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Local properties of complex analytic spaces

1. Introduction

2. Dimension

Definition 2.1. Let X be a complex analytic space and $x \in X$, the *dimension* $\dim_x X$ of X at x is

$$\dim_x X = \dim \mathcal{O}_{X,x}.$$

We also define the *dimension* of the pointed complex analytic space (X, x) and the *dimension* of the complex analytic germ X_x as $\dim_x X$.

When X is connected, the *dimension* of X is defined as

$$\dim X := \sup_{x \in X} \dim_x X.$$

If A is an analytic set in X such that there is a closed analytic subspace of X with $|B| = A$, then $\dim_x B$ does not depend on the choice of B , we define it as $\dim_x A$.

As we will see in [Corollary 6.6](#), B always exists.

Definition 2.2. Let X be a complex analytic space, we say X is *equidimensional* at $x \in X$ if $\mathcal{O}_{X,x}$ is equidimensional.

We also say (X, x) or X_x is *equidimensional*.

We say X is *equidimensional of dimension* $n \in \mathbb{N}$ if X is non-empty and is equidimensional of dimension n at each $x \in X$.

Recall that in general, a local ring R is equidimensional if $\dim R/\mathfrak{p} = \dim R$ for all minimal prime \mathfrak{p} of R .

Definition 2.3. Let X be a complex analytic space and $x \in X$, we say X is *integral* at x if $\mathcal{O}_{X,x}$ is integral.

This corresponds to the notion defined in [Definition 3.12](#) in [Constructions of complex analytic spaces](#).

Theorem 2.4. Let X be a complex analytic space and $n \in \mathbb{N}$, then the set of points $x \in X$ such that X_x is equidimensional of dimension n is open.

This is analogous to the result for noetherian cartenary schemes.

PROOF. Let $x \in X$ be a point such that X_x is equidimensional of dimension n . We want to construct an open neighbourhood V of x in X such that X is equidimensional of dimension n at any $y \in V$.

Step 1. We reduce to the case where X is integral at x .

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal primes of $\mathcal{O}_{X,x}$. The number is finite because $\mathcal{O}_{X,x}$ is noetherian. We have

$$\bigcap_{i=1}^m \mathfrak{p}_i = \text{rad } \mathcal{O}_{X,x}.$$

Take an open neighbourhood U of x in X such that there are ideals of finite type $\mathcal{I}_1, \dots, \mathcal{I}_m$ extending $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Up to shrinking U , we may assume that

$$\bigcap_{i=1}^m \mathcal{I}_i$$

is nilpotent. For each $i = 1, \dots, m$, let U_i denote the closed analytic subspace of U defined by \mathcal{I}_i . Then

$$|U| = \bigcup_{i=1}^m |U_i|$$

by [Corollary 3.17](#) in [Constructions of complex analytic spaces](#). As for any $y \in U$,

$$\bigcap_{i=1}^m \mathcal{I}_{i,y}$$

is nilpotent, we have

$$|\text{Spec } \mathcal{O}_{X,y}| = |\text{Spec } \mathcal{O}_{X,y} / \bigcap_{i=1}^m \mathcal{I}_{i,y}| = \bigcup_{i=1}^m |\text{Spec } \mathcal{O}_{X,y} / \mathcal{I}_{i,y}|.$$

In particular, for any $y \in U$,

$$\dim_y X = \dim_y U = \max_{i=1, \dots, m} \dim_y U_i.$$

It suffices to handle each W_i separately.

Step 2. We assume that X_x is integral. By [Theorem 3.9](#) in [Constructions of complex analytic spaces](#), we may assume that X has the following structure: there is an open neighbourhood W of 0 in \mathbb{C}^n , a morphism $(X, x) \rightarrow (W, 0)$ and a finite \mathcal{O}_W -algebra \mathcal{A} such that $\text{Spec}_W^{\text{an}} \mathcal{A}$ has a unique point x' over 0 and $(\text{Spec}_W^{\text{an}} \mathcal{A}, x')$ is isomorphic to (X, x) over $(W, 0)$. By [Corollary 5.5](#) in [Complex analytic local algebras](#), $\mathcal{O}_{W,0} \rightarrow \mathcal{O}_{X,x}$ is injective, hence $\mathcal{O}_{X,x}$ is torsion-free over $\mathcal{O}_{W,0}$. As the torsion sheaf is coherent, up to shrinking X , we may assume that $\mathcal{O}_{X,y}$ is torsion-free over $\mathcal{O}_{W,z}$, where z denotes the image of y in W . It suffices to apply [Lemma 5.6](#) in [Complex analytic local algebras](#). \square

Corollary 2.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set $\{x \in X : \dim_x X \geq n\}$ is an analytic set in X .

After introducing the analytic Zariski topology, we can reformulate this corollary as follows: the map $x \mapsto \dim_x X$ is upper semi-continuous with respect to the analytic Zariski topology.

PROOF. The problem is local on X . Fix $x \in X$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathcal{O}_{X,x}$. Up to shrinking X , we may assume that

$$|X| = \bigcup_{i=1}^m |W_i|,$$

where W_i is a closed analytic subspace of X defined by a coherent \mathcal{I}_i spreading \mathfrak{p}_i . We can guarantee that

$$\dim_y X = \max_{i=1,\dots,m} \dim_y W_i.$$

This is possible as in the proof of [Theorem 2.4](#). By [Theorem 2.4](#), up to shrinking X , we may assume that W_i is equidimensional of dimension n_i for some $n_i \in \mathbb{N}$ for each $i = 1, \dots, m$. In particular, for each $y \in X$, we have

$$\dim_y X = \sup_{y \in W_i} n_i.$$

So

$$\{x \in X : \dim_x X \geq n\} = \bigcup_{i: n_i \geq n} |W_i|.$$

The corollary follows. \square

Proposition 2.6. Let X, Y be complex analytic spaces and $x \in X, y \in Y$. Then

$$\dim_{(x,y)} X \times Y = \dim_x X + \dim_y Y.$$

PROOF. By [Theorem 5.11](#) in [Complex analytic local algebras](#),

$$\hat{\mathcal{O}}_{X \times Y, (x,y)} = \hat{\mathcal{O}}_{X,x} \hat{\otimes} \hat{\mathcal{O}}_{Y,y}.$$

As dimension is invariant under completion by [\[Stacks, Tag 07NV\]](#), it suffices to show that

$$\dim(\mathcal{O}_{X,x} \hat{\otimes} \mathcal{O}_{Y,y}) = \dim \mathcal{O}_{X,x} + \dim \mathcal{O}_{Y,y},$$

which is well-known. \square

Definition 2.7. Let X_x be an analytic germ and Y_x be a closed analytic subgerm defined by an ideal $I \subseteq \mathcal{O}_{X,x}$.

- (1) When Y_x is irreducible, namely when I is a prime ideal, we define the *codimension* of Y_x in X_x as

$$\text{codim}_x(Y, X) := \text{ht}_{\mathcal{O}_{X,x}}(I).$$

- (2) In general, we define the *codimension* of Y_x in X_x as

$$\text{codim}_x(Y, X) := \inf_{Z_x \subseteq Y_x} \text{codim}_x(Y, X),$$

where Z_x runs over closed analytic subgerms of X_x contained in Y_x .

