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Contents

Topology and bornology

1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A comapct space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

A connected topological is always non-empty.

References to this chapter include [Ber93].

2. Nets

Let X be a set, $Y \subseteq X$ be a subset. Consider a collection τ of subsets of X, we write

$$\tau|_Y := \{V \in \tau : V \subseteq Y\} .$$

Definition 2.1. Let X be a topology space and τ be a collection of subsets of X. We say τ is

(1) *dense* if for any $V \in \tau$ and any $x \in V$, there is a fundamental system of neighbourhoods of x in V consisting of sets from $\tau|_V$;
(2) a *quasi-net* on X if for each $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ such that $x \in V_1 \cap \cdots \cap V_n$ and that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X;
(3) a *net* on X if it is a quasi-net and if for any $U, V \in \tau$, $\tau|_{U\cap V}$ is a quasi-net on $U \cap V$;
(4) *locally finite* if for any $x \in X$, there is a neighbourhood U of x in X such that $\{V \in \tau : V \cap U \neq \emptyset\}$ is finite.

We observe that if τ is a net, $\tau|_{U\cap V}$ is in fact a net.

Lemma 2.2. Let X be a topological space and τ be a quasi-net on X.

(1) A subset $U \subseteq X$ is open if and only if for each $V \in \tau$, $U \cap V$ is open in V.
(2) Suppose that τ consists of compact sets. Then X is Hausdorff if and only if for any $U, V \in \tau$, $U \cap V$ is compact.

We remind the readers that a compact space is Hausdorff by our convention.

Proof. (1) The direct implication is trivial. Suppose that $U \cap V$ is open in V for all $V \in \tau$. We want to show that U is open. Take $x \in U$, we can find $n \in \mathbb{Z}_{>0}$, $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X. By our hypothesis, we can find open sets $W_1, \ldots, W_n$ in W such that $W \cap V_i = U \cap V_i$

for $i = 1, \ldots, n$. Then $W = W_1 \cap \cdots \cap W_n$ is an open neighbourhood of x in X. But then

$$U \cap (V_1 \cup \cdots \cup V_n) \supseteq W \cap (V_1 \cup \cdots \cup V_n),$$

the latter is a neighbourhood of x hence so is the former. It follows that U is open.

(2) The direct implication is trivial. Consider the quasi-net $\tau \times \tau := \{U \times V : U, V \in \tau\}$ on $X \times X$. By (1), it suffices to verify that the intersection of the diagonal with $U \times V$ is closed in $U \times V$ for any $U, V \in \tau$. But this intersection is homeomorphic to $U \cap V$, which is compact by our assumption and hence closed as U, V are both Hausdorff. □

Lemma 2.3. Let X be a Hausdorff space. Assume that X admits a quasi-net τ consisting of compact sets. Then X is locally compact.

Proof. Take $x \in X$. By assumption, we can find $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \tau$ all containing x such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x. This neighbourhood is clearly compact. □

Lemma 2.4. Let X be a Hausdorff space and τ be a collection of compact subsets of X. Then the following are equivalent:

(1) τ is a quasi-net;
(2) For each $x \in X$, there are $n \in \mathbb{N}$ and $V_1, \ldots, V_n \in \tau$ such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in X.

Proof. (1) $\implies$ (2): This is trivial.

(2) $\implies$ (1): Given $x \in X$, take $V_1, \ldots, V_n$ as in (2). We may assume that $x \in V_1, \ldots, V_m$ and $x \notin V_{m+1}, \ldots, V_n$ for some $1 \leq m \leq n$. Then $V_1 \cup \cdots \cup V_m$ is a neighbourhood of x in X: if U is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_n$, then $U \setminus (V_{m+1} \cup \cdots \cup V_n)$ is an open neighbourhood of x in X contained in $V_1 \cup \cdots \cup V_m$. □

Lemma 2.5. Let X be a topological space and τ be a net on X consisting of compact sets. Then

(1) for any pair $U, V \in \tau$, the intersection $U \cap V$ is locally closed in U and in V;
(2) If $n \in \mathbb{Z}_{>0}$, $V, V_1, \ldots, V_n \in \tau$ are such that
$$V \subseteq V_1 \cup \cdots \cup V_n,$$
then there are $m \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_m \in \tau$ such that
$$V = U_1 \cup \cdots \cup U_m$$
and each U_j is contained in some V_i.

