Affinoid algebras

Contents

1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k, | \bullet |)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. We set

$$k\{r^{-1}T\} = k\{r_1^{-1}T_1, \ldots, r_n T_n^{-1}\}$$

$$:= \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \ldots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\}.$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$\|f\|_r = \max_\alpha |a_\alpha| r^\alpha.$$

We call $(k\{r^{-1}T\}, \| \bullet \|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\| \bullet \|_r$ is called the *Gauss norm*.

We omit r from the notation if $r = (1, \ldots, 1)$.

This is a special case of Example 4.15 in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\}, \| \bullet \|_r)$ is a Banach k-algebra and $\| \bullet \|_r$ is a valuation.

PROOF. This is a special case of Proposition 4.16 in the chapter Banach Rings. □

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T \rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T \rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\} \cong k[T_1, \ldots, T_n]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \ldots, T_n]]$ otherwise.

Lemma 2.5. Let A be a Banach k-algebra. For each $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathring{A}$, there is a unique continuous homomorphism $k\{T_1, \ldots, T_n\} \to A$ sending T_i to a_i.

PROOF. This is a special case of Proposition 4.17 in the chapter Banach Rings. □

3. Affinoid algebras

Let $(k, | \bullet |)$ be a complete non-Archimedean valued-field and $H \supseteq |k^\times|$ be a subgroup of $R_{>0}$.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

More generally, a Banach k-algebra A is k_H-*affinoid* if there are $n \in \mathbb{N}$, $r \in H^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

A morphism between k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is a bounded k-algebra homomorphism.

The category of k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is denoted by k-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$ (resp. st-k-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$, resp. k_H-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$). The opposite categories of these categories are denoted by k-$\mathcal{A}\text{ff}$, st-k-$\mathcal{A}\text{ff}$ and k_H-$\mathcal{A}\text{ff}$. For any A in k-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$ (resp. st-k-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$, resp. k_H-$\mathcal{A}\text{ff}\mathcal{A}\text{lg}$), the corresponding image in the opposite category is denoted by $\operatorname{Sp} A$. We can also identify $\operatorname{Sp} A$ with the topological space defined in Definition 6.1 in the chapter Banach Rings.

An affinoid k-algebra is a K-affinoid algebra for some complete non-Archimedean field extension K/k.

For the notion of admissible morphisms, we refer to Definition 2.5 in the chapter Banach rings.

Example 3.2. Let $r \in \mathbb{R}_{>0}$. We let K_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \to 0$ for $i \to \infty$ and $i \to -\infty$. We define a norm $\| \bullet \|_r$ on K_r as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in Proposition 3.3 that K_r is k-affinoid.

Proposition 3.3. Let $r \in \mathbb{R}_{>0}$, then $(K_r, \| \bullet \|_r)$ defined in Example 3.2 is a k-affinoid algebra. Moreover, $\| \bullet \|_r$ is a valuation.

PROOF. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \to K_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$(3.1) \qquad\qquad |a_{i,j}| r^{i-j} \to 0$$

as $i + j \to \infty$. Observe that for each $k \in \mathbb{Z}$, the series

$$c_k := \sum_{i-j=k, i,j \in \mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that $\iota(f) \in K_r$. That is

$$|c_k| r^k \to 0$$

as $k \to \pm\infty$. When $k \geq 0$, we have $|c_k| \leq |a_{k0}|$ by definition of c_k. So $|c_k|r^k \to 0$ as $k \to \infty$ by (3.1). The case $k \to -\infty$ is similar.

We conclude that we have a well-defined map of sets ι. It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g = \sum_{i=-\infty}^{\infty} c_i T^i \in K_r$. We want to show that g lies in the image of ι. As ι is a ring homomorphism, it suffices to treat two cases separately: $g = \sum_{i=0}^{\infty} c_i T^i$ and $g = \sum_{i=-\infty}^{0} c_i T^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$. It is immediate that $\iota(f) = g$.

Next we show that ι is admissible. We first identify the kernel of ι. We claim that the kenrel is the ideal I generated by $T_1 T_2 - 1$. It is obvious that $I \subseteq \ker \iota$. Conversely, consider an element

$$f = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kenrel of ι. Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If $f \in \ker \iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by $T_1 T_2 - 1$. The condition that $f_k \in \ker \iota$ means

$$\sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find $b_{i,j} \in k$ for $i, j \in \mathbb{N}$, $i - j = k$ such that

$$a_{i,j} = b_{i-1,j-1} - b_{i,j}.$$

Then

$$f_k = (T_1 T_2 - 1) \sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$. In particular, the sum of $\sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j$ for various k converges to some $g \in k\{r^{-1}T_1, rT_2\}$ and hence $f_k = (T_1 T_2 - 1)g$. Therefore, we have proved that $\ker \iota$ is generated by $T_1 T_2 - 1$.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \to K_r.$$

To see this, take $f = \sum_{k=-\infty}^{\infty} c_k T^k \in K_r$ and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$, then $\iota(g) = f$ and $\|g\|_{(r,r^{-1})} = \|f\|$. So it suffices to show that for any $h = \sum_{(i,j)\in\mathbb{N}^2} d_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$, we have

$$(3.2) \qquad\qquad \|f\|_r \leq \|g + h(T_1 T_2 - 1)\|_{r,r^{-1}}.$$

We compute

$$g+h(T_1T_2-1) = \sum_{k=1}^{\infty}(c_k-d_{k,0})T_1^k+\sum_{k=1}^{\infty}(c_{-k}-d_{0,k})T_2^k+(c_0-d_0)+\sum_{i,j\geq 1}(d_{i-1,j-1}-d_{i,j})T_1^iT_2^j.$$

So

$$\|g + h(T_1T_2 - 1)\|_{r,r^{-1}} = \max\left\{\max_{k\geq 0}C_{1,k}, \max_{k\geq 1}C_{2,k}\right\},$$

where

$$C_{1,k} = \max\left\{|c_k - d_{k,0}|, \left|\sum_{i-j=k,i,j\geq 1}d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k \geq 0$ and

$$C_{2,k} = \max\left\{|c_{-k} - d_{0,k}|, \left|\sum_{i-j=-k,i,j\geq 1}d_{i-1,j-1}-d_{i,j}\right|\right\}$$

for $k \geq 1$. It follows from the strong triangle inequality that $|c_k| \leq C_{1,k}$ for $k \geq 0$ and $c_{-k} \leq C_{2,k}$ for $k \geq 1$. So (3.2) follows. $\square$

Proposition 3.4. Let $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^{\times}|}$, then $\| \bullet \|_r$ defined in Example 3.2 is a valuation on K_r.

