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Topology and bornology

1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A comapct space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

A connected topological is always non-empty.

References to this chapter include [Ber93].

2. Nets

Let X be a set, $Y \subseteq X$ be a subset. Consider a collection τ of subsets of X, we write

$$\tau|_{Y} := \{V \in \tau : V \subseteq Y\}.$$

Definition 2.1. Let X be a topology space and τ be a collection of subsets of X. We say τ is

- (1) dense if for any $V \in \tau$ and any $x \in V$, there is a fundamental system of neighbourhoods of x in V consisting of sets from $\tau|_V$;
- (2) a quasi-net on X if for each $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \cdots \cap V_n$ and that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X;
- (3) a *net* on X if it is a quasi-net and if for any $U, V \in \tau$, $\tau|_{U \cap V}$ is a quasi-net on $U \cap V$;
- (4) locally finite if for any $x \in X$, there is a neighbourhood U of x in X such that $\{V \in \tau : V \cap U \neq \emptyset\}$ is finite.

We observe that if τ is a net, $\tau|_{U\cap V}$ is in fact a net.

Lemma 2.2. Let X be a topological space and τ be a quasi-net on X.

- (1) A subset $U \subseteq X$ is open if and only if for each $V \in \tau$, $U \cap V$ is open in V.
- (2) Suppose that τ consists of compact sets. Then X is Hausdorff if and only if for any $U, V \in \tau$, $U \cap V$ is compact.

We remind the readers that a compact space is Hausdorff by our convention.

PROOF. (1) The direct implication is trivial. Suppose that $U \cap V$ is open in V for all $V \in \tau$. We want to show that U is open. Take $x \in U$, we can find $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X. By our hypothesis, we can find open sets W_1, \ldots, W_n in W such that $W \cap V_i = U \cap V_i$

for $i=1,\ldots,n$. Then $W=W_1\cap\cdots\cap W_n$ is an open neighbourhood of x in X. But then

$$U \cap (V_1 \cup \cdots \cup V_n) \supseteq W \cap (V_1 \cup \cdots \cup V_n),$$

the latter is a neighbourhood of x hence so is the former. It follows that U is open.

(2) The direct implication is trivial. Consider the quasi-net $\tau \times \tau := \{U \times V : U, V \in \tau\}$ on $X \times X$. By (1), it suffices to verify that the intersection of the diagonal with $U \times V$ is closed in $U \times V$ for any $U, V \in \tau$. But this intersection is homeomorphic to $U \cap V$, which is compact by our assumption and hence closed as U, V are both Hausdorff.

Lemma 2.3. Let X be a Hausdorff space. Assume that X admits a quasi-net τ consisting of compact sets. Then X is locally compact.

PROOF. Take $x \in X$. By assumption, we can find $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x. This neighbourhood is clearly compact.

Lemma 2.4. Let X be a Hausdorff space and τ be a collection of compact subsets of X. Then the following are equivalent:

- (1) τ is a quasi-net;
- (2) For each $x \in X$, there are $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \tau$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X.

PROOF. $(1) \implies (2)$: This is trivial.

(2) \Longrightarrow (1): Given $x \in X$, take V_1, \ldots, V_n as in (2). We may assume that $x \in V_1, \ldots, V_m$ and $x \notin V_{m+1}, \ldots, V_n$ for some $1 \leq m \leq n$. Then $V_1 \cup \cdots \cup V_m$ is a neighbourhood of x in X: if U is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_n$, then $U \setminus (V_{m+1} \cup \cdots \cup V_n)$ is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_m$.

Lemma 2.5. Let X be a topological space and τ be a net on X consisting of compact sets. Then

- (1) for any pair $U, V \in \tau$, the intersection $U \cap V$ is locally closed in U and in V;
- (2) If $n \in \mathbb{Z}_{>0}$, $V, V_1, \ldots, V_n \in \tau$ are such that

$$V \subseteq V_1 \cup \cdots \cup V_n$$
,

then there are $m \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_m \in \tau$ such that

$$V = U_1 \cup \cdots \cup U_m$$

and each U_i is contained in some V_i .

PROOF. (1) It suffices to show that $U \cap V$ is locally compact in the induced topology. This follows from Lemma 2.3.

(2) For each $x \in V$ and each i = 1, ..., n such that $x \in V_i$, we take a neighbourhood of x in $V \cap V_i$ of the form $W_i V_{i1} \cup \cdots \cup V_{im_i}$ for some $m_i \in \mathbb{Z}_{>0}$ and $V_{ij} \in \tau$ for $j = 1, ..., m_i$. Then the union of all W_i 's is a neighbourhood of x of the form $U_1 \cup \cdots \cup U_m$, where U_j belongs to τ and is contained in some V_i . Using the compactness of V, we conclude.

3. Paracompact spaces

Definition 3.1. A topological space X is paracompact if any open covering of X admits a locally finite refinement.

A paracompact space is not necessarily Hausdorff according to our definition.

Proposition 3.2. Let X be a locally compact topological space.

- (1) Assume that each connected component of X is σ -compact, then X is paracompact.
- (2) If X is paracompact and Hausdorff, then each connected component of X is σ -compact.

If the conditions in (2) are satisfied, for any basis of neighbourhoods \mathcal{B} of X, every open covering \mathcal{U} of X can be refined into a locally finite covering \mathcal{V} consisting of elements in \mathcal{B} .

We do not assume that the elements in \mathcal{B} be open. The covering \mathcal{V} is not necessarily open.

Proposition 3.3. Let X be a paracompact space and $Y \subseteq X$ be a closed subspace. Then Y is paracompact.