We also call $\text{codim}_x(Y, X)$ the codimension of Y in X at x .

Observe that

$$\text{codim}_x(Y, X) \leq \dim_x X - \dim_x Y.$$

When X_x is equidimensional, $\text{codim}_x(Y, X)$ is nothing but $\dim_x X - \dim_x Y$.

Observe that

$$(2.1) \quad \text{codim}_x(Y, X) = \text{codim}(Y_x, \text{Spec } \mathcal{O}_{X,x}).$$

Lemma 2.8. Let X be a complex analytic space and T be an analytic set in X . Let Y_1, Y_2 be two closed analytic subspaces of X with underlying set T , then for any $x \in T$,

$$\text{codim}_x(Y_1, X) = \text{codim}_x(Y_2, X).$$

PROOF. This follows from [\(2.1\)](#) and [Corollary 3.14](#) in [Constructions of complex analytic spaces](#). \square

Definition 2.9. Let X be a complex analytic space and T be an analytic set in X . Take $y \in T$. We define the *codimension* $\text{codim}_y(T, X)$ as follows: up to shrinking X , we may take a closed analytic subspace Y of X with underlying set T by [Lemma 4.6](#) in [Constructions of complex analytic spaces](#), we define

$$\text{codim}_y(T, X) := \text{codim}_y(Y, X).$$

This definition does not depend on the choices we made by [Lemma 2.8](#).

Lemma 2.10. Let X be a complex analytic space and Y be a closed analytic subspace of X . Let $y \in Y$ be a point such that Y_y is irreducible. Then there is an open neighbourhood U of y in Y such that

$$\text{codim}_z(Y, X) = \text{codim}_y(Y, X)$$

for any $z \in U$.

PROOF. Let X'_y be an irreducible component of X_y containing Y_y such that

$$\text{codim}_y(Y, X) = \dim_y X' - \dim_y Y.$$

We can then take an open neighbourhood U of x in X such that X'_z is equidimensional of dimension $n := \dim_y X'$ for all $z \in U$ by [Theorem 2.4](#). Then for any $z \in U$, X'_z is a union of some irreducible components of X_z . Up to shrinking U , we may guarantee that for any $z \in U \cap Y$, $Y_z \subseteq X'_z$ and $\dim_z Y = \dim_y Y$. Thereofre, for $z \in Y \cap U$,

$$\text{codim}_z(Y, X) = \text{codim}_z(Y, X') = \dim_z X' - \dim_z Y$$

is a constant. □

Corollary 2.11. Let X be a complex analytic space and Y be an analytic set in X . For any $n \in \mathbb{N}$,

$$\{y \in Y : \text{codim}_y(Y, X) \leq n\}$$

is an analytic set in Y .

PROOF. The problem is local. Let $x \in Y$. Let $Y_{1,x}, \dots, Y_{m,x}$ be the irreducible components of Y_x defined by prime ideals J_1, \dots, J_m in $\mathcal{O}_{Y,x}$. Take an open neighbourhood U of x in X such that for any $y \in Y \cap U$, the ideal

$$\bigcap_{i=1}^m J_{i,y}$$

is nilpotent. By [Lemma 2.10](#), up to shrinking U , we may assume that for any $y \in Y \cap U$,

$$\text{codim}_y(Y_i, X) = \text{codim}_x(Y_i, X) =: c_i$$

for $i = 1, \dots, m$. Then

$$\{y \in Y : \text{codim}_y(Y, X) \leq n\} = \bigcup_{i: c_i \leq n} Y_i.$$

□

Corollary 2.12. Let X be a complex analytic space and Y be an analytic set in X . For any $n \in \mathbb{N}$ and any $y \in Y$,

$$\{y \in Y : \text{codim}_y(Y, X) \leq n\}_y = \{\mathfrak{p} \in \text{Spec } \mathcal{O}_{X,x} : \text{codim}_{\mathfrak{p}}(T_x, \text{Spec } \mathcal{O}_{X,x}) \leq n\}.$$

PROOF. This is immediate from the proof of [Corollary 2.11](#). □

Definition 2.13. Let X be a complex analytic space. A closed subset A of X is *thin* if for any $x \in A$, we can find an open neighbourhood U of x in X such that $A \cap U$ is contained in a nowhere dense analytic subset B of U .

Given $k \in \mathbb{Z}_{>0}$, we say A is *thin of order k* at $x \in A$ if U and B can be chosen so that $\text{codim}_x(B, X) \geq 2$.

We say X is *thin* (*thin of order k*) if X is thin (resp. thin of order k) at all $x \in X$.

The definition in [CAS] Page 132 is not correct when X is not equidimensional. The same happens in several papers of Remmert.

3. Smoothness

Definition 3.1. Let X be a complex analytic space. We say X is *smooth* at $x \in X$ if $\mathcal{O}_{X,x}$ is regular. Otherwise, we say X is *singular* at x .

We also say (X, x) or X_x is *smooth* (resp. *singular*) at x .

We say X is *smooth* if it is smooth at all $x \in X$. In this case, we also say X is a *complex manifold*.

We write X^{sing} and X^{reg} for the set of singular and smooth points of X respectively.

Other common names in the literature include: regular, simple.

Proposition 3.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) X is smooth at x ;
- (2) There is an open neighbourhood U of x in X that is isomorphic to a domain in \mathbb{C}^n with $n = \dim_x X$;
- (3) $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$ -module of rank $\dim_x X$;
- (4) $\Omega_{X,x}$ is generated by $\dim_x X$ elements as an $\mathcal{O}_{X,x}$ -module;
- (5) $\hat{\mathcal{O}}_{X,x}$ is regular;
- (6) $\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[X_1, \dots, X_n]]$ for $n = \dim_x X$.

PROOF. (2) \implies (1): This is obvious.

(1) \implies (2): Let $f_{1,x}, \dots, f_{n,x}$ be a regular system of parameters of $\mathcal{O}_{X,x}$. Up to shrinking X , we may lift them to $f_1, \dots, f_n \in \mathcal{O}_X(X)$. By [Theorem 4.2](#) in [The notion of complex analytic spaces](#), they induce a morphism $f : (U, x) \rightarrow (\mathbb{C}^n, 0)$. Observe that $f_x^\# : \hat{\mathcal{O}}_{\mathbb{C}^n, 0} \rightarrow \hat{\mathcal{O}}_{U,x}$ is an isomorphism, so f is a local isomorphism by [Corollary 3.4](#) in [Constructions of complex analytic spaces](#).

(2) \implies (3): This follows from [Example 8.7](#) in [Constructions of complex analytic spaces](#).

(3) \implies (4): This is trivial.

(4) \implies (1): Recall that Ω_X is coherent by [Corollary 8.2](#) in [Constructions of complex analytic spaces](#). By Nakayama's lemma, the minimal number of generators of $\Omega_{X,x}$ is equal to $\dim_{\mathbb{C}} \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$. By algebraic results, we know that the latter space is $\mathfrak{m}_x / \mathfrak{m}_x^2$. So we find that $\dim \mathfrak{m}_x / \mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$, implying that $\mathcal{O}_{X,x}$ is regular.

