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Constructions of complex analytic spaces

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Proposition 2.1. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Then the presheaf $F_{\mathcal{A}}$ on $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$ defined by

$$F_{\mathcal{A}}(T \xrightarrow{p} S) = \mathrm{Hom}_{\mathcal{O}_T}(p^*\mathcal{A}, \mathcal{O}_T)$$

is representable.

Proof. By the arguments of [Stacks, Tag 01JJ], the problem is local in S. So we may assume that $\mathcal{A}$ has the following form

$$\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and $\mathcal{I} \subseteq \mathcal{O}_S(S)[X_1, \dots, X_n]$ an ideal sheaf of finite type.

Step 1. We first handle the case where $\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]$.

In this case, we claim that $F_{\mathcal{A}}$ is represented by $S \times \mathbb{C}^n$. In fact, it suffices to observe that

$$\begin{aligned} F_{\mathcal{A}}(T \xrightarrow{p} S) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) \xrightarrow{\sim} \mathcal{O}_T(T)^n \\ &= \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}}(T, \mathbb{C}^n) = \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}}(T, S \times \mathbb{C}^n). \end{aligned}$$

From this proof, it is easy to see that the universal morphism is

$$\eta : \mathcal{O}_{S\times\mathbb{C}^n}[X_1, \dots, X_n] \to \mathcal{O}_{S\times\mathbb{C}^n}$$

sending X_i to z_i, the i-th coordinate of $\mathbb{C}^n$.

Step 2. We handle the general case. We have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_S[X_1, \dots, X_n] \to \mathcal{A} \to 0.$$

For any $p : T \to S$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$, we have an exact sequence

$$p^*\mathcal{I} \to \mathcal{O}_T[X_1, \dots, X_n] \to p^*\mathcal{A} \to 0.$$

We then have

$$\begin{aligned} F_{\mathcal{A}}(T) &\xrightarrow{\sim} \{h \in \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) : h|_{p^*\mathcal{I}} = 0\} \\ &\xrightarrow{\sim} \left\{h \in F_{\mathcal{O}_S[X_1,\dots,X_n]}(T) : h|_{p^*\mathcal{I}} = 0\right\}. \end{aligned}$$

Let $\pi : S \times \mathbb{C}^n \to S$ be the projection. Then $F_{\mathcal{A}}(T)$ is represented by the closed subspace of $S \times \mathbb{C}^n$ defined by the ideal $\eta(\pi^*\mathcal{I})$, which is clearly of finite type.

□

Definition 2.2. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Then the complex analytic space representing the functor in Proposition 2.1 is called the *analytic spectrum* of $\mathcal{A}$. We denote it by $\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A}$. By construction, there is a canonical morphism $\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A} \to S$.

It is easy to see that $\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A}$ is contravaraint in $\mathcal{A}$.

Proposition 2.3. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Consider a morphism $g: S' \to S$ of complex analytic spaces. Then we have a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Spec}_{S'}^{\mathrm{an}}\, g^*\mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A} \arrow[d] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

Proof. This is clear at the level of functor of points. $\square$

Corollary 2.4. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Take $s \in S$. Then $\mathrm{Spec}_{\{s\}}^{\mathrm{an}}\,\mathcal{A}_s \xrightarrow{\sim} (\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A})_s$.

Moreover, the universal morphism $\mathcal{A}_{\mathrm{Spec}_{\{s\}}^{\mathrm{an}}\,\mathcal{A}_s} \to \mathcal{O}_{\mathrm{Spec}_{\{s\}}^{\mathrm{an}}\,\mathcal{A}_s}$ is the reduction of the universal morphism $\mathcal{A}_{\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A}} \to \mathcal{O}_{\mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A}}$ modulo $\mathfrak{m}_s$.

