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# Topology and bornology

## 1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A compact space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

A connected topological is always non-empty.

References to this chapter include [\[Ber93\]](#).

## 2. Nets

Let  $X$  be a set,  $Y \subseteq X$  be a subset. Consider a collection  $\tau$  of subsets of  $X$ , we write

$$\tau|_Y := \{V \in \tau : V \subseteq Y\}.$$

**Definition 2.1.** Let  $X$  be a topology space and  $\tau$  be a collection of subsets of  $X$ . We say  $\tau$  is

- (1) *dense* if for any  $V \in \tau$  and any  $x \in V$ , there is a fundamental system of neighbourhoods of  $x$  in  $V$  consisting of sets from  $\tau|_V$ ;
- (2) a *quasi-net* on  $X$  if for each  $x \in X$ , there exist  $n \in \mathbb{Z}_{>0}$ ,  $V_1, \dots, V_n \in \tau$  such that  $x \in V_1 \cap \dots \cap V_n$  and that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ ;
- (3) a *net* on  $X$  if it is a quasi-net and if for any  $U, V \in \tau$ ,  $\tau|_{U \cap V}$  is a quasi-net on  $U \cap V$ ;
- (4) *locally finite* if for any  $x \in X$ , there is a neighbourhood  $U$  of  $x$  in  $X$  such that  $\{V \in \tau : V \cap U \neq \emptyset\}$  is finite.

We observe that if  $\tau$  is a net,  $\tau|_{U \cap V}$  is in fact a net.

**Lemma 2.2.** Let  $X$  be a topological space and  $\tau$  be a quasi-net on  $X$ .

- (1) A subset  $U \subseteq X$  is open if and only if for each  $V \in \tau$ ,  $U \cap V$  is open in  $V$ .
- (2) Suppose that  $\tau$  consists of compact sets. Then  $X$  is Hausdorff if and only if for any  $U, V \in \tau$ ,  $U \cap V$  is compact.

We remind the readers that a compact space is Hausdorff by our convention.

**PROOF.** (1) The direct implication is trivial. Suppose that  $U \cap V$  is open in  $V$  for all  $V \in \tau$ . We want to show that  $U$  is open. Take  $x \in U$ , we can find  $n \in \mathbb{Z}_{>0}$ ,  $V_1, \dots, V_n \in \tau$  all containing  $x$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ . By our hypothesis, we can find open sets  $W_1, \dots, W_n$  in  $W$  such that  $W \cap V_i = U \cap V_i$

for  $i = 1, \dots, n$ . Then  $W = W_1 \cap \dots \cap W_n$  is an open neighbourhood of  $x$  in  $X$ . But then

$$U \cap (V_1 \cup \dots \cup V_n) \supseteq W \cap (V_1 \cup \dots \cup V_n),$$

the latter is a neighbourhood of  $x$  hence so is the former. It follows that  $U$  is open.

(2) The direct implication is trivial. Consider the quasi-net  $\tau \times \tau := \{U \times V : U, V \in \tau\}$  on  $X \times X$ . By (1), it suffices to verify that the intersection of the diagonal with  $U \times V$  is closed in  $U \times V$  for any  $U, V \in \tau$ . But this intersection is homeomorphic to  $U \cap V$ , which is compact by our assumption and hence closed as  $U, V$  are both Hausdorff.  $\square$

**Lemma 2.3.** Let  $X$  be a Hausdorff space. Assume that  $X$  admits a quasi-net  $\tau$  consisting of compact sets. Then  $X$  is locally compact.

PROOF. Take  $x \in X$ . By assumption, we can find  $n \in \mathbb{N}$  and  $V_1, \dots, V_n \in \tau$  all containing  $x$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$ . This neighbourhood is clearly compact.  $\square$

**Lemma 2.4.** Let  $X$  be a Hausdorff space and  $\tau$  be a collection of compact subsets of  $X$ . Then the following are equivalent:

- (1)  $\tau$  is a quasi-net;
- (2) For each  $x \in X$ , there are  $n \in \mathbb{N}$  and  $V_1, \dots, V_n \in \tau$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ .

PROOF. (1)  $\implies$  (2): This is trivial.

