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Global properties of complex analytic spaces

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2. Stein spaces

Definition 2.1. Let X be a complex analytic space. We say X is *holomorphically separable* if for any $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

Here we regard f as a continuous function $X \to \mathbb{C}$. In particular, a holomorphically separable space is Hausdorff.

Definition 2.2. Let X be a complex analytic space. We say X is *holomorphically spreadable* if X is Hausdorff and for any $x \in X$, we can find an open neighbourhood U of x in X such that

$$\{y \in U : f(x) = f(y) \text{ for all } f \in \mathcal{O}_X(X)\} = \{p\}.$$

A holomorphically separable space is clearly holomorphically spreadable.

Proposition 2.3. Let X be an irreducible holomorphically spreadable complex analytic space. Then X has countable basis.

The statement of this proposition in [Fis76, Proposition 0.37] is clearly wrong. I do not understand the argument of either [Jur59] or [Gra55], where they claim that this result holds for connected holomorphically spreadable complex analytic spaces.

Proof. We may assume that X is connected. Recall that by Corollary 8.6 in Local properties of complex analytic spaces, X is locally connected. Let $F : X \to \mathbb{C}^{\mathcal{O}_X(X)}$ be the map sending $x \in X$ to $(f(x))_{f \in \mathcal{O}_X(X)}$. By our assumption, F is continuous and has discrete fibers. In particular, for each $x \in X$, we may assume take finitely many $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ so that the induced morphism $F' : X \to \mathbb{C}^n$ is quasi-finite at x. By Corollary 2.8 in Analytic sets, we can find a nowhere dense analytic set A in X such that the map $X \setminus A \to \mathbb{C}^n$ induced by F' is quasi-finite. Now we endow $\mathcal{O}_X(X)$ with the compact-open topology. It is a metric space. By Proposition 5.2 in Topology and bornology, $X \setminus A$ has countable basis. It follows that $\mathcal{O}_X(X \setminus A)$ is a separable metric space. Hence, so it $\mathcal{O}_X(X)$. In particular, there is a continous map with discrete fibers

$$X \to \mathbb{C}^\omega.$$

It follows again from Proposition 5.2 in Topology and bornology that X has countable basis.

Reproduce the proof of the local connectedness of a complex space □

Definition 2.4. Let X be a complex analytic space. We say X is *holomorphically convex* if $|X|$ is Hausdorff and for any compact set $K \subseteq X$, the set

$$\hat{K}^X := \left\{ x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathcal{O}_X(X) \right\}$$

is compact.

We say X is *weakly holomorphically convex* if for any quasi-compact set $K \subseteq X$, the connected components of $\hat{K}^X$ are all quasi-compact.

Proposition 2.5. Let X be a Hausdorff complex analytic space. Consider the following conditions:

(1) X is holomorphically convex;
(2) For any sequence $x_i \in X$ ($i \in \mathbb{N}$) without accumulation points, there is $f \in \mathcal{O}_X(X)$ such that $|f(x_i)|$ is unbounded.

Then (1) $\implies$ (2). The converse is true if X is Lindelöf.

Proof. (1) $\implies$ (2): Recall that a complex analytic space is always first countable. For a first countable Hausdorff space, compactness implies sequential compactness.

(2) $\implies$ (1): For a Lindelöf Hausdorff space, sequential compactness implies compactness. □

Definition 2.6. Let X be a complex analytic space. We say X is *Stein* if it is holomorphically separable and holomorphically convex.

Theorem 2.7. Let X be a complex analytic space such that X^{red} is Stein and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then

(1) for each $x \in X$, the set
$$\{s_x : s \in H^0(X, \mathcal{F})\}$$
generates $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$;
(2) for each $k \geq 1$,
$$H^k(X, \mathcal{F}) = 0.$$

The two assertions are known as Cartan Theorem A and Cartan Theorem B.

Proof. □

[Stacks]

Bibliography

[Fis76] G. Fischer. Complex analytic geometry. Lecture Notes in Mathematics, Vol. 538. Springer-Verlag, Berlin-New York, 1976, pp. vii+201.

[Gra55] H. Grauert. Charakterisierung der holomorph vollständigen komplexen Räume. *Math. Ann.* 129 (1955), pp. 233–259. URL: https://doi.org/10.1007/BF01362369.

[Jur59] M. Jurchescu. On a theorem of Stoilow. *Math. Ann.* 138 (1959), pp. 332–334. URL: https://doi.org/10.1007/BF01344153.

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