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Complex analytic local algebras

1. Introduction

In this chapter, we study the notion of complex analytic local algberas (*Analytische Stellenalgebren* in German) in the sense of [GR71]. Most of the materials in this chapter are standard, but the proofs are scattered in tons of papers and books.

Results regarding the completion of the analytic tensor products are certainly known, though the author does not know any written references.

2. Ring of convergent power series

Definition 2.1. For any $n \in \mathbb{N}$, let $\mathbb{C}\{z_1, \dots, z_n\}$ denote the subring of $\mathbb{C}[[z_1, \dots, z_n]]$ consisting of

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},$$

which is convergent in a neighbourhood of 0: there is $\epsilon > 0$ such that for any $x_1, \dots, x_n \in \mathbb{C}$ with $|x_i| < \epsilon$, $\sum_\alpha a_\alpha x^\alpha$ is a convergent power series. We will write $f(0)$ for $a_{0,\dots,0}$.

Definition 2.2. Fix $n \in \mathbb{N}$ and $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{>0}$. For any $f \in \mathbb{C}[[z_1, \dots, z_n]]$ with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},$$

we define

$$\|f\|_t := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| t^\alpha \in [0, \infty].$$

We define

$$\mathbb{C}\langle t^{-1} z \rangle = \mathbb{C}\langle t_1^{-1} z_1, \dots, t_n^{-1} z_n \rangle := \{ f \in \mathbb{C}[[z_1, \dots, z_n]] : \|f\|_t < \infty \}.$$

We write

$$|f| = \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| z^\alpha \in \mathbb{R}[[z_1, \dots, z_n]].$$

The construction here is a special case of Example 4.13 in Banach rings.

Observe that $(\mathbb{C}\langle t^{-1} z \rangle, \| \bullet \|_t)$ is a normed $\mathbb{C}$-vector space. We write

$$\mathbb{C}\{z_1, \dots, z_n\} = \bigcup_{t \in \mathbb{R}^n_{>0}} \mathbb{C}\langle t^{-1} z \rangle. \tag{2.1}$$

We write $\epsilon : \mathbb{C}\{z_1, \dots, z_n\} \to \mathbb{C}$ for the unique $\mathbb{C}$-algebra homomorphism sending z_i to 0 for all $i = 1, \dots, n$. We call this map the *augmentation map* of $\mathbb{C}\{z_1, \dots, z_n\}$.

Proposition 2.3. Let $n \in \mathbb{N}$ and $f \in \mathbb{C}[[z_1, \dots, z_n]]$. Then the following are equivalent:

(1) $f \in \mathbb{C}\{z_1, \ldots, z_n\}$;
(2) There exists $C, N \in \mathbb{R}_{>0}$ such that

$$|f|(x_1, \ldots, x_n) \leq C \sum_{\alpha \in \mathbb{N}^n} N^{|\alpha|} x^\alpha$$

for all $x_1, \ldots, x_n \in \mathbb{R}_{\geq 0}$.

If the condition in (2) is satisfied for a specific C and N, we say that g *admits an upper bound of type* (C, N).

Proposition 2.4 (Cauchy coefficients estimate)**.** Let

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in \mathbb{C}\langle t^{-1} z \rangle.$$

Then for any $\alpha \in \mathbb{N}^n$,

$$|a_\alpha| \leq \frac{\|f\|_t}{t^\alpha}.$$

Proof. This follows from the definition. □

Proposition 2.5. $(\mathbb{C}\langle t^{-1} z \rangle, \| \bullet \|_t)$ is a $\mathbb{C}$-Banach algebra.

Proof. This is a special case of Example 4.13 in Banach rings. □

Lemma 2.6. For any $n \in \mathbb{N}$, an element $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ is a unit if and only if $f(0) \neq 0$.

Proof. In fact, as $\mathbb{C}\{z_1, \ldots, z_n\}$ is a subring of $\mathbb{C}[[z_1, \ldots, z_n]]$, a unit f in $\mathbb{C}\{z_1, \ldots, z_n\}$ is also a unit in $\mathbb{C}[[z_1, \ldots, z_n]]$, hence $f(0) \neq 0$.

Conversely, assume that $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ and $f(0) \neq 0$, then

$$\lim_{t \to 0+} \|1 - f(0)^{-1} f\|_t = 0.$$

So by Proposition 4.4 in the chapter Banach Rings, $a^{-1} f$ is a unit in $\mathbb{C}\langle t^{-1} z \rangle$ when t is small enough. As $\mathbb{C}\langle t^{-1} z \rangle$ is a subring of $\mathbb{C}\{z_1, \ldots, z_n\}$, it follows that $f(0)^{-1} f$ is invertible in $\mathbb{C}\{z_1, \ldots, z_n\}$, hence so is f. □

Theorem 2.7. Let $m, n \in \mathbb{N}$.

(1) The ring $\mathbb{C}\{z_1, \ldots, z_n\}$ is local, and the maximal ideal is given by convergent power series without constant terms.
(2) Any homomorphism of $\mathbb{C}$-algebras $F : \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{w_1, \ldots, w_m\}$ is local.

Proof. (1) This follows from Lemma 2.6 and [Stacks, Tag 00E9].

(2) Suppose it is not the case. As $z_1, \ldots, z_n$ generate the maximal ideal in $\mathbb{C}\{z_1, \ldots, z_n\}$ by (1), at least one of $F(z_1), \ldots, F(z_n)$ is not in the maximal ideal of $\mathbb{C}\{w_1, \ldots, w_m\}$. We may assume that it is $F(z_1)$, namely

$$F(z_1) = a + \mathcal{O}(w)$$

with $a \in \mathbb{C}^\times$. As F is a $\mathbb{C}$-algebra homomorphism, $F(z_1 - a) = F(z_1) - a$ is a unit. This contradicts the fact that $z_1 - a$ is a unit. □

Lemma 2.8. Let $m, n \in \mathbb{N}$ and $g_1, \ldots, g_m \in \mathbb{C}\{w_1, \ldots, w_n\}$. Assume that $g_i(0) = 0$ for all $i = 1, \ldots, m$. Then the formal substitution

$$F : \mathbb{C}[[z_1, \ldots, z_m]] \to \mathbb{C}[[w_1, \ldots, w_n]]$$

sending z_i to g_i restricts to a homomorphism of $\mathbb{C}$-algebras

$$F : \mathbb{C}\{z_1, \ldots, z_m\} \to \mathbb{C}\{w_1, \ldots, w_n\}.$$

PROOF. Fix $t \in \mathbb{R}^m_{>0}$. Take $s \in \mathbb{R}^n_{>0}$ so that $\|g_i\|_s \leq t_i$ for $i = 1, \ldots, m$. This is possible as $g_i(0) = 0$. Then we claim that F sends $\mathbb{C}\langle t^{-1}z\rangle$ to $\mathbb{C}\langle s^{-1}w\rangle$. This implies our lemma.

