

Ymir



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# Banach rings

## 1. Introduction

This section concerns the theory of Banach algebras. Our references are [Ber12] and [BGR84].

In this chapter, all rings are assumed to be commutative.

## 2. Semi-normed Abelian groups

**Definition 2.1.** Let  $A$  be an Abelian group. A *semi-norm* on  $A$  is a function  $\|\bullet\| : A \rightarrow [0, \infty]$  satisfying

- (1)  $\|0\| = 0$ ;
- (2)  $\|f - g\| \leq \|f\| + \|g\|$  for all  $f, g \in A$ .

A semi-norm  $\|\bullet\|$  on  $A$  is a *norm* if moreover the following condition is satisfied:

- (0) if  $\|f\| = 0$  for some  $f \in A$ , then  $f = 0$ .

We write

$$\ker \|\bullet\| = \{a \in A : \|a\| = 0\}.$$

A semi-norm  $\|\bullet\|$  on  $A$  is *non-Archimedean* or *ultra-metric* if Condition (2) can be replaced by

$$(2') \quad \|f - g\| \leq \max\{\|f\|, \|g\|\} \text{ for all } f, g \in A.$$

**Definition 2.2.** A *semi-normed Abelian group* (resp. *normed Abelian group*) is a pair  $(A, \|\bullet\|)$  consisting of an Abelian group  $A$  and a semi-norm (resp. norm)  $\|\bullet\|$  on  $A$ . When  $\|\bullet\|$  is clear from the context, we also say  $A$  is a semi-normed Abelian group (resp. normed Abelian group).

**Definition 2.3.** Let  $(A, \|\bullet\|_A)$  be a semi-normed Abelian group and  $B \subseteq A$  be a subgroup. Then we define the *quotient semi-norm*  $\|\bullet\|_{A/B}$  on  $A/B$  as follows:

$$\|a + B\|_{A/B} := \inf\{\|a + b\|_A : b \in B\}$$

for all  $a + B \in A/B$ .

We define the *subgroup semi-norm* on  $B$  as follows:

$$\|b\|_B = \|b\|_A$$

for all  $b \in B$ .

**Definition 2.4.** Let  $A$  be an Abelian group and  $\|\bullet\|, \|\bullet\|'$  be two seminorms on  $A$ . We say  $\|\bullet\|$  and  $\|\bullet\|'$  are *equivalent* if there is a constant  $C > 0$  such that

$$C^{-1}\|f\| \leq \|f\|' \leq C\|f\|$$

for all  $f \in A$ .

**Definition 2.5.** Let  $(A, \|\bullet\|_A)$ ,  $(B, \|\bullet\|_B)$  be semi-normed Abelian groups. A homomorphism  $\varphi : A \rightarrow B$  is said to be

- (1) *bounded* if there is a constant  $C > 0$  such that  $\|\varphi(f)\|_B \leq C\|f\|_A$  for any  $f \in A$ ;
- (2) *admissible* if the quotient semi-norm on  $A/\ker \varphi$  is equivalent to the subspace semi-norm on  $\text{Im } \varphi$ .

Observe that an admissible homomorphism is always bounded.

Next we study the topology defined by a semi-norm.

**Lemma 2.6.** Let  $(A, \|\bullet\|)$  be a semi-normed Abelian group. Define

$$d(a, b) = \|a - b\|$$

for  $a, b \in A$ . Then  $\|\bullet\|$  is a pseudo-metric on  $A$ . This pseudo-metric is a metric if and only if  $\|\bullet\|$  is a norm.

Let  $\hat{A}$  be the metric completion of  $A$ , then there is a norm  $\|\bullet\|$  on  $\hat{A}$  inducing its metric. Moreover, the natural homomorphism  $A \rightarrow \hat{A}$  is an isometric homomorphism with dense image.

PROOF. This is clear from the definitions. □

We always endow  $A$  with the topology induced by the pseudo-metric  $d$ .

**Proposition 2.7.** Let  $f : A \rightarrow B$  be a homomorphism between semi-normed Abelian groups. Assume that  $f$  is bounded, then it is continuous.

The converse is not true.

PROOF. Clear from the definition. □

**Proposition 2.8.** Let  $(A, \|\bullet\|)$  be a normed Abelian group and  $B$  be a subgroup of  $A$ . Assume that there is  $\epsilon \in (0, 1)$  such that for each  $a \in A$ , there is  $b \in B$  such that

$$\|a + b\| \leq \epsilon \|a\|.$$

Then  $B$  is dense in  $A$ .

PROOF. Assume to the contrary that there exists  $a \in A$  so that

$$c := \inf_{b \in B} \|a - b\| > 0.$$

Choose  $b_1 \in B$  so that

$$\|a + b_1\| < \epsilon^{-1}c.$$

By our hypothesis, there is  $b_2 \in B$  such that

$$\|a + b_1 + b_2\| \leq \epsilon \|a + b_1\| < c.$$

This is a contradiction. □

**Definition 2.9.** Let  $(A, \|\bullet\|)$  be a semi-normed Abelian group. The normed Abelian group  $(\hat{A}, \|\bullet\|)$  constructed in [Lemma 2.6](#) is called the *completion* of  $(A, \|\bullet\|)$ .

### 3. Semi-normed rings

**Definition 3.1.** Let  $A$  be a ring. A *semi-norm*  $\|\bullet\|$  on  $A$  is a semi-norm  $\|\bullet\|$  on the underlying additive group satisfying the following extra properties:

- (3)  $\|1\| = 1$ ;
- (4) for any  $f, g \in A$ ,  $\|fg\| \leq \|f\| \cdot \|g\|$ .

A semi-norm  $\|\bullet\|$  on  $A$  is called *power-multiplicative* if  $\|f\|^n = \|f^n\|$  for all  $f \in A$  and  $n \in \mathbb{N}$ .

A semi-norm  $\|\bullet\|$  on  $A$  is called *multiplicative* if  $\|fg\| = \|f\|\|g\|$  for all  $f, g \in A$ .

**Definition 3.2.** A *semi-normed ring* (resp. *normed ring*) is a pair  $(A, \|\bullet\|)$  consisting of a ring  $A$  and a semi-norm (resp. norm)  $\|\bullet\|$  on  $A$ . When  $\|\bullet\|$  is clear from the context, we also say  $A$  is a semi-normed ring (resp. normed ring).

**Definition 3.3.** Let  $(A, \|\bullet\|)$  be a semi-normed ring. An element  $a \in A$  is *multiplicative* if  $a \notin \ker \|\bullet\|$  and for any  $x \in A$ ,

$$\|ax\| = \|a\| \cdot \|x\|.$$

**Definition 3.4.** Let  $(A, \|\bullet\|)$  be a normed ring. An element  $a \in A$  is *power-bounded* if  $\{|a^n| : n \in \mathbb{N}\}$  is bounded in  $\mathbb{R}$ . The set of power-bounded elements in  $A$  is denoted by  $\mathring{A}$ .

An element  $a \in A$  is called *topologically nilpotent* if  $a^n \rightarrow 0$  as  $n \rightarrow \infty$ . The set of topologically nilpotent elements in  $A$  is denoted by  $\check{A}$ .

**Proposition 3.5.** Let  $(A, \|\bullet\|)$  be a non-Archimedean normed ring. Then  $\mathring{A}$  is a subring of  $A$  and  $\check{A}$  is an ideal in  $\mathring{A}$ . Moreover,  $\mathring{A}$ ,  $\check{A}$  are open and closed in  $A$ .

**PROOF.** Choose  $a, b \in \mathring{A}$ , by definition, there is a constant  $C > 0$  so that for any  $n \in \mathbb{N}$ ,

$$\|a^n\| \leq C, \quad \|b^n\| \leq C.$$

It follows that

$$\|(ab)^n\| \leq \|a^n\| \cdot \|b^n\| \leq C^2$$

and

$$\|(a-b)^n\| \leq \max_{i=0, \dots, n} \|a^i b^{n-i}\| \leq C^2.$$

So  $\mathring{A}$  is a subring.

Next we show that  $\check{A}$  is an ideal in  $\mathring{A}$ . On the other hand, take  $c \in \check{A}$ , then

$$\|(ac)^n\| \leq \|a^n\| \cdot \|c^n\| \leq C\|c^n\|$$

But  $\|c^n\| \rightarrow 0$  as  $n \rightarrow \infty$ , hence  $ac \in \check{A}$ .

On the other hand, consider  $c, d \in \check{A}$ , we need to show  $c-d \in \check{A}$ . Choose  $C > 0$  so that

$$\|a^n\| \leq C, \quad \|b^n\| \leq C$$

for all  $n \in \mathbb{N}$ . Fix  $\epsilon > 0$ , then there is  $m \in \mathbb{N}$  so that for any  $k \geq m$ ,

$$\|a^k\| \leq \epsilon C^{-1}, \quad \|b^k\| \leq \epsilon C^{-1}.$$

In particular, for  $k \geq 2m$ , we have

$$\|(a-b)^k\| \leq \max_{i=0, \dots, k} \|a^i\| \cdot \|b^{k-i}\| \leq \epsilon.$$

It follows that  $a-b \in \check{A}$ . This proves that  $\check{A}$  is an ideal in  $\mathring{A}$ .

In order to see  $\check{A}$  is open and closed in  $A$ , observe that it is a subgroup of  $A$ , so it suffices to show that  $\check{A}$  is open in  $A$ . It suffices to show that

$$\{a \in A : \|a\| < 1\} \subseteq \check{A}.$$

But this is obvious, if  $\|a\| < 1$ , then  $\|a^n\| \leq \|a\|^n$  for all  $n \in \mathbb{N}$ , it follows that  $a^n \rightarrow 0$  as  $n \rightarrow \infty$ , namely,  $a \in \check{A}$ .

As  $\check{A}$  is a subgroup of  $\mathring{A}$ , it follows that  $\mathring{A}$  is both open and closed.  $\square$

**Definition 3.6.** Let  $(A, \|\bullet\|)$  be a non-Archimedean normed ring. We define the *reduction* of  $A$  as  $\tilde{A} = \mathring{A}/\check{A}$ . The map  $\mathring{A} \rightarrow \tilde{A}$  is called the *reduction map*. We usually denote the reduction map by  $a \mapsto \tilde{a}$ .

This definition makes sense thanks to [Proposition 3.5](#).

**Definition 3.7.** Let  $A$  be a ring. A *semi-valuation* on  $A$  is a multiplicative semi-norm on  $A$ . A semi-valuation on  $A$  is a *valuation* on  $A$  if its underlying semi-norm of Abelian groups is a norm.

**Definition 3.8.** A *semi-valued ring* (resp. *valued ring*) is a pair  $(A, \|\bullet\|)$  consisting of a ring  $A$  and a semi-valuation (resp. valuation)  $\|\bullet\|$  on  $A$ . When  $\|\bullet\|$  is clear from the context, we also say  $A$  is a semi-valued ring (resp. valued ring).

