Ymir

Contents

Local properties of complex analytic spaces

1. Introduction

2. Dimension

Definition 2.1. Let X be a complex analytic space and $x \in X$, the *dimension* $\dim_x X$ of X at x is

$$\dim_x X = \dim \mathcal{O}_{X,x}.$$

We also define the *dimension* of the pointed complex analytic space (X, x) and the *dimension* of the complex analytic germ X_x as $\dim_x X$.

When X is connected, the *dimension* of X is defined as

$$\dim X := \sup_{x \in X} \dim_x X.$$

If A is an analytic set in X such that there is a closed analytic subspace of X with $|B| = A$, then $\dim_x B$ does not depend on the choice of B, we define it as $\dim_x A$.

As we will see in Corollary 6.6, B always exists.

Definition 2.2. Let X be a complex analytic space, we say X is *equidimensional* at $x \in X$ if $\mathcal{O}_{X,x}$ is equidimensional.

We also say (X, x) or X_x is *equidimensional*.

We say X is *equidimensional of dimension* $n \in \mathbb{N}$ if X is non-empty and is equidimensional of dimension n at each $x \in X$.

Recall that in general, a local ring R is equidimensional if $\dim R/\mathfrak{p} = \dim R$ for all minimal prime $\mathfrak{p}$ of R.

Definition 2.3. Let X be a complex analytic space and $x \in X$, we say X is *integral* at x if $\mathcal{O}_{X,x}$ is integral.

This corresponds to the notion defined in Definition 3.12 in Constructions of complex analytic spaces.

Theorem 2.4. Let X be a complex analytic space and $n \in \mathbb{N}$, then the set of points $x \in X$ such that X_x is equidimensional of dimension n is open.

This is analogous to the result for noetherian cartenary schemes.

Proof. Let $x \in X$ be a point such that X_x is equidimensional of dimension n. We want to construct an open neighbourhood V of x in X such that X is equidimensional of dimension n at any $y \in V$.

Step 1. We reduce to the case where X is integral at x.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal primes of $\mathcal{O}_{X,x}$. The number is finite because $\mathcal{O}_{X,x}$ is noetherian. We have

$$\bigcap_{i=1}^{m} \mathfrak{p}_i = \operatorname{rad} \mathcal{O}_{X,x}.$$

Take an open neighbourhood U of x in X such that there are ideals of finite type $\mathcal{I}_1, \ldots, \mathcal{I}_m$ extending $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Up to shrinking U, we may assume that

$$\bigcap_{i=1}^{m} \mathcal{I}_i$$

is nilpotent. For each $i = 1, \ldots, m$, let U_i denote the closed analytic subspace of U defined by $\mathcal{I}_i$. Then

$$|U| = \bigcup_{i=1}^{m} |U_i|$$

by Corollary 3.17 in Constructions of complex analytic spaces. As for any $y \in U$,

$$\bigcap_{i=1}^{m} \mathcal{I}_{i,y}$$

is nilpotent, we have

$$|\operatorname{Spec} \mathcal{O}_{X,y}| = |\operatorname{Spec} \mathcal{O}_{X,y} / \bigcap_{i=1}^{m} \mathcal{I}_{i,y}| = \bigcup_{i=1}^{m} |\operatorname{Spec} \mathcal{O}_{X,y}/\mathcal{I}_{i,y}|.$$

In particular, for any $y \in U$,

$$\dim_y X = \dim_y U = \max_{i=1,\ldots,m} \dim_y U_i.$$

It suffices to handle each W_i separately.

Step 2. We assume that X_x is integral. By Theorem 3.9 in Constructions of complex analytic spaces, we may assume that X has the following structure: there is an open neighbourhood W of 0 in $\mathbb{C}^n$, a morphism $(X, x) \to (W, 0)$ and a finite $\mathcal{O}_W$-algebra $\mathcal{A}$ such that $\operatorname{Spec}_W^{\mathrm{an}} \mathcal{A}$ has a unique point x' over 0 and $(\operatorname{Spec}_W^{\mathrm{an}} \mathcal{A}, x')$ is isomorphic to (X, x) over $(W, 0)$. By Corollary 5.5 in Complex analytic local algebras, $\mathcal{O}_{W,0} \to \mathcal{O}_{X,x}$ is injective, hence $\mathcal{O}_{X,x}$ is torsion-free over $\mathcal{O}_{W,0}$. As the torsion sheaf is coherent, up to shrinking X, we may assume that $\mathcal{O}_{X,y}$ is torsion-free over $\mathcal{O}_{W,z}$, where z denotes the image of y in W. It suffices to apply Lemma 5.6 in Complex analytic local algebras. □

Corollary 2.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set $\{x \in X : \dim_x X \geq n\}$ is an analytic set in X.

After introducing the analytic Zariski topology, we can reformulate this corollary as follows: the map $x \mapsto \dim_x X$ is upper semi-continuous with respect to the analytic Zariski topology.

Proof. The problem is local on X. Fix $x \in X$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathcal{O}_{X,x}$. Up to shrinking X, we may assume that

$$|X| = \bigcup_{i=1}^{m} |W_i|,$$

where W_i is a closed analytic subspace of X defined by a coherent $\mathcal{I}_i$ spreading $\mathfrak{p}_i$. We can guarantee that

$$\dim_y X = \max_{i=1,\dots,m} \dim_y W_i.$$

This is possible as in the proof of Theorem 2.4. By Theorem 2.4, up to shrinking X, we may assume that W_i is equidimensional of dimension n_i for some $n_i \in \mathbb{N}$ for each $i = 1, \dots, m$. In particular, for each $y \in X$, we have

$$\dim_y X = \sup_{y \in W_i} n_i.$$

So

$$\{x \in X : \dim_x X \geq n\} = \bigcup_{i : n_i \geq n} |W_i|.$$

The corollary follows. □

Proposition 2.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Then

$$\dim_{(x,y)} X \times Y = \dim_x X + \dim_y Y.$$

PROOF. By Theorem 5.11 in Complex analytic local algebras,

$$\hat{\mathcal{O}}_{X\times Y,(x,y)} = \hat{\mathcal{O}}_{X,x} \hat{\otimes} \hat{\mathcal{O}}_{Y,y}.$$

As dimension is invariant under completion by [Stacks, Tag 07NV], it suffices to show that

$$\dim(\mathcal{O}_{X,x} \hat{\otimes} \mathcal{O}_{Y,y}) = \dim \mathcal{O}_{X,x} + \dim \mathcal{O}_{Y,y},$$

which is well-known. □

Definition 2.7. Let X_x be an analytic germ and Y_x be a closed analytic subgerm defined by an ideal $I \subseteq \mathcal{O}_{X,x}$.

(1) When Y_x is irreducible, namely when I is a prime ideal, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \operatorname{ht}_{\mathcal{O}_{X,x}}(I).$$

(2) In general, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \inf_{Z_x \subseteq Y_x} \operatorname{codim}_x(Y, X),$$

where Z_x runs over closed analytic subgerms of X_x contained in Y_x.

