

## Affinoid algebras



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## 1. Introduction

Our references for this chapter include [BGR84], [Ber12].

## 2. Tate algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued-field.

**Definition 2.1.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . We set

$$\begin{aligned} k\{r^{-1}T\} &= k\{r_1^{-1}T_1, \dots, r_n T_n^{-1}\} \\ &:= \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \right\}. \end{aligned}$$

For any  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$ , we set

$$\|f\|_r = \max_{\alpha} |a_\alpha| r^\alpha.$$

We call  $(k\{r^{-1}T\}, \|\bullet\|_r)$  the *Tate algebra* in  $n$ -variables with radii  $r$ . The norm  $\|\bullet\|_r$  is called the *Gauss norm*.

We omit  $r$  from the notation if  $r = (1, \dots, 1)$ .

This is a special case of [Example 4.15](#) in the chapter Banach Rings.

**Proposition 2.2.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . Then the Tate algebra  $(k\{r^{-1}T\}, \|\bullet\|_r)$  is a Banach  $k$ -algebra and  $\|\bullet\|_r$  is a valuation.

PROOF. This is a special case of [Proposition 4.16](#) in the chapter Banach Rings.  $\square$

**Remark 2.3.** One should think of  $k\{r^{-1}T\}$  as analogues of  $\mathbb{C}\langle r^{-1}T \rangle$  in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings  $\mathbb{C}\langle r^{-1}T \rangle$ , as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

**Example 2.4.** Assume that the valuation on  $k$  is trivial.

Let  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Then  $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$  if  $r_i \geq 1$  for all  $i$  and  $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$  otherwise.

**Lemma 2.5.** Let  $A$  be a Banach  $k$ -algebra. For each  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathring{A}$ , there is a unique continuous homomorphism  $k\{T_1, \dots, T_n\} \rightarrow A$  sending  $T_i$  to  $a_i$ .

PROOF. This is a special case of [Proposition 4.18](#) in the chapter Banach Rings.  $\square$

## 3. Affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 3.1.** A Banach  $k$ -algebra  $A$  is  *$k$ -affinoid* (resp. *strictly  $k$ -affinoid*) if there are  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$  and an admissible epimorphism  $k\{r^{-1}T\} \rightarrow A$  (resp. an admissible epimorphism  $k\{T\} \rightarrow A$ ).

More generally, a Banach  $k$ -algebra  $A$  is  *$k_H$ -affinoid* if there are  $n \in \mathbb{N}$ ,  $r \in H^n$  and an admissible epimorphism  $k\{r^{-1}T\} \rightarrow A$ .

A morphism between  $k$ -affinoid (resp. strictly  $k$ -affinoid, resp.  $k_H$ -affinoid) algebras is a bounded  $k$ -algebra homomorphism.

The category of  $k$ -affinoid (resp. strictly  $k$ -affinoid, resp.  $k_H$ -affinoid) algebras is denoted by  $k\text{-AffAlg}$  (resp.  $\text{st-}k\text{-AffAlg}$ , resp.  $k_H\text{-AffAlg}$ ). The opposite categories of these categories are denoted by  $k\text{-Aff}$ ,  $\text{st-}k\text{-Aff}$  and  $k_H\text{-Aff}$ . For any  $A$  in  $k\text{-AffAlg}$  (resp.  $\text{st-}k\text{-AffAlg}$ , resp.  $k_H\text{-AffAlg}$ ), the corresponding image in the opposite category is denoted by  $\text{Sp } A$ . We can also identify  $\text{Sp } A$  with the topological space defined in [Definition 6.1](#) in the chapter Banach Rings.

For the notion of admissible morphisms, we refer to [Definition 2.5](#) in the chapter Banach rings.

**Remark 3.2.** Berkovich also introduced the notion of *affinoid  $k$ -algebras*: it is a  $K$ -affinoid algebra for some complete non-Archimedean field extension  $K/k$ . We will not use this notion.

**Example 3.3.** Let  $r \in \mathbb{R}_{>0}$ . We let  $k_r$  denote the subring of  $k[[T]]$  consisting of  $f = \sum_{i=-\infty}^{\infty} a_i T^i$  satisfying  $|a_i| r^i \rightarrow 0$  for  $i \rightarrow \infty$  and  $i \rightarrow -\infty$ . We define a norm  $\|\bullet\|_r$  on  $k_r$  as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in [Proposition 3.4](#) that  $k_r$  is  $k$ -affinoid.

**Proposition 3.4.** Let  $r \in \mathbb{R}_{>0}$ , then  $(k_r, \|\bullet\|_r)$  defined in [Example 3.3](#) is a  $k$ -affinoid algebra. Moreover,  $\|\bullet\|_r$  is a valuation.

PROOF. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \rightarrow k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$(3.1) \quad |a_{i,j}| r^{i-j} \rightarrow 0$$

as  $i+j \rightarrow \infty$ . Observe that for each  $k \in \mathbb{Z}$ , the series

$$c_k := \sum_{i-j=k, i,j \in \mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image  $\iota(f)$  is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that  $\iota(f) \in k_r$ . That is

$$|c_k|r^k \rightarrow 0$$

as  $k \rightarrow \pm\infty$ . When  $k \geq 0$ , we have  $|c_k| \leq |a_{k0}|$  by definition of  $c_k$ . So  $|c_k|r^k \rightarrow 0$  as  $k \rightarrow \infty$  by (3.1). The case  $k \rightarrow -\infty$  is similar.

We conclude that we have a well-defined map of sets  $\iota$ . It is straightforward to verify that  $\iota$  is a ring homomorphism. Next we show that  $\iota$  is surjective. Take  $g = \sum_{i=-\infty}^{\infty} c_i T^i \in k_r$ . We want to show that  $g$  lies in the image of  $\iota$ . As  $\iota$  is a ring homomorphism, it suffices to treat two cases separately:  $g = \sum_{i=0}^{\infty} c_i T^i$  and  $g = \sum_{i=-\infty}^0 c_i T^i$ . We handle the first case only, as the second case is similar. In this case, it suffices to consider  $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$ . It is immediate that  $\iota(f) = g$ .

Next we show that  $\iota$  is admissible. We first identify the kernel of  $\iota$ . We claim that the kernel is the ideal  $I$  generated by  $T_1 T_2 - 1$ . It is obvious that  $I \subseteq \ker \iota$ . Conversely, consider an element

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kernel of  $\iota$ . Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If  $f \in \ker \iota$ , then so is each  $f_k$  by our construction.

We first show that each  $f_k$  lies in the ideal generated by  $T_1 T_2 - 1$ . The condition that  $f_k \in \ker \iota$  means

$$\sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find  $b_{i,j} \in k$  for  $i, j \in \mathbb{N}$ ,  $i - j = k$  such that

$$a_{i,j} = b_{i-1,j-1} - b_{i,j}.$$

Then

$$f_k = (T_1 T_2 - 1) \sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that  $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$ . In particular, the sum of  $\sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j$  for various  $k$  converges to some  $g \in k\{r^{-1}T_1, rT_2\}$  and hence  $f_k = (T_1 T_2 - 1)g$ . Therefore, we have proved that  $\ker \iota$  is generated by  $T_1 T_2 - 1$ .

It remains to show that  $\iota$  is admissible. In fact, we will prove a stronger result:  $\iota$  induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \rightarrow k_r.$$

To see this, take  $f = \sum_{k=-\infty}^{\infty} c_k T^k \in k_r$  and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set  $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$ , then  $\iota(g) = f$  and  $\|g\|_{(r,r^{-1})} = \|f\|_r$ . So it suffices to show that for any  $h = \sum_{(i,j) \in \mathbb{N}^2} d_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$ , we have

$$(3.2) \quad \|f\|_r \leq \|g + h(T_1 T_2 - 1)\|_{r,r^{-1}}.$$

We compute

$$g + h(T_1 T_2 - 1) = \sum_{k=1}^{\infty} (c_k - d_{k,0}) T_1^k + \sum_{k=1}^{\infty} (c_{-k} - d_{0,k}) T_2^k + (c_0 - d_0) + \sum_{i,j \geq 1} (d_{i-1,j-1} - d_{i,j}) T_1^i T_2^j.$$

So

$$\|g + h(T_1 T_2 - 1)\|_{r,r^{-1}} = \max \left\{ \max_{k \geq 0} C_{1,k}, \max_{k \geq 1} C_{2,k} \right\},$$

where

$$C_{1,k} = \max \left\{ |c_k - d_{k,0}|, \left| \sum_{i-j=k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for  $k \geq 0$  and

$$C_{2,k} = \max \left\{ |c_{-k} - d_{0,k}|, \left| \sum_{i-j=-k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for  $k \geq 1$ . It follows from the strong triangle inequality that  $|c_k| \leq C_{1,k}$  for  $k \geq 0$  and  $c_{-k} \leq C_{2,k}$  for  $k \geq 1$ . So (3.2) follows.  $\square$

**Proposition 3.5.** Let  $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$ , then  $\|\bullet\|_r$  defined in Example 3.3 is a valuation on  $k_r$ .

PROOF. Take  $f, g \in k_r$ , we need to show that

$$\|fg\|_r \geq \|f\|_r \|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_i T^i, \quad g = \sum_{i=-\infty}^{\infty} b_i T^i.$$

Take  $i$  and  $j$  so that

$$(3.3) \quad |a_i| r^i = \|f\|_r, \quad |b_j| r^j = \|g\|_r.$$

By our assumption on  $r$ ,  $i, j$  are unique. Then

$$\|fg\|_r = \max_{k \in \mathbb{Z}} \{ |c_k| r^k \},$$

where

$$c_k := \sum_{u,v \in \mathbb{Z}, u+v=k} a_u b_v.$$

It suffices to show that

$$(3.4) \quad |c_k| r^k = \|f\|_r \|g\|_r.$$

for  $k = i + j$ . Of course, we may assume that  $a_i \neq 0$  and  $b_j \neq 0$  as otherwise there is nothing to prove. For  $u, v \in \mathbb{Z}$ ,  $u + v = i + j$  while  $(u, v) \neq (i, j)$ , we may assume that  $u \neq i$ . Then  $|a_u| r^u < |a_i| r^i$  and  $|b_v| r^v \leq |b_j| r^j$ . So  $|a_u b_v| < |a_i b_j|$  and we conclude (3.4).  $\square$

**Remark 3.6.** The argument of Proposition 4.16 in the chapter Banach Rings does not work here if  $r \in \sqrt{|k^\times|}$ , as in general one can not take minimal  $i, j$  so that (3.3) is satisfied.

**Proposition 3.7.** Assume that  $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$ . Then  $k_r$  is a valuation field and  $\|\bullet\|_r$  is non-trivial.

PROOF. We first show that  $\text{Sp } k_r$  consists of a single point:  $\|\bullet\|_r$ . Assume that  $|\bullet| \in \text{Sp } k_r$ . As  $\|\bullet\|_r$  is a valuation, we find

$$(3.5) \quad |\bullet| \leq \|\bullet\|_r.$$

In particular,  $|\bullet|$  restricted to  $k$  is the given valuation on  $k$ . It suffices to show that  $|T| = r$ . This follows from (3.5) applied to  $T$  and  $T^{-1}$ .

It follows that  $k_r$  does not have any non-zero proper closed ideals: if  $I$  is such an ideal,  $k_r/I$  is a Banach  $k$ -algebra. By Proposition 6.10 in the chapter Banach rings,  $\text{Sp } k_r$  is non-empty. So  $k_r$  has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.7 in the chapter Banach rings, the only maximal ideal of  $k_r$  is 0. It follows that  $k_r$  is a field.

The valuation  $\|\bullet\|_r$  is non-trivial as  $\|T\|_r = r$ .  $\square$

**Definition 3.8.** An element  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$  for some  $n \in \mathbb{N}$  is called a *k-free polyray* if  $r_1, \dots, r_n$  are linearly independent in the  $\mathbb{Q}$ -linear space  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_{>0} / \sqrt{|k^\times|}$ .

Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . Assume that  $r$  is a *k-free polyray*. We define

$$k_r = k_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{r_n}.$$

By an iterated application of Proposition 3.7,  $k_r$  is a complete valuation field. As a general explanation of why  $k_r$  is useful, we prove the following proposition:

**Proposition 3.9.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a *k-free polyray*.

- (1) For any  $k$ -Banach space  $X$ , the natural map

$$X \rightarrow X \hat{\otimes}_k k_r$$

is an isometric embedding.

- (2) Consider a sequence of bounded homomorphisms of  $k$ -Banach spaces  $X \rightarrow Y \rightarrow Z$ . Then the sequence is admissible and exact (resp. coexact) if and only if  $X \hat{\otimes}_k k_r \rightarrow Y \hat{\otimes}_k k_r \rightarrow Z \hat{\otimes}_k k_r$  is admissible and exact (resp. coexact).

