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Contents

The notion of complex analytic spaces

1. Introduction

We introduce the notion of complex analytic spaces in this section.

2. $\mathbb{C}$-ringed space

Definition 2.1. A $\mathbb{C}$-*ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras on X.

A *morphism of* $\mathbb{C}$-*ringed spaces* $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair consisting of a continuous map $f : Y \to X$ and a morphism of sheaves of $\mathbb{C}$-algebras $f^\# : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$.

Given two morphisms of $\mathbb{C}$-ringed spaces $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $g : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$, their *composition* is the morphism $f \circ g : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ consisting of the continuous map $f \circ g : Z \to X$ and a morphism of sheaves $(f \circ g)^\# = g^\# \circ g^{-1}f^\# : (f \circ g)^{-1}\mathcal{O}_X \xrightarrow{\sim} g^{-1}f^{-1}\mathcal{O}_X \to \mathcal{O}_Z$.

When there is no risk of confusion, we say X is a $\mathbb{C}$-ringed space. In this case, we write $|X|$ for the topological space underlying X.

It is straightforward to verify that $\mathbb{C}$-ringed spaces form a category, which we denote by $\mathbb{C}$-$\mathcal{R}$S. Similarly, we denote by $\mathcal{R}$S the category of ringed spaces defined in [Stacks, Tag 0090].

In fact, by definition a $\mathbb{C}$-ringed space is nothing but a morphism in the category of ringed spaces $X \to \mathbb{C}^0$, where $\mathbb{C}^0$ is a single point $*$ endowed with the sheaf of rings $\mathcal{O}_{\mathbb{C}^0}$ with $\mathcal{O}_{\mathbb{C}^0}(*) = \mathbb{C}$. In terms of slice categories, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{R}\mathrm{S} \approx \mathcal{R}\mathrm{S}/\mathbb{C}^0.$$

From this identification, most of the basic results above $\mathbb{C}$-$\mathcal{R}$S follows, which we will use freely.

There is an obvious faithful forget functor $\mathbb{C}$-$\mathcal{R}$S $\to$ $\mathcal{R}$S.

Definition 2.2. A *locally* $\mathbb{C}$-*ringed space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ which when regarded as a ringed space is a locally ringed space.

A *morphism* between two locally $\mathbb{C}$-ringed spaces is a morphism between the underlying $\mathbb{C}$-ringed spaces which is a morphism of locally ringed spaces at the same time.

The category of locally $\mathbb{C}$-ringed spaces is denoted by $\mathbb{C}$-$\mathcal{L}$RS.

We refer to [Stacks, Tag 01HA] for the notion of locally ringed spaces. Similar to the case of $\mathbb{C}$-ringed space, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{L}\mathrm{RS} \approx \mathcal{L}\mathrm{RS}/\mathbb{C}^0.$$

Example 2.3. Let $n \in \mathbb{N}$, we define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{\mathbb{C}^n}$ on $\mathbb{C}^n$ as follows: for any open subset $U \subseteq \mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^n}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on U. It is easy to see that $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf and $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a $\mathbb{C}$-ringed space. Moreover, it is easy to show that $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a locally $\mathbb{C}$-ringed space.

Proposition 2.4. Let $n \in \mathbb{N}$, $w \in \mathbb{C}^n$, then there is a natural isomorphism $\mathcal{O}_{\mathbb{C}^n,w} \cong \mathbb{C}\{z_1, \ldots, z_n\}$.

The ring on the right-hand side is defined in Definition 2.1 in the Complex Analytic Local Algebras.

Proof. This is a well-known result from classical complex analysis. Include details later. $\square$

3. Complex model spaces and complex analytic spaces

Definition 3.1. Given any domain D in $\mathbb{C}^n$, we can define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_D$ on D as the restriction of $\mathcal{O}_{\mathbb{C}^n}$ defined in Example 2.3 to D. Observe that $(D, \mathcal{O}_D)$ is a locally $\mathbb{C}$-ringed space.

Definition 3.2. A *complex model space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that there exist

(1) a domain D in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and
(2) an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_D$ of finite type

such that thre is an isomorphism

$$(X, \mathcal{O}_X) \cong (\operatorname{Supp} \mathcal{O}_D/\mathcal{I}, i^{-1}(\mathcal{O}_D/\mathcal{I}))$$

in the category of $\mathbb{C}$-$\mathcal{R}$S, where $i : \operatorname{Supp} \mathcal{O}_D/\mathcal{I} \to D$ is the inclusion map. Here $\mathcal{O}_D$ is the sheaf of $\mathbb{C}$-algebras defined in Definition 3.1.

Clearly, $(X, \mathcal{O}_X)$ is a locally $\mathbb{C}$-ringed space.

Observe that X is always a Hausdorff space.

Definition 3.3. A *complex analytic space* is a locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that

(1) X is a Hausdorff space.
(2) For any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is isomorphic to a complex model space in the sense of Definition 3.2 in the category $\mathbb{C}$-$\mathcal{L}$RS.