A semi-valued ring (resp. valued ring)  $(A, \|\bullet\|)$  is called a *semi-valued field* (resp. *valued field*) if  $A$  is a field.

#### 4. Banach rings

**Definition 4.1.** A *Banach ring* is a normed ring that is complete with respect to the metric defined in [Lemma 2.6](#).

**Definition 4.2.** A Banach ring  $(A, \|\bullet\|_A)$  is *uniform* if  $\|\bullet\|_A$  is power-multiplicative.

**Definition 4.3.** Let  $A$  be a semi-normed ring. There is an obvious ring structure on the completion  $\hat{A}$  of  $A$  defined in [Definition 2.9](#). We call the resulting Banach ring the *completion* of  $A$ .

**Proposition 4.4.** Let  $(A, \|\bullet\|)$  be a Banach ring and  $f \in A$ . Assume that  $\|f\| < 1$ , then  $1 - f$  is invertible.

PROOF. Define

$$g = \sum_{i=0}^{\infty} f^i.$$

From our assumption, the series converges and  $g \in A$ . It is elementary to check that  $g$  is the inverse of  $1 - f$ .  $\square$

In the non-Archimedean case, we have a stronger result:

**Proposition 4.5.** Let  $(A, \|\bullet\|)$  be a non-Archimedean Banach ring and  $f \in \check{A}$ . Then  $1 - f$  is invertible. Moreover,  $(1 - f)^{-1}$  can be written as  $1 + z$  for some  $z \in \check{A}$ .

PROOF. Define

$$g = \sum_{i=0}^{\infty} f^i.$$

From our assumption, the series converges and  $g \in A$ . It is elementary to check that  $g$  is the inverse of  $1 - f$ . Moreover, in view of [Proposition 3.5](#) as for any  $i \geq 1$ ,  $f^i \in \check{A}$ , the same holds for their sum, we conclude the final assertion.  $\square$



**Corollary 4.6.** Let  $(A, \|\bullet\|)$  be a Banach ring. Then the set of invertible elements in  $A$  is open.

PROOF. Let  $x \in A$  be an invertible element. It suffices to show that for any  $y \in A$ ,  $|y| < 1/(\|x^{-1}\|)$ ,  $y + x$  is invertible. For this purpose, it suffices to show that  $1 + x^{-1}y$  is invertible. But this follows from [Proposition 4.4](#).  $\square$

**Corollary 4.7.** Let  $A$  be a Banach ring and  $\mathfrak{m}$  be a maximal ideal in  $A$ . Then  $\mathfrak{m}$  is closed.

PROOF. The closure  $\bar{\mathfrak{m}}$  is obviously an ideal in  $A$ . We need to show that  $\mathfrak{m} \neq A$ . Namely, 1 is not in the closure of  $\mathfrak{m}$ . But clearly,  $\mathfrak{m}$  is contained in the set of non-invertible elements, the latter being closed by [Corollary 4.6](#). So we conclude.  $\square$

**Lemma 4.8.** Let  $A$  be a non-Archimedean Banach ring. An element  $a \in \mathring{A}$  is a unit in  $\mathring{A}$  if and only if  $\tilde{a}$  is a unit in  $\tilde{A}$ .

PROOF. The direct implication is trivial. Conversely, assume that  $a \in \mathring{A}$  and there is an element  $b \in \mathring{A}$  such that

$$\tilde{a}\tilde{b} = 1.$$

Then  $1 - ab \in \check{A}$ . It follows from [Proposition 4.5](#) that  $ab$  is a unit in  $\mathring{A}$  and hence  $a$  is a unit in  $\mathring{A}$ .  $\square$

**Definition 4.9.** Let  $(A, \|\bullet\|)$  be a Banach ring. We define the *spectral radius*  $\rho = \rho_A : A \rightarrow [0, \infty)$  as follows:

$$\rho(f) = \inf_{n \geq 1} \|f^n\|^{1/n}, \quad f \in A.$$

**Lemma 4.10.** Let  $(A, \|\bullet\|)$  be a Banach ring. Then for any  $f \in A$ , we have

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

PROOF. This follows from the multiplicative version of Fekete's lemma.  $\square$

**Example 4.11.** The ring  $\mathbb{C}$  with its usual norm  $|\bullet|$  is a Banach ring. In fact,  $(\mathbb{C}, |\bullet|)$  is a complete valued field.

**Example 4.12.** Let  $\{(A_i, \|\bullet\|_i)\}_{i \in I}$  be a family of Banach rings. We define their *product*  $\prod_{i \in I} A_i$  as the following Banach ring: as a set it consists of all elements  $f = (f_i)_{i \in I}$  with

$$\|f\| := \sup_{i \in I} \|f_i\|_i < \infty.$$

The norm is given by  $\|\bullet\|$ . It is easy to verify that  $\prod_{i \in I} A_i$  is indeed a Banach ring.

**Example 4.13.** For any Banach ring  $(A, \|\bullet\|)$ , any  $n \in \mathbb{N}$  and any  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ , we define  $A\langle r^{-1}z \rangle = A\langle r_1^{-1}z_1, \dots, r_n^{-1}z_n \rangle$  as the subring of  $A[[z_1, \dots, z_n]]$  consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in A$$

such that

$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha < \infty.$$

We will verify in [Proposition 4.14](#) that  $(A\langle r^{-1}z \rangle, \|\bullet\|_r)$  is a Banach ring.

When  $r = (1, \dots, 1)$ , we omit  $r^{-1}$  from our notations.

**Proposition 4.14.** In the setting of [Example 4.13](#),  $(A\langle r^{-1}z \rangle, \|\bullet\|_r)$  is a Banach ring.

PROOF. By induction, we may assume that  $n = 1$ .

It is obvious that  $\|\bullet\|_r$  is a norm on the underlying Abelian group. To see that  $\|\bullet\|_r$  is a norm on the ring  $A\langle r^{-1}z \rangle$ , we need to verify the condition in [Definition 3.1](#). Condition (3) in [Definition 3.1](#) is obvious. Let us consider Condition (4). Let

$$f = \sum_{i=0}^{\infty} a_i z^i, \quad g = \sum_{j=0}^{\infty} b_j z^j$$

be two elements in  $A\langle r^{-1}z \rangle$ . Then

$$fg = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} a_i b_j \right) z^k.$$

We compute

$$\|fg\|_r = \sum_{k=0}^{\infty} \left\| \sum_{i+j=k} a_i b_j \right\| r^k \leq \sum_{k=0}^{\infty} \left( \sum_{i+j=k} \|a_i\| \cdot \|b_j\| \right) r^k = \|f\|_r \cdot \|g\|_r.$$

It remains to verify that  $A\langle r^{-1}z \rangle$  is complete.

For this purpose, take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b z^i \in A\langle r^{-1}z \rangle$$

for  $b \in \mathbb{N}$ . Then for each  $i$ , the coefficients  $(a_i^b)_b$  is a Cauchy sequence in  $A$ . Let  $a_i$  be the limit of  $a_i^b$  as  $b \rightarrow \infty$  and set

$$f = \sum_{i=0}^{\infty} a_i z^i \in A[[z]].$$

We need to show that  $f \in A\langle r^{-1}z \rangle$  and  $f^b \rightarrow f$ .

Fix a constant  $\epsilon > 0$ . There is  $m = m(\epsilon) > 0$  such that for all  $j \geq m$  and all  $k \geq 0$ , we have

$$\sum_{i=0}^{\infty} \|a_i^{j+k} - a_i^j\| r^i < \epsilon/2.$$

In particular, for any  $s > 0$ , we have

$$\sum_{i=0}^s \|a_i - a_i^j\| r^i \leq \sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i + \sum_{i=0}^s \|a_i^j - a_i^{j+k}\| r^i \leq \sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i + \epsilon/2.$$

When  $k$  is large enough, we can guarantee that

$$\sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i < \epsilon/2.$$

So

$$\sum_{i=0}^s \|a_i - a_i^j\| r^i \leq \epsilon.$$

Let  $s \rightarrow \infty$ , we find

$$\|f - f^j\|_r \leq \sum_{i=0}^{\infty} \|a_i - a_i^j\| r^i \leq \epsilon.$$

In particular,  $\|f\|_r < \infty$  and  $f^j \rightarrow f$  as  $j \rightarrow \infty$ .  $\square$

**Example 4.15.** For any non-Archimedean Banach ring  $(A, \|\bullet\|)$ , any  $n \in \mathbb{N}$  and any  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ , we define  $A\{r^{-1}T\} = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  as the subring of  $A[[T_1, \dots, T_n]]$  consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in A$$

such that  $\|a_\alpha\| r^\alpha \rightarrow 0$  as  $|\alpha| \rightarrow \infty$ . We set

$$\|f\|_r := \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

We will verify in [Proposition 4.16](#) that  $(A\{r^{-1}T\}, \|\bullet\|_r)$  is a Banach ring.

The semi-norm  $\|\bullet\|_r$  is called the *Gauss norm*.

**Proposition 4.16.** In the setting of [Example 4.15](#),  $(A\{r^{-1}T\}, \|\bullet\|_r)$  is a Banach ring.

Moreover, if the norm  $\|\bullet\|$  on  $A$  is a valuation, so is  $\|\bullet\|_r$ .

The second part is usually known as the *Gauss lemma*.

PROOF. By induction on  $n$ , we may assume that  $n = 1$ .

The proof of the fact that  $\|\bullet\|_r$  is a norm is similar to that of [Proposition 4.14](#).

We leave the details to the readers.

Next we argue that  $(A\{r^{-1}T\}, \|\bullet\|_r)$  is complete. Take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b T^i \in A\{r^{-1}T\}$$

for  $b \in \mathbb{N}$ . As

$$\|a_i^b - a_i^{b'}\| r^i \leq \|f^b - f^{b'}\|_r$$

for any  $i, b, b' \geq 0$ , it follows that for any  $i \geq 0$ ,  $\{a_i^b\}_b$  is a Cauchy sequence. Let  $a_i \in A$  be its limit and set

$$f = \sum_{i=0}^{\infty} a_i T^i \in A[[T]].$$

We need to show that  $f \in A\{r^{-1}T\}$  and  $f^b \rightarrow f$ .

Fix  $\epsilon > 0$ . We can find  $m = m(\epsilon) > 0$  such that for all  $j \geq m$  and all  $k \geq 0$ ,

$$\|f^j - f^{j+k}\|_r \leq \epsilon.$$

It follows that  $\|a_i^j - a_i^{j+k}\| r^i \leq \epsilon$  for all  $i \geq 0$ . Let  $k \rightarrow \infty$ , we find

$$\|a_i^j - a_i\| r^i \leq \epsilon$$

for all  $i \geq 0$ . Fix  $j \geq 0$ , take  $i$  large enough so that  $|a_i^j| r^i < \epsilon$ . Then  $\|a_i\| r^i \leq \epsilon$ . So we find  $f \in A\{r^{-1}T\}$ . On the other hand,

$$\|f - f^j\|_r = \max_i \|a_i^j - a_i\| r^i \leq \epsilon.$$

This proves that  $f^j \rightarrow f$ .