PROOF. We may assume that  $n = 1$ .

- (1) We have a more explicit description of  $X \hat{\otimes}_k k_r$ : as a vector space, it is the space of  $f = \sum_{i=-\infty}^{\infty} a_i T^i$  with  $a_i \in X$  and  $\|a_i\| r^i \rightarrow 0$  when  $|i| \rightarrow \infty$ . The norm is given by  $\max_i \|a_i\| r^i$ . From this description, the embedding is obvious.

- (2) This follows easily from the explicit description in (1).  $\square$

When  $X$  is a Banach  $k$ -algebra,  $X \hat{\otimes}_k k_r$  is a Banach  $k_r$ -algebra.

**Example 3.10.** For any  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$ , not necessarily *k-free*. We define  $k_r$  as the completed fraction field of  $k\{r^{-1}T\}$  provided with the extended valuation  $|\bullet|_r$ . Then  $k_r$  is still a valuation field extending  $k$ .

When  $r$  is a *k-free polyray*, we claim that  $k_r$  coincides with  $k_r$  defined in Definition 3.8. To see this, let us temporarily denote the  $k_r$  defined in this example as  $k'_r$ , consider the extension of field:

$$\text{Frac } k\{r^{-1}T\} \rightarrow k_r = k\{r^{-1}T, rS\} / (T_1 S_1 - 1, \dots, T_n S_n - 1)$$

sending  $T_i$  to  $T_i$  for  $i = 1, \dots, n$ . Observe that this is an extension of valuation field as well by the same arguments as in Proposition 3.4. In particular, it induces an



extension of complete valuation fields  $k'_r \rightarrow k_r$ . But the image clearly contains the classes of all polynomials in  $k[T, S]$ , so  $k'_r \rightarrow k_r$  is an isometric isomorphism.

**Proposition 3.11.** Assume that  $k$  is non-trivially valued. Let  $B$  be a strict  $k$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite bounded  $k$ -algebra homomorphism into a  $k$ -Banach algebra  $A$ . Then  $A$  is also strictly  $k$ -affinoid.

PROOF. We may assume that  $B = k\{T_1, \dots, T_n\}$  for some  $n \in \mathbb{N}$ . By assumption, we can find finitely many  $a_1, \dots, a_m \in A$  such that  $A = \sum_{i=1}^m \varphi(B)a_i$ .

We may assume that  $a_i \in \hat{A}$  as  $k$  is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings,  $\varphi$  admits a unique extension to a bounded  $k$ -algebra epimorphism

$$\Phi : k\{T_1, \dots, T_n, S_1, \dots, S_m\} \rightarrow A$$

sending  $S_i$  to  $a_i$ . By [Corollary 7.5](#) in the chapter Banach Rings,  $\Phi$  is admissible. Moreover, the homomorphism  $\Phi$  is surjective by our assumption. It follows that  $A$  is strictly  $k$ -affinoid.  $\square$

**Proposition 3.12.** Assume that  $k$  is non-trivially valued. Let  $B$  be a strict  $k$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite  $k$ -algebra homomorphism into a  $k$ -algebra  $A$ . Then there is a norm on  $A$  such that the morphism is bounded and  $A$  is strictly  $k$ -affinoid.

PROOF. By [Proposition 8.4](#) in the chapter Banach Rings, we can endow  $A$  with a Banach norm such that  $\varphi$  is admissible. Then we can apply [Proposition 3.11](#).  $\square$

**Lemma 3.13.** Assume that  $k$  is non-trivially valued. Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . The algebra  $k\{r^{-1}T\}$  is strictly  $k$ -affinoid if  $r_i \in \sqrt{|k^\times|}$  for all  $i = 1, \dots, n$ .

**Remark 3.14.** The converse is also true.

PROOF. Assume that  $r_i \in \sqrt{|k^\times|}$  for all  $i = 1, \dots, n$ . Take  $s_i \in \mathbb{N}$  and  $c_i \in k^\times$  such that

$$r_i^{s_i} = |c_i^{-1}|$$

for  $i = 1, \dots, n$ . We define a bounded  $k$ -algebra homomorphism  $\varphi : k\{T_1, \dots, T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  by sending  $T_i$  to  $c_i T_i^{s_i}$ . This is possible by [Proposition 4.18](#) in the chapter Banach Rings.

We claim that  $\varphi$  is finite. To see this, it suffices to observe that if we expand  $f \in k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (cT^s)^\gamma,$$

where the product  $\gamma s$  is taken component-wise. For each  $\beta \in \mathbb{N}^n, \beta_i < s_i$ , we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \dots, T_n\}.$$

While  $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$ . So We have shown that  $\varphi$  is finite. Hence,  $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  is  $k$ -affinoid by [Proposition 3.11](#).  $\square$

**Proposition 3.15.** Let  $A$  be a  $k$ -affinoid algebra, then there is  $n \in \mathbb{N}$  and a  $k$ -free polyray  $r = (r_1, \dots, r_n)$  such that  $A \hat{\otimes}_k k_r$  is strictly  $k_r$ -affinoid. Moreover, we can guarantee that  $k_r$  is non-trivially valued.

PROOF. By [Proposition 3.9](#), we may assume that  $A = k\{t^{-1}T\}$  for some  $t \in \mathbb{R}_{>0}^m$ . By [Lemma 3.13](#), it suffices to take  $r$  so that the linear subspace of  $\mathbb{R}_{>0}/\sqrt{|k^\times|}$  generated by  $r_1, \dots, r_n$  contains all components of  $t$ . By taking  $n \geq 1$ , we can guarantee that  $k_r$  is non-trivially valued.  $\square$

**Proposition 3.16.** Let  $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a monomorphism in  $k_H\text{-Aff}$ . Then for any  $y \in \mathrm{Sp} B$  with  $x = \varphi(y)$ , one has  $\varphi^{-1}(x) = \{y\}$  and the natural map  $\mathcal{H}(x) \rightarrow \mathcal{H}(y)$  is an isomorphism of complete valuation rings.

PROOF. It suffices to show that  $\mathcal{H}(x) \rightarrow B \hat{\otimes}_A \mathcal{H}(y)$  is an isomorphism as Banach  $k$ -algebras. [Include details about cofiber products in affalg.](#) By assumption, the codiagonal map  $B \hat{\otimes}_A B \rightarrow B$  is an isomorphism. It follows that the base change with respect to  $A \rightarrow \mathcal{H}(x)$  is also an isomorphism:  $B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$ , where  $B' = B \hat{\otimes}_A \mathcal{H}(x)$ .

[Include the fact that the first map is injective.](#) It follows that the composition  $B' \otimes_{\mathcal{H}(x)} B \rightarrow B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$  is injective. Therefore,  $\mathcal{H}(x) \rightarrow B'$  is an isomorphism of rings. We also know that this map is bounded. But we already know that  $\mathcal{H}(x)$  is a complete valuation ring, so the map  $\mathcal{H}(x) \rightarrow B'$  is an isomorphism of complete valuation rings.  $\square$

#### 4. Weierstrass theory

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued-field.

**Proposition 4.1.** We have canonical identifications

$$\begin{aligned} (k\{T_1, \dots, T_n\})^\circ &\cong \mathring{k}\{T_1, \dots, T_n\}, \\ (k\{T_1, \dots, T_n\})^\sim &\cong \check{k}\{T_1, \dots, T_n\}, \\ k\{\widetilde{T_1, \dots, T_n}\} &\cong \tilde{k}[T_1, \dots, T_n]. \end{aligned}$$

The last identification extends  $\mathring{k} \rightarrow \tilde{k}$  and  $T_i$  is mapped to  $T_i$ .

PROOF. This follows from [Corollary 4.20](#) from the chapter Banach rings.  $\square$

We will denote the reduction map  $\mathring{k}\{T_1, \dots, T_n\} \rightarrow \tilde{k}[T_1, \dots, T_n]$  by  $\tilde{\bullet}$ .

**Definition 4.2.** Let  $n \in \mathbb{N}$ . A system  $f_1, \dots, f_n \in k\{T_1, \dots, T_n\}$  is called an *affinoid chart* of  $k\{T_1, \dots, T_n\}$  if  $f_i \in \mathring{k}\{T_1, \dots, T_n\}$  for each  $i = 1, \dots, n$  and the continuous  $k$ -algebra homomorphism  $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$  sending  $T_i$  to  $f_i$  is an isomorphism.

The map  $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$  is well-defined by [Proposition 4.1](#) and [Lemma 2.5](#).

**Lemma 4.3.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ . Assume that  $\|f\|_1 = 1$ . Then the following are equivalent:

- (1)  $f$  is a unit  $k\{T_1, \dots, T_n\}$ .
- (2)  $\tilde{f}$  is a unit in  $\tilde{k}[T_1, \dots, T_n]$ .

PROOF. As  $\|\bullet\|_1$  is a valuation by [Proposition 3.4](#),  $f$  is a unit in  $k\{T_1, \dots, T_n\}$  if and only if it is a unit in  $(k\{T_1, \dots, T_n\})^\circ$ , which is identified with  $\dot{k}\{T_1, \dots, T_n\}$  by [Proposition 4.1](#). This result then follows from [Corollary 4.21](#) in the chapter Banach Rings.  $\square$

**Definition 4.4.** Let  $n \in \mathbb{N}$ . Consider  $g \in k\{T_1, \dots, T_n\}$ . We expand  $g$  as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

For  $s \in \mathbb{N}$ , we say  $g$  is  $X_n$ -distinguished of degree  $s$  if  $g_s$  is a unit in  $k\{T_1, \dots, T_{n-1}\}$ ,  $\|g_s\|_1 = \|g\|_1$  and  $\|g_s\|_1 > \|g_t\|_1$  for all  $t > s$ .

**Theorem 4.5** (Weierstrass division theorem). Let  $n, s \in \mathbb{N}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished of degree  $s$ . Then for each  $f \in k\{T_1, \dots, T_n\}$ , there exist  $q \in k\{T_1, \dots, T_n\}$  and  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\deg_{T_n} r < s$  such that

$$f = qg + r.$$

Moreover,  $q$  and  $r$  are uniquely determined. We have the following estimates

$$(4.1) \quad \|q\|_1 \leq \|g\|_1^{-1} \|f\|_1, \quad \|r\|_1 \leq \|f\|_1.$$

If in addition,  $f, g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then  $q \in k\{T_1, \dots, T_{n-1}\}[T_n]$  as well.

PROOF. We may assume that  $\|g\|_1 = 1$ .

**Step 1.** Assuming the existence of the division. Let us prove (4.1). We may assume that  $f \neq 0$ , so that one of  $q, r$  is non-zero. Up to replacing  $q, r$  by a scalar multiple, we may assume that  $\max\{\|q\|_1, \|r\|_1\} = 1$ . So  $\|f\|_1 \leq 1$  as well. We need to show that  $\|f\|_1 = 1$ . Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here  $\tilde{\bullet}$  denotes the reduction map. By our assumption,  $\deg_{T_n} g = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$ . From [Proposition 4.1](#), the equality is in  $\dot{k}[T_1, \dots, T_n]$ . From the usual Euclidean division, we have  $\tilde{q} = \tilde{r} = 0$ . This is a contradiction to our assumption.

**Step 2.** Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with  $q$  and  $r$  as in the theorem. The estimate in Step 1 shows that  $q = r = 0$ .

**Step 3.** We prove the existence of the division.

We define

$$B := \{qg + r : r \in k\{T_1, \dots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \dots, T_n\}\}.$$

From Step 1,  $B$  is a closed subgroup of  $k\{T_1, \dots, T_n\}$ . In fact, suppose  $f_i \in B$  is a sequence converging to  $f \in k\{T_1, \dots, T_n\}$ . From Step 1, we can represent  $f_i = q_i g + r_i$ , then from Step 1,  $q_i$  and  $r_i$  are both Cauchy sequences, we may assume that  $q_i \rightarrow q \in k\{T_1, \dots, T_n\}$  and  $r_i \rightarrow r$ . As  $\deg_{T_n} r_i < s$ , it follows that  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  and  $\deg_{T_n} r < s$ . So  $f = qg + r$  and hence  $B$  is closed.

It suffices to show that  $B$  is dense in  $k\{T_1, \dots, T_n\}$ . We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

We may assume that  $\|g\|_1 = 1$ . Define  $\epsilon := \max_{j \geq s} \|g_j\|$ . Then  $\epsilon < 1$  by our assumption. Let  $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$  for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \dots, T_n\})^\circ \rightarrow (\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$$

with kernel  $\{f \in k\{T_1, \dots, T_n\} : \|f\|_1 \leq \epsilon\}$ . We now apply Euclidean division in the ring  $(\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$  to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some  $q \in (k\{T_1, \dots, T_n\})^\circ$  and  $r \in (k\{T_1, \dots, T_{n-1}\})^\circ[T_n]$  with  $\deg_{T_n} r < s$ . So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that  $B$  is dense in  $k\{T_1, \dots, T_n\}$  by [Proposition 2.8](#) in the chapter Banach rings.

