](#).

(1) An isomorphism $Y' \rightarrow Y$ in \mathcal{C} is in $\mathrm{Cov}(Y)$.

This is trivial as an isomorphism in \mathcal{C} is necessarily identity.

(2) If $\{Y_i \rightarrow Y\}_{i \in I}$ and $\{Y_{ij} \rightarrow Y_i\}_{j \in J_i}$ for all $i \in I$ are in $\mathrm{Cov}(Y)$ and $\mathrm{Cov}(Y_i)$ respectively, then $\{Y_{ij} \rightarrow Y\}_{i,j}$ is in $\mathrm{Cov}(Y)$.

By [Proposition 4.7](#), Y_{ij} is a k_H -analytic domain in Y for any $i \in I, j \in J_i$. It suffices to show that $\{Y_{ij}\}_{i,j}$ is a quasi-net on Y . Let $y \in Y$, we can find finitely many elements among $\{Y_i\}_{i \in I}$, say Y_1, \dots, Y_n so that $y \in Y_i$ for each $i = 1, \dots, n$ and $Y_1 \cup \dots \cup Y_n$ is a neighbourhood of y in Y . Similarly, for each $i = 1, \dots, n$, we can find finitely many Y_{i1}, \dots, Y_{ij_i} among $\{Y_{ij}\}_{j \in J_i}$ so that y is contained in each of them and $Y_{i1} \cup \dots \cup Y_{ij_i}$ is a neighbourhood of y in Y_i . Then each element in $\{Y_{ij}\}_{i=1, \dots, n; j=1, \dots, j_i}$ contains y and the union is a neighbourhood of y in Y .

(3) If $\{Y_i \rightarrow Y\}_{i \in I}$ lies in $\mathrm{Cov}(Y)$ and $Z \rightarrow Y$ is a k_H -analytic domain in Y , then the fiber products $Y_i \times_Y Z$ exist and $\{Y_i \times_Y Z \rightarrow Z\}_{i \in I}$ lies in $\mathrm{Cov}(Z)$.

By [Corollary 4.5](#), $Y_i \times_Y Z$ exists and is represented by the inverse image of Z in Y_i , which is a k_H -analytic domain in Y_i by [Proposition 4.3](#). It is clear that $\{Y_i \times_Y Z\}_{i \in I}$ is a quasi-net on Z . \square

Definition 5.2. Let X be a k_H -analytic space. We will write the site constructed in [Lemma 5.1](#) as X and call it the *Berkovich site* of X . The corresponding Grothendieck

topology is called the *Berkovich Grothendieck topology*. The topos $\mathrm{Sh}(X)$ associated with X is called the *Berkovich topos* of X .

Observe that the Berkovich Grothendieck topology is subcanonical by [Proposition 4.10](#).

Definition 5.3. Let X be a k_H -analytic space. We define a sheaf of rings \mathcal{O}_X on X as follows: let Y be a k_H -analtic domain in X , we set

$$\mathcal{O}_X(Y) = \mathrm{Hom}_{k_H\text{-An}}(X, \mathbb{A}_k^1).$$

By [Corollary 4.6](#) and [Proposition 4.10](#), \mathcal{O}_X defines a sheaf of rings. We call \mathcal{O}_X the structure sheaf of X . The corresponding ringed site (X, \mathcal{O}_X) is called the *Berkovich ringed site*. The induced ringed topos $(\mathrm{Sh}(X), \mathcal{O}_X)$ is called the *Berkovich ringed topos*.

Given any morphism $f : Y \rightarrow X$ of k_H -analytic spaces, we have an induced morphism of the corresponding ringed sites, still denoted by φ .

Definition 5.4. Let X be a k_H -analytic space. An \mathcal{O}_X -module \mathcal{M} is *coherent* if there is an admissible covering $\{Y_i\}_{i \in I}$ of X such that $\mathcal{M}|_{Y_i}$ is isomorphic to the cokernel of a homomorphism of finite free \mathcal{O}_{Y_i} -modules.

