

## Affinoid algebras



## Contents

1. Introduction	4
2. Tate algebras	4
3. Affinoid algebras	4
4. Weierstrass theory	10
5. Noetherian normalization and maximal modulus principle	15
6. Properties of affinoid algebras	19
7. $H$ -strict affinoid algebras	22
8. Finite modules over affinoid algebras	24
9. Affinoid domains	27
10. Graded reduction	34
11. Gerritzen–Grauert theorem	38
12. Tate acyclicity theorem	41
Bibliography	49

## 1. Introduction

Our references for this chapter include [BGR84], [Ber12].

## 2. Tate algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued-field.

**Definition 2.1.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . We set

$$\begin{aligned} k\{r^{-1}T\} &= k\{r_1^{-1}T_1, \dots, r_n T_n^{-1}\} \\ &:= \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \right\}. \end{aligned}$$

For any  $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$ , we set

$$\|f\|_r = \max_{\alpha} |a_\alpha| r^\alpha.$$

We call  $(k\{r^{-1}T\}, \|\bullet\|_r)$  the *Tate algebra* in  $n$ -variables with radii  $r$ . The norm  $\|\bullet\|_r$  is called the *Gauss norm*.

We omit  $r$  from the notation if  $r = (1, \dots, 1)$ .

This is a special case of [Example 4.15](#) in the chapter Banach Rings.

**Proposition 2.2.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . Then the Tate algebra  $(k\{r^{-1}T\}, \|\bullet\|_r)$  is a Banach  $k$ -algebra and  $\|\bullet\|_r$  is a valuation.

PROOF. This is a special case of [Proposition 4.16](#) in the chapter Banach Rings.  $\square$

**Remark 2.3.** One should think of  $k\{r^{-1}T\}$  as analogues of  $\mathbb{C}\langle r^{-1}T \rangle$  in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings  $\mathbb{C}\langle r^{-1}T \rangle$ , as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

**Example 2.4.** Assume that the valuation on  $k$  is trivial.

Let  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Then  $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$  if  $r_i \geq 1$  for all  $i$  and  $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$  otherwise.

**Lemma 2.5.** Let  $A$  be a Banach  $k$ -algebra. For each  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathring{A}$ , there is a unique continuous homomorphism  $k\{T_1, \dots, T_n\} \rightarrow A$  sending  $T_i$  to  $a_i$ .

PROOF. This is a special case of [Proposition 4.18](#) in the chapter Banach Rings.  $\square$

## 3. Affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 3.1.** A Banach  $k$ -algebra  $A$  is  *$k$ -affinoid* (resp. *strictly  $k$ -affinoid*) if there are  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$  and an admissible epimorphism  $k\{r^{-1}T\} \rightarrow A$  (resp. an admissible epimorphism  $k\{T\} \rightarrow A$ ).

More generally, a Banach  $k$ -algebra  $A$  is  *$k_H$ -affinoid* if there are  $n \in \mathbb{N}$ ,  $r \in H^n$  and an admissible epimorphism  $k\{r^{-1}T\} \rightarrow A$ .

A morphism between  $k$ -affinoid (resp. strictly  $k$ -affinoid, resp.  $k_H$ -affinoid) algebras is a bounded  $k$ -algebra homomorphism.

The category of  $k$ -affinoid (resp. strictly  $k$ -affinoid, resp.  $k_H$ -affinoid) algebras is denoted by  $k\text{-AffAlg}$  (resp.  $\text{st-}k\text{-AffAlg}$ , resp.  $k_H\text{-AffAlg}$ ).

For the notion of admissible morphisms, we refer to [Definition 2.5](#) in the chapter Banach rings.

Although we have defined strictly  $k$ -affinoid algebra when  $k$  is trivially valued, we will deliberately avoid talking about it in order to avoid pathologies.

**Remark 3.2.** Berkovich also introduced the notion of *affinoid  $k$ -algebras*: it is a  $K$ -affinoid algebra for some complete non-Archimedean field extension  $K/k$ . We will not use this notion.

**Definition 3.3.** The category of  *$k$ -affinoid spectra*  $k\text{-Aff}$  (resp. *strictly  $k$ -affinoid spectra*  $\text{st-}k\text{-Aff}$ , resp.  *$k_H$ -affinoid spectra*  $k_H\text{-Aff}$ ) is the opposite category of  $k\text{-AffAlg}$  (resp.  $\text{st-}k\text{-AffAlg}$ , resp.  $k_H\text{-AffAlg}$ ). An object in these categories are called a  *$k$ -affinoid spectrum*, *strictly  $k$ -affinoid spectrum* and  *$k_H$ -affinoid spectrum* respectively.

Given an object  $A$  of  $k\text{-AffAlg}$  (resp.  $\text{st-}k\text{-AffAlg}$ , resp.  $k_H\text{-AffAlg}$ ), we denote the corresponding object in  $k\text{-Aff}$  (resp.  $\text{st-}k\text{-Aff}$ , resp.  $k_H\text{-Aff}$ ) by  $\text{Sp } A$ . We call  $\text{Sp } A$  the *affinoid spectrum* of  $A$ .

In [Definition 6.1](#) in the chapter Banach Rings, we defined functors  $\text{Sp} : k\text{-Aff} \rightarrow \mathcal{Top}$ ,  $\text{Sp} : \text{st-}k\text{-Aff} \rightarrow \mathcal{Top}$  and  $\text{Sp} : k_H\text{-Aff} \rightarrow \mathcal{Top}$ . This motivates our notation. We will freely view  $\text{Sp } A$  as an object in these categories or as a topological space.

**Proposition 3.4.** Finite limits exist in  $k_H\text{-Aff}$ . Moreover, fiber products in  $k_H\text{-Aff}$  corresponds to completed tensor product in  $k_H\text{-AffAlg}$ .

PROOF. It suffices to prove that finite fibered products exist.

We prove the equivalent statement, finite fibered coproducts exist in  $k_H\text{-AffAlg}$ . Given  $k_H$ -affinoid algebras  $A, B, C$  and morphisms  $A \rightarrow B$ ,  $A \rightarrow C$ , we claim that  $B \hat{\otimes}_A C$  represents the fibered coproduct of  $B$  and  $C$  over  $A$ . By general abstract nonsense, we are reduced to handle the following cases:  $A = k$  and  $A \rightarrow C$  is the codiagonal  $C \hat{\otimes}_k C \rightarrow C$ . In both cases, the proposition is clear.  $\square$

**Example 3.5.** Let  $r \in \mathbb{R}_{>0}$ . We let  $k_r$  denote the subring of  $k[[T]]$  consisting of  $f = \sum_{i=-\infty}^{\infty} a_i T^i$  satisfying  $|a_i| r^i \rightarrow 0$  for  $i \rightarrow \infty$  and  $i \rightarrow -\infty$ . We define a norm  $\|\bullet\|_r$  on  $k_r$  as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in [Proposition 3.6](#) that  $k_r$  is  $k$ -affinoid.

**Proposition 3.6.** Let  $r \in \mathbb{R}_{>0}$ , then  $(k_r, \|\bullet\|_r)$  defined in [Example 3.5](#) is a  $k$ -affinoid algebra. Moreover,  $\|\bullet\|_r$  is a valuation.

PROOF. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \rightarrow k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$(3.1) \quad |a_{i,j}| r^{i-j} \rightarrow 0$$

as  $i + j \rightarrow \infty$ . Observe that for each  $k \in \mathbb{Z}$ , the series

$$c_k := \sum_{i-j=k, i,j \in \mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image  $\iota(f)$  is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that  $\iota(f) \in k_r$ . That is

$$|c_k| r^k \rightarrow 0$$

as  $k \rightarrow \pm\infty$ . When  $k \geq 0$ , we have  $|c_k| \leq |a_{k0}|$  by definition of  $c_k$ . So  $|c_k| r^k \rightarrow 0$  as  $k \rightarrow \infty$  by (3.1). The case  $k \rightarrow -\infty$  is similar.

We conclude that we have a well-defined map of sets  $\iota$ . It is straightforward to verify that  $\iota$  is a ring homomorphism. Next we show that  $\iota$  is surjective. Take  $g = \sum_{i=-\infty}^{\infty} c_i T^i \in k_r$ . We want to show that  $g$  lies in the image of  $\iota$ . As  $\iota$  is a ring homomorphism, it suffices to treat two cases separately:  $g = \sum_{i=0}^{\infty} c_i T^i$  and  $g = \sum_{i=-\infty}^0 c_i T^i$ . We handle the first case only, as the second case is similar. In this case, it suffices to consider  $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$ . It is immediate that  $\iota(f) = g$ .

Next we show that  $\iota$  is admissible. We first identify the kernel of  $\iota$ . We claim that the kernel is the ideal  $I$  generated by  $T_1 T_2 - 1$ . It is obvious that  $I \subseteq \ker \iota$ . Conversely, consider an element

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kernel of  $\iota$ . Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If  $f \in \ker \iota$ , then so is each  $f_k$  by our construction.

We first show that each  $f_k$  lies in the ideal generated by  $T_1 T_2 - 1$ . The condition that  $f_k \in \ker \iota$  means

$$\sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find  $b_{i,j} \in k$  for  $i, j \in \mathbb{N}$ ,  $i - j = k$  such that

$$a_{i,j} = b_{i-1, j-1} - b_{i,j}.$$

Then

$$f_k = (T_1 T_2 - 1) \sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that  $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$ . In particular, the sum of  $\sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j$  for various  $k$  converges to some  $g \in k\{r^{-1}T_1, rT_2\}$  and hence  $f_k = (T_1 T_2 - 1)g$ . Therefore, we have proved that  $\ker \iota$  is generated by  $T_1 T_2 - 1$ .

It remains to show that  $\iota$  is admissible. In fact, we will prove a stronger result:  $\iota$  induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \rightarrow k_r.$$

To see this, take  $f = \sum_{k=-\infty}^{\infty} c_k T^k \in k_r$  and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set  $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$ , then  $\iota(g) = f$  and  $\|g\|_{(r,r^{-1})} = \|f\|_r$ . So it suffices to show that for any  $h = \sum_{(i,j) \in \mathbb{N}^2} d_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$ , we have

$$(3.2) \quad \|f\|_r \leq \|g + h(T_1 T_2 - 1)\|_{r,r^{-1}}.$$

We compute

$$g + h(T_1 T_2 - 1) = \sum_{k=1}^{\infty} (c_k - d_{k,0}) T_1^k + \sum_{k=1}^{\infty} (c_{-k} - d_{0,k}) T_2^k + (c_0 - d_{0,0}) + \sum_{i,j \geq 1} (d_{i-1,j-1} - d_{i,j}) T_1^i T_2^j.$$

So

$$\|g + h(T_1 T_2 - 1)\|_{r,r^{-1}} = \max \left\{ \max_{k \geq 0} C_{1,k}, \max_{k \geq 1} C_{2,k} \right\},$$

where

$$C_{1,k} = \max \left\{ |c_k - d_{k,0}|, \left| \sum_{i-j=k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for  $k \geq 0$  and

$$C_{2,k} = \max \left\{ |c_{-k} - d_{0,k}|, \left| \sum_{i-j=-k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for  $k \geq 1$ . It follows from the strong triangle inequality that  $|c_k| \leq C_{1,k}$  for  $k \geq 0$  and  $c_{-k} \leq C_{2,k}$  for  $k \geq 1$ . So (3.2) follows.  $\square$

**Proposition 3.7.** Let  $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$ , then  $\|\bullet\|_r$  defined in Example 3.5 is a valuation on  $k_r$ .

PROOF. Take  $f, g \in k_r$ , we need to show that

$$\|fg\|_r \geq \|f\|_r \|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_i T^i, \quad g = \sum_{i=-\infty}^{\infty} b_i T^i.$$

Take  $i$  and  $j$  so that

$$(3.3) \quad |a_i| r^i = \|f\|_r, \quad |b_j| r^j = \|g\|_r.$$

By our assumption on  $r$ ,  $i, j$  are unique. Then

$$\|fg\|_r = \max_{k \in \mathbb{Z}} \{|c_k| r^k\},$$

where

$$c_k := \sum_{u, v \in \mathbb{Z}, u+v=k} a_u b_v.$$

It suffices to show that

$$(3.4) \quad |c_k| r^k = \|f\|_r \|g\|_r.$$

for  $k = i + j$ . Of course, we may assume that  $a_i \neq 0$  and  $b_j \neq 0$  as otherwise there is nothing to prove. For  $u, v \in \mathbb{Z}$ ,  $u + v = i + j$  while  $(u, v) \neq (i, j)$ , we may assume that  $u \neq i$ . Then  $|a_u| r^u < |a_i| r^i$  and  $|b_v| r^v \leq |b_j| r^j$ . So  $|a_u b_v| < |a_i b_j|$  and we conclude (3.4).  $\square$

**Remark 3.8.** The argument of [Proposition 4.16](#) in the chapter Banach Rings does not work here if  $r \in \sqrt{|k^\times|}$ , as in general one can not take minimal  $i, j$  so that (3.3) is satisfied.

**Proposition 3.9.** Assume that  $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$ . Then  $k_r$  is a valuation field and  $\|\bullet\|_r$  is non-trivial.

PROOF. We first show that  $\text{Sp } k_r$  consists of a single point:  $\|\bullet\|_r$ . Assume that  $|\bullet| \in \text{Sp } k_r$ . As  $\|\bullet\|_r$  is a valuation, we find

$$(3.5) \quad |\bullet| \leq \|\bullet\|_r.$$

In particular,  $|\bullet|$  restricted to  $k$  is the given valuation on  $k$ . It suffices to show that  $|T| = r$ . This follows from (3.5) applied to  $T$  and  $T^{-1}$ .

It follows that  $k_r$  does not have any non-zero proper closed ideals: if  $I$  is such an ideal,  $k_r/I$  is a Banach  $k$ -algebra. By [Proposition 6.10](#) in the chapter Banach rings,  $\text{Sp } k_r$  is non-empty. So  $k_r$  has to admit bounded semi-valuation with non-trivial kernel.

In particular, by [Corollary 4.7](#) in the chapter Banach rings, the only maximal ideal of  $k_r$  is 0. It follows that  $k_r$  is a field.

The valuation  $\|\bullet\|_r$  is non-trivial as  $\|T\|_r = r$ .  $\square$

**Definition 3.10.** An element  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$  for some  $n \in \mathbb{N}$  is called a *k-free polyray* if  $r_1, \dots, r_n$  are linearly independent in the  $\mathbb{Q}$ -linear space  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_{>0} / \sqrt{|k^\times|}$ .

Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . Assume that  $r$  is a  $k$ -free polyray. We define

$$k_r = k_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{r_n}.$$

By an iterated application of [Proposition 3.9](#),  $k_r$  is a complete valuation field. As a general explanation of why  $k_r$  is useful, we prove the following proposition:

**Proposition 3.11.** Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a  $k$ -free polyray.

- (1) For any  $k$ -Banach space  $X$ , the natural map

$$X \rightarrow X \hat{\otimes}_k k_r$$

is an isometric embedding.



- (2) Consider a sequence of bounded homomorphisms of  $k$ -Banach spaces  $X \rightarrow Y \rightarrow Z$ . Then the sequence is admissible and exact (resp. coexact) if and only if  $X \hat{\otimes}_k k_r \rightarrow Y \hat{\otimes}_k k_r \rightarrow Z \hat{\otimes}_k k_r$  is admissible and exact (resp. coexact).

PROOF. We may assume that  $n = 1$ .

(1) We have a more explicit description of  $X \hat{\otimes}_k k_r$ : as a vector space, it is the space of  $f = \sum_{i=-\infty}^{\infty} a_i T^i$  with  $a_i \in X$  and  $\|a_i\| r^i \rightarrow 0$  when  $|i| \rightarrow \infty$ . The norm is given by  $\max_i \|a_i\| r^i$ . From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1).  $\square$

When  $X$  is a Banach  $k$ -algebra,  $X \hat{\otimes}_k k_r$  is a Banach  $k_r$ -algebra.

**Example 3.12.** For any  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$ , not necessarily  $k$ -free. We define  $k_r$  as the completed fraction field of  $k\{r^{-1}T\}$  provided with the extended valuation  $|\bullet|_r$ . Then  $k_r$  is still a valuation field extending  $k$ .

When  $r$  is a  $k$ -free polyray, we claim that  $k_r$  coincides with  $k_r$  defined in [Definition 3.10](#). To see this, let us temporarily denote the  $k_r$  defined in this example as  $k'_r$  consider the extension of field:

$$\text{Frac } k\{r^{-1}T\} \rightarrow k_r = k\{r^{-1}T, rS\} / (T_1 S_1 - 1, \dots, T_n S_n - 1)$$

sending  $T_i$  to  $T_i$  for  $i = 1, \dots, n$ . Observe that this is an extension of valuation field as well by the same arguments as in [Proposition 3.6](#). In particular, it induces an extension of complete valuation fields  $k'_r \rightarrow k_r$ . But the image clearly contains the classes of all polynomials in  $k[T, S]$ , so  $k'_r \rightarrow k_r$  is an isometric isomorphism.

**Proposition 3.13.** Assume that  $k$  is non-trivially valued. Let  $B$  be a strict  $k$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite bounded  $k$ -algebra homomorphism into a  $k$ -Banach algebra  $A$ . Then  $A$  is also strictly  $k$ -affinoid.

PROOF. We may assume that  $B = k\{T_1, \dots, T_n\}$  for some  $n \in \mathbb{N}$ . By assumption, we can find finitely many  $a_1, \dots, a_m \in A$  such that  $A = \sum_{i=1}^m \varphi(B) a_i$ .

We may assume that  $a_i \in \mathring{A}$  as  $k$  is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings,  $\varphi$  admits a unique extension to a bounded  $k$ -algebra epimorphism

$$\Phi : k\{T_1, \dots, T_n, S_1, \dots, S_m\} \rightarrow A$$

sending  $S_i$  to  $a_i$ . By [Corollary 7.5](#) in the chapter Banach Rings,  $\Phi$  is admissible. Moreover, the homomorphism  $\Phi$  is surjective by our assumption. It follows that  $A$  is strictly  $k$ -affinoid.  $\square$

**Proposition 3.14.** Assume that  $k$  is non-trivially valued. Let  $B$  be a strict  $k$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite  $k$ -algebra homomorphism into a  $k$ -algebra  $A$ . Then there is a norm on  $A$  such that the morphism is bounded and  $A$  is strictly  $k$ -affinoid.

PROOF. By [Proposition 8.4](#) in the chapter Banach Rings, we can endow  $A$  with a Banach norm such that  $\varphi$  is admissible. Then we can apply [Proposition 3.13](#).  $\square$

**Lemma 3.15.** Assume that  $k$  is non-trivially valued. Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ . The algebra  $k\{r^{-1}T\}$  is strictly  $k$ -affinoid if  $r_i \in \sqrt{|k^\times|}$  for all  $i = 1, \dots, n$ .

**Remark 3.16.** The converse is also true.

PROOF. Assume that  $r_i \in \sqrt{|k^\times|}$  for all  $i = 1, \dots, n$ . Take  $s_i \in \mathbb{N}$  and  $c_i \in k^\times$  such that

$$r_i^{s_i} = |c_i^{-1}|$$

for  $i = 1, \dots, n$ . We define a bounded  $k$ -algebra homomorphism  $\varphi : k\{T_1, \dots, T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  by sending  $T_i$  to  $c_i T_i^{s_i}$ . This is possible by [Proposition 4.18](#) in the chapter Banach Rings.

We claim that  $\varphi$  is finite. To see this, it suffices to observe that if we expand  $f \in k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (c T^s)^\gamma,$$

where the product  $\gamma s$  is taken component-wise. For each  $\beta \in \mathbb{N}^n, \beta_i < s_i$ , we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \dots, T_n\}.$$

While  $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$ . So We have shown that  $\varphi$  is finite. Hence,  $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  is  $k$ -affinoid by [Proposition 3.13](#).  $\square$

**Proposition 3.17.** Let  $A$  be a  $k$ -affinoid algebra, then there is  $n \in \mathbb{N}$  and a  $k$ -free polyray  $r = (r_1, \dots, r_n)$  such that  $A \hat{\otimes}_k k_r$  is strictly  $k_r$ -affinoid. Moreover, we can guarantee that  $k_r$  is non-trivially valued.

PROOF. By [Proposition 3.11](#), we may assume that  $A = k\{t^{-1}T\}$  for some  $t \in \mathbb{R}_{>0}^m$ . By [Lemma 3.15](#), it suffices to take  $r$  so that the linear subspace of  $\mathbb{R}_{>0}/\sqrt{|k^\times|}$  generated by  $r_1, \dots, r_n$  contains all components of  $t$ . By taking  $n \geq 1$ , we can guarantee that  $k_r$  is non-trivially valued.  $\square$

**Proposition 3.18.** Let  $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a monomorphism in  $k_H\text{-Aff}$ . Then for any  $y \in \mathrm{Sp} B$  with  $x = \varphi(y)$ , one has  $\varphi^{-1}(x) = \{y\}$  and the natural map  $\mathcal{H}(x) \rightarrow \mathcal{H}(y)$  is an isomorphism of complete valuation rings.