PROOF. Take $f, g \in K_r$, we need to show that

$$\|fg\|_r \geq \|f\|_r\|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_iT^i, \quad g = \sum_{i=-\infty}^{\infty} b_iT^i.$$

Take i and j so that

$$(3.3) \qquad |a_i|r^i = \|f\|_r, \quad |b_j|r^j = \|g\|_r.$$

By our assumption on r, i, j are unique. Then

$$\|fg\|_r = \max_{k\in\mathbb{Z}}\{|c_k|r^k\},$$

where

$$c_k := \sum_{u,v\in\mathbb{Z},u+v=k} a_ub_v.$$

It suffices to show that

$$(3.4) \qquad |c_k|r^k = \|f\|_r\|g\|_r.$$

for $k = i + j$. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. For $u, v \in \mathbb{Z}$, $u + v = i + j$ while $(u, v) \neq (i, j)$, we may assume that $u \neq i$. Then $|a_u|r^u < |a_i|r^i$ and $|b_v|r^v \leq |b_j|r^j$. So $|a_ub_v| < |a_ib_j|$ and we conclude (3.4). $\square$

Remark 3.5. The argument of Proposition 4.16 in the chapter Banch Rings does not work here if $r \in \sqrt{|k^{\times}|}$, as in general one can not take minimal i, j so that (3.3) is satisfied.

Proposition 3.6. Assume that $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^{\times}|}$. Then K_r is a valuation field and $\| \bullet \|_r$ is non-trivial.

PROOF. We first show that $\mathrm{Sp}\, K_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet| \in \mathrm{Sp}\, K_r$. As $\|\bullet\|_r$ is a valuation, we find

$$(3.5) \qquad\qquad\qquad |\bullet| \leq \|\bullet\|_r.$$

In particular, $|\bullet|$ restricted to k is the given valuation on k. It suffices to show that $|T| = r$. This follows from (3.5) applied to T and T^{-1}.

It follows that K_r does not have any non-zero proper closed ideals: if I is such an ideal, K_r/I is a Banach k-algebra. By Proposition 6.10 in the chapter Banach rings, $\mathrm{Sp}\, K_r$ is non-empty. So K_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.7 in the chapter Banach rings, the only maximal ideal of K_r is 0. It follows that K_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r = r$. $\square$

Definition 3.7. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_{\geq 0}^n$. Assume that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$. We define

$$K_r = K_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k K_{r_n}.$$

By an interated application of Proposition 3.6, K_r is a complete valuation field. As a general explanation of why K_r is useful, we prove the following proposition:

Proposition 3.8. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_{\geq 0}^n$. Assume that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$.

(1) For any k-Banach space X, the natural map

$$X \to X \hat{\otimes}_k K_r$$

is an isometric embedding.

(2) Consider a sequence of bounded homomorphisms of k-Banch spaces $X \to Y \to Z$. Then the sequence is admissible and exact (resp. coexact) if and only if $X \hat{\otimes}_k K_r \to Y \hat{\otimes}_k K_r \to Z \hat{\otimes}_k K_r$ is admissible and exact (resp. coexact).

PROOF. We may assume that $n = 1$.

(1) We have a more explicit description of $X \hat{\otimes}_k K_r$: as a vector space, it is the space of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ with $a_i \in X$ and $\|a_i\| r^i \to 0$ when $|i| \to \infty$. The norm is given by $\max_i \|a_i\| r^i$. From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1). $\square$

When X is a Banach k-algebra, $X \hat{\otimes}_k K_r$ is a Banach K_r-algebra.

Proposition 3.9. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite bounded homomorphism into a k-Banach algebra A. Then A is also strictly k-affinoid.

PROOF. We may assume that $B = k\{T_1, \ldots, T_n\}$ for some $n \in \mathbb{N}$. By assumption, we can find finitely many $a_1, \ldots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(B) a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.17 in the chapter Banach Rings, φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{T_1, \ldots, T_n, S_1, \ldots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is strictly k-affinoid. $\square$

Lemma 3.10. Assume that k is non-trivially valued. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. The algebra $k\{r^{-1}T\}$ is strictly k-affinoid if $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$.

Remark 3.11. The converse is also true.

PROOF. Assume that $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$. Take $s_i \in \mathbb{N}$ and $c_i \in k^\times$ such that
$$r_i^{s_i} = |c_i^{-1}|$$
for $i = 1, \ldots, n$. We deifne a bounded k-algebra homomorphism $\varphi : k\{T_1, \ldots, T_n\} \to k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ by sending T_i to $c_i T_i^{s_i}$. This is possible by Proposition 4.17 in the chapter Banach Rings.

We claim that φ is finite. To see this, it suffices to observe that if we expand $f \in k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ as
$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$
we can regroup
$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (cT^s)^\gamma,$$
where the product γs is taken component-wise. For each $\beta \in \mathbb{N}^n, \beta_i < s_i$, we set
$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \ldots, T_n\}.$$

While $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$. So We have shown that φ is finite. Hence, $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is k-affinoid by Proposition 3.9. $\square$

Proposition 3.12. Let A be a k-affinoid algebra, then there is $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ such that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ and such that $A\hat{\otimes}_k K_r$ is strictly K_r-affinoid. Moreover, we can guarantee that K_r is non-trivially valued.

