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Contents

Morphisms between complex analytic spaces	5
1. Introduction	5
2. Open morphism	5
3. Quasi-finite morphisms	5
4. Finite morphisms	5
5. Flat morphisms	18
Bibliography	21

Morphisms between complex analytic spaces

1. Introduction

2. Open morphism

Definition 2.1. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. We say f is *open* at $x \in X$ if for any neighbourhood U of x in X , $f(U)$ is a neighbourhood of $f(x)$ in Y .

Proposition 2.2. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. Assume that f is open at $x \in X$, then the kernel of $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is nilpotent.

The converse fails.

PROOF. Let $g_{f(x)} \in \mathcal{O}_{Y,f(x)}$ be an element in the kernel of $f_x^\#$. Up to shrinking Y , we may spread $g_{f(x)}$ to $g \in \mathcal{O}_Y(Y)$. Then f^*g vanishes in a neighbourhood of x in X . As f is open at x , g vanishes in the neighbourhood $f(U)$ of $f(x)$. By [Corollary 3.18](#) in [Constructions of complex analytic spaces](#), $g_{f(x)}$ is nilpotent. \square

3. Quasi-finite morphisms

Definition 3.1. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. We say f is *quasi-finite* at $x \in X$ if x is isolated in $f^{-1}(f(x))$. We say f is *quasi-finite* if f is quasi-finite at all $x \in X$.

This definition is purely topological. We will show that it is equivalent to an analytic definition.

Proposition 3.2. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

- (1) f is quasi-finite at $x \in X$;
- (2) $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$;
- (3) $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$.

PROOF. (1) \Leftrightarrow (2): By [Corollary 3.16](#) in [Constructions of complex analytic spaces](#), f is quasi-finite at $x \in X$ if and only if $\mathcal{O}_{X_{f(x)},x} = \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is artinian. In other words, $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is finite-dimensional over \mathbb{C} . The latter is equivalent to that $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.

(2) \Leftrightarrow (3): This follows from [Theorem 5.4](#) in [Complex analytic local algebras](#). \square

4. Finite morphisms

Definition 4.1. A morphism of complex analytic spaces $f : X \rightarrow Y$ is *finite* if its underlying map of topological spaces is topologically finite.

We say a morphism of complex analytic spaces $f : X \rightarrow Y$ is *finite at* $x \in X$ if there is an open neighbourhood U of x in X and V of $f(x)$ in Y such that $f(U) \subseteq V$ and the restriction $U \rightarrow V$ of f is finite.

Let S be a complex analytic space. A *finite analytic space over* S is a finite morphism $f : X \rightarrow S$ of complex analytic spaces. A morphism between finite analytic spaces over S is a morphism of complex analytic spaces over S .

Proposition 4.2. Let $n \in \mathbb{N}$ and D be an open neighbourhood of 0 in \mathbb{C}^n . Let X be a closed subspace of D which intersects $\{(0, \dots, 0)\} \times \mathbb{C}$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that the projection $B \times W \rightarrow B$ induces a finite morphism $h : X' \rightarrow B$ with $X' = X \cap (B \times W)$.

PROOF. We will denote the coordinates on $\mathbb{C}^{n-1} \times \mathbb{C}$ as (z, w) .

Let \mathcal{I} be the ideal of X in D . By our assumption, we can choose $f_0 \in \mathcal{I}_0$ such that $\deg_w f_0 < \infty$ and $f_0(0) = 0$. By [Theorem 4.3 in Complex analytic local algebras](#), we can find a Weierstrass polynomial $\omega_0 = w^b + a_1 w^{b-1} + \dots + a_b \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[w]$ such that $f_0 = e\omega_0$ for some unit e in $\mathbb{C}\{z_1, \dots, z_n\}$. We choose a product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that ω_0 can be represented by $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)[w]$ with $\omega|_{B \times W} \in \mathcal{I}(B \times W)$. Let $\pi : A \rightarrow B$ be the Weierstrass map defined by ω . Then π is finite by [Theorem 6.2 in The notion of complex analytic spaces](#). Up to shrinking B and W , we may assume that $A \cap (B \times W) \rightarrow B$ is finite as well. Set $X' := X \cap (B \times W)$. The restriction $h : X' \rightarrow B$ of π is then finite. \square

Corollary 4.3. Let $n, k \in \mathbb{N}$ and D be an open neighbourhood of 0 in \mathbb{C}^n . Let X be a closed subspace of D which intersects $\{(0, \dots, 0)\} \times \mathbb{C}^k$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-k} \times \mathbb{C}^k$ of 0 in D such that the projection $B \times W \rightarrow B$ induces a finite morphism $h : X' \rightarrow B$ with $X' = X \cap (B \times W)$.

PROOF. This follows from a repeated application of [Proposition 4.2](#). \square

Proposition 4.4. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

- (1) f is quasi-finite at x ;
- (2) f is finite at x .

PROOF. (2) \implies (1): This follows from [Proposition 4.5 in Topology and bornology](#).

(1) \implies (2): Write $y = f(x)$. The assertion is local on both X and Y . So we may assume that U and V are complex model spaces in domains $W \subseteq \mathbb{C}^k$ and $B \subseteq \mathbb{C}^d$ respectively with $x = 0$ and $y = 0$. Moreover, we may assume that $\{x\} = f'^{-1}(y)$. We have the following commutative diagram:

$$\begin{array}{ccc}
 & U \times V & \hookrightarrow W \times B \\
 & \uparrow & \uparrow \\
 U & \xrightarrow{\Gamma_f} & \\
 & \downarrow f' & \downarrow \\
 & V & \hookrightarrow B
 \end{array} ,$$

where $\Gamma_{f'}$ denotes the graph of $f' : U \rightarrow V$. As $\{x\} = f'^{-1}(y)$, we have $\mathbb{C}^k \times \{0\}$ intersects Γ_f only at the origin. By [Corollary 4.3](#), up to shrinking W and B , we may guarantee that the projection $W \times B \rightarrow B$ induces a finite morphism $\Gamma_f \rightarrow B$ and the pushforward under this map preserves coherence. Observe that $U \rightarrow \Gamma_f$ is a biholomorphism, we conclude that f' is finite. \square

Corollary 4.5. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. The following are equivalent:

- (1) f is finite;
- (2) f is quasi-finite and proper.

