

Banach rings

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1. Introduction

This section concerns the theory of Banach algebras. Our references are [Ber12] and [BGR84].

In this chapter, all rings are assumed to be commutative.

2. Semi-normed Abelian groups

Definition 2.1. Let A be an Abelian group. A *semi-norm* on A is a function $\|\bullet\| : A \rightarrow [0, \infty]$ satisfying

- (1) $\|0\| = 0$;
- (2) $\|f - g\| \leq \|f\| + \|g\|$ for all $f, g \in A$.

A semi-norm $\|\bullet\|$ on A is a *norm* if moreover the following conditions is satisfied:

- (0) if $\|f\| = 0$ for some $f \in A$, then $f = 0$.

We write

$$\ker \|\bullet\| = \{a \in A : \|a\| = 0\}.$$

A semi-norm $\|\bullet\|$ on A is *non-Archimedean* or *ultra-metric* if Condition (2) can be replaced by

$$(2') \quad \|f - g\| \leq \max\{\|f\|, \|g\|\} \text{ for all } f, g \in A.$$

Definition 2.2. A *semi-normed Abelian group* (resp. *normed Abelian group*) is a pair $(A, \|\bullet\|)$ consisting of an Abelian group A and a semi-norm (resp. norm) $\|\bullet\|$ on A . When $\|\bullet\|$ is clear from the context, we also say A is a semi-normed Abelian group (resp. normed Abelian group).

Definition 2.3. Let $(A, \|\bullet\|_A)$ be a semi-normed Abelian group and $B \subseteq A$ be a subgroup. Then we define the *quotient semi-norm* $\|\bullet\|_{A/B}$ on A/B as follows:

$$\|a + B\|_{A/B} := \inf\{\|a + b\|_A : b \in B\}$$

for all $a + B \in A/B$.

We define the *subgroup semi-norm* on B as follows:

$$\|b\|_B = \|b\|_A$$

for all $b \in B$.

Definition 2.4. Let A be an Abelian group and $\|\bullet\|, \|\bullet\|'$ be two seminorms on A . We say $\|\bullet\|$ and $\|\bullet\|'$ are *equivalent* if there is a constant $C > 0$ such that

$$C^{-1}\|f\| \leq \|f\|' \leq C\|f\|$$

for all $f \in A$.

Definition 2.5. Let $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$ be semi-normed Abelian groups. A homomorphism $\varphi : A \rightarrow B$ is said to be

- (1) *bounded* if there is a constant $C > 0$ such that $\|\varphi(f)\|_B \leq C\|f\|_A$ for any $f \in A$;
- (2) *admissible* if the quotient semi-norm on $A/\ker \varphi$ is equivalent to the subspace semi-norm on $\text{Im } \varphi$.

Observe that an admissible homomorphism is always bounded.

Next we study the topology defined by a semi-norm.

Lemma 2.6. Let $(A, \|\bullet\|)$ be a semi-normed Abelian group. Define

$$d(a, b) = \|a - b\|$$

for $a, b \in A$. Then $\|\bullet\|$ is a pseudo-metric on A . This pseudo-metric is a metric if and only if $\|\bullet\|$ is a norm.

Let \hat{A} be the metric completion of A , then there is a norm $\|\bullet\|$ on \hat{A} inducing its metric. Moreover, the natural homomorphism $A \rightarrow \hat{A}$ is an isometric homomorphism with dense image.

PROOF. This is clear from the definitions. \square

We always endow A with the topology induced by the pseudo-metric d .

Proposition 2.7. Let $f : A \rightarrow B$ be a homomorphism between semi-normed Abelian groups. Assume that f is bounded, then it is continuous.

The converse is not true.

PROOF. Clear from the definition. \square

Proposition 2.8. Let $(A, \|\bullet\|)$ be a normed Abelian group and B be a subgroup of A . Assume that there is $\epsilon \in (0, 1)$ such that for each $a \in A$, there is $b \in B$ such that

$$\|a + b\| \leq \epsilon \|a\|.$$

Then B is dense in A .

PROOF. Assume to the contrary that there exists $a \in A$ so that

$$c := \inf_{b \in B} \|a - b\| > 0.$$

Choose $b_1 \in B$ so that

$$\|a + b_1\| < \epsilon^{-1}c.$$

By our hypothesis, there is $b_2 \in B$ such that

$$\|a + b_1 + b_2\| \leq \epsilon \|a + b_1\| < c.$$

This is a contradiction. \square

Definition 2.9. Let $(A, \|\bullet\|)$ be a semi-normed Abelian group. The normed Abelian group $(\hat{A}, \|\bullet\|)$ constructed in [Lemma 2.6](#) is called the *completion* of $(A, \|\bullet\|)$.

3. Semi-normed rings

Definition 3.1. Let A be a ring. A *semi-norm* $\|\bullet\|$ on A is a semi-norm $\|\bullet\|$ on the underlying additive group satisfying the following extra properties:

- (3) $\|1\| = 1$;
- (4) for any $f, g \in A$, $\|fg\| \leq \|f\| \cdot \|g\|$.

A semi-norm $\|\bullet\|$ on A is called *power-multiplicative* if $\|f\|^n = \|f^n\|$ for all $f \in A$ and $n \in \mathbb{N}$.

A semi-norm $\|\bullet\|$ on A is called *multiplicative* if $\|fg\| = \|f\|\|g\|$ for all $f, g \in A$.

Definition 3.2. A *semi-normed ring* (resp. *normed ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring A and a semi-norm (resp. norm) $\|\bullet\|$ on A . When $\|\bullet\|$ is clear from the context, we also say A is a semi-normed ring (resp. normed ring).

Definition 3.3. Let $(A, \|\bullet\|)$ be a semi-normed ring. An element $a \in A$ is *multiplicative* if $a \notin \ker \|\bullet\|$ and for any $x \in A$,

$$\|ax\| = \|a\| \cdot \|x\|.$$

Definition 3.4. Let $(A, \|\bullet\|)$ be a normed ring. An element $a \in A$ is *power-bounded* if $\{|a^n| : n \in \mathbb{N}\}$ is bounded in \mathbb{R} . The set of power-bounded elements in A is denoted by \mathring{A} .

An element $a \in A$ is called *topologically nilpotent* if $a^n \rightarrow 0$ as $n \rightarrow \infty$. The set of topologically nilpotent elements in A is denoted by \check{A} .

Proposition 3.5. Let $(A, \|\bullet\|)$ be a non-Archimedean normed ring. Then \mathring{A} is a subring of A and \check{A} is an ideal in \mathring{A} . Moreover, \mathring{A} , \check{A} are open and closed in A .

PROOF. Choose $a, b \in \mathring{A}$, by definition, there is a constant $C > 0$ so that for any $n \in \mathbb{N}$,

$$\|a^n\| \leq C, \quad \|b^n\| \leq C.$$

It follows that

$$\|(ab)^n\| \leq \|a^n\| \cdot \|b^n\| \leq C^2$$

and

$$\|(a-b)^n\| \leq \max_{i=0, \dots, n} \|a^i b^{n-i}\| \leq C^2.$$

So \mathring{A} is a subring.

Next we show that \check{A} is an ideal in \mathring{A} . On the other hand, take $c \in \check{A}$, then

$$\|(ac)^n\| \leq \|a^n\| \cdot \|c^n\| \leq C\|c^n\|$$

But $\|c^n\| \rightarrow 0$ as $n \rightarrow \infty$, hence $ac \in \check{A}$.

On the other hand, consider $c, d \in \check{A}$, we need to show $c-d \in \check{A}$. Choose $C > 0$ so that

$$\|a^n\| \leq C, \quad \|b^n\| \leq C$$

for all $n \in \mathbb{N}$. Fix $\epsilon > 0$, then there is $m \in \mathbb{N}$ so that for any $k \geq m$,

$$\|a^k\| \leq \epsilon C^{-1}, \quad \|b^k\| \leq \epsilon C^{-1}.$$

In particular, for $k \geq 2m$, we have

$$\|(a-b)^k\| \leq \max_{i=0, \dots, k} \|a^i\| \cdot \|b^{k-i}\| \leq \epsilon.$$

It follows that $a-b \in \check{A}$. This proves that \check{A} is an ideal in \mathring{A} .

In order to see \check{A} is open and closed in A , observe that it is a subgroup of A , so it suffices to show that \check{A} is open in A . It suffices to show that

$$\{a \in A : \|a\| < 1\} \subseteq \check{A}.$$

But this is obvious, if $\|a\| < 1$, then $\|a^n\| \leq \|a\|^n$ for all $n \in \mathbb{N}$, it follows that $a^n \rightarrow 0$ as $n \rightarrow \infty$, namely, $a \in \check{A}$.