Example 5.5. Let A be a k_H -affinoid algebra and M be a finite A -module. Then M induces a coherent sheaf of $\mathcal{O}_{\mathrm{Sp} A}$ -modules \tilde{M} as follows:

$$\tilde{M}(V) = M \otimes_A A_V.$$

Conversely, we can reformulate Kiehl's theorem.

Theorem 5.6. Let A be a k_H -affinoid algebra and \mathcal{M} be a coherent sheaf of $\mathcal{O}_{\mathrm{Sp} A}$ -modules. Set $M = H^0(X, \mathcal{M})$, then M is a finite A -module and we have a canonical isomorphism

$$\tilde{M} \xrightarrow{\sim} \mathcal{M}.$$

The left-hand side is defined in [Example 5.5](#).

PROOF. This is just a reformulation of ?? in ??. □

Corollary 5.7. Let $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism of k_H -affinoid spaces. Then the following are equivalent:

- (1) $\varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is a coherent $\mathcal{O}_{\mathrm{Sp} A}$ -module;
- (2) B is a finite Banach A -module.

PROOF. Observe that for any k_H -affinoid domain $\mathrm{Sp} C$ in $\mathrm{Sp} A$,

$$\varphi_* \mathcal{O}_{\mathrm{Sp} B}(\mathrm{Sp} C) = \mathcal{O}_{\mathrm{Sp} B}(\varphi^{-1}(\mathrm{Sp} C)) = \mathcal{O}_{\mathrm{Sp} B}(\mathrm{Sp} C \hat{\otimes}_A B) = C \hat{\otimes}_A B \xrightarrow{\sim} C \otimes_A B.$$

Here we applied ?? in ?? and ?? in ??. So $\varphi_* \mathcal{O}_{\mathrm{Sp} B} \cong \tilde{B}$.

From this (2) trivially implies (1).

Conversely, assume (1), let $B = H^0(\mathrm{Sp} A, \varphi_* \mathcal{O}_{\mathrm{Sp} B})$. By [Theorem 5.6](#), B is a finite A -module. Let B' denote the ring B endowed with the finite Banach A -algebra structure as in ?? in ??. We need to show that the identity map $B' \rightarrow B$ is admissible. Observe that the identity map is bounded by ?? in ??. By ?? in ??, it suffices to show that the induced map $\mathrm{Sp} B \rightarrow \mathrm{Sp} B'$ is surjective. Let $\varphi' : \mathrm{Sp} B' \rightarrow \mathrm{Sp} A$ be the natural morphism of k_H -affinoid spaces. Then

$$\varphi_*(\mathcal{O}_{\mathrm{Sp} B}) \xrightarrow{\sim} \varphi'_*(\mathcal{O}_{\mathrm{Sp} B'}).$$

It follows that $\varphi^{-1}(x) = \varphi'^{-1}(x)$ for any $x \in \mathrm{Sp} A$. We conclude. □

Corollary 5.8. Let $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism of k_H -affinoid spaces. Then the following are equivalent:

- (1) $\varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is a coherent $\mathcal{O}_{\mathrm{Sp} A}$ -module and $\mathcal{O}_{\mathrm{Sp} A} \rightarrow \varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is surjective;
- (2) $A \rightarrow B$ is an admissible epimorphism.

PROOF. Assume (2). By [Corollary 5.7](#), $\varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is a coherent $\mathcal{O}_{\mathrm{Sp} A}$ -module. To see that $\mathcal{O}_{\mathrm{Sp} A} \rightarrow \varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is surjective, it suffices to show that for each k_H -affinoid space $\mathrm{Sp} C$ in $\mathrm{Sp} A$,

$$C \rightarrow C \otimes_A B$$

is surjective. This follows from the assumption.

