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Contents

Berkovich analytic spaces

1. Introduction

2. The category of Berkovich analytic spaces

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 2.1. Let X be a locally Hausdorff space and τ be a net of compact subsets. A *k_H-affinoid atlas* $\mathcal{A}$ on X with the net τ is a map which assigns

(1) to each $V \in \tau$, a k_H-affinoid algebra A_V and a homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$;
(2) to each $U, V \in \tau$, $U \subseteq V$, a morphism of k_H-affinoid algebras $\alpha_{V/U} : A_V \to A_U$ representing a k_H-affinoid domain $\operatorname{Sp} A_U$ in $\operatorname{Sp} A_V$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Sp} A_U \arrow[r, "\operatorname{Sp} \alpha_{V/U}"] \arrow[d, "\varphi_U"] & \operatorname{Sp} A_V \arrow[d, "\varphi_V"] \\ U \arrow[r] & V \end{tikzcd} .$$

The triple $(X, \mathcal{A}, \tau)$ as above is called a *k_H-analytic space*.

A *morphism* between atlases $\mathcal{A}$ and $\mathcal{A}'$ on X with the net τ is an assignment that with each $V \in \tau$, one associates a morphism of k_H-affinoid algebras $\beta_V : A_V \to A'_V$ such that

(1) for each $V \in \tau$, the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Sp} A'_V \arrow[r, "\operatorname{Sp} \beta_V"] \arrow[d, "\varphi'_V"] & \operatorname{Sp} A_V \arrow[dl, "\varphi_V"] \\ V & \end{tikzcd} ;$$

(2) for each $U, V \in \tau$, $U \subseteq V$, the following diagram is commutative:

$$\begin{tikzcd} A_V \arrow[r, "\alpha_{V/U}"] \arrow[d, "\beta_V"] & A_U \arrow[d, "\beta_U"] \\ A'_V \arrow[r, "\alpha'_{V/U}"] & A'_U \end{tikzcd}$$

Here we have denoted the data associated with $\mathcal{A}'$ with a prime. In this way, the atlases on X with the net τ form a category.

We remind the readers that by our convention a compact space is Hausdorff.

By Condition (2), it $W \subseteq U \subseteq V$ are three sets in τ, then $\alpha_{V/U} \circ \alpha_{U/W} = \alpha_{V/W}$.

Remark 2.2. As a convention, we will denote the atlas by capital letters in caligraphic font and the affinoid algebras by the same letter in roman font. We will usually omit the maps φ_U's by identifying $\operatorname{Sp} A_U$ with U. We will say U is a k_H-affinoid domain in V.

Remark 2.3. Our definition is a special case of the original definitions in [Ber93]. This seems to be the most important case though.

Lemma 2.4. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space, $U \in \tau$ and W is a k_H-affinoid domain in U. Then for any $V \in \tau$ containing W, W is a k_H-affinoid domain in V.

Proof. As $\tau|_{U\cap V}$ is a net and W is compact, we can find $U_1, \ldots, U_n \in \tau_{U\cap V}$ with $W \subseteq U_1 \cup \cdots \cup U_n$. As W, U_i are k_H-affinoid domains in U, $W_i = W \cap U_i$ is a k_H-affinoid domain in U_i for all $i = 1, \ldots, n$ by Corollary 13.12 in Affinoid algebras. It follows from Corollary 10.7 and Corollary 13.12 in Affinoid algebras that W_i and $W_i \cap W_j$ are both k_H-affinoid domains in V for $i, j = 1, \ldots, n$. So W is a compact k_H-analytic domain in V.

By Proposition 13.25 in Affinoid algebras,

$$A_W := \ker\left(\prod_{i=1}^{n} A_{W_i} \to \prod_{i,j=1}^{n} A_{W_i\cap W_j}\right)$$

is k_H-affinoid and $\operatorname{Sp} A_W \to \operatorname{Sp} A$ induces a hoemomorphism $\operatorname{Sp} A_W \to W$ by Proposition 10.6 in Affinoid algebras. By Proposition 13.25 in Affinoid algebras again, W is affinoid in V. □

Definition 2.5. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. We define $\bar{\tau}$ as the set of all $W \subseteq X$ such that there is $U \in \tau$ containing W and W is k_H-affinoid in U.

Lemma 2.6. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Then $\bar{\tau}$ is a net on X and there is a k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$. Moreover, the k_H-affinoid atlas $\overline{\mathcal{A}}$ on X with the net $\bar{\tau}$ extending $\mathcal{A}$ is unique up to a canonical isomorphism.

Proof. **Step 1**. We first show that $\bar{\tau}$ is a net. Let $U, V \in \bar{\tau}$ and $x \in U \cap V$. Take $U', V' \in \tau$ containing U and V respectively. Take $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n \in \tau$ such that

(1) $x \in W_1 \cap \cdots \cap W_n$;
(2) $W_1 \cup \cdots \cup W_n$ is a neighbourhood of x in $U' \cap V'$.

This is possible because $\tau|_{U'\cap V'}$ is a quasi-net by assumption.

By Lemma 2.4, U (resp. V) and $W_1, \ldots, W_n$ are k_H-affinoid domains in U' (resp. V').

By Corollary 13.12 in Affinoid algebras, $U_i := U \cap W_i$ (resp. $V_i := V \cap W_i$) is a k_H-affinoid domain in W_i for $i = 1, \ldots, n$. By Corollary 13.12 in Affinoid algebras again, $U_i \cap V_i$ is a k_H-affinoid domain in W_i for $i = 1, \ldots, n$. So $U_i \cap V_i \in \bar{\tau}|_{U\cap V}$ for $i = 1, \ldots, n$. But

$$\bigcup_{i=1}^{n} U_i \cap V_i = (U \cap V) \cap \bigcup_{i=1}^{n} W_i,$$

so $\bigcup_{i=1}^{n} U_i \cap V_i$ is a neighbourhood of x in $U \cap V$ and $x \in \bigcap_{i=1}^{n} U_i \cap V_i$. It follows that $\bar{\tau}$ is a net.

Step 2. We extend the k_H-affinoid atlas $\mathcal{A}$.

For each $V \in \bar{\tau}$, we fix a $V' \in \tau$ containing V.

By Lemma 2.4, V is a k_H-affinoid domain in V'. Let $A_{V'} \to A_V$ be the morphism of k_H-affinoid algebras representing the k_H-affinoid domain V in $\operatorname{Sp} A_{V'}$. We define the homeomorphism $\varphi_V : \operatorname{Sp} A_V \to V$ as the morphism induced by $\operatorname{Sp} A_V \to \operatorname{Sp} A$.

For $U, V \in \bar{\tau}$ with $U \subseteq V$, we want to define $\alpha_{V/U} : A_V \to A_U$. We handle two cases. When $V \in \tau$, as $\tau|_{U' \cap V}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \tau|_{U' \cap V}$ such that

$$U = \bigcup_{i=1}^{n} U_i.$$

By Lemma 2.4, $U_1, \dots, U_n$ are k_H-affinoid domains in U' and in V. By Theorem 13.19 in Affinoid algebras,

$$A_U \xrightarrow{\sim} \ker \left(\prod_{i=1}^{n} A_{U_i} \to \prod_{i,j=1}^{n} A_{U_i \cap U_j} \right).$$

So the morphism $\alpha_{V/U_i} : A_V \to A_{U_i}$ and $A_{V/U_i \cap U_j} : \alpha_{V/U_i} : A_V \to A_{U_i \cap U_j}$ for $i = 1, \dots, n$ and $j = 1, \dots, n$ induces a morphism $\alpha_{V/U} : A_V \to A_U$. Observe that $\alpha_{V/U}$ represents the k_H-affinoid domain U in V, so it is independent of the choice of $U_1, \dots, U_n$.

More generally, when $V \in \bar{\tau}$, we have constructed a morphism $\alpha_{V'/U} : A_{V'} \to A_U$ representing the k_H-affinoid domain U in V', it follows that U is a k_H-affinoid domain in V, and we therefore get the desired morphism $\alpha_{V/U} : A_V \to A_U$.

It is easy to verify that the constructions gives a k_H-affinoid atlas with the net $\bar{\tau}$ extending $\mathcal{A}$. The uniqueness of the extension is immediate. ☐

Definition 2.7. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A *strong morphism* $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is a pair consisting of

(1) a continuous map $\varphi : X \to X'$ such that for each $V \in \tau$, there is $V' \in \tau'$ with $\varphi(V) \subseteq V'$;
(2) for each $V \in \tau$, $V' \in \tau'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that for each $V, W \in \tau$, $V', W' \in \tau'$ satisfying $V \subseteq W$, $W' \subseteq W'$, $\varphi(V) \subseteq V'$ and $\varphi(W) \subseteq W'$, the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi_{V/V'}"] \arrow[d] & V' \arrow[d] \\ W \arrow[r, "\varphi_{W/W'}"] & W' \end{tikzcd}.$$

Recall our convention Remark 2.2, the morphism $\varphi_{V/V'}$ means a morphism $A'_{V'} \to A_V$ of k_H-affinoid algebras making the following diagram commutative

$$\begin{tikzcd} \operatorname{Sp} A_V \arrow[r] \arrow[d, "\varphi_V"] & \operatorname{Sp} A'_{V'} \arrow[d, "\varphi'_{V'}"] \\ V \arrow[r, "\varphi"] & V' \end{tikzcd}.$$

We will continue our identifications as in Remark 2.2 to simplify our notations.

Proposition 2.8. Let $(X,\mathcal{A},\tau)$ and $(X',\mathcal{A}',\tau')$ be k_H-analytic spaces. Let $\varphi : (X,\mathcal{A},\tau)\to(X',\mathcal{A}',\tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X,\bar{\mathcal{A}},\bar{\tau})\to(X',\overline{\mathcal{A}'},\overline{\tau'})$.

Proof. Let $U\in\bar{\tau}$, $U'\in\overline{\tau'}$ with $\varphi(U)\subseteq U'$. Take $V\in\tau$ and $V'\in\tau'$ containing U and U' respectively. By Lemma 2.4, U (resp. V) is a k_H-affinoid domain in V (resp. V'). Take $W\in\tau'$ with $\varphi(V)\subseteq W'$. Then in particular, $\varphi(U)\subseteq W'$. As $\tau'|_{V'\cap W'}$ is a quasi-net and $\varphi(U)$ is compact, we can find $n\in\mathbb{Z}_{>0}$ and $W_1,\dots,W_n\in\tau'|_{V'\cap W}$ such that

$$\varphi(U)\subseteq W_1\cup\cdots\cup W_n.$$

Now W_i is a k_H-affinoid domain in W' by Lemma 2.4, so $V_i:=\varphi_{V/W'}^{-1}(W_i)$ is an affinoid domain in V by Corollary 13.12 in Affinoid algebras, and we have an induced morphism $V_i\to W_i$ for $i=1,\dots,n$. This morphism in turn induces a morphism of k_H-affinoid spectra

$$U_i:=U\cap V_i\to U_i':=U'\cap W_i\to U'$$

for $i=1,\dots,n$. These morphisms are compatible on their intersections by construction. So by Theorem 13.19 in Affinoid algebras, they glue together to a morphism of k_H-affinoid spectra $\bar{\varphi}_{U/U'}:U\to U'$. It is easy to see that this construction defines a strong morphism.

As for the uniqueness, it suffices to show that the morphism $U_i\to U_i'$ is uniquely determined for $i=1,\dots,n$. In other words, we need to show that the dotted arrow that makes the following diagram commutes is unique:

$$\begin{tikzcd} U_i \arrow[r, dotted] \arrow[d] & U_i' \arrow[d] \\ V \arrow[r, "\varphi_{V/W'}"] & W' \end{tikzcd}$$

for $i=1,\dots,n$. It suffices to apply the universal property of the k_H-affinoid domain $U_i'\to W'$. □

Definition 2.9. Let $(X,\mathcal{A},\tau)$, $(X',\mathcal{A}',\tau')$, $(X'',\mathcal{A}'',\tau'')$ be k_H-analytic spaces. Let

$$\varphi:(X,\mathcal{A},\tau)\to(X',\mathcal{A}',\tau'),\quad \psi:(X',\mathcal{A}',\tau')\to(X'',\mathcal{A}'',\tau'')$$

be strong morphisms. We will define their *composition* $\chi=\psi\circ\varphi$ as follows. The underlying map of topological spaces is just the composition of the unlerlying maps of topological spaces corresponding to ψ and φ.

Let $\bar{\varphi}$ and $\bar{\psi}$ be the extensions of φ and ψ to $\bar{\tau}$ and $\overline{\tau'}$ as in Proposition 2.8.

