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# Commutative algebras

## 1. Introduction

### 2. Graded commutative algebra

Let  $G$  be an Abelian group. We write the group operation of  $G$  multiplicatively and denote the identity of  $G$  as 1.

**Definition 2.1.** Let  $A$  be an Abelian group. A  $G$ -grading on  $A$  is a coproduct decomposition

$$A = \bigoplus_{g \in G} A_g$$

of Abelian groups such that  $A_g \subseteq A$ . An Abelian group with a  $G$ -grading is called a  $G$ -graded Abelian group.

An element  $a \in A$  is said to be *homogeneous* if there is  $g \in G$  such that  $a \in A_g$ . If  $a$  is furthermore non-zero, we write  $g = \rho(a)$ . We set  $\rho(0) = 0$ . We will write  $\rho(A)$  for the set of  $\rho(a)$  when  $a$  runs over all homogeneous elements in  $A$ .

A  $G$ -graded homomorphism between  $G$ -graded Abelian groups  $A$  and  $B$  is a homogeneous of the underlying Abelian groups  $f : A \rightarrow B$  such that  $f(A_g) \subseteq B_g$  for any  $g \in G$ .

The category of  $G$ -graded Abelian groups is denoted by  $\mathcal{Ab}^G$ .

**Remark 2.2.** A usual Abelian group  $A$  can be given the *trivial  $G$ -grading*:  $A_0 = A$  and  $A_g = 0$  for  $g \in G, g \neq 0$ . In this way, we find a fully faithful embedding

$$\mathcal{Ab} \rightarrow \mathcal{Ab}^G.$$

When we regard an Abelian group as a  $G$ -graded Abelian group and there are no natural gradings, we always understand that we are taking the trivial  $G$ -grading.

More generally, let  $G'$  be a subgroup of  $G$ . Then any  $G'$ -graded Abelian group can be canonically identified with a  $G$ -graded Abelian group: for the extra pieces in the grading, we simply put 0.

Conversely, if  $G'$  be a subgroup of  $G$  and  $A$  is a  $G$ -graded Abelian group, we write

$$A^{G'} := \bigoplus_{g \in G'} A_g.$$

This is a  $G'$ -graded Abelian group.

The same remark applies to all the other constructions in this section, which we will not repeat.

**Definition 2.3.** A  $G$ -graded ring is a commutative ring  $A$  endowed with a  $G$ -grading:

$$A = \bigoplus_{g \in G} A_g$$

as Abelian groups and such that

- (1)  $A_g A_h \subseteq A_{gh}$  for any  $g, h \in G$ ;
- (2)  $1 \in A_1$ .

A *G-graded homomorphism* of *G*-graded rings  $A$  and  $B$  is a ring homomorphism  $f : A \rightarrow B$  such that  $f(A_g) \subseteq B_g$  for each  $g \in G$ . A *G-graded subring* of a *G*-graded ring  $B$  is a subring  $A$  of  $B$  such that the grading on  $B$  restricts to a grading on  $A$ .

The category of *G*-graded rings is denoted by  $\mathcal{R}ing^G$ .

**Example 2.4.** Let  $A$  be a *G*-graded ring,  $n \in \mathbb{N}$  and  $g = (g_1, \dots, g_n) \in G^n$ . Then there is a unique *G*-grading on  $A[T_1, \dots, T_n]$  extending the grading on  $A$  and such that  $\rho(T_i) = g_i$  for  $i = 1, \dots, n$ . We will denote  $A[T_1, \dots, T_n]$  with this grading as  $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n]$  or simply  $A[g^{-1}T]$ .

**Example 2.5.** Let  $A$  be a *G*-graded ring and  $S$  be a multiplicative subset of  $A$  consisting of homogeneous elements, then  $S^{-1}A$  has a natural *G*-grading. To see this, recall the construction of  $S^{-1}A$  in [Stacks, Tag 00CM]. One defines an equivalence relation on  $A \times S$ :  $(x, s) \sim (y, t)$  if there is  $u \in S$  such that  $(xt - ys)u = 0$ . For each  $g \in G$ , we define  $(S^{-1}A)_g$  as the set of  $(x, s)$  for all  $s \in S$  and  $x \in A_{g\rho(s)}$ . It is easy to verify that this is a well-defined *G*-grading on  $S^{-1}A$ . [Add details.](#)

In particular, if  $f \in A$  is a non-zero homogeneous element, then we define  $A_f$  as  $S^{-1}f$  with  $S = \{f^n : n \in \mathbb{N}\}$ .

**Definition 2.6.** Let  $A$  be a *G*-graded ring. A *G-homogeneous ideal* in  $A$  is an ideal  $I$  in  $G$  such that if  $a \in A$  can be written as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all  $a_g = 0$ , then  $a_g \in I$ .

**Example 2.7.** Let  $A$  be a *G*-graded ring and  $n \in \mathbb{N}$  and  $a_1, \dots, a_n$  be homogeneous elements in  $A$ . Then  $a_1, \dots, a_n$  generate a *G*-homogeneous ideal  $(a_1, \dots, a_n)$  as follows:

$$(a_1, \dots, a_n)_g = \sum_{i=1}^n A_{g\rho(a_i)-1} a_i$$

for any  $g \in G$ .

**Lemma 2.8.** Let  $f : A \rightarrow B$  be a *G*-homomorphism of *G*-graded rings. Then  $\ker f$  is a *G*-homogeneous ideal in  $A$ .

PROOF. We need to show that

$$\ker f = \sum_{g \in G} (\ker f) \cap A_g.$$

Take  $x \in \ker f$ , we can write  $x$  as

$$\sum_{g \in G} a_g, \quad a_g \in A_g$$

and almost all  $a_g$ 's are 0. Then

$$f(x) = \sum_{g \in G} f(a_g), \quad f(a_g) \in B_g.$$

It follows that  $f(a_g) = 0$  for each  $g \in G$  and hence  $a_g \in (\ker f) \cap A_g$ . □

**Definition 2.9.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then we define a  $G$ -grading on  $A/I$  as follows: for any  $g \in G$

$$(A/I)_g := (A_g + I)/I.$$

**Proposition 2.10.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then the construction in [Definition 2.9](#) defines a grading on  $A/I$ . The natural map  $\pi : A \rightarrow A/I$  is a  $G$ -homomorphism.

For any  $G$ -graded ring  $B$  and any  $G$ -homomorphism  $f : A \rightarrow B$  such that  $I \subseteq \ker f$ , there is a unique  $G$ -homomorphism  $f' : A/I \rightarrow B$  such that  $f' \circ \pi = f$ .

**PROOF.** We first argue that for different  $g, h \in G$ ,  $(A/I)_g \cap (A/I)_h = 0$ . Suppose  $x \in (A/I)_g \cap (A/I)_h$ , we can lift  $x$  to both  $y_g + i_g \in A$  and  $y_h + i_h \in A$  with  $y_g, y_h \in A$  and  $i_g, i_h \in I$ . It follows that  $y_g - y_h \in I$ . But  $I$  is a  $G$ -homogeneous ideal, so it follows that  $y_g, y_h \in I$  and hence  $x = 0$ .

Next we argue that

$$A/I = \sum_{g \in G} (A/I)_g.$$

Lift an element  $x \in A/I$  by  $a \in A$ , we represent  $a$  as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all  $a_g$ 's equal to 0. Then  $x$  can be represented as

$$x = \sum_{g \in G} \pi(a_g).$$

We have shown that the construction in [Definition 2.9](#) gives a  $G$ -grading on  $A$ . It is clear from the definition that  $\pi$  is a  $G$ -homomorphism.

Next assume that  $B$  and  $f$  are given as in the proposition. Then there is a ring homomorphism  $f' : A/I \rightarrow B$  such that  $f = f' \circ \pi$ . We need to argue that  $f'$  is a  $G$ -homomorphism. For this purpose, take  $g \in G$ ,  $x \in (A/I)_g$ , we need to show that  $f'(x) \in B_g$ . Lift  $x$  to  $y + i$  with  $y \in A_g$  and  $i \in I$ , then we know that  $f'(x) = \pi(y + i) = \pi(y) \in B_g$ .  $\square$

**Definition 2.11.** Let  $A$  be a  $G$ -graded ring.

Let  $M$  an  $A$ -module which is also a  $G$ -graded Abelian group. We say  $M$  is a  $G$ -graded  $A$ -module if for each  $g, h \in G$ , we have

$$A_g M_h \subseteq M_{gh}.$$

A  $G$ -graded homomorphism of  $G$ -graded  $A$ -modules  $M$  and  $N$  is an  $A$ -module homomorphism  $f : M \rightarrow N$  which is at the same time a homomorphism of the underlying  $G$ -graded Abelian groups.

The category of  $G$ -graded  $A$ -modules is denoted by  $\text{Mod}_A^G$ .

A  $G$ -graded  $A$ -algebra is a  $G$ -graded ring  $B$  together with a  $G$ -graded ring homomorphism  $A \rightarrow B$  such that  $B$  is also a  $G$ -graded  $A$ -module.

A  $G$ -graded homomorphism between  $G$ -graded  $A$ -algebras  $B$  and  $C$  is a  $G$ -graded homomorphism between the underlying  $G$ -graded rings that is at the same time a  $G$ -graded homomorphism of  $G$ -graded  $A$ -modules.

Observe that  $G$ -homogeneous ideals of  $A$  are  $G$ -graded submodules of  $A$ . Also observe that  $\text{Mod}_{\mathbb{Z}}^G$  is isomorphic to  $\text{Ab}^G$ .