As \check{A} is a subgroup of \mathring{A} , it follows that \mathring{A} is both open and closed. \square

Definition 3.6. Let $(A, \|\bullet\|)$ be a non-Archimedean normed ring. We define the *reduction* of A as $\tilde{A} = \mathring{A}/\check{A}$. The map $\mathring{A} \rightarrow \tilde{A}$ is called the *reduction map*. We usually denote the reduction map by $a \mapsto \tilde{a}$.

This definition makes sense thanks to [Proposition 3.5](#).

Definition 3.7. Let A be a ring. A *semi-valuation* on A is a multiplicative semi-norm on A . A semi-valuation on A is a *valuation* on A if its underlying semi-norm of Abelian groups is a norm.

Definition 3.8. A *semi-valued ring* (resp. *valued ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring A and a semi-valuation (resp. valuation) $\|\bullet\|$ on A . When $\|\bullet\|$ is clear from the context, we also say A is a semi-valued ring (resp. valued ring).

A semi-valued ring (resp. valued ring) $(A, \|\bullet\|)$ is called a *semi-valued field* (resp. *valued field*) if A is a field.

4. Banach rings

Definition 4.1. A *Banach ring* is a normed ring that is complete with respect to the metric defined in [Lemma 2.6](#).

Definition 4.2. A Banach ring $(A, \|\bullet\|_A)$ is *uniform* if $\|\bullet\|_A$ is power-multiplicative.

Definition 4.3. Let A be a semi-normed ring. There is an obvious ring structure on the completion \hat{A} of A defined in [Definition 2.9](#). We call the resulting Banach ring the *completion* of A .

Proposition 4.4. Let $(A, \|\bullet\|)$ be a Banach ring and $f \in A$. Assume that $\|f\| < 1$, then $1 - f$ is invertible.

PROOF. Define

$$g = \sum_{i=0}^{\infty} f^i.$$

From our assumption, the series converges and $g \in A$. It is elementary to check that g is the inverse of $1 - f$. \square

In the non-Archimedean case, we have a stronger result:

Proposition 4.5. Let $(A, \|\bullet\|)$ be a non-Archimedean Banach ring and $f \in \check{A}$. Then $1 - f$ is invertible. Moreover, $(1 - f)^{-1}$ can be written as $1 + z$ for some $z \in \check{A}$.

PROOF. Define

$$g = \sum_{i=0}^{\infty} f^i.$$

From our assumption, the series converges and $g \in A$. It is elementary to check that g is the inverse of $1 - f$. Moreover, in view of [Proposition 3.5](#) as for any $i \geq 1$, $f^i \in \check{A}$, the same holds for their sum, we conclude the final assertion. \square

Corollary 4.6. Let $(A, \|\bullet\|)$ be a Banach ring. Then the set of invertible elements in A is open.

PROOF. Let $x \in A$ be an invertible element. It suffices to show that for any $y \in A$, $|y| < 1/(\|x^{-1}\|)$, $y + x$ is invertible. For this purpose, it suffices to show that $1 + x^{-1}y$ is invertible. But this follows from [Proposition 4.4](#). \square

Corollary 4.7. Let A be a Banach ring and \mathfrak{m} be a maximal ideal in A . Then \mathfrak{m} is closed.

PROOF. The closure $\bar{\mathfrak{m}}$ is obviously an ideal in A . We need to show that $\mathfrak{m} \neq \bar{\mathfrak{m}}$. Namely, 1 is not in the closure of \mathfrak{m} . But clearly, \mathfrak{m} is contained in the set of non-invertible elements, the latter being closed by [Corollary 4.6](#). So we conclude. \square

Lemma 4.8. Let A be a non-Archimedean Banach ring. An element $a \in \mathring{A}$ is a unit in \mathring{A} if and only if \tilde{a} is a unit in \tilde{A} .

PROOF. The direct implication is trivial. Conversely, assume that $a \in \mathring{A}$ and there is an element $b \in \mathring{A}$ such that

$$\tilde{a}\tilde{b} = 1.$$

Then $1 - ab \in \check{A}$. It follows from [Proposition 4.5](#) that ab is a unit in \mathring{A} and hence a is a unit in \mathring{A} . \square

Definition 4.9. Let $(A, \|\bullet\|)$ be a Banach ring. We define the *spectral radius* $\rho = \rho_A : A \rightarrow [0, \infty)$ as follows:

$$\rho(f) = \inf_{n \geq 1} \|f^n\|^{1/n}, \quad f \in A.$$

Lemma 4.10. Let $(A, \|\bullet\|)$ be a Banach ring. Then for any $f \in A$, we have

$$\rho(f) = \lim_{n \rightarrow \infty} \|f^n\|^{1/n}.$$

PROOF. This follows from the multiplicative version of Fekete's lemma. \square

Example 4.11. The ring \mathbb{C} with its usual norm $|\bullet|$ is a Banach ring. In fact, $(\mathbb{C}, |\bullet|)$ is a complete valued field.

Example 4.12. Let $\{(A_i, \|\bullet\|_i)\}_{i \in I}$ be a family of Banach rings. We define their *product* $\prod_{i \in I} A_i$ as the following Banach ring: as a set it consists of all elements $f = (f_i)_{i \in I}$ with

$$\|f\| := \sup_{i \in I} \|f_i\|_i < \infty.$$

The norm is given by $\|\bullet\|$. It is easy to verify that $\prod_{i \in I} A_i$ is indeed a Banach ring.

Example 4.13. For any Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$, we define $A\langle r^{-1}z \rangle = A\langle r_1^{-1}z_1, \dots, r_n^{-1}z_n \rangle$ as the subring of $A[[z_1, \dots, z_n]]$ consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in A$$

such that

$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha < \infty.$$

We will verify in [Proposition 4.14](#) that $(A\langle r^{-1}z \rangle, \|\bullet\|_r)$ is a Banach ring.

When $r = (1, \dots, 1)$, we omit r^{-1} from our notations.

Proposition 4.14. In the setting of [Example 4.13](#), $(A\langle r^{-1}z \rangle, \|\bullet\|_r)$ is a Banach ring.

PROOF. By induction, we may assume that $n = 1$.

It is obvious that $\|\bullet\|_r$ is a norm on the underlying Abelian group. To see that $\|\bullet\|_r$ is a norm on the ring $A\langle r^{-1}z \rangle$, we need to verify the condition in [Definition 3.1](#). Condition (3) in [Definition 3.1](#) is obvious. Let us consider Condition (4). Let

$$f = \sum_{i=0}^{\infty} a_i z^i, \quad g = \sum_{j=0}^{\infty} b_j z^j$$

be two elements in $A\langle r^{-1}z \rangle$. Then

$$fg = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} a_i b_j \right) z^k.$$

We compute

$$\|fg\|_r = \sum_{k=0}^{\infty} \left\| \sum_{i+j=k} a_i b_j \right\| r^k \leq \sum_{k=0}^{\infty} \left(\sum_{i+j=k} \|a_i\| \cdot \|b_j\| \right) r^k = \|f\|_r \cdot \|g\|_r.$$

It remains to verify that $A\langle r^{-1}z \rangle$ is complete.

For this purpose, take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b z^i \in A\langle r^{-1}z \rangle$$

for $b \in \mathbb{N}$. Then for each i , the coefficients $(a_i^b)_b$ is a Cauchy sequence in A . Let a_i be the limit of a_i^b as $b \rightarrow \infty$ and set

$$f = \sum_{i=0}^{\infty} a_i z^i \in A[[z]].$$

We need to show that $f \in A\langle r^{-1}z \rangle$ and $f^b \rightarrow f$.

Fix a constant $\epsilon > 0$. There is $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$, we have

$$\sum_{i=0}^{\infty} \|a_i^{j+k} - a_i^j\| r^i < \epsilon/2.$$

In particular, for any $s > 0$, we have

$$\sum_{i=0}^s \|a_i - a_i^j\| r^i \leq \sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i + \sum_{i=0}^s \|a_i^j - a_i^{j+k}\| r^i \leq \sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i + \epsilon/2.$$

When k is large enough, we can guarantee that

$$\sum_{i=0}^s \|a_i - a_i^{j+k}\| r^i < \epsilon/2.$$

So

$$\sum_{i=0}^s \|a_i - a_i^j\| r^i \leq \epsilon.$$

Let $s \rightarrow \infty$, we find

$$\|f - f^j\|_r \leq \sum_{i=0}^{\infty} \|a_i - a_i^j\| r^i \leq \epsilon.$$

In particular, $\|f\|_r < \infty$ and $f^j \rightarrow f$ as $j \rightarrow \infty$. \square

Example 4.15. For any non-Archimedean Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$, we define $A\{r^{-1}T\} = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ as the subring of $A[[T_1, \dots, T_n]]$ consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} T^{\alpha}, \quad a_{\alpha} \in A$$

such that $\|a_\alpha\|r^\alpha \rightarrow 0$ as $|\alpha| \rightarrow \infty$. We set

$$\|f\|_r := \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\|r^\alpha.$$

We will verify in [Proposition 4.16](#) that $(A\langle r^{-1}T \rangle, \|\bullet\|_r)$ is a Banach ring.