Given $V\in\tau$ and $V''\in\tau''$ with $\chi(V)\subseteq V''$, we need to define a morphism of k_H-affinoid spectra $\chi_{V/V''}:V\to V''$. Take $V'\in\tau'$ and $U''\in\tau''$ such that $\varphi(V)\subseteq V'$ and $\psi(V')\subseteq U''$. Since $\chi(V)\subseteq U''\cap V''$ and V is compact, we can take $n\in\mathbb{Z}_{>0}$ and $V_1'',\dots,V_n''\in\tau''|_{U''\cap V''}$ with $\chi(V)\subseteq V_1''\cup\cdots\cup V_n''$. Then $V_i':=\psi_{V'/U''}^{-1}(V_i'')$ and $V_i:=\varphi_{V/V'}^{-1}(V_i')$ are k_H-affinoid domains in V' and V respectively for $i=1,\dots,n$ and $V=V_1\cup\cdots\cup V_n$. The morphisms $\bar{\varphi}$ and $\bar{\psi}$ then induce a morphism $V_i\to V_i''\to V$ of k_H-affinoid spectra. These morphisms are clearly compatible on the intersections and hence induce a morphism $V\to V''$ of k_H-affinoid spectra by Theorem 13.19 in Affinoid algebras.

It is easy to verify that $\psi\circ\varphi$ is a strong morphism.

In this way, we get a category k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ of k_H-analytic spaces.

Definition 2.10. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. A strong morphism $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ is said to be a *quasi-isomorphism* if

(1) φ is a homeomorphism between X and X';
(2) for any pair $V \in \tau$ and $V' \in \tau'$ with $\varphi(V) \subseteq V'$, $\operatorname{Sp} \varphi_{V/V'}$ identifies V with an affinoid domain in V'.

Lemma 2.11. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces and $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then for any $V \in \bar{\tau}$ and $V' \in \overline{\tau'}$, the intersection $V \cap \varphi^{-1}(V')$ is a compact k_H-analytic domain in V.

Proof. Take $U' \in \overline{\tau'}$ with $\varphi(V) \subseteq U'$. As $\tau|_{U' \cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $U'_1, \dots, U'_n \in \tau|_{U' \cap V'}$ with $\varphi(V) \subseteq U'_1 \cup \dots \cup U'_n$ and

$$V \cap \varphi^{-1}(V') = \bigcup_{i=1}^{n} \varphi^{-1}_{V/U}(U'_i).$$

□

Lemma 2.12. The system of quasi-isomorphisms in k_H-$\widetilde{\mathcal{A}}$n is a right multiplicative system.

For the notion of right multiplicative system, we refer to [Stacks, Tag 04VC].

Proof. We verify the three axioms as in [Stacks, Tag 04VC].

RMS1. The identity is clear a quasi-isomorphism. It remains to verify that the composition of quasi-isomorphisms is still a quasi-isomorphism.

We take φ, ψ as in Definition 2.9. We will use the same notations as in Definition 2.9. We need to show that $V \to V''$ identifies V with a k_H-affinoid domain in V''. From the construction, we know that φ identifies V_i with a k_H-affinoid domain in V'_i and ψ identifies V'_i with a k_H-affinoid domain in V''_i for $i = 1, \dots, n$. In particular, $\chi(V)$ is a compact k_H-analytic domain in V''. It follows from Proposition 13.25 in Affinoid algebras that $\chi(V)$ is a k_H-affinoid domain in V''.

RMS2. If $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ and $f : (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \to (X', \mathcal{A}', \tau')$ are given strong morphisms of k_H-analytic spaces and g is a quasi-isomorphism, then there are k_H-analytic space $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau})$ and strong morphisms $\tilde{\varphi} : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ and $f : (\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \to (X, \mathcal{A}, \tau)$ such that f is a quasi-isomorphism and the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\tau}) \arrow[r, dotted, "\tilde{\varphi}"] \arrow[d, dotted, "f"] & (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'}) \arrow[d, "g"] \\
(X, \mathcal{A}, \tau) \arrow[r, "\varphi"] & (X', \mathcal{A}', \tau')
\end{tikzcd}.$$

We may assume that $\widetilde{X'} = X'$. Then $\widetilde{\tau'} \subseteq \overline{\tau'}$. We let $\tilde{X} = X$. Let $\tilde{\tau}$ be the family of all $V \in \bar{\tau}$ for which there is $\widetilde{V'} \in \widetilde{\tau'}$ with $\varphi(V) \subseteq \widetilde{V'}$. By Lemma 2.11, $\tilde{\tau}$ is a net on $\tilde{X}$. The k_H-atlas $\bar{A}$ defines a k_H-affinoid atlas $\tilde{A}$ with the net $\tilde{\tau}$. The strong morphism $\bar{\varphi}$ induces $\tilde{\varphi}$. The morphism f is the canonical quasi-isomorphism. It is immediate that these constructions satisfy the desired conditions.

RMS3. If $\varphi, \psi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ are strong morphisms of k_H-analytic spaces and there is a quasi-isomorphism $g : (X', \mathcal{A}', \tau') \to (\widetilde{X'}, \widetilde{\mathcal{A}'}, \widetilde{\tau'})$ of k_H-analytic spaces such that $g \circ \varphi = g \circ \psi$, then there is a quasi-isomorphism $f : (\tilde{X}, \tilde{\mathcal{A}}, \tilde{\tau}) \to (X, \mathcal{A}, \tau)$ with $\varphi \circ f = \psi \circ f$.

We will in fact show that $\varphi = \psi$. It is clear that they coincide as maps of topological spaces. Let $V \in \tau$, $V' \in \tau'$ such that $\varphi(V) \subseteq V'$. Take $\widetilde{V'} \in \widetilde{\tau'}$ with $g(V') \subseteq \widetilde{V'}$. Then we have two morphisms of k-affinoid spectra $\varphi_{V/V'}, \psi_{V/V'} : V \to V'$ such that their compositions with $g_{V'/\widetilde{V'}}$ coincide. As V' is an affinoid domain in $\widetilde{V'}$, it follows that $\varphi_{V/V'} = \psi_{V/V'}$ by the universal property. □

Definition 2.13. The category k_H-$\mathcal{A}$n is the right category of fractions of k_H-$\widetilde{\mathcal{A}\mathrm{n}}$ with respect to the system of quasi-isomorphisms. A morphism in k_H-$\mathcal{A}$n is called a *morphism* between k_H-analytic spaces.

We refer to [Stacks, Tag 04VB] for the definition of right category of fractions. For later references, we explicitly write down the morphisms in k_H-$\mathcal{A}$n.

Lemma 2.14. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of k_H-analytic spaces. We define a partial order on the set of nets on X: $\tau_1 \preceq \tau_0$ if $\tau_1 \subseteq \overline{\tau_0}$. Then the set of nets is a directed set and

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}\left((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau')\right) = \varinjlim_{\sigma \preceq \tau} \mathrm{Hom}_{k_H\text{-}\widetilde{\mathcal{A}\mathrm{n}}}\left((X, \mathcal{A}_\sigma, \sigma), (X', \mathcal{A}', \tau')\right)$$

in the category of sets, where $\mathcal{A}_\sigma$ is induced by $\overline{\mathcal{A}}$. The transition maps are all injective.

Proof. This follows immediately from the definition. □

Definition 2.15. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. We say a subset $W \subseteq X$ is *τ-special* if it is compact and there exist $n \in \mathbb{Z}_{>0}$ and a covering $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$ and the natural map

$$A_{W_i} \hat{\otimes}_k A_{W_j} \to A_{W_i \cap W_j}$$

is an admissible epimorphism.

The covering $W_1, \ldots, W_n$ is called a *τ-special covering* of W.

Under our convention, the assumption means that $W_i \cap W_j \to W_i \times W_j$ is a closed immersion of k_H-affinoid spectra.

Example 2.16. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. Suppose that $V \in \tau$ and W is a compact k_H-analytic domain in V. Let $n \in \mathbb{Z}_{>0}$ and $W = W_1 \cup \cdots \cup W_n$ with $W_i \in \tau$, $W_i \cap W_j \in \tau$ for all $i, j = 1, \ldots, n$. Then $\{W_i\}_i$ is a τ-special covering of W. This follows from Corollary 13.14 in Affinoid algebras.

Lemma 2.17. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space and W be a τ-special subset of X. If $U, V \in \tau|_W$, then $U \cap V \in \bar{\tau}$ and the natural map

$$A_U \hat{\otimes}_k A_V \to A_{U \cap V}$$

is an admissible epimorphism.

Proof. Let $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n$ be a τ-special covering of W. As $U \cap W_i$ and $V \cap W_i$ are compact for $i = 1, \ldots, n$, we can find $m_i \in \mathbb{Z}_{>0}$ (resp. $k_i \in \mathbb{Z}_{>0}$) and finite coverings $U_{i1}, \ldots, U_{im_i} \in \tau$ of $U \cap W_i$ (resp. $V_{i1}, \ldots, V_{ik_i} \in \tau$ of $V \cap W_i$).

Observe that $U_{ik} \cap V_{jl}$ is a k_H-affinoid domain in $U \cap V$, hence $U_{ik} \cap V_{jl} \in \bar{\tau}$ for any $i, j = 1, \ldots, n$, $k = 1, \ldots, m_i$ and $l = 1, \ldots, k_l$. By Proposition 12.3 in Affinoid algebras, $U_{ik}\cap V_{jl}\to U_{ik}\times V_{jl}$ is a closed immersion since $W_i\cap W_j\to W_i\times W_j$ is by our assumption.

Consider the finite convering

$$\mathcal{U}:=\{U_{ik}\times V_{jl}:i,j=1,\dots,n;k=1,\dots,m_i;l=1,\dots,k_l\}$$

of $U\times V$. For each tuple (i,j,k,l), $A_{U_{ik}\cap V_{jl}}$ is a finite $A_{U_{ik}\times V_{jl}}$-algebra. By Theorem 14.1 in Affinoid algebras, we can construct a finite $A_{U\times V}$-algebra $A_{U\cap V}$ inducing all of these $A_{U_{ik}\cap V_{jl}}$'s. By Proposition 8.1 in Affinoid algebras, $A_{U\cap V}$ is k_H-affinoid.

As $\mathcal{U}$ is a finite k_H-affinoid covering of $U\times V$, $\{A_{U_{ik}\cap V_{jl}}\}_{i,k,j,l}$ is a finite k_H-affinoid covering of $U\cap V$ by Corollary 13.12 in Affinoid algebras. In particular, we have a natural homeomorphism

$$\operatorname{Sp}A_{U\cap V}\xrightarrow{\sim}U\cap V.$$

Observe that $A_U\hat{\otimes}_k A_V\to A_{U\cap V}$ is surjective. We endow $A_{U\cap V}$ with the structure of finite $A_U\hat{\otimes}_k A_V$-Banach algebras by Proposition 9.10 in Affinoid algebras. Then $A_U\hat{\otimes}_k A_V\to A_{U\cap V}$ is an admissible epimorphism by Proposition 9.7 in Affinoid algebras.

On the other hand $U\cap V$ is a compact k_H-analytic domain in U, so by Proposition 13.25 in Affinoid algebras, $U\cap V$ is a k_H-affinoid in U. In particular, $U\cap V\in\bar{\tau}$. □

Lemma 2.18. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space and $W\subseteq X$ be a τ-special set. Then for any finite covering $\{W_i\}_{i\in I}$ of W with $W_i\in\tau$ for $i\in I$, the Banach k-algebra

$$A_W:=\ker\left(\prod_{i\in I}A_{W_i}\to A_{W_i\cap W_j}\right)$$

does not depend on the choice of $\{W_i\}_{i\in I}$ up to canonical isomorphisms.

Moreover, we have a canonical map $W\to\operatorname{Sp}A_W$, which does not depend on the choice of the covering modulo the canonical isomorphism between A_W.

Proof. It follows from Lemma 2.17 that the covering $\{W_i\}_{i\in I}$ is τ-special. It suffices to apply the same argument of Lemma 13.22 in Affinoid algebras. □

Definition 2.19. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. Let $\hat{\tau}$ denote the collection of $\bar{\tau}$-special subsets $W\subseteq X$ such that

(1) A_W is k-affinoid;
(2) the natural map $W\to\operatorname{Sp}A_W$ is bijective;
(3) there is a $\bar{\tau}$-special covering $\{W_i\}_{i\in I}$ of W such that W_i is a k-affinoid domain in W for $i\in I$.

The sets from $\hat{\tau}$ are called *k_H-affinoid domains in* $(X,\mathcal{A},\tau)$.

Observe that W is k_H-affinoid and W_i is a k_H-affinoid domain in W by Corollary 13.20 in Affinoid algebras. Condition (3) holds for any $\bar{\tau}$-special covering.

Proposition 2.20. Let $(X,\mathcal{A},\tau)$ be a k_H-analytic space. Then $\hat{\tau}$ is a net. For any net σ on X contained in $\bar{\tau}$, we have $\hat{\sigma}=\hat{\tau}$.

Moreover, $\hat{\hat{\tau}}=\hat{\tau}$.

Proof. Let $U, V \in \hat{\tau}$. Take $\bar{\tau}$-special coverings $\{U_i\}_{i\in I}$, $\{V_j\}_{j\in J}$ of U and V respectively. In order to show that $\hat{\tau}|_{U\cap V}$ is a quasi-net, it suffices to show that $\hat{\tau}|_{U_i\cap V_j}$ is for any $i \in I$ and $j \in J$. This follows simply from the fact that $\bar{\tau}|_{U_i\cap V_j}$ is a quasi-net. Similarly, as $\hat{\tau}$ is a quasi-net as $\bar{\tau}$ is. So $\hat{\tau}$ is a net.