Assume (1). We know that B is a finite Banach A -module. In particular, $A \rightarrow B$ is admissible by ?? in ??. As $\mathcal{O}_{\mathrm{Sp} A} \rightarrow \varphi_* \mathcal{O}_{\mathrm{Sp} B}$ is surjective, by [Theorem 5.6](#), $A \rightarrow B$ is surjective. [Include details](#) \square

Definition 5.9. Let $\mathrm{Sp} A$ be a k_H -affinoid space and $\mathcal{M} = \tilde{M}$ is a coherent sheaf of \mathcal{O}_X -modules on X , where M is a finite A -module. The *support* $\mathrm{Supp} M$ of \mathcal{M} is the closed subset $\mathrm{Sp} A / \mathrm{Ann}_A(M)$ of $\mathrm{Sp} A$.

Let X be a k_H -analytic space and \mathcal{M} be a coherent sheaf of \mathcal{O}_X -modules. Then the *support* $\mathrm{Supp} \mathcal{M}$ of \mathcal{M} is a subset of X such that a point $x \in X$ lies in $\mathrm{Supp} \mathcal{M}$ if and only if for some k_H -affinoid domain V in X containing x , $x \in \mathrm{Supp} \mathcal{M}|_V$.

Here $\mathrm{Ann}_A(M)$ is the annihilator of M in A .

Lemma 5.10. Let X be a k_H -analytic space and \mathcal{M} be a coherent sheaf of \mathcal{O}_X -modules. Take $x \in \mathrm{Supp} \mathcal{M}|_V$ and a k_H -affinoid domain V in X containing x . Then $x \in \mathrm{Supp} \mathcal{M}|_V$.

PROOF. By assumption, there is a k_H -affinoid domain U in X containing x such that $x \in \mathrm{Supp} \mathcal{M}|_U$.

Let $W \subseteq U \cap V$ be a k_H -affinoid domain in X containing x . We claim that $x \in \mathrm{Supp} \mathcal{M}|_W$. Let $M = H^0(U, \mathcal{M})$, then $M \otimes_{A_U} A_W = H^0(W, \mathcal{M})$. By [\[Stacks, Tag 07T8\]](#) and ?? in ??,

$$\mathrm{Ann}_{A_U}(M) \otimes_{A_U} A_W = \mathrm{Ann}_{A_W}(M \otimes_{A_U} A_W)$$

and $\mathrm{Supp}(\mathcal{M}|_W) = \mathrm{Supp}(\mathcal{M}|_U) \cap W$. The claim follows. We may assume that $U \subseteq V$. In this case, the same argument shows that $x \in \mathrm{Supp} \mathcal{M}|_V$. \square

6. Closed immersion

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Lemma 6.1. Let $\varphi : Y \rightarrow X$ be a morphism of k_H -analytic spaces. Then the following are equivalent:

- (1) for any $x \in X$, there are $n \in \mathbb{Z}_{>0}$ and k_H -affinoid domains V_1, \dots, V_n in X containing x such that $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X and the restriction $\varphi^{-1}(V_i) \rightarrow V_i$ is a closed immersion for any $i = 1, \dots, n$;
- (2) for any k_H -affinoid domain V in X , $\varphi^{-1}(V) \rightarrow V$ is a closed immersion.

Recall that closed immersions between k_H -affinoid spaces are defined in ?? in ??.

The statement in [\[Ber93, Lemma 1.3.7\]](#) is not correct.

PROOF. Only (1) \implies (2) is non-trivial. Assume (1). Let τ be the collections of $V \subseteq X$ satisfying (2). Then we claim that τ is a net.

Observe that τ is a quasi-net by our assumption. To see that it is a net, take $U, V \in \tau$ and $x \in U \cap V$, then we can find $n \in \mathbb{Z}_{>0}$ and k_H -affinoid domains W_1, \dots, W_n in $U \cap V$ containing x such that $W_1 \cup \dots \cup W_n$ is a neighbourhood of x in $U \cap V$. In order to show that $\tau|_{U \cap V}$ is a quasi-net, it suffices to show that $\varphi^{-1}(W_i) \rightarrow W_i$ is a closed immersion for $i = 1, \dots, n$. This follows from ?? in ??.

