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# Contents

# Berkovich analytic spaces

## 1. Introduction

## 2. Compact analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Definition 2.1.** Let $A$ be a $k_H$-affinoid algebra. A *compact $k_H$-analytic domain* $V$ in $\operatorname{Sp} A$ is a finite union of $k_H$-affinoid domains in $\operatorname{Sp} A$.

**Lemma 2.2.** Let $A$ be a $k_H$-affinoid algebra and $V$ be a compact $k_H$-analytic domain. Write $\operatorname{Sp} A$ as a finite union of $k_H$-affinoid domains $\operatorname{Sp} A_i$ with $i = 1, \ldots, n$ in $\operatorname{Sp} A$. Define $A_{ij} = A_i \hat{\otimes}_A A_j$ and

$$A_V := \ker \left( \prod_{i=1}^{n} A_i \to \prod_{i,j=1}^{n} A_{ij} \right).$$

Then the Banach $k$-algebra does not depend on the choice of the covering $\{\operatorname{Sp} A_i\}_i$ up to a canonical isomorphism.

The image of the natural continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ contains $V$ and the map does not depend on the choice of the covering up to the canonical isomorphism between $\operatorname{Sp} A_V$ for different coverings.

Proof. We first observe that $A_V$ is a Banach $k$-algebra as it is defined as an equalizer. This follows from **??** in **??**.

Let $\{\operatorname{Sp} B_j\}_{j=1,\ldots,m}$ be another $k_H$-affinoid covering of $\operatorname{Sp} A$. We need to show that $A_V$ defined using the two coverings are canonically isomorphic. We write $A'_V$ for

$$\ker \left( \prod_{j=1}^{m} B_j \to \prod_{i,j=1}^{m} B_{ij} \right)$$

to make a distinction. We write $B_{ij} = B_i \hat{\otimes}_A B_j$.

By **??** in **??**, the colomns in the following commutative diagram are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A_V \arrow[r] \arrow[d, dotted] & \prod_{i=1}^n A_i \arrow[r] \arrow[d, "\eta"] & \prod_{i,i'=1}^n A_{ii'} \arrow[d] \\
0 \arrow[r] & \ker \iota \arrow[r] & \prod_{i=1}^n \prod_{j=1}^m A_i \hat{\otimes}_A B_j \arrow[r, "\iota"] \arrow[d, "\tau"] & \prod_{i,i'=1}^n \prod_{j,j'=1}^m A_{ii'} \hat{\otimes}_A B_{jj'} \\
 & & \prod_{i=1}^n \prod_{j,j'=1}^m A_i \hat{\otimes}_A B_{jj'} &
\end{tikzcd}$$

The rows are exact by definition. By diagram chasing, the dotted arrow is injective. To see it is surjective, it suffices to observe that the factors with $i = i'$ in the lower right corner is exactly the same as the factors of the lower corner, so an element in $\ker \iota$ is necessarily in $\ker \tau$. It follows that the dotted arrow is surjective.

Similarly, we have a natural isomorphism $A'_V \xrightarrow{\sim} \ker \iota$. We conclude the first assertion.

As for the second, observe that $\operatorname{Sp} A_V$ is defined as a colimit in the category of Banach $k$-algebras, so it follows from general abstract nonsense that there is a natural morphism $\operatorname{Sp} A_V \to \operatorname{Sp} A$. It clearly contains $V$ in the image. The compatibility with the isomorphism above follows simply from the fact that the map $\eta$ is an $A$-algebra homomorphism. □

**Remark 2.3.** This is also a natural continuous map $V \to \operatorname{Sp} A_V$, given by the natural map $A_V \to A_i$ for $i = 1, \dots, n$. This map is a section of the continuous map $\operatorname{Sp} A_V \to A$ we just constructed over $V$. In [Ber93], Berkovich always uses this map instead of $\operatorname{Sp} A_V \to A$.

**Definition 2.4.** Let $A$ be a $k$-affinoid algebra and $V$ be a compact $k$-analytic domain in $\operatorname{Sp} A$. We define *the Banach $k$-algebra $A_V$ associated with $V$* as $A_V$ constructed in Lemma 2.2.

The continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ constructed in Lemma 2.2 is called the *structure map* ov $V$.

**Proposition 2.5.** Let $A$ be a $k_H$-affinoid algebra and $V$ be a compact $k_H$-analytic domain in $\operatorname{Sp} A$. Then the following are equivalent:

(1) $V$ is a $k_H$-affinoid domain.
(2) $A_V$ is a $k_H$-affinoid algebra and the image of the structure map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is exactly $V$.

Proof. (1) $\implies$ (2): By **??** in **??**, when $V$ is a $k_H$-affinoid domain, $A_V$ is a $k_H$-affinoid algebra and the structure map corresponds to the inclusion of the $k_H$-affinoid domain. There is nothing to prove.

(2) $\implies$ (1): It suffices to show that the structure map represents the $k_H$-affinoid domain $V$. Take a $k_H$-affinoid algebra $D$ and a morphism $\operatorname{Sp} D \to \operatorname{Sp} A$ of $k_H$-affinoid spectra that factorizes through $V$. We need to construct a morphism

$\operatorname{Sp} D \to \operatorname{Sp} A_V$ making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} A_V \arrow[r] & \operatorname{Sp} A \end{tikzcd}.$$

Take $k_H$-affinoid domains $\operatorname{Sp} B_1, \ldots, \operatorname{Sp} B_n$ in $\operatorname{Sp} A$ that cover $V$. Let $C_i = B_i \hat{\otimes}_A D$ for $i = 1, \ldots, n$, then $\operatorname{Sp} C_i$ is a $k_H$-affinoid domain in $\operatorname{Sp} D$ by **??** in **??**. By **??** in **??** and general abstract nonsense, it suffices to construct the dotted arrow after restricting to $\operatorname{Sp} C_i$ for $i = 1, \ldots, n$. So we could assume that $\operatorname{Sp} D \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B_1$. From the universal property, we therefore have the dotted morphism making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B_1 \arrow[r] & \operatorname{Sp} A \end{tikzcd}.$$

It suffices to show that the natural homomorphism

$$B_1 \to A_V \hat{\otimes}_A B_1$$

is an isomorphism. But this follows from general abstract nonsense as $B_1$ is already a Banach $A_V$-algebra. □

**Remark 2.6.** This proposition is not correctly stated in [Ber12, Corollary 2.2.6]. The corresponding statement in [Ber93, Remark 1.2.1] is slightly weaker than our statement.

## 3. The category of Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Definition 3.1.** Let $X$ be a locally Hausdorff space and $\tau$ be a net of compact subsets. A *$k_H$-affinoid atlas* $\mathcal{A}$ on $X$ with the net $\tau$ is a map which assigns

(1) to each $V \in \tau$, a $k_H$-affinoid algebra $A_V$ and a homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$;
(2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of $k_H$-affinoid algebras $\alpha_{V/U} : A_V \to A_U$ representing a $k_H$-affinoid domain $\operatorname{Sp} A_U$ in $\operatorname{Sp} A_V$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Sp} A_U \arrow[r, "\operatorname{Sp} \alpha_{V/U}"] \arrow[d, "\varphi_U"] & \operatorname{Sp} A_V \arrow[d, "\varphi_V"] \\ U \arrow[r] & V \end{tikzcd}.$$

The triple $(X, \mathcal{A}, \tau)$ as above is called a *$k_H$-analytic space*.

A *morphism* between atlases $\mathcal{A}$ and $\mathcal{A}'$ on $X$ with the net $\tau$ is an assignment that with each $V \in \tau$, one associates a morphism of $k_H$-affinoid algebras $\beta_V : A_V \to A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Sp} A'_V \arrow[r, "\operatorname{Sp}\beta_V"] \arrow[d, "\varphi'_V"] & \operatorname{Sp} A_V \arrow[dl, "\varphi_V"] \\ V & \end{tikzcd};$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{tikzcd} A_V \arrow[r, "\alpha_{V/U}"] \arrow[d, "\beta_V"] & A_U \arrow[d, "\beta_U"] \\ A'_V \arrow[r, "\alpha'_{V/U}"] & A'_U \end{tikzcd}$$

Here we have denoted the data associated with $\mathcal{A}'$ with a prime. In this way, the atlases on $X$ with the net $\tau$ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), it $W \subseteq U \subseteq V$ are three sets in $\tau$, then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

**Remark 3.2.** As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps $\varphi_U$'s by identifying $\operatorname{Sp} A_U$ with $U$. We will say $U$ is a $k_H$-affinoid domain in $V$.

**Remark 3.3.** Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

**Lemma 3.4.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space, $U \in \tau$ and $W$ is a $k_H$-affinoid domain in $U$. Then for any $V \in \tau$ containing $W$, $W$ is a $k_H$-affinoid domain in $V$.

Proof. As $\tau|_{U\cap V}$ is a net and $W$ is compact, we can find $U_1, \ldots, U_n \in \tau_{U\cap V}$ with $W \subseteq U_1 \cup \cdots \cup U_n$. As $W$, $U_i$ are $k_H$-affinoid domains in $U$, $W_i = W \cap U_i$ is a $k_H$-affinoid domain in $U_i$ for all $i = 1, \ldots, n$ by **??** in **??**. It follows from **??** and **??** in **??** that $W_i$ and $W_i \cap W_j$ are both $k_H$-affinoid domains in $V$ for $i, j = 1, \ldots, n$. So $W$ is a compact $k_H$-analytic domain in $V$.

By Proposition 2.5,

$$A_W := \ker \left( \prod_{i=1}^{n} A_{W_i} \to \prod_{i,j=1}^{n} A_{W_i \cap W_j} \right)$$

is $k_H$-affinoid and $\operatorname{Sp} A_W \to \operatorname{Sp} A$ induces a hoemomorphism $\operatorname{Sp} A_W \to W$ by **??** in **??**. By Proposition 2.5 again, $W$ is affinoid in $V$. □

**Definition 3.5.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. We define $\bar{\tau}$ as the set of all $W \subseteq X$ such that there is $U \in \tau$ containing $W$ and $W$ is $k_H$-affinoid in $U$.