Let σ be a net on X contained in $\bar{\tau}$. By Lemma 2.17, it suffices to verify that for any $V \in \bar{\tau}$, there are $n \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_n \in \bar{\sigma}$ with $V = U_1 \cup \cdots \cup U_n$. As σ is a net on X, we can find $m \in \mathbb{Z}_{>0}$, $W_1, \dots, W_m \in \sigma$ such that

$$V \subseteq W_1 \cup \cdots \cup W_m.$$

As $V, W_j \in \bar{\tau}$ for $j = 1, \dots, m$, by **??** in **??**, we can find $U_1, \dots, U_n \in \bar{\tau}$ such that $V = U_1 \cup \cdots \cup U_n$ and each U_i is contained in some W_j. As $W_j \in \sigma$ for $j = 1, \dots, m$, it follows that $U_i \in \bar{\sigma}$ for $i = 1, \dots, n$.

By Lemma 2.17,

$$\bar{\hat{\tau}} = \hat{\tau}.$$

Let $V \in \hat{\hat{V}}$. Let $\{V_i\}_{i\in I}$ be a $\hat{\tau}$-special covering of V. For each $i \in I$, take a $\bar{\tau}$-special covering $\{V_{ij}\}_{j\in J_i}$ of V_i. Then $\{V_{ij}\}_{i,j}$ is a $\bar{\tau}$-special covering of V. It follows that $V \in \hat{\tau}$. □

Proposition 2.21. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. There is a k_H-analytic atlas $\hat{\mathcal{A}}$ on X with the net $\hat{\tau}$ extending $\mathcal{A}$. Moreover, $\hat{\mathcal{A}}$ is unique up to a canonical isomorphism.

Proof. For each $V \in \hat{\tau}$, Fix a $\bar{\tau}$-special covering $\{V_i\}_{i\in I_V}$.

We define A_V using this covering as in Lemma 2.18. By definition, the canonical map $V \to \operatorname{Sp} A_V$ is a homeomorphism.

Next take $U, V \in \hat{\tau}$ with $U \subseteq V$. We want to identify U with a k_H-affinoid domain in V. First assume that $U \in \tau$, then $U \cap V_i$ is a k_H-affinoid domain in V_i for $i \in I_V$ by Lemma 2.17. Hence, U is a k_H-affinoid domain in V. If we only know $U \in \hat{\tau}$, we know that U_i is a k_H-affinoid domain in V for any $i \in I_U$. It follows that U is a k_H-affinoid domain in V by Proposition 13.25 in Affinoid algebras.

The uniqueness is immediate. □

Definition 2.22. Let $(X, \mathcal{A}, \tau)$ be a k_H-analytic space. A $\hat{\tau}$-special set is called a *k_H-special domain* in X.

Observe that a k_H-special domain inherits a structure of k_H-analytic space from $(X, \mathcal{A}, \tau)$.

Proposition 2.23. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism of k_H-analytic spaces. Then for any k_H-affinoid domains $V \subseteq X$ and $V' \subseteq X'$, the intersection $V \cap \varphi^{-1}(V')$ is a k_H-special domain in X.

Proof. By Proposition 2.20, we may assume that φ is a strong morphism. In this case, it suffices to apply Lemma 2.11. □

Lemma 2.24. Let $(X, \mathcal{A}, \tau)$ and $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a strong morphism. Then φ extends uniquely to a strong morphism $\varphi : (X, \hat{\mathcal{A}}, \hat{\tau}) \to (X', \widehat{\mathcal{A}'}, \widehat{\tau'})$.

Proof. Let $V \in \hat{\tau}$ and $V' \in \widehat{\tau'}$ with $\varphi(V) \subseteq V'$. We want to define $\varphi_{V/V'} : V \to V'$ of k_H-affinoid spectra. By Proposition 2.8, we may extend φ uniquely to $\bar{\tau}$. Take a $\bar{\tau}$-special covering of V, we may reduce to the case where $V \in \bar{\tau}$. Take

$W' \in \tau'$ such that $\varphi(V) \subseteq W'$. As $\tau|_{W'\cap V'}$ is a quasi-net, we can find $n \in \mathbb{Z}_{>0}$ and $W_1, \ldots, W_n \in \tau'|_{V'\cap W}$ such that $\varphi(V) \subseteq W_1 \cup \cdots \cup W_n$. Considering the inverse images of W_i's and $W_i \cap W_j$'s using Lemma 2.17, we are reduced to the case where $V' \in \overline{\tau'}$. This is already handled in Proposition 2.8. The uniqueness of the extension is clear. $\square$

Proposition 2.25. Let $(X, \mathcal{A}, \tau)$, $(X', \mathcal{A}', \tau')$ be k_H-analytic spaces.

(1) There is a canonical bijection between

$$\mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}((X, \mathcal{A}, \tau), (X', \mathcal{A}', \tau'))$$

and the set of pairs consisting of

(a) a continuous map $\varphi : X \to X'$ such that for all $x \in X$, there exist $n \in \mathbb{Z}_{>0}$, neighbourhoods $V_1 \cup \cdots \cup V_n$ of x and $V_1' \cup \cdots \cup V_n'$ of $\varphi(x)$ with $x \in V_1 \cap \cdots \cap V_n$ and $\varphi(V_i) \subseteq V_i'$ for $i = 1, \ldots, n$, where $V_i \subseteq X$ and $V_i' \subseteq X'$ are k_H-affinoid domains;

(b) for each pair of k_H-affinoid domains $V \subseteq X$, $V' \subseteq X'$ with $\varphi(V) \subseteq V'$, a morphism of k_H-affinoid spectra $\varphi_{V/V'} : V \to V'$

such that if $V, W \subseteq X$ and $V', W' \subseteq X'$ are k_H-affinoid domains with $\varphi(V) \subseteq V'$, $\varphi(W) \subseteq W'$, the diagram below commutes

$$\begin{array}{ccc} V & \xrightarrow{\varphi_{V/V'}} & V' \\ \downarrow & & \downarrow \\ W & \xrightarrow{\varphi_{W/W'}} & W' \end{array} .$$

(2) Under the bijection in (1), an isomorphism corresponds to the pair where φ is a hoememorphism such that $\varphi(\hat{\tau}) = \widetilde{\tau'}$ and for any $V \in \hat{\tau}$, $\varphi_{V/\varphi(V)}$ is an isomorphism of k_H-affinoid spectra.

Proof. (2) follows immediately from (1). So it suffices to prove (1).

We construct the forward map. Let $\varphi : (X, \mathcal{A}, \tau) \to (X', \mathcal{A}', \tau')$ be a morphism. Take a subnet σ of $\bar{\tau}$ such that φ is represented by a strong morphism

$$\varphi : (X, \mathcal{A}_\sigma, \sigma) \to (X', \mathcal{A}', \tau').$$

By Lemma 2.24, this extends to a strong morphism

$$\varphi : (X, \widehat{\mathcal{A}_\sigma}, \hat{\sigma}) \to (X', \widehat{\mathcal{A}'}, \widehat{\tau'}).$$

We get an injective map from the first set into the second set.

Conversely, we need to show that any given map from the second map comes from the first set. It suffices to show that

$$\sigma := \left\{ V \in \hat{\tau} : \varphi(V) \subseteq V' \text{ for some } V' \in \widehat{\tau'} \right\}$$

is a net. Take $x \in X$ and neighbourhoods $V_1 \cup \cdots \cup V_n$ of x and $V_1' \cup \cdots \cup V_n'$ of $\varphi(x)$ as in the statement of (1). Then $V_i \in \sigma$, so we conclude. $\square$

In practice, we do not distinguish a k_H-analytic space from the isomorphic k_H-analytic spaces. In particular, we will write $(X, \mathcal{A}, \tau)$ as X and always endow it with the strucutre $(X, \hat{\mathcal{A}}, \hat{\tau})$ of k_H-analytic space. If necessarily, we will write $|X|$ for the underlying topological space.

Corollary 2.26. The natural functor $k_H\text{-}\mathcal{A}\mathrm{ff} \to k_H\text{-}\mathcal{A}\mathrm{n}$ is fully faithful.

Proof. Let $X = \operatorname{Sp} A$ be a k_H-affinoid spectrum. We endow it with the net $\tau = \{X\}$. The k_H-atlas with the net τ assigns $X \in \tau$ with A. It is easily verified that this is a functor. By Proposition 2.25, the functor is fully faithful. $\square$

Definition 2.27. A *k_H-affinoid space* is an object of k_H-$\mathcal{A}$n lying in the essential image of the functor k_H-$\mathcal{A}$ff $\to k_H$-$\mathcal{A}$n.

The category of k_H-affinoid spaces is denoted by k_H-$\mathcal{A}$ff.

The notation for the category of k_H-affinoid spaces is the same as the notation for the category of k_H-affinoid spectra, as the two categories are canonically equivalent.

Definition 2.28. A k_H-analytic space X is *good* if any point $x \in X$ admits a k_H-affinoid neighbourhood.

Example 2.29. Fix $n \in \mathbb{N}$. Let $\mathbb{A}^n_k$ denote the set of all semi-valuations on $k[T_1, \dots, T_n]$ whose restriction to k coincides with the given valuation on k. We provide $\mathbb{A}^n_k$ with the weakest topology such that for any $f \in k[T_1, \dots, T_n]$, the map $|\bullet| \mapsto |f|$ is continuous.

Observe that as a topological space,

$$\mathbb{A}^n_k \xrightarrow{\sim} \varinjlim r \in \mathbb{R}^n_{>0} \operatorname{Sp} k\{r^{-1}T\}. \tag{2.1}$$

As a set, this is clear: if $|\bullet| \in \mathbb{A}^n_k$, we take $r = (|T_1|, \dots, |T_n|)$, then $|\bullet| \leq \|\bullet\|_r$, so $|\bullet| \in \operatorname{Sp} k\{r^{-1}T\}$. As

$$\bigcap_{r \in \mathbb{R}^n_{>0}} k\{r^{-1}T\} = k[T_1, \dots, T_n],$$

so the topology on the right-hand side of (2.1) is the weakest topology making $|\bullet| \mapsto |f|$ continuous for any $f \in k[T_1, \dots, T_n]$. It follows immediately that (2.1) is an identification of topological spaces.

It is clear that $\mathbb{A}^n_k$ has a structure of good k_H-analytic space.

Proposition 2.30. Let X be a k_H-analytic space, $x \in X$ and U be a neighbourhood of x in X. Then there is a neighbourhood V of x in X contained in U such that V is open connected locally compact paracompact and Hausdorff. Moreover, we can guarantee that $\bar{V} \subseteq U$ and V is a countable union of k_H-affinoid domains.

Proof. Take $n \in \mathbb{Z}_{>0}$ and k_H-affinoid spaces $V_1, \dots, V_n$ containing x and $V_1 \cup \dots \cup V_n$ is a neighbourhood of x in X. If we have proved the proposition for V_i in place of X and $U \cap V_i$ in place of U for $i = 1, \dots, n$, namely, if we have found open connected locally compact paracompact and Hausdorff sets W_i containing x and contained in $U \cap V_i$ whose closure in V_i is contained in $U \cap V_i$, then we can take $V = W_1 \cup \dots \cup W_n$.

So we may assume that X is a k_H-affinoid space, say $X = \operatorname{Sp} A$. Choose a k_H-rational neighbourhood

$$W = \operatorname{Sp} A\{r^{-1}\frac{f}{g}\}$$

of x in U, where $n \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in A^n$, $r \in \sqrt{|k^\times| \cdot H}^n$, $g \in A$ and $f_1, \dots, f_n, g$ generate the unit ideal in A. This is possible by Corollary 10.9 and Proposition 10.13 in Affinoid algebras. Take $\delta > 0$ so that $x \in \operatorname{Sp} A\{((1-\delta)r)^{-1}\frac{f}{g}\}$.

Choose a strictly increasing sequence $\epsilon_i \in (0,1) \cap \sqrt{|k^\times| \cdot H}$ converging to $1-\delta/2$ for $i \in \mathbb{Z}_{>0}$. Let

$$W_i = \operatorname{Sp} A\left\{(\epsilon_i r)^{-1}\frac{f}{g}\right\}$$

for $i \in \mathbb{Z}_{>0}$. Then W_i lies in the interior of W_{i+1} for $i \in \mathbb{Z}_{>0}$. Choose a connected component V_i of W_i so that $V_1 \subseteq V_2 \subseteq \cdots$ and $x \in V := \bigcup_{i=1}^\infty V_i$. If $x \in V_i$ for some $i \in \mathbb{Z}_{>0}$, then x lies in the topological interior of V_{i+1}. Hence, x lies in the interior of V. By construction, V is open connected paracompact locally compact and Hausdorff. Moreover, $\bar{V} \subseteq U$ by our construction. □

Proposition 2.31. Let $\{X_i\}_{i\in I}$ be a family of k_H-analytic spaces. Suppose that for $i,j \in I$, we are given a k_H-analytic domain $X_{ij} \subseteq X_i$ and an isomorphism $\nu_{ij} : X_{ij} \to X_{ji}$ satisfying the *cocycle condition*: $X_{ii} = X_i$, $\nu_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl}$ and $\nu_{il} = \nu_{jl} \circ \nu_{ij}$ on $X_{ij} \cap X_{il}$ for $i,j,l \in I$.

