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Analytic sets

1. Introduction

2. Remmert–Stein theorem

Lemma 2.1. Let $n \in \mathbb{N}$ and U be a relatively compact open neighbourhood of 0 in \mathbb{C}^n . Let $k \in \{0, 1, \dots, n-1\}$. We write L^k for the intersection of $z_1 = \dots = z_{n-k} = 0$ with U , where z_1, \dots, z_n are the coordinates on \mathbb{C}^n . Let A be an analytic set in $U \setminus L^k$ of dimension $\leq k$. Then for $i = 0, \dots, k$, we can find a linear subspace L' of \mathbb{C}^n of dimension $n - k + i$ such that

$$\dim L' \cap A \leq i, \quad \dim L' \cap L^k \leq i.$$

PROOF. We make an induction on n . When $n = 0, 1$, there is nothing to prove. Let $n > 1$. If $i = k$, we just take $L' = \mathbb{C}^n$. Assume $0 \leq i < k$.

Let M_1, \dots, M_N be the irreducible components of A . We may assume that no components are single points. Take a non-zero base point $p_j \in M_j$ for $j = 1, \dots, N$. Let H be an $(n-1)$ -dimensional linear subspace of \mathbb{C}^n which does not contain L^k or any of the points p_1, \dots, p_N . Without loss of generality, we may guarantee that H is given by $z_n = 0$.

Let k_j denote the dimension of M_j for $j = 1, \dots, N$. Let $M'_j = M_j \cap H$ for $j = 1, \dots, N$. Observe that the dimension of M'_j is either k_j or $k_j - 1$ for $j = 1, \dots, N$. Let

$$M' := \bigcup_{i=1}^N M'_i.$$

Then $\dim M' \leq k - 1$. By the inductive hypothesis, we can find a linear subspace L' of \mathbb{C}^n of dimension $n - k + 1$ with the desired properties. \square

Lemma 2.2. Let $k \leq n$ be two elements in \mathbb{N} and $D = \Delta^k \times \Delta^{n-k}$ be the product of two unit polycylinders. Write L for $\Delta^k \times \{0\}$. Consider a non-empty analytic subset M of $D \setminus L$ of dimension k everywhere. Assume that M does not intersect a neighbourhood of $\Delta^k \times \{y \in \mathbb{C}^{n-k} : \|y\|_{L^\infty} = 1\}$. Then for any $\epsilon > 0$, M meets the polycylinder $\{(x, y) \in D : \|x\|_{L^\infty} < \epsilon, \|y\|_{L^\infty} \in (0, 1)\}$.

PROOF. **Step 1.** We observe that for each $a \in \Delta^k$, the intersection

$$\{(x, y) \in D : x = a\} \cap M$$

is discrete. In fact, by our assumption, the absolute values the coordinate functions of Δ^{n-k} obtain their maxima on each irreducible component of the intersection. By [Corollary 4.20 in Morphisms between complex analytic spaces](#), these coordinates are all constant.

Step 2. Let $(x^1, y^1) \in M$. Then $y^1 \neq 0$ by assumption. We may assume that $x^1 \neq 0$ as otherwise there is nothing to prove. Let us write $x^1 = (x_1^1, \dots, x_k^1)$, $y^1 = (y_1^1, \dots, y_{n-k}^1)$ with $x_1^1 \neq 0$ and $y_1^1 \neq 0$.

Let $b = (x_2^1, \dots, x_k^1)$. Let N be the intersection of M with $\Delta \times \{b\} \times \Delta^{n-k}$. Then N is non-empty and has dimension 1 everywhere. In fact, by Krulls Hauptidealsatz, the dimension of N at each point is at least 1. By Step 1, the dimension is at most 1.

We argue that we can take $|z_1|$ on M as small as we wish. Suppose otherwise,

$$\sup_{z \in M} |z_1| > 0.$$

Take $q \in \mathbb{Z}_{>0}$ with

$$|x_1^1|^q < |y_1^1|.$$

Consider the function $f : N \rightarrow \mathbb{C}$ sending (x, y) to y_1/x_1^q . Then f is a morphism of complex analytic spaces and is bounded, say

$$\sup_{(x,y) \in N} |f(x, y)| = C_0.$$

Then $C_0 > 1$ by our choice of q . But at the boundary of D , $|z_1| = 1$, so we find that $|f(x, y)|$ obtains its maximum on each irreducible component of N . So in particular, $|z_1|$ obtains its infimum on each irreducible component of N . This contradicts the fact that N has dimension 1 everywhere.