PROOF. (1) \implies (2): This follows from [Proposition 4.4](#).

(2) \implies (1): This follows from [Proposition 4.5](#) in [Topology and bornology](#). \square

Corollary 4.6. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces. Then the set

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is open.

PROOF. This follows from [Proposition 4.4](#). \square

Theorem 4.7. Let S be a complex analytic space. Then the functor $\mathrm{Spec}_S^{\mathrm{an}}$ defines an anti-equivalence from the category of finite \mathcal{O}_S -algebras to the category of finite analytic spaces over S .

PROOF. We first observe that the functor is well-defined. This follows from [Corollary 3.8](#) in [Constructions of complex analytic spaces](#).

The functor is fully faithful by [Proposition 2.10](#) in [Constructions of complex analytic spaces](#). Suppose that $f : X \rightarrow S$ is a finite morphism of complex analytic spaces. We need to show that X is isomorphic to $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{A}$ for some finite \mathcal{O}_S -algebra \mathcal{A} in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}/_S$.

By [Proposition 2.8](#) in [Constructions of complex analytic spaces](#), we necessarily have $\mathcal{A} \cong f_* \mathcal{O}_X$. So we need to show that the natural morphism $\mathrm{Spec}_S^{\mathrm{an}} f_* \mathcal{O}_X \rightarrow X$ over S is an isomorphism. The problem is local on S .

Fix $s \in S$. Write x_1, \dots, x_n for the distinct points in $f^{-1}(s)$. Up to shrinking S , we may assume that X is the disjoint union of V_1, \dots, V_n , where V_i is an open neighbourhood of x_i in X . We need to show that X has the form $\mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}$ for some \mathcal{O}_S -algebra \mathcal{B} in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}/_S$.

It suffices to handle each V_i separately, so we may assume that $f^{-1}(s) = \{x\}$ consists of a single point. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by [Proposition 3.2](#). Up to shrinking S , we may assume that $\mathcal{O}_{X,x}$ spreads out to a finite \mathcal{O}_S -algebra \mathcal{B} . Let $X' = \mathrm{Spec}_S^{\mathrm{an}} \mathcal{B}$. There is a unique point x' of X' over s and $X'_{x'}$ is isomorphic to X_x over S_s . By [Lemma 4.2](#) in [Topology and bornology](#), up to shrinking S , we may assume that X is isomorphic to X' over S . We conclude. \square

Corollary 4.8. Let $f : X \rightarrow Y$ be a finite morphism of complex analytic spaces and \mathcal{M} be a coherent sheaf of \mathcal{O}_X -modules, then $f_* \mathcal{M}$ is coherent. Moreover, f_* is exact from $\mathrm{Coh}(\mathcal{O}_X)$ to $\mathrm{Coh}(\mathcal{O}_Y)$.

PROOF. This follows from [Corollary 2.9](#) in [Constructions of complex analytic spaces](#) and [Theorem 4.7](#). \square

Corollary 4.9. Let X be a reduced complex analytic space. Then

- (1) \bar{X} is normal;
- (2) $p : \bar{X} \rightarrow X$ is finite and surjective;
- (3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in \bar{X} and the morphism $\bar{X} \setminus p^{-1}(Y) \rightarrow X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines \bar{X} up to a unique isomorphism in $\mathbb{C}\text{-An}/X$.

PROOF. These properties are established in [Proposition 7.8](#) in [Local properties of complex analytic spaces](#). We need to prove the uniqueness.

Let $p : X' \rightarrow X$ be a morphism satisfying the three conditions. We need to show that X' is canonically isomorphic to \bar{X} in $\mathbb{C}\text{-An}/X$. By (2) and [Theorem 4.7](#), it suffices to show that $p_*\mathcal{O}_{X'}$ is canonically isomorphic to $\bar{\mathcal{O}}_X$. By (1), and the universal property of normalization, there is a canonical morphism

$$p_*\mathcal{O}_{X'} \rightarrow \bar{\mathcal{O}}_X$$

of \mathcal{O}_X -algebras. We will show that this map is an isomorphism.

The problem is local. Let $x \in X$. By (3) and [Corollary 3.14](#) in [Constructions of complex analytic spaces](#), up to shrinking X , we can find $f \in \mathcal{O}_X(X)$ such that $f(y) = 0$ for all $y \in Y$ and f_x is a non-zero divisor in $(p_*\mathcal{O}_{X'})_x$. Up to shrinking X , we may assume that f_y is a non-zero divisor in $(p_*\mathcal{O}_{X'})_y$ for all $y \in X$. By (3), we have

$$\mathcal{O}_X|_{X \setminus Y} \rightarrow (p_*\mathcal{O}_{X'})|_{X \setminus Y}$$

is an isomorphism. It follows that

$$fp_*\mathcal{O}_{X'} \rightarrow \mathcal{O}_X$$

is injective. We then have an injective homomorphism:

$$p_*\mathcal{O}_{X'} \rightarrow \mathcal{O}_X \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we deduce that $(p_*\mathcal{O}_{X'})_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}[f_y^{-1}]$. But $(p_*\mathcal{O}_{X'})_y$ is finite and integral over $\mathcal{O}_{X,y}$, so is isomorphic to $\bar{\mathcal{O}}_{X,y}$ as $\mathcal{O}_{Y,y}$ -algebras. \square

Corollary 4.10. Let $f : X \rightarrow Y$ be a finite morphism of complex analytic spaces. Assume that $x \in X$ is a point such that $(f_*\mathcal{O}_X)_{f(x)}$ is torsion-free as an $\mathcal{O}_{Y,f(x)}$ -module and Y is integral at $f(x)$. Then f is open at x .

