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Contents

Global properties of complex analytic spaces

1. Introduction

2. Holomorphically convex hulls

Definition 2.1. Let X be a complex analytic space and M be a subset of X, we define the *holomorphically convex hull* of M in X as

$$\hat{M}^X := \left\{ x \in X : |f(x)| \leq \sup_{y \in M} |f(y)| \text{ for all } f \in \mathcal{O}_X(X) \right\}.$$

Proposition 2.2. Let X be a complex analytic space and M be a subset of X. Then the following properties hold:

(1) $\hat{M}^X$ is closed in X;
(2) $M \subseteq \hat{M}^X$ and $\widehat{\hat{M}^X}^X = \hat{M}^X$;
(3) If M' is another subset of X containing M, then $\hat{M}^X \subseteq \hat{M'}^X$;
(4) If $f : Y \to X$ is a morphism of complex analytic spaces, then

$$\widehat{f^{-1}(M)}^Y \subseteq f^{-1}(\hat{M}^X);$$

(5) If X' is another complex analytic space and M' is a subset of X', then

$$\widehat{M \times M'}^{X \times X'} \subseteq \hat{M}^X \times \hat{M'}^{X'};$$

(6) If M' is another subset of X and $\hat{M}^X = M$, $\hat{M'}^X = M'$, then

$$\widehat{M \cap M'}^X = M \cap M'.$$

Proof. (1), (2), (3), (4), (5) are obvious by definition.
(6) is a consequence of (3). $\square$

Example 2.3. Let Q be a compact cube in $\mathbb{C}^n$ for some $n \in \mathbb{N}$, then $\hat{Q}^{\mathbb{C}^n} = Q$.

In fact, by Proposition 2.2(5), we may assume that $n = 1$. Given $p \in \mathbb{C} \setminus Q$, we can take a closed disk $T \subseteq \mathbb{C}$ centered at $a \in \mathbb{C}$ such that $Q \subseteq T$ while $p \notin T$. Consider $z - a \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$, then

$$|f(p)| > \sup_{q \in Q} |f(q)|.$$

So $p \notin \hat{Q}^{\mathbb{C}}$.

3. Stones

Definition 3.1. Let X be a complex analytic space. A *stone* in X is a pair (P, π) consisting of

(1) a non-empty compact set P in X and

(2) a morphism $\pi : X \to \mathbb{C}^n$ for some $n \in \mathbb{N}$

such that there is a compact tube Q in $\mathbb{C}^n$ and an open set W in X such that $P = \pi^{-1}(Q) \cap W$.

We call $P^0 := \pi^{-1}(\operatorname{Int} Q) \cap W$ the *analytic interior* of the stone (P, π). It clearly does not depend on the choice of W.

We observe that $\hat{P}^X \cap W = P$. In fact, $P \subseteq \pi^{-1}(Q)$, so

$$\hat{P}^X \subseteq \pi^{-1}(\hat{Q}^{\mathbb{C}^n}) = \pi^{-1}(Q) = P \cap W = P.$$

Here we applied Proposition 2.2 and Example 2.3.

In general, $P^0 \subseteq \operatorname{Int} P$, but they can be different.

Theorem 3.2. Let X be a Hausdorff complex analytic space and $K \subseteq X$ be a compact subset. Then the following are equivalent:

(1) There is an open neighbourhood W of K in X such that $\hat{K}^X \cap W$ is compact;
(2) There is an open relative compact neighbourhood W of K in X such that $\partial W \cap \hat{K} = \emptyset$;
(3) There is a stone (P, π) in X with $K \subseteq P^0$.

Proof. (1) $\implies$ (2): This is trivial, in fact, we may assume that W in (1) is relatively compact in X.

(2) $\implies$ (3): As $\hat{K}^X$ is closed by Proposition 2.2(1) and $\partial W \cap \hat{K}^X = \emptyset$, given $p \in \partial W$, we can find $h \in \mathcal{O}_X(X)$ such that

$$\sup_{x \in K} |h(x)| < 1 < |h(p)|.$$

We will denote the left-hand side by $|h|_K$. Up to raising h to a power, we may assume that

$$\max\{|\operatorname{Re} h(p)|, |\operatorname{Im} h(p)|\} > 1.$$

As ∂W is compact, we can find finitely many sections $h_1, \dots, h_m \in \mathcal{O}_X(X)$ so that

$$\max_{j=1,\dots,m} \{|\operatorname{Re} h_j|_K, |\operatorname{Im} h_j|_K\} < 1, \quad \max_{j=1,\dots,m} \{|\operatorname{Re} h_j(p)|, |\operatorname{Im} h_j(p)|\} > 1.$$

Let

$$Q := \{(z_1, \dots, z_m) \in \mathbb{C}^m : |\operatorname{Re} z_i| \leq 1, |\operatorname{Im} z_i| \leq 1 \text{ for all } i = 1, \dots, m\}.$$

The sections $h_1, \dots, h_m$ defines a homomorphism $\pi : X \to \mathbb{C}^m$ by Theorem 4.2 in The notion of complex analytic spaces. Obviously, $P = \pi^{-1}(Q) \cap W$ satisfies our assumptions.

(3) $\implies$ (1): Let W be the open set as in Definition 3.1. As $\hat{P}^X \cap W = P$ and $K \subseteq P$, we have

$$\hat{K} \cap W \subseteq P \cap W = P.$$

As P is compact, so is $\hat{K} \cap W$. □

Theorem 3.3. Let X be a Hausdorff complex analytic space and $(P, \pi : X \to \mathbb{C}^n)$ be a stone in X. Let Q be the tube in $\mathbb{C}^m$ as in Definition 3.1. Then there are open neighbourhoods U and V of P and Q in X and $\mathbb{C}^n$ respectively with $\pi(U) \subseteq V$ and $P = \pi^{-1}(Q) \cap U$ such that $\pi|_U : U \to V$ is proper.

PROOF. Let $W \subseteq X$ be the open set as in Definition 3.1. We may assume that W is relatively compact. Then ∂W and $\pi(\partial W)$ are also compact. As $\partial W \cap \pi^{-1}(Q)$ is empty, we know that $V := \mathbb{C}^n \setminus \pi(\partial W)$ is an open neighbourhood of Q. The set $U := W \cap \pi^{-1}(V) = W \setminus \pi^{-1}(\pi(\partial W))$ is open in X and $\pi(U) \subseteq V$. Observe that $\pi|_U : U \to V$ is proper by Lemma 4.6 in Topology and bornology.

Furthermore,

$$\pi^{-1}(Q) \cap U = \pi^{-1}(Q) \cap \left(W \setminus \left(\pi^{-1}(Q) \cap \pi^{-1}\pi(\partial W)\right)\right).$$

But $\pi^{-1}Q \cap \pi^{-1}\pi(\partial W)$ is empty as $Q \cap \pi(\partial W)$ is. It follows that $\pi^{-1}(Q) \cap U = P$ and hence U is a neighbourhood of P. □

Definition 3.4. Let X be a complex analytic space. Let $(P, \pi : X \to \mathbb{C}^n)$, $(P', \pi' : X \to \mathbb{C}^{n'})$ be two stones on X. We say (P, π) *is contained in* (P', π') if the following conditions are satisfied:

(1) P lies in the analytic interior of P';
(2) $n' \geq n$ and there is $q \in \mathbb{C}^{n'-n}$ such that if $Q \subseteq \mathbb{C}^n$, $\mathbb{Q}' \subseteq \mathbb{C}^{n'}$ be the tubes as in Definition 3.1, then
$$Q \times \{q\} \subseteq Q'.$$
(3) There is a morphism $\varphi : X \to \mathbb{C}^{n'-n}$ such that
$$\pi' = (\pi, \varphi).$$

We formally write $(P, \pi) \subseteq (P', \pi')$ in this case. Clearly, this defines a partial order on the set of stones on X.