Proof. (1) It suffices to show that $U \cap V$ is locally compact in the induced topology. This follows from Lemma 2.3.

(2) For each $x \in V$ and each $i = 1, \ldots, n$ such that $x \in V_i$, we take a neighbourhood of x in $V \cap V_i$ of the form $W_i V_{i1} \cup \cdots \cup V_{im_i}$ for some $m_i \in \mathbb{Z}_{>0}$ and $V_{ij} \in \tau$ for $j = 1, \ldots, m_i$. Then the union of all W_i's is a neighbourhood of x of the form $U_1 \cup \cdots U_m$, where U_j belongs to τ and is contained in some V_i. Using the compactness of V, we conclude. □

3. Paracompact spaces

Definition 3.1. A topological space X is *paracompact* if any open covering of X admits a locally finite refinement.

A paracompact space is not necessarily Hausdorff according to our definition.

Proposition 3.2. Let X be a locally compact topological space.

(1) Assume that each connected component of X is σ-compact, then X is paracompact.
(2) If X is paracompact and Hausdorff, then each connected component of X is σ-compact.

If the conditions in (2) are satisfied, for any basis of neighbourhoods $\mathcal{B}$ of X, every open covering $\mathcal{U}$ of X can be refined into a locally finite covering $\mathcal{V}$ consisting of elements in $\mathcal{B}$.

We do not assume that the elements in $\mathcal{B}$ be open. The covering $\mathcal{V}$ is not necessarily open.

Proposition 3.3. Let X be a paracompact space and $Y \subseteq X$ be a closed subspace. Then Y is paracompact.

Proposition 3.4. Let X be a locally compact Hausdorff space and $Y \subseteq X$ be a subspace, then the following are equivalent:

(1) Y is locally compact and Hausdorff;
(2) Y is a locally closed subspace of X.

4. Closed maps and topologically finite maps

Definition 4.1 ([Stacks, Tag 004E],[Stacks, Tag 0CY1])**.** A map $f : X \to Y$ of topological spaces is *closed* if for each closed subset Z in X, $f(Z)$ is closed in Y.

A map $f : X \to Y$ of topological spaces is *separated* if it is continuous and the diagonal map $\Delta : X \to X \times_Y X$ is closed.

A closed map is not necessarily continuous.

Lemma 4.2. Let $f : X \to Y$ be a closed map of topological spaces, then for each $y \in Y$ and any open neighbourhood U of $f^{-1}(y)$ in X, there is an open neighbourhood V of y in Y such that $f^{-1}(V) \subseteq U$.

Proof. It suffices to take $V = Y \setminus f(X \setminus U)$, □

Lemma 4.3. Let $f : X \to Y$ be a closed map of topological spaces. Then for any subspace V of Y, the map $f^{-1}(V) \to V$ induced by f is closed.

Proof. Let A be a closed subset of $U := f^{-1}(V)$. We need to show that $f(A)$ is closed in V. Choose a closed subset B of X such that $A = B \cap U$, then $f(B)$ is closed in Y and $f(A) = f(B) \cap V$ is closed in V. □

Definition 4.4. A $f : X \to Y$ of topological spaces is *topologically finite* if

(1) f is separated and closed;
(2) for each $y \in Y$, the set $f^{-1}(y)$ is finite.

A map $f : X \to Y$ of topological spaces is *topologically finite at* $x \in X$ if there is an open neighbourhood U of x in X and an open neighbourhood V of $f(x)$ in Y such that $f(U) \subseteq V$ and the induced map $U \to V$ is topologically finite.

Proposition 4.5. Let $f : X \to Y$ be a map of topological spaces. Then the following are equivalent:

(1) f is topologically finite;
(2) f is proper and all fibers of f are discrete.

Here the properness is defined as in [Stacks, Tag 005O]. In particular, a proper map is always separated and hence continuous.

Proof. Assume that f is topologically finite. As the fibers of f are finite and Hausdorff, they are discrete. We need to show that f is proper. This follows from [Stacks, Tag 005R].

Conversely, assume that f is proper with discrete fibers. By [Stacks, Tag 005R] again, the fibers of f are compact and hence finite. The map f is closed and separated as it is proper. So (1) follows. □

Lemma 4.6. Let $f : X \to Y$ be a continuous map between topologically spaces. Assume that Y is Hausdorff. Let W be an open relative quasi-compact subset of X, then the map

$$W \setminus f^{-1}(f(\partial W)) \to Y \setminus f(\partial W)$$

induced by f is proper.

