Berkovich analytic spaces

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Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 2.1. Let A be a k_H-affinoid algebra. A *compact k_H-analytic domain V* in $\operatorname{Sp} A$ is a finite union of k_H-affinoid domains in $\operatorname{Sp} A$.

Lemma 2.2. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain. Write $\operatorname{Sp} A$ as a finite union of k_H-affinoid domains $\operatorname{Sp} A_i$ with $i = 1, \dots, n$ in $\operatorname{Sp} A$. Define $A_{ij} = A_i \hat{\otimes}_A A_j$ and

$$A_V := \ker\left(\prod_{i=1}^n A_i \to \prod_{i,j=1}^n A_{ij}\right).$$

Then the Banach k-algebra does not depend on the choice of the covering $\{\operatorname{Sp} A_i\}_i$ up to a canonical isomorphism.

The image of the natural continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ contains V and the map does not depend on the choice of the covering up to the canonical isomorphism between $\operatorname{Sp} A_V$ for different coverings.

Proof. We first observe that A_V is a Banach k-algebra as it is defined as an equalizer. This follows from Lemma 4.22.

Let $\{\operatorname{Sp} B_j\}_{j=1,\dots,m}$ be another k_H-affinoid covering of $\operatorname{Sp} A$. We need to show that A_V defined using the two coverings are canonically isomorphic. We write A'_V for

$$\ker\left(\prod_{j=1}^m B_j \to \prod_{i,j=1}^m B_{ij}\right)$$

to make a distinction. We write $B_{ij} = B_i \hat{\otimes}_A B_j$.

By Theorem 12.16 in the chapter Affinoid Algebras, the colomns in the following commutative diagram are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A_V \arrow[r] \arrow[d, dotted] & \prod_{i=1}^n A_i \arrow[r] \arrow[d, "\eta"] & \prod_{i,i'=1}^n A_{ii'} \arrow[d] \\
0 \arrow[r] & \ker \iota \arrow[r] & \prod_{i=1}^n \prod_{j=1}^m A_i \hat{\otimes}_A B_j \arrow[r, "\iota"] \arrow[d, "\tau"] & \prod_{i,i'=1}^n \prod_{j,j'=1}^m A_{ii'} \hat{\otimes}_A B_{jj'} \\
 & & \prod_{i=1}^n \prod_{j,j'=1}^m A_i \hat{\otimes}_A B_{jj'} &
\end{tikzcd}$$

The rows are exact by definition. By diagram chasing, the dotted arrow is injective. To see it is surjective, it suffices to observe that the factors with $i = i'$ in the lower right corner is exactly the same as the factors of the lower corner, so an element in $\ker \iota$ is necessarily in $\ker \tau$. It follows that the dotted arrow is surjective.

Similarly, we have a natural isomorphism $A'_V \xrightarrow{\sim} \ker \iota$. We conclude the first assertion.

As for the second, observe that $\operatorname{Sp} A_V$ is defined as a colimit in the category of Banach k-algebras, so it follows from general abstract nonsense that there is a natural morphism $\operatorname{Sp} A_V \to \operatorname{Sp} A$. It clearly contains V in the image. The compatibility with the isomorphism above follows simply from the fact that the map η is an A-algebra homomorphism. $\square$

Definition 2.3. Let A be a k-affinoid algebra and V be a compact k-analytic domain in $\operatorname{Sp} A$. We define *the Banach k-algebra A_V associated with V* as A_V constructed in Lemma 2.2.

The continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ constructed in Lemma 2.2 is called the *structure map* ov V.

Proposition 2.4. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain in $\operatorname{Sp} A$. Then the following are equivalent:

(1) V is a k_H-affinoid domain.
(2) A_V is a k_H-affinoid algebra and the image of the structure map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is exactly V.

Proof. (1) $\implies$ (2): By Theorem 12.16 in the chapter Affinoid Algebras, when V is a k_H-affinoid domain, A_V is a k_H-affinoid algebra and the structure map corresponds to the inclusion of the k_H-affinoid domain. There is nothing to prove.

