Affinoid algebras

Contents

1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. We set

$$
\begin{aligned}
k\{r^{-1}T\} =& k\{r_1^{-1}T_1, \ldots, r_n T_n^{-1}\} \\
:=& \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \ldots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\}.
\end{aligned}
$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$
\|f\|_r = \max_\alpha |a_\alpha| r^\alpha .
$$

We call $(k\{r^{-1}T\}, \|\bullet\|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\|\bullet\|_r$ is called the *Gauss norm*.

We omit r from the notation if $r = (1, \ldots, 1)$.

This is a special case of Example 4.15 in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach k-algebra and $\|\bullet\|_r$ is a valuation.

PROOF. This is a special case of Proposition 4.16 in the chapter Banach Rings. □

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T\rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T\rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\} \cong k[T_1, \ldots, T_n]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \ldots, T_n]]$ otherwise.

Lemma 2.5. Let A be a Banach k-algebra. For each $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathring{A}$, there is a unique continuous homomorphism $k\{T_1, \ldots, T_n\} \to A$ sending T_i to a_i.

PROOF. This is a special case of Proposition 4.18 in the chapter Banach Rings. □

3. Affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

More generally, a Banach k-algebra A is *k_H-affinoid* if there are $n \in \mathbb{N}$, $r \in H^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

A morphism between k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is a bounded k-algebra homomorphism.

The category of k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is denoted by k-$\mathcal{A}$ff$\mathcal{A}$lg (resp. st-k-$\mathcal{A}$ff$\mathcal{A}$lg, resp. k_H-$\mathcal{A}$ff$\mathcal{A}$lg).

For the notion of admissible morphisms, we refer to Definition 2.5 in the chapter Banach rings.

Although we have defined strictly k-affinoid algebra when k is trivially valued, we will deliberately avoid talking about it in order to avoid pathologies.

Remark 3.2. Berkovich also introduced the notion of *affinoid k-algebras*: it is a K-affinoid algebra for some complete non-Archimedean field extension K/k. We will not use this notion.

Definition 3.3. The category of *k-affinoid spectra* k-$\mathcal{A}$ff (resp. *strictly k-affinoid spectra* st-k-$\mathcal{A}$ff, resp. *k_H-affinoid spectra* k_H-$\mathcal{A}$ff) is the opposite category of k-$\mathcal{A}$ff$\mathcal{A}$lg (resp. st-k-$\mathcal{A}$ff$\mathcal{A}$lg, resp. k_H-$\mathcal{A}$ff$\mathcal{A}$lg). An object in these categories are called a *k-affinoid spectrum*, *strictly k-affinoid spectrum* and *k_H-affinoid spectrum* respectively.

Given an object A of k-$\mathcal{A}$ff$\mathcal{A}$lg (resp. st-k-$\mathcal{A}$ff$\mathcal{A}$lg, resp. k_H-$\mathcal{A}$ff$\mathcal{A}$lg), we denote the corresponding object in k-$\mathcal{A}$ff (resp. st-k-$\mathcal{A}$ff, resp. k_H-$\mathcal{A}$ff) by $\operatorname{Sp} A$. We call $\operatorname{Sp} A$ the *affinoid spectrum* of A.

In Definition 6.1 in the chapter Banach Rings, we defined functors $\operatorname{Sp} : k\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$, $\operatorname{Sp} : \text{st-}k\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$ and $\operatorname{Sp} : k_H\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$. This motivates our notation. We will freely view $\operatorname{Sp} A$ as an object in these categories or as a topological space.

Proposition 3.4. Finite limits exist in k_H-$\mathcal{A}$ff.

Proof. It suffices to prove that finite fibered products exsit.

We prove the equivalent statement, finite fibered coproducts exist in k_H-$\mathcal{A}$ff$\mathcal{A}$lg. Given k_H-affinoid algebras A, B, C and morphisms $A \to B$, $A \to C$, we claim that $B \hat{\otimes}_A C$ represents the fibered coproduct of B and C over A. By general abstract nonsense, we are reduced to handle the following cases: $A = k$ and $A \to C$ is the codiagonal $C \hat{\otimes}_k C \to C$. In both cases, the proposition is clear.

□

Example 3.5. Let $r \in \mathbb{R}_{>0}$. We let k_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \to 0$ for $i \to \infty$ and $i \to -\infty$. We define a norm $\| \bullet \|_r$ on k_r as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in Proposition 3.6 that k_r is k-affinoid.

Proposition 3.6. Let $r \in \mathbb{R}_{>0}$, then $(k_r, \|\bullet\|_r)$ defined in Example 3.5 is a k-affinoid algebra. Moreover, $\| \bullet \|_r$ is a valuation.

Proof. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \to k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f=\sum_{(i,j)\in\mathbb{N}^2} a_{i,j}T_1^iT_2^j\in k\{r^{-1}T_1,rT_2\},$$

namely,

$$|a_{i,j}|r^{i-j}\to 0 \tag{3.1}$$

as $i+j\to\infty$. Observe that for each $k\in\mathbb{Z}$, the series

$$c_k:=\sum_{i-j=k,i,j\in\mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_kT^k.$$

We need to verify that $\iota(f)\in k_r$. That is

$$|c_k|r^k\to 0$$

as $k\to\pm\infty$. When $k\geq 0$, we have $|c_k|\leq |a_{k0}|$ by definition of c_k. So $|c_k|r^k\to 0$ as $k\to\infty$ by (3.1). The case $k\to-\infty$ is similar.

We conclude that we have a well-defined map of sets ι. It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g=\sum_{i=-\infty}^{\infty}c_iT^i\in k_r$. We want to show that g lies in the image of ι. As ι is a ring homomorphism, it suffices to treat two cases separately: $g=\sum_{i=0}^{\infty}c_iT^i$ and $g=\sum_{i=-\infty}^{0}c_iT^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f=\sum_{i=0}^{\infty}c_iT_1^i\in k\{r^{-1}T_1,rT_2\}$. It is immediate that $\iota(f)=g$.

Next we show that ι is admissible. We first identify the kernel of ι. We claim that the kenrel is the ideal I generated by T_1T_2-1. It is obvious that $I\subseteq\ker\iota$. Conversely, consider an element

$$f=\sum_{(i,j)\in\mathbb{N}^2} a_{i,j}T_1^iT_2^j\in k\{r^{-1}T_1,rT_2\}$$

lying in the kenrel of ι. Observe that

$$f=\sum_{k=-\infty}^{\infty} f_k,\quad f_k=\sum_{(i,j)\in\mathbb{N}^2,i-j=k} a_{i,j}T_1^iT_2^j.$$

If $f\in\ker\iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by T_1T_2-1. The condition that $f_k\in\ker\iota$ means

$$\sum_{(i,j)\in\mathbb{N}^2,i-j=k} a_{i,j}=0.$$

It is elementary to find $b_{i,j}\in k$ for $i,j\in\mathbb{N}$, $i-j=k$ such that

$$a_{i,j}=b_{i-1,j-1}-b_{i,j}.$$

Then

$$f_k = (T_1T_2 - 1) \sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j}T_1^iT_2^j.$$

Observe that we can make sure that $|b_{i,j}| \le \max\{|a_{i',j'}| : i-j = i'-j'\}$. In particular, the sum of $\sum_{i,j\in\mathbb{N},i-j=k} b_{i,j}T_1^iT_2^j$ for various k converges to some $g \in k\{r^{-1}T_1, rT_2\}$ and hence $f_k = (T_1T_2 - 1)g$. Therefore, we have proved that $\ker\iota$ is generated by $T_1T_2 - 1$.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \to k_r.$$

To see this, take $f = \sum_{k=-\infty}^{\infty} c_kT^k \in k_r$ and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set $g = \sum_{k=0}^{\infty} c_kT_1^k + \sum_{k=1}^{\infty} c_{-k}T_2^k$, then $\iota(g) = f$ and $\|g\|_{(r,r^{-1})} = \|f\|$. So it suffices to show that for any $h = \sum_{(i,j)\in\mathbb{N}^2} d_{i,j}T_1^iT_2^j \in k\{r^{-1}T_1, rT_2\}$, we have

$$\|f\|_r \le \|g + h(T_1T_2 - 1)\|_{r,r^{-1}}. \tag{3.2}$$

We compute

$$g+h(T_1T_2-1) = \sum_{k=1}^{\infty}(c_k-d_{k,0})T_1^k+\sum_{k=1}^{\infty}(c_{-k}-d_{0,k})T_2^k+(c_0-d_0)+\sum_{i,j\ge1}(d_{i-1,j-1}-d_{i,j})T_1^iT_2^j.$$

So

$$\|g + h(T_1T_2 - 1)\|_{r,r^{-1}} = \max\left\{\max_{k\ge0} C_{1,k}, \max_{k\ge1} C_{2,k}\right\},$$

where

$$C_{1,k} = \max\left\{|c_k - d_{k,0}|, \left|\sum_{i-j=k,i,j\ge1} d_{i-1,j-1} - d_{i,j}\right|\right\}$$

for $k \ge 0$ and

$$C_{2,k} = \max\left\{|c_{-k} - d_{0,k}|, \left|\sum_{i-j=-k,i,j\ge1} d_{i-1,j-1} - d_{i,j}\right|\right\}$$

for $k \ge 1$. It follows from the strong triangle inequality that $|c_k| \le C_{1,k}$ for $k \ge 0$ and $c_{-k} \le C_{2,k}$ for $k \ge 1$. So (3.2) follows. □

Proposition 3.7. Let $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$, then $\|\bullet\|_r$ defined in Example 3.5 is a valuation on k_r.

Proof. Take $f, g \in k_r$, we need to show that

$$\|fg\|_r \ge \|f\|_r\|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_iT^i, \quad g = \sum_{i=-\infty}^{\infty} b_iT^i.$$

Take i and j so that

$$|a_i|r^i = \|f\|_r, \quad |b_j|r^j = \|g\|_r. \tag{3.3}$$

By our assumption on r, i,j are unique. Then

$$\|fg\|_r=\max_{k\in\mathbb{Z}}\{|c_k|r^k\},$$

where

$$c_k:=\sum_{u,v\in\mathbb{Z},u+v=k}a_ub_v.$$

It suffices to show that

$$|c_k|r^k=\|f\|_r\|g\|_r. \tag{3.4}$$

for $k=i+j$. Of course, we may assume that $a_i\neq 0$ and $b_j\neq 0$ as otherwise there is nothing to prove. For $u,v\in\mathbb{Z}$, $u+v=i+j$ while $(u,v)\neq(i,j)$, we may assume that $u\neq i$. Then $|a_u|r^u<|a_i|r^i$ and $|b_v|r^v\leq |b_j|r^j$. So $|a_ub_v|<|a_ib_j|$ and we conclude (3.4). □

Remark 3.8. The argument of Proposition 4.16 in the chapter Banch Rings does not work here if $r\in\sqrt{|k^\times|}$, as in general one can not take minimal i,j so that (3.3) is satisfied.

Proposition 3.9. Assume that $r\in\mathbb{R}_{>0}\setminus\sqrt{|k^\times|}$. Then k_r is a valuation field and $\|\bullet\|_r$ is non-trivial.

Proof. We first show that $\operatorname{Sp}k_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet|\in\operatorname{Sp}k_r$. As $\|\bullet\|_r$ is a valuation, we find

$$|\bullet|\leq\|\bullet\|_r. \tag{3.5}$$

In particular, $|\bullet|$ restricted to k is the given valuation on k. It suffices to show that $|T|=r$. This follows from (3.5) applied to T and T^{-1}.

It follows that k_r does not have any non-zero proper closed ideals: if I is such an ideal, k_r/I is a Banach k-algebra. By Proposition 6.10 in the chapter Banach rings, $\operatorname{Sp}k_r$ is non-empty. So k_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.7 in the chapter Banach rings, the only maximal ideal of k_r is 0. It follows that k_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r=r$. □

Definition 3.10. An element $r=(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$ for some $n\in\mathbb{N}$ is called a *k-free polyray* if $r_1,\dots,r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{R}_{>0}/\sqrt{|k^\times|}$.

Let $n\in\mathbb{N}$ and $r=(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$. Assume that r is a k-free polyray. We define

$$k_r=k_{r_1}\hat{\otimes}_k\cdots\hat{\otimes}_k k_{r_n}.$$

By an interated application of Proposition 3.9, k_r is a complete valuation field.

As a general explanation of why k_r is useful, we prove the following proposition:

Proposition 3.11. Let $n\in\mathbb{N}$ and $r=(r_1,\dots,r_n)$ be a k-free polyray.

(1) For any k-Banach space X, the natural map

$$X\to X\hat{\otimes}_k k_r$$

is an isometric embedding.

(2) Consider a sequence of bounded homomorphisms of k-Banch spaces $X \to Y \to Z$. Then the sequence is admissible and exact (resp. coexact) if and only if $X\hat{\otimes}_k k_r \to Y\hat{\otimes}_k k_r \to Z\hat{\otimes}_k k_r$ is admissible and exact (resp. coexact).

PROOF. We may assume that $n = 1$.

(1) We have a more explicit description of $X\hat{\otimes}_k k_r$: as a vector space, it is the space of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ with $a_i \in X$ and $\|a_i\| r^i \to 0$ when $|i| \to \infty$. The norm is given by $\max_i \|a_i\| r^i$. From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1). □

When X is a Banach k-algebra, $X\hat{\otimes}_k k_r$ is a Banach k_r-algebra.

Example 3.12. For any $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$, not necessarily k-free. We define k_r as the completed fraction field of $k\{r^{-1}T\}$ provided with the extended valuation $|\bullet|_r$. Then k_r is still a valuation field extending k.

When r is a k-free polyray, we claim that k_r coincides with k_r defined in Definition 3.10. To see this, let us temporarily denote the k_r defined in this example as k'_r consider the extension of field:

$$\operatorname{Frac} k\{r^{-1}T\} \to k_r = k\{r^{-1}T, rS\}/(T_1S_1 - 1, \ldots, T_nS_n - 1)$$

sending T_i to T_i for $i = 1, \ldots, n$. Observe that this is an extension of valuation field as well by the same arguments as in Proposition 3.6. In particular, it induces an extension of complete valuation fields $k'_r \to k_r$. But the image clearly contains the classes of all polynomials in $k[T, S]$, so $k'_r \to k_r$ is an isometric isomorphism.