Assume that either of the followinig conditions holds:

(1) X_{ij} is open in X_i for all $i,j \in I$;
(2) for any $i \in I$, all X_{ij}'s are closed in X_i and the number of $j \in I$ with $X_{ij} \neq \emptyset$ is finite.

Then there is a k_H-analytic space X and morphisms $\mu_i : X_i \to X$ for $i \in I$ such that

(1) μ_i is an isomorphism of X_i with a k_H-analytic domain in X;
(2) $X = \bigcup_{i\in I} \mu_i(X_i)$;
(3) $\mu_i(X_{ij}) = \mu_i(X_i) \cap \mu_j(X_j)$ for $i,j \in I$;
(4) $\mu_i = \mu_j \circ \nu_{ij}$ on X_{ij} for $i,j \in I$.

The space X is unique up to a canonical isomorphism. Moreover, under Condition (1), $\mu_i(X_i)$ is open in X for $i \in I$; under Condition (2), $\mu_i(X_i)$ is closed in X for $i \in I$.

Under both conditions, if all X_i's are Hausdorff (resp. paracompact), then so is X.

We will call X the *gluing* of the X_i's along the X_{ij}'s.

Proof. By Proposition 3.11, the uniquenss of X is clear.

Let

$$\tilde{X} = \coprod_{i\in I} X_i$$

in k_H-$\mathcal{A}$n. Observe that

$$|\tilde{X}| = \coprod_{i\in I} |X_i|$$

in the category $\mathcal{T}$op. The system ν_{ij}'s defines an equivalence relation R on $|\tilde{X}|$. Let $|X| = |\tilde{X}|/R$ and $\mu_i : |X_i| \to |X|$ be the induced map for $i \in I$.

Under Condition (1), $\mu_i(|X_i|)$ is open in $|X|$ for $i \in I$. Under Condition (2), $\mu_i(|X_i|)$ is closed in X for $i \in I$.

Under both conditions, the map μ_i induces a homeomorphism $|X_i| \to \mu_i(|X_i|)$ for $i \in I$. If all $|X_i|$'s are Hausdorff (resp. paracompact), so is $|X|$.

All these claims follow from well-known results in general topology.

We will endow $|X|$ with a structure of k_H-analytic space. Let τ be the set of $V \subseteq |X|$ for which there is $i \in I$ such that $V \subseteq \mu_i(X_i)$ and $\mu_i^{-1}(V)$ is a k_H-affinoid domain in X_i. Then τ is a net on X. There is an obvious k-affinoid atlas on X with the net τ. All properties in the proposition are satisfied by $X = (|X|, \mathcal{A}, \tau)$. □

Definition 2.32. Let X be a k_H-analytic space and $x \in X$, take a k_H-affinoid domain $\operatorname{Sp} A$ in X containing x, we define the *completed residue field* $\mathscr{H}(x)$ of x in X as the completed residue field of x in $\operatorname{Sp} A$.

By Corollary 13.16 in Affinoid algebras, $\mathscr{H}(x)$ does not depend on the choice of $\operatorname{Sp} A$ up to an isomorphism of complete valuation fields over k.

3. Analytic domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. Let X be a k_H-analytic space. A subset $Y \subseteq X$ is called a *k_H-analytic domain* if for any $y \in Y$, there exist $n \in \mathbb{Z}_{>0}$, k_H-affinoid domains $V_1, \ldots, V_n$ contained in Y such that

(1) $y \in V_1 \cap \cdots \cap V_n$;
(2) $V_1 \cup \cdots \cup V_n$ is a neighbourhood of y in Y.

Observe that the net of k_H-affinoid domains in X that are contained in Y form a net on Y. In particular, Y inherits a k_H-analytic space structure from X, and we have a canonical morphism $Y \to X$ in k_H-$\mathcal{A}\mathrm{n}$.

Lemma 3.2. Let X be a k_H-analytic space, Y be a k_H-analytic domain in X and $x \in Y$. Then the complete residue field of x in X is the same as the complete residue field of x in Y modulo isomorphisms of completed valuation fields over k.

Proof. This follows immediately from Corollary 13.16 in Affinoid algebras. □

Example 3.3. Let X be a k_H-analytic space. Then any open subset U of X is a k_H-analytic domain.

In fact, for $x \in U$, take $V_1, \ldots, V_n$ as in Definition 3.1. By Proposition 10.13 in Affinoid algebras, up to replacing V_i's by k_H-Laurent domains in them, we may guarantee that $V_i \subseteq U$ for all $i = 1, \ldots, n$.

Proposition 3.4. Let X, X' be k_H-analytic spaces and $\varphi : X' \to X$ a morphism of k_H-analytic spaces.

(1) Let Y, Z be k_H-analytic domains in X, then so is $Y \cap Z$.
(2) Let Y be a k_H-analytic domain in X, then $\varphi^{-1}(Y)$ is a k_H-analytic domain in X'.

Proof. (1) Let $x \in Y \cap Z$. Take k_H-affinoid domains $V_1, \ldots, V_n$ contained in Y and k_H-affinoid domains $W_1, \ldots, W_m$ contained in Z such that

$$x \in V_1 \cap \cdots \cap V_n, \quad x \in W_1 \cap \cdots \cap W_m$$

and $V_1 \cup \cdots \cup V_n$ is a neighbourhood of x in Y, $W_1 \cup \cdots \cup W_m$ is a neighbourhood of x in Z. For each $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $\hat{\tau}|_{V_i \cap W_j}$ is a quasi-net, so we can find a neighbourhood of x in $V_i \cap W_j$ of the form $U_1^{ij} \cup \cdots \cup U_{m_{ij}}^{ij}$ with $U_1^{ij}, \ldots, U_{m_{ij}}^{ij}$ being k_H-affinoid domains in X containing x. Then each element in the collection $\{U_k^{ij}\}$ contains x and the union is a neighbourhood of x in $Y \cap Z$.

(2) Let $x' \in \varphi^{-1}(Y)$ and $x = \varphi(x')$. By Proposition 2.25, we can find $n \in \mathbb{Z}_{>0}$, k_H-affinoid domains $V'_1, \ldots, V'_n$ on X' and k_H-affinoid domains $V_1, \ldots, V_n$ on X such that

$$x' \in V'_1 \cap \cdots \cap V'_n, \quad x \in V_1 \cap \cdots \cap V_m,$$
$$\varphi(V'_i) \subseteq V_i \text{ for } i = 1, \ldots, n,$$

and $V'_1 \cup \cdots \cup V'_n$ (resp. $V_1 \cup \cdots \cup V_n$) is a neighbourhood of x' (resp. x) in X' (resp. X). Take k_H-affinoid domains $W_1, \ldots, W_m$ in X contained in Y, each containing x such that $W_1 \cup \cdots \cup W_m$ is a neighbourhood of x in Y.

Then for each $i = 1, \ldots, n$, $j = 1, \ldots, m$, we can find k_H-affinoid domains W_{ij}^k for $k = 1, \ldots, r_{ij}$ contained in $W_j \cap V_i$ and containing x such that $\cup_k W_{ij}^k$ is a neighbourhood of x in $W_j \cap V_i$. Thus, $\cup_{j,k} W_{ij}^k$ is a neighbourhood of x in $V_i \cap Y$. Then $U_{ij}^k := \varphi^{-1}(V_{ij}^k) \cap V'_i$ is a k_H-affinoid domain in V'_i by Corollary 13.12 in Affinoid algebras. Moreover, $\cup_{j,k} U_{ij}^k$ is a neighbourhood of x' in $V'_i \cap Y'$. So $\cup_{i,j,k} U_{ij}^k$ is a neighbourhood of x' in Y'. □

Proposition 3.5. Let X be a k_H-analytic space and Y be a k_H-analytic domain in X. Then for any k_H-analytic space Z and any morphism $\varphi : Z \to X$ whose image is contained in Y, there is a unique morphism $\psi : Z \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[d, dotted, "\psi"] \arrow[dr, "\varphi"] & \\ Y \arrow[r] & X \end{tikzcd} .$$

Proof. The uniqueness of ψ is obvious. We only need to prove the existence. This is an immediate consequence of Proposition 2.25 and Proposition 3.4.

To be more precise, assume that φ is given by a data as in Proposition 2.25, we only have to show that each k_H-affinoid domain V in X, $V \cap Y$ is a k_H-affinoid domain in Y. This follows from Proposition 3.4. □

Corollary 3.6. Let $\varphi : X' \to X$ be a morphism of k_H-analytic spaces and Y be a k_H-analytic domain in X. Then $X' \times_Y X$ in the category k_H-$\mathcal{A}\mathrm{n}$ exists and $\varphi^{-1}(Y)$ represents $X' \times_Y X$.

Proof. This follows from Proposition 3.5 and Proposition 3.4. □

Corollary 3.7. Let $\operatorname{Sp} B$ be a k_H-affinoid space, then we have a functorial isomorphism

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \mathbb{A}^1_k) \xrightarrow{\sim} B.$$

Proof. As $\operatorname{Sp} B$ is compact as a topological space, its image in $\mathbb{A}^1_k$ is contained in $\operatorname{Sp} k\{r^{-1}T\}$ for some $r > 0$. By Proposition 3.5, we have natural bijections

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \mathbb{A}^1_k) \xrightarrow{\sim} \varinjlim_{r>0} \operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(\operatorname{Sp} B, \operatorname{Sp} k\{r^{-1}T\}) \xrightarrow{\sim} \varinjlim_{r>0} \operatorname{Hom}_{k\text{-}\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}}(k\{r^{-1}T\}, B).$$

By Corollary 6.5 in Affinoid algebras, the right-hand side is identified with B. □

Proposition 3.8. Let X be a k_H-analytic space, Y be a k_H-analytic domain in X. For a subset $Z \subseteq Y$, the following are equivalent:

(1) Z be a k_H-analytic domain in X;
(2) Z is a k_H-analytic domain in Y.

Proof. (1) $\implies$ (2): Let $z \in Z$, we can find $n \in \mathbb{Z}_{>0}$ and k_H-affinoid domains $V_1, \ldots, V_n$ in X containing x and contained in Z such that $V_1 \cup \cdots \cup V_n$ is a neighbourhood of z in Z. But observe that $V_1, \ldots, V_n$ are k_H-affinoid domains in Y as well, so we conclude.

(2) $\implies$ (1): This follows from the same argument. It suffices to observe that a k_H-affinoid domain in Y is necessarily k_H-affinoid in X, as can be seen from Definition 2.19. $\square$

Definition 3.9. Let X, Y be k_H-analytic spaces and $\varphi : Y \to X$ be a morphism. We say φ is an *open immersion* if $\varphi(Y)$ is open in X and φ induces an isomorphism between Y and $\varphi(Y)$ as k_H-analytic spaces.

By Example 3.3, $\varphi(Y)$ is a k_H-analytic domain in X and by Proposition 3.5, we have a morphism of k_H-analytic spaces $Y \to \varphi(Y)$.

Proposition 3.10. Let X be a k_H-analytic space and Y be a k_H-analytic domain in X. Assume that Y is a k_H-affinoid space, then Y is a k_H-affinoid domain in X.

Proof. As Y is a k_H-affinoid space, we know that $|Y|$ is compact. Take finitely many k_H-affinoid domains $V_1, \dots, V_n$ in X such that

$$Y = V_1 \cup \dots \cup V_n.$$

Then $V_1, \dots, V_n$ are k_H-affinoid domains in Y: let $\operatorname{Sp} D \to Y$ be a morphism of k_H-affinoid spectra, whose image lies in V_i for some $i = 1, \dots, n$. Consider the following commutative diagram

$$\begin{tikzcd} & \operatorname{Sp} D \arrow[dl, dotted] \arrow[d] \arrow[dr] & \\ V_i \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

By Proposition 3.5, there is a unique dotted morphism making the outer triangle commutative, hence making the whole diagram commutative. We have therefore shown that V_i is a k_H-affinoid domain in Y.

So the covering $\{V_1, \dots, V_n\}$ of Y satisfies the assumptions in Definition 2.19 and Y is k_H-affinoid. $\square$

Proposition 3.11. Let X be a k_H-analytic space and $\{Y_i\}_{i\in I}$ be a family of k_H-analytic domains in X which forms a quasi-net on X. Then for any k_H-analytic space X', the following sequence is exact

$$\operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(X, X') \to \prod_{i\in I} \operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(Y_i, X') \rightrightarrows \prod_{i,j\in I} \operatorname{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(Y_i \cap Y_j, X').$$

Proof. Let $\{\varphi_i : Y_i \to X'\}_{i\in I}$ be a family of morphisms such that φ_i, φ_j coincides on $Y_i \cap Y_j$ for $i, j \in I$. We need to glue the φ_i's into a single morphism $\varphi : X \to X'$. Clearly, the underlying maps glue together to a continuous map $\varphi : X \to X'$ by **??** in **??**.

