The notion of complex analytic spaces

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1. Introduction

2. $\mathbb{C}$-ringed space

Definition 2.1. A $\mathbb{C}$-*ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras on X.

A *morphism of* $\mathbb{C}$-*ringed spaces* $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair consisting of a continuous map $f : Y \to X$ and a morphism of sheaves of $\mathbb{C}$-algebras $f^\# : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$.

Given two morphisms of $\mathbb{C}$-ringed spaces $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $g : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$, their *composition* is the morphism $f \circ g : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ consisting of the continuous map $f \circ g : Z \to X$ and a morphism of sheaves $(f \circ g)^\# = g^\# \circ g^{-1}f^\# : (f \circ g)^{-1}\mathcal{O}_X \xrightarrow{\sim} g^{-1}f^{-1}\mathcal{O}_X \to \mathcal{O}_Z$.

When there is no risk of confusion, we say X is a $\mathbb{C}$-ringed space.

It is straightforward to verify that $\mathbb{C}$-ringed spaces form a category, which we denote by $\mathbb{C}$-$\mathcal{R}$S. Similarly, we denote by $\mathcal{R}$S the category of ringed spaces defined in [Stacks, Tag 0090].

In fact, by definition a $\mathbb{C}$-ringed space is nothing but a morphism in the category of ringed spaces $X \to \mathbb{C}^0$, where $\mathbb{C}^0$ is a single point $*$ endowed with the sheaf of rings $\mathcal{O}_{\mathbb{C}^0}$ with $\mathcal{O}_{\mathbb{C}^0}(*) = \mathbb{C}$. In terms of slice categories, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{R}\mathrm{S} \approx \mathcal{R}\mathrm{S}/\mathbb{C}^0.$$

From this identification, most of the basic results above $\mathbb{C}$-$\mathcal{R}$S follows, which we will use freely.

There is an obvious faithful forget functor $\mathbb{C}$-$\mathcal{R}$S $\to \mathcal{R}$S.

Definition 2.2. A *locally* $\mathbb{C}$-*ringed space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ which when regarded as a ringed space is a locally ringed space.

A *morphism* between two locally $\mathbb{C}$-ringed spaces is a morphism between the underlying $\mathbb{C}$-ringed spaces which is a morphism of locally ringed spaces at the same time.

The category of locally $\mathbb{C}$-ringed spaces is denoted by $\mathbb{C}$-$\mathcal{L}$RS.

We refer to [Stacks, Tag 01HA] for the notion of locally ringed spaces. Similar to the case of $\mathbb{C}$-ringed space, we have a canonical equivalence of categories

$$\mathbb{C}\text{-}\mathcal{L}\mathrm{RS} \approx \mathcal{L}\mathrm{RS}/\mathbb{C}^0.$$

Example 2.3. *Let $n \in \mathbb{N}$, we define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_{\mathbb{C}^n}$ on $\mathbb{C}^n$ as follows: for any open subset $U \subseteq \mathbb{C}^n$, $\mathcal{O}_{\mathbb{C}^n}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions on U. It is easy to see that $\mathcal{O}_{\mathbb{C}^n}$ is a sheaf and $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a $\mathbb{C}$-ringed space. Moreover, it is easy to show that $(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ is a locally $\mathbb{C}$-ringed space.*

3. Complex model spaces and complex analytic spaces

Definition 3.1. Given any domain D in $\mathbb{C}^n$, we can define a sheaf of $\mathbb{C}$-algebras $\mathcal{O}_D$ on D as the restriction of $\mathcal{O}_{\mathbb{C}^n}$ defined in Example 2.3 to D. Observe that $(D, \mathcal{O}_D)$ is a locally $\mathbb{C}$-ringed space.

Definition 3.2. A *complex model space* is a $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that there exist

(1) a domain D in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ and

(2) an ideal sheaf $\mathcal{I}$ in $\mathcal{O}_D$ of finite type

such that thre is an isomorphism

$$(X, \mathcal{O}_X) \cong (\operatorname{Supp} \mathcal{O}_D/\mathcal{I}, i^{-1}(\mathcal{O}_D/\mathcal{I}))$$

in the category of $\mathbb{C}$-$\mathcal{R}$S, where $i : \operatorname{Supp} \mathcal{O}_D/\mathcal{I} \to D$ is the inclusion map. Here $\mathcal{O}_D$ is the sheaf of $\mathbb{C}$-algebras defined in Definition 3.1.

