Ymir

Contents

Morphisms between complex analytic spaces

1. Introduction

2. Open morphism

Definition 2.1. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. We say f is *open* at $x \in X$ if for any neighbourhood U of x in X, $f(U)$ is a neighbourhood of $f(x)$ in Y.

Proposition 2.2. Let $f : X \to Y$ be a morphism of complex analytic spaces. Assume that f is open at $x \in X$, then the kernel of $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is nilpotent.

The converse fails.

Proof. Let $g_{f(x)} \in \mathcal{O}_{Y,f(x)}$ be an element in the kernel of $f_x^\#$. Up to shrinking Y, we may spread $g_{f(x)}$ to $g \in \mathcal{O}_Y(Y)$. Then f^*g vanishes in a neighbourhood of x in X. As f is open at x, g vanishes in the neighbourhood $f(U)$ of $f(x)$. By Corollary 3.18 in Constructions of complex analytic spaces, $g_{f(x)}$ is nilpotent. □

3. Quasi-finite morphisms

Definition 3.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. We say f is *quasi-finite* at $x \in X$ if x is isolated in $f^{-1}(f(x))$. We say f is *quasi-finite* if f is quasi-finite at all $x \in X$.

This definition is purely topological. We will show that it is equivalent to an analytic definition.

Proposition 3.2. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is quasi-finite at $x \in X$;
(2) $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$;
(3) $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,f(x)}$.

Proof. (1) ⇔ (2): By Corollary 3.16 in Constructions of complex analytic spaces, f is quasi-finite at $x \in X$ if and only if $\mathcal{O}_{X_{f(x)},x} = \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is artinian. In other words, $\mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}\mathcal{O}_{X,x}$ is finite-dimensional over $\mathbb{C}$. The latter is equivalent to that $\mathcal{O}_{X,x}$ is quasi-finite over $\mathcal{O}_{Y,f(x)}$.

(2) ⇔ (3): This follows from Theorem 5.4 in Complex analytic local algebras. □

4. Finite morphisms

Definition 4.1. A morphism of complex analytic spaces $f : X \to Y$ is *finite* if its underlying map of topological spaces is topologically finite.

We say a morphism of complex analytic spaces $f : X \to Y$ is *finite at* $x \in X$ if there is an open neighbourhood U of x in X and V of $f(x)$ in Y such that $f(U) \subseteq V$ and the restriction $U \to V$ of f is finite.

Let S be a complex analytic space. A *finite analytic space over* S is a finite morphism $f : X \to S$ of complex analytic spaces. A morphism between finite analytic spaces over S is a morphism of complex analytic spaces over S.

Proposition 4.2. Let $n \in \mathbb{N}$ and D be an open neighbourhood of 0 in $\mathbb{C}^n$. Let X be a closed subspace of D which intersections $\{(0, \dots, 0)\} \times \mathbb{C}$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that the projection $B \times W \to B$ induces a finite morphism $h : X' \to B$ with $X' = X \cap (B \times W)$.

Proof. We will denote the coordinates on $\mathbb{C}^{n-1} \times \mathbb{C}$ as (z, w).

Let $\mathcal{I}$ be the ideal of X in D. By our assumption, we can choose $f_0 \in \mathcal{I}_0$ such that $\deg_w f_0 < \infty$ and $f_0(0) = 0$. By Theorem 4.3 in Complex analytic local algebras, we can find a Weierstrass polynomial $\omega_0 = w^b + a_1 w^{b-1} + \cdots + a_b \in \mathbb{C}\{z_1, \dots, z_{n-1}\}[w]$ such that $f_0 = e\omega_0$ for some unit e in $\mathbb{C}\{z_1, \dots, z_n\}$. We choose a product neighbourhood $B \times W \subseteq \mathbb{C}^{n-1} \times \mathbb{C}$ of 0 in D such that ω_0 can be represented by $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)[w]$ with $\omega|_{B\times W} \in \mathcal{I}(B \times W)$. Let $\pi : A \to B$ be the Weierstrass map defined by ω. Then π is finite by Theorem 6.2 in The notion of complex analytic spaces. Up to shrinking B and W, we may assume that $A \cap (B \times W) \to B$ is finite as well. Set $X' := X \cap (B \times W)$. The restriction $h : X' \to B$ of π is then finite. □

Corollary 4.3. Let $n, k \in \mathbb{N}$ and D be an open neighbourhood of 0 in $\mathbb{C}^n$. Let X be a closed subspace of D which intersections $\{(0, \dots, 0)\} \times \mathbb{C}^k$ at and only at 0. Then there is a connected open product neighbourhood $B \times W \subseteq \mathbb{C}^{n-k} \times \mathbb{C}^k$ of 0 in D such that the projection $B \times W \to B$ induces a finite morphism $h : X' \to B$ with $X' = X \cap (B \times W)$.