Definition 3.5. Let X be a complex analytic space. An *exhaustion of* X *by stones* is a sequence $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ of stones such that

(1) $(P_i, \pi_i) \subseteq (P_{i+1}, \pi_{i+1})$ for all $i \in \mathbb{Z}_{>0}$;
(2)
$$X = \bigcup_{i=1}^{\infty} P_i^0.$$

We say X is *weakly holomorphically convex* if it there is an exhaustion of X by stones.

Theorem 3.6. Let X be a Hausdorff complex analytic space. Consider the following conditions:

(1) X is weakly holomorphically convex;
(2) For any compact subset $K \subseteq X$, there is an open set $W \subseteq X$ such that $\hat{K}^X \cap W$ is compact.

Then (1) $\implies$ (2). If X is Lindelöf, then (2) $\implies$ (1).

PROOF. (1) $\implies$ (2): It suffices to observe that $K \subseteq P_j^0$ when j is large enough and apply Theorem 3.2.

Assume that X is Lindelöf. (2) $\implies$ (1): Let (K_i) a compact exhaustion of X. We construcct the stones $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ so that

$$K_i \subseteq P_i^0$$

for all $i \in \mathbb{Z}_{>0}$ inductively. Let P_1 be an arbitrary stone in X such that $K_1 \subseteq P_1^0$. The existence of P_1 is guaranteed by Theorem 3.2.

Assume that we have constructed $(P_{i-1}, \pi_{i-1} : X \to \mathbb{C}^{n_{i-1}})$ for $i \geq 2$. Let $Q_{i-1} \subseteq \mathbb{C}^{n_{i-1}}$ be the associated tube. By Theorem 3.2 again, take a stone $(P_i, \pi_i^* : X \to \mathbb{C}^n)$ with $K_i \cup P_{i-1} \subseteq P_i^0$. Let $Q_i^* \subseteq \mathbb{C}^n$ be the associated tube. Let W be an open subset of X with

$$P_i = \pi_i^{*,-1}(Q_i^*) \cap W.$$

Choose a tube $Q_i' \subseteq \mathbb{C}^{n_{i-1}}$ with $Q_{i-1} \subseteq \operatorname{Int} Q_i'$ so that

$$\pi_{i-1}(P_i) \subseteq \operatorname{Int} Q_i'.$$

Let $\pi_i := (\pi_{i-1}, \pi_i^*) : X \to \mathbb{C}^{n_{i-1}+n}$ and $Q_i := Q_i' \times Q_i^*$. Then (P_i, π_i) is a stone and $(P_{i-1}, \pi_{i-1}) \subseteq (P_i, \pi_i)$. □

4. Holomorphical separable spaces

Definition 4.1. Let X be a complex analytic space. We say X is *holomorphically separable* if for any $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

Here we regard f as a continuous function $X \to \mathbb{C}$. In particular, a holomorphically separable space is Hausdorff.

Definition 4.2. Let X be a complex analytic space. We say X is *holomorphically convex* if $|X|$ is Hausdorff and for any compact set $K \subseteq X$, $\hat{K}^X$.

We say X is *weakly holomorphically convex* if for any quasi-compact set $K \subseteq X$, the connected components of $\hat{K}^X$ are all quasi-compact.

Proposition 4.3. Let X be a holomorphically convex complex analytic space. Then X^{red} is holomorphically convex.

Proof. This follows immediately from the definition. □

Proposition 4.4. Let X be a Hausdorff complex analytic space. Consider the following conditions:

(1) X is holomorphically convex;
(2) For any sequence $x_i \in X$ ($i \in \mathbb{Z}_{>0}$) without accumulation points, there is $f \in \mathcal{O}_X(X)$ such that $|f(x_i)|$ is unbounded.

Then (2) $\implies$ (1) if X is Lindelöf.

Proof. (2) $\implies$ (1): For a Lindelöf Hausdorff space, sequential compactness implies compactness.

□

Corollary 4.5. Let $n \in \mathbb{N}$ and Ω be a domain in $\mathbb{C}^n$. Assume that for each $p \in \partial\Omega$, there is a holomorphic function f on an open neighbourhood U of $\bar{\Omega}$ such that $f(p) = 0$ and f is non-zero on Ω. Then Ω is holomorphically convex.

Proof. Let $x_i \in \Omega$ ($i \in \mathbb{Z}_{>0}$) be a sequence without accumulation points in Ω. We need to construct $f \in \mathcal{O}_\Omega(\Omega)$ such that $(|f(x_i)|)_{i \in \mathbb{Z}_{>0}}$ is unbounded. This is clear if x_i itself is unbounded. Assume that x_i is bounded. Then up to passing to a subsequence, we may assume that $x_i \to p \in \partial\Omega$ as $i \to \infty$. The inverse of the function f in our assumption of the corollary works. □

5. Stein sets

Definition 5.1. Let X be a complex analytic space and P be a closed subset of X. We say P is a *Stein set* in X if for any coherent $\mathcal{O}_U$-module $\mathcal{F}$ for some open neighbourhood U of P in X, we have

$$H^i(P, \mathcal{F}) = 0 \quad \text{for all } i \in \mathbb{Z}_{>0}.$$

A *coherent $\mathcal{O}_P$-module* is a coherent $\mathcal{O}_U$-module for some open neighbourhood U of P in X. Two coherent $\mathcal{O}_P$-modules are isomorphic if there is a small enough open neighbourhood V of P in X such that they are isomorphic when restricted to V. In particular, $\mathcal{O}_P$ denotes the coherent $\mathcal{O}_P$-module defined by $\mathcal{O}_X$ on X.

The germ-wise notions obviously make sense for coherent $\mathcal{O}_P$-modules.

The given condition is usually known as *Cartan's Theorem B*. It implies *Cartan's Theorem A*:

Theorem 5.2 (Cartan's Theorem A)**.** Let X be a complex analytic space and P be a Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module for some open neighbourhood U of P in X. Then $H^0(P, \mathcal{F})$ generates $\mathcal{F}_x$ for each $x \in P$.

Proof. Fix $x \in P$. Let $\mathcal{M}$ be the coherent ideal sheaf on U consisting of holomorphic functions vanishing at x. Then $\mathcal{F}\mathcal{M}$ is a coherent $\mathcal{O}_U$-module. It follows from Theorem B that

$$H^0(P, \mathcal{F}) \to H^0(P, \mathcal{F}/\mathcal{F}\mathcal{M})$$

is surjective. Note that we can identify this map with the natural map

$$H^0(P, \mathcal{F}) \to \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x.$$

Let $e_1, \ldots, e_m$ be a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Lift them to $s_1, \ldots, s_m \in H^0(P, \mathcal{F})$. By Nakayama's lemma, $s_{1x}, \ldots, s_{mx}$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. □

Corollary 5.3. Let X be a complex analytic space and P be a quasi-compact Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-module. Then there is $n \in \mathbb{Z}_{>0}$ and an epimorphism

$$\mathcal{O}_P^n \to \mathcal{F}.$$

Proof. By Theorem 5.2, we can find an open covering $\{U_i\}_{i\in I}$ of P such that there are homomorphisms

$$h_i : \mathcal{O}_P^{n_i} \to \mathcal{F}$$

for some $n_i \in \mathbb{Z}_{>0}$, which is surjective on U_i for each $i \in I$. By the quasi-compactness of P, we may assume that I is a finite set. Then it suffices to set $n = \sum_{i\in I} n_i$ and consider the epimorphism $\mathcal{O}_P^n \to \mathcal{F}$ induced by the h_i's. □

Theorem 5.4. Let X be a complex analytic space and $P \subseteq X$ be a set with the following properties:

(1) there is an open neighbourhood U of P in X, a domain V in $\mathbb{C}^m$ for some $m \in \mathbb{N}$ and a finite holomorphic morphism $\tau : U \to V$;
(2) There exists a compact tube in $\mathbb{C}^m$ contained in V such that $P = \tau^{-1}(Q)$.