**Lemma 3.6.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. Then $\bar{\tau}$ is a net on $X$ and there is a $k_H$-affinoid atlas $\overline{\mathcal{A}}$ on $X$ with the net $\bar{\tau}$ extending $\mathcal{A}$. Moreover, the $k_H$-affinoid atlas $\overline{\mathcal{A}}$ on $X$ with the net $\bar{\tau}$ extending $\mathcal{A}$ is unique up to a canonical isomorphism.

Proof. **Step 1**. We first show that $\bar{\tau}$ is a net. Let $U, V \in \bar{\tau}$ and $x \in U \cap V$. Take $U', V' \in \tau$ containing $U$ and $V$ respectively. Take $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n \in \tau$ such that

(1) $x \in W_1 \cap \cdots \cap W_n$;

(2) $W_1 \cup \cdots \cup W_n$ is a neighbourhood of $x$ in $U' \cap V'$.

This is possible because $\tau|_{U' \cap V'}$ is a quasi-net by assumption.

By Lemma 3.4, $U$ (resp. $V$) and $W_1, \ldots, W_n$ are $k_H$-affinoid domains in $U'$ (resp. $V'$).

By **??** in **??**, $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a $k_H$-affinoid domain in $W_i$ for $i = 1, \ldots, n$. By **??** in **??** again, $U_i \cap V_i$ is a $k_H$-affinoid domain in $W_i$ for $i = 1, \ldots, n$. So $U_i \cap V_i \in \bar{\tau}|_{U \cap V}$ for $i = 1, \ldots, n$. But

$$\bigcup_{i=1}^{n} U_i \cap V_i = (U \cap V) \cap \bigcup_{i=1}^{n} W_i,$$

so $\bigcup_{i=1}^{n} U_i \cap V_i$ is a neighbourhood of $x$ in $U \cap V$ and $x \in \bigcap_{i=1}^{n} U_i \cap V_i$. It follows that $\bar{\tau}$ is a net.

**Step 2**. We extend the $k_H$-affinoid atlas $\mathcal{A}$.

For each $V \in \bar{\tau}$, we fix a $V' \in \tau$ containing $V$.

By Lemma 3.4, $V$ is a $k_H$-affinoid domain in $V'$. Let $A_{V'} \to A_V$ be the morphism of $k_H$-affinoid algebras representing the $k_H$-affinoid domain $V$ in $\operatorname{Sp} A_{V'}$. We define the homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$ as the morphism induced by $\operatorname{Sp} A_V \to \operatorname{Sp} A$.

For $U, V \in \bar{\tau}$ with $U \subseteq V$, we want to define $\alpha_{V/U} : A_V \to A_U$. We handle two cases. When $V \in \tau$, as $\tau|_{U' \cap V}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_n \in \tau|_{U' \cap V}$ such that

$$U = \bigcup_{i=1}^{n} U_i.$$

By Lemma 3.4, $U_1, \ldots, U_n$ are $k_H$-affinoid domains in $U'$ and in $V$. By **??** in **??**,

$$A_U \xrightarrow{\sim} \ker \left( \prod_{i=1}^{n} A_{U_i} \to \prod_{i,j=1}^{n} A_{U_i \cap U_j} \right).$$

So the morphism $\alpha_{V/U_i} : A_V \to A_{U_i}$ and $A_{V/U_i \cap U_j} : \alpha_{V/U_i} : A_V \to A_{U_i \cap U_j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$ induces a morphism $\alpha_{V/U} : A_V \to A_U$. Observe that $\alpha_{V/U}$ represents the $k_H$-affinoid domain $U$ in $V$, so it is independent of the choice of $U_1, \ldots, U_n$.

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \to A_U$ representing the $k_H$-affinoid domain $U$ in $V'$, it follows that $U$ is a $k_H$-affinoid domain in $V$, and we therefore get the desired morphism $\alpha_{V/U} : A_V \to A_U$.

It is easy to verify that the constructions gives a $k_H$-affinoid atlas with the net $\bar{\tau}$ extending $\mathcal{A}$. The uniqueness of the extension is immediate. □

**Definition 3.7.** Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be $k_H$-analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of

(1) a continuous map $\varphi : X \to X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
(2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of $k_H$-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq W'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd}.$$

Recall our convention Remark 3.2, the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \to A_V$ of $k_H$-affinoid algebras making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} A_V \arrow[r] \arrow[d, "\varphi_V"] & \operatorname{Sp} A'_{V'} \arrow[d, "\varphi'_{V'}"] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}.$$

We will continue our identifications as in Remark 3.2 to simplify our notations.

**Proposition 3.8.** Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be $k_H$-analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then $\varphi$ extends uniquely to a strong morphism $\varphi : (X, \bar{\mathcal{A}}, \bar{\tau}) \to (X', \overline{\mathcal{A}'}, \overline{\tau'})$.

Proof. Let $U \in \bar{\tau}$, $U' \in \overline{\tau'}$ with $\varphi(U) \subseteq U'$. Take $V \in \tau$ and $V' \in \tau'$ containing $U$ and $U'$ respectively. By Lemma 3.4, $U$ (resp. $V$) is a $k_H$-affinoid domain in $V$ (resp. $V'$). Take $W \in \tau'$ with $\varphi(V) \subseteq W'$. Then in particular, $\varphi(U) \subseteq W'$. As $\tau'|_{V' \cap W'}$ is a quasi-net and $\varphi(U)$ is compact, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n \in \tau'|_{V' \cap W}$ such that

$$\varphi(U) \subseteq W_1 \cup \cdots \cup W_n.$$

Now $W_i$ is a $k_H$-affinoid domain in $W'$ by Lemma 3.4, so $V_i := \varphi^{-1}_{V/W'}(W_i)$ is an affinoid domain in $V$ by **??** in **??**, and we have an induced morphism $V_i \to W_i$ for $i = 1, \ldots, n$. This morphism in turn induces a morphism of $k_H$-affinoid spectra

$$U_i := U \cap V_i \to U'_i := U' \cap W_i \to U'$$

for $i = 1, \ldots, n$. These morphisms are compatible on their intersections by construction. So by **??** in **??**, they glue together to a morphism of $k_H$-affinoid spectra $\bar{\varphi}_{U/U'} : U \to U'$. It is easy to see that this construction defines a strong morphism.

As for the uniqueness, it suffices to show that the morphism $U_i \to U'_i$ is uniquely determined for $i = 1, \ldots, n$. In other words, we need to show that the dotted arrow that makes the following diagram commutes is unique:

$$\begin{tikzcd} U_i \arrow[r, dotted] \arrow[d] & U'_i \arrow[d] \\ V \arrow[r, "\varphi_{V/W'}"] & W' \end{tikzcd}$$

for $i = 1, \ldots, n$. It suffices to apply the universal property of the $k_H$-affinoid domain $U'_i \to W'$. □

**Definition 3.9.** Let $(X, \mathcal{A}, \tau)$, $(X', \mathcal{A}', \tau')$, $(X'', \mathcal{A}'', \tau'')$ be $k_H$-analytic spaces. Let

$$\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau'), \quad \psi : (X', \mathcal{A}', \tau') \to (X'', \mathcal{A}'', \tau'')$$

be strong morphisms. We will define their *composition* $\chi = \psi \circ \varphi$ as follows. The underlying map of topological spaces is just the composition of the unlerlying maps of topological spaces corresponding to $\psi$ and $\varphi$.

Let $\bar{\varphi}$ and $\bar{\psi}$ be the extensions of $\varphi$ and $\psi$ to $\bar{\tau}$ and $\overline{\tau'}$ as in Proposition 3.8.

Given $V \in \tau$ and $V'' \in \tau''$ with $\chi(V) \subseteq V''$, we need to define a morphism of $k_H$-affinoid spectra $\chi_{V/V''} : V \to V''$. Take $V' \in \tau'$ and $U'' \in \tau''$ such that $\varphi(V) \subseteq V'$ and $\psi(V') \subseteq U''$. Since $\chi(V) \subseteq U'' \cap V''$ and $V$ is compact, we can take $n \in \mathbb{Z}_{>0}$ and $V_1'', \ldots, V_n'' \in \tau''|_{U'' \cap V''}$ with $\chi(V) \subseteq V_1'' \cup \cdots \cup V_n''$. Then $V_i' := \psi^{-1}_{V'/U''}(V_i'')$ and $V_i := \varphi^{-1}_{V/V'}(V_i')$ are $k_H$-affinoid domains in $V'$ and $V$ respectively for $i = 1, \ldots, n$ and $V = V_1 \cup \cdots \cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ then induce a morphism $V_i \to V_i'' \to V$ of $k_H$-affinoid spectra. These morphisms are clearly compatible on the intersections and hence induce a morphism $V \to V''$ of $k_H$-affinoid spectra by **??** in **??**.

It is easy to verify that $\psi \circ \varphi$ is a strong morphism.

In this way, we get a category $k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}$ of $k_H$-analytic spaces.

**Definition 3.10.** Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be $k_H$-analytic spaces. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if

(1) $\varphi$ is a homeomorphism between $X$ and $X'$;
(2) for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$, $\mathrm{Sp}\,\varphi_{V/V'}$ identifies $V$ with an affinoid domain in $V'$.

**Lemma 3.11.** Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be $k_H$-analytic spaces and $\varphi$ : $(X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then for any $V \in \bar{\tau}$ and $V' \in \overline{\tau'}$, the intersection $V \cap \varphi^{-1}(V')$ is a compact $k_H$-analytic domain in $V$.

Proof. Take $U' \in \overline{\tau'}$ with $\varphi(V) \subseteq U'$. As $\tau|_{U' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1', \ldots, U_n' \in \tau|_{U' \cap V'}$ with $\varphi(V) \subseteq U_1' \cup \cdots \cup U_n'$ and

$$V \cap \varphi^{-1}(V') = \bigcup_{i=1}^{n} \varphi^{-1}_{V/U}(U_i').$$

□

**Lemma 3.12.** The system of quasi-isomorphisms in $k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}$ is a right multiplicative system.

For the notion of right multiplicative system, we refer to [Stacks, Tag 04VC].

Proof. We verify the three axioms as in [Stacks, Tag 04VC].

**RMS1**. The identity is clear a quasi-isomorphism. It remains to verify that the composition of quasi-isomorphisms is still a quasi-isomorphism.

