Berkovich analytic spaces

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Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 2.1. Let A be a k_H-affinoid algebra. A *compact k_H-analytic domain V* in $\operatorname{Sp} A$ is a finite union of k_H-affinoid domains in $\operatorname{Sp} A$.

Lemma 2.2. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain. Write $\operatorname{Sp} A$ as a finite union of k_H-affinoid domains $\operatorname{Sp} A_i$ with $i = 1, \dots, n$ in $\operatorname{Sp} A$. Define $A_{ij} = A_i \hat{\otimes}_A A_j$ and

$$A_V := \ker\left(\prod_{i=1}^n A_i \to \prod_{i,j=1}^n A_{ij}\right).$$

Then the Banach k-algebra does not depend on the choice of the covering $\{\operatorname{Sp} A_i\}_i$ up to a canonical isomorphism.

The image of the natural continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ contains V and the map does not depend on the choice of the covering up to the canonical isomorphism between $\operatorname{Sp} A_V$ for different coverings.

Proof. We first observe that A_V is a Banach k-algebra as it is defined as an equalizer. This follows from Lemma 4.22 in the chapter Banach Rings.

Let $\{\operatorname{Sp} B_j\}_{j=1,\dots,m}$ be another k_H-affinoid covering of $\operatorname{Sp} A$. We need to show that A_V defined using the two coverings are canonically isomorphic. We write A'_V for

$$\ker\left(\prod_{j=1}^m B_j \to \prod_{i,j=1}^m B_{ij}\right)$$

to make a distinction. We write $B_{ij} = B_i \hat{\otimes}_A B_j$.

By Theorem 12.16 in the chapter Affinoid Algebras, the colomns in the following commutative diagram are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A_V \arrow[r] \arrow[d, dotted] & \prod_{i=1}^n A_i \arrow[r] \arrow[d, "\eta"] & \prod_{i,i'=1}^n A_{ii'} \arrow[d] \\
0 \arrow[r] & \ker\iota \arrow[r] & \prod_{i=1}^n \prod_{j=1}^m A_i\hat{\otimes}_A B_j \arrow[r, "\iota"] \arrow[d, "\tau"] & \prod_{i,i'=1}^n \prod_{j,j'=1}^m A_{ii'}\hat{\otimes}_A B_{jj'} \\
 & & \prod_{i=1}^n \prod_{j,j'=1}^m A_i\hat{\otimes}_A B_{jj'} &
\end{tikzcd}$$

The rows are exact by definition. By diagram chasing, the dotted arrow is injective. To see it is surjective, it suffices to observe that the factors with $i = i'$ in the lower right corner is exactly the same as the factors of the lower corner, so an element in $\ker\iota$ is necessarily in $\ker\tau$. It follows that the dotted arrow is surjective.

Similarly, we have a natural isomorphism $A'_V \xrightarrow{\sim} \ker\iota$. We conclude the first assertion.

As for the second, observe that $\operatorname{Sp} A_V$ is defined as a colimit in the category of Banach k-algebras, so it follows from general abstract nonsense that there is a natural morphism $\operatorname{Sp} A_V \to \operatorname{Sp} A$. It clearly contains V in the image. The compatibility with the isomorphism above follows simply from the fact that the map η is an A-algebra homomorphism. $\square$

Definition 2.3. Let A be a k-affinoid algebra and V be a compact k-analytic domain in $\operatorname{Sp} A$. We define *the Banach k-algebra A_V associated with V* as A_V constructed in Lemma 2.2.

The continuous map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ constructed in Lemma 2.2 is called the *structure map* ov V.

Proposition 2.4. Let A be a k_H-affinoid algebra and V be a compact k_H-analytic domain in $\operatorname{Sp} A$. Then the following are equivalent:

(1) V is a k_H-affinoid domain.
(2) A_V is a k_H-affinoid algebra and the image of the structure map $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is exactly V.

Proof. (1) $\implies$ (2): By Theorem 12.16 in the chapter Affinoid Algebras, when V is a k_H-affinoid domain, A_V is a k_H-affinoid algebra and the structure map corresponds to the inclusion of the k_H-affinoid domain. There is nothing to prove.