Then P is a compact Stein set in X.

Proof. As $P=\tau^{-1}(Q)$ and τ is proper, we see that P is compact.

It remains to show that P is a Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-module.

Step 1. We first reduce to the case where $\mathcal{F}$ is defined by a coherent $\mathcal{O}_U$-module.

Take an open neighbourhood U' of P in X contained in U such that $\mathcal{F}$ is defined by a coherent $\mathcal{O}_{U'}$-module. By Lemma 4.2 in Topology and bornology, we can take an open neighbourhood V' of Q in V such that $\tau^{-1}(V')\subseteq U'$. The restriction of τ to $\tau^{-1}(V')\to V'$ is again finite.

Step 2. By Leray spectral sequence,

$$H^i(P,\mathcal{F})\cong H^i(Q,(\tau|_P)_*\mathcal{F})$$

for all $i\geq 0$. By Corollary 4.8 in Morphisms between complex analytic spaces, $(\tau|_P)_*\mathcal{F}$ is a coherent $\mathcal{O}_Q$-module, so we are reduced to show that Q is a Stein set in $\mathbb{C}^m$, which is well-known. □

Definition 5.5. Let X be a Hausdorff complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. A *Stein exhaustion of X relative to $\mathcal{F}$* is a compact exhaustion $(P_i)_{i\in\mathbb{Z}_{>0}}$ such that the following conditions are satisfied:

(1) P_i is a Stein set in X for each $i\in\mathbb{Z}_{>0}$;

(2) the $\mathbb{C}$-vector space $H^0(P_i,\mathcal{F})$ admits a semi-norm $|\bullet|_i$ such that the restriction map
$$H^0(X,\mathcal{F})\to H^0(P_i,\mathcal{F})$$
has dense image with respect to the topological defined by $|\bullet|_i$ for each $i\in\mathbb{Z}_{>0}$;

(3) The restriction map
$$H^0(P_{i+1},\mathcal{F})\to H^0(P_i,\mathcal{F})$$
is bounded for each $i\in\mathbb{Z}_{>0}$;

(4) Let $i\in\mathbb{Z}_{\geq 2}$. Suppose that $(s_j)_{j\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $H^0(P_i,\mathcal{F})$, then the restricted sequence $s_j|_{P_{i-1}}$ has a limit in $H^0(P_{i-1},\mathcal{F})$;

(5) Let $i\in\mathbb{Z}_{\geq 2}$. If $s\in H^0(P_i,\mathcal{F})$ and $|s|_i=0$, then $s|_{P_{i-1}}=0$.

A *Stein exhaustion of* X is a compact exhaustion of X that is a Stein exhaustion of X relative to any coherent $\mathcal{O}_X$-module.

Theorem 5.6. Let X be a Hausdorff complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $(P_i)_{i\in\mathbb{Z}_{>0}}$ is a Stein exhaustion of X relative to $\mathcal{F}$. Then

$$H^q(X,\mathcal{F})=0\quad\text{for any } q\in\mathbb{Z}_{>0}.$$

Proof. When $q\geq 2$, this follows from the general facts proved in Lemma 5.4 in Topology and bornology. We will assume that $q=1$.

We may assume that X is connected. First observe that X is necessarily paracompact. This follows from Proposition 3.2 in Topology and bornology. In particular, we can take a flabby resolution

$$0\to\mathcal{F}\to\mathcal{G}^0\to\mathcal{G}^1\to\cdots.$$

Taking global sections, we get a complex

$$0\to H^0(X,\mathcal{F})\xrightarrow{i}H^0(X,\mathcal{G}^0)\xrightarrow{d_0}H^0(X,\mathcal{G}^1)\xrightarrow{d_1}H^0(X,\mathcal{G}^2)\xrightarrow{d_2}\cdots.$$

We need to show that $\ker d_1=\operatorname{Im} d_0$. Let $\alpha\in\ker d_1$. We need to construct $\beta\in H^0(X,\mathcal{G}^0)$ with $d_0\beta=\alpha$.

We take semi-norms $|\bullet|_i$ on $H^0(P_i,\mathcal{F})$ for each $i\in\mathbb{Z}_{>0}$ satisfying the conditions in Definition 5.5. We may furthermore assume that the restriction $H^0(P_{i+1},\mathcal{F})\to H^0(P_i,\mathcal{F})$ is a contraction for each $i\in\mathbb{Z}_{>0}$.

For each $j\in\mathbb{Z}_{\geq 2}$, we will construct $\beta_j\in H^0(P_j,\mathcal{G}^0)$ and $\delta_j\in H^0(P_{j-1},\mathcal{F})$ such that

(1) $(d_0|_{P_j})\beta_j=\alpha|_{P_j}$;
(2) $(\beta_{j+1}+\delta_{j+1})|_{P_{j-1}}=(\beta_j+\delta_j)|_{P_{j-1}}$.

It suffices to take $\beta\in H^0(X,\mathcal{G}^0)$ as the section defined by the $\beta_j+\delta_j$'s.

We first construct β_j. Choose a sequence $\beta'_j\in H^0(P_j,\mathcal{G}^0)$ with

$$(d_0|_{P_j})\beta'_j=\alpha|_{P_j}$$

for each $j\in\mathbb{Z}_{>0}$. This is possible because P_j is Stein. We define β_j satisfying Condition (1) for $j\in\mathbb{Z}_{>0}$ inductively. We begin with $\beta_1=\beta'_1$. Assume that $\beta_1,\ldots,\beta_j$ have been constructed. Let

$$\gamma'_j:=\beta'_{j+1}|_{P_j}-\beta_j.$$

Then

$$(d_0|_{P_j})\gamma'_j=0.$$

It follows that $\gamma'_j\in H^0(P_j,\mathcal{F})$. Take $\gamma_j\in H^0(X,\mathcal{F})$ with

$$|\gamma'_j-\gamma_j|_{P_j}|_j\leq 2^{-j}.$$

Define

$$\beta_{j+1}=\beta'_{j+1}-\gamma_i|_{P_{j+1}}.$$

Then clearly β_{j+1} satisfies (1).

Next we construct the sequence δ_j.

We observe that for each $j\in\mathbb{Z}_{>0}$,

$$\left|\beta_{j+1}|_{P_j}-\beta_j\right|_j\leq 2^{-j}.$$

Let

$$s^j_k:=\beta_{j+k}|_{P_j}-\beta_j\in H^0(P_j,\mathcal{F})$$

for all $j\in\mathbb{Z}_{>0}$ and $k\in\mathbb{N}$. By definition,

$$s^j_k-s^{j+1}_{k-1}|_{P_j}=\beta_{j+1}|_{P_j}-\beta_j$$

for all $j\in\mathbb{Z}_{>0}$ and $k\in\mathbb{Z}_{>0}$.

We claim that $(s^j_k|_{P_{j-1}})_k$ converges in $H^0(P_{j-1},\mathcal{F})$ as $k\to\infty$. By our assumption, it suffices to show that $(s^j_k)_k$ is a Cauchy sequence in $H^0(P_j,\mathcal{F})$ for each $j\in\mathbb{Z}_{>1}$. We first compute

$$\left|\beta_{j+l}|_{P_j}-\beta_{j+l-1}|_{P_j}\right|_j\leq\left|\beta_{j+l}|_{P_{j+l-1}}-\beta_{j+l-1}\right|_{j+l-1}\leq 2^{1-j-l}$$

for all $l\in\mathbb{Z}_{>0}$ and $j\in\mathbb{Z}_{>0}$. As a consequence for $k'>k\geq 1$, we have

$$|s^j_k-s^j_{k'}|_j\leq\sum_{l=k+1}^{k}2^{1-j-l}\leq 2^{1-j+k}.$$

So we conclude our claim.