Let τ be the collection of k_H-affinoid domains V in X such that there is $i \in I$ and a k_H-affinoid domain $V' \subseteq X'$ with $V \subseteq Y_i$ and $\varphi_i(V) \subseteq V'$. Then τ is a net on X, and we have a morphism $X \to X'$. $\square$

4. Berkovich site

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Lemma 4.1. Let X be a k_H-analytic space. Consider the category $\mathcal{C}$ of k_H-analytic domains in X, where the morphisms are inclusions of k_H-analytic domains. For each $Y \in \mathcal{C}$, consider the set of coverings $\mathrm{Cov}(Y)$ consisting of all $\{Y_i \to Y\}_{i\in I}$ such that Y_i is a k_H-analytic domain in Y and $\{Y_i\}_{i\in I}$ is a quasi-net on Y. The class of coverings $\{\mathrm{Cov}(Y)\}_Y$ defines a Grothendieck pretopology.

Proof. It suffices to verify the axioms in [Stacks, Tag 03NH].

(1) An isomorphism $Y' \to Y$ in $\mathcal{C}$ is in $\mathrm{Cov}(Y)$.

This is trivial as an isomorphism in $\mathcal{C}$ is necessarily identity.

(2) If $\{Y_i \to Y\}_{i\in I}$ and $\{Y_{ij} \to Y_i\}_{j\in J_i}$ for all $i \in I$ are in $\mathrm{Cov}(Y)$ and $\mathrm{Cov}(Y_i)$ respectively, then $\{Y_{ij} \to Y\}_{i,j}$ is in $\mathrm{Cov}(Y)$.

By Proposition 3.8, Y_{ij} is a k_H-analytic domain in Y for any $i \in I$, $j \in I_j$. It suffices to show that $\{Y_{ij}\}_{i,j}$ is a quasi-net on Y. Let $y \in Y$, we can find finitely many elements among $\{Y_i\}_{i\in I}$, say $Y_1, \ldots, Y_n$ so that $y \in Y_i$ for each $i = 1, \ldots, n$ and $Y_1 \cup \cdots \cup Y_n$ is a neighbourhood of y in Y. Similarly, for each $i = 1, \ldots, n$, we can find finitely many $Y_{i1}, \ldots, Y_{ij_i}$ among $\{Y_{ij}\}_{j\in J_i}$ so that y is contained in each of them and $Y_{i1} \cup \cdots \cup Y_{ij_i}$ is a neighbourhood of y in Y_i. Then each element in $\{Y_{ij}\}_{i=1,\ldots,n;j=1,\ldots,j_i}$ contains y and the union is a neighbourhood of y in Y.

(3) If $\{Y_i \to Y\}_{i\in I}$ lies in $\mathrm{Cov}(Y)$ and $Z \to Y$ is a k_H-analytic domain in Y, then the fiber products $Y_i \times_Y Z$ exist and $\{Y_i \times_Y Z \to Z\}_{i\in I}$ lies in $\mathrm{Cov}(Z)$.

By Corollary 3.6, $Y_i \times_Y Z$ exists and is represented by the inverse image of Z in Y_i, which is a k_H-analytic domain in Y_i by Proposition 3.4. It is clear that $\{Y_i \times_Y Z\}_{i\in I}$ is a quasi-net on Z. □

Definition 4.2. Let X be a k_H-analytic space. We will write the site constructed in Lemma 4.1 as X and call it the *Berkovich site* of X. The corresponding Grothendieck topology is called the *Berkovich Grothencieck topology*. The topos $\mathrm{Sh}(X)$ associated with X is called the *Berkovich topos* of X.

Observe that the Berkovich Grothendieck topology is subcanonical by Proposition 3.11.

Definition 4.3. Let X be a k_H-analytic space. We define a sheaf of rings $\mathcal{O}_X$ on X as follows: let Y be a k_H-analtic domain in X, we set

$$\mathcal{O}_X(Y) = \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}}(X, \mathbb{A}^1_k).$$

By Corollary 3.7 and Proposition 3.11, $\mathcal{O}_X$ defines a sheaf of rings. We call $\mathcal{O}_X$ the structure sheaf of X. The corresponding ringed site $(X, \mathcal{O}_X)$ is called the *Berkovich ringed site*. The induced ringed topos $(\mathrm{Sh}(X), \mathcal{O}_X)$ is called the *Berkovich ringed topos*.

Given any morphism $f : Y \to X$ of k_H-analytic spaces, we have an induced morphism of the corresponding ringed sites, still denoted by φ.

Definition 4.4. Let X be a k_H-analytic space. An $\mathcal{O}_X$-module $\mathcal{M}$ is *coherent* if there is an admissible covering $\{Y_i\}_{i\in I}$ of X such that $\mathcal{M}|_{Y_i}$ is isomorphic to the cokernel of a homomorphism of finite free $\mathcal{O}_{V_i}$-modules.

Example 4.5. Let A be a k_H-affinoid algebra and M be a fintie A-module. Then M induces a coherent sheaf of $\mathcal{O}_{\mathrm{Sp}\,A}$-modules $\tilde{M}$ as follows:

$$\tilde{M}(V) = M \otimes_A A_V.$$

Conversely, we can reformulate Kiehl's theorem.

Theorem 4.6. Let A be a k_H-affinoid algebra and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_{\operatorname{Sp} A}$-modules. Set $M = H^0(X, \mathcal{M})$, then M is a finite A-modue and we have a canonical isomorphism

$$\tilde{M} \xrightarrow{\sim} \mathcal{M}.$$

The left-hand side is defined in Example 4.5.

Proof. This is just a reformulation of Theorem 14.1 in Affinoid algebras. □

Corollary 4.7. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spaces. Then the following are equivalent:

(1) $\varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is a coherent $\mathcal{O}_{\operatorname{Sp} A}$-module;
(2) B is a finite Banach A-module.

Proof. Observe that for any k_H-affinoid domain $\operatorname{Sp} C$ in $\operatorname{Sp} A$,

$$\varphi_* \mathcal{O}_{\operatorname{Sp} B}(\operatorname{Sp} C) = \mathcal{O}_{\operatorname{Sp} B}(\varphi^{-1}(\operatorname{Sp} C)) = \mathcal{O}_{\operatorname{Sp} B}(\operatorname{Sp} C \hat{\otimes}_A B) = C \hat{\otimes}_A B \xrightarrow{\sim} C \otimes_A B.$$

Here we applied Corollary 13.12 in **??** and Proposition 9.6 in **??** . So $\varphi_* \mathcal{O}_{\operatorname{Sp} B} \cong \widetilde{B}$.

From this (2) trivially implies (1).

Conversely, assume (1), let $B = H^0(\operatorname{Sp} A, \varphi_* \mathcal{O}_{\operatorname{Sp} B})$. By Theorem 4.6, B is a finite A-module. Let B' denote the ring B endowed with the finite Banach A-algebra structure as in Proposition 9.10 in Affinoid algebras. We need to show that the identity map $B' \to B$ is admissible. Observe that the identity map is bounded by Proposition 9.4 in **??** . By Proposition 10.5 in **??** , it suffices to show that the induced map $\operatorname{Sp} B \to \operatorname{Sp} B'$ is surjective. Let $\varphi' : \operatorname{Sp} B' \to \operatorname{Sp} A$ be the natural morphism of k_H-affinoid spaces. Then

$$\varphi_*(\mathcal{O}_{\operatorname{Sp} B}) \xrightarrow{\sim} \varphi'_*(\mathcal{O}_{\operatorname{Sp} B'}).$$

It follows that $\varphi^{-1}(x) = \varphi'^{-1}(x)$ for any $x \in \operatorname{Sp} A$. We conclude. □

Corollary 4.8. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spaces. Then the following are equivalent:

(1) $\varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is a coherent $\mathcal{O}_{\operatorname{Sp} A}$-module and $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective;
(2) $A \to B$ is an admissible epimorphism.

Proof. Assume (2). By Corollary 4.7, $\varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is a coherent $\mathcal{O}_{\operatorname{Sp} A}$-module. To see that $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective, it suffices to show that for each k_H-affinoid space $\operatorname{Sp} C$ in $\operatorname{Sp} A$,

$$C \to C \otimes_A B$$

is surjective. This follows from the assumption.

Assume (1). We know that B is a finite Banach A-module. In particular, $A \to B$ is admissible by Proposition 9.7 in Affinoid algebras. As $\mathcal{O}_{\operatorname{Sp} A} \to \varphi_* \mathcal{O}_{\operatorname{Sp} B}$ is surjective, by Theorem 4.6, $A \to B$ is surjective. Include details □

Definition 4.9. Let $\operatorname{Sp} A$ be a k_H-affinoid space and $\mathcal{M} = \tilde{M}$ is a coherent sheaf of $\mathcal{O}_X$-modules on X, where M is a finite A-module. The *support* $\operatorname{Supp} M$ of $\mathcal{M}$ is the closed subset $\operatorname{Sp} A/\operatorname{Ann}_A(M)$ of $\operatorname{Sp} A$.

Let X be a k_H-analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then the *support* $\operatorname{Supp} \mathcal{M}$ of $\mathcal{M}$ is a subset of X such that a point $x \in X$ lies in $\operatorname{Supp} \mathcal{M}$ if and only if for some k_H-affinoid domain V in X containing x, $x \in \operatorname{Supp} \mathcal{M}|_V$.

Here $\operatorname{Ann}_A(M)$ is the annihilator of M in A.

Lemma 4.10. Let X be a k_H-analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Take $x \in \operatorname{Supp} \mathcal{M}|_V$ and a k_H-affinoid domain V in X containing x. Then $x \in \operatorname{Supp} \mathcal{M}|_V$.

Proof. By assumption, there is a k_H-affinoid domain U in X containing x such that $x \in \operatorname{Supp} \mathcal{M}|_U$.

Let $W \subseteq U \cap V$ be a k_H-affinoid domain in X containing x. We claim that $x \in \operatorname{Supp} \mathcal{M}|_W$. Let $M = H^0(U, \mathcal{M})$, then $M \otimes_{A_U} A_W = H^0(W, \mathcal{M})$. By [Stacks, Tag 07T8] and Theorem 13.18 in Affinoid algebras,

$$\operatorname{Ann}_{A_U}(M) \otimes_{A_U} A_W = \operatorname{Ann}_{A_W}(M \otimes_{A_U} A_W)$$

and $\operatorname{Supp}(\mathcal{M}|_W) = \operatorname{Supp}(\mathcal{M}|_U) \cap W$. The claim follows. We may assume that $U \subseteq V$. In this case, the same argument shows that $x \in \operatorname{Supp} \mathcal{M}|_V$. □

Proposition 4.11. Let X be a Hausdorff k_H-analytic space. Then the following are equivalent:

(1) X is paracompact;
(2) X admits a locally finite covering by k_H-affinoid domains.

Note that the covering in (2) is necessarily a G-covering.

Proof. Assume (1). Then (2) follows from **??** in **??**. We take $\mathcal{B}$ to the collection of finite unions of k_H-affinoid domains that contain an open subset of X.

Assume (2). Let $\{X_i\}_{i\in I}$ be a locally finite covering of X by k_H-affinoid domains. Define an equivalence relation on I generated by $i \sim j$ if $X_i \cap X_j \neq \emptyset$. We say X_i and X_j are elementarily linked in this case. Fix $C \in I/\sim$ and $i \in C$. For any $n \in \mathbb{Z}_{>0}$, C_n denotes the union of X_j where j and i are linked through a chain of elementary links of length at most n. As the covering is locally finite, we see that C_n is compact. So

$$X_C = \bigcup_{i=1}^{\infty} C_i$$

is σ-compact. The space X is clearly the coproduct of X_C's, hence paracompact by **??** in **??**. □

Proposition 4.12. The category k_H-$\mathcal{A}$n admits finite limits.

Proof. By general abstract nonsense, it suffices to show that k_H-$\mathcal{A}$n admits finite fiber products.

Let $\varphi : Y \to X$ and $f : X' \to X$ be morphisms of k_H-affinoid spaces. We want to construct $Y \times_X X'$.

Step 1. We assume that X, Y, X' are all paracompact and Hausdorff.

By Proposition 4.11, we can find a locally finite G-covering $\{X_i\}_{i\in I}$ of X consisting of k_H-affinoid domains in X. By Proposition 4.11 again, we can find a locally finite G-covering $\{Y_{ij}\}_j$ $\varphi^{-1}(X_i)$ consisting of k_H-affinoid domains in Y and a locally finite G-covering $\{X'_{il}\}_l$ consisting of k_H-affinoid domains in X' for each $i \in I$.

We can glue $Y_{ij} \times_{X_i} X'_{il}$'s by Proposition 2.31 to get a k_H-analytic space Y'. By Proposition 3.11, Y' represents the fiber product $Y \times_X X'$.

Step 2. Assume only that X is a paracompact and Hausdorff.

Take open paracompact Hausdorff coverings $\{Y_i\}_{i\in I}$ of Y and $\{X'_j\}_{j\in J}$ of X'. The existence of these coverings follows from Proposition 2.30. Similar to Step 1, we glue the $Y_i\times_X X'_j$'s along the open subsets $(Y_i\cap Y_k)\times_X(X'_j\cap X'_l)$'s by Proposition 2.31, we get a locally Hausdorff k_H-analytic space Y'. Then by Proposition 3.11 again, Y' represents the fiber product $Y\times_X X'$.

Step 3. We handle the general case.