We take $\varphi, \psi$ as in Definition 3.9. We will use the same notations as in Definition 3.9. We need to show that $V \to V''$ identifies $V$ with a $k_H$-affinoid domain in $V''$. From the construction, we know that $\varphi$ identifies $V_i$ with a $k_H$-affinoid domain in $V_i'$ and $\psi$ identifies $V_i'$ with a $k_H$-affinoid domain in $V_i''$ for $i = 1, \ldots, n$. In particular, $\chi(V)$ is a compact $k_H$-analytic domain in $V''$. It follows from Proposition 2.5 that $\chi(V)$ is a $k_H$-affinoid domain in $V''$.

**RMS2**. If $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ and $f : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ are given strong morphisms of $k_H$-analytic spaces and $g$ is a quasi-isomorphism, then there are $k_H$-analytic space $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau})$ and strong morphisms $\tilde{\varphi} : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to$

$(\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ and $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ such that $f$ is a quasi-isomorphism and the following diagram commutes:

$$\begin{tikzcd} (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \arrow[r, dotted, "\tilde{\varphi}"] \arrow[d, dotted, "f"] & (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \arrow[d, "g"] \\ (X, \mathcal{A}, \tau) \arrow[r, "\varphi"] & (X', \mathcal{A}', \tau') \end{tikzcd} .$$

We may assume that $\widetilde{X'} = X'$. Then $\widetilde{\tau'} \subseteq \overline{\tau'}$. We let $\tilde{X} = X$. Let $\tilde{\tau}$ be the family of all $V \in \bar{\tau}$ for which there is $\widetilde{V'} \in \widetilde{\tau'}$ with $\varphi(V) \subseteq \widetilde{V'}$. By Lemma 3.11, $\tilde{\tau}$ is a net on $\tilde{X}$. The $k_H$-atlas $\bar{\mathcal{A}}$ defines a $k_H$-affinoid atlas $\tilde{\mathcal{A}}$ with the net $\tilde{\tau}$. The strong morphism $\bar{\varphi}$ induces $\tilde{\varphi}$. The morphism $f$ is the canonical quasi-isomorphism. It is immediate that these constructions satisfy the desired conditions.

**RMS3**. If $\varphi, \psi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ are strong morphisms of $k_H$-analytic spaces and there is a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ of $k_H$-analytic spaces such that $g \circ \varphi = g \circ \psi$, then there is a quasi-isomorphism $f : (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi \circ f = \psi \circ f$.

We will in fact show that $\varphi = \psi$. It is clear that they coincide as maps of topological spaces. Let $V \in \tau$, $V' \in \tau'$ such that $\varphi(V) \subseteq V'$. Take $\widetilde{V'} \in \widetilde{\tau'}$ with $g(V') \subseteq \widetilde{V'}$. Then we have two morphisms of $k$-affinoid spectra $\varphi_{V/V'}, \psi_{V/V'} : V \to V'$ such that their compositions with $g_{V'/\widetilde{V'}}$ coincide. As $V'$ is an affinoid domain in $\widetilde{V'}$, it follows that $\varphi_{V/V'} = \psi_{V/V'}$ by the universal property. □

**Definition 3.13.** The category $k_H$-$\mathcal{A}$n is the right category of fractions of $k_H$-$\widetilde{\mathcal{A}\mathrm{n}}$ with respect to the system of quasi-isomorphisms. A morphism in $k_H$-$\mathcal{A}$n is called a *morphism* between $k_H$-analytic spaces.

We refer to [Stacks, Tag 04VB] for the definition of right category of fractions. For later references, we explicitly write down the morphisms in $k_H$-$\mathcal{A}$n.

**Lemma 3.14.** Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of $k_H$-analytic spaces. We define a partial order on the set of nets on $X$: $\tau_1 \preceq \tau_0$ if $\tau_1 \subseteq \overline{\tau_0}$. Then the set of nets is a directed set and

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}\left((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')\right) = \varinjlim_{\sigma \preceq \tau} \mathrm{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}}\left((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')\right)$$

in the category of sets, where $\mathcal{A}_\sigma$ is induced by $\overline{\mathcal{A}}$. The transition maps are all injective.

Proof. This follows immediately from the definition. □

**Definition 3.15.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. We say a subset $W \subseteq X$ is *$\tau$-special* if it is compact and there exist $n \in \mathbb{Z}_{>0}$ and a covering $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$ and the natural map

$$A_{W_i} \hat{\otimes}_k A_{W_j} \to A_{W_i \cap W_j}$$

is an admissible epimorphism.

The covering $W_1, \ldots, W_n$ is called a *$\tau$-special covering* of $W$.

Under our convention, the assumption means that $W_i \cap W_j \to W_i \times W_j$ is a closed immersion of $k_H$-affinoid spectra.

**Example 3.16.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. Suppose that $V \in \tau$ and $W$ is a compact $k_H$-analytic domain in $V$. Let $n \in \mathbb{Z}_{>0}$ and $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$. Then $\{W_i\}_i$ is a $\tau$-special covering of $W$. This follows from **??** in **??**.

**Lemma 3.17.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space and $W$ be a $\tau$-special subset of $X$. If $U, V \in \tau|_W$, then $U \cap V \in \bar{\tau}$ and the natural map

$$A_U \hat{\otimes}_k A_V \to A_{U\cap V}$$

is an admissible epimorphism.

Proof. Let $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n$ be a $\tau$-special covering of $W$. As $U \cap W_i$ and $V \cap W_i$ are compact for $i = 1, \ldots, n$, we can find $m_i \in \mathbb{Z}_{>0}$ (resp. $k_i \in \mathbb{Z}_{>0}$) and finite coverings $U_{i1}, \ldots, U_{im_i} \in \tau$ of $U \cap W_i$ (resp. $V_{i1}, \ldots, V_{ik_i} \in \tau$ of $V \cap W_i$).

Observe that $U_{ik} \cap V_{jl}$ is a $k_H$-affinoid domain in $U \cap V$, hence $U_{ik} \cap V_{jl} \in \bar{\tau}$ for any $i, j = 1, \ldots, n$, $k = 1, \ldots, m_i$ and $l = 1, \ldots, k_l$. By **??** in **??**, $U_{ik} \cap V_{jl} \to U_{ik} \times V_{jl}$ is a closed immersion since $W_i \cap W_j \to W_i \times W_j$ is by our assumption.

Consider the finite convering

$$\mathcal{U} := \{U_{ik} \times V_{jl} : i, j = 1, \ldots, n; k = 1, \ldots, m_i; l = 1, \ldots, k_l\}$$

of $U \times V$. For each tuple $(i, j, k, l)$, $A_{U_{ik}\cap V_{jl}}$ is a finite $A_{U_{ik}\times V_{jl}}$-algebra. By **??** in **??**, we can construct a finite $A_{U\times V}$-algebra $A_{U\cap V}$ inducing all of these $A_{U_{ik}\cap V_{jl}}$'s. By **??** in **??**, $A_{U\cap V}$ is $k_H$-affinoid.

As $\mathcal{U}$ is a finite $k_H$-affinoid covering of $U \times V$, $\{A_{U_{ik}\cap V_{jl}}\}_{i,k,j,l}$ is a finite $k_H$-affinoid covering of $U \cap V$ by **??** in **??**. In particular, we have a natural homeomorphism

$$\operatorname{Sp} A_{U\cap V} \xrightarrow{\sim} U \cap V.$$

Observe that $A_U \hat{\otimes}_k A_V \to A_{U\cap V}$ is surjective. We endow $A_{U\cap V}$ with the structure of finite $A_U \hat{\otimes}_k A_V$-Banach algebras by **??** in **??**. Then $A_U \hat{\otimes}_k A_V \to A_{U\cap V}$ is an admissible epimorphism by **??** in **??**.

On the other hand $U \cap V$ is a compact $k_H$-analytic domain in $U$, so by Proposition 2.5, $U \cap V$ is a $k_H$-affinoid in $U$. In particular, $U \cap V \in \bar{\tau}$. ☐

**Lemma 3.18.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space and $W \subseteq X$ be a $\tau$-special set. Then for any finite covering $\{W_i\}_{i\in I}$ of $W$ with $W_i \in \tau$ for $i \in I$, the Banach $k$-algebra

$$A_W := \ker\left(\prod_{i\in I} A_{W_i} \to A_{W_i\cap W_j}\right)$$

does not depend on the choice of $\{W_i\}_{i\in I}$ up to canonical isomorphisms.

Moreover, we have a canonical map $W \to \operatorname{Sp} A_W$, which does not depend on the choice of the covering modulo the canonical isomorphism between $A_W$.

Proof. It follows from Lemma 3.17 that the covering $\{W_i\}_{i\in I}$ is $\tau$-special. It suffices to apply the same argument of Lemma 2.2. ☐

**Definition 3.19.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. Let $\hat{\tau}$ denote the collection of $\bar{\tau}$-special subsets $W \subseteq X$ such that

(1) $A_W$ is $k$-affinoid;
(2) the natural map $W \to \operatorname{Sp} A_W$ is bijective;
(3) there is a $\bar{\tau}$-special covering $\{W_i\}_{i\in I}$ of $W$ such that $W_i$ is a $k$-affinoid domain in $W$ for $i \in I$.

The sets from $\hat{\tau}$ are called *$k_H$-affinoid domains in* $(X, \mathcal{A}, \tau)$.

Observe that $W$ is $k_H$-affinoid and $W_i$ is a $k_H$-affinoid domain in $W$ by **??** in **??**. Condition (3) holds for any $\bar{\tau}$-special covering.

**Proposition 3.20.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. Then $\hat{\tau}$ is a net. For any net $\sigma$ on $X$ contained in $\bar{\tau}$, we have $\hat{\sigma} = \hat{\tau}$.

Moreover, $\hat{\hat{\tau}} = \hat{\tau}$.

Proof. Let $U, V \in \hat{\tau}$. Take $\bar{\tau}$-special coverings $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ of $U$ and $V$ respectively. In order to show that $\hat{\tau}|_{U\cap V}$ is a quasi-net, it suffices to show that $\hat{\tau}|_{U_i\cap V_j}$ is for any $i \in I$ and $j \in J$. This follows simply from the fact that $\bar{\tau}|_{U_i\cap V_j}$ is a quasi-net. Similarly, as $\hat{\tau}$ is a quasi-net as $\bar{\tau}$ is. So $\hat{\tau}$ is a net.