To prove the assertion, let $f \in \mathbb{C}\langle t^{-1}z\rangle$, which we expand as

$$f = \sum_{\alpha \in \mathbb{N}^m} a_\alpha z^\alpha.$$

Then

$$\|F(f)\|_s = \sum_{j=0}^{\infty} \| \sum_{|\alpha|=j} a_\alpha g^\alpha \|_s \leq \sum_{\alpha} |a_\alpha| \cdot \|g\|_s^\alpha \leq \|f\|_t.$$

Here the first inequality follows from Proposition 2.5. □

Conversely, we have

Lemma 2.9. Let $m, n \in \mathbb{N}$. Then any homomorphism of $\mathbb{C}$-algebras

$$F : \mathbb{C}\{z_1, \ldots, z_m\} \to \mathbb{C}\{w_1, \ldots, w_n\}$$

is the restriction of a substitution homomorphism

$$\mathbb{C}[[z_1, \ldots, z_m]] \to \mathbb{C}[[w_1, \ldots, w_n]].$$

In particular, F is uniquely determined by $F(z_1), \ldots, F(z_m)$.

PROOF. Let $g_i = F(z_i)$. It follows from Theorem 2.7 that $g_i \in \mathfrak{m}$, the maximal ideal in $\mathbb{C}[[w_1, \ldots, w_n]]$. Let $G : \mathbb{C}\{z_1, \ldots, z_m\} \to \mathbb{C}\{w_1, \ldots, w_n\}$ be the substitution homomorphism sending z_i to g_i. Then F and G agrees on $\mathbb{C}[z_1, \ldots, z_m]$. In particular, for any $f \in \mathbb{C}\{z_1, \ldots, z_m\}$ and $a \in \mathbb{N}$ if we write f_1 the sum of the homogeneous parts of f of degree no more than a and $f_2 = f - f_1$, we see that $f_2 \in \mathfrak{m}_1^{a+1}$, where $\mathfrak{m}_1$ is the maximal ideal of $\mathbb{C}\{z_1, \ldots, z_m\}$. It follows that $(F - G)(f_1) = 0$ and $(F - G)(f_2) \in \mathfrak{m}^{a+1}$, the latter is a consequence of Theorem 2.7. As a is arbitrary, we find that

$$(F - G)(f) \in \bigcap_{a=1}^{\infty} \mathfrak{m}^a \subseteq \bigcap_{a=1}^{\infty} \mathfrak{m}_2^a,$$

where $\mathfrak{m}_2$ is the maximal ideal in $\mathbb{C}[[w_1, \ldots, w_n]]$. As $\mathbb{C}[[w_1, \ldots, w_n]]$ is Noetherian, it follows from Krull's intersection theorem [Stacks, Tag 00IP] that $\bigcap_{a=1}^{\infty} \mathfrak{m}_2^a = 0$, so $F = G$. □

We prove a few elementary results about the structure of the ring $\mathbb{C}\{z_1, \ldots, z_n\}$. In the following sections, we will develop deeper structures after developing the Weierstrass theory.

Proposition 2.10. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ is an integral domain.

PROOF. It suffices to observe that $\mathbb{C}\{z_1, \ldots, z_n\}$ is a subring of $\mathbb{C}[[z_1, \ldots, z_n]]$. □

Theorem 2.11. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\{z_1, \dots, z_n\}$ is strictly Henselian.

PROOF. As the residue field of $\mathbb{C}\{z_1, \dots, z_n\}$ is $\mathbb{C}$ by Theorem 2.7, it suffices to show that $\mathbb{C}\{z_1, \dots, z_n\}$ is Henselian. Take a monic polynomial $f \in \mathbb{C}\{z_1, \dots, z_n\}[w]$, say

$$f = w^b + f_1 w^{b-1} + \cdots + f_b, \quad f_i \in \mathbb{C}\{z_1, \dots, z_n\}.$$

Suppose that $\bar{a} \in \mathbb{C}$ is a simple root of $\bar{f} = w^b + f_1(0)w^{b-1} + \cdots + f_b(0)$. We want to lift find $a \in \mathbb{C}\{z_1, \dots, z_n\}$ such that $f(a) = 0$ and $a(0) = \bar{a}$. As $\mathbb{C}$ is algebraically closed, we can prove a stronger result: suppose that

$$\bar{f} = (w - c_1)^{b_1} \cdots (w - c_t)^{b_t}$$

for some $c_i \in \mathbb{C}$ and $b_i \in \mathbb{Z}_{>0}$. Then we claim that there are monic polynomials $g_i \in \mathbb{C}\{z_1, \dots, z_n\}[w]$ such that $\bar{g_i} = (w - c_i)^{b_i}$ for all $i = 1, \dots, t$ and $f = g_1 \cdots g_t$.

We make an induction on t. When $t = 1$, there is nothing to prove, so assume that $t > 1$ and the theorem has been proved for $t-1$. We may assume that $c_1 = 0$. By Weierstrass preparation theorem Theorem 4.3, we can find a Weierstrass polynomial $h \in \mathbb{C}\{z_1, \dots, z_n\}[w]$ of degree b_1 and a monic polynomial $k \in \mathbb{C}\{z_1, \dots, z_n\}[w]$ such that $f = hk$. By the inductive hypothesis, we can find monic polynomials $g_2, \dots, g_t \in \mathbb{C}\{z_1, \dots, z_n\}[w]$ such that $\bar{g_i} = (w - c_i)^{b_i}$ for all $i = 2, \dots, t$ and $k = g_2 \cdots g_t$. It suffices to take $g_1 = h$. □

3. Weierstrass division and excellence of the ring of formal power series

Definition 3.1. Let $n \in \mathbb{N}$ and $f \in \mathbb{C}\{z_1, \dots, z_n\}$, we say f *has order* e *in* z_n for some $e \in \mathbb{N}$ if we expand f as

$$f = \sum_{i=0}^{\infty} f_i z_n^i$$

with $f_i \in \mathbb{C}\{z_1, \dots, z_{n-1}\}$, then $f_0(0) = \cdots = f_{e-1}(0) = 0$ while $f_e(0) \neq 0$.