When there is no risk of confusion, we also omit $\mathcal{O}_X$ from the notation say X is a complex analytic space.

A morphism between complex analytic spaces is a morphism of the underlying locally $\mathbb{C}$-ringed spaces. Such a morphism is also known as a *holomorphic map*.

The category of complex analytic spaces is denoted as $\mathbb{C}$-$\mathcal{A}$n.

Remark 3.4. It seems that all authors on this subject requires that complex analytic spaces be Hausdorff, which may seem unnatural from the eyes of an algebro-geometrist. Morally, Hausdorffness corresponds to separatedness in the scheme world. However, non-Hausdorff analytic spaces do not seem to play a major role, in contrast to non-separated schemes, so we stick to the current definition.

Remark 3.5. Most of the authors require extra conditions in the definition of a complex analytic space: σ-compactness, paracompactness, having countable basis etc. We will not put these constraints in the definition, instead, we choose to include them into the assumptions of the theorems.

Proposition 3.6. Let X be a complex analytic space, $x \in X$. Then $\mathcal{O}_{X,x}$ is a complex analytic local algebra.

Recall that complex analytic local algebras are defined in Definition 5.1 in the Complex Analytic Local Algebras.

PROOF. The problem is local, so we may assume that X is a complex model space. In this case, the result follows easily from Proposition 2.4. $\square$

4. Open and closed immersions

Definition 4.1. A morphism $f : X \to Y$ of complex analytic spaces is an *open immersion* if it is an open immersion of locally ringed spaces.

Recall that an open immersion of locally ringed spaces is defined in [Stacks, Tag 01HE].

Example 4.2. Let X be a complex analytic space and U be an open subset of U. Then U has a structure of complex anlaytic space induced from X. The inclusion $U \hookrightarrow X$ is an open immersion. We say U is an *open subspace* of X.

Definition 4.3. A morphism $f : X \to Y$ of complex analytic spaces is a *closed immersion* if it is a closed immersion of locally ringed spaces.

The kenrel of the canonical morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is called the *ideal* of X in Y.

Recall that a closed immersion of locally ringed spaces is defined in [Stacks, Tag 01HK]. Note that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0.$$

Later on, we will show that $\mathcal{I}$ is coherent after proving Oka's coherence theorem.

Example 4.4. Let X be a complex analytic space and $\mathcal{I}$ be a subsheaf of $\mathcal{O}_X$ locally generated by sections as a sheaf of $\mathcal{O}_X$-modules in the sense of [Stacks, Tag 01B2]. Set $Z = \operatorname{Supp} \mathcal{O}_X/\mathcal{I}$ and let $i : Z \to X$ be the inclusion map. Let $\mathcal{O}_Z$ be the unique sheaf of rings on Z such that $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$, whose existence and uniqueness is guaranteed by [Stacks, Tag 01AX]. Then $(Z, \mathcal{O}_Z)$ is a complex analytic space and $i : Z \to X$ is a closed immersion of complex analytic spaces. We say Z is the *closed subspace* of X defined by $\mathcal{I}$.

Definition 4.5. A morphism $f : X \to Y$ of complex analytic spaces is an *immersion* if it can be factorized aas $j \circ i$ where i is a closed immersion and j is an open immersion.

5. Weierstrass map

Definition 5.1. Let $d \in \mathbb{N}$ and B be a domain (non-empty open subset) in $\mathbb{C}^d$. Let $\omega_j \in \mathcal{O}_B(B)[w_j] \subseteq \mathcal{O}_B(B)[w_1, \dots, w_k]$ be monic polynomials for $j = 1, \dots, k$ for some $k \in \mathbb{N}$. We let A be the closed subspace of $B \times \mathbb{C}^k$ defined by the ideal generated by $\omega_1, \dots, \omega_k$. The projection map $B \times \mathbb{C}^k \to B$ induces a map $\pi : A \to B$. The map π is called the *Weierstrass* map defined by $\omega_1, \dots, \omega_k$ over B.

THEOREM 5.2. Let $d \in \mathbb{N}$ and B be a domain in $\mathbb{C}^d$. Let $\omega_j \in \mathcal{O}_B(B)[w_j] \subseteq \mathcal{O}_B(B)[w_1, \ldots, w_k]$ be monic polynomials of degree b_j for $j = 1, \ldots, k$ for some $k \in \mathbb{N}$. Then the Weierstrass map $\pi : A \to B$ defined by $\omega_1, \ldots, \omega_k$ over B is topologically finite and open.

