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Global properties of complex analytic spaces

1. Introduction

2. Topological properties of complex analytic spaces

Proposition 2.1. Let X be a Hausdorff complex analytic space. Then the following are equivalent:

- (1) X is paracompact;
- (2) Each connected component of X is σ -compact;
- (3) Each connected component of X is Lindelöf;
- (4) X admits a compact exhaustion.

PROOF. (1) \Leftrightarrow (2): This follows from [Proposition 3.2](#) in [Topology and bornology](#).

(2) \Leftrightarrow (3): This follows from [Proposition 5.2](#) in [Topology and bornology](#).

(3) \Leftrightarrow (4): This follows from [Proposition 5.2](#) in [Topology and bornology](#). \square

Lemma 2.2. Let $f : X \rightarrow Y$ be a proper surjective morphism of complex analytic spaces. Then the following are equivalent:

- (1) X is paracompact and Hausdorff;
- (2) Y is paracompact and Hausdorff.

PROOF. (1) \implies (2): This follows from [Theorem 3.3](#) in [Topology and bornology](#).

(2) \implies (1): We may assume that Y is connected. Then X is Hausdorff as f is separated. By [Proposition 2.1](#), Y is σ -compact. It follows that X is also σ -compact. In particular, each connected component of X is also σ -compact. In particular, X is paracompact. \square

3. Holomorphically convex hulls

Definition 3.1. Let X be a complex analytic space and M be a subset of X , we define the *holomorphically convex hull* of M in X as

$$\hat{M}^X := \left\{ x \in X : |f(x)| \leq \sup_{y \in M} |f(y)| \text{ for all } f \in \mathcal{O}_X(X) \right\}.$$

Proposition 3.2. Let X be a complex analytic space and M be a subset of X . Then the following properties hold:

- (1) \hat{M}^X is closed in X ;
- (2) $M \subseteq \hat{M}^X$ and $\widehat{\hat{M}^X}^X = \hat{M}^X$;
- (3) If M' is another subset of X containing M , then $\hat{M}^X \subseteq \hat{M}'^X$;
- (4) If $f : Y \rightarrow X$ is a morphism of complex analytic spaces, then

$$\widehat{f^{-1}(M)}^Y \subseteq f^{-1}(\hat{M}^X);$$

(5) If X' is another complex analytic space and M' is a subset of X' , then

$$\widehat{M \times M'}^{X \times X'} \subseteq \hat{M}^X \times \hat{M'}^{X'};$$

(6) If M' is another subset of X and $\hat{M}^X = M$, $\hat{M'}^X = M'$, then

$$\widehat{M \cap M'}^X = M \cap M'.$$

PROOF. (1), (2), (3), (4), (5) are obvious by definition.

(6) is a consequence of (3). \square

Example 3.3. Let Q be a compact cube in \mathbb{C}^n for some $n \in \mathbb{N}$, then $\hat{Q}^{\mathbb{C}^n} = Q$.

In fact, by [Proposition 3.2\(5\)](#), we may assume that $n = 1$. Given $p \in \mathbb{C} \setminus Q$, we can take a closed disk $T \subseteq \mathbb{C}$ centered at $a \in \mathbb{C}$ such that $Q \subseteq T$ while $p \notin T$. Consider $z - a \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$, then

$$|f(p)| > \sup_{q \in Q} |f(q)|.$$

So $p \notin \hat{Q}^{\mathbb{C}}$.

4. Stones

Definition 4.1. Let X be a complex analytic space. A *stone* in X is a pair (P, π) consisting of

- (1) a non-empty compact set P in X and
- (2) a morphism $\pi : X \rightarrow \mathbb{C}^n$ for some $n \in \mathbb{N}$

such that there is a compact tube Q in \mathbb{C}^n and an open set W in X such that $P = \pi^{-1}(Q) \cap W$.

We call $P^0 := \pi^{-1}(\text{Int } Q) \cap W$ the *analytic interior* of the stone (P, π) . It clearly does not depend on the choice of W .

We observe that $\hat{P}^X \cap W = P$. In fact, $P \subseteq \pi^{-1}(Q)$, so

$$\hat{P}^X \subseteq \pi^{-1}(\hat{Q}^{\mathbb{C}^n}) = \pi^{-1}(Q) = P \cap W = P.$$

Here we applied [Proposition 3.2](#) and [Example 3.3](#).

In general, $P^0 \subseteq \text{Int } P$, but they can be different.

Theorem 4.2. Let X be a Hausdorff complex analytic space and $K \subseteq X$ be a compact subset. Then the following are equivalent:

- (1) There is an open neighbourhood W of K in X such that $\hat{K}^X \cap W$ is compact;
- (2) There is an open relative compact neighbourhood W of K in X such that $\partial W \cap \hat{K} = \emptyset$;
- (3) There is a stone (P, π) in X with $K \subseteq P^0$.

PROOF. (1) \implies (2): This is trivial, in fact, we may assume that W in (1) is relatively compact in X .

(2) \implies (3): As \hat{K}^X is closed by [Proposition 3.2\(1\)](#) and $\partial W \cap \hat{K}^X = \emptyset$, given $p \in \partial W$, we can find $h \in \mathcal{O}_X(X)$ such that

$$\sup_{x \in K} |h(x)| < 1 < |h(p)|.$$

We will denote the left-hand side by $|h|_K$. Up to raising h to a power, we may assume that

$$\max\{|\operatorname{Re} h(p)|, |\operatorname{Im} h(p)|\} > 1.$$

As ∂W is compact, we can find finitely many sections $h_1, \dots, h_m \in \mathcal{O}_X(X)$ so that

$$\max_{j=1, \dots, m} \{|\operatorname{Re} h_j|_K, |\operatorname{Im} h_j|_K\} < 1, \quad \max_{j=1, \dots, m} \{|\operatorname{Re} h_j(p)|, |\operatorname{Im} h_j(p)|\} > 1.$$

Let

$$Q := \{(z_1, \dots, z_m) \in \mathbb{C}^m : |\operatorname{Re} z_i| \leq 1, |\operatorname{Im} z_i| \leq 1 \text{ for all } i = 1, \dots, m\}.$$

The sections h_1, \dots, h_m defines a homomorphism $\pi : X \rightarrow \mathbb{C}^m$ by [Theorem 4.2](#) in [The notion of complex analytic spaces](#). Obviously, $P = \pi^{-1}(Q) \cap W$ satisfies our assumptions.

(3) \implies (1): Let W be the open set as in [Definition 4.1](#). As $\hat{P}^X \cap W = P$ and $K \subseteq P$, we have

$$\hat{K} \cap W \subseteq P \cap W = P.$$

As P is compact, so is $\hat{K} \cap W$. \square

Theorem 4.3. Let X be a Hausdorff complex analytic space and $(P, \pi : X \rightarrow \mathbb{C}^n)$ be a stone in X . Let Q be the tube in \mathbb{C}^m as in [Definition 4.1](#). Then there are open neighbourhoods U and V of P and Q in X and \mathbb{C}^n respectively with $\pi(U) \subseteq V$ and $P = \pi^{-1}(Q) \cap U$ such that $\pi|_U : U \rightarrow V$ is proper.

PROOF. Let $W \subseteq X$ be the open set as in [Definition 4.1](#). We may assume that W is relatively compact. Then ∂W and $\pi(\partial W)$ are also compact. As $\partial W \cap \pi^{-1}(Q)$ is empty, we know that $V := \mathbb{C}^n \setminus \pi(\partial W)$ is an open neighbourhood of Q . The set $U := W \cap \pi^{-1}(V) = W \setminus \pi^{-1}(\pi(\partial W))$ is open in X and $\pi(U) \subseteq V$. Observe that $\pi|_U : U \rightarrow V$ is proper by [Lemma 4.6](#) in [Topology and bornology](#).

Furthermore,

$$\pi^{-1}(Q) \cap U = \pi^{-1}(Q) \cap (W \setminus (\pi^{-1}(Q) \cap \pi^{-1}(\pi(\partial W)))).$$

But $\pi^{-1}Q \cap \pi^{-1}\pi(\partial W)$ is empty as $Q \cap \pi(\partial W)$ is. It follows that $\pi^{-1}(Q) \cap U = P$ and hence U is a neighbourhood of P . \square

Definition 4.4. Let X be a complex analytic space. Let $(P, \pi : X \rightarrow \mathbb{C}^n)$, $(P', \pi' : X \rightarrow \mathbb{C}^{n'})$ be two stones on X . We say (P, π) is contained in (P', π') if the following conditions are satisfied:

- (1) P lies in the analytic interior of P' ;
- (2) $n' \geq n$ and there is $q \in \mathbb{C}^{n'-n}$ such that if $Q \subseteq \mathbb{C}^n$, $Q' \subseteq \mathbb{C}^{n'}$ be the tubes as in [Definition 4.1](#), then

$$Q \times \{q\} \subseteq Q'.$$

- (3) There is a morphism $\varphi : X \rightarrow \mathbb{C}^{n'-n}$ such that

$$\pi' = (\pi, \varphi).$$

We formally write $(P, \pi) \subseteq (P', \pi')$ in this case. Clearly, this defines a partial order on the set of stones on X .