PROOF. By Proposition 3.8, we may assume that $A = k\{t^{-1}T\}$ for some $t \in \mathbb{R}^m_{>0}$. By Lemma 3.10, it suffices to take r so that the linear subspace of $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ generated by $r_1, \ldots, r_n$ contains all components of t. By taking $n \geq 1$, we can guarantee that K_r is non-trivially valued. $\square$

Proposition 3.13. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a monomorphism in k_H-$\mathcal{A}$ff. Then for any $y \in \operatorname{Sp} B$ with $x = \varphi(y)$, one has $\varphi^{-1}(x) = \{y\}$ and the natural map $\mathscr{H}(x) \to \mathscr{H}(y)$ is an isomorphism of complete valuation rings.

PROOF. It suffices to show that $\mathscr{H}(x) \to B\hat{\otimes}_A \mathscr{H}(y)$ is an isomorphism as Banach k-algebras. Include details about cofiber products in affalg. By assumption, the codiagonal map $B\hat{\otimes}_A B \to B$ is an isomorphism. It follows that the base change with respect to $A \to \mathscr{H}(x)$ is also an isomorphism: $B'\hat{\otimes}_{\mathscr{H}(x)} B' \to B'$, where $B' = B\hat{\otimes}_A \mathscr{H}(x)$.

Include the fact that the first map is injective. It follows that the composition $B' \otimes_{\mathscr{H}(x)} B \to B'\hat{\otimes}_{\mathscr{H}(x)} B' \to B'$ is injective. Therefore, $\mathscr{H}(x) \to B'$ is an

isomorphism of rings. We also know that this map is bounded. But we already know that $\mathscr{H}(x)$ is a complete valuation ring, so the map $\mathscr{H}(x) \to B'$ is an isomorphism of complete valuation rings. $\qquad\square$

4. Weierstrass theory

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 4.1. We have canonical identifications

$$(k\{T_1, \ldots, T_n\})^\circ \cong \mathring{k}\{T_1, \ldots, T_n\},$$
$$(k\{T_1, \ldots, T_n\})^\vee \cong \check{k}\{T_1, \ldots, T_n\},$$
$$\widetilde{k\{T_1, \ldots, T_n\}} \cong \tilde{k}[T_1, \ldots, T_n].$$

The last identification extends $\mathring{k} \to \tilde{k}$ and T_i is mapped to T_i.

PROOF. This follows from Corollary 4.19 from the chapter Banach rings. $\qquad\square$

We will denote the reduction map $\mathring{k}\{T_1, \ldots, T_n\} \to \tilde{k}[T_1, \ldots, T_n]$ by $\tilde{\bullet}$.

Definition 4.2. Let $n \in \mathbb{N}$. A system $f_1, \ldots, f_n \in k\{T_1, \ldots, T_n\}$ is called an *affinoid chart* of $k\{T_1, \ldots, T_n\}$ if $f_i \in \mathring{k}\{T_1, \ldots, T_n\}$ for each $i = 1, \ldots, n$ and the continuous k-algebra homomorphism $k\{T_1, \ldots, T_n\} \to k\{T_1, \ldots, T_n\}$ sending T_i to f_i is an isomorphism.

The map $k\{T_1, \ldots, T_n\} \to k\{T_1, \ldots, T_n\}$ is well-defined by Proposition 4.1 and Lemma 2.5.

Lemma 4.3. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$. Assume that $\|f\|_1 = 1$. Then the following are equivalent:

(1) f is a unit $k\{T_1, \ldots, T_n\}$.
(2) $\tilde{f}$ is a unit in $\tilde{k}[T_1, \ldots, T_n]$.

PROOF. As $\|\bullet\|_1$ is a valuation by Proposition 3.3, f is a unit in $k\{T_1, \ldots, T_n\}$ if and only if it is a unit in $(k\{T_1, \ldots, T_n\})^\circ$, which is identified with $\mathring{k}\{T_1, \ldots, T_n\}$ by Proposition 4.1. This result then follows from Corollary 4.20 in the chapter Banach Rings. $\qquad\square$

Definition 4.4. Let $n \in \mathbb{N}$. Consider $g \in k\{T_1, \ldots, T_n\}$. We expand g as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \ldots, T_{n-1}\}.$$

For $s \in \mathbb{N}$, we say g is X_n-*distinguished of degree s* if g_s is a unit in $k\{T_1, \ldots, T_{n-1}\}$, $\|g_s\|_1 = \|g\|_1$ and $\|g_s\|_1 > \|g_t\|_1$ for all $t > s$.

Theorem 4.5 (Weierstrass division theorem). Let $n, s \in \mathbb{N}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished of degree s. Then for each $f \in k\{T_1, \ldots, T_n\}$, there exist $q \in k\{T_1, \ldots, T_n\}$ and $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$ such that

$$f = qg + r.$$

Moreover, q and r are uniquely determined. We have the following estimates

$$(4.1) \qquad \|q\|_1 \le \|g\|_1^{-1} \|f\|_1, \quad \|r\|_1 \le \|f\|_1.$$

If in addition, $f, g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, then $q \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ as well.

PROOF. We may assume that $\|g\|_1 = 1$.

Step 1. Assuming the existence of the division. Let us prove (4.1). We may assume that $f \neq 0$, so that one of q, r is non-zero. Up to replacing q, r by a scalar multiple, we may assume that $\max\{\|q\|_1, \|r\|_1\} = 1$. So $\|f\|_1 \leq 1$ as well. We need to show that $\|f\|_1 = 1$. Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here $\tilde{\bullet}$ denotes the reduction map. By our assumption, $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$. From Proposition 4.1, the equality is in $\tilde{k}[T_1, \ldots, T_n]$. From the usual Euclidean division, we have $\tilde{q} = \tilde{r} = 0$. This is a contradiction to our assumption.

Step 2. Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with q and r as in the theorem. The estimate in Step 1 shows that $q = r = 0$.

Step 3. We prove the existence of the division.

We define

$$B := \left\{ qg + r : r \in k\{T_1, \ldots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \ldots, T_n\} \right\}.$$

From Step 1, B is a closed subgroup of $k\{T_1, \ldots, T_n\}$. In fact, suppose $f_i \in B$ is a sequence converging to $f \in k\{T_1, \ldots, T_n\}$. From Step 1, we can represent $f_i = q_i g + r_i$, then from Step 1, q_i and r_i are both Cauchy sequences, we may assume that $q_i \to q \in k\{T_1, \ldots, T_n\}$ and $r_i \to r$. As $\deg_{T_n} r_i < s$, it follows that $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ and $\deg_{T_n} r < s$. So $f = qg + r$ and hence B is closed.