(2)  $\implies$  (1): Given  $x \in X$ , take  $V_1, \dots, V_n$  as in (2). We may assume that  $x \in V_1, \dots, V_m$  and  $x \notin V_{m+1}, \dots, V_n$  for some  $1 \leq m \leq n$ . Then  $V_1 \cup \dots \cup V_m$  is a neighbourhood of  $x$  in  $X$ : if  $U$  is an open neighbourhood of  $x$  in  $X$  contained in  $V_1 \cup \dots \cup V_n$ , then  $U \setminus (V_{m+1} \cup \dots \cup V_n)$  is an open neighbourhood of  $x$  in  $X$  contained in  $V_1 \cup \dots \cup V_m$ .  $\square$

**Lemma 2.5.** Let  $X$  be a topological space and  $\tau$  be a net on  $X$  consisting of compact sets. Then

- (1) for any pair  $U, V \in \tau$ , the intersection  $U \cap V$  is locally closed in  $U$  and in  $V$ ;
- (2) If  $n \in \mathbb{Z}_{>0}$ ,  $V, V_1, \dots, V_n \in \tau$  are such that

$$V \subseteq V_1 \cup \dots \cup V_n,$$

then there are  $m \in \mathbb{Z}_{>0}$  and  $U_1, \dots, U_m \in \tau$  such that

$$V = U_1 \cup \dots \cup U_m$$

and each  $U_j$  is contained in some  $V_i$ .

PROOF. (1) It suffices to show that  $U \cap V$  is locally compact in the induced topology. This follows from [Lemma 2.3](#).

(2) For each  $x \in V$  and each  $i = 1, \dots, n$  such that  $x \in V_i$ , we take a neighbourhood of  $x$  in  $V \cap V_i$  of the form  $W_i V_{i1} \cup \dots \cup V_{im_i}$  for some  $m_i \in \mathbb{Z}_{>0}$  and  $V_{ij} \in \tau$  for  $j = 1, \dots, m_i$ . Then the union of all  $W_i$ 's is a neighbourhood of  $x$  of the form  $U_1 \cup \dots \cup U_m$ , where  $U_j$  belongs to  $\tau$  and is contained in some  $V_i$ . Using the compactness of  $V$ , we conclude.  $\square$

### 3. Paracompact spaces

**Definition 3.1.** A topological space  $X$  is *paracompact* if any open covering of  $X$  admits a locally finite refinement.

A paracompact space is not necessarily Hausdorff according to our definition.

**Proposition 3.2.** Let  $X$  be a locally compact topological space.

- (1) Assume that each connected component of  $X$  is  $\sigma$ -compact, then  $X$  is paracompact.
- (2) If  $X$  is paracompact and Hausdorff, then each connected component of  $X$  is  $\sigma$ -compact.

If the conditions in (2) are satisfied, for any basis of neighbourhoods  $\mathcal{B}$  of  $X$ , every open covering  $\mathcal{U}$  of  $X$  can be refined into a locally finite covering  $\mathcal{V}$  consisting of elements in  $\mathcal{B}$ .

We do not assume that the elements in  $\mathcal{B}$  be open. The covering  $\mathcal{V}$  is not necessarily open.

**Proposition 3.3.** Let  $X$  be a paracompact space and  $Y \subseteq X$  be a closed subspace. Then  $Y$  is paracompact.

**Proposition 3.4.** Let  $X$  be a locally compact Hausdorff space and  $Y \subseteq X$  be a subspace, then the following are equivalent:

- (1)  $Y$  is locally compact and Hausdorff;
- (2)  $Y$  is a locally closed subspace of  $X$ .

### 4. Closed maps and topologically finite maps

**Definition 4.1** ([Stacks, Tag 004E], [Stacks, Tag 0CY1]). A map  $f : X \rightarrow Y$  of topological spaces is *closed* if for each closed subset  $Z$  in  $X$ ,  $f(Z)$  is closed in  $Y$ .

A map  $f : X \rightarrow Y$  of topological spaces is *separated* if it is continuous and the diagonal map  $\Delta : X \rightarrow X \times_Y X$  is closed.

A closed map is not necessarily continuous.

**Lemma 4.2.** Let  $f : X \rightarrow Y$  be a closed map of topological spaces, then for each  $y \in Y$  and any open neighbourhood  $U$  of  $f^{-1}(y)$  in  $X$ , there is an open neighbourhood  $V$  of  $y$  in  $Y$  such that  $f^{-1}(V) \subseteq U$ .