The semi-norm $\|\bullet\|_r$ is called the *Gauss norm*.

Proposition 4.16. In the setting of [Example 4.15](#), $(A\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach ring.

Moreover, if the norm $\|\bullet\|$ on A is a valuation, so is $\|\bullet\|_r$.

The second part is usually known as the *Gauss lemma*.

PROOF. By induction on n , we may assume that $n = 1$.

The proof of the fact that $\|\bullet\|_r$ is a norm is similar to that of [Proposition 4.14](#). We leave the details to the readers.

Next we argue that $(A\{r^{-1}T\}, \|\bullet\|_r)$ is complete. Take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b T^i \in A\{r^{-1}T\}$$

for $b \in \mathbb{N}$. As

$$\|a_i^b - a_i^{b'}\|r^i \leq \|f^b - f^{b'}\|_r$$

for any $i, b, b' \geq 0$, it follows that for any $i \geq 0$, $\{a_i^b\}_b$ is a Cauchy sequence. Let $a_i \in A$ be its limit and set

$$f = \sum_{i=0}^{\infty} a_i T^i \in A[[T]].$$

We need to show that $f \in A\{r^{-1}T\}$ and $f^b \rightarrow f$.

Fix $\epsilon > 0$. We can find $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$,

$$\|f^j - f^{j+k}\|_r \leq \epsilon.$$

It follows that $\|a_i^j - a_i^{j+k}\|r^i \leq \epsilon$ for all $i \geq 0$. Let $k \rightarrow \infty$, we find

$$\|a_i^j - a_i\|r^i \leq \epsilon$$

for all $i \geq 0$. Fix $j \geq 0$, take i large enough so that $|a_i^j|r^i < \epsilon$. Then $\|a_i\|r^i \leq \epsilon$. So we find $f \in A\{r^{-1}T\}$. On the other hand,

$$\|f - f^j\|_r = \max_i \|a_i^j - a_i\|r^i \leq \epsilon.$$

This proves that $f^j \rightarrow f$.

Now assume that $\|\bullet\|$ is a valuation, we verify that $\|\bullet\|_r$ is also a valuation. Again, we may assume that $n = 1$. Take two elements $f, g \in A\{r^{-1}T\}$:

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad g = \sum_{j=0}^{\infty} b_j T^j.$$

As we have already shown $|fg|_r \leq |f|_r |g|_r$, it suffices to check the reverse inequality. For this purpose, choose the minimal indices i, j so that

$$\|f\|_r = \|a_i\|r^i, \quad \|g\|_r = \|b_j\|r^j.$$

Write

$$fg = \sum_{k=0}^{\infty} \left(\sum_{p+q=k} a_p b_q \right) T^k.$$

Then we claim that

$$\left\| \sum_{p+q=k} a_p b_q \right\| r^k = \|f\|_r \|g\|_r$$

when $k = i + j$. This implies the desired inequality. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. To verify our claim, it suffices to observe that for $(p, q) \neq (i, j)$, $r + s = i + j$, say $p < i$ and $q > j$, we have

$$\|a_p b_q\| r^k = \|a_p\| r^p \cdot \|b_q\| r^q < \|a_i\| r^i \cdot \|b_j\| r^j.$$

So

$$\|a_p b_q\| < \|a_i b_j\|.$$

Since the valuation on A is non-Archimedean, it follows that

$$\left\| \sum_{p+q=k} a_p b_q \right\| = \|a_i b_j\|.$$

Our claim follows. \square

Proposition 4.17. Let A, B be a non-Archimedean Banach ring and $f : A \rightarrow B$ be a continuous homomorphism. Then for any $b \in \mathring{B}$, there is a unique continuous homomorphism $F : A\{T\} \rightarrow B$ extending f and sending T to b .

PROOF. From the continuity and the fact that $A[T]$ is dense in $A\{T\}$, F is clearly unique. To prove the existence, we define F directly: consider $g = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}$, we define

$$F(g) := \sum_{i=0}^{\infty} f(a_i) f^i.$$

As $f_i \in \mathring{A}$ and $a_i \rightarrow 0$, the right-hand side is well-defined. It is straightforward to check that F is a continuous homomorphism. \square

Proposition 4.18. For any non-Archimedean Banach ring $(A, \|\bullet\|)$, we have

$$(A\{T\})^\circ = \mathring{A}\{T\}, \quad (A\{T\})^\check{} = \check{A}\{T\}.$$

For the definitions of $\mathring{\bullet}$ and $\check{\bullet}$, we refer to [Definition 3.4](#).

PROOF. We first show that

$$\mathring{A}\{T\} \subseteq (A\{T\})^\circ.$$

Let $f \in \mathring{A}\{T\}$. We expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \mathring{A}.$$

Then for each $i, j \in \mathbb{N}$, $\|a_i T^i\|_1^j = \|a_i\|^j$. So for each $i \in \mathbb{N}$, $a_i T^i \in (A\{T\})^\circ$. By [Proposition 3.5](#), it follows that $f \in (A\{T\})^\circ$.

Next we prove the reverse inclusion. Take $f \in (A\{T\})^\circ$, suppose by contrary that $f \notin \mathring{A}\{T\}$. Expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal $m \in \mathbb{N}$ so that $a_m \notin \mathring{A}$. Then $\sum_{i=0}^{m-1} a_i T^i \in \mathring{A}\{T\} \subseteq (A\{T\})^\circ$ by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})^\circ.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any $j \in \mathbb{N}$. It follows that $a_m \in \mathring{A}$, which is a contradiction.

Next we show that

$$\check{A}\{T\} \subseteq (A\{T\})^\vee.$$

Let $f \in \check{A}\{T\}$. We expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \check{A}.$$

Then for each $i, j \in \mathbb{N}$, $\|a_i T^i\|_1^j = \|a_i\|^j$. So for each $i \in \mathbb{N}$, $a_i T^i \in (A\{T\})^\vee$. By [Proposition 3.5](#), it follows that $f \in (A\{T\})^\vee$.

Conversely, take $f \in (A\{T\})^\vee$, suppose by contrary that $f \notin \check{A}\{T\}$. Expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal $m \in \mathbb{N}$ so that $a_m \notin \check{A}$. Then $\sum_{i=0}^{m-1} a_i T^i \in \check{A}\{T\} \subseteq (A\{T\})^\vee$ by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})^\vee.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any $j \in \mathbb{N}$. It follows that $a_m \in \check{A}$, which is a contradiction. \square

Corollary 4.19. For any non-Archimedean Banach ring $(A, \|\bullet\|)$, we have a canonical isomorphism

$$\widetilde{A\{T\}} \cong \check{A}[T].$$

The natural map $A\{T\}^\circ \rightarrow \widetilde{A\{T\}}$ corresponds to a homomorphism $\mathring{A}\{T\} \rightarrow \check{A}[T]$ extending the homomorphism $\mathring{A} \rightarrow \check{A}$ and sending T to T .

PROOF. Let $f = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}^\circ$. Then $a_i \in \mathring{A}$ by [Proposition 4.18](#). But $\|a_i\| \rightarrow 0$ as $i \rightarrow \infty$, so $a_i \in \check{A}$ for almost all i . It follows that the image of f in $\widetilde{A\{T\}}$ is the same as the image of an element from $\mathring{A}[T]$. On the other hand, for each $f \in \check{A}[T]$, we can expand $f = a_N T^N + \cdots + a_1 T^1 + a_0$ with $a_N \in \check{A}$. Lift each a_i to $b_i \in \mathring{A}$. Then the image of $b_N T^N + \cdots + b_1 T^1 + b_0$ under the reduction corresponds to f . The assertions follow. \square

Corollary 4.20. Let $(A, \|\bullet\|)$ be a non-Archimedean Banach ring. An element $f = \sum_{i=0}^{\infty} a_i T^i \in \mathring{A}\{T\}$ is a unit in $\mathring{A}\{T\}$ if and only if a_0 is a unit in \mathring{A} and $a_i \in \mathring{A}$ for all $i > 0$.

