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Contents

| | |
|--|----|
| Constructions of complex analytic spaces | 5 |
| 1. Introduction | 5 |
| 2. Analytic spectra | 5 |
| 3. Analytic germs | 9 |
| Bibliography | 15 |

Constructions of complex analytic spaces

1. Introduction

2. Analytic spectra

Proposition 2.1. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Then the presheaf $F_{\mathcal{A}}$ on $\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}/_S$ defined by

$$F_{\mathcal{A}}(T \xrightarrow{p} S) = \mathrm{Hom}_{\mathcal{O}_T}(p^*\mathcal{A}, \mathcal{O}_T)$$

is representable.

PROOF. By the arguments of [Stacks, Tag 01JJ], the problem is local in S . So we may assume that \mathcal{A} has the following form

$$\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and $\mathcal{I} \subseteq \mathcal{O}_S(S)[X_1, \dots, X_n]$ an ideal sheaf of finite type.

Step 1. We first handle the case where $\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]$.

In this case, we claim that $F_{\mathcal{A}}$ is represented by $S \times \mathbb{C}^n$. In fact, it suffices to observe that

$$\begin{aligned} F_{\mathcal{A}}(T \xrightarrow{p} S) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) \xrightarrow{\sim} \mathcal{O}_T(T)^n \\ &= \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}}(T, \mathbb{C}^n) = \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}/_S}(T, S \times \mathbb{C}^n). \end{aligned}$$

From this proof, it is easy to see that the universal morphism is

$$\eta : \mathcal{O}_{S \times \mathbb{C}^n}[X_1, \dots, X_n] \rightarrow \mathcal{O}_{S \times \mathbb{C}^n}$$

sending X_i to z_i , the i -th coordinate of \mathbb{C}^n .

Step 2. We handle the general case. We have a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_S[X_1, \dots, X_n] \rightarrow \mathcal{A} \rightarrow 0.$$

For any $p : T \rightarrow S$ in $\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}$, we have an exact sequence

$$p^*\mathcal{I} \rightarrow \mathcal{O}_T[X_1, \dots, X_n] \rightarrow p^*\mathcal{A} \rightarrow 0.$$

We then have

$$\begin{aligned} F_{\mathcal{A}}(T) &\xrightarrow{\sim} \{h \in \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T[X_1, \dots, X_n], \mathcal{O}_T) : h|_{p^*\mathcal{I}} = 0\} \\ &\xrightarrow{\sim} \{h \in F_{\mathcal{O}_S[X_1, \dots, X_n]}(T) : h|_{p^*\mathcal{I}} = 0\}. \end{aligned}$$

Let $\pi : S \times \mathbb{C}^n \rightarrow S$ be the projection. Then $F_{\mathcal{A}}(T)$ is represented by the closed subspace of $S \times \mathbb{C}^n$ defined by the ideal $\eta(\pi^*\mathcal{I})$, which is clearly of finite type. \square

Definition 2.2. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Then the complex analytic space representing the functor $F_{\mathcal{A}}$ in Proposition 2.1 is called the *analytic spectrum* of \mathcal{A} . We denote it by $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$. By construction, there is a canonical morphism $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$.

By definition, we have a universal morphism $\xi \in F_{\mathcal{A}}(X) = \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_X, \mathcal{O}_X)$ with $X = \text{Spec}_S^{\text{an}} \mathcal{A}$. It defines a morphism of ringed spaces $X \rightarrow (|S|, \mathcal{A})$. The pull-back of an \mathcal{A} -module \mathcal{M} is denoted by $\tilde{\mathcal{M}}$. The assignment $\mathcal{M} \mapsto \tilde{\mathcal{M}}$ is functorial in \mathcal{M} .

It is easy to see that $\text{Spec}_S^{\text{an}} \mathcal{A}$ is contravariant in \mathcal{A} .

Proposition 2.3. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Consider a morphism $g : S' \rightarrow S$ of complex analytic spaces. Then we have a Cartesian diagram

$$\begin{array}{ccc} \text{Spec}_{S'}^{\text{an}} g^* \mathcal{A} & \longrightarrow & \text{Spec}_S^{\text{an}} \mathcal{A} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

PROOF. This is clear at the level of functor of points. \square

Corollary 2.4. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Take $s \in S$. Then $\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s \xrightarrow{\sim} (\text{Spec}_S^{\text{an}} \mathcal{A})_s$.

Moreover, the universal morphism $\mathcal{A}_{\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s} \rightarrow \mathcal{O}_{\text{Spec}_{\{s\}}^{\text{an}} \mathcal{A}_s}$ is the reduction of the universal morphism $\mathcal{A}_{\text{Spec}_S^{\text{an}} \mathcal{A}} \rightarrow \mathcal{O}_{\text{Spec}_S^{\text{an}} \mathcal{A}}$ modulo \mathfrak{m}_s .