(2) $\implies$ (1): It suffices to show that the structure map represents the k_H-affinoid domain V. Take a k_H-affinoid algebra D and a morphism $\operatorname{Sp} D \to \operatorname{Sp} A$ of k_H-affinoid spaces that factorizes through V. We need to construct a morphism $\operatorname{Sp} D \to \operatorname{Sp} A_V$ making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} A_V \arrow[r] & \operatorname{Sp} A \end{tikzcd} .$$

Take k_H-affinoid domains $\operatorname{Sp} B_1, \ldots, \operatorname{Sp} B_n$ in $\operatorname{Sp} A$ that cover V. Let $C_i = B_i \hat{\otimes}_A D$ for $i = 1, \ldots, n$, then $\operatorname{Sp} C_i$ is a k_H-affinoid domain in $\operatorname{Sp} D$ by Corollary 12.12 in the chapter Affinoid Algebras. By Theorem 12.16 in the chapter Affinoid Algebras and general abstract nonsense, it suffices to construct the dotted arrow after restricting to $\operatorname{Sp} C_i$ for $i = 1, \ldots, n$. So we could assume that $\operatorname{Sp} D \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B_1$. From the universal property, we therefore have the dotted morphism making the following diagram commutative:

$$\begin{tikzcd} \operatorname{Sp} D \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B_1 \arrow[r] & \operatorname{Sp} A \end{tikzcd} .$$

It suffices to show that the natural homomorphism

$$B_1 \to A_V \hat{\otimes}_A B_1$$

is an isomorphism. But this follows from general abstract nonsense as B_1 is already a Banach A_V-algebra. $\square$

Remark 2.5. This proposition is not correctly stated in [Ber12, Corollary 2.2.6]. The corresponding statement in [Ber93, Remark 1.2.1] is slightly weaker than our statement.

3. The category of Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. Let X be a locally Hausdorff space and τ be a net of compact subsets. A k_H-*affinoid atlas* $\mathcal{A}$ on X with the net τ is a map which assigns

(1) to each $V \in \tau$, a k_H-affinoid algebra A_V and a homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$;
(2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of k_H-affinoid algebras $\alpha_{V/U} : A_V \to A_U$ representing a k_H-affinoid domain $\operatorname{Sp} A_U$ in $\operatorname{Sp} A_V$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Sp} A_U \arrow[r, "\operatorname{Sp} \alpha_{V/U}"] \arrow[d, "\varphi_U"] & \operatorname{Sp} A_V \arrow[d, "\varphi_V"] \\ U \arrow[r] & V \end{tikzcd} .$$

The triple $(X, \mathcal{A}, \tau)$ as above is called a k_H-*analytic space*.

A *morphism* between atlases $\mathcal{A}$ and $\mathcal{A}'$ on X with the net τ is an assignment that with each $V \in \tau$, one associates a morphism of k_H-affinoid algebras $\beta_V : A_V \to A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Sp} A'_V \arrow[r, "\operatorname{Sp} \beta_V"] \arrow[d, "\varphi'_V"] & \operatorname{Sp} A_V \arrow[dl, "\varphi_V"] \\ V & \end{tikzcd} ;$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{tikzcd} A_V \arrow[r, "\alpha_{V/U}"] \arrow[d, "\beta_V"] & A_U \arrow[d, "\beta_U"] \\ A'_V \arrow[r, "\alpha'_{V/U}"] & A'_U \end{tikzcd}$$

Here we have denoted the data associated with $\mathcal{A}'$ with a prime. In this way, the atlases on X with the net τ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), it $W \subseteq U \subseteq V$ are three sets in τ, then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

Remark 3.2. As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps φ_U's by identifying $\operatorname{Sp} A_U$ with U. We will say U is a k_H-affinoid domain in V.

Remark 3.3. Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

Lemma 3.4. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space, $U \in \tau$ and W is a k_H-affinoid domain in U. Then for any $V \in \tau$ containing W, W is a k_H-affinoid domain in V.

PROOF. As $\tau|_{U\cap V}$ is a net and W is compact, we can find $U_1,\dots,U_n \in \tau_{U\cap V}$ with $W \subseteq U_1 \cup \dots \cup U_n$. As W, U_i are k_H-affinoid domains in U, $W_i = W \cap U_i$ is a k_H-affinoid domain in U_i for all $i = 1,\dots,n$ by Corollary 12.12 in the chapter Affinoid Algebras. It follows from Corollary 9.6 and Corollary 12.12 in the chapter Affinoid Algebras that W_i and $W_i \cap W_j$ are both k_H-affinoid domains in V for $i,j = 1,\dots,n$. So W is a compact k_H-analytic domain in V.

