Banach rings

Contents

1. Introduction

This section conerns the theory of Banach algebras. Our references are [Ber12] and [BGR84].

In this chapter, all rings are assumed to be commutative.

2. Semi-normed Abelian groups

Definition 2.1. Let A be an Abelian group. A *semi-norm* on A is a function $\|\bullet\| : A \to [0,\infty]$ satisfying

(1) $\|0\| = 0$;
(2) $\|f-g\| \leq \|f\| + \|g\|$ for all $f, g \in A$.

A semi-norm $\|\bullet\|$ on A is a *norm* if moreover the following conditions is satisfied:

(0) if $\|f\| = 0$ for some $f \in A$, then $f = 0$.

We write
$$\ker \|\bullet\| = \{a \in A : \|a\| = 0\}.$$

A semi-norm $\|\bullet\|$ on A is *non-Archimedean* or *ultra-metric* if Condition (2) can be replaced by

(2') $\|f-g\| \leq \max\{\|f\|, \|g\|\}$ for all $f, g \in A$.

Definition 2.2. A *semi-normed Abelian group* (resp. *normed Abelian group*) is a pair $(A, \|\bullet\|)$ consisting of an Abelian group A and a semi-norm (resp. norm) $\|\bullet\|$ on A. When $\|\bullet\|$ is clear from the context, we also say A is a semi-normed Abelian group (resp. normed Abelian group).

Definition 2.3. Let $(A, \|\bullet\|_A)$ be a semi-normed Abelian group and $B \subseteq A$ be a subgroup. Then we define the *quotient semi-norm* $\|\bullet\|_{A/B}$ on A/B as follows:
$$\|a+B\|_{A/B} := \inf\{\|a+b\|_A : b \in B\}$$
for all $a + B \in A/B$.

We define the *subgroup semi-norm* on B as follows:
$$\|b\|_B = \|b\|_A$$
for all $b \in B$.

Definition 2.4. Let A be an Abelian group and $\|\bullet\|$, $\|\bullet\|'$ be two seminorms on A. We say $\|\bullet\|$ and $\|\bullet\|'$ are *equivalent* if there is a constant $C > 0$ such that
$$C^{-1}\|f\| \leq \|f\|' \leq C\|f\|$$
for all $f \in A$.

Definition 2.5. Let $(A, \|\bullet\|_A)$, $(B, \|\bullet\|_B)$ be semi-normed Abelian groups. A homomorphism $\varphi : A \to B$ is said to be

(1) *bounded* if there is a constant $C > 0$ such that $\|\varphi(f)\|_B \leq C\|f\|_A$ for any $f \in A$;
(2) *admissible* if the quotient semi-norm on $A/\ker\varphi$ is equivalent to the subspace semi-norm on $\operatorname{Im}\varphi$.

Observe that an admissible homomorphism is always bounded.

Next we study the topology defined by a semi-norm.

Lemma 2.6. Let $(A, \| \bullet \|)$ be a semi-normed Abelian group. Define

$$d(a, b) = \|a - b\|$$

for $a, b \in A$. Then $\| \bullet \|$ is a pseudo-metric on A. This psuedo-metric is a metric if and only if $\| \bullet \|$ is a norm.

PROOF. This is clear from the definitions. □

We always endow A with the topology induced by the psuedo-metric d.

3. Semi-normed rings

Definition 3.1. Let A be a ring. A *semi-norm* $\| \bullet \|$ on A is a semi-norm $\| \bullet \|$ on the underlying additive group satisfying the following extra properties:

(3) $\|1\| = 1$;
(4) for any $f, g \in A$, $\|fg\| \leq \|f\| \cdot \|g\|$.

A semi-norm $\| \bullet \|$ on A is called *power-multiplicative* if $\|f\|^n = \|f^n\|$ for all $f \in A$ and $n \in \mathbb{N}$.

A semi-norm $\| \bullet \|$ on A is called *multiplicative* if $\|fg\| = \|f\|\|g\|$ for all $f, g \in A$.

Definition 3.2. A *semi-normed ring* (resp. *normed ring*) is a pair $(A, \| \bullet \|)$ consisting of a ring A and a semi-norm (resp. norm) $\| \bullet \|$ on A. When $\| \bullet \|$ is clear from the context, we also say A is a semi-normed ring (resp. normed ring).

Definition 3.3. Let $(A, \| \bullet \|)$ be a semi-normed ring. An element $a \in A$ is *multiplicative* if $a \notin \ker \| \bullet \|$ and for any $x \in A$,

$$\|ax\| = \|a\| \cdot \|x\|.$$

Definition 3.4. Let $(A, \| \bullet \|)$ be a normed ring. An element $a \in A$ is *power-bounded* if $\{|a^n| : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. The set of power-bounded elements in A is denoted by $\mathring{A}$.

An element $a \in A$ is called *topologically nilpotent* if $a^n \to 0$ as $n \to \infty$. The set of topologically nilpotent elements in A is denoted by $\check{A}$.