Let $\sigma$ be a net on $X$ contained in $\bar{\tau}$. By Lemma 3.17, it suffices to verify that for any $V \in \bar{\tau}$, there are $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \bar{\sigma}$ with $V = U_1 \cup \dots \cup U_n$. As $\sigma$ is a net on $X$, we can find $m \in \mathbb{Z}_{>0}$, $W_1, \dots, W_m \in \sigma$ such that

$$V \subseteq W_1 \cup \dots \cup W_m.$$

As $V, W_j \in \bar{\tau}$ for $j = 1, \dots, m$, by **??** in **??**, we can find $U_1, \dots, U_n \in \bar{\tau}$ such that $V = U_1 \cup \dots \cup U_n$ and each $U_i$ is contained in some $W_j$. As $W_j \in \sigma$ for $j = 1, \dots, m$, it follows that $U_i \in \bar{\sigma}$ for $i = 1, \dots, n$.

By Lemma 3.17,

$$\bar{\hat{\tau}} = \hat{\tau}.$$

Let $V \in \hat{\hat{V}}$. Let $\{V_i\}_{i\in I}$ be a $\hat{\tau}$-special covering of $V$. For each $i \in I$, take a $\bar{\tau}$-special covering $\{V_{ij}\}_{j\in J_i}$ of $V_i$. Then $\{V_{ij}\}_{i,j}$ is a $\bar{\tau}$-special covering of $V$. It follows that $V \in \hat{\tau}$. □

**Proposition 3.21.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. There is a $k_H$-analytic atlas $\hat{\mathcal{A}}$ on $X$ with the net $\hat{\tau}$ extending $\mathcal{A}$. Moreover, $\hat{\mathcal{A}}$ is unique up to a canonical isomorphism.

Proof. For each $V \in \hat{\tau}$, Fix a $\bar{\tau}$-special covering $\{V_i\}_{i\in I_V}$.

We define $A_V$ using this covering as in Lemma 3.18. By definition, the canonical map $V \to \operatorname{Sp} A_V$ is a homeomorphism.

Next take $U, V \in \hat{\tau}$ with $U \subseteq V$. We want to identify $U$ with a $k_H$-affinoid domain in $V$. First assume that $U \in \tau$, then $U \cap V_i$ is a $k_H$-affinoid domain in $V_i$ for $i \in I_V$ by Lemma 3.17. Hence, $U$ is a $k_H$-affinoid domain in $V$. If we only know $U \in \hat{\tau}$, we know that $U_i$ is a $k_H$-affinoid domain in $V$ for any $i \in I_U$. It follows that $U$ is a $k_H$-affinoid domain in $V$ by Proposition 2.5.

The uniqueness is immediate. □

**Definition 3.22.** Let $(X, \mathcal{A}, \tau)$ be a $k_H$-analytic space. A $\hat{\tau}$-special set is called a *$k_H$-special domain* in $X$.

Observe that a $k_H$-special domain inherits a structure of $k_H$-analytic space from $(X, \mathcal{A}, \tau)$.

**Proposition 3.23.** Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of $k_H$-analytic spaces. Then for any $k_H$-affinoid domains $V \subseteq X$ and $V' \subseteq X'$, the intersection $V \cap \varphi^{-1}(V')$ is a $k_H$-special domain in $X$.

Proof. By Proposition 3.20, we may assume that $\varphi$ is a strong morphism. In this case, it suffices to apply Lemma 3.11. □

**Lemma 3.24.** Let $(X,\mathcal{A},\tau)$ and $(X',\mathcal{A}',\tau')$ be $k_H$-analytic spaces. Let $\varphi : (X,\mathcal{A},\tau) \to (X',\mathcal{A}',\tau')$ be a strong morphism. Then $\varphi$ extends uniquely to a strong morphism $\varphi : (X,\hat{\mathcal{A}},\hat{\tau}) \to (X',\widehat{\mathcal{A}'},\widehat{\tau'})$.

PROOF. Let $V \in \hat{\tau}$ and $V' \in \widehat{\tau'}$ with $\varphi(V) \subseteq V'$. We want to define $\varphi_{V/V'} : V \to V'$ of $k_H$-affinoid spectra. By Proposition 3.8, we may extend $\varphi$ uniquely to $\bar{\tau}$. Take a $\bar{\tau}$-special covering of $V$, we may reduce to the case where $V \in \bar{\tau}$. Take $W' \in \tau'$ such that $\varphi(V) \subseteq W'$. As $\tau|_{W'\cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $W_1,\dots,W_n \in \tau'|_{V'\cap W}$ such that $\varphi(V) \subseteq W_1 \cup \cdots \cup W_n$. Considering the inverse images of $W_i$'s and $W_i \cap W_j$'s using Lemma 3.17, we are reduced to the case where $V' \in \overline{\tau'}$. This is already handled in Proposition 3.8. The uniqueness of the extension is clear. $\square$

**Proposition 3.25.** Let $(X,\mathcal{A},\tau)$, $(X',\mathcal{A}',\tau')$ be $k_H$-analytic spaces.

(1) There is a canonical bijection between

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}((X,\mathcal{A},\tau),(X',\mathcal{A}',\tau'))$$

and the set of pairs consisting of

(a) a continuous map $\varphi : X \to X'$ such that for all $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, neighbourhoods $V_1 \cup \cdots \cup V_n$ of $x$ and $V_1' \cup \cdots \cup V_n'$ of $\varphi(x)$ with $x \in V_1 \cap \cdots \cap V_n$ and $\varphi(V_i) \subseteq V_i'$ for $i = 1,\dots,n$, where $V_i \subseteq X$ and $V_i' \subseteq X'$ are $k_H$-affinoid domains;

(b) for each pair of $k_H$-affinoid domains $V \subseteq X$, $V' \subseteq X'$ with $\varphi(V) \subseteq V'$, a morphism of $k_H$-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that if $V,W \subseteq X$ and $V',W' \subseteq X'$ are $k_H$-affinoid domains with $\varphi(V) \subseteq V'$, $\varphi(W) \subseteq W'$, the diagram below commutes

$$\begin{array}{ccc} V & \xrightarrow{\varphi_{V/V'}} & V' \\ \downarrow & & \downarrow \\ W & \xrightarrow{\varphi_{W/W'}} & W' \end{array}.$$

(2) Under the bijection in (1), an isomorphism corresponds to the pair where $\varphi$ is a hoemomorphism such that $\varphi(\hat{\tau}) = \widetilde{\tau'}$ and for any $V \in \hat{\tau}$, $\varphi_{V/\varphi(V)}$ is an isomorphism of $k_H$-affinoid spectra.

PROOF. (2) follows immediately from (1). So it suffices to prove (1).

We construct the forward map. Let $\varphi : (X,\mathcal{A},\tau) \to (X',\mathcal{A}',\tau')$ be a morphism. Take a subnet $\sigma$ of $\bar{\tau}$ such that $\varphi$ is represented by a strong morphism

$$\varphi : (X,\mathcal{A}_\sigma,\sigma) \to (X',\mathcal{A}',\tau').$$

By Lemma 3.24, this extends to a strong morphism

$$\varphi : (X,\widehat{\mathcal{A}_\sigma},\hat{\sigma}) \to (X',\widehat{\mathcal{A}'},\widehat{\tau'}).$$

We get an injective map from the first set into the second set.

Conversely, we need to show that any given map from the second map comes from the first set. It suffices to show that

$$\sigma := \left\{ V \in \hat{\tau} : \varphi(V) \subseteq V' \text{ for some } V' \in \widehat{\tau'} \right\}$$

is a net. Take $x \in X$ and neighbourhoods $V_1 \cup \cdots \cup V_n$ of $x$ and $V_1' \cup \cdots \cup V_n'$ of $\varphi(x)$ as in the statement of (1). Then $V_i \in \sigma$, so we conclude. $\square$

In practice, we do not distinguish a $k_H$-analytic space from the isomorphic $k_H$-analytic spaces. In particular, we will write $(X,\mathcal{A},\tau)$ as $X$ and always endow it with the strucutre $(X,\hat{\mathcal{A}},\hat{\tau})$ of $k_H$-analytic space. If necessarily, we will write $|X|$ for the underlying topological space.

**Corollary 3.26.** The natural functor $k_H\text{-}\mathcal{A}\mathrm{ff}\to k_H\text{-}\mathcal{A}\mathrm{n}$ is fully faithful.

Proof. Let $X=\operatorname{Sp}A$ be a $k_H$-affinoid spectrum. We endow it with the net $\tau=\{X\}$. The $k_H$-atlas with the net $\tau$ assigns $X\in\tau$ with $A$. It is easily verified that this is a functor. By Proposition 3.25, the functor is fully faithful. $\square$

**Definition 3.27.** A *$k_H$-affinoid space* is an object of $k_H\text{-}\mathcal{A}\mathrm{n}$ lying in the essential image of the functor $k_H\text{-}\mathcal{A}\mathrm{ff}\to k_H\text{-}\mathcal{A}\mathrm{n}$.

The category of $k_H$-affinoid spaces is denoted by $k_H\text{-}\mathcal{A}\mathrm{ff}$.

The notation for the category of $k_H$-affinoid spaces is the same as the notation for the category of $k_H$-affinoid spectra, as the two categories are canonically equivalent.

**Definition 3.28.** A $k_H$-analytic space $X$ is *good* if any point $x\in X$ admits a $k_H$-affinoid neighbourhood.

**Example 3.29.** Fix $n\in\mathbb{N}$. Let $\mathbb{A}^n_k$ denote the set of all semi-valuations on $k[T_1,\dots,T_n]$ whose restriction to $k$ coincides with the given valuation on $k$. We provide $\mathbb{A}^n_k$ with the weakest topology such that for any $f\in k[T_1,\dots,T_n]$, the map $|\bullet|\mapsto|f|$ is continuous.

Observe that as a topological space,

$$\mathbb{A}^n_k\xrightarrow{\sim}\varinjlim r\in\mathbb{R}^n_{>0}\operatorname{Sp}k\{r^{-1}T\}. \tag{3.1}$$

As a set, this is clear: if $|\bullet|\in\mathbb{A}^n_k$, we take $r=(|T_1|,\dots,|T_n|)$, then $|\bullet|\leq\|\bullet\|_r$, so $|\bullet|\in\operatorname{Sp}k\{r^{-1}T\}$. As

$$\bigcap_{r\in\mathbb{R}^n_{>0}}k\{r^{-1}T\}=k[T_1,\dots,T_n],$$

so the topology on the right-hand side of (3.1) is the weakest topology making $|\bullet|\mapsto|f|$ continuous for any $f\in k[T_1,\dots,T_n]$. It follows immediately that (3.1) is an identification of topological spaces.