Proof. This follows from Proposition 2.3. $\square$

Proposition 2.5. Let S be a complex analytic space and $\mathcal{A}$ be an $\mathcal{O}_S$-module of finite presentation. Take $s \in S$. Write $X = \mathrm{Spec}_S^{\mathrm{an}}\,\mathcal{A}$ and $\mathcal{A}_s := \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,s}$. Then the map from X_s to

$$\{\mathfrak{m} \in \mathrm{Spm}_{\mathbb{C}}\,\mathcal{A}_s : \mathfrak{m} \supseteq \mathfrak{m}_s\}$$

sending $x \in X_s$ to the inverse image of $\mathfrak{m}_x$ with respect to $\mathcal{A}_s \to \mathcal{O}_{X,x}$ is bijective.

If $\mathfrak{m}$ corresponds to $x \in X_s$, then the natural homomorphism $\mathcal{A}_s \to \mathcal{O}_{X,x}$ factorizes through $\mathcal{A}_{s,\mathfrak{m}} \to \mathcal{O}_{X,x}$. The completion of the latter

$$\widehat{\mathcal{A}_{s,\mathfrak{m}}} \to \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism.

Proof. By Corollary 2.4, we have natural bijections

$$X_s \xrightarrow{\sim} \mathrm{Hom}_{\{s\}}(\{s\}, X_s) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathrm{lg}}(\mathcal{A}_s/\mathfrak{m}_s\mathcal{A}_s, \mathbb{C}).$$

This gives the desired bijection.

Next we prove the latter part. The problem is local on S, we may assume that

$$\mathcal{A} = \mathcal{O}_S[X_1, \ldots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and some ideal $\mathcal{I}$ of finite type in $\mathcal{O}_S[X_1, \ldots, X_n]$. Recall that the universal morphism

$$\eta : \mathcal{O}_{S\times\mathbb{C}^n}[X_1, \ldots, X_n] \to \mathcal{O}_{S\times\mathbb{C}^n}$$

sends X_i to z_i, the i-th coordinate of $\mathbb{C}^n$.

By construction, we have

$$\mathcal{A}_s \xrightarrow{\sim} \mathcal{O}_{S,s}[X_1, \ldots, X_n]/\mathcal{I}_s$$

and

$$\mathcal{O}_{X,x} = \mathcal{O}_{S\times\mathbb{C}^n,x}/\mathcal{J}_x,$$

where

$$\mathcal{J}_x = \eta_x \left(\mathcal{I}_s \mathcal{O}_{S\times\mathbb{C}^n,x}[X_1,\ldots,X_n]\right).$$

We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}_s \arrow[r] \arrow[d] & \mathcal{O}_{S,s}[X_1,\ldots,X_n] \arrow[r] \arrow[d] & \mathcal{A}_s \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{J}_x \arrow[r] & \mathcal{O}_{S\times\mathbb{C}^n,x} \arrow[r] & \mathcal{O}_{X,x} \arrow[r] & 0 \end{tikzcd}.$$

The middle vertical map is induced by η_x. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{m}_{S\times\mathbb{C}^n,x}$ under the vertical map in the middle. Then $\mathfrak{p}$ is generated by $\mathfrak{m}_s$ and $X_1 - x_1,\ldots,X_n-x_n$, where $x_i\in\mathbb{C}$ is the value of z_i at x for $i=1,\ldots,n$. By localization and completion, we find a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \widehat{(\mathcal{I}_s)_{\mathfrak{p}}} \arrow[r] \arrow[d] & (\mathcal{O}_{S,s}[X_1,\ldots,X_n])^{\wedge}_{\mathfrak{p}} \arrow[r] \arrow[d] & \widehat{(\mathcal{A}_s)_{\mathfrak{m}}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widehat{\mathcal{J}_x} \arrow[r] & \widehat{\mathcal{O}_{S\times\mathbb{C}^n,x}} \arrow[r] & \widehat{\mathcal{O}_{X,x}} \arrow[r] & 0 \end{tikzcd}.$$