PROOF. It suffices to show that  $\mathcal{H}(x) \rightarrow B \hat{\otimes}_A \mathcal{H}(y)$  is an isomorphism as Banach  $k$ -algebras. [Include details about cofiber products in affalg](#). By assumption, the codiagonal map  $B \hat{\otimes}_A B \rightarrow B$  is an isomorphism. It follows that the base change with respect to  $A \rightarrow \mathcal{H}(x)$  is also an isomorphism:  $B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$ , where  $B' = B \hat{\otimes}_A \mathcal{H}(x)$ .

[Include the fact that the first map is injective](#). It follows that the composition  $B' \otimes_{\mathcal{H}(x)} B \rightarrow B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$  is injective. Therefore,  $\mathcal{H}(x) \rightarrow B'$  is an isomorphism of rings. We also know that this map is bounded. But we already know that  $\mathcal{H}(x)$  is a complete valuation ring, so the map  $\mathcal{H}(x) \rightarrow B'$  is an isomorphism of complete valuation rings.  $\square$

#### 4. Weierstrass theory

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued-field.

**Proposition 4.1.** We have canonical identifications

$$\begin{aligned} (k\{T_1, \dots, T_n\})^\circ &\cong \mathring{k}\{T_1, \dots, T_n\}, \\ (k\{T_1, \dots, T_n\})^\vee &\cong \check{k}\{T_1, \dots, T_n\}, \\ k\{\widetilde{T_1, \dots, T_n}\} &\cong \tilde{k}[T_1, \dots, T_n]. \end{aligned}$$

The last identification extends  $\mathring{k} \rightarrow \tilde{k}$  and  $T_i$  is mapped to  $T_i$ .

PROOF. This follows from [Corollary 4.20](#) from the chapter Banach rings.  $\square$

We will denote the reduction map  $\mathring{k}\{T_1, \dots, T_n\} \rightarrow \tilde{k}[T_1, \dots, T_n]$  by  $\tilde{\bullet}$ .

**Definition 4.2.** Let  $n \in \mathbb{N}$ . A system  $f_1, \dots, f_n \in k\{T_1, \dots, T_n\}$  is called an *affinoid chart* of  $k\{T_1, \dots, T_n\}$  if  $f_i \in \mathring{k}\{T_1, \dots, T_n\}$  for each  $i = 1, \dots, n$  and the continuous  $k$ -algebra homomorphism  $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$  sending  $T_i$  to  $f_i$  is an isomorphism.

The map  $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$  is well-defined by [Proposition 4.1](#) and [Lemma 2.5](#).

**Lemma 4.3.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ . Assume that  $\|f\|_1 = 1$ . Then the following are equivalent:

- (1)  $f$  is a unit in  $k\{T_1, \dots, T_n\}$ .
- (2)  $\tilde{f}$  is a unit in  $\tilde{k}[T_1, \dots, T_n]$ .

PROOF. As  $\|\bullet\|_1$  is a valuation by [Proposition 3.6](#),  $f$  is a unit in  $k\{T_1, \dots, T_n\}$  if and only if it is a unit in  $(k\{T_1, \dots, T_n\})^\circ$ , which is identified with  $\mathring{k}\{T_1, \dots, T_n\}$  by [Proposition 4.1](#). This result then follows from [Corollary 4.21](#) in the chapter Banach Rings.  $\square$

**Definition 4.4.** Let  $n \in \mathbb{N}$ . Consider  $g \in k\{T_1, \dots, T_n\}$ . We expand  $g$  as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

For  $s \in \mathbb{N}$ , we say  $g$  is  *$X_n$ -distinguished of degree  $s$*  if  $g_s$  is a unit in  $k\{T_1, \dots, T_{n-1}\}$ ,  $\|g_s\|_1 = \|g\|_1$  and  $\|g_s\|_1 > \|g_t\|_1$  for all  $t > s$ .

**Theorem 4.5** (Weierstrass division theorem). Let  $n, s \in \mathbb{N}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished of degree  $s$ . Then for each  $f \in k\{T_1, \dots, T_n\}$ , there exist  $q \in k\{T_1, \dots, T_n\}$  and  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\deg_{T_n} r < s$  such that

$$f = qg + r.$$

Moreover,  $q$  and  $r$  are uniquely determined. We have the following estimates

$$(4.1) \quad \|q\|_1 \leq \|g\|_1^{-1} \|f\|_1, \quad \|r\|_1 \leq \|f\|_1.$$

If in addition,  $f, g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then  $q \in k\{T_1, \dots, T_{n-1}\}[T_n]$  as well.

PROOF. We may assume that  $\|g\|_1 = 1$ .

**Step 1.** Assuming the existence of the division. Let us prove (4.1). We may assume that  $f \neq 0$ , so that one of  $q, r$  is non-zero. Up to replacing  $q, r$  by a scalar multiple, we may assume that  $\max\{\|q\|_1, \|r\|_1\} = 1$ . So  $\|f\|_1 \leq 1$  as well. We need to show that  $\|f\|_1 = 1$ . Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here  $\tilde{\bullet}$  denotes the reduction map. By our assumption,  $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$ . From [Proposition 4.1](#), the equality is in  $\tilde{k}[T_1, \dots, T_n]$ . From the usual Euclidean division, we have  $\tilde{q} = \tilde{r} = 0$ . This is a contradiction to our assumption.

**Step 2.** Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with  $q$  and  $r$  as in the theorem. The estimate in Step 1 shows that  $q = r = 0$ .

**Step 3.** We prove the existence of the division.

We define

$$B := \{qg + r : r \in k\{T_1, \dots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \dots, T_n\}\}.$$

From Step 1,  $B$  is a closed subgroup of  $k\{T_1, \dots, T_n\}$ . In fact, suppose  $f_i \in B$  is a sequence converging to  $f \in k\{T_1, \dots, T_n\}$ . From Step 1, we can represent  $f_i = q_i g + r_i$ , then from Step 1,  $q_i$  and  $r_i$  are both Cauchy sequences, we may assume that  $q_i \rightarrow q \in k\{T_1, \dots, T_n\}$  and  $r_i \rightarrow r$ . As  $\deg_{T_n} r_i < s$ , it follows that  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  and  $\deg_{T_n} r < s$ . So  $f = qg + r$  and hence  $B$  is closed.

It suffices to show that  $B$  is dense  $k\{T_1, \dots, T_n\}$ . We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

We may assume that  $\|g\|_1 = 1$ . Define  $\epsilon := \max_{j \geq s} \|g_j\|$ . Then  $\epsilon < 1$  by our assumption. Let  $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$  for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \dots, T_n\})^\circ \rightarrow (\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$$

with kernel  $\{f \in k\{T_1, \dots, T_n\} : \|f\|_1 \leq \epsilon\}$ . We now apply Euclidean division in the ring  $(\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$  to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some  $q \in (k\{T_1, \dots, T_n\})^\circ$  and  $r \in (k\{T_1, \dots, T_{n-1}\})^\circ[T_n]$  with  $\deg_{T_n} r < s$ . So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that  $B$  is dense in  $k\{T_1, \dots, T_n\}$  by [Proposition 2.8](#) in the chapter Banach rings.

**Step 4.** It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring  $k\{T_1, \dots, T_{n-1}\}[T_n]$  and the uniqueness proved in Step 2.  $\square$

**Lemma 4.6.** Let  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  be a Weierstrass polynomial and  $g \in k\{T_1, \dots, T_n\}$ . Assume that  $\omega g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then  $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ .

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some  $q, r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ ,  $\deg_{T_n} r < \deg_{T_n} \omega g$ . But we can write  $\omega g = \omega \cdot g$ . From the uniqueness part of [Theorem 4.5](#), we know that  $q = g$ , so  $g$  is a polynomial in  $T_n$ .  $\square$

As a consequence, we deduce Weierstrass preparation theorem.

**Definition 4.7.** Let  $n \in \mathbb{Z}_{>0}$ . A *Weierstrass polynomial* in  $n$ -variables is a monic polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\|\omega\|_1 = 1$ .

**Lemma 4.8.** Let  $n \in \mathbb{Z}_{>0}$  and  $\omega_1, \omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  be two monic polynomials. If  $\omega_1\omega_2$  is a Weierstrass polynomial then so are  $\omega_1$  and  $\omega_2$ .

PROOF. As  $\omega_1$  and  $\omega_2$  are monic,  $\|\omega_i\|_1 \geq 1$  for  $i = 1, 2$ . On the other hand,  $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1\omega_2\|_1 = 1$ , so  $\|\omega_i\|_1 = 1$  for  $i = 1, 2$ .  $\square$

**Theorem 4.9** (Weierstrass preparation theorem). Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished of degree  $s$ . Then there are a Weierstrass polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  of degree  $s$  and a unit  $e \in k\{T_1, \dots, T_n\}$  such that

$$g = e\omega.$$

Moreover,  $e$  and  $\omega$  are unique. If  $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , then so is  $e$ .

PROOF. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let  $r = T_n^s - \omega$ . Then  $T_n^s = e^{-1}g + r$ . The uniqueness part of [Theorem 4.5](#) implies that  $e$  and  $r$  are uniquely determined, hence so is  $\omega$ .

Next we prove the existence. By Weierstrass division theorem [Theorem 4.5](#), we can write

$$T_n^s = qg + r$$

for some  $q \in k\{T_1, \dots, T_n\}$  and  $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$  with  $\deg_{T_n} r < s$ . Let  $\omega = T_n^s - r$ . From the estimates in [Theorem 4.5](#),  $\|r\|_1 \leq 1$ . So  $\|\omega\|_1 = 1$ . Then  $\omega$  is a Weierstrass polynomial of degree  $s$  and  $\omega = qg$ . It suffices to argue that  $q$  is a unit.

We may assume that  $\|g\|_1 = 1$ . By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As  $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$  and the leading coefficients of both polynomials are units in  $\tilde{k}[T_1, \dots, T_{n-1}]$ , it follows that  $\tilde{q}$  is a unit in  $\tilde{k}[T_1, \dots, T_{n-1}]$ . It follows that  $\tilde{q}$  is also a unit in  $\tilde{k}[T_1, \dots, T_n]$ . By [Lemma 4.3](#),  $q$  is a unit in  $k\{T_1, \dots, T_n\}$ .

The last assertion is already proved in [Theorem 4.5](#).  $\square$

**Definition 4.10.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished. Then the Weierstrass polynomial  $\omega$  constructed in [Theorem 4.9](#) is called the *Weierstrass polynomial* defined by  $g$ .

**Corollary 4.11.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  be  $X_n$ -distinguished. Let  $\omega$  be the Weierstrass polynomial of  $g$ . Then the injection

$$k\{T_1, \dots, T_{n-1}\}[T_n] \rightarrow k\{T_1, \dots, T_n\}$$

induces an isomorphism of  $k$ -algebras

$$k\{T_1, \dots, T_{n-1}\}[T_n]/(\omega) \rightarrow k\{T_1, \dots, T_n\}/(g).$$

PROOF. The surjectivity follows from [Theorem 4.5](#) and the injectivity follows from [Lemma 4.6](#).  $\square$

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

**Lemma 4.12.** Let  $n \in \mathbb{Z}_{>0}$  and  $g \in k\{T_1, \dots, T_n\}$  is non-zero. Then there is a  $k$ -algebra automorphism  $\sigma$  of  $k\{T_1, \dots, T_n\}$  so that  $\sigma(g)$  is  $T_n$ -distinguished.

PROOF. We may assume that  $\|g\|_1 = 1$ . We expand  $g$  as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow  $\mathbb{N}^n$  with the lexicographic order. Take the maximal  $\beta \in \mathbb{N}^n$  so that  $|a_\beta| = 1$ . Take  $t \in \mathbb{Z}_{>0}$  so that  $t \geq \max_{i=1, \dots, n} \alpha_i$  for all  $\alpha \in \mathbb{N}^n$  with  $\tilde{a}_\alpha \neq 0$ .

We will define  $\sigma$  by sending  $T_i$  to  $T_i + T_n^{c_i}$  for all  $i = 1, \dots, n-1$ . The  $c_i$ 's are to be defined. We begin with  $c_n = 1$  and define the other  $c_i$ 's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for  $j = 1, \dots, n-1$ . We claim that  $\sigma(f)$  is  $T_n$ -distinguished of order  $s = \sum_{i=1}^n c_i \beta_i$ .

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^s p_i T_n^i$$

for some  $p_i \in \tilde{k}[T_1, \dots, T_{n-1}]$  and  $p_s = \tilde{a}_\beta$ . Our claim follows.  $\square$

**Proposition 4.13.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is Noetherian.

PROOF. We make induction on  $n$ . The case  $n = 0$  is trivial. Assume that  $n > 0$ . It suffices to show that for any non-zero  $g \in k\{T_1, \dots, T_n\}$ ,  $k\{T_1, \dots, T_n\}/(g)$  is Noetherian. By Lemma 4.12, we may assume that  $g$  is  $T_n$ -distinguished. By Theorem 4.5,  $k\{T_1, \dots, T_n\}/(g)$  is a finite free  $k\{T_1, \dots, T_{n-1}\}$ -module. By the inductive hypothesis and Hilbert basis theorem,  $k\{T_1, \dots, T_n\}/(g)$  is indeed Noetherian.  $\square$

**Proposition 4.14.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is Jacobson.

PROOF. When  $n = 0$ , there is nothing to prove. We make induction on  $n$  and assume that  $n > 0$ . Let  $\mathfrak{p}$  be a prime ideal in  $k\{T_1, \dots, T_n\}$ , we want to show that the Jacobson radical of  $\mathfrak{p}$  is equal to  $\mathfrak{p}$ .

We distinguish two cases. First we assume that  $\mathfrak{p} \neq 0$ . Let  $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \dots, T_{n-1}\}$ . By Lemma 4.12, we may assume that  $\mathfrak{p}$  contains a Weierstrass polynomial  $\omega$ . Observe that

$$k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' \rightarrow k\{T_1, \dots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any  $b \in J(k\{T_1, \dots, T_n\}/\mathfrak{p})$  (where  $J$  denotes the Jacobson radical), we consider a monic integral equation of minimal degree over  $k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'$ :

$$b^n + a_1 b^{n-1} + \dots + a_n = 0, \quad a_i \in k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \dots, T_n\}/\mathfrak{p}) \cap k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that  $n = 1$  and so  $b = 0$ . This proves  $J(k\{T_1, \dots, T_n\}/\mathfrak{p}) = 0$ .

On the other hand, let us consider the case  $\mathfrak{p} = 0$ . As  $k\{T_1, \dots, T_n\}$  is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \dots, T_n\}) = 0.$$

Assume that there is a non-zero element  $f$  in  $J(k\{T_1, \dots, T_n\})$ . We may assume that  $\|f\|_1 = 1$ .

We claim that there is  $c \in k$  with  $|c| = 1$  such that  $c + f$  is not a unit in  $k\{T_1, \dots, T_n\}$ . Assuming this claim for the moment, we can find a maximal ideal  $\mathfrak{m}$  of  $k\{T_1, \dots, T_n\}$  such that  $c + f \in \mathfrak{m}$ . But  $f \in \mathfrak{m}$  by our assumption, so  $c \in \mathfrak{m}$  as well. This contradicts the fact that  $c \in k^\times$ .

It remains to prove the claim. We treat two cases separately. When  $|f(0)| < 1$ , we simply take  $c = 1$ , which works thanks to [Lemma 4.3](#). If  $|f(0)| = 1$ , we just take  $c = -f(0)$ .  $\square$

**Proposition 4.15.** Let  $n \in \mathbb{N}$ . Then  $k\{T_1, \dots, T_n\}$  is UFD. In particular,  $k\{T_1, \dots, T_n\}$  is normal.

PROOF. As  $\|\bullet\|_1$  is a valuation by [Proposition 2.2](#),  $k\{T_1, \dots, T_n\}$  is an integral domain. In order to see that  $k\{T_1, \dots, T_n\}$  has the unique factorization property, we make induction on  $n \geq 0$ . When  $n = 0$ , there is nothing to prove. Assume that  $n > 0$ . Take a non-unit element  $f \in k\{T_1, \dots, T_n\}$ . By [Theorem 4.9](#) and [Lemma 4.12](#), we may assume that  $f$  is a Weierstrass polynomial. By inductive hypothesis,  $k\{T_1, \dots, T_{n-1}\}$  is a UFD, hence so is  $k\{T_1, \dots, T_{n-1}\}[T_n]$  by [[Stacks, Tag 0BC1](#)]. It follows that  $f$  can be decomposed into the products of monic prime elements  $f_1, \dots, f_r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ , which are all Weierstrass polynomials by [Lemma 4.8](#). Then by [Corollary 4.11](#), we see that each  $f_i$  is prime in  $k\{T_1, \dots, T_n\}$ .

Any UFD is normal by [[Stacks, Tag 0AFV](#)].  $\square$

**Corollary 4.16.** Let  $A$  be a strictly  $k$ -affinoid algebra,  $d \in \mathbb{N}$  and  $\varphi : k\{T_1, \dots, T_d\} \rightarrow A$  be an integral torsion-free injective homomorphism of  $k$ -algebras. Then  $\rho$  is a faithful  $k\{T_1, \dots, T_d\}$ -algebra norm on  $A$ . If  $f^n + \varphi(t_1)f^{n-1} + \dots + \varphi(t_n) = 0$  is the minimal integral equation of  $f$  over  $k\{T_1, \dots, T_d\}$ , then

$$|f|_{\sup} = \max_{i=1, \dots, n} |t_i|^{1/i}.$$

PROOF. This follows from [Proposition 9.11](#) in the chapter Banach Rings and [Proposition 4.15](#).  $\square$

## 5. Noetherian normalization and maximal modulus principle

Let  $(k, |\bullet|)$  be a complete non-trivially valued non-Archimedean valued-field.

**Theorem 5.1.** Let  $A$  be a non-zero strictly  $k$ -affinoid algebra,  $n \in \mathbb{N}$  and  $\alpha : k\{T_1, \dots, T_n\} \rightarrow A$  be a finite (resp. integral)  $k$ -algebra homomorphism. Then up to replacing  $T_1, \dots, T_n$  by an affinoid chart, we can guarantee that there exists  $d \in \mathbb{N}$ ,  $d \leq n$  such that  $\alpha$  when restricted to  $k\{T_1, \dots, T_d\}$  is finite (resp. integral) and injective.

PROOF. We make an induction on  $n$ . The case  $n = 0$  is trivial. Assume that  $n > 0$ . If  $\ker \alpha = 0$ , there is nothing to prove, so we may assume that  $\ker \alpha \neq 0$ . By [Lemma 4.12](#) and [Theorem 4.9](#), we may assume that there is a Weierstrass polynomial  $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$  in  $\ker \alpha$ . Then  $\alpha$  induces a finite (resp. integral) homomorphism  $\beta : k\{T_1, \dots, T_n\}/(\omega) \rightarrow A$ . By [Theorem 4.5](#),  $k\{T_1, \dots, T_{n-1}\} \rightarrow k\{T_1, \dots, T_n\}/(\omega)$  is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism  $k\{T_1, \dots, T_{n-1}\} \rightarrow A$ . We can apply the inductive hypothesis to conclude.  $\square$

**Corollary 5.2.** Let  $A$  be a non-zero strictly  $k$ -affinoid algebra, then there is  $d \in \mathbb{N}$  and a finite injective  $k$ -algebra homomorphism:  $k\{T_1, \dots, T_d\} \rightarrow A$ .

PROOF. Take some  $n \in \mathbb{N}$  and a surjective  $k$ -algebra homomorphism  $k\{T_1, \dots, T_n\} \rightarrow A$  and apply [Theorem 5.1](#), we conclude.  $\square$

**Corollary 5.3.** Let  $A$  be a strictly  $k$ -affinoid algebra and  $I$  be an ideal in  $A$  such that  $\sqrt{I}$  is a maximal ideal in  $A$ , then  $A/I$  is finite-dimensional over  $k$ .

In particular,  $\text{Spm } A = \text{Spm}_k A$ .