PROOF. This follows from [Proposition 2.3](#). \square

Proposition 2.5. Let S be a complex analytic space and \mathcal{A} be an \mathcal{O}_S -module of finite presentation. Take $s \in S$. Write $X = \text{Spec}_S^{\text{an}} \mathcal{A}$ and $\mathcal{A}_s := \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_{S,s}$. Then the map from X_s to

$$\{\mathfrak{m} \in \text{Spm}_{\mathbb{C}} \mathcal{A}_s : \mathfrak{m} \supseteq \mathfrak{m}_s\}$$

sending $x \in X_s$ to the inverse image of \mathfrak{m}_x with respect to $\mathcal{A}_s \rightarrow \mathcal{O}_{X,x}$ is bijective.

If \mathfrak{m} corresponds to $x \in X_s$, then the natural homomorphism $\mathcal{A}_s \rightarrow \mathcal{O}_{X,x}$ factorizes through $\mathcal{A}_{s,\mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$. The completion of the latter

$$\widehat{\mathcal{A}_{s,\mathfrak{m}}} \rightarrow \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism.

PROOF. By [Corollary 2.4](#), we have natural bijections

$$X_s \xrightarrow{\sim} \text{Hom}_{\{s\}}(\{s\}, X_s) \xrightarrow{\sim} \text{Hom}_{\mathbb{C}\text{-Alg}}(\mathcal{A}_s/\mathfrak{m}_s \mathcal{A}_s, \mathbb{C}).$$

This gives the desired bijection.

Next we prove the latter part. The problem is local on S , we may assume that

$$\mathcal{A} = \mathcal{O}_S[X_1, \dots, X_n]/\mathcal{I}$$

for some $n \in \mathbb{N}$ and some ideal \mathcal{I} of finite type in $\mathcal{O}_S[X_1, \dots, X_n]$. Recall that the universal morphism

$$\eta : \mathcal{O}_{S \times \mathbb{C}^n}[X_1, \dots, X_n] \rightarrow \mathcal{O}_{S \times \mathbb{C}^n}$$

sends X_i to z_i , the i -th coordinate of \mathbb{C}^n .

By construction, we have

$$\mathcal{A}_s \xrightarrow{\sim} \mathcal{O}_{S,s}[X_1, \dots, X_n]/\mathcal{I}_s$$

and

$$\mathcal{O}_{X,x} = \mathcal{O}_{S \times \mathbb{C}^n, x}/\mathcal{J}_x,$$

where

$$\mathcal{J}_x = \eta_x(\mathcal{I}_s \mathcal{O}_{S \times \mathbb{C}^n, x}[X_1, \dots, X_n]).$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_s & \longrightarrow & \mathcal{O}_{S,s}[X_1, \dots, X_n] & \longrightarrow & \mathcal{A}_s \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{J}_x & \longrightarrow & \mathcal{O}_{S \times \mathbb{C}^n, x} & \longrightarrow & \mathcal{O}_{X,x} \longrightarrow 0 \end{array}.$$

The middle vertical map is induced by η_x . Let \mathfrak{p} be the inverse image of $\mathfrak{m}_{S \times \mathbb{C}^n, x}$ under the vertical map in the middle. Then \mathfrak{p} is generated by \mathfrak{m}_s and $X_1 - x_1, \dots, X_n - x_n$, where $x_i \in \mathbb{C}$ is the value of z_i at x for $i = 1, \dots, n$. By localization and completion, we find a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{(\mathcal{I}_s)}_{\mathfrak{p}} & \longrightarrow & (\mathcal{O}_{S,s}[X_1, \dots, X_n])_{\mathfrak{p}}^{\wedge} & \longrightarrow & \widehat{(\mathcal{A}_s)}_{\mathfrak{m}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{\mathcal{J}}_x & \longrightarrow & \widehat{\mathcal{O}_{S \times \mathbb{C}^n, x}} & \longrightarrow & \widehat{\mathcal{O}_{X,x}} \longrightarrow 0 \end{array}.$$

Observe that

$$(\mathcal{O}_{S,s}[X_1, \dots, X_n])_{\mathfrak{p}}^{\wedge} \cong \widehat{\mathcal{O}_{S,s}}[[X_1 - x_1, \dots, X_n - x_n]]$$

and

$$\widehat{\mathcal{O}_{S \times \mathbb{C}^n, x}} \cong \widehat{\mathcal{O}_{S,s}} \hat{\otimes}_k \widehat{\mathcal{O}_{\mathbb{C}^n, (x_1, \dots, x_n)}} \cong \widehat{\mathcal{O}_{S,s}}[[X_1 - x_1, \dots, X_n - x_n]].$$