PROOF. If not, we can choose open neighbourhoods U of x in X and V of $y := f(x)$ in Y such that $f(U) \subseteq V$ such that the induced morphism $g : U \rightarrow V$ is finite and $f(U)$ is not a neighbourhood of y in Y . Up to shrinking Y , we can find $h \in \mathcal{O}_Y(Y)$ such that $h_y \neq 0$ while h vanishes on $f(U)$. Observe that $f(U)$ is an analytic set in Y by [Corollary 4.8](#). It follows from [Corollary 3.18](#) in [Constructions of complex analytic spaces](#) that there is $t \in \mathbb{Z}_{>0}$ such that

$$h_y^t(g_*\mathcal{O}_U)_y = 0.$$

As $\mathcal{O}_{Y,y}$ is integral, this implies that $(g_*\mathcal{O}_U)_y$ is torsion as an $\mathcal{O}_{Y,f(x)}$ -module. This is a contradiction, as $(f_*\mathcal{O}_X)_y$ as an $\mathcal{O}_{Y,f(x)}$ -module is torsion-free by assumption. \square

Lemma 4.11. Let X be an integral complex analytic space and \mathcal{M} be a coherent sheaf of \mathcal{O}_X -modules. Then

$$\{x \in X : \mathcal{M} \text{ is torsion-free at } x\}$$

is co-analytic in X .

PROOF. It suffices to show that $\text{Supp } \mathcal{T}(\mathcal{M})$ is an analytic set in X . As X is integral, $\mathcal{T}(\mathcal{M})$ is just the kernel of the morphism $\mathcal{M} \rightarrow \mathcal{M}^{\vee\vee}$. \square

Corollary 4.12. Let $f : X \rightarrow Y$ be a finite morphism of complex analytic spaces. Assume that Y is integral. Let $x \in X$ be a point such that X is integral at x and f is open at x , then there is an open neighbourhood U of x in X such that $f|_U : U \rightarrow Y$ is open.

PROOF. Let $y = f(x)$. The problem is local on Y . By [Proposition 4.4](#), we may assume that $\{x\} = f^{-1}(y)$. By [Corollary 4.8](#), $f_*\mathcal{O}_X$ is coherent. By [Lemma 4.11](#), it suffices to show that it is torsion-free.

Observe that $(f_*\mathcal{O}_X)_y \xrightarrow{\sim} \mathcal{O}_{X,x}$. By [Proposition 2.2](#), $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is injective. As $\mathcal{O}_{X,x}$ is integral by our assumption, we conclude. \square

Lemma 4.13. Let $f : X \rightarrow Y$ be a finite morphism of reduced complex analytic spaces and $x \in X$. Assume that $x \in X$, then there is a non-zero divisor $h \in \mathfrak{m}_{f(x)}$ such that $f_x^\#(h)$ is a non-zero divisor in $\mathcal{O}_{X,x}$.

PROOF. By [Proposition 4.4](#), the problem is local on X . We may assume that X can be decomposed into irreducible components at x :

$$X = A_1 \cup \dots \cup A_s.$$

By [Corollary 4.8](#), $B_j := f(A_j)$ is an analytic set in Y for $j = 1, \dots, s$. By our assumption, x is not an isolated point in A_j , so y is not an isolated point in B_j for $j = 1, \dots, s$. Take a non-zero divisor $h \in \mathfrak{m}_{Y,y}$. Up to shrinking Y , we may assume that h spreads to $g \in \mathcal{O}_Y(Y)$. Observe that $W(f^*g) \cap A_j$ is not a neighbourhood of x in A_j for all $j = 1, \dots, s$. So $f_x^\#h$ is not a zero divisor. \square

Theorem 4.14. Let $f : X \rightarrow Y$ be a finite morphism of complex analytic spaces and $y \in Y$. Then

$$\dim_y f(X) = \max_{x \in f^{-1}(y)} \dim_x X.$$

The left-hand side makes sense because $f(X)$ is an analytic set in Y by [Corollary 4.8](#).

PROOF. We may assume that X and Y are reduced and $f(X) = Y$.

Step 1. We reduce to the case where $f^{-1}(y) = \{x\}$ for some $x \in X$.

Let x_1, \dots, x_t be the distinct points in $f^{-1}(y)$. The problem is local on Y . By [Theorem 4.8](#) in [Topology and bornology](#) and [Proposition 4.4](#), up to shrinking Y , we may assume that X is the disjoint union of open neighbourhoods U_1, \dots, U_t of x_1, \dots, x_t and $U_j \rightarrow Y$ is finite for each $j = 1, \dots, t$. It suffices to apply the special case to each $U_j \rightarrow Y$ for $j = 1, \dots, t$.

Step 2. We prove the theorem after the reduction in Step 1.

We make an induction on $d := \dim_x X$. There is nothing to prove when $d = 0$. Assume that $d \geq 1$. By [Lemma 4.13](#), we can choose a non-zero divisor $g_y \in \mathfrak{m}_{Y,y}$ such that $f_x^\#(g_y)$ is a non-zero divisor in $\mathcal{O}_{X,x}$. Up to shrinking Y , we may assume that g spreads to $g \in \mathcal{O}_Y(Y)$. It suffices to apply our inductive hypothesis to $W(f_x^\#(g_y)) \subseteq W(g_y)$. \square

Corollary 4.15. Let $f : X \rightarrow Y$ be a finite open surjective morphism of complex analytic spaces. Assume that A is a thin subset of X of order $k \in \mathbb{Z}_{>0}$, then $f(A)$ is a thin subset of Y of order k .

PROOF. We may assume that X and Y are reduced. By [Proposition 4.4](#) and the fact that f is open, the problem is local on X , we may assume that A is an analytic subset of X . Let $x \in A$. It suffices to handle the case where A is irreducible at x and x is the only point in $f^{-1}(f(x))$. By [Corollary 4.8](#), $f(A)$ is an irreducible analytic subset of Y .