Proof. It is well-known that $f|_{\bar{W}} : \bar{W} \to Y$, as a continuous map from a quasi-compact space to a Hausdorff space is proper. The map in the lemma is a base change of the given map, hence is also proper. We apply [Stacks, Tag 005R]. □

Proposition 4.7. Let $f : X \to Y$ be a topologically finite map of topological spaces. Then for any subspace $V \subseteq Y$, the map $f^{-1}(V) \to V$ induced by f is topologically finite.

Proof. This follows immediately from Lemma 4.3. □

Theorem 4.8. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $y \in f(X)$ and $x_1, \ldots, x_n$ ($n \in \mathbb{Z}_{>0}$) denote the distinct points of $f^{-1}(y)$. Take pairwise disjoint open neighbourhoods $U'_1, \ldots, U'_n$ of $x_1, \ldots, x_n$ in X. Then any neighbourhood V' of y in Y contains an open neighbourhood V of y satisfying the following conditions:

(1) $U_1 := f^{-1}(V) \cap U'_1, \ldots, U_n := f^{-1}(V) \cap U'_n$ are pairwise disjoint open neighbourhoods of $x_1, \ldots, x_n$ in X;
(2) $f^{-1}V = \bigcup_{j=1}^n U_j$;
(3) The maps $U_j \to V$ for $j = 1, \ldots, n$ induced from f are all topologically finite.

Let $\mathcal{F}$ be a sheaf of sets on X, then we have a functorial bijection

$$f_*\mathcal{F}(V) \xrightarrow{\sim} \prod_{j=1}^n \mathcal{F}(U_j).$$

The existence of $U'_1, \ldots, U'_n$ is guaranteed by [Stacks, Tag 0CY2].

Proof. As $\bigcup_{j=1}^n U'_j$ is an open neighbourhood of $f^{-1}(y)$ in X, by Lemma 4.2 and Lemma 4.3, we can find an open neighbourhood $V \subseteq V'$ of y in Y such that

$$f^{-1}V \subseteq \bigcup_{j=1}^n U'_j.$$

The conditions (1) and (2) are therefore satisfied.

In order to prove (3), it remains to show that the induced maps $U_j \to V$ are closed for $j = 1, \ldots, n$. We may take $j = 1$. Let A be a closed subset of U_1. Then A is closed in $f^{-1}(V)$ by (1) and (2). It follows that $f(A)$ is closed in V by Lemma 4.3.

The last assertion follows from (1) and (2). □

Corollary 4.9. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $x \in X$ be U' be an open neighbourhood of x in X such that all other points in $f^{-1}(f(x))$ are in the interior of $X \setminus U'$. Then any neighbourhood V' of $f(x)$ in Y contains an open neighbourhood V of y such that for $U := f^{-1}(V) \cap U'$ the map $g : U \to V$ induced by f is topologically finite and $g^{-1}(g(x)) = \{x\}$.

PROOF. This follows immediately from Theorem 4.8. □

Corollary 4.10. Let $f : X \to Y$ be a topologically finite map of topological spaces. Let $\mathcal{F}$ be a sheaf of sets on X, $y \in f(X)$. Denote by $x_1, \ldots, x_n$ ($n \in \mathbb{Z}_{>0}$) the distinct points of the fiber $f^{-1}(y)$. Then we have a canonical bijection

$$(f_*\mathcal{F})_y \xrightarrow{\sim} \prod_{j=1}^{n} \mathcal{F}_{x_j}.$$

In particular, $f_* : \mathcal{A}\mathrm{b}(X) \to \mathcal{A}\mathrm{b}(Y)$ is exact.

PROOF. This follows immediately from Theorem 4.8. □

5. Exhaustion

Definition 5.1. Let X be a Hausdorff space. A *compact exhaustion* of X is a sequence of compact sets $(K_i)_{i \in \mathbb{Z}_{>0}}$ in X such that

(1) For each $i \in \mathbb{Z}_{>0}$,
$$K_i \subseteq \operatorname{Int} K_{i+1};$$

(2)
$$X = \bigcup_{i=1}^{\infty} K_i.$$

Proposition 5.2. Let X be a locally compact Hausdorff topological space admitting a countable basis, then X admits a compact exhaustion.

Note that in the book of Grauret–Remmert, the condition of being Hausdorff is omitted.