Observe that $\check{A}$ is an ideal in $\mathring{A}$. We write $\tilde{A} = \mathring{A}/\check{A}$.

Definition 3.5. Let A be a ring. A *semi-valuation* on A is a multiplicative semi-norm on A. A semi-valuation on A is a *valuation* on A if its underlying semi-norm of Abelian groups is a norm.

Definition 3.6. A *semi-valued ring* (resp. *valued ring*) is a pair $(A, \| \bullet \|)$ consisting of a ring A and a semi-valuation (resp. valuation) $\| \bullet \|$ on A. When $\| \bullet \|$ is clear from the context, we also say A is a semi-valued ring (resp. valued ring).

A semi-valued ring (resp. valued ring) $(A, \| \bullet \|)$ is called a *semi-valued field* (resp. *valued field*) if A is a field.

4. Banach rings

Definition 4.1. A *Banach ring* is a normed ring that is complete with respect to the metric defined in Lemma 2.6.

Proposition 4.2. Let $(A, \| \bullet \|)$ be a Banach ring and $f \in A$. Assume that $\|f\| < 1$, then $1 - f$ is invertible.

PROOF. Define

$$g=\sum_{i=0}^{\infty} f^i.$$

From our assumption, the series converges and $g \in A$. It is elementary to check that g is the inverse of $1-f$. □

Definition 4.3. Let $(A, \|\bullet\|)$ be a Banach ring. We define the *spectral radius* $\rho=\rho_A : A \to [0,\infty)$ as follows:

$$\rho(f)=\inf_{n\geq 1}\|f^n\|^{1/n}, \quad f \in A.$$

Lemma 4.4. Let $(A, \|\bullet\|)$ be a Banach ring. Then for any $f \in A$, we have

$$\rho(f)=\lim_{n\to\infty}\|f^n\|^{1/n}.$$

PROOF. This follows from the multiplicative version of Fekete's lemma. □

Example 4.5. The ring $\mathbb{C}$ with its usual norm $|\bullet|$ is a Banach ring. In fact, $(\mathbb{C}, |\bullet|)$ is a complete valued field.

Example 4.6. For any Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r=(r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}$, we define $A\langle r^{-1}z\rangle = A\langle r_1^{-1}z_1,\ldots,r_n^{-1}z_n\rangle$ as the subring of $A[[z_1,\ldots,z_n]]$ consisting of formal power series

$$f=\sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in A$$

such that

$$\|f\|_r := \sum_{\alpha\in\mathbb{N}^n}\|a_\alpha\| r^\alpha < \infty.$$

We will verify in Proposition 4.7 that $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

When $r=(1,\ldots,1)$, we omit r^{-1} from our notations.

Proposition 4.7. In the setting of Example 4.6, $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

PROOF. By induction, we may assume that $n=1$.

It is obvious that $\|\bullet\|_r$ is a norm on the undelrying Abelian group. To see that $\|\bullet\|_r$ is a norm on the ring $A\langle r^{-1}z\rangle$, we need to verify the condition in Definition 3.1. Condition (3) in Definition 3.1 is obvious. Let us consider Condition (4). Let

$$f=\sum_{i=0}^{\infty} a_i z^i, \quad g=\sum_{j=0}^{\infty} b_j z^j$$

be two elements in $A\langle r^{-1}z\rangle$. Then

$$fg=\sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i b_j\right) z^k.$$

We compute

$$\|fg\|_r=\sum_{k=0}^{\infty}\left\|\sum_{i+j=k} a_i b_j\right\| r^k \leq \sum_{k=0}^{\infty}\left(\sum_{i+j=k}\|a_i\|\cdot\|b_j\|\right) r^k = \|f\|_r\cdot\|g\|_r.$$

It remains to verify that $A\langle r^{-1}z\rangle$ is complete.

For this purpose, take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b z^i \in A\langle r^{-1}z\rangle$$

for $b \in \mathbb{N}$. Then for each i, the coefficients $(a_i^b)_b$ is a Cauchy sequence in A. Let a_i be the limit of a_i^b as $b \to \infty$ and set

$$f = \sum_{i=0}^{\infty} a_i z^i \in A[[z]].$$

We need to show that $f \in A\langle r^{-1}z\rangle$ and $f^b \to f$.

Fix a constant $\epsilon > 0$. There is $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$, we have

$$\sum_{i=0}^{\infty} \|a_i^{j+k} - a_i^j\| r^i < \epsilon/2.$$

In particular, for any $s > 0$, we have

$$\sum_{i=0}^{s} \|a_i - a_i^j\| r^i \leq \sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i + \sum_{i=0}^{s} \|a_i^j - a_i^{j+k}\| r^i \leq \sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i + \epsilon/2.$$

When k is large enough, we can guarantee that

$$\sum_{i=0}^{s} \|a_i - a_i^{j+k}\| r^i < \epsilon/2.$$

So

$$\sum_{i=0}^{s} \|a_i - a_i^j\| r^i \leq \epsilon.$$

Let $s \to \infty$, we find

$$\|f - f^j\|_r \leq \sum_{i=0}^{\infty} \|a_i - a_i^j\| r^i \leq \epsilon.$$

In particular, $\|f\|_r < \infty$ and $f^j \to f$ as $j \to \infty$. □

Example 4.8. For any non-Archimedean Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$, we define $A\{r^{-1}T\} = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ as the subring of $A[[T_1, \dots, T_n]]$ consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in A$$

such that $\|a_\alpha\| r^\alpha \to 0$ as $|\alpha| \to \infty$. We set

$$\|f\|_r := \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

We will verify in Proposition 4.9 that $(A\langle r^{-1}T\rangle, \|\bullet\|_r)$ is a Banach ring.