Definition 4.5. Let X be a complex analytic space. An *exhaustion of X by stones* is a sequence $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ of stones such that

- (1) $(P_i, \pi_i) \subseteq (P_{i+1}, \pi_{i+1})$ for all $i \in \mathbb{Z}_{>0}$;

(2)

$$X = \bigcup_{i=1}^{\infty} P_i^0.$$

We say X is *weakly holomorphically convex* if there is an exhaustion of X by stones.

Theorem 4.6. Let X be a Hausdorff complex analytic space. Consider the following conditions:

- (1) X is weakly holomorphically convex;
- (2) For any compact subset $K \subseteq X$, there is an open set $W \subseteq X$ such that $\hat{K}^X \cap W$ is compact.

Then (1) \implies (2). If X is paracompact, then (2) \implies (1).

PROOF. (1) \implies (2): It suffices to observe that $K \subseteq P_j^0$ when j is large enough and apply [Theorem 4.2](#).

Assume that X is paracompact. (2) \implies (1): Let (K_i) a compact exhaustion of X . We construct the stones $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ so that

$$K_i \subseteq P_i^0$$

for all $i \in \mathbb{Z}_{>0}$ inductively. Let P_1 be an arbitrary stone in X such that $K_1 \subseteq P_1^0$. The existence of P_1 is guaranteed by [Theorem 4.2](#).

Assume that we have constructed $(P_{i-1}, \pi_{i-1} : X \rightarrow \mathbb{C}^{n_{i-1}})$ for $i \geq 2$. Let $Q_{i-1} \subseteq \mathbb{C}^{n_{i-1}}$ be the associated tube. By [Theorem 4.2](#) again, take a stone $(P_i, \pi_i^* : X \rightarrow \mathbb{C}^n)$ with $K_i \cup P_{i-1} \subseteq P_i^0$. Let $Q_i^* \subseteq \mathbb{C}^n$ be the associated tube. Let W be an open subset of X with

$$P_i = \pi_i^{*, -1}(Q_i^*) \cap W.$$

Choose a tube $Q'_i \subseteq \mathbb{C}^{n_{i-1}}$ with $Q_{i-1} \subseteq \text{Int } Q'_i$ so that

$$\pi_{i-1}(P_i) \subseteq \text{Int } Q'_i.$$

Let $\pi_i := (\pi_{i-1}, \pi_i^*) : X \rightarrow \mathbb{C}^{n_{i-1}+n}$ and $Q_i := Q'_i \times Q_i^*$. Then (P_i, π_i) is a stone and $(P_{i-1}, \pi_{i-1}) \subseteq (P_i, \pi_i)$. \square

5. Holomorphical separable spaces

Definition 5.1. Let X be a complex analytic space. We say X is *holomorphically separable* if for any $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

Here we regard f as a continuous function $X \rightarrow \mathbb{C}$. In particular, a holomorphically separable space is Hausdorff.

Definition 5.2. Let X be a complex analytic space. We say X is *holomorphically convex* if $|X|$ is Hausdorff and for any compact set $K \subseteq X$, \hat{K}^X .

We say X is *weakly holomorphically convex* if for any quasi-compact set $K \subseteq X$, the connected components of \hat{K}^X are all quasi-compact.

Proposition 5.3. Let X be a holomorphically convex complex analytic space. Then X^{red} is holomorphically convex.

PROOF. This follows immediately from the definition. \square

Proposition 5.4. Let X be a Hausdorff complex analytic space. Consider the following conditions:

- (1) X is holomorphically convex;
- (2) For any sequence $x_i \in X$ ($i \in \mathbb{Z}_{>0}$) without accumulation points, there is $f \in \mathcal{O}_X(X)$ such that $|f(x_i)|$ is unbounded.

Then (2) \implies (1) if X is paracompact.

PROOF. (2) \implies (1): By [Proposition 2.1](#), each connected component of X is Lindelöf. For a Lindelöf Hausdorff space, sequential compactness implies compactness. \square

Corollary 5.5. Let $n \in \mathbb{N}$ and Ω be a domain in \mathbb{C}^n . Assume that for each $p \in \partial\Omega$, there is a holomorphic function f on an open neighbourhood U of $\bar{\Omega}$ such that $f(p) = 0$ and f is non-zero on Ω . Then Ω is holomorphically convex.

PROOF. Let $x_i \in \Omega$ ($i \in \mathbb{Z}_{>0}$) be a sequence without accumulation points in Ω . We need to construct $f \in \mathcal{O}_\Omega(\Omega)$ such that $(|f(x_i)|)_{i \in \mathbb{Z}_{>0}}$ is unbounded. This is clear if x_i itself is unbounded. Assume that x_i is bounded. Then up to passing to a subsequence, we may assume that $x_i \rightarrow p \in \partial\Omega$ as $i \rightarrow \infty$. The inverse of the function f in our assumption of the corollary works. \square

6. Stein sets

Definition 6.1. Let X be a complex analytic space and P be a closed subset of X . We say P is a *Stein set* in X if for any coherent \mathcal{O}_U -module \mathcal{F} for some open neighbourhood U of P in X , we have

$$H^i(P, \mathcal{F}) = 0 \quad \text{for all } i \in \mathbb{Z}_{>0}.$$

A *coherent \mathcal{O}_P -module* is a coherent \mathcal{O}_U -module for some open neighbourhood U of P in X . Two coherent \mathcal{O}_P -modules are isomorphic if there is a small enough open neighbourhood V of P in X such that they are isomorphic when restricted to V . In particular, \mathcal{O}_P denotes the coherent \mathcal{O}_P -module defined by \mathcal{O}_X on X .

The germ-wise notions obviously make sense for coherent \mathcal{O}_P -modules.

The given condition is usually known as *Cartan's Theorem B*. It implies *Cartan's Theorem A*:

Theorem 6.2 (Cartan's Theorem A). Let X be a complex analytic space and P be a Stein set in X . Let \mathcal{F} be a coherent \mathcal{O}_U -module for some open neighbourhood U of P in X . Then $H^0(P, \mathcal{F})$ generates \mathcal{F}_x for each $x \in P$.

PROOF. Fix $x \in P$. Let \mathcal{M} be the coherent ideal sheaf on U consisting of holomorphic functions vanishing at x . Then $\mathcal{F}\mathcal{M}$ is a coherent \mathcal{O}_U -module. It follows from Theorem B that

$$H^0(P, \mathcal{F}) \rightarrow H^0(P, \mathcal{F}/\mathcal{F}\mathcal{M})$$

is surjective. Note that we can identify this map with the natural map

$$H^0(P, \mathcal{F}) \rightarrow \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x.$$

Let e_1, \dots, e_m be a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Lift them to $s_1, \dots, s_m \in H^0(P, \mathcal{F})$. By Nakayama's lemma, s_{1x}, \dots, s_{mx} generate the $\mathcal{O}_{X,x}$ -module \mathcal{F}_x . \square

Corollary 6.3. Let X be a complex analytic space and P be a quasi-compact Stein set in X . Let \mathcal{F} be a coherent \mathcal{O}_P -module. Then there is $n \in \mathbb{Z}_{>0}$ and an epimorphism

$$\mathcal{O}_P^n \rightarrow \mathcal{F}.$$

PROOF. By [Theorem 6.2](#), we can find an open covering $\{U_i\}_{i \in I}$ of P such that there are homomorphisms

$$h_i : \mathcal{O}_P^{n_i} \rightarrow \mathcal{F}$$

for some $n_i \in \mathbb{Z}_{>0}$, which is surjective on U_i for each $i \in I$. By the quasi-compactness of P , we may assume that I is a finite set. Then it suffices to set $n = \sum_{i \in I} n_i$ and consider the epimorphism $\mathcal{O}_P^n \rightarrow \mathcal{F}$ induced by the h_i 's. \square

Theorem 6.4. Let X be a complex analytic space and $P \subseteq X$ be a set with the following properties:

- (1) there is an open neighbourhood U of P in X , a domain V in \mathbb{C}^m for some $m \in \mathbb{N}$ and a finite holomorphic morphism $\tau : U \rightarrow V$;
- (2) There exists a compact tube in \mathbb{C}^m contained in V such that $P = \tau^{-1}(Q)$.

Then P is a compact Stein set in X .

PROOF. As $P = \tau^{-1}(Q)$ and τ is proper, we see that P is compact.

It remains to show that P is a Stein set in X . Let \mathcal{F} be a coherent \mathcal{O}_P -module.