By Proposition 2.4,

$$A_W := \ker\left(\prod_{i=1}^n A_{W_i} \to \prod_{i,j=1}^n A_{W_i\cap W_j}\right)$$

is k_H-affinoid and $\operatorname{Sp} A_W \to \operatorname{Sp} A$ induces a hoemomorphism $\operatorname{Sp} A_W \to W$ by Proposition 9.5 in the chatper Affinoid Algebras. By Proposition 2.4 again, W is affinoid in V. $\square$

Definition 3.5. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. We define $\bar{\tau}$ as the set of all $W \subseteq X$ such that there is $U \in \tau$ containing W and W is k_H-affinoid in U.

Lemma 3.6. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. Then $\bar{\tau}$ is a net on X and there is a k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$. Moreover, the k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$ is unique up to a canonical isomorphism.

PROOF. **Step 1**. We first show that $\bar{\tau}$ is a net. Let $U,V \in \bar{\tau}$ and $x \in U \cap V$. Take $U',V' \in \tau$ containing U and V respectively. Take $n \in \mathbb{Z}_{>0}$ and $W_1,\dots,W_n \in \tau$ such that

(1) $x \in W_1 \cap \dots \cap W_n$;
(2) $W_1 \cup \dots \cup W_n$ is a neighbourhood of x in $U' \cap V'$.

This is possible because $\tau|_{U'\cap V'}$ is a quasi-net by assumption.

By Lemma 3.4, U (resp. V) and $W_1,\dots,W_n$ are k_H-affinoid domains in U' (resp. V').

By Corollary 12.12 in the chapter Affinoid Algebras, $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a k_H-affinoid domain in W_i for $i = 1,\dots,n$. By Corollary 12.12 in the chapter Affinoid Algebras again, $U_i \cap V_i$ is a k_H-affinoid domain in W_i for $i = 1,\dots,n$. So $U_i \cap V_i \in \bar{\tau}|_{U\cap V}$ for $i = 1,\dots,n$. But

$$\bigcup_{i=1}^n U_i \cap V_i = (U \cap V) \cap \bigcup_{i=1}^n W_i,$$

so $\bigcup_{i=1}^n U_i \cap V_i$ is a neighbourhood of x in $U \cap V$ and $x \in \bigcap_{i=1}^n U_i \cap V_i$. It follows that $\bar{\tau}$ is a net.

Step 2. We extend the k_H-affinoid atlas $\mathcal{A}$.

For each $V \in \bar{\tau}$, we fix a $V' \in \tau$ containing V.

By Lemma 3.4, V is a k_H-affinoid domain in V'. Let $A_{V'} \to A_V$ be the morphism of k_H-affinoid algebras representing the k_H-affinoid domain V in $\operatorname{Sp} A_{V'}$. We define the homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$ as the morphism induced by $\operatorname{Sp} A_V \to \operatorname{Sp} A$.

For $U,V \in \bar{\tau}$ with $U \subseteq V$, we want to define $\alpha_{V/U} : A_V \to A_U$. We handle two cases. When $V \in \tau$, as $\tau|_{U'\cap V}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1,\dots,U_n \in \tau|_{U'\cap V}$ such that

$$U = \bigcup_{i=1}^n U_i.$$

By Lemma 3.4, $U_1, \dots, U_n$ are k_H-affinoid domains in U' and in V. By Theorem 12.16 in the chapter Affinoid Algebras,

$$A_U \xrightarrow{\sim} \ker\left(\prod_{i=1}^n A_{U_i} \to \prod_{i,j=1}^n A_{U_i\cap U_j}\right).$$

So the morphism $\alpha_{V/U_i} : A_V \to A_{U_i}$ and $A_{V/U_i\cap U_j} : \alpha_{V/U_i} : A_V \to A_{U_i\cap U_j}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ induces a morphism $\alpha_{V/U} : A_V \to A_U$. Observe that $\alpha_{V/U}$ represents the k_H-affinoid domain U in V, so it is independent of the choice of $U_1, \dots, U_n$.