We also call $\operatorname{codim}_x(Y, X)$ the codimension of Y in X at x.

Observe that

$$\operatorname{codim}_x(Y, X) \leq \dim_x X - \dim_x Y.$$

When X_x is equidimensional, $\operatorname{codim}_x(Y, X)$ is nothing but $\dim_x X - \dim_x Y$.

Observe that

$$\operatorname{codim}_x(Y, X) = \operatorname{codim}(Y_x, \operatorname{Spec} \mathcal{O}_{X,x}). \tag{2.1}$$

Lemma 2.8. Let X be a complex analytic space and T be an analytic set in X. Let Y_1, Y_2 be two closed analytic subspaces of X with underlying set T, then for any $x \in T$,

$$\operatorname{codim}_x(Y_1, X) = \operatorname{codim}_x(Y_2, X).$$

PROOF. This follows from (2.1) and Corollary 3.14 in Constructions of complex analytic spaces. □

Definition 2.9. Let X be a complex analytic space and T be an analytic set in X. Take $y \in T$. We define the *codimension* $\operatorname{codim}_y(T, X)$ as follows: up to shrinking X, we may take a closed analytic subspace Y of X with underlying set T by Lemma 4.6 in Constructions of complex analytic spaces, we define

$$\operatorname{codim}_y(T, X) := \operatorname{codim}_y(Y, X).$$

This definition does not depend on the choices we made by Lemma 2.8.

Lemma 2.10. Let X be a complex analytic space and Y be a closed analytic subspace of X. Let $y \in Y$ be a point such that Y_y is irreducible. Then there is an open neighbourhood U of y in Y such that

$$\operatorname{codim}_z(Y, X) = \operatorname{codim}_y(Y, X)$$

for any $z \in U$.

Proof. Let X'_y be an irreducible component of X_y containing Y_y such that

$$\operatorname{codim}_y(Y, X) = \dim_y X' - \dim_y Y.$$

We can then take an open neighbourhood U of x in X such that X'_z is equidimensional of dimension $n := \dim_y X'$ for all $z \in U$ by Theorem 2.4. Then for any $z \in U$, X'_z is a union of some irreducible components of X_z. Up to shrinking U, we may guarantee that for any $z \in U \cap Y$, $Y_z \subseteq X'_z$ and $\dim_z Y = \dim_y Y$. Thereofre, for $z \in Y \cap U$,

$$\operatorname{codim}_z(Y, X) = \operatorname{codim}_z(Y, X') = \dim_z X' - \dim_z Y$$

is a constant. □

Corollary 2.11. Let X be a complex analytic space and Y be an analytic set in X. For any $n \in \mathbb{N}$,

$$\{y \in Y : \operatorname{codim}_y(Y, X) \leq n\}$$

is an analytic set in Y.

Proof. The problem is local. Let $x \in Y$. Let $Y_{1,x}, \dots, Y_{m,x}$ be the irreducible components of Y_x defined by prime ideals $J_1, \dots, J_m$ in $\mathcal{O}_{Y,x}$. Take an open neighbourhood U of x in X such that for any $y \in Y \cap U$, the ideal

$$\bigcap_{i=1}^{m} J_{i,y}$$

is nilpotent. By Lemma 2.10, up to shrinking U, we may assume that for any $y \in Y \cap U$,

$$\operatorname{codim}_y(Y_i, X) = \operatorname{codim}_x(Y_i, X) =: c_i$$

for $i = 1, \dots, m$. Then

$$\{y \in Y : \operatorname{codim}_y(Y, X) \leq n\} = \bigcup_{i : c_i \leq n} Y_i.$$

□

Corollary 2.12. Let X be a complex analytic space and Y be an analytic set in X. For any $n \in \mathbb{N}$ and any $y \in Y$,

$$\{y \in Y : \operatorname{codim}_y(Y, X) \leq n\}_y = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x} : \operatorname{codim}_{\mathfrak{p}}(T_x, \operatorname{Spec} \mathcal{O}_{X,x}) \leq n\}.$$

Proof. This is immediate from the proof of Corollary 2.11. □

Definition 2.13. Let X be a complex analytic space. A closed subset A of X is *thin* if for any $x \in A$, we can find an open neighbourhood U of x in X such that $A \cap U$ is contained in a nowhere dense analytic subset B of U.

Given $k \in \mathbb{Z}_{>0}$, we say A is *thin of order* k at $x \in A$ if U and B can be chosen so that $\operatorname{codim}_x(B, X) \geq 2$.

We say X is *thin* (*thin of order* k) if X is thin (resp. thin of order k) at all $x \in X$.

The definition in [CAS] Page 132 is not correct when X is not equidimensional. The same happens in several papers of Remmert.

3. Smoothness

Definition 3.1. Let X be a complex analytic space. We say X is *smooth* at $x \in X$ if $\mathcal{O}_{X,x}$ is regular. Otherwise, we say X is *singular* at x.

We also say (X, x) or X_x is *smooth* (resp. *singular*) at x.

We say X is *smooth* if it is smooth at all $x \in X$. In this case, we also say X is a *complex manifold*.

We write X^{sing} and X^{reg} for the set of singular and smooth points of X respectively.

Other common names in the literature include: regular, simple.

Proposition 3.2. Let X be a complex analytic space and $x \in X$. THen the following are equivalent:

1. X is smooth at x;
2. There is an open neighbourhood U of x in X that is isomorphic to a domain in $\mathbb{C}^n$ with $n = \dim_x X$;
3. $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$-module of rank $\dim_x X$;
4. $\Omega_{X,x}$ is generated by $\dim_x X$ elements as an $\mathcal{O}_{X,x}$-module;
5. $\hat{\mathcal{O}}_{X,x}$ is regular;
6. $\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[X_1, \ldots, X_n]]$ for $n = \dim_x X$.

Proof. (2) $\implies$ (1): This is obvious.

(1) $\implies$ (2): Let $f_{1,x}, \ldots, f_{n,x}$ be a regular system of parameters of $\mathcal{O}_{X,x}$. Up to shrinking X, we may lift them to $f_1, \ldots, f_n \in \mathcal{O}_X(X)$. By Theorem 4.2 in The notion of complex analytic spaces, they induce a morphism $f : (U, x) \to (\mathbb{C}^n, 0)$. Observe that $f_x^\# : \hat{\mathcal{O}}_{\mathbb{C}^n,0} \to \hat{\mathcal{O}}_{U,x}$ is an isomorphism, so f is a local isomorphism by Corollary 3.4 in Constructions of complex analytic spaces.

(2) $\implies$ (3): This follows from Example 8.7 in Constructions of complex analytic spaces.

(3) $\implies$ (4): This is trivial.