**Proposition 2.12.** Let  $A$  be a  $G$ -graded ring. Then  $\mathcal{Mod}_A^G$  is an Abelian category satisfying AB5.

PROOF. We first show that  $\mathcal{Mod}_A^G$  is preadditive. Given  $M, N \in \mathcal{Mod}_A^G$ , we can regard  $\text{Hom}_{\mathcal{Mod}_A^G}(M, N)$  as a subgroup of  $\text{Hom}_A(M, N)$ . It is easy to see that this gives  $\mathcal{Mod}_A^G$  an enrichment over  $\mathcal{Ab}$ .

Next we show that  $\mathcal{Mod}_A^G$  is additive. The zero object is clearly given by 0 with the trivial grading. Given  $M, N \in \mathcal{Mod}_A^G$ , we define

$$(M \oplus N)_g := M_g \oplus N_g, \quad g \in G.$$

This construction makes  $M \oplus N$  a  $G$ -graded  $A$ -module. It is easy to verify that  $M \oplus N$  is the biproduct of  $M$  and  $N$ .

Next we show that  $\mathcal{Mod}_A^G$  is pre-Abelian. Given an arrow  $f : M \rightarrow N$  in  $\mathcal{Mod}_A^G$ , we need to define its kernel and cokernel. We define

$$(\ker f)_g := (\ker f) \cap M_g$$

and  $(\text{coker } f)_g$  as the image of  $N_g$  for any  $g \in G$ . It is straightforward to verify that these are kernels and cokernels.

Next, given a monomorphism  $f : M \rightarrow N$ , it is obvious that the map  $f$  is injective and  $f$  can be identified with the kernel of the natural map  $N/\text{Im } f$ . A dual argument shows that an epimorphism is the cokernel of some morphism as well.

The AB5 condition is easily verified. **Expand the details of this argument!**  $\square$

Next we define the tensor product of  $G$ -graded modules.

**Definition 2.13.** Let  $A$  be a  $G$ -graded ring and  $M, N$  be  $G$ -graded  $A$ -modules. We define a  $G$ -grading on  $M \otimes_A N$  as follows: for any  $g \in G$ ,  $(M \otimes_A N)_g$  is defined as the image of  $\sum_{h \in G} M_h \times N_{gh^{-1}}$  in  $M \otimes_A N$ . We always endow  $M \otimes_A N$  with this  $G$ -grading.

**Verify the universal property; show that this is indeed a grading**

**Example 2.14.** This is a continuation of **Example 2.5**. Let  $A$  be a  $G$ -graded ring and  $S$  be a multiplicative subset of  $A$  consisting of homogeneous elements. Consider a  $G$ -graded  $A$ -module  $M$ . We define a  $G$ -grading on  $S^{-1}M$ . Recall that  $S^{-1}M$  can be realized as follows: one defines an equivalence relation on  $M \times S$ :  $(x, s) \sim (y, t)$  if there is  $u \in S$  such that  $(xt - ys)u = 0$ . For each  $g \in G$ , we define  $(S^{-1}M)_g$  as the set of  $(x, s)$  for all  $s \in S$  and  $x \in M_{g\rho(s)}$ . It is easy to verify that this is a well-defined  $G$ -grading on  $S^{-1}M$  and  $S^{-1}M$  is a  $G$ -graded  $S^{-1}A$ -module. **Add details.**

**Example 2.15.** Let  $A$  be a  $G$ -graded ring and  $g \in G$ . We define  $g^{-1}A$  as the  $G$ -graded  $A$ -module:

$$(g^{-1}A)_h = A_{g^{-1}h}$$

for any  $h \in G$ . Observe that  $1 \in (g^{-1}A)_g$ .

**Definition 2.16.** Let  $A$  be a  $G$ -graded ring and  $M$  be a  $G$ -graded  $A$ -module. We say  $M$  is *free* if there exists a family  $\{g_i\}_{i \in I}$  in  $G$  such that

$$M = \coprod_{i \in I} g_i^{-1}A.$$



**Definition 2.17.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. We say  $f$  is *finite* (resp. *finitely generated*, resp. *integral*) if it is finite (resp. finitely generated, resp. integral) as a usual ring map.

**Proposition 2.18.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. Then

- (1)  $f$  is finite if and only if there are  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G$  and a surjective  $G$ -graded homomorphism

$$\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$$

of graded  $A$ -modules.

- (2)  $f$  is finitely generated if and only if there are  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G$  and a surjective  $G$ -graded  $A$ -algebra homomorphism

$$A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B.$$

- (3)  $f$  is integral if and only if for any non-zero homogeneous element  $b \in B$ , there is  $n \in \mathbb{N}$  and homogeneous elements  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

- (4) A non-zero homogeneous element  $b \in B$  is integral over  $A$  if there is  $n \in \mathbb{N}$  and homogeneous elements  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

PROOF. (1) The non-trivial direction is the direct implication. Assume that  $f$  is finite. Take  $b_1, \dots, b_n \in B$  so that  $\sum_{i=1}^n f(A)b_i = B$ . Up to replacing the collection  $\{b_i\}_i$  by the finite collection of non-zero homogeneous components of the  $b_i$ 's, we may assume that each  $b_i$  is homogeneous. We define  $g_i = \rho(b_i)$  and the map  $\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$  sends 1 at the  $i$ -th place to  $b_i$ .

(2) The non-trivial direction is the direct implication. Suppose  $f$  is finitely generated, say by  $b_1, \dots, b_n$ . Up to replacing the collection  $\{b_i\}_i$  by the finite collection of non-zero homogeneous components of the  $b_i$ 's, we may assume that each  $b_i$  is homogeneous. Then we define  $g_i = \rho(b_i)$  for  $i = 1, \dots, n$  and the  $A$ -algebra homomorphism  $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B$  sends  $T_i$  to  $b_i$  for  $i = 1, \dots, n$ .

(3) Assume that  $f$  is integral, then for any non-zero homogeneous element  $b \in B$ , we can find  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

Obviously, we can replace  $a_i$  by its component in  $\rho(b)^i$  for  $i = 1, \dots, n$  and the equation remains true.

The reverse direction follows from [Stacks, Tag 00GO].

- (4) This is argued in the same way as (3). □

**Definition 2.19.** A  $G$ -graded ring  $A$  is a  $G$ -graded field if

- (1)  $A \neq 0$ .  
(2)  $A$  does not admit any non-zero proper  $G$ -homogeneous ideals.

**Proposition 2.20.** Let  $A$  be a non-zero  $G$ -graded ring. Then the following conditions are equivalent:

- (1)  $A$  is a  $G$ -graded field.

(2) Any non-zero homogeneous element in  $A$  is invertible.

PROOF. Assume that  $A$  is a  $G$ -graded field. Let  $a \in A$  be a non-zero homogeneous element. Consider the  $G$ -homogeneous ideal  $(a)$  generated by  $a$  as in [Example 2.7](#). As  $a \neq 0$ , it follows that  $(a) = 1$ . Hence,  $a$  is invertible.

Conversely, suppose that any non-zero homogeneous element in  $A$  is invertible. If  $I$  is a non-zero  $G$ -homogeneous ideal in  $A$ . There is a non-zero homogeneous element  $a \in I$ . But we know that  $a$  is invertible and hence  $I = A$ .  $\square$

**Definition 2.21.** A  $G$ -graded ring  $A$  is an *integral domain* if for any non-zero homogeneous elements  $a, b \in A$ ,  $ab \neq 0$ .

**Lemma 2.22.** Let  $A$  be a  $G$ -graded integral domain. Let  $S$  denote the set of non-zero homogeneous elements in  $A$ . Then  $S^{-1}A$  is a graded field. The natural map  $A \rightarrow S^{-1}A$  is injective.

Recall that  $S^{-1}A$  is defined in [Example 2.5](#).

PROOF. By [Proposition 2.20](#), it suffices to show that each non-zero homogeneous element in  $S^{-1}A$  is invertible. Such an element has the form  $a/s$  for some homogeneous element  $a \in A$  and  $s \in S$ . As  $A$  is a  $G$ -graded integral domain,  $a$  is invertible and hence  $s/a \in S^{-1}A$ .

In general, the kernel of the localization map is given by  $\{a \in A : \text{there is } s \in S \text{ such that } sa = 0\}$ . As  $A \rightarrow S^{-1}A$  is a  $G$ -graded homomorphism, the kernel is in addition a  $G$ -homogeneous ideal in  $A$  by [Lemma 2.8](#). So it suffices to show that each homogeneous element in the kernel vanishes: if  $a \in A$  is a homogeneous element and there is  $s \in S$  such that  $sa = 0$ , then  $a = 0$ . Otherwise,  $a$  is invertible by [Proposition 2.20](#), which is a contradiction.  $\square$

**Definition 2.23.** Let  $A$  be a  $G$ -graded integral domain. We call the graded field defined in [Lemma 2.22](#) the *fraction  $G$ -graded field* of  $A$  and denote it by  $\text{Frac}^G A$ .

**Definition 2.24.** Let  $A$  be a  $G$ -graded ring. A proper  $G$ -homogeneous ideal  $I$  in  $A$  is called *prime* if the  $G$ -graded ring  $A/I$  is a  $G$ -graded integral domain.