Proof. This follows from a repeted application of Proposition 4.2. □

Proposition 4.4. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Then the following are equivalent:

(1) f is quasi-finite at x;
(2) f is finite at x.

Proof. (2) $\implies$ (1): This follows from This follows from Proposition 4.5 in Topology and bornology.

(1) $\implies$ (2): Write $y = f(x)$. The assertion is local on both X and Y. So we may assume that U and V are complex model spaces in domains $W \subseteq \mathbb{C}^k$ and $B \subseteq \mathbb{C}^d$ respectively with $x = 0$ and $y = 0$. Moreover, we may assume that $\{x\} = f'^{-1}(y)$. We have the following commutative diagram:

$$\begin{tikzcd}
 & & U \times V \arrow[r, hook] \arrow[dd] & W \times B \arrow[dd] \\
U \arrow[r] \arrow[drr] & \Gamma_f \arrow[ur] \arrow[dr, "f'"'] & & \\
 & & V \arrow[r, hook] & B
\end{tikzcd},$$

where $\Gamma_{f'}$ denotes the graph of $f' : U \to V$. As $\{x\} = f'^{-1}(y)$, we have $\mathbb{C}^k \times \{0\}$ intersects Γ_f only at the origin. By Corollary 4.3, up to shrinking W and B, we may guarantee that the projection $W \times B \to B$ induces a finite morphism $\Gamma_f \to B$ and the pushforward under this map preserves coherence. Observe that $U \to \Gamma_f$ is a biholomorphism, we conclude that f' is finite. □

Corollary 4.5. Let $f : X \to Y$ be a morphism of complex analytic spaces. The following are equivalent:

(1) f is finite;
(2) f is quasi-finite and proper.

Proof. (1) $\implies$ (2): This follows from Proposition 4.4.
(2) $\implies$ (1): This follows from Proposition 4.5 in Topology and bornology. □

Corollary 4.6. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then the set

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is open.

Proof. This follows from Proposition 4.4. □

Theorem 4.7. Let S be a complex analytic space. Then the functor $\mathrm{Spec}^{\mathrm{an}}_S$ defines an anti-equivalence from the category of finite $\mathcal{O}_S$-algebras to the category of finite analytic spaces over S.

Proof. We first observe that the functor is well-defined. This follows from Corollary 3.8 in Constructions of complex analytic spaces.

The functor is fully faithfull by Proposition 2.10 in Constructions of complex analytic spaces. Suppose that $f : X \to S$ is a finite morphism of complex analytic spaces. We need to show that X is isomorphic to $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{A}$ for some finite $\mathcal{O}_S$-algebra $\mathcal{A}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

By Proposition 2.8 in Constructions of complex analytic spaces, we necessarily have $\mathcal{A} \cong f_*\mathcal{O}_X$. So we need to show that the natural morphism $\mathrm{Spec}^{\mathrm{an}}_S f_*\mathcal{O}_X \to X$ over S is an isomorphism. The problem is local on S.

Fix $s \in S$. Write $x_1, \dots, x_n$ for the distinct points in $f^{-1}(s)$. Up to shrinking S, we may assume that X is the disjoint union of $V_1, \dots, V_n$, where V_i is an open neighbourhood of x_i in X. We need to show that X has the form $\mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}$ for some $\mathcal{O}_S$-algebra $\mathcal{B}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/S}$.

It suffices to handle each V_i separately, so we may assume that $f^{-1}(s) = \{x\}$ consists of a single point. Then $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$ by Proposition 3.2. Up to shrinking S, we may assume that $\mathcal{O}_{X,x}$ spreads out to a finite $\mathcal{O}_S$-algebra $\mathcal{B}$. Let $X' = \mathrm{Spec}^{\mathrm{an}}_S \mathcal{B}$. There is a unique point x' of X' over s and $X'_{x'}$ is isomorphic to X_x over S_s. By Lemma 4.2 in Topology and bornology, up to shrinking S, we may assume that X is isomorphic to X' over S. We conclude. □

Corollary 4.8. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules, then $f_*\mathcal{M}$ is coherent. Moreover, f_* is exact from $\mathcal{C}\mathrm{oh}(\mathcal{O}_X)$ to $\mathcal{C}\mathrm{oh}(\mathcal{O}_Y)$.

