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Contents

Local properties of complex analytic spaces

1. Introduction

2. Dimension

Definition 2.1. Let X be a complex analytic space and $x \in X$, the *dimension* $\dim_x X$ of X at x is

$$\dim_x X = \dim \mathcal{O}_{X,x}.$$

We also define the *dimension* of the pointed complex analytic space (X, x) and the *dimension* of the complex analytic germ X_x as $\dim_x X$.

When X is connected, the *dimension* of X is defined as

$$\dim X := \sup_{x \in X} \dim_x X.$$

If A is an analytic set in X such that there is a closed analytic subspace of X with $|B| = A$, then $\dim_x B$ does not depend on the choice of B, we define it as $\dim_x A$.

As we will see in **??**, B always exists.

Definition 2.2. Let X be a complex analytic space, we say X is *equidimensional* at $x \in X$ if $\mathcal{O}_{X,x}$ is equidimensional.

We also say (X, x) or X_x is *equidimensional*.

We say X is *equidimensional of dimension* $n \in \mathbb{N}$ if X is non-empty and is equidimensional of dimension n at each $x \in X$.

Recall that in general, a local ring R is equidimensional if $\dim R/\mathfrak{p} = \dim R$ for all minimal prime $\mathfrak{p}$ of R.

Definition 2.3. Let X be a complex analytic space and $x \in X$, we say X is *integral* at x if $\mathcal{O}_{X,x}$ is integral.

This corresponds to the notion defined in **??** in **??**.

Theorem 2.4. Let X be a complex analytic space and $n \in \mathbb{N}$, then the set of points $x \in X$ such that X_x is equidimensional of dimension n is open.

This is analogous to the result for noetherian cartenary schemes.

Proof. Let $x \in X$ be a point such that X_x is equidimensional of dimension n. We want to construct an open neighbourhood V of x in X such that X is equidimensional of dimension n at any $y \in V$.

Step 1. We reduce to the case where X is integral at x.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal primes of $\mathcal{O}_{X,x}$. The number is finite because $\mathcal{O}_{X,x}$ is noetherian. We have

$$\bigcap_{i=1}^{m} \mathfrak{p}_i = \operatorname{rad} \mathcal{O}_{X,x}.$$

Take an open neighbourhood U of x in X such that there are ideals of finite type $\mathcal{I}_1, \dots, \mathcal{I}_m$ extending $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Up to shrinking U, we may assume that

$$\bigcap_{i=1}^{m} \mathcal{I}_i$$

is nilpotent. For each $i = 1, \dots, m$, let U_i denote the closed analytic subspace of U defined by $\mathcal{I}_i$. Then

$$|U| = \bigcup_{i=1}^{m} |U_i|$$

by **??** in **??**. As for any $y \in U$,

$$\bigcap_{i=1}^{m} \mathcal{I}_{i,y}$$

is nilpotent, we have

$$|\operatorname{Spec} \mathcal{O}_{X,y}| = |\operatorname{Spec} \mathcal{O}_{X,y} / \bigcap_{i=1}^{m} \mathcal{I}_{i,y}| = \bigcup_{i=1}^{m} |\operatorname{Spec} \mathcal{O}_{X,y}/\mathcal{I}_{i,y}|.$$

In particular, for any $y \in U$,

$$\dim_y X = \dim_y U = \max_{i=1,\dots,m} \dim_y U_i.$$

It suffices to handle each W_i separately.

Step 2. We assume that X_x is integral. By **??** in **??**, we may assume that X has the following structure: there is an open neighbourhood W of 0 in $\mathbb{C}^n$, a morphism $(X, x) \to (W, 0)$ and a finite $\mathcal{O}_W$-algebra $\mathcal{A}$ such that $\operatorname{Spec}^{\mathrm{an}}_W \mathcal{A}$ has a unique point x' over 0 and $(\operatorname{Spec}^{\mathrm{an}}_W \mathcal{A}, x')$ is isomorphic to (X, x) over $(W, 0)$. By **??** in **??**, $\mathcal{O}_{W,0} \to \mathcal{O}_{X,x}$ is injective, hence $\mathcal{O}_{X,x}$ is torsion-free over $\mathcal{O}_{W,0}$. As the torsion sheaf is coherent, up to shrinking X, we may assume that $\mathcal{O}_{X,y}$ is torsion-free over $\mathcal{O}_{W,z}$, where z denotes the image of y in W. It suffices to apply **??** in **??**. $\square$

Corollary 2.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set $\{x \in X : \dim_x X \geq n\}$ is an analytic set in X.

After introducing the analytic Zariski topology, we can reformulate this corollary as follows: the map $x \mapsto \dim_x X$ is upper semi-continuous with respect to the analytic Zariski topology.

Proof. The problem is local on X. Fix $x \in X$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathcal{O}_{X,x}$. Up to shrinking X, we may assume that

$$|X| = \bigcup_{i=1}^{m} |W_i|,$$

where W_i is a closed analytic subspace of X defined by a coherent $\mathcal{I}_i$ spreading $\mathfrak{p}_i$. We can guarantee that

$$\dim_y X = \max_{i=1,\dots,m} \dim_y W_i.$$

This is possible as in the proof of **??**. By **??**, up to shrinking X, we may assume that W_i is equidimensional of dimension n_i for some $n_i \in \mathbb{N}$ for each $i = 1, \dots, m$. In particular, for each $y \in X$, we have

$$\dim_y X = \sup_{y \in W_i} n_i.$$

So

$$\{x \in X : \dim_x X \geq n\} = \bigcup_{i : n_i \geq n} |W_i|.$$

The corollary follows. □

Proposition 2.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Then

$$\dim_{(x,y)} X \times Y = \dim_x X + \dim_y Y.$$

PROOF. By **??** in **??**,

$$\hat{\mathcal{O}}_{X\times Y,(x,y)} = \hat{\mathcal{O}}_{X,x} \hat{\otimes} \hat{\mathcal{O}}_{Y,y}.$$

As dimension is invariant under completion by [**stacks-project**], it suffices to show that

$$\dim(\mathcal{O}_{X,x} \hat{\otimes} \mathcal{O}_{Y,y}) = \dim \mathcal{O}_{X,x} + \dim \mathcal{O}_{Y,y},$$

which is well-known. □

Definition 2.7. Let X_x be an analytic germ and Y_x be a closed analytic subgerm defined by an ideal $I \subseteq \mathcal{O}_{X,x}$.

(1) When Y_x is irreducible, namely when I is a prime ideal, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \operatorname{ht}_{\mathcal{O}_{X,x}}(I).$$

(2) In general, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \inf_{Z_x \subseteq Y_x} \operatorname{codim}_x(Y, X),$$

where Z_x runs over closed analytic subgerms of X_x contained in Y_x.

We also call $\operatorname{codim}_x(Y, X)$ the codimension of Y in X at x.