Let δ_j be the limit of $s^j_k|_{P_{j-1}}$ as $k\to\infty$ for each $j\in\mathbb{Z}_{\geq 2}$. Then

$$\lim_{k\to\infty}\left(s^j_k-s^{j+1}_{k-1}\right)|_{P_{j-1}}=\left(\delta_j-\delta_{j+1}\right)|_{P_{j-1}}$$

for each $j\in\mathbb{Z}_{\geq 2}$. The desired identity is clear. $\square$

6. Analytic blocks

Definition 6.1. Let X be a Hausdorff complex analytic space. A stone $(P, \pi : X \to \mathbb{C}^n)$ on X is an *analytic block* in X if there are open neighbourhoods U and V of P and Q in X and Y respectively, where $Q \subseteq \mathbb{C}^n$ denotes the tube associated with the stone, such that

(1) $\pi(U) \subseteq V$;
(2) $P = \pi^{-1}(Q) \cap U$;
(3) $U \to V$ induced by π is a finite morphism.

Recall that by Theorem 3.3, we can always assume that $U \to V$ is proper.

Proposition 6.2. Let X be a Hausdorff complex analytic space and (P, π) be an analytic block in X. Then P is a compact Stein set in X.

Proof. This follows from Theorem 5.4 applied to $U \to V$ in Definition 6.1. □

Proposition 6.3. Let X be a complex analytic space such that each compact analytic set in X is finite, then every stone in X is an analytic block in X.

Proof. Let $(P, \pi : X \to \mathbb{C}^n)$ be a stone in X. We consider the proper morphism $\tau : U \to V$ as in Theorem 3.3. Each fiber of τ is a compact subset of U and hence a compact subset of X. By our assumption, it is finite. It suffices to apply Proposition 4.5 in Topology and bornology to conclude that τ is finite. □

7. Holomorphically spreadable spaces

Definition 7.1. Let X be a complex analytic space. We say X is *holomorphically spreadable* if $|X|$ is Hausdorff and for any $x \in X$, we can find an open neighbourhood U of x in X such that

$$\{y \in U : f(x) = f(y) \text{ for all } f \in \mathcal{O}_X(X)\} = \{x\}.$$

A holomorphically separable space is clearly holomorphically spreadable.

Proposition 7.2. Let X be a holomorphically spreadable complex analytic space and $x \in X$. Then there exist finitely many $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ such that x is an isolated point of $W(f_1, \ldots, f_n)$.

Proof. By induction on $\dim_x X$, it suffices to prove the following claim: if A is an analytic set in X and $a \in A$ such that $\dim_a A \geq 1$. Then there is $f \in \mathcal{O}_X(X)$ such that $\dim_a(A \cap W(f)) = \dim_a A - 1$.

To prove the claim, let $A_1, \ldots, A_k$ be the irreducible components of A. We may assume that all of them contain a. Choose $a_j \in A_j$ for each $j = 1, \ldots, k$ so that $a, a_1, \ldots, a_k$ are pairwise different. Then there is a function $f \in \mathcal{O}_X(X)$ with $f(a) = 0$ while $f(a_j) \neq 0$ for $j = 1, \ldots, k$. Then $a \in W(f)$ while $f|_{A_j}$ is not identically 0. By Krulls Hauptidealsatz, $\dim_a(A_j \cap W(f)) = \dim_a A_j - 1$ for all $j = 1, \ldots, k$. Observe that $A \cap W(f)$ and $\bigcup_{j=1}^k (A_j \cap W(f))$ coincide near a, so

$$\dim_a(A \cap W(f)) = \max_{j=1,\ldots,k} \dim_a(A_j \cap W(f)) = \max_{j=1,\ldots,k} (\dim_a A_j - 1) = \dim_a A - 1.$$

□

Proposition 7.3. Let X be an irreducible holomorphically spreadable complex analytic space. Then X has countable basis.

The statement of this proposition in [Fis76, Proposition 0.37] is clearly wrong. I do not understand the argument of either [Jur59] or [Gra55], where they claim that this result holds for connected holomorphically spreadable complex analytic spaces.

PROOF. We may assume that X is connected. Recall that by Corollary 8.6 in Local properties of complex analytic spaces, X is locally connected. Let $F : X \to \mathbb{C}^{\mathcal{O}_X(X)}$ be the map sending $x \in X$ to $(f(x))_{f\in\mathcal{O}_X(X)}$. By our assumption, F is continuous and has discrete fibers. In particular, for each $x \in X$, we may assume take finitely many $f_1, \dots, f_n \in \mathcal{O}_X(X)$ so that the induced morphism $F' : X \to \mathbb{C}^n$ is quasi-finite at x. By Corollary 2.8 in Analytic sets, we can find a nowhere dense analytic set A in X such that the map $X \setminus A \to \mathbb{C}^n$ induced by F' is quasi-finite. Now we endow $\mathcal{O}_X(X)$ with the compact-open topology. It is a metric space. By Proposition 6.2 in Topology and bornology, $X \setminus A$ has countable basis. It follows that $\mathcal{O}_X(X \setminus A)$ is a separable metric space. Hence, so it $\mathcal{O}_X(X)$. In particular, there is a continous map with discrete fibers

$$X \to \mathbb{C}^{\omega}.$$

It follows again from Proposition 6.2 in Topology and bornology that X has countable basis. □

Proposition 7.4. Let X be a holomorphically spreadable complex analytic space. Then any compact analytic set A in X is finite.

PROOF. Let B be a connected component of A and $p \in B$. We need to show that $B = \{p\}$. Take finitely many $f_1, \dots, f_n \in \mathcal{O}_X(X)$ so that p is an isolated point of $W(f_1, \dots, f_n)$. This is possible by Proposition 7.2. As f_i vanishes on B for each $i = 1, \dots, n$, we have $B = \{p\}$. □

Corollary 7.5. Let X be a complex analytic space and A be a compact analytic subset of X. Suppose that there exists an analytic block $(P, \pi : X \to \mathbb{C}^n)$ in X with $A \subseteq P$, then A is finite.

PROOF. Take $U \subseteq X, V \subseteq \mathbb{C}^n$ as in Definition 6.1 so that $U \to V$ is finite. Then U is clearly holomorphically spreadable. By Proposition 7.4, A is finite. □

8. Holomorphically complete spacs

Definition 8.1. Let X be a complex analytic space. An *exhaustion of X by analytic blocks* is an exhaustion of X by stones $(P_i, \pi_i)_{i\in\mathbb{Z}_{>0}}$ such that (P_i, π_i) is an analytic block for each $i \in \mathbb{Z}_{>0}$.

We say X is *holomorphically complete* if X is Hausdorff and there is an exhaustion of X by analytic stones.

Theorem 8.2. Let X be a Hausdorff complex analytic space. Then the following are equivalent:

(1) X is holomorphically complete;
(2) X is weakly holomorphically convex and every compact analytic subset of X is finite.

PROOF. (1) $\implies$ (2): X is weakly holomorphically convex by definition. Each compact analytic subset A of X is contained in some analytic block, hence finite by Corollary 7.5.

(2) $\implies$ (1): This follows from Proposition 6.3. □

Lemma 8.3. Let X be a complex manifold and $\mathcal{I}$ be a coherent subsheaf of $\mathcal{O}_X^l$ for some $l \in \mathbb{Z}_{>0}$. Then $\mathcal{I}(X)$ is a closed subspace of $\mathcal{O}_X(X)^l$ endowed with the compact-open topology.

Proof. Let $(f_j \in \mathcal{I}(X))_{j \in \mathbb{Z}_{>0}}$ be a sequence with a limit $f \in \mathcal{O}_X^l(X)$. Let $x \in X$. It suffices to show that $f_x \in \mathcal{I}_x$. Observe that f_x is the limit of f_{jx} as $j \to \infty$. As $\mathcal{O}_{X,x}$ is noetherian, the submodule $\mathcal{I}_x$ of $\mathcal{O}_x^l$ is closed by Corollary 7.4 in **??**. We conclude. □

Definition 8.4. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$.