Proof. This follows from Corollary 2.9 in Constructions of complex analytic spaces and Theorem 4.7. □

Corollary 4.9. Let X be a reduced complex analytic space. Then

(1) $\bar{X}$ is normal;
(2) $p : \bar{X} \to X$ is finite and surjective;
(3) There is a nowhere dense analytic set Y in X such that $p^{-1}(Y)$ is nowhere dense in $\bar{X}$ and the morphism $\bar{X} \setminus p^{-1}(Y) \to X \setminus Y$ induced by p is an isomorphism.

Conversely, these conditions determines $\bar{X}$ up to a unique isomorphism in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$.

Proof. These properties are established in Proposition 7.8 in Local properties of complex analytic spaces. We need to prove the uniqueness.

Let $p : X' \to X$ be a morphism satisfying the three conditions. We need to show that X' is canonically isomorphic to $\bar{X}$ in $\mathbb{C}\text{-}\mathcal{A}\mathrm{n}_{/X}$. By (2) and Theorem 4.7, it suffices to show that $p_*\mathcal{O}_{X'}$ is canonically isomorphic to $\bar{\mathcal{O}}_X$. By (1), and the universal property of normalization, there is a canonical morphism

$$p_*\mathcal{O}_{X'} \to \bar{\mathcal{O}}_X$$

of $\mathcal{O}_X$-algebras. We will show that this map is an isomorphism.

The problem is local. Let $x \in X$. By (3) and Corollary 3.14 in Constructions of complex analytic spaces, up to shrinking X, we can find $f \in \mathcal{O}_X(X)$ such that $f(y) = 0$ for all $y \in Y$ and f_x is a non-zero divisor in $(p_*\mathcal{O}_{X'})_x$. Up to shrinking X, we may assume that f_y is a non-zero divsior in $(p_*\mathcal{O}_{X'})_y$ for all $y \in X$. By (3), we have

$$\mathcal{O}_X|_{X\setminus Y} \to (p_*\mathcal{O}_{X'})|_{X\setminus Y}$$

is an isomorphism. It follows that

$$fp_*\mathcal{O}_{X'} \to \mathcal{O}_X$$

is injective. We then have an injective homomorphism:

$$p_*\mathcal{O}_{X'} \to \mathcal{O}_X \xrightarrow{\times f} \mathcal{O}_X \xrightarrow{\times f^{-1}} \mathcal{O}_X[f^{-1}].$$

For each $y \in X$, we deduce that $(p_*\mathcal{O}_{X'})_y$ is in the total ring of fraction of $\mathcal{O}_{X,y}[f_y^{-1}]$. But $(p_*\mathcal{O}_{X'})_y$ is finite and integral over $\mathcal{O}_{X,y}$, so is isomorphic to $\overline{\mathcal{O}_{X,y}}$ as $\mathcal{O}_{Y,y}$-algebras. ☐

Corollary 4.10. Let $f : X \to Y$ be a finite morphism of complex analytic spaces. Assume that $x \in X$ is a point such that $(f_*\mathcal{O}_X)_{f(x)}$ is torsion-free as an $\mathcal{O}_{Y,f(x)}$-module and Y is integral at $f(x)$. Then f is open at x.

Proof. If not, we can choose open neighbourhoods U of x in X and V of $y := f(x)$ in Y such that $f(U) \subseteq V$ such that the induced morphism $g : U \to V$ is finite and $f(U)$ is not a neighbourhood of y in Y. Up to shrinking Y, we can find $h \in \mathcal{O}_Y(Y)$ such that $h_y \neq 0$ while h vanishes on $f(U)$. Observe that $f(U)$ is an analytic set in Y by Corollary 4.8. It follows from Corollary 3.18 in Constructions of complex analytic spaces that there is $t \in \mathbb{Z}_{>0}$ such that

$$h_y^t (g_*\mathcal{O}_U)_y = 0.$$

As $\mathcal{O}_{Y,y}$ is integral, this implies that $(g_*\mathcal{O}_U)_y$ is torsion as an $\mathcal{O}_{Y,f(x)}$-module. This is a contradiction, as $(f_*\mathcal{O}_X)_y$ as an $\mathcal{O}_{Y,f(x)}$-module is torsion-free by assumption. ☐

Lemma 4.11. Let X be an integral complex analytic space and $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then

$$\{x \in X : \mathcal{M} \text{ is torsion-free at } x\}$$

is co-analytic in X.

