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Global properties of complex analytic spaces

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Definition 2.1. Let X be a complex analytic space. We say X is *holomorphically separable* if for any $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

Here we regard f as a continuous function $X \to \mathbb{C}$. In particular, a holomorphically separable space is Hausdorff.

Definition 2.2. Let X be a complex analytic space. We say X is *holomorphically spreadable* if for any $x \in X$, we can find an open neighbourhood U of x in X such that

$$\{y \in U : f(x) = f(y) \text{ for all } f \in \mathcal{O}_X(X)\} = \{p\}.$$

A holomorphically separable space is clearly holomorphically spreadable.

Definition 2.3. Let X be a complex analytic space. We say X is *holomorphically convex* if $|X|$ is Hausdorff and for any compact set $K \subseteq X$, the set

$$\hat{K}^X := \left\{ x \in X : |f(x)| \leq \sup_{y \in K} |f(y)| \text{ for all } f \in \mathcal{O}_X(X) \right\}$$

is compact.

We say X is *weakly holomorphically convex* if for any quasi-compact set $K \subseteq X$, the connected components of $\hat{K}^X$ are all quasi-compact.

Proposition 2.4. Let X be a Hausdorff complex analytic space. Consider the following conditions:

(1) X is holomorphically convex;
(2) For any sequence $x_i \in X$ ($i \in \mathbb{N}$) without accumulation points, there is $f \in \mathcal{O}_X(X)$ such that $|f(x_i)|$ is unbounded.

Then (1) $\implies$ (2). The converse is true if X is Lindelöf.

Proof. (1) $\implies$ (2): Recall that a complex analytic space is always first countable. For a first countable Hausdorff space, compactness implies sequential compactness.

(2) $\implies$ (1): For a Lindelöf Hausdorff space, sequential compactness implies compactness. $\square$

Definition 2.5. Let X be a complex analytic space. We say X is *Stein* if it is holomorphically separable and holomorphically convex.

[Stacks]

Bibliography

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.