Berkovich analytic spaces

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2. Compact analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 2.1. Let A be a k_H-affinoid algebra. A *compact k_H-analytic domain V* in $\operatorname{Sp} A$ is a finite union of k_H-affinoid domains in $\operatorname{Sp} A$.

Lemma 2.2. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain. Write $\operatorname{Sp} A$ as a finite union of k_H-affinoid domains $\operatorname{Sp} A_i$ with $i = 1, \dots, n$ in $\operatorname{Sp} A$. Define $A_{ij} = A_i \hat{\otimes}_A A_j$ and

$$A_V := \ker \left(\prod_{i=1}^n A_i \to \prod_{i,j=1}^n A_{ij} \right).$$

Then the Banach k-algebra does not depend on the choice of the covering $\{\operatorname{Sp} A_i\}_i$ up to a canonical isomorphism.

The image of the natural continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ contains V and the map does not depend on the choice of the covering up to the canonical isomorphism between $\operatorname{Sp} A_V$ for different coverings.

Proof. We first observe that A_V is a Banach k-algebra as it is defined as an equalizer. This follows from **??** in the chapter Banach Rings.

Let $\{\operatorname{Sp} B_j\}_{j=1,\dots,m}$ be another k_H-affinoid covering of $\operatorname{Sp} A$. We need to show that A_V defined using the two coverings are canonically isomorphic. We write A'_V for

$$\ker \left(\prod_{j=1}^m B_j \to \prod_{i,j=1}^m B_{ij} \right)$$

to make a distinction. We write $B_{ij} = B_i \hat{\otimes}_A B_j$.

By **??** in the chapter Affinoid Algebras, the colomns in the following commutative diagram are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A_V \arrow[r] \arrow[d, dotted] & \prod_{i=1}^n A_i \arrow[r] \arrow[d, "\eta"] & \prod_{i,i'=1}^n A_{ii'} \arrow[d] \\
0 \arrow[r] & \ker \iota \arrow[r] & \prod_{i=1}^n \prod_{j=1}^m A_i \hat{\otimes}_A B_j \arrow[r, "\iota"] \arrow[d, "\tau"] & \prod_{i,i'=1}^n \prod_{j,j'=1}^m A_{ii'} \hat{\otimes}_A B_{jj'} \\
 & & \prod_{i=1}^n \prod_{j,j'=1}^m A_i \hat{\otimes}_A B_{jj'} &
\end{tikzcd}$$

The rows are exact by definition. By diagram chasing, the dotted arrow is injective. To see it is surjective, it suffices to observe that the factors with $i = i'$ in the lower right corner is exactly the same as the factors of the lower corner, so an element in $\ker \iota$ is necessarily in $\ker \tau$. It follows that the dotted arrow is surjective.

Similarly, we have a natural isomorphism $A'_V \xrightarrow{\sim} \ker \iota$. We conclude the first assertion.

As for the second, observe that $\operatorname{Sp} A_V$ is defined as a colimit in the category of Banach k-algebras, so it follows from general abstract nonsense that there is a natural morphism $\operatorname{Sp} A_V \to \operatorname{Sp} A$. It clearly contains V in the image. The compatibility with the isomorphism above follows simply from the fact that the map η is an A-algebra homomorphism. □

Remark 2.3. This is also a natural continuous map $V \to \operatorname{Sp} A_V$, given by the natural map $A_V \to A_i$ for $i = 1, \ldots, n$. This map is a section of the continuous map $\operatorname{Sp} A_V \to A$ we just constructed over V. In [Ber93], Berkovich always uses this map instead of $\operatorname{Sp} A_V \to A$.

Definition 2.4. Let A be a k-affinoid algebra and V be a compact k-analytic domain in $\operatorname{Sp} A$. We define *the Banach k-algebra A_V associated with V* as A_V constructed in Lemma 2.2.

The continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ constructed in Lemma 2.2 is called the *structure map* ov V.

Proposition 2.5. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain in $\operatorname{Sp} A$. Then the following are equivalent:

(1) V is a k_H-affinoid domain.
(2) A_V is a k_H-affinoid algebra and the image of the structure map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is exactly V.

Proof. (1) $\implies$ (2): By **??** in the chapter Affinoid Algebras, when V is a k_H-affinoid domain, A_V is a k_H-affinoid algebra and the structure map corresponds to the inclusion of the k_H-affinoid domain. There is nothing to prove.