We may assume that Y is irreducible at $y := f(x)$. Then

$$\text{codim}_y(f(A), Y) = \dim_y Y - \dim_y f(A).$$

By [Theorem 4.14](#), $\dim_y Y = \dim_x X$, $\dim_y f(A) = \dim_x A$. It follows that

$$\text{codim}_y(f(A), Y) = \dim_x X - \dim_x A \geq \text{codim}_x(A, X) \geq k.$$

□

Proposition 4.16. Let $f : X \rightarrow Y$ be a finite morphism of complex analytic spaces and $x \in X$. Assume that Y is unibranch at $f(x)$. Assume that $\dim_x X = \dim_{f(x)} Y$, then f is open at x .

PROOF. We may assume that X and Y are both reduced. Let $y = f(x)$. By [Proposition 4.4](#), we may assume that $\{x\} = f^{-1}(y)$. By [Corollary 4.8](#), $f(X)$ is an analytic set in Y . By [Theorem 4.14](#),

$$\dim_y f(X) = \dim_x X.$$

As Y is irreducible at $f(x)$, we conclude that $f(X)_y = X_y$ and hence $f(X)$ is a neighbourhood of y . □

Corollary 4.17. Let $f : X \rightarrow Y$ be a quasi-finite morphism of equidimensional complex analytic spaces of dimension $d \in \mathbb{N}$. Assume that Y is unibranch. Then f is open.

The corollary fails if Y is not unibranch.

PROOF. By [Proposition 4.4](#), f is finite at all $x \in X$. It suffices to apply [Proposition 4.16](#). □

Lemma 4.18. Let $f : X \rightarrow Y$ be a finite open morphism of reduced complex analytic spaces. Assume that Y is a complex manifold. Then f is a branched covering.

PROOF. The statement is local on Y , so we may assume that Y is an open neighbourhood of 0 in \mathbb{C}^n for some $n \in \mathbb{N}$. By [Proposition 4.4](#), we may assume that $\pi^{-1}\{0\}$ consists of a single point and X is a closed analytic subspace of a domain V in \mathbb{C}^d for some $d \in \mathbb{N}$. Replacing X by the graph of f , we may assume that X is a closed analytic subspace of $V \times Y$ and f is the restriction of the projection map $V \times Y \rightarrow V$. In this case, the result follows from the local description lemma. [Reproduce CAS p72!](#) □

Corollary 4.19. Let X be an equidimensional complex analytic space of dimension d and $x \in X$. Then there is an open neighbourhood U of x in X and a connected domain $V \in \mathbb{C}^d$ such that there is a branched covering $U \rightarrow V$.

In fact, given any system of parameters $f_1, \dots, f_d \in \mathcal{O}_{X,x}$, we can define a morphism sending x to 0 and the corresponding local ring homomorphism at x is

$$\mathcal{O}_{\mathbb{C}^d,0} \rightarrow \mathcal{O}_{X,x}$$

given by f_1, \dots, f_d .

PROOF. This follows from [Theorem 3.9](#) in [Constructions of complex analytic spaces](#), [Lemma 4.18](#) and [Corollary 4.17](#). \square

Corollary 4.20. Let X be a complex analytic space and $x \in X$. Assume that X is unibranch at x . Let $f \in \mathcal{O}_{X,x}$. We assume that f is not constant and $\dim_x X \geq 1$, then for any open neighbourhood U of x in X such that f spreads to $g \in \mathcal{O}_X(U)$, there is $\epsilon > 0$ such that g takes all values $c \in \mathbb{C}$ with $|c - f(x)| < \epsilon$.

PROOF. We may assume that X is reduced and $f(x) = 0$. Then f is a non-zero divisor in $\mathcal{O}_{X,x}$. We can find a system of parameters f, g_1, \dots, g_{n-1} with $n = \dim_x X$ such that f, g_1, \dots, g_{n-1} induce a branched covering $X \rightarrow V$ sending x to 0 after shrinking X , where V is an open neighbourhood of 0 in \mathbb{C}^n . This follows from [Corollary 4.19](#). As the branched covering is open by [Proposition 4.16](#), we conclude. \square

Theorem 4.21. Let $f : X \rightarrow Y$ be a finite open surjective morphism of reduced complex analytic spaces, then f is a branched covering.

PROOF. Let $x \in X$ and $y = f(x)$. As f is open, it suffices to find open neighbourhoods U of x in X and V of y in Y such that the morphism $U \rightarrow V$ induced by f is a branched covering. We first take U small enough so that U can be decomposed into prime components at x :

$$U = X_1 \cup \dots \cup X_s.$$

We can assume that $X_i \cap X_j$ is thin in U for $i, j = 1, \dots, s, i \neq j$. Up to shrinking U , we may assume that $U \rightarrow V$ is finite [Proposition 4.4](#) for some open neighbourhood V of y in Y . As f is open, we may take $V = f(U)$. Observe that $f(X_i)$ is analytic in V for $i = 1, \dots, s$ by [Corollary 4.8](#). Moreover, $f(X_i)$ is irreducible at y for $i = 1, \dots, s$. By [Theorem 2.4](#) in [Local properties of complex analytic spaces](#), we may assume that $f(X_i)$ is equidimensional of dimension $n_i \in \mathbb{N}$ for $i = 1, \dots, s$.

By [Corollary 4.19](#), up to shrinking V , we may assume that there is a branched covering $\eta_i : f(X_i) \rightarrow V_i$, where V_i is a connected domain in \mathbb{C}^{n_i} for $i = 1, \dots, s$. By [Lemma 4.18](#), $\eta_i \circ f|_{X_i}$ is a branched covering for $i = 1, \dots, s$. It follows that $X_i \rightarrow \pi(X_i)$ is a branched covering for $i = 1, \dots, s$. This readily implies that f is a branched covering. \square

Definition 4.22. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X . Take a critical locus T of f containing $f(A)$.