**Proposition 2.25.** Let  $A$  be a  $G$ -graded ring and  $I$  be a proper homogeneous ideal in  $A$ . Then the following are equivalent:

- (1)  $I$  is a  $G$ -graded prime ideal in  $A$ .
- (2) For any homogeneous elements  $a, b \in A$  satisfying  $ab \in I$ , at least one of  $a$  and  $b$  lies in  $I$ .

PROOF. Assume that  $I$  is a  $G$ -graded prime ideal in  $A$ . Let  $a, b \in A$  be homogeneous elements satisfying  $ab \in I$ . Let  $\bar{a}, \bar{b}$  be the images of  $a, b$  in  $A/I$ . Then  $\bar{a}, \bar{b}$  are homogeneous and  $\bar{a}\bar{b} = 0$ . So at least one of  $\bar{a}$  and  $\bar{b}$  is zero. That is,  $a$  or  $b$  lies in  $I$ .

Conversely, assume that the condition in (2) is satisfied. Take  $x, y \in A/I$  with  $xy = 0$ . We need to show that at least one of  $x$  and  $y$  is 0. Lift  $x$  and  $y$  to  $a + i$  and  $b + i'$  in  $A$  with  $a, b$  being homogeneous and  $i, i' \in I$ . Then  $ab \in I$  and hence  $a \in I$  or  $b \in I$ . It follows that  $x = 0$  or  $y = 0$ .  $\square$

**Definition 2.26.** Let  $A$  be a  $G$ -graded ring and  $\mathfrak{p}$  be a prime  $G$ -homogeneous ideal in  $A$ . Then we define the  *$G$ -graded localization*  $A_{\mathfrak{p}}^G$  of  $A$  at  $\mathfrak{p}$  as  $S^{-1}A$ , where  $S$  is the set of homogeneous elements in  $A \setminus \mathfrak{p}$ .

Similarly, let  $M$  be a  $G$ -graded  $A$ -module. We define the  $G$ -graded localization  $M_{\mathfrak{p}}^G$  as  $S^{-1}M$ .

Recall that  $S^{-1}A$  and  $S^{-1}M$  are defined in [Example 2.5](#) and [Example 2.14](#).

**Definition 2.27.** Let  $A$  be a  $G$ -graded ring.

A  $G$ -homogeneous ideal  $I$  in  $A$  is said to be *maximal* if it is proper, and it is not contained in any other proper  $G$ -homogeneous ideals.

We call  $A$  a  $G$ -graded local ring if it has a unique maximal homogeneous ideal. This ideal is called the *maximal  $G$ -homogeneous ideal* of  $A$ .

**Proposition 2.28.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then the following are equivalent:

- (1)  $I$  is a maximal  $G$ -homogeneous ideal in  $A$ ;
- (2)  $A/I$  is a  $G$ -graded field.

In particular, a maximal  $G$ -homogeneous ideal is a prime  $G$ -homogeneous ideal.

PROOF. Assume (1). Then  $I$  is a proper ideal, so  $A/I$  is non-zero. Suppose that  $A/I$  has a proper  $G$ -homogeneous ideal  $J$ , it lifts to an ideal  $J'$  of  $A$ . We claim that  $J'$  is  $G$ -homogeneous. In fact, we set  $J'_g := \{x \in A_g : x + I \in J\}$  for  $g \in G$ , we need to show that

$$J' = \sum_{g \in G} J'_g.$$

For any  $j \in J'$ , we can expand  $j + I$  as  $\sum_{g \in G} a_g + I$  with  $a_g \in A_g$  and almost all  $a_g$ 's are 0. We take  $i \in I$  so that

$$j = i + \sum_{g \in G} a_g.$$

The desired equation follows. But then it follows that  $J' = I$  and hence  $J = 0$ .

Assume (2). Then  $I$  is a proper ideal in  $A$ . If  $J$  is a  $G$ -homogeneous proper ideal of  $A$  containing  $I$ , then  $J/I$  is a  $G$ -homogeneous proper ideal of  $A/I$ . It follows that  $J/I = 0$  and hence  $J = I$ .  $\square$

**Corollary 2.29.** Let  $A$  be a non-zero  $G$ -graded ring, then  $A$  admits a prime  $G$ -homogeneous ideal.

PROOF. By our assumption,  $0$  is a proper ideal in  $A$ . By Zorn's lemma,  $A$  admits a maximal  $G$ -homogeneous ideal, which is prime by [Proposition 2.28](#).  $\square$

**Proposition 2.30.** Let  $A$  be a  $G$ -graded ring and  $a \in A$  be a homogeneous element. Then  $a$  is a unit in  $A$  if and only if  $a$  is not contained in any maximal  $G$ -homogeneous ideal of  $A$ .

PROOF. The direct implication is trivial. Assume that  $a$  is not a unit. Then the ideal  $(a)$  generated by  $a$  is  $G$ -homogeneous. By Zorn's lemma, there is a maximal  $G$ -homogeneous ideal containing  $(a)$ .  $\square$

**Lemma 2.31.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. Let  $b_1, \dots, b_n \in B$  be a finite set of homogeneous elements integral over  $A$ , then there is a  $G$ -graded  $A$ -subalgebra  $B' \subseteq B$  containing  $b_1, \dots, b_n$  such that  $A \rightarrow B'$  is finite.

PROOF. We may assume that none of the  $b_i$ 's is zero. By [Proposition 2.18](#), we can find  $m_1, \dots, m_n \in \mathbb{N}$  and homogeneous elements  $a_{i,j} \in A$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$  such that

$$b_i^{m_i} + f(a_{i,1})b_i^{m_i-1} + \dots + f(a_{i,m_i}) = 0$$

for  $i = 1, \dots, n$ . It suffices to take  $B'$  as the  $A$ -submodule generated by  $a_{i,j}$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$ .  $\square$

**Proposition 2.32.** Let  $f : A \rightarrow B$  be an injective integral  $G$ -graded homomorphism of  $G$ -graded rings. Then for any prime  $G$ -homogeneous ideal  $\mathfrak{p}$  in  $A$ , there is a prime  $G$ -homogeneous ideal  $\mathfrak{p}'$  in  $B$  such that  $\mathfrak{p} = f^{-1}\mathfrak{p}'$ .

PROOF. We may assume that  $A \neq 0$ , as otherwise there is nothing to prove.

It suffices to show that  $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$ . [Include a proof](#) We could localize that  $\mathfrak{p}$  and assume that  $\mathfrak{p}$  is a maximal  $G$ -homogeneous ideal. [Include details about localization](#) It suffices then to show that  $\mathfrak{p}B \neq B$ . Assume by contrary that we can write  $1 = \sum_{i=1}^n f_i b_i$  for some homogeneous elements  $f_i \in \mathfrak{p}$  and some homogeneous elements  $b_i \in B$ . Let  $B'$  be a  $G$ -graded subring of  $B$  containing  $A$  and  $b_1, \dots, b_n$  and such that  $A \rightarrow B'$  is finite. The existence of  $B'$  is guaranteed by [Lemma 2.31](#). Then we find immediately  $B' = \mathfrak{m}_A B'$ . Then  $B' = 0$  by the graded Nakayama's lemma. [Include details](#) So  $A = 0$ , which is a contradiction.  $\square$

**Lemma 2.33.** Let  $A$  be a  $G$ -graded ring. Then the following are equivalent:

- (1)  $A$  is a  $G$ -graded local ring;
- (2) There is a proper homogeneous ideal  $I$  in  $A$  such that any non-invertible homogeneous element in  $A$  is contained in  $I$ .

In fact,  $I$  in (2) is just the maximal  $G$ -homogeneous ideal in  $A$ .

PROOF. Assume that (1) holds, let  $I$  be the maximal  $G$ -homogeneous ideal of  $A$ . Let  $a$  be a non-invertible homogeneous element in  $A$ . Then the image of  $a$  in  $A/I$  is invertible by [Proposition 2.28](#) and [Proposition 2.20](#).

Assume (2). We show that  $I$  is the maximal  $G$ -homogeneous ideal in  $A$ . By [Proposition 2.28](#), it suffices to show that  $A/I$  is a graded field. By [Proposition 2.20](#), we need to show that any non-zero homogeneous element  $b \in A/I$  is invertible. Lift  $b$  to  $a + i \in A$  with  $a \in A$  homogeneous and  $i \in I$ . If  $a$  is not invertible, then  $a \in I$  by the assumption hence  $b = 0$ . This is a contradiction.  $\square$

**Lemma 2.34.** Let  $k$  be a  $G$ -graded field and  $A$  be a graded  $k$ -algebra. Suppose that  $\rho(A) = \rho(k)$ , then

- (1) For any  $g \in G$ , there is a natural isomorphism

$$A_g \cong A_1 \otimes_{k_1} k_g.$$

- (2) The map  $I \mapsto I \cap A_1$  is a bijection between the set of  $G$ -homogeneous ideals (resp. prime  $G$ -homogeneous ideals) in  $A$  and ideals (resp. prime ideals) in  $A_1$ .

PROOF. (1) Take  $g \in \rho(A)$ . As  $\rho(A) = \rho(k)$ , we can take a non-zero homogeneous element  $b \in k_g$ . Then  $b$  and  $b^{-1}$  induces inverse bijections between  $A_1$  and  $A_g$ .

(2) The part about ideals can be proved in the same way as (1). The part about prime ideals follows easily.  $\square$

**Proposition 2.35.** Let  $k$  be a  $G$ -graded field and  $M$  be a  $G$ -graded  $A$ -module. Then  $M$  is free as  $G$ -graded  $A$ -module.