PROOF. By [Corollary 5.2](#), there is  $d \in \mathbb{N}$  and a finite monomorphism  $f : k\{T_1, \dots, T_d\} \rightarrow A/I$ . It suffices to show that  $d = 0$ . Observe that the composition

$$k\{T_1, \dots, T_d\} \xrightarrow{f} A/I \rightarrow A/\sqrt{I}$$

is finite and injective as  $k\{T_1, \dots, T_d\}$  is an integral domain, so  $k\{T_1, \dots, T_d\}$  is a field. This is possible only when  $d = 0$ .  $\square$

**Corollary 5.4.** Let  $B$  be a strictly  $k$ -affinoid algebra and  $A$  be a Noetherian Banach  $k$ -algebra. Let  $f : A \rightarrow B$  a  $k$ -algebra homomorphism. Then  $f$  is bounded.

PROOF. This follows from [Proposition 8.1](#) in the chapter Banach Rings and [Proposition 4.13](#).  $\square$

In particular, we see that the topology of a  $k$ -affinoid algebra is uniquely determined by the algebraic structure.

**Corollary 5.5.** Let  $A, B$  be strictly  $k$ -affinoid algebras. Let  $f$  be a finite  $k$ -algebra homomorphism, then  $f$  is admissible.

PROOF. This follows from [Proposition 3.14](#) and [Corollary 5.4](#),  $\square$

**Definition 5.6.** For any non-Archimedean valuation field  $(K, |\bullet|)$  and  $n \in \mathbb{N}$ , we define the  $n$ -dimensional polydisk with value in  $K$ :

$$B^n(K) := \left\{ (x_1, \dots, x_n) \in K^n : \max_{i=1, \dots, n} |x_i| \leq 1 \right\}.$$

**Definition 5.7.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ , say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function  $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$  as sending  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

**Lemma 5.8.** Let  $n \in \mathbb{N}$  and  $f \in k\{T_1, \dots, T_n\}$ , then  $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$  is continuous and for any  $x \in B^n(k^{\text{alg}})$ ,

$$|f(x)| \leq \|f\|_1.$$

There is  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  such that  $|f(x)| = \|f\|_1$ .



PROOF. To see that  $f$  is continuous, it suffices to observe that  $f$  is a uniform limit of polynomials. For any  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$ , we have

$$|f(x)| = \left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \right| \leq \max_{\alpha \in \mathbb{N}^n} |a_\alpha x^\alpha| \leq \|f\|_1.$$

To prove the last assertion, we may assume that  $\|f\|_1 = 1$ . As the residue field of  $k^{\text{alg}}$  is equal to  $\tilde{k}^{\text{alg}}$ , it has infinitely many elements, so there is a point  $x \in B^n(k^{\text{alg}})$  such that  $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$ . In other words,  $\|f(x)\|_1 = 1$ .  $\square$

**Proposition 5.9.** Let  $n \in \mathbb{N}$ , then the maximal modulus principle holds for  $k\{T_1, \dots, T_n\}$ . Moreover, for any  $f \in k\{T_1, \dots, T_n\}$ ,  $\|f\|_1 = |f|_{\text{sup}}$ .

PROOF. By Lemma 6.3 in the chapter Banach Rings, we have

$$\|f\|_1 \geq |f|_{\text{sup}}$$

for any  $f \in A$ . We only have to show that for any  $f \in k\{T_1, \dots, T_n\}$  there is a maximal ideal  $\mathfrak{m} \subseteq k\{T_1, \dots, T_n\}$  such that  $|f(\mathfrak{m})| = \|f\|_1$ .

By Lemma 5.8 we can take  $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$  such that  $|f(x)| = \|f\|_1$ . Let  $L$  be the field extension of  $k$  generated by  $x_1, \dots, x_n$ , then  $L/k$  is finite. Then we can define a homomorphism

$$\text{ev}_x : k\{T_1, \dots, T_n\} \rightarrow L$$

sending  $g \in k\{T_1, \dots, T_n\}$  to  $g(x)$ . Observe that the image is indeed in  $L$ . Clearly  $\text{ev}_x$  is surjective. So  $\mathfrak{m}_x := \ker \text{ev}_x$  is a  $k$ -algebraic maximal ideal in  $k\{T_1, \dots, T_n\}$ . Then

$$|f(\mathfrak{m}_x)| = |f(x)| = \|f\|_1.$$

$\square$

**Corollary 5.10.** Let  $A$  be a strictly  $k$ -affinoid algebra. Then for any  $f \in A$ ,

$$|f|_{\text{sup}} \subseteq \sqrt{|k^\times|} \cup \{0\}.$$

PROOF. We may assume that  $A \neq 0$ . By Corollary 5.2 and Proposition 9.11 in the chapter Banach Rings, we may assume that  $A = k\{T_1, \dots, T_n\}$  for some  $n \in \mathbb{N}$ . The result then follows from Proposition 5.9.  $\square$

**Corollary 5.11.** Maximal modulus principle holds for any strictly  $k$ -affinoid algebras.

PROOF. This follows from Corollary 5.2, Proposition 9.11 in the chapter Banach Rings and Proposition 5.9.  $\square$

**Proposition 5.12.** Let  $\varphi : B \rightarrow A$  be an integral  $k$ -algebra homomorphism of strictly  $k$ -affinoid algebras. Then for each non-zero  $f \in A$ , there is a moine polynomial  $q(f) = f^n + \varphi(b_1)f^{n-1} + \dots + \varphi(b_n)$  of  $f$  over  $B$ . Then

$$|f|_{\text{sup}} = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

PROOF. One side is simple: choose  $j = 1, \dots, n$  that maximizes  $|\varphi(b_j)f^{n-j}|_{\text{sup}}$ , then

$$|f|_{\text{sup}}^n = |f^n|_{\text{sup}} \leq |\varphi(b_j)f^{n-j}|_{\text{sup}} \leq |b_j|_{\text{sup}} \cdot |f|_{\text{sup}}^{n-j}.$$

So

$$|f|_{\text{sup}} \leq |b_j|_{\text{sup}}^{1/j}.$$

To prove the reverse inequality, let us begin with the case where  $A$  is an integral domain.

We claim that there is  $d \in \mathbb{N}$  and a  $k$ -algebra homomorphism  $\psi : k\{T_1, \dots, T_d\} \rightarrow B$  such that  $\varphi \circ \psi$  is integral and injective. In fact, choosing an epimorphism  $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$ , we can apply [Theorem 5.1](#) to find  $\phi \circ \alpha$  to conclude.

By [Corollary 4.16](#), if  $p$  denotes the minimal polynomial of  $f$  over  $k\{T_1, \dots, T_d\}$ , we have  $|f|_{\text{sup}} = \sigma(p)$ . In particular,  $p(f) = 0$ . Let  $q \in B[X]$  be the polynomial obtained from  $p$  by replacing all coefficients by their  $\psi$ -images in  $B$ . Then clearly,  $|f|_{\text{sup}} = \sigma(q)$ .

In general, let  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  be the minimal primes in  $A$ . The number is finite by [Proposition 4.13](#). For each  $i = 1, \dots, r$ , let  $\pi_i : A \rightarrow A/\mathfrak{p}_i$  denote the natural homomorphism. We know that there are monic polynomials  $q_i \in B[X]$  such that  $q_i(\pi(f)) = 0$  and  $|\pi_i(f)|_{\text{sup}} = \sigma(q_i)$  for  $i = 1, \dots, r$ . We let  $q' = q_1 \cdots q_r$ . Then

$$q'(f) \in \bigcap_{i=1}^r \mathfrak{p}_i.$$

So there is  $e \in \mathbb{Z}_{>0}$  such that  $q'(f)^e = 0$ . Let  $q = q'^e$ . By [Proposition 9.5](#) in the chapter Banach Rings,

$$\sigma(q) \leq \max_{i=1, \dots, r} \sigma(q_i) = \max_{i=1, \dots, r} |\pi_i(f)|_{\text{sup}} = |f|_{\text{sup}}.$$

The last equality follows from [Proposition 9.9](#) in the chapter Banach Rings.  $\square$

**Lemma 5.13.** Let  $\varphi : B \rightarrow A$  be an admissible  $k$ -algebra homomorphism between strictly  $k$ -affinoid algebras. Let  $\tau : \check{B} \rightarrow \check{B}$  be the reduction map, then

$$\tau^{-1}(\ker \tilde{\varphi}) = \sqrt{\check{B} + \ker \tilde{\varphi}}, \quad \ker \tilde{\varphi} = \sqrt{\tau(\ker \tilde{\varphi})}.$$

PROOF. The second equation follows from the first one by applying  $\tau$ . Let us prove the first equation. By assumption,  $\varphi(\check{B})$  is open in  $\varphi(B)$ . Consider  $g \in \tau^{-1}(\ker \tilde{\varphi})$ , we know that

$$\lim_{n \rightarrow \infty} \varphi(g)^n = 0.$$

So  $\varphi(g)^n \in \varphi(\check{B})$  for  $n$  large enough, and hence  $g^n \in \check{B} + \ker \tilde{\varphi}$ .  $\square$

**Lemma 5.14.** Let  $m \in \mathbb{N}$  and  $T = k\{T_1, \dots, T_m\}$ . Let  $A$  be a  $k$ -affinoid algebra and  $\varphi : T\{S_1, \dots, S_n\} \rightarrow A$  be a finite morphism such that  $\tilde{\varphi}(S_i)$  is integral over  $\check{T}$ . Then  $\varphi|_T : T \rightarrow A$  is finite.

PROOF. We make an induction on  $n$ . When  $n = 0$ , there is nothing to prove. So assume  $n > 0$  and the lemma has been proved for smaller values of  $n$ .

Let  $T' = T\{S_1, \dots, S_n\}$ . By assumption, there is a Weierstrass polynomial

$$\omega = S_n^k + a_1 S_n^{k-1} + \cdots + a_k \in \check{T}[S_n]$$

such that  $\tilde{\omega} \in \ker \tilde{\varphi}$ . As  $\varphi$  is admissible by [Corollary 5.5](#), we have  $\omega^q \in \check{T}' + \ker \tilde{\varphi}$  for some  $q \in \mathbb{Z}$  by [Lemma 5.13](#).

In particular, we can find  $r \in (T')^\vee$  such that  $g := \omega^q - r \in \ker \tilde{\varphi}$ . Observe that  $g$  is  $S_n$  distinguished of order  $mq$  as  $\tilde{g} = \tilde{\omega}^q$ . By [Corollary 4.11](#), the restriction of  $\varphi$  to  $T\{S_1, \dots, S_{n-1}\}$  is finite. We can apply the inductive hypothesis to conclude.  $\square$

**Lemma 5.15.** Let  $\varphi : B \rightarrow A$  be a  $k$ -algebra homomorphism of strictly  $k$ -affinoid algebras. Assume that there exist affinoid generators  $f_1, \dots, f_n \in \mathring{A}$  of  $A$  such that  $\tilde{f}_1, \dots, \tilde{f}_n$  are all integral over  $\tilde{B}$ , then  $\varphi$  is finite.

PROOF. By assumption, we can find  $s_i \in \mathbb{Z}_{>0}$ ,  $b_{ij} \in \mathring{B}$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, s_i$  such that

$$\tilde{f}_i^{s_i} + \tilde{\varphi}(\tilde{b}_{i1})\tilde{f}_i^{s_i-1} + \dots + \tilde{\varphi}(\tilde{b}_{is_i}) = 0$$

for  $i = 1, \dots, n$ . Let  $s = s_1 + \dots + s_n$  and define a bounded  $k$ -algebra homomorphism  $\psi : D := k\{T_{ij}\} \rightarrow B$  sending  $T_{ij}$  to  $b_{ij}$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, s_i$ . Observe that  $\tilde{f}_1, \dots, \tilde{f}_n$  are all integral over  $\tilde{D}$ . So it suffices to prove the theorem when  $B = k\{T_1, \dots, T_m\}$ . We extend  $\varphi$  to a bounded  $k$ -algebra epimorphism  $\varphi' : T\{S_1, \dots, S_n\} \rightarrow A$  sending  $S_i$  to  $f_i$  for  $i = 1, \dots, n$ . Then  $\varphi'(\tilde{S}_i)$  is integral over  $\tilde{B}$ . It suffices to apply [Lemma 5.14](#).  $\square$

## 6. Properties of affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Proposition 6.1.** Assume that  $k$  is non-trivially valued. Let  $A$  be a strictly  $k$ -affinoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\text{sup}} \leq 1\}.$$

PROOF. By [Lemma 6.3](#), we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\text{sup}} \leq 1\}.$$

Conversely, let  $f \in A$ ,  $|f|_{\text{sup}} \leq 1$ . Choose  $d \in \mathbb{N}$  and a surjective  $k$ -algebra homomorphism

$$\varphi : k\{T_1, \dots, T_d\} \rightarrow A.$$

Let  $f^n + t_1 f^{n-1} + \dots + t_n = 0$  be the minimal equation of  $f$  over  $k\{T_1, \dots, T_d\}$ . Then  $t_i \in (k\{T_1, \dots, T_d\})^\circ$  by [Proposition 9.11](#) in the chapter Banach Rings. An induction on  $i \geq 0$  shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi((k\{T_1, \dots, T_d\})^\circ) f^j.$$

The right-hand side is clearly bounded.  $\square$

**Corollary 6.2.** Assume that  $k$  is non-trivially valued. Let  $(A, \|\bullet\|)$  be a strictly  $k$ -affinoid algebra. For any  $f \in A$ ,

$$\rho(f) = |f|_{\text{sup}}.$$

PROOF. We have shown that  $\rho(f) \geq |f|_{\text{sup}}$  in [Lemma 6.3](#) from the chapter Banach Rings. Assume that the inverse inequality fails: for some  $f \in A$ ,

$$\rho(f) > |f|_{\text{sup}}.$$

If  $|f|_{\text{sup}} = 0$ , then  $f$  lies in the Jacobson radical of  $A$ , which is equal to the nilradical of  $A$  by [Proposition 4.14](#). But then  $\rho(f) = 0$  as well. We may therefore assume that  $|f|_{\text{sup}} \neq 0$ . By [Corollary 5.10](#), we may assume that  $|f|_{\text{sup}} = 1$  as  $\rho$  is power-multiplicative. Then  $\rho(f) > 1$ . This contradicts [Proposition 6.1](#).  $\square$

**Theorem 6.3.** A  $k$ -affinoid algebra  $A$  is Noetherian and all ideals of  $A$  are closed.

PROOF. Let  $I$  be an ideal in  $A$ . By [Proposition 3.17](#), we can take a suitable  $r \in \mathbb{R}_{>0}^m$  so that  $A \hat{\otimes} k_r$  is strictly  $k_r$ -affinoid. Then  $I(A \hat{\otimes} k_r)$  is an ideal in  $A \hat{\otimes} k_r$ . By [Proposition 4.13](#), the latter ring is Noetherian. So we may take finitely many generators  $f_1, \dots, f_k \in I$ . Each  $f \in I$  can be written as

$$f = \sum_{i=1}^k f_i g_i$$

with  $g_i = \sum_{j=-\infty}^{\infty} g_{i,j} T^j \in A \hat{\otimes} k_r$ . But then

$$f = \sum_{i=1}^k f_i g_{i,0}.$$

So  $I$  is finitely generated.

As  $I = A \cap (I(A \hat{\otimes} k_r))$ , by [Corollary 7.4](#) in the chapter Banach Rings, we see that  $I$  is closed in  $A \hat{\otimes} k_r$  and hence closed in  $A$ .  $\square$

**Proposition 6.4.** Let  $(A, \|\bullet\|)$  be a  $k$ -affinoid algebra and  $f \in A$ . Then there is  $C > 0$  and  $N \geq 1$  such that for any  $n \geq N$ , we have

$$\|f^n\| \leq C\rho(f)^n.$$

Recall that  $\rho$  is the spectral radius map defined in [Definition 4.9](#) in the chapter Banach Rings.

PROOF. By [Proposition 3.11](#), we may assume that  $k$  is non-trivially valued and  $k$  is non-trivially valued.

If  $\rho(f) = 0$ , then  $f$  lies in each maximal ideal of  $A$ . To see this, we may assume that  $A$  is a field, then by [Proposition 6.10](#) in the chapter Banach Rings, there is a bounded valuation  $\|\bullet\|'$  on  $A$ . But then  $\rho(f) = 0$  implies that  $\|f\|' = 0$  and hence  $f = 0$ .

It follows that if  $\rho(f) = 0$  then  $f$  lies in  $J(A)$ , the Jacobson radical of  $A$ . By [Proposition 4.14](#),  $A$  is a Jacobson ring. So  $f$  is nilpotent. The assertion follows.

So we can assume that  $\rho(f) > 0$ . In this case, by [Corollary 5.2](#) and [Proposition 9.11](#) in the chapter Banach Rings, we have  $\rho(f) \in \sqrt{|k^\times|}$ . Take  $a \in k^\times$  and  $d \in \mathbb{Z}_{>0}$  so that  $\rho(f)^d = |a|$ . Then  $\rho(f^d/a) = 1$  and hence it is powerly-bounded by [Proposition 6.1](#). It follows that there is  $C > 0$  so that for  $n \geq 1$ ,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that  $\|f^n\| \leq C\rho(f)^n$  for  $n \geq d$  as long as we enlarge  $C$ .  $\square$

**Corollary 6.5.** Let  $\varphi : A \rightarrow B$  be a bounded homomorphism of  $k$ -affinoid algebras. Let  $n \in \mathbb{N}$  and  $f_1, \dots, f_n \in B$  and  $r_1, \dots, r_n \in \mathbb{R}_{>0}$  with  $r_i \geq \rho(f_i)$  for  $i = 1, \dots, n$ . Write  $r = (r_1, \dots, r_n)$ , then there is a unique bounded homomorphism  $\Phi : A\{r^{-1}T\} \rightarrow B$  extending  $\varphi$  and sending  $T_i$  to  $f_i$ .

PROOF. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from [Proposition 6.4](#) that the right-hand side the series converges. The boundedness of  $\Phi$  is obvious.  $\square$

**Proposition 6.6.** Let  $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$  be  $k$ -affinoid algebras,  $r \in \mathbb{R}_{>0}^n$  and  $\varphi : A\{r^{-1}T\} \rightarrow B$  be an admissible epimorphism. Write  $f_i = \varphi(T_i)$  for  $i = 1, \dots, n$ . Then there is  $\epsilon > 0$  such that for any  $g = (g_1, \dots, g_n) \in B^n$  with  $\|f_i - g_i\|_B < \epsilon$  for all  $i = 1, \dots, n$ , there exists a unique bounded  $k$ -algebra homomorphism  $\psi : A\{r^{-1}T\} \rightarrow B$  that coincides with  $\varphi$  on  $A$  and sends  $T_i$  to  $g_i$ . Moreover,  $\psi$  is also an admissible epimorphism.

PROOF. The uniqueness of  $\psi$  is obvious. We prove the remaining assertions. Taking  $\epsilon > 0$  small enough, we could further guarantee that  $\rho(g_i) \leq r_i$ . It follows from [Corollary 6.5](#) that there exists a bounded homomorphism  $\psi$  as in the statement of the proposition.

As  $\varphi$  is an admissible epimorphism, we may assume that  $\|\bullet\|_B$  is the residue induced by  $\|\bullet\|_r$  on  $A\{r^{-1}T\}$ .

By definition of the residue norm, for any  $\delta > 0$  and any  $h \in B$ , we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1 + \delta) \|h\|_B$$

for any  $\alpha \in \mathbb{N}^n$ . Choose  $\epsilon \in (0, (1 + \delta)^{-1})$ . Now for  $g_1, \dots, g_n$  as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1 + \delta) \epsilon \|h\|_B.$$

Repeating this procedure, we can construct  $k_i \in A\{r^{-1}T\}$  for  $i \in \mathbb{N}$  and  $h_j \in B$  for  $j \in \mathbb{Z}_{>0}$  such that for any  $i \in \mathbb{Z}_{>0}$ , we have

$$h = \psi(k_0 + \dots + k_{i-1}) + h_i,$$

$$\|k_i\|_r \leq ((1 + \delta)\epsilon)^i (1 + \delta) \|h\|_B,$$

$$\|h_i\|_B \leq ((1 + \delta)\epsilon)^i \|h\|_B.$$

In particular,  $k := \sum_{i=0}^{\infty} k_i$  converges in  $A\{r^{-1}T\}$  and

$$\|k\|_r \leq (1 + \delta) \|h\|_B.$$

It follows that  $\psi$  is an admissible epimorphism.  $\square$

**Corollary 6.7.** Let  $A$  be a Banach  $k$ -algebra,  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a  $k$ -free polyray. Assume that  $A \hat{\otimes}_k k_r$  is  $k_r$ -affinoid, then  $A$  is  $k$ -affinoid.

If  $A \hat{\otimes}_k k_r$  is  $k_H$ -affinoid and  $r \in H$ , then  $A$  is also  $k_H$ -affinoid.

PROOF. We may assume that  $r$  has only one component.