Now assume that  $\|\bullet\|$  is a valuation, we verify that  $\|\bullet\|_r$  is also a valuation. Again, we may assume that  $n = 1$ . Take two elements  $f, g \in A\{r^{-1}T\}$ :

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad g = \sum_{j=0}^{\infty} b_j T^j.$$

As we have already shown  $|fg|_r \leq |f|_r |g|_r$ , it suffices to check the reverse inequality. For this purpose, choose the minimal indices  $i, j$  so that

$$\|f\|_r = \|a_i\| r^i, \quad \|g\|_r = \|b_j\| r^j.$$

Write

$$fg = \sum_{k=0}^{\infty} \left( \sum_{p+q=k} a_p b_q \right) T^k.$$

Then we claim that

$$\left\| \sum_{p+q=k} a_p b_q \right\| r^k = \|f\|_r \|g\|_r$$

when  $k = i + j$ . This implies the desired inequality. Of course, we may assume that  $a_i \neq 0$  and  $b_j \neq 0$  as otherwise there is nothing to prove. To verify our claim, it suffices to observe that for  $(p, q) \neq (i, j)$ ,  $r + s = i + j$ , say  $p < i$  and  $q > j$ , we have

$$\|a_p b_q\| r^k = \|a_p\| r^p \cdot \|b_q\| r^q < \|a_i\| r^i \cdot \|b_j\| r^j.$$

So

$$\|a_p b_q\| < \|a_i b_j\|.$$

Since the valuation on  $A$  is non-Archimedean, it follows that

$$\left\| \sum_{p+q=k} a_p b_q \right\| = \|a_i b_j\|.$$

Our claim follows.  $\square$

**Remark 4.17.** More generally, if  $A$  is endowed with a semi-valuation  $\|\bullet\|'$ , then the same procedure and the same proof produces a semi-valuation on  $A\{r^{-1}T\}$ .

**Proposition 4.18.** Let  $A, B$  be a non-Archimedean Banach ring and  $f : A \rightarrow B$  be a continuous homomorphism. Then for any  $b \in \mathring{B}$ , there is a unique continuous homomorphism  $F : A\{T\} \rightarrow B$  extending  $f$  and sending  $T$  to  $b$ .

PROOF. From the continuity and the fact that  $A[T]$  is dense in  $A\{T\}$ ,  $F$  is clearly unique. To prove the existence, we define  $F$  directly: consider  $g = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}$ , we define

$$F(g) := \sum_{i=0}^{\infty} f(a_i) b^i.$$

As  $f_i \in \mathring{A}$  and  $a_i \rightarrow 0$ , the right-hand side is well-defined. It is straightforward to check that  $F$  is a continuous homomorphism.  $\square$

**Proposition 4.19.** For any non-Archimedean Banach ring  $(A, \|\bullet\|)$ , we have

$$(A\{T\})^\circ = \mathring{A}\{T\}, \quad (A\{T\})^\check{} = \check{A}\{T\}.$$

For the definitions of  $\mathring{\bullet}$  and  $\check{\bullet}$ , we refer to [Definition 3.4](#).

PROOF. We first show that

$$\mathring{A}\{T\} \subseteq (A\{T\})^\circ.$$

Let  $f \in \mathring{A}\{T\}$ . We expand  $f$  as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \mathring{A}.$$

Then for each  $i, j \in \mathbb{N}$ ,  $\|a_i T^i\|_1^j = \|a_i\|^j$ . So for each  $i \in \mathbb{N}$ ,  $a_i T^i \in (A\{T\})^\circ$ . By **Proposition 3.5**, it follows that  $f \in (A\{T\})^\circ$ .

Next we prove the reverse inclusion. Take  $f \in (A\{T\})^\circ$ , suppose by contrary that  $f \notin \mathring{A}\{T\}$ . Expand  $f$  as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal  $m \in \mathbb{N}$  so that  $a_m \notin \mathring{A}$ . Then  $\sum_{i=0}^{m-1} a_i T^i \in \mathring{A}\{T\} \subseteq (A\{T\})^\circ$  by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})^\circ.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any  $j \in \mathbb{N}$ . It follows that  $a_m \in \mathring{A}$ , which is a contradiction.

Next we show that

$$\check{A}\{T\} \subseteq (A\{T\})^\vee.$$

Let  $f \in \check{A}\{T\}$ . We expand  $f$  as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \check{A}.$$

Then for each  $i, j \in \mathbb{N}$ ,  $\|a_i T^i\|_1^j = \|a_i\|^j$ . So for each  $i \in \mathbb{N}$ ,  $a_i T^i \in (A\{T\})^\vee$ . By **Proposition 3.5**, it follows that  $f \in (A\{T\})^\vee$ .

Conversely, take  $f \in (A\{T\})^\vee$ , suppose by contrary that  $f \notin \check{A}\{T\}$ . Expand  $f$  as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal  $m \in \mathbb{N}$  so that  $a_m \notin \check{A}$ . Then  $\sum_{i=0}^{m-1} a_i T^i \in \check{A}\{T\} \subseteq (A\{T\})^\vee$  by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})^\vee.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any  $j \in \mathbb{N}$ . It follows that  $a_m \in \check{A}$ , which is a contradiction.  $\square$

**Corollary 4.20.** For any non-Archimedean Banach ring  $(A, \|\bullet\|)$ , we have a canonical isomorphism

$$\widetilde{A\{T\}} \cong \tilde{A}[T].$$

The natural map  $A\{T\}^\circ \rightarrow \widetilde{A\{T\}}$  corresponds to a homomorphism  $\mathring{A}\{T\} \rightarrow \tilde{A}[T]$  extending the homomorphism  $\mathring{A} \rightarrow \tilde{A}$  and sending  $T$  to  $T$ .

PROOF. Let  $f = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}^\circ$ . Then  $a_i \in \mathring{A}$  by [Proposition 4.19](#). But  $\|a_i\| \rightarrow 0$  as  $i \rightarrow \infty$ , so  $a_i \in \tilde{A}$  for almost all  $i$ . It follows that the image of  $f$  in  $\widetilde{A\{T\}}$  is the same as the image of an element from  $\mathring{A}[T]$ . On the other hand, for each  $f \in \tilde{A}[T]$ , we can expand  $f = a_N T^N + \cdots + a_1 T^1 + a_0$  with  $a_N \in \tilde{A}$ . Lift each  $a_i$  to  $b_i \in \mathring{A}$ . Then the image of  $b_N T^N + \cdots + b_1 T^1 + b_0$  under the reduction corresponds to  $f$ . The assertions follow.  $\square$

**Corollary 4.21.** Let  $(A, \|\bullet\|)$  be a non-Archimedean Banach ring. An element  $f = \sum_{i=0}^{\infty} a_i T^i \in \mathring{A}\{T\}$  is a unit in  $\mathring{A}\{T\}$  if and only if  $a_0$  is a unit in  $\mathring{A}$  and  $a_i \in \tilde{A}$  for all  $i > 0$ .

PROOF. By [Proposition 4.16](#), we know that  $A\{T\}$  is complete. According to [Lemma 4.8](#) and [Proposition 4.19](#),  $f$  is a unit in  $\mathring{A}\{T\}$  if and only if  $\sum_{i=0}^{\infty} \tilde{a}_i T^i$  is a unit in  $\tilde{A}[T]$ . By [Lemma 4.8](#) again,  $a_0$  is a unit in  $\mathring{A}$  if and only if  $\tilde{a}_0$  is a unit in  $\tilde{A}$ . So we are reduced to argue that units in  $\tilde{A}[T]$  are exactly units in  $\tilde{A}$ . This follows from the general fact about units in polynomial rings over a reduced ring.  $\square$

**Lemma 4.22.** Let  $A, B$  be Banach rings and  $f, g : A \rightarrow B$  be bounded homomorphisms. Then the equalizer  $\text{Eq}(f, g)$  of  $f$  and  $g$  is a Banach subring of  $A$ .

If  $C$  is a Banach ring,  $A, B$  are moreover Banach  $C$ -algebras and  $f, g$  are moreover bounded homomorphism of Banach  $C$ -algebras, then  $\text{Eq}(f, g)$  is a Banach  $C$ -subalgebra of  $A$ .

PROOF. As an equalizer of ring homomorphisms,  $\text{Eq}(f, g)$  is a subring of  $A$ . We can realize  $\text{Eq}(f, g) = \ker(f - g)$ , so  $\text{Eq}(f, g)$  is a closed subring of  $A$ , hence a Banach subring with respect to the subspace norm.

The second assertion is proved similarly.  $\square$

## 5. Semi-normed modules

**Definition 5.1.** Let  $(A, \|\bullet\|_A)$  be a normed ring. A *semi-normed  $A$ -module* (resp. *normed  $A$ -module*) is a pair  $(M, \|\bullet\|_M)$  consisting of an  $A$ -module  $M$  and a semi-norm (resp. norm) on the underlying Abelian group of  $M$  such that there is a constant  $C > 0$  such that

$$\|fm\|_M \leq C\|f\|_A\|m\|_M$$

for all  $f \in A$  and  $m \in M$ . In case  $\|\bullet\|_A$  is non-Archimedean, we require that  $\|\bullet\|_M$  is also non-Archimedean.

We say the semi-normed  $A$ -module (resp. normed  $A$ -module)  $M$  is faithful if we can take  $C = 1$ .

When  $\|\bullet\|_M$  is clear from the context, we say  $M$  is a semi-normed  $A$ -module (resp. normed  $A$ -module).

An  $A$ -module homomorphism  $\varphi : M \rightarrow N$  between two semi-normed  $A$ -modules  $M$  and  $N$  is *bounded* if the homomorphism of the underlying semi-normed Abelian groups is bounded in the sense of [Definition 2.5](#).

A *Banach  $A$ -module* is a normed  $A$ -module which is complete with respect to the metric [Lemma 2.6](#).

We denote by  $\mathcal{Ban}_A$  the category of Banach  $A$ -modules with bounded  $A$ -module homomorphisms as morphisms.

**Definition 5.2.** Let  $A$  be a Banach ring and  $(M, \|\bullet\|_M), (N, \|\bullet\|_N)$  be two Banach  $A$ -modules. Define their *direct sum* as the Banach  $A$ -module  $(M \oplus N, \|\bullet\|_{M \oplus N})$ , where for  $m \in M, n \in N$ , we set

$$\|(m, n)\|_{M \oplus N} := \max\{\|m\|_M, \|n\|_N\}.$$

This definition extends immediately to finite direct sums of Banach  $A$ -modules.

**Definition 5.3.** Let  $A$  be a Banach ring. A Banach  $A$ -module  $M$  is said to be *finite* if there is  $n \in \mathbb{N}$  and an admissible epimorphism  $A^n \rightarrow M$ .

A morphism between finite  $A$  modules  $M$  and  $N$  is a morphism  $M \rightarrow N$  in  $\mathcal{Ban}_A$ . We write  $\mathcal{Ban}_A^f$  for the category of finite Banach  $A$ -modules.