Consider $g \in \mathcal{O}_X(X \setminus A)$. We define a monic polynomial

$$\chi_g(w)(y) := \prod_{x \in f^{-1}(y)} (w - g(x)) \in \mathcal{O}_Y(Y \setminus T)[w].$$

By [Theorem 3.7](#) in [Local properties of complex analytic spaces](#), χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$. The monic polynomial χ_g is called the *characteristic polynomial* of g (with respect to f).

Proposition 4.23. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $g \in \mathcal{O}_X(X \setminus A)$. Let $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$ be the characteristic polynomial of g . Then $\chi_g(g) = 0$.

If either of the following conditions hold:

- (1) g is locally bounded near A ;
- (2) A is thin of order 2 in Y .

Then χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y)[w]$.

PROOF. Only the second part is non-trivial. By [Corollary 4.15](#), f is open. By [Corollary 4.15](#), $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y . It suffices to apply [Theorem 3.7](#) in [Local properties of complex analytic spaces](#). \square

Proposition 4.24. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $e, g \in \mathcal{O}_X(X \setminus A)$. Take a critical locus T of f containing $f(A)$. Consider the $b \times b$ -matrice

$$M(y) = \begin{bmatrix} 1 & e(x_1) & \dots & e(x_1)^{b-1} \\ 1 & e(x_2) & \dots & e(x_2)^{b-1} \\ & & \ddots & \\ 1 & e(x_b) & \dots & e(x_b)^{b-1} \end{bmatrix}$$

and $M_i(y)$ is $M(y)$ with the i -th column replace by

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_b) \end{bmatrix}$$

for $i = 0, \dots, b-1$, where $y \in Y \setminus T$ and x_1, \dots, x_b are the distinct points in $f^{-1}(y)$. Then there are $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus f(A))$ such that for all $y \in Y \setminus T$,

$$\Delta_e(y) = (\det M(y))^2, \quad c_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. If either of the following conditions holds:

- (1) e and g are locally bounded near A ;
- (2) A is thin of order 2 in X ,

then we can take $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y)$

The function Δ_e is called the *discriminant* of e . We say e is *primitive* with respect to f if Δ is not identically 0.

PROOF. We first observe that $\det M(y)$ and $\det M_i(y)$ are independent of the ordering of x_1, \dots, x_b by elementary linear algebra, where $i = 1, \dots, b$. The entries of $M(y)$ and $M_i(y)$ can all be taken to be holomorphic outside T , so $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus T)$ are defined and the desired equation holds. By [Theorem 3.7](#) in [Local properties of complex analytic spaces](#), these functions can be extended uniquely into $\mathcal{O}_Y(Y \setminus f(A))$.

By [Corollary 4.15](#), $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y . Applying [Theorem 3.7](#) in [Local properties of complex analytic spaces](#), we conclude the last assertion. \square

Corollary 4.25. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a connected complex manifold. A primitive element $e \in \mathcal{O}_X(X)$ exists if X is holomorphically separable.

PROOF. Take a critical locus T of f . Let $y \in X \setminus T$. Let x_1, \dots, x_b be distinct points of $f^{-1}(y)$. For each $i, j = 1, \dots, b$ with $i < j$, we can find a $g_{ij} \in \mathcal{O}_X(X)$ with $g(x_i) \neq g(x_j)$. A suitable linear combination of g_{ij} 's works. \square

Proposition 4.26. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X .

Let $e \in \mathcal{O}_X(X \setminus A)$ primitive element with respect to f . Then for each $g \in \mathcal{O}_X(X \setminus A)$, we have canonical polynomial $\Omega \in \mathcal{O}_Y(Y \setminus \pi(A))[X]$ such that

$$\Delta_e g = \Omega(e) \quad \text{on } X \setminus A.$$

If either of the following conditions holds:

- (1) e and g are locally bounded near A ;
- (2) A is thin of order 2 in X ,

then we can take $\Omega \in \mathcal{O}_Y(Y)[X]$.

In the traditional terminology, Δ_e is a *universal denominator* of the $\mathcal{O}_Y(Y)$ -module $\mathcal{O}_X(X)$ if one of the two assumptions is satisfied.

PROOF. Take a critical locus T of f containing $f(A)$. Consider $y \in Y \setminus T$ with fibers x_1, \dots, x_b . Consider the system of b -linear equations:

$$\Delta_e(y)g(x_i) = c_0(y) + c_1(y)e(x_i) + \dots + c_{b-1}(y)e(x_i)^{b-1}$$

for $j = 1, \dots, b$. By Cramer's rule, if we use the notations of [Proposition 4.24](#), if $\det M(y) \neq 0$, the unique solution is then

$$c_i(y) = (\det M(y))^{-1} \Delta_e(y) \det M_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. From [Proposition 4.24](#), $c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus \pi(A))$. It suffices to take

$$\Omega = c_0 + c_1 X + \dots + c_{b-1} X^{b-1}.$$

It is obvious that on $X \setminus (A \cup W(\Delta))$,

$$\Delta_e g = \Omega(e).$$

The same holds on $X \setminus A$ by continuity. The last assertion follows from [Proposition 4.24](#). \square

Corollary 4.27 (Riemann extension theorem). Let X be a reduced equidimensional complex analytic space of dimension $n \in \mathbb{N}$ and A be a thin set in X . Let $f \in \mathcal{O}_X(X \setminus A)$. Assume one of the following conditions holds:

- (1) f is locally bounded near A ;
- (2) A is thin of order 2.

Then there is an element $g \in \overline{\mathcal{O}_X(X)}$ extending f .

PROOF. The uniqueness is obvious, we prove the existence. The problem is local on X , we may assume that X is holomorphically separable. By [Corollary 4.19](#), we may take a connected complex manifold Y of dimension n , $b \in \mathbb{Z}_{>0}$, a b -sheeted branched covering $f : X \rightarrow Y$. By [Corollary 4.25](#), we can find a primitive element $e \in \mathcal{O}_X(X)$. By [Proposition 4.26](#) and [Proposition 4.23](#), it suffices to take $g = \Omega(e)/\Delta_e$, where Ω_e is the polynomial in [Proposition 4.26](#). \square

Corollary 4.28. Let X be a normal complex analytic space. Then the canonical map

$$\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X^{\text{reg}})$$

is an isomorphism.