Take  $m \in \mathbb{N}$ ,  $p_1, \dots, p_m \in \mathbb{R}_{>0}$  and an admissible epimorphism

$$\pi : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \rightarrow A \hat{\otimes}_k k_r.$$

Let

$$\pi(S_i) = \sum_{j=-\infty}^{\infty} a_{i,j} T^j, \quad a_{i,j} \in A$$

for  $i = 1, \dots, m$ . By [Proposition 6.6](#), we may assume that there is a large integer  $l$  such that  $a_{i,j} = 0$  for  $|j| > l$  and for any  $i = 1, \dots, m$ . We define  $B = k\{p_i^{-1} r^j T_{i,j}\}$ ,  $i = 1, \dots, n$  and  $j = -l, -l+1, \dots, l$ . Let  $\varphi : B \rightarrow A$  be the bounded  $k$ -algebra homomorphism sending  $T_{i,j}$  to  $a_{i,j}$ . The existence of  $\varphi$  is guaranteed by [Corollary 6.5](#).

We claim that  $\varphi$  is an admissible epimorphism. It is clearly an epimorphism. Let us show that  $\varphi$  is admissible. Let  $\eta : k_r\{p_1^{-1} S_1, \dots, p_m^{-1} S_m\} \rightarrow B \hat{\otimes}_k k_r$  be the bounded homomorphism sending  $S_i$  to  $\sum_{j=-l}^l T_{i,j} T^j$ , then we have the following commutative diagram

$$\begin{array}{ccc} k_r\{p^{-1} S\} & & \\ \downarrow \eta & \searrow \pi & \\ B \hat{\otimes}_k k_r & \xrightarrow{\varphi \hat{\otimes}_k k_r} & A \hat{\otimes}_k k_r \end{array}$$

It follows that  $\varphi \hat{\otimes}_k k_r$  is also an admissible epimorphism. By [Proposition 3.11](#),  $\varphi$  is also admissible.  $\square$

## 7. $H$ -strict affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $R_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

We next give a non-strict extension of [Proposition 3.13](#).

**Proposition 7.1.** Let  $B$  be a  $k_H$ -affinoid algebra and  $\varphi : B \rightarrow A$  be a finite bounded homomorphism into a  $k$ -Banach algebra  $A$ . Then  $A$  is also  $k_H$ -affinoid.

PROOF. We first assume that  $k$  is non-trivially valued.

We may assume that  $B = k\{r_1^{-1} T_1, \dots, r_n^{-1} T_n\}$  for some  $n \in \mathbb{N}$  and  $r_1, \dots, r_n \in H$ . By assumption, we can find finitely many  $a_1, \dots, a_m \in A$  such that  $A = \sum_{i=1}^m \varphi(B) a_i$ .

We may assume that  $a_i \in \mathring{A}$  as  $k$  is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings,  $\varphi$  admits a unique extension to a bounded  $k$ -algebra epimorphism

$$\Phi : k\{r_1^{-1} T_1, \dots, r_n^{-1} T_n, S_1, \dots, S_m\} \rightarrow A$$

sending  $S_i$  to  $a_i$ . By [Corollary 7.5](#) in the chapter Banach Rings,  $\Phi$  is admissible. Moreover, the homomorphism  $\Phi$  is surjective by our assumption. It follows that  $A$  is  $k_H$ -affinoid.

If  $k$  is trivially valued, then  $H$  is non-trivial. Take  $s \in H \setminus \{1\}$ . It follows from the previous case applied to  $\varphi \hat{\otimes}_k k_s : B \hat{\otimes}_k k_s \rightarrow A \hat{\otimes}_k k_s$  that  $A \hat{\otimes}_k k_s$  is  $k_H$ -affinoid. By [Corollary 6.7](#),  $A$  is also  $k_H$ -affinoid.  $\square$

**Proposition 7.2.** Let  $A$  be a Banach  $k$ -algebra. Then the following are equivalent:

- (1)  $A$  is  $k_H$ -affinoid;
- (2) there are  $n \in \mathbb{N}$ ,  $r \in \sqrt{|k^\times| \cdot H}$  and an admissible epimorphism  $k\{r^{-1} T\} \rightarrow A$ .

PROOF. The non-trivial direction is (2). Assume (2). Take  $s_1, \dots, s_n \in \mathbb{Z}_{>0}$ ,  $c_1, \dots, c_n \in k^\times$  and  $h_1, \dots, h_n \in H$  such that

$$r_i^{s_i} = |c_i^{-1}| h_i$$

for  $i = 1, \dots, n$ . We define a bounded  $k$ -algebra homomorphism

$$\varphi : k\{h_1^{-1}T_1, \dots, h_n^{-1}T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

by sending  $T_i$  to  $c_i T_i^{s_i}$ . The existence of such a homomorphism is guaranteed by [Corollary 6.5](#). The same proof of [Lemma 3.15](#) shows that  $\varphi$  is finite. By [Proposition 7.1](#),  $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$  is  $k_H$ -affinoid.  $\square$

**Lemma 7.3.** Assume that  $k$  is non-trivially valued. Let  $A$  be a  $k$ -affinoid algebra. Then the following are equivalent:

- (1)  $A$  is strictly  $k$ -affinoid;
- (2) for any  $a \in A$ ,  $\rho(a) \in \sqrt{|k^\times|} \cup \{0\}$ .

PROOF. (1)  $\implies$  (2) by [Corollary 5.10](#) and [Corollary 6.2](#).

(2)  $\implies$  (1): Take  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$  and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Let  $f_i = \varphi(T_i)$  for  $i = 1, \dots, n$ . Suppose  $r_1, \dots, r_m \notin \sqrt{|k^\times|}$  and  $r_{m+1}, \dots, r_n \in \sqrt{|k^\times|}$ . Then  $\rho(f_i) < r_i$  for  $i = 1, \dots, m$  and we can choose  $r'_1, \dots, r'_m \in \sqrt{|k^\times|}$  such that

$$\rho(f_i) \leq r'_i < r_i$$

for  $i = 1, \dots, m$ . Set  $r'_i = r_i$  when  $i = m+1, \dots, n$ . We can then define a bounded  $k$ -algebra homomorphism  $\psi : k\{r'^{-1}T\} \rightarrow A$  sending  $T_i$  to  $f_i$  for  $i = 1, \dots, n$ . The existence of  $\psi$  is guaranteed by [Corollary 6.5](#). Observe that  $\psi$  is surjective and admissible. It follows that  $A$  is strictly  $k$ -affinoid.  $\square$

**Theorem 7.4.** Let  $A$  be a  $k$ -affinoid algebra. Then the following are equivalent:

- (1)  $A$  is  $k_H$ -affinoid;
- (2)  $A$  is  $k_{\sqrt{|k^\times|} \cdot H}$ -affinoid;
- (3) For any non-zero  $a \in A$ ,  $\rho(a) \in \sqrt{|k^\times| \cdot H} \cup \{0\}$ .

PROOF. The equivalence between (1) and (2) follows from [Proposition 7.2](#).

(1)  $\implies$  (3): we may assume that  $H \supseteq |k^\times|$ . Take  $n \in \mathbb{N}$ ,  $r = (r_1, \dots, r_n) \in H^n$  and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Take a  $k$ -free polyray  $s$  with at least one component so that  $|k_s| \supseteq \{r_1, \dots, r_n\}$ . We can apply [Lemma 7.3](#) to  $\varphi \hat{\otimes}_k k_s$ , it follows that  $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$ .

(3)  $\implies$  (2): we may assume that  $H \supseteq |k^\times|$ . It suffices to apply the same argument as (2)  $\implies$  (1) in the proof of [Lemma 7.3](#).  $\square$

## 8. Finite modules over affinoid algebras

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field.

For any  $k$ -affinoid algebra  $A$ , we have defined the category  $\mathcal{B}\text{an}_A^f$  of finite Banach  $A$ -modules in [Definition 5.3](#) in the chapter Banach Rings. We write  $\mathcal{M}\text{od}_A^f$  for the category of finite  $A$ -modules.

**Lemma 8.1.** Let  $A$  be a  $k$ -affinoid algebra,  $(M, \|\bullet\|_M)$  be a finite Banach  $A$ -module and  $(N, \|\bullet\|_N)$  be a Banach  $A$ -module  $N$ . Let  $\varphi : M \rightarrow N$  be an  $A$ -linear homomorphism. Then  $\varphi$  is bounded.

PROOF. Take  $n \in \mathbb{N}$  such that there is an admissible epimorphism

$$\pi : A^n \rightarrow M.$$

It suffices to show that  $\varphi \circ \pi$  is bounded. So we may assume that  $M = A^n$ . For  $i = 1, \dots, n$ , let  $e_i$  be the vector with  $(0, \dots, 0, 1, 0, \dots, 0)$  of  $A^n$  with 1 placed at the  $i$ -th place. Set  $C = \max_{i=1, \dots, n} \|\varphi(e_i)\|_N$ . For a general  $f = \sum_{i=1}^n a_i e_i$  with  $a_i \in A$ , we have

$$\|\varphi(f)\|_N \leq C \|f\|_M.$$

So  $\varphi$  is bounded.  $\square$

**Proposition 8.2.** Let  $A$  be a  $k$ -affinoid algebra. The forgetful functor  $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$  is an equivalence of categories.

PROOF. It suffices to construct the inverse functor. Let  $M$  be a finite  $A$ -module. Choose  $n \in \mathbb{N}$  and an  $A$ -linear epimorphism  $\pi : A^n \rightarrow M$ . By [Theorem 6.3](#),  $\ker \pi$  is closed in  $A^n$ . We can endow  $M$  with the residue norm. By [Lemma 8.1](#), the equivalence class of the norm does not depend on the choice of  $\pi$ .

For any  $A$ -linear homomorphism  $f : M \rightarrow N$  of finite  $A$ -modules, we endow  $M$  and  $N$  with the Banach structures as above. It follows from [Lemma 8.1](#) that  $f$  is bounded. We have defined the inverse functor of the forgetful functor  $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$ .  $\square$

**Remark 8.3.** Let  $A$  be a  $k$ -affinoid algebra. It is not true that a Banach  $A$ -module which is finite as  $A$ -module is finite as Banach  $A$ -module.

As an example, take  $0 < p < q < 1$  and  $A = k\{q^{-1}T\}$ ,  $B = k\{p^{-1}T\}$ . Then  $B$  is a Banach  $A$ -module. By [Example 2.4](#), the underlying rings of  $A$  and  $B$  are both  $k[[T]]$ . So the canonical map  $A \rightarrow B$  is bijective. But  $B$  is not a finite  $A$ -module. As otherwise, the inverse map  $B \rightarrow A$  is bounded by [Lemma 8.1](#), which is not the case.

The correct statement is the following: consider a Banach  $A$ -module  $(M, \|\bullet\|_M)$  which is finite as  $A$ -module, then there is a norm on  $M$  such that  $M$  becomes a finite Banach  $A$ -module. The new norm is not necessarily equivalent to the given norm  $\|\bullet\|_M$ .

**Proposition 8.4.** Let  $A$  be a  $k$ -affinoid algebra and  $M, N$  be finite Banach  $A$ -modules. Then the natural map

$$M \otimes_A N \rightarrow M \hat{\otimes}_A N$$

is an isomorphism of Banach  $A$ -modules and  $M \hat{\otimes}_A N$  is a finite Banach  $A$ -module.

Here the Banach  $A$ -module structure on  $M \otimes_A N$  is given by [Proposition 8.2](#).



PROOF. Choose  $m, m' \in \mathbb{N}$  an admissibly coexact sequence

$$A^{m'} \rightarrow A^m \rightarrow M \rightarrow 0$$

of Banach  $A$ -modules. Then we have a commutative diagram of  $A$ -modules:

$$\begin{array}{ccccccc} A^{m'} \otimes_A N & \longrightarrow & A^m \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{m'} \hat{\otimes}_A N & \longrightarrow & A^m \hat{\otimes}_A N & \longrightarrow & M \hat{\otimes}_A N & \longrightarrow & 0 \end{array}$$

with exact rows. By 5-lemma, in order to prove  $M \otimes_A N \xrightarrow{\sim} M \hat{\otimes}_A N$  and  $M \hat{\otimes}_A N$  is a finite Banach  $A$ -module, we may assume that  $M = A^m$  for some  $m \in \mathbb{N}$ . Similarly, we can assume  $N = A^n$  for some  $n \in \mathbb{N}$ . In this case, the isomorphism is immediate and  $M \hat{\otimes}_A N$  is clearly a finite Banach  $A$ -module. By [Lemma 8.1](#), the Banach  $A$ -module structure on  $M \hat{\otimes}_A N$  coincides with the Banach  $A$ -module structure on  $M \otimes_A N$  induced by [Proposition 8.2](#).  $\square$

**Proposition 8.5.** Let  $A, B$  be a  $k$ -affinoid algebra and  $A \rightarrow B$  be a bounded  $k$ -algebra homomorphism. Let  $M$  be a finite Banach  $A$ -module, then the natural map

$$M \otimes_A B \rightarrow M \hat{\otimes}_A B$$

is an isomorphism of Banach  $B$ -modules and  $M \hat{\otimes}_A B$  is a finite Banach  $B$ -module.

PROOF. By the same argument as [Proposition 8.4](#), we may assume that  $M = A^n$  for some  $n \in \mathbb{N}$ . In this case, the assertions are trivial.  $\square$

**Proposition 8.6.** Let  $A$  be a  $k$ -affinoid algebra and  $M, N$  be finite Banach  $A$ -modules. Let  $\varphi : M \rightarrow N$  be an  $A$ -linear map. Then  $\varphi$  is admissible.

PROOF. By [Lemma 8.1](#),  $\varphi$  is always bounded. By [Proposition 8.5](#) and [Proposition 3.11](#), we may assume that  $k$  is non-trivially valued. By [Theorem 6.3](#),  $N$  is a Noetherian  $A$ -module. It follows from [Corollary 7.4](#) in the chapter Banach Rings that  $\text{Im } \varphi$  is closed in  $N$  and is finite as an  $A$  module. In particular, the norm induced from  $N$  and from  $M$  are equivalent by [Lemma 8.1](#). It follows that  $\varphi$  is admissible.  $\square$

**Proposition 8.7.** Let  $A$  be a  $k$ -affinoid algebra. Let  $n \in \mathbb{N}$  and  $r = (r_1, \dots, r_n)$  be a  $k$ -free polyray. Then  $M$  is a finite Banach  $A$ -module if and only if  $M \hat{\otimes}_k k_r$  is a finite Banach  $A \hat{\otimes}_k k_r$ -module.

PROOF. We may assume that  $r$  has only one component and write  $r_1 = r$ . The direct implication is trivial. Let us assume that  $M \hat{\otimes}_k k_r$  is a finite Banach  $A \hat{\otimes}_k k_r$ -module. Take  $n \in \mathbb{N}$  and an admissible epimorphism of  $A \hat{\otimes}_k k_r$ -modules

$$\varphi : (A \hat{\otimes}_k k_r)^n \rightarrow M \hat{\otimes}_k k_r.$$

Let  $e_1, \dots, e_n$  denotes the standard basis of  $(A \hat{\otimes}_k k_r)^n$ . We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By [Proposition 6.6](#), we can assume that there is  $l > 0$  such that  $m_{i,j} = 0$  for all  $i = 1, \dots, n$  and  $|j| > l$ . It follows that

$$A^{n(2l+1)} \rightarrow M$$

sending the standard basis to  $m_{i,j}$  with  $i = 1, \dots, n$  and  $j = -l, -l+1, \dots, l$  is an admissible epimorphism.  $\square$

**Proposition 8.8.** Let  $\phi : A \rightarrow B$  be a morphism of  $k$ -affinoid algebras,  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Then the following are equivalent:

- (1)  $\phi$  is finite and admissible.
- (2)  $\phi \hat{\otimes}_k k_r$  is finite and admissible.

This is [Tem04, Lemma 3.2]. I do not understand Temkin's argument. The following proof is a modification of the argument of Temkin's.

PROOF. (1)  $\implies$  (2): This is straightforward.

(2)  $\implies$  (1): The admissible part is straightforward. Let us prove that  $\phi$  is finite. We may assume that  $n = 1$ . When  $r$  is not in  $\sqrt{|k^\times|}$ , we just apply [Proposition 8.7](#). Now suppose  $r \in \sqrt{|k^\times|}$ . Let us take  $m \in \mathbb{Z}_{>0}$  such that  $r^m = |c^{-1}|$  for some  $c \in k^\times$ . Define a bounded  $k$ -algebra homomorphism

$$\varphi : k\{T\} \rightarrow k\{r^{-1}T\}$$

sending  $T$  to  $cT^m$ . Observe that  $\varphi$  is injective. We have argued in the proof of [Lemma 3.15](#) that this homomorphism is finite.

Then  $\varphi$  induces a finite extension of ring  $\text{Frac } k\{r^{-1}T\} / \text{Frac } k\{T\}$ . In particular, the closure of  $\text{Frac } k\{T\}$  in  $k_r$  is a subfield over which  $k_r$  is finite. But this valuation field is isomorphic to  $k\{T\}$ . By [Proposition 8.5](#) and fpqc descent [[Stacks](#), [Tag 02LA](#)], we may assume that  $r = 1$ .

Recall that  $k_1$  is the completion of  $\text{Frac } k\{T\}$ . Let  $\{\tilde{f}_i\}_{i \in I}$  be the set of irreducible monic polynomials in  $\tilde{k}[T]$ . Lift each  $\tilde{f}_i$  to  $f_i \in k[T]$ . Let  $a \in A \hat{\otimes}_k k_1$ , we represent  $a$  as

$$a = \sum_{l=0}^{\infty} a_l T^l + \sum_{i \in I, j \geq 1, 0 \leq k < \deg f_i} a_{ijk} T^k / f_i^j.$$

A similar expression exists for elements in  $B \hat{\otimes}_k k_1$  as well. Moreover, the representation is unique.

As  $B \hat{\otimes}_k k_1$  is finite over  $A \hat{\otimes}_k k_1$ , we can find  $b_1, \dots, b_m$  such that any  $b \in B$  can be written as

$$b = \sum_{j=1}^m \phi \hat{\otimes}_k k_1(a_j) b_j,$$

where  $a_j \in A \hat{\otimes}_k k'$ . We can replace  $b_j$  by  $b_{j,0}$  and  $a_j$  by  $a_{j,0}$ . It follows that  $B$  is generated  $b_{1,0}, \dots, b_{m,0}$  over  $A$ .  $\square$

For any ring  $A$ ,  $\text{Alg}_A^f$  denotes the category of finitely generated  $A$ -algebras.

**Proposition 8.9.** Let  $A$  be a  $k$ -affinoid algebra. Then the forgetful functor  $\text{BanAlg}_A^f \rightarrow \text{Alg}_A^f$  is an equivalence of categories.

Recall that  $\text{BanAlg}_A^f$  is defined in [Definition 5.9](#) in the chapter Banach Rings.

PROOF. It suffices to construct an inverse functor. Let  $B$  be a finite  $A$ -algebra. We endow  $B$  with the norm  $\|\bullet\|_B$  as in [Proposition 8.2](#). We claim that  $B$  is a Banach  $A$ -algebra.

Let us recall the definition of the norm. Take  $n \in \mathbb{N}$ , an epimorphism  $\varphi : A^n \rightarrow B$  of  $A$ -modules. Then  $\|\bullet\|_B$  is the residue norm induced by  $\varphi$ .

Consider the  $A$ -linear epimorphism  $\psi : A^n \otimes_A A^n \rightarrow B \otimes_A B$ . By [Proposition 8.6](#), when both sides are endowed with the norms  $\|\bullet\|_{A^n \otimes_A A^n}$  and  $\|\bullet\|_{B \otimes_A B}$  as in [Proposition 8.2](#),  $\psi$  is admissible. It follows that there is  $C > 0$  such that for any  $f, g \in B$ ,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by [Proposition 8.2](#), the natural map  $B \otimes_A B \rightarrow B$  is bounded. It follows that there is a constant  $C' > 0$  such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in  $B$  is bounded and hence  $B$  is a finite Banach algebra. Given any morphism  $B \rightarrow B'$  in  $\text{Alg}_A^f$ , we endow  $B$  and  $B'$  with the norms given by [Proposition 8.2](#). It follows from [Lemma 8.1](#) that  $B \rightarrow B'$  is a bounded homomorphism of finite Banach  $A$ -algebras. So we have defined an inverse functor to the forgetful functor  $\text{BanAlg}_A^f \rightarrow \text{Alg}_A^f$ .  $\square$

**Remark 8.10.** It is not true that any homomorphism of  $k$ -affinoid algebras is bounded. For example, if the valuation on  $k$  is trivial. Take  $0 < p < q < 1$  and consider the natural homomorphism  $k_p \rightarrow k_q$ . This homomorphism is bijective but not bounded.