**Step 4.** It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring  $k\{T_1, \dots, T_{n-1}\}[T_n]$  and the uniqueness proved in Step 2.  $\square$

**Lemma 4.6.** Let  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  be a Weierstrass polynomial and  $g \in k\{T_1, \dots, T_n\}$ . Assume that  $\omega g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then  $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ .

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some  $q, r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ ,  $\deg_{T_n} r < \deg_{T_n} \omega g$ . But we can write  $\omega g = \omega \cdot g$ . From the uniqueness part of [Theorem 4.5](#), we know that  $q = g$ , so  $g$  is a polynomial in  $T_n$ .  $\square$

As a consequence, we deduce Weierstrass preparation theorem.

**Definition 4.7.** Let  $n \in \mathbb{Z}_{>0}$ . A *Weierstrass polynomial* in  $n$ -variables is a monic polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\|\omega\|_1 = 1$ .

**Lemma 4.8.** Let  $n \in \mathbb{Z}_{>0}$  and  $\omega_1, \omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  be two monic polynomials. If  $\omega_1\omega_2$  is a Weierstrass polynomial then so are  $\omega_1$  and  $\omega_2$ .

PROOF. As  $\omega_1$  and  $\omega_2$  are monic,  $\|\omega_i\|_1 \geq 1$  for  $i = 1, 2$ . On the other hand,  $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1\omega_2\|_1 = 1$ , so  $\|\omega_i\|_1 = 1$  for  $i = 1, 2$ .  $\square$

**Theorem 4.9** (Weierstrass preparation theorem). Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished of degree  $s$ . Then there are a Weierstrass polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  of degree  $s$  and a unit  $e \in k\{T_1, \dots, T_n\}$  such that

$$g = e\omega.$$

Moreover,  $e$  and  $\omega$  are unique. If  $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then so is  $e$ .

PROOF. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let  $r = T_n^s - \omega$ . Then  $T_n^s = e^{-1}g + r$ . The uniqueness part of [Theorem 4.5](#) implies that  $e$  and  $r$  are uniquely determined, hence so is  $\omega$ .

Next we prove the existence. By Weierstrass division theorem [Theorem 4.5](#), we can write

$$T_n^s = qg + r$$

for some  $q \in k\{T_1, \dots, T_n\}$  and  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\deg_{T_n} r < s$ . Let  $\omega = T_n^s - r$ . From the estimates in [Theorem 4.5](#),  $\|r\|_1 \leq 1$ . So  $\|\omega\|_1 = 1$ . Then  $\omega$  is a Weierstrass polynomial of degree  $s$  and  $\omega = qg$ . It suffices to argue that  $q$  is a unit.

We may assume that  $\|g\|_1 = 1$ . By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As  $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$  and the leading coefficients of both polynomials are units in  $\tilde{k}[T_1, \dots, T_{n-1}]$ , it follows that  $\tilde{q}$  is a unit in  $\tilde{k}[T_1, \dots, T_{n-1}]$ . It follows that  $\tilde{q}$  is also a unit in  $\tilde{k}[T_1, \dots, T_n]$ . By [Lemma 4.3](#),  $q$  is a unit in  $k\{T_1, \dots, T_n\}$ .

The lsat assertion is already proved in [Theorem 4.5](#). □

**Definition 4.10.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished. Then the Weierstrass polynomial  $\omega$  constructed in [Theorem 4.9](#) is called the *Weierstrass polynomial* defined by  $g$ .

**Corollary 4.11.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished. Let  $\omega$  be the Weierstrass polynomial of  $g$ . Then the injection

$$k\{T_1, \dots, T_{n-1}\}[T_n] \rightarrow k\{T_1, \dots, T_n\}$$

induces an isomorphism of  $k$ -algebras

$$k\{T_1, \dots, T_{n-1}\}[T_n]/(\omega) \rightarrow k\{T_1, \dots, T_n\}/(g).$$

PROOF. The surjectivity follows from [Theorem 4.5](#) and the injectivity follows from [Lemma 4.6](#). □

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

**Lemma 4.12.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  is non-zero. Then there is a  $k$ -algebra automorphism  $\sigma$  of  $k\{T_1, \dots, T_n\}$  so that  $\sigma(g)$  is  $T_n$ -distinguished.

PROOF. We may assume that  $\|g\|_1 = 1$ . We expand  $g$  as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow  $\mathbb{N}^n$  with the lexicographic order. Take the maximal  $\beta \in \mathbb{N}^n$  so that  $|a_\beta| = 1$ . Take  $t \in \mathbb{Z}_{>0}$  so that  $t \geq \max_{i=1, \dots, n} \alpha_i$  for all  $\alpha \in \mathbb{N}^n$  with  $\tilde{a}_\alpha \neq 0$ .

We will define  $\sigma$  by sending  $T_i$  to  $T_i + T_n^{c_i}$  for all  $i = 1, \dots, n-1$ . The  $c_i$ 's are to be defined. We begin with  $c_n = 1$  and define the other  $c_i$ 's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for  $j = 1, \dots, n-1$ . We claim that  $\sigma(f)$  is  $T_n$ -distinguished of order  $s = \sum_{i=1}^n c_i \beta_i$ .

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^s p_i T_n^i$$

for some  $p_i \in \tilde{k}[T_1, \dots, T_{n-1}]$  and  $p_s = \tilde{a}_\beta$ . Our claim follows. □

**Proposition 4.13.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is Noetherian.

PROOF. We make induction on  $n$ . The case  $n = 0$  is trivial. Assume that  $n > 0$ . It suffices to show that for any non-zero  $g \in k\{T_1, \dots, T_n\}$ ,  $k\{T_1, \dots, T_n\}/(g)$  is Noetherian. By [Lemma 4.12](#), we may assume that  $g$  is  $T_n$ -distinguished. By [Theorem 4.5](#),  $k\{T_1, \dots, T_n\}/(g)$  is a finite free  $k\{T_1, \dots, T_{n-1}\}$ -module. By the inductive hypothesis and Hilbert basis theorem,  $k\{T_1, \dots, T_n\}/(g)$  is indeed Noetherian.  $\square$

**Proposition 4.14.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is Jacobson.

PROOF. When  $n = 0$ , there is nothing to prove. We make induction on  $n$  and assume that  $n > 0$ . Let  $\mathfrak{p}$  be a prime ideal in  $k\{T_1, \dots, T_n\}$ , we want to show that the Jacobson radical of  $\mathfrak{p}$  is equal to  $\mathfrak{p}$ .

We distinguish two cases. First we assume that  $\mathfrak{p} \neq 0$ . Let  $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \dots, T_{n-1}\}$ . By [Lemma 4.12](#), we may assume that  $\mathfrak{p}$  contains a Weierstrass polynomial  $\omega$ . Observe that

$$k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' \rightarrow k\{T_1, \dots, T_n\}/\mathfrak{p}$$

is finite by [Theorem 4.5](#). For any  $b \in J(k\{T_1, \dots, T_n\}/\mathfrak{p})$  (where  $J$  denotes the Jacobson radical), we consider a monic integral equation of minimal degree over  $k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'$ :

$$b^n + a_1 b^{n-1} + \dots + a_n = 0, \quad a_i \in k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \dots, T_n\}/\mathfrak{p}) \cap k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that  $n = 1$  and so  $b = 0$ . This proves  $J(k\{T_1, \dots, T_n\}/\mathfrak{p}) = 0$ .

On the other hand, let us consider the case  $\mathfrak{p} = 0$ . As  $k\{T_1, \dots, T_n\}$  is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \dots, T_n\}) = 0.$$

Assume that there is a non-zero element  $f$  in  $J(k\{T_1, \dots, T_n\})$ . We may assume that  $\|f\|_1 = 1$ .

We claim that there is  $c \in k$  with  $|c| = 1$  such that  $c + f$  is not a unit in  $k\{T_1, \dots, T_n\}$ . Assuming this claim for the moment, we can find a maximal ideal  $\mathfrak{m}$  of  $k\{T_1, \dots, T_n\}$  such that  $c + f \in \mathfrak{m}$ . But  $f \in \mathfrak{m}$  by our assumption, so  $c \in \mathfrak{m}$  as well. This contradicts the fact that  $c \in k^\times$ .

It remains to prove the claim. We treat two cases separately. When  $|f(0)| < 1$ , we simply take  $c = 1$ , which works thanks to [Lemma 4.3](#). If  $|f(0)| = 1$ , we just take  $c = -f(0)$ .  $\square$

**Proposition 4.15.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is UFD. In particular,  $k\{T_1, \dots, T_n\}$  is normal.

PROOF. As  $\|\bullet\|_1$  is a valuation by [Proposition 2.2](#),  $k\{T_1, \dots, T_n\}$  is an integral domain. In order to see that  $k\{T_1, \dots, T_n\}$  has the unique factorization property, we make induction on  $n \geq 0$ . When  $n = 0$ , there is nothing to prove. Assume that  $n > 0$ . Take a non-unit element  $f \in k\{T_1, \dots, T_n\}$ . By [Theorem 4.9](#) and [Lemma 4.12](#), we may assume that  $f$  is a Weierstrass polynomial. By inductive hypothesis,  $k\{T_1, \dots, T_{n-1}\}$  is a UFD, hence so is  $k\{T_1, \dots, T_{n-1}\}[T_n]$  by [[Stacks, Tag 0BC1](#)]. It follows that  $f$  can be decomposed into the products of monic prime elements  $f_1, \dots, f_r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , which are all Weierstrass polynomials by [Lemma 4.8](#). Then by [Corollary 4.11](#), we see that each  $f_i$  is prime in  $k\{T_1, \dots, T_n\}$ .

Any UFD is normal by [Stacks, Tag 0AFV].  $\square$

**Corollary 4.16.** Let  $A$  be a strictly  $k$ -affinoid algebra,  $d \in \mathbb{N}$  and  $\varphi : k\{T_1, \dots, T_d\} \rightarrow A$  be an integral torsion-free injective homomorphism of  $k$ -algebras. Then  $\rho$  is a faithful  $k\{T_1, \dots, T_d\}$ -algebra norm on  $A$ . If  $f^n + \varphi(t_1)f^{n-1} + \dots + \varphi(t_n) = 0$  is the minimal integral equation of  $f$  over  $k\{T_1, \dots, T_d\}$ , then

$$|f|_{\sup} = \max_{i=1, \dots, n} |t_i|^{1/i}.$$

PROOF. This follows from Proposition 9.11 in the chapter Banach Rings and Proposition 4.15.  $\square$

## 5. Noetherian normalization and maximal modulus principle

Let  $(k, |\bullet|)$  be a complete non-trivially valued non-Archimedean valued-field.

**Theorem 5.1.** Let  $A$  be a non-zero strictly  $k$ -affinoid algebra,  $n \in \mathbb{N}$  and  $\alpha : k\{T_1, \dots, T_n\} \rightarrow A$  be a finite (resp. integral)  $k$ -algebra homomorphism. Then up to replacing  $T_1, \dots, T_n$  by an affinoid chart, we can guarantee that there exists  $d \in \mathbb{N}$ ,  $d \leq n$  such that  $\alpha$  when restricted to  $k\{T_1, \dots, T_d\}$  is finite (resp. integral) and injective.

PROOF. We make an induction on  $n$ . The case  $n = 0$  is trivial. Assume that  $n > 0$ . If  $\ker \alpha = 0$ , there is nothing to prove, so we may assume that  $\ker \alpha \neq 0$ . By Lemma 4.12 and Theorem 4.9, we may assume that there is a Weierstrass polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  in  $\ker \alpha$ . Then  $\alpha$  induces a finite (resp. integral) homomorphism  $\beta : k\{T_1, \dots, T_n\}/(\omega) \rightarrow A$ . By Theorem 4.5,  $k\{T_1, \dots, T_{n-1}\} \rightarrow k\{T_1, \dots, T_n\}/(\omega)$  is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism  $k\{T_1, \dots, T_{n-1}\} \rightarrow A$ . We can apply the inductive hypothesis to conclude.  $\square$

**Corollary 5.2.** Let  $A$  be a non-zero strictly  $k$ -affinoid algebra, then there is  $d \in \mathbb{N}$  and a finite injective  $k$ -algebra homomorphism:  $k\{T_1, \dots, T_d\} \rightarrow A$ .