(2) $\implies$ (1): It suffices to show that the structure map represents the k_H-affinoid domain V. Take a k_H-affinoid algebra D and a morphism $\operatorname{Sp} D \to \operatorname{Sp} A$ of k_H-affinoid spectra that factorizes through V. We need to construct a morphism $\operatorname{Sp} D \to \operatorname{Sp} A_V$ making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} A_V \arrow[r] & \operatorname{Sp} A \end{tikzcd}.$$

Take k_H-affinoid domains $\operatorname{Sp} B_1, \ldots, \operatorname{Sp} B_n$ in $\operatorname{Sp} A$ that cover V. Let $C_i = B_i \hat{\otimes}_A D$ for $i = 1, \ldots, n$, then $\operatorname{Sp} C_i$ is a k_H-affinoid domain in $\operatorname{Sp} D$ by **??** in the chapter Affinoid Algebras. By **??** in the chapter Affinoid Algebras and general abstract nonsense, it suffices to construct the dotted arrow after restricting to $\operatorname{Sp} C_i$ for $i = 1, \ldots, n$. So we could assume that $\operatorname{Sp} D \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B_1$. From the universal property, we therefore have the dotted morphism making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B_1 \arrow[r] & \operatorname{Sp} A \end{tikzcd}.$$

It suffices to show that the natural homomorphism

$$B_1 \to A_V \hat{\otimes}_A B_1$$

is an isomorphism. But this follows from general abstract nonsense as B_1 is already a Banach A_V-algebra. $\square$

Remark 2.6. This proposition is not correctly stated in [Ber12, Corollary 2.2.6]. The corresponding statement in [Ber93, Remark 1.2.1] is slightly weaker than our statement.

3. The category of Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. Let X be a locally Hausdorff space and τ be a net of compact subsets. A *k_H-affinoid atlas* $\mathcal{A}$ on X with the net τ is a map which assigns

(1) to each $V \in \tau$, a k_H-affinoid algebra A_V and a homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$;
(2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of k_H-affinoid algebras $\alpha_{V/U} : A_V \to A_U$ representing a k_H-affinoid domain $\operatorname{Sp} A_U$ in $\operatorname{Sp} A_V$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Sp} A_U \arrow[r, "\operatorname{Sp}\alpha_{V/U}"] \arrow[d, "\varphi_U"] & \operatorname{Sp} A_V \arrow[d, "\varphi_V"] \\ U \arrow[r] & V \end{tikzcd}.$$

The triple $(X, \mathcal{A}, \tau)$ as above is called a *k_H-analytic space*.

A *morphism* between atlases $\mathcal{A}$ and $\mathcal{A}'$ on X with the net τ is an assignment that with each $V \in \tau$, one associates a morphism of k_H-affinoid algebras $\beta_V : A_V \to A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Sp} A'_V \arrow[r, "\operatorname{Sp}\beta_V"] \arrow[d, "\varphi'_V"] & \operatorname{Sp} A_V \arrow[dl, "\varphi_V"] \\ V & \end{tikzcd};$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{tikzcd} A_V \arrow[r, "\alpha_{V/U}"] \arrow[d, "\beta_V"] & A_U \arrow[d, "\beta_U"] \\ A'_V \arrow[r, "\alpha'_{V/U}"] & A'_U \end{tikzcd}$$

Here we have denoted the data associated with $\mathcal{A}'$ with a prime. In this way, the atlases on X with the net τ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), it $W \subseteq U \subseteq V$ are three sets in τ, then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

Remark 3.2. As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps φ_U's by identifying $\operatorname{Sp} A_U$ with U. We will say U is a k_H-affinoid domain in V.

Remark 3.3. Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

Lemma 3.4. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space, $U \in \tau$ and W is a k_H-affinoid domain in U. Then for any $V \in \tau$ containing W, W is a k_H-affinoid domain in V.

PROOF. As $\tau|_{U\cap V}$ is a net and W is compact, we can find $U_1, \dots, U_n \in \tau_{U\cap V}$ with $W \subseteq U_1 \cup \cdots \cup U_n$. As W, U_i are k_H-affinoid domains in U, $W_i = W \cap U_i$ is a k_H-affinoid domain in U_i for all $i = 1, \dots, n$ by **??** in the chapter Affinoid Algebras. It follows from **??** and **??** in the chapter Affinoid Algebras that W_i and $W_i \cap W_j$ are both k_H-affinoid domains in V for $i, j = 1, \dots, n$. So W is a compact k_H-analytic domain in V.

By Proposition 2.5,

$$A_W := \ker\left(\prod_{i=1}^n A_{W_i} \to \prod_{i,j=1}^n A_{W_i \cap W_j}\right)$$

is k_H-affinoid and $\operatorname{Sp} A_W \to \operatorname{Sp} A$ induces a hoemomorphism $\operatorname{Sp} A_W \to W$ by **??** in the chatper Affinoid Algebras. By Proposition 2.5 again, W is affinoid in V. □

Definition 3.5. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. We define $\bar{\tau}$ as the set of all $W \subseteq X$ such that there is $U \in \tau$ containing W and W is k_H-affinoid in U.