It is easy to see that the middle map is an isomorphism. As \mathcal{J}_x is generated by \mathcal{I}_s , the first vertical map is also an isomorphism. Our assertion follows. \square

Corollary 2.6. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $X = \text{Spec}_S^{\text{an}} \mathcal{A}$. Take $s \in S$. Then the fiber X_s is finite and is in bijection with $\text{Spm}_{\mathbb{C}} \mathcal{A}_s$. If \mathfrak{m} corresponds to $x \in X_s$, then we have a natural isomorphism

$$\mathcal{A}_{s, \mathfrak{m}} \xrightarrow{\sim} \mathcal{O}_{X,x}.$$

PROOF. As $\mathcal{O}_{S,s} \rightarrow \mathcal{A}_s$ is finite, \mathcal{A}_s is semi-local. On the other hand, by [Proposition 2.5](#),

$$\mathcal{A}_{s, \mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$$

is injective and $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{S,s}$. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by [Theorem 5.4](#) in [Complex analytic local algebras](#). It follows from Nakayama's lemma that $\mathcal{A}_{s, \mathfrak{m}} \rightarrow \mathcal{O}_{X,x}$ is also surjective. \square

Corollary 2.7. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Then the image of $\text{Spec}_S^{\text{an}} \mathcal{A} \rightarrow S$ is $\text{Supp } \mathcal{A}$.

PROOF. This follows from [Corollary 2.6](#) and the fact that $\text{Spm}_{\mathbb{C}} \mathcal{A}_s = \text{Spm } \mathcal{A}_s$ for all $s \in S$. \square

Proposition 2.8. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $f : \text{Spec}_S^{\text{an}} \mathcal{A}$ for the structure map. Then we have the following assertions:

- (1) for all \mathcal{A} -module \mathcal{M} , the natural morphism

$$\mathcal{M} \rightarrow f_* \tilde{\mathcal{M}}$$

is an isomorphism,

In particular, $\mathcal{A} \xrightarrow{\sim} f_* \mathcal{O}_X$.

(2) for all \mathcal{O}_X -module \mathcal{F} , the canonical morphism

$$\widehat{f_*\mathcal{F}} \rightarrow \mathcal{F}$$

is an isomorphism.

In particular, the category of \mathcal{A} -modules is equivalent to the category of \mathcal{O}_X -modules.

PROOF. By [Corollary 3.5](#), f is topologically finite. Take $s \in S$. Let x_1, \dots, x_n be the distinct points of $f^{-1}(s)$ and $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ denote the maximal ideals of \mathcal{A}_s corresponding to x_1, \dots, x_n .

(1) By [Corollary 4.9](#) in [Topology and bornology](#) and [Corollary 2.6](#),

$$(f_*\tilde{\mathcal{M}})_s \cong \prod_{i=1}^n \widehat{M}_{x_i} \cong \prod_{i=1}^n \widehat{\mathcal{M}}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \mathcal{M}_s \otimes_{\mathcal{A}_s} \prod_{i=1}^n \mathcal{A}_{s,\mathfrak{m}_i} \xrightarrow{\sim} \mathcal{M}_s.$$

(2) By [Corollary 4.9](#) in [Topology and bornology](#),

$$f_*\mathcal{F}_s \cong \prod_{i=1}^n \mathcal{F}_{x_i}.$$

It follows that

$$\widehat{f_*\mathcal{M}}_{x_i} \cong f_*\mathcal{F}_s \otimes_{\mathcal{A}_s} \mathcal{O}_{X,x_i} \cong \prod_{j=1}^n \mathcal{F}_{x_j} \otimes_{\mathcal{A}_s} \mathcal{A}_{s,\mathfrak{m}_i}$$

for $i = 1, \dots, n$. But the only non-zero term is when $j = i$, so

$$\widehat{f_*\mathcal{M}}_{x_i} \cong \mathcal{F}_{x_i}$$

for $i = 1, \dots, n$. □

Corollary 2.9. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra. Write $f : \mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$ for the structure map. Then for any coherent \mathcal{O}_X -module \mathcal{M} , $f_*\mathcal{F}$ is coherent.

Moreover, f_* is exact from $\mathrm{Coh}(\mathcal{O}_X)$ to $\mathrm{Coh}(\mathcal{O}_Y)$.

PROOF. The exactness of f_* follows from [Proposition 2.8](#).

We claim that up to shrinking S , we may assume that \mathcal{M} has a global presentation. Fix $s \in S$ and let x_1, \dots, x_n be the distinct points of $f^{-1}(s)$.