PROOF. It suffices to show that $\operatorname{Supp} \mathcal{T}(\mathcal{M})$ is an analytic set in X. As X is integral, $\mathcal{T}(\mathcal{M})$ is just the kernel of the morphism $\mathcal{M} \to \mathcal{M}^{\vee\vee}$. $\square$

Corollary 4.12. Let $f : X \to Y$ be a finite morphism of complex analytic spaces. Assume that Y is integral. Let $x \in X$ be a point such that X is integral at x and f is open at x, then there is an open neighbourhood U of x in X such that $f|_U : U \to Y$ is open.

PROOF. Let $y = f(x)$. The problem is local on Y. By Proposition 4.4, we may assume that $\{x\} = f^{-1}(y)$. By Corollary 4.8, $f_*\mathcal{O}_X$ is coherent. By Lemma 4.11, it suffices to show that it is torsion-free.

Observe that $(f_*\mathcal{O}_X)_y \xrightarrow{\sim} \mathcal{O}_{X,x}$. By Proposition 2.2, $f_x^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. As $\mathcal{O}_{X,x}$ is integral by our assumption, we conclude. $\square$

Lemma 4.13. Let $f : X \to Y$ be a finite morphism of reduced complex analytic spaces and $x \in X$. Assume that $x \in X$, then there is a non-zero divisor $h \in \mathfrak{m}_{f(x)}$ such that $f_x^\#(h)$ is a non-zero divisor in $\mathcal{O}_{X,x}$.

PROOF. By Proposition 4.4, the problem is local on X. We may assume that X can be decomposed into irreducible components at x:

$$X = A_1 \cup \cdots \cup A_s.$$

By Corollary 4.8, $B_j := f(A_j)$ is an ananlytic set in Y for $j = 1, \ldots, s$. By our assumption, x is not an isolated point in A_j, so y is not an isolated point in B_j for $j = 1, \ldots, s$. Take a non-zero divisor $h \in \mathfrak{m}_{Y,y}$. Up to shrinking Y, we may assume that h spreads to $g \in \mathcal{O}_Y(Y)$. Observe that $W(f^*g) \cap A_j$ is not a neighbourhood of x in A_j for all $j = 1, \ldots, s$. So $f_x^\# h$ is not a zero divisor. $\square$

Theorem 4.14. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $y \in Y$. Then

$$\dim_y f(X) = \max_{x \in f^{-1}(y)} \dim_x X.$$

The left-hand side makes sense because $f(X)$ is an analytic set in Y by Corollary 4.8.

PROOF. We may assume that X and Y are reduced and $f(X) = Y$.

Step 1. We reduce to the case where $f^{-1}(y) = \{x\}$ for some $x \in X$.

Let $x_1, \ldots, x_t$ be the distinct points in $f^{-1}(y)$. The problem is local on Y. By Theorem 4.7 in Topology and bornology and Proposition 4.4, up to shrinking Y, we may assume that X is the disjoint union of open neighbourhoods $U_1, \ldots, U_t$ of $x_1, \ldots, x_t$ and $U_j \to V$ is finite for each $j = 1, \ldots, t$. It suffices to apply the special case to each $U_j \to V$ for $j = 1, \ldots, t$.

Step 2. We prove the theorem after the reduction in Step 1.

We make an induction on $d := \dim_x X$. There is nothing to prove when $d = 0$. Assume that $d \geq 1$. By Lemma 4.13, we can choose a non-zero divisor $g_y \in \mathfrak{m}_{Y,g_y}$ such that $f_x^\#(g_y)$ is a non-zero divisor in $\mathcal{O}_{X,x}$. Up to shrinking Y, we may assume that g spreads to $g \in \mathcal{O}_Y(Y)$. It suffices to apply our inductive hypothesis to $W(f_x^\#(g_y)) \subseteq W(g_y)$. $\square$

Corollary 4.15. Let $f : X \to Y$ be a finite open surjective morphism of complex analytic spaces. Assume that A is a thin subset of X of order $k \in \mathbb{Z}_{>0}$, then $f(A)$ is a thin subset of Y of order k.