PROOF. We may assume that  $M \neq 0$ . Let  $\{m_i\}_{i \in I}$  be a maximal set of non-zero homogeneous elements in  $M$  such that the corresponding homomorphism

$$F := \bigoplus_{i \in I} (\rho(f))^{-1} k \rightarrow M$$

is injective. The existence of  $\{m_i\}_{i \in I}$  follows from Zorn's lemma.

If  $f \in M/F$  is a non-zero homogeneous element, then we get a homomorphism  $(\rho(f))^{-1} k \rightarrow M/F$ . This map is necessarily injective as  $(\rho(f))^{-1} k$  does not have non-zero proper graded submodules. This contradicts the definition of  $F$ .  $\square$

**Corollary 2.36.** Let  $k$  be a  $G$ -graded field,  $C$  be a  $G$ -graded  $k$ -algebra. Consider a  $G$ -graded homomorphism of  $G$ -graded  $k$ -algebras  $f : A \rightarrow B$ . Then the following are equivalent:

- (1)  $f$  is finite (resp. finitely generated);
- (2)  $f \otimes_k C$  is finite (resp. finitely generated).

PROOF. (1)  $\implies$  (2): This implication is trivial.

(2)  $\implies$  (1): By [Proposition 2.35](#), this implication follows from fpqc descent [\[Stacks, Tag 02YJ\]](#).  $\square$

**Definition 2.37.** Let  $K$  be a  $G$ -graded field. A  $G$ -graded subring  $A \subseteq K$  is a  *$G$ -graded valuation ring* in  $K$  if

- (1)  $A$  is a local  $G$ -graded ring;
- (2) the natural map  $\text{Frac}^G A \rightarrow K$  is an isomorphism;
- (3) For any non-zero homogeneous element  $f \in K$ , either  $f \in A$  or  $f^{-1} \in A$ .

**Definition 2.38.** Let  $K$  be a  $G$ -graded field and  $A, B$  be  $G$ -graded local subrings of  $K$ . We say  $B$  *dominates*  $A$  if  $A \subseteq B$  and  $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ , where  $\mathfrak{m}_A$  and  $\mathfrak{m}_B$  are the maximal  $G$ -homogeneous ideals in  $A$  and  $B$ .

**Proposition 2.39.** Let  $K$  be a  $G$ -graded field and  $A \subseteq K$  be a  $G$ -graded local subring. Then the following are equivalent:

- (1)  $A$  is a  $G$ -graded valuation ring in  $K$ .
- (2)  $A$  is maximal among the  $G$ -graded local subrings of  $K$  with respect to the order of domination.

PROOF. Assume (1). We may assume that  $A \neq K$ . Then  $A$  is not a  $G$ -graded field as  $\text{Frac}^G A = K$ . Let  $\mathfrak{m}$  be a maximal  $G$ -homogeneous ideal in  $A$ . Then  $\mathfrak{m} \neq 0$ .

We argue first that  $A$  is a  $G$ -graded local ring. Assume the contrary. Let  $\mathfrak{m}' \neq \mathfrak{m}$  be maximal  $G$ -homogeneous ideal in  $A$ . Choose non-zero homogeneous elements  $x, y \in A$  with  $x \in \mathfrak{m}' \setminus \mathfrak{m}$ ,  $y \in \mathfrak{m} \setminus \mathfrak{m}'$ . Then  $x/y \notin A$  as otherwise  $x = (x/y)y \in \mathfrak{m}$ . Similarly,  $y/x \notin A$ . This is a contradiction.

Next suppose that  $A'$  is a  $G$ -graded local subring of  $K$  dominating  $A$ . Let  $x \in A'$  be a non-zero homogeneous element, we need to show that  $x \in A$ . If not, we have  $x^{-1} \in A$  and as  $x^{-1}$  is not a unit,  $x^{-1} \in \mathfrak{m}_{A'}$ . But then  $x^{-1} \in \mathfrak{m}_{A'}$ , the maximal  $G$ -homogeneous ideal in  $A'$ . This contradicts the fact that  $x \in A'$ .

Assume (2). Take a homogeneous element  $x \in K \setminus A$ , we need to argue that  $x^{-1} \in A$ . Let  $A'$  denote the minimal  $G$ -homogeneous subring of  $K$  containing  $A$  and  $x$ . It is easy to see that  $A'$  is the usual subring generated by  $A$  and  $x$ .

By our assumption, there is no  $G$ -graded prime ideal of  $A'$  lying over  $\mathfrak{m}_A$ , as otherwise, if  $\mathfrak{p}$  is such an ideal, the  $G$ -graded local subring  $A'_{\mathfrak{p}}^G$  of  $K$  dominates  $A$ .

In other words, the  $G$ -graded ring  $A'/\mathfrak{m}_A A'$  does not have any homogeneous prime ideals and hence  $A' = \mathfrak{m}_A A'$  by [Corollary 2.29](#).

We can therefore write

$$1 = \sum_{i=0}^d t_i x^i$$

with some homogeneous elements  $t_i \in \mathfrak{m}_A$ . In particular,

$$(1 - t_0)(x^{-1})^d - \sum_{i=1}^d t_i (x^{-1})^{d-i} = 0.$$

So  $x^{-1}$  is integral over  $A$ . Let  $A''$  be the subring of  $K$  generated by  $A$  and  $x^{-1}$ . Then  $A \rightarrow A''$  is finite and there is a  $G$ -homogeneous prime ideal  $\mathfrak{m}''$  of  $A''$  lying over  $\mathfrak{m}_A$  by [Proposition 2.32](#). By our assumption,  $A = A''_{\mathfrak{m}''}^G$  and hence  $x^{-1} \in A$ .

It remains to verify that  $\text{Frac}^G A = K$ . Suppose that it is not the case, let  $B \subseteq K$  be a  $G$ -graded local subring dominating  $A$ . Take a homogeneous element  $t \in K$  that is not in  $\text{Frac}^G A$ . Observe that  $t$  can not be transcendental over  $A$ , as otherwise  $A[t] \subseteq K$  is a  $G$ -graded subring, and we can localize it at the prime  $G$ -homogeneous ideal generated by  $t$  and  $\mathfrak{m}_A$ . We get a  $G$ -graded local ring dominating  $A$  that is different from  $A$ .

So  $t$  is algebraic over  $A$ . We can then take a non-zero homogeneous  $a \in A$  such that  $at$  is integral over  $A$ . The ring  $A' \subseteq K$  generated by  $A$  and  $ta$  is a  $G$ -graded subring and  $A \rightarrow A'$  is finite. By [Proposition 2.32](#), there is a prime  $G$ -homogeneous ideal  $\mathfrak{m}'$  of  $A'$  lifting  $\mathfrak{m}_A$ . But then  $A'_{\mathfrak{m}'}^G$  dominates  $A$  and so  $A = A'_{\mathfrak{m}'}^G$ . It follows that  $t \in \text{Frac}^G A$ , which is a contradiction.  $\square$

**Corollary 2.40.** Let  $K$  be a  $G$ -graded field. Any  $G$ -graded local subring  $B \subseteq K$  is dominated by a  $G$ -graded valuation subring of  $K$ .

PROOF. This follows from [Proposition 2.39](#) and Zorn's lemma.  $\square$

In the next lemma, graded rings are written additively.

**Lemma 2.41.** Let  $n \in \mathbb{N}$  and  $R = \mathbb{Z}[1^{-1}A_1, \dots, n^{-1}A_n]$  be the  $\mathbb{Z}$ -graded polynomial ring in  $n$ -variables. Consider a ring homomorphism

$$\Phi : R[T_0, n^{-1}T_1, (n+1)^{-1}T_2, \dots, (2n-1)^{-1}T_n] \rightarrow R[T]$$

sending  $T_0$  to  $T$  and  $T_i$  to  $T^{i-1}(T^n + A_1T^{n-1} + \dots + A_n)$  for  $i = 1, \dots, n$ . Then for all  $l \in \mathbb{N}$ , there are homogeneous polynomials  $G_l \in R[n^{-1}T_1, \dots, (2n-1)^{-1}T_n]$  and  $H_l \in R[T_0]$  of degree  $l$  such that  $\deg_{T_0} H_l \leq n-1$  and  $T_0^l - G_l - H_l \in \ker \Phi$ .

PROOF. Fix  $l \geq 0$ , consider a polynomial  $G_l \in R[n^{-1}T_1, \dots, (2n-1)^{-1}T_n]$  homogeneous of degree  $l$  such that  $\Phi(T_0^l - G_l)$  has the minimal possible degree. We have to show that this degree is less than  $n$ . If not, say the leading term is  $cT^a$  with  $a \geq n$  and  $c \in R$  is a homogeneous element. Observe the leading term of the image of  $T_i$  in  $R[T]$  is  $T^{n+i-1}$  for  $i = 1, \dots, n$ . We can always find a monomial  $Q$  in  $T_1, \dots, T_n$  such that the leading term of its image in  $R[T]$  is  $T^a$ . Then set  $G'_l = G_l - cQ$ , we find that  $\deg \Phi(G'_l) < \deg \Phi(G_l)$ . This is a contradiction.

Now we can write

$$\Phi(T_0^l - G_l) = c_{n-1}T^{n-1} + \dots + c_0.$$

It suffices to take  $H_l = c_{n-1}T_0^{n-1} + \cdots + c_0$ .  $\square$

### 3. Graded algebraic geometry

Let  $G$  be an Abelian group. We write the group operation of  $G$  multiplicatively and denote the identity of  $G$  as 1.