Take a covering $\{X_i\}_{i\in I}$ by open paracompact Hausdorff subsets. Let Y' be the gluing of $\varphi^{-1}(X_i)\times_{X_i} f^{-1}(X_i)$'s along $\varphi^{-1}(X_i\cap X_j)\times_{X_i\cap X_j} f^{-1}(X_i\cap X_j)$'s by Proposition 2.31. Then by Proposition 3.11 again, Y' represents the fiber product $Y\times_X X'$. □

Remark 4.13. The original proof in [Ber93] doees not make any sense to me. Please contact me if you understand the details of Berkovich's argument.

In a similar vein, we prove

Proposition 4.14. If K/k is an analytic field extension, then there is a natural functor of base extension k_H-$\mathcal{A}$n $\to K_H$-$\mathcal{A}$n extending the functor k_H-$\mathcal{A}$ff $\to K_H$-$\mathcal{A}$ff defined by $\mathrm{Sp}\,A\mapsto \mathrm{Sp}\,A\hat{\otimes}_k K$.

We will denote the image of a k_H-analytic space X by X_K.

Proof. Fix a k_H-analytic space X, we want to construct functorially a K_H-analytic space X_K.

Step 1. We assume that X is paracompact and Hausdorff.

By Proposition 4.11, we can find a locally finite G-covering $\{X_i\}_{i\in I}$ of X consisting of k_H-affinoid domains in X. We can glue $X_{i,K}$'s by Proposition 2.31 to get X_K.

Step 2. In general, let $\{Y_i\}_{i\in I}$ be an open covering of X by paracompact Hausdorff subsets. We glue $Y_{i,K}$'s by Proposition 2.31 to get X_K.

These constructions are clearly functorial and defines a functor k_H-$\mathcal{A}$n $\to$ K_H-$\mathcal{A}$n. □

5. Closed immersions

Let $(k,|\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times|\cdot H\neq\{1\}$.

Lemma 5.1. Let $\varphi: Y\to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) for any $x\in X$, there are $n\in\mathbb{Z}_{>0}$ and k_H-affinoid domains $V_1,\ldots,V_n$ in X containing x such that $V_1\cup\cdots\cup V_n$ is a neighbourhood of x in X and the restriction $\varphi^{-1}(V_i)\to V_i$ is a closed immersion for any $i=1,\ldots,n$;
(2) for any k_H-affinoid domain V in X, $\varphi^{-1}(V)\to V$ is a closed immersion.

Recall that closed immersions between k_H-affinoid spaces are defined in Definition 12.1 in Affinoid algebras.

The statement in [Ber93, Lemma 1.3.7] is not correct.

Proof. Only (1) $\implies$ (2) is non-trivial. Assume (1). Let τ be the collections of $V\subseteq X$ satisfying (2). Then we claim that τ is a net.

Observe that τ is a quasi-net by our assumption. To see that it is a net, take $U,V\in\tau$ and $x\in U\cap V$, then we can find $n\in\mathbb{Z}_{>0}$ and k_H-affinoid domains $W_1,\ldots,W_n$ in $U\cap V$ containing x such that $W_1\cup\cdots\cup W_n$ is a neighbourhood of x in $U\cap V$. In order to show that $\tau|_{U\cap V}$ is a quasi-net, it suffices to show that $\varphi^{-1}(W_i) \to W_i$ is a closed immersion for $i = 1, \dots, n$. This follows from Proposition 12.3 in Affinoid algebras.

Let V be a k_H-affinoid domain in X. By (1) and the compactness of V, we can find $n \in \mathbb{Z}_{>0}$ and $V_1, \dots, V_n \in \tau$ such that $V \subseteq V_1 \cup \cdots \cup V_n$. By **??** in **??**, we can find $m \in \mathbb{Z}_{>0}$ and $U_1, \dots, U_m \in \tau$ such that

$$V = U_1 \cup \cdots \cup U_m$$

and each U_j is contained in some V_i, where $j = 1, \dots, m$ and $i = 1, \dots, n$. By Proposition 12.3 in Affinoid algebras again, $U_j \in \tau$ for each $j = 1, \dots, m$. It suffices to apply Corollary 4.8 to conclude that $V \in \tau$. □

Definition 5.2. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. We say φ is a *closed immersion* if the equivalent conditions in Lemma 5.1 are satisfied.

Observe that this definition extends Definition 12.1 in Affinoid algebras.

Proposition 5.3. Let $\varphi : Y \to X$, $\psi : Z \to X$ be a morphism of k_H-analytic spaces. Assume that $\varphi : Y \to X$ is a closed immersion. Consider the Cartesian diagram

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\varphi"] \\ Z \arrow[r, "\psi"] & X \end{tikzcd}.$$

Then $Z \times_X Y \to Z$ is a closed immersion.

Proof. Taking a G-covering of Z, we may assume that Z is compact. We could cover the images of Z in X by finitely many k_H-affinoid domains $V_1, \dots, V_n$ in X, considering their preimages in Z, we could reduce to the case where the image of Z in X is contained in a k_H-affinoid domain. We could then assume that X is a k_H-affinoid space and hence so is Y. By taking a G-covering of Z again, we may assume that Z is affinoid. It suffices to apply Proposition 12.3 in Affinoid algebras. □

Proposition 5.4. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) φ is a closed immersion;
(2) for any G-covering $\{X_i\}_{i \in I}$ of X, the restriction of φ to $\varphi^{-1}(X_i) \to X_i$ is a closed immersion for all $i \in I$;
(3) for some G-covering $\{X_i\}_{i \in I}$ of X, the restriction of φ to $\varphi^{-1}(X_i) \to X_i$ is a closed immersion for all $i \in I$.

In other words, being a closed immersion is a G-local property on the target.

Proof. Assume (1). Let $\{X_i\}_{i \in I}$ be a G-covering of X. Then the restriction of φ to $\varphi^{-1}(X_i) \to X_i$ is a closed immersion for all $i \in I$ by Proposition 5.3. So (2) holds.

(2) trivially implies (3).

Assume (3). Using the fact that (1) implies (2) as we already proved, we may refine the G-covering $\{X_i\}_{i \in I}$ and assume that each X_i is k_H-affinoid. It follows from Lemma 5.1 that φ is a closed immersion, so (1) holds. □

Corollary 5.5. Let $H' \supseteq H$ is a subgroup of $\mathbb{R}_{>0}$. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) φ is a closed immersion;
(2) φ is a closed immersion when view as a morphism of $k_{H'}$-affinoid spaces.

Proof. By Proposition 5.4, we may assume that X is k_H-affinoid. In this case, Y is also k_H-affinoid and the result is clear. □

Corollary 5.6. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces and K/k be an analytic field extension.

(1) If φ is a closed immersion, so is φ_K;
(2) If $K = k_r$ for some k-free polyray r and φ_K is a closed immersion, then so is φ.

Proof. By Proposition 5.4, we may assume that X is a k_H-affinoid space in both cases. Then so is Y. Now (1) is obvious and (2) follows from Proposition 3.11 in Affinoid algebras. □

Proposition 5.7. Let $\varphi : X \to Y$, $\psi : Y \to Z$ be closed immersions of k_H-affinoid spaces. Then $\psi \circ \varphi : X \to Z$ is also a closed immersion.

Proof. By Proposition 5.4, we may assume that Z is k_H-affinoid, so Y and X are also k_H-affinoid. In this case, the result is clear, as the composition of admissible epimorphisms is clearly admissible epimorphic. □

Proposition 5.8. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) φ is a closed immersion;
(2) $\varphi_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module and $\mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ is surjective.

Proof. As both properties are G-local on X, we may assume that X is a k_H-affinoid space and hence so is Y. This result then follows from Corollary 4.8. □

6. Separated morphisms

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 6.1. Let $\varphi : X \to Y$ be a morphism of k_H-analytic spaces. The *diagonal morphism* of f is the morphism $\Delta_\varphi = \Delta_{X/Y} : X \to X \times_Y X$ defined as follows: let $\{Y_i\}_{i\in I}$ be a G-covering of Y by k_H-affinoid domains and $\{X_{ij}\}_{j\in J_i}$ be a G-covering of $\varphi^{-1}(Y_i)$ by k_H-affinoid domains in X. Then we have a diagonal morphism $\Delta_{X_{ij}/Y_i} : X_{ij} \to X_{ij} \times_{Y_i} X_{ij}$ defined by the codiagonal morphism of k_H-affinoid algebras. The induced morphisms $X_{ij} \to X \times_Y X$ can be glued together by Proposition 3.11 to get Δ_φ. By Proposition 3.11 does not depend on the choices of the G-coverings.

Definition 6.2. A morphism $\varphi : X \to Y$ of k_H-analytic spaces is *separated* if $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion.

Example 6.3. A morphism between k_H-affinoid spaces is always separated. This follows from Example 12.2 in Affinoid algebras by base change.

Proposition 6.4. Let $\varphi : Y \to X$, $\psi : Z \to X$ be a morphism of k_H-analytic spaces. Assume that $\varphi : Y \to X$ is separated. Consider the Cartesian diagram

$$\begin{tikzcd} Z \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\varphi"] \\ Z \arrow[r, "\psi"] & X \end{tikzcd}.$$

Then $Z \times_X Y \to Z$ is separated.

Proof. By general abstract nonsense, we have a Cartesian diagram

$$\begin{tikzcd} Z \times_X Y \arrow[r, "\Delta_{Z\times_X Y/Z}"] \arrow[d] \arrow[dr, phantom, "\square"] & (Z \times_X Y) \times_Z (Z \times_X Y) = Z \times_X (Y \times_X Y) \arrow[d] \\ Y \arrow[r, "\Delta_{Y/X}"] & Y \times_X Y \end{tikzcd}.$$

So the assertion follows from Proposition 5.3. $\square$

Proposition 6.5. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) φ is separated;
(2) for any G-covering $\{X_i\}_{i\in I}$ of X, the restriction of φ to $\varphi^{-1}(X_i) \to X_i$ is separated for all $i \in I$;
(3) for some G-covering $\{X_i\}_{i\in I}$ of X, the restriction of φ to $\varphi^{-1}(X_i) \to X_i$ is separated for all $i \in I$.

Proof. (1) $\implies$ (2) by Proposition 6.4.
(2) $\implies$ (3) is trivial.
Assume (3). Let $Y_i = \varphi^{-1}(X_i)$. Then $Y_i \times_{X_i} Y_i$ is a G-covering of $Y \times_X Y$, and we have a Cartesian diagram

$$\begin{tikzcd} Y_i \arrow[r, "\Delta_{Y_i/X_i}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_i \times_{X_i} Y_i \arrow[d] \\ Y \arrow[r, "\Delta_{Y/X}"] & Y \times_X Y \end{tikzcd}$$

for $i \in I$. So the assertion follows from Proposition 5.4. $\square$

Proposition 6.6. Let $\varphi : X \to Y$, $\psi : Y \to Z$ be separated morphisms of k_H-affinoid spaces. Then $\psi \circ \varphi : X \to Z$ is also separated.

Proof. We have a Cartesian diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_Z X \arrow[d] \\ Y \arrow[r, "\Delta_{Y/Z}"] & Y \times_Z Y \end{tikzcd}.$$

By Proposition 6.4, $\psi : X \times_Y X \to X \times_Z X$ is a closed immersion. On the other hand, $\Delta_{X/Z} : X \to X \times_Z X$ factorizes as $\psi \circ \Delta_{X/Y}$. It follows from Proposition 5.7 that $\Delta_{X/Z}$ is a closed immersion. $\square$

Proposition 6.7. Let $H' \supseteq H$ is a subgroup of $\mathbb{R}_{>0}$. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces. Then the following are equivalent:

(1) φ is separated;
(2) φ is separated when view as a morphism of $k_{H'}$-affinoid spaces.

PROOF. This follows immediately from Corollary 5.5. $\square$

Proposition 6.8. Let $\varphi : Y \to X$ be a morphism of k_H-analytic spaces and K/k be an analytic field extension.

(1) If φ is separated, so is φ_K;
(2) If $K = k_r$ for some k-free polyray r and φ_K is separated, then so is φ.

We will prove later on that the assumption in (2) is unnecessary.

PROOF. This follows immediately from Corollary 5.6. $\square$

7. Reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 7.1. A *punctured k_H-analytic space* (X, x) is a k_H-analytic space X together with a point $x \in X$.

A morphism between punctured k_H-analytic spaces (X, x) and (Y, y) is a morphism $\varphi : X \to Y$ of k_H-analytic spaces sending x to y.

The category of punctured k_H-analytic spaces is denoted by $k_H\text{-}\mathcal{A}\mathrm{n}_*$.

Definition 7.2. A morphism of punctured k_H-analytic spaces $(X, x) \to (Y, y)$ is said to be *separated* (resp. *a closed immersion*) is the underlying morphism of k_H-analytic spaces is separated (resp. a closed immersion).

Definition 7.3. The category $k_H\text{-}\mathcal{G}\mathrm{er}$ is the category of right fractions of $k_H\text{-}\mathcal{A}\mathrm{n}_*$ with respect to the system of morphisms

$$\varphi : (X, x) \to (Y, y)$$

that induces an isomorphism of X with an open neighbourhood of y in Y in $k\text{-}\mathcal{G}\mathrm{er}$.