Clearly, $(X, \mathcal{O}_X)$ is a locally $\mathbb{C}$-ringed space.

Observe that X is always a Hausdorff space.

Definition 3.3. A *complex analytic space* is a locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that

(1) X is a Hausdorff space.
(2) For any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is isomorphic to a complex model space in the sense of Definition 3.2 in the category $\mathbb{C}$-$\mathcal{L}$RS.

When there is no risk of confusion, we also omit $\mathcal{O}_X$ from the notation say X is a complex analytic space.

A morphism between complex analytic spaces is a morphism of the underlying locally $\mathbb{C}$-ringed spaces. Such a morphism is also known as a *holomorphic map*.

Remark 3.4. It seems that all authors on this subject requires that complex analytic spaces be Hausdorff, which may seem unnatural from the eyes of an algebro-geometrist. Morally, Hausdorffness corresponds to separatedness in the scheme world. However, non-Hausdorff analytic spaces do not seem to play a major role, in contrast to non-separated schemes, so we stick to the current definition.

Remark 3.5. Most of the authors require extra conditions in the definition of a complex analytic space: σ-compactness, paracompactness, having countable basis etc. We will not put these constraints in the definition, instead, we choose to include them into the assumptions of the theorems.

4. Weierstrass map

5. Oka's coherence theorem

This lemma needs to be placed elsewhere. Proof at CAS p58 needs to be included

Lemma 5.1. *Let X be a topological space and $\mathcal{A}$ be a Hausdorff sheaf of rings on X (in the sense that the espace étalé of $\mathcal{A}$ is Hausdorff) such that all stalks of $\mathcal{A}$ are integral domains. Then $\mathcal{A}$ is coherent if and only if for any open set $V \subseteq X$ and any section $s \in \mathcal{A}(X)$, $\mathcal{A}_V/s\mathcal{A}_V$ is coherent at every $x \in V$ where $s_x \neq 0$.*

Lemma 5.2 (Oka)**.** *For any $n \in \mathbb{N}$, $\mathcal{O}_{\mathcal{C}^n}$ is coherent.*

Proof. As a preparation, observe that $\mathcal{O}_{\mathbb{C}^n}$ is a Hausdorff sheaf.

For any two germs $s_i \in \mathcal{O}_{\mathbb{C}^n, a_i}$ $(i = 1, 2)$, we need to construct disjoint open neighbourhoods U_i in the espace étalé of $\mathcal{O}_{\mathbb{C}^n}$ of s_i. If $a_1 \neq a_2$, the assertion is clear. So assume that $a_1 = a_2 = 0$. We extend s_i to $f_i \in \mathcal{O}_{\mathbb{C}^n}(U)$ for a connected open neighbourhood $U \subseteq \mathbb{C}^n$ of 0. Then $\{f_x : x \in U\}$ and $\{g_x : x \in U\}$ are disjoint: if for some $z \in U$, $f_z = g_z$, then the same holds in a neighbourhood of z and so $f = g$ on U by Identitätssatz. Include the proof

We will prove the coherence of $\mathcal{O}_{\mathcal{C}^n}$ by induction on n. The case $n = 0$ is trivial. Assume that $n > 0$ and the theorem has been proved for all smaller n. We will apply Lemma 5.1. Take an open set $U \subseteq \mathbb{C}^n$ and $g \in \mathcal{O}_{\mathbb{C}^n}(U)$. We need to show that $\mathcal{O}_U/g\mathcal{O}_U$ is coherent at all $x \in U$ with $g_x \neq 0$.