PROOF. It suffices to take  $V = Y \setminus f(X \setminus U)$ , □

**Lemma 4.3.** Let  $f : X \rightarrow Y$  be a closed map of topological spaces. Then for any subspace  $V$  of  $Y$ , the map  $f^{-1}(V) \rightarrow V$  induced by  $f$  is closed.

PROOF. Let  $A$  be a closed subset of  $U := f^{-1}(V)$ . We need to show that  $f(A)$  is closed in  $V$ . Choose a closed subset  $B$  of  $X$  such that  $A = B \cap U$ , then  $f(B)$  is closed in  $Y$  and  $f(A) = f(B) \cap V$  is closed in  $V$ . □

**Definition 4.4.** A  $f : X \rightarrow Y$  of topological spaces is *topologically finite* if

- (1)  $f$  is separated and closed;
- (2) for each  $y \in Y$ , the set  $f^{-1}(y)$  is finite.

A map  $f : X \rightarrow Y$  of topological spaces is *topologically finite at*  $x \in X$  if there is an open neighbourhood  $U$  of  $x$  in  $X$  and an open neighbourhood  $V$  of  $f(x)$  in  $Y$  such that  $f(U) \subseteq V$  and the induced map  $U \rightarrow V$  is topologically finite.

**Proposition 4.5.** Let  $f : X \rightarrow Y$  be a map of topological spaces. Then the following are equivalent:

- (1)  $f$  is topologically finite;
- (2)  $f$  is proper and all fibers of  $f$  are discrete.

Here the properness is defined as in [Stacks, Tag 005O]. In particular, a proper map is always separated and hence continuous.

PROOF. Assume that  $f$  is topologically finite. As the fibers of  $f$  are finite and Hausdorff, they are discrete. We need to show that  $f$  is proper. This follows from [Stacks, Tag 005R].

Conversely, assume that  $f$  is proper with discrete fibers. By [Stacks, Tag 005R] again, the fibers of  $f$  are compact and hence finite. The map  $f$  is closed and separated as it is proper. So (1) follows.  $\square$

**Lemma 4.6.** Let  $f : X \rightarrow Y$  be a continuous map between topological spaces. Assume that  $Y$  is Hausdorff. Let  $W$  be an open relative quasi-compact subset of  $X$ , then the map

$$W \setminus f^{-1}(f(\partial W)) \rightarrow Y \setminus f(\partial W)$$

induced by  $f$  is proper.

PROOF. It is well-known that  $f|_{\bar{W}} : \bar{W} \rightarrow Y$ , as a continuous map from a quasi-compact space to a Hausdorff space is proper. The map in the lemma is a base change of the given map, hence is also proper. We apply [Stacks, Tag 005R].  $\square$

**Proposition 4.7.** Let  $f : X \rightarrow Y$  be a topologically finite map of topological spaces. Then for any subspace  $V \subseteq Y$ , the map  $f^{-1}(V) \rightarrow V$  induced by  $f$  is topologically finite.

PROOF. This follows immediately from Lemma 4.3.  $\square$

**Theorem 4.8.** Let  $f : X \rightarrow Y$  be a topologically finite map of topological spaces. Let  $y \in f(X)$  and  $x_1, \dots, x_n$  ( $n \in \mathbb{Z}_{>0}$ ) denote the distinct points of  $f^{-1}(y)$ . Take pairwise disjoint open neighbourhoods  $U'_1, \dots, U'_n$  of  $x_1, \dots, x_n$  in  $X$ . Then any neighbourhood  $V'$  of  $y$  in  $Y$  contains an open neighbourhood  $V$  of  $y$  satisfying the following conditions:

- (1)  $U_1 := f^{-1}(V) \cap U'_1, \dots, U_n := f^{-1}(V) \cap U'_n$  are pairwise disjoint open neighbourhoods of  $x_1, \dots, x_n$  in  $X$ ;
- (2)  $f^{-1}V = \bigcup_{j=1}^n U_j$ ;
- (3) The maps  $U_j \rightarrow V$  for  $j = 1, \dots, n$  induced from  $f$  are all topologically finite.

Let  $\mathcal{F}$  be a sheaf of sets on  $X$ , then we have a functorial bijection

$$f_*\mathcal{F}(V) \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}(U_j).$$

The existence of  $U'_1, \dots, U'_n$  is guaranteed by [Stacks, Tag 0CY2].