PROOF. Take some  $n \in \mathbb{N}$  and a surjective  $k$ -algebra homomorphism  $k\{T_1, \dots, T_n\} \rightarrow A$  and apply Theorem 5.1, we conclude.  $\square$

**Corollary 5.3.** Let  $A$  be a strictly  $k$ -affinoid algebra and  $I$  be an ideal in  $A$  such that  $\sqrt{I}$  is a maximal ideal in  $A$ , then  $A/I$  is finite-dimensional over  $k$ .

In particular,  $\text{Spm } A = \text{Spm}_k A$ .

PROOF. By Corollary 5.2, there is  $d \in \mathbb{N}$  and a finite monomorphism  $f : k\{T_1, \dots, T_d\} \rightarrow A/I$ . It suffices to show that  $d = 0$ . Observe that the composition

$$k\{T_1, \dots, T_d\} \xrightarrow{f} A/I \rightarrow A/\sqrt{I}$$

is finite and injective as  $k\{T_1, \dots, T_d\}$  is an integral domain, so  $k\{T_1, \dots, T_d\}$  is a field. This is possible only when  $d = 0$ .  $\square$

**Corollary 5.4.** Let  $B$  be a strictly  $k$ -affinoid algebra and  $A$  be a Noetherian Banach  $k$ -algebra. Let  $f : A \rightarrow B$  a  $k$ -algebra homomorphism. Then  $f$  is bounded.

PROOF. This follows from Proposition 8.1 in the chapter Banach Rings and Proposition 4.13.  $\square$

In particular, we see that the topology of a  $k$ -affinoid algebra is uniquely determined by the algebraic structure.

**Corollary 5.5.** Let  $A, B$  be strictly  $k$ -affinoid algebras. Let  $f$  be a finite  $k$ -algebra homomorphism, then  $f$  is admissible.

PROOF. This follows from [Proposition 3.12](#) and [Corollary 5.4](#), □

**Definition 5.6.** For any non-Archimedean valuation field  $(K, |\bullet|)$  and  $n \in \mathbb{N}$ , we define the  $n$ -dimensional polydisk with value in  $K$ :

$$B^n(K) := \left\{ (x_1, \dots, x_n) \in K^n : \max_{i=1, \dots, n} |x_i| \leq 1 \right\}.$$

**Definition 5.7.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ , say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function  $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$  as sending  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

**Lemma 5.8.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ , then  $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$  is continuous and for any  $x \in B^n(k^{\text{alg}})$ ,

$$|f(x)| \leq \|f\|_1.$$

There is  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  such that  $|f(x)| = \|f\|_1$ .

PROOF. To see that  $f$  is continuous, it suffices to observe that  $f$  is a uniform limit of polynomials. For any  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$ , we have

$$|f(x)| = \left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \right| \leq \max_{\alpha \in \mathbb{N}^n} |a_\alpha x^\alpha| \leq \|f\|_1.$$

To prove the last assertion, we may assume that  $\|f\|_1 = 1$ . As the residue field of  $k^{\text{alg}}$  is equal to  $\tilde{k}^{\text{alg}}$ , it has infinitely many elements, so there is a point  $x \in B^n(k^{\text{alg}})$  such that  $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$ . In other words,  $\|f(x)\|_1 = 1$ . □

**Proposition 5.9.** Let  $n \in \mathbb{N}$ , then the maximal modulus principle holds for  $k\{T_1, \dots, T_n\}$ . Moreover, for any  $f \in k\{T_1, \dots, T_n\}$ ,  $\|f\|_1 = |f|_{\text{sup}}$ .

PROOF. By [Lemma 6.3](#) in the chapter Banach Rings, we have

$$\|f\|_1 \geq |f|_{\text{sup}}$$

for any  $f \in A$ . We only have to show that for any  $f \in k\{T_1, \dots, T_n\}$  there is a maximal ideal  $\mathfrak{m} \subseteq k\{T_1, \dots, T_n\}$  such that  $|f(\mathfrak{m})| = \|f\|_1$ .

By [Lemma 5.8](#) we can take  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  such that  $|f(x)| = \|f\|_1$ . Let  $L$  be the field extension of  $k$  generated by  $x_1, \dots, x_n$ , then  $L/k$  is finite. Then we can define a homomorphism

$$\text{ev}_x : k\{T_1, \dots, T_n\} \rightarrow L$$



sending  $g \in k\{T_1, \dots, T_n\}$  to  $g(x)$ . Observe that the image is indeed in  $L$ . Clearly  $\text{ev}_x$  is surjective. So  $\mathfrak{m}_x := \ker \text{ev}_x$  is a  $k$ -algebraic maximal ideal in  $k\{T_1, \dots, T_n\}$ . Then

$$|f(\mathfrak{m}_x)| = |f(x)| = \|f\|_1.$$

□

**Corollary 5.10.** Let  $A$  be a strictly  $k$ -affinoid algebra. Then for any  $f \in A$ ,

$$|f|_{\text{sup}} \subseteq \sqrt{|k^\times|} \cup \{0\}.$$

PROOF. We may assume that  $A \neq 0$ . By [Corollary 5.2](#) and [Proposition 9.11](#) in the chapter Banach Rings, we may assume that  $A = k\{T_1, \dots, T_n\}$  for some  $n \in \mathbb{N}$ . The result then follows from [Proposition 5.9](#). □

**Corollary 5.11.** Maximal modulus principle holds for any strictly  $k$ -affinoid algebras.

PROOF. This follows from [Corollary 5.2](#), [Proposition 9.11](#) in the chapter Banach Rings and [Proposition 5.9](#). □

**Proposition 5.12.** Let  $\varphi : B \rightarrow A$  be an integral  $k$ -algebra homomorphism of strictly  $k$ -affinoid algebras. Then for each non-zero  $f \in A$ , there is a monic polynomial  $q(f) = f^n + \varphi(b_1)f^{n-1} + \dots + \varphi(b_n)$  of  $f$  over  $B$ . Then

$$|f|_{\text{sup}} = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

PROOF. One side is simple: choose  $j = 1, \dots, n$  that maximizes  $|\varphi(b_j)f^{n-j}|_{\text{sup}}$ , then

$$|f|_{\text{sup}}^n = |f^n|_{\text{sup}} \leq |\varphi(b_j)f^{n-j}|_{\text{sup}} \leq |b_j|_{\text{sup}} \cdot |f|_{\text{sup}}^{n-j}.$$

So

$$|f|_{\text{sup}} \leq |b_j|_{\text{sup}}^{1/j}.$$

To prove the reverse inequality, let us begin with the case where  $A$  is an integral domain.

We claim that there is  $d \in \mathbb{N}$  and a  $k$ -algebra homomorphism  $\psi : k\{T_1, \dots, T_d\} \rightarrow B$  such that  $\varphi \circ \psi$  is integral and injective. In fact, choosing an epimorphism  $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$ , we can apply [Theorem 5.1](#) to find  $\phi \circ \alpha$  to conclude.

By [Corollary 4.16](#), if  $p$  denotes the minimal polynomial of  $f$  over  $k\{T_1, \dots, T_d\}$ , we have  $|f|_{\text{sup}} = \sigma(p)$ . In particular,  $p(f) = 0$ . Let  $q \in B[X]$  be the polynomial obtained from  $p$  by replacing all coefficients by their  $\psi$ -images in  $B$ . Then clearly,  $|f|_{\text{sup}} = \sigma(q)$ .

In general, let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the minimal primes in  $A$ . The number is finite by [Proposition 4.13](#). For each  $i = 1, \dots, r$ , let  $\pi_i : A \rightarrow A/\mathfrak{p}_i$  denote the natural homomorphism. We know that there are monic polynomials  $q_i \in B[X]$  such that  $q_i(\pi_i(f)) = 0$  and  $|\pi_i(f)|_{\text{sup}} = \sigma(q_i)$  for  $i = 1, \dots, r$ . We let  $q' = q_1 \cdots q_r$ . Then

$$q'(f) \in \bigcap_{i=1}^r \mathfrak{p}_i.$$

So there is  $e \in \mathbb{Z}_{>0}$  such that  $q'(f)^e = 0$ . Let  $q = q'^e$ . By [Proposition 9.5](#) in the chapter Banach Rings,

$$\sigma(q) \leq \max_{i=1, \dots, r} \sigma(q_i) = \max_{i=1, \dots, r} |\pi_i(f)|_{\text{sup}} = |f|_{\text{sup}}.$$

The last equality follows from [Proposition 9.9](#) in the chapter Banach Rings. □

**Lemma 5.13.** Let  $\varphi : B \rightarrow A$  be an admissible  $k$ -algebra homomorphism between strictly  $k$ -affinoid algebras. Let  $\tau : \mathring{B} \rightarrow \check{B}$  be the reduction map, then

$$\tau^{-1}(\ker \tilde{\varphi}) = \sqrt{\check{B} + \ker \tilde{\varphi}}, \quad \ker \tilde{\varphi} = \sqrt{\tau(\ker \tilde{\varphi})}.$$

PROOF. The second equation follows from the first one by applying  $\tau$ . Let us prove the first equation. By assumption,  $\varphi(\mathring{B})$  is open in  $\varphi(B)$ . Consider  $g \in \tau^{-1}(\ker \tilde{\varphi})$ , we know that

$$\lim_{n \rightarrow \infty} \varphi(g)^n = 0.$$

So  $\varphi(g)^n \in \varphi(\mathring{B})$  for  $n$  large enough, and hence  $g^n \in \check{B} + \ker \tilde{\varphi}$ .  $\square$

**Lemma 5.14.** Let  $m \in \mathbb{N}$  and  $T = k\{T_1, \dots, T_m\}$ . Let  $A$  be a  $k$ -affinoid algebra and  $\varphi : T\{S_1, \dots, S_n\} \rightarrow A$  be a finite morphism such that  $\tilde{\varphi}(S_i)$  is integral over  $\check{T}$ . Then  $\varphi|_T : T \rightarrow A$  is finite.

PROOF. We make an induction on  $n$ . When  $n = 0$ , there is nothing to prove. So assume  $n > 0$  and the lemma has been proved for smaller values of  $n$ .

Let  $T' = T\{S_1, \dots, S_n\}$ . By assumption, there is a Weierstrass polynomial

$$\omega = S_n^k + a_1 S_n^{k-1} + \dots + a_k \in \mathring{T}[S_n]$$

such that  $\tilde{\omega} \in \ker \tilde{\varphi}$ . As  $\varphi$  is admissible by [Corollary 5.5](#), we have  $\omega^q \in \check{T}' + \ker \tilde{\varphi}$  for some  $q \in \mathbb{Z}$  by [Lemma 5.13](#).

In particular, we can find  $r \in (T')^\vee$  such that  $g := \omega^q - r \in \ker \tilde{\varphi}$ . Observe that  $g$  is  $S_n$  distinguished of order  $mq$  as  $\tilde{g} = \tilde{\omega}^q$ . By [Corollary 4.11](#), the restriction of  $\varphi$  to  $T\{S_1, \dots, S_{n-1}\}$  is finite. We can apply the inductive hypothesis to conclude.  $\square$

**Lemma 5.15.** Let  $\varphi : B \rightarrow A$  be a  $k$ -algebra homomorphism of strictly  $k$ -affinoid algebras. Assume that there exist affinoid generators  $f_1, \dots, f_n \in \mathring{A}$  of  $A$  such that  $\tilde{f}_1, \dots, \tilde{f}_n$  are all integral over  $\check{B}$ , then  $\varphi$  is finite.