Observe that

$$(\mathcal{O}_{S,s}[X_1,\ldots,X_n])^{\wedge}_{\mathfrak{p}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1-x_1,\ldots,X_n-x_n]]$$

and

$$\widehat{\mathcal{O}_{S\times\mathbb{C}^n,x}} \cong \widehat{\mathcal{O}_{S,s}}\hat{\otimes}_k \widehat{\mathcal{O}_{\mathbb{C}^n,(x_1,\ldots,x_n)}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1-x_1,\ldots,X_n-x_n]].$$

It is easy to see that the middle map is an isomorphism. As $\mathcal{J}_x$ is generated by $\mathcal{I}_s$, the first vertical map is also an isomorphism. Our assertion follows. □

Corollary 2.6. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Write $X = \operatorname{Spec}^{\mathrm{an}}_S \mathcal{A}$. Take $s\in S$. Then the fiber X_s is finite and is in bijection with $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_s$. If $\mathfrak{m}$ corresponds to $x\in X_s$, then we have a natural isomorphism

$$\mathcal{A}_{s,\mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}.$$

Proof. As $\mathcal{O}_{S,s}\to\mathcal{A}_s$ is finite, $\mathcal{A}_s$ is semi-local. On the other hand, by Proposition 2.5,

$$\mathcal{A}_{s,\mathfrak{m}} \to \mathcal{O}_{X,x}$$

is injective and $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{S,s}$. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by Theorem 5.4 in Complex analytic local algebras. It follows from Nakayama's lemma that $\mathcal{A}_{s,\mathfrak{m}}\to\mathcal{O}_{X,x}$ is also surjective. □

Corollary 2.7. Let S be a complex analytic space and $\mathcal{A}$ be a finite $\mathcal{O}_S$-algebra. Then the image of $\operatorname{Spec}^{\mathrm{an}}_S \mathcal{A}\to S$ is $\operatorname{Supp} A$.

Proof. This follows from Corollary 2.6 and the fact that $\operatorname{Spm}_{\mathbb{C}}\mathcal{A}_s = \operatorname{Spm}\mathcal{A}_s$ for all $s\in S$. □

Definition 2.8. Let S be a complex analytic space and $\mathcal{E}$ be an $\mathcal{O}_S$-module of finite presentation. We define the *vector bundle* $\mathbf{V}(\mathcal{E})$ generated by $\mathcal{E}$ as

$$\mathbf{V}(\mathcal{E}) = \operatorname{Spec}^{\mathrm{an}}_S \operatorname{Sym}\mathcal{E}.$$

We have a natural projection $\mathbf{V}(\mathcal{E})\to S$.

We remind the readers that we are following Grothendieck's convention for $\mathbf{V}(\mathcal{E})$, which is different from Fulton's.

3. Analytic germs

Definition 3.1. A *pointed complex analytic space* is a pair (X, x) consisting of a complex analytic space X and a point $x \in X$. A morphism between pointed complex analytic spaces (X, x) and (Y, y) is a morphism $f : X \to Y$ of complex analytic spaces such that $f(x) = y$. The category of pointed complex analytic spaces is denoted by $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_*$.

The category of *complex analytic germs* $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is the right category of fractions of $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$ with respect to the system of morphisms $f : (X, x) \to (Y, y)$ such that $f : X \to Y$ is an open immersion. An element in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is called a *complex analytic germ*. A complex analytic germ represented by (X, x) is denoted by X_x.

Given a complex analytic germ X_x, we write $\mathcal{O}_{X,x}$ for the local ring of X at x. Clearly, it does not depend on the choice of (X, x). Given any morphism $f : X_x \to Y_y$ of complex analytic germs, we have an obvious local homomorphism $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

Definition 3.2. Given a complex analytic germ X_x, a *closed subgerm* of X_x is an isomorphism class in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}_{/X_x}$ of Y_x represented by a closed analytic subspace of X containing x for any representation (X, x) of X_x.

In particular, X_x is a closed subgerm of X_x. A closed subgerm Y_y of X_x is *proper* if Y_y is different from X_x as subgerms.