When we view (X, x) as an object in $k_H\text{-}\mathcal{G}\mathrm{er}$, we write it as X_x. An object in $k_H\text{-}\mathcal{G}\mathrm{er}$ is called a k_H-analytic germ.

Be careful, we require φ to induce an isomorphism in $k\text{-}\mathcal{G}\mathrm{er}$ instead of $k_H\text{-}\mathcal{G}\mathrm{er}$, although eventually, we will show that these notions coincide.

By definition,

$$\mathrm{Hom}_{k_H\text{-}\mathcal{G}\mathrm{er}}(X_x, Y_y) = \varinjlim_{U} \mathrm{Hom}_{k_H\text{-}\mathcal{A}\mathrm{n}_*}((U, x), (Y, y)),$$

where U runs over all open neighbourhoods of x in X.

Definition 7.4. A k_H-analytic germ X_x is *good* if x admits an affinoid neighbourhood in X.

Note that this condition does not depend on the representative (X, x). To see this, let $U \subseteq x$ be an open subset containing x. We need to show that if x admits a k_H-affinoid neighbourhood in X, then it admits one in U. This follows from Proposition 10.13 in Affinoid algebras.

Definition 7.5. A morphism of k_H-analytic germs $\varphi : X_x \to Y_y$ is saied to be *separated* (resp. *boundaryless*) if it is induced by a separated morphism (resp. boundaryless) of punctured k_H-analytic spaces $(U, x) \to (Y, y)$, where U is an open neighbourhood of x in X.

Definition 7.6. Let X_x be a k_H-analytic germ. A *k_H-analytic domain* in X_x is a k_H-analytic germ V_x, where V is a k_H-analytic domain in X containing x.

We say a finite family of k_H-analytic germs $\{V_{ix}\}_{i\in I}$ *covers* X_x if there is a representative (X,x) of X_x such that V_{ix} can be represented by a k_H-analytic domain $V_i \in X$ for $i \in I$ and

$$X = \bigcup_{i\in I} V_i.$$

Definition 7.7. Let $\phi : Y_y \to X_x$ be a morphism of k_H-analytic germs and V_x be a k_H-analytic domain in X_x. Represent ϕ by a morphism $\phi : (Y,y) \to (X,x)$ and represent V_x by a k_H-analytic domain in X. Then the k_H-analytic domain $\phi^{-1}(V)$ in Y determines a k_H-analytic germ $\phi^{-1}(V)_y$, which does not depend on the choices we made. This k_H-analytic germ is denoted by $\phi^{-1}(V_x)$.

Recall that $\phi^{-1}(V)$ is a k_H-analytic domain in Y by Proposition 3.4.

Definition 7.8. Let X be a good k_H-analytic space and $x \in X$, we define

$$\mathcal{O}_{X,x} := \varinjlim_{V} A_V ,$$

where V runs over all k_H-affinoid neighbourhoods of x in X. Include the definition of affinoid neighbourhoods

Observe that k_H-affinoid neighbourhoods of x in X are cofinal in the directed set of k-affinoid neighbourhoods of x in X. This follows from Proposition 10.13 in Affinoid algebras. So we may let V runs over all k-affinoid neighbourhoods of x in X as well.

Example 7.9. Let X_x be a k_H-analytic germ. Take a k_H-affinoid domain $\operatorname{Sp} A$ of X containing x. Given $r \in \sqrt{|k^\times| \cdot H}^n$ and $f \in A^n$, we write

$$X_x\{r^{-1}f\} := \left(\operatorname{Sp} A\{r^{-1}f\}\right)_x .$$

Then $X_x\{r^{-1}f\}$ is a k_H-analytic germ. Observe that $X_x\{r^{-1}f\}$ is independent of the choice of $\operatorname{Sp} A$. This construction depends only on the classes of f in $\mathcal{O}_{X,x}^n$. Given $\bar{f} \in \mathcal{O}_{X,x}^n$, we define $X_x\{r^{-1}\bar{f}\} = X_x\{r^{-1}f\}$ for any $f \in A^n$ lifting $\bar{f}$ as above.

Next we define the reduction.

Definition 7.10. Let $X = \operatorname{Sp} A$ be a k_H-affinoid space and $x \in X$, we define the *reduction* $\widetilde{(X,x)}$ of X at x as follows: let $\chi_x : A \to \mathscr{H}(x)$ be the character corresponding to x, we define

$$\widetilde{(X,x)}^H := \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\tilde{k}^H} \left\{\widetilde{\chi_x}(\tilde{A}^H)\right\} .$$

Observe that this construction defines a functor

$$k_H\text{-}\mathcal{A}\mathrm{n}_* \to \mathrm{bir}_{\tilde{k}^H} .$$

Observe that $\widetilde{(X,x)}^H$ is an affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\tilde{k}^H}$. This follows from Corollary 11.11 in Affinoid algebras.

Lemma 7.11. Let $X=\operatorname{Sp} A$ be a k_H-affinoid space and $x\in X$. Let $U=\operatorname{Sp} B$ be a k_H-affinoid space. Let $\iota : U\to X$ be an isomorphism of U with an open neighbourhood of x in k-$\mathcal{A}$n. We still write $\iota^{-1}(x)\in U$ as x. Then the natural morphism

$$\widetilde{(U,x)}^H \to \widetilde{(X,x)}^H$$

is an isomorphism in $\mathrm{bir}_{\tilde{k}^H}$.

Proof. We first recall that $\mathscr{H}(x)$ does not depend on if we view x as in $\operatorname{Sp} A$ or in $\operatorname{Sp} B$ by Corollary 13.16 in Affinoid algebras.

Observe that the morphism $\chi_x : B\to \mathscr{H}(x)$ is boundaryless with respect to A by Proposition 15.9 in Affinoid algebras. By Proposition 15.2 in Affinoid algebras, $\widetilde{\chi_x}(\tilde{B}^H)$ is finite over $\widetilde{\chi_x}(\tilde{A}^H)$. By Lemma 4.5 in Commutative algebras, we have

$$\mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\tilde{k}^H}\left\{\widetilde{\chi_x}(\tilde{A}^H)\right\} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\tilde{k}^H}\left\{\widetilde{\chi_x}(\tilde{B}^H)\right\}.$$

□

Definition 7.12. Let X_x be a good k_H-analytic germs. Take an affinoid neighbourhood U of x in X, then we define

$$\widetilde{X_x}^H := \widetilde{(U,x)}^H \subseteq \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\tilde{k}^H}.$$

By Lemma 7.11, $\widetilde{X_x}^H$ depends only on X_x.

The construction is clearly functorial in X_x.

Lemma 7.13. Let X_x be a good k_H-analytic germ and Y_x be a k_H-analytic domain in X_x. Then Y_x can be covered by finitely many k_H-analytic domains in X_x of the form

$$X_x\left\{r^{-1}f\right\},$$

where $n\in\mathbb{N}$, $f=(f_1,\dots,f_n)\in\mathcal{O}_{X,x}^{\times}$ is a tuple of invertible elements and $r_i=|f_i(x)|\in\sqrt{|k^\times|\cdot H}$.

Proof. We may assume that X is k_H-affinoid, say $X=\operatorname{Sp} A$. By Corollary 12.8 in Affinoid algebras, Y can be covered by finitely many k_H-rational domains in X, say of the form $\operatorname{Sp} A\{r^{-1}g/h\}$, where $m\in\mathbb{N}$, $r=(r_1,\dots,r_m)\in\sqrt{|k^\times|\cdot H}^m$, $g=(g_1,\dots,g_m)\in A^m$, $h\in A$ and $g_1,\dots,g_m,h$ generates the unit ideal. We may assume that $Y=\operatorname{Sp} A\{r^{-1}g/h\}$.

By shrinking X, we may assume that h is invertible. Set $f_i=g_i/h$, then

$$Y=\operatorname{Sp} A\{r_1^{-1}f_1,\dots,r_m^{-1}f_m\}.$$

By further shrinking X, it suffices to consider those i with $|f_i(x)|=r_i$. □

Lemma 7.14. Let X_x be a good k_H-analytic germ. Given $n\in\mathbb{N}$ and $f=(f_1,\dots,f_n)\in\mathcal{O}_{X,x}^{\times}$, then

$$\widetilde{X_x\{r^{-1}f\}}^H = \widetilde{X_x}^H\left\{\widetilde{\chi_x}(\tilde{f}_1),\dots,\widetilde{\chi_x}(\tilde{f}_n)\right\},$$

where $r=(r_1,\dots,r_n)$ and $r_i=|f_i(x)|$ for $i=1,\dots,n$.

Proof. We may assume that X is k_H-affinoid, say $X = \operatorname{Sp} A$. By induction on n, we may assume that $n = 1$. Consider the admissible epimorphism

$$\phi : A\{r^{-1}T\} \to A\{r^{-1}f\}$$

sending T to f. By Theorem 11.10 in Affinoid algebras,

$$\tilde{\phi} : \tilde{A}^H[r^{-1}T] \to \widetilde{A\{r^{-1}f\}}^H$$

is finite. Let $\chi_x : A\{r^{-1}f\} \to \mathscr{H}(x)$ be the character defined by x.

Then $\widetilde{\chi_x}(\widetilde{A\{r^{-1}f\}}^H)$ is finite over $\widetilde{\chi_x}(\tilde{A}^H)[\tilde{f}]$. So the assertion follows from Lemma 4.5 in Commutative algebras. $\square$

Lemma 7.15. Let X_x be a good k_H-analytic germs and Y_x be a good k_H-analytic domain in X_x. Then we can find $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{O}_{X,x}^\times$ such that

$$\widetilde{Y_x}^H = \widetilde{X_x}^H \left\{\widetilde{\chi_x}(\tilde{f}_1), \ldots, \widetilde{\chi_x}(\tilde{f}_n)\right\}.$$

In particular, we can identify $\widetilde{Y_x}^H$ with an open susbet of $\widetilde{X_x}^H$.

Proof. The same argument as in Lemma 7.13 that we can assume that $X = \operatorname{Sp} A$ and $Y = \operatorname{Sp} A\{r^{-1}f\}$ for some $n \in \mathbb{N}$, $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times| \cdot H}^n$, $f = (f_1, \ldots, f_n) \in A^n$ with $r_i = |f_i(x)|$ for $i = 1, \ldots, n$. So the assertion follows from Lemma 7.14. $\square$

Lemma 7.16. Let X_x be a good k_H-analytic germ, $n \in \mathbb{Z}_{>0}$ and $Y_{1x}, \ldots, Y_{nx}$ be a covering of X_x by good k_H-analytic domains. Then

$$\widetilde{X_x}^H = \bigcup_{i=1}^n \widetilde{Y_{ix}}^H.$$

Proof. Observe that we are free to replace $\{Y_{ix}\}_i$ by its refinements by coverings by good k_H-analytic domains. We may assume that X is k_H-affinoid, say $X = \operatorname{Sp} A$. Then by Lemma 13.3 in Affinoid algebras, we may assume that the covering is k_H-rational and is generated by $r_1^{-1}f_1, \ldots, r_n^{-1}f_n$. Up to shrinking X, we may guarantee that $|f_i(x)| = r_i$ for $i = 1, \ldots, n$. In this case, the assertion follows from Lemma 7.14. $\square$

Lemma 7.17. Let $\phi : Y_y \to X_x$ be a morphism of good k_H-analytic germs. Let X'_x be a good k_H-analytic domain in X_x and set $Y'_y = \phi^{-1}(X'_x)$, then

$$\widetilde{Y'_y}^H = \tilde{\phi}^{-1}(\widetilde{X'_x}^H).$$

Proof. By Lemma 7.13, we may find $m \in \mathbb{Z}_{>0}$, $n_1, \ldots, n_m \in \mathbb{N}$, $g_{i1}, \ldots, g_{in_i} \in \mathcal{O}_{X,x}^\times$ for $i = 1, \ldots, m$ such that X'_x is covered by $X\{r_{i1}^{-1}g_{i1}, \ldots, r_{in_i}^{-1}g_{in_i}\}$ for $i = 1, \ldots, m$, where $r_{ij} = |g_{ij}(x)|$ for $i = 1, \ldots, m$, $j = 1, \ldots, n_i$.

Then Y'_x is covered by $Y\{r_{i1}^{-1}g'_{i1}, \ldots, r_{in_i}^{-1}g'_{in_i}\}$ for $i = 1, \ldots, m$, where g'_{ij} is the image of g_{ij} in $\mathcal{O}_{Y,y}^\times$ for $i = 1, \ldots, m$, $j = 1, \ldots, n_i$.

By Lemma 7.14, we have

$$\widetilde{X'_x}^H = \bigcup_{i=1}^m \widetilde{X_x}\left\{\widetilde{\chi_x}(\widetilde{g_{i1}}), \ldots, \widetilde{\chi_x}(\widetilde{g_{in_i}})\right\}$$

and
$$\widetilde{Y'_y}^H = \bigcup_{i=1}^{m} \widetilde{Y_y}\left\{\widetilde{\chi_y}(\widetilde{g'_{i1}}), \ldots, \widetilde{\chi_y}(\widetilde{g'_{in_i}})\right\}.$$
Our assertion is now clear. □

Definition 7.18. Let X_x be a k_H-analytic germ. By Lemma 7.15, the reduction defines a functor from the category of good k_H-analytic germs in X_x (with inclusions as the morphisms) to the category of open affine subsets of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\widetilde{k}^H}$. We define
$$\widetilde{X_x}^H := \varinjlim_{Y_x} \widetilde{Y_x}^H,$$
where Y_x runs over the filtered category of good k_H-analytic germs in X_x and the colimit is taken in the category $\mathrm{bir}_{\tilde{k}^H}$.