The semi-norm $\|\bullet\|_r$ is called the *Gauss norm*.

Proposition 4.9. In the setting of Example 4.8, $(A\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach ring.

Moreover, if the norm $\|\bullet\|$ on A is a valuation, so is $\|\bullet\|_r$.

The second part is usually known as the *Gauss lemma*.

PROOF. By induction on n, we may assume that $n = 1$.

The proof of the fact that $\| \bullet \|_r$ is a norm is similar to that of Proposition 4.7. We leave the details to the readers.

Next we argue that $(A\{r^{-1}T\}, \| \bullet \|_r)$ is complete. Take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b T^i \in A\{r^{-1}T\}$$

for $b \in \mathbb{N}$. As

$$\|a_i^b - a_i^{b'}\| r^i \leq \|f^b - f^{b'}\|_r$$

for any $i, b, b' \geq 0$, it follows that for any $i \geq 0$, $\{a_i^b\}_b$ is a Cauchy sequence. Let $a_i \in A$ be its limit and set

$$f = \sum_{i=0}^{\infty} a_i T^i \in A[[T]].$$

We need to show that $f \in A\{r^{-1}T\}$ and $f^b \to f$.

Fix $\epsilon > 0$. We can find $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$,

$$\|f^j - f^{j+k}\|_r \leq \epsilon.$$

It follows that $\|a_i^j - a_i^{j+k}\| r^i \leq \epsilon$ for all $i \geq 0$. Let $k \to \infty$, we find

$$\|a_i^j - a_i\| r^i \leq \epsilon$$

for all $i \geq 0$. Fix $j \geq 0$, take i large enough so that $|a_i^j| r^i < \epsilon$. Then $\|a_i\| r^i \leq \epsilon$. So we find $f \in A\{r^{-1}T\}$. On the other hand,

$$\|f - f^j\|_r = \max_i \|a_i^j - a_i\| r^i \leq \epsilon.$$

This proves that $f^j \to f$.

Now assume that $\| \bullet \|$ is a valuation, we verify that $\| \bullet \|_r$ is also a valuation. Again, we may assume that $n = 1$. Take two elements $f, g \in A\{r^{-1}T\}$:

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad g = \sum_{j=0}^{\infty} b_j T^j.$$

As we have already shown $|fg|_r \leq |f|_r |g|_r$, it suffices to check the reverse inequality. For this purpose, choose the minimal indices i, j so that

$$\|f\|_r = \|a_i\| r^i, \quad \|g\|_r = \|b_j\| r^j.$$

Write

$$fg = \sum_{k=0}^{\infty} \left(\sum_{p+q=k} a_p b_q \right) T^k.$$

Then we claim that

$$\| \sum_{p+q=k} a_p b_q \| r^k = \|f\|_r \|g\|_r.$$

This implies the desired inequality. To verify our claim, it suffices to observe that for $(p, q) \neq (i, j)$, $r + s = k$, say $p < i$ and $q > j$, we have

$$\|a_p b_q\| r^k = \|a_p\| r^p \cdot \|b_q\| r^q < \|a_i\| r^i \cdot \|b_j\| r^j.$$

So

$$\|a_p b_q\| < \|a_i b_j\|.$$

Since the valuation on A is non-Archimedean, it follows that

$$\|\sum_{p+q=k} a_p b_q\| = \|a_i b_j\|.$$

Our claim follows. $\square$

5. Semi-normed modules

Definition 5.1. Let $(A, \| \bullet \|_A)$ be a normed ring. A *semi-normed A-module* (resp. *normed A-module*) is a pair $(M, \| \bullet \|_M)$ consisting of a A-module M and a semi-norm (resp. norm) on the underlying Abelian group of M such that there is a constant $C > 0$ such that

$$\|fm\|_M \leq C\|f\|_A\|m\|_M$$

for all $f \in A$ and $m \in M$. When $\| \bullet \|_M$ is clear from the context, we say M is a semi-normed A-module (resp. normed A-module).

A *Banach $\mathcal{A}$-module* is a normed A-module which is complete with respect to the metric Lemma 2.6.

Definition 5.2. Let $(A, \| \bullet \|_A)$ be a normed ring. A *Banach A-algebra* is a pair $(B, \| \bullet \|_B)$ such that $(B, \| \bullet \|_B)$ is a Banach A-module and $(B, \| \bullet \|_B)$ is a Banach ring.

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