Given a closed subgerm Y_x of X_x, we have an induced surjective homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$. The kernel is denoted by $I(Y, x)$ or $I_X(Y, x)$.

Theorem 3.3. The functor $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}^{\mathrm{op}} \to \mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ defined in Definition 3.1 is an equivalence.

Proof. **Step 1**. We show that the functor is faithfully.

In order words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $f, g : (X, x) \to (Y, y)$ be two morphsims inducing the same map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, then f and g coincide on a neighbourhood of x in X.

The question is open on Y, so we may reduce to the case where Y is a complex model space. We then further reduce to the case where Y is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and then to $Y = \mathbb{C}^n$.

By Theorem 4.2 in The notion of complex analytic spaces, f and g can be identified with systems $(f_1, \dots, f_n) \in \mathcal{O}_X(X)^n$ and $(g_1, \dots, g_n) \in \mathcal{O}_X(X)^n$. The assumption $f_x^\# = g_x^\#$ menas $f_{i,x} = g_{i,x}$ for $i = 1, \dots, n$. So $f_i = g_i$ after shrinking X. We conclude by Theorem 4.2 in The notion of complex analytic spaces again.

Step 2. We show that the functor is fully faithful.

In other words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $\varphi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ be a morphism in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$. Then we can find an open neighbourhood U of x in X and a morphism $(U, x) \to (Y, y)$ inducing φ.

The problem is local on Y, so we may assuem that Y is a complex model space, say Y is a closed subspace of a domain V in $\mathbb{C}^n$ defined by a coherent ideal $\mathcal{I}$. We write $\psi : \mathcal{O}_{V,y} \to \mathcal{O}_{X,x}$ the homomorphism induced by φ, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{V,y} \arrow[r, "\psi"] \arrow[d, two heads] & \mathcal{O}_{Y,y} \\ \mathcal{O}_{X,x} \arrow[ur, "\varphi"] & \end{tikzcd} .$$

Let $z_1, \dots, z_n$ be the coordinates on V. Let $f_{i,x}$ be the image of $z_{i,x}$ under ψ for $i = 1, \dots, n$. Take an open neighbourhood U of x in X so that $f_{i,x}$ lifts to $f_i \in \mathcal{O}_X(U)$ for $i = 1, \dots, n$. By Theorem 4.2 in The notion of complex analytic spaces, $f_1, \dots, f_n$ then defines a morphism $g : U \to \mathbb{C}^n$. Clearly $g(x) = y$. But $g_x^{\#}$ and ψ coincide on $z_{i,y}$ so $g_x^{\#} = \psi$ as $\mathcal{O}_{V,y} = \mathbb{C}\{z_{1,y} - a_1, \dots, z_{n,y} - a_n\}$ with $a_i = \epsilon(z_{i,y})$ for $i = 1, \dots, n$. Therefore, $g_x^{\#}(\mathcal{I}_y) = 0$. Up to shrinking U, we may guarantee that $g(U) \subseteq V$ and $g^*(\mathcal{I}) = 0$ on U. Namely, g factorizes through $f : U \to Y$ and $f_x^* = \varphi$.

Step 3. We show that the functor is essentially surjective.

In other words, let A be a complex analytic local algebra, then there is a pointed complex analytic space (X, x) with $\mathcal{O}_{X,x} \cong A$ in $\mathbb{C}$-$\mathcal{L}$A.

We may assume that $A = \mathbb{C}\{z_1, \dots, z_n\}/I$ for some $n \in \mathbb{N}$ and ideal I in $\mathbb{C}\{z_1, \dots, z_n\}$. Then I is finitely generated as $\mathbb{C}\{z_1, \dots, z_n\}$ is noetherian. Take finitely many generators $f_1, \dots, f_m \in I$. We extend $f_1, \dots, f_m$ to $g_1, \dots, g_m \in \mathcal{O}_{\mathbb{C}^n}(U)$ for some open neighbourhood U of 0 in $\mathbb{C}^n$. Then the closed subspace X of U defined by $f_1, \dots, f_m$ satisfies the required conditions. □

Lemma 3.4. Let S be a complex analytic space and $s \in S$. For any finite $\mathcal{O}_{S,s}$-algebra A, there is an open neighbourhood U of s in S and a finite $\mathcal{O}_U$-algebra such that $\mathcal{A}_s \cong A$.