We can now assume that $|x_1^1| < \epsilon$. Now we can replace M by $\{x_1^1\} \times \Delta^{k-1} \times \Delta^{n-k}$ and reduce the value of k by 1. By induction, we conclude. \square

Lemma 2.3 (Fundamental lemma). Let X be a complex manifold and F be a nowhere dense analytic set of dimension $\leq k$, where $k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ such that for any $x \in E$,

$$\dim_x E = k.$$

Then

$$\{x \in F : \bar{E} \text{ is analytic at } x\}$$

is clopen in X .

PROOF. The given set is clearly open. It suffices to show that it is closed.

Let $p \in F$ be a point in the closure of the given set. We need to show that \bar{E} is analytic at p . The problem is local on X , we may assume that X is a complex model space. Then it is immediate that we can reduce to the case where X is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$. By enlarging F , we may assume that F is defined by $y = 0$, where x, y denote the first k and the last $n - k$ coordinates on $X \subseteq \mathbb{C}^n$. Finally, we may assume that $p = 0$.

By [Lemma 2.1](#), we can take a linear subspace L of \mathbb{C}^n which meets F and E only at discrete points. We may arrange that L is defined by the condition $x = 0$.

Take $\epsilon, \delta > 0$ so that

(1)

$$S := \{(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \|x\|_{L^\infty} < \epsilon, \|y\|_{L^\infty} < \delta\} \subseteq D;$$

(2)

$$\{(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \|x\|_{L^\infty} < \epsilon, \|y\|_{L^\infty} = \delta\} \cap E = \emptyset.$$

Observe that for all $a \in \mathbb{C}^k$, $\|a\|_{L^\infty} < \epsilon$, the intersection

$$(\{a\} \times \mathbb{C}^{n-k}) \cap E \cap \text{Int } S$$

is discrete. In fact, the intersection is an analytic set in $S \setminus F$ and the absolute values of y_1, \dots, y_{n-k} take their maxima on each irreducible components by (2). So they are in fact constant.

By our assumption, there are points at which \bar{E} is analytic on $Z := \{|x| < \epsilon, y = 0\}$. Let B_0 be a connected component of the set of such points. We can equivalently view B_0 as an open subset of $\{|x| < \epsilon\}$. Then for any $a \in B_0$, the set

$$F_a := \{(x, y) \in \mathbb{C}^n : x = a\} \cap \bar{E} \cap \text{Int } S$$

is discrete. Let (x^1, y^1) be a point in this set, then \bar{E} is equidimensional of dimension k at this point. Each irreducible component K_j at (x^1, y^1) is a ramified covering of order m_j . We define the order $m(x^1, y^1)$ as this sum.

For each $a \in B_0$, we define $s(a)$ as the sum of multiplicities of points of F_a . Then $s(a)$ is locally constant on B_0 and by (2), $s(a)$ is actually constant. Let s be this common value.

Assume that \bar{E} is not analytic at 0. Then B_0 meets $|x| = \epsilon$, say at x' . Let s' be the number of intersection points of $\{x = x'\} \cap E$ counting multiplicity.

Observe that $s' \leq s$, as otherwise, there will be more than s points of E over points of B_0 close to x' . But $s' \neq s$ as otherwise, we contradict [Lemma 2.2](#).

So $s' < s$. If $x \in B_0$ converges to x' , then at least one of the s points of \bar{E} over x converges to $(x', 0)$ and the coordinates y_1, \dots, y_{n-k} of this point converge to 0. The same holds for all boundary points of B_0 in $\{\|x\|_{L^\infty} < \epsilon\}$.

We introduce $n - k$ unknowns X_1, \dots, X_{n-k} and set

$$z = \sum_{j=1}^{n-k} y_j X_j.$$

If (x, y^i) ($i = 1, \dots, s$) denotes the s -points of \bar{E} lying over $x \in B_0$, then we set

$$z^i := \sum_{j=1}^{n-k} y_j^i X_j$$

for $i = 1, \dots, s$. Then $z^1 \cdots z^s$ is a homogeneous polynomial of degree s . The coefficients are holomorphic on B_0 by Riemann extension theorem. As B_0 is not contained in \bar{E} , the coefficients are not all 0.

If $x \in B_0$ converges to a boundary point of B_0 in $\{\|x\|_{L^\infty} < \epsilon\}$, then all coefficients converge to 0.

By [Proposition 4.32](#) in [Morphisms between complex analytic spaces](#), we conclude that the boundary points of B_0 in the interior of $\{\|x\|_{L^\infty} < \epsilon\}$ lie in an analytic subset of codimension 1.