It suffices to show that B is dense $k\{T_1, \ldots, T_n\}$. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \ldots, T_{n-1}\}.$$

We may assume that $\|g\|_1 = 1$. Define $\epsilon := \max_{j \geq s} \|g_j\|$. Then $\epsilon < 1$ by our assumption. Let $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$ for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \ldots, T_n\})^\circ \to (\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$$

with kernel $\{f \in k\{T_1, \ldots, T_n\} : \|f\|_1 \leq \epsilon\}$. We now apply Euclidean division in the ring $(\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$ to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some $q \in (k\{T_1, \ldots, T_n\})^\circ$ and $r \in (k\{T_1, \ldots, T_{n-1}\})^\circ[T_n]$ with $\deg_{T_n} r < s$. So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that B is dense in $k\{T_1, \ldots, T_n\}$ by Proposition 2.8 in the chapter Banach rings.

Step 4. It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring $k\{T_1, \ldots, T_{n-1}\}[T_n]$ and the uniqueness proved in Step 2. $\square$

Lemma 4.6. Let $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ be a Weierstrass polynomial and $g \in k\{T_1, \ldots, T_n\}$. Assume that $\omega g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, then $g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$.

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, $\deg_{T_n} r < \deg_{T_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of Theorem 4.5, we know that $q = g$, so g is a polynomial in T_n. □

As a consequence, we deduce Weierstrass preparation theorem.

Definition 4.7. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n-variables is a monic polynomial $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\|\omega\|_1 = 1$.

Lemma 4.8. Let $n \in \mathbb{Z}_{>0}$ and $\omega_1, \omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ be two monic polynomials. If $\omega_1 \omega_2$ is a Weierstrass polynomial then so are ω_1 and ω_2.

PROOF. As ω_1 and ω_2 are monic, $\|\omega_i\|_1 \geq 1$ for $i = 1, 2$. On the other hand, $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1 \omega_2\|_1 = 1$, so $\|\omega_i\|_1 = 1$ for $i = 1, 2$. □

Theorem 4.9 (Weierstrass preparation theorem)**.** Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished of degree s. Then there are a Weierstrass polynomial $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ of degree s and a unit $e \in k\{T_1, \ldots, T_n\}$ such that

$$g = e\omega.$$

Moreover, e and ω are unique. If $g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, then so is e.

PROOF. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let $r = T_n^s - \omega$. Then $T_n^s = e^{-1}g + r$. The uniqueness part of Theorem 4.5 implies that e and r are uniquely determined, hence so is ω.

Next we prove the existence. By Weierstrass division theorem Theorem 4.5, we can write

$$T_n^s = qg + r$$

for some $q \in k\{T_1, \ldots, T_n\}$ and $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$. Let $\omega = T_n^s - r$. From the estimates in Theorem 4.5, $\|r\|_1 \leq 1$. So $\|\omega\|_1 = 1$. Then ω is a Weierstrass polynomial of degree s and $\omega = qg$. It suffices to argue that q is a unit.

We may assume that $\|g\|_1 = 1$. By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$ and the leading coefficients of both polynomials are units in $\tilde{k}[T_1, \ldots, T_{n-1}]$, it follows that $\tilde{q}$ is a unit in $\tilde{k}[T_1, \ldots, T_{n-1}]$. It follows that $\tilde{q}$ is also a unit in $\tilde{k}[T_1, \ldots, T_n]$. By Lemma 4.3, q is a unit in $k\{T_1, \ldots, T_n\}$.

The lsat assertion is already proved in Theorem 4.5.

□

Definition 4.10. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished. Then the Weierstrass polynomial ω constructed in Theorem 4.9 is called the *Weierstrass polynomial* defined by g.

Corollary 4.11. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ be X_n-distinguished. Let ω be the Weierstrass polynomial of g. Then the injection

$$k\{T_1, \ldots, T_{n-1}\}[T_n] \to k\{T_1, \ldots, T_n\}$$

induces an isomorphism of k-algebras

$$k\{T_1, \ldots, T_{n-1}\}[T_n]/(\omega) \to k\{T_1, \ldots, T_n\}/(g).$$

PROOF. The surjectivity follows from Theorem 4.5 and the injectivity follows from Lemma 4.6. $\qquad\square$

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

Lemma 4.12. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \ldots, T_n\}$ is non-zero. Then there is a k-algebra automorphism σ of $k\{T_1, \ldots, T_n\}$ so that $\sigma(g)$ is T_n-distinguished.

PROOF. We may assume that $\|g\|_1 = 1$. We expand g as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow $\mathbb{N}^n$ with the lexicographic order. Take the maximal $\beta \in \mathbb{N}^n$ so that $|a_\beta| = 1$. Take $t \in \mathbb{Z}_{>0}$ so that $t \geq \max_{i=1,\ldots,n} \alpha_i$ for all $\alpha \in \mathbb{N}^n$ with $\tilde{a}_\alpha \neq 0$.

We will define σ by sending T_i to $T_i + T_n^{c_i}$ for all $i = 1, \ldots, n-1$. The c_i's are to be defined. We begin with $c_n = 1$ and define the other c_i's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for $j = 1, \ldots, n-1$. We claim that $\sigma(f)$ is T_n-distinguished of order $s = \sum_{i=1}^{n} c_i \beta_i$.

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^{s} p_i T_n^i$$

for some $p_i \in \tilde{k}[T_1, \ldots, T_{n-1}]$ and $p_s = \tilde{a}_\beta$. Our claim follows. $\qquad\square$

Proposition 4.13. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Noetherian.

