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# Morphisms between complex analytic spaces

## 1. Introduction

## 2. Open morphism

**Definition 2.1.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces and  $x \in X$ . We say  $f$  is *open* at  $x \in X$  if for any neighbourhood  $U$  of  $x$  in  $X$ ,  $f(U)$  is a neighbourhood of  $f(x)$  in  $Y$ .

**Proposition 2.2.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces. Assume that  $f$  is open at  $x \in X$ , then the kernel of  $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$  is nilpotent.

The converse fails.

PROOF. Let  $g_{f(x)} \in \mathcal{O}_{Y,f(x)}$  be an element in the kernel of  $f_x^\#$ . Up to shrinking  $Y$ , we may spread  $g_{f(x)}$  to  $g \in \mathcal{O}_Y(Y)$ . Then  $f^*g$  vanishes in a neighbourhood of  $x$  in  $X$ . As  $f$  is open at  $x$ ,  $g$  vanishes in the neighbourhood  $f(U)$  of  $f(x)$ . By [Corollary 3.18](#) in [Constructions of complex analytic spaces](#),  $g_{f(x)}$  is nilpotent.  $\square$

## 3. Quasi-finite morphisms

**Definition 3.1.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces. We say  $f$  is *quasi-finite* at  $x \in X$  if  $x$  is isolated in  $f^{-1}(f(x))$ . We say  $f$  is *quasi-finite* if  $f$  is quasi-finite at all  $x \in X$ .

This definition is purely topological. We will show that it is equivalent to an analytic definition.

**Proposition 3.2.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces and  $x \in X$ . Then the following are equivalent:

- (1)  $f$  is quasi-finite at  $x \in X$ ;
- (2)  $\mathcal{O}_{X,x}$  is quasi-finite over  $\mathcal{O}_{Y,f(x)}$ ;
- (3)  $\mathcal{O}_{X,x}$  is finite over  $\mathcal{O}_{Y,f(x)}$ .

PROOF. (1)  $\Leftrightarrow$  (2): By [Corollary 3.16](#) in [Constructions of complex analytic spaces](#),  $f$  is quasi-finite at  $x \in X$  if and only if  $\mathcal{O}_{X_{f(x)},x} = \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$  is artinian. In other words,  $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$  is finite-dimensional over  $\mathbb{C}$ . The latter is equivalent to that  $\mathcal{O}_{X,x}$  is quasi-finite over  $\mathcal{O}_{Y,f(x)}$ .

(2)  $\Leftrightarrow$  (3): This follows from [Theorem 5.4](#) in [Complex analytic local algebras](#).  $\square$

## 4. Finite morphisms

**Definition 4.1.** A morphism of complex analytic spaces  $f : X \rightarrow Y$  is *finite* if its underlying map of topological spaces is topologically finite.

We say a morphism of complex analytic spaces  $f : X \rightarrow Y$  is *finite at*  $x \in X$  if there is an open neighbourhood  $U$  of  $x$  in  $X$  and  $V$  of  $f(x)$  in  $Y$  such that  $f(U) \subseteq V$  and the restriction  $U \rightarrow V$  of  $f$  is finite.

Let  $S$  be a complex analytic space. A *finite analytic space over*  $S$  is a finite morphism  $f : X \rightarrow S$  of complex analytic spaces. A morphism between finite analytic spaces over  $S$  is a morphism of complex analytic spaces over  $S$ .

**Proposition 4.2.** Let  $n \in \mathbb{N}$  and  $D$  be an open neighbourhood of 0 in  $\mathbb{C}^n$ . Let  $X$  be a closed subspace of  $D$  which intersects  $\{(0, \dots, 0)\} \times \mathbb{C}$  at and only at 0. Then there is a connected open product neighbourhood  $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$  of 0 in  $D$  such that the projection  $B \times W \rightarrow B$  induces a finite morphism  $h : X' \rightarrow B$  with  $X' = X \cap (B \times W)$ .

PROOF. We will denote the coordinates on  $\mathbb{C}^{n-1} \times \mathbb{C}$  as  $(z, w)$ .

Let  $\mathcal{I}$  be the ideal of  $X$  in  $D$ . By our assumption, we can choose  $f_0 \in \mathcal{I}_0$  such that  $\deg_w f_0 < \infty$  and  $f_0(0) = 0$ . By [Theorem 4.3 in Complex analytic local algebras](#), we can find a Weierstrass polynomial  $\omega_0 = w^b + a_1 w^{b-1} + \dots + a_b \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[w]$  such that  $f_0 = e\omega_0$  for some unit  $e$  in  $\mathbb{C}\{z_1, \dots, z_n\}$ . We choose a product neighbourhood  $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$  of 0 in  $D$  such that  $\omega_0$  can be represented by  $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)[w]$  with  $\omega|_{B \times W} \in \mathcal{I}(B \times W)$ . Let  $\pi : A \rightarrow B$  be the Weierstrass map defined by  $\omega$ . Then  $\pi$  is finite by [Theorem 6.2 in The notion of complex analytic spaces](#). Up to shrinking  $B$  and  $W$ , we may assume that  $A \cap (B \times W) \rightarrow B$  is finite as well. Set  $X' := X \cap (B \times W)$ . The restriction  $h : X' \rightarrow B$  of  $\pi$  is then finite.  $\square$

**Corollary 4.3.** Let  $n, k \in \mathbb{N}$  and  $D$  be an open neighbourhood of 0 in  $\mathbb{C}^n$ . Let  $X$  be a closed subspace of  $D$  which intersects  $\{(0, \dots, 0)\} \times \mathbb{C}^k$  at and only at 0. Then there is a connected open product neighbourhood  $B \times W \subseteq \mathbb{C}^{n-k} \times \mathbb{C}^k$  of 0 in  $D$  such that the projection  $B \times W \rightarrow B$  induces a finite morphism  $h : X' \rightarrow B$  with  $X' = X \cap (B \times W)$ .

PROOF. This follows from a repeated application of [Proposition 4.2](#).  $\square$

**Proposition 4.4.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces and  $x \in X$ . Then the following are equivalent:

- (1)  $f$  is quasi-finite at  $x$ ;
- (2)  $f$  is finite at  $x$ .

PROOF. (2)  $\implies$  (1): This follows from [Proposition 4.5 in Topology and bornology](#).