For each $j = 1, \dots, n$, we can find an open neighbourhood U_j of x_j in X , pairwise disjoint and an exact sequence

$$\mathcal{O}_{U_j}^{p_j} \rightarrow \mathcal{O}_{U_j}^{q_j} \rightarrow \mathcal{M}|_{U_j} \rightarrow 0$$

for some $p_j, q_j \in \mathbb{Z}_{>0}$. We may assume that $p_1 = \dots = p_n$ and $q_1 = \dots = q_n$. We denote the common values by p and q . Then $U = U_1 \cup \dots \cup U_n$ is a neighbourhood of $f^{-1}(s)$, and we have an exact sequence

$$\mathcal{O}_U^p \rightarrow \mathcal{O}_U^q \rightarrow \mathcal{M}|_U \rightarrow 0.$$

By [Lemma 4.2](#) in [Topology and bornology](#), we may assume that $U = \pi^{-1}(V)$ for some open neighbourhood V of s in S . The induced map $f' : U \rightarrow V$ is finite and by [Corollary 4.9](#) in [Topology and bornology](#).

Now let us take a presentation

$$\mathcal{O}^p \rightarrow \mathcal{O}^q \rightarrow \mathcal{M} \rightarrow 0.$$

By [Proposition 2.8](#), we have an exact sequence

$$f_*\mathcal{O}^p \rightarrow f_*\mathcal{O}^q \rightarrow f_*\mathcal{M} \rightarrow 0.$$

By [Proposition 2.8](#) again, this can be written as

$$\mathcal{A}^p \rightarrow \mathcal{A}^q \rightarrow f_*\mathcal{M} \rightarrow 0.$$

It follows that $f_*\mathcal{M}$ is coherent. \square

Proposition 2.10. Let S be a complex analytic space and \mathcal{A}, \mathcal{B} be \mathcal{O}_S -algebras of finite presentation. Assume that \mathcal{A} is finite. Then we have a natural bijection

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}/S}(\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}, \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}).$$

PROOF. Let $f : X := \mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$ be the natural map. We construct the bijection as

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{B}, f_*\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{B}_X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}/S}(\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}, \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}).$$

The first map is a bijection by [Proposition 2.8](#) \square

Definition 2.11. Let S be a complex analytic space and \mathcal{E} be an \mathcal{O}_S -module of finite presentation. We define the *vector bundle* $\mathbf{V}(\mathcal{E})$ generated by \mathcal{E} as

$$\mathbf{V}(\mathcal{E}) = \mathrm{Spec}_S^{\mathrm{an}} \mathrm{Sym} \mathcal{E}.$$

We have a natural projection $\mathbf{V}(\mathcal{E}) \rightarrow S$.

We remind the readers that we are following Grothendieck's convention for $\mathbf{V}(\mathcal{E})$, which is different from Fulton's.

3. Analytic germs

Definition 3.1. A *pointed complex analytic space* is a pair (X, x) consisting of a complex analytic space X and a point $x \in X$. A morphism between pointed complex analytic spaces (X, x) and (Y, y) is a morphism $f : X \rightarrow Y$ of complex analytic spaces such that $f(x) = y$. The category of pointed complex analytic spaces is denoted by $\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}_*$.

The category of *complex analytic germs* $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is the right category of fractions of $\mathbb{C}\text{-}\mathcal{A}\mathfrak{n}$ with respect to the system of morphisms $f : (X, x) \rightarrow (Y, y)$ such that $f : X \rightarrow Y$ is an open immersion. An element in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}$ is called a *complex analytic germ*. A complex analytic germ represented by (X, x) is denoted by X_x .

Given a complex analytic germ X_x , we write $\mathcal{O}_{X,x}$ for the local ring of X at x . Clearly, it does not depend on the choice of (X, x) . Given any morphism $f : X_x \rightarrow Y_y$ of complex analytic germs, we have an obvious local homomorphism $f^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$.

Definition 3.2. Given a complex analytic germ X_x , a *closed subgerm* of X_x is an isomorphism class in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}_{/X_x}$ of Y_x represented by a closed analytic subspace of X containing x for any representation (X, x) of X_x .

In particular, X_x is a closed subgerm of X_x . A closed subgerm Y_y of X_x is *proper* if Y_y is different from X_x as subgerms.

Given a closed subgerm Y_x of X_x , we have an induced surjective homomorphism $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$. The kernel is denoted by $I(Y, x)$ or $I_X(Y, x)$.

Theorem 3.3. The functor $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}^{\mathrm{op}} \rightarrow \mathbb{C}\text{-}\mathcal{L}\mathcal{A}$ defined in [Definition 3.1](#) is an equivalence.

PROOF. Step 1. We show that the functor is faithfully.