Lemma 3.6. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Then $\bar{\tau}$ is a net on X and there is a k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$. Moreover, the k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$ is unique up to a canonical isomorphism.

PROOF. **Step 1**. We first show that $\bar{\tau}$ is a net. Let $U, V \in \bar{\tau}$ and $x \in U \cap V$. Take $U', V' \in \tau$ containing U and V respectively. Take $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau$ such that

(1) $x \in W_1 \cap \cdots \cap W_n$;
(2) $W_1 \cup \cdots \cup W_n$ is a neighbourhood of x in $U' \cap V'$.

This is possible because $\tau|_{U'\cap V'}$ is a quasi-net by assumption.

By Lemma 3.4, U (resp. V) and $W_1, \dots, W_n$ are k_H-affinoid domains in U' (resp. V').

By **??** in the chapter Affinoid Algebras, $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a k_H-affinoid domain in W_i for $i = 1, \dots, n$. By **??** in the chapter Affinoid Algebras again, $U_i \cap V_i$ is a k_H-affinoid domain in W_i for $i = 1, \dots, n$. So $U_i \cap V_i \in \bar{\tau}|_{U\cap V}$ for $i = 1, \dots, n$. But

$$\bigcup_{i=1}^n U_i \cap V_i = (U \cap V) \cap \bigcup_{i=1}^n W_i,$$

so $\bigcup_{i=1}^n U_i \cap V_i$ is a neighbourhood of x in $U \cap V$ and $x \in \bigcap_{i=1}^n U_i \cap V_i$. It follows that $\bar{\tau}$ is a net.

Step 2. We extend the k_H-affinoid atlas $\mathcal{A}$.

For each $V \in \bar{\tau}$, we fix a $V' \in \tau$ containing V.

By Lemma 3.4, V is a k_H-affinoid domain in V'. Let $A_{V'} \to A_V$ be the morphism of k_H-affinoid algebras representing the k_H-affinoid domain V in $\operatorname{Sp} A_{V'}$. We define the homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$ as the morphism induced by $\operatorname{Sp} A_V \to \operatorname{Sp} A$.

For $U, V \in \bar{\tau}$ with $U \subseteq V$, we want to define $\alpha_{V/U} : A_V \to A_U$. We handle two cases. When $V \in \tau$, as $\tau|_{U' \cap V}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \tau|_{U' \cap V}$ such that

$$U = \bigcup_{i=1}^{n} U_i.$$

By Lemma 3.4, $U_1, \dots, U_n$ are k_H-affinoid domains in U' and in V. By **??** in the chapter Affinoid Algebras,

$$A_U \xrightarrow{\sim} \ker \left(\prod_{i=1}^{n} A_{U_i} \to \prod_{i,j=1}^{n} A_{U_i \cap U_j} \right).$$

So the morphism $\alpha_{V/U_i} : A_V \to A_{U_i}$ and $A_{V/U_i \cap U_j} : \alpha_{V/U_i} : A_V \to A_{U_i \cap U_j}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ induces a morphism $\alpha_{V/U} : A_V \to A_U$. Observe that $\alpha_{V/U}$ represents the k_H-affinoid domain U in V, so it is independent of the choice of $U_1, \dots, U_n$.

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \to A_U$ representing the k_H-affinoid domain U in V', it follows that U is a k_H-affinoid domain in V, and we therefore get the desired morphism $\alpha_{V/U} : A_V \to A_U$.

It is easy to verify that the constructions gives a k_H-affinoid atlas with the net $\bar{\tau}$ extending $\mathcal{A}$. The uniqueness of the extension is immediate. $\square$

Definition 3.7. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of

(1) a continuous map $\varphi : X \to X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
(2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq W'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd}.$$

Recall our convention Remark 3.2, the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \to A_V$ of k_H-affinoid algebras making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} A_V \arrow[r] \arrow[d, "\varphi_V"] & \operatorname{Sp} A'_{V'} \arrow[d, "\varphi'_{V'}"] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}.$$

We will continue our identifications as in Remark 3.2 to simplify our notations.

Proposition 3.8. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \bar{\mathcal{A}}, \bar{\tau}) \to (X', \overline{\mathcal{A}'}, \overline{\tau'})$.

