Affinoid algebras

Contents

1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. We set

$$\begin{aligned} k\{r^{-1}T\} =& k\{r_1^{-1}T_1, \dots, r_n T_n^{-1}\} \\ :=& \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\}. \end{aligned}$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$\|f\|_r = \max_\alpha |a_\alpha| r^\alpha.$$

We call $(k\{r^{-1}T\}, \|\bullet\|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\|\bullet\|_r$ is called the *Gauss norm.*

We omit r from the notation if $r = (1, \dots, 1)$.

This is a special case of Example 4.11 in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach k-algebra and $\|\bullet\|_r$ is a valuation.

PROOF. This is a special case of Proposition 4.12 in the chapter Banach Rings. $\square$

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T\rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T\rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\} \cong k[T_1, \dots, T_n]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$ otherwise.

3. Affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

An affinoid k-algebra is a K-affinoid algebra for some complete non-Archimedean field extension K/k.

For the notion of admissible morphisms, we refer to Definition 2.5 in the chapter Banach rings.

Example 3.2. Let $r \in \mathbb{R}_{>0}$. We let K_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \to 0$ for $i \to \infty$ and $i \to -\infty$. We define a norm $\| \bullet \|_r$ on K_r as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in Proposition 3.3 that K_r is k-affinoid.

Proposition 3.3. Let $r \in \mathbb{R}_{>0}$, then $(K_r, \| \bullet \|_r)$ defined in Example 3.2 is a k-affinoid algebra. Moreover, $\| \bullet \|_r$ is a valuation.

Proof. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \to K_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$|a_{i,j}| r^{i-j} \to 0 \tag{3.1}$$

as $i + j \to \infty$. Observe that for each $k \in \mathbb{Z}$, the series

$$c_k := \sum_{i-j=k, i,j \in \mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that $\iota(f) \in K_r$. That is

$$|c_k| r^k \to 0$$

as $k \to \pm\infty$. When $k \geq 0$, we have $|c_k| \leq |a_{k0}|$ by definition of c_k. So $|c_k| r^k \to 0$ as $k \to \infty$ by (3.1). The case $k \to -\infty$ is similar.

We conclude that we have a well-defined map of sets ι. It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g = \sum_{i=-\infty}^{\infty} c_i T^i \in K_r$. We want to show that g lies in the image of ι. As ι is a ring homomorphism, it suffices to treat two cases separately: $g = \sum_{i=0}^{\infty} c_i T^i$ and $g = \sum_{i=-\infty}^{0} c_i T^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$. It is immediate that $\iota(f) = g$.

Next we show that ι is admissible. We first identify the kernel of ι. We claim that the kenrel is the ideal I generated by $T_1T_2 - 1$. It is obvious that $I \subseteq \ker \iota$. Conversely, consider an element

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kenrel of ι. Observe that

$$f=\sum_{k=-\infty}^{\infty} f_k, \quad f_k=\sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j}T_1^iT_2^j.$$

If $f\in\ker\iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by T_1T_2-1. The condition that $f_k\in\ker\iota$ means

$$\sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j}=0.$$

It is elementary to find $b_{i,j}\in k$ for $i,j\in\mathbb{N}$, $i-j=k$ such that

$$a_{i,j}=b_{i-1,j-1}-b_{i,j}.$$

Then

$$f_k=(T_1T_2-1)\sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j}T_1^iT_2^j.$$

Observe that we can make sure that $|b_{i,j}|\leq\max\{|a_{i',j'}|: i-j=i'-j'\}$. In particular, the sum of $\sum_{i,j\in\mathbb{N},i-j=k} b_{i,j}T_1^iT_2^j$ for various k converges to some $g\in k\{r^{-1}T_1,rT_2\}$ and hence $f_k=(T_1T_2-1)g$. Therefore, we have proved that $\ker\iota$ is generated by T_1T_2-1.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1,rT_2\}/I\to K_r.$$