(1) \Leftrightarrow (5): This follows from [\[Stacks, Tag 07NY\]](#).

(2) \implies (6): This is clear.

(6) \implies (5): This is clear. □

Theorem 3.3. Let X be a complex analytic space, then X^{Sing} is an analytic set in X .

PROOF. The problem is local. Let $x \in X$.

Step 1. We reduce to the case where X is equidimensional of dimension n .
Let

$$0 = \bigcap_{i=1}^r \mathfrak{p}_i$$

be the primary decomposition of 0. Up to shrinking X , we may assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ spread to coherent ideals $\mathcal{I}_1, \dots, \mathcal{I}_r$ on X and

$$\bigcap_{i=1}^r \mathcal{I}_i = 0.$$

Let X_i be the closed analytic subspace of X defined by \mathcal{I}_i for $i = 1, \dots, r$. Then

$$X = \bigcup_{i=1}^r X_i.$$

As each X_i is equidimensional at x , say of dimension n_i for $i = 1, \dots, r$. By [Theorem 2.4](#), up to shrinking X , we may assume that X_i is equidimensional of dimension n_i for $i = 1, \dots, r$. For each

Let $y \in X^{\text{reg}}$, as $\mathcal{O}_{X,y}$ is regular hence integral, from

$$\bigcap_{i=1}^r \mathcal{I}_{i,y} = 0$$

we find that at least one $\mathcal{I}_{i,y}$ vanishes. Then

$$\mathcal{O}_{X_i,y} = \mathcal{O}_{X,y}$$

is regular. Namely, $y \in X_i^{\text{reg}}$. Conversely, if for some $i = 1, \dots, r$, we have $\mathcal{I}_{i,y} = 0$ and $y \in X_i^{\text{reg}}$, X_i is a neighbourhood of y in X , so $y \in X^{\text{reg}}$. It follows that

$$X^{\text{sing}} = \bigcap_{i=1}^r \left(\text{Supp } \mathcal{I}_i \cup X_i^{\text{Sing}} \right).$$

Recall that $\text{Supp } \mathcal{I}_i$ is analytic for each $i = 1, \dots, r$ by [Example 4.2](#) in [Constructions of complex analytic spaces](#).

By [Proposition 4.3](#) in [Constructions of complex analytic spaces](#), in order to show that X^{sing} is an analytic set in X , it suffices to know that X_i^{Sing} is an analytic set in X_i for $i = 1, \dots, r$.

Step 2. Assume that X is equidimensional of dimension n . We need to show that the locus where Ω_X is locally free of rank n is co-analytic in X .

When $n = 0$, the locus where Ω_X is not locally free of rank 0 is exactly $\text{Supp } \Omega_X$, which is analytic in X by [Example 4.2](#) and [Corollary 8.2](#) in [Constructions of complex analytic spaces](#).

Assume that $n \geq 1$. Let $\Omega_X^n := \bigwedge^n \Omega_X$. Then the locus where Ω_X is locally free of rank n is exactly the locus where Ω_X^n is invertible. The invertible locus of Ω_X^n is exactly the locus where the canonical map

$$(\Omega_X^n)^\vee \otimes_{\mathcal{O}_X} \Omega_X^n \rightarrow \mathcal{O}_X$$

is an isomorphism. It follows that the complement of the locus is analytic in X . \square

Theorem 3.4 (Generic smoothness). Let X be a complex analytic space and $x \in X$. Assume that X is integral at x , then $X_x^{\text{Sing}} \neq |X|_x$.

PROOF. Let $n = \dim_x X$. The problem is local on X . By [Theorem 3.9 in Constructions of complex analytic spaces](#), we may assume that there is a finite morphism $\varphi : (X, x) \rightarrow (V, 0)$, where V is an open neighbourhood of 0 in \mathbb{C}^n and there is a finite \mathcal{O}_V -algebra \mathcal{A} with $\mathcal{A}_0 = \mathcal{O}_{X,x}$ such that there is unique point x' of $\text{Spec}_V^{\text{an}} \mathcal{A}$ over 0 and (X, x) can be identified with $(\text{Spec}_V^{\text{an}} \mathcal{A}, x')$.

Take $\xi \in \mathcal{O}_{X,x} = \mathcal{A}_0$ such that

$$\text{Frac } \mathcal{O}_{X,x} = \text{Frac } \mathcal{O}_{\mathbb{C}^n,0}(\xi).$$

Let $P_0 \in \mathcal{O}_{\mathbb{C}^n,0}[X]$ be the minimal polynomial of ξ . Up to shrinking V , we may assume that ξ spreads to a section $f \in \mathcal{A}(V)$. Then $\mathcal{B} = \mathcal{O}_V[f]$ is a finite sub- \mathcal{O}_V -algebra of \mathcal{A} . Up to shrinking V , we may assume that the kernel of $\mathcal{O}_V[X] \rightarrow \mathcal{B}$ sending X to f is generated by a unitary polynomial $P \in \mathcal{O}_V(V)[X]$ of degree $d := [\text{Frac } \mathcal{O}_{X,x} : \text{Frac } \mathcal{O}_{\mathbb{C}^n,0}]$ that extends P_0 . Therefore,

$$\mathcal{B} \cong \mathcal{O}_V[X]/(P).$$

Let $T = \text{Supp } \mathcal{A}/\mathcal{B}$. We endow T with the structure of closed analytic subspace of V induced by the annihilator of \mathcal{A}/\mathcal{B} . Observe that $\mathcal{A}_0/\mathcal{B}_0 = \mathcal{O}_{X,x}/\mathcal{O}_{\mathbb{C}^n,0}$ is torsion, so $|T|_0 = \text{Supp } \mathcal{A}_0/\mathcal{B}_0 \neq \text{Spec } \mathcal{O}_{\mathbb{C}^n,0}$. That is, $T_0 \neq \mathbb{C}_0^n$ by [Theorem 3.13 in Constructions of complex analytic spaces](#). Observe that $X \setminus \varphi^{-1}(T) = \text{Spec}_{V \setminus T}^{\text{an}} \mathcal{B}|_{V \setminus T}$.

On the other hand, $P'_0(\xi) \neq 0$ as ξ is separable. So $W(P'(f)) \neq |X|_x$. Let $Z = \text{Supp } \mathcal{O}_X/(P'(f))$, then φ is unramified outside T . [Include the parts regarding unramified morphisms and étale morphisms before this section](#) In particular, φ is étale outside T and hence a local isomorphism by [Corollary 3.4 in Constructions of complex analytic spaces](#). In particular,

$$X^{\text{Sing}} \subseteq Z \cup \varphi^{-1}(T)$$

and hence

$$X_x^{\text{Sing}} \subseteq Z_x \cup \varphi^{-1}(T)_x.$$

The latter is not equal to $|X|_x$ by [Corollary 3.14 in Constructions of complex analytic spaces](#) and the fact that $\mathcal{O}_{X,x}$ is integral. \square

Theorem 3.5 (Abhyankar). Let X be a complex analytic space and $x \in X$, then

$$X_x^{\text{Sing}} = (\text{Spec } \mathcal{O}_{X,x})^{\text{Sing}}.$$

PROOF. Let $\mathfrak{p} \in \text{Spec } \mathcal{O}_{X,x}$. In concrete terms, we need to show that $W(\mathfrak{p}) \not\subset X_x^{\text{Sing}}$ if and only if $\text{Spec } \mathcal{O}_{X,x}$ is regular at \mathfrak{p} .