PROOF. By assumption, we can find  $s_i \in \mathbb{Z}_{>0}$ ,  $b_{ij} \in \mathring{B}$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, s_i$  such that

$$\tilde{f}_i^{s_i} + \tilde{\varphi}(\tilde{b}_{i1})\tilde{f}_i^{s_i-1} + \dots + \tilde{\varphi}(\tilde{b}_{is_i}) = 0$$

for  $i = 1, \dots, n$ . Let  $s = s_1 + \dots + s_n$  and define a bounded  $k$ -algebra homomorphism  $\psi : D := k\{T_{ij}\} \rightarrow B$  sending  $T_{ij}$  to  $b_{ij}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, s_i$ . Observe that  $\tilde{f}_1, \dots, \tilde{f}_n$  are all integral over  $\check{D}$ . So it suffices to prove the theorem when  $B = k\{T_1, \dots, T_m\}$ . We extend  $\varphi$  to a bounded  $k$ -algebra epimorphism  $\varphi' : T\{S_1, \dots, S_n\} \rightarrow A$  sending  $S_i$  to  $f_i$  for  $i = 1, \dots, n$ . Then  $\varphi'(\tilde{S}_i)$  is integral over  $\check{B}$ . It suffices to apply [Lemma 5.14](#).  $\square$

## 6. Properties of affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $R_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Proposition 6.1.** Assume that  $k$  is non-trivially valued. Let  $A$  be a strictly  $k$ -affinoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\sup} \leq 1\}.$$

PROOF. By [Lemma 6.3](#), we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\sup} \leq 1\}.$$

Conversely, let  $f \in A$ ,  $|f|_{\sup} \leq 1$ . Choose  $d \in \mathbb{N}$  and a surjective  $k$ -algebra homomorphism

$$\varphi : k\{T_1, \dots, T_d\} \rightarrow A.$$

Let  $f^n + t_1 f^{n-1} + \dots + t_n = 0$  be the minimal equation of  $f$  over  $k\{T_1, \dots, T_d\}$ . Then  $t_i \in (k\{T_1, \dots, T_d\})^\circ$  by [Proposition 9.11](#) in the chapter Banach Rings. An induction on  $i \geq 0$  shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi((k\{T_1, \dots, T_d\})^\circ) f^j.$$

The right-hand side is clearly bounded.  $\square$

**Corollary 6.2.** Assume that  $k$  is non-trivially valued. Let  $(A, \|\bullet\|)$  be a strictly  $k$ -affinoid algebra. For any  $f \in A$ ,

$$\rho(f) = |f|_{\sup}.$$

PROOF. We have shown that  $\rho(f) \geq |f|_{\sup}$  in [Lemma 6.3](#) from the chapter Banach Rings. Assume that the inverse inequality fails: for some  $f \in A$ ,

$$\rho(f) > |f|_{\sup}.$$

If  $|f|_{\sup} = 0$ , then  $f$  lies in the Jacobson radical of  $A$ , which is equal to the nilradical of  $A$  by [Proposition 4.14](#). But then  $\rho(f) = 0$  as well. We may therefore assume that  $|f|_{\sup} \neq 0$ . By [Corollary 5.10](#), we may assume that  $|f|_{\sup} = 1$  as  $\rho$  is power-multiplicative. Then  $\rho(f) > 1$ . This contradicts [Proposition 6.1](#).  $\square$

**Theorem 6.3.** A  $k$ -affinoid algebra  $A$  is Noetherian and all ideals of  $A$  are closed.

PROOF. Let  $I$  be an ideal in  $A$ . By [Proposition 3.15](#), we can take a suitable  $r \in \mathbb{R}_{>0}^m$  so that  $A \hat{\otimes} k_r$  is strictly  $k_r$ -affinoid. Then  $I(A \hat{\otimes} k_r)$  is an ideal in  $A \hat{\otimes} k_r$ . By [Proposition 4.13](#), the latter ring is Noetherian. So we may take finitely many generators  $f_1, \dots, f_k \in I$ . Each  $f \in I$  can be written as

$$f = \sum_{i=1}^k f_i g_i$$

with  $g_i = \sum_{j=-\infty}^{\infty} g_{i,j} T^j \in A \hat{\otimes} k_r$ . But then

$$f = \sum_{i=1}^k f_i g_{i,0}.$$

So  $I$  is finitely generated.

As  $I = A \cap (I(A \hat{\otimes} k_r))$ , by [Corollary 7.4](#) in the chapter Banach Rings, we see that  $I$  is closed in  $A \hat{\otimes} k_r$  and hence closed in  $A$ .  $\square$

**Proposition 6.4.** Let  $(A, \|\bullet\|)$  be a  $k$ -affinoid algebra and  $f \in A$ . Then there is  $C > 0$  and  $N \geq 1$  such that for any  $n \geq N$ , we have

$$\|f^n\| \leq C \rho(f)^n.$$

Recall that  $\rho$  is the spectral radius map defined in [Definition 4.9](#) in the chapter Banach Rings.

PROOF. By [Proposition 3.9](#), we may assume that  $k$  is non-trivially valued and  $k$  is non-trivially valued.

If  $\rho(f) = 0$ , then  $f$  lies in each maximal ideal of  $A$ . To see this, we may assume that  $A$  is a field, then by [Proposition 6.10](#) in the chapter Banach Rings, there is a bounded valuation  $\|\bullet\|'$  on  $A$ . But then  $\rho(f) = 0$  implies that  $\|f\|' = 0$  and hence  $f = 0$ .

It follows that if  $\rho(f) = 0$  then  $f$  lies in  $J(A)$ , the Jacobson radical of  $A$ . By [Proposition 4.14](#),  $A$  is a Jacobson ring. So  $f$  is nilpotent. The assertion follows.

So we can assume that  $\rho(f) > 0$ . In this case, by [Corollary 5.2](#) and [Proposition 9.11](#) in the chapter Banach Rings, we have  $\rho(f) \in \sqrt{|k^\times|}$ . Take  $a \in k^\times$  and  $d \in \mathbb{Z}_{>0}$  so that  $\rho(f)^d = |a|$ . Then  $\rho(f^d/a) = 1$  and hence it is powerly-bounded by [Proposition 6.1](#). It follows that there is  $C > 0$  so that for  $n \geq 1$ ,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that  $\|f^n\| \leq C\rho(f)$  for  $n \geq d$  as long as we enlarge  $C$ .  $\square$

**Corollary 6.5.** Let  $\varphi : A \rightarrow B$  be a bounded homomorphism of  $k$ -affinoid algebras. Let  $n \in \mathbb{N}$  and  $f_1, \dots, f_n \in B$  and  $r_1, \dots, r_n \in \mathbb{R}_{>0}$  with  $r_i \geq \rho(f_i)$  for  $i = 1, \dots, n$ . Write  $r = (r_1, \dots, r_n)$ , then there is a unique bounded homomorphism  $\Phi : A\{r^{-1}T\} \rightarrow B$  extending  $\varphi$  and sending  $T_i$  to  $f_i$ .

PROOF. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from [Proposition 6.4](#) that the right-hand side the series converges. The boundedness of  $\Phi$  is obvious.  $\square$

**Definition 6.6.** Let  $A$  be an affinoid algebra,  $f \in A$  is a non-zero element and  $r \in \mathbb{R}_{>0}$ , we define the *localization*  $A\{rf^{-1}\}$  of  $A$  at  $r^{-1}f$  as follows:

$$A\{rf^{-1}\} := A\{rT\}/(Tf - 1).$$

Observe that  $A\{rf^{-1}\}$  is  $k$ -affinoid by [Theorem 6.3](#).

**Proposition 6.7.** Let  $A$  be an affinoid algebra,  $f \in A$  is a non-zero element and  $r \in \mathbb{R}_{>0}$ . Consider the natural map  $\iota : A \rightarrow A\{rf^{-1}\}$ , then  $\mathrm{Sp} \iota : \mathrm{Sp} A\{rf^{-1}\} \rightarrow \mathrm{Sp} A$  is injective. We will identify  $\mathrm{Sp} A\{rf^{-1}\}$  with a subset of  $\mathrm{Sp} A$ . Then

$$\mathrm{Sp} A\{rf^{-1}\} = \{x \in \mathrm{Sp} A : |f(x)| \geq r\}.$$

For any  $x \in \mathrm{Sp} A\{rf^{-1}\}$ , we have

$$|f(x)| \geq r.$$

PROOF. The first assertion means that each bounded semi-valuation on  $A$  admits at most one bounded extension to  $A\{r^{-1}T\}$ . This is obvious as the image of  $A$  in  $A\{r^{-1}T\}$  is dense.

For the second statement, let  $\|\bullet\|_x$  be the bounded semi-norm on  $A\{r^{-1}T\}$  corresponding to  $x$ . We need to show that

$$\|f\|_x \geq r.$$

We know that

$$\|T\|_{r^{-1}} = r^{-1}$$

so

$$\|T\|_x \leq r^{-1}.$$

From  $Tf = 1$ , we find

$$1 \leq \|f\|_x \cdot \|T\|_x \leq r^{-1} \|f\|_x.$$

Conversely, let  $x \in \text{Sp } A$  with  $|f(x)| \geq r$ . Let  $\|\bullet\|_x$  be the bounded semi-valuation on  $A$  corresponding to  $x$ . We can extend  $\|\bullet\|_x$  to a semi-valuation  $\|\bullet\|'_x$  on  $B$  by [Remark 4.17](#) in the chapter Banach Rings. The assumption  $|f(x)| \geq r$  guarantees exactly that  $\|\bullet\|'_x$  is bounded.  $\square$

**Proposition 6.8.** Let  $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$  be  $k$ -affinoid algebras,  $r \in \mathbb{R}_{>0}^n$  and  $\varphi : A\{r^{-1}T\} \rightarrow B$  be an admissible epimorphism. Write  $f_i = \varphi(T_i)$  for  $i = 1, \dots, n$ . Then there is  $\epsilon > 0$  such that for any  $g = (g_1, \dots, g_n) \in B^n$  with  $\|f_i - g_i\|_B < \epsilon$  for all  $i = 1, \dots, n$ , there exists a unique bounded  $k$ -algebra homomorphism  $\psi : A\{r^{-1}T\} \rightarrow B$  that coincides with  $\varphi$  on  $A$  and sends  $T_i$  to  $g_i$ . Moreover,  $\psi$  is also an admissible epimorphism.

PROOF. The uniqueness of  $\psi$  is obvious. We prove the remaining assertions. Taking  $\epsilon > 0$  small enough, we could further guarantee that  $\rho(g_i) \leq r_i$ . It follows from [Corollary 6.5](#) that there exists a bounded homomorphism  $\psi$  as in the statement of the proposition.

As  $\varphi$  is an admissible epimorphism, we may assume that  $\|\bullet\|_B$  is the residue induced by  $\|\bullet\|_r$  on  $A\{r^{-1}T\}$ .

By definition of the residue norm, for any  $\delta > 0$  and any  $h \in B$ , we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1 + \delta) \|h\|_B$$

for any  $\alpha \in \mathbb{N}^n$ . Choose  $\epsilon \in (0, (1 + \delta)^{-1})$ . Now for  $g_1, \dots, g_n$  as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1 + \delta) \epsilon \|h\|_B.$$

Repeating this procedure, we can construct  $k_i \in A\{r^{-1}T\}$  for  $i \in \mathbb{N}$  and  $h_j \in B$  for  $j \in \mathbb{Z}_{>0}$  such that for any  $i \in \mathbb{Z}_{>0}$ , we have

$$h = \psi(k_0 + \dots + k_{i-1}) + h_i,$$

$$\|k_i\|_r \leq ((1 + \delta)\epsilon)^i (1 + \delta) \|h\|_B,$$

$$\|h_i\|_B \leq ((1 + \delta)\epsilon)^i \|h\|_B.$$

In particular,  $k := \sum_{i=0}^{\infty} k_i$  converges in  $A\{r^{-1}T\}$  and

$$\|k\|_r \leq (1 + \delta) \|h\|_B.$$

It follows that  $\psi$  is an admissible epimorphism.  $\square$

**Corollary 6.9.** Let  $A$  be a Banach  $k$ -algebra,  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a  $k$ -free polyray. Assume that  $A \hat{\otimes}_k k_r$  is  $k_r$ -affinoid, then  $A$  is  $k$ -affinoid.

If  $A \hat{\otimes}_k k_r$  is  $k_H$ -affinoid and  $r \in H$ , then  $A$  is also  $k_H$ -affinoid.

PROOF. We may assume that  $r$  has only one component.

Take  $m \in \mathbb{N}$ ,  $p_1, \dots, p_m \in \mathbb{R}_{>0}$  and an admissible epimorphism

$$\pi : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \rightarrow A \hat{\otimes}_k k_r.$$

Let

$$\pi(S_i) = \sum_{j=-\infty}^{\infty} a_{i,j} T^j, \quad a_{i,j} \in A$$

for  $i = 1, \dots, m$ . By [Proposition 6.8](#), we may assume that there is a large integer  $l$  such that  $a_{i,j} = 0$  for  $|j| > l$  and for any  $i = 1, \dots, m$ . We define  $B = k\{p_i^{-1}r^j T_{i,j}\}$ ,  $i = 1, \dots, n$  and  $j = -l, -l+1, \dots, l$ . Let  $\varphi : B \rightarrow A$  be the bounded  $k$ -algebra homomorphism sending  $T_{i,j}$  to  $a_{i,j}$ . The existence of  $\varphi$  is guaranteed by [Corollary 6.5](#).

We claim that  $\varphi$  is an admissible epimorphism. It is clearly an epimorphism. Let us show that  $\varphi$  is admissible. Let  $\eta : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \rightarrow B \hat{\otimes}_k k_r$  be the bounded homomorphism sending  $S_i$  to  $\sum_{j=-l}^l T_{i,j} T^j$ , then we have the following commutative diagram

$$\begin{array}{ccc} k_r\{p^{-1}S\} & & \\ \downarrow \eta & \searrow \pi & \\ B \hat{\otimes}_k k_r & \xrightarrow{\varphi \hat{\otimes}_k k_r} & A \hat{\otimes}_k k_r \end{array}$$

It follows that  $\varphi \hat{\otimes}_k k_r$  is also an admissible epimorphism. By [Proposition 3.9](#),  $\varphi$  is also admissible.  $\square$

## 7. $H$ -strict affinoid algebras

Let  $(k, |\cdot|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $R_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

We next give a non-strict extension of [Proposition 3.11](#).