Proposition 3.13. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite bounded k-algebra homomorphism into a k-Banach algebra A. Then A is also strictly k-affinoid.

PROOF. We may assume that $B = k\{T_1, \ldots, T_n\}$ for some $n \in \mathbb{N}$. By assumption, we can find finitely many $a_1, \ldots, a_m \in A$ such that $A = \sum_{i=1}^{m} \varphi(B) a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.18 in the chapter Banach Rings, φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{T_1, \ldots, T_n, S_1, \ldots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is strictly k-affinoid. □

Proposition 3.14. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite k-algebra homomorphism into a k-algebra A. Then there is a norm on A such that the morphism is bounded and A is strictly k-affinoid.

PROOF. By Proposition 8.4 in the chapter Banach Rings, we can endow A with a Banach norm such that φ is admissible. Then we can apply Proposition 3.13. □

Lemma 3.15. Assume that k is non-trivially valued. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. The algebra $k\{r^{-1}T\}$ is strictly k-affinoid if $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$.

Remark 3.16. The converse is also true.

PROOF. Assume that $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$. Take $s_i \in \mathbb{N}$ and $c_i \in k^\times$ such that

$$r_i^{s_i} = |c_i^{-1}|$$

for $i = 1, \ldots, n$. We deifne a bounded k-algebra homomorphism $\varphi : k\{T_1, \ldots, T_n\} \to k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ by sending T_i to $c_i T_i^{s_i}$. This is possible by Proposition 4.18 in the chapter Banach Rings.

We claim that φ is finite. To see this, it suffices to observe that if we expand $f \in k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (cT^s)^\gamma,$$

where the product γs is taken component-wise. For each $\beta \in \mathbb{N}^n, \beta_i < s_i$, we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \ldots, T_n\}.$$

While $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$. So We have shown that φ is finite. Hence, $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is k-affinoid by Proposition 3.13. □

Proposition 3.17. Let A be a k-affinoid algebra, then there is $n \in \mathbb{N}$ and a k-free polyray $r = (r_1, \ldots, r_n)$ such that $A \hat{\otimes}_k k_r$ is strictly k_r-affinoid. Moreover, we can guarantee that k_r is non-trivially valued.

PROOF. By Proposition 3.11, we may assume that $A = k\{t^{-1}T\}$ for some $t \in \mathbb{R}_{>0}^m$. By Lemma 3.15, it suffices to take r so that the linear subspace of $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ generated by $r_1, \ldots, r_n$ contains all components of t. By taking $n \geq 1$, we can guarantee that k_r is non-trivially valued. □

Proposition 3.18. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a monomorphism in k_H-$\mathcal{A}$ff. Then for any $y \in \operatorname{Sp} B$ with $x = \varphi(y)$, one has $\varphi^{-1}(x) = \{y\}$ and the natural map $\mathscr{H}(x) \to \mathscr{H}(y)$ is an isomorphism of complete valuation rings.

PROOF. It suffices to show that $\mathscr{H}(x) \to B \hat{\otimes}_A \mathscr{H}(y)$ is an isomorphism as Banach k-algebras. Include details about cofiber products in affalg. By assumption, the codiagonal map $B \hat{\otimes}_A B \to B$ is an isomorphism. It follows that the base change with respect to $A \to \mathscr{H}(x)$ is also an isomorphism: $B' \hat{\otimes}_{\mathscr{H}(x)} B' \to B'$, where $B' = B \hat{\otimes}_A \mathscr{H}(x)$.

Include the fact that the first map is injective. It follows that the composition $B' \otimes_{\mathscr{H}(x)} B \to B' \hat{\otimes}_{\mathscr{H}(x)} B' \to B'$ is injective. Therefore, $\mathscr{H}(x) \to B'$ is an isomorphism of rings. We also know that this map is bounded. But we already know that $\mathscr{H}(x)$ is a complete valuation ring, so the map $\mathscr{H}(x) \to B'$ is an isomorphism of complete valuation rings. □

4. Weierstrass theory

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 4.1. We have canonical identifications

$$(k\{T_1,\ldots,T_n\})^\circ \cong \mathring{k}\{T_1,\ldots,T_n\},$$
$$(k\{T_1,\ldots,T_n\})^\vee \cong \check{k}\{T_1,\ldots,T_n\},$$
$$\widetilde{k\{T_1,\ldots,T_n\}} \cong \tilde{k}[T_1,\ldots,T_n].$$

The last identification extends $\mathring{k} \to \tilde{k}$ and T_i is mapped to T_i.

Proof. This follows from Corollary 4.20 from the chapter Banach rings. $\square$

We will denote the reduction map $\mathring{k}\{T_1,\ldots,T_n\} \to \tilde{k}[T_1,\ldots,T_n]$ by $\tilde{\bullet}$.

Definition 4.2. Let $n \in \mathbb{N}$. A system $f_1,\ldots,f_n \in k\{T_1,\ldots,T_n\}$ is called an *affinoid chart* of $k\{T_1,\ldots,T_n\}$ if $f_i \in \mathring{k}\{T_1,\ldots,T_n\}$ for each $i = 1,\ldots,n$ and the continuous k-algebra homomorphism $k\{T_1,\ldots,T_n\} \to k\{T_1,\ldots,T_n\}$ sending T_i to f_i is an isomorphism.

The map $k\{T_1,\ldots,T_n\} \to k\{T_1,\ldots,T_n\}$ is well-defined by Proposition 4.1 and Lemma 2.5.

Lemma 4.3. Let $n \in \mathbb{N}$ and $f \in k\{T_1,\ldots,T_n\}$. Assume that $\|f\|_1 = 1$. Then the following are equivalent:

(1) f is a unit $k\{T_1,\ldots,T_n\}$.
(2) $\tilde{f}$ is a unit in $\tilde{k}[T_1,\ldots,T_n]$.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 3.6, f is a unit in $k\{T_1,\ldots,T_n\}$ if and only if it is a unit in $(k\{T_1,\ldots,T_n\})^\circ$, which is identified with $\mathring{k}\{T_1,\ldots,T_n\}$ by Proposition 4.1. This result then follows from Corollary 4.21 in the chapter Banach Rings. $\square$

Definition 4.4. Let $n \in \mathbb{N}$. Consider $g \in k\{T_1,\ldots,T_n\}$. We expand g as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1,\ldots,T_{n-1}\}.$$

For $s \in \mathbb{N}$, we say g is X_n*-distinguished of degree* s if g_s is a unit in $k\{T_1,\ldots,T_{n-1}\}$, $\|g_s\|_1 = \|g\|_1$ and $\|g_s\|_1 > \|g_t\|_1$ for all $t > s$.

Theorem 4.5 (Weierstrass division theorem)**.** Let $n, s \in \mathbb{N}$ and $g \in k\{T_1,\ldots,T_n\}$ be X_n-distinguished of degree s. Then for each $f \in k\{T_1,\ldots,T_n\}$, there exist $q \in k\{T_1,\ldots,T_n\}$ and $r \in k\{T_1,\ldots,T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$ such that

$$f = qg + r.$$

Moreover, q and r are uniquely determined. We have the following estimates

$$\|q\|_1 \le \|g\|_1^{-1}\|f\|_1, \quad \|r\|_1 \le \|f\|_1. \tag{4.1}$$

If in addition, $f, g \in k\{T_1,\ldots,T_{n-1}\}[T_n]$, then $q \in k\{T_1,\ldots,T_{n-1}\}[T_n]$ as well.

Proof. We may assume that $\|g\|_1 = 1$.

Step 1. Assuming the existence of the division. Let us prove (4.1). We may assume that $f \neq 0$, so that one of q, r is non-zero. Up to replacing q, r by a scalar multiple, we may assume that $\max\{\|q\|_1, \|r\|_1\} = 1$. So $\|f\|_1 \le 1$ as well. We need to show that $\|f\|_1 = 1$. Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here $\tilde{\bullet}$ denotes the reduction map. By our assumption, $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$. From Proposition 4.1, the equality is in $\tilde{k}[T_1, \dots, T_n]$. From the usual Euclidean division, we have $\tilde{q} = \tilde{r} = 0$. This is a contradiction to our assumption.

Step 2. Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with q and r as in the theorem. The estimate in Step 1 shows that $q = r = 0$.

Step 3. We prove the existence of the division.

We define

$$B := \left\{qg + r : r \in k\{T_1, \dots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \dots, T_n\}\right\}.$$

From Step 1, B is a closed subgroup of $k\{T_1, \dots, T_n\}$. In fact, suppose $f_i \in B$ is a sequence converging to $f \in k\{T_1, \dots, T_n\}$. From Step 1, we can represent $f_i = q_i g + r_i$, then from Step 1, q_i and r_i are both Cauchy sequences, we may assume that $q_i \to q \in k\{T_1, \dots, T_n\}$ and $r_i \to r$. As $\deg_{T_n} r_i < s$, it follows that $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ and $\deg_{T_n} r < s$. So $f = qg + r$ and hence B is closed.

It suffices to show that B is dense $k\{T_1, \dots, T_n\}$. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

We may assume that $\|g\|_1 = 1$. Define $\epsilon := \max_{j \geq s} \|g_j\|$. Then $\epsilon < 1$ by our assumption. Let $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$ for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \dots, T_n\})^\circ \to (\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$$

with kernel $\{f \in k\{T_1, \dots, T_n\} : \|f\|_1 \leq \epsilon\}$. We now apply Euclidean division in the ring $(\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$ to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some $q \in (k\{T_1, \dots, T_n\})^\circ$ and $r \in (k\{T_1, \dots, T_{n-1}\})^\circ[T_n]$ with $\deg_{T_n} r < s$. So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that B is dense in $k\{T_1, \dots, T_n\}$ by Proposition 2.8 in the chapter Banach rings.

Step 4. It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring $k\{T_1, \dots, T_{n-1}\}[T_n]$ and the uniqueness proved in Step 2. □

Lemma 4.6. Let $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ be a Weierstrass polynomial and $g \in k\{T_1, \dots, T_n\}$. Assume that $\omega g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$.

Proof. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in k\{T_1, \dots, T_{n-1}\}[T_n]$, $\deg_{T_n} r < \deg_{T_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of Theorem 4.5, we know that $q = g$, so g is a polynomial in T_n. □

As a consequence, we deduce Weierstrass preparation theorem.

Definition 4.7. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n-variables is a monic polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\|\omega\|_1 = 1$.

Lemma 4.8. Let $n \in \mathbb{Z}_{>0}$ and $\omega_1, \omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ be two monic polynomials. If $\omega_1\omega_2$ is a Weierstrass polynomial then so are ω_1 and ω_2.

Proof. As ω_1 and ω_2 are monic, $\|\omega_i\|_1 \geq 1$ for $i = 1, 2$. On the other hand, $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1\omega_2\|_1 = 1$, so $\|\omega_i\|_1 = 1$ for $i = 1, 2$. □

Theorem 4.9 (Weierstrass preparation theorem)**.** Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished of degree s. Then there are a Weierstrass polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ of degree s and a unit $e \in k\{T_1, \dots, T_n\}$ such that

$$g = e\omega.$$

Moreover, e and ω are unique. If $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then so is e.

Proof. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let $r = T_n^s - \omega$. Then $T_n^s = e^{-1}g + r$. The uniqueness part of Theorem 4.5 implies that e and r are uniquely determined, hence so is ω.

Next we prove the existence. By Weierstrass division theorem Theorem 4.5, we can write

$$T_n^s = qg + r$$

for some $q \in k\{T_1, \dots, T_n\}$ and $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$. Let $\omega = T_n^s - r$. From the estimates in Theorem 4.5, $\|r\|_1 \leq 1$. So $\|\omega\|_1 = 1$. Then ω is a Weierstrass polynomial of degree s and $\omega = qg$. It suffices to argue that q is a unit.

We may assume that $\|g\|_1 = 1$. By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$ and the leading coefficients of both polynomials are units in $\tilde{k}[T_1, \dots, T_{n-1}]$, it follows that $\tilde{q}$ is a unit in $\tilde{k}[T_1, \dots, T_{n-1}]$. It follows that $\tilde{q}$ is also a unit in $\tilde{k}[T_1, \dots, T_n]$. By Lemma 4.3, q is a unit in $k\{T_1, \dots, T_n\}$.

The lsat assertion is already proved in Theorem 4.5.

□

Definition 4.10. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished. Then the Weierstrass polynomial ω constructed in Theorem 4.9 is called the *Weierstrass polynomial* defined by g.

Corollary 4.11. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished. Let ω be the Weierstrass polynomial of g. Then the injection

$$k\{T_1, \dots, T_{n-1}\}[T_n] \to k\{T_1, \dots, T_n\}$$

induces an isomorphism of k-algebras

$$k\{T_1, \dots, T_{n-1}\}[T_n]/(\omega) \to k\{T_1, \dots, T_n\}/(g).$$

Proof. The surjectivity follows from Theorem 4.5 and the injectivity follows from Lemma 4.6. □

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

Lemma 4.12. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ is non-zero. Then there is a k-algebra automorphism σ of $k\{T_1, \dots, T_n\}$ so that $\sigma(g)$ is T_n-distinguished.

Proof. We may assume that $\|g\|_1 = 1$. We expand g as

$$g = \sum_{\alpha\in\mathbb{N}^n} a_\alpha T^\alpha.$$

Endow $\mathbb{N}^n$ with the lexicographic order. Take the maximal $\beta \in \mathbb{N}^n$ so that $|a_\beta| = 1$. Take $t \in \mathbb{Z}_{>0}$ so that $t \geq \max_{i=1,\ldots,n} \alpha_i$ for all $\alpha \in \mathbb{N}^n$ with $\tilde{a}_\alpha \neq 0$.

We will define σ by sending T_i to $T_i + T_n^{c_i}$ for all $i = 1, \ldots, n-1$. The c_i's are to be defined. We begin with $c_n = 1$ and define the other c_i's inductively:

$$c_{n-j} = 1 + t\sum_{d=0}^{j-1} c_{n-d}$$

for $j = 1, \ldots, n-1$. We claim that $\sigma(f)$ is T_n-distinguished of order $s = \sum_{i=1}^n c_i\beta_i$.

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^s p_i T_n^i$$

for some $p_i \in \tilde{k}[T_1, \ldots, T_{n-1}]$ and $p_s = \tilde{a_\beta}$. Our claim follows. □

Proposition 4.13. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Noetherian.