Observe that

$$\operatorname{codim}_x(Y, X) \leq \dim_x X - \dim_x Y.$$

When X_x is equidimensional, $\operatorname{codim}_x(Y, X)$ is nothing but $\dim_x X - \dim_x Y$.

Observe that

$$\operatorname{codim}_x(Y, X) = \operatorname{codim}(Y_x, \operatorname{Spec} \mathcal{O}_{X,x}). \tag{2.1}$$

Lemma 2.8. Let X be a complex analytic space and T be an analytic set in X. Let Y_1, Y_2 be two closed analytic subspaces of X with underlying set T, then for any $x \in T$,

$$\operatorname{codim}_x(Y_1, X) = \operatorname{codim}_x(Y_2, X).$$

PROOF. This follows from (**??**) and **??** in **??**. □

Definition 2.9. Let X be a complex analytic space and T be an analytic set in X. Take $y \in T$. We define the *codimension* $\operatorname{codim}_y(T, X)$ as follows: up to shrinking X, we may take a closed analytic subspace Y of X with underlying set T by **??** in **??**, we define

$$\operatorname{codim}_y(T, X) := \operatorname{codim}_y(Y, X).$$

This definition does not depend on the choices we made by **??**.

Lemma 2.10. Let X be a complex analytic space and Y be a closed analytic subspace of X. Let $y \in Y$ be a point such that Y_y is irreducible. Then there is an open neighbourhood U of y in Y such that

$$\operatorname{codim}_z(Y,X) = \operatorname{codim}_y(Y,X)$$

for any $z \in U$.

Proof. Let X'_y be an irreducible component of X_y containing Y_y such that

$$\operatorname{codim}_y(Y,X) = \dim_y X' - \dim_y Y.$$

We can then take an open neighbourhood U of x in X such that X'_z is equidimensional of dimension $n := \dim_y X'$ for all $z \in U$ by **??**. Then for any $z \in U$, X'_z is a union of some irreducible components of X_z. Up to shrinking U, we may guarantee that for any $z \in U \cap Y$, $Y_z \subseteq X'_z$ and $\dim_z Y = \dim_y Y$. Thereofre, for $z \in Y \cap U$,

$$\operatorname{codim}_z(Y,X) = \operatorname{codim}_z(Y,X') = \dim_z X' - \dim_z Y$$

is a constant. □

Corollary 2.11. Let X be a complex analytic space and Y be an analytic set in X. For any $n \in \mathbb{N}$,

$$\{y \in Y : \operatorname{codim}_y(Y,X) \leq n\}$$

is an analytic set in Y.

Proof. The problem is local. Let $x \in Y$. Let $Y_{1,x},\ldots,Y_{m,x}$ be the irreducible components of Y_x defined by prime ideals $J_1,\ldots,J_m$ in $\mathcal{O}_{Y,x}$. Take an open neighbourhood U of x in X such that for any $y \in Y \cap U$, the ideal

$$\bigcap_{i=1}^{m} J_{i,y}$$

is nilpotent. By **??**, up to shrinking U, we may assume that for any $y \in Y \cap U$,

$$\operatorname{codim}_y(Y_i,X) = \operatorname{codim}_x(Y_i,X) =: c_i$$

for $i = 1,\ldots,m$. Then

$$\{y \in Y : \operatorname{codim}_y(Y,X) \leq n\} = \bigcup_{i:c_i \leq n} Y_i.$$

□

Corollary 2.12. Let X be a complex analytic space and Y be an analytic set in X. For any $n \in \mathbb{N}$ and any $y \in Y$,

$$\{y \in Y : \operatorname{codim}_y(Y,X) \leq n\}_y = \{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,x} : \operatorname{codim}_{\mathfrak{p}}(T_x, \operatorname{Spec}\mathcal{O}_{X,x}) \leq n\}.$$

Proof. This is immediate from the proof of **??**. □

Definition 2.13. Let X be a complex analytic space. A closed subset A of X is *thin* if for any $x \in A$, we can find an open neighbourhood U of x in X such that $A \cap U$ is contained in a nowhere dense analytic subset B of U.

Given $k \in \mathbb{Z}_{>0}$, we say A is *thin of order* k at $x \in A$ if U and B can be chosen so that $\operatorname{codim}_x(B,X) \geq 2$.

We say X is *thin* (*thin of order* k) if X is thin (resp. thin of order k) at all $x \in X$.

The definition in [**CAS**] Page 132 is not correct when X is not equidimensional. The same happens in several papers of Remmert.

3. Smoothness

Definition 3.1. Let X be a complex analytic space. We say X is *smooth* at $x \in X$ if $\mathcal{O}_{X,x}$ is regular. Otherwise, we say X is *singular* at x.

We also say (X, x) or X_x is *smooth* (resp. *singular*) at x.

We say X is *smooth* if it is smooth at all $x \in X$. In this case, we also say X is a *complex manifold*.

We write X^{sing} and X^{reg} for the set of singular and smooth points of X respectively.

Other common names in the literature include: regular, simple.

Proposition 3.2. Let X be a complex analytic space and $x \in X$. THen the following are equivalent:

(1) X is smooth at x;
(2) There is an open neighbourhood U of x in X that is isomorphic to a domain in $\mathbb{C}^n$ with $n = \dim_x X$;
(3) $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$-module of rank $\dim_x X$;
(4) $\Omega_{X,x}$ is generated by $\dim_x X$ elements as an $\mathcal{O}_{X,x}$-module;
(5) $\hat{\mathcal{O}}_{X,x}$ is regular;
(6) $\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[X_1, \dots, X_n]]$ for $n = \dim_x X$.

Proof. (2) $\implies$ (1): This is obvious.

(1) $\implies$ (2): Let $f_{1,x}, \dots, f_{n,x}$ be a regular system of parameters of $\mathcal{O}_{X,x}$. Up to shrinking X, we may lift them to $f_1, \dots, f_n \in \mathcal{O}_X(X)$. By **??** in **??**, they induce a morphism $f : (U, x) \to (\mathbb{C}^n, 0)$. Observe that $f_x^\# : \hat{\mathcal{O}}_{\mathbb{C}^n,0} \to \hat{\mathcal{O}}_{U,x}$ is an isomorphism, so f is a local isomorphism by **??** in **??**.

(2) $\implies$ (3): This follows from **??** in **??**.

(3) $\implies$ (4): This is trivial.

(4) $\implies$ (1): Recall that Ω_X is coherent by **??** in **??**. By Nakayama's lemma, the minimal number of generators of $\Omega_{X,x}$ is equal to $\dim_{\mathbb{C}} \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$. By algebraic results, we know that the latter space is $\mathfrak{m}_x/\mathfrak{m}_x^2$. So we find that $\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$, implying that $\mathcal{O}_{X,x}$ is regular.

(1) $\Leftrightarrow$ (5): This follows from [**stacks-project**].