Choose $U \subseteq X, V \subseteq \mathbb{C}^n$ as in Definition 6.1 so that $\tau : U \to V$ induced by π is finite. Then $\mathcal{G} := \tau_*(\mathcal{F}|_U)$ is a coherent $\mathcal{O}_V$-module. By Corollary 5.3, we can find $l \in \mathbb{Z}_{>0}$ and an epimorphism $\mathcal{O}_Q^l \to \mathcal{G}|_Q$. It induces an epimorphism $\epsilon : H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l \to H^0(Q, \mathcal{G}) \xrightarrow{\sim} H^0(P, \mathcal{F})$. We define a semi-norm $|\bullet|$ on $H^0(P, \mathcal{F})$ as the quotient semi-norm induced by the sup seminorm on $H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$.

A seminorm on $H^0(P, \mathcal{F})$ defined in this way is called a *good semi-norm* on $H^0(P, \mathcal{F})$ with respect to (P, π).

Lemma 8.5. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let (P, π) be an analytic block on X. A good semi-norm on $H^0(P, \mathcal{F})$ induces a metric on $H^0(P^0, \mathcal{F})$.

Proof. We need to show that if $|s| = 0$ for some $s \in H^0(P, \mathcal{F})$, then $s|_{P^0} = 0$, where P^0 is the analytic interior of P.

We use the same notations as in Definition 8.4. We can take $h \in H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$ and $h_j \in \ker \epsilon$ for each $j \in \mathbb{Z}_{>0}$ so that $\epsilon(h) = s$ and $\|h_j - h\|_{L^\infty} \to 0$. So $h_j|_Q \to h|_Q$ with respect to the compact-open topology. From Lemma 8.3, we conclude that the image of $h|_{\operatorname{Int} Q}$ is 0. Namely, s vanishes on $P^0 = \tau^{-1}(\operatorname{Int} Q)$. □

Lemma 8.6. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$. Consider the epimorphism of sheaves

$$\mathcal{O}_Q^l \to \pi_*(\mathcal{F}|_P)$$

as in Definition 8.4 and endow $H^0(P^0, \mathcal{F})$ with the metric induced by the corresponding good semi-norm. Let

$$Q_1 \subseteq Q_2 \subseteq \cdots$$

be a compact exhaustion of $\operatorname{Int} Q$ by tubes with the same centers in $\mathbb{C}^n$. We get an induced map

$$\epsilon_j : H^0(Q_j, \mathcal{O}_{\mathbb{C}^n}^l) \to \pi_*(\mathcal{F}|_P)(Q_j)$$

for each $j \in \mathbb{Q}_{>0}$. We therefore get good semi-norms $|\bullet|_j$ on $H^0(P^0, \mathcal{F})$ for each $j \in \mathbb{Z}_{>0}$. Let

$$d(s_1, s_2) := \sum_{j=1}^{\infty} 2^{-j} \frac{|s_1 - s_2|_j}{1 + |s_1 - s_2|_j}$$

for each $s_1, s_2 \in H^0(P^0, \mathcal{F})$. Then d is a metric on $H^0(P^0, \mathcal{F})$ and $H^0(P^0, \mathcal{F})$ is a Fréchet space with respect to this topology.

Moreover, the topology does not depend on the choice of π, ϵ and the exhaustion.

PROOF. By Lemma 8.5, each $|\bullet|_\nu$ is a norm on $H^0(P^0,\mathcal{F})$. It follows that d is a metric. Next we show that $H^0(P^0,\mathcal{F})$ is Fréchet. Let $(s_j)_{j\in\mathbb{Z}_{>0}}$ be a Cauchy sequence in $H^0(P^0,\mathcal{F})$. We can find bounded sequences $(f_{jk}\in H^0(Q_k,\mathcal{O}^l_{\mathbb{C}^n}))_{k\in\mathbb{Z}_{>0}}$ so that $\epsilon_k(f_{jk})=s_j|_{\pi^{-1}(Q_k)\cap P}$ $(k\in\mathbb{Z}_{>0})$ for each $j\mathbb{Z}_{>0}$. By Montel's theorem, there is a subsequence of $(f_{jk})_j$ which converges uniformly on Q_{k-1} to $f_k\in H^0(Q_{k-1},\mathcal{O}^l_{\mathbb{C}^n})$. Then $\epsilon_{k-1}(f_{k+1})|_{\operatorname{Int}Q_{k-1}}=\epsilon_{k-1}(f_k)|_{\operatorname{Int}Q_{k-1}}$ for each $k\in\mathbb{Z}_{\geq 2}$. So we can glue the f_k's to $s\in H^0(P^0,\mathcal{F})$. Clearly, $s_k\to s$ as $k\to\infty$.

Next we show that the topology is independent of the choice of π, ϵ and the exhaustion. The independence of the exhaustion is obvious. We prove the other two independence. Let $(P,\pi':X\to\mathbb{C}^{n'})$ be another analytic block with $\pi'=(\pi,\varphi):X\to\mathbb{C}^n\times\mathbb{C}^m$, $n'=n+m$. Let $Q^*\subseteq\mathbb{C}^m$ be a tube such that $\varphi(P)\subseteq Q^*$. Then $P=\pi'^{-1}(Q\times Q^*)\cap U$. We can find an open neighbourhood U' of P in X and V' of $Q\times Q^*$ in $\mathbb{C}^{n'}$ for which the induced map $\tau':U'\to V'$ is finite by Definition 6.1. Fix an epimorphism $\mathcal{O}^{l'}_{\mathbb{C}^{n'}}|_{Q\times Q^*}\to\pi'_*(\mathcal{F}|_P)$ for some $l'\in\mathbb{Z}_{>0}$. Construct an exhanstion of $\operatorname{Int}Q\times\operatorname{Int}Q^*$ of the product type: $(Q_j\times Q_j^*)_{j\in\mathbb{Z}_{>0}}$ as in the lemma. Let d' denote the induced metric on $H^0(\operatorname{Int}P,\mathcal{F})$.

We will show that d' and d induce the same topology. Let $e_1,\ldots,e_l\in H^0(Q,\mathcal{O}^l_{\mathbb{C}^n})$ be the standard basis. Let $e'_1,\ldots,e'_l$ be the preimages of $\epsilon(e_1),\ldots,\epsilon(e_l)\in\pi_*(\mathcal{F}|P)(Q)=\pi'_*(\mathcal{F}|_P)(Q\times Q^*)$ in $\mathcal{O}_{\mathbb{C}^{n'}}(Q\times Q^*)^{l'}$ under ϵ'. Further, for $f\in\mathcal{O}_{\mathbb{C}^n}(Q_j)$, we denote by $f'\in\mathcal{O}_{\mathbb{C}^{n'}}(Q_j\times Q_j^*)$ the holomorphic extension of f to $Q_j\times Q_j^*$ constant along $\{q\}\times Q_j^*$ for each $q\in Q_j$ for each $j\in\mathbb{Z}_{>0}$. The norms of

$$\mathcal{O}_{\mathbb{C}^n}(Q_j)^l\to\mathcal{O}_{\mathbb{C}^{n'}}(Q_j\times Q_j^*)^l,\quad \sum_{i=1}^l f_ie_i\mapsto\sum_{i=1}^l f'_ie'_i$$

for $j\in\mathbb{Z}_{>0}$ are bounded by a constant independent of j. Therefore, the identity map

$$(H^0(P^0,\mathcal{F}),d)\to(H^0(P^0,\mathcal{F}),d')$$

is continuous. By open mapping theorem, the map is a homeomorphism. □

Theorem 8.7. Let X be a complex analytic space and $(P,\pi)\subseteq(P',\pi')$ be two analytic blocks on X and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, then the restriction map

$$H^0(P',\mathcal{F})\to H^0(P,\mathcal{F})$$

with respect to any good semi-norms.