It is clear that $\mathbb{A}^n_k$ has a structure of good $k_H$-analytic space.

**Proposition 3.30.** Let $X$ be a $k_H$-analytic space, $x\in X$ and $U$ be a neighbourhood of $x$ in $X$. Then there is a neighbourhood $V$ of $x$ in $X$ contained in $U$ such that $V$ is open connected locally compact paracompact and Hausdorff. Moreover, we can guarantee that $\bar{V}\subseteq U$ and $V$ is a countable union of $k_H$-affinoid domains.

Proof. Take $n\in\mathbb{Z}_{>0}$ and $k_H$-affinoid spaces $V_1,\dots,V_n$ containing $x$ and $V_1\cup\dots\cup V_n$ is a neighbourhood of $x$ in $X$. If we have proved the proposition for $V_i$ in place of $X$ and $U\cap V_i$ in place of $U$ for $i=1,\dots,n$, namely, if we have found open connected locally compact paracompact and Hausdorff sets $W_i$ containing $x$ and contained in $U\cap V_i$ whose closure in $V_i$ is contained in $U\cap V_i$, then we can take $V=W_1\cup\dots\cup W_n$.

So we may assume that $X$ is a $k_H$-affinoid space, say $X=\operatorname{Sp}A$. Choose a $k_H$-rational neighbourhood

$$W=\operatorname{Sp}A\{r^{-1}\frac{f}{g}\}$$

of $x$ in $U$, where $n \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in A^n$, $r \in \sqrt{|k^\times| \cdot H}^n$, $g \in A$ and $f_1, \dots, f_n, g$ generate the unit ideal in $A$. This is possible by **??** and **??** in **??**. Take $\delta > 0$ so that $x \in \operatorname{Sp} A\{((1-\delta)r)^{-1}\frac{f}{g}\}$. Choose a strictly increasing sequence $\epsilon_i \in (0,1) \cap \sqrt{|k^\times| \cdot H}$ converging to $1 - \delta/2$ for $i \in \mathbb{Z}_{>0}$. Let

$$W_i = \operatorname{Sp} A\left\{(\epsilon_i r)^{-1}\frac{f}{g}\right\}$$

for $i \in \mathbb{Z}_{>0}$. Then $W_i$ lies in the interior of $W_{i+1}$ for $i \in \mathbb{Z}_{>0}$. Choose a connected component $V_i$ of $W_i$ so that $V_1 \subseteq V_2 \subseteq \cdots$ and $x \in V := \bigcup_{i=1}^\infty V_i$. If $x \in V_i$ for some $i \in \mathbb{Z}_{>0}$, then $x$ lies in the topological interior of $V_{i+1}$. Hence, $x$ lies in the interior of $V$. By construction, $V$ is open connected paracompact locally compact and Hausdorff. Moreover, $\bar{V} \subseteq U$ by our construction. □

**Proposition 3.31.** Let $\{X_i\}_{i\in I}$ be a family of $k_H$-analytic spaces. Suppose that for $i, j \in I$, we are given a $k_H$-analytic domain $X_{ij} \subseteq X_i$ and an isomorphism $\nu_{ij} : X_{ij} \to X_{ji}$ satisfying the *cocycle condition*: $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $X_{ij} \cap X_{il}$ for $i, j, l \in I$.

Assume that either of the followinig conditions holds:

(1) $X_{ij}$ is open in $X_i$ for all $i, j \in I$;
(2) for any $i \in I$, all $X_{ij}$'s are closed in $X_i$ and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite.

Then there is a $k_H$-analytic space $X$ and morphisms $\mu_i : X_i \to X$ for $i \in I$ such that

(1) $\mu_i$ is an isomorphism of $X_i$ with a $k_H$-analytic domain in $X$;
(2) $X = \bigcup_{i\in I} \mu_i(X_i)$;
(3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$ for $i, j \in I$;
(4) $\mu_i = \mu_j \circ \nu_{ij}$ on $X_{ij}$ for $i, j \in I$.

The space $X$ is unique up to a canonical isomorphism. Moreover, under Condition (1), $\mu_i(X_i)$ is open in $X$ for $i \in I$; under Condition (2), $\mu_i(X_i)$ is closed in $X$ for $i \in I$.

Under both conditions, if all $X_i$'s are Hausdorff (resp. paracompact), then so is $X$.

We will call $X$ the *gluing* of the $X_i$'s along the $X_{ij}$'s.

Proof. By Proposition 4.10, the uniquenss of $X$ is clear.
Let

$$\tilde{X} = \coprod_{i\in I} X_i$$

in $k_H$-$\mathcal{A}$n. Observe that

$$|\tilde{X}| = \coprod_{i\in I} |X_i|$$

in the category $\mathcal{T}$op. The system $\nu_{ij}$'s defines an equivalence relation $R$ on $|\tilde{X}|$. Let $|X| = |\tilde{X}|/R$ and $\mu_i : |X_i| \to |X|$ be the induced map for $i \in I$.

Under Condition (1), $\mu_i(|X_i|)$ is open in $|X|$ for $i \in I$. Under Condition (2), $\mu_i(|X_i|)$ is closed in $X$ for $i \in I$.

Under both conditions, the map $\mu_i$ induces a homeomorphism $|X_i| \to \mu_i(|X_i|)$ for $i \in I$. If all $|X_i|$'s are Hausdorff (resp. paracompact), so is $|X|$.

All these claims follow from well-known results in general topology.

We will endow $|X|$ with a structure of $k_H$-analytic space. Let $\tau$ be the set of $V \subseteq |X|$ for which there is $i \in I$ such that $V \subseteq \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is a $k_H$-affinoid domain in $X_i$. Then $\tau$ is a net on $X$. There is an obvious $k$-affinoid atlas on $X$ with the net $\tau$. All properties in the proposition are satisfied by $X = (|X|, \mathcal{A}, \tau)$. □

**Proposition 3.32.** Let $(X, \mathcal{A}, \tau)$ be a paracompact Hausdorff $k_H$-analytic space. Then the locally finite nets are cofinal in the directed set of nets $\sigma$ on $X$ contained in $\hat{\tau}$.

Proof. Let $\sigma$ be a net on $X$ contained in $\bar{\tau}$. We need to find a refinement $\eta$ of $\sigma$ contained in $\bar{\tau}$ that is locally finite.

We first choose an open covering $\mathcal{U}$ of $X$ by open $\sigma$-compact Hausdorff and quasi-compact subsets. This is possible by Proposition 3.30. Recall that open subsets of $\sigma$-compact Hausdorff and quasi-compact sets are $\sigma$-compact Hausdorff and quasi-compact. So up to replacing $\mathcal{U}$ by a refinement, we may assume that $\mathcal{U}$ is locally finite.

Now we can choose a locally finite covering $\mathcal{V}_U$ of each $U \in \mathcal{U}$ by $k_H$-affinoid domains in $X$ which are contained in some element in $\sigma$. Let $\mathcal{V} = \bigcup_{U \in \mathcal{U}} \mathcal{V}_U$. Then $\mathcal{V}$ is a locally finite covering of $X$.

Refining $\mathcal{V}$, we may assume that each pair of elements in $\mathcal{V}$ are either disjoint or both contained in a common $k_H$-affinoid domain in $X$. Then the collection of all finite intersections in $\mathcal{V}$ is a net.

□

**Proposition 3.33.** The category $k_H$-$\mathcal{A}$n admits finite limits.

Many details need to be included in this proof!

Proof. By general abstract nonsense, it suffices to show that $k_H$-$\mathcal{A}$n admits finite fiber products.

Let $\varphi : Y \to X$ and $f : X' \to X$ be morphisms of $k_H$-affinoid spaces.

**Step 1**. We show assume that $H = \mathbb{R}_{>0}$. We show that $Y \times_X X'$ exists in $k$-$\mathcal{A}$n.

**Step 1.1**. Assume that $Y, X, X'$ are all paracompact and Hausdorff.

We first claim that we can represent $\varphi$ and $f$ by strong morphisms $(Y, \mathcal{B}, \sigma) \to (X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau') \to (X, \mathcal{A}, \tau)$ with $\sigma$, $\tau'$, $\tau$ locally finite.

We first choose a locally finite net $\tau$ consisting of $k$-affinoid domains on $X$ by Proposition 3.32. Consider the net $\sigma'$ consisting of all $k$-affinoid domains on $Y$ that are mapped into some element in $\tau$. Then $\varphi$ can be represented by a strong morphism $(Y, \mathcal{B}, \sigma') \to (X, \mathcal{A}, \tau)$. We now refine $\sigma'$ using Proposition 3.32 again to construct $\sigma$. The locally finite net $\tau'$ is constructed in the same way.

Let

$$\mathcal{S} := \{(V, U, U') \in \sigma \times \tau \times \tau' : \varphi(V) \subseteq U, f(U') \subseteq U\}.$$

For any $\alpha = (V, U, U') \in \mathcal{S}$, we write

$$W_\alpha := V \times_U U', \quad \Sigma_\alpha = |V| \times_{|U|} |U'|,$$

and we have an obvious continuous map of topological spaces $W_\alpha \to \Sigma_\alpha$. Let

$$\Sigma = |Y| \times_{|X|} |X'|.$$

Then we have a canonical map $\pi_\alpha : W_\alpha \to \Sigma$ for any $\alpha \in \mathcal{S}$.

We claim that for any $\alpha, \beta \in S$,

$$W_{\alpha\beta} := \pi_\alpha^{-1}(\Sigma_\alpha \cap \Sigma_\beta)$$

is a $k$-special domain in $W_\alpha$ and there is a canonical isomorphism $\nu_{\alpha\beta} : W_{\alpha\beta} \to W_{\beta\alpha}$ satisfying the cocycle condition.