(2) $\implies$ (6): This is clear.

(6) $\implies$ (5): This is clear. $\square$

Theorem 3.3. Let X be a complex analytic space, then X^{Sing} is an analytic set in X.

Proof. The problem is local. Let $x \in X$.

Step 1. We reduce to the case where X is equidimensional of dimension n.

Let

$$0 = \bigcap_{i=1}^{r} \mathfrak{p}_i$$

be the primary decomposition of 0. Up to shrinking X, we may assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ spread to coherent ideals $\mathcal{I}_1, \dots, \mathcal{I}_r$ on X and

$$\bigcap_{i=1}^{r} \mathcal{I}_i = 0.$$

Let X_i be the closed analytic subspace of X defined by $\mathcal{I}_i$ for $i = 1, \dots, n$. Then

$$X = \bigcup_{i=1}^{r} X_i.$$

As each X_i is equidimensional at x, say of dimension n_i for $i = 1, \dots, r$. By **??**, up to shrinking X, we may assume that X_i is equidimensional of dimension n_i for $i = 1, \dots, r$. For each

Let $y \in X^{\mathrm{reg}}$, as $\mathcal{O}_{X,y}$ is regular hence integral, from

$$\bigcap_{i=1}^{r} \mathcal{I}_{i,y} = 0$$

we find that at least one $\mathcal{I}_{i,y}$ vanishes. Then

$$\mathcal{O}_{X_i,y} = \mathcal{O}_{X,y}$$

is regular. Namely, $y \in X_i^{\mathrm{reg}}$. Conversely, if for some $i = 1, \dots, n$, we have $\mathcal{I}_{i,y} = 0$ and $y \in X_i^{\mathrm{reg}}$, X_i is a neighbourhood of y in X, so $y \in X^{\mathrm{reg}}$. It follows that

$$X^{\mathrm{sing}} = \bigcap_{i=1}^{r} \left(\operatorname{Supp} \mathcal{I}_i \cup X_i^{\mathrm{Sing}} \right).$$

Recall that $\operatorname{Supp} \mathcal{I}_i$ is analytic for each $i = 1, \dots, n$ by **??** in **??**.

By **??** in **??**, in order to show that X^{sing} is an analytic set in X, it suffices to know that X_i^{Sing} is an analytic set in X_i for $i = 1, \dots, n$.

Step 2. Assume that X is equidimensional of dimension n. We need to show that the locus where Ω_X is locally free of rank n is co-analytic in X.

When $n = 0$, the locus where Ω_X is not locally free of rank 0 is exactly $\operatorname{Supp} \Omega_X$, which is analytic in X by **??** and **??** in **??**.

Assume that $n \geq 1$. Let $\Omega_X^n := \bigwedge^n \Omega_X$. Then the locus where Ω_X is locally free of rank n is exactly the locus where Ω_X^n is invertible. The invertible locus of Ω_X^n is exactly the locus where the canonical map

$$(\Omega_X^n)^\vee \otimes_{\mathcal{O}_X} \Omega_X^n \to \mathcal{O}_X$$

is an isomorphism. It follows that the complement of the locus is analytic in X. □

Theorem 3.4 (Generic smoothness)**.** *Let X be a complex analytic space and $x \in X$. Assume that X is integral at x, then $X_x^{\mathrm{Sing}} \neq |X|_x$.*

Proof. Let $n = \dim_x X$. The problem is local on X. By **??** in **??**, we may assume that there is a finite morphism $\varphi : (X, x) \to (V, 0)$, where V is an open neighbourhood of 0 in $\mathbb{C}^n$ and there is a finite $\mathcal{O}_V$-algebra $\mathcal{A}$ with $\mathcal{A}_0 = \mathcal{O}_{X,x}$ such that there is unique point x' of $\operatorname{Spec}_V^{\mathrm{an}} \mathcal{A}$ over 0 and (X, x) can be identified with $(\operatorname{Spec}_V^{\mathrm{an}} \mathcal{A}, x')$.

Take $\xi \in \mathcal{O}_{X,x} = \mathcal{A}_0$ such that

$$\operatorname{Frac} \mathcal{O}_{X,x} = \operatorname{Frac} \mathcal{O}_{\mathbb{C}^n,0}(\xi).$$

Let $P_0 \in \mathcal{O}_{\mathbb{C}^n,0}[X]$ be the minimal polynomial of ξ. Up to shrinking V, we may assume that ξ spreads to a section $f \in \mathcal{A}(V)$. Then $\mathcal{B} = \mathcal{O}_V[f]$ is a finite sub-$\mathcal{O}_V$-algebra of $\mathcal{A}$. Up to shrinking V, we may assume that the kernel of $\mathcal{O}_V[X] \to \mathcal{B}$ sending X to f is generated by a unitary polynomial $P \in \mathcal{O}_V(V)[X]$ of degree $d := [\operatorname{Frac} \mathcal{O}_{X,x} : \operatorname{Frac} \mathcal{O}_{\mathbb{C}^n,0}]$ that extends P_0. Therefore,

$$\mathcal{B} \cong \mathcal{O}_V[X]/(P).$$

Let $T = \operatorname{Supp} \mathcal{A}/\mathcal{B}$. We endow T with the structure of closed analytic subspace of V induced by the annihilator of $\mathcal{A}/\mathcal{B}$. Observe that $\mathcal{A}_0/\mathcal{B}_0 = \mathcal{O}_{X,x}/\mathcal{O}_{\mathbb{C}^n,0}$ is torsion, so $|T|_0 = \operatorname{Supp} \mathcal{A}_0/\mathcal{B}_0 \neq \operatorname{Spec} \mathcal{O}_{\mathbb{C}^n,0}$. That is, $T_0 \neq \mathbb{C}^n_0$ by **??** in **??**. Observe that $X \setminus \varphi^{-1}(T) = \operatorname{Spec}^{\mathrm{an}}_{V\setminus T} \mathcal{B}|_{V\setminus T}$.

On the other hand, $P_0'(\xi) \neq 0$ as ξ is separable. So $W(P'(f)) \neq |X|_x$. Let $Z = \operatorname{Supp} \mathcal{O}_X/(P'(f))$, then φ is unramified outside T. Include the parts regarding unramified morphisms and étale morphisms before this section In particular, φ is étale outside T and hence a local isomorphism by **??** in **??**. In particular,

$$X^{\mathrm{sing}} \subseteq Z \cup \varphi^{-1}(T)$$

and hence

$$X^{\mathrm{sing}}_x \subseteq Z_x \cup \varphi^{-1}(T)_x.$$

The latter is not equal to $|X|_x$ by **??** in **??** and the fact that $\mathcal{O}_{X,x}$ is integral. □

Theorem 3.5 (Abhyankar)**.** Let X be a complex analytic space and $x \in X$, then

$$X^{\mathrm{Sing}}_x = (\operatorname{Spec} \mathcal{O}_{X,x})^{\mathrm{Sing}}.$$

Proof. Let $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x}$. In concrete terms, we need to show that $W(\mathfrak{p}) \not\subseteq X^{\mathrm{Sing}}_x$ if and only if $\operatorname{Spec} \mathcal{O}_{X,x}$ is regular at $\mathfrak{p}$.