**Definition 3.1.** Let  $A$  be a  $G$ -graded ring. We define the  $G$ -graded affine spectrum  $\text{Spec}^G(A)$  as follows: as a set  $\text{Spec}^G(A)$  consists of all prime  $G$ -homogeneous ideals of  $A$ ; we endow  $\text{Spec}^G(A)$  with the *Zariski topology*, whose base consists of sets of the form

$$D(f) := \left\{ \mathfrak{p} \in \text{Spec}^G(A) : f \notin \mathfrak{p} \right\}$$

for all homogeneous elements  $f \in A$ .

**Lemma 3.2.** Let  $k$  be a  $G$ -graded field and  $A$  be a finitely generated  $G$ -graded  $k$ -algebra. Then  $\text{Spec}^G(A)$  has only finitely many maximal points.

PROOF. Take a  $G$ -graded field  $K/k$  such that  $\rho(A) \subseteq \rho(K)$ . By [Lemma 2.34](#), the statement of the lemma holds for  $A \otimes_k k'$ . But each generic point of an irreducible component of  $\text{Spec}^G(A)$  can be lifted to a generic point of an irreducible component in  $\text{Spec}^G(A \otimes_k k')$ .  $\square$

### 4. Graded Riemann–Zariski spaces

Let  $G$  be an Abelian group. Let  $k$  be a  $G$ -graded field and  $K/k$  be a  $G$ -graded field extension.

**Definition 4.1.** We let  $\mathbf{P}_{K/k}$  denote the set of  $G$ -graded valuation rings  $\mathcal{O}$  of  $K$  with  $G$ -graded fraction field  $K$  such that  $k \subseteq \mathcal{O}$ . If there is a risk of confusion, we write  $\mathbf{P}_{K/k}^G$  instead.

We endow  $\mathbf{P}_{K/k}$  with the weakest topology with respect to which  $\{\mathcal{O} \in \mathbf{P}_{K/k} : f \in \mathcal{O}\}$  is open for any homogeneous element  $f \in K$ .

The space  $\mathbf{P}_{K/k}$  is called the *Riemann–Zariski space* of  $K/k$ .

Given an inclusion of  $G$ -graded fields  $i : L \rightarrow K$  over  $k$ , we have a natural continuous map  $i^\# : \mathbf{P}_{K/k} \rightarrow \mathbf{P}_{L/k}$  sending  $\mathcal{O}$  to  $i^{-1}(\mathcal{O}) \cap K$ .

Given  $X \subseteq \mathbf{P}_{K/k}$  and  $A \subseteq K$  consisting of homogeneous elements, we write

$$\begin{aligned} X\{A\} &:= \{\mathcal{O} \in X : f \in \mathcal{O} \text{ for all non-zero } f \in A\}, \\ X\{\{A\}\} &:= \{\mathcal{O} \in X : f \in \mathfrak{m}_{\mathcal{O}} \text{ for all non-zero } f \in A\}, \end{aligned}$$

where  $\mathfrak{m}_{\mathcal{O}}$  is the maximal  $G$ -homogeneous ideal of  $\mathcal{O}$ . When  $A$  consists of finitely many elements  $f_1, \dots, f_n$ , we will write  $X\{f_1, \dots, f_n\}$  and  $X\{\{f_1, \dots, f_n\}\}$  instead. When  $A \subseteq K$  consists of non-homogeneous elements as well,  $X\{A\}$  means  $X\{B\}$ , where  $B$  is the set of homogeneous elements in  $A$ .

**Definition 4.2.** An *affine subset* of  $\mathbf{P}_{K/k}$  is a subset of  $\mathbf{P}_{K/k}$  of the form:  $\mathbf{P}_{K/k}\{F\}$  for some finite set  $F$  of homogeneous elements in  $K$ .

**Lemma 4.3.** Let  $X \subseteq \mathbf{P}_{K/k}$  and  $f \in K$  be a non-zero homogeneous element. Then

$$X \setminus X\{f\} = X\{\{f^{-1}\}\}.$$

PROOF. We first observe that  $X\{f\} \cap X\{f^{-1}\} = \emptyset$ . Otherwise, let  $\mathcal{O}$  be a  $G$ -graded valuation ring in this intersection, then  $f \in \mathcal{O}$  and  $f^{-1} \in \mathfrak{m}_{\mathcal{O}}$ . So  $1 \in \mathfrak{m}_{\mathcal{O}}$ , which is a contradiction.

To show that  $X\{f\} \cup X\{f^{-1}\} = X$ , we may assume that  $X = \mathbf{P}_{K/k}$ . Let  $\mathcal{O} \in \mathbf{P}_{K/k}$ . We need to show that  $f \in \mathcal{O}$  or  $f^{-1} \in \mathfrak{m}_{\mathcal{O}}$ .

By definition, either  $f \in \mathcal{O}$  or  $f^{-1} \in \mathcal{O}$ . We may assume that  $f \notin \mathcal{O}$  and  $f^{-1} \in \mathcal{O}$ . If  $f^{-1} \notin \mathfrak{m}_{\mathcal{O}}$ , then  $f^{-1}$  is invertible in  $\mathcal{O}$  by [Lemma 2.33](#). In particular,  $f \in \mathcal{O}$ , which is a contradiction.  $\square$

**Lemma 4.4.** Let  $A \subseteq K$  be a subset of  $K$ , then  $\mathbf{P}_{K/k}\{A\}$  is quasi-compact.

PROOF. By our convention, we may assume that  $A$  consists of homogeneous elements.

We may replace  $A$  by the  $G$ -graded subring generated of  $K$  generated by  $A$ . So we may assume that  $A$  is a  $G$ -graded subring of  $K$ .

Write  $X = \mathbf{P}_{K/k}\{A\}$ . By definition, a sub-base for the topology on  $X$  is given by  $X\{f\}$  for all non-zero homogeneous elements  $f \in K$ .

By Alexander sub-base theorem and [Lemma 4.3](#), in order to show that  $X$  is quasi-compact, it suffices to show that if  $F \subseteq K$  consists of homogeneous elements and if for any finite subset  $F_0 \subseteq F$ ,  $X\{F_0\} \neq \emptyset$ , then  $X\{F\}$  is non-empty. We assume by contrary that  $X\{F\}$  is empty.

Let  $B$  be the  $G$ -graded subring of  $K$  generated by  $A$  and  $F$ . Let  $\mathfrak{m}$  be the  $G$ -homogeneous ideal of  $B$  generated by elements in  $F$ . We claim that  $\mathfrak{m} = B$ . Otherwise, let  $\mathfrak{p}$  be a maximal  $G$ -homogeneous ideal of  $B$  containing  $\mathfrak{m}$ , then we can find a  $G$ -graded valuation subring  $\mathcal{O}$  of  $K$  dominating  $B_{\mathfrak{p}}^G$ . The existence of  $\mathcal{O}$  is guaranteed by [Proposition 2.39](#). It follows that  $\mathcal{O} \in \{F\}$ .

We write  $1 = b_1 f_1 + \dots + b_n f_n$  for some  $n \in \mathbb{Z}_{>0}$ ,  $b_1, \dots, b_n \in B$  and  $f_1, \dots, f_n \in F$ . Then  $X\{f_1, \dots, f_n\}$  is empty.  $\square$

**Lemma 4.5.** Let  $A \subseteq B \subseteq K$  be  $G$ -graded  $k$ -subalgebras of  $K$ . Assume that both  $A$  and  $B$  are finitely generated over  $k$ . Then the following are equivalent:

- (1)  $\mathbf{P}_{K/k}\{A\} = \mathbf{P}_{K/k}\{B\}$ ;
- (2)  $B$  is finite over  $A$ ;
- (3)  $B$  is integral over  $A$ .

PROOF. (3)  $\implies$  (1): Let  $\mathcal{O} \in \mathbf{P}_{K/k}\{A\}$  and  $x \in B$  a non-zero homogeneous element, we need to show that  $x \in \mathcal{O}$ . If not,  $x^{-1} \in \mathfrak{m}_{\mathcal{O}}$  by [Lemma 4.3](#). As  $x$  is integral over  $A$ , we can find  $n \in \mathbb{Z}_{>0}$ , homogeneous elements  $a_1, \dots, a_n \in A$  such that

$$b^n + a_1 b^{n-1} + \dots + a_n = 0$$

by [Proposition 2.18](#). So

$$1 = -b^{-n} (a_1 b^{n-1} + \dots + a_n) \in \mathfrak{m}_{\mathcal{O}},$$

which is a contradiction.

(1)  $\implies$  (3): Suppose  $x \in B$  is a homogeneous element which is not integral over  $A$ . The existence of  $x$  is guaranteed by [Proposition 2.18](#). Then  $x^{-1}$  is not invertible in  $C = A[1/x]$ : otherwise, we can find  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in A$  such that

$$(a_n x^{-n} + a_{n-1} x^{1-n} + \dots + a_0) x^{-1} = 1$$

or equivalently,

$$x^{n+1} = a_0 x^n + \dots + a_n.$$



This contradicts the fact that  $x$  is not integral. In particular, there is a maximal  $G$ -homogeneous ideal  $\mathfrak{p}$  containing  $x^{-1}$  by [Proposition 2.30](#). Let  $\mathcal{O}$  be a  $G$ -graded valuation ring of  $K$  dominating  $C_{\mathfrak{p}}^G$ , whose existence is guaranteed by [Corollary 2.40](#). But then  $x^{-1}$  lies in the maximal ideal of  $\mathcal{O}$  and hence  $x \notin \mathcal{O}$  by [Lemma 4.3](#). It follows that  $B \not\subseteq \mathcal{O}$ .