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \to A_U$ representing the k_H-affinoid domain U in V', it follows that U is a k_H-affinoid domain in V, and we therefore get the desired morphism $\alpha_{V/U} : A_V \to A_U$.

It is easy to verify that the constructions gives a k_H-affinoid atlas with the net $\bar{\tau}$ extending $\mathcal{A}$. The uniqueness of the extension is immediate. □

Definition 3.7. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of

(1) a continuous map $\varphi : X \to X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
(2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq W'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd}.$$

Recall our convention Remark 3.2, the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \to A_V$ of k_H-affinoid algebras making the following diagram commutative

$$\begin{tikzcd} \mathrm{Sp}\, A_V \arrow[r] \arrow[d, "\varphi_V"] & \mathrm{Sp}\, A'_{V'} \arrow[d, "\varphi'_{V'}"] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}.$$

We will continue our identifications as in Remark 3.2 to simplify our notations.

Proposition 3.8. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \bar{\mathcal{A}}, \bar{\tau}) \to (X', \overline{\mathcal{A}'}, \overline{\tau'})$.

Proof. Let $U \in \bar{\tau}$, $U' \in \overline{\tau'}$ with $\varphi(U) \subseteq U'$. Take $V \in \tau$ and $V' \in \tau'$ containing U and U' respectively. By Lemma 3.4, U (resp. V) is a k_H-affinoid domain in V (resp. V'). Take $W \in \tau'$ with $\varphi(V) \subseteq W'$. Then in particular, $\varphi(U) \subseteq W'$. As $\tau'|_{V'\cap W'}$ is a quasi-net and $\varphi(U)$ is compact, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \dots, W_n \in \tau'|_{V'\cap W}$ such that

$$\varphi(U) \subseteq W_1 \cup \dots \cup W_n.$$

Now W_i is a k_H-affinoid domain in W' by Lemma 3.4, so $V_i := \varphi^{-1}_{V/W'}(W_i)$ is an affinoid domain in V by Corollary 12.12 in the chatper Affinoid Algebras and we have an induced morphism $V_i \to W_i$ for $i = 1, \ldots, n$. This morphism in turn induces a morphism of k_H-affinoid spaces

$$U_i := U \cap V_i \to U_i' := U' \cap W_i \to U'$$

for $i = 1, \ldots, n$. These morphisms are compatible on their intersections by construction. So by Theorem 12.16 in the chapter Affinoid Algebras, they glue together to a morphism of k_H-affinoid spectra $\bar{\varphi}_{U/U'} : U \to U'$. It is easy to see that this construction defines a strong morphism.

As for the uniqueness, it suffices to show that the morphism $U_i \to U_i'$ is uniquely determined for $i = 1, \ldots, n$. In other words, we need to show that the dotted arrow that makes the following diagram commutes is unique:

$$\begin{tikzcd} U_i \arrow[r, dotted] \arrow[d] & U_i' \arrow[d] \\ V \arrow[r, "\varphi_{V/W'}"] & W' \end{tikzcd}$$

for $i = 1, \ldots, n$. It suffices to apply the universal property of the k_H-affinoid domain $U_i' \to W'$. $\square$

Definition 3.9. Let $(X, \mathcal{A}, \tau)$, $(X', \mathcal{A}', \tau')$, $(X'', \mathcal{A}'', \tau'')$ be k_H-analytic spaces. Let

$$\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau'), \quad \psi : (X', \mathcal{A}', \tau') \to (X'', \mathcal{A}'', \tau'')$$

be strong morphisms. We will define their *composition* $\chi = \psi \circ \varphi$ as follows. The underlying map of topological spaces is just the composition of the unlerlying maps of topological spaces corresponding to ψ and φ.

Let $\bar{\varphi}$ and $\bar{\psi}$ be the extensions of φ and ψ to $\bar{\tau}$ and $\overline{\tau'}$ as in Proposition 3.8.