(1)  $\implies$  (2): Write  $y = f(x)$ . The assertion is local on both  $X$  and  $Y$ . So we may assume that  $U$  and  $V$  are complex model spaces in domains  $W \subseteq \mathbb{C}^k$  and  $B \subseteq \mathbb{C}^d$  respectively with  $x = 0$  and  $y = 0$ . Moreover, we may assume that  $\{x\} = f'^{-1}(y)$ . We have the following commutative diagram:

$$\begin{array}{ccc}
 & U \times V & \hookrightarrow W \times B \\
 & \uparrow & \downarrow \\
 U & \xrightarrow{\Gamma_f} & V \\
 & \uparrow f' & \downarrow \\
 & & B
 \end{array}
 ,$$

where  $\Gamma_{f'}$  denotes the graph of  $f' : U \rightarrow V$ . As  $\{x\} = f'^{-1}(y)$ , we have  $\mathbb{C}^k \times \{0\}$  intersects  $\Gamma_f$  only at the origin. By [Corollary 4.3](#), up to shrinking  $W$  and  $B$ , we may guarantee that the projection  $W \times B \rightarrow B$  induces a finite morphism  $\Gamma_f \rightarrow B$  and the pushforward under this map preserves coherence. Observe that  $U \rightarrow \Gamma_f$  is a biholomorphism, we conclude that  $f'$  is finite.  $\square$

**Corollary 4.5.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces. The following are equivalent:

- (1)  $f$  is finite;
- (2)  $f$  is quasi-finite and proper.

PROOF. (1)  $\implies$  (2): This follows from [Proposition 4.4](#).

(2)  $\implies$  (1): This follows from [Proposition 4.5](#) in [Topology and bornology](#).  $\square$

**Corollary 4.6.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces. Then the set

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is open.

PROOF. This follows from [Proposition 4.4](#).  $\square$

**Theorem 4.7.** Let  $S$  be a complex analytic space. Then the functor  $\mathrm{Spec}_S^{\mathrm{an}}$  defines an anti-equivalence from the category of finite  $\mathcal{O}_S$ -algebras to the category of finite analytic spaces over  $S$ .

PROOF. We first observe that the functor is well-defined. This follows from [Corollary 3.8](#) in [Constructions of complex analytic spaces](#).

The functor is fully faithful by [Proposition 2.10](#) in [Constructions of complex analytic spaces](#). Suppose that  $f : X \rightarrow S$  is a finite morphism of complex analytic spaces. We need to show that  $X$  is isomorphic to  $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$  for some finite  $\mathcal{O}_S$ -algebra  $\mathcal{A}$  in  $\mathbb{C}\text{-An}_S$ .

By [Proposition 2.8](#) in [Constructions of complex analytic spaces](#), we necessarily have  $\mathcal{A} \cong f_* \mathcal{O}_X$ . So we need to show that the natural morphism  $\mathrm{Spec}_S^{\mathrm{an}} f_* \mathcal{O}_X \rightarrow X$  over  $S$  is an isomorphism. The problem is local on  $S$ .

Fix  $s \in S$ . Write  $x_1, \dots, x_n$  for the distinct points in  $f^{-1}(s)$ . Up to shrinking  $S$ , we may assume that  $X$  is the disjoint union of  $V_1, \dots, V_n$ , where  $V_i$  is an open neighbourhood of  $x_i$  in  $X$ . We need to show that  $X$  has the form  $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}$  for some  $\mathcal{O}_S$ -algebra  $\mathcal{B}$  in  $\mathbb{C}\text{-An}_S$ .

It suffices to handle each  $V_i$  separately, so we may assume that  $f^{-1}(s) = \{x\}$  consists of a single point. Then  $\mathcal{O}_{X,x}$  is finite over  $\mathcal{O}_{S,s}$  by [Proposition 3.2](#). Up to shrinking  $S$ , we may assume that  $\mathcal{O}_{X,x}$  spreads out to a finite  $\mathcal{O}_S$ -algebra  $\mathcal{B}$ . Let  $X' = \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}$ . There is a unique point  $x'$  of  $X'$  over  $s$  and  $X'_{x'}$  is isomorphic to  $X_x$  over  $S_s$ . By [Lemma 4.2](#) in [Topology and bornology](#), up to shrinking  $S$ , we may assume that  $X$  is isomorphic to  $X'$  over  $S$ . We conclude.  $\square$

**Corollary 4.8.** Let  $f : X \rightarrow Y$  be a finite morphism of complex analytic spaces and  $\mathcal{M}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules, then  $f_* \mathcal{M}$  is coherent. Moreover,  $f_*$  is exact from  $\mathrm{Coh}(\mathcal{O}_X)$  to  $\mathrm{Coh}(\mathcal{O}_Y)$ .

PROOF. This follows from [Corollary 2.9](#) in [Constructions of complex analytic spaces](#) and [Theorem 4.7](#).  $\square$

**Corollary 4.9.** Let  $X$  be a reduced complex analytic space. Then

- (1)  $\bar{X}$  is normal;
- (2)  $p : \bar{X} \rightarrow X$  is finite and surjective;
- (3) There is a nowhere dense analytic set  $Y$  in  $X$  such that  $p^{-1}(Y)$  is nowhere dense in  $\bar{X}$  and the morphism  $\bar{X} \setminus p^{-1}(Y) \rightarrow X \setminus Y$  induced by  $p$  is an isomorphism.

Conversely, these conditions determines  $\bar{X}$  up to a unique isomorphism in  $\mathbb{C}\text{-An}/X$ .

PROOF. These properties are established in [Proposition 7.8](#) in [Local properties of complex analytic spaces](#). We need to prove the uniqueness.

Let  $p : X' \rightarrow X$  be a morphism satisfying the three conditions. We need to show that  $X'$  is canonically isomorphic to  $\bar{X}$  in  $\mathbb{C}\text{-An}/X$ . By (2) and [Theorem 4.7](#), it suffices to show that  $p_*\mathcal{O}_{X'}$  is canonically isomorphic to  $\bar{\mathcal{O}}_X$ . By (1), and the universal property of normalization, there is a canonical morphism

$$p_*\mathcal{O}_{X'} \rightarrow \bar{\mathcal{O}}_X$$

of  $\mathcal{O}_X$ -algebras. We will show that this map is an isomorphism.

The problem is local. Let  $x \in X$ . By (3) and [Corollary 3.14](#) in [Constructions of complex analytic spaces](#), up to shrinking  $X$ , we can find  $f \in \mathcal{O}_X(X)$  such that  $f(y) = 0$  for all  $y \in Y$  and  $f_x$  is a non-zero divisor in  $(p_*\mathcal{O}_{X'})_x$ . Up to shrinking  $X$ , we may assume that  $f_y$  is a non-zero divisor in  $(p_*\mathcal{O}_{X'})_y$  for all  $y \in X$ . By (3), we have

$$\mathcal{O}_X|_{X \setminus Y} \rightarrow (p_*\mathcal{O}_{X'})|_{X \setminus Y}$$

is an isomorphism. It follows that

$$fp_*\mathcal{O}_{X'} \rightarrow \mathcal{O}_X$$

is injective. We then have an injective homomorphism:

$$p_*\mathcal{O}_{X'} \rightarrow \mathcal{O}_X \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each  $y \in X$ , we deduce that  $(p_*\mathcal{O}_{X'})_y$  is in the total ring of fraction of  $\mathcal{O}_{X,y}[f_y^{-1}]$ . But  $(p_*\mathcal{O}_{X'})_y$  is finite and integral over  $\mathcal{O}_{X,y}$ , so is isomorphic to  $\bar{\mathcal{O}}_{X,y}$  as  $\mathcal{O}_{Y,y}$ -algebras.  $\square$

**Corollary 4.10.** Let  $f : X \rightarrow Y$  be a finite morphism of complex analytic spaces. Assume that  $x \in X$  is a point such that  $(f_*\mathcal{O}_X)_{f(x)}$  is torsion-free as an  $\mathcal{O}_{Y,f(x)}$ -module and  $Y$  is integral at  $f(x)$ . Then  $f$  is open at  $x$ .