The problem is local on X . Up to shrinking X , we may assume that \mathfrak{p} spreads to a coherent ideal \mathcal{I} on X . Let Y be the closed analytic subspace of X defined by \mathcal{I} . By [Lemma 2.10](#), up to shrinking X , we may assume that $\text{codim}_y(Y, X)$ is constant for $y \in Y$. We denote this common value as p , which is necessarily equal to the height of \mathfrak{p} .

As Y_x is irreducible by assumption, for an analytic set Z in Y satisfying $Z_x \neq |Y|_x$, the following conditions are equivalent:

- (1) $|Y|_x \not\subset X_x^{\text{Sing}}$;
- (2) $(|Y| \setminus Z)_x \not\subset X_x^{\text{Sing}}$.

(2) \implies (1) is trivial. If (2) fails, then

$$|Y|_x = (|Y| \cup X^{\text{Sing}})_x \cup Z_x.$$

So $|Y|_x = (|Y| \cup X^{\text{Sing}})_x$, namely (1) holds. We apply this remark to

$$Z = Y^{\text{Sing}} \cup S_{p'}(\mathcal{I}/\mathcal{I}^2),$$

where p' is the dimension of the Zariski tangent space of $\text{Spec } \mathcal{O}_{X,x}$ at \mathfrak{p} and $S_{p'}(\mathcal{I}/\mathcal{I}^2)$ is the locus where $\mathcal{I}/\mathcal{I}^2$ is not locally free of rank p' . Note that neither part of Z is equal to $|Y|_x$, the former follows from [Theorem 3.4](#) and the latter follows from [Theorem 3.13](#) in [Constructions of complex analytic spaces](#) as clearly $\mathfrak{p} \notin S_{p'}(\mathcal{I}/\mathcal{I}^2)$. We find that $W(\mathfrak{p}) \not\subset X_x^{\text{Sing}}$ if and only if $(|Y| \setminus Z)_x \not\subset X_x^{\text{Sing}}$.

If $y \in |Y| \setminus Z$, then y is a regular point of Y and $\text{codim}_y(Y, X) = p$. On the other hand, $\mathcal{I}/\mathcal{I}^2$ is free of rank p' around y . But given the regularity of $\mathcal{O}_{Y,y}$, the regularity of $\mathcal{O}_{X,y}$ is equivalent to the fact that $\mathcal{I}/\mathcal{I}^2$ is free of rank p . Or equivalently to $p = p'$. The latter is equivalent to the regularity of \mathfrak{p} in $\text{Spec } \mathcal{O}_{X,x}$. The theorem is established. \square

Proposition 3.6. Let X, Y be complex analytic spaces and $x \in X, y \in Y$. Then the following are equivalent:

- (1) X is regular at x and Y is regular at y ;
- (2) $X \times Y$ is regular at (x, y) .

This follows from [Corollary 8.6](#) in [Constructions of complex analytic spaces](#) and [Proposition 3.2](#).

Theorem 3.7. Let X be a complex manifold and A be a thin subset of X . Let $f \in \mathcal{O}_X(X \setminus A)$. Assume that either of the following conditions hold:

- (1) f is locally bounded near A ;
- (2) A is thin of order 2 in X .

Then f admits a unique extension to an element in $\mathcal{O}_X(X)$.

PROOF. The problem is local on X . By [Proposition 3.2](#), we may assume that X is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$. In this case, the results are the classical Riemann extension theorem. \square

Corollary 3.8. Let X be a connected complex manifold and A be a thin set in X . Then $X \setminus A$ is connected.

PROOF. Assume that $X \setminus A$ can be written as the disjoint union of two open subsets U_0, U_1 . Then the function $f \in \mathcal{O}_X(X \setminus A) = \mathcal{O}_X(U_0) \times \mathcal{O}_X(U_1)$ given by $0 \in \mathcal{O}_X(U_0)$ and $1 \in \mathcal{O}_X(U_1)$ is locally bounded near A . By [Theorem 3.7](#), f admits a unique extension to $g \in \mathcal{O}_X(X)$. As X is connected and the image of f is contained in $\{0, 1\} = \{0, 1\}$, it follows that f is constant, so U_0 or U_1 has to be empty. \square

4. Serre's condition R_n

Fix $n \in \mathbb{N}$ in this section.

Definition 4.1. Let X be a complex analytic space, we say X satisfies R_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n . We also say (X, x) or X_x satisfies R_n at $x \in X$.

We say X satisfies R_n if X satisfies R_n at all points $x \in X$.

Proposition 4.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

- (1) X satisfies R_n at x ;
- (2) $\hat{\mathcal{O}}_{X,x}$ satisfies R_n .

PROOF. This follows from [Stacks, Tag 07NY]. \square

Proposition 4.3. Let X be a complex analytic space, $x \in X$ and $n \in \mathbb{N}$. Then the following are equivalent:

- (1) X satisfies R_n at x ;
- (2) $\text{codim}_x(X^{\text{Sing}}, X) > n$.

PROOF. It follows from Theorem 3.5 that (1) holds if and only if $\text{codim}_x(X_x^{\text{Sing}}, \text{Spec } \mathcal{O}_{X,x}) > n$. The latter condition is equivalent to (2) by definition. \square

Corollary 4.4. Let X be a complex analytic space and $n \in \mathbb{N}$. The

$$\{x \in X : X \text{ satisfies } R_n \text{ at } x\}$$

is co-analytic in X .

PROOF. This follows from Proposition 4.3 and Corollary 2.11. \square

Proposition 4.5. Let X, Y be complex analytic spaces and $x \in X, y \in Y$. Take $n \in \mathbb{N}$. Then the following are equivalent:

- (1) X satisfies R_n at x and Y satisfies R_n at y ;
- (2) $X \times Y$ satisfies R_n at (x, y) .

PROOF. By Proposition 3.6,

$$(X \times Y)^{\text{Sing}} = (X^{\text{Sing}} \times Y) \cup (X \times Y^{\text{Sing}}).$$

It follows that

$$\text{codim}_{(x,y)}((X \times Y)^{\text{Sing}}, X \times Y) = \min \{ \text{codim}_x(X^{\text{Sing}}, X), \text{codim}_y(Y^{\text{Sing}}, Y) \}$$

We conclude by Proposition 4.3. \square

5. Serre's condition S_n

Fix $n \in \mathbb{N}$ in this section.

Definition 5.1. Let X be a complex analytic space, we say X satisfies S_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n . We also say (X, x) or X_x satisfies S_n at $x \in X$.

We say X satisfies S_n if X satisfies S_n at all points $x \in X$.

Proposition 5.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

- (1) X satisfies S_n at x ;
- (2) $\hat{\mathcal{O}}_{X,x}$ satisfies S_n .