Proof. Let $U \in \bar{\tau}$, $U' \in \overline{\tau'}$ with $\varphi(U) \subseteq U'$. Take $V \in \tau$ and $V' \in \tau'$ containing U and U' respectively. By Lemma 3.4, U (resp. V) is a k_H-affinoid domain in V (resp. V'). Take $W \in \tau'$ with $\varphi(V) \subseteq W'$. Then in particular,

$\varphi(U) \subseteq W'$. As $\tau'|_{V'\cap W'}$ is a quasi-net and $\varphi(U)$ is compact, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau'|_{V'\cap W}$ such that

$$\varphi(U) \subseteq W_1 \cup \cdots \cup W_n.$$

Now W_i is a k_H-affinoid domain in W' by Lemma 3.4, so $V_i := \varphi_{V/W'}^{-1}(W_i)$ is an affinoid domain in V by **??** in the chatper Affinoid Algebras and we have an induced morphism $V_i \to W_i$ for $i = 1, \dots, n$. This morphism in turn induces a morphism of k_H-affinoid spectra

$$U_i := U \cap V_i \to U_i' := U' \cap W_i \to U'$$

for $i = 1, \dots, n$. These morphisms are compatible on their intersections by construction. So by **??** in the chapter Affinoid Algebras, they glue together to a morphism of k_H-affinoid spectra $\bar{\varphi}_{U/U'} : U \to U'$. It is easy to see that this construction defines a strong morphism.

As for the uniqueness, it suffices to show that the morphism $U_i \to U_i'$ is uniquely determined for $i = 1, \dots, n$. In other words, we need to show that the dotted arrow that makes the following diagram commutes is unique:

$$\begin{tikzcd} U_i \arrow[r, dotted] \arrow[d] & U_i' \arrow[d] \\ V \arrow[r, "\varphi_{V/W'}"] & W' \end{tikzcd}$$

for $i = 1, \dots, n$. It suffices to apply the universal property of the k_H-affinoid domain $U_i' \to W'$. □

Definition 3.9. Let $(X, \mathcal{A}, \tau)$, $(X', \mathcal{A}', \tau')$, $(X'', \mathcal{A}'', \tau'')$ be k_H-analytic spaces. Let

$$\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau'), \quad \psi : (X', \mathcal{A}', \tau') \to (X'', \mathcal{A}'', \tau'')$$

be strong morphisms. We will define their *composition* $\chi = \psi \circ \varphi$ as follows. The underlying map of topological spaces is just the composition of the unlerlying maps of topological spaces corresponding to ψ and φ.

Let $\bar{\varphi}$ and $\bar{\psi}$ be the extensions of φ and ψ to $\bar{\tau}$ and $\overline{\tau'}$ as in Proposition 3.8.

Given $V \in \tau$ and $V'' \in \tau''$ with $\chi(V) \subseteq V''$, we need to define a morphism of k_H-affinoid spectra $\chi_{V/V''} : V \to V''$. Take $V' \in \tau'$ and $U'' \in \tau''$ such that $\varphi(V) \subseteq V'$ and $\psi(V') \subseteq U''$. Since $\chi(V) \subseteq U'' \cap V''$ and V is compact, we can take $n \in \mathbb{Z}_{>0}$ and $V_1'', \dots, V_n'' \in \tau''|_{U''\cap V''}$ with $\chi(V) \subseteq V_1'' \cup \cdots \cup V_n''$. Then $V_i' := \psi_{V'/U''}^{-1}(V_i'')$ and $V_i := \varphi_{V/V'}^{-1}(V_i')$ are k_H-affinoid domains in V' and V respectively for $i = 1, \dots, n$ and $V = V_1 \cup \cdots \cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ then induce a morphism $V_i \to V_i'' \to V$ of k_H-affinoid spectra. These morphisms are clearly compatible on the intersections and hence induce a morphism $V \to V''$ of k_H-affinoid spectra by **??** in the chapter Affinoid Algebras.

It is easy to verify that $\psi \circ \varphi$ is a strong morphism.

In this way, we get a category k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ of k_H-analytic spaces.

Definition 3.10. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if

(1) φ is a homeomorphism between X and X';
(2) for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$, $\mathrm{Sp}\,\varphi_{V/V'}$ identifies V with an affinoid domain in V'.

Lemma 3.11. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces and $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then for any $V \in \bar{\tau}$ and $V' \in \overline{\tau'}$, the intersection $V \cap \varphi^{-1}(V')$ is a compact k_H-analytic domain in V.

Proof. Take $U' \in \overline{\tau'}$ with $\varphi(V) \subseteq U'$. As $\tau|_{U' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U'_1, \ldots, U'_n \in \tau|_{U' \cap V'}$ with $\varphi(V) \subseteq U'_1 \cup \cdots \cup U'_n$ and

$$V \cap \varphi^{-1}(V') = \bigcup_{i=1}^{n} \varphi_{V/U}^{-1}(U'_i).$$

$\square$

Lemma 3.12. The system of quasi-isomorphisms in k_H-$\widetilde{\mathcal{A}}$n is a right multiplicative system.