PROOF. We may assume that X and Y are reduced. By Proposition 4.4 and the fact that f is open, the problem is local on X, we may assume that A is an analytic subset of X. Let $x \in A$. It suffices to handle the case where A is irreducible at x and x is the only point in $f^{-1}(f(x))$. By Corollary 4.8, $f(A)$ is an irreducible analytic subset of Y.

We may assume that Y is irreducible at $y := f(x)$. Then

$$\operatorname{codim}_y(f(A), Y) = \dim_y Y - \dim_y f(A).$$

By Theorem 4.14, $\dim_y Y = \dim_x X$, $\dim_y f(A) = \dim_x A$. It follows that

$$\operatorname{codim}_y(f(A), Y) = \dim_x X - \dim_x A \geq \operatorname{codim}_x(A, X) \geq k.$$

□

Proposition 4.16. Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $x \in X$. Assume that Y is irreducible at $f(x)$. Assume that $\dim_x X = \dim_{f(x)} Y$, then f is open at x.

PROOF. We may assume that X and Y are both reduced. Let $y = f(x)$. By Proposition 4.4, we may assume that $\{x\} = f^{-1}(y)$. By Corollary 4.8, $f(X)$ is an analytic set in Y. By Theorem 4.14,

$$\dim_y f(X) = \dim_x X.$$

As Y is irreducible at $f(x)$, we conclude that $f(X)_y = X_y$ and hence $f(X)$ is a neighbourhood of y. □

Corollary 4.17. Let $f : X \to Y$ be a quasi-finite morphism of equidimensional complex analytic spaces of dimension $d \in \mathbb{N}$. Assume that Y is unibranch. Then f is open.

The corollary fails if Y is not unibranch.

PROOF. By Proposition 4.4, f is finite at all $x \in X$. It suffices to apply Proposition 4.16. □

Lemma 4.18. Let $f : X \to Y$ be a finite open morphism of reduced complex analytic spaces. Assume that Y is a complex manifold. Then f is a branched covering.

PROOF. The statement is local on Y, so we may assume that Y is an open neighbourhood of 0 in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. By Proposition 4.4, we may assume that $\pi^{-1}\{0\}$ consists of a single point and X is a closed analytic subspace of a domain V in $\mathbb{C}^d$ for some $d \in \mathbb{N}$. Replacing X by the graph of f, we may assume that X is a closed analytic subspace of $V \times Y$ and f is the restriction of the projection map $V \times Y \to V$. In this case, the result follows from the local description lemma. Reproduce CAS p72! □

Corollary 4.19. Let X be a equidimensional complex analytic space of dimension d and $x \in X$. Then there is an open neighbourhood U of x in X and a connected domain $V \in \mathbb{C}^d$ such that there is a branched covering $U \to V$.

In fact, given any system of parameters $f_1, \ldots, f_d \in \mathcal{O}_{X,x}$, we can define sch a morphism sending x to 0 and the corresponding local ring homomorphism at x is

$$\mathcal{O}_{\mathbb{C}^d,0} \to \mathcal{O}_{X,x}$$

given by $f_1, \ldots, f_d$.

PROOF. This follows from Theorem 3.9 in Constructions of complex analytic spaces, Lemma 4.18 and Corollary 4.17. □

Corollary 4.20. Let X be a complex analytic space and $x \in X$. Assume that X is unibranch at x. Let $f \in \mathcal{O}_{X,x}$. We assume that f is not constant and $\dim_x X \geq 1$, then for any open neighbourhood U of x in X such that f spreads to $g \in \mathcal{O}_X(U)$, there is $\epsilon > 0$ such that g takes all values $c \in \mathbb{C}$ with $|c - f(x)| < \epsilon$.

PROOF. We may assume that X is reduced and $f(x) = 0$. Then f is a non-zero divisor in $\mathcal{O}_{X,x}$. We can find a system of parameters $f, g_1, \ldots, g_{n-1}$ with $n = \dim_x X$ such that $f, g_1, \ldots, g_{n-1}$ induce a branched covering $X \to V$ sending x to 0 after shrinking X, where V is an open neighbourhood of 0 in $\mathbb{C}^n$. This follows from Corollary 4.19. As the branched covering is open by Proposition 4.16, we conclude. □

Theorem 4.21. Let $f : X \to Y$ be an open, finite surjective morphism of reduced complex analytic spaces, then f is a branched covering.