PROOF. This follows from the fact that $\mathcal{O}_{X,x}$ is the quotient of a regular local ring. Include a reference \square

Proposition 5.3. Let X be a complex analytic space, \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules and $n \in \mathbb{N}$. Then

$$\left\{x \in X : \text{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\}$$

is an analytic subset of X . Moreover, the germ of this set in $\text{Spec } \mathcal{O}_{X,x}$ is equal to

$$\left\{\mathfrak{p} \in \text{Spec } \mathcal{O}_{X,x} : \text{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}} > n\right\}.$$

PROOF. Step 1. We reduce to the case where X is smooth and equidimensional of dimension N .

The problem is local in X , so we may assume that X is a complex model space. Assume that X is a closed analytic subspace of a domain U in \mathbb{C}^m for some $m \in \mathbb{N}$. For any $x \in X$, we have

$$\text{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \text{codep}_{\mathcal{O}_{U,x}} \mathcal{G}_x,$$

where \mathcal{G} is the zero-extension of \mathcal{F} to U . A similar formula holds for $\text{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}}$. So it suffices to handle U instead of X .

Step 2. We prove the result after the reduction in Step 1.

By Auslander–Buchsbaum formula, for $x \in X$,

$$\text{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x + \text{dep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \text{dep } \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}.$$

So the condition $\text{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n$ is equivalent to

$$\text{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \dim \mathcal{O}_{X,x} - \dim_x \text{Supp } \mathcal{F}.$$

As $\mathcal{O}_{X,x}$ is regular hence equidimensional, the condition just means

$$\text{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \text{codim}_x(\text{Supp } \mathcal{F}, X).$$

As $\mathcal{O}_{X,x}$ is regular, this condition is equivalent to the existence of an integer $r > n + \text{codim}_x(\text{Supp } \mathcal{F}, X)$ such that

$$\mathcal{E}\text{xt}_{\mathcal{O}_X}^r(\mathcal{F}, \mathcal{O}_X)_x \neq 0.$$

For each $p \in \mathbb{N}$, we introduce

$$T_p(\mathcal{F}) := \bigcup_{r=p+1}^N \text{Supp } \mathcal{E}\text{xt}_{\mathcal{O}_X}^r(\mathcal{F}, \mathcal{O}_X).$$

Then the proceeding analysis shows that

$$\left\{x \in X : \text{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\} = \bigcup_{s=0}^N T_{n+s}(\mathcal{F}) \cap \{y \in \text{Supp } \mathcal{F} : \text{codim}_y(\text{Supp } \mathcal{F}, X) \leq s\}.$$

Observe that the right-hand side is an analytic set in X by [Example 4.2 in Constructions of complex analytic spaces](#) and [Corollary 2.11](#), hence so is the left-hand side.

It remains to compute the germ at $y \in X$. For $p \in \mathbb{N}$, we compute

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^N \text{Supp } \mathcal{E}\text{xt}_{\mathcal{O}_X}^r(\mathcal{F}, \mathcal{O}_X)_y.$$

But observe that

$$\mathcal{E}\text{xt}_{\mathcal{O}_X}^r(\mathcal{F}, \mathcal{O}_X)_y = \text{Ext}_{\mathcal{O}_{X,y}}^r(\mathcal{F}_y, \mathcal{O}_{X,y}).$$

Let $\widetilde{\mathcal{F}}_y$ be the coherent module on $\operatorname{Spec} \mathcal{O}_{X,x}$ associated with \mathcal{F}_y . Let $X_y = \operatorname{Spec} \mathcal{O}_{X,y}$. Then

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^N \operatorname{Supp} \mathcal{E}xt_{\mathcal{O}_{X_y}}^r(\widetilde{\mathcal{F}}_y, \mathcal{O}_{X_y})_y.$$

On the other hand, by [Corollary 2.12](#), for $s \in \mathbb{N}$,

$$\{x \in \operatorname{Supp} \mathcal{F} : \operatorname{codim}_x(\operatorname{Supp} \mathcal{F}, X) \leq s\}_y = \left\{ \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,y} : \operatorname{codim}_{\mathfrak{p}}(\operatorname{Supp} \widetilde{\mathcal{F}}_y, \operatorname{Spec} \mathcal{O}_{X,y}) \leq s \right\}.$$

The same argument as above shows that

$$\left\{ x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n \right\}_y = \left\{ \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,y} : \operatorname{codep}_{\mathcal{O}_{X,y,\mathfrak{p}}} \mathcal{F}_{y,\mathfrak{p}} > n \right\}.$$

□

Proposition 5.4. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is the complement of

$$\bigcup_{m=0}^{\infty} \{y \in Z_m : \operatorname{codim}_y(Z_m, X) \leq n + m\},$$

where

$$Z_m = \{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > m\}.$$

PROOF. It suffices to observe that for $x \in X$, X satisfies S_n at x if and only if

$$\operatorname{codim}(\{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x} : \operatorname{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}} > n + m\}, \operatorname{Spec} \mathcal{O}_{X,x}) > n + m$$

for all $m \in \mathbb{N}$. □

Corollary 5.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is co-analytic.

PROOF. This follows from [Proposition 5.4](#) and [Proposition 5.3](#). □

Proposition 5.6. Let X, Y be complex analytic spaces and $x \in X, y \in Y$. Take $n \in \mathbb{N}$. Assume that X satisfies S_n at x and Y satisfies S_n at y , then $X \times Y$ satisfies S_n at (x, y) .

PROOF. By [Theorem 5.11](#) in [Complex analytic local algebras](#),

$$\hat{\mathcal{O}}_{X \times Y, (x,y)} = \hat{\mathcal{O}}_{X,x} \hat{\otimes} \hat{\mathcal{O}}_{Y,y}.$$

As being S_n is invariant under completion by [\[Stacks, Tag 07NW\]](#) and [\[Stacks, Tag 07NV\]](#), it suffices to prove the corresponding algebraic result, which is known. □

6. Reducedness

Definition 6.1. Let X be a complex analytic space, we say X is *reduced* at $x \in X$ if $\mathcal{O}_{X,x}$ is reduced. We also say (X, x) or X_x is *reduced* at $x \in X$.

We say X is *reduced* if X is reduced at all points $x \in X$.

Proposition 6.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) X is reduced at x ;
- (2) $\hat{\mathcal{O}}_{X,x}$ is reduced.

PROOF. This follows from [Proposition 4.2](#) and [Proposition 5.2](#).

Otherwise, one can also argue as follows: Recall that an excellent ring is Nagata by [\[Stacks, Tag 07QV\]](#). A Nagata noetherian local ring is reduced if and only if its completion is by [\[Stacks, Tag 07NZ\]](#). \square

Theorem 6.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is reduced is co-analytic.

PROOF. This follows from [Corollary 5.5](#) and [Corollary 4.4](#) as reduceness is equivalent to S_1 and R_0 . \square

Corollary 6.4. Let X be a complex analytic space, then the nilradical $\text{rad } \mathcal{O}_X$ is coherent.