PROOF. As  $\bigcup_{j=1}^n U'_j$  is an open neighbourhood of  $f^{-1}(y)$  in  $X$ , by Lemma 4.2 and Lemma 4.3, we can find an open neighbourhood  $V \subseteq V'$  of  $y$  in  $Y$  such that

$$f^{-1}V \subseteq \bigcup_{j=1}^n U'_j.$$



The conditions (1) and (2) are therefore satisfied.

In order to prove (3), it remains to show that the induced maps  $U_j \rightarrow V$  are closed for  $j = 1, \dots, n$ . We may take  $j = 1$ . Let  $A$  be a closed subset of  $U_1$ . Then  $A$  is closed in  $f^{-1}(V)$  by (1) and (2). It follows that  $f(A)$  is closed in  $V$  by [Lemma 4.3](#).

The last assertion follows from (1) and (2).  $\square$

**Corollary 4.9.** Let  $f : X \rightarrow Y$  be a topologically finite map of topological spaces. Let  $x \in X$  be  $U'$  be an open neighbourhood of  $x$  in  $X$  such that all other points in  $f^{-1}(f(x))$  are in the interior of  $X \setminus U'$ . Then any neighbourhood  $V'$  of  $f(x)$  in  $Y$  contains an open neighbourhood  $V$  of  $y$  such that for  $U := f^{-1}(V) \cap U'$  the map  $g : U \rightarrow V$  induced by  $f$  is topologically finite and  $g^{-1}(g(x)) = \{x\}$ .

PROOF. This follows immediately from [Theorem 4.8](#).  $\square$

**Corollary 4.10.** Let  $f : X \rightarrow Y$  be a topologically finite map of topological spaces. Let  $\mathcal{F}$  be a sheaf of sets on  $X$ ,  $y \in f(X)$ . Denote by  $x_1, \dots, x_n$  ( $n \in \mathbb{Z}_{>0}$ ) the distinct points of the fiber  $f^{-1}(y)$ . Then we have a canonical bijection

$$(f_*\mathcal{F})_y \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}_{x_j}.$$

In particular,  $f_* : \mathcal{A}b(X) \rightarrow \mathcal{A}b(Y)$  is exact.

PROOF. This follows immediately from [Theorem 4.8](#).  $\square$

## 5. Exhaustion

**Definition 5.1.** Let  $X$  be a Hausdorff space. A *compact exhaustion* of  $X$  is a sequence of compact sets  $(K_i)_{i \in \mathbb{Z}_{>0}}$  in  $X$  such that

(1) For each  $i \in \mathbb{Z}_{>0}$ ,

$$K_i \subseteq \text{Int } K_{i+1};$$

(2)

$$X = \bigcup_{i=1}^{\infty} K_i.$$

**Proposition 5.2.** Let  $X$  be a locally compact Hausdorff topological space admitting a countable basis, then  $X$  admits a compact exhaustion.

Note that in the book of Grauert–Remmert, the condition of being Hausdorff is omitted.

PROOF. [Include a proof](#)  $\square$

**Lemma 5.3.** Let  $X$  be a paracompact Hausdorff topological space and  $\mathcal{F}$  be a sheaf of Abelian groups on  $X$ . Let  $q \in \mathbb{Z}_{\geq 2}$  and  $(K_i)_{i \in \mathbb{Z}_{>0}}$  be a compact exhaustion of  $X$  with the following property:

$$H^{q-1}(K_i, \mathcal{F}) = H^q(K_i, \mathcal{F}) = 0$$

for all  $i \in \mathbb{Z}_{>0}$ . Then  $H^q(X, \mathcal{F}) = 0$ .

PROOF. Grauert–Remmert P103.  $\square$

## 6. Maps with discrete fibers

**Lemma 6.1.** Let  $X$  be a locally connected locally compact Hausdorff topological space and  $X_0$  be a Hausdorff space with a basis  $\beta_0$ . Consider a continuous map  $f : X \rightarrow X_0$  with discrete fiber. Then there is a basis of  $X$  made up of connected components of  $f^{-1}U_0$  with  $U_0 \in \beta_0$ .

PROOF. Let  $x \in X$  and  $V$  be an open neighbourhood of  $x$  in  $X$ . We need to find  $U_0 \in \beta_0$  and a component  $U$  of  $f^{-1}(U_0)$  such that  $U \subseteq V$ .

