Commutative algebra

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2. Graded commutative algebra

Let G be an Abelian group. We write the group operation of G multiplicatively and denote the identity of G as 1.

Definition 2.1. Let A be an Abelian group. A *G-grading* on A is a coproduct decomposition

$$A = \bigoplus_{g \in G} A_g$$

of Abelian groups such that $A_g \subseteq A$. An Abelian group with a G-grading is called a *G-graded Abelian group*.

An element $a \in A$ is said to be *homogeneous* if there is $g \in G$ such that $a \in A_g$. If a is furthermore non-zero, we write $g = \rho(a)$. We set $\rho(0) = 0$. We will write $\rho(A)$ for the set of $\rho(a)$ when a runs over all homogeneous elements in A.

A *G-graded homomorphism* between G-graded Abelian groups A and B is a homogeneous of the underlying Abelian groups $f : A \to B$ such that $f(A_g) \subseteq B_g$ for any $g \in G$.

The category of G-graded Abelian groups is denoted by $\mathcal{A}\mathrm{b}^G$.

A usual Abelian group A can be given the *trivial G-grading*: $A_0 = A$ and $A_g = 0$ for $g \in G$, $g \neq 0$. In this way, we find a fully faithful embedding

$$\mathcal{A}\mathrm{b} \to \mathcal{A}\mathrm{b}^G.$$

When we regard an Abelian group as a G-graded Abelian group and there are no natural gradings, we always understand that we are taking the trivial G-grading.

Definition 2.2. A *G-graded ring* is a commutative ring A endowed with a G-grading:

$$A = \bigoplus_{g \in G} A_g$$

as Abelian groups and such that

(1) $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$;
(2) $1 \in A_1$.

A *G-homomorphism* of G-graded rings A and B is a ring homomorphism $f : A \to B$ such that $f(A_g) \subseteq B_g$ for each $g \in G$.

The category of G-graded rings is denoted by $\mathcal{R}\mathrm{ing}^G$.

Example 2.3. Let A be a G-graded ring, $n \in \mathbb{N}$ and $g = (g_1, \ldots, g_n) \in G^n$. Then there is a unique G-grading on $A[T_1, \ldots, T_n]$ extending the grading on A and such that $\rho(T_i) = g_i$ for $i = 1, \ldots, n$. We will denote $A[T_1, \ldots, T_n]$ with this grading as $A[g_1^{-1}T_1, \ldots, g_n^{-1}T_n]$ or simply $A[g^{-1}T]$.

Example 2.4. Let A be a G-graded ring and S be a multiplicative subset of A consisting of homogeneous elements, then $S^{-1}A$ has a natural G-grading. To see this, recall the construction of $S^{-1}A$ in [Stacks, Tag 00CM]. One defines an equivalence relation on $A \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}A)_g$ as the set of (x, s) for all $s \in S$ and $x \in A_{g\rho(s)}$. It is easy to verify that this is a well-defined G-grading on $S^{-1}A$. Add details.

Definition 2.5. Let A be a G-graded ring. A *G-homogeneous ideal* in A is an ideal I in G such that if $a \in A$ can be written as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all $a_g = 0$, then $a_g \in I$.

Example 2.6. Let A be a G-graded ring and $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ be homogeneous elements in A. Then $a_1, \ldots, a_n$ generate a G-homogeneous ideal $(a_1, \ldots, a_n)$ as follows:

$$(a_1, \ldots, a_n)_g = \sum_{i=1}^{n} A_{g\rho(a_i)^{-1}} a_i$$

for any $g \in G$.

Lemma 2.7. Let $f : A \to B$ be a G-homomorphism of G-graded rings. Then $\ker f$ is a G-homogeneous ideal in A.

Proof. We need to show that

$$\ker f = \sum_{g \in G} (\ker f) \cap A_g.$$

Take $x \in \ker f$, we can write x as

$$\sum_{g \in G} a_g, \quad a_g \in A_g$$

and almost all a_g's are 0. Then

$$f(x) = \sum_{g \in G} f(a_g), \quad f(a_g) \in B_g.$$

It follows that $f(a_g) = 0$ for each $g \in G$ and hence $a_g \in (\ker f) \cap A_g$. $\square$

Definition 2.8. Let A be a G-graded ring and I be a G-homogeneous ideal in A. Then we define a G-grading on A/I as follows: for any $g \in G$

$$(A/I)_g := (A_g + I)/I.$$

Proposition 2.9. Let A be a G-graded ring and I be a G-homogeneous ideal in A. Then the construction in Definition 2.8 defines a grading on A/I. The natural map $\pi : A \to A/I$ is a G-homomorphism.