PROOF. By [Proposition 6.9](#) in [Local properties of complex analytic spaces](#), the map is injective. Take $f \in \mathcal{O}_X(X^{\text{reg}})$, we need to extend it to $g \in \mathcal{O}_X(X)$. The problem is local on X . As X is normal, it is equidimensional at all points. By shrinking X , we may assume that X is equidimensional of some dimension $n \in \mathbb{N}$. Recall that X^{Sing} is thin of order 2 in X by [Proposition 7.4](#) in [Local properties of complex analytic spaces](#), so we can apply [Corollary 4.27](#). \square

Corollary 4.29. Let X be a connected normal complex analytic space then X^{reg} is connected.

PROOF. If not, we can find a continuous function $f : X^{\text{reg}} \rightarrow \{0, 1\}$ which is not constant. By [Corollary 4.28](#), f can be extended to $g \in \mathcal{O}_X(X)$. This contradicts the fact that X is connected. \square

Corollary 4.30. Let X be an irreducible complex analytic space and A be an analytic set in X . Suppose that there is $x \in A$ with $\dim_x A = \dim_x X$, then $A = X$.

PROOF. We may assume that X is irreducible. By [Theorem 4.14](#), we may assume that X is normal.

Endow A with the reduced induced structure. As $\dim_x A = \dim_x X$, $\text{Spec } \mathcal{O}_{X,x} = \text{Spec } \mathcal{O}_{A,x}$ has a common irreducible component. By Nullstellensatz, $\text{Int } A$ is non-empty. So $A' := A \setminus X^{\text{Sing}}$ is non-empty and open in X^{reg} . We need to show that $A' = X^{\text{reg}}$, taking closure we then conclude.

Suppose that $A' \neq X^{\text{reg}}$. Then $\overline{A'} \cap X^{\text{reg}}$ is a non-empty closed in X^{reg} , which is connected by [Corollary 4.29](#). So

$$\overline{A'} \cap X^{\text{reg}} \neq A',$$

as otherwise, $X^{\text{reg}} = (\overline{A'} \cap X^{\text{reg}}) \cup (X^{\text{reg}} \setminus A')$. Take $a \in (\overline{A'} \cap X^{\text{reg}}) \setminus A'$. Take a connected neighbourhood U of a in X^{reg} and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_X(U)$ so that $U \cap A = W(f_1, \dots, f_k)$. As $U \cap A' \neq \emptyset$, f_1, \dots, f_k vanishes identically in U by Identitätssatz. In particular, $a \in A'$, which is a contradiction. \square

Corollary 4.31. Let $f : X \rightarrow Y$ be a morphism of reduced complex analytic spaces. Let $Z \subseteq Y$ be the non-normal locus. Assume that $f^{-1}(Z)$ is nowhere dense in X (for example when X is irreducible and f is surjective), then there is a unique morphism $\bar{f} : \bar{X} \rightarrow \bar{Y}$ such that the following diagram commutes:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Recall that Z is an analytic set in Y by [Theorem 7.3](#) in [Local properties of complex analytic spaces](#).

PROOF. The uniqueness is clear. Let Z' be the inverse image of Z in \bar{Y} and Z'' be the inverse image of Z in \bar{X} . By our assumption, Z'' is thin in \bar{X} . By construction, $\eta : \bar{Y} \setminus Z' \rightarrow Y \setminus Z$ is an isomorphism, so we get a morphism $g : \bar{X} \setminus Z'' \rightarrow \bar{Y} \setminus Z'$ completion the commutative diagram

$$\begin{array}{ccc} \bar{X} \setminus Z'' & \longrightarrow & \bar{Y} \setminus Z' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

Let $p \in Z''$. We need to extend g to a neighbourhood of p . Choose an open neighbourhood $V \subseteq \bar{Y}$ of the preimage of p in \bar{X} which admits a closed immersion into a bounded domain $D \subseteq \mathbb{C}^n$ for some $n \in \mathbb{N}$. There is an open neighbourhood $U \subseteq \bar{X}$ of p such that g maps $U \setminus Z'' \rightarrow V$. The induced morphism $U \setminus Z'' \rightarrow D$ is given by bounded holomorphic functions in $\mathcal{O}_{U \setminus Z''}(U \setminus Z'')$. By [Corollary 4.27](#), we get an extension $U \rightarrow D$. But this morphism factorizes through $U \rightarrow V$ as U is reduced, we conclude. \square

Corollary 4.32. Let X be a complex analytic space. Then the following are equivalent:

- (1) X is irreducible;
- (2) If we write $X = Y_1 \cup Y_2$ with Y_1, Y_2 being analytic sets in X , then $X = Y_1$ or $X = Y_2$.

PROOF. We may assume that X is reduced.

(1) \implies (2): We may assume that X is normal. Suppose $X = Y_1 \cup Y_2$ with Y_1, Y_2 being analytic sets in X . Then $Y_1 \cap Y_2$ is not empty, as otherwise, X is not even connected. Let $x \in Y_1 \cap Y_2$. We then have $X_x = Y_{1,x} \cup Y_{2,x}$. This contradicts the fact that $\mathcal{O}_{X,x}$ is integral unless $Y_{1,x} \subseteq Y_{2,x}$ or $Y_{1,x} \subseteq Y_{2,x}$, which is impossible by [Corollary 4.30](#).