PROOF. Include a proof □

Lemma 5.3. Let X be a paracompact Hausdorff topological space and $\mathcal{F}$ be a sheaf of Abelian groups on X. Let $q \in \mathbb{Z}_{\geq 2}$ and $(K_i)_{i \in \mathbb{Z}_{>0}}$ be a compact exhaustion of X with the following property:

$$H^{q-1}(K_i, \mathcal{F}) = H^q(K_i, \mathcal{F}) = 0$$

for all $i \in \mathbb{Z}_{>0}$. Then $H^q(X, \mathcal{F}) = 0$.

PROOF. Grauert–Remmert P103. □

6. Maps with discrete fibers

Lemma 6.1. Let X be a locally connected locally compact Hausdorff topological space and X_0 be a Hausdorff space with a basis β_0. Consider a continuous map $f : X \to X_0$ with discrete fiber. Then there is a basis of X made up of connected components of $f^{-1}U_0$ with $U_0 \in \beta_0$.

PROOF. Let $x \in X$ and V be an open neighbourhood of x in X. We need to find $U_0 \in \beta_0$ and a component U of $f^{-1}(U_0)$ such that $U \subseteq V$.

For this purpose, we may assume that X is connected. Set $x_0 = f(x)$. Choose an open neighbourhood W of x in V with $\bar{W}$ compact and $B \cap f^{-1}(x_0) = \emptyset$, where $B = \bar{W} \setminus W$. Let $B_0 = f(B)$, then $x_0 \notin B_0$. As B_0 is compact, we can find $U_0 \in \beta_0$ containing x_0 such that $B_0 \cap U_0 = \emptyset$. Let U be the connected component of $f^{-1}(U_0)$ containing x. Then $B \cap U = \emptyset$ and hence $U \subseteq W \cup (X \setminus \bar{W})$. As X is connected and $W \cap U$ is non-empty, we find that $U \subseteq W$. □

Proposition 6.2. Let X be a connected, locally connected, first countable, locally compact Hausdorff space and X_0 be a topological space with countable basis. If there is a map $f : X \to X_0$ with discrete fibers, then X has countable topology as well.

This result is proved in [Jur59].

PROOF. Let β_0 be a countable basis for the topology on X_0. Let β be the collection of open sets U in X such that

(1) There is $U_0 \in \beta_0$ such that U is a connected component of $f^{-1}(U_0)$;
(2) U is relatively compact in X.

By our assumption, any $U \in \beta$ has countable basis. By Lemma 6.1, β is a basis for the topology on X. It remains to show that β is countable.

Let $V \in \beta$. For each $n \in \mathbb{Z}_{>0}$, $\beta^{(n)}$ denotes the collection of $U \in \beta$ with the following property: there is a map $\{1, \ldots, n\} \to \beta$, say assigning $U_i \in \beta$ to i such that $U_1 = V$, $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$. As X is connected,

$$\beta = \bigcup_{n=1}^{\infty} \beta^{(n)}.$$

It remains to show that for each $n \in \mathbb{Z}_{>0}$, $\beta^{(n)}$ is countable. We make an induction. The case $n = 1$ is obvious. Assume that $n \geq 2$ and the assertion has been proved for $n-1$. Let $U_0 \in \beta_0$ and $U' \in \beta^{(n-1)}$. Let $\alpha^{(n)}(U_0, U')$ denote the collection of $U \in \beta^{(n)}$ such that U is a connected component of $f^{-1}(U_0)$ and $U \cap U'$ is non-empty. Then

$$\beta^{(n)} = \bigcup_{U_0 \in \beta_0, U' \in \beta^{(n-1)}} \alpha^{(n)}(U_0, U').$$

But each $\alpha_{(n)}(U_0, U')$ is countable as U' has countable basis. It follows that $\beta^{(n)}$ is countable. □

7. Bornology

Definition 7.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

(1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;

(3) $\mathcal{B}$ is stable under finite union.

The pair $(X, \mathcal{B})$ is called a *bornological set*. The elements of $\mathcal{B}$ are called the *bounded subsets* of $(X, \mathcal{B})$. When $\mathcal{B}$ is obvious from the context, we omit it from the notations.

A morphism between bornological sets $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ is a map of sets $f : X \to Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 7.2. Let $(X, \mathcal{B})$ be a bornological set. A *basis* for $\mathcal{B}$ is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \cdots \cup A_n$.

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