Fix such a point x, which may be assumed to be 0. We may assume that $g(0) = 0$ as otherwise, the stalk of $\mathcal{O}_U/g\mathcal{O}_U$ at 0 is trivial. By perturbing the coordinates, we may guarantee that $g_0(0, w)$ is not identically 0 for $w \in \mathbb{C}$. By Weierstrass preparation theorem Include a proof, there is a monic polynomial $\omega_0 \in \mathcal{O}_{\mathbb{C}^{n-1},0}[w]$ such that $g_0\mathcal{O}_{\mathbb{C}^n,0} = \omega_0\mathcal{O}_{\mathbb{C}^n,0}$. Lift ω_0 to $\omega \in \mathcal{O}_{\mathbb{C}^{n-1}}(B)$ for some neighbourhood $B \subseteq \mathbb{C}^{n-1}$ of 0. In order to show the coherence of $\mathcal{O}_U/g\mathcal{O}_U$ near 0, it suffices to show that $\mathcal{O}_{B\times\mathbb{C}}/\omega\mathcal{O}_{B\times\mathbb{C}}$ near 0. Let $A = Z(\omega) \subseteq B \times \mathbb{C}$ be the closed subspace defined by the coherent sheaf generated by ω, then it suffices to show that $\mathcal{O}_A$ is coherent near 0. Now we have the finite Weierstrass morphism $A \to B$Include, it suffices to prove the coherence of $\mathcal{O}_B$, which follows from inductive hypothesis. □

As a corollary, we have the important Oka's coherence theorem.

Theorem 5.3. *Let X be a complex analytic space, then $\mathcal{O}_X$ is coherent.*

Proof. The problem is local on X, so we may assume that X is a complex model space, say there is a closed immersion into a domain D in $\mathbb{C}^n$ defined by an ideal of finite type $\mathcal{I}$. By Lemma 5.2, $\mathcal{O}_D$ is coherent and hence $\mathcal{I}$ is coherent. It follows that $\mathcal{O}_D/\mathcal{I}$ is coherent and hence $\mathcal{O}_X$ is coherent. □

6. Implicit function theorem

Theorem 6.1. *Let X be a complex analytic space and $x \in X$. Then $\mathcal{O}_{X,x}$ is strictly Henselian.*

7. Rückert Nullstellensatz

Let X be a complex analytic space. It is a sheaf of $\mathbb{C}$-algebras. For any sheaf of local $\mathbb{C}$-algebras $\mathcal{A}$ on X, any open set $U \subseteq X$ and any $s \in \mathcal{A}_X(U)$. We want to construct a function $[s] : U \to \mathbb{C}$.

Take $x \in U$, there is a canonical splitting

$$\mathcal{A}_x \cong \mathbb{C} \oplus \mathfrak{m}_x, \tag{7.1}$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{A}_x$. Then we define $[s](x)$ as the image of s_x in the $\mathbb{C}$-factor in (7.1).

Definition 7.1. Let $X, \mathcal{A}, U, x, s$ be as above. The value $[s](x) \in \mathbb{C}$ is called the *value* of s at x. We sometimes denote it by $s(x)$ as well.

Lemma 7.2. *Let X be a complex analytic space. We denote by $\mathcal{C}_X$ the sheaf of continuous functions on X. The association $s \mapsto [s]$ in Definition 7.1 defines a homomorphism of sheaves of $\mathbb{C}$-algebras $\mathcal{O}_X \to \mathcal{C}_X$.*

When there is no risk of confusion, we also write s instead of $[s]$.

Proof. We need to show that for any open set $U \subseteq X$ and any $s \in \mathcal{O}_X(U)$, $[s]$ is a continuous function on U.

We may clearly assume that $U = X$. The problem is local on X, so we may assume that X is a complex model space in the sense of Definition 3.2 defined by a coherent ideal $\mathcal{I}$ in a domain D in $\mathbb{C}^n$. By further localizing, we may assume that s

can be lifted to a section $f \in \mathcal{O}_D(D)$. Then $[s] = f|_X$ by definition. So the assertion follows from the fact that a holomorphic function on a domain is continuous. $\square$

THEOREM 7.3 (Rückert Nullstellensatz). *Let X be a complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $f \in \mathcal{O}_X(X)$ be a function that vanishes on $\operatorname{Supp} \mathcal{F}$. Then for any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x and $m \in \mathbb{Z}_{>0}$ such that $f^m \mathcal{F}|_U = 0$.*

PROOF. We may assume that $x \in \operatorname{Supp} \mathcal{F}$ as otherwise there is nothing to prove. In particular, $f(x) = 0$.

Step 1. We first reduce the problem to a relatively simple situation.