PROOF. If not, we can choose open neighbourhoods  $U$  of  $x$  in  $X$  and  $V$  of  $y := f(x)$  in  $Y$  such that  $f(U) \subseteq V$  such that the induced morphism  $g : U \rightarrow V$  is finite and  $f(U)$  is not a neighbourhood of  $y$  in  $Y$ . Up to shrinking  $Y$ , we can find  $h \in \mathcal{O}_Y(Y)$  such that  $h_y \neq 0$  while  $h$  vanishes on  $f(U)$ . Observe that  $f(U)$  is an analytic set in  $Y$  by [Corollary 4.8](#). It follows from [Corollary 3.18](#) in [Constructions of complex analytic spaces](#) that there is  $t \in \mathbb{Z}_{>0}$  such that

$$h_y^t(g_*\mathcal{O}_U)_y = 0.$$

As  $\mathcal{O}_{Y,y}$  is integral, this implies that  $(g_*\mathcal{O}_U)_y$  is torsion as an  $\mathcal{O}_{Y,f(x)}$ -module. This is a contradiction, as  $(f_*\mathcal{O}_X)_y$  as an  $\mathcal{O}_{Y,f(x)}$ -module is torsion-free by assumption.  $\square$

**Lemma 4.11.** Let  $X$  be an integral complex analytic space and  $\mathcal{M}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules. Then

$$\{x \in X : \mathcal{M} \text{ is torsion-free at } x\}$$



is co-analytic in  $X$ .

PROOF. It suffices to show that  $\text{Supp } \mathcal{T}(\mathcal{M})$  is an analytic set in  $X$ . As  $X$  is integral,  $\mathcal{T}(\mathcal{M})$  is just the kernel of the morphism  $\mathcal{M} \rightarrow \mathcal{M}^{\vee\vee}$ .  $\square$

**Corollary 4.12.** Let  $f : X \rightarrow Y$  be a finite morphism of complex analytic spaces. Assume that  $Y$  is integral. Let  $x \in X$  be a point such that  $X$  is integral at  $x$  and  $f$  is open at  $x$ , then there is an open neighbourhood  $U$  of  $x$  in  $X$  such that  $f|_U : U \rightarrow Y$  is open.

PROOF. Let  $y = f(x)$ . The problem is local on  $Y$ . By [Proposition 4.4](#), we may assume that  $\{x\} = f^{-1}(y)$ . By [Corollary 4.8](#),  $f_*\mathcal{O}_X$  is coherent. By [Lemma 4.11](#), it suffices to show that it is torsion-free.

Observe that  $(f_*\mathcal{O}_X)_y \xrightarrow{\sim} \mathcal{O}_{X,x}$ . By [Proposition 2.2](#),  $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$  is injective. As  $\mathcal{O}_{X,x}$  is integral by our assumption, we conclude.  $\square$

**Lemma 4.13.** Let  $f : X \rightarrow Y$  be a finite morphism of reduced complex analytic spaces and  $x \in X$ . Assume that  $x \in X$ , then there is a non-zero divisor  $h \in \mathfrak{m}_{f(x)}$  such that  $f_x^\#(h)$  is a non-zero divisor in  $\mathcal{O}_{X,x}$ .

PROOF. By [Proposition 4.4](#), the problem is local on  $X$ . We may assume that  $X$  can be decomposed into irreducible components at  $x$ :

$$X = A_1 \cup \dots \cup A_s.$$

By [Corollary 4.8](#),  $B_j := f(A_j)$  is an analytic set in  $Y$  for  $j = 1, \dots, s$ . By our assumption,  $x$  is not an isolated point in  $A_j$ , so  $y$  is not an isolated point in  $B_j$  for  $j = 1, \dots, s$ . Take a non-zero divisor  $h \in \mathfrak{m}_{Y,y}$ . Up to shrinking  $Y$ , we may assume that  $h$  spreads to  $g \in \mathcal{O}_Y(Y)$ . Observe that  $W(f^*g) \cap A_j$  is not a neighbourhood of  $x$  in  $A_j$  for all  $j = 1, \dots, s$ . So  $f_x^\#h$  is not a zero divisor.  $\square$

**Theorem 4.14.** Let  $f : X \rightarrow Y$  be a finite morphism of complex analytic spaces and  $y \in Y$ . Then

$$\dim_y f(X) = \max_{x \in f^{-1}(y)} \dim_x X.$$

The left-hand side makes sense because  $f(X)$  is an analytic set in  $Y$  by [Corollary 4.8](#).

PROOF. We may assume that  $X$  and  $Y$  are reduced and  $f(X) = Y$ .

**Step 1.** We reduce to the case where  $f^{-1}(y) = \{x\}$  for some  $x \in X$ .

Let  $x_1, \dots, x_t$  be the distinct points in  $f^{-1}(y)$ . The problem is local on  $Y$ . By [Theorem 4.8](#) in [Topology and bornology](#) and [Proposition 4.4](#), up to shrinking  $Y$ , we may assume that  $X$  is the disjoint union of open neighbourhoods  $U_1, \dots, U_t$  of  $x_1, \dots, x_t$  and  $U_j \rightarrow Y$  is finite for each  $j = 1, \dots, t$ . It suffices to apply the special case to each  $U_j \rightarrow Y$  for  $j = 1, \dots, t$ .

**Step 2.** We prove the theorem after the reduction in Step 1.

We make an induction on  $d := \dim_x X$ . There is nothing to prove when  $d = 0$ . Assume that  $d \geq 1$ . By [Lemma 4.13](#), we can choose a non-zero divisor  $g_y \in \mathfrak{m}_{Y,y}$  such that  $f_x^\#(g_y)$  is a non-zero divisor in  $\mathcal{O}_{X,x}$ . Up to shrinking  $Y$ , we may assume that  $g$  spreads to  $g \in \mathcal{O}_Y(Y)$ . It suffices to apply our inductive hypothesis to  $W(f_x^\#(g_y)) \subseteq W(g_y)$ .  $\square$

**Corollary 4.15.** Let  $f : X \rightarrow Y$  be a finite open surjective morphism of complex analytic spaces. Assume that  $A$  is a thin subset of  $X$  of order  $k \in \mathbb{Z}_{>0}$ , then  $f(A)$  is a thin subset of  $Y$  of order  $k$ .