In fact, if $\alpha = (V, U, U')$ and $\beta = (\bar{U}, \bar{U}, \bar{U}')$, we can write

$$U \cap \bar{U} = U_1 \cup \cdots \cup U_n$$

for some $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in \tau$. For each $i = 1, \ldots, n$, we write

$$\varphi_{V/U}^{-1}(U_i) \cap \varphi_{\bar{V}/\bar{U}}^{-1}(U_i) = \bigcup_{j=1}^{p_i} V_{ij}, \quad f_{U'/U}^{-1}(U_i) \cap f_{\bar{U}'/\bar{U}}^{-1}(U_i) = \bigcup_{l=1}^{q_i} U'_{il},$$

where $p_i, q_i \in \mathbb{N}$, $V_{ij} \in \sigma$ and $U'_{il} \in \tau'$ for $j = 1, \ldots, p_i$ and $l = 1, \ldots, q_i$. Then

$$W_{\alpha\beta} = \bigcup_{i,j,l} V_{ij} \times_{U_i} U'_{il}.$$

So $W_{\alpha\beta}$ is a $k$-special domain in $W_\alpha$. We need to add a few lemmas We have an obvious isomorphism $\nu_{\alpha\beta} : W_{\alpha\beta} \to W_{\beta\alpha}$ satisfying the cocycle condition. We can then glue the $W_\alpha$'s along the $W_{\alpha\beta}$'s by **??** to obtain a $k$-analytic space $W$. By Proposition 4.10, $W$ represents the fiber product $Y \times_X X'$.

**Step 1.2**. Now we only assume $X$ to be paracompact and Hausdorff.

Take open paracompact coverings $\{Y_i\}_{i\in I}$ of $Y$ and $\{X'_j\}_{j\in J}$ of $X'$. The existence of these coverings follows from Proposition 3.30. Similar to Step 1.1, we glue the $Y_i \times_X X'_j$'s along the open subsets $(Y_i \cap Y_k) \times_X (X'_j \cap X'_l)$'s, we get a locally Hausdorff $k_H$-analytic space $Y'$. Then by Proposition 4.10 again, $Y'$ represents the fiber product $Y \times_X X'$.

**Step 1.3**. We do not assume that $X$ is paracompact. Take a covering $\{X_i\}_{i\in I}$ by open paracompact subsets. Let $Y'$ be the gluing of $\varphi^{-1}(X_i) \times_{X_i} f^{-1}(X_i)$'s along $\varphi^{-1}(X_i \cap X_j) \times_{X_i\cap X_j} f^{-1}(X_i \cap X_j)$'s. Then by Proposition 4.10 again, $Y'$ represents the fiber product $Y \times_X X'$.

**Step 2**. We handle the general case. Let $\sigma'$ be the collection of all $k_H$-affinoid domains of the form $V \times_U U'$, where $V \subseteq Y$, $U \subseteq X$, $U' \subseteq X'$ are $k_H$-affinoid domains with $\varphi(V) \subseteq U, f(U') \subseteq U$. Then $\sigma'$ is a net on $Y'$ and there is an obvious $k_H$-affinoid atlas on $Y'$ with teh net $\sigma'$. This defines a structure a $k_H$-affinoid spaces on $Y'$. It is clear that $Y'$ represents the fiber product $Y \times_X X'$ in the category of $k_H$-$\mathcal{A}$n. $\square$

## 4. Analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Definition 4.1.** Let $X$ be a $k_H$-analytic space. A subset $Y \subseteq X$ is called a *$k_H$-analytic domain* if for any $y \in Y$, there exist $n \in \mathbb{Z}_{>0}$, $k_H$-affinoid domains $V_1, \ldots, V_n$ contained in $Y$ such that

(1) $y \in V_1 \cap \cdots \cap V_n$;
(2) $V_1 \cup \cdots \cup V_n$ is a neighbourhood of $y$ in $Y$.

Observe that the net of $k_H$-affinoid domains in $X$ that are contained in $Y$ form a net on $Y$. In particular, $Y$ inherits a $k_H$-analytic space structure from $X$, and we have a canonical morphism $Y \to X$ in $k_H$-$\mathcal{A}$n.

**Example 4.2.** Let $X$ be a $k_H$-analytic space. Then any open subset $U$ of $X$ is a $k_H$-analytic domain.

In fact, for $x \in U$, take $V_1, \ldots, V_n$ as in Definition 4.1. By **??** in **??**, up to replacing $V_i$'s by $k_H$-Laurent domains in them, we may guarantee that $V_i \subseteq U$ for all $i = 1, \ldots, n$.

**Proposition 4.3.** Let $X$, $X'$ be $k_H$-analytic spaces and $\varphi : X' \to X$ a morphism of $k_H$-analytic spaces.

(1) Let $Y, Z$ be $k_H$-analytic domains in $X$, then so is $Y \cap Z$.
(2) Let $Y$ be a $k_H$-analytic domain in $X$, then $\varphi^{-1}(Y)$ is a $k_H$-analytic domain in $X'$.

Proof. (1) Let $x \in Y \cap Z$. Take $k_H$-affinoid domains $V_1, \ldots, V_n$ contained in $Y$ and $k_H$-affinoid domains $W_1, \ldots, W_m$ contained in $Z$ such that

$$x \in V_1 \cap \cdots \cap V_n, \quad x \in W_1 \cap \cdots \cap W_m$$

and $V_1 \cup \cdots \cup V_n$ is a neighbourhood of $x$ in $Y$, $W_1 \cup \cdots \cup W_m$ is a neighbourhood of $x$ in $Z$. For each $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $\hat{\tau}|_{V_i \cap W_j}$ is a quasi-net, so we can find a neighbourhood of $x$ in $V_i \cap W_j$ of the form $U_1^{ij} \cup \cdots \cup U_{m_{ij}}^{ij}$ with $U_1^{ij}, \ldots, U_{m_{ij}}^{ij}$ being $k_H$-affinoid domains in $X$ containing $x$. Then each element in the collection $\{U_k^{ij}\}$ contains $x$ and the union is a neighbourhood of $x$ in $Y \cap Z$.

(2) Let $x' \in \varphi^{-1}(Y)$ and $x = \varphi(x')$. By Proposition 3.25, we can find $n \in \mathbb{Z}_{>0}$, $k_H$-affinoid domains $V_1', \ldots, V_n'$ on $X'$ and $k_H$-affinoid domains $V_1, \ldots, V_n$ on $X$ such that

$$x' \in V_1' \cap \cdots \cap V_n', \quad x \in V_1 \cap \cdots \cap V_m,$$
$$\varphi(V_i') \subseteq V_i \text{ for } i = 1, \ldots, n,$$

and $V_1' \cup \cdots \cup V_n'$ (resp. $V_1 \cup \cdots \cup V_n$) is a neighbourhood of $x'$ (resp. $x$) in $X'$ (resp. $X$). Take $k_H$-affinoid domains $W_1, \ldots, W_m$ in $X$ contained in $Y$, each containing $x$ such that $W_1 \cup \cdots \cup W_m$ is a neighbourhood of $x$ in $Y$.

Then for each $i = 1, \ldots, n$, $j = 1, \ldots, m$, we can find $k_H$-affinoid domains $W_{ij}^k$ for $k = 1, \ldots, r_{ij}$ contained in $W_j \cap V_i$ and containing $x$ such that $\cup_k W_{ij}^k$ is a neighbourhood of $x$ in $W_j \cap V_i$. Thus, $\cup_{j,k} W_{ij}^k$ is a neighbourhood of $x$ in $V_i \cap Y$. Then $U_{ij}^k := \varphi^{-1}(V_{ij}^k) \cap V_i'$ is a $k_H$-affinoid domain in $V_i'$ by **??** in **??**. Moreover, $\cup_{j,k} U_{ij}^k$ is a neighbourhood of $x'$ in $V_i' \cap Y'$. So $\cup_{i,j,k} U_{ij}^k$ is a neighbourhood of $x'$ in $Y'$. □

**Proposition 4.4.** Let $X$ be a $k_H$-analytic space and $Y$ be a $k_H$-analytic domain in $X$. Then for any $k_H$-analytic space $Z$ and any morphism $\varphi : Z \to X$ whose image is contained in $Y$, there is a unique morphism $\psi : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[d, dotted, "\psi"'] \arrow[dr, "\varphi"] & \\ Y \arrow[r] & X \end{tikzcd} .$$

Proof. The uniqueness of $\psi$ is obvious. We only need to prove the existence. This is an immediate consequence of Proposition 3.25 and Proposition 4.3.

To be more precise, assume that $\varphi$ is given by a data as in Proposition 3.25, we only have to show that each $k_H$-affinoid domain $V$ in $X$, $V \cap Y$ is a $k_H$-affinoid domain in $Y$. This follows from Proposition 4.3. □

**Corollary 4.5.** Let $\varphi : X' \to X$ be a morphism of $k_H$-analytic spaces and $Y$ be a $k_H$-analytic domain in $X$. Then $X' \times_Y X$ in the category $k_H\text{-}\mathcal{A}\mathrm{n}$ exists and $\varphi^{-1}(Y)$ represents $X' \times_Y X$.

Proof. This follows from Proposition 4.4 and Proposition 4.3. $\square$

**Corollary 4.6.** Let $\operatorname{Sp} B$ be a $k_H$-affinoid space, then we have a functorial isomorphism

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \mathbb{A}^1_k) \xrightarrow{\sim} B.$$

Proof. As $\operatorname{Sp} B$ is compact as a topological space, its image in $\mathbb{A}^1_k$ is contained in $\operatorname{Sp} k\{r^{-1}T\}$ for some $r > 0$. By Proposition 4.4, we have natural bijections

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \mathbb{A}^1_k) \xrightarrow{\sim} \varinjlim_{r>0} \operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \operatorname{Sp} k\{r^{-1}T\}) \xrightarrow{\sim} \varinjlim_{r>0} \operatorname{Hom}_{k\text{-}\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}}(k\{r^{-1}T\}, B).$$

By **??** in **??**, the right-hand side is identified with $B$. $\square$

**Proposition 4.7.** Let $X$ be a $k_H$-analytic space, $Y$ be a $k_H$-analytic domain in $X$. For a subset $Z \subseteq Y$, the following are equivalent:

(1) $Z$ be a $k_H$-analytic domain in $X$;
(2) $Z$ is a $k_H$-analytic domain in $Y$.