## 9. Affinoid domains

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 9.1.** Let  $A$  be a  $k_H$ -affinoid algebra. A closed subset  $V \subseteq \text{Sp } A$  is said to be a  $k_H$ -affinoid domain in  $X$  if there is an object  $\text{Sp } A_V \in k_H\text{-Aff}$  and a morphism  $\phi : \text{Sp } A_V \rightarrow \text{Sp } A$  in  $k_H\text{-Aff}$  such that

- (1) the image of  $\phi$  in  $\text{Sp } A$  is  $V$ ;
- (2) given any object  $\text{Sp } B \in k_H\text{-Aff}$  and a morphism  $\text{Sp } B \rightarrow \text{Sp } A$  whose image lies in  $V$ , there is a unique morphism  $\text{Sp } B \rightarrow \text{Sp } A_V$  in  $k_H\text{-Aff}$  such that the following diagram commutes

$$\begin{array}{ccc} \text{Sp } B & & \\ \downarrow \scriptstyle ! & \searrow & \\ \text{Sp } A_V & \xrightarrow{\phi} & \text{Sp } A \end{array}$$

We say  $V$  is *represented by* the morphism  $\phi$  or by the corresponding morphism  $A \rightarrow A_V$ .

When  $H = \mathbb{R}_{>0}$ , we say  $V$  is a  $k$ -affinoid domain in  $X$ . When  $H = |k^\times|$ , we say  $V$  is a *strict  $k$ -affinoid domain* in  $X$ .

We observe that  $A_V$  is canonically determined by the universal property.

**Remark 9.2.** This definition differs from the original definition of [\[Ber12\]](#), we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the original definition of Berkovich when  $H = \mathbb{R}_{>0}$ .

A priori, this does not seem to be a good definition, as it is not easy to see that it is preserved by base field extension. But we will prove that it is the case after establishing the Gerritzen–Grauert theorem.

We begin with a few examples.

**Example 9.3.** Let  $A$  be a  $k_H$ -affinoid domain. Let  $n, m \in \mathbb{N}$  and  $f = (f_1, \dots, f_n) \in A^n$ ,  $g = (g_1, \dots, g_m) \in A^m$ . Let  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times|} \cdot H^n$  and  $s = (s_1, \dots, s_m) \in \sqrt{|k^\times|} \cdot H^m$ . We define

$$(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\} := \{x \in \mathrm{Sp} A : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

We claim that  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$  is a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ . These domains are called  *$k_H$ -Laurent domains* in  $\mathrm{Sp} A$ . When  $m = 0$ , the domains  $\mathrm{Sp} A \{r^{-1}f\}$  are called  *$k_H$ -Weierstrass domains* in  $\mathrm{Sp} A$ .

To see this, we define

$$A \{r^{-1}f, sg^{-1}\} := A \{r^{-1}T, sS\} / (T_1 - f_1, \dots, T_n - f_n, g_1 S_1 - 1, \dots, g_m S_m - 1).$$

By [Theorem 6.3](#), this defines a Banach  $k$ -algebra structure. We write  $\|\bullet\|'$  for the quotient norm. By definition,  $A \{r^{-1}f, sg^{-1}\}$  is a  $k_H$ -affinoid algebra and there is a natural morphism  $A \rightarrow A \{r^{-1}f, sg^{-1}\}$ . We claim that this morphism represents  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ .

For this purpose, we first compute  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ . We observe that  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\} \rightarrow \mathrm{Sp} A$  is injective since  $A[f, g^{-1}]$  is dense in  $A \{r^{-1}f, sg^{-1}\}$ . We will therefore identify  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$  with a subset of  $\mathrm{Sp} A$ .

Next we show that the image of  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$  in  $\mathrm{Sp} A$  is contained in  $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$ . Take  $\|\bullet\| \in \mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ . Then there is a constant  $C > 0$  such that

$$\|\bullet\| \leq C \|\bullet\|'.$$

Applying this to  $f_i^k$  for some  $k \in \mathbb{Z}_{>0}$  and  $i = 1, \dots, n$ , we find that

$$\|f_i\|^k = \|f_i^k\| \leq C \|f_i^k\|' \leq C \|T_i^k\|_{r, s^{-1}} = C r_i^k.$$

It follows that

$$\|f_i\| \leq r_i.$$

Similarly, we deduce  $|g_j| \geq s_j$  for  $j = 1, \dots, m$ . Namely,  $\|\bullet\| \in (\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$ .

Next we verify the universal property: let  $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a morphism of  $k_H$ -affinoid domains that factorizes through  $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$ . We write  $\psi : A \rightarrow B$  for the corresponding morphism of  $k_H$ -affinoid algebras. By [Corollary 6.12](#) in the chapter Banach Rings, we have

$$\rho_B(f_i) = \sup_{x \in \mathrm{Sp} B} |f_i(x)| \leq \sup_{y \in (\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}} |f_i(y)| \leq r_i$$

for  $i = 1, \dots, n$ . Similarly, one deduces that  $\rho(g_j) \leq s_j^{-1}$  for  $j = 1, \dots, m$ .

We will construct the dotted arrows:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow & \searrow \eta & \uparrow \\ A \{r^{-1}T, sS\}^\tau & & \\ \downarrow & \nearrow & \\ A \{r^{-1}f, sg\} & & \end{array}$$

so that this diagram commutes. We define  $\eta$  as the unique morphism sending  $T_i$  to  $f_i$  and  $S_j$  to  $g_j$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . The existence of such a morphism is guaranteed by [Corollary 6.5](#). In order to descend this morphism to  $\eta'$ , it suffices to

show that  $T_i - f_i$  and  $g_j S_j - 1$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$  lie in the kernel of  $\eta$ . But this is immediate from our definition. Moreover, it is clear that  $\eta'$  is necessarily unique.

It remains to show that each point in  $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$  lies in  $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ .

It suffices to treat the cases  $(n, m) = (1, 0)$  and  $(n, m) = (0, 1)$ . We will only handle the former case, as the latter is similar. In concrete terms, we need to show that for any  $x \in \mathrm{Sp} A$  corresponding to a bounded semi-valuation  $|\bullet|_x$  on  $A$  satisfying  $|f(x)| \leq r$ , we can always extend  $|\bullet|_x$  to a bounded semi-valuation  $\|\bullet\|$  on  $A\{r^{-1}f\}$ . Replacing  $A$  by  $A/\ker |\bullet|_x$ , we may assume that  $|\bullet|_x$  is a valuation on  $A$ . We endow  $A\{r^{-1}T\}$  with the Gauss norm  $\|\bullet\|_{x,r}$  induced by  $|\bullet|_x$  and  $A\{r^{-1}T\}$  with the quotient norm  $\|\bullet\|$ . This norm is bounded by construction. It suffices to show that it is a valuation and it extends the given valuation on  $A$ . The former is a consequence of the latter, as  $A$  is dense in  $A\{r^{-1}T\}$ . Now suppose  $a \in A$ . A general preimage of  $a$  in  $A\{r^{-1}T\}$  is

$$a + (T - f) \sum_{j=0}^{\infty} b_j T^j = a - fb_0 + \sum_{j=1}^{\infty} (b_{j-1} - fb_j) T^j$$

with  $\|b_j\|_A r^j \rightarrow 0$  as  $j \rightarrow \infty$ . Now we compute

$$\begin{aligned} \|a - fb_j + \sum_{j=1}^{\infty} (b_{j-1} - fb_j)\|_{x,r} &= \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |b_{j-1} - fb_j|_x r^j \right\} \\ &\geq \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |b_{j-1} - fb_j|_x |f|_x^j \right\} \\ &= \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |f^j b_{j-1} - f^{j+1} b_j|_x \right\} \geq |a|_x. \end{aligned}$$

So  $\|a\| \geq |a|_x$ . The reverse inequality is trivial. We conclude.

**Example 9.4.** Let  $A$  be a  $k_H$ -affinoid domain. Let  $n \in \mathbb{N}$ ,  $g \in A$ ,  $f = (f_1, \dots, f_n) \in A^n$ ,  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times|} \cdot H^n$ . Assume that  $g, f_1, \dots, f_n$  generates the unit ideal. Define

$$(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\} = \{x \in \mathrm{Sp} A : |f_i(x)| \leq r_i |g(x)| \text{ for } i = 1, \dots, n\}.$$

Then we claim that  $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$  is a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ . Domains of this form are called  *$k_H$ -rational domains*.

To see this, we define

$$A \left\{ r^{-1} \frac{f}{g} \right\} := A\{r^{-1}T\} / (gT_1 - f_1, \dots, gT_n - f_n).$$

By [Theorem 5.1](#), this is indeed a  $k_H$ -affinoid domain. We will denote by  $\|\bullet\|'$  the residue norm. We will prove that the natural map  $A \rightarrow A \left\{ r^{-1} \frac{f}{g} \right\}$  represents the affinoid domain  $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ . Observe that

$$\mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$$

is injective as elements of the form  $a/g$  with  $a \in A$  is dense in  $A \left\{ r^{-1} \frac{f}{g} \right\}$ . Next we show that

$$(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\} \supseteq \mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}.$$

Let  $x \in \mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$ , take  $|\bullet|_x$  as the corresponding bounded semi-valuation on  $A \left\{ r^{-1} \frac{f}{g} \right\}$ . Then there is a constant  $C > 0$  such that for any  $k \in \mathbb{Z}_{>0}$ ,

$$|f_i|_x^k = |f_i^k|_x = |g|_x^k \cdot |T_i^k|_x \leq C |g|_x^k r_i^k.$$

for all  $i = 1, \dots, n$ . In particular,

$$|f_i|_x \leq r_i |g|_x.$$

Hence,  $x \in (\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ .

Next we verify the universal property. Let  $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a morphism of  $k_H$ -affinoid spectra factorizing through  $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ . Observe that  $g(x) \neq 0$  for all  $x \in (\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ . As otherwise,  $f_i(x) = 0$  for all  $i = 1, \dots, n$ . This contradicts our assumption on  $g, f_1, \dots, f_n$ . It follows that  $\psi(g)$  is invertible by [Corollary 6.11](#) in the chapter Banach Rings. From the definition of  $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ , it is clear that  $\rho(\psi(f_i)) \leq r\rho(\psi(g))$  for  $i = 1, \dots, n$ .

We construct

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow & \searrow \eta & \uparrow \\ A\{r^{-1}T\} & \xrightarrow{\tau} & A\{r^{-1}\frac{f}{g}\} \\ \downarrow & \nearrow & \\ A\{r^{-1}\frac{f}{g}\} & & \end{array}$$

successively. The morphism  $\eta$  sends  $T_i$  to  $\psi(f_i)/\psi(g)$  for  $i = 1, \dots, n$ . The existence of such a morphism is guaranteed by [Corollary 6.5](#). Clearly  $gT_i - f_i$  is contained in  $\ker \eta$ , so  $\eta$  descends to  $\tau$ . The morphism  $\tau$  is clearly unique.

It remains to verify that the image of  $\mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$  in  $\mathrm{Sp} A$  is exactly  $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ . In other words, we need to verify that if  $|\bullet|_x$  is a bounded semi-valuation on  $A$  satisfying  $|f_i|_x \leq r_i |g|_x$ , then  $|\bullet|_x$  extends to a bounded semi-valuation on  $A \left\{ r^{-1} \frac{f}{g} \right\}$ . Replacing  $A$  by  $A/\ker |\bullet|_x$ , we may assume that  $|\bullet|_x$  is a valuation on  $A$ . Consider the Gauss valuation  $|\bullet|_{x,r}$  on  $A\{r^{-1}T\}$  and the residue norm  $\|\bullet\|$  on  $A \left\{ r^{-1} \frac{f}{g} \right\}$ . It suffices to show that  $\|\bullet\|$  is a valuation extending the valuation  $|\bullet|_x$  on  $A$ . The former is a consequence of the latter. Take  $a \in A$ , we need to show that  $|a|_x = \|a\|$ .

A general preimage of  $a$  in  $A\{r^{-1}T\}$  has the form

$$a + \sum_{i=1}^n (gT_i - f_i) \sum_{\alpha \in \mathbb{N}^n} b_{i,\alpha} T^\alpha$$

with  $\|b_{i,\alpha}\|_A r^\alpha$ , where  $\|\bullet\|_A$  denotes the initial norm on  $A$ . The same argument as in [Example 9.3](#) shows that

$$\|a + \sum_{i=1}^n (gT_i - f_i) \sum_{\alpha \in \mathbb{N}^n} b_{i,\alpha} T^\alpha\|_{x,r} \geq |a|_x.$$

So  $\|a\|_x \geq |a_x|$ , the reverse inequality is trivial.

**Proposition 9.5.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V \subseteq \mathrm{Sp} A$  be a  $k_H$ -affinoid domain represented by  $\varphi : A \rightarrow A_V$ . Then  $\mathrm{Sp} \varphi$  induces a homeomorphism  $\mathrm{Sp} A_V \rightarrow V$ .

In particular, we will identify  $V$  with  $\mathrm{Sp} A_V$  and say  $\mathrm{Sp} A_V$  is a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ .

PROOF. We observe that  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  is a monomorphism in the category  $k_H\text{-Aff}$ . In other words,  $A \rightarrow A_V$  is an epimorphism in the category  $k_H\text{-AffAlg}$ . To see this, let  $\eta_1, \eta_2 : A_V \rightarrow B$  be two arrows in  $k_H\text{-AffAlg}$  such that  $\eta_1 \circ \varphi = \eta_2 \circ \varphi$ . It follows from the universal property in [Definition 9.1](#) that  $\eta_1 = \eta_2$ . By [Proposition 3.18](#),  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  is a bijection. But  $\mathrm{Sp} A_V$  and  $\mathrm{Sp} A$  are both compact and Hausdorff by [Theorem 6.13](#) in the chapter Banach rings, so  $\mathrm{Sp} A_V \rightarrow V$  is a homeomorphism.  $\square$

**Proposition 9.6.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V, W$  be  $k_H$ -Weierstrass domains (resp.  $k_H$ -Laurent domains, resp.  $k_H$ -rational domains) in  $\mathrm{Sp} A$ . Then  $V \cap W$  is also a  $k_H$ -Weierstrass domain (resp.  $k_H$ -Laurent domain, resp.  $k_H$ -rational domain).

PROOF. This is clear in the Weierstrass and Laurent cases. We will prove therefore assume that  $V$  and  $W$  are  $k_H$ -rational.

We take  $f_1, \dots, f_n \in A$ ,  $g_1, \dots, g_m \in A$  both generating the unit ideal and  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H^n}$ ,  $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H^m}$  such that

$$V = \mathrm{Sp} A \left\{ r^{-1} \frac{f}{f_m} \right\}, \quad W = \mathrm{Sp} A \left\{ s^{-1} \frac{g}{g_n} \right\}.$$

We may assume that  $r_n = s_m = 1$ . Now let  $R = (R_{i,j}) \in \sqrt{|k^\times| \cdot H^{mn}}$  where  $R_{i,j} = r_i s_j$  and  $F = (F_{i,j})$  with  $F_{i,j} = f_i g_j$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . Observe that the  $F_{i,j}$ 's generate the unit ideal. We consider the  $k_H$ -rational domain

$$Z = \mathrm{Sp} A \left\{ R^{-1} \frac{F}{f_n g_m} \right\}.$$

We clearly have  $V \cap W \subseteq Z$ . We need to prove the reverse inequality. Let  $x \in Z$ , so we have

$$|f_i g_j(x)| \leq r_i s_j |f_n g_m(x)|$$

for any  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . In particular, when  $j = m$ , we have

$$|f_i g_m(x)| \leq r_i |f_n g_m(x)|$$

for any  $i = 1, \dots, n$ . But  $f_n g_m$  is invertible, so we can cancel  $g_m(x)$  to find

$$|f_i(x)| \leq r_i |f_n(x)|.$$

So  $x \in V$ . Similarly, we have  $x \in W$ .  $\square$

**Corollary 9.7.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V$  be a  $k_H$ -Laurent domain in  $\mathrm{Sp} A$ . Then  $V$  is also a  $k_H$ -rational domain.

PROOF. By [Proposition 9.6](#), it suffices to show consider  $k_H$ -Laurent domains of the following form:

$$\mathrm{Sp} A\{r^{-1}f\}, \quad \mathrm{Sp} A\{sg^{-1}\}$$

where  $r, s \in \sqrt{|k^\times| \cdot H}$  and  $f, g \in A$ . Both domains are  $k_H$ -rational by definition.  $\square$

**Proposition 9.8.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathrm{Sp} B$  be a  $k_H$ -rational domain in  $\mathrm{Sp} A$ . Then there is a  $k_H$ -Laurent domain  $\mathrm{Sp} C$  in  $\mathrm{Sp} A$  such that  $\mathrm{Sp} B \subseteq \mathrm{Sp} C$  and  $\mathrm{Sp} B$  is a  $k_H$ -Weierstrass domain in  $\mathrm{Sp} C$ .

PROOF. We write

$$B = A \left\{ r^{-1} \frac{f}{g} \right\}$$

for some  $n \in \mathbb{N}$ ,  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H^n}$ ,  $f = (f_1, \dots, f_n) \in A^n$  and  $g \in A$  such that  $f_1, \dots, f_n, g$  generate the unit ideal. Let  $g''$  be the image of  $g$  in  $B$ , which is a unit. Choose  $c \in \sqrt{|k^\times| \cdot H}$  such that  $\rho_B(g^{-1}) < c^{-1}$ . Set  $C = A\{cg^{-1}\}$ , then  $\mathrm{Sp} B \subseteq \mathrm{Sp} C$ . Moreover,

$$\mathrm{Sp} B \cap \mathrm{Sp} C = \emptyset.$$

Let  $f'_1, \dots, f'_n, g'$  be the images of  $f_1, \dots, f_n, g$  in  $C$ . Write  $f' = (f'_1, \dots, f'_n)$ . Then by [Corollary 6.11](#) in the chapter Banach Rings,  $g'$  is a unit and

$$\mathrm{Sp} B = \mathrm{Sp} C\{r^{-1}g'^{-1}f'\}.$$

$\square$

**Proposition 9.9.** Let  $A$  be a  $k_H$ -affinoid algebra,  $\mathrm{Sp} B$  be a  $k_H$ -Weierstrass domain (resp.  $k_H$ -rational domain) in  $\mathrm{Sp} A$  and  $\mathrm{Sp} C$  be a  $k_H$ -Weierstrass domain (resp.  $k_H$ -rational domain) in  $\mathrm{Sp} B$ . Then  $\mathrm{Sp} C$  is a  $k_H$ -Weierstrass domain (resp.  $k_H$ -rational domain) in  $\mathrm{Sp} A$ .

PROOF. We first handle the Weierstrass case. Write

$$B = \mathrm{Sp} A\{r^{-1}f\}, C = \mathrm{Sp} B\{s^{-1}g\}$$

for some  $n, m \in \mathbb{N}$ ,  $r \in \sqrt{|k^\times| \cdot H^n}$ ,  $s \in \sqrt{|k^\times| \cdot H^m}$  and  $f = (f_1, \dots, f_n) \in A^n$ ,  $g = (g_1, \dots, g_m) \in B^m$ . Observe that if we replace  $g$  with a small perturbation, the domain  $\mathrm{Sp} C$  in  $\mathrm{Sp} B$  remains the same, so we may assume that  $g_1, \dots, g_m \in A$ . Then

$$\mathrm{Sp} C = \mathrm{Sp} A\{r^{-1}f\} \cap \mathrm{Sp} A\{s^{-1}g\}$$

is a  $k_H$ -Weierstrass domain by [Proposition 9.6](#).

Next we handle the rational case. Write

$$B = A \left\{ s^{-1} \frac{f}{g} \right\}$$

for some  $m \in \mathbb{N}$ ,  $f = (f_1, \dots, f_m) \in A^m$ ,  $r = (r_1, \dots, r_m) \in \sqrt{|k^\times| \cdot H^m}$  and  $g \in A$  such that  $f_1, \dots, f_m, g$  generate the unit ideal.

By [Proposition 9.8](#) and [Proposition 9.6](#), it suffices to handle the special cases  $C = B\{r^{-1}h\}$  and  $C = B\{rh^{-1}\}$  for some  $r \in \sqrt{|k^\times| \cdot H}$  and  $h \in B$ . Observe that making a small perturbation on  $h$  does not change the domain. As  $A[g^{-1}]$  is



dense in  $B$ , we may assume that there is  $n \in \mathbb{Z}_{>0}$  such that  $h' = g^n h \in A$ . As  $g$  is invertible on  $\mathrm{Sp} B$ , we can find  $c \in \sqrt{|k^\times| \cdot H}$  so that

$$|g(x)|^n > c^{-1}$$

for  $x \in \mathrm{Sp} B$ .