Step 1. We first reduce to the case where \mathcal{F} is defined by a coherent \mathcal{O}_U -module.

Take an open neighbourhood U' of P in X contained in U such that \mathcal{F} is defined by a coherent $\mathcal{O}_{U'}$ -module. By [Lemma 4.2](#) in [Topology and bornology](#), we can take an open neighbourhood V' of Q in V such that $\tau^{-1}(V') \subseteq U'$. The restriction of τ to $\tau^{-1}(V') \rightarrow V'$ is again finite.

Step 2. By Leray spectral sequence,

$$H^i(P, \mathcal{F}) \cong H^i(Q, (\tau|_P)_* \mathcal{F})$$

for all $i \geq 0$. By [Corollary 4.8](#) in [Morphisms between complex analytic spaces](#), $(\tau|_P)_* \mathcal{F}$ is a coherent \mathcal{O}_Q -module, so we are reduced to show that Q is a Stein set in \mathbb{C}^m , which is well-known. \square

Definition 6.5. Let X be a Hausdorff complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. A *Stein exhaustion of X relative to \mathcal{F}* is a compact exhaustion $(P_i)_{i \in \mathbb{Z}_{>0}}$ such that the following conditions are satisfied:

- (1) P_i is a Stein set in X for each $i \in \mathbb{Z}_{>0}$;
- (2) the \mathbb{C} -vector space $H^0(P_i, \mathcal{F})$ admits a semi-norm $|\bullet|_i$ such that the restriction map

$$H^0(X, \mathcal{F}) \rightarrow H^0(P_i, \mathcal{F})$$

has dense image with respect to the topological defined by $|\bullet|_i$ for each $i \in \mathbb{Z}_{>0}$;

- (3) The restriction map

$$H^0(P_{i+1}, \mathcal{F}) \rightarrow H^0(P_i, \mathcal{F})$$

is bounded for each $i \in \mathbb{Z}_{>0}$;

- (4) Let $i \in \mathbb{Z}_{\geq 2}$. Suppose that $(s_j)_{j \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in $H^0(P_i, \mathcal{F})$, then the restricted sequence $s_j|_{P_{i-1}}$ has a limit in $H^0(P_{i-1}, \mathcal{F})$;
- (5) Let $i \in \mathbb{Z}_{\geq 2}$. If $s \in H^0(P_i, \mathcal{F})$ and $|s|_i = 0$, then $s|_{P_{i-1}} = 0$.

A *Stein exhaustion* of X is a compact exhaustion of X that is a Stein exhaustion of X relative to any coherent \mathcal{O}_X -module.

Theorem 6.6. Let X be a Hausdorff complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Assume that $(P_i)_{i \in \mathbb{Z}_{>0}}$ is a Stein exhaustion of X relative to \mathcal{F} . Then

$$H^q(X, \mathcal{F}) = 0 \quad \text{for any } q \in \mathbb{Z}_{>0}.$$

PROOF. When $q \geq 2$, this follows from the general facts proved in [Lemma 5.4](#) in [Topology and bornology](#). We will assume that $q = 1$.

We may assume that X is connected. First observe that X is necessarily paracompact. This follows from [Proposition 3.2](#) in [Topology and bornology](#). In particular, we can take a flabby resolution

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}^0 \rightarrow \mathcal{G}^1 \rightarrow \dots$$

Taking global sections, we get a complex

$$0 \rightarrow H^0(X, \mathcal{F}) \xrightarrow{i} H^0(X, \mathcal{G}^0) \xrightarrow{d_0} H^0(X, \mathcal{G}^1) \xrightarrow{d_1} H^0(X, \mathcal{G}^2) \xrightarrow{d_2} \dots$$

We need to show that $\ker d_1 = \text{Im } d_0$. Let $\alpha \in \ker d_1$. We need to construct $\beta \in H^0(X, \mathcal{G}^0)$ with $d_0\beta = \alpha$.

We take semi-norms $|\bullet|_i$ on $H^0(P_i, \mathcal{F})$ for each $i \in \mathbb{Z}_{>0}$ satisfying the conditions in [Definition 6.5](#). We may furthermore assume that the restriction $H^0(P_{i+1}, \mathcal{F}) \rightarrow H^0(P_i, \mathcal{F})$ is a contraction for each $i \in \mathbb{Z}_{>0}$.

For each $j \in \mathbb{Z}_{\geq 2}$, we will construct $\beta_j \in H^0(P_j, \mathcal{G}^0)$ and $\delta_j \in H^0(P_{j-1}, \mathcal{F})$ such that

- (1) $(d_0|_{P_j})\beta_j = \alpha|_{P_j}$;
- (2) $(\beta_{j+1} + \delta_{j+1})|_{P_{j-1}} = (\beta_j + \delta_j)|_{P_{j-1}}$.

It suffices to take $\beta \in H^0(X, \mathcal{G}^0)$ as the section defined by the $\beta_j + \delta_j$'s.

We first construct β_j . Choose a sequence $\beta'_j \in H^0(P_j, \mathcal{G}^0)$ with

$$(d_0|_{P_j})\beta'_j = \alpha|_{P_j}$$

for each $j \in \mathbb{Z}_{>0}$. This is possible because P_j is Stein. We define β_j satisfying Condition (1) for $j \in \mathbb{Z}_{>0}$ inductively. We begin with $\beta_1 = \beta'_1$. Assume that β_1, \dots, β_j have been constructed. Let

$$\gamma'_j := \beta'_{j+1}|_{P_j} - \beta_j.$$

Then

$$(d_0|_{P_j})\gamma'_j = 0.$$

It follows that $\gamma'_j \in H^0(P_j, \mathcal{F})$. Take $\gamma_j \in H^0(X, \mathcal{F})$ with

$$|\gamma'_j - \gamma_j|_{P_j}|_j \leq 2^{-j}.$$

Define

$$\beta_{j+1} = \beta'_{j+1} - \gamma_j|_{P_{j+1}}.$$

Then clearly β_{j+1} satisfies (1).

Next we construct the sequence δ_j .

We observe that for each $j \in \mathbb{Z}_{>0}$,

$$|\beta_{j+1}|_{P_j} - \beta_j|_j \leq 2^{-j}.$$

Let

$$s_k^j := \beta_{j+k}|_{P_j} - \beta_j \in H^0(P_j, \mathcal{F})$$

for all $j \in \mathbb{Z}_{>0}$ and $k \in \mathbb{N}$. By definition,

$$s_k^j - s_{k-1}^{j+1}|_{P_j} = \beta_{j+1}|_{P_j} - \beta_j$$

for all $j \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{>0}$.

We claim that $(s_k^j|_{P_{j-1}})_k$ converges in $H^0(P_{j-1}, \mathcal{F})$ as $k \rightarrow \infty$. By our assumption, it suffices to show that $(s_k^j)_k$ is a Cauchy sequence in $H^0(P_j, \mathcal{F})$ for each $j \in \mathbb{Z}_{>1}$. We first compute

$$|\beta_{j+l}|_{P_j} - \beta_{j+l-1}|_{P_j}|_j \leq |\beta_{j+l}|_{P_{j+l-1}} - \beta_{j+l-1}|_{P_{j+l-1}}|_{j+l-1} \leq 2^{1-j-l}$$

for all $l \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{>0}$. As a consequence for $k' > k \geq 1$, we have

$$|s_k^j - s_{k'}^j|_j \leq \sum_{l=k+1}^{k'} 2^{1-j-l} \leq 2^{1-j+k}.$$

So we conclude our claim.

Let δ_j be the limit of $s_k^j|_{P_{j-1}}$ as $k \rightarrow \infty$ for each $j \in \mathbb{Z}_{\geq 2}$. Then

$$\lim_{k \rightarrow \infty} (s_k^j - s_{k-1}^{j+1})|_{P_{j-1}} = (\delta_j - \delta_{j+1})|_{P_{j-1}}$$

for each $j \in \mathbb{Z}_{\geq 2}$. The desired identity is clear. \square

7. Analytic blocks

Definition 7.1. Let X be a Hausdorff complex analytic space. A stone $(P, \pi : X \rightarrow \mathbb{C}^n)$ on X is an *analytic block* in X if there are open neighbourhoods U and V of P and Q in X and Y respectively, where $Q \subseteq \mathbb{C}^n$ denotes the tube associated with the stone, such that

- (1) $\pi(U) \subseteq V$;
- (2) $P = \pi^{-1}(Q) \cap U$;
- (3) $U \rightarrow V$ induced by π is a finite morphism.

Recall that by [Theorem 4.3](#), we can always assume that $U \rightarrow V$ is proper.

Proposition 7.2. Let X be a Hausdorff complex analytic space and (P, π) be an analytic block in X . Then P is a compact Stein set in X .