PROOF. We may assume that  $X$  and  $Y$  are reduced. By [Proposition 4.4](#) and the fact that  $f$  is open, the problem is local on  $X$ , we may assume that  $A$  is an analytic subset of  $X$ . Let  $x \in A$ . It suffices to handle the case where  $A$  is irreducible at  $x$  and  $x$  is the only point in  $f^{-1}(f(x))$ . By [Corollary 4.8](#),  $f(A)$  is an irreducible analytic subset of  $Y$ .

We may assume that  $Y$  is irreducible at  $y := f(x)$ . Then

$$\text{codim}_y(f(A), Y) = \dim_y Y - \dim_y f(A).$$

By [Theorem 4.14](#),  $\dim_y Y = \dim_x X$ ,  $\dim_y f(A) = \dim_x A$ . It follows that

$$\text{codim}_y(f(A), Y) = \dim_x X - \dim_x A \geq \text{codim}_x(A, X) \geq k.$$

□

**Proposition 4.16.** Let  $f : X \rightarrow Y$  be a finite morphism of complex analytic spaces and  $x \in X$ . Assume that  $Y$  is unibranch at  $f(x)$ . Assume that  $\dim_x X = \dim_{f(x)} Y$ , then  $f$  is open at  $x$ .

PROOF. We may assume that  $X$  and  $Y$  are both reduced. Let  $y = f(x)$ . By [Proposition 4.4](#), we may assume that  $\{x\} = f^{-1}(y)$ . By [Corollary 4.8](#),  $f(X)$  is an analytic set in  $Y$ . By [Theorem 4.14](#),

$$\dim_y f(X) = \dim_x X.$$

As  $Y$  is irreducible at  $f(x)$ , we conclude that  $f(X)_y = X_y$  and hence  $f(X)$  is a neighbourhood of  $y$ . □

**Corollary 4.17.** Let  $f : X \rightarrow Y$  be a quasi-finite morphism of equidimensional complex analytic spaces of dimension  $d \in \mathbb{N}$ . Assume that  $Y$  is unibranch. Then  $f$  is open.

The corollary fails if  $Y$  is not unibranch.

PROOF. By [Proposition 4.4](#),  $f$  is finite at all  $x \in X$ . It suffices to apply [Proposition 4.16](#). □

**Lemma 4.18.** Let  $f : X \rightarrow Y$  be a finite open morphism of reduced complex analytic spaces. Assume that  $Y$  is a complex manifold. Then  $f$  is a branched covering.

PROOF. The statement is local on  $Y$ , so we may assume that  $Y$  is an open neighbourhood of 0 in  $\mathbb{C}^n$  for some  $n \in \mathbb{N}$ . By [Proposition 4.4](#), we may assume that  $\pi^{-1}\{0\}$  consists of a single point and  $X$  is a closed analytic subspace of a domain  $V$  in  $\mathbb{C}^d$  for some  $d \in \mathbb{N}$ . Replacing  $X$  by the graph of  $f$ , we may assume that  $X$  is a closed analytic subspace of  $V \times Y$  and  $f$  is the restriction of the projection map  $V \times Y \rightarrow V$ . In this case, the result follows from the local description lemma. [Reproduce CAS p72!](#) □

**Corollary 4.19.** Let  $X$  be an equidimensional complex analytic space of dimension  $d$  and  $x \in X$ . Then there is an open neighbourhood  $U$  of  $x$  in  $X$  and a connected domain  $V \in \mathbb{C}^d$  such that there is a branched covering  $U \rightarrow V$ .

In fact, given any system of parameters  $f_1, \dots, f_d \in \mathcal{O}_{X,x}$ , we can define a morphism sending  $x$  to 0 and the corresponding local ring homomorphism at  $x$  is

$$\mathcal{O}_{\mathbb{C}^d,0} \rightarrow \mathcal{O}_{X,x}$$

given by  $f_1, \dots, f_d$ .

PROOF. This follows from [Theorem 3.9](#) in [Constructions of complex analytic spaces](#), [Lemma 4.18](#) and [Corollary 4.17](#).  $\square$

**Corollary 4.20.** Let  $X$  be a complex analytic space and  $x \in X$ . Assume that  $X$  is unibranch at  $x$ . Let  $f \in \mathcal{O}_{X,x}$ . We assume that  $f$  is not constant and  $\dim_x X \geq 1$ , then for any open neighbourhood  $U$  of  $x$  in  $X$  such that  $f$  spreads to  $g \in \mathcal{O}_X(U)$ , there is  $\epsilon > 0$  such that  $g$  takes all values  $c \in \mathbb{C}$  with  $|c - f(x)| < \epsilon$ .

PROOF. We may assume that  $X$  is reduced and  $f(x) = 0$ . Then  $f$  is a non-zero divisor in  $\mathcal{O}_{X,x}$ . We can find a system of parameters  $f, g_1, \dots, g_{n-1}$  with  $n = \dim_x X$  such that  $f, g_1, \dots, g_{n-1}$  induce a branched covering  $X \rightarrow V$  sending  $x$  to 0 after shrinking  $X$ , where  $V$  is an open neighbourhood of 0 in  $\mathbb{C}^n$ . This follows from [Corollary 4.19](#). As the branched covering is open by [Proposition 4.16](#), we conclude.  $\square$

**Theorem 4.21.** Let  $f : X \rightarrow Y$  be a finite open surjective morphism of reduced complex analytic spaces, then  $f$  is a branched covering.

PROOF. Let  $x \in X$  and  $y = f(x)$ . As  $f$  is open, it suffices to find open neighbourhoods  $U$  of  $x$  in  $X$  and  $V$  of  $y$  in  $Y$  such that the morphism  $U \rightarrow V$  induced by  $f$  is a branched covering. We first take  $U$  small enough so that  $U$  can be decomposed into prime components at  $x$ :

$$U = X_1 \cup \dots \cup X_s.$$

We can assume that  $X_i \cap X_j$  is thin in  $U$  for  $i, j = 1, \dots, s, i \neq j$ . Up to shrinking  $U$ , we may assume that  $U \rightarrow V$  is finite [Proposition 4.4](#) for some open neighbourhood  $V$  of  $y$  in  $Y$ . As  $f$  is open, we may take  $V = f(U)$ . Observe that  $f(X_i)$  is analytic in  $V$  for  $i = 1, \dots, s$  by [Corollary 4.8](#). Moreover,  $f(X_i)$  is irreducible at  $y$  for  $i = 1, \dots, s$ . By [Theorem 2.4](#) in [Local properties of complex analytic spaces](#), we may assume that  $f(X_i)$  is equidimensional of dimension  $n_i \in \mathbb{N}$  for  $i = 1, \dots, s$ .

By [Corollary 4.19](#), up to shrinking  $V$ , we may assume that there is a branched covering  $\eta_i : f(X_i) \rightarrow V_i$ , where  $V_i$  is a connected domain in  $\mathbb{C}^{n_i}$  for  $i = 1, \dots, s$ . By [Lemma 4.18](#),  $\eta_i \circ f|_{X_i}$  is a branched covering for  $i = 1, \dots, s$ . It follows that  $X_i \rightarrow \pi(X_i)$  is a branched covering for  $i = 1, \dots, s$ . This readily implies that  $f$  is a branched covering.  $\square$

**Definition 4.22.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a connected complex manifold. Let  $A$  be a thin set in  $X$ . Take a critical locus  $T$  of  $f$  containing  $f(A)$ .