PROOF. We make induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. It suffices to show that for any non-zero $g \in k\{T_1, \ldots, T_n\}$, $k\{T_1, \ldots, T_n\}/(g)$ is Noetherian. By Lemma 4.12, we may assume that g is T_n-distinguished. By Theorem 4.5, $k\{T_1, \ldots, T_n\}/(g)$ is a finite free $k\{T_1, \ldots, T_{n-1}\}$-module. By the inductive hypothesis and Hilbert basis theorem, $k\{T_1, \ldots, T_n\}/(g)$ is indeed Noetherian. $\quad\square$

Proposition 4.14. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Jacobson.

PROOF. When $n = 0$, there is nothing to prove. We make induction on n and assume that $n > 0$. Let $\mathfrak{p}$ be a prime ideal in $k\{T_1, \ldots, T_n\}$, we want to show that the Jacobson radical of $\mathfrak{p}$ is equal to $\mathfrak{p}$.

We distinguish two cases. First we assume that $\mathfrak{p} \neq 0$. Let $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \ldots, T_{n-1}\}$. By Lemma 4.12, we may assume that $\mathfrak{p}$ contains a Weierstrass polynomial ω. Observe that

$$k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' \to k\{T_1, \ldots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any $b \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p})$ (where J denotes the Jacobson radical), we consider a monic integral equation of minimal degree over $k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'$:

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0, \quad a_i \in k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) \cap k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that $n = 1$ and so $b = 0$. This proves $J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) = 0$.

On the other hand, let us consider the case $\mathfrak{p} = 0$. As $k\{T_1, \ldots, T_n\}$ is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \ldots, T_n\}) = 0.$$

Assume that there is a non-zero element f in $J(k\{T_1, \ldots, T_n\})$. We may assume that $\|f\|_1 = 1$.

We claim that there is $c \in k$ with $|c| = 1$ such that $c + f$ is not a unit in $k\{T_1, \ldots, T_n\}$. Assuming this claim for the moment, we can find a maximal ideal $\mathfrak{m}$ of $k\{T_1, \ldots, T_n\}$ such that $c + f \in \mathfrak{m}$. But $f \in \mathfrak{m}$ by our assumption, so $c \in \mathfrak{m}$ as well. This contradicts the fact that $c \in k^\times$.

It remains to prove the claim. We treat two cases separately. When $|f(0)| < 1$, we simply take $c = 1$, which works thanks to Lemma 4.3. If $|f(0)| = 1$, we just take $c = -f(0)$. □

Proposition 4.15. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is UFD. In particular, $k\{T_1, \ldots, T_n\}$ is normal.

PROOF. As $\| \bullet \|_1$ is a valuation by Proposition 2.2, $k\{T_1, \ldots, T_n\}$ is an integral domain. In order to see that $k\{T_1, \ldots, T_n\}$ has the unique factorization property, we make induction on $n \geq 0$. When $n = 0$, there is nothing to prove. Assume that $n > 0$. Take a non-unit element $f \in k\{T_1, \ldots, T_n\}$. By Theorem 4.9 and Lemma 4.12, we may assume that f is a Weierstrass polynomial. By inductive hypothesis, $k\{T_1, \ldots, T_{n-1}\}$ is a UFD, hence so is $k\{T_1, \ldots, T_{n-1}\}[T_n]$ by [Stacks, Tag 0BC1]. It follows that f can be decomposed into the products of monic prime elements $f_1, \ldots, f_r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, which are all Weierstrass polynomials by Lemma 4.8. Then by Corollary 4.11, we see that each f_i is prime in $k\{T_1, \ldots, T_n\}$.

Any UFD is normal by [Stacks, Tag 0AFV]. □

5. Noetherian normalization

Let $(k, | \bullet |)$ be a complete non-trivially valued non-Archimedean valued-field.

Theorem 5.1. Let A be a non-zero strictly k-affinoid algebra, $n \in \mathbb{N}$ and $\alpha : k\{T_1, \ldots, T_n\} \to A$ be a finite (resp. integral) k-algebra homomorphism. Then up to replacing $T_1, \ldots, T_n$ by an affinoid chart, we can guarantee that there exists $d \in \mathbb{N}$, $d \leq n$ such that α when restricted to $k\{T_1, \ldots, T_d\}$ is finite (resp. integral) and injective.

PROOF. We make an induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. If $\ker \alpha = 0$, there is nothing to prove, so we may assume that $\ker \alpha \neq 0$. By Lemma 4.12 and Theorem 4.9, we may assume that there is a Weierstrass polynomial $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ in $\ker \alpha$. Then α induces a finite (resp. integral) homomorphism $\beta : k\{T_1, \ldots, T_n\}/(\omega) \to A$. By Theorem 4.5, $k\{T_1, \ldots, T_{n-1}\} \to k\{T_1, \ldots, T_n\}/(\omega)$ is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism $k\{T_1, \ldots, T_{n-1}\} \to A$. We can apply the inductive hypothesis to conclude. □

Corollary 5.2. Let A be a non-zero strictly k-affinoid algebra, then there is $d \in \mathbb{N}$ and a finite injective k-algebra homomorphism: $k\{T_1, \ldots, T_d\} \to A$.

PROOF. Take some $n \in \mathbb{N}$ and a surjective k-algebra homomorphism $k\{T_1, \ldots, T_n\} \to A$ and apply Theorem 5.1, we conclude. $\qquad\square$

6. Properties of affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 6.1. Assume that k is non-trivially valued. Let A be a strictly k-afifnoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\}.$$

PROOF. It is clear that $\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\}$. Conversely, let $f \in A$, $\rho(f) \leq 1$. Choose $d \in \mathbb{N}$ and a surjective k-algebra homomorphism

$$\varphi : k\{T_1, \ldots, T_d\} \to A.$$

Let $f^n + t_1 f^{n-1} + \cdots + t_n = 0$ be the minimal equation of f over $k\{T_1, \ldots, T_d\}$. Then $t_i \in (k\{T_1, \ldots, T_d\})^\circ$ by Proposition 8.10 in the chapter Banach Rings. An induction on $i \geq 0$ shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi\left((k\{T_1, \ldots, T_d\})^\circ\right) f^j.$$

The right-hand side is clearly bounded. $\qquad\square$

Theorem 6.2. A k-affinoid algebra A is Noetherian and all ideals of A are closed.