PROOF. This follows from [Theorem 6.4](#) applied to $U \rightarrow V$ in [Definition 7.1](#). \square

Proposition 7.3. Let X be a complex analytic space such that each compact analytic set in X is finite, then every stone in X is an analytic block in X .

PROOF. Let $(P, \pi : X \rightarrow \mathbb{C}^n)$ be a stone in X . We consider the proper morphism $\tau : U \rightarrow V$ as in [Theorem 4.3](#). Each fiber of τ is a compact subset of U and hence a compact subset of X . By our assumption, it is finite. It suffices to apply [Proposition 4.5](#) in [Topology and bornology](#) to conclude that τ is finite. \square

8. Holomorphically spreadable spaces

Definition 8.1. Let X be a complex analytic space. We say X is *holomorphically spreadable* if $|X|$ is Hausdorff and for any $x \in X$, we can find an open neighbourhood U of x in X such that

$$\{y \in U : f(x) = f(y) \text{ for all } f \in \mathcal{O}_X(X)\} = \{x\}.$$

A holomorphically separable space is clearly holomorphically spreadable.

Proposition 8.2. Let X be a holomorphically spreadable complex analytic space and $x \in X$. Then there exist finitely many $f_1, \dots, f_n \in \mathcal{O}_X(X)$ such that x is an isolated point of $W(f_1, \dots, f_n)$.

PROOF. By induction on $\dim_x X$, it suffices to prove the following claim: if A is an analytic set in X and $a \in A$ such that $\dim_a A \geq 1$. Then there is $f \in \mathcal{O}_X(X)$ such that $\dim_a(A \cap W(f)) = \dim_a A - 1$.

To prove the claim, let A_1, \dots, A_k be the irreducible components of A . We may assume that all of them contain a . Choose $a_j \in A_j$ for each $j = 1, \dots, k$ so that a, a_1, \dots, a_k are pairwise different. Then there is a function $f \in \mathcal{O}_X(X)$ with $f(a) = 0$ while $f(a_j) \neq 0$ for $j = 1, \dots, k$. Then $a \in W(f)$ while $f|_{A_j}$ is not identically 0. By Krulls Hauptidealsatz, $\dim_a(A_j \cap W(f)) = \dim_a A_j - 1$ for all $j = 1, \dots, k$. Observe that $A \cap W(f)$ and $\bigcup_{j=1}^k (A_j \cap W(f))$ coincide near a , so

$$\dim_a(A \cap W(f)) = \max_{j=1, \dots, k} \dim_a(A_j \cap W(f)) = \max_{j=1, \dots, k} (\dim_a A_j - 1) = \dim_a A - 1.$$

□

Proposition 8.3. Let X be an irreducible holomorphically spreadable complex analytic space. Then X has countable basis.

The statement of this proposition in [Fis76, Proposition 0.37] is clearly wrong. I do not understand the argument of either [Jur59] or [Gra55], where they claim that this result holds for connected holomorphically spreadable complex analytic spaces.

PROOF. We may assume that X is connected. Recall that by Corollary 8.6 in Local properties of complex analytic spaces, X is locally connected. Let $F : X \rightarrow \mathbb{C}^{\mathcal{O}_X(X)}$ be the map sending $x \in X$ to $(f(x))_{f \in \mathcal{O}_X(X)}$. By our assumption, F is continuous and has discrete fibers. In particular, for each $x \in X$, we may assume take finitely many $f_1, \dots, f_n \in \mathcal{O}_X(X)$ so that the induced morphism $F' : X \rightarrow \mathbb{C}^n$ is quasi-finite at x . By Corollary 2.8 in Analytic sets, we can find a nowhere dense analytic set A in X such that the map $X \setminus A \rightarrow \mathbb{C}^n$ induced by F' is quasi-finite. Now we endow $\mathcal{O}_X(X)$ with the compact-open topology. It is a metric space. By Proposition 6.2 in Topology and bornology, $X \setminus A$ has countable basis. It follows that $\mathcal{O}_X(X \setminus A)$ is a separable metric space. Hence, so it $\mathcal{O}_X(X)$. In particular, there is a continuous map with discrete fibers

$$X \rightarrow \mathbb{C}^\omega.$$

It follows again from Proposition 6.2 in Topology and bornology that X has countable basis. □

Proposition 8.4. Let X be a holomorphically spreadable complex analytic space. Then any compact analytic set A in X is finite.

PROOF. Let B be a connected component of A and $p \in B$. We need to show that $B = \{p\}$. Take finitely many $f_1, \dots, f_n \in \mathcal{O}_X(X)$ so that p is an isolated point of $W(f_1, \dots, f_n)$. This is possible by Proposition 8.2. As f_i vanishes on B for each $i = 1, \dots, n$, we have $B = \{p\}$. □

Corollary 8.5. Let X be a complex analytic space and A be a compact analytic subset of X . Suppose that there exists an analytic block $(P, \pi : X \rightarrow \mathbb{C}^n)$ in X with $A \subseteq P$, then A is finite.

PROOF. Take $U \subseteq X, V \subseteq \mathbb{C}^n$ as in [Definition 7.1](#) so that $U \rightarrow V$ is finite. Then U is clearly holomorphically spreadable. By [Proposition 8.4](#), A is finite. \square

9. Holomorphically complete spaces

Definition 9.1. Let X be a complex analytic space. An *exhaustion of X by analytic blocks* is an exhaustion of X by stones $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ such that (P_i, π_i) is an analytic block for each $i \in \mathbb{Z}_{>0}$.

We say X is *holomorphically complete* if X is Hausdorff and there is an exhaustion of X by analytic stones.

Theorem 9.2. Let X be a Hausdorff complex analytic space. Then the following are equivalent:

- (1) X is holomorphically complete;
- (2) X is weakly holomorphically convex and every compact analytic subset of X is finite.

PROOF. (1) \implies (2): X is weakly holomorphically convex by definition. Each compact analytic subset A of X is contained in some analytic block, hence finite by [Corollary 8.5](#).

(2) \implies (1): This follows from [Proposition 7.3](#). \square

Lemma 9.3. Let X be a complex manifold and \mathcal{I} be a coherent subsheaf of \mathcal{O}_X^l for some $l \in \mathbb{Z}_{>0}$. Then $\mathcal{I}(X)$ is a closed subspace of $\mathcal{O}_X(X)^l$ endowed with the compact-open topology.

PROOF. Let $(f_j \in \mathcal{I}(X))_{j \in \mathbb{Z}_{>0}}$ be a sequence with a limit $f \in \mathcal{O}_X^l(X)$. Let $x \in X$. It suffices to show that $f_x \in \mathcal{I}_x$. Observe that f_x is the limit of f_{j_x} as $j \rightarrow \infty$. As $\mathcal{O}_{X,x}$ is noetherian, the submodule \mathcal{I}_x of \mathcal{O}_x^l is closed by [Corollary 7.4](#) in ???. We conclude. \square

Definition 9.4. Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $(P, \pi : X \rightarrow \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$.

Choose $U \subseteq X, V \subseteq \mathbb{C}^n$ as in [Definition 7.1](#) so that $\tau : U \rightarrow V$ induced by π is finite. Then $\mathcal{G} := \tau_*(\mathcal{F}|_U)$ is a coherent \mathcal{O}_V -module. By [Corollary 6.3](#), we can find $l \in \mathbb{Z}_{>0}$ and an epimorphism $\mathcal{O}_Q^l \rightarrow \mathcal{G}|_Q$. It induces an epimorphism $\epsilon : H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l \rightarrow H^0(Q, \mathcal{G}) \xrightarrow{\sim} H^0(P, \mathcal{F})$. We define a semi-norm $|\bullet|$ on $H^0(P, \mathcal{F})$ as the quotient semi-norm induced by the sup seminorm on $H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$.

A seminorm on $H^0(P, \mathcal{F})$ defined in this way is called a *good semi-norm* on $H^0(P, \mathcal{F})$ with respect to (P, π) .

Lemma 9.5. Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let (P, π) be an analytic block on X . A good semi-norm on $H^0(P, \mathcal{F})$ induces a metric on $H^0(P^0, \mathcal{F})$.

PROOF. We need to show that if $|s| = 0$ for some $s \in H^0(P, \mathcal{F})$, then $s|_{P^0} = 0$, where P^0 is the analytic interior of P .