(4) $\implies$ (1): Recall that Ω_X is coherent by Corollary 8.2 in Constructions of complex analytic spaces. By Nakayama's lemma, the minimal number of generators of $\Omega_{X,x}$ is equal to $\dim_{\mathbb{C}} \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$. By algebraic results, we know that the latter space is $\mathfrak{m}_x/\mathfrak{m}_x^2$. So we find that $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$, implying that $\mathcal{O}_{X,x}$ is regular.

(1) $\Leftrightarrow$ (5): This follows from [Stacks, Tag 07NY].

(2) $\implies$ (6): This is clear.

(6) $\implies$ (5): This is clear. $\square$

Theorem 3.3. Let X be a complex analytic space, then X^{Sing} is an analytic set in X.

Proof. The problem is local. Let $x \in X$.

Step 1. We reduce to the case where X is equidimensional of dimension n.

Let

$$0 = \bigcap_{i=1}^{r} \mathfrak{p}_i$$

be the primary decomposition of 0. Up to shrinking X, we may assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ spread to coherent ideals $\mathcal{I}_1, \dots, \mathcal{I}_r$ on X and

$$\bigcap_{i=1}^{r} \mathcal{I}_i = 0.$$

Let X_i be the closed analytic subspace of X defined by $\mathcal{I}_i$ for $i = 1, \dots, n$. Then

$$X = \bigcup_{i=1}^{r} X_i.$$

As each X_i is equidimensional at x, say of dimension n_i for $i = 1, \dots, r$. By Theorem 2.4, up to shrinking X, we may assume that X_i is equidimensional of dimension n_i for $i = 1, \dots, r$. For each

Let $y \in X^{\mathrm{reg}}$, as $\mathcal{O}_{X,y}$ is regular hence integral, from

$$\bigcap_{i=1}^{r} \mathcal{I}_{i,y} = 0$$

we find that at least one $\mathcal{I}_{i,y}$ vanishes. Then

$$\mathcal{O}_{X_i,y} = \mathcal{O}_{X,y}$$

is regular. Namely, $y \in X_i^{\mathrm{reg}}$. Conversely, if for some $i = 1, \dots, n$, we have $\mathcal{I}_{i,y} = 0$ and $y \in X_i^{\mathrm{reg}}$, X_i is a neighbourhood of y in X, so $y \in X^{\mathrm{reg}}$. It follows that

$$X^{\mathrm{sing}} = \bigcap_{i=1}^{r} \left(\operatorname{Supp} \mathcal{I}_i \cup X_i^{\mathrm{Sing}} \right).$$

Recall that $\operatorname{Supp} \mathcal{I}_i$ is analytic for each $i = 1, \dots, n$ by Example 4.2 in Constructions of complex analytic spaces.

By Proposition 4.3 in Constructions of complex analytic spaces, in order to show that X^{sing} is an analytic set in X, it suffices to know that X_i^{Sing} is an analytic set in X_i for $i = 1, \dots, n$.

Step 2. Assume that X is equidimensional of dimension n. We need to show that the locus where Ω_X is locally free of rank n is co-analytic in X.

When $n = 0$, the locus where Ω_X is not locally free of rank 0 is exactly $\operatorname{Supp} \Omega_X$, which is analytic in X by Example 4.2 and Corollary 8.2 in Constructions of complex analytic spaces.

Assume that $n \geq 1$. Let $\Omega_X^n := \bigwedge^n \Omega_X$. Then the locus where Ω_X is locally free of rank n is exactly the locus where Ω_X^n is invertible. The invertible locus of Ω_X^n is exactly the locus where the canonical map

$$(\Omega_X^n)^\vee \otimes_{\mathcal{O}_X} \Omega_X^n \to \mathcal{O}_X$$

is an isomorphism. It follows that the complement of the locus is analytic in X. □

Theorem 3.4 (Generic smoothness)**.** Let X be a complex analytic space and $x \in X$. Assume that X is integral at x, then $X_x^{\mathrm{Sing}} \neq |X|_x$.

Proof. Let $n = \dim_x X$. The problem is local on X. By Theorem 3.9 in Constructions of complex analytic spaces, we may assume that there is a finite morphism $\varphi : (X,x) \to (V,0)$, where V is an open neighbourhood of 0 in $\mathbb{C}^n$ and there is a finite $\mathcal{O}_V$-algebra $\mathcal{A}$ with $\mathcal{A}_0 = \mathcal{O}_{X,x}$ such that there is unique point x' of $\mathrm{Spec}_V^{\mathrm{an}} \mathcal{A}$ over 0 and (X,x) can be identified with $(\mathrm{Spec}_V^{\mathrm{an}} \mathcal{A}, x')$.

Take $\xi \in \mathcal{O}_{X,x} = \mathcal{A}_0$ such that

$$\mathrm{Frac}\, \mathcal{O}_{X,x} = \mathrm{Frac}\, \mathcal{O}_{\mathbb{C}^n,0}(\xi).$$

Let $P_0 \in \mathcal{O}_{\mathbb{C}^n,0}[X]$ be the minimal polynomial of ξ. Up to shrinking V, we may assume that ξ spreads to a section $f \in \mathcal{A}(V)$. Then $\mathcal{B} = \mathcal{O}_V[f]$ is a finite sub-$\mathcal{O}_V$-algebra of $\mathcal{A}$. Up to shrinking V, we may assume that the kernel of $\mathcal{O}_V[X] \to \mathcal{B}$ sending X to f is generated by a unitary polynomial $P \in \mathcal{O}_V(V)[X]$ of degree $d := [\mathrm{Frac}\, \mathcal{O}_{X,x} : \mathrm{Frac}\, \mathcal{O}_{\mathbb{C}^n,0}]$ that extends P_0. Therefore,

$$\mathcal{B} \cong \mathcal{O}_V[X]/(P).$$

Let $T = \mathrm{Supp}\, \mathcal{A}/\mathcal{B}$. We endow T with the structure of closed analytic subspace of V induced by the annihilator of $\mathcal{A}/\mathcal{B}$. Observe that $\mathcal{A}_0/\mathcal{B}_0 = \mathcal{O}_{X,x}/\mathcal{O}_{\mathbb{C}^n,0}$ is torsion, so $|T|_0 = \mathrm{Supp}\, \mathcal{A}_0/\mathcal{B}_0 \neq \mathrm{Spec}\, \mathcal{O}_{\mathbb{C}^n,0}$. That is, $T_0 \neq \mathbb{C}^n_0$ by Theorem 3.13 in Constructions of complex analytic spaces. Observe that $X \setminus \varphi^{-1}(T) = \mathrm{Spec}_{V\setminus T}^{\mathrm{an}} \mathcal{B}|_{V \setminus T}$.