PROOF. By Proposition 4.16, we know that $A\{T\}$ is complete. By Lemma 4.8 and Proposition 4.18, f is a unit in $\mathring{A}\{T\}$ if and only if $\sum_{i=0}^{\infty} \tilde{a}_i T^i$ is a unit in $\tilde{A}[T]$. By Lemma 4.8 again, a_0 is a unit in A if and only if \tilde{a}_0 is a unit in \tilde{A} . So we are reduced to argue that units in $\tilde{A}[T]$ are exactly units in \tilde{A} . This follows from the general fact about units in polynomial rings over a reduced ring. \square

The lemma needs to be places elsewhere.

Lemma 4.21. Let R be a commutative ring. A polynomial $a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is a unit if and only if a_0 is a unit in R and a_1, \dots, a_n are nilpotents.

5. Semi-normed modules

Definition 5.1. Let $(A, \|\bullet\|_A)$ be a normed ring. A *semi-normed A -module* (resp. *normed A -module*) is a pair $(M, \|\bullet\|_M)$ consisting of a A -module M and a semi-norm (resp. norm) on the underlying Abelian group of M such that there is a constant $C > 0$ such that

$$\|fm\|_M \leq C\|f\|_A\|m\|_M$$

for all $f \in A$ and $m \in M$. In case $\|\bullet\|_A$ is non-Archimedean, we require that $\|\bullet\|_M$ is also non-Archimedean.

We say the semi-normed A -module (resp. normed A -module) M is faithful if we can take $C = 1$.

When $\|\bullet\|_M$ is clear from the context, we say M is a semi-normed A -module (resp. normed A -module).

An A -module homomorphism $\varphi : M \rightarrow N$ between two semi-normed A -modules M and N is *bounded* if the homomorphism of the underlying semi-normed Abelian groups is bounded in the sense of Definition 2.5.

A *Banach A -module* is a normed A -module which is complete with respect to the metric Lemma 2.6.

We denote by $\mathcal{B}an_A$ the category of Banach A -modules with bounded A -module homomorphisms as morphisms.

Definition 5.2. Let A be a Banach ring and $(M, \|\bullet\|_M), (N, \|\bullet\|_N)$ be two Banach A -modules. Define their *direct sum* as the Banach A -module $(M \oplus N, \|\bullet\|_{M \oplus N})$, where for $m \in M, n \in N$, we set

$$\|(m, n)\|_{M \oplus N} := \max\{\|m\|_M, \|n\|_N\}.$$

This definition extends immediately to finite direct sums of Banach A -modules.

Definition 5.3. Let A be a Banach ring. A Banach A -module M is said to be *finite* if there is $n \in \mathbb{N}$ and an admissible epimorphism $A^n \rightarrow M$.

A morphism between finite A modules M and N is a morphism $M \rightarrow N$ in $\mathcal{B}an_A$. We write $\mathcal{B}an_A^f$ for the category of finite Banach A -modules.

Definition 5.4. Let A be a semi-normed ring and M be a semi-normed A -module. There is an obvious \hat{A} -module structure on the completion \hat{M} of A defined in Definition 2.9. We call the resulting Banach module the *completion* of M .

Definition 5.5. Let A be a non-Archimedean semi-normed ring. Consider semi-normed A -modules $(M, \|\bullet\|_M)$ and $(N, \|\bullet\|_N)$. We define the *tensor product* of $(M, \|\bullet\|_M)$ and $(N, \|\bullet\|_N)$ as the semi-normed A -module $(M \otimes N, \|\bullet\|_{M \otimes N})$, where

$$\|x\|_{M \otimes N} = \inf \max_i (\|m_i\|_M \cdot \|n_i\|_N),$$

where the infimum is taken over all decompositions $x = \sum_i m_i \otimes n_i$.

Definition 5.6. Let A be a non-Archimedean Banach ring. Consider semi-normed A -modules M and N , we define the *complete tensor product* of M and N as the metric completion $M \hat{\otimes}_A N$ of the tensor product of M and N defined in [Definition 5.5](#).

Theorem 5.7. Let $(A, \|\bullet\|_A)$ be a normed ring. Then $\mathcal{B}an_A$ is a quasi-Abelian category.

PROOF. We first observe that $\mathcal{B}an_A$ is preadditive, as for any $M, N \in \mathcal{B}an_A$, $\text{Hom}_{\mathcal{B}an_A}(M, N)$ can be given the group structure inherited from the Abelian group $\text{Hom}_A(M, N)$. It is obvious that $\mathcal{B}an_A$ is preadditive.

Next we show that finite biproducts exist in $\mathcal{B}an_A$. Given $(M, \|\bullet\|_M), (N, \|\bullet\|_N) \in \mathcal{B}an_A$, we set

$$(5.1) \quad (M, \|\bullet\|_M) \oplus (N, \|\bullet\|_N) := (M \oplus N, \|\bullet\|_{M \oplus N}),$$

where $\|(m, n)\|_{M \oplus N} := \|m\|_M + \|n\|_N$ for $m \in M$ and $n \in N$. It is easy to verify that this gives the biproduct in $\mathcal{B}an_A$.

We have shown that $\mathcal{B}an_A$ is an additive category.

Next given a morphism $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$ in $\mathcal{B}an_A$, we construct its kernel $(\ker \varphi, \|\bullet\|_{\ker \varphi})$ as the kernel of the underlying homomorphism of A -modules of φ endowed with the subgroup semi-norm induced from $\|\bullet\|_M$ as in [Definition 2.3](#). It is easy to verify that $(\ker \varphi, \|\bullet\|_{\ker \varphi})$ is the kernel of φ in $\mathcal{B}an_A$.

We can similarly construct the cokernels. To be more precise, let $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$ be a morphism in $\mathcal{B}an_A$, then the coker $\varphi = \{N/\overline{\varphi(M)}\}$ with quotient norm.

We have shown that $\mathcal{B}an_A$ is a pre-Abelian category.

Observe that given a morphism $\varphi : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$ in $\mathcal{B}an_A$, its image is given by $\text{Im } \varphi = \overline{\varphi(M)}$ with the subspace norm induced from N ; its coimage is $M/\ker \varphi$ with the residue norm. The morphism φ is admissible if the natural map

$$M/\ker \varphi \rightarrow \overline{\varphi(M)}$$

is an isomorphism in $\mathcal{B}an_A$.

It remains to show that pull-backs preserve admissible epimorphisms and pushouts preserve admissible monomorphisms. We first handle the case of admissible epimorphisms. Consider a Cartesian square in $\mathcal{B}an_A$:

$$\begin{array}{ccc} M & \xrightarrow{p} & U \\ \downarrow q & \square & \downarrow f \\ V & \xrightarrow{g} & W \end{array}$$

with g being an admissible epimorphism. We need to show that p is also an admissible epimorphism, namely $U \cong M/\ker p$.

We define $\alpha : U \oplus V \rightarrow W$, $\alpha = (f, -g)$, then there is a natural isomorphism $j : M \rightarrow \ker \alpha$. Let us write $i : \ker \alpha \rightarrow U \oplus V$ the natural morphism. Then

$$q = \pi_V \circ i \circ j, \quad p = \pi_U \circ i \circ j,$$

where $\pi_U : U \oplus V \rightarrow U$, $\pi_V : U \oplus V \rightarrow V$ are the natural morphisms. We may assume that $M = \ker \alpha$ and j is the identity. Then it is obvious that p is surjective on the underlying sets. In order to compute the quotient norm on $M/\ker p$, we need a more explicit description of $\ker p \subseteq \ker \alpha$. We know that

$$\ker \alpha = \{(u, v) \in U \oplus V : f(u) = g(v)\}$$

with the subspace norm induced from the product norm on $U \oplus V$ defined in (5.1). Then

$$\ker p = \{(u, v) \in U \oplus V : u = 0, g(v) = 0\}.$$

It follows that for $(u, v) \in \ker \alpha$,

$$\inf_{(u', v') \in \ker p} \|(u, v) + (u', v')\|_{U \oplus V} = \inf_{v' \in \ker g} (\|v + v'\|_V) + \|u\|_U,$$

where $\|\bullet\|_U$ and $\|\bullet\|_V$ denote the norms on U and V respectively. By our assumption that g is an admissible epimorphism, there is a constant $C > 0$ so that

$$\inf_{v' \in \ker g} (\|v + v'\|_V) \leq C\|g(v)\|_W$$

for any $v \in V$. As f is bounded, we can also find a constant $C' > 0$ so that for any $(u, v) \in \ker \alpha$,

$$\|g(v)\|_W = \|f(u)\|_W \leq C'\|u\|_U.$$

It follows that p is admissible epimorphism.