If $f_i(0) = 0$ for all i, we say f *has order* ∞ *in* z_n. We will write $\operatorname{ord}_{z_n} f$ for the order of f in z_n.

Theorem 3.2 (Weierstrass division theorem)**.** Let $n \in \mathbb{N}$ and $g \in \mathbb{C}\{z_1, \dots, z_n\}$. Assume that $\operatorname{ord}_{z_n} g < \infty$. Then for any $f \in \mathbb{C}\{z_1, \dots, z_n\}$ there is $q \in \mathbb{C}\{z_1, \dots, z_n\}$ and $r \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[w]$ with $\deg_{z_n} r < \operatorname{ord}_{z_n} g$ such that

$$f = qg + r.$$

Moreoer, q and r are uniquely determined.

Remark 3.3. The restriction $\operatorname{ord}_{z_n} g < \infty$ is not too severe. In fact, given any non-zero g, we can always find an invertible $n \times n$ matrix A, so that if we consider gA^{-1} defined in the obvious way, we have $\operatorname{ord}_{z_n} gA^{-1} < \infty$.

PROOF. Fix $\epsilon \in (0, 1)$.

Choose a small enough $t = (t_1, \dots, t_n) \in \mathbb{R}^n_{>0}$ so that $f, g \in \mathbb{C}\langle t^{-1}z \rangle$. This is possible by (2.1). We expand g as

$$g = \sum_{i=0}^{\infty} a_i z_n^i \tag{3.1}$$

with $a_i \in \mathbb{C}\{z_1, \dots, z_{n-1}\}$. We decompose g into

$$g = g_1 + z_n^{\operatorname{ord}_{z_n} g} g_2, \quad g_1 = \sum_{i=0}^{\operatorname{ord}_{z_n} g - 1} a_i z_n^i, \quad g_2 = \sum_{i=\operatorname{ord}_{z_n} g}^{\infty} a_i z_n^{i - \operatorname{ord}_{z_n} g}.$$

Then our assumption implies that g_2 is a unit in $\mathbb{C}\{z_1, \dots, z_n\}$ and by (2.1) again, choosing t small enough, we may assume that $g_2^{-1} \in \mathbb{C}\langle t^{-1}z\rangle$. Then

$$\|gg_2^{-1} - z_n^{\operatorname{ord}_{z_n} g}\|_t = \|g_1 g_2^{-1}\|_t \le \|g_1\|_t \cdot \|g_2^{-1}\|_t.$$

As $\lim_{t\to 0+} \|g_1\|_t = 0$ by our assumption, taking t small enough, we can also guarantee that

$$\|gg_2^{-1} - z_n^{\operatorname{ord}_{z_n} g}\|_t \le t_n^{\operatorname{ord}_{z_n} g} \epsilon.$$

We define $v_j \in \mathbb{C}\langle t^{-1}z\rangle$ for $j \in \mathbb{N}$ as follows: $v_0 = f$ and

$$v_{j+1} = (z_n^{\operatorname{ord}_{z_n} g} - gg_2^{-1}) v_{j,1},$$

where $v_{j,1}$ is defined from v_j in the same way g_1 is defined from g.

Observe that

$$\|v_{j,1}\|_t \le t_n^{-\operatorname{ord}_{z_n} g} \|v_j\|_t.$$

It follows that

$$\|v_{j+1}\|_t \le \epsilon \|v_j\|_t.$$

In particular,

$$w = \sum_{j=0}^{\infty} v_j$$

converges in $\mathbb{C}\langle t^{-1}z\rangle$ by Proposition 2.5. Now we can define $q = g_2^{-1} w_2$ and $r = w_1$. Again w_1, w_2 are defined from w using the same way g_1, g_2 are defined from g. It follows that

$$f = \sum_{j=0}^{\infty} (v_j - v_{j+1}) = \sum_{j=0}^{\infty} (gg_2^{-1} w_2 + w_1) = qg + r.$$

We conclude the existence part.

As for the uniqueness, suppose that $qg + r = 0$ for some $q \in \mathbb{C}\{z_1, \dots, z_n\}$ and $r \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$ with $\deg r < \operatorname{ord}_{z_n} g$. We want to deduce $q = r = 0$. Take $t \in \mathbb{R}^n_{>0}$ small enough, we may assume that $q, g, r \in \mathbb{C}\langle t^{-1}z\rangle$. Expand g as in (3.1), we may assume that $a_{\operatorname{ord}_{z_n} g}^{-1} \in \mathbb{C}\langle t_1^{-1}z_1, \dots, t_{n-1}^{-1} z_{n-1}\rangle$. We can then write

$$a_{\operatorname{ord}_{z_n} g}^{-1} g = z_n^{\operatorname{ord}_{z_n} g} + h$$

for some $h \in \mathbb{C}\langle t^{-1}z\rangle$, $h(0) = 0$. Fix $\epsilon \in (0, 1)$. Choose t small enough, we can then guarantee that

$$\|h\|_t \le t_n^b \epsilon.$$

Now

$$q a_{\operatorname{ord}_{z_n} g} z_n^{\operatorname{ord}_{z_n} g} + r = -qh a_{\operatorname{ord}_{z_n} g}.$$

If we set $M = \|q a_{\operatorname{ord}_{z_n} g}\|_t t_n^{\operatorname{ord}_{z_n} g}$, then we see immediately

$$M = \|q a_{\operatorname{ord}_{z_n} g} z_n^{\operatorname{ord}_{z_n} g}\|_t \le \|q a_{\operatorname{ord}_{z_n} g} z_n^{\operatorname{ord}_{z_n} g} + r\|_t = \|qh a_{\operatorname{ord}_{z_n} g}\|_t \le M\epsilon.$$

It follows that $M = 0$ and hence $q a_{\operatorname{ord}_{z_n} g} = 0$. It follows that $q = 0$ by Proposition 2.10. Therefore, $r = 0$ as well. □

Proposition 3.4. Let $n \in \mathbb{N}$ and $g \in \mathbb{C}\{z_1, \ldots, z_n\}$. Assume that $\operatorname{ord}_{z_n} g < \infty$. Then there is a canonical isomorphism

$$\mathbb{C}\{z_1, \ldots, z_n\}/g\mathbb{C}\{z_1, \ldots, z_n\} \xrightarrow{\sim} \mathbb{C}\{z_1, \ldots, z_{n-1}\}^{\operatorname{ord}_{z_n} g}$$

of $\mathbb{C}\{z_1, \ldots, z_{n-1}\}$-modules.