PROOF. The problem is local on X . Take $x \in X$. Up to shrinking X , we may assume that $\mathcal{O}_{X,x}/(\text{rad } \mathcal{O}_X)_x$ spreads to a finite \mathcal{O}_X -algebra \mathcal{A} by [Lemma 3.7](#) in [Constructions of complex analytic spaces](#). Up to further shrinking X , we may assume that \mathcal{A} is the quotient of \mathcal{O}_X , say $\mathcal{A} \cong \mathcal{O}_X/\mathcal{I}$ for some coherent ideal \mathcal{I} on X . As \mathcal{I}_x is nilpotent by assumption, up to shrinking X , we may assume that \mathcal{I} is also nilpotent, namely

$$\mathcal{I} \subseteq \text{rad } \mathcal{O}_X.$$

Let Y be the closed analytic subspace of X defined by the ideal \mathcal{I} . Then $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}/(\text{rad } \mathcal{O}_X)_x$ is reduced. Up to shrinking X , by [Theorem 6.3](#), we may assume that Y is reduced. But then for any $y \in Y$,

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y$$

is reduced, so

$$\mathcal{I}_y \supseteq (\text{rad } \mathcal{O}_X)_y.$$

It follows that $\text{rad } \mathcal{O}_X = \mathcal{I}$, hence the nilradical is coherent. \square

Corollary 6.5 (Cartan–Oka). Let X be a complex analytic space and A be an analytic subset of X , then the sheaf \mathcal{J}_A is coherent.

Recall that the sheaf \mathcal{J}_A is introduced in [Definition 4.4](#) in [Constructions of complex analytic spaces](#).

PROOF. By [Lemma 4.6](#) in [Constructions of complex analytic spaces](#), we may assume that A is a closed analytic subspace of X defined by a coherent ideal \mathcal{I} . By [Corollary 3.14](#) in [Constructions of complex analytic spaces](#), the sheaf \mathcal{J}_A is nothing but $\sqrt{\mathcal{I}}$, which is coherent by [Corollary 6.4](#). \square

Corollary 6.6. Let X be a complex analytic space and A be an analytic subset of X , then there is a unique reduced closed analytic space Y of X with underlying set A .

PROOF. The existence follows from [Corollary 6.5](#). The uniqueness follows from [Corollary 3.14](#) in [Constructions of complex analytic spaces](#). \square

Definition 6.7. Let X be a complex analytic space and A be an analytic subset of X . The analytic space structure on A defined in [Corollary 6.6](#) is called the *reduced induced structure* on A . In particular, $|X|$ with the reduced induced structure is denoted by X^{red} and is called the *reduced space underlying X* .

Theorem 6.8 (Generic smoothness). Let X be a reduced complex analytic space and $x \in X$, then $X_x^{\text{Sing}} \neq |X|_x$. In other words, X^{Sing} is nowhere dense in $|X|$.

PROOF. The problem is local. Take $x \in X$. As in the proof of [Theorem 3.3](#), up to shrinking X , we may assume that there are finitely many closed analytic subsets X_1, \dots, X_m in X which are irreducible at x such that

$$X = X_1 \cup \dots \cup X_m.$$

As X is reduced, we may also assume that X_1, \dots, X_m are all reduced. Then X_1, \dots, X_m are all integral at x . It follows from [Theorem 3.4](#) that

$$X_i^{\text{Sing}} \neq |X_i|_x$$

for $i = 1, \dots, m$. Let \mathcal{I}_i be the coherent ideal sheaf of X_i in X for $i = 1, \dots, m$. It follows from the proof of [Theorem 3.3](#) that

$$X^{\text{Sing}} = \bigcap_{i=1}^m \left(\text{Supp } \mathcal{I}_i \cup X_i^{\text{Sing}} \right).$$

This implies $X_x^{\text{Sing}} \neq |X|_x$: otherwise, for each $i = 1, \dots, m$, we have

$$(\text{Supp } \mathcal{I}_i)_x \cup (X_i^{\text{Sing}})_x = |X|_x.$$

So

$$(\text{Supp } \mathcal{I}_i)_x = |X|_x$$

for each $i = 1, \dots, m$. In other words,

$$\text{Spec } \mathcal{O}_{X,x} = \bigcup_{i=1}^m \text{Supp } \mathcal{I}_{i,x}.$$

Observe that $\mathcal{I}_{1,x}, \dots, \mathcal{I}_{m,x}$ are exactly the minimal primes of $\text{Spec } \mathcal{O}_{X,x}$. This is possible if and only if $m = 1$. So we are reduced to the case where X is integral at x . But this case is handled in [Theorem 3.4](#). \square

Proposition 6.9. Let X be a reduced complex analytic space and $f, g \in \mathcal{O}_X(X)$. Assume that $[f] = [g]$, then $f = g$.

PROOF. It follows from [Corollary 3.18](#) in [Constructions of complex analytic spaces](#) that $f - g$ is locally nilpotent. As X is reduced, $f = g$. \square

In particular, on a reduced complex analytic space X , a holomorphic function f is uniquely determined by the continuous map $[f] : X \rightarrow \mathbb{C}$ associated with it. In this case, we will say $[f]$ is holomorphic.

Definition 6.10. Let X be a reduced complex analytic space. A *continuous weakly holomorphic function* on X is a continuous map $f : X \rightarrow \mathbb{C}$ such that $f|_{X^{\text{reg}}}$ is holomorphic.

A *weakly holomorphic function* on X is $f \in \mathcal{O}_X(X^{\text{reg}})$ which is locally bounded on X .

Definition 6.11. Let $f : X \rightarrow Y$ be a topologically finite surjective morphism of reduced complex analytic spaces. We say f is a *branched covering* if there is a thin subset T of Y satisfying the following properties:

- (1) $\pi^{-1}(T)$ is thin in X ;
- (2) $X \setminus \pi^{-1}(T) \rightarrow Y \setminus T$ induced by f is a local isomorphism.

The set T is called a *critical locus*.

The set of points $x \in X$ where f is not a local isomorphism at x is called the *branch locus* of f . The image of the branch locus in Y is called the *minimal critical locus* of f .

Observe that the number of points in the fiber is locally constant outside the critical locus. When this number is actually constant say $b \in \mathbb{N}$ (e.g. when Y is a connected complex manifold by [Corollary 3.8](#)), we say f is a *b-sheeted branched covering*.

7. Normalness

Definition 7.1. Let X be a complex analytic space, we say X is *normal* at $x \in X$ if $\mathcal{O}_{X,x}$ is normal. We also say (X, x) or X_x is *normal* at $x \in X$.

We say X is *normal* if X is normal at all points $x \in X$.

Proposition 7.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) X is normal at x ;
- (2) $\hat{\mathcal{O}}_{X,x}$ is normal.

Condition (2) is usually known as the *analytic normality* of $\mathcal{O}_{X,x}$.

PROOF. This follows from [Proposition 4.2](#) and [Proposition 5.2](#). \square

Theorem 7.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is normal is co-analytic.

PROOF. This follows from [Corollary 5.5](#) and [Corollary 4.4](#) as reduceness is equivalent to S_2 and R_1 . \square

Proposition 7.4. Let X be a normal complex analytic space. Then for any $x \in X^{\text{Sing}}$,

$$\text{codim}_x(X^{\text{Sing}}, X) \geq 2.$$

PROOF. This follows from [Theorem 3.5](#) and the corresponding algebraic result. \square

Proposition 7.5. Let X be a reduced complex analytic space. Then there is a finite \mathcal{O}_X -algebra $\overline{\mathcal{O}}_X$ such that for each $x \in X$, $\overline{\mathcal{O}}_{X,x}$ is isomorphism to the inclusion of the integral closure $\overline{\mathcal{O}_{X,x}}$ as $\mathcal{O}_{X,x}$ -algebras.