PROOF. Let I be an ideal in A. By Proposition 3.12, we can take a suitable $r \in \mathbb{R}_{>0}^m$ so that $A\hat{\otimes}K_r$ is strictly K_r-affinoid. Then $I(A\hat{\otimes}K_r)$ is an ideal in $A\hat{\otimes}K_r$. By Proposition 4.13, the latter ring is Noetherian. So we may take finitely many generators $f_1, \ldots, f_k \in I$. Each $f \in I$ can be written as

$$f = \sum_{i=1}^{k} f_i g_i$$

with $g_i = \sum_{j=-\infty}^{\infty} g_{i,j} T^j \in A\hat{\otimes}K_r$. But then

$$f = \sum_{i=1}^{k} f_i g_{i,0}.$$

So I is finitely generated.

As $I = A \cap (I(A\hat{\otimes}K_r))$, by Corollary 7.4 in the chapter Banach Rings, we see that I is closed in $A\hat{\otimes}K_r$ and hence closed in A. $\qquad\square$

Proposition 6.3. Let $(A, \|\bullet\|)$ be a k-affinoid algebra and $f \in A$. Then there is $C > 0$ and $N \geq 1$ such that for any $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Recall that ρ is the spectral radius map defined in Definition 4.9 in the chatper Banach Rings.

PROOF. By Proposition 3.8, we may assume that k is non-trivially valued and k is non-trivially valued.

If $\rho(f) = 0$, then f lies in each maximal ideal of A. To see this, we may assume that A is a field, then by Proposition 6.10 in the chapter Banach Rings, there is a bounded valuation $\|\bullet\|'$ on A. But then $\rho(f) = 0$ implies that $\|f\|' = 0$ and hence $f = 0$.

It follows that if $\rho(f) = 0$ then f lies in $J(A)$, the Jacobson radical of A. By Proposition 4.14, A is a Jacobson ring. So f is nilpotent. The assertion follows.

So we can assume that $\rho(f) > 0$. In this case, by Corollary 5.2 and Proposition 8.10 in the chapter Banach Rings, we have $\rho(f) \in \sqrt{|k^\times|}$. Take $a \in k^\times$ and $d \in \mathbb{Z}_{>0}$ so that $\rho(f)^d = |a|$. Then $\rho(f^d/a) = 1$ and hence it is powerly-bounded by Proposition 6.1. It follows that there is $C > 0$ so that for $n \geq 1$,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that $\|f^n\| \leq C\rho(f)$ for $n \geq d$ as long as we enlarge C. $\square$

Corollary 6.4. Let $\varphi : A \to B$ be a bounded homomorphism of k-affinoid algebras. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in B$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$ for $i = 1, \ldots, n$. Write $r = (r_1, \ldots, r_n)$, then there is a unique bounded homomorphism $\Phi : A\{r^{-1}T\} \to B$ extending φ and sending T_i to f_i.

PROOF. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from Proposition 6.3 that the right-hand side the series converges. The boundedness of Φ is obvious. $\square$

Proposition 6.5. Let $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$ be k-affinoid algebras, $r \in \mathbb{R}_{>0}^n$ and $\varphi : A\{r^{-1}T\} \to B$ be an admissible epimorphism. Write $f_i = \varphi(T_i)$ for $i = 1, \ldots, n$. Then there is $\epsilon > 0$ such that for any $g = (g_1, \ldots, g_n) \in B^n$ with $\|f_i - g_i\|_B < \epsilon$ for all $i = 1, \ldots, n$, there exists a unique bounded k-algebra homomorphism $\psi : A\{r^{-1}T\} \to B$ that coincides with φ on A and sends T_i to g_i. Moreover, ψ is also an admissible epimorphism.

PROOF. The uniqueness of ψ is obvious. We prove the remaining assertions. Taking $\epsilon > 0$ small enough, we could further guarantee that $\rho(g_i) \leq r_i$. It follows from Corollary 6.4 that there exists a bounded homomorphism ψ as in the statement of the proposition.

As φ is an admissible epimorphism, we may assume that $\|\bullet\|_B$ is the residue induced by $\|\bullet\|_r$ on $A\{r^{-1}T\}$.

By definition of the residue norm, for any $\delta > 0$ and any $h \in B$, we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1+\delta)\|h\|_B$$

for any $\alpha \in \mathbb{N}^n$. Choose $\epsilon \in (0, (1 + \delta)^{-1})$. Now for $g_1, \ldots, g_n$ as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1 + \delta)\epsilon \|h\|_B.$$

Repeating this procedure, we can construct $k_i \in A\{r^{-1}T\}$ for $i \in \mathbb{N}$ and $h_j \in B$ for $j \in \mathbb{Z}_{>0}$ such that for any $i \in \mathbb{Z}_{>0}$, we have

$$h = \psi(k_0 + \cdots + k_{i-1}) + h_i,$$
$$\|k_i\|_r \leq ((1 + \delta)\epsilon)^i (1 + \delta)\|h\|_B,$$
$$\|h_i\|_B \leq ((1 + \delta)\epsilon)^i \|h\|_B.$$

In particular, $k := \sum_{i=0}^{\infty} k_i$ converges in $A\{r^{-1}T\}$ and

$$\|k\|_r \leq (1 + \delta)\|h\|_B.$$

It follows that ψ is an admissible epimorphism. $\qquad\square$

7. Finite modules over affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

For any k-affinoid algebra A, we have defined the category $\mathcal{B}an_A^f$ of finite Banach A-modules in Definition 5.3 in the chapter Banach Rings. We write $\mathcal{M}od_A^f$ for the category of finite A-modules.

Lemma 7.1. Let A be a k-affinoid algebra, $(M, \|\bullet\|_M)$ be a finite Banach A-module and $(N, \|\bullet\|_N)$ be a Banach A-module N. Let $\varphi : M \to N$ be an A-linear homomorphism. Then φ is bounded.