Proof. Given any $f \in \mathbb{C}\{z_1, \ldots, z_n\}$, we consider the Weierstrass division $f = qg + r$ as in Theorem 3.2. Write

$$r = r_0 + r_1 z_n + \cdots + r_{\operatorname{ord}_{z_n} g - 1} z_n^{\operatorname{ord}_{z_n} g - 1}.$$

Then we map f to $(r_0, \ldots, r_{\operatorname{ord}_{z_n} g - 1})$. Clearly, this defines the isomorphism as in the proposition. □

As an application of Weierstrass theory, we prove a few results about the structure of the ring $\mathbb{C}\{z_1, \ldots, z_n\}$.

Theorem 3.5. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ is a noetherian integral domain.

Proof. We make an induction on n to prove that $\mathbb{C}\{z_1, \ldots, z_n\}$ is noetherian. The case $n = 0$ is trivial. So assume $n > 0$ and that the theorem has been proved for all smaller values of n. Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a non-zero element. It suffices to show that $\mathbb{C}\{z_1, \ldots, z_n\}/f\mathbb{C}\{z_1, \ldots, z_n\}$ is noetherian. By Remark 3.3, we may assume that $\operatorname{ord}_{z_n} f < \infty$. According to Proposition 3.4, we know that

$$\mathbb{C}\{z_1, \ldots, z_n\}/f\mathbb{C}\{z_1, \ldots, z_n\} \cong \mathbb{C}\{z_1, \ldots, z_{n-1}\}^c$$

for some $c \in \mathbb{N}$ as $\mathbb{C}\{z_1, \ldots, z_{n-1}\}$-modules. By the inductive hypothesis and Hilbert basis theorem, the quotient ring is noetherian. □

Proposition 3.6. Let $n \in \mathbb{N}$. Then the $\mathfrak{m}$-adic completion of the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ is canonically isomorphic to $\mathbb{C}[[z_1, \ldots, z_n]]$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\{z_1, \ldots, z_n\}$.

Proof. Let $\mathfrak{m}_1$ be the maximal ideal in $\mathbb{C}[z_1, \ldots, z_n]$. It suffices to observe that we have canonical identifications

$$\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{m}_1^n \xrightarrow{\sim} \mathbb{C}\{z_1, \ldots, z_n\}/\mathfrak{m}^n$$

for any $n \in \mathbb{N}$. So in particular, the $\mathfrak{m}$-adic completion of $\mathbb{C}\{z_1, \ldots, z_n\}$ is isomorphic to the $\mathfrak{m}_1$-adic completion of $\mathbb{C}[z_1, \ldots, z_n]$, which is $\mathbb{C}\{z_1, \ldots, z_n\}$. □

Corollary 3.7. Let $n \in \mathbb{N}$. Then the Krull dimension of $\mathbb{C}\{z_1, \ldots, z_n\}$ is n.

Proof. This follows from Proposition 3.6 and [Stacks, Tag 07NV]. □

Theorem 3.8. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ is regular. In particular, it is a UFD.

Proof. We have computed that the completion of $\mathbb{C}\{z_1, \ldots, z_n\}$ is $\mathbb{C}[[z_1, \ldots, z_n]]$ in Proposition 3.6. The regularity of $\mathbb{C}\{z_1, \ldots, z_n\}$ follows from the regularity of the completion by descent [Stacks, Tag 07NY]. It is a UFD by Auslander–Buchsbaum theorem. □

Proposition 3.9. Let A be a complex analytic local algebra. Then A is an excellent, strictly Henselian.

PROOF. This follows from the corresponding results in Theorem 3.11 and Theorem 2.11. □

We recall the following criterion.

Theorem 3.10. Let k be a field of characteristic 0 and R be a regular ring containing k. Suppose that there is $n \in \mathbb{N}$ such that

(1) for any maixmal ideal $\mathfrak{m}$ of R, the residue $R/\mathfrak{m}$ is algebraic over k and the height of $\mathfrak{m}$ is n;
(2) there exists $D_1, \dots, D_n \in \mathrm{Der}_k(R)$ and $x_1, \dots, x_n \in R$ such that $D_i x_j = \delta_{ij}$ for all $i, j = 1, \dots, n$.

Then R is excellent.

PROOF. We refer to [Mat80, Theorem 102]. □

Theorem 3.11. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\{z_1, \dots, z_n\}$ is excellent.

PROOF. This is an immediate consequence of Theorem 3.10. □

4. Weierstrass preparation theorem

Definition 4.1. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n variables is a monic polynomial

$$\omega = z_n^b + a_1 z_n^{b-1} + \cdots + a_b \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$$

such that $a_i(0) = 0$ for all $i = 0, \dots, b$.

Observe that by definition, $\mathrm{ord}_{z_n} \omega = b$.

Lemma 4.2. Let $\omega \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$ be a Weierstrass polynomial and $g \in \mathbb{C}\{z_1, \dots, z_n\}$. Assume that $\omega g \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$, then $g \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$.

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$, $\deg_{z_n} r < \deg_{z_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of Theorem 3.2, we know that $q = g$, so g is a polynomial in z_n. □

THEOREM 4.3 (Weierstrass preparation theorem). Let $n \in \mathbb{Z}_{>0}$ and $g \in \mathbb{C}\{z_1, \dots, z_n\}$. Assume that $\mathrm{ord}_{z_n} g < \infty$, then there is a unique Weierstrass polynomial $\omega \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$ of degree $\mathrm{ord}_{z_n} g$ and a unit $e \in \mathbb{C}\{z_1, \dots, z_n\}$ such that $g = e\omega$.

Moreover, if $g \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$, then so is e.