For any G-graded ring B and any G-homomorphism $f : A \to B$ such that $I \subseteq \ker A$, there is a unique G-homomorphism $f' : A/I \to B$ such that $f' \circ \pi = f$.

Proof. We first argue that for different $g, h \in G$, $(A/I)_g \cap (A/I)_h = 0$. Suppose $x \in (A/I)_g \cap (A/I)_h$, we can lift x to both $y_g + i_g \in A$ and $y_h + i_h \in A$ with $y_g, y_h \in A$ and $i_g, i_h \in I$. It follows that $y_g - y_h \in I$. But I is a G-homogeneous ideal, so it follows that $y_g, y_h \in I$ and hence $x = 0$.

Next we argue that

$$A/I = \sum_{g \in G} (A/I)_g.$$

Lift an element $x \in A/I$ by $a \in A$, we represent a as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all a_g's equal to 0. Then x can be represented as

$$x = \sum_{g \in G} \pi(a_g).$$

We have shown that the construction in Definition 2.8 gives a G-grading on A. It is clear from the definition that π is a G-homomorphism.

Next assume that B and f are given as in the proposition. Then there is a ring homomorphism $f' : A/I \to B$ such that $f = f' \circ \pi$. We need to argue that f' is a G-homomorphism. For this purpose, take $g \in G$, $x \in (A/I)_g$, we need to show that $f'(x) \in B_g$. Lift x to $y + i$ with $y \in A_g$ and $i \in I$, then we know that $f'(x) = \pi(y + i) = \pi(y) \in B_g$. □

Definition 2.10. Let A be a G-graded ring.

Let M an A-module which is also a G-graded Abelian group. We say M is a *G-graded A-module* if for each $g, h \in G$, we have

$$A_g M_h \subseteq M_{gh}.$$

A *G-graded homomorphism* of G-graded A-modules M and N is an A-module homomorphism $f : M \to N$ which is at the same time a homomorphism of the underlying G-graded Abelian groups.

The category of G-graded A-modules is denoted by $\mathcal{M}\mathrm{od}_A^G$.

A *G-graded A-algebra* is a G-graded ring B together with a G-graded ring homomorphism $A \to B$ such that B is also a G-graded A-module.

Observe that G-homogeneous ideals of A are G-graded submodules of A. Also observe that $\mathcal{M}\mathrm{od}_{\mathbb{Z}}^G$ is isomorphic to $\mathcal{A}\mathrm{b}^G$.

Proposition 2.11. Let A be a G-graded ring. Then $\mathcal{M}\mathrm{od}_A^G$ is an Abelian category satisfying AB5.

Proof. We first show that $\mathcal{M}\mathrm{od}_A^G$ is preadditive. Given $M, N \in \mathcal{M}\mathrm{od}_A^G$, we can regard $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_A^G}(M, N)$ as a subgroup of $\mathrm{Hom}_A(M, N)$. It is easy to see that this gives $\mathcal{M}\mathrm{od}_A^G$ an enrichment over $\mathcal{A}\mathrm{b}$.

Next we show that $\mathcal{M}\mathrm{od}_A^G$ is additive. The zero object is clearly given by 0 with the trivial grading. Given $M, N \in \mathcal{M}\mathrm{od}_A^G$, we define

$$(M \oplus N)_g := M_g \oplus N_g, \quad g \in G.$$

This construction makes $M \oplus N$ a G-graded A-module. It is easy to verify that $M \oplus N$ is the biproduct of M and N.

Next we show that $\mathcal{M}\mathrm{od}_A^G$ is pre-Abelian. Given an arrow $f : M \to N$ in $\mathcal{M}\mathrm{od}_A^G$, we need to define its kernel and cokernel. We define

$$(\ker f)_g := (\ker f) \cap M_g$$

and $(\mathrm{coker}\, f)_g$ as the image of N_g for any $g \in G$. It is straightforward to verify that these are kernels and cokernels.