PROOF. Let $x \in X$ and $y = f(x)$. As f is open, it suffices to find open neighbourhoods U of x in X and V of y in Y such that the morphism $U \to V$ induced by f is a branched covering. We first take U small enough so that U can be decomposed into prime components at x:

$$U = X_1 \cup \cdots \cup X_s.$$

We can assume that $X_i \cap X_j$ is thin in U for $i, j = 1, \ldots, s$, $i \neq j$. Up to shrinking U, we may assume that $U \to V$ is finite Proposition 4.4 for some open neighbourhood V of y in Y. As f is open, we may take $V = f(U)$. Observe that $f(X_i)$ is analytic in V for $i = 1, \ldots, s$ by Corollary 4.8. Moreover, $f(X_i)$ is irreducible at y for $i = 1, \ldots, s$. By Theorem 2.4 in Local properties of complex analytic spaces, we may assume that $f(X_i)$ is equidimensional of dimension $n_i \in \mathbb{N}$ for $i = 1, \ldots, s$.

By Corollary 4.19, up to shrinking V, we may assume that there is a branched covering $\eta_i : f(X_i) \to V_i$, where V_i is a connected domain in $\mathbb{C}^{n_i}$ for $i = 1, \ldots, s$. By Lemma 4.18, $\eta_i \circ f|_{X_i}$ is a branched covering for $i = 1, \ldots, s$. It follows that $X_i \to \pi(X_i)$ is a branched covering for $i = 1, \ldots, s$. This readily implies that f is a branched covering. □

Definition 4.22. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X. Take a critical locus T of f containing $f(A)$.

Consider $g \in \mathcal{O}_X(X \setminus A)$. We define a monic polynomial

$$\chi_g(w)(y) := \prod_{x \in f^{-1}(y)} (w - g(x)) \in \mathcal{O}_Y(Y \setminus T)[w].$$

By Theorem 3.7 in Local properties of complex analytic spaces, χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$. The monic polynomial χ_g is called the *characteristic polynomial* of g (with respect to f).

Proposition 4.23. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $g \in \mathcal{O}_X(X \setminus A)$. Let $\chi_g \in \mathcal{O}_Y(Y \setminus f(A))[w]$ be the characteristic polynomial of g. Then $\chi_g(g) = 0$.

If either of the following conditions hold:

(1) f is locally bounded near A;
(2) A is thin of order 2 in Y.

Then χ_g can be uniquely extended to $\chi_g \in \mathcal{O}_Y(Y)[w]$.

Proof. Only the second part is non-trivial. By Corollary 4.15, f is open. By Corollary 4.15, $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y. It suffices to apply Theorem 3.7 in Local properties of complex analytic spaces. □

Proposition 4.24. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X and $e, g \in \mathcal{O}_X(X \setminus A)$. Take a critical locus T of f containing $f(A)$. Consider the $b \times b$-matrice

$$M(y) = \begin{bmatrix} 1 & e(x_1) & \dots & e(x_1)^{b-1} \\ 1 & e(x_2) & \dots & e(x_2)^{b-1} \\ & & \ddots & \\ 1 & e(x_b) & \dots & e(x_b)^{b-1} \end{bmatrix}$$

and $M_i(y)$ is $M(y)$ with the i-th colomn replace by

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_b) \end{bmatrix}$$

for $i = 0, \dots, b-1$, where $y \in Y \setminus T$ and $x_1, \dots, x_b$ are the distinct points in $f^{-1}(y)$. Then there are $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus f(A))$ such that for all $y \in Y \setminus T$,

$$\Delta_e(y) = (\det M(y))^2, \quad c_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. If either of the following conditions holds:

(1) e and g are locally bounded near A;
(2) A is thin of order 2 in X,

then we can take $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y)$

The function Δ_e is called the *discriminant* of e. We say e is *primitive* with respect to f if Δ is not identically 0.

Proof. We first observe that $\det M(y)$ and $\det M_i(y)$ are independent of the ordering of $x_1, \dots, x_b$ by elementary lineary algebra, where $i = 1, \dots, b$. The entries of $M(y)$ and $M_i(y)$ can all be taken to be holomorphic outside T, so $\Delta_e, c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus T)$ are defined and the desired equation holds. By Theorem 3.7 in Local properties of complex analytic spaces, these functions can be extended uniquely into $\mathcal{O}_Y(Y \setminus f(A))$.