For the notion of right multiplicative system, we refer to [Stacks, Tag 04VC].

Proof. We verify the three axioms as in [Stacks, Tag 04VC].

RMS1. The identity is clear a quasi-isomorphism. It remains to verify that the composition of quasi-isomorphisms is still a quasi-isomorphism.

We take φ, ψ as in Definition 3.9. We will use the same notations as in Definition 3.9. We need to show that $V \to V''$ identifies V with a k_H-affinoid domain in V''. From the construction, we know that φ identifies V_i with a k_H-affinoid domain in V'_i and ψ identifies V'_i with a k_H-affinoid domain in V''_i for $i = 1, \ldots, n$. In particular, $\chi(V)$ is a compact k_H-analytic domain in V''. It follows from Proposition 2.5 that $\chi(V)$ is a k_H-affinoid domain in V''.

RMS2. If $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ and $f : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ are given strong morphisms of k_H-analytic spaces and g is a quasi-isomorphism, then there are k_H-analytic space $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau})$ and strong morphisms $\tilde{\varphi} : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ and $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ such that f is a quasi-isomorphism and the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \arrow[r, dotted, "\tilde{\varphi}"] \arrow[d, dotted, "f"] & (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \arrow[d, "g"] \\
(X, \mathcal{A}, \tau) \arrow[r, "\varphi"] & (X', \mathcal{A}', \tau')
\end{tikzcd}.$$

We may assume that $\widetilde{X'} = X'$. Then $\widetilde{\tau'} \subseteq \overline{\tau'}$. We let $\tilde{X} = X$. Let $\tilde{\tau}$ be the family of all $V \in \bar{\tau}$ for which there is $\widetilde{V'} \in \widetilde{\tau'}$ with $\varphi(V) \subseteq \widetilde{V'}$. By Lemma 3.11, $\tilde{\tau}$ is a net on $\tilde{X}$. The k_H-atlas $\bar{\mathcal{A}}$ defines a k_H-affinoid atlas $\tilde{\mathcal{A}}$ with the net $\tilde{\tau}$. The strong morphism $\bar{\varphi}$ induces $\tilde{\varphi}$. The morphism f is the canonical quasi-isomorphism. It is immediate that these constructions satisfy the desired conditions.

RMS3. If $\varphi, \psi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ are strong morphisms of k_H-analytic spaces and there is a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ of k_H-analytic spaces such that $g \circ \varphi = g \circ \psi$, then there is a quasi-isomorphism $f : (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi \circ f = \psi \circ f$.

We will in fact show that $\varphi = \psi$. It is clear that they coincide as maps of topological spaces. Let $V \in \tau$, $V' \in \tau'$ such that $\varphi(V) \subseteq V'$. Take $\widetilde{V'} \in \widetilde{\tau'}$ with $g(V') \subseteq \widetilde{V'}$. Then we have two morphisms of k-affinoid spectra $\varphi_{V/V'}, \psi_{V/V'} : V \to V'$ such that their compositions with $g_{V'/\widetilde{V'}}$ coincide. As V' is an affinoid domain in $\widetilde{V'}$, it follows that $\varphi_{V/V'} = \psi_{V/V'}$ by the universal property. $\square$

Definition 3.13. The category k_H-$\mathcal{A}$n is the right category of fractions of k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ with respect to the system of quasi-isomorphisms. A morphism in k_H-$\mathcal{A}$n is called a *morphism* between k_H-analytic spaces.

We refer to [Stacks, Tag 04VB] for the definition of right category of fractions. For later references, we explicitly write down the morphisms in k_H-$\mathcal{A}$n.

Lemma 3.14. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of k_H-analytic spaces. We define a partial order on the set of nets on X: $\tau_1 \preceq \tau_0$ if $\tau_1 \subseteq \overline{\tau_0}$. Then the set of nets is a directed set and

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}\left((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')\right) = \varinjlim_{\sigma \preceq \tau} \mathrm{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}}\left((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')\right)$$

in the category of sets, where $\mathcal{A}_\sigma$ is induced by $\overline{\mathcal{A}}$. The transition maps are all injective.

Proof. This follows immediately from the definition. $\square$

Definition 3.15. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. We say a subset $W \subseteq X$ is *τ-special* if it is compact and there exist $n \in \mathbb{Z}_{>0}$ and a covering $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$ and the natural map

$$A_{W_i} \hat{\otimes}_k A_{W_j} \to A_{W_i \cap W_j}$$

is an admissible epimorphism.