It remains to check that the pushforwards preserve admissible monomorphisms. Consider a co-Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & U \\ \downarrow f & & \downarrow q \\ V & \xrightarrow{p} & M \end{array}$$

with g being an admissible monomorphism. We need to show that p is an admissible monomorphism. This boils down to the following: p is injective with closed image and the norms on $p(V)$ obtained in the obvious ways are equivalent. As in the case of pull-backs, we may let $\alpha : W \rightarrow U \oplus V$ be the morphism $(g, -f)$ and assume that $M = \operatorname{coker} \alpha$. It is then easy to see that p is injective. The proof that the two norms on $p(V)$ are equivalent is parallel to the argument in the pull-back case and we omit it.

It remains to verify that $p(V)$ is closed in W . Consider the admissibly coexact sequence in $\mathcal{B}\mathbf{an}_A$:

$$W \xrightarrow{\alpha} U \oplus V \xrightarrow{\pi} M \rightarrow 0.$$

It is also admissibly coexact in the category of semi-normed A -modules. **Include details later.** Let $x_n \in V$ be a sequence so that $p(x_n) \rightarrow y \in M$. We may write $y = \pi(u, v)$ for some $(u, v) \in U \oplus V$. Then

$$\pi(-u, x_n - v) \rightarrow 0$$

as $n \rightarrow \infty$. From the strict coexact sequence, we can find a sequence $w_n \in W$ so that

$$(-u - g(w_n), x_n - v + f(w_n)) \rightarrow 0$$

as $n \rightarrow \infty$. Then $g(w_n) \rightarrow -u$ in U and hence there is $w \in W$ so that $w_n \rightarrow w \in W$ and $g(w) = -u$. But then $x_n \rightarrow x$ and $p(x) = y$. \square

Definition 5.8. Let $(A, \|\bullet\|_A)$ be a normed ring. A *Banach A -algebra* is a pair $(B, \|\bullet\|_B)$ such that $(B, \|\bullet\|_B)$ is a Banach A -module and $(B, \|\bullet\|_B)$ is a Banach ring.

A morphism of Banach A -algebras is a bounded A -algebra homomorphism. The category of Banach A -algebras is denoted by \mathcal{BanAlg}_A .

Definition 5.9. Let A be a normed ring. A Banach A -algebra B is said to be *finite* if B is finite as a Banach A -module. A morphism of finite Banach A -algebras is a morphism in \mathcal{BanAlg}_A . The category of finite Banach A -algebras is denoted by \mathcal{BanAlg}_A^f .

6. Berkovich spectra

Definition 6.1. Let $(A, \|\bullet\|_A)$ be a Banach ring. A semi-norm $|\bullet|$ on A is *bounded* if there is a constant $C > 0$ such that for any $f \in A$, $|f| \leq C\|f\|_A$.

We write $\mathrm{Sp} A$ for the set of bounded semi-valuations on A . We call $\mathrm{Sp} A$ the *Berkovich spectrum* of A .

We endow $\mathrm{Sp} A$ with the weakest topology such that for each $f \in A$, the map $\mathrm{Sp} A \rightarrow \mathbb{R}_{\geq 0}$ sending $\|\bullet\|$ to $\|f\|$ is continuous.

It is sometimes preferable to denote an element $\|\bullet\|$ in $\mathrm{Sp} A$ by a single letter x . In this case, we write $|f(x)| = \|f\|$ for any $f \in A$.

Given a bounded homomorphism $\varphi : A \rightarrow B$ of Banach rings, we define $\mathrm{Sp} \varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ as follows: given a bounded semi-valuation $\|\bullet\|$ on B , we define $\mathrm{Sp} \varphi(\|\bullet\|)$ as the bounded semi-valuation on A sending $f \in A$ to $\|\varphi(f)\|$.

Observe that there is a natural map of sets:

$$(6.1) \quad \mathrm{Sp} A \rightarrow \{\mathfrak{p} \in \mathrm{Spec} A : \mathfrak{p} \text{ is closed.}\}$$

sending each bounded semi-valuation to its kernel. The fiber over a closed ideal $\mathfrak{p} \in \mathrm{Spec} A$ is identified with the set of bounded valuations on A/\mathfrak{p} . Here boundedness is with respect to the residue norm.

Remark 6.2. In the literature, it is more common to denote $\mathrm{Sp} A$ by $\mathcal{M}(A)$.

Lemma 6.3. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then for any $x \in \mathrm{Sp} A$, we have

$$|f(x)| \leq \rho(f) \leq \|f\|_A.$$

PROOF. Let $\|\bullet\|_x$ be the bounded semi-valuation corresponding to x . Then there is a constant $C > 0$ such that

$$\|\bullet\|_x \leq C\|\bullet\|_A.$$

It follows that for any $n \in \mathbb{N}$,

$$\|f\|_x^n = \|f^n\|_x \leq C\|f^n\|_A.$$

Taking n -th root and letting $n \rightarrow \infty$, we find

$$\|f\|_x \leq \rho(f).$$

The inequality $\rho(f) \leq \|f\|_A$ follows from the definition of ρ . \square

Example 6.4. If $(k, |\bullet|)$ is a complete valuation field, then $\mathrm{Sp} k$ is a single point $|\bullet|$.

To see this, let $\|\bullet\| \in \mathrm{Sp} k$, then by [Lemma 6.3](#),

$$\|f\| \leq |f|$$

for any $f \in k$. If $f \neq 0$, the same inequality applied to f^{-1} implies that $\|f\| = |f|$. When $f = 0$, the equality is trivial.

Example 6.5. Let $\{K_i\}_{i \in I}$ be a family of complete valuation fields. Recall that $\prod_{i \in I} K_i$ is defined in [Example 4.12](#). Then $\text{Sp } \prod_{i \in I} K_i$ is homeomorphic to the Stone–Čech compactification of the discrete set I .

To see this, we first identify the set of proper closed ideals in $\prod_{i \in I} K_i$ with the set of filters on I .

We first introduce a notation: for each $J \subseteq I$, we write $a_J \in \prod_{i \in I} K_i$ for the element

$$a_{J,i} = \begin{cases} 0, & \text{if } i \in J; \\ 1, & \text{if } i \notin J. \end{cases}$$

Given a proper closed ideal $\mathfrak{a} \subseteq \prod_{i \in I} K_i$, we define a filter $\Phi_{\mathfrak{a}} = \{J \subseteq I : a_J \in \mathfrak{a}\}$. Conversely, given a filter Φ on I , we denote by \mathfrak{a}_{Φ} the closed ideal of $\prod_{i \in I} K_i$ generated by a_J for all $J \in \Phi$. These maps are inverse to each other and order preserving. In particular, the maximal ideals of $\prod_{i \in I} K_i$ are identified with ultrafilters of I by [Corollary 4.7](#).

Next we show that all prime ideals of $\prod_{i \in I} K_i$ are maximal. In fact, take $\mathfrak{p} \in \text{Spec } \prod_{i \in I} K_i$ and suppose that there is a maximal ideal \mathfrak{m} properly containing \mathfrak{p} . Let $J \in \Phi_{\mathfrak{m}} \setminus \Phi_{\mathfrak{p}}$ so that $a_J \in \mathfrak{m} \setminus \mathfrak{p}$. As $I \setminus J \notin \Phi_{\mathfrak{m}}$, we have $a_{I \setminus J} \notin \mathfrak{m}$. But $a_J \cdot a_{I \setminus J} = 0$. This contradicts the fact that $a_J \notin \mathfrak{p}$ and $a_{I \setminus J} \notin \mathfrak{p}$.

So we have shown that as a set $\text{Spec } \prod_{i \in I} K_i$ is identified with the Stone–Čech compactification of I .

Next we show that if $\mathfrak{m} \in \text{Spec } \prod_{i \in I} K_i$, then the residue norm on $\prod_{i \in I} K_i / \mathfrak{m}$ is multiplicative. In fact, for each $f \in \prod_{i \in I} K_i$, we have

$$\|\pi(f)\|_{\prod_{i \in I} K_i / \mathfrak{m}} = \inf_{J \in \Phi_{\mathfrak{m}}} \sup_{i \in J} \|f\|.$$

Here $\pi : \prod_{i \in I} K_i \rightarrow \prod_{i \in I} K_i / \mathfrak{m}$ is the natural map and $\|\bullet\|$ denotes the norm on $\prod_{i \in I} K_i$ defined in [Example 4.12](#). It follows immediately that the residue norm on $\prod_{i \in I} K_i / \mathfrak{m}$ is multiplicative. In particular, by [Example 6.4](#), $\text{Sp } \prod_{i \in I} K_i$ and $\text{Spec } \prod_{i \in I} K_i$ are identified as sets under the natural map [\(6.1\)](#).