We need to treat the cases  $C = B\{r^{-1}h\}$  and  $C = B\{rh^{-1}\}$  separately. In the first case, we write

$$\mathrm{Sp} C = \mathrm{Sp} B \cap \mathrm{Sp} A \left\{ (r, c)^{-1} \frac{(h', 1)}{g^n} \right\}.$$

In the second case,

$$\mathrm{Sp} C = \mathrm{Sp} B \cap \mathrm{Sp} A \left\{ (r, c)^{-1} \frac{(g^n, 1)}{h'} \right\}.$$

□

**Lemma 9.10.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathrm{Sp} B$  be a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ . Let  $\mathrm{Sp} C$  be a rational domain in  $\mathrm{Sp} A$ , then  $(\mathrm{Sp} C) \cap (\mathrm{Sp} B)$  is a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$  represented by  $A \rightarrow B \hat{\otimes}_A C$ .

PROOF. We first recall that  $B \hat{\otimes}_A C$  is  $k_H$ -affinoid by [Proposition 3.4](#).

We may assume that

$$C = A \left\{ s \frac{f}{g} \right\}$$

for some  $m \in \mathbb{N}$ ,  $f = (f_1, \dots, f_m) \in A^m$ ,  $r = (r_1, \dots, r_m) \in \sqrt{|k^\times| \cdot H^m}$  and  $g \in A$  such that  $f_1, \dots, f_m, g$  generate the unit ideal.

We observe that there is a natural isomorphism

$$B \hat{\otimes}_A C \cong B \left\{ s^{-1} \frac{f}{g} \right\}.$$

Hence,

$$\mathrm{Sp} B \hat{\otimes}_A C = \{x \in \mathrm{Sp} B : |f_i(x)| \leq s|g(x)| \text{ for } i = 1, \dots, m\}.$$

On the other hand,

$$\mathrm{Sp} C = \{x \in \mathrm{Sp} A : |f_i(x)| \leq s|g(x)| \text{ for } i = 1, \dots, m\}.$$

So  $\mathrm{Sp} B \hat{\otimes}_A C = B \hat{\otimes}_A C$ . By [Proposition 3.4](#), we have the Cartesian square in the diagram below:

$$\begin{array}{ccc} \mathrm{Sp} D & & \\ \swarrow \text{dotted} & \searrow & \\ & \mathrm{Sp} B \hat{\otimes}_A C & \longrightarrow \mathrm{Sp} C \\ & \downarrow & \downarrow \\ & \mathrm{Sp} B & \longrightarrow \mathrm{Sp} A \end{array} \quad \square$$

It remains to verify the universal property. Let  $\mathrm{Sp} D \rightarrow \mathrm{Sp} C$  be a morphism of  $k_H$ -affinoid spectra that factorizes through  $(\mathrm{Sp} C) \cap (\mathrm{Sp} B)$ . Then by the universal property of  $\mathrm{Sp} B$  in  $\mathrm{Sp} A$ , we find the dotted morphism  $\mathrm{Sp} D \rightarrow \mathrm{Sp} B$  making the diagram commutes. Then as the square is Cartesian, we get the desired morphism  $\mathrm{Sp} D \rightarrow \mathrm{Sp} B \hat{\otimes}_A C$ . This morphism is clearly unique. □

## 10. Graded reduction

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 10.1.** Let  $A$  be a Banach  $k$ -algebra, we define the *graded reduction* of  $A$  as

$$\tilde{A} := \bigoplus_{h \in \mathbb{R}_{>0}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any  $f \in A$  with  $\rho(f) \neq 0$ , we define  $\tilde{f}$  as the image of  $f$  in the  $\rho(f)$ -graded piece of  $\tilde{A}$ .

**Definition 10.2.** Let  $A$  be a  $k_H$ -affinoid algebra. We define the  *$k_H$ -graded reduction* of  $A$  as the  $\sqrt{|k^\times|} \cdot H$ -graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times|} \cdot H} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any  $f \in A$  with  $\rho(f) \neq 0$ , we define  $\tilde{f}$  as the image of  $f$  in the  $\rho(f)$ -graded piece of  $\tilde{A}^H$ .

For any morphism  $f : A \rightarrow B$  of  $k_H$ -affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \rightarrow \tilde{B}^H$$

as the map induced by sending the class of  $x \in A$  with  $\rho(x) \leq h$  for any  $h \in \sqrt{|k^\times|} \cdot H$  to the class of  $f(x) \in B$ .

Recall that  $\rho(A) = \sqrt{|k^\times|} \cdot H \cup \{0\}$  by [Theorem 7.4](#), so  $\tilde{f}$  is well-defined. This definition is compatible with [Definition 10.1](#) in the sense that if we regard a  $\sqrt{|k^\times|} \cdot H$ -graded ring as a  $\mathbb{R}_{>0}$ -graded ring, the two definitions give the same object.

**Example 10.3.** If  $K$  is a  $k_H$ -affinoid algebra which is a field as well, then  $\tilde{K}^H$  is a  $\sqrt{|k^\times|} \cdot H$ -graded field. This is immediate from the definition.

**Lemma 10.4.** Let  $(A, |\bullet|)$  be a  $k$ -affinoid algebra,  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$ . Let  $f \in k\{r^{-1}T\}$ . Expand  $f$  as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha.$$

**PROOF.** By induction, we may assume that  $n = 1$  and write  $r = r_1$ . As  $\rho$  is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j) \rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j.$$

Observe that  $\rho(a_j)$  is not ambiguous: when interpreted as in  $A$  and in  $A\{r^{-1}T\}$ , it has the same value.

Conversely, we need to show that for any  $j \in \mathbb{N}$ ,

$$\rho(f) \geq \rho(a_j) r^j.$$

Equivalently, this means for any  $k \in \mathbb{Z}_{>0}$  and any  $j \in \mathbb{N}$ , we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk}.$$

Fix  $j$  and  $k$  as above. We compute the left-hand side:

$$f^k = \sum_{\beta=(\beta_1, \dots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^k a_{\beta_l}.$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|}.$$

Take  $\beta = (j, j, \dots, j)$ , we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk}.$$

□

**Lemma 10.5.** Assume that  $k$  is non-trivially valued. Let  $A$  be a strictly  $k$ -affinoid algebra. Then for any  $a, f \in A$ , the set of non-zero values  $\rho(f^n a)$  for  $n \in \mathbb{N}$  is a discrete subset of  $\mathbb{R}_{>0}$ .

PROOF. As  $A$  is noetherian [Theorem 6.3](#), it has only finitely many minimal prime ideals, say  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ . It follows that

$$\mathrm{Sp} A = \bigcup_{i=1}^m \mathrm{Sp} A/\mathfrak{p}_i.$$

Here we make the obvious identification by identifying  $\mathrm{Sp} A/\mathfrak{p}_i$  with a subset of  $\mathrm{Sp} A$ .

By [Corollary 6.12](#) in the chapter Banach Rings, it suffices to consider each of  $\mathrm{Sp} A/\mathfrak{p}_i$  separately, so we may assume that  $A$  is an integral domain.

By [Corollary 5.2](#), we can take  $d \in \mathbb{N}$  and a finite injective homomorphism of  $k$ -algebras  $\iota : k\{T_1, \dots, T_d\} \rightarrow A$ . According to [Proposition 9.11](#) in the chapter Banach Rings,  $\rho_A$  is the restriction of the norm  $\|\bullet\|_{\mathrm{Frac} A}$  on  $\mathrm{Frac} A$  induced by the finite extension  $\mathrm{Frac} A/\mathrm{Frac} k\{T_1, \dots, T_d\}$  from the Gauss valuation. But it is well-known that  $\|\bullet\|_{\mathrm{Frac} A}$  is the maximum of finitely many valuations on  $\mathrm{Frac} A$ . [Reproduce BGR3.3.3.1 somewhere](#). The assertion is by now obvious. □

**Lemma 10.6.** Let  $(A, \|\bullet\|)$  be a  $k$ -affinoid algebra,  $f \in A$  with  $r = \rho(f) > 0$ . Let  $B = A\{r^{-1}f\}$ . Then for any  $a \in A$ , we have

$$\rho_B(a) = \lim_{n \rightarrow \infty} r^{-n} \rho_A(f^n a).$$

If moreover,  $\rho_B(a) > 0$ , then there is  $n_0 > 0$  such that for  $n \geq n_0$ ,

$$\rho_B(a) = r^{-n} \rho_A(f^n a), \quad \rho_B(f^n a) = r^{-n} \rho_A(a).$$

PROOF. We observe that for any  $a \in A$ ,  $n \in \mathbb{Z}_{>0}$ , we have

$$\rho_B(f^n a) = r^n \rho_B(a).$$

So the last two assertions are equivalent.

Take a  $k$ -free polyray  $s$  such that  $A \hat{\otimes}_k k_s$  and  $B \hat{\otimes}_k k_s$  are both strictly  $k_s$ -affinoid. By [Proposition 3.11](#),  $A \hat{\otimes}_k k_s \{r^{-1}f\} \xrightarrow{\sim} B \hat{\otimes}_k k_s$ . Moreover,  $\rho_A$  and  $\rho_B$  are both preserved after base change to  $k_s$ . So we may assume that  $k$  is non-trivially valued and  $A$  and  $B$  are strictly  $k$ -affinoid.

Observe that for  $n \in \mathbb{Z}_{>0}$ ,

$$\rho_A(f^{n+1}a) \leq \rho_A(f) \rho_A(f^n a) = r \rho_A(f^n a).$$

So  $r^{-n}\rho_A(f^n a)$  is decreasing in  $n$ . Moreover, for any  $x \in \operatorname{Sp} A\{r^{-1}f\}$ , by [Example 9.3](#), we have

$$|f(x)| \geq r.$$

By [Corollary 6.12](#) in the chapter Banach Rings, we have

$$|f(x)| = r$$

for any  $x \in \operatorname{Sp} A\{r^{-1}f\}$ . It follows from [Corollary 6.12](#) in the chapter Banach Rings that for any  $n \in \mathbb{Z}_{>0}$ ,

$$\rho_A(f^n a) = \sup_{x \in \operatorname{Sp} A} |f^n a(x)| \geq r^n \sup_{x \in \operatorname{Sp} A\{rf^{-1}\}} |a(x)| = r^n \rho_B(a).$$

By [Lemma 10.5](#), the decreasing sequence  $\{r^{-n}\rho_A(f^n a)\}_n$  either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is  $\alpha \in \mathbb{R}_{>0}$  and  $n_0 > 0$  such that for  $n \geq n_0$ , we have

$$r^{-n}\rho_A(f^n a) = \alpha.$$

We have to show that  $\alpha \leq \rho_B(a)$ . Assume the contrary  $\alpha > \rho_B(a)$ . Then for all  $x \in \operatorname{Sp} A$ , we have

$$|f^n a(x)| \leq r^n |a(x)|.$$

So  $f^n a$  must obtain its maximum on  $U := \{x \in \operatorname{Sp} A : |a(x)| \geq \alpha\}$ . But  $U$  is disjoint from  $\operatorname{Sp} A\{r^{-1}f\}$  as

$$\alpha > \rho_B(a).$$

It follows from [Example 9.3](#) that

$$\beta := \sup_{x \in U} |f(x)| = \max_{x \in U} |f(x)| < r.$$

So

$$\rho(f^n a) = \sup_{x \in \operatorname{Sp} A} |f^n a(x)| = \sup_{x \in U} |f^n a(x)| \leq \beta^n \sup_{x \in U} |a(x)|.$$

This contradicts the fact that  $\alpha > 0$ . □

**Proposition 10.7.** Let  $A$  be a  $k_H$ -affinoid algebra and  $r \in \mathbb{R}_{>0}^n$ , then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of  $\sqrt{|k^\times|} \cdot H$ -graded rings.

Recall that  $k_r$  is defined in [Example 3.12](#).

PROOF. By [Lemma 10.4](#), we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}_s^H \xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{N}^n} \tilde{A}_{sr^{-\alpha}}^H$$

for any  $s \in \sqrt{|k^\times|} \cdot H$ . This establishes the desired isomorphism. □

**Proposition 10.8.** Let  $A$  be a  $k_H$ -affinoid algebra and  $f \in A$  with  $r = \rho(f) > 0$ . Then there is a natural isomorphism

$$\tilde{A}_f^H \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of  $\sqrt{|k^\times|} \cdot H$ -graded rings.

Recall that  $A\{rf^{-1}\}$  is defined in [Example 9.3](#), by [Theorem 7.4](#), it is  $k_H$ -affinoid.

PROOF. Let  $B = A\{rf^{-1}\}$  and denote by  $\phi : \tilde{A}^H \rightarrow \tilde{A}_{\tilde{f}}^H$  the natural  $\sqrt{|k^\times|} \cdot \bar{H}$ -graded homomorphism. From the universal property [add details](#), we can factor the natural map  $\tilde{A}^H \rightarrow \tilde{B}^H$  as  $\psi : \tilde{A}_{\tilde{f}}^H \rightarrow \tilde{B}^H$ . We have a commutative diagram:

$$\begin{array}{ccc} \tilde{A}^H & \longrightarrow & \tilde{B}^H \\ \phi \downarrow & \nearrow \psi & \\ \tilde{A}_{\tilde{f}}^H & & \end{array}$$

We claim that  $\psi$  is bijective. Let  $\tilde{a}/\tilde{f}^m$  be an element in  $\ker \psi$ , where  $\tilde{a} \in \tilde{A}^H$  is homogeneous. Lift  $\tilde{a}$  to  $a \in A$ . Then  $\rho_B(a) < \rho_A(a)$ . By [Lemma 10.6](#),  $\rho_A(f^n a) < r^n \rho_A(a)$  when  $n$  is large enough, so

$$\tilde{f}^n \tilde{a} = 0$$

in  $\tilde{A}$ . Therefore,  $\tilde{a}/\tilde{f}^m = 0$  in  $\tilde{A}_{\tilde{f}}^H$ . We have shown that  $\psi$  is injective.

It remains to show that  $\psi$  is surjective. Let  $\tilde{b} \in \tilde{B}^H$  be a non-zero homogeneous element. Lift  $\tilde{b}$  to  $b \in B$  of the form  $f^{-n}a$  for some  $a \in A$ . By [Lemma 10.6](#) again, up to enlarging  $n$ , we can assume that  $\rho_B(a) = \rho_A(a)$ . Then  $\tilde{a} = \tilde{f}^n \tilde{b}$  has a preimage in  $\tilde{A}$ . □

**Corollary 10.9.** Let  $A$  be a  $k_H$ -affinoid algebra and  $r \in \mathbb{R}_{>0}^n$ , then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}_H} \tilde{k}_r^H \cong \widetilde{A \hat{\otimes}_k k_r}^H$$

of  $\sqrt{|k^\times|} \cdot \bar{H}$ -graded rings.

PROOF. We can write

$$A \hat{\otimes}_k k_r = \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}.$$

Taking graded reduction, we find

$$\begin{aligned} \widetilde{A \hat{\otimes}_k k_r}^H &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \widetilde{A\{r^{-1}T\}\{\rho(g)g^{-1}\}}^H \\ &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \widetilde{A\{r^{-1}T\}}_{\tilde{g}}^H \\ &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \tilde{A}^H[r^{-1}T]_{\tilde{g}} \\ &= \tilde{A}^H \otimes_{\tilde{k}_H} \tilde{k}_r^H. \end{aligned}$$

Here we have applied [Proposition 10.8](#) in the second equality and [Proposition 10.7](#) in the third equality. The first equality follows from the simple observation that graded reduction commutes with filtered colimits. □

**Theorem 10.10.** Let  $\phi : A \rightarrow B$  be a morphism of  $k_H$ -affinoid algebras. Then the following are equivalent:

- (1)  $\phi$  is finite and admissible.
- (2)  $\tilde{\phi} : \tilde{A}^H \rightarrow \tilde{B}^H$  is finite.

PROOF. Take  $n \in \mathbb{N}$  and  $r \in \mathbb{R}_{>0}^n$  so that

$$\rho(A \hat{\otimes}_k k_r) = \rho(B \hat{\otimes}_k k_r) = |k_r|$$

and  $k_r$  is non-trivially valued. **Proof that this is possible.**

By ?? in the chapter Commutative Algebra and **Proposition 8.8**, we may assume that  $k$  is non-trivially valued and  $\rho(A) = \rho(B) = |k|$ . By ?? in the chapter Commutative Algebra, we have  $\tilde{A} = \tilde{A}_1 \otimes_{\tilde{k}_1} \tilde{k}$ . By **Corollary 5.5**,  $\phi$  is automatically admissible if it is finite.

So it suffices to argue that  $\phi$  is finite if and only if  $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$  is finite.

Assume that  $\varphi$  is finite. We show that  $\tilde{\varphi}$  is finite.

First consider the case where  $A$  is an integral domain.

We claim that there is  $d \in \mathbb{N}$  and a  $k$ -algebra homomorphism  $\psi : k\{T_1, \dots, T_d\} \rightarrow A$  such that  $\phi \circ \psi$  is finite and injective. In fact, choosing an epimorphism  $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$ , we can apply **Theorem 5.1** to find  $\phi \circ \alpha$  to conclude.

It suffices to show that  $\widetilde{\phi \circ \psi}$  is finite in order to conclude that  $\tilde{\phi}$  is finite. So we are reduced to the case  $A = k\{T_1, \dots, T_d\}$  and  $\ker \phi = 0$ .

We will show that the conditions of **Lemma 10.1** in the chapter Banach Rings is satisfied with  $\rho_B$  as the norm  $B$ . We have shown that  $\rho_B$  is a faithful  $k\{T_1, \dots, T_d\}$ -algebra norm in **Corollary 4.16**. As  $B$  is of finite over  $k\{T_1, \dots, T_d\}$ , the rank condition is clearly satisfied. It remains to establish that  $\phi$  is integral.

By **Proposition 5.12**, for  $f \in B$ , there is an integral equation

$$f^n + \phi(a_1)f^{n-1} + \dots + \phi(a_n) = 0$$

over  $A$  such that  $\rho_B(f) = \max_{i=1, \dots, n} |b_i|_{\sup}^{1/i}$ . If  $f \in \mathring{B}$ , then  $|b_i|_{\sup} \leq 1$ , hence  $b_i \in \mathring{B}$ . **Add a ref**

Conversely, assume that  $\tilde{\phi}$  is finite. It suffices to apply **Lemma 5.15** to conclude that  $\phi$  is finite.  $\square$

**Corollary 10.11.** Let  $A$  be a  $k_H$ -affinoid algebra, then  $\tilde{A}^H$  is finitely generated over  $\tilde{k}^H$ .

PROOF. Take  $n \in \mathbb{N}$ ,  $r \in \mathbb{R}_{>0}^n$  and an admissible epimorphism

$$\pi : k\{r^{-1}T\} \rightarrow A.$$

Applying **Theorem 10.10**, we find that it suffices to prove that  $\widetilde{k\{r^{-1}T\}}^H$  is finitely generated over  $\tilde{k}^H$ . But this follows from **Proposition 10.7**.  $\square$

**Definition 10.12.** Let  $A$  be a  $k_H$ -affinoid algebra, we define the *reduction map*

$$\mathrm{Sp} \tilde{A}^H := \mathrm{Spec} \sqrt{|k^\times| \cdot H} \tilde{A}^H.$$

We have a natural map  $\pi^H : \mathrm{Sp} A \rightarrow \mathrm{Sp} \tilde{A}^H$ .

## 11. Gerritzen–Grauert theorem

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 11.1.** Let  $A$  be a  $k_H$ -affinoid algebra. A morphism  $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  in  $k_H\text{-Aff}$  is a *closed immersion* if the corresponding morphism  $A \rightarrow B$  in  $k_H\text{-AffAlg}$  is an admissible epimorphism.

**Definition 11.2.** Let  $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a morphism in  $k_H\text{-Aff}$ . We call  $\varphi$  a  *$k_H$ -Runge immersion* if there is a factorization in  $k_H\text{-Aff}$  of  $\varphi$ :

$$\mathrm{Sp} B \rightarrow \mathrm{Sp} C \rightarrow \mathrm{Sp} A,$$

such that  $\mathrm{Sp} B \rightarrow \mathrm{Sp} C$  is a closed immersion and  $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$  is a  $k_H$ -Weierstrass domain.

Add a prop rational domains form basis

**Lemma 11.3.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V$  be a  $k_H$ -Laurent domain in  $\mathrm{Sp} A$  represented by  $A \rightarrow B = A\{r^{-1}f, sg\}$  for some  $n, m \in \mathbb{N}$ ,  $f = (f_1, \dots, f_n) \in A^n$  and  $g = (g_1, \dots, g_m) \in A^m$ ,  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$  and  $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H}^m$ . Then

- (1)  $\tilde{B}^H$  is finite over the subalgebra generated by  $\tilde{A}^H$  and  $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1^{-1}, \dots, \tilde{g}_m^{-1}$ ;
- (2) if  $V$  is a neighbourhood of a point  $x \in \mathrm{Sp} A$ , then  $\tilde{\chi}_x(\tilde{B}^H)$  is finite over  $\tilde{\chi}_x(\tilde{A}^H)$ .