Proof. (1) $\implies$ (2): Let $z \in Z$, we can find $n \in \mathbb{Z}_{>0}$ and $k_H$-affinoid domains $V_1, \dots, V_n$ in $X$ containing $x$ and contained in $Z$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of $z$ in $Z$. But observe that $V_1, \dots, V_n$ are $k_H$-affinoid domains in $Y$ as well, so we conclude.

(2) $\implies$ (1): This follows from the same argument. It suffices to observe that a $k_H$-affinoid domain in $Y$ is necessarily $k_H$-affinoid in $X$, as can be seen from Definition 3.19. $\square$

**Definition 4.8.** Let $X, Y$ be $k_H$-analytic spaces and $\varphi : Y \to X$ be a morphism. We say $\varphi$ is an *open immersion* if $\varphi(Y)$ is open in $X$ and $\varphi$ induces an isomorphism between $Y$ and $\varphi(Y)$ as $k_H$-analytic spaces.

By Example 4.2, $\varphi(Y)$ is a $k_H$-analytic domain in $X$ and by Proposition 4.4, we have a morphism of $k_H$-analytic spaces $Y \to \varphi(Y)$.

**Proposition 4.9.** Let $X$ be a $k_H$-analytic space and $Y$ be a $k_H$-analytic domain in $X$. Assume that $Y$ is a $k_H$-affinoid space, then $Y$ is a $k_H$-affinoid domain in $X$.

Proof. As $Y$ is a $k_H$-affinoid space, we know that $|Y|$ is compact. Take finitely many $k_H$-affinoid domains $V_1, \dots, V_n$ in $X$ such that

$$Y = V_1 \cup \cdots \cup V_n.$$

Then $V_1, \dots, V_n$ are $k_H$-affinoid domains in $Y$: let $\operatorname{Sp} D \to Y$ be a morphism of $k_H$-affinoid spectra, whose image lies in $V_i$ for some $i = 1, \dots, n$. Consider the following commutative diagram

$$\begin{tikzcd}
& \operatorname{Sp} D \arrow[dl, dotted] \arrow[d] \arrow[dr] & \\
V_i \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

By Proposition 4.4, there is a unique dotted morphism making the outer triangle commutative, hence making the whole diagram commutative. We have therefore shown that $V_i$ is a $k_H$-affinoid domain in $Y$.

So the covering $\{V_1, \ldots, V_n\}$ of $Y$ satisfies the assumptions in Definition 3.19 and $Y$ is $k_H$-affinoid. $\square$

**Proposition 4.10.** Let $X$ be a $k_H$-analytic space and $\{Y_i\}_{i\in I}$ be a family of $k_H$-analytic domains in $X$ which forms a quasi-net on $X$. Then for any $k_H$-analytic space $X'$, the following sequence is exact

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(X, X') \to \prod_{i\in I} \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(Y_i, X') \rightrightarrows \prod_{i,j\in I} \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(Y_i \cap Y_j, X').$$

Proof. Let $\{\varphi_i : Y_i \to X'\}_{i\in I}$ be a family of morphisms such that $\varphi_i$, $\varphi_j$ coincides on $Y_i \cap Y_j$ for $i, j \in I$. We need to glue the $\varphi_i$'s into a single morphism $\varphi : X \to X'$. Clearly, the underlying maps glue together to a continuous map $\varphi : X \to X'$ by **??** in **??**.

Let $\tau$ be the collection of $k_H$-affinoid domains $V$ in $X$ such that there is $i \in I$ and a $k_H$-affinoid domain $V' \subseteq X'$ with $V \subseteq Y_i$ and $\varphi_i(V) \subseteq V'$. Then $\tau$ is a net on $X$, and we have a morphism $X \to X'$. $\square$

## 5. Berkovich site

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Lemma 5.1.** Let $X$ be a $k_H$-analytic space. Consider the category $\mathcal{C}$ of $k_H$-analytic domains in $X$, where the morphisms are inclusions of $k_H$-analytic domains. For each $Y \in \mathcal{C}$, consider the set of coverings $\mathrm{Cov}(Y)$ consisting of all $\{Y_i \to Y\}_{i\in I}$ such that $Y_i$ is a $k_H$-analytic domain in $Y$ and $\{Y_i\}_{i\in I}$ is a quasi-net on $Y$. The class of coverings $\{\mathrm{Cov}(Y)\}_Y$ defines a Grothendieck pretopology.

Proof. It suffices to verify the axioms in [Stacks, Tag 03NH].

(1) An isomorphism $Y' \to Y$ in $\mathcal{C}$ is in $\mathrm{Cov}(Y)$.

This is trivial as an isomorphism in $\mathcal{C}$ is necessarily identity.

(2) If $\{Y_i \to Y\}_{i\in I}$ and $\{Y_{ij} \to Y_i\}_{j\in J_i}$ for all $i \in I$ are in $\mathrm{Cov}(Y)$ and $\mathrm{Cov}(Y_i)$ respectively, then $\{Y_{ij} \to Y\}_{i,j}$ is in $\mathrm{Cov}(Y)$.

By Proposition 4.7, $Y_{ij}$ is a $k_H$-analytic domain in $Y$ for any $i \in I$, $j \in I_j$. It suffices to show that $\{Y_{ij}\}_{i,j}$ is a quasi-net on $Y$. Let $y \in Y$, we can find finitely many elements among $\{Y_i\}_{i\in I}$, say $Y_1, \ldots, Y_n$ so that $y \in Y_i$ for each $i = 1, \ldots, n$ and $Y_1 \cup \cdots \cup Y_n$ is a neighbourhood of $y$ in $Y$. Similarly, for each $i = 1, \ldots, n$, we can find finitely many $Y_{i1}, \ldots, Y_{ij_i}$ among $\{Y_{ij}\}_{j\in J_i}$ so that $y$ is contained in each of them and $Y_{i1} \cup \cdots \cup Y_{ij_i}$ is a neighbourhood of $y$ in $Y_i$. Then each element in $\{Y_{ij}\}_{i=1,\ldots,n;j=1,\ldots,j_i}$ contains $y$ and the union is a neighbourhood of $y$ in $Y$.

(3) If $\{Y_i \to Y\}_{i\in I}$ lies in $\mathrm{Cov}(Y)$ and $Z \to Y$ is a $k_H$-analytic domain in $Y$, then the fiber products $Y_i \times_Y Z$ exist and $\{Y_i \times_Y Z \to Z\}_{i\in I}$ lies in $\mathrm{Cov}(Z)$.

By Corollary 4.5, $Y_i \times_Y Z$ exists and is represented by the inverse image of $Z$ in $Y_i$, which is a $k_H$-analytic domain in $Y_i$ by Proposition 4.3. It is clear that $\{Y_i \times_Y Z\}_{i\in I}$ is a quasi-net on $Z$. $\square$

**Definition 5.2.** Let $X$ be a $k_H$-analytic space. We will write the site constructed in Lemma 5.1 as $X$ and call it the *Berkovich site* of $X$. The corresponding Grothendieck topology is called the *Berkovich Grothencieck topology.* The topos $\mathrm{Sh}(X)$ associated with $X$ is called the *Berkovich topos* of $X$.

Observe that the Berkovich Grothendieck topology is subcanonical by Proposition 4.10.

**Definition 5.3.** Let $X$ be a $k_H$-analytic space. We define a sheaf of rings $\mathcal{O}_X$ on $X$ as follows: let $Y$ be a $k_H$-analtic domain in $X$, we set

$$\mathcal{O}_X(Y) = \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(X, \mathbb{A}^1_k).$$

By Corollary 4.6 and Proposition 4.10, $\mathcal{O}_X$ defines a sheaf of rings. We call $\mathcal{O}_X$ the structure sheaf of $X$. The corresponding ringed site $(X, \mathcal{O}_X)$ is called the *Berkovich ringed site.* The induced ringed topos $(\mathrm{Sh}(X), \mathcal{O}_X)$ is called the *Berkovich ringed topos.*

Given any morphism $f : Y \to X$ of $k_H$-analytic spaces, we have an induced morphism of the corresponding ringed sites, still denoted by $\varphi$.

**Definition 5.4.** Let $X$ be a $k_H$-analytic space. An $\mathcal{O}_X$-module $\mathcal{M}$ is *coherent* if there is an admissible covering $\{Y_i\}_{i\in I}$ of $X$ such that $\mathcal{M}|_{Y_i}$ is isomorphic to the cokernel of a homomorphism of finite free $\mathcal{O}_{V_i}$-modules.

**Example 5.5.** Let $A$ be a $k_H$-affinoid algebra and $M$ be a fintie $A$-module. Then $M$ induces a coherent sheaf of $\mathcal{O}_{\mathrm{Sp}\,A}$-modules $\tilde{M}$ as follows:

$$\tilde{M}(V) = M \otimes_A A_V.$$

Conversely, we can reformulate Kiehl's theorem.

**Theorem 5.6.** Let $A$ be a $k_H$-affinoid algebra and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_{\mathrm{Sp}\,A}$-modules. Set $M = H^0(X, \mathcal{M})$, then $M$ is a finite $A$-modue and we have a canonical isomorphism

$$\tilde{M} \xrightarrow{\sim} \mathcal{M}.$$

The left-hand side is defined in Example 5.5.

PROOF. This is just a reformulation of **??** in **??**. □

**Corollary 5.7.** Let $\varphi : \mathrm{Sp}\,B \to \mathrm{Sp}\,A$ be a morphism of $k_H$-affinoid spaces. Then the following are equivalent:

(1) $\varphi_*\mathcal{O}_{\mathrm{Sp}\,B}$ is a coherent $\mathcal{O}_{\mathrm{Sp}\,A}$-module;
(2) $B$ is a finite Banach $A$-module.

PROOF. Observe that for any $k_H$-affinoid domain $\mathrm{Sp}\,C$ in $\mathrm{Sp}\,A$,

$$\varphi_*\mathcal{O}_{\mathrm{Sp}\,B}(\mathrm{Sp}\,C) = \mathcal{O}_{\mathrm{Sp}\,B}(\varphi^{-1}(\mathrm{Sp}\,C)) = \mathcal{O}_{\mathrm{Sp}\,B}(\mathrm{Sp}\,C\hat{\otimes}_A B) = C\hat{\otimes}_A B \xrightarrow{\sim} C \otimes_A B.$$

Here we applied **??** in **??** and **??** in **??**. So $\varphi_*\mathcal{O}_{\mathrm{Sp}\,B} \cong \widetilde{B}$.