On the other hand, $P_0'(\xi) \neq 0$ as ξ is separable. So $W(P'(f)) \neq |X|_x$. Let $Z = \mathrm{Supp}\, \mathcal{O}_X/(P'(f))$, then φ is unramified outside T. Include the parts regarding unramified morphisms and étale morphisms before this section In particular, φ is étale outside T and hence a local isomorphism by Corollary 3.4 in Constructions of complex analytic spaces. In particular,

$$X^{\mathrm{sing}} \subseteq Z \cup \varphi^{-1}(T)$$

and hence

$$X_x^{\mathrm{sing}} \subseteq Z_x \cup \varphi^{-1}(T)_x.$$

The latter is not equal to $|X|_x$ by Corollary 3.14 in Constructions of complex analytic spaces and the fact that $\mathcal{O}_{X,x}$ is integral. $\square$

Theorem 3.5 (Abhyankar)**.** Let X be a complex analytic space and $x \in X$, then

$$X_x^{\mathrm{Sing}} = (\mathrm{Spec}\, \mathcal{O}_{X,x})^{\mathrm{Sing}}.$$

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_{X,x}$. In concrete terms, we need to show that $W(\mathfrak{p}) \not\subset X_x^{\mathrm{Sing}}$ if and only if $\mathrm{Spec}\, \mathcal{O}_{X,x}$ is regular at $\mathfrak{p}$.

The problem is local on X. Up to shrinking X, we may assume that $\mathfrak{p}$ spreads to a coherent ideal $\mathcal{I}$ on X. Let Y be the closed analytic subspace of X defined by $\mathcal{I}$. By Lemma 2.10, up to shrinking X, we may assume that $\mathrm{codim}_y(Y,X)$ is constant for $y \in Y$. We denote this common value as p, which is necessarily equal to the height of $\mathfrak{p}$.

As Y_x is irreducible by assumption, for an analytic set Z in Y satisfying $Z_x \neq |Y|_x$, the following conditions are equivalent:

(1) $|Y|_x \not\subset X_x^{\mathrm{Sing}}$;
(2) $(|Y| \setminus Z)_x \not\subset X_x^{\mathrm{Sing}}$.

(2) $\implies$ (1) is trivial. If (2) fails, then

$$|Y|_x = (|Y| \cup X^{\mathrm{Sing}})_x \cup Z_x.$$

So $|Y|_x = (|Y| \cup X^{\mathrm{Sing}})_x$, namely (1) holds. We apply this remark to

$$Z = Y^{\mathrm{Sing}} \cup S_{p'}(\mathcal{I}/\mathcal{I}^2),$$

where p' is the dimension of the Zariski tangent space of $\operatorname{Spec} \mathcal{O}_{X,x}$ at $\mathfrak{p}$ and $S_{p'}(\mathcal{I}/\mathcal{I}^2)$ is the locus where $\mathcal{I}/\mathcal{I}^2$ is not locally free of rank p'. Note that neither part of Z is equal to $|Y|_x$, the former follows from Theorem 3.4 and the latter follows from Theorem 3.13 in Constructions of complex analytic spaces as clearly $\mathfrak{p} \notin S_{p'}(\mathcal{I}/\mathcal{I}^2)$. We find that $W(\mathfrak{p}) \not\subset X_x^{\mathrm{Sing}}$ if and only if $(|Y| \setminus Z)_x \not\subset X_x^{\mathrm{Sing}}$.

If $y \in |Y| \setminus Z$, then y is a regular point of Y and $\operatorname{codim}_y(Y, X) = p$. On the other hand, $\mathcal{I}/\mathcal{I}^2$ is free of rank p' around y. But given the regularity of $\mathcal{O}_{Y,y}$, the regularity of $\mathcal{O}_{X,y}$ is equivalent to the fact that $\mathcal{I}/\mathcal{I}^2$ is free of rank p. Or equivalently to $p = p'$. The latter is equivalent to the regularity of $\mathfrak{p}$ in $\operatorname{Spec} \mathcal{O}_{X,x}$. The theorem is established. □

Proposition 3.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Then the following are equivalent:

(1) X is regular at x and Y is regular at y;
(2) $X \times Y$ is regular at (x, y).

This follows from Corollary 8.6 in Constructions of complex analytic spaces and Proposition 3.2.

Theorem 3.7. Let X be a complex manifold and A be a thin subset of X. Let $f \in \mathcal{O}_X(X \setminus A)$. Assume that either of the following conditions hold:

(1) f is locally bounded near A;
(2) A is thin of order 2 in X.

Then f admits a unique extension to an element in $\mathcal{O}_X(X)$.

Proof. The problem is local on X. By Proposition 3.2, we may assume that X is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In this case, the results are the classical Riemann extension theorem. □

Corollary 3.8. Let X be a connected complex manifold and A be a thin set in X. Then $X \setminus A$ is connected.

Proof. Assume that $X \setminus A$ can be written as the disjoint union of two open subsets U_0, U_1. Then the function $f \in \mathcal{O}_X(X \setminus A) = \mathcal{O}_X(U_0) \times \mathcal{O}_X(U_1)$ given by $0 \in \mathcal{O}_X(U_0)$ and $1 \in \mathcal{O}_X(U_1)$ is locally bounded near A. By Theorem 3.7, f admits a unique extension to $g \in \mathcal{O}_X(X)$. As X is connected and the image of f is contained in $\overline{\{0,1\}} = \{0,1\}$, it follows that f is constant, so U_0 or U_1 has to be empty. □

4. Serre's condition R_n

Fix $n \in \mathbb{N}$ in this section.

Definition 4.1. Let X be a complex analytic space, we say X *satisfies* R_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* R_n at $x \in X$.

We say X *satisfies* R_n if X satisfies R_n at all points $x \in X$.

Proposition 4.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x;
(2) $\hat{\mathcal{O}}_{X,x}$ satisfies R_n.

PROOF. This follows from [Stacks, Tag 07NY]. $\square$

Proposition 4.3. Let X be a complex analytic space, $x \in X$ and $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x;
(2) $\operatorname{codim}_x(X^{\mathrm{Sing}}, X) > n$.

PROOF. It follows from Theorem 3.5 that (1) holds if and only if $\operatorname{codim}_x(X_x^{\mathrm{Sing}}, \operatorname{Spec} \mathcal{O}_{X,x}) > n$, The latter condition is equivalent to (2) by definition. $\square$

Corollary 4.4. Let X be a complex analytic space and $n \in \mathbb{N}$. The

$$\{x \in X : X \text{ satisfies } R_n \text{ at } x\}$$

is co-analytic in X.

PROOF. This follows from Proposition 4.3 and Corollary 2.11. $\square$

Proposition 4.5. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x and Y satisfies R_n at y;
(2) $X \times Y$ satisfies R_n at (x, y).

PROOF. By Proposition 3.6,

$$(X \times Y)^{\mathrm{Sing}} = (X^{\mathrm{Sing}} \times Y) \cup (X \times Y^{\mathrm{Sing}}).$$

It follows that

$$\operatorname{codim}_{(x,y)}((X \times Y)^{\mathrm{Sing}}, X \times Y) = \min \left\{ \operatorname{codim}_x(X^{\mathrm{Sing}}, X), \operatorname{codim}_y(Y^{\mathrm{Sing}}, Y) \right\}$$

We conclude by Proposition 4.3. $\square$

5. Serre's condition S_n

Fix $n \in \mathbb{N}$ in this section.