The problem is local on X. Up to shrinking X, we may assume that $\mathfrak{p}$ spreads to a coherent ideal $\mathcal{I}$ on X. Let Y be the closed analytic subspace of X defined by $\mathcal{I}$. By **??**, up to shrinking X, we may assume that $\operatorname{codim}_y(Y, X)$ is constant for $y \in Y$. We denote this common value as p, which is necessarily equal to the height of $\mathfrak{p}$.

As Y_x is irreducible by assumption, for an analytic set Z in Y satisfying $Z_x \neq |Y|_x$, the following conditions are equivalent:

(1) $|Y|_x \not\subseteq X^{\mathrm{Sing}}_x$;
(2) $(|Y| \setminus Z)_x \not\subseteq X^{\mathrm{Sing}}_x$.

(2) $\implies$ (1) is trivial. If (2) fails, then

$$|Y|_x = (|Y| \cup X^{\mathrm{Sing}})_x \cup Z_x.$$

So $|Y|_x = (|Y| \cup X^{\mathrm{Sing}})_x$, namely (1) holds. We apply this remark to

$$Z = Y^{\mathrm{Sing}} \cup S_{p'}(\mathcal{I}/\mathcal{I}^2),$$

where p' is the dimension of the Zariski tangent space of $\operatorname{Spec} \mathcal{O}_{X,x}$ at $\mathfrak{p}$ and $S_{p'}(\mathcal{I}/\mathcal{I}^2)$ is the locus where $\mathcal{I}/\mathcal{I}^2$ is not locally free of rank p'. Note that neither part of Z is equal to $|Y|_x$, the former follows from **??** and the latter follows from **??** in **??** as clearly $\mathfrak{p} \notin S_{p'}(\mathcal{I}/\mathcal{I}^2)$. We find that $W(\mathfrak{p}) \not\subseteq X^{\mathrm{Sing}}_x$ if and only if $(|Y| \setminus Z)_x \not\subseteq X^{\mathrm{Sing}}_x$.

If $y \in |Y| \setminus Z$, then y is a regular point of Y and $\operatorname{codim}_y(Y, X) = p$. On the other hand, $\mathcal{I}/\mathcal{I}^2$ is free of rank p' around y. But given the regularity of $\mathcal{O}_{Y,y}$, the regularity of $\mathcal{O}_{X,y}$ is equivalent to the fact that $\mathcal{I}/\mathcal{I}^2$ is free of rank p. Or equivalently to $p = p'$. The latter is equivalent to the regularity of $\mathfrak{p}$ in $\operatorname{Spec} \mathcal{O}_{X,x}$. The theorem is established. □

Proposition 3.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Then the following are equivalent:

(1) X is regular at x and Y is regular at y;
(2) $X \times Y$ is regular at (x, y).

This follows from **??** in **??** and **??**.

Theorem 3.7. Let X be a complex manifold and A be a thin subset of X. Let $f \in \mathcal{O}_X(X \setminus A)$. Assume that either of the following conditions hold:

(1) f is locally bounded near A;
(2) A is thin of order 2 in X.

Then f admits a unique extension to an element in $\mathcal{O}_X(X)$.

Proof. The problem is local on X. By **??**, we may assume that X is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In this case, the results are the classical Riemann extension theorem. □

Corollary 3.8. Let X be a connected complex manifold and A be a thin set in X. Then $X \setminus A$ is connected.

Proof. Assume that $X \setminus A$ can be written as the disjoint union of two open subsets U_0, U_1. Then the function $f \in \mathcal{O}_X(X \setminus A) = \mathcal{O}_X(U_0) \times \mathcal{O}_X(U_1)$ given by $0 \in \mathcal{O}_X(U_0)$ and $1 \in \mathcal{O}_X(U_1)$ is locally bounded near A. By **??**, f admits a unique extension to $g \in \mathcal{O}_X(X)$. As X is connected and the image of f is contained in $\overline{\{0,1\}} = \{0, 1\}$, it follows that f is constant, so U_0 or U_1 has to be empty. □

4. Serre's condition R_n

Fix $n \in \mathbb{N}$ in this section.

Definition 4.1. Let X be a complex analytic space, we say X *satisfies* R_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* R_n at $x \in X$.

We say X *satisfies* R_n if X satisfies R_n at all points $x \in X$.

Proposition 4.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x;
(2) $\hat{\mathcal{O}}_{X,x}$ satisfies R_n.

Proof. This follows from [**stacks-project**]. □

Proposition 4.3. Let X be a complex analytic space, $x \in X$ and $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x;
(2) $\operatorname{codim}_x(X^{\mathrm{Sing}}, X) > n$.

Proof. It follows from **??** that (1) holds if and only if $\operatorname{codim}_x(X_x^{\mathrm{Sing}}, \operatorname{Spec} \mathcal{O}_{X,x}) > n$, The latter condition is equivalent to (2) by definition. □

Corollary 4.4. Let X be a complex analytic space and $n \in \mathbb{N}$. The

$$\{x \in X : X \text{ satisfies } R_n \text{ at } x\}$$

is co-analytic in X.

Proof. This follows from **??** and **??**. □

Proposition 4.5. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies R_n at x and Y satisfies R_n at y;
(2) $X \times Y$ satisfies R_n at (x, y).

Proof. By **??**,

$$(X \times Y)^{\mathrm{Sing}} = (X^{\mathrm{Sing}} \times Y) \cup (X \times Y^{\mathrm{Sing}}).$$

It follows that

$$\operatorname{codim}_{(x,y)}((X \times Y)^{\mathrm{Sing}}, X \times Y) = \min \left\{ \operatorname{codim}_x(X^{\mathrm{Sing}}, X), \operatorname{codim}_y(Y^{\mathrm{Sing}}, Y) \right\}$$

We conclude by **??**. □

5. Serre's condition S_n

Fix $n \in \mathbb{N}$ in this section.

Definition 5.1. Let X be a complex analytic space, we say X *satisfies* S_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* S_n at $x \in X$.

We say X *satisfies* S_n if X satisfies S_n at all points $x \in X$.

Proposition 5.2. Let X be a complex analytic space and $x \in X$. Take $n \in \mathbb{N}$. Then the following are equivalent:

(1) X satisfies S_n at x;
(2) $\hat{\mathcal{O}}_{X,x}$ satisfies S_n.