In other words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $f, g : (X, x) \rightarrow (Y, y)$ be two morphisms inducing the same map $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$, then f and g coincide on a neighbourhood of x in X .

The question is open on Y , so we may reduce to the case where Y is a complex model space. We then further reduce to the case where Y is a domain in \mathbb{C}^n for some $n \in \mathbb{N}$ and then to $Y = \mathbb{C}^n$.

By [Theorem 4.2](#) in [The notion of complex analytic spaces](#), f and g can be identified with systems $(f_1, \dots, f_n) \in \mathcal{O}_X(X)^n$ and $(g_1, \dots, g_n) \in \mathcal{O}_X(X)^n$. The assumption $f_x^\# = g_x^\#$ means $f_{i,x} = g_{i,x}$ for $i = 1, \dots, n$. So $f_i = g_i$ after shrinking X . We conclude by [Theorem 4.2](#) in [The notion of complex analytic spaces](#) again.

Step 2. We show that the functor is fully faithful.

In other words, let (X, x) and (Y, y) be two pointed complex analytic spaces and $\varphi : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ be a morphism in $\mathbb{C}\text{-}\mathcal{LA}$. Then we can find an open neighbourhood U of x in X and a morphism $(U, x) \rightarrow (Y, y)$ inducing φ .

The problem is local on Y , so we may assume that Y is a complex model space, say Y is a closed subspace of a domain V in \mathbb{C}^n defined by a coherent ideal \mathcal{I} . We write $\psi : \mathcal{O}_{V,y} \rightarrow \mathcal{O}_{X,x}$ the homomorphism induced by φ , we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{V,y} & \xrightarrow{\psi} & \mathcal{O}_{Y,y} \\ \downarrow & \nearrow \varphi & \\ \mathcal{O}_{X,x} & & \end{array}.$$

Let z_1, \dots, z_n be the coordinates on V . Let $f_{i,x}$ be the image of $z_{i,x}$ under ψ for $i = 1, \dots, n$. Take an open neighbourhood U of x in X so that $f_{i,x}$ lifts to $f_i \in \mathcal{O}_X(U)$ for $i = 1, \dots, n$. By [Theorem 4.2](#) in [The notion of complex analytic spaces](#), f_1, \dots, f_n then defines a morphism $g : U \rightarrow \mathbb{C}^n$. Clearly $g(x) = y$. But $g_x^\#$ and ψ coincide on $z_{i,y}$ so $g_x^\# = \psi$ as $\mathcal{O}_{V,y} = \mathbb{C}\{z_{1,y} - a_1, \dots, z_{n,y} - a_n\}$ with $a_i = \epsilon(z_{i,y})$ for $i = 1, \dots, n$. Therefore, $g_x^\#(\mathcal{I}_y) = 0$. Up to shrinking U , we may guarantee that $g(U) \subseteq V$ and $g^*(\mathcal{I}) = 0$ on U . Namely, g factorizes through $f : U \rightarrow Y$ and $f_x^* = \varphi$.

Step 3. We show that the functor is essentially surjective.

In other words, let A be a complex analytic local algebra, then there is a pointed complex analytic space (X, x) with $\mathcal{O}_{X,x} \cong A$ in $\mathbb{C}\text{-}\mathcal{LA}$.

We may assume that $A = \mathbb{C}\{z_1, \dots, z_n\}/I$ for some $n \in \mathbb{N}$ and ideal I in $\mathbb{C}\{z_1, \dots, z_n\}$. Then I is finitely generated as $\mathbb{C}\{z_1, \dots, z_n\}$ is noetherian. Take finitely many generators $f_1, \dots, f_m \in I$. We extend f_1, \dots, f_m to $g_1, \dots, g_m \in \mathcal{O}_{\mathbb{C}^n}(U)$ for some open neighbourhood U of 0 in \mathbb{C}^n . Then the closed subspace X of U defined by f_1, \dots, f_m satisfies the required conditions. \square

Lemma 3.4. Let S be a complex analytic space and $s \in S$. For any finite $\mathcal{O}_{S,s}$ -algebra A , there is an open neighbourhood U of s in S and a finite \mathcal{O}_U -algebra such that $\mathcal{A}_s \cong A$.