Proof. We make induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. It suffices to show that for any non-zero $g \in k\{T_1, \ldots, T_n\}$, $k\{T_1, \ldots, T_n\}/(g)$ is Noetherian. By Lemma 4.12, we may assume that g is T_n-distinguished. By Theorem 4.5, $k\{T_1, \ldots, T_n\}/(g)$ is a finite free $k\{T_1, \ldots, T_{n-1}\}$-module. By the inductive hypothesis and Hilbert basis theorem, $k\{T_1, \ldots, T_n\}/(g)$ is indeed Noetherian. □

Proposition 4.14. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Jacobson.

Proof. When $n = 0$, there is nothing to prove. We make induction on n and assume that $n > 0$. Let $\mathfrak{p}$ be a prime ideal in $k\{T_1, \ldots, T_n\}$, we want to show that the Jacobson radical of $\mathfrak{p}$ is equal to $\mathfrak{p}$.

We distinguish two cases. First we assume that $\mathfrak{p} \neq 0$. Let $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \ldots, T_{n-1}\}$. By Lemma 4.12, we may assume that $\mathfrak{p}$ contains a Weierstrass polynomial ω. Observe that

$$k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' \to k\{T_1, \ldots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any $b \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p})$ (where J denotes the Jacobson radical), we consider a monic integral equation of minimal degree over $k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'$:

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0, \quad a_i \in k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) \cap k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that $n = 1$ and so $b = 0$. This proves $J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) = 0$.

On the other hand, let us consider the case $\mathfrak{p} = 0$. As $k\{T_1, \ldots, T_n\}$ is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \ldots, T_n\}) = 0.$$

Assume that there is a non-zero element f in $J(k\{T_1,\dots,T_n\})$. We may assume that $\|f\|_1 = 1$.

We claim that there is $c \in k$ with $|c| = 1$ such that $c + f$ is not a unit in $k\{T_1,\dots,T_n\}$. Assuming this claim for the moment, we can find a maximal ideal $\mathfrak{m}$ of $k\{T_1,\dots,T_n\}$ such that $c + f \in \mathfrak{m}$. But $f \in \mathfrak{m}$ by our assumption, so $c \in \mathfrak{m}$ as well. This contradicts the fact that $c \in k^\times$.

It remains to prove the claim. We treat two cases separately. When $|f(0)| < 1$, we simply take $c = 1$, which works thanks to Lemma 4.3. If $|f(0)| = 1$, we just take $c = -f(0)$. □

Proposition 4.15. Let $n \in \mathbb{N}$. Then $k\{T_1,\dots,T_n\}$ is UFD. In particular, $k\{T_1,\dots,T_n\}$ is normal.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 2.2, $k\{T_1,\dots,T_n\}$ is an integral domain. In order to see that $k\{T_1,\dots,T_n\}$ has the unique factorization property, we make induction on $n \geq 0$. When $n = 0$, there is nothing to prove. Assume that $n > 0$. Take a non-unit element $f \in k\{T_1,\dots,T_n\}$. By Theorem 4.9 and Lemma 4.12, we may assume that f is a Weierstrass polynomial. By inductive hypothesis, $k\{T_1,\dots,T_{n-1}\}$ is a UFD, hence so is $k\{T_1,\dots,T_{n-1}\}[T_n]$ by [Stacks, Tag 0BC1]. It follows that f can be decomposed into the products of monic prime elements $f_1,\dots,f_r \in k\{T_1,\dots,T_{n-1}\}[T_n]$, which are all Weierstrass polynomials by Lemma 4.8. Then by Corollary 4.11, we see that each f_i is prime in $k\{T_1,\dots,T_n\}$.

Any UFD is normal by [Stacks, Tag 0AFV]. □

Corollary 4.16. Let A be a strictly k-affinoid algebra, $d \in \mathbb{N}$ and $\varphi : k\{T_1,\dots,T_d\} \to A$ be an integral torsion-free injective homomorphism of k-algebras. Then ρ is a faithful $k\{T_1,\dots,T_d\}$-algebra norm on A. If $f^n + \varphi(t_1)f^{n-1} + \dots + \varphi(t_n) = 0$ is the minimal integral equation of f over $k\{T_1,\dots,T_d\}$, then

$$|f|_{\sup} = \max_{i=1,\dots,n} |t_i|^{1/i}.$$

Proof. This follows from Proposition 9.11 in the chapter Banach Rings and Proposition 4.15. □

5. Noetherian normalization and maximal modulus principle

Let $(k, |\bullet|)$ be a complete non-trivially valued non-Archimedean valued-field.

Theorem 5.1. Let A be a non-zero strictly k-affinoid algebra, $n \in \mathbb{N}$ and $\alpha : k\{T_1,\dots,T_n\} \to A$ be a finite (resp. integral) k-algebra homomorphism. Then up to replacing $T_1,\dots,T_n$ by an affinoid chart, we can guarantee that there exists $d \in \mathbb{N}$, $d \leq n$ such that α when restricted to $k\{T_1,\dots,T_d\}$ is finite (resp. integral) and injective.

Proof. We make an induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. If $\ker\alpha = 0$, there is nothing to prove, so we may assume that $\ker\alpha \neq 0$. By Lemma 4.12 and Theorem 4.9, we may assume that there is a Weierstrass polynomial $\omega \in k\{T_1,\dots,T_{n-1}\}[T_n]$ in $\ker\alpha$. Then α induces a finite (resp. integral) homomorphism $\beta : k\{T_1,\dots,T_n\}/(\omega) \to A$. By Theorem 4.5, $k\{T_1,\dots,T_{n-1}\} \to k\{T_1,\dots,T_n\}/(\omega)$ is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism $k\{T_1,\dots,T_{n-1}\} \to A$. We can apply the inductive hypothesis to conclude. □

Corollary 5.2. Let A be a non-zero strictly k-affinoid algebra, then there is $d \in \mathbb{N}$ and a finite injective k-algebra homomorphism: $k\{T_1, \ldots, T_d\} \to A$.

PROOF. Take some $n \in \mathbb{N}$ and a surjective k-algebra homomorphism $k\{T_1, \ldots, T_n\} \to A$ and apply Theorem 5.1, we conclude. $\square$

Corollary 5.3. Let A be a strictly k-affinoid algebra and I be an ideal in A such that $\sqrt{I}$ is a maximal ideal in A, then A/I is finite-dimensional over k.

In particular, $\operatorname{Spm} A = \operatorname{Spm}_k A$.

PROOF. By Corollary 5.2, there is $d \in \mathbb{N}$ and a finite monomorphism $f : k\{T_1, \ldots, T_d\} \to A/I$. It suffices to show that $d = 0$. Observe that the composition

$$k\{T_1, \ldots, T_d\} \xrightarrow{f} A/I \to A/\sqrt{I}$$

is finite and injective as $k\{T_1, \ldots, T_d\}$ is an integral domain, so $k\{T_1, \ldots, T_d\}$ is a field. This is possible only when $d = 0$. $\square$

Corollary 5.4. Let B be a strictly k-affinoid algebra and A be a Noetherian Banach k-algebra. Let $f : A \to B$ a k-algebra homomorphism. Then f is bounded.

PROOF. This follows from Proposition 8.1 in the chapter Banach Rings and Proposition 4.13. $\square$

In particular, we see that the topology of a k-affinoid algebra is uniquely determined by the algebraic structure.

Corollary 5.5. Let A, B be strictly k-affinoid algebras. Let f be a finite k-algebra homomorphism, then f is admissible.

PROOF. This follows from Proposition 3.14 and Corollary 5.4, $\square$

Definition 5.6. For any non-Archimedean valuation field $(K, |\bullet|)$ and $n \in \mathbb{N}$, we define the *n-dimensional polydisk* with value in K:

$$B^n(K) := \left\{ (x_1, \ldots, x_n) \in K^n : \max_{i=1,\ldots,n} |x_i| \leq 1 \right\}.$$

Definition 5.7. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ as sending $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

Lemma 5.8. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, then $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ is continuous and for any $x \in B^n(k^{\mathrm{alg}})$,

$$|f(x)| \leq \|f\|_1.$$

There is $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ such that $|f(x)| = \|f\|_1$.

PROOF. To see that f is continuous, it suffices to observe that f is a uniform limit of polynomials. For any $x = (x_1, \dots, x_n) \in B^n(k^{\mathrm{alg}})$, we have

$$|f(x)| = \left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \right| \leq \max_{\alpha \in \mathbb{N}^n} |a_\alpha x^\alpha| \leq \|f\|_1.$$

To prove the last assertion, we may assume that $\|f\|_1 = 1$. As the residue field of k^{alg} is equal to $\tilde{k}^{\mathrm{alg}}$, it has infinitely many elements, so there is a point $x \in B^n(k^{\mathrm{alg}})$ such that $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$. In other words, $\|f(x)\|_1 = 1$. $\square$

Proposition 5.9. Let $n \in \mathbb{N}$, then the maximal modulus principle holds for $k\{T_1, \dots, T_n\}$. Moreover, for any $f \in k\{T_1, \dots, T_n\}$, $\|f\|_1 = |f|_{\sup}$.

PROOF. By Lemma 6.3 in the chapter Banach Rings, we have

$$\|f\|_1 \geq |f|_{\sup}$$

for any $f \in A$. We only have to show that for any $f \in k\{T_1, \dots, T_n\}$ there is a maximal ideal $\mathfrak{m} \subseteq k\{T_1, \dots, T_n\}$ such that $|f(\mathfrak{m})| = \|f\|_1$.

By Lemma 5.8 we can take $x = (x_1, \dots, x_n) \in B^n(k^{\mathrm{alg}})$ such that $|f(x)| = \|f\|_1$. Let L be the field extension of k generated by $x_1, \dots, x_n$, then L/k is finite. Then we can define a homomorphism

$$\mathrm{ev}_x : k\{T_1, \dots, T_n\} \to L$$

sending $g \in k\{T_1, \dots, T_n\}$ to $g(x)$. Observe that the image is indeed in L. Clearly ev_x is surjective. So $\mathfrak{m}_x := \ker \mathrm{ev}_x$ is a k-algebraic maximal ideal in $k\{T_1, \dots, T_n\}$. Then

$$|f(\mathfrak{m}_x)| = |f(x)| = \|f\|_1.$$

$\square$

Corollary 5.10. Let A be a strictly k-affinoid algebra. Then for any $f \in A$,

$$|f|_{\sup} \subseteq \sqrt{|k^\times|} \cup \{0\}.$$

PROOF. We may assume that $A \neq 0$. By Corollary 5.2 and Proposition 9.11 in the chapter Banach Rings, we may assume that $A = k\{T_1, \dots, T_n\}$ for some $n \in \mathbb{N}$. The result then follows from Proposition 5.9. $\square$

Corollary 5.11. Maximal modulus principle holds for any strictly k-affinoid algebras.

PROOF. This follows from Corollary 5.2, Proposition 9.11 in the chapter Banach Rings and Proposition 5.9. $\square$

Proposition 5.12. Let $\varphi : B \to A$ be an integral k-algebra homomorphism of strictly k-affinoid algebras. Then for each non-zero $f \in A$, there is a moinc polynomial $q(f) = f^n + \varphi(b_1) f^{n-1} + \cdots + \varphi(b_n)$ of f over B. Then

$$|f|_{\sup} = \max_{i=1,\dots,n} |b_i|_{\sup}^{1/i}.$$

PROOF. One side is simple: choose $j = 1, \dots, n$ that maximizes $|\varphi(b_j) f^{n-j}|_{\sup}$, then

$$|f|_{\sup}^n = |f^n|_{\sup} \leq |\varphi(b_j) f^{n-j}|_{\sup} \leq |b_j|_{\sup} \cdot |f|_{\sup}^{n-j}.$$

So

$$|f|_{\sup} \leq |b_j|_{\sup}^{1/j}.$$

To prove the reverse inequality, let us begin with the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k-algebra homomorphism $\psi : k\{T_1, \dots, T_d\} \to B$ such that $\varphi \circ \psi$ is integral and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \dots, T_d\} \to A$, we can apply Theorem 5.1 to find $\phi \circ \alpha$ to conclude.

By Corollary 4.16, if p denotes the minimal polynomial of f over $k\{T_1, \dots, T_d\}$, we have $|f|_{\sup} = \sigma(p)$. In particular, $p(f) = 0$. Let $q \in B[X]$ be the polynomial obtained from p by replacing all coefficients by their ψ-images in B. Then clearly, $|f|_{\sup} = \sigma(q)$.

In general, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes in A. The number is finite by Proposition 4.13. For each $i = 1, \dots, r$, let $\pi_i : A \to A/\mathfrak{p}_i$ denote the natural homomorphism. We know that there are monic polynomials $q_i \in B[X]$ such that $q_i(\pi(f)) = 0$ and $|\pi_i(f)|_{\sup} = \sigma(q_i)$ for $i = 1, \dots, r$. We let $q' = q_1 \cdots q_r$. Then

$$q'(f) \in \bigcap_{i=1}^{r} \mathfrak{p}_i.$$

So there is $e \in \mathbb{Z}_{>0}$ such that $q'(f)^e = 0$. Let $q = q'^e$. By Proposition 9.5 in the chapter Banach Rings,

$$\sigma(q) \leq \max_{i=1,\dots,r} \sigma(q_i) = \max_{i=1,\dots,r} |\pi_i(f)|_{\sup} = |f|_{\sup}.$$

The last equality follows from Proposition 9.9 in the chapter Banach Rings. □

Lemma 5.13. Let $\varphi : B \to A$ be an admissible k-algebra homomorphism between strictly k-affinoid algebras. Let $\tau : \mathring{B} \to \tilde{B}$ be the reduction map, then

$$\tau^{-1}(\ker \tilde{\varphi}) = \sqrt{\check{B} + \ker \mathring{\varphi}}, \quad \ker \tilde{\varphi} = \sqrt{\tau(\ker \mathring{\varphi})}.$$

Proof. The second equation follows from the first one by applying τ. Let us prove the first equation. By assumption, $\varphi(\check{B})$ is open in $\varphi(B)$. Consider $g \in \tau^{-1}(\ker \tilde{\varphi})$, we know that

$$\lim_{n \to \infty} \varphi(g)^n = 0.$$

So $\varphi(g)^n \in \varphi(\check{B})$ for n large enough, and hence $g^n \in \check{B} + \ker \mathring{\varphi}$. □

Lemma 5.14. Let $m \in \mathbb{N}$ and $T = k\{T_1, \dots, T_m\}$. Let A be a k-affinoid algebra and $\varphi : T\{S_1, \dots, S_n\} \to A$ be a finite morphism such that $\tilde{\varphi}(S_i)$ is integral over $\tilde{T}$. Then $\varphi|_T : T \to A$ is finite.