**Proposition 7.1.** Let  $B$  be a  $k_H$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite bounded homomorphism into a  $k$ -Banach algebra  $A$ . Then  $A$  is also  $k_H$ -affinoid.

PROOF. We first assume that  $k$  is non-trivially valued.

We may assume that  $B = k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  for some  $n \in \mathbb{N}$  and  $r_1, \dots, r_n \in H$ . By assumption, we can find finitely many  $a_1, \dots, a_m \in A$  such that  $A = \sum_{i=1}^m \varphi(B)a_i$ .

We may assume that  $a_i \in \mathring{A}$  as  $k$  is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings,  $\varphi$  admits a unique extension to a bounded  $k$ -algebra epimorphism

$$\Phi : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, S_1, \dots, S_m\} \rightarrow A$$

sending  $S_i$  to  $a_i$ . By [Corollary 7.5](#) in the chapter Banach Rings,  $\Phi$  is admissible. Moreover, the homomorphism  $\Phi$  is surjective by our assumption. It follows that  $A$  is  $k_H$ -affinoid.

If  $k$  is trivially valued, then  $H$  is non-trivial. Take  $s \in H \setminus \{1\}$ . It follows from the previous case applied to  $\varphi \hat{\otimes}_k k_s : B \hat{\otimes}_k k_s \rightarrow A \hat{\otimes}_k k_s$  that  $A \hat{\otimes}_k k_s$  is  $k_H$ -affinoid. By [Corollary 6.9](#),  $A$  is also  $k_H$ -affinoid.  $\square$

**Proposition 7.2.** Let  $A$  be a Banach  $k$ -algebra. Then the following are equivalent:

- (1)  $A$  is  $k_H$ -affinoid;
- (2) there are  $n \in \mathbb{N}$ ,  $r \in \sqrt{|k^\times|} \cdot H$  and an admissible epimorphism  $k\{r^{-1}T\} \rightarrow A$ .

PROOF. The non-trivial direction is (2). Assume (2). Take  $s_1, \dots, s_n \in \mathbb{Z}_{>0}$ ,  $c_1, \dots, c_n \in k^\times$  and  $h_1, \dots, h_n \in H$  such that

$$r_i^{s_i} = |c_i^{-1}|h_i$$

for  $i = 1, \dots, n$ . We define a bounded  $k$ -algebra homomorphism

$$\varphi : k\{h_1^{-1}T_1, \dots, h_n^{-1}T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

by sending  $T_i$  to  $c_i T_i^{s_i}$ . The existence of such a homomorphism is guaranteed by [Corollary 6.5](#). The same proof of [Lemma 3.13](#) shows that  $\varphi$  is finite. By [Proposition 7.1](#),  $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  is  $k_H$ -affinoid.  $\square$

**Lemma 7.3.** Assume that  $k$  is non-trivially valued. Let  $A$  be a  $k$ -affinoid algebra. Then the following are equivalent:

- (1)  $A$  is strictly  $k$ -affinoid;
- (2) for any  $a \in A$ ,  $\rho(a) \in \sqrt{|k^\times|} \cup \{0\}$ .

PROOF. (1)  $\implies$  (2) by [Corollary 5.10](#) and [Corollary 6.2](#).

(2)  $\implies$  (1): Take  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$  and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Let  $f_i = \varphi(T_i)$  for  $i = 1, \dots, n$ . Suppose  $r_1, \dots, r_m \notin \sqrt{|k^\times|}$  and  $r_{m+1}, \dots, r_n \in \sqrt{|k^\times|}$ . Then  $\rho(f_i) < r_i$  for  $i = 1, \dots, m$  and we can choose  $r'_1, \dots, r'_m \in \sqrt{|k^\times|}$  such that

$$\rho(f_i) \leq r'_i < r_i$$

for  $i = 1, \dots, m$ . Set  $r'_i = r_i$  when  $i = m+1, \dots, n$ . We can then define a bounded  $k$ -algebra homomorphism  $\psi : k\{r'^{-1}T\} \rightarrow A$  sending  $T_i$  to  $f_i$  for  $i = 1, \dots, n$ . The existence of  $\psi$  is guaranteed by [Corollary 6.5](#). Observe that  $\psi$  is surjective and admissible. It follows that  $A$  is strictly  $k$ -affinoid.  $\square$

**Theorem 7.4.** Let  $A$  be a  $k$ -affinoid algebra. Then the following are equivalent:

- (1)  $A$  is  $k_H$ -affinoid;
- (2)  $A$  is  $k_{\sqrt{|k^\times|} \cdot H}$ -affinoid;
- (3) For any non-zero  $a \in A$ ,  $\rho(a) \in \sqrt{|k^\times|} \cdot H \cup \{0\}$ .

PROOF. The equivalence between (1) and (2) follows from [Proposition 7.2](#).

(1)  $\implies$  (3): we may assume that  $H \supseteq |k^\times|$ . Take  $n \in \mathbb{N}$ ,  $r = (r_1, \dots, r_n) \in H^n$  and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Take a  $k$ -free polyray  $s$  with at least one component so that  $|k_s| \supseteq \{r_1, \dots, r_n\}$ . We can apply [Lemma 7.3](#) to  $\varphi \hat{\otimes}_k k_s$ , it follows that  $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$ .

(3)  $\implies$  (2): we may assume that  $H \supseteq |k^\times|$ . It suffices to apply the same argument as (2)  $\implies$  (1) in the proof of [Lemma 7.3](#).  $\square$

## 8. Finite modules over affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field.

For any  $k$ -affinoid algebra  $A$ , we have defined the category  $\mathcal{B}\text{an}_A^f$  of finite Banach  $A$ -modules in [Definition 5.3](#) in the chapter Banach Rings. We write  $\mathcal{M}\text{od}_A^f$  for the category of finite  $A$ -modules.

**Lemma 8.1.** Let  $A$  be a  $k$ -affinoid algebra,  $(M, \|\bullet\|_M)$  be a finite Banach  $A$ -module and  $(N, \|\bullet\|_N)$  be a Banach  $A$ -module  $N$ . Let  $\varphi : M \rightarrow N$  be an  $A$ -linear homomorphism. Then  $\varphi$  is bounded.

PROOF. Take  $n \in \mathbb{N}$  such that there is an admissible epimorphism

$$\pi : A^n \rightarrow M.$$

It suffices to show that  $\varphi \circ \pi$  is bounded. So we may assume that  $M = A^n$ . For  $i = 1, \dots, n$ , let  $e_i$  be the vector with  $(0, \dots, 0, 1, 0, \dots, 0)$  of  $A^n$  with 1 placed at the  $i$ -th place. Set  $C = \max_{i=1, \dots, n} \|\varphi(e_i)\|_N$ . For a general  $f = \sum_{i=1}^n a_i e_i$  with  $a_i \in A$ , we have

$$\|\varphi(f)\|_N \leq C \|f\|_M.$$

So  $\varphi$  is bounded.  $\square$

**Proposition 8.2.** Let  $A$  be a  $k$ -affinoid algebra. The forgetful functor  $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$  is an equivalence of categories.

PROOF. It suffices to construct the inverse functor. Let  $M$  be a finite  $A$ -module. Choose  $n \in \mathbb{N}$  and an  $A$ -linear epimorphism  $\pi : A^n \rightarrow M$ . By [Theorem 6.3](#),  $\ker \pi$  is closed in  $A^n$ . We can endow  $M$  with the residue norm. By [Lemma 8.1](#), the equivalence class of the norm does not depend on the choice of  $\pi$ .

For any  $A$ -linear homomorphism  $f : M \rightarrow N$  of finite  $A$ -modules, we endow  $M$  and  $N$  with the Banach structures as above. It follows from [Lemma 8.1](#) that  $f$  is bounded. We have defined the inverse functor of the forgetful functor  $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$ .  $\square$

**Remark 8.3.** Let  $A$  be a  $k$ -affinoid algebra. It is not true that a Banach  $A$ -module which is finite as  $A$ -module is finite as Banach  $A$ -module.

As an example, take  $0 < p < q < 1$  and  $A = k\{q^{-1}T\}$ ,  $B = k\{p^{-1}T\}$ . Then  $B$  is a Banach  $A$ -module. By [Example 2.4](#), the underlying rings of  $A$  and  $B$  are both  $k[[T]]$ . So the canonical map  $A \rightarrow B$  is bijective. But  $B$  is not a finite  $A$ -module. As otherwise, the inverse map  $B \rightarrow A$  is bounded by [Lemma 8.1](#), which is not the case.

The correct statement is the following: consider a Banach  $A$ -module  $(M, \|\bullet\|_M)$  which is finite as  $A$ -module, then there is a norm on  $M$  such that  $M$  becomes a finite Banach  $A$ -module. The new norm is not necessarily equivalent to the given norm  $\|\bullet\|_M$ .

**Proposition 8.4.** Let  $A$  be a  $k$ -affinoid algebra and  $M, N$  be finite Banach  $A$ -modules. Then the natural map

$$M \otimes_A N \rightarrow M \hat{\otimes}_A N$$

is an isomorphism of Banach  $A$ -modules and  $M \hat{\otimes}_A N$  is a finite Banach  $A$ -module.

Here the Banach  $A$ -module structure on  $M \otimes_A N$  is given by [Proposition 8.2](#).



PROOF. Choose  $m, m' \in \mathbb{N}$  an admissibly coexact sequence

$$A^{m'} \rightarrow A^m \rightarrow M \rightarrow 0$$

of Banach  $A$ -modules. Then we have a commutative diagram of  $A$ -modules:

$$\begin{array}{ccccccc} A^{m'} \otimes_A N & \longrightarrow & A^m \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{m'} \hat{\otimes}_A N & \longrightarrow & A^m \hat{\otimes}_A N & \longrightarrow & M \hat{\otimes}_A N & \longrightarrow & 0 \end{array}$$

with exact rows. By 5-lemma, in order to prove  $M \otimes_A N \xrightarrow{\sim} M \hat{\otimes}_A N$  and  $M \hat{\otimes}_A N$  is a finite Banach  $A$ -module, we may assume that  $M = A^m$  for some  $m \in \mathbb{N}$ . Similarly, we can assume  $N = A^n$  for some  $n \in \mathbb{N}$ . In this case, the isomorphism is immediate and  $M \hat{\otimes}_A N$  is clearly a finite Banach  $A$ -module. By [Lemma 8.1](#), the Banach  $A$ -module structure on  $M \hat{\otimes}_A N$  coincides with the Banach  $A$ -module structure on  $M \otimes_A N$  induced by [Proposition 8.2](#).  $\square$

**Proposition 8.5.** Let  $A, B$  be a  $k$ -affinoid algebra and  $A \rightarrow B$  be a bounded  $k$ -algebra homomorphism. Let  $M$  be a finite Banach  $A$ -module, then the natural map

$$M \otimes_A B \rightarrow M \hat{\otimes}_A B$$

is an isomorphism of Banach  $B$ -modules and  $M \hat{\otimes}_A B$  is a finite Banach  $B$ -module.

PROOF. By the same argument as [Proposition 8.4](#), we may assume that  $M = A^n$  for some  $n \in \mathbb{N}$ . In this case, the assertions are trivial.  $\square$

**Proposition 8.6.** Let  $A$  be a  $k$ -affinoid algebra and  $M, N$  be finite Banach  $A$ -modules. Let  $\varphi : M \rightarrow N$  be an  $A$ -linear map. Then  $\varphi$  is admissible.

PROOF. By [Lemma 8.1](#),  $\varphi$  is always bounded. By [Proposition 8.5](#) and [Proposition 3.9](#), we may assume that  $k$  is non-trivially valued. By [Theorem 6.3](#),  $N$  is a Noetherian  $A$ -module. It follows from [Corollary 7.4](#) in the chapter Banach Rings that  $\text{Im } \varphi$  is closed in  $N$  and is finite as an  $A$  module. In particular, the norm induced from  $N$  and from  $M$  are equivalent by [Lemma 8.1](#). It follows that  $\varphi$  is admissible.  $\square$

**Proposition 8.7.** Let  $A$  be a  $k$ -affinoid algebra. Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a  $k$ -free polyray. Then  $M$  is a finite Banach  $A$ -module if and only if  $M \hat{\otimes}_k k_r$  is a finite Banach  $A \hat{\otimes}_k k_r$ -module.