This construction defines a covariant functor
$$k_H\text{-}\mathcal{G}\mathrm{er} \to \mathrm{bir}_{\tilde{k}^H}.$$
This is called the *reduction functor* and for a k_H-analytic germ X_x, $\widetilde{X_x}^H$ is called its *reduction*.

In more concrete terms, we define an equivalence relation $\sim$ on $\coprod_{Y_x} \widetilde{Y_x}^H$ by identifying two points in $\widetilde{Y_x}^H$ and $\widetilde{Y'_x}^H$ if there is a good k_H-analytic domain Z_x in Y_x and in Y'_x such that there is a point in $\widetilde{Z_x}^H$ whose images in $\widetilde{Y_x}^H$ and $\widetilde{Y'_x}^H$ are these points respectively. The quotient $\coprod_{Y_x} \widetilde{Y_x}^H/\sim$ defines the underlying topological space of $\widetilde{Y_x}^H$. The valuation field is defined as $\widetilde{\mathscr{H}(x)}^H$. The local homeomorphis $\phi : \widetilde{Y_x}^H \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\widetilde{k}^H}$ is defined by gluing the inclusions $\widetilde{Y_x}^H \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H/\widetilde{k}^H}$.

Theorem 7.19. Let X_x be a k_H-analytic germ. Then the reduction functor
$$k_H\text{-}\mathcal{G}\mathrm{er} \to \mathrm{bir}_{\tilde{k}^H}$$
establishes a bijection between the k_H-analytic domains and quasi-compact open subsets of $\widetilde{X_x}^H$.

This bijection commutes with finite unions and finite intersections.

Proof. The last assertion is obvious by construction.

Step 1. We prove the theorem under the additional assumption that X_x is good.

Step 1.1. Let $l, m \in \mathbb{N}$ and $f = (f_1, \ldots, f_l) \in \mathcal{O}^l_{X,x}$, $g = (g_1, \ldots, g_m) \in \mathcal{O}^m_{X,x}$. Assume that
$$\widetilde{X_x}^H\{\tilde{f}\} \subseteq \widetilde{X_x}^H\{\tilde{g}\},$$
then we prove that
$$X_x\{r^{-1}f\} \subseteq X_x\{s^{-1}g\},$$
where $r = (r_1, \ldots, r_l)$, $s = (s_1, \ldots, s_m)$,
$$r_i = |f_i(x)|, \quad s_j = |g_j(x)|$$
for $i = 1, \ldots, l$, $j = 1, \ldots, m$.

We may assume that X is k_H-affinoid, say $X = \operatorname{Sp} A$ and $f_1, \ldots, f_l, g_1, \ldots, g_m \in A$. Let $\chi_x : A \to \mathscr{H}(x)$ be the character of x. Let

$$B = \widetilde{\chi_x}(\tilde{A}^H) \subseteq \widetilde{\mathscr{H}(x)}^H.$$

By definition,

$$\widetilde{X_x}^H = \mathbf{P}_{\widetilde{\mathscr{H}(x)}^H}\{B\}.$$

By Lemma 7.14, we have

$$\begin{aligned} \widetilde{X_x\{r^{-1}f\}}^H &= \widetilde{X_x}^H\{B[\tilde{f}]\}, \\ \widetilde{X_x\{r^{-1}f, s^{-1}g\}}^H &= \widetilde{X_x}^H\{B[\tilde{f}, \tilde{g}]\}. \end{aligned}$$

The right-hand sides are equal by our assumption, so by Lemma 4.5 in Commutative algebras, $B[\tilde{f}, \tilde{g}]$ is finite over $B[\tilde{f}]$. We take monic polynomials of $\tilde{g}_j$ over $B[\tilde{f}]$:

$$T^{n_j} + \tilde{a}_{j,1}T^{n-1} + \cdots + \tilde{a}_{j,n_j} \in B[\tilde{f}][T]$$

with $\tilde{a}_{j,1}, \ldots, \tilde{a}_{j,n_j}$ homogeneous of degree $|g_j(x)|^1, \ldots, |g_j(x)|^{n_j}$ respectively. This is possible by Proposition 2.18 in Commutative algebras. We lift $\tilde{a}_{j,k}$ to $a_{j,k} \in A\{r^{-1}f\}$ with $\rho(a_{j,k}) = \rho(g_j)^k$ for $j = 1, \ldots, m$, $k = 1, \ldots, n_j$. It follows that

$$\left|\left(g_j^{n_j} + a_{j,1}g_j^{n-1} + \cdots + a_{j,n_j}\right)(x)\right| < |g_j(x)|^n$$

for $j = 1, \ldots, m$. Up to shrinking X, we may assume that this inequality holds everywhere on $X\{r^{-1}f\}$.

By then $|g_j(y)| \leq |g_j(x)|$ for any $y \in X\{r^{-1}f\}$. Our assertion follows.

Step 1.2. Suppose that Y_x is a k_H-analytic domain in X_x with $\tilde{Y}_x = \tilde{X}_x$, then $Y_x = X_x$.

We may assume that X is k_H-affioid, say $X = \operatorname{Sp} A$.

By Lemma 7.13, we can write Y_x as a finite union of $V_{i,x} := X_x\{r_i^{-1}f_i\}$ for $i = 1, \ldots, m$, where $n_i \in \mathbb{N}$, $f_i = (f_{i1}, \ldots, f_{in_i}) \in \mathcal{O}_{X,x}^{\times, n_i}$ and $r_i = (r_{i1}, \ldots, r_{in_i})$ with $r_{ij} = |f_{ij}(x)| \in \sqrt{|k^\times| \cdot H}$.

By Lemma 7.16, $\widetilde{V_{i,x}}^H$ for $i = 1, \ldots, m$ covers $\widetilde{X_x}^H$.

By Lemma 4.8 in Commutative algebras, we can refine this covering to a Laurent covering

$$\mathcal{U} := \left\{\tilde{U}_j = \widetilde{X_x}^H\{\tilde{g}_1^{j_1}, \ldots, \tilde{g}_l^{j_l}\}\right\}_{j=(j_1,\ldots,j_l)\in\{\pm1\}^l},$$

where $l \in \mathbb{N}$ and $\tilde{g}_1, \ldots, \tilde{g}_l$ are homogeneous elements in $\widetilde{\mathscr{H}(x)}^H$. Lift $\tilde{g}_1, \ldots, \tilde{g}_l$ to $g_1, \ldots, g_l \in A$. We consider the k_H-Laurent covering of X generated by

$$\rho(\tilde{g}_1)^{-1}g_1, \ldots, \rho(\tilde{g}_l)^{-1}g_l.$$

The reduction of this covering is clearly $\mathcal{U}$. By Step 1.1, the germs of $\mathcal{U}$ at x is a refinement of $\{V_{1,x}, \ldots, V_{m,x}\}$, so the latter is a covering of X_x, namely $X_x \xrightarrow{\sim} Y_x$.

Step 1.3. We prove that each quasi-compact open subset $\widetilde{Y_x}^H$ of $\widetilde{X_x}^H$ is the reduction of some k_H-analytic domain Y_x in X_x.

We can write

$$\widetilde{Y_x}^H = \bigcup_{i=1}^m \widetilde{X_x}\{\widetilde{f_{i1}}, \ldots, \widetilde{f_{in_i}}\},$$

where $m \in \mathbb{Z}_{>0}$, $n_1, \dots, n_m \in \mathbb{N}$, $\widetilde{f_{ij}} \in \widetilde{\mathscr{H}(x)}^H$ are homogeneous elements for $i = 1, \dots, m$, $j = 1, \dots, n_i$. We lift $\widetilde{f_{ij}}$ to $f_{ij} \in \mathcal{O}_{X,x}$, it suffices to take

$$\bigcup_{i=1}^m X_x \left\{ r_{i1}^{-1} f_{i1}, \dots, r_{in_i}^{-1} f_{in_i} \right\},$$

where $r_{ij} = |f_{ij}(x)|$ for $i = 1, \dots, m$, $j = 1, \dots, n_i$.

Step 1.4. Suppose that Y_x, Z_x are k_H-analytic domains in X_x with $\widetilde{Y_x} = \widetilde{Z_x}$. Then we prove that $Y_x = Z_x$.

Take $p, q \in \mathbb{N}$, good k_H-analytic domains $Y_x^1, \dots, Y_x^p$ in Y_x and good k_H-analytic domains $Z_x^1, \dots, Z_x^p$ in Z_x such that

$$\widetilde{Y_x}^H = \bigcup_{i=1}^p \widetilde{Y_x^i}^H = \bigcup_{i=1}^q \widetilde{Z_x^i}^H.$$

Therefore, for any $i = 1, \dots, p$, $\{\widetilde{Y_x^i \cap Z_x^j}^H\}_{j=1,\dots,q}$ is a covering of $\widetilde{Y_x^i}$. By Step 1.2, $\{Y_x^i \cap Z_x^j\}_{j=1,\dots,q}$ is a covering of Y_x^i for $i = 1, \dots, p$. So $Y_x \subseteq Z_x$. By symmetry $Y_x = Z_x$.

We have finshed the proof when X_x is good.

Step 2. We handle the general case.

Step 2.1. We prove that each quasi-compact open subset $\widetilde{Y_x}^H$ of $\widetilde{X_x}^H$ is the reduction of some k_H-analytic domain Y_x in X_x.

Take $p \in \mathbb{N}$, good k_H-analytic domains $X_x^1, \dots, X_x^p$ in X_x such that

$$\widetilde{X_x}^H = \bigcup_{i=1}^p \widetilde{X_x^i}^H.$$

By Step 1, $\widetilde{Y_x}^H \cap \widetilde{X_x^i}^H$ ca be lifted to a k_H-analytic domain Y_x^i in X_x^i for $i = 1, \dots, p$. The union of Y_x^i's for $i = 1, \dots, p$ is a lifting of $\widetilde{Y_x}^H$.

Step 2.2. Suppose that Y_x, Z_x are k_H-analytic domains in X_x with $\widetilde{Y_x} = \widetilde{Z_x}$. Then we prove that $Y_x = Z_x$.

For each $i = 1, \dots, p$, we have

$$\widetilde{Y_x \cap X_x^i}^H = \widetilde{Y_x}^H \cap \widetilde{X_x^i}^H = \widetilde{Z_x}^H \cap \widetilde{X_x^i}^H = \widetilde{Z_x \cap X_x^i}^H.$$

By Step 1, $Y_x \cap X_x^i = Z_x \cap X_x^i$ coincides for $i = 1, \dots, p$, so $Y_x = Z_x$.

□

Theorem 7.20. Let $H' \supseteq H$ be a subgroup of $\mathbb{R}_{>0}$. The natural functor

$$k_H\text{-}\mathcal{A}\mathrm{n} \to k_{H'}\text{-}\mathcal{A}\mathrm{n}$$

is fully faithful.

Proof. Let X, Y be k_H-analytic spaces and $\varphi : Y \to X$ a morphism in $k_{H'}$-$\mathcal{A}$n. We need to show that φ is induced by a morphism in k_H-$\mathcal{A}$n.

This amounts to showing that if V is a k_H-analytic domain in X, then $W = \varphi^{-1}(V)$ is a k_H-analytic domain. Let $y \in W$ and $x = \varphi(y)$. Let

$$\tilde{\varphi}^{H'} : \widetilde{Y_y}^{H'} \to \widetilde{X_x}^{H'}$$

be the reduction map. Then

$$\widetilde{W_y}^{H'} = \tilde{\varphi}^{H',-1}\widetilde{V_x}^{H'}.$$

As V is a k_H-analytic domain, the non-zero degrees of $\widetilde{V_x}^{H'}$ are contained in $\sqrt{|k^\times| \cdot H'}$, hence so is $\widetilde{W_y}^{H'}$. We write the corresponding $\sqrt{|k^\times| \cdot H'}$-graded object as α. By Theorem 7.19, there is a k_H-analytic domain U_y of X_x with

$$\widetilde{U_y}^{H} = \alpha.$$

By our construction of reduction, we know that $\widetilde{U_y}^{H'}$ is the $\sqrt{|k^\times| \cdot H'}$-graded object corresponding to $\widetilde{U_y}^{H}$, namely $\widetilde{W_y}^{H'}$. By Theorem 7.19 again, we conclude that $W_y = U_y$. In particular, W_y is k_H-analytic. The theorem follows. □

Corollary 7.21. Let X be a k-analytic space. Then there is at most one k_H-analytic space X' up to isomorphisms in k_H-$\mathcal{A}$n whose image under

$$k_H\text{-}\mathcal{A}\mathrm{n} \to k\text{-}\mathcal{A}\mathrm{n}$$

is isomorphic to X.

In particular, we can and will view k_H-analytic spaces as k-analytic spaces that admit (necessarily unique) structures of k_H-analytic spaces.

Proof. This follows immediately from Theorem 7.20. □

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