Proof. We make an induction on n. When $n = 0$, there is nothing to prove. So assume $n > 0$ and the lemma has been proved for smaller values of n.

Let $T' = T\{S_1, \dots, S_n\}$. By assumption, there is a Weierstrass polynomial

$$\omega = S_n^k + a_1 S_n^{k-1} + \dots + a_k \in \mathring{T}[S_n]$$

such that $\tilde{\omega} \in \ker \tilde{\varphi}$. As φ is admissible by Corollary 5.5, we have $\omega^q \in \check{T}' + \ker \mathring{\varphi}$ for some $q \in \mathbb{Z}$ by Lemma 5.13.

In particular, we can find $r \in (T')^{\check{}}$ such that $g := \omega^q - r \in \ker \mathring{\varphi}$. Observe that g is S_n distinguished of order mq as $\tilde{g} = \tilde{\omega}^q$. By Corollary 4.11, the restriction of φ to $T\{S_1, \dots, S_{n-1}\}$ is finite. We can apply the inductive hypothesis to conclude. □

Lemma 5.15. Let $\varphi : B \to A$ be a k-algebra homomorphism of strictly k-affinoid algebras. Assume that there exist affinoid generators $f_1, \dots, f_n \in \mathring{A}$ of A such that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over $\tilde{B}$, then φ is finite.

Proof. By assumption, we can find $s_i \in \mathbb{Z}_{>0}$, $b_{ij} \in \mathring{B}$ for $i = 1, \dots, n$, $j = 1, \dots, s_i$ such that

$$\tilde{f}_i^{s_i} + \tilde{\varphi}(\tilde{b}_{i1})\tilde{f}_i^{s_i-1} + \cdots + \tilde{\varphi}(\tilde{b}_{is_i}) = 0$$

for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$ and define a bounded k-algebra homomorphism $\psi : D := k\{T_{ij}\} \to B$ sending T_{ij} to b_{ij}, for $i = 1, \dots, n$ and $j = 1, \dots, s_i$. Observe that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over $\tilde{D}$. So it suffices to prove the theorem when $B = k\{T_1, \dots, T_m\}$. We extend φ to a bounded k-algebra epimorphism $\varphi' : T\{S_1, \dots, S_n\} \to A$ sending S_i to f_i for $i = 1, \dots, n$. Then $\varphi'(\tilde{S_i})$ is integral over $\tilde{B}$. It suffices to apply Lemma 5.14. □

6. Properties of affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Proposition 6.1. Assume that k is non-trivially valued. Let A be a strictly k-afifnoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\sup} \leq 1\}.$$

Proof. By Lemma 6.3, we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\sup} \leq 1\}.$$

Conversely, let $f \in A$, $|f|_{\sup} \leq 1$. Choose $d \in \mathbb{N}$ and a surjective k-algebra homomorphism

$$\varphi : k\{T_1, \dots, T_d\} \to A.$$

Let $f^n + t_1 f^{n-1} + \cdots + t_n = 0$ be the minimal equation of f over $k\{T_1, \dots, T_d\}$. Then $t_i \in (k\{T_1, \dots, T_d\})^\circ$ by Proposition 9.11 in the chapter Banach Rings. An induction on $i \geq 0$ shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi\left((k\{T_1, \dots, T_d\})^\circ\right) f^j.$$

The right-hand side is clearly bounded. □

Corollary 6.2. Assume that k is non-trivially valued. Let $(A, \|\bullet\|)$ be a strictly k-affinoid algebra. For any $f \in A$,

$$\rho(f) = |f|_{\sup}.$$

Proof. We have shown that $\rho(f) \geq |f|_{\sup}$ in Lemma 6.3 from the chapter Banach Rings. Assume tha tthe inverse inequality fails: for some $f \in A$,

$$\rho(f) > |f|_{\sup}.$$

If $|f|_{\sup} = 0$, then f lies in the Jacobson radical of A, which is equal to the nilradial of A by Proposition 4.14. But then $\rho(f) = 0$ as well. We may therefore assume that $|f|_{\sup} \neq 0$. By Corollary 5.10, we may assume that $|f|_{\sup} = 1$ as ρ is power-multiplicative. Then $\rho(f) > 1$. This contradicts Proposition 6.1. □

Theorem 6.3. A k-affinoid algebra A is Noetherian and all ideals of A are closed.

Proof. Let I be an ideal in A. By Proposition 3.17, we can take a suitable $r \in \mathbb{R}_{>0}^m$ so that $A\hat{\otimes}k_r$ is strictly k_r-affinoid. Then $I(A\hat{\otimes}k_r)$ is an ideal in $A\hat{\otimes}k_r$. By Proposition 4.13, the latter ring is Noetherian. So we may take finitely many generators $f_1, \ldots, f_k \in I$. Each $f \in I$ can be written as

$$f = \sum_{i=1}^{k} f_i g_i$$

with $g_i = \sum_{j=-\infty}^{\infty} g_{i,j}T^j \in A\hat{\otimes}k_r$. But then

$$f = \sum_{i=1}^{k} f_i g_{i,0}.$$

So I is finitely generated.

As $I = A \cap (I(A\hat{\otimes}k_r))$, by Corollary 7.4 in the chapter Banach Rings, we see that I is closed in $A\hat{\otimes}k_r$ and hence closed in A. □

Proposition 6.4. Let $(A, \|\bullet\|)$ be a k-affinoid algebra and $f \in A$. Then there is $C > 0$ and $N \geq 1$ such that for any $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Recall that ρ is the spectral radius map defined in Definition 4.9 in the chatper Banach Rings.

Proof. By Proposition 3.11, we may assume that k is non-trivially valued and k is non-trivially valued.

If $\rho(f) = 0$, then f lies in each maximal ideal of A. To see this, we may assume that A is a field, then by Proposition 6.10 in the chapter Banach Rings, there is a bounded valuation $\|\bullet\|'$ on A. But then $\rho(f) = 0$ implies that $\|f\|' = 0$ and hence $f = 0$.

It follows that if $\rho(f) = 0$ then f lies in $J(A)$, the Jacobson radical of A. By Proposition 4.14, A is a Jacobson ring. So f is nilpotent. The assertion follows.

So we can assume that $\rho(f) > 0$. In this case, by Corollary 5.2 and Proposition 9.11 in the chapter Banach Rings, we have $\rho(f) \in \sqrt{|k^\times|}$. Take $a \in k^\times$ and $d \in \mathbb{Z}_{>0}$ so that $\rho(f)^d = |a|$. Then $\rho(f^d/a) = 1$ and hence it is powerly-bounded by Proposition 6.1. It follows that there is $C > 0$ so that for $n \geq 1$,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that $\|f^n\| \leq C\rho(f)$ for $n \geq d$ as long as we enlarge C. □

Corollary 6.5. Let $\varphi : A \to B$ be a bounded homomorphism of k-affinoid algebras. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in B$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$ for $i = 1, \ldots, n$. Write $r = (r_1, \ldots, r_n)$, then there is a unique bounded homomorphism $\Phi : A\{r^{-1}T\} \to B$ extending φ and sending T_i to f_i.

Proof. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from Proposition 6.4 that the right-hand side the series converges. The boundedness of Φ is obvious. $\square$

Proposition 6.6. Let $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$ be k-affinoid algebras, $r \in \mathbb{R}^n_{>0}$ and $\varphi : A\{r^{-1}T\} \to B$ be an admissible epimorphism. Write $f_i = \varphi(T_i)$ for $i = 1, \dots, n$. Then there is $\epsilon > 0$ such that for any $g = (g_1, \dots, g_n) \in B^n$ with $\|f_i - g_i\|_B < \epsilon$ for all $i = 1, \dots, n$, there exists a unique bounded k-algebra homomorphism $\psi : A\{r^{-1}T\} \to B$ that coincides with φ on A and sends T_i to g_i. Moreover, ψ is also an admissible epimorphism.

Proof. The uniqueness of ψ is obvious. We prove the remaining assertions. Taking $\epsilon > 0$ small enough, we could further guarantee that $\rho(g_i) \leq r_i$. It follows from Corollary 6.5 that there exists a bounded homomorphism ψ as in the statement of the proposition.

As φ is an admissible epimorphism, we may assume that $\|\bullet\|_B$ is the residue induced by $\|\bullet\|_r$ on $A\{r^{-1}T\}$.

By definition of the residue norm, for any $\delta > 0$ and any $h \in B$, we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1+\delta)\|h\|_B$$

for any $\alpha \in \mathbb{N}^n$. Choose $\epsilon \in (0, (1+\delta)^{-1})$. Now for $g_1, \dots, g_n$ as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1+\delta)\epsilon \|h\|_B.$$

Repeating this procedure, we can construct $k_i \in A\{r^{-1}T\}$ for $i \in \mathbb{N}$ and $h_j \in B$ for $j \in \mathbb{Z}_{>0}$ such that for any $i \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} h =& \psi(k_0 + \dots + k_{i-1}) + h_i, \\ \|k_i\|_r \leq& ((1+\delta)\epsilon)^i (1+\delta) \|h\|_B, \\ \|h_i\|_B \leq& ((1+\delta)\epsilon)^i \|h\|_B. \end{aligned}$$

In particular, $k := \sum_{i=0}^\infty k_i$ converges in $A\{r^{-1}T\}$ and

$$\|k\|_r \leq (1+\delta)\|h\|_B.$$

It follows that ψ is an admissible epimorphism. $\square$

Corollary 6.7. Let A be a Banach k-algebra, $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k-free polyray. Assume that $A \hat{\otimes}_k k_r$ is k_r-affinoid, then A is k-affinoid.

If $A \hat{\otimes}_k k_r$ is k_H-affinoid and $r \in H$, then A is also k_H-affinoid.

Proof. We may assume that r has only one component.

Take $m \in \mathbb{N}$, $p_1, \dots, p_m \in \mathbb{R}_{>0}$ and an admissible epimorphism

$$\pi : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \to A \hat{\otimes}_k k_r.$$

Let

$$\pi(S_i) = \sum_{j=-\infty}^{\infty} a_{i,j} T^j, \quad a_{i,j} \in A$$

for $i = 1, \ldots, m$. By Proposition 6.6, we may assume that there is a large integer l such that $a_{i,j} = 0$ for $|j| > l$ and for any $i = 1, \ldots, m$. We define $B = k\{p_i^{-1} r^j T_{i,j}\}$, $i = 1, \ldots, n$ and $j = -l, -l+1, \ldots, l$. Let $\varphi : B \to A$ be the bounded k-algebra homomorphism sending $T_{i,j}$ to $a_{i,j}$. The existence of φ is guaranteed by Corollary 6.5.

We claim that φ is an admissible epimorphism. It is clearly an epimorphism. Let us show that φ is admissible. Let $\eta : k_r\{p_1^{-1} S_1, \ldots, p_m^{-1} S_m\} \to B \hat{\otimes}_k k_r$ be the bounded homomorphism sending S_i to $\sum_{j=-l}^{l} T_{i,j} T^j$, then we have the following commutative diagram

$$\begin{tikzcd}
k_r\{p^{-1}S\} \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\
B\hat{\otimes}_k k_r \arrow[r, "\varphi\hat{\otimes}_k k_r"] & A\hat{\otimes}_k k_r
\end{tikzcd}$$

It follows that $\varphi \hat{\otimes}_k k_r$ is also an admissible epimorphism. By Proposition 3.11, φ is also admissible. $\square$

7. H-strict affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $R_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

We next give a non-strict extension of Proposition 3.13.

Proposition 7.1. Let B be a k_H-affinoid algebra and $\varphi : B \to A$ be a finite bounded homomorphism into a k-Banach algebra A. Then A is also k_H-affinoid.

Proof. We first assume that k is non-trivially valued.

We may assume that $B = k\{r_1^{-1} T_1, \ldots, r_n^{-1} T_n\}$ for some $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in H$. By assumption, we can find finitely many $a_1, \ldots, a_m \in A$ such that $A = \sum_{i=1}^{m} \varphi(B) a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.18 in the chapter Banach Rings, φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{r_1^{-1} T_1, \ldots, r_n^{-1} T_n, S_1, \ldots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is k_H-affinoid.

If k is trivially valued, then H is non-trivial. Take $s \in H \setminus \{1\}$. It follows from the previous case applied to $\varphi \hat{\otimes} k_s : B \hat{\otimes} k_s \to A \hat{\otimes} k_s$ that $A \hat{\otimes} k_s$ is k_H-affinoid. By Corollary 6.7, A is also k_H-affinoid. $\square$

Proposition 7.2. Let A be a Banach k-algebra. Then the following are equivalent:

(1) A is k_H-affinoid;
(2) there are $n \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

Proof. The non-trivial direction is (2). Assume (2). Take $s_1, \ldots, s_n \in \mathbb{Z}_{>0}$, $c_1, \ldots, c_n \in k^\times$ and $h_1, \ldots, h_n \in H$ such that

$$r_i^{s_i} = |c_i^{-1}| h_i$$

for $i = 1, \ldots, n$. We define a bounded k-algebra homomorphism

$$\varphi : k\{h_1^{-1}T_1, \ldots, h_n^{-1}T_n\} \to k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$$

by sending T_i to $c_i T_i^{s_i}$. The existence of such a homomorphism is guaranteed by Corollary 6.5. The same proof of Lemma 3.15 shows that φ is finite. By Proposition 7.1, $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is k_H-affinoid. □

Lemma 7.3. Assume that k is non-trivially valued. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is strictly k-affinoid;
(2) for any $a \in A$, $\rho(a) \in \sqrt{|k^\times|} \cup \{0\}$.