**Definition 5.4.** Let  $A$  be a semi-normed ring and  $M$  be a semi-normed  $A$ -module. There is an obvious  $\hat{A}$ -module structure on the completion  $\hat{M}$  of  $A$  defined in [Definition 2.9](#). We call the resulting Banach module the *completion* of  $M$ .

**Definition 5.5.** Let  $A$  be a non-Archimedean semi-normed ring. Consider semi-normed  $A$ -modules  $(M, \|\bullet\|_M)$  and  $(N, \|\bullet\|_N)$ . We define the *tensor product* of  $(M, \|\bullet\|_M)$  and  $(N, \|\bullet\|_N)$  as the semi-normed  $A$ -module  $(M \otimes N, \|\bullet\|_{M \otimes N})$ , where

$$\|x\|_{M \otimes N} = \inf \max_i (\|m_i\|_M \cdot \|n_i\|_N),$$

where the infimum is taken over all decompositions  $x = \sum_i m_i \otimes n_i$ .

**Definition 5.6.** Let  $A$  be a non-Archimedean Banach ring. Consider semi-normed  $A$ -modules  $M$  and  $N$ , we define the *complete tensor product* of  $M$  and  $N$  as the metric completion  $M \hat{\otimes}_A N$  of the tensor product of  $M$  and  $N$  defined in [Definition 5.5](#).

**Theorem 5.7.** Let  $(A, \|\bullet\|_A)$  be a normed ring. Then  $\mathcal{Ban}_A$  is a quasi-Abelian category.

**PROOF.** We first observe that  $\mathcal{Ban}_A$  is preadditive, as for any  $M, N \in \mathcal{Ban}_A$ ,  $\text{Hom}_{\mathcal{Ban}_A}(M, N)$  can be given the group structure inherited from the Abelian group  $\text{Hom}_A(M, N)$ . It is obvious that  $\mathcal{Ban}_A$  is preadditive.

Next we show that finite biproducts exist in  $\mathcal{Ban}_A$ . Given  $(M, \|\bullet\|_M), (N, \|\bullet\|_N) \in \mathcal{Ban}_A$ , we set

$$(5.1) \quad (M, \|\bullet\|_M) \oplus (N, \|\bullet\|_N) := (M \oplus N, \|\bullet\|_{M \oplus N}),$$

where  $\|(m, n)\|_{M \oplus N} := \|m\|_M + \|n\|_N$  for  $m \in M$  and  $n \in N$ . It is easy to verify that this gives the biproduct in  $\mathcal{Ban}_A$ .

We have shown that  $\mathcal{Ban}_A$  is an additive category.

Next given a morphism  $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$  in  $\mathcal{Ban}_A$ , we construct its kernel  $(\ker \varphi, \|\bullet\|_{\ker \varphi})$  as the kernel of the underlying homomorphism of  $A$ -modules of  $\varphi$  endowed with the subgroup semi-norm induced from  $\|\bullet\|_M$  as in [Definition 2.3](#). It is easy to verify that  $(\ker \varphi, \|\bullet\|_{\ker \varphi})$  is the kernel of  $\varphi$  in  $\mathcal{Ban}_A$ .

We can similarly construct the cokernels. To be more precise, let  $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$  be a morphism in  $\mathcal{Ban}_A$ , then the coker  $\varphi = \{N/\overline{\varphi(M)}\}$  with quotient norm.

We have shown that  $\mathcal{Ban}_A$  is a pre-Abelian category.

Observe that given a morphism  $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$  in  $\mathcal{Ban}_A$ , its image is given by  $\text{Im } \varphi = \overline{\varphi(M)}$  with the subspace norm induced from  $N$ ; its coimage is  $M/\ker f$  with the residue norm. The morphism  $\varphi$  is admissible if the natural map

$$M/\ker f \rightarrow \overline{\varphi(M)}$$

is an isomorphism in  $\mathcal{Ban}_A$ .

It remains to show that pullbacks preserve admissible epimorphisms and pushouts preserve admissible monomorphisms. We first handle the case of admissible epimorphisms. Consider a Cartesian square in  $\mathcal{Ban}_A$ :

$$\begin{array}{ccc} M & \xrightarrow{p} & U \\ \downarrow q & \square & \downarrow f \\ V & \xrightarrow{g} & W \end{array}$$

with  $g$  being an admissible epimorphism. We need to show that  $p$  is also an admissible epimorphism, namely  $U \cong M/\ker p$ .

We define  $\alpha : U \oplus V \rightarrow W$ ,  $\alpha = (f, -g)$ , then there is a natural isomorphism  $j : M \rightarrow \ker \alpha$ . Let us write  $i : \ker \alpha \rightarrow U \oplus V$  the natural morphism. Then

$$q = \pi_V \circ i \circ j, \quad p = \pi_U \circ i \circ j,$$

where  $\pi_U : U \oplus V \rightarrow U$ ,  $\pi_V : U \oplus V \rightarrow V$  are the natural morphisms. We may assume that  $M = \ker \alpha$  and  $j$  is the identity. Then it is obvious that  $p$  is surjective on the underlying sets. In order to compute the quotient norm on  $M/\ker p$ , we need a more explicit description of  $\ker p \subseteq \ker \alpha$ . We know that

$$\ker \alpha = \{(u, v) \in U \oplus V : f(u) = g(v)\}$$

with the subspace norm induced from the product norm on  $U \oplus V$  defined in (5.1). Then

$$\ker p = \{(u, v) \in U \oplus V : u = 0, g(v) = 0\}.$$

It follows that for  $(u, v) \in \ker \alpha$ ,

$$\inf_{(u', v') \in \ker p} \|(u, v) + (u', v')\|_{U \oplus V} = \inf_{v' \in \ker g} (\|v + v'\|_V) + \|u\|_U,$$

where  $\|\bullet\|_U$  and  $\|\bullet\|_V$  denote the norms on  $U$  and  $V$  respectively. By our assumption that  $g$  is an admissible epimorphism, there is a constant  $C > 0$  so that

$$\inf_{v' \in \ker g} (\|v + v'\|_V) \leq C \|g(v)\|_W$$

for any  $v \in V$ . As  $f$  is bounded, we can also find a constant  $C' > 0$  so that for any  $(u, v) \in \ker \alpha$ ,

$$\|g(v)\|_W = \|f(u)\|_W \leq C' \|u\|_U.$$

It follows that  $p$  is admissible epimorphism.

It remains to check that the pushforwards preserve admissible monomorphisms. Consider a co-Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & U \\ \downarrow f & & \downarrow q \\ V & \xrightarrow{p} & M \end{array}$$

with  $g$  being an admissible monomorphism. We need to show that  $p$  is an admissible monomorphism. This boils down to the following:  $p$  is injective with closed image



and the norms on  $p(V)$  obtained in the obvious ways are equivalent. As in the case of pullbacks, we may let  $\alpha : W \rightarrow U \oplus V$  be the morphism  $(g, -f)$  and assume that  $M = \text{coker } \alpha$ . It is then easy to see that  $p$  is injective. The proof that the two norms on  $p(V)$  are equivalent is parallel to the argument in the pull-back case, and we omit it.

It remains to verify that  $p(V)$  is closed in  $W$ . Consider the admissibly coexact sequence in  $\mathcal{B}\text{an}_A$ :

$$W \xrightarrow{\alpha} U \oplus V \xrightarrow{\pi} M \rightarrow 0.$$

It is also admissibly coexact in the category of semi-normed  $A$ -modules. **Include details later.** Let  $x_n \in V$  be a sequence so that  $p(x_n) \rightarrow y \in M$ . We may write  $y = \pi(u, v)$  for some  $(u, v) \in U \oplus V$ . Then

$$\pi(-u, x_n - v) \rightarrow 0$$

as  $n \rightarrow \infty$ . From the strict coexact sequence, we can find a sequence  $w_n \in W$  so that

$$(-u - g(w_n), x_n - v + f(w_n)) \rightarrow 0$$

as  $n \rightarrow \infty$ . Then  $g(w_n) \rightarrow -u$  in  $U$  and hence there is  $w \in W$  so that  $w_n \rightarrow w \in W$  and  $g(w) = -u$ . But then  $x_n \rightarrow x$  and  $p(x) = y$ .  $\square$

**Definition 5.8.** Let  $(A, \|\bullet\|_A)$  be a normed ring. A *Banach  $A$ -algebra* is a pair  $(B, \|\bullet\|_B)$  such that  $(B, \|\bullet\|_B)$  is a Banach  $A$ -module and  $(B, \|\bullet\|_B)$  is a Banach ring.

A morphism of Banach  $A$ -algebras is a bounded  $A$ -algebra homomorphism. The category of Banach  $A$ -algebras is denoted by  $\mathcal{B}\text{anAlg}_A$ .

**Definition 5.9.** Let  $A$  be a normed ring. A Banach  $A$ -algebra  $B$  is said to be *finite* if  $B$  is finite as a Banach  $A$ -module. A morphism of finite Banach  $A$ -algebras is a morphism in  $\mathcal{B}\text{anAlg}_A$ . The category of finite Banach  $A$ -algebras is denoted by  $\mathcal{B}\text{anAlg}_A^f$ .

## 6. Berkovich spectra

**Definition 6.1.** Let  $(A, \|\bullet\|_A)$  be a Banach ring. A semi-norm  $|\bullet|$  on  $A$  is *bounded* if there is a constant  $C > 0$  such that for any  $f \in A$ ,  $|f| \leq C\|f\|_A$ .

Write  $\text{Sp } A$  for the set of bounded semi-valuations on  $A$ . We call  $\text{Sp } A$  the *Berkovich spectrum* of  $A$ .

We endow  $\text{Sp } A$  with the weakest topology such that for each  $f \in A$ , the map  $\text{Sp } A \rightarrow \mathbb{R}_{\geq 0}$  sending  $\|\bullet\|$  to  $\|f\|$  is continuous.

It is sometimes preferable to denote an element  $\|\bullet\|$  in  $\text{Sp } A$  by a single letter  $x$ . In this case, we write  $|f(x)| = \|f\|$  for any  $f \in A$ .

Given a bounded homomorphism  $\varphi : A \rightarrow B$  of Banach rings, we define  $\text{Sp } \varphi : \text{Sp } B \rightarrow \text{Sp } A$  as follows: given a bounded semi-valuation  $\|\bullet\|$  on  $B$ , we define  $\text{Sp } \varphi(\|\bullet\|)$  as the bounded semi-valuation on  $A$  sending  $f \in A$  to  $\|\varphi(f)\|$ .

Observe that there is a natural map of sets:

$$(6.1) \quad \text{Sp } A \rightarrow \{\mathfrak{p} \in \text{Spec } A : \mathfrak{p} \text{ is closed.}\}$$

sending each bounded semi-valuation to its kernel. The fiber over a closed ideal  $\mathfrak{p} \in \text{Spec } A$  is identified with the set of bounded valuations on  $A/\mathfrak{p}$ . Here boundedness is with respect to the residue norm.