PROOF. By Theorem 3.2, we can write

$$z_n^{\mathrm{ord}_{z_n} g} = qg + r$$

for $q \in \mathbb{C}\{z_1, \dots, z_n\}$ and $r \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$ with $\deg_{z_n} r < \mathrm{ord}_{z_n} g$. Write $g = \sum_{i=0}^{\infty} a_i z_n^i$ for some $a_i \in \mathbb{C}\{z_1, \dots, z_{n-1}\}$. Define $\bar{g} = \sum_{i=0}^{\infty} a_i(0) z_n^i$, set $e' := z_n^{-\mathrm{ord}_{z_n} g} \bar{g} \in \mathbb{C}[z_n]$. Then $e'(0) \neq 0$. Similarly define $\bar{q}$ and $\bar{r}$, then we have

$$z_n^{\mathrm{ord}_{z_n} g} = \bar{q}\bar{g} + \bar{r}.$$

Or

$$1 = \bar{q}e' + \bar{r}.$$

From the uniqueness part of Theorem 3.2, we conclude that $\bar{q} = e'^{-1}$, namely q is a unit. Now

$$g = q^{-1}(z_n^{\mathrm{ord}_{z_n} g} - r)$$

is the desired decomposition. When $g \in \mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]$, so is e, as can be seen from Lemma 4.2.

It remains to prove the uniqueness: if e is a unit in $\mathbb{C}\{z_1, \ldots, z_n\}$, ω, ω' are two Weierstrass polynomials and $e\omega = \omega'$, then we need to show that $e = 1$. It follows from Lemma 4.2 that e is a polynomial in z_n. Setting $z_1 = \cdots = z_{n-1} = 0$, we find that $\bar{e}$ is a power of z_n. As e is a unit, it follows that $\bar{e} = 1$. On the other hand, clearly e is a monic polynomial, it follows that $e = 1$. □

Definition 4.4. Let $n \in \mathbb{Z}_{>0}$ and $g \in \mathbb{C}\{z_1, \ldots, z_n\}$. Assume that $\mathrm{ord}_{z_n} g < \infty$. The Weierstrass polynomial constructed in Theorem 4.3 is called the *Weierstrass polynomial* of g.

Corollary 4.5. Let $n \in \mathbb{Z}_{>0}$ and $g \in \mathbb{C}\{z_1, \ldots, z_n\}$. Assume that $\mathrm{ord}_{z_n} g < \infty$. Let ω be the Weierstrass polynomial of g. Then the injection

$$\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n] \to \mathbb{C}\{z_1, \ldots, z_n\}$$

induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n]/\omega\mathbb{C}\{z_1, \ldots, z_{n-1}\}[z_n] \to \mathbb{C}\{z_1, \ldots, z_n\}/g\mathbb{C}\{z_1, \ldots, z_n\}.$$

Proof. The morphism is surjective by Theorem 3.2 and injective by Lemma 4.2. □

5. Complex analytic local algebras

Definition 5.1. A *complex analytic local algebra* is a $\mathbb{C}$-algebra A such that $A \neq 0$ and there exists some $n \in \mathbb{N}$ and an ideal I in $\mathbb{C}\{z_1, \ldots, z_n\}$ such that

$$A \cong \mathbb{C}\{z_1, \ldots, z_n\}/I$$

as $\mathbb{C}$-algebras. The augmentation map of $\mathbb{C}\{z_1, \ldots, z_n\}$ induces a $\mathbb{C}$-algebra homomorphism $\epsilon : A \to \mathbb{C}$ called the *augmentation map* of A.

A morphism between complex analytic local algebras A and B is a $\mathbb{C}$-algebra homomorphism $A \to B$.

The category of complex analytic local algebras is denoted by $\mathbb{C}$-$\mathcal{L}$A.

Proposition 5.2. Let A be an object in $\mathbb{C}$-$\mathcal{L}$A and $f : B \to C$ be a morphism in $\mathbb{C}$-$\mathcal{L}$A. Then

(1) A is local with residue field $\mathbb{C}$.
(2) f is a local homomorphism.

Proof. (1) This follows immediately from Theorem 2.7.
(2) This follows from the same arguments as Theorem 2.7 (2). □

Observe that a complex analytic local algebra is always local with residue field $\mathbb{C}$ and a morphism in $\mathbb{C}$-$\mathcal{L}$A is always a local homomorphism. We will write $\mathfrak{m}_A$ for the maximal ideal in A.

Lemma 5.3. Let A be a complex analytic local algebra and $n \in \mathbb{N}$, then there is a natural bijection

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\{z_1, \ldots, z_n\}, A) \cong \mathfrak{m}_A^n$$

sending a morphism f to $(f(z_1), \ldots, f(z_n))$.

PROOF. As a morphism $f : \mathbb{C}\{z_1, \dots, z_n\} \to A$ is necessarily local, we see that $f(z_i) \in A$ for all $i = 1, \dots, n$. So the map $\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\{z_1, \dots, z_n\}, A) \to \mathfrak{m}_A^n$ is well-defined. Conversely, given $w_1, \dots, w_n \in \mathfrak{m}_A$, we claim that there is a unique morphism $f : \mathbb{C}\{z_1, \dots, z_n\} \to A$ in $\mathbb{C}$-$\mathcal{L}\mathrm{A}$ sending z_i to w_i.

The uniqueness follows from Lemma 2.9, so let us consider only the existence. Let $\mathbb{C}\{z_1, \dots, z_m\} \to A$ be a surjective morphism. Lift w_i to $w_i' \in \mathbb{C}\{z_1, \dots, z_m\}$, it suffices to construct a morphism $\mathbb{C}\{z_1, \dots, z_n\} \to \mathbb{C}\{z_1, \dots, z_m\}$ sending z_i to w_i'. So we may assume that $A = \mathbb{C}\{z_1, \dots, z_m\}$. In this case, the result follows from Lemma 2.8. □

Theorem 5.4. Let $\varphi : A \to B$ be a morphism of complex analytic local algebras. Then the following are equivalent:

(1) B is a quasi-finite over A;
(2) B is finite over A.

Reproduce the part on SGA1 about quasi-finite noetherian local ring homomorphisms

PROOF. (2) $\implies$ (1): This is trivial.

(1) $\implies$ (2): It suffices to show that $\mathfrak{m}_A B$ is an ideal of definition of B.