By Corollary 4.15, $f(A)$ is thin in Y and under assumption (2), $f(A)$ is thin of order 2 in Y. Applying Theorem 3.7 in Local properties of complex analytic spaces, we conclude the last assertion. □

Corollary 4.25. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. A primitive element $e \in \mathcal{O}_X(X)$ exists if X is holomorphically separable.

PROOF. Take a critical locus T of f. Let $y \in X \setminus T$. Let $x_1, \dots, x_b$ be distinct points of $f^{-1}(y)$. For each $i, j = 1, \dots, b$ with $i < j$, we can find a $g_{ij} \in \mathcal{O}_X(X)$ with $g(x_i) \neq g(x_j)$. A suitable linear combination of g_{ij}'s works. □

Proposition 4.26. Let $b \in \mathbb{Z}_{>0}$, $f : X \to Y$ be a b-sheeted branched covering with Y being a connected complex manifold. Let A be a thin set in X.

Let $e \in \mathcal{O}_X(X \setminus A)$ primitive element with respect to f. Then for each $g \in \mathcal{O}_X(X \setminus A)$, we have canonical polynomial $\Omega \in \mathcal{O}_Y(Y \setminus \pi(A))[X]$ such that

$$\Delta_e g = \Omega(e) \quad \text{on } X \setminus A.$$

If either of the following conditions holds:

(1) e and g are locally bounded near A;
(2) A is thin of order 2 in X,

then we can take $\Omega \in \mathcal{O}_Y(Y)[X]$.

In the traditional terminology, Δ_e is a *universal denominator* of the $\mathcal{O}_Y(Y)$-module $\mathcal{O}_X(X)$ if one of the two assumptons is satisfied.

PROOF. Take a critical locus T of f containing $f(A)$. Consider $y \in Y \setminus T$ with fibers $x_1, \dots, x_b$. Consider the system of b-linear equations:

$$\Delta_e(y) g(x_i) = c_0(y) + c_1(y) e(x_i) + \cdots + c_{b-1}(y) e(x_i)^{b-1}$$

for $j = 1, \dots, b$. By Cramer's rule, if we use the notations of Proposition 4.24, if $\det M(y) \neq 0$, the unique solution is then

$$c_i(y) = (\det M(y))^{-1} \Delta(y) \det M_i(y) = \det M(y) \cdot \det M_i(y)$$

for $i = 0, \dots, b-1$. From Proposition 4.24, $c_0, \dots, c_{b-1} \in \mathcal{O}_Y(Y \setminus \pi(A))$. It suffices to take

$$\Omega = c_0 + c_1 X + \cdots + c_{b-1} w^{b-1}.$$

It is obvious that on $X \setminus (A \cup W(\Delta))$,

$$\Delta_e g = \Omega(e).$$

The same holds on $X \setminus A$ by continuity. The last asertion follows from Proposition 4.24. □

Corollary 4.27 (Riemann extension theorem)**.** Let X be a reduced equidimensional complex analytic space of dimension $n \in \mathbb{N}$ and A be a thin set in X. Let $f \in \mathcal{O}_X(X \setminus A)$. Assume one of the following conditions holds:

(1) f is locally bounded near A;
(2) A is thin of order 2.

Then there is an element $g \in \overline{O}_X(X)$ extending f.

PROOF. The uniquenss is obvious, we prove the existence. The problem is local on X, we may assume that X is holomorphically separable. By Corollary 4.19, we may take a connected complex manifold Y of dimension Y, $b \in \mathbb{Z}_{>0}$,a b-sheeted branched covering $f : X \to Y$. By Corollary 4.25, we can find a primitive element $e \in \mathcal{O}_X(X)$. By Proposition 4.26 and Proposition 4.23, it suffices to take $g = \Omega(e)/\Delta_e$, where Ω_e is the polynomial in Proposition 4.26. □

Corollary 4.28. Let X be a normal complex analytic space. Then the canonical map

$$\mathcal{O}_X(X) \to \mathcal{O}_X(X^{\mathrm{reg}})$$

is an isomorphism.