Let $Q(z) = (z - z^1) \cdots (z - z^s)$. Then Q is a homogeneous polynomial of degree s with respect to the u_j 's. The coefficients are holomorphic on $x \in B_0$ and are polynomials in the y_i 's. The vanishing of the coordinates defines exactly the part of \bar{E} over B_0 in the interior of S . But the coefficients are bounded at the boundary, so they extend to holomorphic functions everywhere and in particular on $\{\|x\|_{L^\infty} < \epsilon\}$. The vanishing of the coefficients define an analytic set E' in $B_0 \times \{\|y\|_{L^\infty} < \delta\}$. Each point of E' belongs to the part of \bar{E} lying over B_0 . So \bar{E} is analytic at each

point of $\{\{(x, 0) : \|x\|_{L^\infty} < \epsilon\} < \epsilon\}$. In particular, $B_0 = \{\{\|x\|_{L^\infty} < \epsilon\} < \epsilon\}$. This is a contradiction. \square

Theorem 2.4. Let X be a complex manifold and F be a nowhere-dense analytic set in X of dimension $\leq k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ all of whose irreducible components are of dimension $\geq k$ on each point. Consider a point $x \in F$ with $\dim_x F < k$. Then \bar{E} is analytic at x .

PROOF. Let $r = \dim_x F$. The problem is local. By [Theorem 2.4](#) in [Local properties of complex analytic spaces](#), we may assume that F is of dimension $\leq r$ everywhere. We need to show that \bar{E} is an analytic set in X . By induction on r , we may clearly assume that F is a complex manifold of equidimension r with respect to the reduced induced structure.

Again, as the problem is local, we may reduce to the case where X is a complex model space and then to the case where X is an open neighbourhood of $0 \in \mathbb{C}^n$ for some $n \in \mathbb{N}$. Let $p \in F$, we want to show that \bar{E} is analytic at p . We may then assume that $p = 0$. We can then rearrange F so that F is a linear subspace of dimension r_0 . We can take a closed subspace V of X such that $V \setminus F$ intersects E at an analytic subset of dimension $< k$. Let $E_1 = E \setminus V$. Then

$$\overline{E_1} = \bar{E}.$$

As $\overline{E_1}$ is analytic at all points in $V \setminus F$, it follows from [Lemma 2.3](#) that $\overline{E_1}$ is analytic on all points of V . So \bar{E} is analytic at points in F . \square

Theorem 2.5 (Remmert–Stein). Let X be a complex analytic space and F be a nowhere-dense analytic set in X of dimension $\leq k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ all of whose irreducible components are of dimension $\geq k$ on each point. Then

$$\{x \in F : \bar{E} \text{ is not analytic at } x\}$$

is an analytic set of dimension k at each point.

PROOF. The problem is local on X , so we may assume that X is a complex model space. Then we reduce immediately to the case where X is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$. In particular, we may assume that X is a complex manifold.

Let F' be set of regular points of F of dimension k and $F'_0 \subseteq F'$ be the set of points where \bar{E} is analytic. Then F'_0 is the union of some connected components of F_0 by [Lemma 2.3](#).

Let F'_1 be the union of the other connected components of F' . Observe that $G := \overline{F'_1}$ is an analytic subset of F . Observe that \bar{E} is analytic at no points of G . It suffices to show that \bar{E} is analytic at all points of $F \setminus G$.

Let $p \in F \setminus G$. We show that \bar{E} is analytic at p . If $\dim_p F < k$, we just apply [Theorem 2.4](#). So we may assume that $\dim_p F = k$. By our choice, $p \in \overline{F'_0}$. In a neighbourhood of p , the subset of F consisting of points where \bar{E} is not analytic is contained in $\overline{F'_0} \setminus F'_0$, which is an analytic set of dimension $< k$. We conclude again by [Theorem 2.4](#). \square

Corollary 2.6. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $n \in \mathbb{N}$. Assume that X is a complex manifold. Then

$$\{x \in X : \dim_x f^{-1}(x) \geq n\}$$

is closed.

PROOF. Let $x \in X$, $\dim_x f^{-1}(x) = n$. We need to show that the fiber dimension in a neighbourhood of x is at most n .

The problem is local, so we may assume that Y is Hausdorff. Suppose our assertion is false, then we can find a sequence $x_i \in X$ converging to x such that $\dim_{x_i} f^{-1}(x_i) > d$ for all $i \in \mathbb{Z}_{>0}$. Let E_i be the irreducible component of $f^{-1}(x_i)$ containing x_i such that $\dim_{x_i} E_i = \dim_{x_i} f^{-1}(x_i)$ for $i \in \mathbb{Z}_{>0}$.

