Ymir

Contents

Morphisms between complex analytic spaces

1. Introduction

2. Quasi-finite morphisms

Definition 2.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. We say f is *quasi-finite* at $x \in X$ if x is isolated in $f^{-1}(f(x))$. We say f is *quasi-finite* if f is quasi-finite at all $x \in X$.

This definition is purely topological. We will show that it is equivalent to an analytic definition.

Proposition 2.2. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is quasi-finite at $x \in X$;
(2) $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$;
(3) $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$.

Proof. (1) $\Leftrightarrow$ (2): By Corollary 3.16 in Constructions of complex analytic spaces, f is quasi-finite at $x \in X$ if and only if $\mathcal{O}_{X_{f(x)},x} = \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is artinian. In other words, $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is finite-dimensional over $\mathbb{C}$. The latter is equivalent to that $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.

(2) $\Leftrightarrow$ (3): This follows from Theorem 5.4 in Complex analytic local algebras. $\square$

3. Finite morphisms

Definition 3.1. A morphism of complex analytic spaces $f : X \to Y$ is *finite* if its underlying map of topological spaces is topologically finite.

We say a morphism of complex analytic spaces $f : X \to Y$ is *finite at $x \in X$* if there is an open neighbourhood U of x in X and V of $f(x)$ in Y such that $f(U) \subseteq V$ and the restriction $U \to V$ of f is finite.

Let S be a complex analytic space. A *finite analytic space over S* is a finite morphism $f : X \to S$ of complex analytic spaces. A morphism between finite analytic spaces over S is a morphism of complex analytic spaces over S.

Proposition 3.2. Let $f : X \to Y$ be a finite morphism of complex analytic spaces. Then f is quasi-finite.

Proof. This follows from Proposition 4.5 in Topology and bornology. $\square$

Theorem 3.3. Let S be a complex analytic space. Then the functor $\mathrm{Spec}_S^{\mathrm{an}}$ defines an anti-equivalence from the category of finite $\mathcal{O}_S$-algebras to the category of finite analytic spaces over S.

PROOF. We first observe that the functor is well-defined. This follows from Corollary 3.8 in Constructions of complex analytic spaces.

The functor is fully faithfull by Proposition 2.10 in Constructions of complex analytic spaces. Suppose that $f : X \to S$ is a finite morphism of complex analytic spaces. We need to show that X is isomorphic to $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A}$ for some finite $\mathcal{O}_S$-algebra $\mathcal{A}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

By Proposition 2.8 in Constructions of complex analytic spaces, we necessarily have $\mathcal{A} \cong f_*\mathcal{O}_X$. So we need to show that the natural morphism $\operatorname{Spec}^{\mathrm{an}}_S f_*\mathcal{O}_X \to X$ over S is an isomorphism. The problem is local on S.

Fix $s \in S$. Write $x_1, \dots, x_n$ for the distinct points in $f^{-1}(s)$. Up to shrinking S, we may assume that X is the disjoint union of $V_1, \dots, V_n$, where V_i is an open neighbourhood of x_i in X. We need to show that X has the form $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{B}$ for some $\mathcal{O}_S$-algebra $\mathcal{B}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

It suffices to handle each V_i separately, so we may assume that $f^{-1}(s) = \{x\}$ consists of a single point. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by Proposition 2.2. Up to shrinking S, we may assume that $\mathcal{O}_{X,x}$ spreads out to a finite $\mathcal{O}_S$-algebra $\mathcal{B}$. Let $X' = \operatorname{Spec}^{\mathrm{an}}_S \mathcal{B}$. There is a unique point x' of X' over s and $X'_{x'}$ is isomorphic to X_x over S_s. By Lemma 4.2 in Topology and bornology, up to shrinking S, we may assume that X is isomorphic to X' over S. We conclude. □

Corollary 3.4. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{M}$ is coherent. Moreover, f_* is exact from $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$ to $\mathcal{C}\mathrm{oh}(\mathcal{O}_Y)$.

PROOF. This follows from Corollary 2.9 in Constructions of complex analytic spaces and Theorem 3.3. □

Corollary 3.5. Let X be a reduced complex analytic space. Then

(1) $\bar{X}$ is normal;
(2) $p : \bar{X} \to X$ is finite and surjective;
(3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in $\bar{X}$ and the morphism $\bar{X} \setminus p^{-1}(Y) \to X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines $\bar{X}$ up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$.

PROOF. These properties are established in Proposition 7.8 in Properties of complex analytic spaces. We need to prove the uniqueness.

Let $p : X' \to X$ be a morphism satisfying the three conditions. We need to show that X' is canonically isomorphic to $\bar{X}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$. By (2) and Theorem 3.3, it suffices to show that $p_*\mathcal{O}_{X'}$ is canonically isomorphic to $\bar{\mathcal{O}}_X$. By (1), and the universal property of normalization, there is a canonical morphism

$$p_*\mathcal{O}_{X'} \to \bar{\mathcal{O}}_X$$

of $\mathcal{O}_X$-algebras. We will show that this map is an isomorphism.

The problem is local. Let $x \in X$. By (3) and Corollary 3.14 in Constructions of complex analytic spaces, up to shrinking X, we can find $f \in \mathcal{O}_X(X)$ such that $f(y) = 0$ for all $y \in Y$ and f_x is a non-zero divisor in $(p_*\mathcal{O}_{X'})_x$. Up to shrinking X, we may assume that f_y is a non-zero divsior in $(p_*\mathcal{O}_{X'})_y$ for all $y \in X$. By (3), we have

$$\mathcal{O}_X|_{X\setminus Y} \to (p_*\mathcal{O}_{X'})|_{X\setminus Y}$$

is an isomorphism. It follows that

$$fp_*\mathcal{O}_{X'} \to \mathcal{O}_X$$

is injective. We then have an injective homomorphism:

$$p_*\mathcal{O}_{X'} \to \mathcal{O}_X \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we deduce that $(p_*\mathcal{O}_{X'})_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}[f_y^{-1}]$. But $(p_*\mathcal{O}_{X'})_y$ is finite and integral over $\mathcal{O}_{X,y}$, so is isomorphic to $\overline{\mathcal{O}_{X,y}}$ as $\mathcal{O}_{Y,y}$-algebras. $\square$