Consider  $g \in \mathcal{O}_X(X \setminus A)$ . We define a monic polynomial

$$\chi_g(w)(y) := \prod_{x \in f^{-1}(y)} (w - g(x)) \in \mathcal{O}_Y(Y \setminus T)[w].$$

By [Theorem 3.7](#) in [Local properties of complex analytic spaces](#),  $\chi_g$  can be uniquely extended to  $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$ . The monic polynomial  $\chi_g$  is called the *characteristic polynomial* of  $g$  (with respect to  $f$ ).

**Proposition 4.23.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a connected complex manifold. Let  $A$  be a thin set in  $X$  and  $g \in \mathcal{O}_X(X \setminus A)$ . Let  $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$  be the characteristic polynomial of  $g$ . Then  $\chi_g(g) = 0$ .

If either of the following conditions hold:

- (1)  $g$  is locally bounded near  $A$ ;
- (2)  $A$  is thin of order 2 in  $Y$ .

Then  $\chi_g$  can be uniquely extended to  $\chi_g \in \mathcal{O}_Y(Y)[w]$ .

PROOF. Only the second part is non-trivial. By [Corollary 4.15](#),  $f$  is open. By [Corollary 4.15](#),  $f(A)$  is thin in  $Y$  and under assumption (2),  $f(A)$  is thin of order 2 in  $Y$ . It suffices to apply [Theorem 3.7](#) in [Local properties of complex analytic spaces](#).  $\square$

**Proposition 4.24.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a connected complex manifold. Let  $A$  be a thin set in  $X$  and  $e, g \in \mathcal{O}_X(X \setminus A)$ . Take a critical locus  $T$  of  $f$  containing  $f(A)$ . Consider the  $b \times b$ -matrice

$$M(y) = \begin{bmatrix} 1 & e(x_1) & \dots & e(x_1)^{b-1} \\ 1 & e(x_2) & \dots & e(x_2)^{b-1} \\ & & \ddots & \\ 1 & e(x_b) & \dots & e(x_b)^{b-1} \end{bmatrix}$$

and  $M_i(y)$  is  $M(y)$  with the  $i$ -th column replace by

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_b) \end{bmatrix}$$

for  $i = 0, \dots, b-1$ , where  $y \in Y \setminus T$  and  $x_1, \dots, x_b$  are the distinct points in  $f^{-1}(y)$ . Then there are  $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus f(A))$  such that for all  $y \in Y \setminus T$ ,

$$\Delta_e(y) = (\det M(y))^2, \quad c_i(y) = \det M(y) \cdot \det M_i(y)$$

for  $i = 0, \dots, b-1$ . If either of the following conditions holds:

- (1)  $e$  and  $g$  are locally bounded near  $A$ ;
- (2)  $A$  is thin of order 2 in  $X$ ,

then we can take  $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y)$

The function  $\Delta_e$  is called the *discriminant* of  $e$ . We say  $e$  is *primitive* with respect to  $f$  if  $\Delta$  is not identically 0.

PROOF. We first observe that  $\det M(y)$  and  $\det M_i(y)$  are independent of the ordering of  $x_1, \dots, x_b$  by elementary linear algebra, where  $i = 1, \dots, b$ . The entries of  $M(y)$  and  $M_i(y)$  can all be taken to be holomorphic outside  $T$ , so  $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus T)$  are defined and the desired equation holds. By [Theorem 3.7](#) in [Local properties of complex analytic spaces](#), these functions can be extended uniquely into  $\mathcal{O}_Y(Y \setminus f(A))$ .

By [Corollary 4.15](#),  $f(A)$  is thin in  $Y$  and under assumption (2),  $f(A)$  is thin of order 2 in  $Y$ . Applying [Theorem 3.7](#) in [Local properties of complex analytic spaces](#), we conclude the last assertion.  $\square$

**Corollary 4.25.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a connected complex manifold. A primitive element  $e \in \mathcal{O}_X(X)$  exists if  $X$  is holomorphically separable.

PROOF. Take a critical locus  $T$  of  $f$ . Let  $y \in X \setminus T$ . Let  $x_1, \dots, x_b$  be distinct points of  $f^{-1}(y)$ . For each  $i, j = 1, \dots, b$  with  $i < j$ , we can find a  $g_{ij} \in \mathcal{O}_X(X)$  with  $g(x_i) \neq g(x_j)$ . A suitable linear combination of  $g_{ij}$ 's works.  $\square$

**Proposition 4.26.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a connected complex manifold. Let  $A$  be a thin set in  $X$ .

Let  $e \in \mathcal{O}_X(X \setminus A)$  primitive element with respect to  $f$ . Then for each  $g \in \mathcal{O}_X(X \setminus A)$ , we have canonical polynomial  $\Omega \in \mathcal{O}_Y(Y \setminus \pi(A))[X]$  such that

$$\Delta_e g = \Omega(e) \quad \text{on } X \setminus A.$$

If either of the following conditions holds:

- (1)  $e$  and  $g$  are locally bounded near  $A$ ;
- (2)  $A$  is thin of order 2 in  $X$ ,

then we can take  $\Omega \in \mathcal{O}_Y(Y)[X]$ .

In the traditional terminology,  $\Delta_e$  is a *universal denominator* of the  $\mathcal{O}_Y(Y)$ -module  $\mathcal{O}_X(X)$  if one of the two assumptions is satisfied.

PROOF. Take a critical locus  $T$  of  $f$  containing  $f(A)$ . Consider  $y \in Y \setminus T$  with fibers  $x_1, \dots, x_b$ . Consider the system of  $b$ -linear equations:

$$\Delta_e(y)g(x_i) = c_0(y) + c_1(y)e(x_i) + \dots + c_{b-1}(y)e(x_i)^{b-1}$$

for  $j = 1, \dots, b$ . By Cramer's rule, if we use the notations of [Proposition 4.24](#), if  $\det M(y) \neq 0$ , the unique solution is then

$$c_i(y) = (\det M(y))^{-1} \Delta_e(y) \det M_i(y) = \det M(y) \cdot \det M_i(y)$$

for  $i = 0, \dots, b-1$ . From [Proposition 4.24](#),  $c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus \pi(A))$ . It suffices to take

$$\Omega = c_0 + c_1 X + \dots + c_{b-1} X^{b-1}.$$

It is obvious that on  $X \setminus (A \cup W(\Delta))$ ,

$$\Delta_e g = \Omega(e).$$

The same holds on  $X \setminus A$  by continuity. The last assertion follows from [Proposition 4.24](#).  $\square$

**Corollary 4.27** (Riemann extension theorem). Let  $X$  be a reduced equidimensional complex analytic space of dimension  $n \in \mathbb{N}$  and  $A$  be a thin set in  $X$ . Let  $f \in \mathcal{O}_X(X \setminus A)$ . Assume one of the following conditions holds:

- (1)  $f$  is locally bounded near  $A$ ;
- (2)  $A$  is thin of order 2.