The covering $W_1, \ldots, W_n$ is called a *τ-special covering* of W.

Under our convention, the assumption means that $W_i \cap W_j \to W_i \times W_j$ is a closed immersion of k_H-affinoid spectra.

Example 3.16. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Suppose that $V \in \tau$ and W is a compact k_H-analytic domain in V. Let $n \in \mathbb{Z}_{>0}$ and $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$. Then $\{W_i\}_i$ is a τ-special covering of W. This follows from **??** in the chapter Affinoid Algebras.

Lemma 3.17. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space and W be a τ-special subset of X. If $U, V \in \tau|_W$, then $U \cap V \in \bar{\tau}$ and the natural map

$$A_U \hat{\otimes}_k A_V \to A_{U \cap V}$$

is an admissible epimorphism.

Proof. Let $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n$ be a τ-special covering of W. As $U \cap W_i$ and $V \cap W_i$ are compact for $i = 1, \ldots, n$, we can find $m_i \in \mathbb{Z}_{>0}$ (resp. $k_i \in \mathbb{Z}_{>0}$) and finite coverings $U_{i1}, \ldots, U_{im_i} \in \tau$ of $U \cap W_i$ (resp. $V_{i1}, \ldots, V_{ik_i} \in \tau$ of $V \cap W_i$).

Observe that $U_{ik} \cap V_{jl}$ is a k_H-affinoid domain in $U \cap V$, hence $U_{ik} \cap V_{jl} \in \bar{\tau}$ for any $i, j = 1, \ldots, n$, $k = 1, \ldots, m_i$ and $l = 1, \ldots, k_l$. By **??** in the chapter Affinoid Algebras, $U_{ik} \cap V_{jl} \to U_{ik} \times V_{jl}$ is a closed immersion since $W_i \cap W_j \to W_i \times W_j$ is by our assumption.

Consider the finite convering

$$\mathcal{U} := \{U_{ik} \times V_{jl} : i, j = 1, \ldots, n; k = 1, \ldots, m_i; l = 1, \ldots, k_l\}$$

of $U \times V$. For each tuple (i, j, k, l), $A_{U_{ik} \cap V_{jl}}$ is a finite $A_{U_{ik} \times V_{jl}}$-algebra. By **??** in the chapter Affinoid Algebras, we can construct a finite $A_{U \times V}$-algebra $A_{U \cap V}$ inducing all of these $A_{U_{ik} \cap V_{jl}}$'s. By **??** in the chapter Affinoid Algebras, $A_{U \cap V}$ is k_H-affinoid.

As $\mathcal{U}$ is a finite k_H-affinoid covering of $U \times V$, $\{A_{U_{ik}\cap V_{jl}}\}_{i,k,j,l}$ is a finite k_H-affinoid covering of $U \cap V$ by **??** in the chapter Affinoid Algebras. In particular, we have a natural homeomorphism

$$\operatorname{Sp} A_{U\cap V} \xrightarrow{\sim} U \cap V.$$

Observe that $A_U \hat{\otimes}_k A_V \to A_{U\cap V}$ is surjective. We endow $A_{U\cap V}$ with the structure of finite $A_U \hat{\otimes}_k A_V$-Banach algebras by **??** in the chapter Affinoid Algebras. Then $A_U \hat{\otimes}_k A_V \to A_{U\cap V}$ is an admissible epimorphism by **??** in the chapter Affinoid Algebras.

On the other hand $U \cap V$ is a compact k_H-analytic domain in U, so by Proposition 2.5, $U \cap V$ is a k_H-affinoid in U. In particular, $U \cap V \in \bar{\tau}$. □

Lemma 3.18. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space and $W \subseteq X$ be a τ-special set. Then for any finite covering $\{W_i\}_{i\in I}$ of W with $W_i \in \tau$ for $i \in I$, the Banach k-algebra

$$A_W := \ker\left(\prod_{i\in I} A_{W_i} \to A_{W_i \cap W_j}\right)$$

does not depend on the choice of $\{W_i\}_{i\in I}$ up to canonical isomorphisms.

Moreover, we have a canonical map $W \to \operatorname{Sp} A_W$, which does not depend on the choice of the covering modulo the canonical isomorphism between A_W.

Proof. It follows from Lemma 3.17 that the covering $\{W_i\}_{i\in I}$ is τ-special. It suffices to apply the same argument of Lemma 2.2. □

Definition 3.19. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Let $\hat{\tau}$ denote the collection of $\bar{\tau}$-special subsets $W \subseteq X$ such that

(1) A_W is k-affinoid;
(2) the natural map $W \to \operatorname{Sp} A_W$ is bijective;
(3) there is a $\bar{\tau}$-special covering $\{W_i\}_{i\in I}$ of W such that W_i is a k-affinoid domain in W for $i \in I$.