Proposition 3.4. Let X be a locally compact Hausdorff space and $Y \subseteq X$ be a subspace, then the following are equivalent:

- (1) Y is locally compact and Hausdorff;
- (2) Y is a locally closed subspace of X.

4. Closed maps and topologically finite maps

Definition 4.1. A map $f: X \to Y$ of topological spaces is *closed* if for each closed subset Z in X, f(Z) is closed in Y.

A closed map is not necessarily continuous.

Lemma 4.2. Let $f: X \to Y$ be a closed map of topological spaces, then for each $y \in Y$ and any open neighbourhood U of $f^{-1}(y)$ in X, there is an open neighbourhood V of y in Y such that $f^{-1}(V) \subseteq U$.

Proof. 1	It suffices t	so take V	$Y = Y \setminus$	f($X \setminus$	V),		
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Lemma 4.3. Let $f: X \to Y$ be a closed map of topological spaces. Then for any subspace V of Y, the map $f^{-1}(V) \to V$ induced by f is closed.

PROOF. Let A be a closed subset of $U := f^{-1}(V)$. We need to show that f(A) is closed in V. Choose a closed subset B of X such that $A = B \cap U$, then f(B) is closed in Y and $f(A) = f(B) \cap V$ is closed in V.

Definition 4.4. A continuous and closed map $f: X \to Y$ of topological spaces is topologically finite if for each $y \in Y$, the set $f^{-1}(y)$ is finite.

A map $f: X \to Y$ of topological spaces is topologically finite at $x \in X$ if there is an open neighbourhood U of x in X and an open neighbourhood Y of f(x) in Y such that $f(U) \subseteq V$ and the induced map $U \to V$ is topologically finite.

Proposition 4.5. Let $f: X \to Y$ be a topologically finite map of topological spaces. Then for any subspace $V \subseteq Y$, the map $f^{-1}(V) \to V$ induced by f is topologically finite.

PROOF. This follows immediately from Lemma 4.3.

Theorem 4.6. Let $f: X \to Y$ be a topologically finite map of topological spaces. Assume that X is Hausdorff. Let $y \in f(X)$ and x_1, \ldots, x_n $(n \in \mathbb{Z}_{>0})$ denote the distinct points of $f^{-1}(y)$. Take pairwise disjoint open neighbourhoods U'_1, \ldots, U'_n of x_1, \ldots, x_n in X. Then any neighbourhood V' of y in Y contains an open neighbourhood V of y satisfying the following conditions:

- (1) $U_1 := f^{-1}(V) \cap U'_1, \dots, U_n := f^{-1}(V) \cap U'_n$ are pairwise disjoint open neighbourhoods of x_1, \dots, x_n in X;
- (2) $f^{-1}V = \bigcup_{j=1}^{n} U_j;$ (3) The maps $U_j \to V$ for j = 1, ..., n induced from f are all topologically

Let \mathcal{F} be a sheaf of sets on X, then we have a functorial bijection

$$f_*\mathcal{F}(V) \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}(U_j).$$

PROOF. As $\bigcup_{j=1}^n U_j'$ is an open neighbourhood of $f^{-1}(y)$ in X, by Lemma 4.2 and Lemma 4.3, we can find an open neighbourhood $V \subseteq V'$ of y in Y such that

$$f^{-1}V \subseteq \bigcup_{j=1}^{n} U'_{j}.$$

The conditions (1) and (2) are therefore satisfied.

In order to prove (3), it remains to show that the induced maps $U_i \to V$ are closed for j = 1, ..., n. We may take j = 1. Let A be a closed subset of U_1 . Then A is closed in $f^{-1}(V)$ by (1) and (2). It follows that f(A) is closed in V by Lemma 4.3. The last assertion follows from (1) and (2).

Corollary 4.7. Let $f: X \to Y$ be a topologically finite map of topological spaces. Assume that X is Hausdorff. Let $x \in X$ be U' be an open neighbourhood of x in X such that all other points in $f^{-1}(f(x))$ are in the interior of $X \setminus U'$. Then any neighbourhood V' of f(x) in Y contains an open neighbourhood V of y such that for $U := f^{-1}(V) \cap U'$ the map $g: U \to V$ induced by f is topologically finite and $g^{-1}(g(x)) = \{x\}.$

PROOF. This follows immediately from Theorem 4.6.

Corollary 4.8. Let $f: X \to Y$ be a topologically finite map of topological spaces. Assume that X is Hausdorff. Let \mathcal{F} be a sheaf of sets on $X, y \in f(X)$. Denote by x_1, \ldots, x_n $(n \in \mathbb{Z}_{>0})$ the distinct points of the fiber $f^{-1}(y)$. Then we have a canonical bijection

$$(f_*\mathcal{F})_y \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}_{x_j}.$$

In particular, $f_*: Ab(X) \to Ab(Y)$ is exact.

PROOF. This follows immediately from Theorem 4.6.

5. Bornology

Definition 5.1. Let X be a set. A *bornology* on X is a collection \mathcal{B} of subsets of X such that

- (1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
- (2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;
- (3) \mathcal{B} is stable under finite union.

The pair (X, \mathcal{B}) is called a *bornological set*. The elements of \mathcal{B} are called the *bounded subsets* of (X, \mathcal{B}) . When \mathcal{B} is obvious from the context, we omit it from the notations

A morphism between bornological sets (X, \mathcal{B}_X) and (Y, \mathcal{B}_Y) is a map of sets $f: X \to Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 5.2. Let (X, \mathcal{B}) be a bornological set. A *basis* for \mathcal{B} is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \ldots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \cdots \cup A_n$.

Bibliography

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