Definition 5.1. Let X be a complex analytic space, we say X *satisfies* S_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* S_n at $x \in X$.

We say X *satisfies* S_n if X satisfies S_n at all points $x \in X$.

Proposition 5.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies S_n at x;
(2) $\hat{\mathcal{O}}_{X,x}$ satisfies S_n.

PROOF. This follows from the fact that $\mathcal{O}_{X,x}$ is the quotient of a regular local ring. Include a reference $\square$

Proposition 5.3. Let X be a complex analytic space, $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and $n \in \mathbb{N}$. Then

$$\left\{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\}$$

is an analytic subset of X. Moreover, the germ of this set in $\operatorname{Spec} \mathcal{O}_{X,x}$ is equal to

$$\left\{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x} : \operatorname{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}} > n\right\}.$$

Proof. **Step 1**. We reduce to the case where X is smooth and equidimensional of dimension N.

The problem is local in X, so we may assume that X is a complex model space. Assume that X is a closed analytic subspace of a domain U in $\mathbb{C}^m$ for some $m \in \mathbb{N}$. For any $x \in X$, we have

$$\operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \operatorname{codep}_{\mathcal{O}_{U,x}} \mathcal{G}_x,$$

where $\mathcal{G}$ is the zero-extension of $\mathcal{F}$ to U. A similar formula holds for $\operatorname{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}}$. So it suffices to handle U instead of X.

Step 2. We prove the result after the reduction in Step 1.

By Auslander–Buchsbaum formula, for $x \in X$,

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x + \operatorname{dep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \operatorname{dep} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}.$$

So the condition $\operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n$ is equivalent to

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \dim \mathcal{O}_{X,x} - \dim_x \operatorname{Supp} \mathcal{F}.$$

As $\mathcal{O}_{X,x}$ is regular hence equidimensional, the condition just means

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \operatorname{codim}_x(\operatorname{Supp} \mathcal{F}, X).$$

As $\mathcal{O}_{X,x}$ is regular, this condition is equivalent to the existence of an integer $r > n + \operatorname{codim}_x(\operatorname{Supp} \mathcal{F}, X)$ such that

$$\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_x \neq 0.$$

For each $p \in \mathbb{N}$, we introduce

$$T_p(\mathcal{F}) := \bigcup_{r=p+1}^{N} \operatorname{Supp} \mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

Then the proceeding analysis shows that

$$\left\{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\} = \bigcup_{s=0}^{N} T_{n+s}(\mathcal{F}) \cap \{y \in \operatorname{Supp} \mathcal{F} : \operatorname{codim}_y(\operatorname{Supp} \mathcal{F}, X) \leq s\}.$$

Observe that the right-hand side is an analyitc set in X by Example 4.2 in Constructions of complex analytic spaces and Corollary 2.11, hence so is the left-hand side.

It remains to compute the germ at $y \in X$. For $p \in \mathbb{N}$, we compute

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^{N} \operatorname{Supp} \mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_y.$$

But observe that

$$\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_y = \operatorname{Ext}^r_{\mathcal{O}_{X,y}}(\mathcal{F}_y, \mathcal{O}_{X,y}).$$

Let $\widetilde{\mathcal{F}_y}$ be the coherent module on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated with $\mathcal{F}_y$. Let $X_y = \operatorname{Spec}\mathcal{O}_{X,y}$ Then

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^{N} \operatorname{Supp}\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_{X_y}}(\widetilde{\mathcal{F}_y}, \mathcal{O}_{X_y})_y.$$

On the other hand, by Corollary 2.12, for $s \in \mathbb{N}$,

$$\{x \in \operatorname{Supp}\mathcal{F} : \operatorname{codim}_x(\operatorname{Supp}\mathcal{F}, X) \leq s\}_y = \left\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,y} : \operatorname{codim}_{\mathfrak{p}}(\operatorname{Supp}\widetilde{F_y}, \operatorname{Spec}\mathcal{O}_{X,y})\right\}.$$

The same argument as above shows that

$$\left\{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\}_y = \left\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,y} : \operatorname{codep}_{\mathcal{O}_{X,y,\mathfrak{p}}} \mathcal{F}_{y,\mathfrak{p}} > n\right\}.$$

□

Proposition 5.4. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is the complement of

$$\bigcup_{m=0}^{\infty} \{y \in Z_m : \operatorname{codim}_y(Z_m, X) \leq n+m\},$$

where

$$Z_m = \{x \in X : \operatorname{codep}\mathcal{O}_{X,x}\mathcal{F}_x > m\}.$$

Proof. It suffices to observe that for $x \in X$, X satisfies S_n at x if and only if

$$\operatorname{codim}\left(\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,x} : \operatorname{codep}\mathcal{O}_{X,x,\mathfrak{p}}\}, \operatorname{Spec}\mathcal{O}_{X,x}\right) > n+m$$

for all $m \in \mathbb{N}$. □

Corollary 5.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is co-analytic.

Proof. This follows from Proposition 5.4 and Proposition 5.3. □

Proposition 5.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Take $n \in \mathbb{N}$. Assume that X satisfies S_n at x and Y satisfies S_n at y, then $X \times Y$ satisfies S_n at (x, y).

Proof. By Theorem 5.11 in Complex analytic local algebras,

$$\hat{\mathcal{O}}_{X\times Y,(x,y)} = \hat{\mathcal{O}}_{X,x}\hat{\otimes}\hat{\mathcal{O}}_{Y,y}.$$

As being S_n is invariant under completion by [Stacks, Tag 07NW] and [Stacks, Tag 07NV], it suffices to prove the corresponding algebraic result, which is known. □

6. Reducedness

Definition 6.1. Let X be a complex analytic space, we say X is *reduced* at $x \in X$ if $\mathcal{O}_{X,x}$ is reduced. We also say (X, x) or X_x is *reduced* at $x \in X$.

We say X is *reduced* if X is reduced at all points $x \in X$.

Proposition 6.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is reduced x;
(2) $\hat{\mathcal{O}}_{X,x}$ is reduced.

Proof. This follows from Proposition 4.2 and Proposition 5.2.

Otherwise, one can also argue as follows: Recall that an excellent ring is Nagata by [Stacks, Tag 07QV]. A Nagata noetherian local ring is reduced if and only if its completion is by [Stacks, Tag 07NZ]. □

Theorem 6.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is reduced is co-analytic.

Proof. This follows from Corollary 5.5 and Corollary 4.4 as reduceness is equivalent to S_1 and R_0. □

Corollary 6.4. Let X be a complex analytic space, then the nilradical $\operatorname{rad}\mathcal{O}_X$ is coherent.