Next, given a monomorphism $f : M \to N$, it is obvious that the map f is injective and f can be identified with the kenrel of the natural map $N/\operatorname{Im} f$. A dual argument shows that an epimorphism is the cokernel of some morphism as well.

The AB5 condition is easily verified. Expand the details of this argument! □

Example 2.12. This is a continuition of Example 2.4. Let A be a G-graded ring and S be a multiplicative subset of A consisting of homogeneous elements. Consider a G-graded A-module M. We define a G-grading on $S^{-1}M$. Recall that $S^{-1}M$ can be realized as follows: one defines an equivalence relation on $M \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}M)_g$ as the set of (x, s) for all $s \in S$ and $x \in M_{g\rho(s)}$. It is easy to verify that this is a well-defined G-grading on $S^{-1}M$ and $S^{-1}M$ is a G-graded $S^{-1}A$-module. Add details.

Example 2.13. Let A be a G-graded ring and $g \in G$. We define $g^{-1}A$ as the G-graded A-module:

$$(g^{-1}A)_h = A_{g^{-1}h}$$

for any $h \in G$. Observe that $1 \in (g^{-1}A)_g$.

Definition 2.14. Let A be a G-graded ring and M be a G-graded A-module. We say M is *free* if there exists a family $\{g_i\}_{i\in I}$ in G such that

$$M = \coprod_{i\in I} g_i^{-1}A.$$

Definition 2.15. Let $f : A \to B$ be a G-graded homomorphism of G-graded rings. We say f is *finite* (resp. *finitely generated*, resp. *integral*) if it is finite (resp. finitely generated, resp. integral) as a usual ring map.

Proposition 2.16. Let $f : A \to B$ be a G-graded homomorphism of G-graded rings. Then

(1) f is finite if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G-graded homomorphism

$$\bigoplus_{i=1}^{n} (g_i^{-1}A)^n \to B$$

of graded A-modules.

(2) f is finitely generated if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G-graded A-algebra homomorphism

$$A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \to B.$$

(3) f is integral if and only if for any non-zero homogeneous element $b \in B$, there in $n \in \mathbb{N}$ and homogeneous elements $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \cdots + f(a_n) = 0.$$

Proof. (1) The non-trivial direction is the direct implication. Assume that f is finite. Take $b_1, \dots, b_n \in B$ so that $\sum_{i=1}^{n} f(A)b_i = B$. Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i's, we may assume that each b_i is homogeneous. We define $g_i = \rho(b_i)$ and the map $\bigoplus_{i=1}^{n}(g_i^{-1}A)^n \to B$ sends 1 at the i-th place to b_i.

(2) The non-trivial direction is the direct implication. Suppose f is finitely generated, say by $b_1, \dots, b_n$. Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i's, we may assume that each b_i is homogeneous. Then we define $g_i = \rho(b_i)$ for $i = 1, \dots, n$ and the A-algebra homomorphism $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \to B$ sends T_i to b_i for $i = 1, \dots, n$.

(3) Assume that f is integral, then for any non-zero homogeneous element $b \in B$, we can find $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \cdots + f(a_n) = 0.$$

Obviously, we can replace a_i by its component in $\rho(b)^i$ for $i = 1, \dots, n$ and the equation remains true.

The reverse direction follows from [Stacks, Tag 00GO]. □

Definition 2.17. A G-graded ring A is a *G-graded field* if

(1) $A \neq 0$.
(2) A does not admit any non-zero proper G-homogeneous ideals.

Proposition 2.18. Let A be a non-zero G-graded ring. Then the following conditions are equivalent:

(1) A is a G-graded field.
(2) Any non-zero homogeneous element in A is invertible.

Proof. Assume that A is a G-graded field. Let $a \in A$ be a non-zero homogeneous element. Consider the G-homogeneous ideal (a) generated by a as in Example 2.6. As $a \neq 0$, it follows that $(a) = 1$. Hence, a is invertible.

Conversely, suppose that any non-zero homogeneous element in A is invertible. If I is a non-zero G-homogeneous ideal in A. There is a non-zero homogeneous element $a \in I$. But we know that a is invertible and hence $I = A$. □

Definition 2.19. A G-graded ring A is an *integral domain* if for any non-zero homogeneous elements $a, b \in A$, $ab \neq 0$.