(2)  $\equiv$  (3): This followss from [\[Stacks, Tag 02JJ\]](#).

□

**Definition 4.6.** Let  $X$  be an open subset of  $\mathbf{P}_{K/k}$ . A *Laurent covering* of  $X$  is a covering of  $X$  of the form

$$\{X\{f_1^{\epsilon_1}, \dots, f_n^{\epsilon_n}\} : \epsilon_i = \pm 1 \text{ for } i = 1, \dots, n\},$$

where  $n \in \mathbb{Z}_{>0}$ ,  $f_1, \dots, f_n \in K$  are homogeneous. We say the Laurent covering is *generated by*  $f_1, \dots, f_n$ .

**Definition 4.7.** Let  $X$  be an open subset of  $\mathbf{P}_{K/k}$ . A *rational covering* of  $X$  is a covering of the form:

$$\left\{X \left\{ \frac{f_1}{f_i}, \dots, \frac{f_n}{f_i} \right\} : i = 1, \dots, n \right\},$$

where  $n \in \mathbb{Z}_{>0}$ ,  $f_1, \dots, f_n \in K$  are non-zero homogeneous elements. We say the rational covering is *generated by*  $f_1, \dots, f_n$ .

**Lemma 4.8.** Let  $X$  be an open subset of  $\mathbf{P}_{K/k}$ . Any finite covering  $\mathcal{U}$  of  $X$  by open subsets of the form  $X\{A\}$  for some finite set of homogeneous elements  $A \subseteq K$  has a refinement which is a Laurent covering of  $X$ .

**PROOF. Step 1.** We show that  $\mathcal{U}$  admits a refinement by a rational covering. We may assume that there is  $n \in \mathbb{Z}_{>0}$  such that  $\mathcal{U}$  consists of  $U_1, \dots, U_m$  below:

$$U_i = X\{f_{i1}, \dots, f_{in}\}$$

with  $f_{ij} \in K$  being non-zero and homogeneous for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . In addition, we may assume that  $f_{in} = 1$  for  $i = 1, \dots, m$ .

Let

$$J := \left\{ \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m : 1 \leq \alpha_i \leq n \text{ for } i = 1, \dots, m; \max_{i=1, \dots, m} \alpha_i = n \right\}.$$

We claim that the rational covering generated by  $g_\alpha = f_{1\alpha_1} \cdots f_{m\alpha_m}$  with  $\alpha = (\alpha_1, \dots, \alpha_m) \in J$  refines  $\mathcal{U}$ .

Given  $\alpha = (\alpha_1, \dots, \alpha_m) \in J$ , we consider the set

$$V_\alpha = X\{g_\beta/g_\alpha : \beta \in J\}.$$

Let  $i \in \{1, \dots, m\}$  such that  $j_i = n$ . We claim that

$$V_\alpha \subseteq U_i.$$

Suppose it is not the case, let  $\mathcal{O} \in V_\alpha$  not lying in  $U_i$ , we need to verify that  $f_{ik} \in \mathcal{O}$  for  $k = 1, \dots, n$ . Take  $l \neq i$  so that  $\mathcal{O} \in U_l$ . So  $f_{lj_l} \in \mathcal{O}$ . On the other hand, if  $\beta \in J$  with  $\beta_l = n$  and  $\beta_k = \alpha_k$  for  $k \neq l$ , we have  $f_{lj_l}^{-1} = g_\beta/g_\alpha \in \mathcal{O}$ , so  $f_{lj_l}$  is invertible in  $\mathcal{O}$ .

Fix  $k = 1, \dots, n$ , consider  $\gamma \in J$  given by  $\gamma_i = k$ ,  $\gamma_l = n$  and  $\gamma_p = \alpha_p$  otherwise. Then  $g_\gamma/g_\alpha = f_{ik}/f_{lj_l} \in \mathcal{O}$  and  $f_{ik} \in \mathcal{O}$ .

**Step 2.** It remains to show that each rational covering generated by non-zero homogeneous elements  $f_1, \dots, f_n \in K$  admits a refinement by Laurent coverings.

We claim that the Laurent covering of  $X$  generated by  $g_{ij} = f_i/f_j$  with  $1 \leq i < j \leq n$  refines the given covering. Let  $V$  be a subset of the form

$$V = X\{g_{ij}^{\epsilon_{ij}} : 1 \leq i < j \leq n\}$$

for some  $\epsilon_{ij} = \pm 1$  for  $1 \leq i < j \leq n$ . We need to show that  $V$  is contained in a set in  $\mathcal{U}$ .

For  $1 \leq i, j \leq n$  and  $i \neq j$ , we write  $i \preceq j$  if  $i < j$  and  $\epsilon_{ij} = 1$  or  $i > j$  and  $\epsilon_{ji} = -1$ . This is an ordering on  $\{1, \dots, n\}$ . Choose a maximal element  $i$ . Then  $f_j/f_i \in \mathcal{O}$  for all  $\mathcal{O} \in V$ , so

$$V \subseteq X\{f_1/f_i, \dots, f_n/f_i\}.$$

□

### 5. Strictness in graded Riemann–Zariski spaces

In this section, graded will mean  $\mathbb{R}_{>0}$ -graded unless we explicitly state the grading. Similarly, a graded Riemann–Zariski space without specifying the grading means the  $\mathbb{R}_{>0}$ -graded one.

Let  $H$  be a subgroup of  $\mathbb{R}_{>0}$ . Fix graded field extension  $K/k$ .

**Definition 5.1.** Consider the natural continuous map

$$\psi_{K/k}^H : \mathbf{P}_{K/k} \rightarrow \mathbf{P}_{K^H/k^H}$$

defined by sending a graded valuation ring  $\mathcal{O}$  to  $\mathcal{O}^H$ .

A quasi-compact open subset  $U$  of  $\mathbf{P}_{K/k}$  is *H-strict* if  $U$  is the preimage of some quasi-compact open subset  $U'$  of  $\mathbf{P}_{K^H/k^H}$ .

**Lemma 5.2.** Let  $U \subseteq \mathbf{P}_{K/k}$  be a quasi-compact open subset. The following are equivalent:

- (1)  $U$  is *H-strict*;
- (2)  $U$  is  $\sqrt{\rho(k^\times) \cdot H}$ -strict.

PROOF. (1)  $\implies$  (2): This is trivial.

(2)  $\implies$  (1): Let  $S \subseteq L^{\sqrt{\rho(k^\times) \cdot H}}$  be a finite set. For any  $f \in S$ , we can find a non-zero homogeneous element  $a_f \in k$  and  $n_f \in \mathbb{Z}_{>0}$  such that  $a_f f^{n_f} \in L^H$ . We let  $S' = \{a_f f^{n_f}\}_{f \in S}$ , then

$$\mathbf{P}_{L/K}\{S\} = \mathbf{P}_{L/K}\{S'\}.$$

□

**Lemma 5.3.** Let  $A$  be a graded  $k$ -subalgebra of  $K$ . Then

$$\psi_{K/k}^H(\mathbf{P}_{K/k}\{A\}) = \mathbf{P}_{K^H/k^H}\{A^H\}.$$

In particular,  $\psi_{K/k}^H$  is a surjective open map.

PROOF. The inclusion

$$\psi_{K/k}^H(\mathbf{P}_{K/k}\{A\}) \subseteq \mathbf{P}_{K^H/k^H}\{A^H\}$$

is obvious. Conversely, let  $\mathcal{O} \in \mathbf{P}_{K^H/k^H}\{A^H\}$ . We set

$$B = A \cdot \mathcal{O}.$$

Observe that for  $h \in H$ ,  $B_h = A_h$ , so

$$B \cap K^H = \mathcal{O}, \quad \mathfrak{m}_{\mathcal{O}} B \cap \mathcal{O} = \mathfrak{m}_{\mathcal{O}}.$$

In particular, there is a prime homogeneous ideal  $\mathfrak{p}$  of  $B$  with  $\mathfrak{p} \cap \mathcal{O} = \mathfrak{m}_{\mathcal{O}}$ . Take a graded valuation ring  $\mathcal{O}' \in \mathbf{P}_{K/k}$  dominating  $B_{\mathfrak{p}}^{\mathbb{R}_{>0}}$ . The existence of  $\mathcal{O}'$  follows from [Corollary 2.40](#). Moreover,  $\mathcal{O}' \cap K_H = \mathcal{O}$  the left-hand side dominates  $\mathcal{O}$ . But then

$$\mathcal{O}' \in \mathbf{P}_{K/k}\{A\}$$

and  $\psi_{K/k}^H(\mathcal{O}') = \mathcal{O}$ .  $\square$

**Corollary 5.4.** Suppose that  $n \in \mathbb{N}$  and  $f_1, \dots, f_n$  is a finite set of non-zero homogeneous elements in  $K$ . Then the following are equivalent:

- (1)  $\mathbf{P}_K\{f_1, \dots, f_n\}$  is  $H$ -strict;
- (2)  $\rho(f) \in \sqrt{\rho(k^\times) \cdot H}$ .

PROOF. (2)  $\implies$  (1): This follows from [Lemma 5.2](#).