**Remark 6.2.** In the literature, it is more common to denote  $\mathrm{Sp} A$  by  $\mathcal{M}(A)$ .

**Lemma 6.3.** Let  $(A, \|\bullet\|_A)$  be a Banach ring. Then for any  $x \in \mathrm{Sp} A$ , we have

$$|f(x)| \leq \rho(f) \leq \|f\|_A.$$

PROOF. Let  $\|\bullet\|_x$  be the bounded semi-valuation corresponding to  $x$ . Then there is a constant  $C > 0$  such that

$$\|\bullet\|_x \leq C\|\bullet\|_A.$$

It follows that for any  $n \in \mathbb{N}$ ,

$$\|f\|_x^n = \|f^n\|_x \leq C\|f^n\|_A.$$

Taking  $n$ -th root and letting  $n \rightarrow \infty$ , we find

$$\|f\|_x \leq \rho(f).$$

The inequality  $\rho(f) \leq \|f\|_A$  follows from the definition of  $\rho$ .  $\square$

**Example 6.4.** If  $(k, |\bullet|)$  is a complete valuation field, then  $\mathrm{Sp} k$  is a single point  $|\bullet|$ .

To see this, let  $\|\bullet\| \in \mathrm{Sp} k$ , then by [Lemma 6.3](#),

$$\|f\| \leq |f|$$

for any  $f \in k$ . If  $f \neq 0$ , the same inequality applied to  $f^{-1}$  implies that  $\|f\| = |f|$ . When  $f = 0$ , the equality is trivial.

**Example 6.5.** Let  $\{K_i\}_{i \in I}$  be a family of complete valuation fields. Recall that  $\prod_{i \in I} K_i$  is defined in [Example 4.12](#). Then  $\mathrm{Sp} \prod_{i \in I} K_i$  is homeomorphic to the Stone–Čech compactification of the discrete set  $I$ .

To see this, we first identify the set of proper closed ideals in  $\prod_{i \in I} K_i$  with the set of filters on  $I$ .

We first introduce a notation: for each  $J \subseteq I$ , we write  $a_J \in \prod_{i \in I} K_i$  for the element

$$a_{J,i} = \begin{cases} 0, & \text{if } i \in J; \\ 1, & \text{if } i \notin J. \end{cases}$$

Given a proper closed ideal  $\mathfrak{a} \subseteq \prod_{i \in I} K_i$ , we define a filter  $\Phi_{\mathfrak{a}} = \{J \subseteq I : a_J \in \mathfrak{a}\}$ . Conversely, given a filter  $\Phi$  on  $I$ , we denote by  $\mathfrak{a}_{\Phi}$  the closed ideal of  $\prod_{i \in I} K_i$  generated by  $a_J$  for all  $J \in \Phi$ . These maps are inverse to each other and order preserving. In particular, the maximal ideals of  $\prod_{i \in I} K_i$  are identified with ultrafilters of  $I$  by [Corollary 4.7](#).

Next we show that all prime ideals of  $\prod_{i \in I} K_i$  are maximal. In fact, take  $\mathfrak{p} \in \mathrm{Spec} \prod_{i \in I} K_i$  and suppose that there is a maximal ideal  $\mathfrak{m}$  properly containing  $\mathfrak{p}$ . Let  $J \in \Phi_{\mathfrak{m}} \setminus \Phi_{\mathfrak{p}}$  so that  $a_J \in \mathfrak{m} \setminus \mathfrak{p}$ . As  $I \setminus J \notin \Phi_{\mathfrak{m}}$ , we have  $a_{I \setminus J} \notin \mathfrak{m}$ . But  $a_J \cdot a_{I \setminus J} = 0$ . This contradicts the fact that  $a_J \notin \mathfrak{p}$  and  $a_{I \setminus J} \notin \mathfrak{p}$ .

So we have shown that as a set  $\mathrm{Spec} \prod_{i \in I} K_i$  is identified with the Stone–Čech compactification of  $I$ .

Next we show that if  $\mathfrak{m} \in \mathrm{Spec} \prod_{i \in I} K_i$ , then the residue norm on  $\prod_{i \in I} K_i / \mathfrak{m}$  is multiplicative. In fact, for each  $f \in \prod_{i \in I} K_i$ , we have

$$\|\pi(f)\|_{\prod_{i \in I} K_i / \mathfrak{m}} = \inf_{J \in \Phi_{\mathfrak{m}}} \sup_{i \in J} \|f\|.$$

Here  $\pi : \prod_{i \in I} K_i \rightarrow \prod_{i \in I} K_i / \mathfrak{m}$  is the natural map and  $\|\bullet\|$  denotes the norm on  $\prod_{i \in I} K_i$  defined in [Example 4.12](#). It follows immediately that the residue norm

on  $\prod_{i \in I} K_i / \mathfrak{m}$  is multiplicative. In particular, by [Example 6.4](#),  $\mathrm{Sp} \prod_{i \in I} K_i$  and  $\mathrm{Spec} \prod_{i \in I} K_i$  are identified as sets under the natural map [\(6.1\)](#).

It remains to identify the topologies. But this is easy: for any ultrafilter  $\Phi$  on  $I$ , let  $\mathfrak{m} = \mathfrak{m}_\Phi$ , then  $\|\pi(a_J)\| = 0$  for  $J \in \Phi$  and  $\|\pi(a_J)\| = 1$  otherwise.

**Proposition 6.6.** Let  $\varphi : A \rightarrow B$  be a bounded homomorphism of Banach rings, then  $\mathrm{Sp} \varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  is continuous.

PROOF. For each  $f \in A$ , we define  $\mathrm{ev}_f : \mathrm{Sp} A \rightarrow \mathbb{R}$  by sending  $\|\bullet\|$  to  $\|f\|$ . It suffices to show that for any  $f \in A$ , the map  $\mathrm{Sp} \varphi \circ \mathrm{ev}_f$  is continuous. But the composition is just the map sending  $\|\bullet\| \in \mathrm{Sp} B$  to  $\|\varphi(f)\|$ . It is continuous by definition of the topology on  $\mathrm{Sp} B$  as  $\varphi$  is bounded.  $\square$

**Definition 6.7.** Let  $(A, \|\bullet\|_A)$  be a Banach ring. For each  $x \in \mathrm{Sp} A$  corresponding to a bounded semi-valuation  $\|\bullet\|_x$  on  $A$ , there is a natural induced valuation on  $\mathrm{Frac} \ker \|\bullet\|_x$ . We write  $\mathcal{H}(x)$  for the completion of  $\mathrm{Frac} \ker \|\bullet\|_x$  with the induced valuation. The complete valuation field  $\mathcal{H}(x)$  is called the *complete residue field* of  $A$  at  $x$ . We write  $\chi_x : A \rightarrow \mathcal{H}(x)$  the canonical map.

We will write  $f(x)$  for the residue class of  $f$  in  $\mathcal{H}(x)$ .

Observe that for any  $f \in A$ ,  $|f(x)|$  is exactly the valuation of  $f(x)$  with respect to the valuation on  $\mathcal{H}(x)$ .

**Definition 6.8.** Let  $A$  be a Banach ring. The *Gelfand transform* of  $A$  is the homomorphism

$$A \rightarrow \prod_{x \in \mathrm{Sp} A} \mathcal{H}(x).$$

Here the product is defined in [Example 4.12](#).

We will denote the Gelfand transform as  $f \mapsto \hat{f} = (f(x))_{x \in \mathrm{Sp} A}$ .

By [Lemma 6.3](#), the Gelfand transform is well-defined.

**Proposition 6.9.** Let  $(A, \|\bullet\|_A)$  be a Banach ring. Then the Gelfand transform

$$A \rightarrow \prod_{x \in \mathrm{Sp} A} \mathcal{H}(x)$$

is bounded. In fact, the Gelfand transform is contractive.

PROOF. This follows simply from [Lemma 6.3](#).  $\square$

**Proposition 6.10.** Let  $(A, \|\bullet\|)$  be a Banach ring. Then  $\mathrm{Sp} A$  is empty if and only if  $A = 0$ .

PROOF. If  $A = 0$ ,  $\mathrm{Sp} A$  is clearly empty. Conversely, suppose that  $\mathrm{Sp} A$  is empty. Assume that  $A \neq 0$ . For any maximal ideal  $\mathfrak{m}$ , by [Corollary 4.7](#),  $A/\mathfrak{m}$  is a Banach ring and  $\mathrm{Sp} A/\mathfrak{m}$  is a subset of  $\mathrm{Sp} A$ . So we may assume that  $A$  is a field. Let  $S$  be the set of bounded semi-norms on  $A$ . Then  $S$  is non-empty as  $\|\bullet\| \in S$ . By Zorn's lemma, we can take a minimal element  $|\bullet| \in S$ . Up to replacing  $A$  by the completion with respect to  $|\bullet|$ , we may assume that  $|\bullet|$  is a norm on  $A$ . As  $A$  is a field, we may further assume that  $|\bullet| = \|\bullet\|$ .

We claim that  $\|\bullet\|$  is multiplicative. As  $A$  is a field, it suffices to show that  $\|f^{-1}\| = \|f\|^{-1}$  for any non-zero  $f \in A$ . We may assume that  $\|f\|^{-1} < \|f^{-1}\|$ .

Let  $r$  be a positive real number. Let  $\varphi : A \rightarrow A\{r^{-1}T\}/(T - f)$  be the natural map. The map is injective as  $A$  is a field. We endow  $A\{r^{-1}T\}/(T - f)$  with the quotient semi-norm induced by  $\|\bullet\|_r$  and still denote this semi-norm by  $\|\bullet\|_r$ .

We claim that  $f - T$  is not invertible in  $A\{r^{-1}T\}$  for the choice  $r = \|f^{-1}\|^{-1}$ . From this, it follows that

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

The last step is our assumption. This contradicts our choice of  $\|\bullet\|$ .

In order to prove the claim, we need to show that  $\|\bullet\|$  is power multiplicative first. Assuming this, it is obvious that

$$\sum_{i=0}^{\infty} |f^{-i}| r^i = \sum_{i=0}^{\infty} |f^{-1}|^i |f^{-1}|^{-i}$$

diverges.

It remains to show that  $\|\bullet\|$  is power multiplicative. Suppose that is  $f \in A$  so that  $\|f^n\| < \|f\|^n$  for some  $n > 1$ . We claim that  $f - T$  is not invertible in  $A\{r^{-1}T\}$  for the choice  $r = \|f^n\|^{1/n}$ . From this,

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

This contradicts our choice of  $\|\bullet\|$ . The claim amounts to the divergence of

$$\sum_{i=0}^{\infty} \|f^{-i}\| r^i.$$

For a general  $i \geq 0$ , we write  $i = pn + q$  for  $p, q \in \mathbb{N}$  and  $q \leq n - 1$ . Then  $\|f^i\| \leq \|f^n\|^p \|f^q\|$ . So

$$\|f^{-i}\| r^i \geq \|f^i\|^{-1} \|f^n\|^{p+n^{-1}q} \geq \|f^n\|^{n^{-1}q} \|f^q\|^{-1}.$$

It therefore follows that  $|f^{-i}| r^i$  admits a positive lower bound, and we conclude.  $\square$

**Corollary 6.11.** Let  $A$  be a Banach ring. Then an element  $f \in A$  is invertible if and only if  $f(x) \neq 0$  for all  $x \in \text{Sp } A$ .