Proof. This follows from the fact that $\mathcal{O}_{X,x}$ is the quotient of a regular local ring. Include a reference □

Proposition 5.3. Let X be a complex analytic space, $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and $n \in \mathbb{N}$. Then

$$\left\{ x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n \right\}$$

is an analytic subset of X. Moreover, the germ of this set in $\operatorname{Spec} \mathcal{O}_{X,x}$ is equal to

$$\left\{ \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X,x} : \operatorname{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}} > n \right\}.$$

Proof. **Step 1**. We reduce to the case where X is smooth and equidimensional of dimension N.

The problem is local in X, so we may assume that X is a complex model space. Assume that X is a closed analytic subspace of a domain U in $\mathbb{C}^m$ for some $m \in \mathbb{N}$. For any $x \in X$, we have

$$\operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \operatorname{codep}_{\mathcal{O}_{U,x}} \mathcal{G}_x,$$

where $\mathcal{G}$ is the zero-extension of $\mathcal{F}$ to U. A similar formula holds for $\operatorname{codep}_{\mathcal{O}_{X,x,\mathfrak{p}}} \mathcal{F}_{x,\mathfrak{p}}$. So it suffices to handle U instead of X.

Step 2. We prove the result after the reduction in Step 1.

By Auslander–Buchsbaum formula, for $x \in X$,

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x + \operatorname{dep}_{\mathcal{O}_{X,x}} \mathcal{F}_x = \operatorname{dep} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x}.$$

So the condition $\operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n$ is equivalent to

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \dim \mathcal{O}_{X,x} - \dim_x \operatorname{Supp} \mathcal{F}.$$

As $\mathcal{O}_{X,x}$ is regular hence equidimensional, the condition just means

$$\operatorname{pd}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n + \operatorname{codim}_x(\operatorname{Supp}\mathcal{F}, X).$$

As $\mathcal{O}_{X,x}$ is regular, this condition is equivalent to the existence of an integer $r > n + \operatorname{codim}_x(\operatorname{Supp}\mathcal{F}, X)$ such that

$$\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_x \neq 0.$$

For each $p \in \mathbb{N}$, we introduce

$$T_p(\mathcal{F}) := \bigcup_{r=p+1}^{N} \operatorname{Supp}\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

Then the proceeding analysis shows that

$$\left\{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\} = \bigcup_{s=0}^{N} T_{n+s}(\mathcal{F}) \cap \{y \in \operatorname{Supp}\mathcal{F} : \operatorname{codim}_y(\operatorname{Supp}\mathcal{F}, X) \leq s\}.$$

Observe that the right-hand side is an analyitc set in X by **??** in **??** and **??**, hence so is the left-hand side.

It remains to compute the germ at $y \in X$. For $p \in \mathbb{N}$, we compute

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^{N} \operatorname{Supp}\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_y.$$

But observe that

$$\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)_y = \mathrm{Ext}^r_{\mathcal{O}_{X,y}}(\mathcal{F}_y, \mathcal{O}_{X,y}).$$

Let $\widetilde{\mathcal{F}_y}$ be the coherent module on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated with $\mathcal{F}_y$. Let $X_y = \operatorname{Spec}\mathcal{O}_{X,y}$ Then

$$T_p(\mathcal{F})_y = \bigcup_{r=p+1}^{N} \operatorname{Supp}\mathcal{E}\mathrm{xt}^r_{\mathcal{O}_{X_y}}(\widetilde{\mathcal{F}_y}, \mathcal{O}_{X_y})_y.$$

On the other hand, by **??**, for $s \in \mathbb{N}$,

$$\{x \in \operatorname{Supp}\mathcal{F} : \operatorname{codim}_x(\operatorname{Supp}\mathcal{F}, X) \leq s\}_y = \left\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,y} : \operatorname{codim}_{\mathfrak{p}}(\operatorname{Supp}\widetilde{\mathcal{F}_y}, \operatorname{Spec}\mathcal{O}_{X,y})\right\}.$$

The same argument as above shows that

$$\left\{x \in X : \operatorname{codep}_{\mathcal{O}_{X,x}} \mathcal{F}_x > n\right\}_y = \left\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,y} : \operatorname{codep}_{\mathcal{O}_{X,y,\mathfrak{p}}} \mathcal{F}_{y,\mathfrak{p}} > n\right\}.$$

□

Proposition 5.4. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is the complement of

$$\bigcup_{m=0}^{\infty} \{y \in Z_m : \operatorname{codim}_y(Z_m, X) \leq n + m\},$$

where

$$Z_m = \{x \in X : \operatorname{codep}\mathcal{O}_{X,x}\mathcal{F}_x > m\}.$$

Proof. It suffices to observe that for $x \in X$, X satisfies S_n at x if and only if

$$\operatorname{codim}\left(\{\mathfrak{p} \in \operatorname{Spec}\mathcal{O}_{X,x} : \operatorname{codep}\mathcal{O}_{X,x,\mathfrak{p}}\}, \operatorname{Spec}\mathcal{O}_{X,x}\right) > n + m$$

for all $m \in \mathbb{N}$. □

Corollary 5.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set of $x \in X$ such that X satisfies S_n at x is co-analytic.

PROOF. This follows from **??** and **??**. □

Proposition 5.6. Let X, Y be complex analytic spaces and $x \in X$, $y \in Y$. Take $n \in \mathbb{N}$. Assume that X satisfies S_n at x and Y satisfies S_n at y, then $X \times Y$ satisfies S_n at (x, y).

PROOF. By **??** in **??**,

$$\hat{\mathcal{O}}_{X\times Y,(x,y)} = \hat{\mathcal{O}}_{X,x} \hat{\otimes} \hat{\mathcal{O}}_{Y,y}.$$

As being S_n is invariant under completion by [**stacks-project**] and [**stacks-project**], it suffices to prove the corresponding algebraic result, which is known. □

6. Reducedness

Definition 6.1. Let X be a complex analytic space, we say X is *reduced* at $x \in X$ if $\mathcal{O}_{X,x}$ is reduced. We also say (X, x) or X_x is *reduced* at $x \in X$.

We say X is *reduced* if X is reduced at all points $x \in X$.

Proposition 6.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is reduced x;
(2) $\hat{\mathcal{O}}_{X,x}$ is reduced.

PROOF. This follows from **??** and **??**.

Otherwise, one can also argue as follows: Recall that an excellent ring is Nagata by [**stacks-project**]. A Nagata noetherian local ring is reduced if and only if its completion is by [**stacks-project**]. □

Theorem 6.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is reduced is co-analytic.

PROOF. This follows from **??** and **??** as reduceness is equivalent to S_1 and R_0. □

Corollary 6.4. Let X be a complex analytic space, then the nilradical $\operatorname{rad} \mathcal{O}_X$ is coherent.