(2) \implies (1): Suppose that X is not irreducible. Then the normalization \bar{X} is not connected, say $\bar{X} = Y'_1 \cup Y'_2$, where Y'_1, Y'_2 are disjoint clopen sets in \bar{X} . Let $\pi : \bar{X} \rightarrow X$ be the normalization morphism. Then

$$X = \pi(Y'_1) \cup \pi(Y'_2).$$

By our assumption, either $X = \pi(Y'_1)$ or $X = \pi(Y'_2)$. We assume that the former holds. From [Proposition 7.8](#) in [Local properties of complex analytic spaces](#), we conclude that $Y'_1 = \bar{X}$, which is a contradiction. \square

Corollary 4.33. Let X be a connected complex analytic space. Then X is path-connected.

PROOF. We may assume that X is reduced.

If X is irreducible, after passing to the normalization, we may assume that X is normal. Then clearly X^{reg} is connected. So it suffices to apply [Proposition 7.12](#) in [Local properties of complex analytic spaces](#).

In general, take $x \in X$ and let X' be the set of all points of X that can be joined to x by a path. Then from the previous case, X' is the union of certain irreducible components of X . So is the complement $X \setminus X'$. As X is connected, we find that $X = X'$. \square

Corollary 4.34. Let X be an irreducible complex analytic space. Then there is $n \in \mathbb{N}$ such that X is equidimensional of dimension n .

PROOF. We may assume that X is reduced. By [Theorem 4.14](#), we can even assume that X is normal. Then X is connected. In particular, X^{reg} is connected by [Corollary 4.29](#). But X^{reg} is then equidimensional of some dimension $n \in \mathbb{N}$. If $\dim_x X \neq n$ for some $x \in X^{\text{Sing}}$, by [Theorem 2.4](#) in [Local properties of complex analytic spaces](#), $\dim_y X = \dim_x X$ whenever y is close to x . This is a contradiction. \square

Corollary 4.35. Let $f : X \rightarrow Y$ be a finite surjective morphism between irreducible reduced complex analytic spaces. Then f is a branched covering.

PROOF. By [Corollary 4.31](#), we have an obvious commutative diagram:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

It suffices to show that \bar{f} is a branched covering, so we may assume that X and Y are normal.

By [Proposition 4.16](#) and [Corollary 4.34](#), f is open. So it suffices to apply [Theorem 4.21](#). \square

Corollary 4.36. Let $f : X \rightarrow Y$ be a finite surjective morphism between reduced complex analytic spaces. Then the following are equivalent:

- (1) f is a branched covering;
- (2) The image of each irreducible component of X has an interior point;
- (3) The image of each irreducible component of X is an irreducible component of Y .

PROOF. (1) \implies (2): Let $T \subseteq Y$ be a critical locus of f . Then $f^{-1}(T)$ is thin in X . Each irreducible component X' of X meets $X \setminus f^{-1}(T)$. It follows that $f(X' \setminus f^{-1}(T))$ is non-empty and open in Y .

(2) \implies (3): Let X' be an irreducible component of X . Then $f(X')$ is an analytic set in Y . It is clearly irreducible. So $f(X')$ is contained in an irreducible component Y' of Y . But as $f(X')$ has an interior point, we find that $f(X') = Y'$ by [Corollary 4.30](#).

(3) \implies (1): The assertion is local, we may assume that the number of irreducible components of X is finite. Let X_1, \dots, X_s be the irreducible components of X . For each $i = 1, \dots, s$, the induced map $X_i \rightarrow \pi(X_i)$ is finite and hence a branched covering by [Corollary 4.35](#). It is enough to verify that $\pi^{-1}(\pi(X_i \cap X_j))$ is thin in X for $i, j = 1, \dots, s$ and $i \neq j$. If this fails, this set contains an interior point in X_k for some $k \in \{1, \dots, s\}$. But then

$$X_k \subseteq \pi^{-1}(\pi(X_i \cap X_j)).$$

It follows that

$$\pi(X_i \cap X_j) \supseteq \pi(X_k).$$

This is impossible as $X_i \cap X_j \cap X_k$ is thin in X_k . \square

Definition 4.37. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a normal complex analytic space. Take a critical locus $T \subseteq Y$ of f containing Y^{Sing} .

Consider $g \in \mathcal{O}_X(X)$. We define the *characteristic polynomial* $\chi_g \in \mathcal{O}_Y(Y)[w]$ of g (with respect to f) as follows: When Y is connected, by [Corollary 4.29](#), Y^{reg} is a connected complex manifold. We define $\chi_g \in \mathcal{O}_Y(Y^{\text{reg}})[w]$ as in [Definition 4.22](#). We then extend χ_g to $\mathcal{O}_Y(Y^{\text{reg}})[w]$ using [Corollary 4.28](#). It is a monic polynomial of degree b . When Y is not connected, we just glue the characteristic polynomials defined using each connected components. Then we find a monic polynomial $\chi_g \in \mathcal{O}_Y(Y)[w]$ of degree b .

Proposition 4.38. Let $b \in \mathbb{Z}_{>0}$, $f : X \rightarrow Y$ be a b -sheeted branched covering with Y being a normal complex analytic space. Let $g \in \mathcal{O}_X(X)$. Let $\chi_g \in \mathcal{O}_Y(Y)[w]$ be the characteristic polynomial of g . Then $\chi_g(g) = 0$.

PROOF. This follows immediately from [Proposition 4.23](#). \square

We give an alternative characterization of $\overline{\mathcal{O}}_X$.

Proposition 4.39. Let X be a reduced complex analytic space. Then for any open set $U \subseteq X$,

$$\overline{\mathcal{O}}_X(U) \xrightarrow{\sim} \{f : U \rightarrow \mathbb{C} : f \text{ is weakly holomorphic}\}.$$

PROOF. We temporarily denote the sheaf stated in the proposition by \mathcal{O}' . From the uniqueness in ??, it suffices to show that \mathcal{O}'_x is isomorphic to $\overline{\mathcal{O}}_{X,x}$ as $\mathcal{O}_{X,x}$ -algebras for any $x \in X$.