The sheaf $\overline{\mathcal{O}}_X$ is unique up to a unique isomorphism.

PROOF. The uniqueness is obvious, as there are no non-trivial automorphisms of $\overline{\mathcal{O}}_{X,x}$ as an $\mathcal{O}_{X,x}$ -algebra.

We prove the existence. The problem is then local on X . Let $x \in X$. By [Lemma 3.7](#) in [Constructions of complex analytic spaces](#), up to shrinking X , $\overline{\mathcal{O}}_{X,x}$ spreads to a finite \mathcal{O}_X -algebra \mathcal{A} . Let $X' = \text{Spec}_X^{\text{an}} \mathcal{A}$. Let x'_1, \dots, x'_m be the distinct points on the fiber over x of $X' \rightarrow X$. By [Corollary 2.6](#) in [Constructions of complex analytic spaces](#), the points corresponds to $\text{Spm}_{\mathbb{C}} \mathcal{A}_x$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_{m'}$ be the minimal primes of $\mathcal{O}_{X,x}$, then

$$\mathcal{A}_x = \overline{\mathcal{O}_{X,x}} \cong \prod_{i=1}^{m'} \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}.$$

As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is Henselian, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is in fact local for each $i = 1, \dots, m'$. As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is finite over $\mathcal{O}_{X,x}/\mathfrak{p}_i$. It follows that $\mathrm{Spm}_{\mathbb{C}} \mathcal{A}_x = \mathrm{Spm} \mathcal{A}_x$. So we find that $m' = m$. Up to a renumbering, we may assume that \mathfrak{p}_i corresponds to x'_i for $i = 1, \dots, m$. Then by [Corollary 2.6](#) in [Constructions of complex analytic spaces](#),

$$\mathcal{O}_{X',x'_i} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$$

for $i = 1, \dots, m$. In particular, X' is normal at x'_i for all $i = 1, \dots, m$. By [Theorem 7.3](#), [Corollary 3.8](#) in [Constructions of complex analytic spaces](#) and [Lemma 4.2](#) in [Constructions of complex analytic spaces](#), up to shrinking X , we may assume that X' is normal. We observe that for each $y \in X$, \mathcal{A}_y is the product of the local rings of points on the fiber hence normal.

For $i = 1, \dots, m$, as $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, its conductor is non-zero. We can find a non-zero $f_{i,x} \in \mathcal{O}_{X,x}/\mathfrak{p}_i$ such that $f_{i,x} \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i} \subseteq \mathcal{O}_{X,x}/\mathfrak{p}_i$. Take

$$f_x = \prod_{i=1}^m f_{i,x}.$$

Then f_x is a non-zero divisor in $\mathcal{O}_{X,x}$ and $f_x \mathcal{A}_x \subseteq \mathcal{O}_{X,x}$. Up to shrinking X , we may assume that f_x spreads to $f \in \mathcal{O}_X(X)$, and we have an injection

$$f\mathcal{A} \subseteq \mathcal{O}_X.$$

Up to shrinking X , we may also assume that $\mathcal{O}_X \rightarrow \mathcal{A}$ is injective. We therefore get an injective map

$$\mathcal{A} \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we get an injective map

$$\mathcal{A}_y \rightarrow \mathcal{O}_{X,y}[f_y^{-1}].$$

In particular, \mathcal{A}_y is in the total ring of fraction of $\mathcal{O}_{X,y}$. As \mathcal{A}_y is finite over $\mathcal{O}_{X,y}$, we have

$$\mathcal{A}_y \subseteq \overline{\mathcal{O}_{X,y}}.$$

On the other hand, \mathcal{A}_y is normal, so equality holds. \square

Definition 7.6. Let X be a reduced complex analytic space. Then $\mathrm{Spec}_X^{\mathrm{an}} \overline{\mathcal{O}_X}$ constructed in [Proposition 7.5](#) is called the *normalization* of X . We denote it by \bar{X} . Note that we have a canonical morphism $\bar{X} \rightarrow X$.

The normalization of X is well-defined up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}/X$.

Proposition 7.7. Let X be a reduced complex analytic space. For each $x \in X$, the fiber of $\bar{X} \rightarrow X$ over x is in bijection with the set of minimal prime ideals in $\mathcal{O}_{X,x}$. Moreover, if y corresponds to \mathfrak{p} , we have

$$\mathcal{O}_{\bar{X},y} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}}$$

as $\mathcal{O}_{X,x}$ -algebras.

PROOF. This follows from the proof of [Proposition 7.5](#). \square

Proposition 7.8. Let X be a reduced complex analytic space. Then

- (1) \bar{X} is normal;
- (2) $p : \bar{X} \rightarrow X$ is topologically finite and surjective;

- (3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in \bar{X} and the morphism $\bar{X} \setminus p^{-1}(Y) \rightarrow X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines \bar{X} up to a unique isomorphism in $\mathbb{C}\text{-An}/X$. We will establish this result later.

PROOF. That \bar{X} is normal follows from [Corollary 2.6](#) in [Constructions of complex analytic spaces](#). The morphism $\bar{X} \rightarrow X$ is topologically finite by [Corollary 3.8](#) in [Constructions of complex analytic spaces](#). It is surjective by [Corollary 2.7](#) in [Constructions of complex analytic spaces](#).

Let Y be the non-normal locus of X . It is in particular contained in X^{Sing} . By [Proposition 7.4](#) and [Theorem 7.3](#), Y is a nowhere dense analytic set in X . It is clear that p is an isomorphism outside Y .

We prove that $p^{-1}(Y)$ is nowhere dense. Let $x \in X$ and x' be a point on the fiber of $\bar{X} \rightarrow X$ over x . Let \mathfrak{p}' be the minimal prime ideal of $\mathcal{O}_{X,x}$ corresponding to x' . Then the morphism $\text{Spec } \mathcal{O}_{\bar{X},x'} \rightarrow \text{Spec } \mathcal{O}_{X,x}$ factorizes through $\text{Spec } \mathcal{O}_{\bar{X},x'} \rightarrow \text{Spec } \mathcal{O}_{X,x}/\mathfrak{p}'$. The map is finite and surjective. The latter is because $\mathcal{O}_{X,x}/\mathfrak{p}' \rightarrow \mathcal{O}_{\bar{X},x'}$ is injective. If $p^{-1}(Y)$ contains a neighbourhood of x' in \bar{X} , then $|p^{-1}(Y)|_{x'} = \text{Spec } \mathcal{O}_{\bar{X},x'}$. Then $|Y|_x = |\text{Spec } \mathcal{O}_{X,x}/\mathfrak{p}'|$, which is a contradiction. \square

Definition 7.9. Let X be a complex analytic space and A be an analytic set in X . We say A is irreducible if A cannot be written as the union of two analytic sets B and C in X with $B \not\subset C$ and $C \not\subset B$.