Proof. (1) $\implies$ (2) by Corollary 5.10 and Corollary 6.2.
(2) $\implies$ (1): Take $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \to A.$$

Let $f_i = \varphi(T_i)$ for $i = 1, \ldots, n$. Suppose $r_1, \ldots, r_m \notin \sqrt{|k^\times|}$ and $r_{m+1}, \ldots, r_n \in \sqrt{|k^\times|}$. Then $\rho(f_i) < r_i$ for $i = 1, \ldots, m$ and we can choose $r'_1, \ldots, r'_m \in \sqrt{|k^\times|}$ such that

$$\rho(f_i) \leq r'_i < r_i$$

for $i = 1, \ldots, m$. Set $r'_i = r_i$ when $i = m+1, \ldots, n$. We can then define a bounded k-algebra homomorphism $\psi : k\{r'^{-1}T\} \to A$ sending T_i to f_i for $i = 1, \ldots, n$. The existence of ψ is guaranteed by Corollary 6.5. Observe that ψ is surjective and admissible. It follows that A is strictly k-affinoid. □

Theorem 7.4. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is k_H-affinoid;
(2) A is $k_{\sqrt{|k^\times| \cdot H}}$-affinoid;
(3) For any non-zero $a \in A$, $\rho(a) \in \sqrt{|k^\times| \cdot H} \cup \{0\}$.

Proof. The equivalence between (1) and (2) follows from Proposition 7.2.
(1) $\implies$ (3): we may assume that $H \supseteq |k^\times|$. Take $n \in \mathbb{N}$, $r = (r_1, \ldots, r_n) \in H^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \to A.$$

Take a k-free polyray s with at least one component so that $|k_s| \supseteq \{r_1, \ldots, r_n\}$. We can apply Lemma 7.3 to $\varphi \hat{\otimes}_k k_s$, it follows that $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$.
(3) $\implies$ (2): we may assume that $H \supseteq |k^\times|$. It suffices to apply the same argument as (2) $\implies$ (1) in the proof of Lemma 7.3.

□

8. Finite modules over affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

For any k-affinoid algebra A, we have defined the category $\mathcal{B}\mathrm{an}_A^f$ of finite Banach A-modules in Definition 5.3 in the chapter Banach Rings. We write $\mathcal{M}\mathrm{od}_A^f$ for the category of finite A-modules.

Lemma 8.1. Let A be a k-affinoid algebra, $(M, \|\bullet\|_M)$ be a finite Banach A-module and $(N, \|\bullet\|_N)$ be a Banach A-module N. Let $\varphi : M \to N$ be an A-linear homomorphism. Then φ is bounded.

Proof. Take $n \in \mathbb{N}$ such that there is an admissible epimorphism

$$\pi : A^n \to M.$$

It suffices to show that $\varphi \circ \pi$ is bounded. So we may assume that $M = A^n$. For $i = 1, \ldots, n$, let e_i be the vector with $(0, \ldots, 0, 1, 0, \ldots, 0)$ of A^n with 1 placed at the i-th place. Set $C = \max_{i=1,\ldots,n} \|\varphi(e_i)\|_N$. For a general $f = \sum_{i=1}^n a_i e_i$ with $a_i \in A$, we have

$$\|\varphi(f)\|_N \leq C\|f\|_M.$$

So φ is bounded. $\square$

Proposition 8.2. Let A be a k-affinoid algebra. The forgetful functor $\mathcal{B}\mathrm{an}_A^f \to \mathcal{M}\mathrm{od}_A^f$ is an equivalence of categories.

Proof. It suffices to construct the inverse functor. Let M be a finite A-module. Choose $n \in \mathbb{N}$ and an A-linear epimorphism $\pi : A^n \to M$. By Theorem 6.3, $\ker \pi$ is closed in A^n. We can endow M with the residue norm. By Lemma 8.1, the equivalence class of the norm does not depend on the choice of π.

For any A-linear homomorphism $f : M \to N$ of finite A-modules, we endow M and N with the Banach structures as above. It follows from Lemma 8.1 that f is bounded. We have defined the inverse functor of the forgetful functor $\mathcal{B}\mathrm{an}_A^f \to \mathcal{M}\mathrm{od}_A^f$. $\square$

Remark 8.3. Let A be a k-affinoid algebra. It is not true that a Banach A-module which is finite as A-module is finite as Banach A-module.

As an example, take $0 < p < q < 1$ and $A = k\{q^{-1}T\}$, $B = k\{p^{-1}T\}$. Then B is a Banach A-module. By Example 2.4, the underlying rings of A and B are both $k[[T]]$. So the canonical map $A \to B$ is bijective. But B is not a finite A-module. As otherwise, the inverse map $B \to A$ is bounded by Lemma 8.1, which is not the case.

The correct statement is the following: consider a Banach A-module $(M, \|\bullet\|_M)$ which is finite as A-module, then there is a norm on M such that M becomes a finite Banach A-module. The new norm is not necessarily equivalent to the given norm $\|\bullet\|_M$.

Proposition 8.4. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Then the natural map

$$M \otimes_A N \to M\hat{\otimes}_A N$$

is an isomorphism of Banach A-modules and $M\hat{\otimes}_A N$ is a finite Banach A-module.

Here the Banach A-module structure on $M \otimes_A N$ is given by Proposition 8.2.

PROOF. Choose $m, m' \in \mathbb{N}$ an admissibly coexact sequence

$$A^{m'} \to A^m \to M \to 0$$

of Banach A-modules. Then we have a commutative diagram of A-modules:

$$\begin{tikzcd} A^{m'} \otimes_A N \arrow[r] \arrow[d] & A^m \otimes_A N \arrow[r] \arrow[d] & M \otimes_A N \arrow[r] \arrow[d] & 0 \\ A^{m'} \hat{\otimes}_A N \arrow[r] & A^m \hat{\otimes}_A N \arrow[r] & M \hat{\otimes}_A N \arrow[r] & 0 \end{tikzcd}$$

with exact rows. By 5-lemma, in order to prove $M \otimes_A N \xrightarrow{\sim} M\hat{\otimes}_A N$ and $M\hat{\otimes}_A N$ is a finite Banach A-module, we may assume that $M = A^m$ for some $m \in \mathbb{N}$. Similarly, we can assume $N = A^n$ for some $n \in \mathbb{N}$. In this case, the isomorphism is immediate and $M\hat{\otimes}_A N$ is clearly a finite Banach A-module. By Lemma 8.1, the Banach A-module structure on $M\hat{\otimes}_A N$ coincides with the Banach A-module strucutre on $M \otimes_A N$ induced by Proposition 8.2. □

Proposition 8.5. Let A, B be a k-affinoid algebra and $A \to B$ be a bounded k-algebra homomorphism. Let M be a finite Banach A-module, then the natural map

$$M \otimes_A B \to M\hat{\otimes}_A B$$

is an isomorphism of Banach B-modules and $M\hat{\otimes}_A B$ is a finite Banach B-module.

PROOF. By the same argument as Proposition 8.4, we may assume that $M = A^n$ for some $n \in \mathbb{N}$. In this case, the assertions are trivial. □

Proposition 8.6. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Let $\varphi : M \to N$ be an A-linear map. Then φ is admissible.

PROOF. By Lemma 8.1, φ is always bounded. By Proposition 8.5 and Proposition 3.11, we may assume that k is non-trivially valued. By Theorem 6.3, N is a Noetherian A-module. It follows from Corollary 7.4 in the chapter Banach Rings that $\operatorname{Im}\varphi$ is closed in N and is finite as an A module. In particular, the norm induced from N and from M are equivalent by Lemma 8.1. It follows that φ is admissible. □

Proposition 8.7. Let A be a k-affinoid algebra. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n)$ be a k-free polyray. Then M is a finite Banach A-module if and only if $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module.

PROOF. We may assume that r has only one component and write $r_1 = r$. The direct implication is trivial. Let us assume that $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module. Take $n \in \mathbb{N}$ and an admissible epimorphism of $A\hat{\otimes}_k k_r$-modules

$$\varphi : (A\hat{\otimes}_k k_r)^n \to M\hat{\otimes}_k k_r.$$

Let $e_1, \ldots, e_n$ denotes the standard basis of $(A\hat{\otimes}_k k_r)^n$. We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By Proposition 6.6, we can assume that there is $l > 0$ such that $m_{i,j} = 0$ for all $i = 1, \ldots, n$ and $|j| > l$. It follows that

$$A^{n(2l+1)} \to M$$

sending the standard basis to $m_{i,j}$ with $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$ is an admissible epimorphism. $\square$

Proposition 8.8. Let $\phi : A \to B$ be a morphism of k-affinoid algebras, $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then the following are equivalent:

(1) ϕ is finite and admissible.
(2) $\phi \hat{\otimes}_k k_r$ is finite and admissible.

This is [Tem04, Lemma 3.2]. I do not understand Temkin's argument. The following proof is a modification of the argument of Temkin's.

Proof. (1) $\implies$ (2): This is straightforward.

(2) $\implies$ (1): The admissible part is straightforward. Let us prove that ϕ is finite. We may assume that $n = 1$. When r is not in $\sqrt{|k^\times|}$, we just apply Proposition 8.7. Now suppose $r \in \sqrt{|k^\times|}$. Let us take $m \in \mathbb{Z}_{>0}$ such that $r^m = |c^{-1}|$ for some $c \in k^\times$. Define a bounded k-algebra homomorphism

$$\varphi : k\{T\} \to k\{r^{-1}T\}$$

sending T to cT^m. Observe that φ is injective. We have argued in the proof of Lemma 3.15 that this homomorphism is finite.

Then φ induces a finite extension of ring $\operatorname{Frac} k\{r^{-1}T\} / \operatorname{Frac} k\{T\}$. In particular, the closure of $\operatorname{Frac} k\{T\}$ in k_r is a subfield over which k_r is finite. But this valuation field is isomorphic to $k\{T\}$. By Proposition 8.5 and fpqc descent [Stacks, Tag 02LA], we may assume that $r = 1$.

Recall that k_1 is the completion of $\operatorname{Frac} k\{T\}$. Let $\{\tilde{f}_i\}_{i \in I}$ be the set of irreducible monic polynomials in $\tilde{k}[T]$. Lift each $\tilde{f}_i$ to $f_i \in \mathring{k}[T]$. Let $a \in A \hat{\otimes}_k k_1$, we represent a as

$$a = \sum_{l=0}^{\infty} a_l T^l + \sum_{i \in I, j \geq 1, 0 \leq k < \deg f_i} a_{ijk} T^k / f_i^j .$$

A similar expression exists for elements in $B \hat{\otimes}_k k_1$ as well. Moreover, the representation is unique.

As $B \hat{\otimes}_k k_1$ is finite over $A \hat{\otimes}_k k_1$, we can find $b_1, \dots, b_m$ such that any $b \in B$ can be written as

$$b = \sum_{j=1}^{m} \phi \hat{\otimes}_k k_1 (a_j) b_j ,$$

where $a_j \in A \hat{\otimes}_k k'$. We can replace b_j by $b_{j,0}$ and a_j by $a_{j,0}$. It follows that B is generated $b_{1,0}, \dots, b_{m,0}$ over A. $\square$

For any ring A, $\mathcal{A}\mathrm{lg}^f_A$ denotes the category of finitely generated A-algebras.

Proposition 8.9. Let A be a k-affinoid algebra. Then the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A \to \mathcal{A}\mathrm{lg}^f_A$ is an equivalence of categories.

Recall that $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A$ is defined in Definition 5.9 in the chapter Banach Rings.

Proof. It suffices to construct an inverse functor. Let B be a finite A-algebra. We endow B with the norm $\| \bullet \|_B$ as in Proposition 8.2. We claim that B is a Banach A-algebra.

Let us recall the definition of the norm. Take $n \in \mathbb{N}$, an epimorphism $\varphi : A^n \to B$ of A-modules. Then $\| \bullet \|_B$ is the residue norm induced by φ.

Consider the A-linear epimorphism $\psi : A^n \otimes_A A^n \to B \otimes_A B$. By Proposition 8.6, when both sides are endowed with the norms $\| \bullet \|_{A^n \otimes_A A^n}$ and $\| \bullet \|_{B \otimes_A B}$ as in Proposition 8.2, ψ is admissible. It follows that there is $C > 0$ such that for any $f, g \in B$,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by Proposition 8.2, the natural map $B \otimes_A B \to B$ is bounded. It follows that there is a constant $C' > 0$ such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in B is bounded and hence B is a finite Banach algebra. Given any morphism $B \to B'$ in $\mathcal{A}\mathrm{lg}_A^f$, we endow B and B' with the norms given by Proposition 8.2. It follows from Lemma 8.1 that $B \to B'$ is a bounded homomorphism of finite Banach A-algebras. So we have defined an inverse functor to the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_A^f \to \mathcal{A}\mathrm{lg}_A^f$. □

Remark 8.10. It is not true that any homomorphism of k-affinoid algebras is bounded. For example, if the valuation on k is trivial. Take $0 < p < q < 1$ and consider the natural homomorphism $k_p \to k_q$. This homomorphism is bijective but not bounded.

9. Affinoid domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 9.1. Let A be a k_H-affinoid algebra. A closed subset $V \subseteq \mathrm{Sp}\, A$ is said to be a *k_H-affinoid domain* in X if there is an object $\mathrm{Sp}\, A_V \in k_H$-$\mathcal{A}\mathrm{ff}$ and a morphism $\phi : \mathrm{Sp}\, A_V \to \mathrm{Sp}\, A$ in k_H-$\mathcal{A}\mathrm{ff}$ such that

1. the image of ϕ in $\mathrm{Sp}\, A$ is V;
2. given any object $\mathrm{Sp}\, B \in k_H$-$\mathcal{A}\mathrm{ff}$ and a morphism $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$ whose image lies in V, there is a unique morphism $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$ in k_H-$\mathcal{A}\mathrm{ff}$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Sp}\, B \arrow[d, dotted, "!"] \arrow[dr] & \\ \mathrm{Sp}\, A_V \arrow[r, "\phi"] & \mathrm{Sp}\, A \end{tikzcd}$$

We say V is *represented by* the morphism ϕ or by the corresponding morphism $A \to A_V$.

When $H = \mathbb{R}_{>0}$, we say V is a *k-affinoid domain* in X. When $H = |k^\times|$, we say V is a *strict k-affinoid domain* in X.

We observe that A_V is canonically determined by the universal property.

Remark 9.2. This definition differs from the original definition of [Ber12], we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the orignal definition of Berkovich when $H = \mathbb{R}_{>0}$.

We begin with a few examples.