PROOF. We may assume that  $r$  has only one component and write  $r_1 = r$ . The direct implication is trivial. Let us assume that  $M \hat{\otimes}_k k_r$  is a finite Banach  $A \hat{\otimes}_k k_r$ -module. Take  $n \in \mathbb{N}$  and an admissible epimorphism of  $A \hat{\otimes}_k k_r$ -modules

$$\varphi : (A \hat{\otimes}_k k_r)^n \rightarrow M \hat{\otimes}_k k_r.$$

Let  $e_1, \dots, e_n$  denotes the standard basis of  $(A \hat{\otimes}_k k_r)^n$ . We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By [Proposition 6.8](#), we can assume that there is  $l > 0$  such that  $m_{i,j} = 0$  for all  $i = 1, \dots, n$  and  $|j| > l$ . It follows that

$$A^{n(2l+1)} \rightarrow M$$

sending the standard basis to  $m_{i,j}$  with  $i = 1, \dots, n$  and  $j = -l, -l+1, \dots, l$  is an admissible epimorphism.  $\square$

**Proposition 8.8.** Let  $\phi : A \rightarrow B$  be a morphism of  $k$ -affinoid algebras,  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Then the following are equivalent:

- (1)  $\phi$  is finite and admissible.
- (2)  $\phi \hat{\otimes}_k k_r$  is finite and admissible.

This is [Tem04, Lemma 3.2]. I do not understand Temkin's argument. The following proof is a modification of the argument of Temkin's.

PROOF. (1)  $\implies$  (2): This is straightforward.

(2)  $\implies$  (1): The admissible part is straightforward. Let us prove that  $\phi$  is finite. We may assume that  $n = 1$ . When  $r$  is not in  $\sqrt{|k^\times|}$ , we just apply [Proposition 8.7](#). Now suppose  $r \in \sqrt{|k^\times|}$ . Let us take  $m \in \mathbb{Z}_{>0}$  such that  $r^m = |c^{-1}|$  for some  $c \in k^\times$ . Define a bounded  $k$ -algebra homomorphism

$$\varphi : k\{T\} \rightarrow k\{r^{-1}T\}$$

sending  $T$  to  $cT^m$ . Observe that  $\varphi$  is injective. We have argued in the proof of [Lemma 3.13](#) that this homomorphism is finite.

Then  $\varphi$  induces a finite extension of ring  $\text{Frac } k\{r^{-1}T\} / \text{Frac } k\{T\}$ . In particular, the closure of  $\text{Frac } k\{T\}$  in  $k_r$  is a subfield over which  $k_r$  is finite. But this valuation field is isomorphic to  $k\{T\}$ . By [Proposition 8.5](#) and fpqc descent [[Stacks](#), [Tag 02LA](#)], we may assume that  $r = 1$ .

Recall that  $k_1$  is the completion of  $\text{Frac } k\{T\}$ . Let  $\{\tilde{f}_i\}_{i \in I}$  be the set of irreducible monic polynomials in  $\tilde{k}[T]$ . Lift each  $\tilde{f}_i$  to  $f_i \in k[T]$ . Let  $a \in A \hat{\otimes}_k k_1$ , we represent  $a$  as

$$a = \sum_{l=0}^{\infty} a_l T^l + \sum_{i \in I, j \geq 1, 0 \leq k < \deg f_i} a_{ijk} T^k / f_i^j.$$

A similar expression exists for elements in  $B \hat{\otimes}_k k_1$  as well. Moreover, the representation is unique.

As  $B \hat{\otimes}_k k_1$  is finite over  $A \hat{\otimes}_k k_1$ , we can find  $b_1, \dots, b_m$  such that any  $b \in B$  can be written as

$$b = \sum_{j=1}^m \phi \hat{\otimes}_k k_1(a_j) b_j,$$

where  $a_j \in A \hat{\otimes}_k k'$ . We can replace  $b_j$  by  $b_{j,0}$  and  $a_j$  by  $a_{j,0}$ . It follows that  $B$  is generated  $b_{1,0}, \dots, b_{m,0}$  over  $A$ .  $\square$

For any ring  $A$ ,  $\text{Alg}_A^f$  denotes the category of finitely generated  $A$ -algebras.

**Proposition 8.9.** Let  $A$  be a  $k$ -affinoid algebra. Then the forgetful functor  $\text{BanAlg}_A^f \rightarrow \text{Alg}_A^f$  is an equivalence of categories.

Recall that  $\text{BanAlg}_A^f$  is defined in [Definition 5.9](#) in the chapter Banach Rings.

PROOF. It suffices to construct an inverse functor. Let  $B$  be a finite  $A$ -algebra. We endow  $B$  with the norm  $\|\bullet\|_B$  as in [Proposition 8.2](#). We claim that  $B$  is a Banach  $A$ -algebra.

Let us recall the definition of the norm. Take  $n \in \mathbb{N}$ , an epimorphism  $\varphi : A^n \rightarrow B$  of  $A$ -modules. Then  $\|\bullet\|_B$  is the residue norm induced by  $\varphi$ .

Consider the  $A$ -linear epimorphism  $\psi : A^n \otimes_A A^n \rightarrow B \otimes_A B$ . By [Proposition 8.6](#), when both sides are endowed with the norms  $\|\bullet\|_{A^n \otimes_A A^n}$  and  $\|\bullet\|_{B \otimes_A B}$  as in [Proposition 8.2](#),  $\psi$  is admissible. It follows that there is  $C > 0$  such that for any  $f, g \in B$ ,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by [Proposition 8.2](#), the natural map  $B \otimes_A B \rightarrow B$  is bounded. It follows that there is a constant  $C' > 0$  such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in  $B$  is bounded and hence  $B$  is a finite Banach algebra. Given any morphism  $B \rightarrow B'$  in  $\mathcal{Alg}_A^f$ , we endow  $B$  and  $B'$  with the norms given by [Proposition 8.2](#). It follows from [Lemma 8.1](#) that  $B \rightarrow B'$  is a bounded homomorphism of finite Banach  $A$ -algebras. So we have defined an inverse functor to the forgetful functor  $\mathcal{BanAlg}_A^f \rightarrow \mathcal{Alg}_A^f$ .  $\square$

**Remark 8.10.** It is not true that any homomorphism of  $k$ -affinoid algebras is bounded. For example, if the valuation on  $k$  is trivial. Take  $0 < p < q < 1$  and consider the natural homomorphism  $k_p \rightarrow k_q$ . This homomorphism is bijective but not bounded.

## 9. Graded reduction

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 9.1.** Let  $A$  be a Banach  $k$ -algebra, we define the *graded reduction* of  $A$  as

$$\tilde{A} := \bigoplus_{h \in \mathbb{R}_{>0}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any  $f \in A$  with  $\rho(f) \neq 0$ , we define  $\tilde{f}$  as the image of  $f$  in the  $\rho(f)$ -graded piece of  $\tilde{A}$ .

**Definition 9.2.** Let  $A$  be a  $k_H$ -affinoid algebra. We define the  *$k_H$ -graded reduction* of  $A$  as the  $\sqrt{|k^\times| \cdot H}$ -graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times| \cdot H}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any  $f \in A$  with  $\rho(f) \neq 0$ , we define  $\tilde{f}$  as the image of  $f$  in the  $\rho(f)$ -graded piece of  $\tilde{A}^H$ .

For any morphism  $f : A \rightarrow B$  of  $k_H$ -affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \rightarrow \tilde{B}^H$$

as the map induced by sending the class of  $x \in A$  with  $\rho(x) \leq h$  for any  $h \in \sqrt{|k^\times| \cdot H}$  to the class of  $f(x) \in B$ .

Recall that  $\rho(A) = \sqrt{|k^\times| \cdot H} \cup \{0\}$  by [Theorem 7.4](#), so  $\tilde{f}$  is well-defined. This definition is compatible with [Definition 9.1](#) in the sense that if we regard a  $\sqrt{|k^\times| \cdot H}$ -graded ring as a  $\mathbb{R}_{>0}$ -graded ring, the two definitions give the same object.

**Example 9.3.** If  $K$  is a  $k_H$ -affinoid algebra which is a field as well, then  $\tilde{K}^H$  is a  $\sqrt{|k^\times|} \cdot H$ -graded field. This is immediate from the definition.

**Lemma 9.4.** Let  $(A, \|\bullet\|)$  be a  $k$ -affinoid algebra,  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Let  $f \in k\{r^{-1}T\}$ . Expand  $f$  as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha.$$

PROOF. By induction, we may assume that  $n = 1$  and write  $r = r_1$ . As  $\rho$  is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j) \rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j.$$

Observe that  $\rho(a_j)$  is not ambiguous: when interpreted as in  $A$  and in  $A\{r^{-1}T\}$ , it has the same value.

Conversely, we need to show that for any  $j \in \mathbb{N}$ ,

$$\rho(f) \geq \rho(a_j) r^j.$$

Equivalently, this means for any  $k \in \mathbb{Z}_{>0}$  and any  $j \in \mathbb{N}$ , we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk}.$$

Fix  $j$  and  $k$  as above. We compute the left-hand side:

$$f^k = \sum_{\beta = (\beta_1, \dots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^k a_{\beta_l}.$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|}.$$

Take  $\beta = (j, j, \dots, j)$ , we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk}.$$

□

**Lemma 9.5.** Assume that  $k$  is non-trivially valued. Let  $A$  be a strictly  $k$ -affinoid algebra. Then for any  $a, f \in A$ , the set of non-zero values  $\rho(f^n a)$  for  $n \in \mathbb{N}$  is a discrete subset of  $\mathbb{R}_{>0}$ .

PROOF. As  $A$  is noetherian [Theorem 6.3](#), it has only finitely many minimal prime ideals, say  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ . It follows that

$$\mathrm{Sp} A = \bigcup_{i=1}^m \mathrm{Sp} A/\mathfrak{p}_i.$$

Here we make the obvious identification by identifying  $\mathrm{Sp} A/\mathfrak{p}_i$  with a subset of  $\mathrm{Sp} A$ .

By [Corollary 6.12](#) in the chapter Banach Rings, it suffices to consider each of  $\mathrm{Sp} A/\mathfrak{p}_i$  separately, so we may assume that  $A$  is an integral domain.

By [Corollary 5.2](#), we can take  $d \in \mathbb{N}$  and a finite injective homomorphism of  $k$ -algebras  $\iota : k\{T_1, \dots, T_d\} \rightarrow A$ . According to [Proposition 9.11](#) in the chapter Banach Rings,  $\rho_A$  is the restriction of the norm  $\|\bullet\|_{\mathrm{Frac} A}$  on  $\mathrm{Frac} A$  induced by

the finite extension  $\text{Frac } A / \text{Frac } k\{T_1, \dots, T_d\}$  from the Gauss valuation. But it is well-known that  $\|\bullet\|_{\text{Frac } A}$  is the maximum of finitely many valuations on  $\text{Frac } A$ . **Reproduce BGR3.3.3.1 somewhere.** The assertion is by now obvious.  $\square$

**Lemma 9.6.** Let  $(A, \|\bullet\|)$  be a  $k$ -affinoid algebra,  $f \in A$  with  $r = \rho(f) > 0$ . Let  $B = A\{r^{-1}f\}$ . Then for any  $a \in A$ , we have

$$\rho_B(a) = \lim_{n \rightarrow \infty} r^{-n} \rho_A(f^n a).$$

If moreover,  $\rho_B(a) > 0$ , then there is  $n_0 > 0$  such that for  $n \geq n_0$ ,

$$\rho_B(a) = r^{-n} \rho_A(f^n a), \quad \rho_B(f^n a) = r^{-n} \rho_A(a).$$

PROOF. We observe that for any  $a \in A$ ,  $n \in \mathbb{Z}_{>0}$ , we have

$$\rho_B(f^n a) = r^n \rho_B(a).$$

So the last two assertions are equivalent.

Take a  $k$ -free polyradius  $s$  such that  $A \hat{\otimes}_k k_s$  and  $B \hat{\otimes}_k k_s$  are both strictly  $k_s$ -affinoid. By **Proposition 3.9**,  $A \hat{\otimes}_k k_s\{r^{-1}f\} \xrightarrow{\sim} B \hat{\otimes}_k k_s$ . Moreover,  $\rho_A$  and  $\rho_B$  are both preserved after base change to  $k_s$ . So we may assume that  $k$  is non-trivially valued and  $A$  and  $B$  are strictly  $k$ -affinoid.