PROOF. Let $s \in S$, as \mathcal{A}_s is a finite $\mathcal{O}_{S,s}$ -algebra, we can find finitely many generators $\sigma_{1,s}, \dots, \sigma_{n,s}$. As \mathcal{A}_s is integral over $\mathcal{O}_{S,s}$, we can find unitary polynomials $F_{i,s} \in \mathcal{O}_{S,s}[X_i]$ such that $F_{i,s}(\sigma_{i,s}) = 0$ for $i = 1, \dots, n$. Take a sufficient small neighbourhood U of s so that $\sigma_{i,s}$ lifts to $\sigma_i \in \mathcal{O}_S(U)$ and $F_{i,s}$ lifts to a unitary polynomial $F_i \in H^0(U, \mathcal{O}_S[X_i])$ for $i = 1, \dots, n$. Up to shrinking U ,

we may guarantee that $\sigma_1, \dots, \sigma_n$ generate $\mathcal{A}|_U$ at all points and $F_i(\sigma_i) = 0$ for $i = 1, \dots, n$. Then $\mathcal{B} := \mathcal{O}_U[X_1, \dots, X_n]/(F_1, \dots, F_n)$ is coherent and we have a surjective homomorphism $\mathcal{B} \rightarrow \mathcal{A}|_U$ sending X_i to σ_i for $i = 1, \dots, n$. As the kernel of this homomorphism is of finite type, up to shrinking U , we may take finitely many $G_1, \dots, G_m \in \mathcal{B}(U)$ that generate the kernel. Lift G_1, \dots, G_m to $H_1, \dots, H_m \in H^0(U, \mathcal{O}_S[X_1, \dots, X_n])$, then

$$\mathcal{A}|_U \cong \mathcal{O}_U[X_1, \dots, X_n]/(F_1, \dots, F_n, G_1, \dots, G_m).$$

This follows from the same arguments of the proof of [Theorem 3.3](#) Step 3. \square

Corollary 3.5. Let S be a complex analytic space and \mathcal{A} be a finite \mathcal{O}_S -algebra, then the map $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$ is topologically finite.

PROOF. By [Corollary 2.6](#), the fibers of $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$ is finite. The map $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow S$ is separated by construction. It remains to show that the map is closed.

The problem is local on S . By the proof of [Lemma 3.4](#), we can find a closed immersion over S : $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A} \rightarrow \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}$, where $\mathcal{B} = \mathcal{O}_S[X_1, \dots, X_n]/(F_1, \dots, F_n)$ for some $n \in \mathbb{N}$, where F_i is a unitary polynomial in $\mathcal{O}_S(S)[X_i]$ for $i = 1, \dots, n$. It suffices to show that $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{B} \rightarrow S$ is closed.

Observe that

$$\mathrm{Spec}_S^{\mathrm{an}} \mathcal{B} \cong \mathrm{Spec}_S^{\mathrm{an}} \prod_{j=1}^n \mathcal{O}_S[X_j]/(F_j)$$

in $\mathcal{A}\mathrm{n}/_S$ as can be seen from the functor of points. So the problem reduces to showing that

$$\mathrm{Spec}_S^{\mathrm{an}} \mathcal{O}_S[X]/(F) \rightarrow S$$

for a unitary polynomial is closed. This is the classical continuity of roots. \square

Corollary 3.6. Let X_x be a complex analytic germ, $n \in \mathbb{Z}_{>0}$ and $f_1, \dots, f_n \in \mathcal{O}_{X,x}$ be a system of parameters. We have a morphism $X_x \rightarrow \mathbb{C}_0^n$ induced by f_1, \dots, f_n . Then there is an open neighbourhood U of 0 in \mathbb{C}^n and a finite \mathcal{O}_U -algebra \mathcal{A} such that $\mathcal{A}_0 \cong \mathcal{O}_{X,x}$. The space $\mathrm{Spec}_U^{\mathrm{an}}(\mathcal{A})$ admits a unique point x' over 0 and X_x is isomorphic to $\mathrm{Spec}_U^{\mathrm{an}}(\mathcal{A})_{x'}$ in $\mathbb{C}\text{-}\mathcal{G}\mathrm{er}/\mathbb{C}_0^n$.

PROOF. As f_1, \dots, f_n is a system of parameters, $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\mathbb{C}^n,0}$ is finite. By [Lemma 3.4](#), we can spread $\mathcal{O}_{X,x}$ to a finite \mathcal{O}_U -algebra on an open neighbourhood U of 0 in \mathbb{C}^n . Let $Y = \mathrm{Spec}_U^{\mathrm{an}}(\mathcal{A})$. It follows from [Corollary 2.6](#) that Y has a unique point x' over 0. By [Theorem 3.3](#), up to shrinking U , we may guarantee that X_x and $Y_{x'}$ are isomorphic over \mathbb{C}_0^n . \square

Proposition 3.7. Let X_x be a complex analytic germ. The map $Y_x \mapsto I_X(Y, x)$ defines a bijection between the set of closed subgerms of X_x and the set of ideals of $\mathcal{O}_{X,x}$.