Step 1. We reduce to the case where $A = \mathbb{C}\{t_1, \dots, t_m\}$ and $B = \{x_1, \dots, x_n\}$ for some $m, n \in \mathbb{N}$.

Write $A = \mathbb{C}\{t_1, \dots, t_m\}/I$ for some $m \in \mathbb{N}$ and some ideal I. Then B is finite (resp. quasi-finite) over A if and only if it is finite (resp. quasi-finite) over $\mathbb{C}\{t_1, \dots, t_m\}$. So we may assume that $A = \mathbb{C}\{t_1, \dots, t_m\}$.

Write $B = \mathbb{C}\{x_1, \dots, x_n\}/(f_1, \dots, f_p)$ for some $n, p \in \mathbb{N}$ and $f_1, \dots, f_p \in \mathbb{C}\{x_1, \dots, x_n\}$. Let $u_1, \dots, u_m$ be the images of $t_1, \dots, t_m$ in B. By assumption, $(u_1, \dots, u_m)$ is an ideal of definition of B. Lift $u_1, \dots, u_m$ to $v_1, \dots, v_m \in \mathbb{C}\{x_1, \dots, x_n\}$. We define a $\mathbb{C}$-algebra homomorphism

$$\Phi : \mathbb{C}\{t_1, \dots, t_m, s_1, \dots, s_p\} \to \mathbb{C}\{x_1, \dots, x_n\}$$

sending t_i to v_i and s_j to f_j for $i = 1, \dots, m$ and $j = 1, \dots, p$. The existence of Φ is guaranteed by Lemma 5.3.

We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}\{t_1, \dots, t_m, s_1, \dots, s_p\} & \xrightarrow{\Phi} & \mathbb{C}\{x_1, \dots, x_n\} \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi} & B \end{array} .$$

In order to show that φ is finite, it suffices to show that Φ is. But we know that Φ is also quasi-finite, so we have finished the reduction.

Step 2. Assume that we have done the reduction as in Step 1. We will prove the theorem.

Take $r \in \mathbb{Z}_{>0}$ so that

$$\mathfrak{m}_B^r \subseteq \mathfrak{m}_A B.$$

For $i = 1, \dots, n$, we can then write

$$x_i^r = \sum_{j=1}^{m} u_j \lambda_{ji}, \quad \lambda_{1i}, \dots, \lambda_{mi} \in B, \tag{5.1}$$

where u_j is the image of t_j in B for $j = 1, \dots, m$.

We claim that x^α with $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ satisfying $\alpha_j < r$ for all $j = 1, \dots, m$ generates B as A-module. We temporarily write $\sum'_{\alpha \in \mathbb{N}^m}$ for summation over such indices.

Let

$$f = \sum_{\alpha \in \mathbb{N}^m} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{C}$$

be a general element in B. We want to express f as an A-linear combination of the desired basis. Let

$$\rho(f) := \sum_{\alpha \in \mathbb{N}^m}{}' a_\alpha x^\alpha.$$

Then we can write

$$\begin{aligned} f =& \rho(f) + \sum_{i=1}^n x_i^r \sigma_i(f), \\ \sigma_i(f) =& \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m \\ \alpha_1<r, \quad ,\alpha_{i-1}<r, \alpha_i \geq r}} a_\alpha x_1^{\alpha_1} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - r} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \text{ for } i = 1, \dots, n. \end{aligned}$$

Substituting (5.1), we find

$$f = \rho(f) + \sum_{j=1}^m u_j s_j(f), \quad s_j(f) = \sum_{i=1}^n \lambda_{ji} \sigma_i(f) \text{ for } j = 1, \dots, m. \tag{5.2}$$

Applying (5.2) to $s_j(f)$, we obtain that

$$s_j(f) = \rho(s_j(f)) + \sum_{j'=1}^m u_{j'} s_{j'}(s_j(f))$$

for $j = 1, \dots, m$. Substituting back to (5.2), we find

$$f = \rho(f) + \sum_{j=1}^m u_j \rho(s_j(f)) + \sum_{j_1, j_2 = 1}^m u_{j_1} u_{j_2} s_{j_1}(s_{j_2}(f)).$$

By iterating the same procedure, we find that for each $p \in \mathbb{N}$,

$$f = \rho(f) + \sum_{j_1=1}^m u_{j_1} \rho(s_{j_1}(f)) + \cdots + \sum_{j_1, \dots, j_p = 1}^m u_{j_1} \cdots u_{j_p} \rho(s_{j_1} \cdots s_{j_p}(f)) + R_p,$$

where

$$R_p = \sum_{j_1, \dots, j_p, j_{p+1} = 1}^m u_{j_1} \cdots u_{j_{p+1}} s_{j_1} \cdots s_{j_p}(f) \in \mathfrak{m}_B^{p+1}.$$

Now we can write $f - R_p$ as

$$\sum_{\alpha \in \mathbb{N}^m}{}' u_\alpha^{(p)}(f) x^\alpha, \quad u_\alpha^{(p)}(f) \in \mathbb{C}[u_1, \dots, u_m]$$

is a polynomial of degree p. Moreover, we can lift $u_\alpha^{(p)}(f)$ to a polynomial $v_\alpha^{(p)}(f)$ of degree p in $t_1, \dots, t_m$ and $v_\alpha^{(p+1)}(f) - v_\alpha^{(p)}(f)$ is homogeneous of degree $p+1$. So we can define

$$v_\alpha(f) := \lim_{p \to \infty} v_\alpha^{(p)}(f)$$

in $\mathbb{C}[[t_1,\dots,t_m]]$ and

$$f=\sum_{\alpha\in\mathbb{N}^m}{}'\varphi(v_\alpha^{(p)}(f))x^\alpha$$

in $\mathbb{C}[[x_1,\dots,x_n]]$. It suffices to show that $v_\alpha(f)\in A$.

We first reduce to the case λ_{ji} admits an upper bound of type $(1,1)$ for all $j=1,\dots,m$, $i=1,\dots,n$.