Example 9.3. Let A be a k_H-affinoid domain. Let $n, m \in \mathbb{N}$ and $f = (f_1, \dots, f_n) \in A^n$, $g = (g_1, \dots, g_m) \in A^m$. Let $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ and $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H}^m$. We define

$$(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\} := \{x \in \operatorname{Sp} A : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

We claim that $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ is a k_H-affinoid domain in $\operatorname{Sp} A$. These domains are called *k_H-Laurent domains* in $\operatorname{Sp} A$. When $m = 0$, the domains $\operatorname{Sp} A\{r^{-1}f\}$ are called *k_H-Weierstrass domains* in $\operatorname{Sp} A$.

To see this, we define

$$A\left\{r^{-1}f, sg^{-1}\right\} := A\{r^{-1}T, sS\}/(T_1 - f_1, \dots, T_n - f_n, g_1S_1 - 1, \dots, g_mS_m - 1).$$

By Theorem 6.3, this defines a Banach k-algebra structure. We write $\|\bullet\|'$ for the quotient norm. By definition, $A\left\{r^{-1}f, sg^{-1}\right\}$ is a k_H-affinoid algebra and there is a natural morphism $A \to A\left\{r^{-1}f, sg^{-1}\right\}$. We claim that this morphism represents $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$.

For this purpose, we first compute $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$. We observe that $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\} \to \operatorname{Sp} A$ is injective since $A[f, g^{-1}]$ is dense in $A\left\{r^{-1}f, sg^{-1}\right\}$. We will therefore identify $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ with a subset of $\operatorname{Sp} A$.

Next we show that the image of $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ in $\operatorname{Sp} A$ is contained in $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$. Take $\|\bullet\| \in \operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$. Then there is a constant $C > 0$ such that

$$\|\bullet\| \leq C\|\bullet\|'.$$

Applying this to f_i^k for some $k \in \mathbb{Z}_{>0}$ and $i = 1, \dots, n$, we find that

$$\|f_i\|^k = \|f_i^k\| \leq C\|f_i^k\|' \leq C\|T_i^i\|_{r,s^{-1}} = Cr_i^k.$$

It follows that

$$\|f_i\| \leq r_i.$$

Similarly, we deduce $|g_j| \geq s_j$ for $j = 1, \dots, m$. Namely, $\|\bullet\| \in (\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$.

Next we verify the universal property: let $\operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid domains that factorizes through $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$. We write $\psi : A \to B$ for the corresponding morphism of k_H-affinoid algebras. By Corollary 6.12 in the chapter Banach Rings, we have

$$\rho_B(f_i) = \sup_{x \in \operatorname{Sp} B} |f_i(x)| \leq \sup_{y \in (\operatorname{Sp} A)\{r^{-1}f, sg^{-1}\}} |f_i(y)| \leq r_i$$

for $i = 1, \dots, n$. Similarly, one deduces that $\rho(g_j) \leq s_j^{-1}$ for $j = 1, \dots, m$.

We will construct the dotted arrows:

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d] & B \\
A\{r^{-1}T, sS\} \arrow[ur, dotted, "\eta"] \arrow[d] & \\
A\{r^{-1}f, sg\} \arrow[uur, dotted, "\tau"'] &
\end{tikzcd}$$

so that this diagram commutes. We define η as the unique morphism sending T_i to f_i and S_j to g_j for $i = 1, \dots, n$, $j = 1, \dots, m$. The existence of such a morphism is guaranteed by Proposition 6.6. In order to descend this morphism to η', it suffices to show that $T_i - f_i$ and $g_j S_j - 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ lie in the kernel of η. But this is immediate from our definition. Moreover, it is clear that η' is necessarily unique.

It remains to show that each point in $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$ lies in $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$.

It suffices to treat the cases $(n,m) = (1,0)$ and $(n,m) = (0,1)$. We will only handle the former case, as the latter is similar. In concrete terms, we need to show that for any $x \in \operatorname{Sp} A$ corresponding to a bounded semi-valuation $|\bullet|_x$ on A satisfying $|f(x)| \leq r$, we can always extend $|\bullet|_x$ to a bounded semi-valuation $\|\bullet\|$ on $A\{r^{-1}f\}$. Replacing A by $A/\ker|\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A. We endow $A\{r^{-1}T\}$ with the Gauss norm $\|\bullet\|_{x,r}$ induced by $|\bullet|_x$ and $A\{r^{-1}T\}$ with the quotient norm $\|\bullet\|$. This norm is bounded by construction. It suffices to show that it is a valuation and it extends the given valuation on A. The former is a consequence of the latter, as A is dense in $A\{r^{-1}f\}$. Now suppose $a \in A$. A general preimage of a in $A\{r^{-1}T\}$ is

$$a + (T-f)\sum_{j=0}^{\infty} b_j T^j = a - fb_0 + \sum_{j=1}^{\infty}(b_{j-1} - fb_j)T^j$$

with $\|b_j\|_A r^j \to 0$ as $j \to \infty$. Now we compute

$$\begin{aligned}
\|a - fb_j + \sum_{j=1}^{\infty}(b_{j-1} - fb_j)\|_{x,r} &= \max\left\{|a - fb_0|_x, \max_{j\geq 1}|b_{j-1} - fb_j|_x r^j\right\} \\
&\geq \max\left\{|a - fb_0|_x, \max_{j\geq 1}|b_{j-1} - fb_j|_x |f|_x^j\right\} \\
&= \max\left\{|a - fb_0|_x, \max_{j\geq 1}|f^j b_{j-1} - f^{j+1}b_j|_x\right\} \geq \quad |a|_x.
\end{aligned}$$

So $\|a\| \geq |a|_x$. The reverse inequality is trivial. We conclude.

Example 9.4. Let A be a k_H-affinoid domain. Let $n \in \mathbb{N}$, $g \in A$, $f = (f_1, \ldots, f_n) \in A^n$, $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times| \cdot H}^n$. Assume that $g, f_1, \ldots, f_n$ generates the unit ideal. Define

$$(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\} = \left\{x \in \operatorname{Sp} A : |f_i(x)| \leq r_i|g(x)| \text{ for } i = 1, \ldots, n\right\}.$$

Then we claim that $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$ is a k_H-affinoid domain in $\operatorname{Sp} A$. Domains of this form are called *k_H-rational domains*.

To see this, we define

$$A\left\{r^{-1}\frac{f}{g}\right\} := A\{r^{-1}T\}/(gT_1 - f_1, \ldots, gT_n - f_n).$$

By Theorem 5.1, this is indeed a k_H-affinoid domain. We will denote by $\|\bullet\|'$ the residue norm. We will prove that the natural map $A \to A\left\{r^{-1}\frac{f}{g}\right\}$ represents the affinoid domain $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. Observe that

$$\operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$$

is injective as elemnts of the form a/g with $a \in A$ is dense in $A\left\{r^{-1}\frac{f}{g}\right\}$. Next we show that

$$(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\} \supseteq \operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}.$$

Let $x \in \operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$, take $|\bullet|_x$ as the correspoding bounded semi-valuation on $A\left\{r^{-1}\frac{f}{g}\right\}$. Then there is a constant $C > 0$ such that for any $k \in \mathbb{Z}_{>0}$,

$$|f_i|_x^k = |f_i^k|_x = |g|_x^k \cdot |T_i^k|_x \leq C|g|_x^k r_i^k.$$

for all $i = 1, \ldots, n$. In particular,

$$|f_i|_x \leq r_i|g|_x.$$

Hence, $x \in (\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$.

Next we verify the universal property. Let $\operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spectra factorizing through $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. Observe that $g(x) \neq 0$ for all $x \in (\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. As otherwise, $f_i(x) = 0$ for all $i = 1, \ldots, n$. This contradicts our assumption on $g, f_1, \ldots, f_n$. It follows that $\psi(g)$ is invertible by Corollary 6.11 int the chapter Banach Rings. From the definition of $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$, it is clear that $\rho(\psi(f_i)) \leq r\rho(\psi(g))$ for $i = 1, \ldots, n$.

We construct

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d] & B \\
A\{r^{-1}T\} \arrow[ur, dotted, "\eta"] \arrow[d] & \\
A\left\{r^{-1}\frac{f}{g}\right\} \arrow[uur, dotted, "\tau"'] &
\end{tikzcd}$$

successively. The morphism η sends T_i to $\psi(f_i)/\psi(g)$ for $i = 1, \ldots, n$. The existence of such a morphism is guaranteed by Proposition 6.6. Clearly $gT_i - f_i$ is contained in $\ker \eta$, so η descends to τ. The morphism τ is clearly unique.

It remains to verify that the image of $\operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$ in $\operatorname{Sp} A$ is exactly $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. In other words, we need to verify that if $|\bullet|_x$ is a bounded semi-valuation on A satisfying $|f_i|_x \leq r_i|g|_x$, then $|\bullet|_x$ extends to a bounded semi-valuation on $A\left\{r^{-1}\frac{f}{g}\right\}$. Replacing A by $A/\ker|\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A. Consider the Gauss valuation $|\bullet|_{x,r}$ on $A\{r^{-1}T\}$ and the residue norm $\|\bullet\|$ on $A\left\{r^{-1}\frac{f}{g}\right\}$. It suffices to show that $\|\bullet\|$ is a valuation extending the valuation $|\bullet|_x$ on A. The former is a consequence of the latter. Take $a \in A$, we need to show that $|a|_x = \|a\|$.

A general preimage of a in $A\{r^{-1}T\}$ has the form

$$a + \sum_{i=1}^{n}(gT_i - f_i)\sum_{\alpha \in \mathbb{N}^n}^{\infty} b_{i,\alpha}T^\alpha$$

with $\|b_{i,\alpha}\|_A r^\alpha$, where $\| \bullet \|_A$ denotes the initial norm on A. The same argument as in Example 9.3 shows that

$$\|a + \sum_{i=1}^{n}(gT_i - f_i) \sum_{\alpha\in\mathbb{N}^n}^{\infty} b_{i,\alpha}T^\alpha\|_{x,r} \geq |a|_x.$$

So $\|a\|_x \geq |a_x|$, the reverse inequality is trivial.

Proposition 9.5. Let A be a k_H-affinoid algebra and $V \subseteq \operatorname{Sp} A$ be a k_H-affinoid domain represented by $\varphi : A \to A_V$. Then $\operatorname{Sp}\varphi$ induces a homeomorphism $\operatorname{Sp} A_V \to V$.

In particular, we will identify V with $\operatorname{Sp} A_V$ and say $\operatorname{Sp} A_V$ is a k_H-affinoid domain in $\operatorname{Sp} A$.

PROOF. We observe that $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a monomorphism in the category k_H-$\mathcal{A}\mathrm{ff}$. In other words, $A \to A_V$ is an epimorphism in the category k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$. To see this, let $\eta_1, \eta_2 : A_V \to B$ be two arrows in k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$ such that $\eta_1 \circ \varphi = \eta_2 \circ \varphi$. It follows from the universal property in Definition 9.1 that $\eta_1 = \eta_2$. By Proposition 3.18, $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a bijection. But $\operatorname{Sp} A_V$ and Sp_A are both compact and Hausdorff by Theorem 6.13 in the chapter Banach rings, so $\operatorname{Sp} A_V \to V$ is a homeomorphism. □

Proposition 9.6. Let A be a k_H-affinoid algebra and V, W be k_H-Weierstrass domains (resp. k_H-Laurent domains, resp. k_H-rational domains) in $\operatorname{Sp} A$. Then $V \cap W$ is also a k_H-Weierstrass domain (resp. k_H-Laurent domain, resp. k_H-rational domain).

PROOF. This is clear in the Weierstrass and Laurent cases. We will prove therefore assume that V and W are k_H-rational.

We take $f_1, \ldots, f_n \in A$, $g_1, \ldots, g_m \in A$ both generating the unit ideal and $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times| \cdot H}^n$, $s = (s_1, \ldots, s_m) \in \sqrt{|k^\times| \cdot H}^m$ such that

$$V = \operatorname{Sp} A\left\{r^{-1}\frac{f}{f_m}\right\}, \quad W = \operatorname{Sp} A\left\{s^{-1}\frac{g}{g_n}\right\}.$$

We may assume that $r_n = s_m = 1$. Now let $R = (R_{i,j}) \in \sqrt{|k^\times| \cdot H}^{mn}$ where $R_{i,j} = r_i s_j$ and $F = (F_{i,j})$ with $F_{i,j} = f_i g_j$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$. Observe that the $F_{i,j}$'s generate the unit ideal. We consider the k_H-rational domain

$$Z = \operatorname{Sp} A\left\{R^{-1}\frac{F}{f_n g_m}\right\}.$$

We clearly have $V \cap W \subseteq Z$. We need to prove the reverse inequality. Let $x \in Z$, so we have

$$|f_i g_j(x)| \leq r_i s_j |f_n g_m(x)|$$

for any $i = 1, \ldots, n$, $j = 1, \ldots, m$. In particular, when $j = m$, we have

$$|f_i g_m(x)| \leq r_i |f_n g_m(x)|$$

for any $i = 1, \ldots, n$. But $f_n g_m$ is invertible, so we can cancel $g_m(x)$ to find

$$|f_i(x)| \leq r_i |f_n(x)|.$$

So $x \in V$. Similarly, we have $x \in W$. □

Corollary 9.7. Let A be a k_H-affinoid algebra and V be a k_H-Laurent domain in $\operatorname{Sp} A$. Then V is also a k_H-rational domain.

PROOF. By Proposition 9.6, it suffices to show consider k_H-Laurent domains of the following form:

$$\operatorname{Sp} A\{r^{-1}f\}, \quad \operatorname{Sp} A\{sg^{-1}\}$$

where $r, s \in \sqrt{|k^\times| \cdot H}$ and $f, g \in A$. Both domains are k_H-rational by definition. □

Proposition 9.8. Let A be a k_H-affinoid algebra and $\operatorname{Sp} B$ be a k_H-rational domain in $\operatorname{Sp} A$. Then there is a k_H-Laurent domain $\operatorname{Sp} C$ in $\operatorname{Sp} A$ such that $\operatorname{Sp} B \subseteq \operatorname{Sp} C$ and $\operatorname{Sp} B$ is a k_H-Weierstrass domain in $\operatorname{Sp} C$.