It remains to identify the topologies. But this is easy: for any ultrafilter Φ on I , let $\mathfrak{m} = \mathfrak{m}_{\Phi}$, then $\|\pi(a_J)\| = 0$ for $J \in \Phi$ and $\|\pi(a_J)\| = 1$ otherwise.

Proposition 6.6. Let $\varphi : A \rightarrow B$ be a bounded homomorphism of Banach rings, then $\text{Sp } \varphi : \text{Sp } B \rightarrow \text{Sp } A$ is continuous.

PROOF. For each $f \in A$, we define $\text{ev}_f : \text{Sp } A \rightarrow \mathbb{R}$ by sending $\|\bullet\|$ to $\|f\|$. It suffices to show that for any $f \in A$, the map $\text{Sp } \varphi \circ \text{ev}_f$ is continuous. But the composition is just the map sending $\|\bullet\| \in \text{Sp } B$ to $\|\varphi(f)\|$. It is continuous by definition of the topology on $\text{Sp } B$ as φ is bounded. \square

Definition 6.7. Let $(A, \|\bullet\|_A)$ be a Banach ring. For each $x \in \text{Sp } A$ corresponding to a bounded semi-valuation $\|\bullet\|_x$ on A , there is a natural induced valuation on $\text{Frac } A$ $\|\bullet\|_x$. We write $\mathcal{H}(x)$ for the completion of $\text{Frac } A$ with the induced valuation. The complete valuation field $\mathcal{H}(x)$ is called the *complete residue field* of A at x .

We will write $f(x)$ for the residue class of f in $\mathcal{H}(x)$.

Observe that for any $f \in A$, $|f(x)|$ is exactly the valuation of $f(x)$ with respect to the valuation on $\mathcal{H}(x)$.

Definition 6.8. Let A be a Banach ring. The *Gelfand transform* of A is the homomorphism

$$A \rightarrow \prod_{x \in \operatorname{Sp} A} \mathcal{H}(x).$$

Here the product is defined in [Example 4.12](#).

We will denote the Gelfand transform as $f \mapsto \hat{f} = (f(x))_{x \in \operatorname{Sp} A}$.

By [Lemma 6.3](#), the Gelfand transform is well-defined.

Proposition 6.9. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then the Gelfand transform

$$A \rightarrow \prod_{x \in \operatorname{Sp} A} \mathcal{H}(x).$$

is bounded. In fact, the Gelfand transform is contractive.

PROOF. This follows simply from [Lemma 6.3](#). \square

Proposition 6.10. Let $(A, \|\bullet\|)$ be a Banach ring. Then $\operatorname{Sp} A$ is empty if and only if $A = 0$.

PROOF. If $A = 0$, $\operatorname{Sp} A$ is clearly empty. Conversely, suppose that $\operatorname{Sp} A$ is empty. Assume that $A \neq 0$. For any maximal ideal \mathfrak{m} , by [Corollary 4.7](#), A/\mathfrak{m} is a Banach ring and $\operatorname{Sp} A/\mathfrak{m}$ is a subset of $\operatorname{Sp} A$. So we may assume that A is a field. Let S be the set of bounded semi-norms on A . Then S is non-empty as $\|\bullet\| \in S$. By Zorn's lemma, we can take a minimal element $|\bullet| \in S$. Up to replacing A by the completion with respect to $|\bullet|$, we may assume that $|\bullet|$ is a norm on A . As A is a field, we may further assume that $|\bullet| = \|\bullet\|$.

We claim that $\|\bullet\|$ is multiplicative. As A is a field, it suffices to show that $\|f^{-1}\| = \|f\|^{-1}$ for any non-zero $f \in A$. We may assume that $\|f\|^{-1} < \|f^{-1}\|$.

Let r be a positive real number. Let $\varphi : A \rightarrow A\{r^{-1}T\}/(T - f)$ be the natural map. The map is injective as A is a field. We endow $A\{r^{-1}T\}/(T - f)$ with the quotient semi-norm induced by $\|\bullet\|_r$. We still denote this semi-norm by $\|\bullet\|_r$.

We claim that $f - T$ is not invertible in $A\{r^{-1}T\}$ for the choice $r = \|f^{-1}\|^{-1}$. From this, it follows that

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

The last step is our assumption. This contradicts our choice of $\|\bullet\|$.

In order to prove the claim, we need to show that $\|\bullet\|$ is power multiplicative first. Assuming this, it is obvious that

$$\sum_{i=0}^{\infty} \|f^{-i}\| r^i = \sum_{i=0}^{\infty} \|f^{-1}\|^i \|f^{-1}\|^{-i}$$

diverges.

It remains to show that $\|\bullet\|$ is power multiplicative. Suppose that is $f \in A$ so that $\|f^n\| < \|f\|^n$ for some $n > 1$. We claim that $f - T$ is not invertible in $A\{r^{-1}T\}$ for the choice $r = \|f^n\|^{1/n}$. From this,

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

This contradicts our choice of $\|\bullet\|$. The claim amounts to the divergence of

$$\sum_{i=0}^{\infty} \|f^{-i}\| r^i.$$

For a general $i \geq 0$, we write $i = pn + q$ for $p, q \in \mathbb{N}$ and $q \leq n - 1$. Then $\|f^i\| \leq \|f^n\|^p \|f^q\|$. So

$$\|f^{-i}\| r^i \geq \|f^i\|^{-1} \|f^n\|^{p+n^{-1}q} \geq \|f^n\|^{n^{-1}q} \|f^q\|^{-1}.$$

It therefore follows that $|f^{-i}| r^i$ admits a positive lower bound, and we conclude. \square

Corollary 6.11. Let A be a Banach ring. Then an element $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in \operatorname{Sp} A$.

PROOF. The direct implication is trivial. Assume that $f(x) \neq 0$ for all $x \in \operatorname{Sp} A$. We claim that $f \notin \mathfrak{m}$ for any maximal ideal \mathfrak{m} in A . From this, it follows that f is invertible in A .

By [Corollary 4.7](#), A/\mathfrak{m} is a Banach ring. It follows from [Proposition 6.10](#) that there is a non-trivial bounded semi-valuation on A/\mathfrak{m} , which lifts to a bounded semi-valuation on A . \square

Corollary 6.12. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then for any $f \in A$, we have

$$\rho(f) = \sup_{x \in \operatorname{Sp} A} |f(x)|.$$

PROOF. We have already shown $\rho(f) \geq \sup_{x \in \operatorname{Sp} A} |f(x)|$ in [Lemma 6.3](#). To verify the reverse inequality, take $f \in A$ and $r \in \mathbb{R}_{>0}$, it suffices to show that if $|f(x)| < r$ for all $x \in \operatorname{Sp} A$, then $\rho(f) \leq r$.

Consider the Banach ring $B = A\{rT\}$. By [Lemma 6.3](#) again, $|T(x)| \leq \|T\|_{r^{-1}} = r^{-1}$ for all $x \in \operatorname{Sp} B$. Therefore, for any $x \in \operatorname{Sp} B$, $|(fT)(x)| < 1$. Hence, $(1 - fT)(x) \neq 0$ for all $x \in \operatorname{Sp} B$. By [Corollary 6.11](#), $1 - fT$ is invertible in B . But this happens exactly when

$$\sum_{i=0}^{\infty} \|f^i\|_A r^{-i}$$

is convergent. It follows that $\rho(f) \leq r$. \square

Theorem 6.13. Let $(A, \|\bullet\|)$ be a Banach ring. Then $\operatorname{Sp} A$ is a compact Hausdorff space.

PROOF. We first show that $\operatorname{Sp} A$ is Hausdorff. Take $x_1, x_2 \in A$, $x_1 \neq x_2$. In other words, we can find $f \in A$ so that $|f(x_1)| \neq |f(x_2)|$. We may assume that $|f(x_1)| < |f(x_2)|$. Take a real number $r > 0$ so that

$$|f(x_1)| < r < |f(x_2)|.$$

Then $\{x \in \operatorname{Sp} A : |f(x)| < r\}$ and $\{x \in \operatorname{Sp} A : |f(x)| > r\}$ are disjoint neighbourhoods of x_1 and x_2 .