Lemma 2.20. Let A be a G-graded integral domain. Let S denote the set of non-zero homogeneous elemnts in A. Then $S^{-1}A$ is a graded field. The natural map $A \to S^{-1}A$ is injective.

Recall that $S^{-1}A$ is defined in Example 2.4.

Proof. By Proposition 2.18, it suffices to show that each non-zero homogeneous element in $S^{-1}A$ is invertible. Such an element has the form a/s for some homogeneous element $a \in A$ and $s \in S$. As A is a G-graded integral domain, a is invertible and hence $s/a \in S^{-1}A$.

In general, the kernel of the localzation map is given by $\{a \in A : \text{ there is } s \in S \text{ such that } sa = 0\}$. As $A \to S^{-1}A$ is a G-graded homomorphism, the kernel is in addition a G-homogeneous ideal in A by Lemma 2.7. So it suffices to show that each homogeneous element in the kenrel vanishes: if $a \in A$ is a homogeneous element and there is $s \in S$ such that $sa = 0$, then $a = 0$. Otherwise, a is invertible by Proposition 2.18, which is a contradiction. □

Definition 2.21. Let A be a G-graded integral domain. We call the graded field defined in Lemma 2.20 the *fraction G-graded field* of A and denote it by $\mathrm{Frac}^G A$.

Definition 2.22. Let A be a G-graded ring. A proper G-homogeneous ideal I in A is called *prime* if the G-graded ring A/I is a G-graded integral domain.

Proposition 2.23. Let A be a G-graded ring and I be a proper homogeneous ideal in A. Then the following are equivalent:

(1) I is a G-graded prime ideal in A.

(2) For any homogeneous elements $a, b \in A$ satisfying $ab \in I$, at least one of a and b lies in I.

Proof. Assume that I is a G-graded prime ideal in A. Let $a, b \in A$ be homogeneous elements satisfying $ab \in I$. Let $\bar{a}, \bar{b}$ be the images of a, b in A/I. Then $\bar{a}, \bar{b}$ are homogeneous and $\bar{a}\bar{b} = 0$. So at least one of $\bar{a}$ and $\bar{b}$ is zero. That is, a or b lies in I.

Conversely, assume that the conditon in (2) is satisfied. Take $x, y \in A/I$ with $xy = 0$. We need to show that at least one of x and y is 0. Lift x and y to $a + i$ and $b + i'$ in A with a, b being homogeneous and $i, i' \in I$. Then $ab \in I$ and hence $a \in I$ or $b \in I$. It follows that $x = 0$ or $y = 0$. □

Lemma 2.24. Let k be a G-graded field and A be a graded k-algebra. Suppose that $\rho(A) = \rho(k)$, then

(1) For any $g \in G$, there is a natural isomorphism
$$A_g \cong A_1 \otimes_{k_1} k_g.$$
(2) The map $I \mapsto I \cap A_1$ is a bijection between the set of homogeneous ideals (resp. homogeneous prime ideals) in A and ideals (resp. prime ideals) in A_1.

Proof. (1) Take $g \in \rho(A)$. As $\rho(A) = \rho(k)$, we can take a non-zero homogeneous element $b \in k_g$. Then b and b^{-1} induces inverse bijections between A_1 and A_g.

(2) The part about ideals can be proved in the same way as (1). The part about prime ideals follows easily. □

Proposition 2.25. Let k be a G-graded field and M be a G-graded A-module. Then M is free as G-graded A-module.

Proof. We may assume that $M \neq 0$. Let $\{m_i\}_{i \in I}$ be a maximal set of non-zero homogeneous elements in M such that the corresponding homomorphism
$$F := \bigoplus_{i \in I} (\rho(f))^{-1} k \to M$$
is injective. The existence of $\{m_i\}_{i \in I}$ follows from Zorn's lemma.

If $f \in M/F$ is a non-zero homogeneous element, then we get a homomorphism $(\rho(f))^{-1} k \to M/F$. This map is necessarily injective as $(\rho(f))^{-1} k$ does not have non-zero proper graded submodules. This contradicts the definition of F. □

Bibliography

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