Let V be a k_H -affinoid domain in X . By (1) and the compactness of V , we can find $n \in \mathbb{Z}_{>0}$ and $V_1, \dots, V_n \in \tau$ such that $V \subseteq V_1 \cup \dots \cup V_n$. By ?? in ??, we can find $m \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_m \in \tau$ such that

$$V = U_1 \cup \dots \cup U_m$$

and each U_j is contained in some V_i , where $j = 1, \dots, m$ and $i = 1, \dots, n$. By ?? in ?? again, $U_j \in \tau$ for each $j = 1, \dots, m$. It suffices to apply [Corollary 5.8](#) to conclude that $V \in \tau$. \square

Definition 6.2. Let $\varphi : Y \rightarrow X$ be a morphism of k_H -analytic spaces. We say φ is a *closed immersion* if the equivalent conditions in [Lemma 6.1](#) are satisfied.

Observe that this definition extends ?? in ??. Moreover, [Lemma 6.1](#) shows that being a closed immersion is a G-local property on the target.

[Include the proof of the existence of finite limits](#)

Proposition 6.3. Let $\varphi : Y \rightarrow X$, $\psi : Z \rightarrow X$ be a morphism of k_H -analytic spaces. Assume that $\varphi : Y \rightarrow X$ is a closed immersion. Consider the Cartesian diagram

$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Y \\ \downarrow & \square & \downarrow \varphi \\ Z & \xrightarrow{\psi} & X \end{array}$$

Then $Z \times_X Y \rightarrow Z$ is a closed immersion.

PROOF. Taking a G -covering of Z , we may assume that Z is compact. We could cover the images of Z in X by finitely many k_H -affinoid domains V_1, \dots, V_n in X , considering their preimages in Z , we could reduce to the case where the image of Z in X is contained in a k_H -affinoid domain. We could then assume that X is a k_H -affinoid space and hence so is Y . By taking a G -covering of Z again, we may assume that Z is affinoid. It suffices to apply ?? in ??. \square

7. Reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 7.1. A *punctured k_H -analytic space* (X, x) is a k_H -analytic space X together with a point $x \in X$.

A morphism between punctured k_H -analytic spaces (X, x) and (Y, y) is a morphism $\varphi : X \rightarrow Y$ of k_H -analytic spaces sending x to y .

The category of punctured k_H -analytic spaces is denoted by $k_H\text{-An}_*$.

Definition 7.2. The category $k_H\text{-Ger}$ is the category of right fractions of $k_H\text{-An}_*$ with respect to the system of morphisms

$$\varphi : (X, x) \rightarrow (Y, y)$$

that induces an isomorphism of X with an open neighbourhood of y in Y in the category of $k_H\text{-Ger}$.

When we view (X, x) as an object in $k_H\text{-Ger}$, we write it as X_x . An object in $k_H\text{-Ger}$ is called a k_H -analytic germ.

By definition,

$$\text{Hom}_{k_H\text{-Ger}}(X_x, Y_y) = \varinjlim_U \text{Hom}_{k_H\text{-An}^*}((U, x), (Y, y)),$$

where U runs over all open neighbourhoods of x in X .

Definition 7.3. A k_H -analytic germ X_x is *good* if x admits an affinoid neighbourhood in X .

Note that this condition does not depend on the representative (X, x) . To see this, let $U \subseteq x$ be an open subset containing x . We need to show that if x admits a k_H -affinoid neighbourhood in X , then it admits one in U . This follows from ?? in ??.

Definition 7.4. A morphism of k_H -analytic germs $\varphi : X_x \rightarrow Y_y$ is said to be *separated* (resp. *closed*) if it is induced

Theorem 7.5. Let $H' \supseteq H$ be a subgroup of $\mathbb{R}_{>0}$. The natural functor

$$k_H\text{-An} \rightarrow k_{H'}\text{-An}$$

is fully faithful.

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