PROOF. The direct implication is trivial. Assume that  $f(x) \neq 0$  for all  $x \in \text{Sp } A$ . We claim that  $f \notin \mathfrak{m}$  for any maximal ideal  $\mathfrak{m}$  in  $A$ . From this, it follows that  $f$  is invertible in  $A$ .

By [Corollary 4.7](#),  $A/\mathfrak{m}$  is a Banach ring. It follows from [Proposition 6.10](#) that there is a non-trivial bounded semi-valuation on  $A/\mathfrak{m}$ , which lifts to a bounded semi-valuation on  $A$ .  $\square$

**Corollary 6.12.** Let  $(A, \|\bullet\|_A)$  be a Banach ring. Then for any  $f \in A$ , we have

$$\rho(f) = \sup_{x \in \text{Sp } A} |f(x)|.$$

PROOF. We have already shown  $\rho(f) \geq \sup_{x \in \text{Sp } A} |f(x)|$  in [Lemma 6.3](#). To verify the reverse inequality, take  $f \in A$  and  $r \in \mathbb{R}_{>0}$ , it suffices to show that if  $|f(x)| < r$  for all  $x \in \text{Sp } A$ , then  $\rho(f) \leq r$ .

Consider the Banach ring  $B = A\{rT\}$ . By [Lemma 6.3](#) again,  $|T(x)| \leq \|T\|_{r^{-1}} = r^{-1}$  for all  $x \in \text{Sp } B$ . Therefore, for any  $x \in \text{Sp } B$ ,  $|(fT)(x)| < 1$ . Hence,  $(1 -$

$fT)(x) \neq 0$  for all  $x \in \text{Sp } B$ . By [Corollary 6.11](#),  $1 - fT$  is invertible in  $B$ . But this happens exactly when

$$\sum_{i=0}^{\infty} \|f^i\|_A r^{-i}$$

is convergent. It follows that  $\rho(f) \leq r$ .  $\square$

**Theorem 6.13.** Let  $(A, \|\bullet\|)$  be a Banach ring. Then  $\text{Sp } A$  is a compact Hausdorff space.

PROOF. We first show that  $\text{Sp } A$  is Hausdorff. Take  $x_1, x_2 \in A$ ,  $x_1 \neq x_2$ . In other words, we can find  $f \in A$  so that  $|f(x_1)| \neq |f(x_2)|$ . We may assume that  $|f(x_1)| < |f(x_2)|$ . Take a real number  $r > 0$  so that

$$|f(x_1)| < r < |f(x_2)|.$$

Then  $\{x \in \text{Sp } A : |f(x)| < r\}$  and  $\{x \in \text{Sp } A : |f(x)| > r\}$  are disjoint neighbourhoods of  $x_1$  and  $x_2$ .

Next we show that  $\text{Sp } A$  is compact. By [Proposition 6.9](#) and [Proposition 6.6](#), we can define a continuous map

$$\text{Sp } \prod_{x \in \text{Sp } A} \mathcal{H}(x) \rightarrow \text{Sp } A.$$

The map is clearly surjective: for any  $x \in \text{Sp } A$ , the valuation on  $\mathcal{H}(x)$  induces a semi-valuation on  $\prod_{x \in \text{Sp } A} \mathcal{H}(x)$ , which is clearly bounded. The image of this semi-valuation in  $\text{Sp } A$  is just  $x$ .

So it suffices to show that  $\text{Sp } \prod_{x \in \text{Sp } A} \mathcal{H}(x)$  is compact. This follows from [Example 6.5](#).  $\square$

## 7. Open mapping theorem

Let  $(k, |\bullet|)$  be a complete non-trivially valued field. All results in this section fail when  $k$  is trivially valued.

**Proposition 7.1.** Let  $A$  be a normed  $k$ -algebra and  $f : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$  be an  $A$ -homomorphism of normed  $A$ -modules. Then  $f$  is bounded if and only if  $f$  is continuous.

PROOF. The direct implication follows from [Proposition 2.7](#). Assume that  $f$  is continuous. We may assume that  $A = k$ .

Assume that  $f$  is not bounded. Fix  $a \in k$  with  $|a| \in (0, 1)$ . This is possible as  $k$  is non-trivially valued. Then we can find a sequence  $m_i \in M$  such that  $\|f(m_i)\|_N > |a|^{-i} \|m_i\|_M$ . Up to replace  $m_i$  by a scalar multiple, we may assume that  $\|m_i\|_M \in [1, |a|^{-1})$ : if  $\|m_i\|_M \geq 1$ , choose  $n \in \mathbb{N}$  such that  $|a|^{-n} \leq \|m_i\|_M < |a|^{-n-1}$ , then replace  $m_i$  with  $a^n m_i$ . The case  $|x| < 1$  is similar. Then  $\|f(a^i m_i)\|_N > \|m_i\|_M \geq 1$  while  $\|a^i m_i\|_M < |a|^n |a|^{-1} \rightarrow 0$ . This is a contradiction.  $\square$

**Theorem 7.2** (Open mapping theorem). Let  $(V, \|\bullet\|_V), (W, \|\bullet\|_W)$  be Banach  $k$ -spaces and  $L : V \rightarrow W$  be a bounded and surjective  $k$ -homomorphism. Then  $L$  is open.

PROOF. We write  $V_0 = \{v \in V : \|v\|_V < 1\}$ . Similarly define  $W_0$ .

**Step 1.** We claim that there is a constant  $C > 0$  such that for all  $w' \in W$ , there is  $v' \in V$  such that

$$\|v'\|_V \leq C\|w'\|_W, \quad \|w' - L(v')\|_W < 1/2.$$

As  $k$  is non-trivially valued, we can take  $c \in k$  with  $|c| \in (0, 1)$ , so

$$V = \bigcup_{n \in \mathbb{N}} c^n V_0.$$

As  $L$  is surjective, we have

$$W = \bigcup_{n \in \mathbb{N}} c^n L(V_0).$$

By Baire's category theorem, we may assume that  $\overline{L(V_0)}$  has non-empty interior. Take  $w \in W$  and  $r > 0$  so that

$$\{w' \in W : \|w - w'\|_W < r\} \subseteq \overline{L(V_0)}.$$

Take  $d \in W_0$  and  $c' \in k^\times$  so that  $|c'| < r$ , then  $w + c'd \in \overline{L(V_0)}$ . It follows that

$$c'd \in \overline{L(V_0)} + \overline{L(V_0)} \subseteq \overline{L(V_0) + L(V_0)} = \overline{L(V_0)}.$$

So

$$W_0 \subseteq \overline{L(c'^{-1}V_0)}.$$

It suffices to take  $C = |c'^{-1}|$ .

**Step 2.** Now given  $w \in W_0$ , we want to show that  $w \in L(\{v \in V : \|v\|_V < C\})$ . This will finish the argument: as  $k$  is non-trivially valued, this implies that  $L(V_0)$  contains an open neighbourhood of 0.

From Step 1, we can construct  $v_1 \in V$  with  $\|v_1\|_V < C$  and  $\|w - L(v_1)\|_W < 1/2$ . Repeat this process, we can  $v_n \in V$  inductively so that

$$\|v_n\|_V < 2^{1-n}C, \quad \|w - L(v_1 + \cdots + v_n)\|_W < 2^{-n}.$$

We set  $v = \sum_{i=1}^{\infty} v_i$ . Then  $v \in V$  and  $Av = w$  by continuity. Moreover,

$$\|v\|_V \leq \max_n \|v_n\|_V < C.$$

□

**Corollary 7.3.** Let  $A$  be a Banach  $k$ -algebra and  $M$  be a normed  $A$ -module. Assume that  $\hat{M}$  is a finite  $A$ -module, then  $M$  is complete.

PROOF. Take  $x_1, \dots, x_n \in \hat{M}$  so that  $\pi : A^n \rightarrow \hat{M}$  sending  $(a_1, \dots, a_n)$  to  $\sum_{i=1}^n a_i x_i$  is surjective. By open mapping theorem [Theorem 7.2](#),  $\sum_{i=1}^n \check{A}x_i$  is a neighbourhood of 0 in  $\hat{M}$ . So

$$x_j \in M + \sum_{i=1}^n \check{A}x_i.$$

It follows from (a version of) Nakayama's lemma that  $M = \hat{M}$ . □

**Corollary 7.4.** Let  $A$  be a Banach  $k$ -algebra and  $M$  be a Noetherian Banach  $A$ -module. Let  $N$  be a submodule of  $M$ . Then  $N$  is closed in  $M$ .

In particular, if  $A$  is Noetherian, then all ideals of  $A$  are closed.

PROOF. As  $M$  is noetherian,  $\bar{N}$  is a finite  $A$ -module. In particular,  $N$  is complete by [Corollary 7.3](#). Hence,  $N$  is closed in  $M$ . □

**Corollary 7.5.** A bounded epimorphism of Banach  $k$ -algebras  $f : A \rightarrow B$  is admissible.

PROOF. Replacing  $A$  by  $A/\ker f$ , we may assume that  $f$  is bijective. It follows from [Theorem 7.2](#) that  $f$  is a homeomorphism. The inverse of  $f$  is therefore continuous, and hence bounded by [Proposition 7.1](#).  $\square$

**Corollary 7.6** (Closed graph theorem). Let  $L : V \rightarrow W$  be a  $k$ -linear map between  $k$ -Banach spaces. The following are equivalent:

- (1)  $L$  is bounded.
- (2) The graph of  $L$  is closed.

PROOF. (1)  $\implies$  (2) is trivial.

Assume (2). Let  $p_1 : V \times W \rightarrow V$ ,  $p_2 : V \times W \rightarrow W$  be the natural projections and  $q : G \rightarrow V$  the restriction of  $p_1$  to the graph  $G$  of  $L$ . Observe that  $L$  is a closed subspace of  $V \times W$ , hence a Banach space. By open mapping theorem [Theorem 7.2](#),  $q$  is an open mapping. In particular, the map  $r : V \rightarrow G$  sending  $v \in V$  to  $(v, Lv)$  is bounded. It follows that  $L = p_2 \circ r$  is also bounded.  $\square$

## 8. Properties of Banach algebras over a field

Let  $(k, |\bullet|)$  be a complete non-trivially valued non-Archimedean valued field.

**Proposition 8.1.** Let  $A, B$  be Banach  $k$ -algebras and  $\varphi : A \rightarrow B$  be a  $k$ -algebra homomorphism. Assume that there is a family  $\{I_i\}$  of ideals in  $B$  satisfying

- (1) Each  $I_i$  is closed in  $B$  and each inverse image  $\varphi^{-1}(I_i)$  is closed in  $A$ .
- (2) For each  $I_i$ ,  $\dim_k B/I_i$  is finite.
- (3)  $\bigcap_{i \in I} I_i = 0$ .