PROOF. We claim that there exists an analytic block (P_1,π) such that

$$(P,\pi)\subseteq(P_1,\pi)\subseteq(P',\pi').$$

Assume this claim, then we have a decomposition of the restriction map

$$H^0(P',\mathcal{F})\to H^0(P_1^0,\mathcal{F})\to H^0(P,\mathcal{F}).$$

The first map is continuous if we endow $H^0(P_1^0,\mathcal{F})$ with the topology induced by π', the second is continuous if we endow $H^0(P_1^0,\mathcal{F})$ with the topology induced by π. These topologies are identical by Lemma 8.6. Our assertion follows.

To argue the claim, let us write $\pi:X\to\mathbb{C}^n$ and $\pi'=(\pi,\varphi):X\to\mathbb{C}^n\times\mathbb{C}^m$. Take $q\in\mathbb{C}^m$ with $Q\times\{q\}\subseteq\operatorname{Int}Q'$. Let $Q'':=Q'\cap(\mathbb{C}^n\times\{q\})$ and identify it with a subset of $\mathbb{C}^n$. Let Q^* be the image of Q' under the projection $\mathbb{C}^{n+m}\to\mathbb{C}^m$.

Choose open neighbourhoods $U \subseteq P'^0$, $V \subseteq Q'$ of P and Q respectively such that $\tau : U \to V$ is finite and $U \cap \pi^{-1}(Q) = P$. Take a tube $Q_1 \subseteq \mathbb{C}^n$ such that

$$Q \subseteq \operatorname{Int} Q_1 \subseteq Q_1 \subseteq \operatorname{Int} Q''.$$

Now it suffices to set $P_1 := \pi^{-1}(Q_1) \cap U$. □

Corollary 8.8. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi) \subseteq (P', \pi')$ be analytic blocks in X. Then for any Cauchy sequence $(s_j)_{j\in\mathbb{Z}_{>0}}$ in $H^0(P', \mathcal{F})$, the restriction sequence $(s_j|_P)_{j\in\mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$.

Proof. Choose an analytic block (P_1, π) such that

$$(P, \pi) \subseteq (P_1, \pi) \subseteq (P', \pi').$$

The existence of the block (P_1, π) is argued in the proof of Theorem 8.7. We have a decomposition of the restriction map

$$H^0(P', \mathcal{F}) \to H^0(P_1^0, \mathcal{F}) \to H^0(P, \mathcal{F}).$$

The first map is bounded, so the images of $(s_j)_{j\in\mathbb{Z}_{>0}}$ in $H^0(P_1^0, \mathcal{F})$ is a Cauchy sequence. As we have shown that $H^0(P_1^0, \mathcal{F})$ is a Fréchet space in Lemma 8.6, the sequence converges. As the second map is also continuous, it follows that $(s_j|_P)_{j\in\mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$. □

Lemma 8.9. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n) \subseteq (P', \pi' : X \to \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q'. Choose $U' \subseteq X$ and $V' \subseteq \mathbb{C}^{n+m}$ of P' and Q' respectively as in Definition 6.1 such that $U' \to V'$ is finite. Set

$$Q_1 := (Q \times \mathbb{C}^m) \cap Q', \quad P_1 = \pi'^{-1}(Q_1) \cap U'.$$

Then $(P_1.\pi')$ is an analytic block in X with block Q_1 and $H^0(P', \mathcal{F}) \to H^0(P_1, \mathcal{F})$ has dense image. Here we take an epsimorphism

$$\mathcal{O}^{l'}_{\mathbb{C}^{n+m}}|_{Q'} \to (\tau'(\mathcal{F}|_{U'}))_{Q'}$$

and it induces

$$\mathcal{O}^{l'}_{\mathbb{C}^{n+m}}|_{Q_1} \to (\tau'(\mathcal{F}|_{U'}))_{Q_1},$$

which in turn induces a good semi-norm on $H^0(P_1, \mathcal{F})$. This is the semi-norm we are using.

Moreover, there is a compact set $\tilde{P} \subseteq X$ disjoint from P such that

$$P_1 = P \cup \tilde{P}.$$

Proof. We have a commutative diagram in the category of topological linear spaces:

$$\begin{array}{ccc} H^0(Q', \mathcal{O}^l_{\mathbb{C}^{m+n}}) & \longrightarrow & H^0(P', \mathcal{F}) \\ \downarrow & & \downarrow \\ H^0(Q_1, \mathcal{O}^l_{\mathbb{C}^{m+n}}) & \longrightarrow & H^0(P_1, \mathcal{F}) \end{array}.$$

In order to show that the right vertical map has dense image, it is enough to show that the map on the left-hand side has dense images, which is the Runge approximation.

For the last assertion, as $Q_1 = (Q \times \mathbb{C}^m) \cap Q'$, we have

$$P_1 = \pi^{-1}(Q) \cap P'.$$

As $P \subseteq P'$ and $P \subseteq \pi^{-1}(Q)$, it follows that $P \subseteq P_1$. But there is an open neighbourhood U of P in X so that $P = \pi^{-1}(Q) \cap U$. Hence, $\tilde{P} = P_1 \setminus P$ is compact. □

Theorem 8.10 (Runge approximation)**.** Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n) \subseteq (P', \pi' : X \to \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q'. Then the map

$$H^0(P', \mathcal{F}) \to H^0(P, \mathcal{F})$$

has dense image with respect to a good semi-norm.

Proof. We use the notations of Lemma 8.9. We extend Q, Q_1, Q' to tubes $\hat{Q}, \hat{Q}_1, \hat{Q}'$ and get $\hat{P}, \hat{P}_1, \hat{P}'$ corresponding to the original P, P_1, P'. The restriction map

$$H^0(\hat{P_1}^0, \mathcal{F}) \to H^0(\hat{P}^0, \mathcal{F})$$

is a continuous morphism of Fréchet spaces.

Let $s \in H^0(P, \mathcal{F})$ be a section. Lift s to $s_1 \in H^0(P_1, \mathcal{F})$. Up to a suitable modification of the tubes, we can extend s_1 to $\hat{s_1} \in H^0(\hat{P}_1, \mathcal{F})$. Then there is a sequence $(s^j \in H^0(\hat{P}', \mathcal{F}))_{j \in \mathbb{Z}_{>0}}$ such that $s^j|_{\hat{P}_1} \to \hat{s_1}$ as $j \to \infty$ in $H^0(\hat{P}_1, \mathcal{F})$. It follows that $s^j|_{\hat{P}^0} \to \hat{s}_1|_{\hat{P}^0}$ in $H^0(\hat{P}^0, \mathcal{F})$. It follows that $s^j|_P \to s_1|_P = s$ sa $j \to \infty$. □

Theorem 8.11. Let X be a complex analytic space. Each exhaustion of X by analytic blocks is a Stein exhaustion.

Proof. Let $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ be an exhaustion of X by analytic blocks. Take a coherent $\mathcal{O}_X$-module $\mathcal{F}$.

We verify the conditions in Definition 5.5. By Theorem 5.4, P_i is a compact Stein set for each $i \in \mathbb{Z}_{>0}$. So (1) is satisfied.

On $H^0(P_i, \mathcal{F})$, we fix a good semi-norm $|\bullet|_i$ for each $i \in \mathbb{Z}_{>0}$. We may assume that $H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$ is contractive for $i \in \mathbb{Z}_{>0}$.

We have already verified (3), (4) and (5).

We verify (2). It suffices to show that

$$H^0(X, \mathcal{F}) \to H^0(P_1, \mathcal{F})$$

has dense image. Let $s \in H^0(P_1, \mathcal{F})$ and $\delta > 0$. By Theorem 8.10, we can find $s_i \in H^0(P_i, \mathcal{F})$ for $i \in \mathbb{Z}_{>0}$ such that $s_1 = s$,

$$\left|s_{i+1}|_{P_i} - s_i\right|_i < 2^{-i}\delta$$

for $i \in \mathbb{Z}_{>0}$. By Corollary 8.8, $(s_j|_{P_i})_{j \in \mathbb{Z}_{>0}}$ has a limit $t_i \in H^0(P_i, \mathcal{F})$ for each $i \in \mathbb{Z}_{>0}$. As $H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$ is continuous for $i \in \mathbb{Z}_{>0}$, the $t_{i+1}|_{P_i}$'s are compatible and defines $t \in H^0(X, \mathcal{F})$. It is easy to see that $|t|_{P_1} - s|_1 < \delta$. Thus condition (2) is satisfied. □

9. Stein spaces

Definition 9.1. Let X be a complex analytic space. We say that X is a Stein space if X is a Stein set in X and $|X|$ is paracompact and Hausdorff.