PROOF. We write

$$B = A\left\{r^{-1}\frac{f}{g}\right\}$$

for some $n \in \mathbb{N}$, $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times| \cdot H}^n$, $f = (f_1, \ldots, f_n) \in A^n$ and $g \in A$ such that $f_1, \ldots, f_n, g$ generate the unit ideal. Let g'' be the image of g in B, which is a unit. Choose $c \in \sqrt{|k^\times| \cdot H}$ such that $\rho_B(g^{-1}) < c^{-1}$. Set $C = A\{cg^{-1}\}$, then $\operatorname{Sp} B \subseteq \operatorname{Sp} C$. Moreover,

$$\operatorname{Sp} B \cap \operatorname{Sp} C = \emptyset.$$

Let $f'_1, \ldots, f'_n, g'$ be the images of $f_1, \ldots, f_n, g$ in C. Write $f' = (f'_1, \ldots, f'_n)$. Then by Corollary 6.11 in the chapter Banach Rings, g' is a unit and

$$\operatorname{Sp} B = \operatorname{Sp} C\{r^{-1}g'^{-1}f'\}.$$

□

Proposition 9.9. Let A be a k_H-affinoid algebra, $\operatorname{Sp} B$ be a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} A$ and $\operatorname{Sp} C$ be a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} B$. Then $\operatorname{Sp} C$ is a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} A$.

PROOF. We first handle the Weierstrass case. Write

$$B = \operatorname{Sp} A\{r^{-1}f\}, C = \operatorname{Sp} B\{s^{-1}g\}$$

for some $n, m \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H}^n$, $s \in \sqrt{|k^\times| \cdot H}^m$ and $f = (f_1, \ldots, f_n) \in A^n$, $g = (g_1, \ldots, g_m) \in B^m$. Observe that if we replace g with a small perturbation, the domain $\operatorname{Sp} C$ in $\operatorname{Sp} B$ remains the same, so we may assume that $g_1, \ldots, g_m \in A$. Then

$$\operatorname{Sp} C = \operatorname{Sp} A\{r^{-1}f\} \cap \operatorname{Sp} A\{s^{-1}g\}$$

is a k_H-Weierstrass domain by Proposition 9.6.

Next we handle the rational case. Write

$$B = A\left\{s\frac{f}{g}\right\}$$

for some $m \in \mathbb{N}$, $f = (f_1, \ldots, f_m) \in A^m$, $r = (r_1, \ldots, r_m) \in \sqrt{|k^\times| \cdot H}^m$ and $g \in A$ such that $f_1, \ldots, f_m, g$ generate the unit ideal.

By Proposition 9.8 and Proposition 9.6, it suffices to handle the special cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ for some $r \in \sqrt{|k^\times| \cdot H}$ and $h \in B$. Observe that making a small perturbation on h does not change the domain. As $A[g^{-1}]$ is dense in B, we may assume that there is $n \in \mathbb{Z}_{>0}$ such that $h' = g^n h \in A$. As g is invertible on $\operatorname{Sp} B$, we can find $c \in \sqrt{|k^\times| \cdot H}$ so that

$$|g(x)|^n > c^{-1}$$

for $x \in \operatorname{Sp} B$.

We need to treat the cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ separately. In the first case, we write

$$\operatorname{Sp} C = \operatorname{Sp} B \cap \operatorname{Sp} A \left\{ (r,c)^{-1} \frac{(h',1)}{g^n} \right\}.$$

In the second case,

$$\operatorname{Sp} C = \operatorname{Sp} B \cap \operatorname{Sp} A \left\{ (r,c)^{-1} \frac{(g^n,1)}{h'} \right\}.$$

□

10. Graded reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 10.1. Let A be a Banach k-algebra, we define the *graded reduction* of A as

$$\tilde{A} := \bigoplus_{h \in \mathbb{R}_{>0}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define $\tilde{f}$ as the image of f in the $\rho(f)$-graded piece of $\tilde{A}$.

Definition 10.2. Let A be a k_H-affinoid algebra. We define the *k_H-graded reduction* of A as the $\sqrt{|k^\times| \cdot H}$-graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times| \cdot H}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define $\tilde{f}$ as the image of f in the $\rho(f)$-graded piece of $\tilde{A}^H$.

For any morphism $f : A \to B$ of k_H-affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \to \tilde{B}^H$$

as the map induced by sending the class of $x \in A$ with $\rho(x) \leq h$ for any $h \in \sqrt{|k^\times| \cdot H}$ to the class of $f(x) \in B$.

Recall that $\rho(A) = \sqrt{|k^\times| \cdot H} \cup \{0\}$ by Theorem 7.4, so $\tilde{f}$ is well-defined. This definition is compatible with Definition 10.1 in the sense that if we regard a $\sqrt{|k^\times| \cdot H}$-graded ring as a $\mathbb{R}_{>0}$-graded ring, the two definitions give the same object.

Example 10.3. If K is a k_H-affinoid algebra which is a field as well, then $\tilde{K}^H$ is a $\sqrt{|k^\times| \cdot H}$-graded field. This is immediate from the definition.

Lemma 10.4. Let $(A, \|\bullet\|)$ be a k-affinoid algebra, $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Let $f \in k\{r^{-1}T\}$. Expand f as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha.$$

PROOF. By induction, we may assume that $n = 1$ and write $r = r_1$. As ρ is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j)\rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j.$$

Observe that $\rho(a_j)$ is not ambiguous: when intepreted as in A and in $A\{r^{-1}T\}$, it has the same value.

Conversely, we need to show that for any $j \in \mathbb{N}$,

$$\rho(f) \geq \rho(a_j) r^j.$$

Equivalently, this means for any $k \in \mathbb{Z}_{>0}$ and any $j \in \mathbb{N}$, we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk}.$$

Fix j and k as above. We compute the left-hand side:

$$f^k = \sum_{\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^k a_{\beta_l}.$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|}.$$

Take $\beta = (j, j, \ldots, j)$, we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk}.$$

□

Lemma 10.5. Assume that k is non-trivially valued. Let A be a strictly k-affinoid algebra. Then for any $a, f \in A$, the set of non-zero values $\rho(f^n a)$ for $n \in \mathbb{N}$ is a discrete subset of $\mathbb{R}_{>0}$.

PROOF. As A is noetherian Theorem 6.3, it has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. It follows that

$$\operatorname{Sp} A = \bigcup_{i=1}^m \operatorname{Sp} A/\mathfrak{p}_i.$$

Here we make the obvious identification by identifying $\operatorname{Sp} A/\mathfrak{p}_i$ with a subset of $\operatorname{Sp} A$.

By Corollary 6.12 in the chapter Banach Rings, it suffices to consider each of $\operatorname{Sp} A/\mathfrak{p}_i$ separately, so we may assume that A is an integral domain.

By Corollary 5.2, we can take $d \in \mathbb{N}$ and a finite injective homomorphism of k-algebras $\iota : k\{T_1, \ldots, T_d\} \to A$. According to Proposition 9.11 in the chapter Banach Rings, ρ_A is the restriction of the norm $\|\bullet\|_{\operatorname{Frac} A}$ on $\operatorname{Frac} A$ induced by the finite extension $\operatorname{Frac} A / \operatorname{Frac} k\{T_1, \ldots, T_d\}$ from the Gauss valuation. But it is well-known that $\|\bullet\|_{\operatorname{Frac} A}$ is the maximum of finitely many valuations on $\operatorname{Frac} A$. Reproduce BGR3.3.3.1 somewhere. The assertion is by now obvious. □

Lemma 10.6. Let $(A, \|\bullet\|)$ be a k-affinoid algebra, $f \in A$ with $r = \rho(f) > 0$. Let $B = A\{r^{-1}f\}$. Then for any $a \in A$, we have

$$\rho_B(a) = \lim_{n\to\infty} r^{-n}\rho_A(f^n a).$$

If moreover, $\rho_B(a) > 0$, then there is $n_0 > 0$ such that for $n \geq n_0$,

$$\rho_B(a) = r^{-n}\rho_A(f^n a), \quad \rho_B(f^n a) = r^{-n}\rho_A(a).$$

Proof. We observe that for any $a \in A$, $n \in \mathbb{Z}_{>0}$, we have

$$\rho_B(f^n a) = r^n \rho_B(a).$$

So the last two assertions are equivalent.

Take a k-free polyradius s such that $A\hat{\otimes}_k k_s$ and $B\hat{\otimes}_k k_s$ are both strictly k_s-affinoid. By Proposition 3.11, $A\hat{\otimes}_k k_s\{r^{-1}f\} \xrightarrow{\sim} B\hat{\otimes}_k k_s$. Moreover, ρ_A and ρ_B are both preserved after base change to k_s. So we may assume that k is non-trivially valued and A and B are strictly k-affinoid.

Observe that for $n \in \mathbb{Z}_{>0}$,

$$\rho_A(f^{n+1}a) \leq \rho_A(f)\rho_A(f^n a) = r\rho_A(f^n a).$$

So $r^{-n}\rho_A(f^n a)$ is decreasing in n. Moreover, for any $x \in \operatorname{Sp} A\{r^{-1}f\}$, by Example 9.3, we have

$$|f(x)| \geq r.$$

By Corollary 6.12 in the chapter Banach Rings, we have

$$|f(x)| = r$$

for any $x \in \operatorname{Sp} A\{r^{-1}f\}$. It follows from Corollary 6.12 in the chapter Banach Rings that for any $n \in \mathbb{Z}_{>0}$,

$$\rho_A(f^n a) = \sup_{x\in \operatorname{Sp} A} |f^n a(x)| \geq r^n \sup_{x \in \operatorname{Sp} A\{rf^{-1}\}} |a(x)| = r^n \rho_B(a).$$

By Lemma 10.5, the decreasing sequence $\{r^{-n}\rho_A(f^n a)\}_n$ either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is $\alpha \in \mathbb{R}_{>0}$ and $n_0 > 0$ such that for $n \geq n_0$, we have

$$r^{-n}\rho_A(f^n a) = \alpha.$$

We have to show that $\alpha \leq \rho_B(a)$. Assume the contrary $\alpha > \rho_B(a)$. Then for all $x \in \operatorname{Sp} A$, we have

$$|f^n a(x)| \leq r^n |a(x)|.$$

So $f^n a$ must obtain its maximum on $U := \{x \in \operatorname{Sp} A : |a(x)| \geq \alpha\}$. But U is disjoint from $\operatorname{Sp} A\{r^{-1}f\}$ as

$$\alpha > \rho_B(a).$$

It follows from Example 9.3 that

$$\beta := \sup_{x\in U} |f(x)| = \max_{x\in U} |f(x)| < r.$$

So

$$\rho(f^n a) = \sup_{x\in \operatorname{Sp} A} |f^n a(x)| = \sup_{x\in U} |f^n a(x)| \leq \beta^n \sup_{x\in U} |a(x)|.$$

This contradicts the fact that $\alpha > 0$. $\square$

Proposition 10.7. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that k_r is defined in Example 3.12.

Proof. By Lemma 10.4, we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}^H_s \xrightarrow{\sim} \bigoplus_{\alpha\in\mathbb{N}^n} \tilde{A}^H_{sr^{-\alpha}}$$

for any $s \in \sqrt{|k^\times| \cdot H}$. This establishes the desired isomorphism. $\square$

Proposition 10.8. Let A be a k_H-affinoid algebra and $f \in A$ with $r = \rho(f) > 0$. Then there is a natural isomorphism

$$\tilde{A}^H_{\tilde{f}} \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that $A\{rf^{-1}\}$ is defined in Example 9.3, by Theorem 7.4, it is k_H-affinoid.

Proof. Let $B = A\{rf^{-1}\}$ and denote by $\phi : \tilde{A}^H \to \tilde{A}^H_{\tilde{f}}$ the natural $\sqrt{|k^\times| \cdot H}$-graded homomorphism. From the universal property add details, we can factor the natural map $\tilde{A}^H \to \tilde{B}^H$ as $\psi : \tilde{A}^H_{\tilde{f}} \to \tilde{B}^H$. We have a commutative diagram:

$$\begin{tikzcd} \tilde{A}^H \arrow[r] \arrow[d, "\phi"'] & \tilde{B}^H \\ \tilde{A}^H_{\tilde{f}} \arrow[ur, "\psi"] & \end{tikzcd}$$

We claim that ψ is bijective. Let $\tilde{a}/\tilde{f}^m$ be an element in $\ker\psi$, where $\tilde{a} \in \tilde{A}^H$ is homogeneous. Lift $\tilde{a}$ to $a \in A$. Then $\rho_B(a) < \rho_A(a)$. By Lemma 10.6, $\rho_A(f^n a) < r^n \rho_A(a)$ when n is large enough, so

$$\tilde{f}^n \tilde{a} = 0$$

in $\tilde{A}$. Therefore, $\tilde{a}/f^m = 0$ in $\tilde{A}^H_{\tilde{f}}$. We have shown that ψ is injective.

It remains to show that ψ is surjective. Let $\tilde{b} \in \tilde{B}^H$ be a non-zero homogeneous element. Lift $\tilde{b}$ to $b \in B$ of the form $f^{-n}a$ for some $a \in A$. By Lemma 10.6 again, up to enlarging n, we can assume that $\rho_B(a) = \rho_A(a)$. Then $\tilde{a} = \tilde{f}^n\tilde{b}$ has a preimage in $\tilde{A}$.

$\square$

Corollary 10.9. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H \cong \widetilde{A\hat{\otimes}_k k_r}^H$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

PROOF. We can write

$$A\hat{\otimes}_k k_r = \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}.$$

Taking graded reduction, we find

$$\begin{aligned}\widetilde{A\hat{\otimes}_k k_r}^H &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \widetilde{A\{r^{-1}T\}\{\rho(g)g^{-1}\}}^H \\ &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \widetilde{A\{r^{-1}T\}}^H_{\tilde{g}} \\ &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \tilde{A}^H[r^{-1}T]_{\tilde{g}} \\ &= \tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H.\end{aligned}$$

Here we have applied Proposition 10.8 in the second equality and Proposition 10.7 in the third equality. The first equality follows from the simple observation that graded reduction commutes with filtered colimits. □

Theorem 10.10. Let $\phi : A \to B$ be a morphism of k_H-affinoid algebras. Then the following are equivalent:

(1) ϕ is finite and admissible.
(2) $\tilde{\phi} : \tilde{A}^H \to \tilde{B}^H$ is finite.