Next we show that $\operatorname{Sp} A$ is compact. By [Proposition 6.9](#) and [Proposition 6.6](#), we can define a continuous map

$$\operatorname{Sp} \prod_{x \in \operatorname{Sp} A} \mathcal{H}(x) \rightarrow \operatorname{Sp} A.$$

The map is clearly surjective: for any $x \in \operatorname{Sp} A$, the valuation on $\mathcal{H}(x)$ induces a semi-valuation on $\prod_{x \in \operatorname{Sp} A} \mathcal{H}(x)$, which is clearly bounded. The image of this semi-valuation in $\operatorname{Sp} A$ is just x .

So it suffices to show that $\operatorname{Sp} \prod_{x \in \operatorname{Sp} A} \mathcal{H}(x)$ is compact. This follows from [Example 6.5](#). \square

7. Open mapping theorem

Let $(k, |\cdot|)$ be a complete non-trivially valued field. All results in this section fail when k is trivially valued.

Proposition 7.1. Let A be a normed k -algebra and $f : (M, \|\bullet\|_M) \rightarrow (N, \|\bullet\|_N)$ be an A -homomorphism of normed A -modules. Then f is bounded if and only if f is continuous.

PROOF. The direct implication follows from [Proposition 2.7](#). Assume that f is continuous. We may assume that $A = k$.

Assume that f is not bounded. Fix $a \in k$ with $|a| \in (0, 1)$. This is possible as k is non-trivially valued. Then we can find a sequence $m_i \in M$ such that $\|f(m_i)\|_N > |a|^{-i} \|m_i\|_M$. Up to replace m_i by a scalar multiple, we may assume that $\|m_i\|_M \in [1, |a|^{-1}]$: if $\|m_i\|_M \geq 1$, choose $n \in \mathbb{N}$ such that $|a|^{-n} \leq \|m_i\|_M < |a|^{-n-1}$, then replace m_i with $a^n m_i$. The case $|x| < 1$ is similar. Then $\|f(a^i m_i)\|_N > \|m_i\|_M \geq 1$ while $\|a^i m_i\|_M < |a|^n |a|^{-1} \rightarrow 0$. This is a contradiction. \square

Theorem 7.2 (Open mapping theorem). Let $(V, \|\bullet\|_V), (W, \|\bullet\|_W)$ be k -Banach spaces and $L : V \rightarrow W$ be a bounded and surjective k -homomorphism. Then L is open.

PROOF. We write $V_0 = \{v \in V : \|v\|_V < 1\}$. Similarly define W_0 .

Step 1. We claim that there is a constant $C > 0$ such that for all $w' \in W$, there is $v' \in V$ such that

$$\|v'\|_V \leq C \|w'\|_W, \quad \|w' - L(v')\|_W < 1/2.$$

As k is non-trivially valued, we can take $c \in k$ with $|c| \in (0, 1)$, so

$$V = \bigcup_{n \in \mathbb{N}} c^n V_0.$$

As L is surjective, we have

$$W = \bigcup_{n \in \mathbb{N}} c^n L(V_0).$$

By Baire's category theorem, we may assume that $\overline{L(V_0)}$ has non-empty interior. Take $w \in W$ and $r > 0$ so that

$$\{w' \in W : \|w - w'\|_W < r\} \subseteq \overline{L(V_0)}.$$

Take $d \in W_0$ and $c' \in k^\times$ so that $|c'| < r$, then $w + c'd \in \overline{L(V_0)}$. It follows that

$$c'd \in \overline{L(V_0)} + \overline{L(V_0)} \subseteq \overline{L(V_0) + L(V_0)} = \overline{L(V_0)}.$$

So

$$W_0 \subseteq \overline{L(c'^{-1}V_0)}.$$

It suffices to take $C = |c'^{-1}|$.

Step 2. Now given $w \in W_0$, we want to show that $w \in L(\{v \in V : \|v\|_V < C\})$. This will finish the argument: as k is non-trivially valued, this implies that $L(V_0)$ contains an open neighbourhood of 0.

From Step 1, we can construct $v_1 \in V$ with $\|v_1\|_V < C$ and $\|w - L(v_1)\|_W < 1/2$. Repeat this process, we can $v_n \in V$ inductively so that

$$\|v_n\|_V < 2^{1-n}C, \quad \|w - L(v_1 + \cdots + v_n)\|_W < 2^{-n}.$$

We set $v = \sum_{i=1}^{\infty} v_i$. Then $v \in V$ and $Av = w$ by continuity. Moreover,

$$\|v\|_V \leq \max_n \|v_n\|_V < C.$$

□

Corollary 7.3. Let A be a k -Banach algebra and M be a normed A -module. Assume that \hat{M} is a finite A -module, then M is complete.

PROOF. Take $x_1, \dots, x_n \in \hat{M}$ so that $\pi : A^n \rightarrow \hat{M}$ sending (a_1, \dots, a_n) to $\sum_{i=1}^n a_i x_i$ is surjective. By open mapping theorem [Theorem 7.2](#), $\sum_{i=1}^n \check{A}x_i$ is a neighbourhood of 0 in \hat{M} . So

$$x_j \in M + \sum_{i=1}^n \check{A}x_i.$$

It follows from (a version of) Nakayama's lemma that $M = \hat{M}$. □

Corollary 7.4. Let A be a k -Banach algebra and M be a Noetherian Banach A -module. Let N be a submodule of M . Then N is closed in M .

In particular, if A is Noetherian, then all ideals of A are closed.

PROOF. As M is noetherian, \bar{N} is a finite A -module. In particular, N is complete by [Corollary 7.3](#). Hence, N is closed in M . □

Corollary 7.5. A bounded epimorphism of k -Banach algebras $f : A \rightarrow B$ is admissible.

PROOF. Replacing A by $A/\ker f$, we may assume that f is bijective. It follows from [Theorem 7.2](#) that f is a homeomorphism. The inverse of f is therefore continuous, and hence bounded by [Proposition 7.1](#). □

8. Maximum spectra

Let $(k, |\bullet|)$ a complete non-Archimedean valued field.

Definition 8.1. For any k -algebra A , we write

$$\mathrm{Spm}_k A := \{\mathfrak{m} \in \mathrm{Spm} A : A/\mathfrak{m} \text{ is algebraic over } k\}.$$

For any $x \in \mathrm{Spm}_k A$ and any $f \in A$, we write $f(x)$ for the residue of f in A/\mathfrak{m}_x , where \mathfrak{m}_x is the maximal ideal corresponding to x . We write $|f(x)|$ for the valuation of $f(x)$ with respect to the extended valuation induced from the given valuation on k .

Definition 8.2. Let A be a k -algebra. For each $f \in A$, we write $|f|_{\mathrm{sup}}$ for the supremum of $|f(x)|$ for all $x \in \mathrm{Spm}_k A$ if $\mathrm{Spm}_k A$ is non-empty and 0 otherwise.

Definition 8.3. Let f be a monic polynomial in $k[X]$, we expand $f = X^n + a_1 X^{n-1} + \dots + a_n \in k[X]$, then we define $\sigma(f) := \max_{i=1, \dots, n} |a_i|^{1/i}$.

Definition 8.4. Let L be a reduced integral k -algebra. We define the *spectral norm* $|\bullet|_{\mathrm{sp}}$ on L as follows: given a non-zero $x \in L$, take a minimal polynomial $X^n + a_1 X^{n-1} + \dots + a_n \in k[X]$ of x over k . Then we set

$$|x|_{\mathrm{sp}} := \max_{i=1, \dots, n} |a_i|^{1/i}.$$

Proposition 8.5. Let f, g be monic polynomials in $k[X]$, then

$$\sigma(fg) = \max\{\sigma(f), \sigma(g)\}.$$

PROOF. Replacing k by a finite extension, we may assume that f and g split into linear factors a_i and b_j . Then it is straightforward to show that

$$\sigma(f) = \prod_i a_i, \quad \sigma(g) = \prod_j b_j, \quad \sigma(fg) = \prod_i a_i \cdot \prod_j b_j.$$

The assertion follows. \square

Proposition 8.6. Let L be a reduced integral k -algebra. Then $|\bullet|_{\text{sp}}$ is a power-multiplicative norm on L , and it extends the norm on k .