We use the same notations as in [Definition 9.4](#). We can take $h \in H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$ and $h_j \in \ker \epsilon$ for each $j \in \mathbb{Z}_{>0}$ so that $\epsilon(h) = s$ and $\|h_j - h\|_{L^\infty} \rightarrow 0$. So $h_j|_Q \rightarrow h|_Q$ with respect to the compact-open topology. From [Lemma 9.3](#), we conclude that the image of $h|_{\text{Int } Q}$ is 0. Namely, s vanishes on $P^0 = \tau^{-1}(\text{Int } Q)$. \square

Lemma 9.6. Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $(P, \pi : X \rightarrow \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$. Consider the epimorphism of sheaves

$$\mathcal{O}_Q^l \rightarrow \pi_*(\mathcal{F}|_P)$$

as in [Definition 9.4](#) and endow $H^0(P^0, \mathcal{F})$ with the metric induced by the corresponding good semi-norm. Let

$$Q_1 \subseteq Q_2 \subseteq \cdots$$

be a compact exhaustion of $\text{Int } Q$ by tubes with the same centers in \mathbb{C}^n . We get an induced map

$$\epsilon_j : H^0(Q_j, \mathcal{O}_{\mathbb{C}^n}^l) \rightarrow \pi_*(\mathcal{F}|_P)(Q_j)$$

for each $j \in \mathbb{Q}_{>0}$. We therefore get good semi-norms $|\bullet|_j$ on $H^0(P^0, \mathcal{F})$ for each $j \in \mathbb{Z}_{>0}$. Let

$$d(s_1, s_2) := \sum_{j=1}^{\infty} 2^{-j} \frac{|s_1 - s_2|_j}{1 + |s_1 - s_2|_j}$$

for each $s_1, s_2 \in H^0(P^0, \mathcal{F})$. Then d is a metric on $H^0(P^0, \mathcal{F})$ and $H^0(P^0, \mathcal{F})$ is a Fréchet space with respect to this topology.

Moreover, the topology does not depend on the choice of π , ϵ and the exhaustion.

PROOF. By [Lemma 9.5](#), each $|\bullet|_\nu$ is a norm on $H^0(P^0, \mathcal{F})$. It follows that d is a metric. Next we show that $H^0(P^0, \mathcal{F})$ is Fréchet. Let $(s_j)_{j \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in $H^0(P^0, \mathcal{F})$. We can find bounded sequences $(f_{jk} \in H^0(Q_k, \mathcal{O}_{\mathbb{C}^n}^l))_{k \in \mathbb{Z}_{>0}}$ so that $\epsilon_k(f_{jk}) = s_j|_{\pi^{-1}(Q_k) \cap P}$ ($k \in \mathbb{Z}_{>0}$) for each $j \in \mathbb{Z}_{>0}$. By Montel's theorem, there is a subsequence of $(f_{jk})_j$ which converges uniformly on Q_{k-1} to $f_k \in H^0(Q_{k-1}, \mathcal{O}_{\mathbb{C}^n}^l)$. Then $\epsilon_{k-1}(f_{k+1})|_{\text{Int } Q_{k-1}} = \epsilon_{k-1}(f_k)|_{\text{Int } Q_{k-1}}$ for each $k \in \mathbb{Z}_{\geq 2}$. So we can glue the f_k 's to $s \in H^0(P^0, \mathcal{F})$. Clearly, $s_k \rightarrow s$ as $k \rightarrow \infty$.

Next we show that the topology is independent of the choice of π , ϵ and the exhaustion. The independence of the exhaustion is obvious. We prove the other two independence. Let $(P, \pi' : X \rightarrow \mathbb{C}^{n'})$ be another analytic block with $\pi' = (\pi, \varphi) : X \rightarrow \mathbb{C}^n \times \mathbb{C}^m$, $n' = n + m$. Let $Q^* \subseteq \mathbb{C}^m$ be a tube such that $\varphi(P) \subseteq Q^*$. Then $P = \pi'^{-1}(Q \times Q^*) \cap U$. We can find an open neighbourhood U' of P in X and V' of $Q \times Q^*$ in $\mathbb{C}^{n'}$ for which the induced map $\tau' : U' \rightarrow V'$ is finite by [Definition 7.1](#). Fix an epimorphism $\mathcal{O}_{\mathbb{C}^{n'}}^{l'}|_{Q \times Q^*} \rightarrow \pi'_*(\mathcal{F}|_P)$ for some $l' \in \mathbb{Z}_{>0}$. Construct an exhaustion of $\text{Int } Q \times \text{Int } Q^*$ of the product type: $(Q_j \times Q_j^*)_{j \in \mathbb{Z}_{>0}}$ as in the lemma. Let d' denote the induced metric on $H^0(\text{Int } P, \mathcal{F})$.

We will show that d' and d induce the same topology. Let $e_1, \dots, e_l \in H^0(Q, \mathcal{O}_{\mathbb{C}^n}^l)$ be the standard basis. Let e'_1, \dots, e'_l be the preimages of $\epsilon(e_1), \dots, \epsilon(e_l)$ in $\pi_*(\mathcal{F}|_P)(Q) = \pi'_*(\mathcal{F}|_P)(Q \times Q^*)$ in $\mathcal{O}_{\mathbb{C}^{n'}}(Q \times Q^*)^{l'}$ under ϵ' . Further, for $f \in \mathcal{O}_{\mathbb{C}^n}(Q_j)$, we denote by $f' \in \mathcal{O}_{\mathbb{C}^{n'}}(Q_j \times Q_j^*)$ the holomorphic extension of f to $Q_j \times Q_j^*$ constant along $\{q\} \times Q_j^*$ for each $q \in Q_j$ for each $j \in \mathbb{Z}_{>0}$. The norms of

$$\mathcal{O}_{\mathbb{C}^n}(Q_j)^l \rightarrow \mathcal{O}_{\mathbb{C}^{n'}}(Q_j \times Q_j^*)^l, \quad \sum_{i=1}^l f_i e_i \mapsto \sum_{i=1}^l f'_i e'_i$$

for $j \in \mathbb{Z}_{>0}$ are bounded by a constant independent of j . Therefore, the identity map

$$(H^0(P^0, \mathcal{F}), d) \rightarrow (H^0(P^0, \mathcal{F}), d')$$

is continuous. By open mapping theorem, the map is a homeomorphism. \square

Theorem 9.7. Let X be a complex analytic space and $(P, \pi) \subseteq (P', \pi')$ be two analytic blocks on X and \mathcal{F} be a coherent \mathcal{O}_X -module, then the restriction map

$$H^0(P', \mathcal{F}) \rightarrow H^0(P, \mathcal{F})$$

with respect to any good semi-norms.

PROOF. We claim that there exists an analytic block (P_1, π) such that

$$(P, \pi) \subseteq (P_1, \pi) \subseteq (P', \pi').$$

Assume this claim, then we have a decomposition of the restriction map

$$H^0(P', \mathcal{F}) \rightarrow H^0(P_1^0, \mathcal{F}) \rightarrow H^0(P, \mathcal{F}).$$

The first map is continuous if we endow $H^0(P_1^0, \mathcal{F})$ with the topology induced by π' , the second is continuous if we endow $H^0(P_1^0, \mathcal{F})$ with the topology induced by π . These topologies are identical by [Lemma 9.6](#). Our assertion follows.

To argue the claim, let us write $\pi : X \rightarrow \mathbb{C}^n$ and $\pi' = (\pi, \varphi) : X \rightarrow \mathbb{C}^n \times \mathbb{C}^m$. Take $q \in \mathbb{C}^m$ with $Q \times \{q\} \subseteq \text{Int } Q'$. Let $Q'' := Q' \cap (\mathbb{C}^n \times \{q\})$ and identify it with a subset of \mathbb{C}^n . Let Q^* be the image of Q' under the projection $\mathbb{C}^{n+m} \rightarrow \mathbb{C}^m$.

Choose open neighbourhoods $U \subseteq P'^0$, $V \subseteq Q'$ of P and Q respectively such that $\tau : U \rightarrow V$ is finite and $U \cap \pi^{-1}(Q) = P$. Take a tube $Q_1 \subseteq \mathbb{C}^n$ such that

$$Q \subseteq \text{Int } Q_1 \subseteq Q_1 \subseteq \text{Int } Q''.$$

Now it suffices to set $P_1 := \pi^{-1}(Q_1) \cap U$. \square

Corollary 9.8. Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $(P, \pi) \subseteq (P', \pi')$ be analytic blocks in X . Then for any Cauchy sequence $(s_j)_{j \in \mathbb{Z}_{>0}}$ in $H^0(P', \mathcal{F})$, the restriction sequence $(s_j|_P)_{j \in \mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$.