We may assume that E_i 's have the same dimension $d > n$ and x_i and x are all different. Let M be the union of the E_i 's, then M is an analytic set in $X \setminus f^{-1}(x)$. By [Theorem 2.5](#), \bar{M} is analytic near x . This is absurd. \square

Corollary 2.7 (Remmert). Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $n \in \mathbb{N}$. Then

$$\{x \in X : \dim_x f^{-1}(x) \geq n\}$$

is an analytic set in X .

This result is not stated in the correct way in Remmert's paper. In most of Remmert's papers, the notion of codimension is misused.

PROOF. By [Corollary 2.6](#), the given set is closed. It suffices to show that it is analytic along each point on X . In particular, we may assume that X is connected.

Step 1. We reduce to the case where Y is a complex manifold.

The problem is local on Y , so we may assume that Y is a complex model space. Then clearly, we can assume that Y is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$. In particular, Y is a complex manifold.

Step 2. We first handle the case where X is a complex manifold and the rank of $\Omega_{X/Y}$ is constant.

In this case, we simply observe that $\dim_x f^{-1}(x) = \text{rank}_x \Omega_{X/Y}$ and our assertion is obvious.

Step 3. The problem is local on X , so we may assume that $\dim X < \infty$.

Let

$$B = \{x \in X^{\text{reg}} : \text{rank}_x \Omega_{X/Y} > \tau\},$$

where

$$\tau := \min_{x' \in X} \text{rank}_{x'} \Omega_{X/Y}.$$

Then B is an analytic set in X^{reg} by Step 2. The closure \bar{B} is an analytic set in X , as this can be characterized by the condition that $\text{rank}_x \Omega_{X/Y} > \tau$. Moreover, $\dim \bar{B} < N$.

We may assume that $n > \tau$, as there is nothing to prove otherwise. In particular,

$$\{x \in X : \dim_x f^{-1}(x) \geq n\} \subseteq \bar{B} \cup X^{\text{Sing}}.$$

We write $D = \bar{B} \cup X^{\text{Sing}}$ and endow it with the reduced induced structure.

We make induction on $N := \dim X$. The problem is trivial when $N = 0$. Assume that $N \geq 1$. Then

$$\{x \in D_0 : \dim_x f^{-1}(x) \geq n\}$$

is an analytic set in D for each connected component D_0 of D .

We observe that

$$\{x \in X : \dim_x f^{-1}(x) \geq n\} = \bigcup_{D_0} \{x \in D_0 : \dim_x f^{-1}(x) \geq n\},$$

where D_0 runs over all connected components of D and $N - s_0$ is the dimension of D_0 . From this it follows that $\{x \in X : \dim_x f^{-1}(x) \geq n\}$ is analytic, as the formula union on the right-hand side is locally finite. \square

Corollary 2.8. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. Then

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is co-analytic.

PROOF. This follows immediately from [Corollary 2.7](#). \square

As an application of Remmert–Stein theorem, we prove Chow’s theorem.

Theorem 2.9. Let $n \in \mathbb{N}$ and X be a closed analytic subspace of \mathbb{P}^n . Then X is the analytification of a closed subvariety of \mathbb{P}^n .

PROOF. We may assume that X is non-empty. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the projection and $Y = \pi^{-1}(X)$. Then X is analytic in $\mathbb{C}^{n+1} \setminus \{0\}$. By [Theorem 2.5](#), \bar{X} is an analytic set in \mathbb{C}^{n+1} .

Choose an open ball U in \mathbb{C}^{n+1} centered at 0 and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^{n+1}}(U)$ such that $\bar{X} \cap U = W(f_1, \dots, f_k)$. Let \mathcal{P} be the collection of homogeneous components of the f_i ’s. Then

$$X = \bigcap_{f \in \mathcal{P}} W(f).$$

In fact, let us denote the right-hand side by Y for the moment. It is clear that $\bar{X} \cap U$ contains $Y \cap U$ and hence $\bar{X} \supseteq Y$. Conversely, take $x \in \bar{X} \cap U$, from the fact that $\lambda x \in \bar{X} \cap U$ for all $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we find easily that all homogeneous components of the f_i ’s vanishes at x . So $x \in Y$. We conclude that $\bar{X} \subseteq Y$.

Now as $\mathbb{C}[X_0, \dots, X_n]$ is noetherian, we may take a finite subcollection \mathcal{P}' of \mathcal{P} such that

$$X = \bigcap_{f \in \mathcal{P}'} W(f).$$

\square

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Bibliography

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