PROOF. It is clear that $|\bullet|_{\text{sp}}$ extends the valuation on k . In order to show that $|\bullet|_{\text{sp}}$ is a power-multiplicative norm on L , we may assume that L is finite dimensional over k . Then we can find finite field extensions L_1, \dots, L_t of k such that $L = \bigoplus_{i=1}^t L_i$. By [Proposition 8.5](#), we can immediately reduce to the case where L/k is a finite field extension. In this case, the result is well-known. [Expand](#). \square

Proposition 8.7. Let L be a reduced integral k -algebra. For any $\mathfrak{p} \in \text{Spec } L$, write $\pi_{\mathfrak{p}} : L \rightarrow L/\mathfrak{p}$ the residue map. Then for any $y \in L$,

$$|y|_{\text{sp}} = \max_{\mathfrak{p} \in \text{Spec } L} |\pi_{\mathfrak{p}}(y)|_{\text{sp}}.$$

PROOF. Fix $y \in L$. For any $\mathfrak{p} \in \text{Spec } L$, let $q_{\mathfrak{p}} \in k[X]$ be the minimal polynomial of $\pi_{\mathfrak{p}}(y)$ over k . Let $q \in k[X]$ be the minimal polynomial of y over k . Then clearly $q_{\mathfrak{p}}$ divides q for all $\mathfrak{p} \in \text{Spec } L$. In particular, there are only finitely many different polynomials among $q_{\mathfrak{p}}$ ($\mathfrak{p} \in \text{Spec } L$), say q_1, \dots, q_r . Define $q' = q_1 \cdots q_r \in k[X]$. Then for $f \in k[X]$, $f(y) = 0$ if and only if $\pi_{\mathfrak{p}}(f(y)) = 0$ for all $\mathfrak{p} \in \text{Spec } L$ as L is reduced. The latter condition is equivalent to that $q' | f$. It follows that $q' = q$. Now by [Proposition 8.5](#),

$$|y|_{\text{sp}} = \sigma(q) = \max_{i=1, \dots, r} \sigma(q_i) = \max_{\mathfrak{p} \in \text{Spec } L} |\pi_{\mathfrak{p}}(y)|_{\text{sp}}.$$

\square

Proposition 8.8. Let $\varphi : B \rightarrow A$ be a homomorphism of commutative k -algebras. Then for any $f \in B$,

$$|\varphi(f)|_{\text{sup}} \leq |f|_{\text{sup}}.$$

PROOF. Of course, we can assume that $\text{Spm}_k A \neq \emptyset$. Let $x \in \text{Spm}_k A$, then $\varphi^{-1}x \in \text{Spm}_k B$. But for any $f \in B$, $|\varphi(f)(x)| = |f(\varphi^{-1}x)|$. We conclude. \square

Proposition 8.9. Let A be a k -algebra. Let \mathfrak{M} be the set of minimal prime ideals in A and let $\pi_{\mathfrak{p}} : A \rightarrow A/\mathfrak{p}$ be the canonical residue map for all $\mathfrak{p} \in \mathfrak{M}$. Then for any $f \in A$,

$$(8.1) \quad |f|_{\text{sup}} = \sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}}.$$

In particular, if A be a reduced integral k -algebra. Then $|\bullet|_{\text{sup}} = |\bullet|_{\text{sp}}$ on A .

PROOF. By [Proposition 8.8](#),

$$\sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}} \leq |f|_{\text{sup}}.$$

In order to show the reverse inequality, let $x \in \text{Spm}_k A$. Take $\mathfrak{p} \in \mathfrak{M}$ such that $x \supseteq \mathfrak{p}$. Clearly, $\pi_{\mathfrak{p}}(x) \in \text{Spm}_k A/\mathfrak{p}$ and

$$|f(x)| = |\pi_{\mathfrak{p}}(f)(\pi_{\mathfrak{p}}(x))|.$$

In particular,

$$|f(x)| \leq |\pi_{\mathfrak{p}}(f)|_{\text{sup}} \leq \sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\text{sup}}.$$

Take sup with respect to x , we conclude [\(8.1\)](#).

When A is a reduced and integral k -algebra, all prime ideals of A are minimal. The final assertion follows from [Proposition 8.7](#). \square

Proposition 8.10. Let $\varphi : B \rightarrow A$ be an injective integral homomorphism of k -algebras. Assume that B is a normal integral domain. Fix $f \in A$. Let $f^n + \varphi(b_1)f^{n-1} + \dots + \varphi(b_n) = 0$ be the minimal equation of f over A . Then

$$|f|_{\text{sup}} = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

PROOF. We first show the inequality

$$|f|_{\text{sup}} \leq \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

Of course, we can assume that $\text{Spm}_k A \neq \emptyset$. For all $x \in \text{Spm}_k A$, we have

$$0 = f(x)^n + \varphi(b_1)f(x)^{n-1} + \dots + \varphi(b_n) = f(x)^n + b_1(\varphi^{-1}x)f(x)^{n-1} + \dots + b_n(\varphi^{-1}x).$$

Then we in fact have that

$$|f(x)| \leq \max_{i=1, \dots, n} |b_i(\varphi^{-1}x)|_{\text{sup}}^{1/i}.$$

Assume that to the contrary that

$$|f(x)|^i > |b_i(\varphi^{-1}x)|$$

for all $i = 1, \dots, n$. Then

$$|b_i(\varphi^{-1}x)f(x)^{n-i}| < |f(x)|^n = |f(x)^n|.$$

It follows that

$$|b_1(\varphi^{-1}x)f(x)^{n-1} + \dots + b_n(\varphi^{-1}x)| < |f(x)^n|.$$

This is a contradiction.

It remains to argue that

$$(8.2) \quad |f|_{\text{sup}} \geq \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

Next let $A' = B[f]$. We argue that $A' \rightarrow A$ is an isometry with respect to $|\bullet|_{\text{sup}}$. If $\text{Spm}_k A'$ is empty, then the assertion follows from [Proposition 8.8](#). Assume that $\text{Spm}_m A'$ is non-empty. Take $y \in \text{Spm}_k A'$. By [\[Stacks, Tag 00GQ\]](#), there is a maximal ideal $x \in \text{Spm } A$ lying over y . As the induced map $A'/y \rightarrow A/x$ is integral, we find $x \in \text{Spm}_k A$. So the map $\text{Spm}_k A \rightarrow \text{Spm}_k A'$ is surjective. It follows that $A' \rightarrow A$ is an isometry with respect to $|\bullet|_{\text{sup}}$.

In order to argue [\(8.2\)](#), we may assume that $A = B[f]$. Let $q \in B[X]$ denote the minimal polynomial of f over A . Then $A = B[X]/(q)$. Let $y \in \text{Spm}_k B$, we write f_y for the residue class of f in A/yA and write f_y for the residue class in $(A/yA)^{\text{red}}$.

Similarly, let q_y denote the residue class of q in $B/y[X]$. As y is contained in some $\text{Spm}_k A$, we see that

$$|f|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} |f_y|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} |\bar{f}_y|_{\text{sup}}.$$

For $y \in \text{Spm}_k B$, we decompose q_y into prime factors $q_1^{n_1} \cdots q_r^{n_r}$ in $B/y[X]$. Then

$$A/yA \cong B/y[X]/(q_y)$$

and

$$(A/yA)^{\text{red}} \cong \bigoplus_{i=1}^r B/y[X]/(q_i).$$

We endow $\bigoplus_{i=1}^r B/y[X]/(q_i)$ with the spectral norm over B/y . If \bar{f}_i denotes the residue class of \bar{f}_y in $B/y[X]/(q_i)$, by [Proposition 8.9](#) and [Proposition 8.5](#),

$$|\bar{f}_y|_{\text{sup}} = \max_{i=1, \dots, r} |\bar{f}_i|_{\text{sp}} = \max_{i=1, \dots, r} \sigma(q_i) = \sigma(q_y).$$

Therefore,

$$|f|_{\text{sup}} = \sup_{y \in \text{Spm}_k B} \sigma(q_y) = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/n}.$$

□

9. Bornology

[This section may be placed elsewhere.](#)

Definition 9.1. Let X be a set. A *bornology* on X is a collection \mathcal{B} of subsets of X such that

- (1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in B$;
- (2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;
- (3) \mathcal{B} is stable under finite union.

The pair (X, \mathcal{B}) is called a *bornological set*. The elements of \mathcal{B} are called the *bounded subsets* of (X, \mathcal{B}) . When \mathcal{B} is obvious from the context, we omit it from the notations.

A morphism between bornological sets (X, \mathcal{B}_X) and (Y, \mathcal{B}_Y) is a map of sets $f : X \rightarrow Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 9.2. Let (X, \mathcal{B}) be a bornological set. A *basis* for \mathcal{B} is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \cdots \cup A_n$.

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