Definition 9.2. Let X be a complex analytic space. An *effective formal* 0*-cycle* on X consists of

(1) A disrete set $D \subseteq X$;
(2) An integer n_x for each $x \in D$.

We write the effective formal 0-cycle as $\sum_{x\in D} n_x x$. We define the *ideal sheaf* $\mathcal{O}_X(-\sum_{x\in D} n_x x)$ of an effective formal 0-cycle as $\sum_{x\in D} n_x x$ as

$$\mathcal{O}_X(-\sum_{x\in D} n_x x)(U) = \left\{ f \in H^0(U, \mathcal{O}_X) : f_x \in \mathfrak{m}_x^{n_x} \text{ for each } x \in D \cap U \right\}$$

for each open subset $U \subseteq X$.

Observe that $\mathcal{O}_X(-\sum_{x\in D} n_x x)$ is a coherent $\mathcal{O}_X$-module. In fact, the problem is local, so we may assume that D is finite. In this case, D is an effective 0-cycle and the result is clear.

Lemma 9.3. Let X be a complex analytic space and $\sum_{x\in D} n_x x$ be an effective formal 0-cycle on X. Assume that

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X/\mathcal{O}_X(-\sum_{x\in D} n_x x))$$

is surjective. Suppose that for each $x \in D$, we assign $g_x \in \mathcal{O}_{X,x}$. Then there is $f \in H^0(X, \mathcal{O}_X)$ such that

$$f_x - g_x \in \mathfrak{m}_x^{n_x}$$

for all $x \in D$.

Proof. We define $s \in H^0(X, \mathcal{O}_X/\mathcal{O}_X(-\sum_{x\in D} n_x x))$ by $s_x = g_x$ for each $x \in D$. Lift s to $f \in H^0(X, \mathcal{O}_X)$. Then f clearly satisfies the required properties. □

Proposition 9.4. Let X be a complex analytic space. Assume that $H^1(X, \mathcal{I}) = 0$ for each coherent ideal sheaf $\mathcal{I}$ on X. Let $(x_i \in X)_{i\in\mathbb{Z}_{>0}}$ be a sequence without accumulation points and $(c_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{C}$. Then there is $f \in \mathcal{O}_X(X)$ with $f(x_i) = c_i$ for each $i \in \mathbb{Z}_{>0}$.

Proof. Consider the formal cycle $\sum_{i=1}^{\infty} x_i$. Apply Lemma 9.3 with $g_{x_i} = c_i$. □

Theorem 9.5. Let X be a paracompact Hausdorff complex analytic space. Then the following are equivalent:

(1) X is a Stein space;
(2) For any coherent ideal sheaf $\mathcal{I}$ on X, we have $H^1(X, \mathcal{I}) = 0$;
(3) X is holomorphically separable and holomorphically convex;
(4) X is holomorphically spreadable and weakly holomorphically convex;
(5) X is holomorphically complete;
(6) X is weakly holomorphically convex and every compact analytic subset of X is finite.

Proof. (1) $\implies$ (2): This is trivial.

(2) $\implies$ (3): X is holomorphicaly convex by Proposition 9.4 and Proposition 4.4. X is holomorphically separable by Proposition 9.4.

(3) $\implies$ (4): X is holomorphically spreadable and weakly holomorphically convex by definition.

(4) $\implies$ (5): This follows from Theorem 8.2 and Proposition 7.4.

(5) $\implies$ (1): This follows from Theorem 8.11 and Theorem 5.6.

(5) $\Leftrightarrow$ (6): This is just Theorem 8.2. $\square$

Lemma 9.6. Let $b \in \mathbb{Z}_{>0}$ and $f : X \to Y$ be a b-sheeted branched covering of complex analytic spaces. Assume that Y is normal, paracompact and Hausdorff. Then the following are equivalent:

(1) X is Stein;
(2) Y is Stein.

The corresponding statement in Narasimhan is not correct. It is not clear to me if this holds for a general finite surjective morphism between paracompact normal Hausdorff complex analytic spaces.

Proof. First observe that X is necessarily Hausdorff. We may assume that Y is connected. Then Y is σ-compact and hence X is σ-compact. It follows that X is paracompact by Proposition 3.2 in Topology and bornology.

(2) $\implies$ (1): This follows from Leray's spectral sequence.

(1) $\implies$ (2): We may assume that X is connected. By Theorem 9.5, it suffices to verify that Y is holomorphically convex and every analytic set in Y is finite.

Let $(y_i \in Y)_{i\in\mathbb{Z}_{>0}}$ be a sequence without accumulation points. We can lift the sequence to $(x_i \in X)_{i\in\mathbb{Z}_{>0}}$ without accumulation points. By Proposition 9.4, we can find $g \in \mathcal{O}_X(X)$ such that $(|g(x_i)|)_{i\in\mathbb{Z}_{>0}}$ is unbounded. Let $\chi_g \in \mathcal{O}_Y(Y)[w]$ be the characteristic polynomial of g. As $\chi_g(g) = 0$, it follows that at least one coefficient of χ_g is unbounded along $(y_i)_{i\in\mathbb{Z}_{>0}}$. By Proposition 4.4, we conclude that Y is holomorphically convex.

Let T be an analytic set in Y. Then so is $f^{-1}(T)$. As X is Stein, $f^{-1}(T)$ is finite, hence so is T. $\square$

Corollary 9.7. Let $f : X \to Y$ be a finite surjective morphism of normal complex analytic spaces with Y is paracompact and Hausdorff. Then the following are equivalent:

(1) X is Stein;
(2) Y is Stein.

Proof. We may assume that Y is connected. Again, X is paracompact and Hausdorff as in the proof of Lemma 9.6.

(2) $\implies$ (1): This follows from Leray's spectral sequence.

(1) $\implies$ (2): Observe that Y is irreducible, so there is a connected component X' of X so that the restriction $X' \to Y$ is surjective. Then $X' \to Y$ is a branched covering by Corollary 4.36 in Morphisms between complex analytic spaces. But X' is Stein as it is a connected component of a Stein space. We conclude using Lemma 9.6. $\square$

Lemma 9.8. Let X be a paracompact Hausdorff reduced complex analytic space whose normalization $\bar{X}$ is Stein. Then for any reduced closed analytic subspace Y of X, $\bar{Y}$ is also Stein.

Proof. We write $\pi : \bar{X} \to X$ for the normalization morphism. Let $Y^1 = \pi^{-1}(Y)$, the preimage is endowed with a structure of a closed analytic subspace of X. It follows that Y^1 is Stein. Its normalization $\overline{Y^1}$ is then Stein, as the normalization morphism is finite. We have commutative diagram induced by the universal property of the normalization:

$$\begin{tikzcd} \overline{Y^1} \arrow[r] \arrow[d] & \overline{Y} \arrow[dl] \\ Y & \end{tikzcd}.$$

The natural morphism $\overline{Y^1} \to Y$ is a finite as it is the composition of two finite coverings. Then morphism $\bar{Y} \to Y$ is finite, so $\overline{Y^1} \to \overline{Y}$ is finite. But its image contains a dense open subset of $\bar{Y}$, so $\overline{Y^1} \to \bar{Y}$ is surjective. Observe that $\bar{Y}$ is paracompact and Hausdorff by the same arguments as in Lemma 9.6. Now we can apply Corollary 9.7 to conclude that $\bar{Y}$ is Stein. $\square$

Corollary 9.9. Let X be a paracompact Hausdorff complex analytic space. Then the following are equivalent:

(1) X is Stein;
(2) X^{red} is Stein;
(3) The normalization $\overline{X^{\mathrm{red}}}$ is Stein.