Then  $\varphi$  is continuous.

Observe that when  $A$  and  $B$  are both noetherian, Condition (1) is automatically satisfied.

PROOF. For each  $i \in I$ , we write  $\pi_i : B \rightarrow B/I_i$  the projection. Let  $\psi_i : A \rightarrow B/I_i$  denote  $\pi_i \circ \varphi$ . Let  $\bar{\psi}_i : A/\ker \psi_i \rightarrow B/I_i$  the injective map induced by  $\psi_i$ . We know that  $A/\ker \psi_i$  and  $B/I_i$  are both finite dimensional. We endow them with the residue norm. Then  $\bar{\psi}_i$  is continuous. It follows that  $\psi_i$  is also continuous.

By the closed graph theorem [Corollary 7.6](#), it suffices to verify the following claim: let  $a_i \in A$  be a sequence with limit 0 and  $\varphi(a_i) \rightarrow b \in B$ , then  $b = 0$ . From the continuity of  $\bar{\psi}_i$ , we know that  $b \in I_i$  for all  $i \in I$ , it follows that  $b = 0$  by our assumption.  $\square$

**Lemma 8.2.** Let  $A$  be a Noetherian  $k$ -Banach algebra and  $M, N$  be Banach  $A$ -modules, which are finite as  $A$ -modules. Let  $f : M \rightarrow N$  be an  $A$ -linear map. Then  $f$  is bounded.

PROOF. Choose  $n \in \mathbb{N}$  and an  $A$ -linear epimorphism  $\pi : A^n \rightarrow M$ . It is clear that  $\pi$  is bounded. Similarly,  $\pi \circ f$  is also bounded. By open mapping theorem [Theorem 7.2](#),  $\pi$  is open, so  $\varphi$  is continuous and hence bounded by [Proposition 7.1](#).  $\square$

**Proposition 8.3.** Let  $A$  be a Noetherian  $k$ -Banach algebra. Then any finite  $A$ -module  $M$  admits a complete  $A$ -module norm. Such norms are unique up to equivalence.

PROOF. The uniqueness follows from [Lemma 8.2](#). As for the existence, take  $n \in \mathbb{N}$  and an  $A$ -linear epimorphism  $\pi : A^n \rightarrow M$ . By [Corollary 7.4](#),  $\ker A^n$  is closed in  $A^n$ , it suffices to take the residue norm on  $M$ .  $\square$

**Proposition 8.4.** Let  $(A, \|\bullet\|_A)$  be a Noetherian  $k$ -Banach algebra and  $\varphi : A \rightarrow B$  be a finite  $k$ -algebra homomorphism from  $A$  to a  $k$ -algebra  $B$ . Then  $B$  is Noetherian and admits a complete  $A$ -algebra norm such that  $\varphi$  is admissible. All complete  $k$ -algebra norms on  $B$  such that  $\varphi$  is bounded are equivalent.

PROOF. The uniqueness follows from [Proposition 8.3](#).

As  $\varphi$  is finite,  $B$  is a finite  $A$ -module. So by [Proposition 8.3](#), we can endow  $B$  with a complete  $A$ -module norm  $|\bullet|$  such that  $\varphi$  is contractive.

We claim that there is a constant  $C > 0$  such that

$$|xy| \leq C|x| \cdot |y|$$

for all  $x, y \in B$ .

Assuming this claim, it suffices to define

$$\|x\| := \sup_{y \in B, y \neq 0} \frac{|xy|}{|y|}$$

for  $x \in B$ .

It remains to establish the claim. Let  $b_1, \dots, b_n$  be generators of  $B$  as an  $A$ -module. Let  $C' = \max_{i,j=1,\dots,n} |b_i b_j|$ . Choose  $\eta > 1$  such that for each  $x \in B$ , there is an equation

$$x = \sum_{j=1}^n \varphi(a_j) b_j, \quad \max_{j=1,\dots,n} \|a_j\|_A \leq \eta |x|.$$

The existence of  $\eta$  follows from the construction of  $|\bullet|$  in [Proposition 8.3](#). Let  $C = C'\eta^2$ . Then for any  $x_1, x_2 \in B$ , we write

$$x_i = \sum_{j=1}^n \varphi(a_{ij}) b_j, \quad i = 1, 2.$$

We compute

$$|x_1 x_2| \leq \max_{i,j=1,\dots,n} |\varphi(a_{1i}) \varphi(a_{2j}) b_i b_j| \leq C' \max_{i=1,\dots,n} |a_{1i}| \max_{j=1,\dots,n} |a_{2j}| \leq C |x_1| \cdot |x_2|.$$

$\square$

**Proposition 8.5.** Let  $k$  be a complete valuation field and  $A$  be a Banach  $k$ -algebra. Let  $K$  be a finite normal extension of  $k$ .

- (1) If  $K/k$  is separable (or equivalently Galois), we have a natural homeomorphism

$$\mathrm{Sp}(A \otimes_k K) / \mathrm{Gal}(K/k) \xrightarrow{\sim} \mathrm{Sp} A.$$

- (2) If  $K/k$  is purely inseparable, we have a natural homeomorphism

$$\mathrm{Sp}(A \otimes_k K) \xrightarrow{\sim} \mathrm{Sp} A.$$

PROOF. In both cases, the inclusion  $A \rightarrow A \otimes_k K$  induces a morphism

$$\mathrm{Sp}(A \otimes_k K) \rightarrow \mathrm{Sp} A.$$

In the first case,  $\mathrm{Gal}(K/k)$  clearly acts on  $\mathrm{Sp} A \otimes_k K$  preserving each fiber: given a bounded semi-valuation  $\|\bullet\|_x$  on  $A \otimes_k K$  corresponding to a point  $x \in \mathrm{Sp}(A \otimes_k K)$



and  $\sigma \in \text{Gal}(K/k)$ , then  $\sigma x$  corresponds to the bounded semi-valuation  $\|\bullet\|_{\sigma x}$  on  $A \otimes_k K$  so that

$$\|f\|_{\sigma x} = \|\sigma^{-1}(f)\|_x.$$

It follows that the maps in both cases are well-defined. We observe that the fiber of a point  $x \in \text{Sp } A$  under  $\text{Sp}(A \otimes_k K) \rightarrow \text{Sp } A$  can be identified with the image of  $\text{Sp } \mathcal{H}(x) \otimes_k K$  in  $\text{Sp } A \otimes_k K$ . Observe that  $\text{Sp } \mathcal{H}(x) \otimes_k K$  is just the set of maximal ideals. When  $K/k$  is Galois,  $\text{Gal}(K/k)$  acts transitively. If  $K/k$  is purely inseparable, then  $\mathcal{H}(x) \otimes_k K$  is a local ring.  $\square$

**Corollary 8.6.** Let  $k$  be a complete valuation field and  $A$  be a Banach  $k$ -algebra. Then we have a canonical identification

$$\text{Sp } A \widehat{\otimes}_k \widehat{k^{\text{alg}}} / \text{Gal}(k^{\text{sep}}/k) \xrightarrow{\sim} \text{Sp } A.$$

PROOF. This follows from [Proposition 8.5](#). [Add details later.](#)  $\square$

**Definition 8.7.** Let  $A$  be a Banach  $k$ -algebra. A closed subset  $\Gamma \subseteq \text{Sp } A$  is called a *boundary* of  $A$  if for any  $f \in A$ ,

$$\sup_{x \in \text{Sp } A} |f(x)| = \sup_{x \in \Gamma} |f(x)|.$$

If there is a minimal boundary of  $A$ , we call it the *Shilov boundary* of  $A$ .

## 9. Maximum spectra

Let  $(k, |\bullet|)$  a complete non-Archimedean valued field.

**Definition 9.1.** For any  $k$ -algebra  $A$ , we write

$$\text{Spm}_k A := \{\mathfrak{m} \in \text{Spm } A : A/\mathfrak{m} \text{ is algebraic over } k\}.$$

For any  $x \in \text{Spm}_k A$  and any  $f \in A$ , we write  $f(x)$  for the residue of  $f$  in  $A/\mathfrak{m}_x$ , where  $\mathfrak{m}_x$  is the maximal ideal corresponding to  $x$ . We write  $|f(x)|$  for the valuation of  $f(x)$  with respect to the extended valuation induced from the given valuation on  $k$ .

**Definition 9.2.** Let  $A$  be a  $k$ -algebra. For each  $f \in A$ , we write  $|f|_{\text{sup}}$  for the supremum of  $|f(x)|$  for all  $x \in \text{Spm}_k A$  if  $\text{Spm}_k A$  is non-empty and 0 otherwise.

**Definition 9.3.** Let  $f$  be a monic polynomial in  $k[X]$ , we expand  $f = X^n + a_1 X^{n-1} + \dots + a_n \in k[X]$ , then we define  $\sigma(f) := \max_{i=1, \dots, n} |a_i|^{1/i}$ .

**Definition 9.4.** Let  $L$  be a reduced integral  $k$ -algebra. We define the *spectral norm*  $|\bullet|_{\text{sp}}$  on  $L$  as follows: given a non-zero  $x \in L$ , take a minimal polynomial  $X^n + a_1 X^{n-1} + \dots + a_n \in k[X]$  of  $x$  over  $k$ . Then we set

$$|x|_{\text{sp}} := \max_{i=1, \dots, n} |a_i|^{1/i}.$$

**Proposition 9.5.** Let  $f, g$  be monic polynomials in  $k[X]$ , then

$$\sigma(fg) = \max\{\sigma(f), \sigma(g)\}.$$

PROOF. Replacing  $k$  by a finite extension, we may assume that  $f$  and  $g$  split into linear factors  $a_i$  and  $b_j$ . Then it is straightforward to show that

$$\sigma(f) = \prod_i a_i, \quad \sigma(g) = \prod_j b_j, \quad \sigma(fg) = \prod_i a_i \cdot \prod_j b_j.$$

The assertion follows.  $\square$

**Proposition 9.6.** Let  $L$  be a reduced integral  $k$ -algebra. Then  $|\bullet|_{\text{sp}}$  is a power-multiplicative norm on  $L$ , and it extends the norm on  $k$ .