Proof. The problem is local on X. Take $x \in X$. Up to shrinking X, we may assume that $\mathcal{O}_{X,x}/(\operatorname{rad}\mathcal{O}_X)_x$ spreads to a finite $\mathcal{O}_X$-algebra $\mathcal{A}$ by Lemma 3.7 in Constructions of complex analytic spaces. Up to further shrinking X, we may assume that $\mathcal{A}$ is the quotient of $\mathcal{O}_X$, say $\mathcal{A} \cong \mathcal{O}_X/\mathcal{I}$ for some coherent ideal $\mathcal{I}$ on X. As $\mathcal{I}_x$ is nilpotent by assumption, up to shrinking X, we may assume that $\mathcal{I}$ is also nilpotent, namely

$$\mathcal{I} \subseteq \operatorname{rad}\mathcal{O}_X.$$

Let Y be the closed analytic subspace of X defined by the ideal $\mathcal{I}$. Then $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}/(\operatorname{rad}\mathcal{O}_X)_x$ is reduced. Up to shrinking X, by Theorem 6.3, we may assume that Y is reduced. But then for any $y \in Y$,

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y$$

is reduced, so

$$\mathcal{I}_y \supseteq (\operatorname{rad}\mathcal{O}_X)_y.$$

It follows that $\operatorname{rad}\mathcal{O}_X = \mathcal{I}$, hence the nilradical is coherent. □

Corollary 6.5 (Cartan–Oka)**.** Let X be a complex analytic space and A be an analytic subset of X, then the sheaf $\mathcal{J}_A$ is coherent.

Recall that the sheaf $\mathcal{J}_A$ is introduced in Definition 4.4 in Constructions of complex analytic spaces.

Proof. By Lemma 4.6 in Constructions of complex analytic spaces, we may assume that A is a closed analytic subspace of X defined by a coherent ideal $\mathcal{I}$. By Corollary 3.14 in Constructions of complex analytic spaces, the sheaf $\mathcal{J}_A$ is nothing but $\sqrt{I}$, which is coherent by Corollary 6.4. □

Corollary 6.6. Let X be a complex analytic space and A be an analytic subset of X, then there is a unique reduced closed analytic space Y of X with underlying set A.

Proof. The existence follows from Corollary 6.5. The uniqueness follows from Corollary 3.14 in Constructions of complex analytic spaces. □

Definition 6.7. Let X be a complex analytic space and A be an analytic subset of X. The analytic space structure on A defined in Corollary 6.6 is called the *reduced induced structure* on A. In particular, $|X|$ with the reduced induced structure is denoted by X^{red} and is called the *reduced space underlying* X.

Theorem 6.8 (Generic smoothness)**.** Let X be a reduced complex analytic space and $x \in X$, then $X^{\mathrm{Sing}}_x \neq |X|_x$. In other words, X^{Sing} is nowhere dense in $|X|$.

PROOF. The problem is local. Take $x \in X$. As in the proof of Theorem 3.3, up to shrinking X, we may assume that there are finitely many closed analytic subsets $X_1, \ldots, X_m$ in X which are irreducible at x such that

$$X = X_1 \cup \cdots \cup X_m.$$

As X is reduced, we may also assume that $X_1, \ldots, X_m$ are all reduced. Then $X_1, \ldots, X_m$ are all integral at x. It follows from Theorem 3.4 that

$$X_i^{\mathrm{Sing}} \neq |X_i|_x$$

for $i = 1, \ldots, m$. Let $\mathcal{I}_i$ be the coherent ideal sheaf of X_i in X for $i = 1, \ldots, m$. It follows from the proof of Theorem 3.3 that

$$X^{\mathrm{sing}} = \bigcap_{i=1}^{m} \left(\operatorname{Supp} \mathcal{I}_i \cup X_i^{\mathrm{Sing}} \right).$$

This implies $X^{\mathrm{Sing}}_x \neq |X|_x$: otherwise, for each $i = 1, \ldots, m$, we have

$$(\operatorname{Supp} \mathcal{I}_i)_x \cup (X_i^{\mathrm{Sing}})_x = |X|_x.$$

So

$$(\operatorname{Supp} \mathcal{I}_i)_x = |X|_x$$

for each $i = 1, \ldots, m$. In other words,

$$\operatorname{Spec} \mathcal{O}_{X,x} = \bigcup_{i=1}^{m} \operatorname{Supp} \mathcal{I}_{i,x}.$$

Observe that $\mathcal{I}_{1,x}, \ldots, \mathcal{I}_{m,x}$ are exactly the minimal primes of $\operatorname{Spec} \mathcal{O}_{X,x}$. This is possible if and only if $m = 1$. So we are reduced to the case where X is integral at x. But this case is handled in Theorem 3.4. □

Proposition 6.9. Let X be a reduced complex analytic space and $f, g \in \mathcal{O}_X(X)$. Assume that $[f] = [g]$, then $f = g$.

PROOF. It follows from Corollary 3.18 in Constructions of complex analytic spaces that $f - g$ is locally nilpotent. As X is reduced, $f = g$. □

In particular, on a reduced complex analytic space X, a holomorphic function f is uniquely determined by the continuous map $[f] : X \to \mathbb{C}$ associated with it. In this case, we will say $[f]$ is holomorphic.

Definition 6.10. Let X be a reduced complex analytic space. A *continuous weakly holomorphic function* on X is a continuous map $f : X \to \mathbb{C}$ such that $f|_{X^{\mathrm{reg}}}$ is holomorphic.

A *weakly holomorphic function* on X is $f \in \mathcal{O}_X(X^{\mathrm{reg}})$ which is locally bounded on X.

Definition 6.11. Let $f : X \to Y$ be a topologically finite surjective morphism of reduced complex analytic spaces. We say f is a *branched covering* if there is a thin subset T of Y satisfying the following properties:

(1) $\pi^{-1}(T)$ is thin in X;
(2) $X \setminus \pi^{-1}(T) \to Y \setminus T$ induced by f is a local isomorphism.

The set T is called a *critial locus*.

The set of points $x \in X$ where f is not a local isomorphism at x is called the *branch locus* of f. The image of the branch locus in Y is called the *minimal critical locus* of f.

Observe that the number of points in the fiber is locally constant outside the critical locus. When this number is actually constant say $b \in \mathbb{N}$ (e.g. when Y is a connected complex manifold by Corollary 3.8), we say f is a *b-sheeted branched covering*.

7. Normalness

Definition 7.1. Let X be a complex analytic space, we say X is *normal* at $x \in X$ if $\mathcal{O}_{X,x}$ is normal. We also say (X,x) or X_x is *normal* at $x \in X$.

We say X is *normal* if X is normal at all points $x \in X$.

Proposition 7.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is normal x;
(2) $\hat{\mathcal{O}}_{X,x}$ is normal.

Condition (2) is usually known as the *analytic normality* of $\mathcal{O}_{X,x}$.

Proof. This follows from Proposition 4.2 and Proposition 5.2. □

Theorem 7.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is normal is co-analytic.