PROOF. We first prove that $\pi : A \to B$ is topologically finite. The only non-trivial point is to show that π is closed. Let M be a closed subset in A and y be a point in the closure of $\pi(M)$ in B. Then we can find a sequence $(y_i, c_{1i}, \ldots, c_{ki}) \in M$ with $y_i \in B$ and $(c_{1i}, \ldots, c_{ki}) \in \mathbb{C}^k$ for $i \in \mathbb{N}$ such that $y_i \to y$ in B as $i \to \infty$. Then for each $i \in \mathbb{N}$ and $j = 1, \ldots, k$, (y_i, c_{ji}) is a solution to $\omega_j(y_i, \bullet)$. By continuity of roots, up to extracting a subsequence, we can find $c_j \in \mathbb{C}$ such that $c_{ji} \to c_j$ when $i \to \infty$ for $j = 1, \ldots, k$. It follows that $y = \pi(y, c_1, \ldots, c_k) \in \pi(M)$.

It remains to show that π is open. Take $p = (q, c) \in A$ with $q \in B$ and $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$. Let U be an open neighbourhood of p in A. We need to show that $\pi(U)$ contains an open neighbourhood of $\pi(p)$. We may assume that $U = A \cap (D \times W)$, where D is an open neighbourhood of q in B and W is an open neighbourhood of c in $\mathbb{C}^k$.

By the proof of Theorem 2.10 in Complex analytic local algebras, chossing D small enough, we may guarantee that there are monic polynomials $\omega_j', \omega_j'' \in \mathcal{O}(B)[w_j]$ and $n_j \in \mathbb{Z}_{>0}$ such that

$$\omega_j|_{D \times \mathbb{C}^k} = \omega_j' \omega_j'', \quad \omega_j'(q, w) = (w - c_j)^{n_j}$$

for $j = 1, \ldots, k$. Let A' be the closed subspace of $D \times \mathbb{C}^k$ defined by $\omega_1', \ldots, \omega_k'$. Then $A' \subseteq A$ by construction. Let $\pi' : A' \to D$ be the natural projection. Then p is the only point on $\pi'^{-1}(q)$. We claim that there is an open neighbourhood $V \subseteq D$ of q such that

$$A' \cap (V \times \mathbb{C}^k) \subseteq V \times W.$$

In fact, by **??** in **??**, we can find an open neighbourhood $V \subseteq D$ of q such that

$$\pi'^{-1}(V) \subseteq A' \cap (D \times W).$$

But $\pi'^{-1}(V) = A' \cap (V \times \mathbb{C}^k)$. So our claim follows.

It follows that $V \subseteq \pi(U)$ and our assertion follows. □

Lemma 5.3. Let $d \in \mathbb{N}$, B be a domain in $\mathbb{C}^d$ and

$$\omega = w^b + a_1 w^{b-1} + \cdots + a_b \in \mathcal{O}_B(B)[w]$$

be a monic polynomial. Let $\pi : A \to B$ be the Weierstrass map defined by ω over B. For any $y \in B$ and $x_1, \ldots, x_n$ be the distinct points in the fiber $\pi^{-1}(y)$. Then for any $f_j \in \mathcal{O}_{x_j}$ for $j = 1, \ldots, n$, there exist germs $q_j \in \mathcal{O}_{x_j}$ for $j = 1, \ldots, n$ and a polynomial $r \in \mathcal{O}_y[w]$ with $\deg r < b$ such that

$$f_j = \omega_{x_j} q_j + r_{x_j}$$

for $j = 1, \ldots, n$. The polynomial r and the germs $q_1, \ldots, q_n$ are uniquely determined.

PROOF. We write

$$\omega(y, w) = (w - c_1)^{b_1} \cdots (w - c_n)^{b_n}$$

with $c_1, \ldots, c_n$ with $x_i = (y, c_i)$ for $i = 1, \ldots, n$ and $b_1, \ldots, b_n \in \mathbb{Z}_{>0}$.

By Theorem 2.10 in Complex analytic local algebras, we can find $\omega_1, \ldots, \omega_n \in \mathcal{O}_y[w]$ such that

$$\omega_{x_j} = \omega_{1x_j} \cdots \omega_{nx_j}, \quad \omega_j(y, w) = (w - c_j)^{b_j}$$

for $j = 1, \dots, n$. We define

$$e_j := \prod_{i \neq j} \omega_i \in \mathcal{O}_y[w]$$

for $j = 1, \dots, n$. Then e_{jx_j} is a unit in $\mathcal{O}_{x_j}$ for $j = 1, \dots, n$ as $e_j(x_j) = \prod_{i \neq j}(c_j - c_i)^{b_i} \neq 0$.

By Theorem 3.2 in Complex analytic local algebras, each germ $f_j e_{jx_j}^{-1} \in \mathcal{O}_{x_j}$ can be written as

$$f_j e_{jx_j}^{-1} = \omega_{jx_j} q_j' + r_{jx_j},$$

where $q_j' \in \mathcal{O}_{x_j}$ and $r_j \in \mathcal{O}_y[w - c_j]$ with $\deg r_j < b_j$ for $j = 1, \dots, n$. Set $e_{ij} := \prod_{k \neq i,j} \omega_k \in \mathcal{O}_y[w]$ for any $i, j = 1, \dots, n$ with $i \neq j$. For $j = 1, \dots, n$, we define

$$q_j := q_j' - \sum_{i \neq j} r_{ix_j} e_{ijx_j}$$

and

$$r = r_1 e_1 + \cdots + r_t e_t \in \mathcal{O}_y[w].$$

Then $f_j = \omega_{x_j} q_j + r_{x_j}$ for $j = 1, \dots, n$. This proves the uniqueness part.