PROOF. Take $n \in \mathbb{N}$ such that there is an admissible epimorphism

$$\pi : A^n \to M.$$

It suffices to show that $\varphi \circ \pi$ is bounded. So we may assume that $M = A^n$. For $i = 1, \ldots, n$, let e_i be the vector with $(0, \ldots, 0, 1, 0, \ldots, 0)$ of A^n with 1 placed at the i-th place. Set $C = \max_{i=1,\ldots,n} \|\varphi(e_i)\|_N$. For a general $f = \sum_{i=1}^n a_i e_i$ with $a_i \in A$, we have

$$\|\varphi(f)\|_N \leq C \|f\|_M.$$

So φ is bounded. $\qquad\square$

Proposition 7.2. Let A be a k-affinoid algebra. The forgetful functor $\mathcal{B}an_A^f \to \mathcal{M}od_A^f$ is an equivalence of categories.

PROOF. It suffices to construct the inverse functor. Let M be a finite A-module. Choose $n \in \mathbb{N}$ and an A-linear epimorphism $\pi : A^n \to M$. By Theorem 6.2, $\ker \pi$ is closed in A^n. We can endow M with the residue norm. By Lemma 7.1, the equivalence class of the norm does not depend on the choice of π.

For any A-linear homomorphism $f : M \to N$ of finite A-modules, we endow M and N with the Banach structures as above. It follows from Lemma 7.1 that f

is bounded. We have defined the inverse functor of the forgetful functor $\mathcal{B}\mathrm{an}_A^f \to \mathcal{M}\mathrm{od}_A^f$. $\qquad\square$

Remark 7.3. Let A be a k-affinoid algebra. It is not true that a Banach A-module which is finite as A-module is finite as Banach A-module.

As an example, take $0 < p < q < 1$ and $A = k\{q^{-1}T\}$, $B = k\{p^{-1}T\}$. Then B is a Banach A-module. By Example 2.4, the underlying rings of A and B are both $k[[T]]$. So the canonical map $A \to B$ is bijective. But B is not a finite A-module. As otherwise, the inverse map $B \to A$ is bounded by Lemma 7.1, which is not the case.

The correct statement is the following: consider a Banach A-module $(M, \|\bullet\|_M)$ which is finite as A-module, then there is a norm on M such that M becomes a finite Banach A-module. The new norm is not necessarily equivalent to the given norm $\|\bullet\|_M$.

Proposition 7.4. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Then the natural map

$$M \otimes_A N \to M\hat{\otimes}_A N$$

is an isomorphism of Banach A-modules and $M\hat{\otimes}_A N$ is a finite Banach A-module.

Here the Banach A-module structure on $M \otimes_A N$ is given by Proposition 7.2.

PROOF. Choose $m, m' \in \mathbb{N}$ an admissibly coexact sequence

$$A^{m'} \to A^m \to M \to 0$$

of Banach A-modules. Then we have a commutative diagram of A-modules:

$$
\begin{array}{ccccccc}
A^{m'} \otimes_A N & \longrightarrow & A^m \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{m'}\hat{\otimes}_A N & \longrightarrow & A^m\hat{\otimes}_A N & \longrightarrow & M\hat{\otimes}_A N & \longrightarrow & 0
\end{array}
$$

with exact rows. By 5-lemma, in order to prove $M \otimes_A N \xrightarrow{\sim} M\hat{\otimes}_A N$ and $M\hat{\otimes}_A N$ is a finite Banach A-module, we may assume that $M = A^m$ for some $m \in \mathbb{N}$. Similarly, we can assume $N = A^n$ for some $n \in \mathbb{N}$. In this case, the isomorphism is immediate and $M\hat{\otimes}_A N$ is clearly a finite Banach A-module. By Lemma 7.1, the Banach A-module structure on $M\hat{\otimes}_A N$ coincides with the Banach A-module strucutre on $M \otimes_A N$ induced by Proposition 7.2. $\qquad\square$

Proposition 7.5. Let A, B be a k-affinoid algebra and $A \to B$ be a bounded k-algebra homomorphism. Let M be a finite Banach A-module, then the natural map

$$M \otimes_A B \to M\hat{\otimes}_A B$$

is an isomorphism of Banach B-modules and $M\hat{\otimes}_A B$ is a finite Banach B-module.

PROOF. By the same argument as Proposition 7.4, we may assume that $M = A^n$ for some $n \in \mathbb{N}$. In this case, the assertions are trivial. $\qquad\square$

Proposition 7.6. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Let $\varphi : M \to N$ be an A-linear map. Then φ is admissible.

PROOF. By Lemma 7.1, φ is always bounded. By Proposition 7.5 and Proposition 3.8, we may assume that k is non-trivially valued. By Theorem 6.2, N is a Noetherian A-module. It follows from Corollary 7.4 in the chapter Banach Rings that $\operatorname{Im} \varphi$ is closed in N and is finite as an A module. In particular, the norm induced from N and from M are equivalent by Lemma 7.1. It follows that φ is admissible. □

Proposition 7.7. Let A be a k-affinoid algebra. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_{>0}^n$. Assume that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$. Then M is a finite Banach A-module if and only if $M \hat{\otimes}_k K_r$ is a finite Banach $A \hat{\otimes}_k K_r$-module.

PROOF. We may assume that r has only one component and write $r_1 = r$. The direct implication is trivial. Let us assume that $M \hat{\otimes}_k K_r$ is a finite Banach $A \hat{\otimes}_k K_r$-module. Take $n \in \mathbb{N}$ and an admissible epimorphism of $A \hat{\otimes}_k K_r$-modules

$$\varphi : (A \hat{\otimes}_k K_r)^n \to M \hat{\otimes}_k K_r.$$

Let $e_1, \ldots, e_n$ denotes the standard basis of $(A \hat{\otimes}_k K_r)^n$. We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By Proposition 6.5, we can assume that there is $l > 0$ such that $m_{i,j} = 0$ for all $i = 1, \ldots, n$ and $|j| > l$. It follows that

$$A^{n(2l+1)} \to M$$

sending the standard basis to $m_{i,j}$ with $i = 1, \ldots, n$ and $j = -l, -l+1, \ldots, l$ is an admissible epimorphism. □

For any ring A, $\mathcal{A}\mathrm{lg}_A^f$ denotes the category of finitely generated A-algebras.