Proof. This follows from the same arguments of the proof of Theorem 3.3 Step 3. □

Corollary 3.5. Let X_x be a complex analytic germ, $n \in \mathbb{Z}_{>0}$ and $f_1, \dots, f_n \in \mathcal{O}_{X,x}$ be a system of parameters. We have a morphism $X_x \to \mathbb{C}_0^n$ induced by $f_1, \dots, f_n$. Then there is an open neighbourhood U of 0 in $\mathbb{C}^n$ and a finite $\mathcal{O}_U$-algebra $\mathcal{A}$ such that $\mathcal{A}_0 \cong \mathcal{O}_{X,x}$. The space $\operatorname{Spec}_U^{\mathrm{an}}(\mathcal{A})$ admits a unique point x' over 0 and X_x is isomorphic to $\operatorname{Spec}_U^{\mathrm{an}}(\mathcal{A})_{x'}$ in $\mathbb{C}$-$\mathcal{G}\mathrm{er}_{/\mathbb{C}_0^n}$.

Proof. As $f_1, \dots, f_n$ is a system of parameters, $\mathcal{O}_{X,x} \to \mathcal{O}_{\mathbb{C}^n,0}$ is finite. By Lemma 3.4, we can spread $\mathcal{O}_{X,x}$ to a finite $\mathcal{O}_U$-algebra on an open neighbourhood U of 0 in $\mathbb{C}^n$. Let $Y = \operatorname{Spec}_U^{\mathrm{an}}(\mathcal{A})$. It follows from Corollary 2.6 that Y has a unique point x' over 0. By Theorem 3.3, up to shrinking U, we may guarantee that X_x and $Y_{x'}$ are isomorphic over $\mathbb{C}_0^n$. □

Proposition 3.6. Let X_x be a complex analytic germ. The map $Y_x \mapsto I_X(Y, x)$ defines a bijection between the set of closed subgerms of X_x and the set of ideals of $\mathcal{O}_{X,x}$.

Proof. We construct a reverse map. Given an ideal I of $\mathcal{O}_{X,x}$, as $\mathcal{O}_{X,x}$ is noetherian, I is finitely generated. We can find an open neighbourhood U of x in X and an ideal sheaf of finite type $\mathcal{I}$ of U with $\mathcal{I}_x = I$. Let Y be the closed analytic subspace of X defined by $\mathcal{I}$. We associated Y_x with I.

It is easy to verify that this map is the inverse of the given map. □

Definition 3.7. Let X_x be a complex analytic germ and Y_x, Z_x be two closed subgerms of X_x. We say Y_x is contained in Z_x and write $Y_x \subseteq Z_x$ if $I(Y, x) \supseteq I_X(Z, x)$. This defines a partial order on the set of closed subgerms of X_x.

Definition 3.8. A complex analytic germ X_x is *integral* if $\mathcal{O}_{X,x}$ is integral.

Theorem 3.9 (Nullstellensatz)**.** Let X_x be an integral complex analytic germ and Y_y be a closed subgerm of X_x. Then the following are equivalent:

(1) Y_x is a proper closed subgerm of X_x;
(2) $|Y|_x$ is a proper closed subgerm of $|X|_x$.

PROOF. (2) $\Longrightarrow$ (1): This is obvious.

(1) $\Longrightarrow$ (2): Consider a proper closed subgerm Y_x of X_x. By Proposition 3.6, $I(Y,x) \neq 0$.

Step 1. We reduce to the case $I(Y,x) = (f)$ for some non-zero element $f \in \mathcal{O}_{X,x}$.