(2) $\implies$ (1): It suffices to show that the structure map represents the k_H-affinoid domain V. Take a k_H-affinoid algebra D and a morphism $\operatorname{Sp} D \to \operatorname{Sp} A$ of k_H-affinoid spaces that factorizes through V. We need to construct a morphism $\operatorname{Sp} D \to \operatorname{Sp} A_V$ making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} A_V \arrow[r] & \operatorname{Sp} A \end{tikzcd} .$$

Take k_H-affinoid domains $\operatorname{Sp} B_1, \ldots, \operatorname{Sp} B_n$ in $\operatorname{Sp} A$ that cover V. Let $C_i = B_i \hat{\otimes}_A D$ for $i = 1, \ldots, n$, then $\operatorname{Sp} C_i$ is a k_H-affinoid domain in $\operatorname{Sp} D$ by Corollary 12.12 in the chapter Affinoid Algebras. By Theorem 12.16 in the chapter Affinoid Algebras and general abstract nonsense, it suffices to construct the dotted arrow after restricting to $\operatorname{Sp} C_i$ for $i = 1, \ldots, n$. So we could assume that $\operatorname{Sp} D \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B_1$. From the universal property, we therefore have the dotted morphism making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B_1 \arrow[r] & \operatorname{Sp} A \end{tikzcd} .$$

It suffices to show that the natural homomorphism

$$B_1 \to A_V \hat{\otimes}_A B_1$$

is an isomorphism. But this follows from general abstract nonsense as B_1 is already a Banach A_V-algebra. $\square$

Remark 2.5. This proposition is not correctly stated in [Ber12, Corollary 2.2.6]. The corresponding statement in [Ber93, Remark 1.2.1] is slightly weaker than our statement.

3. Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. Let X be a locally Hausdorff space and τ be a net of compact subsets. A *k_H-affinoid atlas* $\mathcal{A}$ on X with the net τ is a map which assigns

(1) to each $V \in \tau$, a k_H-affinoid algebra A_V and a homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$;
(2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of k_H-affinoid algebras $\alpha_{V/U} : A_V \to A_U$ representing a k_H-affinoid domain $\operatorname{Sp} A_U$ in $\operatorname{Sp} A_V$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Sp} A_U \arrow[r, "\operatorname{Sp} \alpha_{V/U}"] \arrow[d, "\varphi_U"] & \operatorname{Sp} A_V \arrow[d, "\varphi_V"] \\ U \arrow[r] & V \end{tikzcd} .$$

The triple $(X, \mathcal{A}, \tau)$ as above is called a *k_H-analytic space*.

A *morphism* between atlases $\mathcal{A}$ and $\mathcal{A}'$ on X with the net τ is an assignment that with each $V \in \tau$, one associates a morphism of k_H-affinoid algebras $\beta_V : A_V \to A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Sp} A'_V \arrow[r, "\operatorname{Sp} \beta_V"] \arrow[d, "\varphi'_V"] & \operatorname{Sp} A_V \arrow[dl, "\varphi_V"] \\ V & \end{tikzcd} ;$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{tikzcd} A_V \arrow[r, "\alpha_{V/U}"] \arrow[d, "\beta_V"] & A_U \arrow[d, "\beta_U"] \\ A'_V \arrow[r, "\alpha'_{V/U}"] & A'_U \end{tikzcd}$$

Here we have denoted the data associated with $\mathcal{A}'$ with a prime. In this way, the atlases on X with the net τ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), it $W \subseteq U \subseteq V$ are three sets in τ, then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

Remark 3.2. As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps φ_U's by identifying $\operatorname{Sp} A_U$ with U. We will say U is a k_H-affinoid domain in V.

Remark 3.3. Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

Lemma 3.4. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space, $U \in \tau$ and W is a k_H-affinoid domain in U. Then for any $V \in \tau$ containing W, W is a k_H-affinoid domain in V.

Proof. As $\tau|_{U\cap V}$ is a net and W is compact, we can find $U_1,\dots,U_n\in\tau_{U\cap V}$ with $W\subseteq U_1\cup\cdots\cup U_n$. As W, U_i are k_H-affinoid domains in U, $W_i=W\cap U_i$ is a k_H-affinoid domain in U_i for all $i=1,\dots,n$ by Corollary 12.12 in the chapter Affinoid Algebras. It follows from Corollary 9.6 and Corollary 12.12 in the chapter Affinoid Algebras that W_i and $W_i\cap W_j$ are both k_H-affinoid domains in V for $i,j=1,\dots,n$. So W is a compact k_H-analytic domain in V.

By Proposition 2.4,

$$A_W:=\ker\left(\prod_{i=1}^n A_{W_i}\to\prod_{i,j=1}^n A_{W_i\cap W_j}\right)$$

is k_H-affinoid and $\operatorname{Sp}A_W\to\operatorname{Sp}A$ induces a hoemomorphism $\operatorname{Sp}A_W\to W$ by Proposition 9.5 in the chatper Affinoid Algebras. By Proposition 2.4 again, W is affinoid in V. □

Definition 3.5. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. We define $\bar\tau$ as the set of all $W\subseteq X$ such that there is $U\in\tau$ containing W and W is k_H-affinoid in U.