(2)  $\implies$  (1): Let  $A$  be the graded  $k$ -algebra generated by  $f_1, \dots, f_n$ . By [Lemma 5.3](#),  $\mathbf{P}_{K/k}\{f_1, \dots, f_n\} = \mathbf{P}_{K/k}\{A\}$  is  $H$ -strict if and only if it coincides with  $\mathbf{P}_{K/k}\{A^H\}$ . By [Lemma 4.5](#), the latter is equivalent to that  $A$  is integral over  $A_H$ . But then it is clear that

$$\rho(A) \subseteq \sqrt{\rho(A_H)}$$

and (1) follows.  $\square$

**Corollary 5.5.** The natural map

$$\mathbf{P}_{K\sqrt{\rho(k^\times) \cdot H}/k} \rightarrow \mathbf{P}_{K^H/k^H}$$

is a homeomorphism.

PROOF. We may assume that  $\rho(K^\times) \subseteq \sqrt{\rho(k^\times) \cdot H}$  and identify the given map with  $\psi_{K/k}^H$ . We know that  $\psi_{K/k}^H$  is open by [Lemma 5.3](#). It is continuous by construction. So it remains to argue that  $\psi_{K/k}^H$  is bijective.

By [Lemma 5.3](#) again, we know that  $\psi_{K/k}^H$  is surjective. On the other hand, if  $|\bullet|$  is the valuation corresponding to  $\mathcal{O} \in \mathbf{P}_{K^H/k^H}$ , it is clear that the extension to  $K = K\sqrt{\rho(k^\times) \cdot H}$  is unique: if  $f$  is a non-zero homogeneous element in  $K$ , then we can find a non-zero homogeneous element  $a \in K$ , a non-zero homogeneous element  $g \in K^H$  and an integer  $n \in \mathbb{Z}_{>0}$  such that

$$f^n = ag.$$

The valuation is uniquely determined at  $a$  and  $g$ , hence so is its value at  $f$ .  $\square$

**Proposition 5.6.** The fibers of

$$\psi_{K/k}^H : \mathbf{P}_{K/k} \rightarrow \mathbf{P}_{K^H/k^H}$$

are connected.

PROOF. Let  $\mathcal{O} \in \mathbf{P}_{K^H/k^H}$  and  $X = \psi_{K/k}^{H,-1}(\{\mathcal{O}\})$ . We need to show that  $X$  is connected.

Assume to the contrary that  $X$  is the disjoint union of two non-empty open subsets  $U$  and  $V$ .

**Step 1.** We show that  $X$  is quasi-compact.

We set

$$\begin{aligned} Y &= \{\mathcal{O}' \in \mathbf{P}_{K^H/k^H} : \mathcal{O}' \supseteq \mathcal{O}\}, \\ X_Y &= \{\mathcal{O}' \in \mathbf{P}_{K/k} : \mathcal{O}' \supseteq \mathcal{O}\}. \end{aligned}$$

In other words,  $X_Y = \psi_{K/k}^{H,-1}(Y)$ . Observe that  $Y$  and  $X_Y$  are both quasi-compact by [Lemma 4.4](#).

Observe that  $\mathcal{O}$  is a closed point in  $Y$ :

$$Y \setminus \{\mathcal{O}\} = \bigcup_f Y\{f\},$$

where  $f$  runs over all non-zero homogeneous elements of  $K$  not lying in  $\mathcal{O}$ . It follows that  $X$  is closed in  $X_Y$ . In particular,  $X$  is quasi-compact.

In particular,  $U$  and  $V$  are both quasi-compact.

**Step 2.** We reduce to the case where  $H \supseteq \rho(k^\times)$ ,  $\sqrt{H} = H$  and  $\rho(K^\times)/H$  is finitely generated.

By Step 1,  $U$  and  $V$  can be both covered by finitely many sets of the form  $X\{f_1, \dots, f_n\}$  with  $f_1, \dots, f_n \in K$  being some homogeneous elements. Let  $H'$  be the subgroup of  $\mathbb{R}_{>0}$  generated by  $H$  and  $\rho(f_i)$  for the  $f_i$ 's to guarantee that the inverse image of  $\{\mathcal{O}\}$  in  $\mathbf{P}_{K^{H'}/k}$  is not connected. It suffices to prove the theorem with  $K^{H'}$  in place of  $K$ . Moreover, by [Corollary 5.5](#), we may assume that  $H \supseteq \rho(k^\times)$  and  $\sqrt{H} = H$ .

In particular,  $\rho(K^\times)/H$  is locally free of finite rank. Let  $n$  be its rank.

**Step 3.** We provide a contradiction when  $n = 1$ .

We claim that in this case,  $K = K^H[g^{-1}T, gT^{-1}]$  for some  $g \in \rho(K^\times) \setminus H$ .

In fact, let  $g \in \rho(K^\times) \setminus H$  and take  $f \in K_g$ . Then we have an obvious  $\rho(K^\times)$ -graded ring homomorphism

$$K^H[g^{-1}T, gT^{-1}] \rightarrow K$$

sending  $T$  to  $f$ . But  $K^H[g^{-1}T, gT^{-1}]$  is clearly a  $\rho(K^\times)$ -graded field. So this map is injective. It is an isomorphism by [Lemma 2.34](#).

Observe that a basis of the topology on  $X$  is given by sets of the form  $X\{aT^i, bT^{-j}\}$  with  $a, b \in K^H$  and  $i, j \in \mathbb{Z}_{>0}$ . Up to replacing  $aT^i$  by  $a^jT^{ij}$  and  $bT^{-j}$  by  $b^iT^{-ij}$ , we may assume that  $i = j$ .

We cover  $U$  and  $V$  by  $W_1, \dots, W_l$  and  $W_{l+1}, \dots, W_m$  respectively such that

$$W_i = X\{a_iT^{k_i}, b_iT^{-k_i}\},$$

where  $a_i, b_i \in K^H$  and  $k_i \in \mathbb{Z}_{>0}$  for  $i = 1, \dots, m$ . By the same argument as above, we may guarantee that  $k_1 = \dots = k_m$ , and we call this common value  $k$ .

Let  $|\bullet|$  be the valuation determined by  $\mathcal{O}$ , we may assume that  $|a_1| \leq |a_i|$  for all  $i = 1, \dots, m$ . Then  $U, V$  are both contained in  $X\{a_1T^k\}$  so  $X\{a_1T^k\} = X$  and  $a_1 = 0$ .

If  $W_i \cap W_1 \neq \emptyset$  for some  $i = 2, \dots, m$ , then  $a_ib_1 \in \mathcal{O}$  and  $W'_1 := W_1 \cup W_i$  is the same as  $X\{\max\{b_1, b_i\}T^{-k}\}$ , where  $\max\{b_1, b_i\}$  is one of  $b_1$  and  $b_i$  such that the valuation under  $|\bullet|$  is smaller. We could remove  $W_i$  from the list and replace  $W_1$  with  $W'_1$ . Repeating this process, we may guarantee that  $W_1$  is disjoint with all other  $W_i$ 's.

But then  $W_1$  is an open and closed subset of  $X$  contained in  $U$ . So  $b_1 \neq 0$ . We claim that the set  $W_1 = X\{b_1T^{-k}\}$  is not closed: consider

$$\mathcal{O}'_1 := \{\mathcal{O}[xT^k, yT^{-k}] : x, y \text{ are homogeneous elements in } K \text{ such that } |y| < |b_1|, |x| \leq |b_1|^{-1}\}$$

and

$$\mathcal{O}'_2 := \{\mathcal{O}[xT^k, yT^{-k}] : x, y \text{ are homogeneous elements in } K \text{ such that } |y| \leq |b_1|, |x| \leq |b_1|^{-1}\}.$$

Extend them to graded valuation rings  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of  $K$ . These extensions exist and are unique by [Corollary 5.5](#). Then  $\mathcal{O}_1 \notin W_1$  while  $\mathcal{O}_2 \in W_1$ . But  $\mathcal{O}_2$  contains  $\mathcal{O}_1$  so  $\mathcal{O}_1$  is a specialization of  $\mathcal{O}_2$ . This is a contradiction.

**Step 4.** We argue the general case by induction on  $n > 1$ . Assume that we have shown the result for smaller  $n$ .

Choose a subgroup  $L$  of  $\rho(K^\times)$  such that  $L \supseteq H$  and  $\rho(K^\times)/L \cong \mathbb{Z}$ . We can factorize  $\psi_{K/k}^H$  as

$$\mathbf{P}_{K/k} \xrightarrow{\psi_{K/k}^L} \mathbf{P}_{K^L/k} \xrightarrow{\psi_{K^L/k}^H} \mathbf{P}_{K^H/k}.$$

Both maps have connected fibers, hence so is their composition.  $\square$

## 6. The birational categories

In this section, graded will mean  $\mathbb{R}_{>0}$ -graded unless we explicitly state the grading.

Fix graded field extension  $K/k$ . Let  $H$  be a subgroup of  $\mathbb{R}_{>0}$ .

**Definition 6.1.** The category  $\mathcal{T}_{K/k}$  is the full subcategory of the category  $\mathcal{Top}/\mathbf{P}_{K/k}$  consisting of objects  $X \rightarrow \mathbf{P}_{K/k}$  satisfying

- (1)  $X$  is quasi-compact and quasi-separated;
- (2)  $X \rightarrow \mathbf{P}_{K/k}$  is a local homeomorphism.

We will also say  $X$  is an object of  $\mathcal{T}_{K/k}$ .

Let  $X$  be an object of  $\mathcal{T}_{K/k}$ . A *chart* of  $X$  is a quasi-compact open subset  $U$  of  $X$  such that  $U \rightarrow \mathbf{P}_{K/k}$  is a homeomorphism onto an open subset of  $\mathbf{P}_{K/k}$ . An *atlas* is a finite covering of  $X$  by charts.