PROOF. The problem is local on X. Take $x \in X$. Up to shrinking X, we may assume that $\mathcal{O}_{X,x}/(\operatorname{rad} \mathcal{O}_X)_x$ spreads to a finite $\mathcal{O}_X$-algebra $\mathcal{A}$ by **??** in **??**. Up to further shrinking X, we may assume that $\mathcal{A}$ is the quotient of $\mathcal{O}_X$, say $\mathcal{A} \cong \mathcal{O}_X/\mathcal{I}$ for some coherent ideal $\mathcal{I}$ on X. As $\mathcal{I}_x$ is nilpotent by assumption, up to shrinking X, we may assume that $\mathcal{I}$ is also nilpotent, namely

$$\mathcal{I} \subseteq \operatorname{rad} \mathcal{O}_X.$$

Let Y be the closed analytic subspace of X defined by the ideal $\mathcal{I}$. Then $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}/(\operatorname{rad} \mathcal{O}_X)_x$ is reduced. Up to shrinking X, by **??**, we may assume that Y is reduced. But then for any $y \in Y$,

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y$$

is reduced, so

$$\mathcal{I}_y \supseteq (\operatorname{rad} \mathcal{O}_X)_y.$$

It follows that $\operatorname{rad} \mathcal{O}_X = \mathcal{I}$, hence the nilradical is coherent. □

Corollary 6.5 (Cartan–Oka)**.** Let X be a complex analytic space and A be an analytic subset of X, then the sheaf $\mathcal{J}_A$ is coherent.

Recall that the sheaf $\mathcal{J}_A$ is introduced in **??** in **??**.

Proof. By **??** in **??**, we may assume that A is a closed analytic subspace of X defined by a coherent ideal $\mathcal{I}$. By **??** in **??**, the sheaf $\mathcal{J}_A$ is nothing but $\sqrt{I}$, which is coherent by **??**. $\square$

Corollary 6.6. Let X be a complex analytic space and A be an analytic subset of X, then there is a unique reduced closed analytic space Y of X with underlying set A.

Proof. The existence follows from **??**. The uniqueness follows from **??** in **??**. $\square$

Definition 6.7. Let X be a complex analytic space and A be an analytic subset of X. The analytic space structure on A defined in **??** is called the *reduced induced structure* on A. In particular, $|X|$ with the reduced induced structure is denoted by X^{red} and is called the *reduced space underlying* X.

Theorem 6.8 (Generic smoothness)**.** Let X be a reduced complex analytic space and $x \in X$, then $X^{\mathrm{Sing}}_x \neq |X|_x$. In other words, X^{Sing} is nowhere dense in $|X|$.

Proof. The problem is local. Take $x \in X$. As in the proof of **??**, up to shrinking X, we may assume that there are finitely many closed analytic subsets $X_1, \ldots, X_m$ in X which are irreducible at x such that

$$X = X_1 \cup \cdots \cup X_m.$$

As X is reduced, we may also assume that $X_1, \ldots, X_m$ are all reduced. Then $X_1, \ldots, X_m$ are all integral at x. It follows from **??** that

$$X_i^{\mathrm{Sing}} \neq |X_i|_x$$

for $i = 1, \ldots, m$. Let $\mathcal{I}_i$ be the coherent ideal sheaf of X_i in X for $i = 1, \ldots, m$. It follows from the proof of **??** that

$$X^{\mathrm{sing}} = \bigcap_{i=1}^{m} \left(\operatorname{Supp} \mathcal{I}_i \cup X_i^{\mathrm{Sing}} \right).$$

This implies $X^{\mathrm{Sing}}_x \neq |X|_x$: otherwise, for each $i = 1, \ldots, m$, we have

$$(\operatorname{Supp} \mathcal{I}_i)_x \cup (X_i^{\mathrm{Sing}})_x = |X|_x.$$

So

$$(\operatorname{Supp} \mathcal{I}_i)_x = |X|_x$$

for each $i = 1, \ldots, m$. In other words,

$$\operatorname{Spec} \mathcal{O}_{X,x} = \bigcup_{i=1}^{m} \operatorname{Supp} \mathcal{I}_{i,x}.$$

Observe that $\mathcal{I}_{1,x}, \ldots, \mathcal{I}_{m,x}$ are exactly the minimal primes of $\operatorname{Spec} \mathcal{O}_{X,x}$. This is possible if and only if $m = 1$. So we are reduced to the case where X is integral at x. But this case is handled in **??**. $\square$

Proposition 6.9. Let X be a reduced complex analytic space and $f, g \in \mathcal{O}_X(X)$. Assume that $[f] = [g]$, then $f = g$.

PROOF. It follows from **??** in **??** that $f-g$ is locally nilpotent. As X is reduced, $f=g$. $\square$

In particular, on a reduced complex analytic space X, a holomorphic function f is uniquely determined by the continuous map $[f]: X \to \mathbb{C}$ associated with it. In this case, we will say $[f]$ is holomorphic.

Definition 6.10. Let X be a reduced complex analytic space. A *continuous weakly holomorphic function* on X is a continuous map $f: X \to \mathbb{C}$ such that $f|_{X^{\mathrm{reg}}}$ is holomorphic.

A *weakly holomorphic function* on X is $f \in \mathcal{O}_X(X^{\mathrm{reg}})$ which is locally bounded on X.

Definition 6.11. Let $f: X \to Y$ be a topologically finite surjective morphism of reduced complex analytic spaces. We say f is a *branched covering* if there is a thin subset T of Y satisfying the following properties:

(1) $\pi^{-1}(T)$ is thin in X;
(2) $X \setminus \pi^{-1}(T) \to Y \setminus T$ induced by f is a local isomorphism.

The set T is called a *critial locus*.

The set of points $x \in X$ where f is not a local isomorphism at x is called the *branch locus* of f. The image of the branch locus in Y is called the *minimal critical locus* of f.

Observe that the number of points in the fiber is locally constant outside the critical locus. When this number is actually constant say $b \in \mathbb{N}$ (e.g. when Y is a connected complex manifold by **??**), we say f is a *b-sheeted branched covering*.

7. Normalness

Definition 7.1. Let X be a complex analytic space, we say X is *normal* at $x \in X$ if $\mathcal{O}_{X,x}$ is normal. We also say (X,x) or X_x is *normal* at $x \in X$.

We say X is *normal* if X is normal at all points $x \in X$.

Proposition 7.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is normal x;
(2) $\hat{\mathcal{O}}_{X,x}$ is normal.

Condition (2) is usually known as the *analytic normality* of $\mathcal{O}_{X,x}$.

PROOF. This follows from **??** and **??**. $\square$

Theorem 7.3. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is normal is co-analytic.