PROOF. Take $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$ so that

$$\rho(A\hat{\otimes}_k k_r) = \rho(B\hat{\otimes}_k k_r) = |k_r|$$

and k_r is non-trivially valued. Proof that this is possible.

By **??** in the chapter Commutative Algebra and Proposition 8.8, we may assume that k is non-trivially valued and $\rho(A) = \rho(B) = |k|$. By **??** in the chapter Commutative Algebra, we have $\tilde{A} = \tilde{A}_1 \otimes_{\tilde{k}_1} \tilde{k}$. By Corollary 5.5, ϕ is automatically admissible if it is finite.

So it suffices to argue that ϕ is finite if and only if $\tilde{\phi} : \tilde{A} \to \tilde{B}$ is finite.

Assume that φ is finite. We show that $\tilde{\varphi}$ is finite.

First consider the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k-algebra homomorphism $\psi : k\{T_1, \ldots, T_d\} \to A$ such that $\phi \circ \psi$ is finite and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \ldots, T_d\} \to A$, we can apply Theorem 5.1 to find $\phi \circ \alpha$ to conclude.

It suffices to show that $\widetilde{\phi \circ \psi}$ is finite in order to conclude that $\tilde{\phi}$ is finite. So we are reduced to the case $A = k\{T_1, \ldots, T_d\}$ and $\ker \phi = 0$.

We will show that the conditions of Lemma 10.1 in the chapter Banach Rings is satisfied with ρ_B as the norm B. We have shown that ρ_B is a faithful $k\{T_1, \ldots, T_d\}$-algebra nrom in Corollary 4.16. As B is of finite over $k\{T_1, \ldots, T_d\}$, the rank condition is clearly satisfied. It remains to establish that $\mathring{\phi}$ is integral.

By Proposition 5.12, for $f \in B$, there is an integral equation

$$f^n + \phi(a_1) f^{n-1} + \cdots + \phi(a_n) = 0$$

over A such that $\rho_B(f) = \max_{i=1,\ldots,n} |b_i|^{1/i}_{\mathrm{sup}}$. If $f \in \mathring{B}$, then $|b_i|_{\mathrm{sup}} \leq 1$, hence $b_i \in \mathring{B}$. Add a ref

Conversely, assume that $\tilde{\phi}$ is finite. It suffices to apply Lemma 5.15 to conclude that ϕ is finite. $\square$

Corollary 10.11. Let A be a k_H-affinoid algebra, then $\tilde{A}^H$ is finitely generated over $\tilde{k}^H$.

Proof. Take $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism

$$\pi : k\{r^{-1}T\} \to A.$$

Applying Theorem 10.10, we find that it suffices to prove that $\widetilde{k\{r^{-1}T\}}^H$ is finitely generated over $\tilde{k}^H$. But this follows from Proposition 10.7. $\square$

Definition 10.12. Let A be a k_H-affinoid algebra, we define the *reduction map*

$$\tilde{\operatorname{Sp}} A^H := \operatorname{Spec}^{\sqrt{|k^\times|\cdot H}} \tilde{A}^H.$$

We have a natural map $\pi^H : \operatorname{Sp} A \to \tilde{\operatorname{Sp}} A^H$.

11. Gerritzen–Grauert theorem

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 11.1. Let A be a k_H-affinoid algebra. A morphism $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ in k_H-$\mathcal{A}$ff is a *closed immersion* if the corresponding morphism $A \to B$ in k_H-$\mathcal{A}$ff$\mathcal{A}$lg is an admissible epimorphism.

Definition 11.2. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism in k_H-$\mathcal{A}$ff. We call φ a *k_H-Runge immersion* if there is a factorization in k_H-$\mathcal{A}$ff of φ:

$$\operatorname{Sp} B \to \operatorname{Sp} C \to \operatorname{Sp} A,$$

such that $\operatorname{Sp} B \to \operatorname{Sp} C$ is a closd immersion and $\operatorname{Sp} C \to \operatorname{Sp} A$ is a k_H-Weierstrass domain.

Add a prop rational domains form basis

Lemma 11.3. Let A be a k_H-affinoid algebra and V be a k_H-Laurent domain in $\operatorname{Sp} A$ represented by $A \to B = A\{r^{-1}f, sg\}$ for some $n, m \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in A^n$ and $g = (g_1, \dots, g_m) \in A^m$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ and $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H}^m$. Then

(1) $\tilde{B}^H$ is finite over the subalgebra generated by $\tilde{A}^H$ and $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1^{-1}, \dots, \tilde{g}_m^{-1}$;
(2) if V is a neighbourhood of a point $x \in \operatorname{Sp} A$, then $\tilde{\chi_x}(\tilde{B}^H)$ is finite over $\tilde{\chi_x}(\tilde{A}^H)$.

Proof. (1) Consider the admissible epimomorphism

$$A\{r^{-1}T, sS\} \to B.$$

By Theorem 10.10, it induces a finite homomorphism

$$\widetilde{A\{r^{-1}T, sS\}}^H \to \tilde{B}^H.$$

The former is computed in Proposition 10.7 and our assertion follows.

(2) This is a special case of (1). $\square$

THEOREM 11.4 (Gerritzen–Grauert, Temkin). Let $\varphi : \operatorname{Sp} A \to \operatorname{Sp} B$ be a monomorphism in k_H-$\mathcal{A}$ff. Then there is a finite cover of X by k_H-rational domains $W_1, \ldots, W_k$ such that the restrictions $\varphi_i : \varphi^{-1}(W_i) \to W_i$ are k_H-Runge immersions for $i = 1, \ldots, k$.

PROOF. **Step 1**. We reduce to the following claim: for each $x \in \operatorname{Sp} A$, there is a k_H-rational domain U in $\operatorname{Sp} B$ containing $y = \varphi(x)$ such that $V = \varphi^{-1}U$ is a neighbourhood of x in $\operatorname{Sp} A$ and the induced map $V \to U$ is a closed immersion.

Assume this holds. Write $U = \operatorname{Sp} B \left\{ r \frac{f}{g} \right\}$ for some $n \in \mathbb{N}$, $f = (f_1, \ldots, f_n) \in B^n$ and $g \in B$ such that $f_1, \ldots, f_n, g$ generates the unit ideal and $r \in \sqrt{|k^\times| \cdot H}^n$. As g is invertible on U, we can find a small k_H-rational domain W in $\operatorname{Sp} B$ containing y such that

(1) g is invertible on W;
(2) $\varphi^{-1}W \subseteq \varphi^{-1}U$.

Then $U \cap W$ is a k_H-Weierstrass domain in W and $\varphi^{-1}W \to W$ is therefore a k_H-Runge immersion. From the compactness of $\operatorname{Sp} A$, this implies that we can find k_H-rational domains $W_1, \ldots, W_m$ of $\operatorname{Sp} B$ such that $\varphi^{-1}(W_i) \to W_i$ is a k_H-Runge immersion for $i = 1, \ldots, m$ and $X_1 \cup \cdots \cup X_m$ contains an open neighbourhood U of $\varphi(\operatorname{Sp} A)$. As $\operatorname{Sp} B$ is compact, we can find finitely many k_H-rational domains $W_{m+1}, \ldots, W_k$ which do not intersection $\varphi(\operatorname{Sp} A)$ that covers $\operatorname{Sp} B \setminus U$. Then the covering $W_1, \ldots, W_k$ satisfies all the requirements.

We have reduced the problem to a local one on $\operatorname{Sp} B$.

Step 2. We show that we may assume that $\widetilde{\chi_x}(\tilde{A}^H)$ is finite over $\widetilde{\chi_y}(\tilde{B}^H)$. Here the notation χ_y is defined in **??** in the chapter Banach Rings.

By Corollary 10.11, $\widetilde{\chi_x}(\tilde{A}^H)$ is finitely generated over $\widetilde{\chi_y}(\tilde{B}^H)$. Take generators $h_1, \ldots, h_l \in A$. By Proposition 3.18, $\mathscr{H}(x) \xrightarrow{\sim} \mathscr{H}(y)$, so we can find $f_1, \ldots, f_l, g \in B$ with $|g(y)| = 1$ such that

$$\left| \left(\frac{f_i}{g} - h_i \right)(x) \right| < \rho(h_i)$$

for all $i = 1, \ldots, l$.

In fact, we can take $g = 1$. This can be seen as follows. Let $B' = B\{ag^{-1}\}$ for some $a \in \sqrt{|k^\times| \cdot H}$ with $a < 1$. Then by Lemma 11.3, $\tilde{\chi}_y(\tilde{B'}^H)$ is finite over $\tilde{\chi}_y(\tilde{B}^H)$. So up to replacing B by the B' and $\operatorname{Sp} A$ by the inverse image of $\operatorname{Sp} B'$, we may assume that g is invertible. Replacing f_i by f_i/g, we could then assume that $g = 1$.

Up to replacing $\operatorname{Sp} B$ by $\operatorname{Sp} B\{\rho(h_1)^{-1} f_1, \ldots, \rho(h_l)^{-1} f_l\}$, we can guarantee that $\tilde{f}_i = \tilde{h}_i$ for $i = 1, \ldots, l$. So our assertion follows.

Step 3. We may assume that $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$ for any $x' \in \operatorname{Sp} A$ and $y' = \varphi(x')$.

Let $\pi : \operatorname{Sp} A \to \widetilde{\operatorname{Sp} A}^H$ be the reduction map. Let $\mathcal{X}$ denote the Zariski closure of $\pi(x)$. Then for any $x' \in \operatorname{Sp} A$ with $\pi(x') \in \mathcal{X}$, we have

$$\ker \widetilde{\chi_x} \subseteq \ker \widetilde{\chi_{x'}}.$$

It follows that $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$.

Since $\pi^{-1}\mathcal{X}$ is open in $\operatorname{Sp} A$ Include the proof, we can find a k_H-Laurent neighbourhood $\operatorname{Sp} B\{rf, sg^{-1}\}$ for soem suitable tuples r, f, s, g of y such that $\varphi^{-1} \operatorname{Sp} B\{rf, sg^{-1}\} \subseteq \pi^{-1}\mathcal{X}$. Observe that for each $x' \in \operatorname{Sp} A$, $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$. This follows simply from Lemma 11.3. So up to replacing B with $B\{rf, sg^{-1}\}$, we conclude.

Step 4. We claim that after all of these reductions, φ becomes a closed immersion. By our assumptions, for any minimal homogeneous prime ideal $\mathfrak{p}$ of $\tilde{A}^H$, there is a point $x \in \operatorname{Sp} A$ with $\ker \widetilde{\chi_y} = \mathfrak{p}$ and $\tilde{A}^H/\mathfrak{p}$ is finite over $\tilde{A}^H$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the list of minimal homogeneous prime ideals of $\tilde{A}^H$ prove finiteness, then

$$\tilde{A}^H \to \bigoplus_{i=1}^{k} \tilde{A}^H/\mathfrak{p}_i$$

is injective. Since $\tilde{B}^H$ is graded noetherian Introduce this notion, we find that $\tilde{A}^H$ is finite over $\tilde{B}^H$. So $B \to A$ is finite by Theorem 10.10. It follows that the natural map $A \otimes_B A \to A \hat{\otimes}_B A$ is an isomorphism by Proposition 8.4. As φ is a monomorphism, from general abstract nonsense, the codiagonal $A \hat{\otimes}_B A \xrightarrow{\sim} A$ is an isomorphism. In particular, the codiagonal $A \otimes_B A \to A$ is an isomorphism. This implies that $A \to B$ is surjective. □

Lemma 11.5. Let A be a k_H-affinoid domain and V be a k_H-affinoid domain in A represented by $A \to A_V$. Assume that $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a closed immersion, then V is a k_H-affinoid domain.

Proof. As $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a closed immersion, we can find an ideal $I \subseteq A$ and assume that $A_V = A/I$. Consider the morphism of k_H-affinoid spectra $\psi : \operatorname{Sp} A/I^2 \to \operatorname{Sp} A$ induced by the natural map A/I^2. By the universal property of V, we have a commutative diagram:

$$\begin{tikzcd} \operatorname{Sp} A/I^2 \arrow[d] \arrow[dr] & \\ \operatorname{Sp} A/I \arrow[r] & \operatorname{Sp} A \end{tikzcd}$$

On the other hand, the natural map $A/I^2 \to A/I$ induces a morphism of k_H-affinoid spectra $\varphi : \operatorname{Sp} A/I \to \operatorname{Sp} A/I^2$. From the universal property again, the composition $\psi \circ \varphi$ is the identity. In particular, $A/I^2 \to A/I$ is injective and hence $I = I^2$. It follows that I is the principal ideal generated by an idempotent element e. We may assume that $e \neq 0$, $e \neq 1$. Take $c \in \sqrt{|k^\times| \cdot H}$ such that $0 < c < 1$, then $V = (\operatorname{Sp} A)\{c^{-1}e\}$. □

Corollary 11.6.

Corollary 11.7 (?)**.** Let $\operatorname{Sp} A$ be a k-affinoid spectrum and $\operatorname{Sp} B$ be an affinoid domain. Then for any complete non-Archimedean field extension K/k, any K-affinoid algebra C and any bounded algebra homomorphism $A \to C$ such that $\operatorname{Sp} C \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B$, there is a unique bounded homomorphism $B \to C$ making the following diagram commutes:

$$\begin{tikzcd} \operatorname{Sp} C \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B \arrow[r] & \operatorname{Sp} A \end{tikzcd}$$

Proof. □

The following propositions are a priori not clear with the current definition, we need Gerritzen–Grauert first

Proposition 11.8 (?)**.** Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spectra. Let $V \subseteq \operatorname{Sp} A$ be a k_H-affinoid domain in $\operatorname{Sp} A$, then $\varphi^{-1}(V)$ is a k_H-affinoid domain in $\operatorname{Sp} B$.

In fact, suppose that V is represented by $A \to A_V$, then $B \to B \hat{\otimes}_A A_V$ represents $\varphi^{-1} V$.

Proof. $\square$

Proposition 11.9 (?)**.** Let A be a k_H-affinoid algebra. Let V, W be k_H-affinoid domains in $\operatorname{Sp} A$ represented by $A \to A_V$ and $A \to A_W$ respectively. Then $V \cap W$ is a k_H-affinoid domain represented by $A \to A_V \hat{\otimes}_A A_W$.

Proof. $\square$

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