We can take $(C,N)\in\mathbb{Z}^2_{>0}$ so that λ_{ji} admits an upper bound of type (C,N) for all $j=1,\dots,m$, $i=1,\dots,n$. Define automorphisms $\psi:B\to B$ and $\chi:A\to A$ by

$$\psi(x_i)=N^{-1}x_i,\quad \chi(t_j)=CN^rt_j$$

for all $j=1,\dots,m$, $i=1,\dots,n$. Let $\varphi':A\to B$ be $\psi\circ\varphi\circ\chi$. Denote $u'_j=\varphi'(t_j)$ for $j=1,\dots,m$ and

$$x_i^r=\sum_{j=1}^m u'_j\lambda'_{ji}$$

for all $i=1,\dots,n$. Then λ'_{ji} admits an upper bounded of type $(1,1)$. So we can deduce the results for φ from the results for φ'.

Now after the reduction, we observe that for $c_1,\dots,c_n\in\mathbb{R}_{\geq 0}$,

$$|s_j(f)|(c_1,\dots,c_n)\leq\sum_{i=1}^n|\lambda_{ji}|\cdot|\sigma_i(f)|\leq\sum_{i=1}^n\left(\sum_{\gamma\in\mathbb{N}^n}c^\gamma\sum_{\beta\in\mathbb{N}^n//\beta_1,\dots,\beta_{i-1}<r,\beta_i\geq r}|a_\beta|c^\beta c_i^{-r}\right)$$

for $j=1,\dots,m$. Take $(D,M)\in\mathbb{Z}^2_{>0}$ such that f admits an upper bound of type (D,M). Then we can futher bound the coefficients of c^α in $|s_j(f)|(c_1,\dots,c_n)$ by

$$\sum_{i=1}^m\sum_{\substack{r\leq\beta_i\leq\alpha_i+r\\0\leq\beta_q\leq\alpha_q\text{ for }q\neq i}}|a_\beta|\leq nD\left(M^{|\alpha|+r}+nM^{|\alpha|+r-1}+\cdots+n^{|\alpha|}M^r\right)=DnM^r\frac{M^{|\alpha|+1}-n^{|\alpha|_1}}{M-n}.$$

We may take $M\geq n+1$, so that the last expression is bounded from above by $DnM^{r+1+|\alpha|}$. Then we find that $s_j(f)$ admits an upper bound of type (DnM^{r+1},M) for all $j=1,\dots,m$. So $s_{j_1}\cdots s_{j_p}(f)$ admits an upper bound of type $((DnM^{r+1})^p,M)$ for all $p\in\mathbb{N}$ and $j_1,\dots,j_p=1,\dots,m$. It follows that $v_\alpha(f)$ admits an upper bound of type $(DM^{|\alpha|},nM^{r+1})$ for all $\alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m$, $\alpha_i=0,\dots,r-1$ for $i=1,\dots,m$. We conclude by Proposition 2.3.

□

Corollary 5.5. Let A be a complex local analytic algebra, $n\in\mathbb{N}$ and $x_1,\dots,x_n$ a system of parameters in A. Let $\varphi:\mathbb{C}\{X_1,\dots,X_n\}\to A$ be the $\mathbb{C}$-algebra homomorphism sending X_i to x_i for $i=1,\dots,n$. Then

(1) φ is surjective if and only if $x_1,\dots,x_n$ generates $\mathfrak{m}_A$;
(2) φ is injective if and only if $\dim A=n$;
(3) φ is bijective if and only if A is a regular local ring and $x_1,\dots,x_n$ is a regular system of parameters.

The existence of φ is guaranteed by Lemma 5.3.

Proof. Observe that A is quasi-finite over $\mathbb{C}\{X_1,\dots,X_n\}$, hence finite by Theorem 5.4.

(1) The direct implication is trivial. Assume that $x_1, \dots, x_n$ generates $\mathfrak{m}_A$. Then φ is surjective modulo the maximal ideal of $\mathbb{C}\{X_1, \dots, X_n\}$. It follows from Nakayama's lemma that φ is surjective.

(2) As φ is finite, it is integral. By going up, φ is injective if and only if $\dim \mathbb{C}\{X_1, \dots, X_n\} = \dim A$.

(3) This follows from (1) and (2). □

Lemma 5.6. Let A be an integral complex analytic local algebra and B be a finite torsion-free A-algebra. Then for any maximal ideal $\mathfrak{m}$ of B, $B_{\mathfrak{m}}$ is equidimensional of dimension $\dim A$.

Proof. By Theorem 2.11, $B_{\mathfrak{m}}$ is finite over A by [Stacks, Tag 04GG](10). It is clearly torsion-free as A-module. So we may assume that B is local. We will identify A with a subring of B.

Let $\mathfrak{p}$ be a minimal prime ideal of B and $\mathfrak{q} = \mathfrak{p} \cap A$. Then $B_{\mathfrak{p}}$ is finite torsion-free over $A_{\mathfrak{q}}$ and

$$\mathfrak{p}B_{\mathfrak{p}} \cap A_{\mathfrak{q}} = \mathfrak{q}A_{\mathfrak{q}}$$

is the maximal ideal in $A_{\mathfrak{q}}$. By going-up, $\mathfrak{p}B_{\mathfrak{p}}$ is the maximal ideal in $B_{\mathfrak{p}}$. As it is clearly minimal, we find that $\dim B_{\mathfrak{p}} = 0$. So $\dim A_{\mathfrak{q}} = 0$. So $\mathfrak{p}$ is necessarily the minimal prime ideal 0 in A. Namely, $\mathfrak{p} \cap A = 0$. But then $B/\mathfrak{p}$ is finite over A and $A \to B/\mathfrak{p}$ is injective. In particular, $\dim B/\mathfrak{p} = \dim A$. □

Definition 5.7. Let A_1, A_2 be complex analytic local algebras, *an analytic tensor product* of A_1 and A_2 is a complex analytic local algebra A together with morphisms $A_1 \to A$ and $A_2 \to A$ such that for any complex analytic local algebra C, the induced map

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A, C) \to \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_1, C) \times \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_2, C)$$

is bijective.

As analytic tensor product is unique up to unique isomorphism, so we can choose a specific analytic tensor product $A_1 \overline{\otimes} A_2$ and call it *the analytic tensor product* of A_1 and A_2.

More generally, let B be a complex analytic local algebra and $A_1, A_2 \in \mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\setminus B}$ (the under-slice of B). *An analytic tensor product* of A_1 and A_2 over B is a complex analytic local algebra A over B together with morphisms $A_1 \to A$ and $A_2 \to A$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\setminus B}$ such that the induced map

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A, C) \to \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\setminus B}}(A_1, C) \times \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\setminus B}}(A_2, C)$$

is bijective.