Proof. This follows from Corollary 5.5 and Corollary 4.4 as reduceness is equivalent to S_2 and R_1. □

Proposition 7.4. Let X be a normal complex analytic space. Then for any $x \in X^{\mathrm{Sing}}$,

$$\operatorname{codim}_x(X^{\mathrm{Sing}}, X) \geq 2.$$

Proof. This follows from Theorem 3.5 and the corresponding algebraic result. □

Proposition 7.5. Let X be a reduced complex analytic space. Then there is a finite $\mathcal{O}_X$-algebra $\overline{\mathcal{O}}_X$ such that for each $x \in X$, $\overline{\mathcal{O}}_{X,x}$ is isomorphism to the inclusion of the integral closure $\overline{\mathcal{O}_{X,x}}$ as $\mathcal{O}_{X,x}$-algebras.

The sheaf $\overline{\mathcal{O}}_X$ is unique up to a unique isomorphism.

Proof. The uniqueness is obvious, as there are no non-trivial automorphisms of $\overline{\mathcal{O}}_{X,x}$ as an $\mathcal{O}_{X,x}$-algebra.

We prove the existence. The problem is then local on X. Let $x \in X$. By Lemma 3.7 in Constructions of complex analytic spaces, up to shrinking X, $\overline{\mathcal{O}_{X,x}}$ spreads to a finite $\mathcal{O}_X$-algebra $\mathcal{A}$. Let $X' = \operatorname{Spec}^{\mathrm{an}}_X \mathcal{A}$. Let $x'_1, \ldots, x'_m$ be the distinct points on the fiber over x of $X' \to X$. By Corollary 2.6 in Constructions of complex analytic spaces, the points corresponds to $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_x$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{m'}$ be the minimal primes of $\mathcal{O}_{X,x}$, then

$$\mathcal{A}_x = \overline{\mathcal{O}_{X,x}} \cong \prod_{i=1}^{m'} \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}.$$

As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is Henselian, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is in fact local for each $i = 1, \dots, m'$. As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is finite over $\mathcal{O}_{X,x}/\mathfrak{p}_i$. It follows that $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_x = \operatorname{Spm} \mathcal{A}_x$. So we find that $m' = m$. Up to a renumbering, we may assume that $\mathfrak{p}_i$ corresponds to x'_i for $i = 1, \dots, m$. Then by Corollary 2.6 in Constructions of complex analytic spaces,

$$\mathcal{O}_{X',x'_i} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$$

for $i = 1, \dots, m$. In particular, X' is normal at x'_i for all $i = 1, \dots, m$. By Theorem 7.3, Corollary 3.8 in Constructions of complex analytic spaces and Lemma 4.2 in Constructions of complex analytic spaces, up to shrinking X, we may assume that X' is normal. We observe that for each $y \in X$, $\mathcal{A}_y$ is the product of the local rings of points on the fiber hence normal.

For $i = 1, \dots, m$, as $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, its conductor is non-zero. We can find a non-zero $f_{i,x} \in \mathcal{O}_{X,x}/\mathfrak{p}_i$ such that $f_{i,x}\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i} \subseteq \mathcal{O}_{X,x}/\mathfrak{p}_i$. Take

$$f_x = \prod_{i=1}^{m} f_{i,x}.$$

Then f_x is a non-zero divisor in $\mathcal{O}_{X,x}$ and $f_x \mathcal{A}_x \subseteq \mathcal{O}_{X,x}$. Up to shrinking X, we may assume that f_x spreads to $f \in \mathcal{O}_X(X)$, and we have an injection

$$f\mathcal{A} \subseteq \mathcal{O}_X.$$

Up to shrinking X, we may also assume that $\mathcal{O}_X \to \mathcal{A}$ is injective. We therefore get an injective map

$$\mathcal{A} \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we get an injective map

$$\mathcal{A}_y \to \mathcal{O}_{X,y}[f_y^{-1}].$$

In particular, $\mathcal{A}_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}$. As $\mathcal{A}_y$ is finite over $\mathcal{O}_{X,y}$, we have

$$\mathcal{A}_y \subseteq \overline{\mathcal{O}_{X,y}}.$$

On the other hand, $\mathcal{A}_y$ is normal, so equality holds. □

Definition 7.6. Let X be a reduced complex analytic space. Then $\operatorname{Spec}^{\mathrm{an}}_X \overline{\mathcal{O}}_X$ constructed in Proposition 7.5 is called the *normalization* of X. We denote it by $\bar{X}$. Note that we have a canonical morphism $\bar{X} \to X$.

The normalization of X is well-defined up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$.

Proposition 7.7. Let X be a reduced complex analytic space. For each $x \in X$, the fiber of $\bar{X} \to X$ over x is in bijection with the set of minimal prime ideals in $\mathcal{O}_{X,x}$. Moreover, if y corresponds to $\mathfrak{p}$, we have

$$\mathcal{O}_{\bar{X},y} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}}$$

as $\mathcal{O}_{X,x}$-algebras.

Proof. This follows from the proof of Proposition 7.5. □

Proposition 7.8. Let X be a reduced complex analytic space. Then

(1) $\bar{X}$ is normal;
(2) $p : \bar{X} \to X$ is topologically finite and surjective;

(3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in $\bar{X}$ and the morphism $\bar{X} \setminus p^{-1}(Y) \to X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines $\bar{X}$ up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$. We will establish this result later.

Proof. That $\bar{X}$ is normal follows from Corollary 2.6 in Constructions of complex analytic spaces. The morphism $\bar{X} \to X$ is topologically finite by Corollary 3.8 in Constructions of complex analytic spaces. It is surjective by Corollary 2.7 in Constructions of complex analytic spaces.

Let Y be the non-normal locus of X. It is in particular contained in X^{Sing}. By Proposition 7.4 and Theorem 7.3, Y is a nowhere dense analytic set in X. It is clear that p is an isomorphism outside Y.

We prove that $p^{-1}(Y)$ is nowhere dense. Let $x \in X$ and x' be a point on the fiber of $\bar{X} \to X$ over x. Let $\mathfrak{p}'$ be the minimal prime ideal of $\mathcal{O}_{X,x}$ corresponding to x'. Then the morphism $\operatorname{Spec} \mathcal{O}_{\bar{X},x'} \to \operatorname{Spec} \mathcal{O}_{X,x}$ factorizes through $\operatorname{Spec} \mathcal{O}_{\bar{X},x'} \to \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{p}'$. The map is finite and surjective. The latter is because $\mathcal{O}_{X,x}/\mathfrak{p}' \to \mathcal{O}_{\bar{X},x'}$ is injective. If $p^{-1}(Y)$ contains a neighbourhood of x' in $\bar{X}$, then $|p^{-1}(Y)|_{x'} = \operatorname{Spec} \mathcal{O}_{\bar{X},x'}$. Then $|Y|_x = |\operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{p}'|$, which is a contradiction.

□

Definition 7.9. Let X be a complex analytic space and A be an analytic set in X. We say A is irreducible if A cannot be written as the union of two analytic sets B and C in X with $B \not\subseteq C$ and $C \not\subseteq B$.

Lemma 7.10. Let X be a connected normal complex analytic space. Then X is irreducible.