Proposition 7.8. Let A be a k-affinoid algebra. Then the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_A^f \to \mathcal{A}\mathrm{lg}_A^f$ is an equivalence of categories.

Recall that $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_A^f$ is defined in Definition 5.9 in the chapter Banach Rings.

PROOF. It suffices to construct an inverse functor. Let B be a finite A-algebra. We endow B with the norm $\| \bullet \|_B$ as in Proposition 7.2. We claim that B is a Banach A-algebra.

Let us recall the definition of the norm. Take $n \in \mathbb{N}$ an epimorphism $\varphi : A^n \to B$ of A-modules. Then $\| \bullet \|_B$ is the residue norm induced by φ.

Consider the A-linear epimorphism $\psi : A^n \otimes_A A^n \to B \otimes_A B$. By Proposition 7.6, when both sides are endowed with the norms $\| \bullet \|_{A^n \otimes_A A^n}$ and $\| \bullet \|_{B \otimes_A B}$ as in Proposition 7.2, ψ is admissible. It follows that there is $C > 0$ such that for any $f, g \in B$,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by Proposition 7.2, the natural map $B \otimes_A B \to B$ is bounded. It follows that there is a constant $C' > 0$ such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in B is bounded and hence B is a finite Banach algebra. Given any morphism $B \to B'$ in $\mathcal{A}\mathrm{lg}_A^f$, we endow B and B' with the norms

given by Proposition 7.2. It follows from Lemma 7.1 that $B \to B'$ is a bounded homomorphism of finite Banach A-algebras. So we have defined an inverse functor to the forgetful functor $\mathcal{B}an\mathcal{A}lg_A^f \to \mathcal{A}lg_A^f$. $\qquad\square$

Remark 7.9. It is not true that any homomorphism of affinoid k-algebras is bounded. For example, if the valuation on k is trivial. Take $0 < p < q < 1$ and consider the natural homomorphism $K_p \to K_q$. This homomorphism is bijective but not bounded.

8. Affinoid domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and $H \supseteq |k^\times|$ be a subgroup of $R_{>0}$.

Definition 8.1. Let A be a k_H-affinoid algebra. A subset $V \subseteq \mathrm{Sp}\,A$ is said to be a k_H-*affinoid domain* in X if there is a bounded homomorphism of k_H-affinoid algebras $\varphi : A \to A_V$ satisfying

(1) $\mathrm{Im}\,\mathrm{Sp}\,\varphi = V$;
(2) given a bounded homomorphism of k_H-affinoid algebras $\psi : A \to B$ such that $\mathrm{Sp}\,\psi : \mathrm{Sp}\,B \to \mathrm{Sp}\,A$ factorizes through V, there is a unique bounded homomorphism $A_V \to B$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \varphi\ } & A_V \\
{\scriptstyle\psi}\downarrow & \swarrow & \\
B & &
\end{array}
\quad.
$$

We say V is *represented by* the morphism φ.

When $k_H = \mathbb{R}_{>0}$, we say V is a k-*affinoid domain* in X. When $k_H = |k^\times|$, we say V is a *strict k-affinoid domain* in X.

Remark 8.2. This definition differs from the original definition of [Ber12], we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the orignal definition of Berkovich when $H = \mathbb{R}_{>0}$.

Proposition 8.3. Let A be a k_H-affinoid algebra and $V \subseteq \mathrm{Sp}\,A$ be a k_H-affinoid domain represented by $\varphi : A \to A_V$. Then $\mathrm{Sp}\,\varphi$ induces a bijection $\mathrm{Sp}\,A_V \to \mathrm{Sp}\,A$.

PROOF. We observe that $\mathrm{Sp}\,A_V \to \mathrm{Sp}\,A$ is a monomorphism in the category k_H-$\mathcal{A}ff$. In other words, $A \to A_V$ is an epimorhism in the category k_H-$\mathcal{A}ff\mathcal{A}lg$. To see this, let $\eta_1, \eta_2 : A_V \to B$ be two arrows in k_H-$\mathcal{A}ff\mathcal{A}lg$ such that $\eta_1 \circ \varphi = \eta_2 \circ \varphi$. It follows from the universal property in Definition 8.1 that $\eta_1 = \eta_2$. We claim that $\mathrm{Sp}\,A_V \to V$ is a bijection. $\qquad\square$

It is not immediately clear that A_V is canonically assocaited with V. We will prove this now.

Proposition 8.4. Let A be a k_H-affinoid algebra and V be an affinoid domain in X represented by $\varphi : A \to A_V$. Then $\mathrm{Sp}\,\varphi : \mathrm{Sp}\,A_V \to \mathrm{Sp}\,A$ induces a homeomorphism $\mathrm{Sp}\,A_V \to V$.

In particular, A_V is uniquely determined by V up to isomorphisms of Banach k-algebras.

PROOF. Let us reduce the problem to the case where k is non-trivially valued and A and A_V are both strictly k-affinoid.

By Proposition 3.12, taking a suitable $r = r(r_1, \ldots, r_n) \in \mathbb{R}_{>0}^n$ such that $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{R}_{>0}/\sqrt{|k^\times|}$, we may guarantee that $A\hat{\otimes}_k K_r$ and $A_V\hat{\otimes}_k K_r$ are both strictly K_r-affinoid.

Let V' be the inverse image of V in $\operatorname{Sp} A\hat{\otimes}_k K_r$. We claim that V' is a strict K_r-affinoid domain in $\operatorname{Sp} A\hat{\otimes}_k K_r$ represented by $A\hat{\otimes}_k K_r \to A_V\hat{\otimes}_k K_r$. $\square$

9. Graded reduction

Bibliography

[Ber12]   V. G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. 33. American Mathematical Soc., 2012.

[BGR84]   S. Bosch, U. Güntzer, and R. Remmert. Non-Archimedean analysis. Vol. 261. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436. URL: https://doi.org/10.1007/978-3-642-52229-1.

[Stacks]   T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.