Given $V \in \tau$ and $V'' \in \tau''$ with $\chi(V) \subseteq V''$, we need to define a morphism of k_H-affinoid spectra $\chi_{V/V''} : V \to V''$. Take $V' \in \tau'$ and $U'' \in \tau''$ such that $\varphi(V) \subseteq V'$ and $\psi(V') \subseteq U''$. Since $\chi(V) \subseteq U'' \cap V''$ and V is compact, we can take $n \in \mathbb{Z}_{>0}$ and $V_1'', \ldots, V_n'' \in \tau''|_{U'' \cap V''}$ with $\chi(V) \subseteq V_1'' \cup \cdots \cup V_n''$. Then $V_i' := \psi^{-1}_{V'/U''}(V_i'')$ and $V_i := \varphi^{-1}_{V/V'}(V_i')$ are k_H-affinoid domains in V' and V respectively for $i = 1, \ldots, n$ and $V = V_1 \cup \cdots \cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ then induce a morphism $V_i \to V_i'' \to V$ of k_H-affinoid spectra. These morphisms are clearly compatible on the intersections and hence induce a morphism $V \to V''$ of k_H-affinoid spectra by Theorem 12.16 in the chapter Affinoid Algebras.

It is easy to verify that $\psi \circ \varphi$ is a strong morphism.

In this way, we get a category k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ of k_H-analytic spaces.

Definition 3.10. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if

(1) φ is a homeomorphism between X and X';
(2) for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$, $\operatorname{Sp} \varphi_{V/V'}$ identifies V with an affinoid domain in V'.

Lemma 3.11. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces and $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then for any $V \in \bar{\tau}$ and $V' \in \overline{\tau'}$, the intersection $V \cap \varphi^{-1}(V')$ is a compact k_H-analytic domain in V.

Proof. Take $U' \in \overline{\tau'}$ with $\varphi(V) \subseteq U'$. As $\tau|_{U'\cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U'_1, \ldots, U'_n \in \tau|_{U'\cap V'}$ with $\varphi(V) \subseteq U'_1 \cup \cdots \cup U'_n$ and

$$V \cap \varphi^{-1}(V') = \bigcup_{i=1}^{n} \varphi^{-1}_{V/U}(U'_i).$$

□

Lemma 3.12. The system of quasi-isomorphisms in k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ is a right multiplicative system.

For the notion of right multiplicative system, we refer to [Stacks, Tag 04VC].

Proof. We verify the three axioms as in [Stacks, Tag 04VC].

RMS1. The identity is clear a quasi-isomorphism. It remains to verify that the composition of quasi-isomorphisms is still a quasi-isomorphism.

We take φ, ψ as in Definition 3.9. We will use the same notations as in Definition 3.9. We need to show that $V \to V''$ identifies V with a k_H-affinoid domain in V''. From the construction, we know that φ identifies V_i with a k_H-affinoid domain in V'_i and ψ identifies V'_i with a k_H-affinoid domain in V''_i for $i = 1, \ldots, n$. In particular, $\chi(V)$ is a compact k_H-analytic domain in V''. It follows from Proposition 2.4 that $\chi(V)$ is a k_H-affinoid domain in V''.

RMS2. If $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ and $f : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ are given strong morphisms of k_H-analytic spaces and g is a quasi-isomorphism, then there are k_H-analytic space $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau})$ and strong morphisms $\tilde{\varphi} : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ and $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ such that f is a quasi-isomorphism and the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \arrow[r, dotted, "\tilde{\varphi}"] \arrow[d, dotted, "f"] & (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \arrow[d, "g"] \\
(X, \mathcal{A}, \tau) \arrow[r, "\varphi"] & (X', \mathcal{A}', \tau')
\end{tikzcd}.$$

We may assume that $\widetilde{X'} = X'$. Then $\widetilde{\tau'} \subseteq \overline{\tau'}$. We let $\tilde{X} = X$. Let $\tilde{\tau}$ be the family of all $V \in \bar{\tau}$ for which there is $\widetilde{V'} \in \widetilde{\tau'}$ with $\varphi(V) \subseteq \widetilde{V'}$. By Lemma 3.11, $\tilde{\tau}$ is a net on $\tilde{X}$. The k_H-atlas $\bar{\mathcal{A}}$ defines a k_H-affinoid atlas $\tilde{\mathcal{A}}$ with the net $\tilde{\tau}$. The strong morphism $\bar{\varphi}$ induces $\tilde{\varphi}$. The morphism f is the canonical quasi-isomorphism. It is immediate that these constructions satisfy the desired conditions.