The sets from $\hat{\tau}$ are called *k_H-affinoid domains in* $(X, \mathcal{A}, \tau)$.

Observe that W is k_H-affinoid and W_i is a k_H-affinoid domain in W by **??** in the chapter Affinoid Algebras. Condition (3) holds for any $\bar{\tau}$-special covering.

Proposition 3.20. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Then $\hat{\tau}$ is a net. For any net σ on X contained in $\bar{\tau}$, we have $\hat{\sigma} = \hat{\tau}$.

Moreover, $\hat{\hat{\tau}} = \hat{\tau}$.

Proof. Let $U, V \in \hat{\tau}$. Take $\bar{\tau}$-special coverings $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ of U and V respectively. In order to show that $\hat{\tau}|_{U\cap V}$ is a quasi-net, it suffices to show that $\hat{\tau}|_{U_i\cap V_j}$ is for any $i \in I$ and $j \in J$. This follows simply from the fact that $\bar{\tau}|_{U_i\cap V_j}$ is a quasi-net. Similarly, as $\hat{\tau}$ is a quasi-net as $\bar{\tau}$ is. So $\hat{\tau}$ is a net.

Let σ be a net on X contained in $\bar{\tau}$. By Lemma 3.17, it suffices to verify that for any $V \in \bar{\tau}$, there are $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \bar{\sigma}$ with $V = U_1 \cup \cdots \cup U_n$. As σ is a net on X, we can find $m \in \mathbb{Z}_{>0}$, $W_1, \ldots, W_m \in \sigma$ such that

$$V \subseteq W_1 \cup \cdots \cup W_m.$$

As $V, W_j \in \bar{\tau}$ for $j = 1, \ldots, m$, by **??** in the chapter Topology and Bornology, we can find $U_1, \ldots, U_n \in \bar{\tau}$ such that $V = U_1 \cup \cdots \cup U_n$ and each U_i is contained in some W_j. As $W_j \in \sigma$ for $j = 1, \ldots, m$, it follows that $U_i \in \bar{\sigma}$ for $i = 1, \ldots, n$.

By Lemma 3.17,

$$\bar{\bar{\tau}} = \hat{\tau}.$$

Let $V \in \hat{\hat{V}}$. Let $\{V_i\}_{i\in I}$ be a $\hat{\tau}$-special covering of V. For each $i \in I$, take a $\bar{\tau}$-special covering $\{V_{ij}\}_{j\in J_i}$ of V_i. Then $\{V_{ij}\}_{i,j}$ is a $\bar{\tau}$-special covering of V. It follows that $V \in \hat{\tau}$. □

Proposition 3.21. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. There is a k_H-analytic atlas $\hat{\mathcal{A}}$ on X with the net $\hat{\tau}$ extending $\mathcal{A}$. Moreover, $\hat{\mathcal{A}}$ is unique up to a canonical isomorphism.

Proof. For each $V \in \hat{\tau}$, Fix a $\bar{\tau}$-special covering $\{V_i\}_{i\in I_V}$.

We define A_V using this covering as in Lemma 3.18. By definition, the canonical map $V \to \operatorname{Sp} A_V$ is a homeomorphism.

Next take $U, V \in \hat{\tau}$ with $U \subseteq V$. We want to identify U with a k_H-affinoid domain in V. First assume that $U \in \tau$, then $U \cap V_i$ is a k_H-affinoid domain in V_i for $i \in I_V$ by Lemma 3.17. Hence, U is a k_H-affinoid domain in V. If we only know $U \in \hat{\tau}$, we know that U_i is a k_H-affinoid domain in V for any $i \in I_U$. It follows that U is a k_H-affinoid domain in V by Proposition 2.5.

The uniqueness is immediate. □

Definition 3.22. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. A $\hat{\tau}$-special set is called a *k_H-special domain* in X.

Observe that a k_H-special domain inherits a structure of k_H-analytic space from $(X, \mathcal{A}, \tau)$.

Proposition 3.23. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of k_H-analytic spaces. Then for any k_H-affinoid domains $V \subseteq X$ and $V' \subseteq X'$, the intersection $V \cap \varphi^{-1}(V')$ is a k_H-special domain in X.

Proof. By Proposition 3.20, we may assume that φ is a strong morphism. In this case, it suffices to apply Lemma 3.11. □

Lemma 3.24. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \hat{\mathcal{A}}, \hat{\tau}) \to (X', \widehat{\mathcal{A}'}, \widehat{\tau'})$.