Next we show the uniqueness. Assume that

$$0 = \omega_{x_j} q_j + r_{x_j}$$

with $q_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$ and $r \in \mathcal{O}_y[w]$ with degree less than b. We need to show that $r = 0$. Assume by contrary that $r \neq 0$, then $p_j := r(\omega_1 \cdots \omega_j)^{-1} \neq 0$ for $j = 1, \dots, n$. Now $-r_{x_j} = (\omega_1 \cdots \omega_n)_{x_j} q_j$ implies that $p_{jx_j} = -q_j(\omega_{j+1} \cdots \omega_n)_{x_j} \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$. Since $\omega_{jx_j} \in \mathcal{O}_y[w - c_j]$ is a Weierstrass polynomial, it follows from Lemma 4.2 in Complex analytic local algebras and

$$r = p_1 \omega_1, \quad p_{j-1} = p_j \omega_j \text{ for } j = 2, \dots, n$$

that $p_1, \dots, p_n$ are all polynomials in w. As $r = p_n \omega$, we have a contradiction as $\deg r < b$. □

Theorem 5.4. Let $d \in \mathbb{N}$, B be a domain in $\mathbb{C}^d$ and

$$\omega = w^b + a_1 w^{b-1} + \cdots + a_b \in \mathcal{O}_B(B)[w]$$

be a monic polynomial. Let $\pi : A \to B$ be the Weierstrass map defined by ω over B. Then we have a natural isomorphism of $\mathcal{O}_B$-modules

$$\mathcal{O}_B^b \xrightarrow{\sim} \pi_* \mathcal{O}_A.$$

Proof. We first define the map. Let $V \subseteq B$ be an open subset and $s = (s_0, \dots, s_{b-1}) \in \mathcal{O}_B(V)^b$. The polynomial $\sum_{j=0}^{b-1} s_j w^j$ determines a section $s' \in \mathcal{O}_A(\pi^{-1}(V)) = \pi_* \mathcal{O}_A(V)$. The map $s \mapsto s'$ is clearly defines a map of $\mathcal{O}_B$-modules $\mathcal{O}_B^b \xrightarrow{\sim} \pi_* \mathcal{O}_A$. In order to prove that this map is an isomorphism, it suffices to do so for each germ. Let $y \in B$ and $x_1, \dots, x_n$ denote the points in the fiber $\pi^{-1}(b)$. By Theorem 5.2 and **??** in **??**, we have a natural identification

$$(\pi_* \mathcal{O}_A)_y \xrightarrow{\sim} \prod_{j=1}^{n} \mathcal{O}_{A,x_j}.$$

A germ $g \in (\pi_* \mathcal{O}_A)_y$ corresponds to $(g_1, \dots, g_n) \in \prod_{j=1}^n \mathcal{O}_{A,x_j}$. By Lemma 5.3, the latter can be uniquely lifted to $f_j \in \mathcal{O}_{x_j}$ for $j = 1, \dots, n$ such that if we define

$$r := \sum_{j=0}^{b-1} r_j w^j \in \mathcal{O}_y[w],$$

then r_{x_j} restricts to g_j for $j = 1, \dots, n$. This shows that the map of germs

$$\mathcal{O}_y^b \to (\pi_* \mathcal{O}_A)_y$$

is bijective. □

6. Oka's coherence theorem

This lemma needs to be placed elsewhere. Proof at CAS p58 needs to be included

Lemma 6.1. Let X be a topological space and $\mathcal{A}$ be a Hausdorff sheaf of rings on X (in the sense that the espace étalé of $\mathcal{A}$ is Hausdorff) such that all stalks of $\mathcal{A}$ are integral domains. Then $\mathcal{A}$ is coherent if and only if for any open set $V \subseteq X$ and any section $s \in \mathcal{A}(X)$, $\mathcal{A}_V / s\mathcal{A}_V$ is coherent at every $x \in V$ where $s_x \neq 0$.

Lemma 6.2 (Oka)**.** For any $n \in \mathbb{N}$, $\mathcal{O}_{\mathcal{C}^n}$ is coherent.

Proof. As a preparation, observe that $\mathcal{O}_{\mathbb{C}^n}$ is a Hausdorff sheaf.