RMS3. If $\varphi, \psi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ are strong morphisms of k_H-analytic spaces and there is a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ of k_H-analytic spaces such that $g \circ \varphi = g \circ \psi$, then there is a quasi-isomorphism $f : (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi \circ f = \psi \circ f$.

We will in fact show that $\varphi = \psi$. It is clear that they coincide as maps of topological spaces. Let $V \in \tau$, $V' \in \tau'$ such that $\varphi(V) \subseteq V'$. Take $\widetilde{V'} \in \widetilde{\tau'}$ with $g(V') \subseteq \widetilde{V'}$. Then we have two morphisms of k-affinoid spectra $\varphi_{V/V'}, \psi_{V/V'} : V \to V'$ such that their compositions with $g_{V'/\widetilde{V'}}$ coincide. As V' is an affinoid domain in $\widetilde{V'}$, it follows that $\varphi_{V/V'} = \psi_{V/V'}$ by the universal property. □

Definition 3.13. The category k_H-$\mathcal{A}\mathrm{n}$ is the right category of fractions of k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ with respect to the system of quasi-isomorphisms. A morphism in k_H-$\mathcal{A}\mathrm{n}$ is called a *morphism* between k_H-analytic spaces.

We refer to [Stacks, Tag 04VB] for the definition of right category of fractions. For later references, we explicitly write down the morphisms in k_H-$\mathcal{A}$n.

Lemma 3.14. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of k_H-analytic spaces. We define a partial order on the set of nets on X: $\tau_1 \preceq \tau_0$ if $\tau_1 \subseteq \overline{\tau_0}$. Then the set of nets is a directed set and

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}\left((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')\right) = \varinjlim_{\sigma \preceq \tau} \mathrm{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}}\left((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')\right)$$

in the category of sets, where $\mathcal{A}_\sigma$ is induced by $\overline{\mathcal{A}}$. The transition maps are all injective.

PROOF. This follows immediately from the definition. $\square$

Definition 3.15. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. We say a subset $W \subseteq X$ is *τ-special* if it is compact and there exist $n \in \mathbb{Z}_{>0}$ and a covering $W = W_1 \cup \cdots W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$ and the natural map

$$A_{W_i} \hat{\otimes}_k A_{W_j} \to A_{W_i \cap W_j}$$

is an admissible epimorphism.

The covering $W_1, \ldots, W_n$ is called a *τ-special covering* of W.

Lemma 3.16. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space and W be a τ-special subset of X. If $U, V \in \tau|_W$, then $U \cap V \in \bar{\tau}$ and the natural map

$$A_U \hat{\otimes}_k A_V \to A_{U \cap V}$$

is an admissible epimorphism.

PROOF. Let $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n$ be a τ-special covering of W. As $U \cap W_i$ and $V \cap W_i$ are compact for $i = 1, \ldots, n$, we can find $m_i \in \mathbb{Z}_{>0}$ (resp. $s_i \in \mathbb{Z}_{>0}$) and finite coverings $U_{i1}, \ldots, U_{im_i} \in \tau$ of $U \cap W_i$ (resp. $V_{i1}, \ldots, V_{ik_i} \in \tau$ of $V \cap W_i$).

Observe that $U_{ik} \cap V_{jl}$ is a k_H-affinoid domain in $U \cap V$, hence $U_{ik} \cap V_{jl} \in \bar{\tau}$ for any $i, j = 1, \ldots, n$, $k = 1, \ldots, m_i$ and $l = 1, \ldots, k_l$. Observe that $U_{ik} \cap V_{jl} \to U_{ik} \times V_{jl}$ is a closed immersion as $W_i \cap W_j \to W_i \times W_j$ is by our assumption. Consider the finite convering

$$\{U_{ik} \times V_{jl} : i, j = 1, \ldots, n; k = 1, \ldots, m_i; l = 1, \ldots, k_l\}$$

of $U \times V$. For each tuple (i, j, k, l), $A_{U_{ik} cap V_{jl}}$ is a finite $A_{U_{ik} \times V_{jl}}$-algebra. By Theorem 13.1 in the chapter Affinoid Algebras, we can construct a finite $A_{U \times V}$-algebra $A_{U \cap V}$ inducing all of these $A_{U_{ik} \cap V_{jl}}$'s.

To be continued

$\square$

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[Stacks] T. Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`. 2020.