PROOF. Choose an analytic block (P_1, π) such that

$$(P, \pi) \subseteq (P_1, \pi) \subseteq (P', \pi').$$

The existence of the block (P_1, π) is argued in the proof of [Theorem 9.7](#). We have a decomposition of the restriction map

$$H^0(P', \mathcal{F}) \rightarrow H^0(P_1^0, \mathcal{F}) \rightarrow H^0(P, \mathcal{F}).$$

The first map is bounded, so the images of $(s_j)_{j \in \mathbb{Z}_{>0}}$ in $H^0(P_1^0, \mathcal{F})$ is a Cauchy sequence. As we have shown that $H^0(P_1^0, \mathcal{F})$ is a Fréchet space in [Lemma 9.6](#), the sequence converges. As the second map is also continuous, it follows that $(s_j|_P)_{j \in \mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$. \square

Lemma 9.9. Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $(P, \pi : X \rightarrow \mathbb{C}^n) \subseteq (P', \pi' : X \rightarrow \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q' . Choose $U' \subseteq X$ and $V' \subseteq \mathbb{C}^{n+m}$ of P' and Q' respectively as in [Definition 7.1](#) such that $U' \rightarrow V'$ is finite. Set

$$Q_1 := (Q \times \mathbb{C}^m) \cap Q', \quad P_1 = \pi'^{-1}(Q_1) \cap U'.$$

Then (P_1, π') is an analytic block in X with block Q_1 and $H^0(P', \mathcal{F}) \rightarrow H^0(P_1, \mathcal{F})$ has dense image. Here we take an isomorphism

$$\mathcal{O}_{\mathbb{C}^{n+m}}^{I'}|_{Q'} \rightarrow (\tau'(\mathcal{F}|_{U'}))_{Q'}$$

and it induces

$$\mathcal{O}'_{\mathbb{C}^{n+m}|_{Q_1}} \rightarrow (\tau'(\mathcal{F}|_{U'}))_{Q_1},$$

which in turn induces a good semi-norm on $H^0(P_1, \mathcal{F})$. This is the semi-norm we are using.

Moreover, there is a compact set $\tilde{P} \subseteq X$ disjoint from P such that

$$P_1 = P \cup \tilde{P}.$$

PROOF. We have a commutative diagram in the category of topological linear spaces:

$$\begin{array}{ccc} H^0(Q', \mathcal{O}_{\mathbb{C}^{m+n}}^l) & \longrightarrow & H^0(P', \mathcal{F}) \\ \downarrow & & \downarrow \\ H^0(Q_1, \mathcal{O}_{\mathbb{C}^{m+n}}^l) & \longrightarrow & H^0(P_1, \mathcal{F}) \end{array}.$$

In order to show that the right vertical map has dense image, it is enough to show that the map on the left-hand side has dense images, which is the Runge approximation.

For the last assertion, as $Q_1 = (Q \times \mathbb{C}^m) \cap Q'$, we have

$$P_1 = \pi^{-1}(Q) \cap P'.$$

As $P \subseteq P'$ and $P \subseteq \pi^{-1}(Q)$, it follows that $P \subseteq P_1$. But there is an open neighbourhood U of P in X so that $P = \pi^{-1}(Q) \cap U$. Hence, $\tilde{P} = P_1 \setminus P$ is compact. \square

Theorem 9.10 (Runge approximation). Let X be a complex analytic space and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $(P, \pi : X \rightarrow \mathbb{C}^n) \subseteq (P', \pi' : X \rightarrow \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q' . Then the map

$$H^0(P', \mathcal{F}) \rightarrow H^0(P, \mathcal{F})$$

has dense image with respect to a good semi-norm.

PROOF. We use the notations of [Lemma 9.9](#). We extend Q, Q_1, Q' to tubes $\hat{Q}, \hat{Q}_1, \hat{Q}'$ and get $\hat{P}, \hat{P}_1, \hat{P}'$ corresponding to the original P, P_1, P' . The restriction map

$$H^0(\hat{P}_1^0, \mathcal{F}) \rightarrow H^0(\hat{P}^0, \mathcal{F})$$

is a continuous morphism of Fréchet spaces.

Let $s \in H^0(P, \mathcal{F})$ be a section. Lift s to $s_1 \in H^0(P_1, \mathcal{F})$. Up to a suitable modification of the tubes, we can extend s_1 to $\hat{s}_1 \in H^0(\hat{P}_1, \mathcal{F})$. Then there is a sequence $(s^j \in H^0(\hat{P}', \mathcal{F}))_{j \in \mathbb{Z}_{>0}}$ such that $s^j|_{\hat{P}_1} \rightarrow \hat{s}_1$ as $j \rightarrow \infty$ in $H^0(\hat{P}_1, \mathcal{F})$. It follows that $s^j|_{\hat{P}^0} \rightarrow \hat{s}_1|_{\hat{P}^0}$ in $H^0(\hat{P}^0, \mathcal{F})$. It follows that $s^j|_P \rightarrow s_1|_P = s$ as $j \rightarrow \infty$. \square

Theorem 9.11. Let X be a complex analytic space. Each exhaustion of X by analytic blocks is a Stein exhaustion.

PROOF. Let $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ be an exhaustion of X by analytic blocks. Take a coherent \mathcal{O}_X -module \mathcal{F} .

We verify the conditions in [Definition 6.5](#). By [Theorem 6.4](#), P_i is a compact Stein set for each $i \in \mathbb{Z}_{>0}$. So (1) is satisfied.

On $H^0(P_i, \mathcal{F})$, we fix a good semi-norm $|\bullet|_i$ for each $i \in \mathbb{Z}_{>0}$. We may assume that $H^0(P_{i+1}, \mathcal{F}) \rightarrow H^0(P_i, \mathcal{F})$ is contractive for $i \in \mathbb{Z}_{>0}$.

We have already verified (3), (4) and (5).

We verify (2). It suffices to show that

$$H^0(X, \mathcal{F}) \rightarrow H^0(P_1, \mathcal{F})$$

has dense image. Let $s \in H^0(P_1, \mathcal{F})$ and $\delta > 0$. By [Theorem 9.10](#), we can find $s_i \in H^0(P_i, \mathcal{F})$ for $i \in \mathbb{Z}_{>0}$ such that $s_1 = s$,

$$|s_{i+1}|_{P_i} - s_i|_i < 2^{-i}\delta$$

for $i \in \mathbb{Z}_{>0}$. By [Corollary 9.8](#), $(s_j|_{P_i})_{j \in \mathbb{Z}_{>0}}$ has a limit $t_i \in H^0(P_i, \mathcal{F})$ for each $i \in \mathbb{Z}_{>0}$. As $H^0(P_{i+1}, \mathcal{F}) \rightarrow H^0(P_i, \mathcal{F})$ is continuous for $i \in \mathbb{Z}_{>0}$, the $t_{i+1}|_{P_i}$'s are compatible and defines $t \in H^0(X, \mathcal{F})$. It is easy to see that $|t|_{P_1} - s|_1 < \delta$. Thus condition (2) is satisfied. \square

10. Stein spaces

Definition 10.1. Let X be a complex analytic space. We say that X is a Stein space if X is a Stein set in X and $|X|$ is paracompact and Hausdorff.

Definition 10.2. Let X be a complex analytic space. An *effective formal 0-cycle* on X consists of

- (1) A discrete set $D \subseteq X$;
- (2) An integer n_x for each $x \in D$.

We write the effective formal 0-cycle as $\sum_{x \in D} n_x x$. We define the *ideal sheaf* $\mathcal{O}_X(-\sum_{x \in D} n_x x)$ of an effective formal 0-cycle as $\sum_{x \in D} n_x x$ as

$$\mathcal{O}_X(-\sum_{x \in D} n_x x)(U) = \{f \in H^0(U, \mathcal{O}_X) : f_x \in \mathfrak{m}_x^{n_x} \text{ for each } x \in D \cap U\}$$

for each open subset $U \subseteq X$.

Observe that $\mathcal{O}_X(-\sum_{x \in D} n_x x)$ is a coherent \mathcal{O}_X -module. In fact, the problem is local, so we may assume that D is finite. In this case, D is an effective 0-cycle and the result is clear.

Lemma 10.3. Let X be a complex analytic space and $\sum_{x \in D} n_x x$ be an effective formal 0-cycle on X . Assume that

$$H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X / \mathcal{O}_X(-\sum_{x \in D} n_x x))$$

is surjective. Suppose that for each $x \in D$, we assign $g_x \in \mathcal{O}_{X,x}$. Then there is $f \in H^0(X, \mathcal{O}_X)$ such that

$$f_x - g_x \in \mathfrak{m}_x^{n_x}$$

for all $x \in D$.

PROOF. We define $s \in H^0(X, \mathcal{O}_X / \mathcal{O}_X(-\sum_{x \in D} n_x x))$ by $s_x = g_x$ for each $x \in D$. Lift s to $f \in H^0(X, \mathcal{O}_X)$. Then f clearly satisfies the required properties. \square

Proposition 10.4. Let X be a complex analytic space. Assume that $H^1(X, \mathcal{I}) = 0$ for each coherent ideal sheaf \mathcal{I} on X . Let $(x_i \in X)_{i \in \mathbb{Z}_{>0}}$ be a sequence without accumulation points and $(c_i)_{i \in \mathbb{Z}_{>0}}$ be a sequence in \mathbb{C} . Then there is $f \in \mathcal{O}_X(X)$ with $f(x_i) = c_i$ for each $i \in \mathbb{Z}_{>0}$.