For any two germs $s_i \in \mathcal{O}_{\mathbb{C}^n, a_i}$ $(i = 1, 2)$, we need to construct disjoint open neighbourhoods U_i in the espace étalé of $\mathcal{O}_{\mathbb{C}^n}$ of s_i. If $a_1 \neq a_2$, the assertion is clear. So assume that $a_1 = a_2 = 0$. We extend s_i to $f_i \in \mathcal{O}_{\mathbb{C}^n}(U)$ for a connected open neighbourhood $U \subseteq \mathbb{C}^n$ of 0. Then $\{f_x : x \in U\}$ and $\{g_x : x \in U\}$ are disjoint: if for some $z \in U$, $f_z = g_z$, then the same holds in a neighbourhood of z and so $f = g$ on U by Identitätssatz. Include the proof

We will prove the coherence of $\mathcal{O}_{\mathcal{C}^n}$ by induction on n. The case $n = 0$ is trivial. Assume that $n > 0$ and the theorem has been proved for all smaller n. We will apply Lemma 6.1. Take an open set $U \subseteq \mathbb{C}^n$ and $g \in \mathcal{O}_{\mathbb{C}^n}(U)$. We need to show that $\mathcal{O}_U / g\mathcal{O}_U$ is coherent at all $x \in U$ with $g_x \neq 0$.

Fix such a point x, which may be assumed to be 0. We may assume that $g(0) = 0$ as otherwise, the stalk of $\mathcal{O}_U / g\mathcal{O}_U$ at 0 is trivial. By perturbing the coordinates, we may guarantee that $g_0(0, w)$ is not identically 0 for $w \in \mathbb{C}$. By Weierstrass preparation theorem **??**, there is a Weierstrass polynomial $\omega_0 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ such that $g_0 \mathcal{O}_{\mathbb{C}^n,0} = \omega_0 \mathcal{O}_{\mathbb{C}^n,0}$. Lift ω_0 to $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)$ for some neighbourhood $B \subseteq \mathbb{C}^{n-1}$ of 0. In order to show the coherence of $\mathcal{O}_U / g\mathcal{O}_U$ near 0, it suffices to show that $\mathcal{O}_{B \times \mathbb{C}} / \omega \mathcal{O}_{B \times \mathbb{C}}$ near 0. Let $A \subseteq B \times \mathbb{C}$ be the closed subspace defined by ω and $\pi : A \to B$ be the Weierstrass map, then it suffices to show that $\mathcal{O}_A$ is coherent near 0. By our inductive hypothesis, $\mathcal{O}_B$ is coherent. We claim that $\mathcal{O}_A$ is also coherent. Let b be the degree of ω. We recall that π is topologically finite by Theorem 5.2.

We first prove a special case: let $p \in \mathbb{N}$ and $\varphi : \mathcal{O}_A^p \to \mathcal{O}_A$ be an $\mathcal{O}_A$-homomorphism. We show that $\ker \varphi$ is of finite type. By Theorem 5.4, $\pi_* \mathcal{O}_A^p$ is coherent. So $\pi_* \ker \varphi$ is coherent by **??** in **??**. It follows that $\ker \varphi$ is of finite type.

Next let $U \subseteq A$ be an open subset and $s_1, \dots, s_p \in \mathcal{O}_A(U)$. We need to show that the kernel of the assocaited map

$$\mathcal{O}_U^p \to \mathcal{O}_A|_U$$

is of finite type. By **??** in **??**, for each $x \in U$, we can find an open neighbourhood V of x in U such that $\pi^{-1}(V)$ is the disjoint union of open neighbourhoods $U_1, \ldots, U_n$ of the points in $\pi^{-1}(\pi(x))$. We may assume that $x \in U_1$. Extend $s_j|_{U_1}$ to $s'_j \in \mathcal{O}_A(\pi^{-1}(V))$ by setting its values to be 0 on $U_2, \ldots, U_n$ for $j = 1, \ldots, p$. Then φ extends to $\varphi' : \mathcal{O}^p_{\pi^{-1}(V)} \to \mathcal{O}_{\pi^{-1}(V)}$ with the same kernel over U_1. By what we have proved, $\ker \varphi'$ is of finite type. Hence so is $\ker \varphi$. □

As a corollary, we have the important Oka's coherence theorem.

Theorem 6.3. Let X be a complex analytic space, then $\mathcal{O}_X$ is coherent.

Proof. The problem is local on X, so we may assume that X is a complex model space, say there is a closed immersion into a domain D in $\mathbb{C}^n$ defined by an ideal of finite type $\mathcal{I}$. By Lemma 6.2, $\mathcal{O}_D$ is coherent and hence $\mathcal{I}$ is coherent. It follows that $\mathcal{O}_D/\mathcal{I}$ is coherent and hence $\mathcal{O}_X$ is coherent. □

7. Rückert Nullstellensatz

Let X be a complex analytic space. It is a sheaf of $\mathbb{C}$-algebras. For any sheaf of local $\mathbb{C}$-algebras $\mathcal{A}$ on X, any open set $U \subseteq X$ and any $s \in \mathcal{A}_X(U)$. We want to construct a function $[s] : U \to \mathbb{C}$.

Take $x \in U$, there is a canonical splitting

$$\mathcal{A}_x \cong \mathbb{C} \oplus \mathfrak{m}_x, \tag{7.1}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{A}_x$. Then we define $[s](x)$ as the image of s_x in the $\mathbb{C}$-factor in (7.1).