Then there is an element  $g \in \overline{\mathcal{O}_X(X)}$  extending  $f$ .

PROOF. The uniqueness is obvious, we prove the existence. The problem is local on  $X$ , we may assume that  $X$  is holomorphically separable. By [Corollary 4.19](#), we may take a connected complex manifold  $Y$  of dimension  $n$ ,  $b \in \mathbb{Z}_{>0}$ , a  $b$ -sheeted branched covering  $f : X \rightarrow Y$ . By [Corollary 4.25](#), we can find a primitive element  $e \in \mathcal{O}_X(X)$ . By [Proposition 4.26](#) and [Proposition 4.23](#), it suffices to take  $g = \Omega(e)/\Delta_e$ , where  $\Omega_e$  is the polynomial in [Proposition 4.26](#).  $\square$

**Corollary 4.28.** Let  $X$  be a normal complex analytic space. Then the canonical map

$$\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X^{\text{reg}})$$

is an isomorphism.

PROOF. By [Proposition 6.9](#) in [Local properties of complex analytic spaces](#), the map is injective. Take  $f \in \mathcal{O}_X(X^{\text{reg}})$ , we need to extend it to  $g \in \mathcal{O}_X(X)$ . The problem is local on  $X$ . As  $X$  is normal, it is equidimensional at all points. By shrinking  $X$ , we may assume that  $X$  is equidimensional of some dimension  $n \in \mathbb{N}$ . Recall that  $X^{\text{Sing}}$  is thin of order 2 in  $X$  by [Proposition 7.4](#) in [Local properties of complex analytic spaces](#), so we can apply [Corollary 4.27](#).  $\square$

**Corollary 4.29.** Let  $X$  be a connected normal complex analytic space then  $X^{\text{reg}}$  is connected.

PROOF. If not, we can find a continuous function  $f : X^{\text{reg}} \rightarrow \{0, 1\}$  which is not constant. By [Corollary 4.28](#),  $f$  can be extended to  $g \in \mathcal{O}_X(X)$ . This contradicts the fact that  $X$  is connected.  $\square$

**Corollary 4.30.** Let  $X$  be an irreducible complex analytic space and  $A$  be an analytic set in  $X$ . Suppose that there is  $x \in A$  with  $\dim_x A = \dim_x X$ , then  $A = X$ .

PROOF. We may assume that  $X$  is irreducible. By [Theorem 4.14](#), we may assume that  $X$  is normal.

Endow  $A$  with the reduced induced structure. As  $\dim_x A = \dim_x X$ ,  $\text{Spec } \mathcal{O}_{X,x} = \text{Spec } \mathcal{O}_{A,x}$  has a common irreducible component. By Nullstellensatz,  $\text{Int } A$  is non-empty. So  $A' := A \setminus X^{\text{Sing}}$  is non-empty and open in  $X^{\text{reg}}$ . We need to show that  $A' = X^{\text{reg}}$ , taking closure we then conclude.

Suppose that  $A' \neq X^{\text{reg}}$ . Then  $\overline{A'} \cap X^{\text{reg}}$  is a non-empty closed in  $X^{\text{reg}}$ , which is connected by [Corollary 4.29](#). So

$$\overline{A'} \cap X^{\text{reg}} \neq A',$$

as otherwise,  $X^{\text{reg}} = (\overline{A'} \cap X^{\text{reg}}) \cup (X^{\text{reg}} \setminus A')$ . Take  $a \in (\overline{A'} \cap X^{\text{reg}}) \setminus A'$ . Take a connected neighbourhood  $U$  of  $a$  in  $X^{\text{reg}}$  and finitely many holomorphic functions  $f_1, \dots, f_k \in \mathcal{O}_X(U)$  so that  $U \cap A = W(f_1, \dots, f_k)$ . As  $U \cap A' \neq \emptyset$ ,  $f_1, \dots, f_k$  vanishes identically in  $U$  by Identitätssatz. In particular,  $a \in A'$ , which is a contradiction.  $\square$

**Corollary 4.31.** Let  $f : X \rightarrow Y$  be a morphism of reduced complex analytic spaces. Let  $Z \subseteq Y$  be the non-normal locus. Assume that  $f^{-1}(Z)$  is nowhere dense in  $X$  (for example when  $X$  is irreducible and  $f$  is surjective), then there is a unique morphism  $\bar{f} : \bar{X} \rightarrow \bar{Y}$  such that the following diagram commutes:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Recall that  $Z$  is an analytic set in  $Y$  by [Theorem 7.3](#) in [Local properties of complex analytic spaces](#).

PROOF. The uniqueness is clear. Let  $Z'$  be the inverse image of  $Z$  in  $\bar{Y}$  and  $Z''$  be the inverse image of  $Z$  in  $\bar{X}$ . By our assumption,  $Z''$  is thin in  $\bar{X}$ . By construction,  $\eta : \bar{Y} \setminus Z' \rightarrow Y \setminus Z$  is an isomorphism, so we get a morphism  $g : \bar{X} \setminus Z'' \rightarrow \bar{Y} \setminus Z'$  completion the commutative diagram

$$\begin{array}{ccc} \bar{X} \setminus Z'' & \longrightarrow & \bar{Y} \setminus Z' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Let  $p \in Z''$ . We need to extend  $g$  to a neighbourhood of  $p$ . Choose an open neighbourhood  $V \subseteq \bar{Y}$  of the preimage of  $p$  in  $\bar{X}$  which admits a closed immersion into a bounded domain  $D \subseteq \mathbb{C}^n$  for some  $n \in \mathbb{N}$ . There is an open neighbourhood  $U \subseteq \bar{X}$  of  $p$  such that  $g$  maps  $U \setminus Z'' \rightarrow V$ . The induced morphism  $U \setminus Z'' \rightarrow D$  is given by bounded holomorphic functions in  $\mathcal{O}_{U \setminus Z''}(U \setminus Z'')$ . By [Corollary 4.27](#), we get an extension  $U \rightarrow D$ . But this morphism factorizes through  $U \rightarrow V$  as  $U$  is reduced, we conclude.  $\square$

**Corollary 4.32.** Let  $X$  be a complex analytic space. Then the following are equivalent:

- (1)  $X$  is irreducible;
- (2) If we write  $X = Y_1 \cup Y_2$  with  $Y_1, Y_2$  being analytic sets in  $X$ , then  $X = Y_1$  or  $X = Y_2$ .

PROOF. We may assume that  $X$  is reduced.

(1)  $\implies$  (2): We may assume that  $X$  is normal. Suppose  $X = Y_1 \cup Y_2$  with  $Y_1, Y_2$  being analytic sets in  $X$ . Then  $Y_1 \cap Y_2$  is not empty, as otherwise,  $X$  is not even connected. Let  $x \in Y_1 \cap Y_2$ . We then have  $X_x = Y_{1,x} \cup Y_{2,x}$ . This contradicts the fact that  $\mathcal{O}_{X,x}$  is integral unless  $Y_{1,x} \subseteq Y_{2,x}$  or  $Y_{1,x} \subseteq Y_{2,x}$ , which is impossible by [Corollary 4.30](#).