PROOF. This follows from **??** and **??** as reduceness is equivalent to S_2 and R_1. $\square$

Proposition 7.4. Let X be a normal complex analytic space. Then for any $x \in X^{\mathrm{Sing}}$,

$$\operatorname{codim}_x(X^{\mathrm{Sing}}, X) \geq 2.$$

PROOF. This follows from **??** and the corresponding algebraic result. $\square$

Proposition 7.5. Let X be a reduced complex analytic space. Then there is a finite $\mathcal{O}_X$-algebra $\overline{\mathcal{O}}_X$ such that for each $x \in X$, $\overline{\mathcal{O}}_{X,x}$ is isomorphism to the inclusion of the integral closure $\overline{\mathcal{O}_{X,x}}$ as $\mathcal{O}_{X,x}$-algebras.

The sheaf $\overline{\mathcal{O}}_X$ is unique up to a unique isomorphism.

Proof. The uniqueness is obvious, as there are no non-trivial automorphisms of $\overline{\mathcal{O}}_{X,x}$ as an $\mathcal{O}_{X,x}$-algebra.

We prove the existence. The problem is then local on X. Let $x \in X$. By **??** in **??**, up to shrinking X, $\overline{\mathcal{O}_{X,x}}$ spreads to a finite $\mathcal{O}_X$-algebra $\mathcal{A}$. Let $X' = \operatorname{Spec}_X^{\mathrm{an}} \mathcal{A}$. Let $x'_1, \dots, x'_m$ be the distinct points on the fiber over x of $X' \to X$. By **??** in **??**, the points corresponds to $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_x$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_{m'}$ be the minimal primes of $\mathcal{O}_{X,x}$, then

$$\mathcal{A}_x = \overline{\mathcal{O}_{X,x}} \cong \prod_{i=1}^{m'} \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}.$$

As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is Henselian, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is in fact local for each $i = 1, \dots, m'$. As $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, $\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$ is finite over $\mathcal{O}_{X,x}/\mathfrak{p}_i$. It follows that $\operatorname{Spm}_{\mathbb{C}} \mathcal{A}_x = \operatorname{Spm} \mathcal{A}_x$. So we find that $m' = m$. Up to a renumbering, we may assume that $\mathfrak{p}_i$ corresponds to x'_i for $i = 1, \dots, m$. Then by **??** in **??**,

$$\mathcal{O}_{X',x'_i} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}_i}$$

for $i = 1, \dots, m$. In particular, X' is normal at x'_i for all $i = 1, \dots, m$. By **??**, **??** in **??** and **??** in **??**, up to shrinking X, we may assume that X' is normal. We observe that for each $y \in X$, $\mathcal{A}_y$ is the product of the local rings of points on the fiber hence normal.

For $i = 1, \dots, m$, as $\mathcal{O}_{X,x}/\mathfrak{p}_i$ is excellent, its conductor is non-zero. We can find a non-zero $f_{i,x} \in \mathcal{O}_{X,x}/\mathfrak{p}_i$ such that $f_{i,x}\overline{\mathcal{O}_{X,x}/\mathfrak{p}_i} \subseteq \mathcal{O}_{X,x}/\mathfrak{p}_i$. Take

$$f_x = \prod_{i=1}^{m} f_{i,x}.$$

Then f_x is a non-zero divisor in $\mathcal{O}_{X,x}$ and $f_x\mathcal{A}_x \subseteq \mathcal{O}_{X,x}$. Up to shrinking X, we may assume that f_x spreads to $f \in \mathcal{O}_X(X)$, and we have an injection

$$f\mathcal{A} \subseteq \mathcal{O}_X.$$

Up to shrinking X, we may also assume that $\mathcal{O}_X \to \mathcal{A}$ is injective. We therefore get an injective map

$$\mathcal{A} \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we get an injective map

$$\mathcal{A}_y \to \mathcal{O}_{X,y}[f_y^{-1}].$$

In particular, $\mathcal{A}_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}$. As $\mathcal{A}_y$ is finite over $\mathcal{O}_{X,y}$, we have

$$\mathcal{A}_y \subseteq \overline{\mathcal{O}_{X,y}}.$$

On the other hand, $\mathcal{A}_y$ is normal, so equality holds. □

Definition 7.6. Let X be a reduced complex analytic space. Then $\operatorname{Spec}_X^{\mathrm{an}} \overline{\mathcal{O}}_X$ constructed in **??** is called the *normalization* of X. We denote it by $\bar{X}$. Note that we have a canonical morphism $\bar{X} \to X$.

The normalization of X is well-defined up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$.

Proposition 7.7. Let X be a reduced complex analytic space. For each $x \in X$, the fiber of $\bar{X} \to X$ over x is in bijection with the set of minimal prime ideals in $\mathcal{O}_{X,x}$. Moreover, if y corresponds to $\mathfrak{p}$, we have

$$\mathcal{O}_{\bar{X},y} \cong \overline{\mathcal{O}_{X,x}/\mathfrak{p}}$$

as $\mathcal{O}_{X,x}$-algebras.

Proof. This follows from the proof of **??**. $\square$

Proposition 7.8. Let X be a reduced complex analytic space. Then

(1) $\bar{X}$ is normal;
(2) $p : \bar{X} \to X$ is topologically finite and surjective;
(3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in $\bar{X}$ and the morphism $\bar{X} \setminus p^{-1}(Y) \to X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines $\bar{X}$ up to a unique isomorphism in $\mathbb{C}$-$\mathcal{A}\mathrm{n}_{/X}$. We will establish this result later.

Proof. That $\bar{X}$ is normal follows from **??** in **??**. The morphism $\bar{X} \to X$ is topologically finite by **??** in **??**. It is surjective by **??** in **??**.

Let Y be the non-normal locus of X. It is in particular contained in X^{Sing}. By **??** and **??**, Y is a nowhere dense analytic set in X. It is clear that p is an isomorphism outside Y.

We prove that $p^{-1}(Y)$ is nowhere dense. Let $x \in X$ and x' be a point on the fiber of $\bar{X} \to X$ over x. Let $\mathfrak{p}'$ be the minimal prime ideal of $\mathcal{O}_{X,x}$ corresponding to x'. Then the morphism $\operatorname{Spec} \mathcal{O}_{\bar{X},x'} \to \operatorname{Spec} \mathcal{O}_{X,x}$ factorizes through $\operatorname{Spec} \mathcal{O}_{\bar{X},x'} \to \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{p}'$. The map is finite and surjective. The latter is because $\mathcal{O}_{X,x}/\mathfrak{p}' \to \mathcal{O}_{\bar{X},x'}$ is injective. If $p^{-1}(Y)$ contains a neighbourhood of x' in $\bar{X}$, then $|p^{-1}(Y)|_{x'} = \operatorname{Spec} \mathcal{O}_{\bar{X},x'}$. Then $|Y|_x = |\operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{p}'|$, which is a contradiction. $\square$

Definition 7.9. Let X be a complex analytic space and A be an analytic set in X. We say A is irreducible if A cannot be written as the union of two analytic sets B and C in X with $B \not\subset C$ and $C \not\subset B$.

Lemma 7.10. Let X be a connected normal complex analytic space. Then X is irreducible.