Recall that a topological space is quasi-compact if each quasi-compact open subset is retrocompact.

**Example 6.2.** Each quasi-compact open subset  $U$  of  $\mathbf{P}_{K/k}$  can be identified with an object in  $\mathcal{T}_{K/k}$  using the natural inclusion.

The only non-trivial point is to show that  $U$  is quasi-separated. For this, it suffices to show that  $\mathbf{P}_{K/k}$  is quasi-separated. This follows from the fact that the intersection of two affine open subsets is still affine.

**Definition 6.3.** Let  $X$  be an object of  $\mathcal{T}_{K/k}$ .

A chart  $U$  of  $X$  is *H-strict* if its image in  $\mathbf{P}_{K/k}$  is *H-strict*.

An atlas  $\{U_i\}_{i \in I}$  of  $X$  is *H-strict* if  $U_i, U_i \cap U_j$  are *H-strict* for all  $i, j \in I$ .

We say  $X$  is *H-strict* if it admits an *H-strict* atlas.

A quasi-compact open subset of  $X$  is *H-strict* if it is *H-strict* as an object of  $\mathcal{T}_{K/k}$ .

The notion of *H-strictness* of a quasi-compact open subset of  $\mathbf{P}_{K/k}$  in [Definition 6.3](#) is the same as in [Definition 5.1](#).

Observe that  $X$  is *H-strict* if and only if

$$X = Y \times_{\mathbf{P}_{K^H/k^H}} \mathbf{P}_{K/k}$$

for some object  $Y$  in  $\mathbf{P}_{K^H/k^H}$ . The fiber product is taken in  $\mathcal{Top}$ . We will say  $X$  is the *base change* of  $Y$  to  $K/k$  in this case. The base change is clearly functorial. The natural map  $X \rightarrow Y$  is denoted by  $\psi_{K/k}^H$ .

**Proposition 6.4.** The base change  $\mathcal{T}_{K^H/k^H} \rightarrow \mathcal{T}_{K/k}$  is fully faithful.

In particular, for each  $H$ -strict object  $X \in \mathcal{T}_{K/k}$ , there is a unique object (up to canonical isomorphisms)  $\mathcal{T}_{K^H/k^H}$  whose base change to  $K/k$  is  $X$ . We will denote this object by  $X^H$ .

PROOF. Let  $Y, Y' \in \mathcal{T}_{K^H/k^H}$  and  $X, X'$  be their base changes to  $K/k$ . We let  $f : X \rightarrow X'$  be a morphism in  $\mathcal{T}_{K/k}$ .

We are looking for a dotted morphism making the diagram below commutative:

$$\begin{array}{ccccc} X & \xrightarrow{\psi_{K/k}^H} & Y & \longrightarrow & \mathbf{P}_{K^H/k^H} \\ \downarrow f & & \vdots & \nearrow & \\ X' & \xrightarrow{\psi_{K/k}^H} & Y' & & \end{array}$$

As  $\psi_{K/k}^H : X \rightarrow Y$  is open and surjective by [Lemma 5.3](#), it suffices to show that set-theoretically the dotted map exists and is unique.

Let  $Z$  be a fiber of  $\psi_{K/k}^H : X \rightarrow Y$ , then  $Z$  is connected by [Proposition 5.6](#). The image of  $Z$  in  $\mathbf{P}_{K^H/k^H}$  is a single point, so the image of  $Z$  in  $Y'$  is a discrete set. As  $Z$  is connected, its image in  $Y'$  is a single point. This shows the existence and uniqueness of the dotted map.  $\square$

**Proposition 6.5.** Let  $F/K$  be a graded extension of fields. Consider an object  $Y$  (resp.  $X$ ) in  $\mathbf{P}_{F/k}$  (resp.  $\mathbf{P}_{K/k}$ ) and a continuous map  $Y \rightarrow X$  making the following diagram commutative:

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{P}_{F/k} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{P}_{K/k} \end{array},$$

where the horizontal maps are the maps in the definition of  $X$  and  $Y$ , the map on the right-hand side is the restriction map. We assume that  $Y$  and  $X$  are  $H$ -strict, then there is a unique continuous map  $Y^H \rightarrow X^H$  making the following diagram commutative:

$$\begin{array}{ccccc} & Y & \longrightarrow & \mathbf{P}_{F/k} & \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ Y^H & \longrightarrow & \mathbf{P}_{F^H/k^H} & & \\ & \downarrow & \downarrow & & \downarrow \\ & X & \longrightarrow & \mathbf{P}_{K/k} & \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ & X^H & \longrightarrow & \mathbf{P}_{K^H/k^H} & \end{array}.$$

PROOF. The uniqueness follows from the surjectivity of  $Y \rightarrow Y^H$ . The latter follows from [Lemma 5.3](#).

The existence follows from the same arguments as [Proposition 6.4](#).  $\square$

**Proposition 6.6.** Let  $X$  be an  $H$ -strict object of  $\mathcal{T}_{K/k}$ , then there is a bijection from the set of quasi-compact open subsets of  $X^H$  to the set of  $H$ -strict quasi-compact open subsets of  $X$ .

The forward direction sends  $V$  to  $V \times_{X^H} X$ .

PROOF. It suffices to establish a quasi-inverse functor.

Let  $U$  be an  $H$ -strict quasi-compact open subset of  $X$ . By [Proposition 6.4](#), we can find a continuous map  $U^H \rightarrow X^H$  inducing the inclusion map  $U \rightarrow X$  by base change. Consider the commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & X & \longrightarrow & \mathbf{P}_{K/k} \\ \downarrow & & \downarrow & & \downarrow \\ U^H & \longrightarrow & X^H & \longrightarrow & \mathbf{P}_{K/k}^H \end{array} .$$

Observe that  $U^H \rightarrow X^H$  is injective as  $U \rightarrow X$  is. So  $U^H$  is a quasi-compact open subset of  $X^H$ . This construction is functorial in  $U$ .

This functor is clearly quasi-inverse to the given functor.  $\square$

## 7. The birational category à la Temkin

The gradings refer to  $\mathbb{R}_{>0}$ -grading in this section. Let  $k$  be a graded field.

**Definition 7.1.** The category  $\text{bir}_k$  is defined as follows:

- (1) the objects are pairs  $(X, K)$ , where  $K$  is a graded field extension of  $k$  and  $X$  is an object in  $\mathcal{T}_{K/k}$ ;
- (2) a morphism

$$(X, K)$$

to  $(Y, L)$  is a pair  $(h, i)$ , where  $h : X \rightarrow Y$  is a continuous map and  $i : L \rightarrow K$  is an embedding of  $G$ -graded fields such that the following diagram commutes:

$$\begin{array}{ccc} X & \longrightarrow & \mathbf{P}_{K/k} \\ \downarrow h & & \downarrow i^\# \\ Y & \longrightarrow & \mathbf{P}_{L/k} \end{array} ;$$

- (3) the composition of morphisms  $(h, i)$  and  $(h', i')$  is  $(h \circ h', i \circ i')$ .

We will always omit  $i$  from our notations.

We observe that there is a final object in  $\text{bir}_k$ :  $X$  is a single point,  $K = k$ .

**Definition 7.2.** Let  $(X, K), (Y, L) \in \text{bir}_k$  and  $h : (X, K) \rightarrow (Y, L)$  be a morphism. We say the morphism is *separated* (resp. *proper*) if  $X \rightarrow Y \times_{\mathbb{P}_{L/k}} \mathbb{P}_{K/k}$  is injective (resp. bijective).

Here the fiber product is in the category of topological spaces.

We say  $(X, K) \in \text{bir}_k$  is *separated* (resp. *proper*) if the morphism to the final object is separated (resp. proper).

Observe that  $X \rightarrow Y \times_{\mathbb{P}_{L/k}} \mathbb{P}_{K/k}$  is automatically an open embedding (resp. a homeomorphism).

### 8. Miscellany

**Proposition 8.1.** Let  $R$  be a noetherian N-2 integral domain. Let  $\psi : R \rightarrow S$  be a ring homomorphism such that  $S$  is reduced, torsion-free as  $R$ -module and has finite rank as  $R$ -module. Then  $\psi$  is finite.

[BGR84, Page 122]. Reproduce the argument later.

PROOF. As  $\psi$  is injective by assumption, we may assume that  $R$  is a subring of  $S$  and  $\psi$  is identity. The ring  $S_{R \setminus \{0\}} = \text{Frac } S$  is a finite-dimensional reduced  $\text{Frac } R$ -algebra, hence as a ring,  $\text{Frac } S$  is the product of finitely many finite field extensions of  $\text{Frac } R$ , say  $K_1, \dots, K_t$ . As  $R$  is N-2, the integral closure  $R_i$  of  $R$  in  $K_i$  is finite as  $R$ -module for  $i = 1, \dots, t$ . As  $S$  is integral over  $R$ , we have

$$S \subseteq R_1 \times \cdots \times R_t.$$

Since  $R$  is noetherian, we conclude that  $S$  is finite as  $R$ -module.  $\square$

**Lemma 8.2.** Let  $R$  be a commutative ring. A polynomial  $a_0 + a_1X + \cdots + a_nX^n \in R[X]$  is a unit if and only if  $a_0$  is a unit in  $R$  and  $a_1, \dots, a_n$  are nilpotents.



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