The equivalence of (1) and (2) is due to Grauert [Gra60]. Here we follow the simplified approach in [GR77]. The difficult direction (3) implies (2) is claimed in [GR77], where the proof is nonsense. We follow the argument of Narasimhan [Nar62]. We remind the readers that the statements and the arguments in [Nar62] contain several (fixable) mistakes.

Proof. (1) $\implies$ (2): This follows from Leray's spectral sequence.

(2) $\implies$ (1): By Theorem 9.5(3), it suffices to show that the restriction map $H^0(X, \mathcal{O}_X) \to H^0(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$ is surjective.

Let $\mathcal{I}$ be the nilradical of $\mathcal{O}_X$. It is coherent by Cartan–Oka theorem. For each $i \in \mathbb{Z}_{>0}$, we have a short exact sequence

$$0 \to \mathcal{I}^i/\mathcal{I}^{i+1} \to \mathcal{O}_X/\mathcal{I}^{i+1} \to \mathcal{O}_X/\mathcal{I}^i \to 0.$$

As $\mathcal{I}^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{O}_{X^{\mathrm{red}}}$-module, we conclude that

$$\varphi_i : H^0(X, \mathcal{O}_X/\mathcal{I}^{i+1}) \to H^0(X, \mathcal{O}_X/\mathcal{I}^i)$$

is surjective for each $i \in \mathbb{Z}_{>0}$. Let $h_1 \in H^0(X, \mathcal{O}_X/\mathcal{I}) = H^0(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$. We want to lift it to $h \in H^0(X, \mathcal{O}_X)$.

We successively lift h_1 to $h_i \in H^0(X, \mathcal{O}_X/\mathcal{I}^i)$ for each $i \in \mathbb{Z}_{>0}$. Let $X_i = X \setminus \operatorname{Supp} \mathcal{I}^i$ of each $i \in \mathbb{Z}_{>0}$. Then clearly

$$X = \bigcup_{i=1}^{\infty} X_i.$$

It is easy to see that

$$h_{i+1}|_{X_i} = h_i|_{X_i}$$

for each $i \in \mathbb{Z}_{>0}$. It follows that we can glue the $h_i|_{X_i}$'s to $h \in H^0(X, \mathcal{O}_X)$ which restricts to h_1.

(2) $\implies$ (3): This follows from Leray's spectral sequence as $\overline{X^{\mathrm{red}}} \to X^{\mathrm{red}}$ is finite by Proposition 7.8 in Local properties of complex analytic spaces.

(3) $\implies$ (2): We may assume that X is reduced.

Step 1. We first observe that it suffices to prove in the case where $\dim X < \infty$. For each $k \in \mathbb{Z}_{>0}$, we let X_k denote the union of the irreducible components of dimension $\leq k$. Then clearly, X_k is an analytic set in X. We endow it with the reduced induced structure. Then $\dim X_k \leq k$. The normalization $\overline{X_k}$ of X_k is a disjoint union of certain connected components of $\bar{X}$ and hence Stein for each $k \in \mathbb{Z}_{>0}$. It follows that X_k is Stein if the special case is established.

Let $D \subseteq X$ be a countable infinite set without accumulation points. For each $k \in \mathbb{Z}_{>0}$, we set $D_k = D \cap X_k$ and $E_{k+1} = D_{k+1} \setminus D_k$. Further we let $E_1 = D_1$. We write the points of D as $(x_i \in X)_{i \in \mathbb{Z}_{>0}}$. Let $h : D \to \mathbb{C}$ be the map sending x_i to i for each $i \in \mathbb{Z}_{>0}$. For each $k \in \mathbb{Z}_{>0}$, h_k denotes the restriction of h to D_k.

As X_1 is Stein, we can construct $f_1 \in \mathcal{O}_{X_1}(X_1)$ with $f_1|_{E_1} = h_1$ by Proposition 9.4. As $E_2 \cup X_1$ is an analytic subset in X_2, we can find $f_2 \in \mathcal{O}_{X_2}(X_2)$ extending f_1 and such that $f_2|_{E_2} = h_2$. We continue in the obvious way and construct $f_k \in \mathcal{O}_{X_k}(X_k)$ for each $k \in \mathbb{Z}_{>0}$ compatible with each other. Then the f_k's glue to give $f \in \mathcal{O}_X(X)$ unbounded on D. We conclude that X is Stein by Proposition 4.4.

Step 2. We assume that $\dim X < \infty$.

Let $\mathcal{I}$ be a coherent ideal sheaf on X. By Theorem 9.5, it suffices to show that

$$H^1(X, \mathcal{I}) = 0.$$

We may assume that X is connected. We make an induction on $\dim X$. There is nothing to prove if $\dim X = 0$. Assume that $\dim X > 0$.

We write $\pi : \bar{X} \to X$ for the normalization morphism. Let $\mathcal{W}$ be the conductor ideal of $\mathcal{O}_X$. Let $\mathcal{F} := \pi^*(\mathcal{W}\mathcal{I})$. Observe that $\mathcal{F}$ is a coherent $\mathcal{O}_{\bar{X}}$-module. By Leray spectral sequence,

$$H^1(X, \pi_*\mathcal{F}) \cong H^1(\bar{X}, \mathcal{F}) = 0.$$

Let $Y := \operatorname{Supp} \mathcal{O}_X/\mathcal{W} \subseteq X^{\mathrm{Sing}}$. Then Y is an analytic set in X. We endow Y with the reduced induced structure, then Y is Stein by Lemma 9.8 and our inductive hypothesis.

Observe that $\pi_*\mathcal{F}$ can be identified with a subsheaf of $\mathcal{W} \cdot \overline{\mathcal{O}_X} \subseteq \mathcal{I}$. Let $\mathcal{S} = (\mathcal{I}/\pi_*\mathcal{F})|_Y$. Then we have

$$H^1(X, \mathcal{I}/\pi_*\mathcal{F}) \cong H^1(Y, \mathcal{S}) = 0.$$

Consider the short exact sequence

$$0 \to \pi_*\mathcal{F} \to \mathcal{I} \to \mathcal{I}/\pi_*\mathcal{F} \to 0.$$

We conclude that

$$H^1(X, \mathcal{I}) = 0.$$

□

Corollary 9.10. Let X be a paracompact Hausdorff complex analytic space. Then the following are equivalent:

(1) X is Stein;

(2) Each irreducible component of X^{red} is Stein if we endow it with the reduced induced structure.

PROOF. This follows immediately from Corollary 9.9. $\square$

Corollary 9.11. Let $f : X \to Y$ be a finite morphism between complex analytic spaces. Assume that Y is paracompact and Hausdorff. Then

(1) if Y is Stein, so is X;
(2) if f is surjective and X is Stein, then Y is also Stein.

This result is due to Narasimhan [Nar62], although the statement and the proof in [Nar62] are both incorrect.

PROOF. Observe that X is paracompact and Hausdorff as in the proof of Lemma 9.6. By Corollary 9.9, we may assume that X and Y are reduced.

(1) This follows from Leray's spectral sequence.

(2) We may assume that Y is irreducible by Corollary 9.10. Up to replacing X by one of its irreducible components whose image under f is Y, we may assume that X is also irreducible.

By Corollary 4.31 in Morphisms between complex analytic spaces, we can find a commutative diagram

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{f}"] \arrow[d] & \bar{Y} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}.$$

By Corollary 9.9, we are reduced to show that $\bar{X}$ is Stein if and only if $\bar{Y}$ is. But $\bar{f} : \bar{X} \to \bar{Y}$ is clearly finite and surjective. So it suffices to apply Corollary 9.7. $\square$

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