Take a non-zero element $f \in I(Y,x)$. Let Y'_x be the subgerm of X_x corresponding to the ideal (f) of $\mathcal{O}_{X,x}$. Then $Y_x \subseteq Y'_x$. It suffices to show that $|Y'|_x \neq |X|_x$. We may replace Y by Y'.

Step 2. We prove that $|Y|_x \neq |X|_x$.

Note that f is not a zero-divisor as $\mathcal{O}_{X,x}$ is integral. Write $n = \dim \mathcal{O}_{X,x}$. By Krulls Hauptidealsatz, $\dim \mathcal{O}_{X,x}/(f) = n-1$. Let $\overline{f_1}, \ldots, \overline{f_{n-1}}$ be a system of parameters ([Stacks, Tag 00KU]) of $\mathcal{O}_{X,x}/(f)$. Lift them to $f_1, \ldots, f_{n-1} \in \mathcal{O}_{X,x}$. Then $(f_1, \ldots, f_{n-1}, f)$ is a system of parameters of $\mathcal{O}_{X,x}$. Let $\varphi : X_x \to \mathbb{C}^n_0$ and $\psi : Y_x \to \mathbb{C}^{n-1}_0$ be the morphisms defined by these systems of parameters. We then have a commutative diagram in $\mathbb{C}$-$\mathcal{G}$er:

$$\begin{tikzcd} Y_x \arrow[r, hook] \arrow[d, "\psi"] & X_x \arrow[d, "\varphi"] \\ \mathbb{C}^{n-1}_0 \arrow[r, hook] & \mathbb{C}^n_0 \end{tikzcd}$$

It induces a commutative diagram of topological germs:

$$\begin{tikzcd} {|Y|_x} \arrow[r, hook] \arrow[d, "|\psi|"] & {|X|_x} \arrow[d, "|\varphi|"] \\ \mathbb{C}^{n-1}_0 \arrow[r, hook] & \mathbb{C}^n_0 \end{tikzcd}$$

The morphism of topological germs of $\mathbb{C}^{n-1}_0 \to \mathbb{C}^n_0$ is clearly not an isomorphism, so it suffices to show that $|\varphi| : |X|_x \to \mathbb{C}^n_0$ is surjective, in the sense that if we represent $|\varphi|$ by a morphism $(U,x) \to (\mathbb{C}^n, 0)$ from an open neighbourhood U of x in X to $\mathbb{C}^n$, then its image contains an open neighbourhood of 0 in $\mathbb{C}^n$.

By Corollary 3.5, we may assume that $X = \operatorname{Spec}^{\mathrm{an}}_X \mathcal{A}$ for some finite $\mathcal{O}_X$-algebra $\mathcal{A}$ and X has a unique point over 0. Then by Corollary 2.6, we have $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{O}_{X,x}$. By Corollary 5.5 in Complex analytic local algebras, the natural homomorphism

$$\varphi^{\#} : \mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{X_1, \ldots, X_n\} \to \mathcal{A}_0$$

is injective.

By Corollary 2.7, it remains to show that $\operatorname{Supp} \mathcal{A}$ is a neighbourhood of s in S. But the kernel of $\mathcal{O}_S \to \mathcal{A}$ is 0 at s hence 0 in a neighbourhood of s since both $\mathcal{O}_S$ and $\mathcal{A}$ are coherent by Corollary 7.4 in The notion of complex analytic spaces. □

Corollary 3.10. Let X_x be a complex analytic germ and I, J be two ideals in $\mathcal{O}_{X,x}$. We let $W(I)$, $W(J)$ denote the topological germs of the closed analytic subgerms of X_x defined by I and J respectively. Then the following are equivalent:

(1) $W(I) \subseteq W(J)$;

(2) $J \subseteq \sqrt{I}$.