Proof. Let $V \in \hat{\tau}$ and $V' \in \widehat{\tau'}$ with $\varphi(V) \subseteq V'$. We want to define $\varphi_{V/V'} : V \to V'$ of k_H-affinoid spectra. By Proposition 3.8, we may extend φ uniquely to $\bar{\tau}$. Take a $\bar{\tau}$-special covering of V, we may reduce to the case where $V \in \bar{\tau}$. Take $W' \in \tau'$ such that $\varphi(V) \subseteq W'$. As $\tau|_{W' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n \in \tau'|_{V' \cap W}$ such that $\varphi(V) \subseteq W_1 \cup \cdots \cup W_n$. Considering the inverse images of W_i's and $W_i \cap W_j$'s using Lemma 3.17, we are reduced to the case where $V' \in \overline{\tau'}$. This is already handled in Proposition 3.8. The uniqueness of the extension is clear. □

Proposition 3.25. Let $(X, \mathcal{A}, \tau)$, $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces.

(1) There is a canonical bijection between

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau'))$$

and the set of pairs consisting of

(a) a continuous map $\varphi : X \to X'$ such that for all $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, neighbourhoods $V_1 \cup \cdots \cup V_n$ of x and $V'_1 \cup \cdots \cup V'_n$ of $\varphi(x)$ with $x \in V_1 \cap \cdots \cap V_n$ and $\varphi(V_i) \subseteq V'_i$ for $i = 1, \ldots, n$, where $V_i \subseteq X$ and $V'_i \subseteq X'$ are k_H-affinoid domains;

(b) for each pair of k_H-affinoid domains $V \subseteq X$, $V' \subseteq X'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that if $V, W \subseteq X$ and $V', W' \subseteq X'$ are k_H-affinoid domains with $\varphi(V) \subseteq V'$, $\varphi(W) \subseteq W'$, the diagram below commutes

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd} .$$

(2) Under the bijection in (1), an isomorphism corresponds to the pair where φ is a hoemomorphism such that $\varphi(\hat{\tau}) = \widetilde{\tau'}$ and for any $V \in \hat{\tau}$, $\varphi_{V/\varphi(V)}$ is an isomorphism of k_H-affinoid spectra.

Proof. (2) follows immediately from (1). So it suffices to prove (1).

We construct the forward map. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism. Take a subnet σ of $\bar{\tau}$ such that φ is represented by a strong morphism

$$\varphi : (X, \mathcal{A}_\sigma, \sigma) \to (X', \mathcal{A}', \tau').$$

By Lemma 3.24, this extends to a strong morphism

$$\varphi : (X, \widehat{\mathcal{A}_\sigma}, \hat{\sigma}) \to (X', \widehat{\mathcal{A}'}, \widehat{\tau'}).$$

We get an injective map from the first set into the second set.

Conversely, we need to show that any given map from the second map comes from the first set. It suffices to show that

$$\sigma := \left\{ V \in \hat{\tau} : \varphi(V) \subseteq V' \text{ for some } V' \in \widehat{\tau'} \right\}$$

is a net. Take $x \in X$ and neighbourhoods $V_1 \cup \cdots \cup V_n$ of x and $V'_1 \cup \cdots \cup V'_n$ of $\varphi(x)$ as in the statement of (1). Then $V_i \in \sigma$, so we conclude. □

In practice, we do not distinguish a k_H-analytic space from the isomorphic k_H-analytic spaces. In particular, we will write $(X, \mathcal{A}, \tau)$ as X and always endow it with the strucutre $(X, \hat{\mathcal{A}}, \hat{\tau})$ of k_H-analytic space. If necessarily, we will write $|X|$ for the underlying topological space.

Corollary 3.26. The natural functor k_H-$\mathcal{A}$ff $\to k_H$-$\mathcal{A}$n is fully faithful.

Proof. Let $X = \operatorname{Sp} A$ be a k_H-affinoid spectrum. We endow it with the net $\tau = \{X\}$. The k_H-atlas with the net τ assigns $X \in \tau$ with A. It is easily verified that this is a functor. By Proposition 3.25, the functor is fully faithful. □

Definition 3.27. A *k_H-affinoid space* is an object of k_H-$\mathcal{A}$n lying in the essential image of the functor k_H-$\mathcal{A}$ff $\to k_H$-$\mathcal{A}$n.

The category of k_H-affinoid spaces is denoted by k_H-$\mathcal{A}$ff.

The notation for the category of k_H-affinoid spaces is the same as the notation for the category of k_H-affinoid spectra, as the two categories are canonically equivalent.

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