Lemma 7.10. Let X be a connected normal complex analytic space. Then X is irreducible.

PROOF. Suppose otherwise, X can be written as the union of A, B , two analytic sets in X not containing each other. As X is connected, $A \cap B$ is non-trivial. Take $x \in A \cap B$. We endow A and B with the reduced induced structure. Then

$$\text{Spec } \mathcal{O}_{X,x} = \text{Spec } \mathcal{O}_{A,x} \cup \text{Spec } \mathcal{O}_{B,x}.$$

This is impossible as $\mathcal{O}_{X,x}$ is unibranch. \square

Definition 7.11. Let X be a reduced complex analytic space. An *irreducible component* of X is the image of a connected component of \bar{X} .

We say X is *irreducible* if X^{red} is non-empty and has only one irreducible component.

By [Lemma 7.10](#), each irreducible component is irreducible. Moreover, by [Proposition 7.8](#), the decomposition of $|X|$ into the union of its irreducible components is locally finite. No irreducible component is contained in the union of the others.

Proposition 7.12. Let X be a reduced complex analytic space and $x \in X$. Then x can be joined by a path to a point in X^{reg} .

PROOF. We may assume that $x \in X^{\text{Sing}}$.

Step 1. We reduce to the case where X is normal.

Let $p : \bar{X} \rightarrow X$ be the normalization. Take $y \in \bar{X}$ with $p(y) = x$.

We claim that it suffices to show that there is a path connecting y to a regular point of \bar{X} . In fact, let $T \subseteq X$ containing X^{Sing} be a thin analytic set such

that $p^{-1}(T)$ is thin and $\bar{X} \setminus p^{-1}(T) \rightarrow X \setminus T$ induced by p is an isomorphism by [Proposition 7.8](#). If our claim holds, then all neighbourhood points of y are regular and in particular, we may connect y to a regular point in $\bar{X} \setminus p^{-1}(T)$. The image of this path is the desired path.

Step 2. We proceed by induction on $d := \dim_x X$.

When $d = 1$, x is necessarily regular by [Proposition 7.4](#). Assume $d > 1$. Up to shrinking X , we can take $f \in \mathcal{O}_X(X)$ such that $\dim_x W(f) = d - 1$. We may assume that $W(f)$ is equidimensional of dimension $d - 1$ by [Theorem 2.4](#). Then we can find a path from x to a regular point $x' \in W(f)$. By [Proposition 7.4](#), up to perturbation, we may assume that $x' \in X^{\text{reg}}$. \square

8. Unibranchness

Definition 8.1. Let X be a complex analytic space, we say X is *unibranch* at $x \in X$ if $\mathcal{O}_{X,x}$ is unibranch. We also say (X, x) or X_x is *unibranch* at $x \in X$.

We say X is *unibranch* if X is unibranch at all points $x \in X$.

Proposition 8.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) X is unibranch at x ;
- (2) X^{red} is unibranch at x ;
- (3) $\mathcal{O}_{X,x}$ is geometrically unibranch;
- (4) $\mathcal{O}_{X,x}^{\text{red}}$ is geometrically unibranch;
- (5) $\mathcal{O}_{X,x}$ has a unique minimal prime ideal;
- (6) The fiber of $\bar{X}^{\text{red}} \rightarrow X^{\text{red}}$ over x consists of a single point.

PROOF. (1) \Leftrightarrow (3): As $\mathcal{O}_{X,x}$ is excellent, $\overline{\mathcal{O}_{X,x}^{\text{red}}}$ is a finite $\mathcal{O}_{X,x}^{\text{red}}$ -algebra, so the residue field extension is finite. But the residue field of $\mathcal{O}_{X,x}$ is \mathbb{C} , so the residue field extension is the trivial extension.

(1) \Leftrightarrow (5): This follows from [\[Stacks, Tag 0BQ0\]](#) and the fact that $\mathcal{O}_{X,x}$ is Henselian.

(1) \Leftrightarrow (2): This follows from the observation that (5) holds for $\mathcal{O}_{X,x}$ if and only if (5) holds for $\mathcal{O}_{X,x}^{\text{red}}$.

(3) \Leftrightarrow (4): This follows from the same argument as (1) \Leftrightarrow (2).

(5) \Leftrightarrow (6): This follows from [Proposition 7.7](#). \square

Lemma 8.3. Let X be a complex analytic space, \mathcal{M} be a coherent \mathcal{O}_X -module, $n \in \mathbb{N}$. Then the set

$$\{x \in X : \text{rank}_x \mathcal{M} \leq n\}$$

is an analytic set in X .

PROOF. The problem is local on X , we may assume that \mathcal{M} admits a presentation

$$\mathcal{O}_X^p \rightarrow \mathcal{O}_X^q \rightarrow \mathcal{M} \rightarrow 0,$$

where $p, q \in \mathbb{N}$. Up to shrinking X , we may assume that the first map is given by a $p \times q$ matrix M in $\mathcal{O}_X(X)$. The condition that $\text{rank}_x \mathcal{M} \leq n$ is the same as $\text{rank } M_x \leq n$, which defines an analytic set in X . \square

Lemma 8.4. Let X be a reduced complex analytic space and $x \in X$. Then for any neighbourhood V of x in X , we can find an open neighbourhood U of x in X

contained in V such that U has only finitely many irreducible components and all irreducible components of U contain x .

PROOF. Take an open neighbourhood W of x in X contained in V such that \bar{W} is compact and decompose W into irreducible components $W_1, \dots, W_k, W_{k+1}, \dots, W_n$, where W_1, \dots, W_k contain x and W_{k+1}, \dots, W_n do not. It suffices to take

$$U = \left(\bigcup_{i=1}^k W_i \right) \setminus \left(\bigcup_{j=k+1}^n W_j \right).$$

□

Proposition 8.5. Let X be a reduced complex analytic space and $x \in X$. Assume that X is unibranch at x . Then for any neighbourhood V of x in X , there is an open neighbourhood U of x in X contained in V such that U is unibranch and hence connected.

In particular, the unibranch locus is open.

PROOF. The assertion follows from [Lemma 8.4](#).

□

Corollary 8.6. Let X be a complex analytic space. Then X is locally connected.

PROOF. We may assume that X is reduced. The assertion follows from [Lemma 8.4](#) and

□

9. Cohen–Macaulay property

Definition 9.1. Let X be a complex analytic space, we say X is *Cohen–Macaulay* at $x \in X$ if $\mathcal{O}_{X,x}$ is Cohen–Macaulay. We also say (X, x) or X_x is *Cohen–Macaulay* at $x \in X$.

We say X is *Cohen–Macaulay* if X is Cohen–Macaulay at all points $x \in X$.

The reduction and normalization of a Cohen–Macaulay space are not necessarily Cohen–Macaulay.

Theorem 9.2. Let X be a complex analytic space. Then the set

$$\{x \in X : X \text{ is Cohen–Macaulay at } x\}$$

is co-analytic.

PROOF. The set is exactly where $\{x \in X : \text{codep}_x \mathcal{O}_{X,x} = 0\}$, which is co-analytic by [Proposition 5.3](#).

□

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