The problem is local on X, so we may assume that there is a domain D containing 0 in $\mathbb{C}^n$ and a closed immersion $\iota : X \to D$ sending x to 0. Consider the closed immersion $g : V \to D \times \mathbb{C}$ induced by ι and f. Assume that this theorem has been proved for w, $B \times \mathbb{C}$, $g_*\mathcal{F}$ in place of f, X, $\mathcal{F}$ respectively, then we would find an integer $m \in \mathbb{Z}_{>0}$ such that $w^m (g_*\mathcal{F})_0 = 0$. In particular, $f^m \mathcal{F}_x = 0$. As $\mathcal{F}$ is coherent, there is an open neighbourhood $U \subseteq X$ of x such that $f^t \mathcal{F}|_U = 0$.

Step 2. We are reduced to prove the following special case: let D be a domain in $\mathbb{C}^n$ containing 0, $\mathcal{F}$ is a coherent sheaf on D whose support is contained in $\{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : (z, w) \in D, w = 0\}$. Then there is $m \in \mathbb{Z}_{>0}$ such that $w^m \mathcal{F}_0 = 0$.

Let $\mathcal{G}$ be the annihilator sheaf of $\mathcal{F}$:

$$\mathcal{G} := \ker \left(\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F})\right),$$

where the map $\mathcal{O}_D \to \mathcal{H}\mathrm{om}_{\mathcal{O}_D}(\mathcal{F}, \mathcal{F})$ sends a local section f of $\mathcal{O}_D$ to the endo-homomorphism of multiplying by f of $\mathcal{F}$. Then $\mathcal{G}$ is a coherent sheaf by Oka's coherence theorem Theorem 5.3. So it has closed supports. But by our assumtion, the support of $\mathcal{G}$ contains all $w \neq 0$, so $\operatorname{Supp} \mathcal{G} = D$.

Let $f \in \mathcal{G}_0$ be a non-zero element. We write The structure of the local ring needs to be presented earlier

$$f = \sum_{j=b}^{\infty} a_j w^j, \quad a_j \in \mathcal{O}_{\mathbb{C}^{n-1},0}, a_b \neq 0$$

for some $b \in \mathbb{N}$. We may assume that $b = 0$ by replacing f and $\mathcal{F}$ with $w^{-b} f$ and $w^b \mathcal{F}$ respectively. We want to show that $w^m \mathcal{F}_0 = 0$ for some positive integer m.

When a_0 is a unit, namely when $a_0(0) \neq 0$, then f is a unit, so $\mathcal{F}_0 = 0$. We make an induction on n. The case $n = 1$ is trivial, as a_0 is always a unit. So we may assume that $a_0(0) = 0$ and $n > 1$. By perturbing the coordinates in $\mathbb{C}^{n-1}$, we may assume that a_0 is not identically zero in the variable z_1. We need to finish the Weierstrass theory first.

Shrinking D, we may assume that f can be lifted to a holomorphic function $g \in \mathcal{O}_D(D)$ with $g\mathcal{F} = 0$. By our assumption on a_0, we may assume that $Z(g) \cap \{(z_1, 0, \ldots, 0) \in D\} = \{0\}$. Hence, $D \cap \operatorname{Supp} \mathcal{F}$, which is a subset of $Z(g)$ also intersects the z_1-axis only at the origin.

By To be included, we can find a product domain $B \times W \subseteq D$ with $B \subseteq \mathbb{C}$ and $W \subseteq \mathbb{C}^{n-1}$ containing 0 such that the projection $h : (B \times W) \cap \operatorname{Supp} \mathcal{F} \to B$ is finite and $\mathcal{F}' := h_*(\mathcal{F}|_{B \times W})$ is a coherent sheaf of $\mathcal{O}_B$-modules. Observe that $\operatorname{Supp} \mathcal{F}' \subseteq \{(z_2, \ldots, z_{n-1}, w) \in B : w = 0\}$, we can apply the induction hypothesis to obtain $m \in \mathbb{Z}_{>0}$ such that $w^m \mathcal{F}'_0 = 0$. It follows that $w^m \mathcal{F}_0 = 0$. $\square$

8. Fiber products

The goal of this section is to show that the category of complex analytic spaces admits finite limits.

9. Complex analytic topos

Bibliography

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.