Definition 7.1. Let $X, \mathcal{A}, U, x, s$ be as above. The value $[s](x) \in \mathbb{C}$ is called the *value* of s at x. We sometimes denote it by $s(x)$ as well.

Lemma 7.2. Let X be a complex analytic space. We denote by $\mathcal{C}_X$ the sheaf of continuous functions on X. The association $s \mapsto [s]$ in Definition 7.1 defines a homomorphism of sheaves of $\mathbb{C}$-algebras $\mathcal{O}_X \to \mathcal{C}_X$.

When there is no risk of confusion, we also write s instead of $[s]$.

Proof. We need to show that for any open set $U \subseteq X$ and any $s \in \mathcal{O}_X(U)$, $[s]$ is a continuous function on U.

We may clearly assume that $U = X$. The problem is local on X, so we may assume that X is a complex model space in the sense of Definition 3.2 defined by a coherent ideal $\mathcal{I}$ in a domain D in $\mathbb{C}^n$. By further localizing, we may assume that s can be lifted to a section $f \in \mathcal{O}_D(D)$. Then $[s] = f|_X$ by definition. So the assertion follows from the fact that a holomorphic function on a domain is continuous. □

Theorem 7.3 (Rückert Nullstellensatz)**.** Let X be a complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\operatorname{Supp} \mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m \mathcal{F}|_U = 0$.

Proof. We may assume that $x \in \operatorname{Supp} \mathcal{F}$ as otherwise there is nothing to prove. In particular, $f(x) = 0$.

Step 1. We first reduce the problem to a relatively simple situation.

The problem is local on X, so we may assume that there is a domain D containing 0 in $\mathbb{C}^n$ and a closed immersion $\iota : X \to D$ sending x to 0. Consider the closed immersion $g : V \to D \times \mathbb{C}$ induced by ι and f. Assume that this theorem has been proved for w, $B \times \mathbb{C}$, $g_*\mathcal{F}$ in place of f, X, $\mathcal{F}$ respectively, then we would find an integer $m \in \mathbb{Z}_{>0}$ such that $w^m(g_*\mathcal{F})_0 = 0$. In particular, $f^m\mathcal{F}_x = 0$. As $\mathcal{F}$ is coherent, there is an open neighbourhood $U \subseteq X$ of x such that $f^t\mathcal{F}|_U = 0$.

Step 2. We are reduced to prove the following special case: let D be a domain in $\mathbb{C}^n$ containing 0, $\mathcal{F}$ is a coherent sheaf on D whose support is contained in $\{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : (z,w) \in D, w = 0\}$. Then there is $m \in \mathbb{Z}_{>0}$ such that $w^m\mathcal{F}_0 = 0$.

Let $\mathcal{G}$ be the annihilator sheaf of $\mathcal{F}$:

$$\mathcal{G} := \ker\left(\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F},\mathcal{F})\right),$$

where the map $\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F},\mathcal{F})$ sends a local section f of $\mathcal{O}_D$ to the endohomomorphism of multiplying by f of $\mathcal{F}$. Then $\mathcal{G}$ is a coherent sheaf by Oka's coherence theorem Theorem 6.3. So it has closed supports. But by our assumtion, the support of $\mathcal{G}$ contains all $w \neq 0$, so $\operatorname{Supp}\mathcal{G} = D$.

Let $f \in \mathcal{G}_0$ be a non-zero element. Write

$$f = \sum_{j=b}^{\infty} a_j w^j, \quad a_j \in \mathcal{O}_{\mathbb{C}^{n-1},0}, a_b \neq 0$$

for some $b \in \mathbb{N}$. We may assume that $b = 0$ by replacing f and $\mathcal{F}$ with $w^{-b}f$ and $w^b\mathcal{F}$ respectively. We want to show that $w^m\mathcal{F}_0 = 0$ for some positive integer m.

When a_0 is a unit, namely when $a_0(0) \neq 0$, then f is a unit, so $\mathcal{F}_0 = 0$. We make an induction on n. The case $n = 1$ is trivial, as a_0 is always a unit. So we may assume that $a_0(0) = 0$ and $n > 1$. By perturbing the coordinates in $\mathbb{C}^{n-1}$, we may assume that a_0 is not identically zero in the variable z_1.

Shrinking D, we may assume that f can be lifted to a holomorphic function $g \in \mathcal{O}_D(D)$ with $g\mathcal{F} = 0$. By our assumption on a_0, we may assume that $Z(g) \cap \{(z_1, 0, \ldots, 0) \in D\} = \{0\}$. Hence, $D \cap \operatorname{Supp}\mathcal{F}$, which is a subset of $Z(g)$ also intersects the z_1-axis only at the origin.