PROOF. It is clear that  $|\bullet|_{\text{sp}}$  extends the valuation on  $k$ . In order to show that  $|\bullet|_{\text{sp}}$  is a power-multiplicative norm on  $L$ , we may assume that  $L$  is finite dimensional over  $k$ . Then we can find finite field extensions  $L_1, \dots, L_t$  of  $k$  such that  $L = \bigoplus_{i=1}^t L_i$ . By [Proposition 9.5](#), we can immediately reduce to the case where  $L/k$  is a finite field extension. In this case, the result is well-known. [Expand](#).  $\square$

**Proposition 9.7.** Let  $L$  be a reduced integral  $k$ -algebra. For any  $\mathfrak{p} \in \text{Spec } L$ , write  $\pi_{\mathfrak{p}} : L \rightarrow L/\mathfrak{p}$  the residue map. Then for any  $y \in L$ ,

$$|y|_{\text{sp}} = \max_{\mathfrak{p} \in \text{Spec } L} |\pi_{\mathfrak{p}}(y)|_{\text{sp}}.$$

PROOF. Fix  $y \in L$ . For any  $\mathfrak{p} \in \text{Spec } L$ , let  $q_{\mathfrak{p}} \in k[X]$  be the minimal polynomial of  $\pi_{\mathfrak{p}}(y)$  over  $k$ . Let  $q \in k[X]$  be the minimal polynomial of  $y$  over  $k$ . Then clearly  $q_{\mathfrak{p}}$  divides  $q$  for all  $\mathfrak{p} \in \text{Spec } L$ . In particular, there are only finitely many different polynomials among  $q_{\mathfrak{p}}$  ( $\mathfrak{p} \in \text{Spec } L$ ), say  $q_1, \dots, q_r$ . Define  $q' = q_1 \cdots q_r \in k[X]$ . Then for  $f \in k[X]$ ,  $f(y) = 0$  if and only if  $\pi_{\mathfrak{p}}(f(y)) = 0$  for all  $\mathfrak{p} \in \text{Spec } L$  as  $L$  is reduced. The latter condition is equivalent to that  $q' | f$ . It follows that  $q' = q$ . Now by [Proposition 9.5](#),

$$|y|_{\text{sp}} = \sigma(q) = \max_{i=1, \dots, r} \sigma(q_i) = \max_{\mathfrak{p} \in \text{Spec } L} |\pi_{\mathfrak{p}}(y)|_{\text{sp}}.$$

$\square$

**Proposition 9.8.** Let  $\varphi : B \rightarrow A$  be a homomorphism of commutative  $k$ -algebras. Then for any  $f \in B$ ,

$$|\varphi(f)|_{\text{sup}} \leq |f|_{\text{sup}}.$$

PROOF. Of course, we can assume that  $\text{Spm}_k A \neq \emptyset$ . Let  $x \in \text{Spm}_k A$ , then  $\varphi^{-1}x \in \text{Spm}_k B$ . But for any  $f \in B$ ,  $|\varphi(f)(x)| = |f(\varphi^{-1}x)|$ . We conclude.  $\square$

**Proposition 9.9.** Let  $A$  be a  $k$ -algebra. Let  $\mathfrak{M}$  be the set of minimal prime ideals in  $A$  and let  $\pi_{\mathfrak{p}} : A \rightarrow A/\mathfrak{p}$  be the canonical residue map for all  $\mathfrak{p} \in \mathfrak{M}$ . Then for any  $f \in A$ ,

$$(9.1) \quad |f|_{\text{sup}} = \sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}}.$$

In particular, if  $A$  be a reduced integral  $k$ -algebra. Then  $|\bullet|_{\text{sup}} = |\bullet|_{\text{sp}}$  on  $A$ .

PROOF. By [Proposition 9.8](#),

$$\sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}} \leq |f|_{\text{sup}}.$$

In order to show the reverse inequality, let  $x \in \text{Spm}_k A$ . Take  $\mathfrak{p} \in \mathfrak{M}$  such that  $x \supseteq \mathfrak{p}$ . Clearly,  $\pi_{\mathfrak{p}}(x) \in \text{Spm}_k A/\mathfrak{p}$  and

$$|f(x)| = |\pi_{\mathfrak{p}}(f)(\pi_{\mathfrak{p}}(x))|.$$

In particular,

$$|f(x)| \leq |\pi_{\mathfrak{p}}(f)|_{\text{sup}} \leq \sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}}.$$

Take sup with respect to  $x$ , we conclude [\(9.1\)](#).

When  $A$  is a reduced and integral  $k$ -algebra, all prime ideals of  $A$  are minimal. The final assertion follows from [Proposition 9.7](#).  $\square$

**Definition 9.10.** Let  $A$  be a Banach  $k$ -algebra. We say that *maximal modulus principle* holds for  $A$  if for any  $f \in A$ , there is  $x \in \text{Spm}_k A$  such that  $|f(x)| = |f|_{\text{sup}}$ .

**Proposition 9.11.** Let  $\varphi : B \rightarrow A$  be an injective integral torsion-free homomorphism of Banach  $k$ -algebras. Assume that  $B$  is a normal integral domain.

- (1) Fix  $f \in A$ . Let  $f^n + \varphi(b_1)f^{n-1} + \cdots + \varphi(b_n) = 0$  be the minimal equation of  $f$  over  $A$ . Then

$$|f|_{\text{sup}} = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

- (2) Assume that maximal modulus principle holds for  $B$ , then it holds for  $A$  as well.

- (3) Suppose that  $|bb'|_{\text{sup}} = |b|_{\text{sup}}|b'|_{\text{sup}}$  for all  $b, b' \in B$ . Then  $|\varphi(b)f|_{\text{sup}} = |b|_{\text{sup}}|f|_{\text{sup}}$  for all  $b \in B$  and  $f \in A$ .

PROOF. (1) We first show the inequality

$$|f|_{\text{sup}} \leq \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

Of course, we can assume that  $\text{Spm}_k A \neq \emptyset$ . For all  $x \in \text{Spm}_k A$ , we have

$$0 = f(x)^n + \varphi(b_1)f(x)^{n-1} + \cdots + \varphi(b_n) = f(x)^n + b_1(\varphi^{-1}x)f(x)^{n-1} + \cdots + b_n(\varphi^{-1}x).$$

Then we in fact have that

$$|f(x)| \leq \max_{i=1, \dots, n} |b_i(\varphi^{-1}x)|^{1/i}.$$

Assume that to the contrary that

$$|f(x)|^i > |b_i(\varphi^{-1}x)|$$

for all  $i = 1, \dots, n$ . Then

$$|b_i(\varphi^{-1}x)f(x)^{n-i}| < |f(x)|^n = |f(x)|^n.$$

It follows that

$$|b_1(\varphi^{-1}x)f(x)^{n-1} + \cdots + b_n(\varphi^{-1}x)| < |f(x)|^n.$$

This is a contradiction.

It remains to argue that

$$(9.2) \quad |f|_{\text{sup}} \geq \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

Next let  $A' = B[f]$ . We argue that  $A' \rightarrow A$  is an isometry with respect to  $|\bullet|_{\text{sup}}$ . If  $\text{Spm}_k A'$  is empty, then the assertion follows from [Proposition 9.8](#). Assume that  $\text{Spm}_m A'$  is non-empty. Take  $y \in \text{Spm}_k A'$ . By [\[Stacks, Tag 00GQ\]](#), there is a maximal ideal  $x \in \text{Spm} A$  lying over  $y$ . As the induced map  $A'/y \rightarrow A/x$  is integral, we find  $x \in \text{Spm}_k A$ . So the map  $\text{Spm}_k A \rightarrow \text{Spm}_k A'$  is surjective. It follows that  $A' \rightarrow A$  is an isometry with respect to  $|\bullet|_{\text{sup}}$ .

In order to argue (9.2), we may assume that  $A = B[f]$ . Let  $q \in B[X]$  denote the minimal polynomial of  $f$  over  $A$ . Then  $A = B[X]/(q)$ . Let  $y \in \text{Spm}_k B$ , we write  $f_y$  for the residue class of  $f$  in  $A/yA$  and write  $\bar{f}_y$  for the residue class in  $(A/yA)^{\text{red}}$ . Similarly, let  $q_y$  denote the residue class of  $q$  in  $B/y[X]$ . As  $y$  is contained in some  $\text{Spm}_k A$ , we see that

$$|f|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} |f_y|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} |\bar{f}_y|_{\text{sup}}.$$

For  $y \in \text{Spm}_k B$ , we decompose  $q_y$  into prime factors  $q_1^{n_1} \cdots q_r^{n_r}$  in  $B/y[X]$ . Then

$$A/yA \cong B/y[X]/(q_y)$$

and

$$(A/yA)^{\text{red}} \cong \bigoplus_{i=1}^r B/y[X]/(q_i).$$

We endow  $\bigoplus_{i=1}^r B/y[X]/(q_i)$  with the spectral norm over  $B/y$ . If  $\bar{f}_i$  denotes the residue class of  $\bar{f}_y$  in  $B/y[X]/(q_i)$ , by [Proposition 9.9](#) and [Proposition 9.5](#),

$$|\bar{f}_y|_{\text{sup}} = \max_{i=1, \dots, r} |\bar{f}_i|_{\text{sp}} = \max_{i=1, \dots, r} \sigma(q_i) = \sigma(q_y).$$

Therefore,

$$|f|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} \sigma(q_y) = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/n}.$$

(2) Take a non-zero  $f \in A$ . Using the notations in (1), we can find  $y \in \text{Spm}_k B$  such that

$$|\bar{f}_y|_{\text{sup}} = \sigma(q_y) = |f|_{\text{sup}}.$$

As  $A/yA$  contains only finitely many maximal ideals, there is  $x \in \text{Spm}_k A$  such that  $|\bar{f}_y|_{\text{sup}} = |f(x)|$ . So

$$|f|_{\text{sup}} = |f(x)|.$$

(3) Consider  $f \in A$  and let  $f^n + b_1 f^{n-1} + \cdots + b_n = 0$  be its minimal integral equation over  $B$ . Then  $f$  is of degree  $n$  over  $\text{Frac } B$  as well, hence so is  $bf$  for any non-zero  $b \in B$ . So the minimal integral equation of  $bf$  is

$$(bf)^n + bb_1(bf)^{n-1} + \cdots + b^n b_n = 0.$$

By (1), we compute

$$|bf|_{\text{sup}} = \max_{i=1, \dots, n} |b^i b_i|_{\text{sup}}^{1/i} = |b|_{\text{sup}} \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i} = |b|_{\text{sup}} |f|_{\text{sup}}.$$

Also,  $|bf|_{\text{sup}} = |b|_{\text{sup}} |f|_{\text{sup}}$  is trivial for  $b = 0$ . We conclude.  $\square$

## 10. Miscellany

**Lemma 10.1.** Let  $(A, |\bullet|)$  be a valued integral domain such that  $\tilde{A}$  is Noetherian and N-2. Assume that  $|A^\times|$  is a group. Let  $(B, \|\bullet\|)$  be a faithfully normed  $A$ -algebra such that

- (1)  $\|\bullet\|$  is power-multiplicative.
- (2) The  $A$ -rank of  $B$  is finite.
- (3)  $\mathring{B}$  is integral over  $\mathring{A}$ .

Then  $\tilde{B}$  is finite as  $\tilde{A}$ -module.

PROOF. We want to apply [Proposition 8.1](#) in [Commutative algebras](#) to the canonical injection map  $\psi : \tilde{A} \rightarrow \tilde{B}$ . The map  $\psi$  is integral as  $\mathring{B}$  is integral over  $\mathring{A}$ . The conditions are easily verified. [Add details.](#)  $\square$

## Bibliography

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