Proof. Suppose otherwise, X can be written as the union of A, B, two analytic sets in X not containing each other. As X is connected, $A \cap B$ is non-trivial. Take $x \in A \cap B$. We endow A and B with the reduced induced structure. Then

$$\operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} \mathcal{O}_{A,x} \cup \operatorname{Spec} \mathcal{O}_{B,x}.$$

This is impossible as $\mathcal{O}_{X,x}$ is unibranch. □

Definition 7.11. Let X be a reduced complex analytic space. An *irreducible component* of X is the image of a connected component of $\bar{X}$.

We say X is *irreducible* if it is non-empty has only one irreducible component.

By Lemma 7.10, each irreducible component is irreducible. Moreover, by Proposition 7.8, the decomposition of $|X|$ into the union of its irreducible components is locally finite. No irreducible component is contained in the union of the others.

Proposition 7.12. Let X be a reduced complex analytic space and $x \in X$. Then x be be joined by a path to a point in X^{reg}.

Proof. We may assume that $x \in X^{\mathrm{Sing}}$.

Step 1. We reduce to the case where X is normal.

Let $p : \bar{X} \to X$ be the normalization. Take $y \in \bar{X}$ with $p(y) = x$.

We claim that it suffices to show that there is a path connecting y to a regular point of $\bar{X}$. In fact, let $T \subseteq X$ containing X^{Sing} be a thin analytic set such that $p^{-1}(T)$ is thin and $\bar{X} \setminus p^{-1}(T) \to X \setminus T$ induced by p is an isomorphism by

Proposition 7.8. If our claim holds, then all neighbourhood points of y are regular and in particular, we may connect y to a regular poiint in $\tilde{X} \setminus p^{-1}(T)$. The image of this path is the desired path.

Step 2. We proceed by induction on $d := \dim_x X$.

When $d = 1$, x is necessarily regular by Proposition 7.4. Assume $d > 1$. Up to shrinking X, we can take $f \in \mathcal{O}_X(X)$ such that $\dim_x W(f) = d - 1$. We may assume that $W(f)$ is equidimensional of dimension $d-1$ by Theorem 2.4. Then we can find a path from x to a regular point $x' \in W(f)$. By Proposition 7.4, up to perturbation, we may assume that $x' \in X^{\mathrm{reg}}$. $\square$

8. Unibranchness

Definition 8.1. Let X be a complex analytic space, we say X is *unibranch* at $x \in X$ if $\mathcal{O}_{X,x}$ is unibranch. We also say (X, x) or X_x is *unibranch* at $x \in X$.

We say X is *unibranch* if X is unibranch at all points $x \in X$.

Proposition 8.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is unibranch at x;
(2) X^{red} is unibranch at x;
(3) $\mathcal{O}_{X,x}$ is geometrically unibranch;
(4) $\mathcal{O}_{X,x}^{\mathrm{red}}$ is geometrically unibranch;
(5) $\mathcal{O}_{X,x}$ has a unique minimal prime ideal;
(6) The fiber of $\overline{X^{\mathrm{red}}} \to X^{\mathrm{red}}$ over x consists of a single point.

Proof. (1) $\Leftrightarrow$ (3): As $\mathcal{O}_{X,x}$ is excellent, $\overline{\mathcal{O}_{X,x}^{\mathrm{red}}}$ is a finite $\mathcal{O}_{X,x}^{\mathrm{red}}$-algebra, so the residue field extension is finite. But the residue field of $\mathcal{O}_{X,x}$ is $\mathbb{C}$, so the residue field extension is the trivial extension.

(1) $\Leftrightarrow$ (5): This follows from [Stacks, Tag 0BQ0] and the fact that $\mathcal{O}_{X,x}$ is Henselian.

(1) $\Leftrightarrow$ (2): This follows from the observation that (5) holds for $\mathcal{O}_{X,x}$ if and only if (5) holds for $\mathcal{O}_{X,x}^{\mathrm{red}}$.

(3) $\Leftrightarrow$ (4): This follows from the same argument as (1) $\Leftrightarrow$ (2).

(5) $\Leftrightarrow$ (6): This follows from Proposition 7.7. $\square$

Lemma 8.3. Let X be a complex analytic space, $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, $n \in \mathbb{N}$. Then the set

$$\{x \in X : \operatorname{rank}_x \mathcal{M} \le n\}$$

is an analytic set in X.

Proof. The problem is local on X, we may assume that $\mathcal{M}$ admits a presentation

$$\mathcal{O}_X^p \to \mathcal{O}_X^q \to \mathcal{M} \to 0,$$

where $p, q \in \mathbb{N}$. Up to shrinking X, we may assume that the first map is given by a $p \times q$ matrix M in $\mathcal{O}_X(X)$. The condition that $\operatorname{rank}_x \mathcal{M} \le n$ is the same as $\operatorname{rank} M_x \le n$, which is defines an analytic set in X. $\square$

Lemma 8.4. Let X be a reduced complex analytic space and $x \in X$. Then for any neighbourhood V of x in X, we can find an open neighbourhood U of x in X contained in V such that U has only finitely many irreducible componenets and all irreducible componenets of U contain x.

PROOF. Take an open neighbourhood W of x in X contained in V such that $\bar{W}$ is compact and decompose W into irreducible components $W_1, \dots, W_k, W_{k+1}, \dots, W_n$, where $W_1, \dots, W_k$ contain x and $W_{k+1}, \dots, W_n$ do not. It suffices to take

$$U = \left(\bigcup_{i=1}^{k} W_i \right) \setminus \left(\bigcup_{j=k+1}^{n} W_j \right).$$

□

Proposition 8.5. Let X be a reduced complex analytic space and $x \in X$. Assume that X is unibranch at x. Then for any neighbourhood V of x in X, there is an open neighbourhood U of x in X contained in V such that U is unibranch and hence connected.

In particular, the unibranch locus is open.

PROOF. The assertion follows from Lemma 8.4. □

Corollary 8.6. Let X be a complex analytic space. Then X is locally connected.

PROOF. We may assume that X is reduced. The assertion follows from Lemma 8.4 and □

9. Cohen–Macaulay property

Definition 9.1. Let X be a complex analytic space, we say X is *Cohen–Macaulay* at $x \in X$ if $\mathcal{O}_{X,x}$ is Cohen–Macaulay. We also say (X, x) or X_x is *Cohen–Macaulay* at $x \in X$.

We say X is *Cohen–Macaulay* if X is Cohen–Macaulay at all points $x \in X$.

The reduction and normalization of a Cohen–Macaylay space are not necessarily Cohen–Macaulay.

Theorem 9.2. Let X be a complex analytic space. Then the set

$$\{x \in X : X \text{ is Cohen–Macaulay at } x\}$$

is co-analytic.

PROOF. The set is exactly where $\{x \in X : \operatorname{codep}_x \mathcal{O}_{X,x} = 0\}$, which is co-analytic by Proposition 5.3. □

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