PROOF. We construct a reverse map. Given an ideal I of $\mathcal{O}_{X,x}$, as $\mathcal{O}_{X,x}$ is noetherian, I is finitely generated. We can find an open neighbourhood U of x in X and an ideal sheaf of finite type \mathcal{I} of U with $\mathcal{I}_x = I$. Let Y be the closed analytic subspace of X defined by \mathcal{I} . We associated Y_x with I .

It is easy to verify that this map is the inverse of the given map. \square

Definition 3.8. Let X_x be a complex analytic germ and Y_x, Z_x be two closed subgerms of X_x . We say Y_x is contained in Z_x and write $Y_x \subseteq Z_x$ if $I(Y, x) \supseteq I_X(Z, x)$. This defines a partial order on the set of closed subgerms of X_x .

Definition 3.9. A complex analytic germ X_x is *integral* if $\mathcal{O}_{X,x}$ is integral.

Theorem 3.10 (Nullstellensatz). Let X_x be an integral complex analytic germ and Y_x be a closed subgerm of X_x . Then the following are equivalent:

- (1) Y_x is a proper closed subgerm of X_x ;
- (2) $|Y|_x$ is a proper closed subgerm of $|X|_x$.

PROOF. (2) \implies (1): This is obvious.

(1) \implies (2): Consider a proper closed subgerm Y_x of X_x . By [Proposition 3.7](#), $I(Y, x) \neq 0$.

Step 1. We reduce to the case $I(Y, x) = (f)$ for some non-zero element $f \in \mathcal{O}_{X,x}$.

Take a non-zero element $f \in I(Y, x)$. Let Y'_x be the subgerm of X_x corresponding to the ideal (f) of $\mathcal{O}_{X,x}$. Then $Y_x \subseteq Y'_x$. It suffices to show that $|Y'|_x \neq |X|_x$. We may replace Y by Y' .

Step 2. We prove that $|Y|_x \neq |X|_x$.

Note that f is not a zero-divisor as $\mathcal{O}_{X,x}$ is integral. Write $n = \dim \mathcal{O}_{X,x}$. By Krull's Hauptidealsatz, $\dim \mathcal{O}_{X,x}/(f) = n - 1$. Let $\overline{f_1}, \dots, \overline{f_{n-1}}$ be a system of parameters ([\[Stacks, Tag 00KU\]](#)) of $\mathcal{O}_{X,x}/(f)$. Lift them to $f_1, \dots, f_{n-1} \in \mathcal{O}_{X,x}$. Then (f_1, \dots, f_{n-1}, f) is a system of parameters of $\mathcal{O}_{X,x}$. Let $\varphi : X_x \rightarrow \mathbb{C}_0^n$ and $\psi : Y_x \rightarrow \mathbb{C}_0^{n-1}$ be the morphisms defined by these systems of parameters. We then have a commutative diagram in $\mathbb{C}\text{-}\mathcal{G}$ er:

$$\begin{array}{ccc} Y_x & \hookrightarrow & X_x \\ \downarrow \psi & & \downarrow \varphi \\ \mathbb{C}_0^{n-1} & \hookrightarrow & \mathbb{C}_0^n \end{array}$$

It induces a commutative diagram of topological germs:

$$\begin{array}{ccc} |Y|_x & \hookrightarrow & |X|_x \\ \downarrow |\psi| & & \downarrow |\varphi| \\ \mathbb{C}_0^{n-1} & \hookrightarrow & \mathbb{C}_0^n \end{array}$$

The morphism of topological germs of $\mathbb{C}_0^{n-1} \rightarrow \mathbb{C}_0^n$ is clearly not an isomorphism, so it suffices to show that $|\varphi| : |X|_x \rightarrow \mathbb{C}_0^n$ is surjective, in the sense that if we represent $|\varphi|$ by a morphism $(U, x) \rightarrow (\mathbb{C}^n, 0)$ from an open neighbourhood U of x in X to \mathbb{C}^n , then its image contains an open neighbourhood of 0 in \mathbb{C}^n .

By [Corollary 3.6](#), we may assume that $X = \operatorname{Spec}_X^{\text{an}} \mathcal{A}$ for some finite \mathcal{O}_X -algebra \mathcal{A} and X has a unique point over 0. Then by [Corollary 2.6](#), we have $\mathcal{A}_0 \xrightarrow{\sim} \mathcal{O}_{X,x}$. By [Corollary 5.5](#) in [Complex analytic local algebras](#), the natural homomorphism

$$\varphi^\# : \mathcal{O}_{\mathbb{C}^n, 0} = \mathbb{C}\{X_1, \dots, X_n\} \rightarrow \mathcal{A}_0$$

is injective.