Proof. Suppose otherwise, X can be written as the union of A, B, two analytic sets in X not containing each other. As X is connected, $A \cap B$ is non-trivial. Take $x \in A \cap B$. We endow A and B with the reduced induced structure. Then

$$\operatorname{Spec} \mathcal{O}_{X,x} = \operatorname{Spec} \mathcal{O}_{A,x} \cup \operatorname{Spec} \mathcal{O}_{B,x}.$$

This is impossible as $\mathcal{O}_{X,x}$ is unibranch. $\square$

Definition 7.11. Let X be a reduced complex analytic space. An *irreducible component* of X is the image of a connected component of $\bar{X}$.

We say X is *irreducible* if it is non-empty has only one irreducible component.

By **??**, each irreducible component is irreducible. Moreover, by **??**, the decomposition of $|X|$ into the union of its irreducible components is locally finite. No irreducible component is contained in the union of the others.

Proposition 7.12. Let X be a reduced complex analytic space and $x \in X$. Then x be be joined by a path to a point in X^{reg}.

Proof. We may assume that $x \in X^{\mathrm{Sing}}$.

Step 1. We reduce to the case where X is normal.

Let $p : \bar{X} \to X$ be the normalization. Take $y \in \bar{X}$ with $p(y) = x$.

We claim that it suffices to show that there is a path connecting y to a regular point of $\bar{X}$. In fact, let $T \subseteq X$ containing X^{Sing} be a thin analytic set such that $p^{-1}(T)$ is thin and $\bar{X} \setminus p^{-1}(T) \to X \setminus T$ induced by p is an isomorphism by **??**. If our claim holds, then all neighbourhood points of y are regular and in particular, we may connect y to a regular poiint in $\bar{X} \setminus p^{-1}(T)$. The image of this path is the desired path.

Step 2. We proceed by induction on $d := \dim_x X$.

When $d = 1$, x is necessarily regular by **??**. Assume $d > 1$. Up to shrinking X, we can take $f \in \mathcal{O}_X(X)$ such that $\dim_x W(f) = d - 1$. We may assume that $W(f)$ is equidimensional of dimension $d - 1$ by **??**. Then we can find a path from x to a regular point $x' \in W(f)$. By **??**, up to perturbation, we may assume that $x' \in X^{\mathrm{reg}}$. □

8. Unibranchness

Definition 8.1. Let X be a complex analytic space, we say X is *unibranch* at $x \in X$ if $\mathcal{O}_{X,x}$ is unibranch. We also say (X, x) or X_x is *unibranch* at $x \in X$.

We say X is *unibranch* if X is unibranch at all points $x \in X$.

Proposition 8.2. Let X be a complex analytic space and $x \in X$. Then the following are equivalent:

(1) X is unibranch at x;
(2) X^{red} is unibranch at x;
(3) $\mathcal{O}_{X,x}$ is geometrically unibranch;
(4) $\mathcal{O}_{X,x}^{\mathrm{red}}$ is geometrically unibranch;
(5) $\mathcal{O}_{X,x}$ has a unique minimal prime ideal;
(6) The fiber of $\overline{X^{\mathrm{red}}} \to X^{\mathrm{red}}$ over x consists of a single point.

Proof. (1) ⇔ (3): As $\mathcal{O}_{X,x}$ is excellent, $\overline{\mathcal{O}_{X,x}^{\mathrm{red}}}$ is a finite $\mathcal{O}_{X,x}^{\mathrm{red}}$-algebra, so the residue field extension is finite. But the residue field of $\mathcal{O}_{X,x}$ is $\mathbb{C}$, so the residue field extension is the trivial extension.

(1) ⇔ (5): This follows from [**stacks-project**] and the fact that $\mathcal{O}_{X,x}$ is Henselian.

(1) ⇔ (2): This follows from the observation that (5) holds for $\mathcal{O}_{X,x}$ if and only if (5) holds for $\mathcal{O}_{X,x}^{\mathrm{red}}$.

(3) ⇔ (4): This follows from the same argument as (1) ⇔ (2).

(5) ⇔ (6): This follows from **??**. □

Lemma 8.3. Let X be a complex analytic space, $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module, $n \in \mathbb{N}$. Then the set

$$\{x \in X : \operatorname{rank}_x \mathcal{M} \le n\}$$

is an analytic set in X.

PROOF. The problem is local on X, we may assume that $\mathcal{M}$ admits a presentation

$$\mathcal{O}_X^p \to \mathcal{O}_X^q \to \mathcal{M} \to 0,$$

where $p, q \in \mathbb{N}$. Up to shrinking X, we may assume that the first map is given by a $p \times q$ matrix M in $\mathcal{O}_X(X)$. The condition that $\operatorname{rank}_x \mathcal{M} \leq n$ is the same as $\operatorname{rank} M_x \leq n$, which is defines an analytic set in X. □

Lemma 8.4. Let X be a reduced complex analytic space and $x \in X$. Then for any neighbourhood V of x in X, we can find an open neighbourhood U of x in X contained in V such that U has only finitely many irreducible componenets and all irreducible componenets of U contain x.

PROOF. Take an open neighbourhood W of x in X contained in V such that $\bar{W}$ is compact and decompose W into irreducible components $W_1, \ldots, W_k, W_{k+1}, \ldots, W_n$, where $W_1, \ldots, W_k$ contain x and $W_{k+1}, \ldots, W_n$ do not. It suffices to take

$$U = \left(\bigcup_{i=1}^{k} W_i \right) \setminus \left(\bigcup_{j=k+1}^{n} W_j \right).$$

□

Proposition 8.5. Let X be a reduced complex analytic space and $x \in X$. Assume that X is unibranch at x. Then for any neighbourhood V of x in X, there is an open neighbourhood U of x in X contained in V such that U is unibranch and hence connected.

In particular, the unibranch locus is open.

PROOF. The assertion follows from **??**. □

Corollary 8.6. Let X be a complex analytic space. Then X is locally connected.

PROOF. We may assume that X is reduced. The assertion follows from **??** and □

9. Cohen–Macaulay property

Definition 9.1. Let X be a complex analytic space, we say X is *Cohen–Macaulay* at $x \in X$ if $\mathcal{O}_{X,x}$ is Cohen–Macaulay. We also say (X, x) or X_x is *Cohen–Macaulay* at $x \in X$.

We say X is *Cohen–Macaulay* if X is Cohen–Macaulay at all points $x \in X$.

The reduction and normalization of a Cohen–Macaylay space are not necessarily Cohen–Macaulay.

Theorem 9.2. Let X be a complex analytic space. Then the set

$$\{x \in X : X \text{ is Cohen–Macaulay at } x\}$$

is co-analytic.

PROOF. The set is exactly where $\{x \in X : \operatorname{codep}_x \mathcal{O}_{X,x} = 0\}$, which is co-analytic by **??**. □