(2)  $\implies$  (1): Suppose that  $X$  is not irreducible. Then the normalization  $\bar{X}$  is not connected, say  $\bar{X} = Y'_1 \cup Y'_2$ , where  $Y'_1, Y'_2$  are disjoint clopen sets in  $\bar{X}$ . Let  $\pi : \bar{X} \rightarrow X$  be the normalization morphism. Then

$$X = \pi(Y'_1) \cup \pi(Y'_2).$$

By our assumption, either  $X = \pi(Y'_1)$  or  $X = \pi(Y'_2)$ . We assume that the former holds. From [Proposition 7.8](#) in [Local properties of complex analytic spaces](#), we conclude that  $Y'_1 = \bar{X}$ , which is a contradiction.  $\square$

**Corollary 4.33.** Let  $X$  be a connected complex analytic space. Then  $X$  is path-connected.

PROOF. We may assume that  $X$  is reduced.

If  $X$  is irreducible, after passing to the normalization, we may assume that  $X$  is normal. Then clearly  $X^{\text{reg}}$  is connected. So it suffices to apply [Proposition 7.12](#) in [Local properties of complex analytic spaces](#).

In general, take  $x \in X$  and let  $X'$  be the set of all points of  $X$  that can be joined to  $x$  by a path. Then from the previous case,  $X'$  is the union of certain irreducible components of  $X$ . So is the complement  $X \setminus X'$ . As  $X$  is connected, we find that  $X = X'$ .  $\square$

**Corollary 4.34.** Let  $X$  be an irreducible complex analytic space. Then there is  $n \in \mathbb{N}$  such that  $X$  is equidimensional of dimension  $n$ .

PROOF. We may assume that  $X$  is reduced. By [Theorem 4.14](#), we can even assume that  $X$  is normal. Then  $X$  is connected. In particular,  $X^{\text{reg}}$  is connected by [Corollary 4.29](#). But  $X^{\text{reg}}$  is then equidimensional of some dimension  $n \in \mathbb{N}$ . If  $\dim_x X \neq n$  for some  $x \in X^{\text{Sing}}$ , by [Theorem 2.4](#) in [Local properties of complex analytic spaces](#),  $\dim_y X = \dim_x X$  whenever  $y$  is close to  $x$ . This is a contradiction.  $\square$

**Corollary 4.35.** Let  $f : X \rightarrow Y$  be a finite surjective morphism between irreducible reduced complex analytic spaces. Then  $f$  is a branched covering.

PROOF. By [Corollary 4.31](#), we have an obvious commutative diagram:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

It suffices to show that  $\bar{f}$  is a branched covering, so we may assume that  $X$  and  $Y$  are normal.

By [Proposition 4.16](#) and [Corollary 4.34](#),  $f$  is open. So it suffices to apply [Theorem 4.21](#).  $\square$

**Corollary 4.36.** Let  $f : X \rightarrow Y$  be a finite surjective morphism between reduced complex analytic spaces. Then the following are equivalent:

- (1)  $f$  is a branched covering;
- (2) The image of each irreducible component of  $X$  has an interior point;
- (3) The image of each irreducible component of  $X$  is an irreducible component of  $Y$ .

PROOF. (1)  $\implies$  (2): Let  $T \subseteq Y$  be a critical locus of  $f$ . Then  $f^{-1}(T)$  is thin in  $X$ . Each irreducible component  $X'$  of  $X$  meets  $X \setminus f^{-1}(T)$ . It follows that  $f(X' \setminus f^{-1}(T))$  is non-empty and open in  $Y$ .

(2)  $\implies$  (3): Let  $X'$  be an irreducible component of  $X$ . Then  $f(X')$  is an analytic set in  $Y$ . It is clearly irreducible. So  $f(X')$  is contained in an irreducible component  $Y'$  of  $Y$ . But as  $f(X')$  has an interior point, we find that  $f(X') = Y'$  by [Corollary 4.30](#).

(3)  $\implies$  (1): The assertion is local, we may assume that the number of irreducible components of  $X$  is finite. Let  $X_1, \dots, X_s$  be the irreducible components of  $X$ . For each  $i = 1, \dots, s$ , the induced map  $X_i \rightarrow \pi(X_i)$  is finite and hence a branched covering by [Corollary 4.35](#). It is enough to verify that  $\pi^{-1}(\pi(X_i \cap X_j))$  is thin in  $X$  for  $i, j = 1, \dots, s$  and  $i \neq j$ . If this fails, this set contains an interior point in  $X_k$  for some  $k \in \{1, \dots, s\}$ . But then

$$X_k \subseteq \pi^{-1}(\pi(X_i \cap X_j)).$$

It follows that

$$\pi(X_i \cap X_j) \supseteq \pi(X_k).$$

This is impossible as  $X_i \cap X_j \cap X_k$  is thin in  $X_k$ .  $\square$



**Definition 4.37.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a normal complex analytic space. Take a critical locus  $T \subseteq Y$  of  $f$  containing  $Y^{\text{Sing}}$ .

Consider  $g \in \mathcal{O}_X(X)$ . We define the *characteristic polynomial*  $\chi_g \in \mathcal{O}_Y(Y)[w]$  of  $g$  (with respect to  $f$ ) as follows: When  $Y$  is connected, by [Corollary 4.29](#),  $Y^{\text{reg}}$  is a connected complex manifold. We define  $\chi_g \in \mathcal{O}_Y(Y^{\text{reg}})[w]$  as in [Definition 4.22](#). We then extend  $\chi_g$  to  $\mathcal{O}_Y(Y^{\text{reg}})[w]$  using [Corollary 4.28](#). It is a monic polynomial of degree  $b$ . When  $Y$  is not connected, we just glue the characteristic polynomials defined using each connected components. Then we find a monic polynomial  $\chi_g \in \mathcal{O}_Y(Y)[w]$  of degree  $b$ .

**Proposition 4.38.** Let  $b \in \mathbb{Z}_{>0}$ ,  $f : X \rightarrow Y$  be a  $b$ -sheeted branched covering with  $Y$  being a normal complex analytic space. Let  $g \in \mathcal{O}_X(X)$ . Let  $\chi_g \in \mathcal{O}_Y(Y)[w]$  be the characteristic polynomial of  $g$ . Then  $\chi_g(g) = 0$ .

PROOF. This follows immediately from [Proposition 4.23](#).  $\square$

We give an alternative characterization of  $\overline{\mathcal{O}}_X$ .