Lemma 3.6. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. Then $\bar\tau$ is a net on X and there is a k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar\tau$ extending $\mathcal{A}$. Moreover, the k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar\tau$ extending $\mathcal{A}$ is unique up to a canonical isomorphism.

Proof. **Step 1**. We first show that $\bar\tau$ is a net. Let $U,V\in\bar\tau$ and $x\in U\cap V$. Take $U',V'\in\tau$ containing U and V respectively. Take $n\in\mathbb{Z}_{>0}$ and $W_1,\dots,W_n\in\tau$ such that

(1) $x\in W_1\cap\cdots\cap W_n$;
(2) $W_1\cup\cdots\cup W_n$ is a neighbourhood of x in $U'\cap V'$.

This is possible because $\tau|_{U'\cap V'}$ is a quasi-net by assumption.

By Lemma 3.4, U (resp. V) and $W_1,\dots,W_n$ are k_H-affinoid domains in U' (resp. V').

By Corollary 12.12 in the chapter Affinoid Algebras, $U_i:=U\cap W_i$ (resp. $V_i:=V\cap W_i$) is a k_H-affinoid domain in W_i for $i=1,\dots,n$. By Corollary 12.12 in the chapter Affinoid Algebras again, $U_i\cap V_i$ is a k_H-affinoid domain in W_i for $i=1,\dots,n$. So $U_i\cap V_i\in\bar\tau|_{U\cap V}$ for $i=1,\dots,n$. But

$$\bigcup_{i=1}^n U_i\cap V_i=(U\cap V)\cap\bigcup_{i=1}^n W_i,$$

so $\bigcup_{i=1}^n U_i\cap V_i$ is a neighbourhood of x in $U\cap V$ and $x\in\bigcap_{i=1}^n U_i\cap V_i$. It follows that $\bar\tau$ is a net.

Step 2. We extend the k_H-affinoid atlas $\mathcal{A}$.

For each $V\in\bar\tau$, we fix a $V'\in\tau$ containing V.

By Lemma 3.4, V is a k_H-affinoid domain in V'. Let $A_{V'}\to A_V$ be the morphism of k_H-affinoid algebras representing the k_H-affinoid domain V in $\operatorname{Sp}A_{V'}$. We define the homeomorphism $\varphi_V:\operatorname{Sp}A_V\to V$ as the morphism induced by $\operatorname{Sp}A_V\to\operatorname{Sp}A$.

For $U,V\in\bar\tau$ with $U\subseteq V$, we want to define $\alpha_{V/U}:A_V\to A_U$. We handle two cases. When $V\in\tau$, as $\tau|_{U'\cap V}$ is a quasi-net, we can find $n\in\mathbb{Z}_{>0}$ and $U_1,\dots,U_n\in\tau|_{U'\cap V}$ such that

$$U=\bigcup_{i=1}^n U_i.$$

By Lemma 3.4, $U_1, \dots, U_n$ are k_H-affinoid domains in U' and in V. By Theorem 12.16 in the chapter Affinoid Algebras,

$$A_U \xrightarrow{\sim} \ker\left(\prod_{i=1}^{n} A_{U_i} \to \prod_{i,j=1}^{n} A_{U_i \cap U_j}\right).$$

So the morphism $\alpha_{V/U_i} : A_V \to A_{U_i}$ and $A_{V/U_i \cap U_j} : \alpha_{V/U_i} : A_V \to A_{U_i \cap U_j}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ induces a morphism $\alpha_{V/U} : A_V \to A_U$. Observe that $\alpha_{V/U}$ represents the k_H-affinoid domain U in V, so it is independent of the choice of $U_1, \dots, U_n$.

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \to A_U$ representing the k_H-affinoid domain U in V', it follows that U is a k_H-affinoid domain in V, and we therefore get the desired morphism $\alpha_{V/U} : A_V \to A_U$.

It is easy to verify that the constructions gives a k_H-affinoid atlas with the net $\bar{\tau}$ extending $\mathcal{A}$. The uniqueness of the extension is immediate. □

Definition 3.7. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of

(1) a continuous map $\varphi : X \to X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
(2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq W'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd}.$$

Recall our convention Remark 3.2, the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \to A_V$ of k_H-affinoid algebras making the following diagram commutative

$$\begin{tikzcd} \mathrm{Sp}\, A_V \arrow[r] \arrow[d, "\varphi_V"] & \mathrm{Sp}\, A'_{V'} \arrow[d, "\varphi'_{V'}"] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}.$$

We will continue our identifications as in Remark 3.2 to simplify our notations.

Bibliography

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