Proof. By Proposition 6.9 in Local properties of complex analytic spaces, the map is injective. Take $f \in \mathcal{O}_X(X^{\mathrm{reg}})$, we need to extend it to $g \in \mathcal{O}_X(X)$. The problem is local on X. As X is normal, it is equidimensional at all points. By shrinking X, we may assume that X is equidimensional of some dimension $n \in \mathbb{N}$. Recall that X^{Sing} is thin of order 2 in X by Proposition 7.4 in Local properties of complex analytic spaces, so we can apply Corollary 4.27. □

Corollary 4.29. Let X be a connected normal complex analytic space then X^{reg} is connected.

Proof. If not, we can find a continuous function $f : X^{\mathrm{reg}} \to \{0, 1\}$ which is not constant. By Corollary 4.28, f can be extended to $g \in \mathcal{O}_X(X)$. This contradicts the fact taht X is connected. □

Corollary 4.30. Let X be a connected complex analytic space. Then X is path-connected.

Proof. We may assume that X is reduced.

If X is irreducible, after passing to the normalization, we may assume that X is normal. Then clearly X^{reg} is connected. So it suffices to apply Proposition 7.12 in Local properties of complex analytic spaces.

In general, take $x \in X$ and let X' be the set of all points of X that can be joined to x by a path. Then from the previous case, X' is the union of certain irreducible componenets of X. So is the complement $X \setminus X'$. As X is connected, we find that $X = X'$. □

We given an alternative characterization of $\overline{\mathcal{O}}_X$.

Proposition 4.31. Let X be a reduced complex analytic space. Then for any open set $U \subseteq X$,

$$\overline{\mathcal{O}}_X(U) \xrightarrow{\sim} \{f : U \to \mathbb{C} : f \text{ is weakly holomorphic}\}.$$

Proof. We temporarily denote the sheaf stated in theproposition by $\mathcal{O}'$. From the uniqueness in **??**, it suffices to show that $\mathcal{O}'_x$ is isomorphic to $\overline{\mathcal{O}_{X,x}}$ as $\mathcal{O}_{X,x}$-algebras for any $x \in X$.

We first observe that $\overline{\mathcal{O}}_X$ is a subsheaf of $\mathcal{O}'$. Let $U \subseteq X$ be an open subset and $f \in \overline{\mathcal{O}}_X(U)$. We need to show that f is locally bounded around $y \in U \cap X^{\mathrm{Sing}}$. Take an integral equation

$$f_y^n + a_{1,y} f_y^{n-1} + \cdots + a_{n,y} = 0$$

with $a_{1,y}, \ldots, a_{n,y} \in \mathcal{O}_{X,x}$. Take an open neighbourhood V of y in U such that $a_{1,y}, \ldots, a_{n,y}$ lift to $a_1, \ldots, a_n \in \mathcal{O}_X(V)$ and

$$(f|_V)^n + a_1 f|_V^{n-1} + \cdots + a_n = 0.$$

Then for any $z \in V \setminus X^{\mathrm{Sing}}$,

$$|f(z)| \leq \max\{1, |a_1(z)| + \ldots + |a_n(z)|\}.$$

So $f \in \mathcal{O}'$.

Conversely, let $U \subseteq X$ be an open subset and $f \in \mathcal{O}'(U)$. By Proposition 7.8 in Local properties of complex analytic spaces, $p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_X$, where $p : \overline{X} \to X$ is the normalization morphism. It follows from Proposition 7.8 in Local properties of complex analytic spaces and Corollary 4.27 that f can be uniquely extended to $g \in \mathcal{O}_{\overline{X}}(p^{-1}U) = \mathcal{O}_X(U)$. □

Proposition 4.32 (Rado, Cartan)**.** Let X be a normal complex analytic space and $f : X \to \mathbb{C}$ be a continuous map. Let $Z = f^{-1}(0)$. Assume that there is $g \in \mathcal{O}_X(X \setminus Z)$ such that $[g] = f|_{X\setminus Z}$, then $f = [g]$.

This result is proved in [Car52].

Proof. By Corollary 4.28, we may assume that X is a complex manifold. The problem is local on X, we may assume that X is the unit polydisk in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. By Hartogs theorem, we may assume that $n = 1$.

It remains to show that a continuous function $f : \{z \in \mathbb{C} : |z| < 1\}$ which is holomorphic outside $Z := \{f = 0\}$ is holomorphic. This result is well-known. □

Bibliography

[Car52] H. Cartan. Sur une extension d'un théorème de Radó. *Math. Ann.* 125 (1952), pp. 49–50. URL: https://doi.org/10.1007/BF01343105.