**Proposition 4.39.** Let  $X$  be a reduced complex analytic space. Then for any open set  $U \subseteq X$ ,

$$\overline{\mathcal{O}}_X(U) \xrightarrow{\sim} \{f : U \rightarrow \mathbb{C} : f \text{ is weakly holomorphic}\}.$$

PROOF. We temporarily denote the sheaf stated in the proposition by  $\mathcal{O}'$ . From the uniqueness in ??, it suffices to show that  $\mathcal{O}'_x$  is isomorphic to  $\overline{\mathcal{O}}_{X,x}$  as  $\mathcal{O}_{X,x}$ -algebras for any  $x \in X$ .

We first observe that  $\overline{\mathcal{O}}_X$  is a subsheaf of  $\mathcal{O}'$ . Let  $U \subseteq X$  be an open subset and  $f \in \overline{\mathcal{O}}_X(U)$ . We need to show that  $f$  is locally bounded around  $y \in U \cap X^{\text{Sing}}$ . Take an integral equation

$$f_y^n + a_{1,y}f_y^{n-1} + \cdots + a_{n,y} = 0$$

with  $a_{1,y}, \dots, a_{n,y} \in \mathcal{O}_{X,x}$ . Take an open neighbourhood  $V$  of  $y$  in  $U$  such that  $a_{1,y}, \dots, a_{n,y}$  lift to  $a_1, \dots, a_n \in \mathcal{O}_X(V)$  and

$$(f|_V)^n + a_1(f|_V)^{n-1} + \cdots + a_n = 0.$$

Then for any  $z \in V \setminus X^{\text{Sing}}$ ,

$$|f(z)| \leq \max\{1, |a_1(z)| + \cdots + |a_n(z)|\}.$$

So  $f \in \mathcal{O}'$ .

Conversely, let  $U \subseteq X$  be an open subset and  $f \in \mathcal{O}'(U)$ . By [Proposition 7.8](#) in [Local properties of complex analytic spaces](#),  $p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_X$ , where  $p : \overline{X} \rightarrow X$  is the normalization morphism. It follows from [Proposition 7.8](#) in [Local properties of complex analytic spaces](#) and [Corollary 4.27](#) that  $f$  can be uniquely extended to  $g \in \mathcal{O}_{\overline{X}}(p^{-1}U) = \mathcal{O}_X(U)$ .  $\square$

**Proposition 4.40** (Rado, Cartan). Let  $X$  be a normal complex analytic space and  $f : X \rightarrow \mathbb{C}$  be a continuous map. Let  $Z = f^{-1}(0)$ . Assume that there is  $g \in \mathcal{O}_X(X \setminus Z)$  such that  $[g] = f|_{X \setminus Z}$ , then  $f = [g]$ .

This result is proved in [\[Car52\]](#).

PROOF. By [Corollary 4.28](#), we may assume that  $X$  is a complex manifold. The problem is local on  $X$ , we may assume that  $X$  is the unit polydisk in  $\mathbb{C}^n$  for some  $n \in \mathbb{N}$ . By Hartogs theorem, we may assume that  $n = 1$ .

It remains to show that a continuous function  $f : \{z \in \mathbb{C} : |z| < 1\}$  which is holomorphic outside  $Z := \{f = 0\}$  is holomorphic. This result is well-known.  $\square$

## 5. Flat morphisms

The notion of flat morphisms is defined for all ringed spaces. See [\[Stacks, Tag 02N2\]](#). We will make use of these notions directly.

**Proposition 5.1.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces and  $x \in X$ . Write  $y = f(x)$ . Let  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Then the following are equivalent:

- (1)  $\mathcal{F}$  is  $f$ -flat at  $x$ ;
- (2)  $\mathcal{F}_x$  is a flat  $\mathcal{O}_{Y,y}$ -module;
- (3) For all  $n \in \mathbb{N}$ ,

$$\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$$

is a flat  $\hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$ -module;

- (4) We have

$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathbb{C}, \mathcal{F}_x) = 0.$$

PROOF. (1)  $\Leftrightarrow$  (2): This is the definition of flatness.

(2)  $\Leftrightarrow$  (3): This follows from [\[Stacks, Tag 0523\]](#).

(3)  $\Leftrightarrow$  (4): This follows from [\[Stacks, Tag 00MK\]](#).  $\square$

**Proposition 5.2.** Let  $f : X \rightarrow Y$  be a morphism of complex analytic spaces and  $\mathcal{F}$  be a coherent  $\mathcal{O}_X$ -module. Let  $g : Y' \rightarrow Y$  be a morphism of complex analytic spaces and consider the following Cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \square & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Consider a point  $x' \in X'$  defined by  $x \in X$  and  $y' \in Y'$  with common image  $y \in Y$ .

- (1) If  $\mathcal{F}$  is  $f$ -flat at  $x$ , then  $g'^*\mathcal{F}$  is  $f'$ -flat at  $x'$ .
- (2) If  $g'^*\mathcal{F}$  is  $f'$ -flat at  $x'$  and  $\hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{Y',y'}$  is injective, then  $\mathcal{F}$  is  $f$ -flat at  $x$ .

PROOF. (1) Recall that

$$\hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y',y'}.$$

Let  $n \in \mathbb{N}$ , we then find

$$\hat{\mathcal{O}}_{X',x'} / \hat{\mathfrak{m}}_{y'}^{n+1} \hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}} \left( \hat{\mathcal{O}}_{Y',y'} / \hat{\mathfrak{m}}_{y'}^{n+1} \right) \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \otimes_{\hat{\mathcal{O}}_{Y,y}} \left( \hat{\mathcal{O}}_{Y',y'} / \hat{\mathfrak{m}}_{y'}^{n+1} \right).$$

By [Proposition 5.1](#),  $\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$  is a flat  $\mathcal{O}_{Y',y'}$ -module for each  $n \in \mathbb{N}$ . By [Proposition 5.1](#) again,  $\mathcal{F}$  is  $f'$ -flat at  $x'$ .

(2) For each  $n \in \mathbb{N}$ , let  $I_n$  be the inverse image of  $\hat{\mathfrak{m}}_{y'}^{n+1}$  with respect to  $\hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{Y',y'}$ . As the latter map is assumed to be injective, by Krull's intersection theorem, we find that

$$\bigcap_{n \in \mathbb{N}} I_n = 0.$$

It follows that the  $I_n$ 's form a basis at 0 in  $\hat{\mathcal{O}}_{Y,y}$ . By [Proposition 5.1](#), we are reduced to show that  $\hat{\mathcal{F}}_x/I_n\hat{\mathcal{F}}_x$  is flat over  $\hat{\mathcal{O}}_{Y,y}/I_n$ . But by [Proposition 5.1](#) again, we know that its base change along  $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{Y',y'}/\hat{\mathfrak{m}}_{y'}^{n+1}$  is flat. So we are reduced to the well-known algebraic case.  $\square$



## Bibliography

- [Car52] H. Cartan. Sur une extension d'un théorème de Radó. *Math. Ann.* 125 (1952), pp. 49–50. URL: <https://doi.org/10.1007/BF01343105>.
- [Stacks] T. Stacks Project Authors. Stacks Project. <http://stacks.math.columbia.edu>. 2020.