Proof. If (2) is true, as $\mathcal{O}_{X,x}$ is noetherian, we can find $n \in \mathbb{Z}_{>0}$ such that $J^n \subseteq I$. Extend I, J to coherent ideals $\mathcal{I}, \mathcal{J}$ on X up to shrinking X. Then $\operatorname{Supp} \mathcal{O}_X/\mathcal{J} \subseteq \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$. Hence, (1) holds.

Suppose that (1) holds. In order to prove (2), we may assume that I is prime. Then the closed analytic subgerm Y_x of X_x defined by I is integral. Let Z_x denote the closed analytic subgerm of X_x defined by J. The intersection $Y_x \cap Z_x$ of the germs Y_x and Z_x is by definition the closed analytic subgerm of X_x defined by $I + J$. Then

$$|Y_x \cap Z_x| = |Y|_x \cap |Z|_x = W(I).$$

By Theorem 3.9, $Y_x \subseteq Z_x$. Namely, (2) holds. □

Corollary 3.11. Let X_x be a complex analytic germ and Y_x be a closed analytic subgerm. Then the following are equivalent:

(1) $|X|_x = |Y|_x$;
(2) $I_X(Y, x)$ is nilpotent.

In particular, if these conditions hold, $\dim \mathcal{O}_{Y,x} = \dim \mathcal{O}_{X,x}$.

Proof. This follows immediately from Corollary 3.10. □

Corollary 3.12. Let X be a complex analytic space and $x \in X$. Then the following are equvalent:

(1) x is isolated in X;
(2) $\mathcal{O}_{X,x}$ is artinian.

Proof. (1) simply means that $X_x = \{x\}_x$. By Corollary 3.11, this holds if and only if $\mathfrak{m}_x$ is nilpotent. As $\mathcal{O}_{X,x}$ is noetherian, the latter is equivalent to that $\mathcal{O}_{X,x}$ is artinian. □

Corollary 3.13. Let X be a complex analytic space and Y be a closed analytic subspace defined by a coherent ideal $\mathcal{I}$. Then the following are equivalent:

(1) $|X| = |Y|$;
(2) $\mathcal{I}$ is locally nilpotent.

Proof. This follows immediately from Corollary 3.11. □

Corollary 3.14. Let X be a complex analytic space and $f \in \mathcal{O}_X(X)$. Then the following are equivalent:

(1) $f(x) = 0$ for all $x \in X$;
(2) f is locally nilpotent.

Proof. This follows from Corollary 3.13, where we take $\mathcal{I}$ as the coherent ideal generated by f. □

Corollary 3.15 (Rückert Nullstellensatz)**.** Let X be a complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\operatorname{Supp} \mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m \mathcal{F}|_U = 0$.

Proof. Let $\mathcal{G}$ be the annihilator sheaf of $\mathcal{F}$:

$$\mathcal{G} := \ker\left(\mathcal{O}_X \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})\right),$$

where the map $\mathcal{O}_X \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ sends a local section f of $\mathcal{O}_X$ to the endohomomorphism of multiplying by f of $\mathcal{F}$. Then $\mathcal{G}$ is a coherent sheaf by Oka's coherence theorem Theorem 7.3 in The notion of complex analytic spaces. Let Y be the closed analytic subspace defined by $\mathcal{G}$. By our assumption, f is everywhere zero on Y, so f is locally nilpotent in $\mathcal{O}_X/\mathcal{G} \subseteq \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$. □

Corollary 3.16. Let X be a complex analytic space and $\mathcal{I}$ and $\mathcal{J}$ be coherent ideal sheaves on X. Then the following are equivalent:

(1) $\operatorname{Supp}\mathcal{O}_X/\mathcal{I} \subseteq \operatorname{Supp}\mathcal{O}_X/\mathcal{J}$;
(2) For any $x \in X$, there is an open neighbourhood U of x in X and $n \in \mathbb{Z}_{>0}$ such that
$$J^n|_U \subseteq I|_U.$$

Proof. This follows immediately from Corollary 3.10. □

Bibliography

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