Observe that for  $n \in \mathbb{Z}_{>0}$ ,

$$\rho_A(f^{n+1}a) \leq \rho_A(f) \rho_A(f^n a) = r \rho_A(f^n a).$$

So  $r^{-n} \rho_A(f^n a)$  is decreasing in  $n$ . Moreover, for any  $x \in \text{Sp } A\{r^{-1}f\}$ , by **Proposition 6.7**, we have

$$|f(x)| \geq r.$$

By **Corollary 6.12** in the chapter Banach Rings, we have

$$|f(x)| = r$$

for any  $x \in \text{Sp } A\{r^{-1}f\}$ . It follows from **Corollary 6.12** in the chapter Banach Rings that for any  $n \in \mathbb{Z}_{>0}$ ,

$$\rho_A(f^n a) = \sup_{x \in \text{Sp } A} |f^n a(x)| \geq r^n \sup_{x \in \text{Sp } A\{rf^{-1}\}} |a(x)| = r^n \rho_B(a).$$

By **Lemma 9.5**, the decreasing sequence  $\{r^{-n} \rho_A(f^n a)\}_n$  either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is  $\alpha \in \mathbb{R}_{>0}$  and  $n_0 > 0$  such that for  $n \geq n_0$ , we have

$$r^{-n} \rho_A(f^n a) = \alpha.$$

We have to show that  $\alpha \leq \rho_B(a)$ . Assume the contrary  $\alpha > \rho_B(a)$ . Then for all  $x \in \text{Sp } A$ , we have

$$|f^n a(x)| \leq r^n |a(x)|.$$

So  $f^n a$  must obtain its maximum on  $U := \{x \in \text{Sp } A : |a(x)| \geq \alpha\}$ . But  $U$  is disjoint from  $\text{Sp } A\{r^{-1}f\}$  as

$$\alpha > \rho_B(a).$$

It follows from **Proposition 6.7** that

$$\beta := \sup_{x \in U} |f(x)| = \max_{x \in U} |f(x)| < r.$$

So

$$\rho(f^n a) = \sup_{x \in \text{Sp } A} |f^n a(x)| = \sup_{x \in U} |f^n a(x)| \leq \beta^n \sup_{x \in U} |a(x)|.$$

This contradicts the fact that  $\alpha > 0$ .  $\square$

**Proposition 9.7.** Let  $A$  be a  $k_H$ -affinoid algebra and  $r \in \mathbb{R}_{>0}^n$ , then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of  $\sqrt{|k^\times| \cdot H}$ -graded rings.

Recall that  $k_r$  is defined in [Example 3.10](#).

PROOF. By [Lemma 9.4](#), we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}_s^H \xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{N}^n} \tilde{A}_{sr^{-\alpha}}^H$$

for any  $s \in \sqrt{|k^\times| \cdot H}$ . This establishes the desired isomorphism.  $\square$

**Proposition 9.8.** Let  $A$  be a  $k_H$ -affinoid algebra and  $f \in A$  with  $r = \rho(f) > 0$ . Then there is a natural isomorphism

$$\tilde{A}_f^H \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of  $\sqrt{|k^\times| \cdot H}$ -graded rings.

Recall that  $A\{rf^{-1}\}$  is defined in [Definition 6.6](#), by [Theorem 7.4](#), it is  $k_H$ -affinoid.

PROOF. Let  $B = A\{rf^{-1}\}$  and denote by  $\phi : \tilde{A}^H \rightarrow \tilde{A}_f^H$  the natural  $\sqrt{|k^\times| \cdot H}$ -graded homomorphism. From the universal property [add details](#), we can factor the natural map  $\tilde{A}^H \rightarrow \tilde{B}^H$  as  $\psi : \tilde{A}_f^H \rightarrow \tilde{B}^H$ . We have a commutative diagram:

$$\begin{array}{ccc} \tilde{A}^H & \xrightarrow{\quad} & \tilde{B}^H \\ \phi \downarrow & \nearrow \psi & \\ \tilde{A}_f^H & & \end{array}$$

We claim that  $\psi$  is bijective. Let  $\tilde{a}/\tilde{f}^m$  be an element in  $\ker \psi$ , where  $\tilde{a} \in \tilde{A}^H$  is homogeneous. Lift  $\tilde{a}$  to  $a \in A$ . Then  $\rho_B(a) < \rho_A(a)$ . By [Lemma 9.6](#),  $\rho_A(f^n a) < r^n \rho_A(a)$  when  $n$  is large enough, so

$$\tilde{f}^n \tilde{a} = 0$$

in  $\tilde{A}$ . Therefore,  $\tilde{a}/\tilde{f}^m = 0$  in  $\tilde{A}_f^H$ . We have shown that  $\psi$  is injective.

It remains to show that  $\psi$  is surjective. Let  $\tilde{b} \in \tilde{B}^H$  be a non-zero homogeneous element. Lift  $\tilde{b}$  to  $b \in B$  of the form  $f^{-n}a$  for some  $a \in A$ . By [Lemma 9.6](#) again, up to enlarging  $n$ , we can assume that  $\rho_B(a) = \rho_A(a)$ . Then  $\tilde{a} = \tilde{f}^n \tilde{b}$  has a preimage in  $\tilde{A}$ .  $\square$

**Corollary 9.9.** Let  $A$  be a  $k_H$ -affinoid algebra and  $r \in \mathbb{R}_{>0}^n$ , then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}_H} \tilde{k}_r^H \cong \widetilde{A \hat{\otimes}_k k_r}^H$$

of  $\sqrt{|k^\times| \cdot H}$ -graded rings.

PROOF. We can write

$$A \hat{\otimes}_k k_r = \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}.$$

Taking graded reduction, we find

$$\begin{aligned} \widetilde{A \hat{\otimes}_k k_r}^H &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}^H \\ &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \widetilde{A\{r^{-1}T\}}_{\tilde{g}}^H \\ &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \tilde{A}^H[r^{-1}T]_{\tilde{g}} \\ &= \tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k}_r^H. \end{aligned}$$

Here we have applied [Proposition 9.8](#) in the second equality and [Proposition 9.7](#) in the third equality. The first equality follows from the simple observation that graded reduction commutes with filtered colimits.  $\square$

**Proposition 9.10.** Let  $\phi : A \rightarrow B$  be a morphism of  $k_H$ -affinoid algebras. Then the following are equivalent:

- (1)  $\phi$  is finite and admissible.
- (2)  $\tilde{\phi} : \tilde{A}^H \rightarrow \tilde{B}^H$  is finite.

PROOF. Take  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$  so that

$$\rho(A \hat{\otimes}_k k_r) = \rho(B \hat{\otimes}_k k_r) = |k_r|$$

and  $k_r$  is non-trivially valued. [Proof that this is possible.](#)

By [Corollary 2.35](#) in the chapter Commutative Algebra and [Proposition 8.8](#), we may assume that  $k$  is non-trivially valued and  $\rho(A) = \rho(B) = |k|$ . By [Lemma 2.32](#) in the chapter Commutative Algebra, we have  $\tilde{A} = \tilde{A}_1 \otimes_{\tilde{k}_1} \tilde{k}$ . By [Corollary 5.5](#),  $\phi$  is automatically admissible if it is finite.

So it suffices to argue that  $\phi$  is finite if and only if  $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$  is finite.

Assume that  $\phi$  is finite. We show that  $\tilde{\phi}$  is finite.

First consider the case where  $A$  is an integral domain.

We claim that there is  $d \in \mathbb{N}$  and a  $k$ -algebra homomorphism  $\psi : k\{T_1, \dots, T_d\} \rightarrow A$  such that  $\phi \circ \psi$  is finite and injective. In fact, choosing an epimorphism  $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$ , we can apply [Theorem 5.1](#) to find  $\phi \circ \alpha$  to conclude.

It suffices to show that  $\widetilde{\phi \circ \psi}$  is finite in order to conclude that  $\tilde{\phi}$  is finite. So we are reduced to the case  $A = k\{T_1, \dots, T_d\}$  and  $\ker \phi = 0$ .

We will show that the conditions of [Lemma 10.1](#) in the chapter Banach Rings is satisfied with  $\rho_B$  as the norm  $B$ . We have shown that  $\rho_B$  is a faithful  $k\{T_1, \dots, T_d\}$ -algebra norm in [Corollary 4.16](#). As  $B$  is of finite over  $k\{T_1, \dots, T_d\}$ , the rank condition is clearly satisfied. It remains to establish that  $\tilde{\phi}$  is integral.

By [Proposition 5.12](#), for  $f \in B$ , there is an integral equation

$$f^n + \phi(a_1)f^{n-1} + \dots + \phi(a_n) = 0$$

over  $A$  such that  $\rho_B(f) = \max_{i=1, \dots, n} |b_i|_{\sup}^{1/i}$ . If  $f \in \mathring{B}$ , then  $|b_i|_{\sup} \leq 1$ , hence  $b_i \in \mathring{B}$ . [Add a ref](#)

Conversely, assume that  $\tilde{\phi}$  is finite. It suffices to apply [Lemma 5.15](#) to conclude that  $\phi$  is finite.  $\square$

## 10. Affinoid domains

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H \supseteq |k^\times|$  be a subgroup of  $R_{>0}$ .

**Definition 10.1.** Let  $A$  be a  $k_H$ -affinoid algebra. A subset  $V \subseteq \mathrm{Sp} A$  is said to be a  $k_H$ -affinoid domain in  $X$  if there is a bounded homomorphism of  $k_H$ -affinoid algebras  $\varphi : A \rightarrow A_V$  satisfying

- (1)  $\mathrm{Im} \mathrm{Sp} \varphi = V$ ;
- (2) given a bounded homomorphism of  $k_H$ -affinoid algebras  $\psi : A \rightarrow B$  such that  $\mathrm{Sp} \psi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  factorizes through  $V$ , there is a unique bounded homomorphism  $A_V \rightarrow B$  such that the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A_V \\ \downarrow \psi & \swarrow & \\ B & & \end{array}.$$

We say  $V$  is *represented* by the morphism  $\varphi$ .

When  $k_H = \mathbb{R}_{>0}$ , we say  $V$  is a  $k$ -affinoid domain in  $X$ . When  $k_H = |k^\times|$ , we say  $V$  is a *strict*  $k$ -affinoid domain in  $X$ .

**Remark 10.2.** This definition differs from the original definition of [\[Ber12\]](#), we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the original definition of Berkovich when  $H = \mathbb{R}_{>0}$ .

**Proposition 10.3.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V \subseteq \mathrm{Sp} A$  be a  $k_H$ -affinoid domain represented by  $\varphi : A \rightarrow A_V$ . Then  $\mathrm{Sp} \varphi$  induces a bijection  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ .

PROOF. We observe that  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  is a monomorphism in the category  $k_H\text{-Aff}$ . In other words,  $A \rightarrow A_V$  is an epimorphism in the category  $k_H\text{-AffAlg}$ . To see this, let  $\eta_1, \eta_2 : A_V \rightarrow B$  be two arrows in  $k_H\text{-AffAlg}$  such that  $\eta_1 \circ \varphi = \eta_2 \circ \varphi$ . It follows from the universal property in [Definition 10.1](#) that  $\eta_1 = \eta_2$ . We claim that  $\mathrm{Sp} A_V \rightarrow V$  is a bijection.  $\square$

It is not immediately clear that  $A_V$  is canonically associated with  $V$ . We will prove this now.

**Proposition 10.4.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V$  be an affinoid domain in  $X$  represented by  $\varphi : A \rightarrow A_V$ . Then  $\mathrm{Sp} \varphi : \mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  induces a homeomorphism  $\mathrm{Sp} A_V \rightarrow V$ .

In particular,  $A_V$  is uniquely determined by  $V$  up to isomorphisms of Banach  $k$ -algebras.

PROOF. Let us reduce the problem to the case where  $k$  is non-trivially valued and  $A$  and  $A_V$  are both strictly  $k$ -affinoid.

By [Proposition 3.15](#), taking a suitable  $r = r(r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$  such that  $r_1, \dots, r_n$  are linearly independent in the  $\mathbb{Q}$ -linear space  $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ , we may guarantee that  $A \hat{\otimes}_k k_r$  and  $A_V \hat{\otimes}_k k_r$  are both strictly  $k_r$ -affinoid.

Let  $V'$  be the inverse image of  $V$  in  $\mathrm{Sp} A \hat{\otimes}_k k_r$ . We claim that  $V'$  is a strict  $k_r$ -affinoid domain in  $\mathrm{Sp} A \hat{\otimes}_k k_r$  represented by  $A \hat{\otimes}_k k_r \rightarrow A_V \hat{\otimes}_k k_r$ .  $\square$



## Bibliography

- [Ber12] V. G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. 33. American Mathematical Soc., 2012.
- [BGR84] S. Bosch, U. Güntzer, and R. Remmert. Non-Archimedean analysis. Vol. 261. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436. URL: <https://doi.org/10.1007/978-3-642-52229-1>.
- [Stacks] T. Stacks Project Authors. Stacks Project. <http://stacks.math.columbia.edu>. 2020.
- [Tem04] M. Temkin. On local properties of non-Archimedean analytic spaces. II. *Israel J. Math.* 140 (2004), pp. 1–27. URL: <https://doi.org/10.1007/BF02786625>.