For this purpose, we may assume that  $X$  is connected. Set  $x_0 = f(x)$ . Choose an open neighbourhood  $W$  of  $x$  in  $V$  with  $\bar{W}$  compact and  $B \cap f^{-1}(x_0) = \emptyset$ , where  $B = \bar{W} \setminus W$ . Let  $B_0 = f(B)$ , then  $x_0 \notin B_0$ . As  $B_0$  is compact, we can find  $U_0 \in \beta_0$  containing  $x_0$  such that  $B_0 \cap U_0 = \emptyset$ . Let  $U$  be the connected component of  $f^{-1}(U_0)$  containing  $x$ . Then  $B \cap U = \emptyset$  and hence  $U \subseteq W \cup (X \setminus \bar{W})$ . As  $X$  is connected and  $W \cap U$  is non-empty, we find that  $U \subseteq W$ .  $\square$

**Proposition 6.2.** Let  $X$  be a connected, locally connected, first countable, locally compact Hausdorff space and  $X_0$  be a topological space with countable basis. If there is a map  $f : X \rightarrow X_0$  with discrete fibers, then  $X$  has countable topology as well.

This result is proved in [Jur59].

PROOF. Let  $\beta_0$  be a countable basis for the topology on  $X_0$ . Let  $\beta$  be the collection of open sets  $U$  in  $X$  such that

- (1) There is  $U_0 \in \beta_0$  such that  $U$  is a connected component of  $f^{-1}(U_0)$ ;
- (2)  $U$  is relatively compact in  $X$ .

By our assumption, any  $U \in \beta$  has countable basis. By Lemma 6.1,  $\beta$  is a basis for the topology on  $X$ . It remains to show that  $\beta$  is countable.

Let  $V \in \beta$ . For each  $n \in \mathbb{Z}_{>0}$ ,  $\beta^{(n)}$  denotes the collection of  $U \in \beta$  with the following property: there is a map  $\{1, \dots, n\} \rightarrow \beta$ , say assigning  $U_i \in \beta$  to  $i$  such that  $U_1 = V$ ,  $U_i \cap U_{i+1} \neq \emptyset$  for  $i = 1, \dots, n-1$ . As  $X$  is connected,

$$\beta = \bigcup_{n=1}^{\infty} \beta^{(n)}.$$

It remains to show that for each  $n \in \mathbb{Z}_{>0}$ ,  $\beta^{(n)}$  is countable. We make an induction. The case  $n = 1$  is obvious. Assume that  $n \geq 2$  and the assertion has been proved for  $n-1$ . Let  $U_0 \in \beta_0$  and  $U' \in \beta^{(n-1)}$ . Let  $\alpha^{(n)}(U_0, U')$  denote the collection of  $U \in \beta^{(n)}$  such that  $U$  is a connected component of  $f^{-1}(U_0)$  and  $U \cap U'$  is non-empty. Then

$$\beta^{(n)} = \bigcup_{U_0 \in \beta_0, U' \in \beta^{(n-1)}} \alpha^{(n)}(U_0, U').$$

But each  $\alpha^{(n)}(U_0, U')$  is countable as  $U'$  has countable basis. It follows that  $\beta^{(n)}$  is countable.  $\square$

## 7. Bornology

**Definition 7.1.** Let  $X$  be a set. A *bornology* on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  such that

- (1) For any  $x \in X$ , there is  $B \in \mathcal{B}$  such that  $x \in B$ ;
- (2) For any  $B \in \mathcal{B}$  and any subset  $A \subseteq B$ ,  $A \in \mathcal{B}$ ;

(3)  $\mathcal{B}$  is stable under finite union.

The pair  $(X, \mathcal{B})$  is called a *bornological set*. The elements of  $\mathcal{B}$  are called the *bounded subsets* of  $(X, \mathcal{B})$ . When  $\mathcal{B}$  is obvious from the context, we omit it from the notations.

A morphism between bornological sets  $(X, \mathcal{B}_X)$  and  $(Y, \mathcal{B}_Y)$  is a map of sets  $f : X \rightarrow Y$  such that for any  $A \in \mathcal{B}_X$ ,  $f(A) \in \mathcal{B}_Y$ . Such a map is called a *bounded map*.

**Definition 7.2.** Let  $(X, \mathcal{B})$  be a bornological set. A *basis* for  $\mathcal{B}$  is a subset  $\mathcal{A} \subseteq \mathcal{B}$  such that for any  $B \in \mathcal{B}$ , there are  $A_1, \dots, A_n \in \mathcal{A}$  such that  $B \subseteq A_1 \cup \dots \cup A_n$ .



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