By [Corollary 2.7](#), it remains to show that $\operatorname{Supp} \mathcal{A}$ is a neighbourhood of s in S . But the kernel of $\mathcal{O}_S \rightarrow \mathcal{A}$ is 0 at s hence 0 in a neighbourhood of s since both \mathcal{O}_S and \mathcal{A} are coherent by [Corollary 7.4](#) in [The notion of complex analytic spaces](#). \square

Corollary 3.11. Let X_x be a complex analytic germ and I, J be two ideals in $\mathcal{O}_{X,x}$. We let $W(I), W(J)$ denote the topological germs of the closed analytic subgerms of X_x defined by I and J respectively. Then the following are equivalent:

- (1) $W(I) \subseteq W(J)$;
- (2) $J \subseteq \sqrt{I}$.

PROOF. If (2) is true, as $\mathcal{O}_{X,x}$ is noetherian, we can find $n \in \mathbb{Z}_{>0}$ such that $J^n \subseteq I$. Extend I, J to coherent ideals \mathcal{I}, \mathcal{J} on X up to shrinking X . Then $\text{Supp } \mathcal{O}_X/\mathcal{J} \subseteq \text{Supp } \mathcal{O}_X/\mathcal{I}$. Hence, (1) holds.

Suppose that (1) holds. In order to prove (2), we may assume that I is prime. Then the closed analytic subgerm Y_x of X_x defined by I is integral. Let Z_x denote the closed analytic subgerm of X_x defined by J . The intersection $Y_x \cap Z_x$ of the germs Y_x and Z_x is by definition the closed analytic subgerm of X_x defined by $I + J$. Then

$$|Y_x \cap Z_x| = |Y|_x \cap |Z|_x = W(I).$$

By Theorem 3.10, $Y_x \subseteq Z_x$. Namely, (2) holds. \square

Corollary 3.12. Let X_x be a complex analytic germ and Y_x be a closed analytic subgerm. Then the following are equivalent:

- (1) $|X|_x = |Y|_x$;
- (2) $I_X(Y, x)$ is nilpotent.

In particular, if these conditions hold, $\dim \mathcal{O}_{Y,x} = \dim \mathcal{O}_{X,x}$.

PROOF. This follows immediately from Corollary 3.11. \square

Corollary 3.13. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

- (1) x is isolated in X ;
- (2) $\mathcal{O}_{X,x}$ is artinian.

PROOF. (1) simply means that $X_x = \{x\}_x$. By Corollary 3.12, this holds if and only if \mathfrak{m}_x is nilpotent. As $\mathcal{O}_{X,x}$ is noetherian, the latter is equivalent to that $\mathcal{O}_{X,x}$ is artinian. \square

Corollary 3.14. Let X be a complex analytic space and Y be a closed analytic subspace defined by a coherent ideal \mathcal{I} . Then the following are equivalent:

- (1) $|X| = |Y|$;
- (2) \mathcal{I} is locally nilpotent.

PROOF. This follows immediately from Corollary 3.12. \square

Corollary 3.15. Let X be a complex analytic space and $f \in \mathcal{O}_X(X)$. Then the following are equivalent:

- (1) $f(x) = 0$ for all $x \in X$;
- (2) f is locally nilpotent.

PROOF. This follows from Corollary 3.14, where we take \mathcal{I} as the coherent ideal generated by f . \square

Corollary 3.16 (Rückert Nullstellensatz). Let X be a complex analytic space and \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\text{Supp } \mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m \mathcal{F}|_U = 0$.

PROOF. Let \mathcal{G} be the annihilator sheaf of \mathcal{F} :

$$\mathcal{G} := \ker(\mathcal{O}_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})),$$

where the map $\mathcal{O}_X \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ sends a local section f of \mathcal{O}_X to the endomorphism of multiplying by f of \mathcal{F} . Then \mathcal{G} is a coherent sheaf by Oka's coherence theorem [Theorem 7.3](#) in [The notion of complex analytic spaces](#). Let Y be the closed analytic subspace defined by \mathcal{G} . By our assumption, f is everywhere zero on Y , so f is locally nilpotent in $\mathcal{O}_X/\mathcal{G} \subseteq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. \square

Corollary 3.17. Let X be a complex analytic space and \mathcal{I} and \mathcal{J} be coherent ideal sheaves on X . Then the following are equivalent:

- (1) $\text{Supp } \mathcal{O}_X/\mathcal{I} \subseteq \text{Supp } \mathcal{O}_X/\mathcal{J}$;
- (2) For any $x \in X$, there is an open neighbourhood U of x in X and $n \in \mathbb{Z}_{>0}$ such that

$$\mathcal{J}^n|_U \subseteq \mathcal{I}|_U.$$

PROOF. This follows immediately from [Corollary 3.11](#). \square

Bibliography

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