PROOF. Consider the formal cycle $\sum_{i=1}^{\infty} x_i$. Apply [Lemma 10.3](#) with $g_{x_i} = c_i$. \square

Theorem 10.5. Let X be a paracompact Hausdorff complex analytic space. Then the following are equivalent:

- (1) X is a Stein space;
- (2) For any coherent ideal sheaf \mathcal{I} on X , we have $H^1(X, \mathcal{I}) = 0$;
- (3) X is holomorphically separable and holomorphically convex;
- (4) X is holomorphically spreadable and weakly holomorphically convex;
- (5) X is holomorphically complete;
- (6) X is weakly holomorphically convex and every compact analytic subset of X is finite.

PROOF. (1) \implies (2): This is trivial.

(2) \implies (3): X is holomorphically convex by [Proposition 10.4](#) and [Proposition 5.4](#). X is holomorphically separable by [Proposition 10.4](#).

(3) \implies (4): X is holomorphically spreadable and weakly holomorphically convex by definition.

(4) \implies (5): This follows from [Theorem 9.2](#) and [Proposition 8.4](#).

(5) \implies (1): This follows from [Theorem 9.11](#) and [Theorem 6.6](#).

(5) \Leftrightarrow (6): This is just [Theorem 9.2](#). \square

Lemma 10.6. Let $b \in \mathbb{Z}_{>0}$ and $f : X \rightarrow Y$ be a b -sheeted branched covering of complex analytic spaces. Assume that Y is normal. Then the following are equivalent:

- (1) X is Stein;
- (2) Y is Stein.

The corresponding statement in Narasimhan is not correct. It is not clear to me if this holds for a general finite surjective morphism between paracompact normal Hausdorff complex analytic spaces.

PROOF. By [Lemma 2.2](#), X is paracompact and Hausdorff if and only if Y is paracompact and Hausdorff.

(2) \implies (1): This follows from Leray's spectral sequence.

(1) \implies (2): We may assume that X is connected. By [Theorem 10.5](#), it suffices to verify that Y is holomorphically convex and every analytic set in Y is finite.

Let $(y_i \in Y)_{i \in \mathbb{Z}_{>0}}$ be a sequence without accumulation points. We can lift the sequence to $(x_i \in X)_{i \in \mathbb{Z}_{>0}}$ without accumulation points. By [Proposition 10.4](#), we can find $g \in \mathcal{O}_X(X)$ such that $(|g(x_i)|)_{i \in \mathbb{Z}_{>0}}$ is unbounded. Let $\chi_g \in \mathcal{O}_Y(Y)[w]$ be the characteristic polynomial of g . As $\chi_g(g) = 0$, it follows that at least one coefficient of χ_g is unbounded along $(y_i)_{i \in \mathbb{Z}_{>0}}$. By [Proposition 5.4](#), we conclude that Y is holomorphically convex.

Let T be an analytic set in Y . Then so is $f^{-1}(T)$. As X is Stein, $f^{-1}(T)$ is finite, hence so is T . \square

Corollary 10.7. Let $f : X \rightarrow Y$ be a finite surjective morphism of normal complex analytic spaces. Then the following are equivalent:

- (1) X is Stein;
- (2) Y is Stein.

PROOF. By [Lemma 2.2](#), X is paracompact and Hausdorff if and only if Y is paracompact and Hausdorff. We may assume that Y is connected.

(2) \implies (1): This follows from Leray's spectral sequence.

(1) \implies (2): Observe that Y is irreducible, so there is a connected component X' of X so that the restriction $X' \rightarrow Y$ is surjective. Then $X' \rightarrow Y$ is a branched covering by [Corollary 4.37](#) in [Morphisms between complex analytic spaces](#). But X' is Stein as it is a connected component of a Stein space. We conclude using [Lemma 10.6](#). \square

Lemma 10.8. Let X be a reduced complex analytic space whose normalization \bar{X} is Stein. Then for any reduced closed analytic subspace Y of X , \bar{Y} is also Stein.

PROOF. By [Lemma 2.2](#), X is paracompact and Hausdorff. We write $\pi : \bar{X} \rightarrow X$ for the normalization morphism. Let $Y^1 = \pi^{-1}(Y)$, the preimage is endowed with a structure of a closed analytic subspace of \bar{X} . It follows that Y^1 is Stein. Its normalization \bar{Y}^1 is then Stein, as the normalization morphism is finite. We have commutative diagram induced by the universal property of the normalization:

$$\begin{array}{ccc} \bar{Y}^1 & \longrightarrow & \bar{Y} \\ \downarrow & \swarrow & \\ Y & & \end{array} .$$

The natural morphism $\bar{Y}^1 \rightarrow Y$ is a finite as it is the composition of two finite coverings. Then morphism $\bar{Y} \rightarrow Y$ is finite, so $\bar{Y}^1 \rightarrow \bar{Y}$ is finite. But its image contains a dense open subset of \bar{Y} , so $\bar{Y}^1 \rightarrow \bar{Y}$ is surjective. Observe that \bar{Y} is paracompact and Hausdorff by the same arguments as in [Lemma 10.6](#). Now we can apply [Corollary 10.7](#) to conclude that \bar{Y} is Stein. \square

Corollary 10.9. Let X be a complex analytic space. Then the following are equivalent:

- (1) X is Stein;
- (2) X^{red} is Stein;
- (3) The normalization $\overline{X^{\text{red}}}$ is Stein.

The equivalence of (1) and (2) is due to Grauert [[Gra60](#)]. Here we follow the simplified approach in [[GR77](#)]. The difficult direction (3) implies (2) is claimed in [[GR77](#)], where the proof is nonsense. We follow the argument of Narasimhan [[Nar62](#)]. We remind the readers that the statements and the arguments in [[Nar62](#)] contain several (fixable) mistakes.

PROOF. By [Lemma 2.2](#), X is paracompact and Hausdorff if and only if $\overline{X^{\text{red}}}$ is.

(1) \implies (2): This follows from Leray's spectral sequence.

(2) \implies (1): By [Theorem 10.5\(3\)](#), it suffices to show that the restriction map $H^0(X, \mathcal{O}_X) \rightarrow H^0(X^{\text{red}}, \mathcal{O}_{X^{\text{red}}})$ is surjective.

Let \mathcal{I} be the nilradical of \mathcal{O}_X . It is coherent by Cartan–Oka theorem. For each $i \in \mathbb{Z}_{>0}$, we have a short exact sequence

$$0 \rightarrow \mathcal{I}^i / \mathcal{I}^{i+1} \rightarrow \mathcal{O}_X / \mathcal{I}^{i+1} \rightarrow \mathcal{O}_X / \mathcal{I}^i \rightarrow 0.$$

As $\mathcal{I}^i / \mathcal{I}^{i+1}$ is a coherent $\mathcal{O}_{X^{\text{red}}}$ -module, we conclude that

$$\varphi_i : H^0(X, \mathcal{O}_X / \mathcal{I}^{i+1}) \rightarrow H^0(X, \mathcal{O}_X / \mathcal{I}^i)$$

is surjective for each $i \in \mathbb{Z}_{>0}$. Let $h_1 \in H^0(X, \mathcal{O}_X/\mathcal{I}) = H^0(X^{\text{red}}, \mathcal{O}_{X^{\text{red}}})$. We want to lift it to $h \in H^0(X, \mathcal{O}_X)$.

We successively lift h_1 to $h_i \in H^0(X, \mathcal{O}_X/\mathcal{I}^i)$ for each $i \in \mathbb{Z}_{>0}$. Let $X_i = X \setminus \text{Supp } \mathcal{I}^i$ of each $i \in \mathbb{Z}_{>0}$. Then clearly

$$X = \bigcup_{i=1}^{\infty} X_i.$$

It is easy to see that

$$h_{i+1}|_{X_i} = h_i|_{X_i}$$

for each $i \in \mathbb{Z}_{>0}$. It follows that we can glue the $h_i|_{X_i}$'s to $h \in H^0(X, \mathcal{O}_X)$ which restricts to h_1 .

(2) \implies (3): This follows from Leray's spectral sequence as $\overline{X^{\text{red}}} \rightarrow X^{\text{red}}$ is finite by [Proposition 7.8](#) in [Local properties of complex analytic spaces](#).

(3) \implies (2): We may assume that X is reduced.

Step 1. We first observe that it suffices to prove in the case where $\dim X < \infty$. For each $k \in \mathbb{Z}_{>0}$, we let X_k denote the union of the irreducible components of dimension $\leq k$. Then clearly, X_k is an analytic set in X . We endow it with the reduced induced structure. Then $\dim X_k \leq k$. The normalization \bar{X}_k of X_k is a disjoint union of certain connected components of \bar{X} and hence Stein for each $k \in \mathbb{Z}_{>0}$. It follows that X_k is Stein if the special case is established.