PROOF. (1) Consider the admissible epimorphism

$$A\{r^{-1}T, sS\} \rightarrow B.$$

By [Theorem 10.10](#), it induces a finite homomorphism

$$A\{\widetilde{r^{-1}T}, \widetilde{sS}\}^H \rightarrow \tilde{B}^H.$$

The former is computed in [Proposition 10.7](#) and our assertion follows.  $\square$

(2) This is a special case of (1).

**Theorem 11.4** (Gerritzen–Grauert, Temkin). Let  $\varphi : \mathrm{Sp} A \rightarrow \mathrm{Sp} B$  be a monomorphism in  $k_H\text{-Aff}$ . Then there is a finite cover of  $X$  by  $k_H$ -rational domains  $W_1, \dots, W_k$  such that the restrictions  $\varphi_i : \varphi^{-1}(W_i) \rightarrow W_i$  are  $k_H$ -Runge immersions for  $i = 1, \dots, k$ .

PROOF. **Step 1.** We reduce to the following claim: for each  $x \in \mathrm{Sp} A$ , there is a  $k_H$ -rational domain  $U$  in  $\mathrm{Sp} B$  containing  $y = \varphi(x)$  such that  $V = \varphi^{-1}U$  is a neighbourhood of  $x$  in  $\mathrm{Sp} A$  and the induced map  $V \rightarrow U$  is a closed immersion.

Assume this holds. Write  $U = \mathrm{Sp} B \left\{ r \frac{f}{g} \right\}$  for some  $n \in \mathbb{N}$ ,  $f = (f_1, \dots, f_n) \in B^n$  and  $g \in B$  such that  $f_1, \dots, f_n, g$  generates the unit ideal and  $r \in \sqrt{|k^\times| \cdot H}^n$ . As  $g$  is invertible on  $U$ , we can find a small  $k_H$ -rational domain  $W$  in  $\mathrm{Sp} B$  containing  $y$  such that

- (1)  $g$  is invertible on  $W$ ;
- (2)  $\varphi^{-1}W \subseteq \varphi^{-1}U$ .

Then  $U \cap W$  is a  $k_H$ -Weierstrass domain in  $W$  and  $\varphi^{-1}W \rightarrow W$  is therefore a  $k_H$ -Runge immersion. From the compactness of  $\mathrm{Sp} A$ , this implies that we can find  $k_H$ -rational domains  $W_1, \dots, W_m$  of  $\mathrm{Sp} B$  such that  $\varphi^{-1}(W_i) \rightarrow W_i$  is a  $k_H$ -Runge immersion for  $i = 1, \dots, m$  and  $X_1 \cup \dots \cup X_m$  contains an open neighbourhood  $U$  of  $\varphi(\mathrm{Sp} A)$ . As  $\mathrm{Sp} B$  is compact, we can find finitely many  $k_H$ -rational domains  $W_{m+1}, \dots, W_k$  which do not intersect  $\varphi(\mathrm{Sp} A)$  that covers  $\mathrm{Sp} B \setminus U$ . Then the covering  $W_1, \dots, W_k$  satisfies all the requirements.

We have reduced the problem to a local one on  $\mathrm{Sp} B$ .

**Step 2.** We show that we may assume that  $\tilde{\chi}_x(\tilde{A}^H)$  is finite over  $\tilde{\chi}_y(\tilde{B}^H)$ . Here the notation  $\tilde{\chi}_y$  is defined in [Definition 6.7](#) in the chapter Banach Rings.

By [Corollary 10.11](#),  $\widetilde{\chi}_x(\tilde{A}^H)$  is finitely generated over  $\widetilde{\chi}_y(\tilde{B}^H)$ . Take generators  $h_1, \dots, h_l \in A$ . By [Proposition 3.18](#),  $\mathcal{H}(x) \xrightarrow{\sim} \mathcal{H}(y)$ , so we can find  $f_1, \dots, f_l, g \in B$  with  $|g(y)| = 1$  such that

$$\left| \left( \frac{f_i}{g} - h_i \right) (x) \right| < \rho(h_i)$$

for all  $i = 1, \dots, l$ .

In fact, we can take  $g = 1$ . This can be seen as follows. Let  $B' = B\{ag^{-1}\}$  for some  $a \in \sqrt{|k^\times| \cdot H}$  with  $a < 1$ . Then by [Lemma 11.3](#),  $\tilde{\chi}_y(\tilde{B}'^H)$  is finite over  $\tilde{\chi}_y(\tilde{B}^H)$ . So up to replacing  $B$  by the  $B'$  and  $\mathrm{Sp} A$  by the inverse image of  $\mathrm{Sp} B'$ , we may assume that  $g$  is invertible. Replacing  $f_i$  by  $f_i/g$ , we could then assume that  $g = 1$ .

Up to replacing  $\mathrm{Sp} B$  by  $\mathrm{Sp} B\{\rho(h_1)^{-1}f_1, \dots, \rho(h_l)^{-1}f_l\}$ , we can guarantee that  $\tilde{f}_i = \tilde{h}_i$  for  $i = 1, \dots, l$ . So our assertion follows.

**Step 3.** We may assume that  $\widetilde{\chi}_{x'}(\tilde{A}^H)$  is finite over  $\widetilde{\chi}_{y'}(\tilde{B}^H)$  for any  $x' \in \mathrm{Sp} A$  and  $y' = \varphi(x')$ .

Let  $\pi : \mathrm{Sp} A \rightarrow \widetilde{\mathrm{Sp}} \tilde{A}^H$  be the reduction map. Let  $\mathcal{X}$  denote the Zariski closure of  $\pi(x)$ . Then for any  $x' \in \mathrm{Sp} A$  with  $\pi(x') \in \mathcal{X}$ , we have

$$\ker \widetilde{\chi}_x \subseteq \ker \widetilde{\chi}_{x'}.$$

It follows that  $\widetilde{\chi}_{x'}(\tilde{A}^H)$  is finite over  $\widetilde{\chi}_{y'}(\tilde{B}^H)$ .

Since  $\pi^{-1}\mathcal{X}$  is open in  $\mathrm{Sp} A$  [Include the proof](#), we can find a  $k_H$ -Laurent neighbourhood  $\mathrm{Sp} B\{rf, sg^{-1}\}$  for soem suitable tuples  $r, f, s, g$  of  $y$  such that  $\varphi^{-1}\mathrm{Sp} B\{rf, sg^{-1}\} \subseteq \pi^{-1}\mathcal{X}$ . Observe that for each  $x' \in \mathrm{Sp} A$ ,  $\widetilde{\chi}_{x'}(\tilde{A}^H)$  is finite over  $\widetilde{\chi}_{y'}(\tilde{B}^H)$ . This follows simply from [Lemma 11.3](#). So up to replacing  $B$  with  $B\{rf, sg^{-1}\}$ , we conclude.

**Step 4.** We claim that after all of these reductions,  $\varphi$  becomes a closed immersion. By our assumptions, for any minimal homogeneous prime ideal  $\mathfrak{p}$  of  $\tilde{A}^H$ , there is a point  $x \in \mathrm{Sp} A$  with  $\ker \widetilde{\chi}_x = \mathfrak{p}$  and  $\tilde{A}^H/\mathfrak{p}$  is finite over  $\tilde{A}^H$ .

Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_k$  be the list of minimal homogeneous prime ideals of  $\tilde{A}^H$  [prove finiteness](#), then

$$\tilde{A}^H \rightarrow \bigoplus_{i=1}^k \tilde{A}^H/\mathfrak{p}_i$$

is injective. Since  $\tilde{B}^H$  is graded noetherian [Introduce this notion](#), we find that  $\tilde{A}^H$  is finite over  $\tilde{B}^H$ . So  $B \rightarrow A$  is finite by [Theorem 10.10](#). It follows that the natural map  $A \otimes_B A \rightarrow A \hat{\otimes}_B A$  is an isomorphism by [Proposition 8.4](#). As  $\varphi$  is a monomorphism, from general abstract nonsense, the codiagonal  $A \hat{\otimes}_B A \xrightarrow{\sim} A$  is an isomorphism. In particular, the codiagonal  $A \otimes_B A \rightarrow A$  is an isomorphism. This implies that  $A \rightarrow B$  is surjective.  $\square$

**Lemma 11.5.** Let  $A$  be a  $k_H$ -affinoid domain and  $V$  be a  $k_H$ -affinoid domain in  $A$  represented by  $A \rightarrow A_V$ . Assume that  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  is a closed immersion, then  $V$  is a  $k_H$ -Weierstrass domain.

PROOF. As  $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$  is a closed immersion, we can find an ideal  $I \subseteq A$  and assume that  $A_V = A/I$ . Consider the morphism of  $k_H$ -affinoid spectra  $\psi : \mathrm{Sp} A/I^2 \rightarrow \mathrm{Sp} A$  induced by the natural map  $A/I^2$ . By the universal property of  $V$ ,



we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp} A/I^2 & & \\ \downarrow & \searrow & \\ \mathrm{Sp} A/I & \longrightarrow & \mathrm{Sp} A \end{array}$$

On the other hand, the natural map  $A/I^2 \rightarrow A/I$  induces a morphism of  $k_H$ -affinoid spectra  $\varphi : \mathrm{Sp} A/I \rightarrow \mathrm{Sp} A/I^2$ . From the universal property again, the composition  $\psi \circ \varphi$  is the identity. In particular,  $A/I^2 \rightarrow A/I$  is injective and hence  $I = I^2$ . It follows that  $I$  is the principal ideal generated by an idempotent element  $e$ . We may assume that  $e \neq 0$ ,  $e \neq 1$ . Take  $c \in \sqrt{|k^\times| \cdot H}$  such that  $0 < c < 1$ , then  $V = (\mathrm{Sp} A)\{c^{-1}e\}$ .  $\square$

**Corollary 11.6.** Let  $A$  be a  $k_H$ -affinoid algebra and  $V$  be a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ . Then there are finitely many  $k_H$ -affinoid domains  $W_1, \dots, W_n$  in  $\mathrm{Sp} A$  such that

$$V = \bigcup_{i=1}^n W_i.$$

PROOF. By [Theorem 11.4](#), we can find finitely many  $k_H$ -rational domains  $U_1, \dots, U_m$  in  $\mathrm{Sp} A$  such that  $V \cap U_i \rightarrow U_i$  is a  $k_H$ -Runge immersion for each  $i = 1, \dots, m$ . By [Proposition 9.9](#), it suffices to prove that  $V \cap U_i$  is a  $k_H$ -rational domain in  $U_i$ . Observe that  $V \cap U_i$  is a  $k_H$ -affinoid domain in  $U_i$  by [Lemma 9.10](#). So we are reduced to the case where  $V \rightarrow \mathrm{Sp} A$  is also a Runge immersion.

By [Lemma 9.10](#) and [Proposition 9.9](#) again, we may assume that  $V \rightarrow \mathrm{Sp} A$  is a Runge immersion.

In this case, the result follows from [Lemma 11.5](#).  $\square$

## 12. Tate acyclicity theorem

Let  $(k, |\bullet|)$  be a complete non-Archimedean valued field and  $H$  be a subgroup of  $\mathbb{R}_{>0}$  such that  $|k^\times| \cdot H \neq \{1\}$ .

**Definition 12.1.** Let  $A$  be a  $k_H$ -affinoid algebra. Let  $\mathcal{V} = \{V_i\}_{i \in 1, \dots, n}$  be a finite covering of  $\mathrm{Sp} A$  by  $k_H$ -affinoid domains. Let  $M$  be an  $A$ -module. We define the *augmented Čech complex*  $\check{C}(\mathcal{V}, M)$  as the following cochain complex with  $M$  placed at the place 0:

$$\check{C}(\mathcal{V}, M) = 0 \rightarrow M \rightarrow \prod_{i=1}^n M \otimes_A A_{V_i} \rightarrow \prod_{1 \leq i < j \leq n} M \otimes_A A_{V_i} \hat{\otimes}_A A_{V_j} \rightarrow \dots$$

**Definition 12.2.** Let  $A$  be a  $k_H$ -affinoid algebra. A *finite  $k_H$ -affinoid covering* of  $\mathrm{Sp} A$  is a finite covering of  $A$  by  $k_H$ -affinoid domains.

A finite  $k_H$ -affinoid covering  $\mathcal{U}$  is a

- (1)  *$k_H$ -Laurent covering* if there are  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n \in A$  and  $r_1, \dots, r_n \in \sqrt{|k^\times| \cdot H}$  such that  $\mathcal{U}$  consists of

$$\mathrm{Sp} A \{r_1^{-\epsilon_1} f_1^{\epsilon_1}, \dots, r_1^{-\epsilon_n} f_1^{\epsilon_n}\}$$

for all  $\epsilon_i = \pm 1$ ,  $i = 1, \dots, n$ . In this case, we say that  $\mathcal{U}$  is the  $k_H$ -Laurent covering generated by  $r_1^{-1} f_1, \dots, r_n^{-1} f_n$ .

- (2)  $k_H$ -rational covering if there are  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n \in A$  generating the unit ideal,  $r = (r_1, \dots, r_n) \in \sqrt{|k^\times|} \cdot H^n$  such that  $\mathcal{U}$  consists of

$$\mathrm{Sp} A \left\{ (r/r_j)^{-1} \frac{f}{f_j} \right\}$$

for  $j = 1, \dots, n$ . In this case, we say that  $\mathcal{U}$  is the  $k_H$ -rational covering generated by  $r_1^{-1}f_1, \dots, r_n^{-1}f_n$ .

In both cases, if  $f_1, \dots, f_n$  are all units in  $A$ , we say the covering is *generated by units in  $A$* .

**Lemma 12.3.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathcal{V} = \{V_i\}_{i \in 1, \dots, m}$  be a finite  $k_H$ -affinoid covering of  $\mathrm{Sp} A$ . Then there is a  $k_H$ -rational covering refining  $\mathcal{V}$ .

PROOF. By [Corollary 11.6](#), we may assume that all  $V_i$ 's are  $k_H$ -rational domains in  $\mathrm{Sp} A$ . Take  $n_i \in \mathbb{N}$ ,  $g_1^{(i)}, \dots, g_{n_i}^{(i)} \in A$  generating the unit ideal,  $r^{(i)} = (r_1^{(i)}, \dots, r_{n_i-1}^{(i)}, r_{n_i}^{(i)}) \in \sqrt{|k^\times|} \cdot H^{n_i}$  for each  $i = 1, \dots, m$  such that if we write  $g^{(i)} = (g_1^{(i)}, \dots, g_{n_i}^{(i)})$ , then

$$V_i = \mathrm{Sp} A \left\{ \left( r^{(i)} / r_{n_i}^{(i)} \right)^{-1} \frac{g^{(i)}}{g_{n_i}^{(i)}} \right\}$$

for  $i = 1, \dots, m$ . Let  $\mathcal{B}^i$  be the  $k_H$ -rational covering generated by

$$(r^{(i)})^{-1} f_1^{(i)}, \dots, (r^{(i)})^{-1} f_{n_i}^{(i)}$$

for  $i = 1, \dots, m$ . We denote the elements in  $\mathcal{B}^i$  by  $V_j^i$ ,  $j = 1, \dots, n_i$ :

$$V_j^i := \mathrm{Sp} A \left\{ \left( r^{(i)} / r_j^{(i)} \right)^{-1} \frac{g^{(i)}}{g_j^{(i)}} \right\}.$$

Let

$$I := \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m : 1 \leq \alpha_i \leq n_i \text{ for } i = 1, \dots, m\}$$

and

$$I' := \{\alpha = (\alpha_1, \dots, \alpha_m) \in I : \alpha_i = n_i \text{ for some } i = 1, \dots, m\}.$$

Next for  $\beta = (\beta_1, \dots, \beta_m) \in I$ , we let

$$g_\beta = g_{\beta_1}^{(1)} \cdots g_{\beta_m}^{(m)}, \quad r_\beta = r_{\beta_1}^{(1)} \cdots r_{\beta_m}^{(m)}$$

and we have

$$V_\beta := V_{\beta_1}^1 \cap \cdots \cap V_{\beta_m}^m = \mathrm{Sp} A \left\{ ((r_\alpha)_{\alpha \in I} / r_\beta)^{-1} \frac{(g_\alpha)_{\alpha \in I}}{g_\beta} \right\}$$

as in the proof of [Proposition 9.6](#).

When  $\beta \in I'$ , we claim that

$$V_\beta = \mathrm{Sp} A \left\{ ((r_\alpha)_{\alpha \in I'} / r_\beta)^{-1} \frac{(g_\alpha)_{\alpha \in I'}}{g_\beta} \right\}.$$

It is clear that the left-hand side is contained in the right-hand side. Conversely,  $x$  in the right-hand side. By rearranging  $U_1, \dots, U_m$ , we may assume that  $x \in U_1$ . Let  $\gamma = (\gamma_1, \dots, \gamma_m) \in I \setminus I'$ . Then

$$r_\gamma^{-1} |g_\gamma(x)| \leq (r_{n_1}^{(1)})^{-1} (r_{\gamma_2}^{(2)})^{-1} \cdots (r_{\gamma_m}^{(m)})^{-1} \left| g_{n_1}^{(1)} g_{\gamma_2}^{(2)} \cdots g_{\gamma_m}^{(m)} \right| \leq r_\beta^{-1} |g_\beta(x)|.$$

The claim follows. Now  $\{V_\beta\}_{\beta \in I'}$  is the  $k_H$ -rational covering generated by  $r_\beta^{-1}g_\beta$  for  $\beta \in I'$ . It is clear that this covering refines  $\mathcal{V}$ .  $\square$

**Lemma 12.4.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathcal{U}$  be a  $k_H$ -rational covering of  $\mathrm{Sp} A$ . Then there is a  $k_H$ -Laurent covering  $\mathcal{V}$  of  $\mathrm{Sp} A$  such that for each  $\mathrm{Sp} C \in \mathcal{V}$ , the restriction  $\mathcal{U}|_{\mathrm{Sp} C}$  is a  $k_H$ -rational covering of  $\mathrm{Sp} C$  generated by units in  $C$ .

PROOF. We take  $n \in \mathbb{N}$ ,  $f_1, \dots, f_n \in A$  generating the unit ideal and  $r_1, \dots, r_n \in \sqrt{|k^\times| \cdot H}$  such that  $\mathcal{U}$  is generated by  $r_1^{-1}f_1, \dots, r_n^{-1}f_n$ . Choose  $c \in \sqrt{|k^\times| \cdot H}$  such that

$$c < \inf_{x \in \mathrm{Sp} A} \max_{i=1, \dots, n} r_i^{-1}|f_i(x)|.$$

Let  $\mathcal{V}$  be the  $k_H$ -Laurent covering of  $\mathrm{Sp} A$  generated by  $(cr_1)^{-1}f_1, \dots, (cr_n)^{-1}f_n$ . We claim that  $\mathcal{V}$  satisfies our requirements.

Take

$$V = \mathrm{Sp} A \{ (cr_1)^{-\epsilon_1} f_1^{\epsilon_1}, \dots, (cr_n)^{-\epsilon_n} f_n^{\epsilon_n} \}$$

be an element in  $\mathcal{V}$ ,  $\epsilon_i = \pm 1$  for  $i = 1, \dots, n$ . We may assume that there is  $s \in [0, n]$  such that  $\epsilon_1 = \dots = \epsilon_s = 1$  and  $\epsilon_{s+1} = \dots = \epsilon_n = -1$ . We claim that  $\mathcal{U}|_V$  is the  $k_H$ -rational covering generated by the images of  $r_{s+1}^{-1}f_{s+1}, \dots, r_n^{-1}f_n$  in

$$A \{ (cr_1)^{-1}f_1, \dots, (cr_s)^{-1}f_s, (cr_{s+1})f_{s+1}^{-1}, \dots, (cr_n)f_n^{-1} \}$$

and these elements are units.

In fact, by our assumption, for  $x \in V$ ,

$$\begin{aligned} |f_i(x)| &\leq cr_i, & \text{for } i = 1, \dots, s; \\ |f_i(x)| &\geq cr_i, & \text{for } i = s+1, \dots, n. \end{aligned}$$

In particular,

$$\max_{i=1, \dots, s} r_i^{-1}|f_i(x)| \leq c < \max_{i=1, \dots, n} r_i^{-1}|f_i(x)|.$$

Hence,

$$\max_{i=1, \dots, s} r_i^{-1}|f_i(x)| = \max_{i=s+1, \dots, n} r_i^{-1}|f_i(x)|.$$

Our claim follows.  $\square$

**Lemma 12.5.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathcal{U}$  be a  $k_H$ -rational covering of  $\mathrm{Sp} A$  generated by units in  $A$ . Then there is a  $k_H$ -Laurent covering  $\mathcal{V}$  of  $\mathrm{Sp} A$  refining  $\mathcal{U}$ .