We first observe that $\overline{\mathcal{O}}_X$ is a subsheaf of \mathcal{O}' . Let $U \subseteq X$ be an open subset and $f \in \overline{\mathcal{O}}_X(U)$. We need to show that f is locally bounded around $y \in U \cap X^{\text{Sing}}$. Take an integral equation

$$f_y^n + a_{1,y}f_y^{n-1} + \cdots + a_{n,y} = 0$$

with $a_{1,y}, \dots, a_{n,y} \in \mathcal{O}_{X,x}$. Take an open neighbourhood V of y in U such that $a_{1,y}, \dots, a_{n,y}$ lift to $a_1, \dots, a_n \in \mathcal{O}_X(V)$ and

$$(f|_V)^n + a_1(f|_V)^{n-1} + \cdots + a_n = 0.$$

Then for any $z \in V \setminus X^{\text{Sing}}$,

$$|f(z)| \leq \max\{1, |a_1(z)| + \cdots + |a_n(z)|\}.$$

So $f \in \mathcal{O}'$.

Conversely, let $U \subseteq X$ be an open subset and $f \in \mathcal{O}'(U)$. By [Proposition 7.8](#) in [Local properties of complex analytic spaces](#), $p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_X$, where $p : \overline{X} \rightarrow X$ is the normalization morphism. It follows from [Proposition 7.8](#) in [Local properties of complex analytic spaces](#) and [Corollary 4.27](#) that f can be uniquely extended to $g \in \mathcal{O}_{\overline{X}}(p^{-1}U) = \mathcal{O}_X(U)$. \square

Proposition 4.40 (Rado, Cartan). Let X be a normal complex analytic space and $f : X \rightarrow \mathbb{C}$ be a continuous map. Let $Z = f^{-1}(0)$. Assume that there is $g \in \mathcal{O}_X(X \setminus Z)$ such that $[g] = f|_{X \setminus Z}$, then $f = [g]$.

This result is proved in [\[Car52\]](#).

PROOF. By [Corollary 4.28](#), we may assume that X is a complex manifold. The problem is local on X , we may assume that X is the unit polydisk in \mathbb{C}^n for some $n \in \mathbb{N}$. By Hartogs theorem, we may assume that $n = 1$.

It remains to show that a continuous function $f : \{z \in \mathbb{C} : |z| < 1\}$ which is holomorphic outside $Z := \{f = 0\}$ is holomorphic. This result is well-known. \square

5. Flat morphisms

The notion of flat morphisms is defined for all ringed spaces. See [\[Stacks, Tag 02N2\]](#). We will make use of these notions directly.

Proposition 5.1. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and $x \in X$. Write $y = f(x)$. Let \mathcal{F} be a coherent \mathcal{O}_X -module. Then the following are equivalent:

- (1) \mathcal{F} is f -flat at x ;
- (2) \mathcal{F}_x is a flat $\mathcal{O}_{Y,y}$ -module;
- (3) For all $n \in \mathbb{N}$,

$$\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$$

is a flat $\hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$ -module;

- (4) We have

$$\mathrm{Tor}_1^{\mathcal{O}_{Y,y}}(\mathbb{C}, \mathcal{F}_x) = 0.$$

PROOF. (1) \Leftrightarrow (2): This is the definition of flatness.

(2) \Leftrightarrow (3): This follows from [\[Stacks, Tag 0523\]](#).

(3) \Leftrightarrow (4): This follows from [\[Stacks, Tag 00MK\]](#). \square

Proposition 5.2. Let $f : X \rightarrow Y$ be a morphism of complex analytic spaces and \mathcal{F} be a coherent \mathcal{O}_X -module. Let $g : Y' \rightarrow Y$ be a morphism of complex analytic spaces and consider the following Cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \square & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

Consider a point $x' \in X'$ defined by $x \in X$ and $y' \in Y'$ with common image $y \in Y$.

- (1) If \mathcal{F} is f -flat at x , then $g'^*\mathcal{F}$ is f' -flat at x' .
- (2) If $g'^*\mathcal{F}$ is f' -flat at x' and $\hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{Y',y'}$ is injective, then \mathcal{F} is f -flat at x .

PROOF. (1) Recall that

$$\hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y',y'}.$$

Let $n \in \mathbb{N}$, we then find

$$\hat{\mathcal{O}}_{X',x'} / \hat{\mathfrak{m}}_{y'}^{n+1} \hat{\mathcal{O}}_{X',x'} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \hat{\otimes}_{\hat{\mathcal{O}}_{Y,y}} \left(\hat{\mathcal{O}}_{Y',y'} / \hat{\mathfrak{m}}_{y'}^{n+1} \right) \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x} \otimes_{\hat{\mathcal{O}}_{Y,y}} \left(\hat{\mathcal{O}}_{Y',y'} / \hat{\mathfrak{m}}_{y'}^{n+1} \right).$$

By [Proposition 5.1](#), $\hat{\mathcal{F}}_x \otimes_{\hat{\mathcal{O}}_{Y,y}} \hat{\mathcal{O}}_{Y,y} / \hat{\mathfrak{m}}_y^{n+1}$ is a flat $\mathcal{O}_{Y',y'}$ -module for each $n \in \mathbb{N}$. By [Proposition 5.1](#) again, \mathcal{F} is f' -flat at x' .

(2) For each $n \in \mathbb{N}$, let I_n be the inverse image of $\hat{\mathfrak{m}}_{y'}^{n+1}$ with respect to $\hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{Y',y'}$. As the latter map is assumed to be injective, by Krull's intersection theorem, we find that

$$\bigcap_{n \in \mathbb{N}} I_n = 0.$$

It follows that the I_n 's form a basis at 0 in $\hat{\mathcal{O}}_{Y,y}$. By [Proposition 5.1](#), we are reduced to show that $\hat{\mathcal{F}}_x/I_n\hat{\mathcal{F}}_x$ is flat over $\hat{\mathcal{O}}_{Y,y}/I_n$. But by [Proposition 5.1](#) again, we know that its base change along $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{Y',y'}/\hat{\mathfrak{m}}_{y'}^{n+1}$ is flat. So we are reduced to the well-known algebraic case. \square

Bibliography

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