To see this, take $f=\sum_{k=-\infty}^{\infty} c_kT^k\in K_r$ and we need to show that

$$\|f\|_r=\inf\{\|g\|_{(r,r^{-1})}:\iota(g)=f\}.$$

Observe that if we set $g=\sum_{k=0}^{\infty} c_kT_1^k+\sum_{k=1}^{\infty} c_{-k}T_2^k$, then $\iota(g)=f$ and $\|g\|_{(r,r^{-1})}=\|f\|$. So it suffices to show that for any $h=\sum_{(i,j)\in\mathbb{N}^2} d_{i,j}T_1^iT_2^j\in k\{r^{-1}T_1,rT_2\}$, we have

$$\|f\|_r\leq\|g+h(T_1T_2-1)\|_{r,r^{-1}}. \tag{3.2}$$

We compute

$$g+h(T_1T_2-1)=\sum_{k=1}^{\infty}(c_k-d_{k,0})T_1^k+\sum_{k=1}^{\infty}(c_{-k}-d_{0,k})T_2^k+(c_0-d_0)+\sum_{i,j\geq1}(d_{i-1,j-1}-d_{i,j})T_1^iT_2^j.$$

So

$$\|g+h(T_1T_2-1)\|_{r,r^{-1}}=\max\left\{\max_{k\geq0} C_{1,k},\max_{k\geq1} C_{2,k}\right\},$$

where

$$C_{1,k}=\max\left\{|c_k-d_{k,0}|,\left|\sum_{i-j=k,i,j\geq1} d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k\geq0$ and

$$C_{2,k}=\max\left\{|c_{-k}-d_{0,k}|,\left|\sum_{i-j=-k,i,j\geq1} d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k\geq1$. It follows from the strong triangle inequality that $|c_k|\leq C_{1,k}$ for $k\geq0$ and $c_{-k}\leq C_{2,k}$ for $k\geq1$. So (3.2) follows. □

Proposition 3.4. Let $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$, then $\|\bullet\|_r$ defined in Example 3.2 is a valuation on K_r.

Proof. Take $f, g \in K_r$, we need to show that

$$\|fg\|_r \geq \|f\|_r \|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_i T^i, \quad g = \sum_{i=-\infty}^{\infty} b_i T^i.$$

Take i and j so that

$$|a_i| r^i = \|f\|_r, \quad |b_j| r^j = \|g\|_r. \tag{3.3}$$

By our assumption on r, i, j are unique. Then

$$\|fg\|_r = \max_{k \in \mathbb{Z}} \{|c_k| r^k\},$$

where

$$c_k := \sum_{u,v \in \mathbb{Z}, u+v=k} a_u b_v.$$

It suffices to show that

$$|c_k| r^k = \|f\|_r \|g\|_r. \tag{3.4}$$

for $k = i + j$. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. For $u, v \in \mathbb{Z}$, $u + v = i + j$ while $(u, v) \neq (i, j)$, we may assume that $u \neq i$. Then $|a_u| r^u < |a_i| r^i$ and $|b_v| r^v \leq |b_j| r^j$. So $|a_u b_v| < |a_i b_j|$ and we conclude (3.4). □

Remark 3.5. The argument of Proposition 4.12 in the chapter Banch Rings does not work here if $r \in \sqrt{|k^\times|}$, as in general one can not take minimal i, j so that (3.3) is satisfied.

Proposition 3.6. Assume that $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$. Then K_r is a valuation field and $\|\bullet\|_r$ is non-trivial.

Proof. We first show that $\operatorname{Sp} K_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet| \in \operatorname{Sp} K_r$. As $\|\bullet\|_r$ is a valuation, we find

$$|\bullet| \leq \|\bullet\|_r. \tag{3.5}$$

In particular, $|\bullet|$ restricted to k is the given valuation on k. It suffices to show that $|T| = r$. This follows from (3.5) applied to T and T^{-1}.

It follows that K_r does not have any non-zero proper closed ideals: if I is such an ideal, K_r/I is a Banach k-algebra. By Proposition 6.2 in the chapter Banach rings, $\operatorname{Sp} K_r$ is non-empty. So K_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.5 in the chapter Banach rings, the only maximal ideal of K_r is 0. It follows that K_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r = r$. □

Definition 3.7. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Assume that $r_1, \dots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$. We define

$$K_r = K_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k K_{r_n}.$$

By an interated application of Proposition 3.6, K_r is a complete valuation field.

4. Properties of affinoid algebras

Bibliography

[Ber12] V. G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. 33. American Mathematical Soc., 2012.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. Non-Archimedean analysis. Vol. 261. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436. URL: https://doi.org/10.1007/978-3-642-52229-1.