Let $D \subseteq X$ be a countable infinite set without accumulation points. For each $k \in \mathbb{Z}_{>0}$, we set $D_k = D \cap X_k$ and $E_{k+1} = D_{k+1} \setminus D_k$. Further we let $E_1 = D_1$. We write the points of D as $(x_i \in X)_{i \in \mathbb{Z}_{>0}}$. Let $h : D \rightarrow \mathbb{C}$ be the map sending x_i to i for each $i \in \mathbb{Z}_{>0}$. For each $k \in \mathbb{Z}_{>0}$, h_k denotes the restriction of h to D_k .

As X_1 is Stein, we can construct $f_1 \in \mathcal{O}_{X_1}(X_1)$ with $f_1|_{E_1} = h_1$ by [Proposition 10.4](#). As $E_2 \cup X_1$ is an analytic subset in X_2 , we can find $f_2 \in \mathcal{O}_{X_2}(X_2)$ extending f_1 and such that $f_2|_{E_2} = h_2$. We continue in the obvious way and construct $f_k \in \mathcal{O}_{X_k}(X_k)$ for each $k \in \mathbb{Z}_{>0}$ compatible with each other. Then the f_k 's glue to give $f \in \mathcal{O}_X(X)$ unbounded on D . We conclude that X is Stein by [Proposition 5.4](#).

Step 2. We assume that $\dim X < \infty$.

Let \mathcal{I} be a coherent ideal sheaf on X . By [Theorem 10.5](#), it suffices to show that

$$H^1(X, \mathcal{I}) = 0.$$

We may assume that X is connected. We make an induction on $\dim X$. There is nothing to prove if $\dim X = 0$. Assume that $\dim X > 0$.

We write $\pi : \bar{X} \rightarrow X$ for the normalization morphism. Let \mathcal{W} be the conductor ideal of \mathcal{O}_X . Let $\mathcal{F} := \pi^*(\mathcal{W}\mathcal{I})$. Observe that \mathcal{F} is a coherent $\mathcal{O}_{\bar{X}}$ -module. By Leray spectral sequence,

$$H^1(X, \pi_*\mathcal{F}) \cong H^1(\bar{X}, \mathcal{F}) = 0.$$

Let $Y := \text{Supp } \mathcal{O}_X/\mathcal{W} \subseteq X^{\text{Sing}}$. Then Y is an analytic set in X . We endow Y with the reduced induced structure, then Y is Stein by [Lemma 10.8](#) and our inductive hypothesis.

Observe that $\pi_*\mathcal{F}$ can be identified with a subsheaf of $\mathcal{W} \cdot \overline{\mathcal{O}_X} \subseteq \mathcal{I}$. Let $\mathcal{S} = (\mathcal{I}/\pi_*\mathcal{F})|_Y$. Then we have

$$H^1(X, \mathcal{I}/\pi_*\mathcal{F}) \cong H^1(Y, \mathcal{S}) = 0.$$

Consider the short exact sequence

$$0 \rightarrow \pi_* \mathcal{F} \rightarrow \mathcal{I} \rightarrow \mathcal{I}/\pi_* \mathcal{F} \rightarrow 0.$$

We conclude that

$$H^1(X, \mathcal{I}) = 0.$$

□

Corollary 10.10. Let X be a complex analytic space. Then the following are equivalent:

- (1) X is Stein;
- (2) Each irreducible component of X^{red} is Stein if we endow it with the reduced induced structure.

PROOF. This follows immediately from [Corollary 10.9](#). □

Corollary 10.11. Let $f : X \rightarrow Y$ be a finite morphism between complex analytic spaces. Then

- (1) if Y is Stein, so is X ;
- (2) if f is surjective and X is Stein, then Y is also Stein.

This result is due to Narasimhan [\[Nar62\]](#), although the statement and the proof in [\[Nar62\]](#) are both incorrect.

PROOF. Observe that X is paracompact and Hausdorff as in the proof of [Lemma 10.6](#). By [Corollary 10.9](#), we may assume that X and Y are reduced.

(1) Observe that X is paracompact and Hausdorff as f is proper. The fact that X is Stein follows from Leray's spectral sequence.

(2) Observe that Y is by paracompact and Hausdorff by [Lemma 2.2](#). We may assume that Y is irreducible by [Corollary 10.10](#). Up to replacing X by one of its irreducible components whose image under f is Y , we may assume that X is also irreducible.

By [Corollary 4.31](#) in [Morphisms between complex analytic spaces](#), we can find a commutative diagram

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

By [Corollary 10.9](#), we are reduced to show that \bar{X} is Stein if and only if \bar{Y} is. But $\bar{f} : \bar{X} \rightarrow \bar{Y}$ is clearly finite and surjective. So it suffices to apply [Corollary 10.7](#). □

11. Previlaged neighbourhoods

Definition 11.1. Let X be a topological space, $x \in X$ and \mathcal{F} be a sheaf of sets on X . We say a neighbourhood U of x in X is \mathcal{F} -*previlaged* at x if the map

$$H^0(U, \mathcal{F}) \rightarrow \mathcal{F}_x$$

is injective. We also say U is an \mathcal{F} -*previlaged neighbourhood* of x in X .

Proposition 11.2. Let X be a topological space, $x \in X$ and

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{H}$$

be an exact sequence of sheaves of Abelian groups on X . Let U be a neighbourhood of x in X .

- (1) If U is \mathcal{G} -previlaged at x and \mathcal{H} -previlaged at x , then it is \mathcal{F} -previlaged at x ;
- (2) If U is \mathcal{F} -previlaged at x , then it is \mathcal{G} -previlaged at x .

PROOF. We have a commutative diagram of \mathbb{C} -linear spaces

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(X, \mathcal{G}) & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{H}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{G}_x & \longrightarrow & \mathcal{F}_x & \longrightarrow & \mathcal{H}_x
 \end{array}$$

Both assertions follow from simple diagram chasing. \square

Proposition 11.3. Let X be a reduced complex analytic space, $x \in X$ and U be an open neighbourhood of x in X . Consider the following conditions:

- (1) All irreducible components of U pass through x ;
- (2) U is \mathcal{O}_X -previlaged at x .

Then (1) implies (2).

Frisch67 also claims that if U is Stein, then (2) implies (1). I cannot figure out a proof.

PROOF. (1) \implies (2): Let $s \in H^0(U, \mathcal{F})$ with $s_x = 0$. We want to show that $s = 0$. By (1), we may assume that X is irreducible. Then X^{reg} is connected by [Corollary 4.35 in Morphisms between complex analytic spaces](#). As $s_x = 0$, s vanishes on a non-empty open subset of X^{reg} by [Theorem 6.8 in Local properties of complex analytic spaces](#). It follows that $s|_{X^{\text{reg}}} = 0$ by Identitätssatz. Hence, $s = 0$. \square

Proposition 11.4. Let X be a complex analytic space, $x \in X$ and \mathcal{F} be a coherent \mathcal{O}_X -module. There is an open neighbourhood U of x in X and finitely many analytic sets Y_1, \dots, Y_m in X containing x having the following property: a neighbourhood V of x in X contained in U is \mathcal{F} -previlaged at x if $U \cap Y_i$ is $\mathcal{F}|_{Y_i}$ -previlaged at x for each $i = 1, \dots, m$.

PROOF. **Step 1.** Let

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{H}$$

be an exact sequence of coherent \mathcal{O}_X -modules. Suppose that we have proved the proposition with \mathcal{G} and \mathcal{H} in place of \mathcal{F} , let us show that the proposition also holds for \mathcal{F} . Let $U', Y'_1, \dots, Y'_{m'}$ and $U'', Y''_1, \dots, Y''_{m''}$ be the data in the proposition with respect to \mathcal{G} and \mathcal{H} respectively. We let $U := U' \cap U''$, $m = m' + m''$ and

$$Y_1 = Y'_1 \cap U, \dots, Y_{m'} = Y'_{m'} \cap U, Y_{m'+1} = Y''_1 \cap U, \dots, Y_{m'+m''} = Y''_{m''} \cap U.$$

It follows from [Proposition 11.2](#) that these data have the desired property.

Step 2. By Jordan–Hölder theorem, we can find an open neighbourhood U of x in X and a finite chain of coherent \mathcal{O}_U -modules

$$0 = \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_n = \mathcal{F}|_U$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to $\mathcal{O}_{Y_i \cap U}$ for some irreducible reduced closed analytic subspace of X passing through x for $i = 1, \dots, n$. By Step 1, it suffices to handle the case $\mathcal{F} = \mathcal{O}_{Y_i}$ for some $i = 1, \dots, n$.

Step 3. Let Y be an analytic set in X endowed with the reduced induced structure passing through x . Let V be a neighbourhood of x in X . We need to show that V is \mathcal{O}_Y -previlaged if $V \cap Y$ is \mathcal{O}_Y -previlaged.

But this special case follows from **Proposition 11.3 Why?**.

□

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