Topology and bornology

Contents

1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A comapct space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

References to this chapter include [Ber93].

2. Nets

Let X be a set, $Y \subseteq X$ be a subset. Consider a collection τ of subsets of X, we write

$$\tau|_Y := \{V \in \tau : V \subseteq Y\}.$$

Definition 2.1. Let X be a topology space and τ be a collection of subsets of X.

(1) We say τ is *dense* if for any $V \in \tau$ and any $x \in V$, there is a fundamental system of neighbourhoods of x in V consisting of sets from $\tau|_V$.
(2) We say τ is a *quasi-net* on X if for each $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \cdots \cap V_n$ and that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X.
(3) We say τ is a *net* on X if it is a quasi-net and if for any $U, V \in \tau$, $\tau|_{U\cap V}$ is a quasi-net on $U \cap V$.

Lemma 2.2. Let X be a topological space and τ be a quasi-net on X.

(1) A subset $U \subseteq X$ is open if and only if for each $V \in \tau$, $U \cap V$ is open in V.
(2) Suppose that τ consists of compact sets. Then X is Hausdorff if and only if for any $U, V \in \tau$, $U \cap V$ is compact.

We remind the readers that a compact space is Hausdorff by our convention.

Proof. (1) The direct implication is trivial. Suppose that $U \cap V$ is open in V for all $V \in \tau$. We want to show that U is open. Take $x \in U$, we can find $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X. By our hypothesis, we can find open sets $W_1, \ldots, W_n$ in W such that $W \cap V_i = U \cap V_i$ for $i = 1, \ldots, n$. Then $W = W_1 \cap \cdots \cap W_n$ is an open neighbourhood of x in X. But then

$$U \cap (V_1 \cup \cdots \cup V_n) \supseteq W \cap (V_1 \cup \cdots \cup V_n),$$

the latter is a neighbourhood of x hence so is the former. It follows that U is open.

(2) The direct implication is trivial. Consider the quasi-net $\tau \times \tau := \{U \times V : U, V \in \tau\}$ on $X \times X$. By (1), it suffices to verify that the intersection of the diagonal with $U \times V$ is closed in $U \times V$ for any $U, V \in \tau$. But this intersection is homeomorphic to $U \cap V$, which is compact by our assumption and hence closed as U, V are both Hausdorff. □

Lemma 2.3. Let X be a Hausdorff space. Assume that X admits a quasi-net τ consisting of compact sets. Then X is locally compact.

Proof. Take $x \in X$. By assumption, we can find $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x. This neighbourhood is clearly compact. □

Lemma 2.4. *Let X be a Hausdorff space and τ be a collection of compact subsets of X. Then the following are equivalent:*

(1) *τ is a quasi-net;*
(2) *For each $x \in X$, there are $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \tau$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X.*

Proof. (1) $\implies$ (2): This is trivial.

(2) $\implies$ (1): Given $x \in X$, take $V_1, \dots, V_n$ as in (2). We may assume that $x \in V_1, \dots, V_m$ and $x \notin V_{m+1}, \dots, V_n$ for some $1 \leq m \leq n$. Then $V_1 \cup \cdots \cup V_m$ is a neighbourhood of x in X: if U is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_n$, then $U \setminus (V_{m+1} \cup \cdots \cup V_n)$ is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_m$. □

Lemma 2.5. *Let X be a topological space and τ be a net on X consisting of compact sets. Then*

(1) *for any pair $U, V \in \tau$, the intersection $U \cap V$ is lcoally closed in U and in V;*
(2) *If $n \in \mathbb{Z}_{>0}$, $V, V_1, \dots, V_n \in \tau$ are such that*
$$V \subseteq V_1 \cup \cdots \cup V_n,$$
then there are $m \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_m \in \tau$ such that
$$V = U_1 \cup \cdots \cup U_m$$
and each U_j is contained in some V_i.

Proof. (1) It suffices to show that $U \cap V$ is locally compact in the induced topology. This follows from Lemma 2.3.

(2) For each $x \in V$ and each $i = 1, \dots, n$ such that $x \in V_i$, we take a neighbourhood of x in $V \cap V_i$ of the form $W_i V_{i1} \cup \cdots \cup V_{im_i}$ for some $m_i \in \mathbb{Z}_{>0}$ and $V_{ij} \in \tau$ for $j = 1, \dots, m_i$. Then the union of all W_i's is a neighbourhood of x of the form $U_1 \cup \cdots U_m$, where U_j belongs to τ and is contained in some V_i. Using the compactness of V, we conclude. □

3. Bornology

Definition 3.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

(1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;
(3) $\mathcal{B}$ is stable under finite union.

The pair $(X, \mathcal{B})$ is called a *bornological set*. The elements of $\mathcal{B}$ are called the *bounded subsets* of $(X, \mathcal{B})$. When $\mathcal{B}$ is obvious from the context, we omit it from the notations.

A morphism between bornological sets $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ is a map of sets $f : X \to Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 3.2. Let $(X, \mathcal{B})$ be a bornological set. A *basis* for $\mathcal{B}$ is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \cdots \cup A_n$.

Bibliography

[Ber93] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 78.1 (1993), pp. 5–161.