As analytic tensor product is unique up to unique isomorphism, so we can choose a specific analytic tensor product $A_1 \overline{\otimes}_B A_2$ and call it *the analytic tensor product* of A_1 and A_2 over B.

By definition, there are natural morphisms

$$A_1 \otimes A_2 \to A_1 \overline{\otimes} A_2.$$

The simplest example is

Lemma 5.8. For any $m, n \in \mathbb{N}$, we have

$$\mathbb{C}\{z_1, \dots, z_m\} \overline{\otimes} \mathbb{C}\{z_1, \dots, z_n\} \cong \mathbb{C}\{z_1, \dots, z_{m+n}\}$$

as complex analytic local algebras.

PROOF. This is a simple consequence of Lemma 5.3 and Yoneda's lemma. □

Lemma 5.9. Assume that $f_i : A_i \to B_i$ are surjective (i.e. the underlying homomorphisms of algebras are surjective) morphisms in $\mathbb{C}$-$\mathcal{L}$A for $i = 1, 2$. Let I_i be the kernel of f_i as homomorphisms of algebras. If $A_1 \overline{\otimes} A_2$ exists, then so is $B_1 \overline{\otimes} B_2$ and

$$B_1 \overline{\otimes} B_2 \cong A_1 \overline{\otimes} A_2 / (I_1 \otimes 1 + 1 \otimes I_2)(A_1 \overline{\otimes} A_2).$$

PROOF. That $A_1 \overline{\otimes} A_2 / (I_1 \otimes 1 + 1 \otimes I_2)(A_1 \overline{\otimes} A_2)$ is a complex analytic local algebra follows from our assumption. That it represents $B_1 \overline{\otimes} B_2$ follows from general abstract nonsense. Include details □

Corollary 5.10. Let A_1, A_2 be complex analytic local algebras, then $A_1 \overline{\otimes} A_2$ exists.

PROOF. This is a consequence of Lemma 5.9 and Lemma 5.8. □

Theorem 5.11. Let A_1, A_2 be complex analytic local algebras, then there are natural isomorphisms

$$A_1 \hat{\otimes}_{\mathbb{C}} A_2 \xrightarrow{\sim} (A_1 \overline{\otimes} A_2)\hat{}.$$

Here on the right-hand side, we take the adic completion with respect to the maximal ideal in $A_1 \overline{\otimes} A_2$.

PROOF. Observe that the existence of a morphism $A_1 \hat{\otimes} A_2 \to (A_1 \overline{\otimes} A_2)\hat{}$ follows from the universal property.

When A_1 and A_2 are both rings of convergent power series, this result follows from Lemma 5.8 and Proposition 3.6.

In general, represent

$$A_1 = \mathbb{C}\{z_1, \ldots, z_m\}/I_1, \quad A_2 = \mathbb{C}\{w_1, \ldots, w_n\}/I_2.$$

Then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}\{z_1, \ldots, z_m\} \hat{\otimes}_{\mathbb{C}} \mathbb{C}\{w_1, \ldots, w_n\} & \longrightarrow & (\mathbb{C}\{z_1, \ldots, z_m\} \overline{\otimes} \mathbb{C}\{w_1, \ldots, w_n\})\hat{} \\ \downarrow & & \downarrow \\ A_1 \hat{\otimes}_{\mathbb{C}} A_2 & \longrightarrow & (A_1 \overline{\otimes} A_2)\hat{} \end{array}.$$

We already know tha the upper arrow is an isomorphism, it suffices to identify the kernels of the two vertical maps. The kernel of the right vertical map before completion is given by Lemma 5.9, namely, the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$. Accordingly, the kernel of the right vertical map is the closure of the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$ by [Stacks, Tag 0ARZ]. On the other hand, it follows from [Stacks, Tag 0ARZ] that the kenrel of the left vertical map is the closure of the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$. We conclude. □

Theorem 5.12. Let B be a complex analytic local algebra and $A_1, A_2 \in \mathbb{C}$-$\mathcal{L}\mathrm{A}_{\setminus B}$. Then $A_1 \overline{\otimes}_B A_2$ exists. Moreover, there is a natural identification of adic rings

$$A_1 \hat{\otimes}_B A_2 \xrightarrow{\sim} (A_1 \overline{\otimes}_B A_2)\hat{}.$$

Here on the right-hand side, we take the adic completion with respect to the maximal ideal in $A_1 \overline{\otimes}_B A_2$.

PROOF. Observe that we have a natural map $B\overline{\otimes}B \to B$: at the level of functor of points, we simply define

$$h^B \to h^{B\overline{\otimes}B}$$

by sending an arrow $f : B \to C$ in $\mathbb{C}$-$\mathcal{L}$A to $(f, f) \in h^{B\overline{\otimes}B}(C)$. We claim that $B\overline{\otimes}B \to B$ is surjective. In fact, it is easy to construct a section $B \to B\overline{\otimes}B$, which at the level of functor of points, sends a pair of morphisms $(f_1 : B \to C, f_2 : B \to C)$ in $\mathbb{C}$-$\mathcal{L}$A to f_1.

It follows from general abstract nonsense that the tensor product

$$(A_1\overline{\otimes}A_2)\overline{\otimes}_{B\overline{\otimes}B}B$$

represents $A_1\overline{\otimes}_B A_2$. So we are reduced to the case where $B \to A_2$ is surjective. Let I denote the kernel of the map $B \to A_2$. We denote by J the image of I in A_1. It is obvious that A_1/JA_1 is the desired tensor product.

In order to compute the completed local ring, we similarly reduce to the case where $B \to A_2$ is surjective with kernel I. In this case, $A_1\overline{\otimes}_B A_2$ is the quotient of A_1 by the ideal generated by I. So after taking completion, $(A_1\overline{\otimes}_B A_2)\hat{}$ is the quotient of $\hat{A}_1$ by the closed ideal generated by I by [Stacks, Tag 0ARZ]. On the other hand, $A_1\hat{\otimes}_B A_2$ is also the quotient of $\hat{A}_1$ by the closed ideal generated by I by [Stacks, Tag 0AMZ]. We conclude. □

Remark 5.13. One should remark that in general, the completed tensor products of local rings are not local.

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