By To be included, we can find a product domain $B \times W \subseteq D$ with $B \subseteq \mathbb{C}$ and $W \subseteq \mathbb{C}^{n-1}$ containing 0 such that the projection $h : (B \times W) \cap \operatorname{Supp}\mathcal{F} \to B$ is finite and $\mathcal{F}' := h_*(\mathcal{F}|_{B\times W})$ is a coherent sheaf of $\mathcal{O}_B$-modules. Observe that $\operatorname{Supp}\mathcal{F}' \subseteq \{(z_2, \ldots, z_{n-1}, w) \in B : w = 0\}$, we can apply the induction hypothesis to obtain $m \in \mathbb{Z}_{>0}$ such that $w^m\mathcal{F}'_0 = 0$. It follows that $w^m\mathcal{F}_0 = 0$. $\square$

8. Finite limits in the category of complex analytic spaces

The goal of this section is to show that the category of complex analytic spaces admits finite limits.

As the category $\mathbb{C}$-$\mathcal{A}$n admits a final object, namely $\mathbb{C}^0$, the existence of finite limits is the same as the existence of fiber products by general abstract nonsense [Stacks, Tag 002O].

We begin by considering direct products, namely fiber products over $\mathbb{C}^0$.

Lemma 8.1. Let $m, n \in \mathbb{N}$. Then

$$\mathbb{C}^m \times \mathbb{C}^n \cong \mathbb{C}^{m+n}.$$

Here $\times$ denotes the product in $\mathbb{C}$-$\mathcal{A}$n.

PROOF. By Yoneda lemma [Stacks, Tag 001P], it suffices to establish

$$h_{\mathbb{C}^m\times\mathbb{C}^n}\cong h_{\mathbb{C}^{m+n}},$$

where $h_\bullet$ denotes the functor of points [Stacks, Tag 001O]. Take $T\in\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$, then there are isomorphisms

$$h_{\mathbb{C}^m\times\mathbb{C}^n}(T)\xrightarrow{\sim}h_{\mathbb{C}^m}(T)\times h_{\mathbb{C}^m}(T)\xrightarrow{\sim}(\mathcal{O}_T(T))^{m+n}\xrightarrow{\sim}h_{\mathbb{C}^{m+n}}(T),$$

which are all functorial in T. We conclude. □

Lemma 8.2. Let $f:X\to Y$ be a morphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}$. Let $i:Z\to Y$ be a closed (resp. an open) immersion. Then the fiber product $X\times_Y Z$ exists. Moreover, $X\times_Y Z\to X$ is a closed (resp. an open) immersion and there is a natural identification $|X\times_Y Z|\cong|X|\times_{|Y|}|Z|$.

We can draw a Cartesian diagram

$$\begin{tikzcd} X\times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Z \arrow[r, "i"] & Y \end{tikzcd}$$

PROOF. When i is an open immersion, it suffices to take $X\times_Y Z$ as the open subspace of X defined by $f^{-1}(i(Z))$.

Let us consider the case where i is a closed immersion defined by a coherent ideal sheaf $\mathcal{I}$. It is a general result that $X\times_Y Z$ in the category $\mathcal{L}\mathrm{RS}$ exists [Stacks, Tag 01HQ]. Let us show that $X\times_Y Z$ is a closed complex analytic subspace of X and conclude. To do so, recall that $X\times_Y Z$ is by construction a closed subspace of X defined by $\mathcal{J}:=\mathrm{Im}\,(f^*\mathcal{I}\to f^*\mathcal{O}_Y=\mathcal{O}_X)$. It suffices to show that $\mathcal{J}$ is of finite type. By this is clear as $\mathcal{I}$ is of finite type.

The identification of the underlying topological space is obvious. □

Lemma 8.3. Let X, Y be complex analytic spaces. Consider open (resp. closed) immersions $X'\to X$ and $Y'\to Y$. If $X\times Y$ exists, then so is $X'\times Y'$ and the natural morphism $X'\times Y'\to X\times Y$ is an open (resp. a closed) immersion.

PROOF. We form the following large Cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\ Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X\times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y' \arrow[r] & Y \arrow[r] & \mathbb{C}^0 \end{tikzcd}$$

The existences of all but the lower right square are guaranteed by Lemma 8.2. More precisely, we first define the upper right square and the lower left square by Lemma 8.2. It follows from Lemma 8.2 that $X''\to X\times Y$ is an open (resp. a closed) immersion. So we can apply Lemma 8.2 again to construct the upper left square.

It follows from general abstract nonsense that the big square is also Cartesian. Moreover, by Lemma 8.2 again, $Z\to Y''$ and $Y''\to X\times Y$ are both open (resp. closed) immersions. It follows that $Z\to X\times Y$ is also an open (resp. a closed) immersion. □

Corollary 8.4. Let X, Y be complex model spaces. Then $X \times Y$ exists.