From this (2) trivially implies (1).

Conversely, assume (1), let $B = H^0(\mathrm{Sp}\,A, \varphi_*\mathcal{O}_{\mathrm{Sp}\,B})$. By Theorem 5.6, $B$ is a finite $A$-module. Let $B'$ denote the ring $B$ endowed with the finite Banach $A$-algebra structure as in **??** in **??**. We need to show that the identity map $B' \to B$ is admissible. Observe that the identity map is bounded by **??** in **??**. By **??** in **??**, it suffices to show that the induced map $\mathrm{Sp}\,B \to \mathrm{Sp}\,B'$ is surjective. Let $\varphi' : \mathrm{Sp}\,B' \to \mathrm{Sp}\,A$ be the natural morphism of $k_H$-affinoid spaces. Then

$$\varphi_*(\mathcal{O}_{\mathrm{Sp}\,B}) \xrightarrow{\sim} \varphi'_*(\mathcal{O}_{\mathrm{Sp}\,B'}).$$

It follows that $\varphi^{-1}(x) = \varphi'^{-1}(x)$ for any $x \in \mathrm{Sp}\,A$. We conclude. □

**Corollary 5.8.** Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of $k_H$-affinoid spaces. Then the following are equivalent:

(1) $\varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is a coherent $\mathcal{O}_{\operatorname{Sp} A}$-module and $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective;
(2) $A \to B$ is an admissible epimorphism.

Proof. Assume (2). By Corollary 5.7, $\varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is a coherent $\mathcal{O}_{\operatorname{Sp} A}$-module. To see that $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective, it suffices to show that for each $k_H$-affinoid space $\operatorname{Sp} C$ in $\operatorname{Sp} A$,

$$C \to C \otimes_A B$$

is surjective. This follows from the assumption.

Assume (1). We know that $B$ is a finite Banach $A$-module. In particular, $A \to B$ is admissible by **??** in **??**. As $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective, by Theorem 5.6, $A \to B$ is surjective. Include details □

**Definition 5.9.** Let $\operatorname{Sp} A$ be a $k_H$-affinoid space and $\mathcal{M} = \tilde{M}$ is a coherent sheaf of $\mathcal{O}_X$-modules on $X$, where $M$ is a finite $A$-module. The *support* $\operatorname{Supp} M$ of $\mathcal{M}$ is the closed subset $\operatorname{Sp} A/\operatorname{Ann}_A(M)$ of $\operatorname{Sp} A$.

Let $X$ be a $k_H$-analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the *support* $\operatorname{Supp} \mathcal{M}$ of $\mathcal{M}$ is a subset of $X$ such that a point $x \in X$ lies in $\operatorname{Supp} \mathcal{M}$ if and only if for some $k_H$-affinoid domain $V$ in $X$ containing $x$, $x \in \operatorname{Supp} \mathcal{M}|_V$.

Here $\operatorname{Ann}_A(M)$ is the annihilator of $M$ in $A$.

**Lemma 5.10.** Let $X$ be a $k_H$-analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Take $x \in \operatorname{Supp} \mathcal{M}|_V$ and a $k_H$-affinoid domain $V$ in $X$ containing $x$. Then $x \in \operatorname{Supp} \mathcal{M}|_V$.

Proof. By assumption, there is a $k_H$-affinoid domain $U$ in $X$ containing $x$ such that $x \in \operatorname{Supp} \mathcal{M}|_U$.

Let $W \subseteq U \cap V$ be a $k_H$-affinoid domain in $X$ containing $x$. We claim that $x \in \operatorname{Supp} \mathcal{M}|_W$. Let $M = H^0(U, \mathcal{M})$, then $M \otimes_{A_U} A_W = H^0(W, \mathcal{M})$. By [Stacks, Tag 07T8] and **??** in **??**,

$$\operatorname{Ann}_{A_U}(M) \otimes_{A_U} A_W = \operatorname{Ann}_{A_W}(M \otimes_{A_U} A_W)$$

and $\operatorname{Supp}(\mathcal{M}|_W) = \operatorname{Supp}(\mathcal{M}|_U) \cap W$. The claim follows. We may assume that $U \subseteq V$. In this case, the same argument shows that $x \in \operatorname{Supp} \mathcal{M}|_V$. □

## 6. Closed immersion

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Lemma 6.1.** Let $\varphi : Y \to X$ be a morphism of $k_H$-analytic spaces. Then the following are equivalent:

(1) for any $x \in X$, there are $n \in \mathbb{Z}_{>0}$ and $k_H$-affinoid domains $V_1, \dots, V_n$ in $X$ containing $x$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of $x$ in $X$ and the restriction $\varphi^{-1}(V_i) \to V_i$ is a closed immersion for any $i = 1, \dots, n$;
(2) for any $k_H$-affinoid domain $V$ in $X$, $\varphi^{-1}(V) \to V$ is a closed immersion.

Recall that closed immersions between $k_H$-affinoid spaces are defined in **??** in **??**.

The statement in [Ber93, Lemma 1.3.7] is not correct.

PROOF. Only (1) $\implies$ (2) is non-trivial. Assume (1). Let $\tau$ be the collections of $V \subseteq X$ satisfying (2). Then we claim that $\tau$ is a net.

Observe that $\tau$ is a quasi-net by our assumption. To see that it is a net, take $U, V \in \tau$ and $x \in U \cap V$, then we can find $n \in \mathbb{Z}_{>0}$ and $k_H$-affinoid domains $W_1, \ldots, W_n$ in $U \cap V$ containing $x$ such that $W_1 \cup \cdots \cup W_n$ is a neighbourhood of $x$ in $U \cap V$. In order to show that $\tau|_{U \cap V}$ is a quasi-net, it suffices to show that $\varphi^{-1}(W_i) \to W_i$ is a closed immersion for $i = 1, \ldots, n$. This follows from **??** in **??**.

Let $V$ be a $k_H$-affinoid domain in $X$. By (1) and the compactness of $V$, we can find $n \in \mathbb{Z}_{>0}$ and $V_1, \ldots, V_n \in \tau$ such that $V \subseteq V_1 \cup \cdots \cup V_n$. By **??** in **??**, we can find $m \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_m \in \tau$ such that

$$V = U_1 \cup \cdots \cup U_m$$

and each $U_j$ is contained in some $V_i$, where $j = 1, \ldots, m$ and $i = 1, \ldots, n$. By **??** in **??** again, $U_j \in \tau$ for each $j = 1, \ldots, m$. It suffices to apply Corollary 5.8 to conclude that $V \in \tau$. $\square$

**Definition 6.2.** Let $\varphi : Y \to X$ be a morphism of $k_H$-analytic spaces. We say $\varphi$ is a *closed immersion* if the equivalent conditions in Lemma 6.1 are satisfied.

Observe that this definition extends **??** in **??**. Moreover, Lemma 6.1 shows that being a closed immersion is a G-local property on the target.

Include the proof of the existence of finite limits

**Proposition 6.3.** Let $\varphi : Y \to X$, $\psi : Z \to X$ be a morphism of $k_H$-analytic spaces. Assume that $\varphi : Y \to X$ is a closed immersion. Consider the Cartesian diagram

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\varphi"] \\ Z \arrow[r, "\psi"] & X \end{tikzcd}.$$

Then $Z \times_X Y \to Z$ is a closed immersion.

PROOF. Taking a $G$-covering of $Z$, we may assume that $Z$ is compact. We could cover the images of $Z$ in $X$ by finitely many $k_H$-affinoid domains $V_1, \ldots, V_n$ in $X$, considering their preimages in $Z$, we could reduce to the case where the image of $Z$ in $X$ is contained in a $k_H$-affinoid domain. We could then assume that $X$ is a $k_H$-affinoid space and hence so is $Y$. By taking a $G$-covering of $Z$ again, we may assume that $Z$ is affinoid. It suffices to apply **??** in **??**. $\square$

## 7. Reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H$ be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

**Definition 7.1.** A *punctured $k_H$-analytic space* $(X, x)$ is a $k_H$-analytic space $X$ together with a point $x \in X$.

A morphism between punctured $k_H$-analytic spaces $(X, x)$ and $(Y, y)$ is a morphism $\varphi : X \to Y$ of $k_H$-analytic spaces sending $x$ to $y$.

The category of punctured $k_H$-analytic spaces is denoted by $k_H$-$\mathcal{A}\mathrm{n}_*$.

**Definition 7.2.** The category $k_H$-$\mathcal{G}\mathrm{er}$ is the category of right fractions of $k_H$-$\mathcal{A}\mathrm{n}_*$ with respect to the system of morphisms

$$\varphi : (X, x) \to (Y, y)$$

that induces an isomorphism of $X$ with an open neighbourhood of $y$ in $Y$ in the category of $k_H$-$\mathcal{G}$er.

When we view $(X, x)$ as an object in $k_H$-$\mathcal{G}$er, we write it as $X_x$. An object in $k_H$-$\mathcal{G}$er is called a $k_H$-analytic germ.

By definition,

$$\mathrm{Hom}_{k_H\text{-}\mathcal{G}\mathrm{er}}(X_x, Y_y) = \varinjlim_{U} \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}_*}((U, x), (Y, y)),$$

where $U$ runs over all open neighbourhoods of $x$ in $X$.

**Definition 7.3.** A $k_H$-analytic germ $X_x$ is *good* if $x$ admits an affinoid neighbourhood in $X$.

Note that this condition does not depend on the representative $(X, x)$. To see this, let $U \subseteq x$ be an open subset containing $x$. We need to show that if $x$ admits a $k_H$-affinoid neighbourhood in $X$, then it admits one in $U$. This follows from **??** in **??**.

**Definition 7.4.** A morphism of $k_H$-analytic germs $\varphi : X_x \to Y_y$ is saied to be *separated* (resp. *closed*) if it is induced

**Theorem 7.5.** Let $H' \supseteq H$ be a subgroup of $\mathbb{R}_{>0}$. The natural functor

$$k_H\text{-}\mathcal{A}\mathrm{n} \to k_{H'}\text{-}\mathcal{A}\mathrm{n}$$

is fully faithful.

# Bibliography


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