PROOF. We take  $n \in \mathbb{N}$ , units  $f_1, \dots, f_n \in A$  and  $r_1, \dots, r_n \in \sqrt{|k^\times| \cdot H}$  such that  $\mathcal{U}$  is generated by  $r_1^{-1}f_1, \dots, r_n^{-1}f_n$ .

We take  $\mathcal{V}$  as the Laurent covering generated by  $(r_i r_j^{-1})^{-1} f_i f_j^{-1}$  for  $1 \leq i < j \leq n$ . We claim that  $\mathcal{V}$  refines  $\mathcal{U}$ . Write  $I = \{(i, j) \in \mathbb{N}^2 : 1 \leq i < j \leq n\}$ . To see this, consider  $V \in \mathcal{V}$ , say

$$V = \bigcap_{(i,j) \in I_1} \mathrm{Sp} A \{ (r_i r_j^{-1})^{-1} f_i f_j^{-1} \} \cap \bigcap_{(i,j) \in I_2} \mathrm{Sp} A \{ (r_i r_j^{-1})^{+1} f_i^{-1} f_j \},$$

where  $I_1, I_2$  is a partition of  $I$ . For  $i, j \in \{1, \dots, n\}$ , we write  $i \preceq j$  if  $(i, j) \in I_1$  and  $j \preceq i$  if  $(i, j) \in I_2$ . Consider a maximal chain

$$i_1 \preceq i_2 \preceq \dots \preceq i_s$$

on the set  $\{1, \dots, n\}$ . Then  $i \preceq i_s$  for each  $i = 1, \dots, n$ . In other words, for  $x \in X$ , we have

$$|f_i f_{i_s}^{-1}(x)| \leq r_i r_{i_s}^{-1}.$$

The right-hand side defines an element in  $\mathcal{U}$ .  $\square$

We first prove Tate acyclicity theorem in a special case.

**Lemma 12.6.** Let  $A$  be a  $k_H$ -affinoid algebra. Let  $\mathcal{V} = \{V_i\}_{i \in 1, \dots, n}$  be a finite  $k_H$ -affinoid covering of  $\mathrm{Sp} A$ . Assume that each  $V_i$  is a  $k_H$ -rational domain. Then  $\check{C}(\mathcal{V}, A)$  is exact and admissible.

**PROOF. Step 1.** We reduce to the case where

$$\mathcal{V} = \{\{\mathrm{Sp} A\{r^{-1}f\}\}, \{\mathrm{Sp} A\{rf^{-1}\}\}\}$$

for some  $r \in \sqrt{|k^\times| \cdot H}$  and  $f \in A$ .

Take a  $k$ -free polyray  $s$  with at least one component. By [Proposition 3.11](#), we can make the base change to  $k_s$  and assume that  $k$  is non-trivially valued. In this case, by open mapping theorem [Theorem 7.2](#) in the chapter Banach Rings, the admissibility is automatic. It suffices to prove the exactness.

In this case, we can define a presheaf  $\mathcal{O}_X$  on  $X$  on the family of  $k_H$ -rational domains in  $\mathrm{Sp} A$ :  $\mathcal{O}_X(\mathrm{Sp} C) = C$ . From the general comparison theorem of Čech cohomology [BGR P327 reproduce in the topology part](#) and [Lemma 12.3](#), we may assume that the covering  $\mathcal{V}$  is  $k_H$ -rational covering. But then we need to show that for each  $k_H$ -rational domain  $W$  in  $\mathrm{Sp} A$ ,  $\check{C}(\mathcal{V}|_W, A)$  is exact. Similarly, by [Lemma 12.4](#), we may assume that the  $k_H$ -rational covering is generated by units. Again, by [Lemma 12.5](#), we can reduce to the case where  $\mathcal{V}$  is a  $k_H$ -Laurent covering.

We need to show that for each  $k_H$ -affinoid domain  $\mathrm{Sp} C$  in  $\mathrm{Sp} A$ ,  $\check{C}(\mathcal{V}|_W, A)$  is exact. But  $\mathcal{V}|_W$  is also a  $k_H$ -Laurent covering. In particular, it suffices to show that  $\check{C}(\mathcal{V}, A)$  is exact. By induction on the number of generators of  $\mathcal{V}$ , we can reduce the case stated in the beginning.

**Step 2.** After the reduction, we need to show that the following sequence is exact:

$$0 \rightarrow A \xrightarrow{i} A\{r^{-1}f\} \times A\{rf^{-1}\} \xrightarrow{d^0} A\{r^{-1}f, rf^{-1}\} \rightarrow 0,$$

where  $i(a) = (a, a)$  and  $d^0(f, g) = f - g$ . We extend the sequence to the following commutative diagram in  $k_H\text{-AffAlg}$ :

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & (\zeta - f)A\{r^{-1}\zeta\} \times (1 - f\eta)A\{r\eta\} & \xrightarrow{\lambda'} & (\zeta - f)A\{r^{-1}\zeta, r\zeta^{-1}\} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & A\{r^{-1}\zeta\} \times A\{r\eta\} & \xrightarrow{\lambda} & A\{r^{-1}\zeta, r\eta\}/(\zeta\eta - 1) \longrightarrow 0 \\ & & \Downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\epsilon} & A\{r^{-1}f\} \times A\{rf^{-1}\} & \xrightarrow{d^0} & A\{r^{-1}f, rf^{-1}\} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

where  $\iota(a) = (a, a)$  and  $\lambda$  sends  $\zeta$  to  $\zeta$  and  $\eta$  to  $\eta$ . The two columns are clearly exact. It is straightforward to see that everywhere the first non-zero row is exact. The second non-zero row is also exact. The non-trivial part is to show that if  $\sum_{i=0}^{\infty} a_i \zeta^i \in A\{r^{-1}\zeta\} \in A\{r^{-1}\zeta\}$  and  $\sum_{i=0}^{\infty} b_i \zeta^i \in A\{r^{-1}\eta\} \in A\{r\eta\}$  are such that their pair lies in the kernel of  $\lambda$ , then

$$0 = \sum_{i=0}^{\infty} a_i \zeta^i - \sum_{i=0}^{\infty} b_i \zeta^{-i}.$$

It follows that  $a_i = 0 = b_i$  for  $i > 0$  and  $a_i = b_i$ . So we find that the second row is also exact. By diagram chasing, the third row is also exact.  $\square$

**Corollary 12.7.** Let  $A$  be a  $k_H$ -affinoid algebra and  $\mathrm{Sp} B$  be a  $k$ -affinoid domain in  $\mathrm{Sp} A$ . Then for any complete non-Archimedean field extension  $K/k$ , any  $K$ -affinoid algebra  $C$  and any bounded ring homomorphism  $A \rightarrow C$  such that  $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$  factorizes through  $\mathrm{Sp} B$ , there is a unique bounded ring homomorphism  $B \rightarrow C$  making the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sp} C & & \\ \downarrow & \searrow & \\ \mathrm{Sp} B & \longrightarrow & \mathrm{Sp} A \end{array}$$

PROOF. The proof is the same as in [Example 9.4](#) when  $\mathrm{Sp} B$  is an affinoid domain in  $\mathrm{Sp} A$ .

In general, by [Corollary 11.6](#), we can cover  $\mathrm{Sp} B$  by finitely many affinoid domains  $\mathrm{Sp} B_1, \dots, \mathrm{Sp} B_n$  in  $\mathrm{Sp} A$ . Let  $\mathrm{Sp} C_i$  be the rational domain in  $\mathrm{Sp} C$  defined by the preimage of  $\mathrm{Sp} B_i$  for  $i = 1, \dots, n$ . In other words, we have Cartesian diagrams for  $i = 1, \dots, n$ :

$$\begin{array}{ccc} \mathrm{Sp} C_i & \longrightarrow & \mathrm{Sp} C \\ \downarrow & \square & \downarrow \\ \mathrm{Sp} B_i & \longrightarrow & \mathrm{Sp} A \end{array}$$

It follows from [Lemma 12.6](#) that we have an admissible exact sequence

$$0 \rightarrow C \rightarrow \prod_{i=1}^n C_i \rightarrow \prod_{1 \leq i < j \leq n} C_i \hat{\otimes}_C C_j.$$

From general abstract nonsense, to construct bounded  $A$ -homomorphisms  $\varphi : B \rightarrow C$  is the same as to construct bounded homomorphisms  $\varphi_i : B \rightarrow C_i$  over  $A$  such that the induced maps  $B \rightarrow C_i \hat{\otimes}_C C_j$  are compatible. On the other hand, by our definition of  $B_i$ , in order to construct the morphisms  $\varphi_i$ , it suffices to construct  $\psi_i : B_i \rightarrow C_i$  over  $A$ . This reduces to the known case.  $\square$

**Corollary 12.8.** Let  $A$  be a  $k_H$ -affinoid algebra and  $H' \supseteq H$  is a subgroup of  $\mathbb{R}_{>0}$ . Let  $V = \mathrm{Sp} B$  be a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ , then  $\mathrm{Sp} B$  is a  $k_{H'}$ -affinoid domain in  $\mathrm{Sp} A$ .

PROOF. This follows immediately from [Corollary 12.7](#).  $\square$

[Introduce the Shilov point](#)

**Proposition 12.9.** Let  $A$  be a  $k$ -affinoid algebra and  $V \subseteq X$  is a closed subset. Let  $f : A \rightarrow B$  be a morphism of  $k$ -affinoid algebras. Assume that for any complete non-Archimedean field extension  $K/k$ , any  $K$ -affinoid algebra  $C$  and any bounded ring homomorphism  $A \rightarrow C$  such that  $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$  factorizes through  $V$ , there is a unique bounded ring homomorphism  $B \rightarrow C$  making the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sp} C & & \\ \downarrow & \searrow & \\ \mathrm{Sp} B & \longrightarrow & \mathrm{Sp} A \end{array}.$$

Then  $V$  is an affinoid domain represented by the given  $A \rightarrow B$ .

**PROOF.** The only non-trivial thing is to show that the image of  $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$  is  $V$ .

**Step 1.** We reduce to the case where  $k$  is non-trivially valued and  $A, B$  are both strictly  $k$ -affinoid.

Let  $r$  be a  $k$ -free polyray with at least one component such that  $A \hat{\otimes}_k k_r$  and  $B \hat{\otimes}_k k_r$  are both strictly  $k_r$ -affinoid. Let  $V'$  be the inverse image of  $V$  in  $\mathrm{Sp} A \hat{\otimes}_k k_r$ . Then clearly,  $V'$  has the same universal property. Assume that we have already shown that the image of

$$\mathrm{Sp} B \hat{\otimes}_k k_r \rightarrow A \hat{\otimes}_k k_r$$

is exactly  $V'$ . We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp} B \hat{\otimes}_k k_r & \longrightarrow & \mathrm{Sp} A \hat{\otimes}_k k_r \\ \downarrow & & \downarrow \\ \mathrm{Sp} B & \longrightarrow & \mathrm{Sp} A \end{array}$$

From the existence of the Shilov points, both vertical sections are surjective. Hence, the image of  $\mathrm{Sp} B$  in  $\mathrm{Sp} A$  is exactly  $V$ .

**Step 2.** After the reduction, it suffices to argue that each point in  $V \cap \mathrm{Spm} A$  lies in the image. Let  $y$  be such a point corresponding to a maximal ideal  $\mathfrak{m}_y$  of  $A$ . Consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \pi & \searrow \alpha & \downarrow \pi' \\ A/\mathfrak{m}_y & \xrightarrow{\sigma} & B/\mathfrak{m}_y B \end{array}.$$

The two vertical maps are the natural projections and  $\sigma$  is the map induced by  $f$ . The existence of  $\alpha$  and the commutativity of the diagram follow from the universal property. Observe that  $\sigma$  is surjective as  $\pi'$  is. Similarly,  $\alpha$  is surjective as  $\pi$  is. Moreover,  $\mathfrak{m}_y B = \ker \pi' \subseteq \ker \alpha$ . In particular,  $\sigma$  is bijection. So  $\mathfrak{m}_y B$  is a maximal ideal in  $B$  and the corresponding point  $x \in \mathrm{Spm} B$  sends  $x$  to  $y$ .  $\square$

**Remark 12.10.** In fact, the proof proves the following result: assume that the valuation on  $k$  is non-trivial and  $A$  is a strictly  $k$ -affinoid algebra. Let  $\mathrm{Sp} B$  be a strictly  $k$ -affinoid domain. Then for each  $x \in \mathrm{Spm} B$  corresponding to a maximal ideal  $\mathfrak{m}_x$  in  $B$  and any  $n \in \mathbb{Z}_{>0}$ , we have a natural isomorphism

$$A/\mathfrak{m}_y^n \xrightarrow{\sim} B/\mathfrak{m}_x^n,$$

where  $y$  is the image of  $x$  in  $\mathrm{Sp} A$  and  $\mathfrak{m}_y$  is the corresponding maximal ideal in  $A$ . Moreover,  $\mathfrak{m}_x = \mathfrak{m}_y B$ .

In particular, the natural map  $\hat{A}_{\mathfrak{m}_y} \rightarrow \hat{B}_{\mathfrak{m}_x}$  is an isomorphism.

**Corollary 12.11.** Let  $A$  be a  $k$ -affinoid algebra and  $\mathrm{Sp} B$  be a  $k$ -affinoid domain in  $\mathrm{Sp} A$ . Assume that  $K/k$  is an extension of complete valued field. Then  $\mathrm{Sp} B \hat{\otimes}_k K$  is a  $K$ -affinoid domain in  $\mathrm{Sp} A \hat{\otimes}_k K$ . Moreover, the image of  $\mathrm{Sp} B \hat{\otimes}_k K$  in  $\mathrm{Sp} A \hat{\otimes}_k K$  is the inverse image of the image of  $\mathrm{Sp} B$  in  $\mathrm{Sp} A$ .

PROOF. This is an immediate consequence of [Proposition 12.9](#) and [Corollary 12.7](#).  $\square$

**Corollary 12.12.** Let  $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$  be a morphism of  $k_H$ -affinoid spectra. Let  $V \subseteq \mathrm{Sp} A$  be a  $k_H$ -affinoid domain in  $\mathrm{Sp} A$ , then  $\varphi^{-1}(V)$  is a  $k_H$ -affinoid domain in  $\mathrm{Sp} B$ .

In fact, suppose that  $V$  is represented by  $A \rightarrow A_V$ , then  $B \rightarrow B \hat{\otimes}_A A_V$  represents  $\varphi^{-1}V$ .

PROOF. This is an immediate consequence of [Proposition 12.9](#) and [Corollary 12.7](#).  $\square$

**Corollary 12.13.** Let  $A$  be a  $k_H$ -affinoid algebra. Let  $V, W$  be  $k_H$ -affinoid domains in  $\mathrm{Sp} A$  represented by  $A \rightarrow A_V$  and  $A \rightarrow A_W$  respectively. Then  $V \cap W$  is a  $k_H$ -affinoid domain represented by  $A \rightarrow A_V \hat{\otimes}_A A_W$ .

PROOF. This is an immediate consequence of [Corollary 12.12](#).  $\square$

**Definition 12.14.** Let  $X = \mathrm{Sp} A$  be a  $k$ -affinoid spectra, we define a presheaf  $\mathcal{O}_X$  of Banach rings on the family of  $k$ -affinoid domains in  $X$  as follows: for any  $k$ -affinoid domain  $\mathrm{Sp} B$ , we set

$$\mathcal{O}_X(\mathrm{Sp} B) = B.$$

Given an inclusion of affinoid domains,  $\mathrm{Sp} C \rightarrow \mathrm{Sp} B$ , we define the corresponding restriction map as the given morphism  $B \rightarrow C$ .

**Theorem 12.15.** Let  $A$  be a  $k$ -affinoid algebra and  $V' = \mathrm{Sp} B$  be a  $k$ -affinoid domain in  $\mathrm{Sp} A$ . Then  $B$  is a flat  $A$ -algebra.

PROOF. **Step 1.** We reduce to the case where  $k$  is non-trivially valued and  $A$  is strictly  $k$ -affinoid.

Let  $r$  be a  $k$ -free polyray with at least one component. Let  $\varphi : M \rightarrow N$  be an injective  $A$ -module homomorphism. We endow  $M$  and  $N$  with the structures of finite Banach  $A$ -modules by [Proposition 8.2](#) and then  $\varphi$  is admissible by [Proposition 8.6](#). By [Proposition 3.11](#), the induced homomorphism

$$M \hat{\otimes}_k k_r \rightarrow N \hat{\otimes}_k k_r$$

is injective and admissible. Let  $V'$  be the inverse image of  $V$  in  $\mathrm{Sp} A \hat{\otimes}_k k_r$ . By [Corollary 12.11](#),  $V'$  is a  $k_r$ -affinoid domain represented by  $A \hat{\otimes}_k k_r \rightarrow B \hat{\otimes}_k k_r$ .

If we have shown the result in the special case, we know that

$$(M \hat{\otimes}_k k_r) \otimes_{A \hat{\otimes}_k k_r} (B \hat{\otimes}_k k_r) \rightarrow (N \hat{\otimes}_k k_r) \otimes_{A \hat{\otimes}_k k_r} (B \hat{\otimes}_k k_r)$$

is injective. By [Proposition 8.5](#), this map can be identified with

$$(M \hat{\otimes}_k k_r) \hat{\otimes}_{A \hat{\otimes}_k k_r} (B \hat{\otimes}_k k_r) \rightarrow (N \hat{\otimes}_k k_r) \hat{\otimes}_{A \hat{\otimes}_k k_r} (B \hat{\otimes}_k k_r).$$

The latter map is easily identified with

$$M \hat{\otimes}_A B \rightarrow N \hat{\otimes}_A B.$$

By [Proposition 8.5](#) again, the latter map is identified with

$$M \otimes_A B \rightarrow N \otimes_A B.$$

We conclude that  $A \rightarrow B$  is flat.

**Step 2.** After the reduction, we take a maximal ideal  $\mathfrak{m}_x$  in  $B$  corresponding to a point  $x \in \mathrm{Sp} B$ . Let  $y$  be the image of  $y$  in  $\mathrm{Sp} A$  and  $\mathfrak{m}_y$  denotes the corresponding maximal ideal. Then by [Remark 12.10](#),  $\hat{A}_{\mathfrak{m}_y} \rightarrow \hat{B}_{\mathfrak{m}_y}$  is an isomorphism. By [\[Stacks, Tag 0C4G\]](#) and [\[Stacks, Tag 0399\]](#), we conclude that  $A \rightarrow B$  is flat.  $\square$

**Theorem 12.16** (Tate acyclicity theorem). Let  $A$  be a  $k$ -affinoid algebra. Let  $\mathcal{V} = \{V_i\}_{i \in 1, \dots, n}$  be a finite  $k$ -affinoid covering of  $\mathrm{Sp} A$ . Let  $M$  be an  $A$ -module. Then the complex  $\check{C}(\mathcal{V}, A)$  is exact. It is exact and admissible if  $M$  is finite as  $A$ -module.

**PROOF.** We first observe that the admissibility follows from the same argument as in [Lemma 12.6](#). We will only concentrate on the exactness.

**Step 1.** We first reduce to the case  $M = A$ .

As the covering  $\mathcal{V}$  is finite, we can find  $N \in \mathbb{N}$  such that  $\check{H}^j(\mathcal{V}, M'') = 0$  for all  $j \geq N$  and all  $A$ -module  $M''$ . We take the minimum of such  $N$ . Assume that  $N > 0$ .

Assume we have proved the theorem in this case, then the case where  $M$  is free is immediate. In general, choose an exact sequence of  $A$ -modules:

$$0 \rightarrow M' \rightarrow F \rightarrow M \rightarrow 0$$

with  $F$  free. In this case, we have a short exact sequence

$$0 \rightarrow \check{C}(\mathcal{V}, M') \rightarrow \check{C}(\mathcal{V}, F) \rightarrow \check{C}(\mathcal{V}, M) \rightarrow 0.$$

The exactness follows from [Theorem 12.15](#).

From the long exact sequence, we find that

$$H^{q-1}(\mathcal{V}, M) \cong H^q(\mathcal{V}, M').$$

for all  $q \in \mathbb{Z}$ . It follows that  $H^q(\mathcal{V}, M) = 0$  for all  $q \geq N - 1$ . This argument works for any  $A$ -module  $M$  and we get a contradiction with our choice of  $N$ .

**Step 2.** After the reduction in Step 1 and the successful definition of  $\mathcal{O}_X$  in [Definition 12.14](#), the remaining of the argument is exactly the same as [Lemma 12.6](#).  $\square$



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