Proof. By Lemma 8.3, we may assume that X and Y are both domains in some $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. Then applying Lemma 8.3 again, we reduce to the case where $X = \mathbb{C}^m$ and $Y = \mathbb{C}^n$. This case is handled in Lemma 8.1. □

Corollary 8.5. Let X, Y be complex analytic spaces. Then $X \times Y$ exists in $\mathbb{C}$-$\mathcal{A}$n. Moreover, there is a natural identification $|X \times Y| \cong |X| \times |Y|$.

Proof. Let

$$X = \bigcup_{i \in I_X} X_i, \quad X = \bigcup_{j \in I_Y} Y_j$$

be open coverings of X by complex model spaces. Let $K = I_X \times I_Y$. For each $k = (i, j) \in K$, we let $Z_k = X_i \times Y_j$, whose existence is guaranteed by Corollary 8.4. Take another $k' = (i', j') \in K$, then

$$Z_{kk'} := Z_k \cap Z_{k'} = (X_i \times X_{i'}) \cap (Y_j \times Y_{j'})$$

is an open subspace of Z_k. It is clear that $Z_{kk'}$ forms a glueing data. From the general result [Stacks, Tag 01JB], we can glue Z_k's into a locally ringed space Z. From the construction, $|Z| = |X| \times |Y|$ in the category of topological spaces, so $|Z|$ is Hausdorff. On the other hand, from the construction, locally Z is isomorphic to some Z_k, so Z is a complex analytic space. As Z is clearly the product in the category of locally $\mathbb{C}$-ringed spaces, we conclude that $Z = X \times Y$ in $\mathbb{C}$-$\mathcal{A}$n. □

Corollary 8.6. The category $\mathbb{C}$-$\mathcal{A}$n admits all finite limits. Moreover, finite limits commute with the forgetful functor $\mathbb{C}$-$\mathcal{A}\text{n} \to \mathcal{T}\text{op}$.

Proof. By [Stacks, Tag 002O], Corollary 8.5 and the existence of a final object in $\mathbb{C}$-$\mathcal{A}$n (namely, $\mathbb{C}^0$), it suffices to show the existence of fiber products. In other words, suppose that we are given three complex analytic spaces Z, X, Y and morphisms $X \to Z$ and $Y \to Z$ in $\mathbb{C}$-$\mathcal{A}$n, we need to prove the existence of $X \times_Z Y$. From the general abstract nonsense, we can define $X \times_Z Y = (X \times Z)_{Y \times Y, \Delta_Y} Y$:

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times Z \arrow[d] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd},$$

where $\Delta_Y : Y \to Y \times Y$ is the diagonal morphism, which is a closed immersion, the existence of $X \times Z$ is guaranteed by Corollary 8.5 and the existence of the fiber product is guaranteed by Lemma 8.2.

In order to verify that finite limits commute with the forgetful functor $\mathbb{C}$-$\mathcal{A}\text{n} \to \mathcal{T}\text{op}$, it suffices to consider fiber products. By Lemma 8.2, we reduced to the case of finite products. In this case, the result is proved in Corollary 8.5. □

Remark 8.7. It is important to remember that the forgetful functor $\mathbb{C}$-$\mathcal{A}\text{n} \to \mathbb{C}$-$\mathcal{L}$RS does *not* commute with finite limits, in contrast to the case of schemes [Stacks, Tag 01JN]. While the forgetful functor from the category of schemes $\mathcal{S}$ch to $\mathcal{T}$op does not commute with finite limits.

These facts indicate that there are essential differences between the theory of analytic spaces and the theory of schemes.

Next we study the local rings of fiber products.

THEOREM 8.8. Let Y be an object in $\mathbb{C}$-$\mathcal{A}$n and $X_1, X_2 \in \mathbb{C}$-$\mathcal{A}\mathrm{n}_{/Y}$. Let (x_1, x_2) be a point of $X_1 \times_Y X_2$, namely, $x_i \in X_i$ for $i = 1, 2$ and the images of x_1 and x_2 in Y coincide, say $y \in Y$. Then there is a caonical isomorphism

$$\mathcal{O}_{X_1 \times_Y X_2, (x_1, x_2)} \cong \mathcal{O}_{X_1, x_1} \overline{\otimes}_{\mathcal{O}_{Y,y}} \mathcal{O}_{X_2, x_2}.$$

The analytic tensor product here is defined Definition 5.4 in the Complex Analytic Local Algebras. We have shown its existence in Theorem 5.9 in the same chapter.

PROOF. Comparing the constructions of both sides, we see that it suffices to prove the theorem in two special cases: when $Y = \mathbb{C}^0$ and when $X_2 \to Y$ is a closed immersion.

We first consider the case where $Y = \mathbb{C}^0$. As our problem is local, we may assume that X_1 and X_2 are both complex model spaces. From the constructions, we easily reduce to the case where X_1 and X_2 are both domains in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. In this case, the result is proved in Lemma 5.5 in the Complex Analytic Local Algebras and Proposition 2.4.

Next we handle the case where $X_2 \to Y$ is a closed immersion. This case is immediately clear from the constructions of both sides. □

9. Complex analytic topos

Bibliography

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