Banach rings

Contents

1. Introduction

This section conerns the theory of Banach algebras. Our references are [Ber12] and [BGR84].

In this chapter, all rings are assumed to be commutative.

2. Semi-normed Abelian groups

Definition 2.1. Let A be an Abelian group. A *semi-norm* on A is a function $\|\bullet\| : A \to [0,\infty]$ satisfying

(1) $\|0\| = 0$;
(2) $\|f - g\| \leq \|f\| + \|g\|$ for all $f, g \in A$.

A semi-norm $\|\bullet\|$ on A is a *norm* if moreover the following conditions is satisfied:

(0) if $\|f\| = 0$ for some $f \in A$, then $f = 0$.

We write

$$\ker \|\bullet\| = \{a \in A : \|a\| = 0\}.$$

A semi-norm $\|\bullet\|$ on A is *non-Archimedean* or *ultra-metric* if Condition (2) can be replaced by

(2') $\|f - g\| \leq \max\{\|f\|, \|g\|\}$ for all $f, g \in A$.

Definition 2.2. A *semi-normed Abelian group* (resp. *normed Abelian group*) is a pair $(A, \|\bullet\|)$ consisting of an Abelian group A and a semi-norm (resp. norm) $\|\bullet\|$ on A. When $\|\bullet\|$ is clear from the context, we also say A is a semi-normed Abelian group (resp. normed Abelian group).

Definition 2.3. Let $(A, \|\bullet\|_A)$ be a semi-normed Abelian group and $B \subseteq A$ be a subgroup. Then we define the *quotient semi-norm* $\|\bullet\|_{A/B}$ on A/B as follows:

$$\|a + B\|_{A/B} := \inf\{\|a + b\|_A : b \in B\}$$

for all $a + B \in A/B$.

We define the *subgroup semi-norm* on B as follows:

$$\|b\|_B = \|b\|_A$$

for all $b \in B$.

Definition 2.4. Let A be an Abelian group and $\|\bullet\|$, $\|\bullet\|'$ be two seminorms on A. We say $\|\bullet\|$ and $\|\bullet\|'$ are *equivalent* if there is a constant $C > 0$ such that

$$C^{-1}\|f\| \leq \|f\|' \leq C\|f\|$$

for all $f \in A$.

Definition 2.5. Let $(A, \|\bullet\|_A)$, $(B, \|\bullet\|_B)$ be semi-normed Abelian groups. A homomorphism $\varphi : A \to B$ is said to be

(1) *bounded* if there is a constant $C > 0$ such that $\|\varphi(f)\|_B \leq C\|f\|_A$ for any $f \in A$;
(2) *admissible* if the quotient semi-norm on $A/\ker\varphi$ is equivalent to the subspace semi-norm on $\operatorname{Im}\varphi$.

Observe that an admissible homomorphism is always bounded.

Next we study the topology defined by a semi-norm.

Lemma 2.6. Let $(A, \|\bullet\|)$ be a semi-normed Abelian group. Define

$$d(a,b) = \|a-b\|$$

for $a, b \in A$. Then $\|\bullet\|$ is a pseudo-metric on A. This psuedo-metric is a metric if and only if $\|\bullet\|$ is a norm.

Let $\hat{A}$ be the metric completion of A, then there is a norm $\|\bullet\|$ on $\hat{A}$ inducing its metric. Moreover, the natural homomorphism $A \to \hat{A}$ is an isometric homomorphism with dense image.

Proof. This is clear from the definitions. $\square$

We always endow A with the topology induced by the psuedo-metric d.

Proposition 2.7. Let $f : A \to B$ be a homomorphism between semi-normed Abelian groups. Assume that f is bounded, then it is continuous.

The converse is not true.

Proof. Clear from the definition. $\square$

Proposition 2.8. Let $(A, \|\bullet\|)$ be a normed Abelian group and B be a subgroup of A. Assume that there is $\epsilon \in (0,1)$ such that for each $a \in A$, there is $b \in B$ such that

$$\|a+b\| \leq \epsilon \|a\|.$$

Then B is dense in A.

Proof. Assume to the contrary that there exists $a \in A$ so that

$$c := \inf_{b \in B} \|a-b\| > 0.$$

Choose $b_1 \in B$ so that

$$\|a+b_1\| < \epsilon^{-1} c.$$

By our hypothesis, there is $b_2 \in B$ such that

$$\|a+b_1+b_2\| \leq \epsilon \|a+b_1\| < c.$$

This is a contradiction. $\square$

Definition 2.9. Let $(A, \|\bullet\|)$ be a semi-normed Abelian group. The normed Abelian group $(\hat{A}, \|\bullet\|)$ constructed in Lemma 2.6 is called the *completion* of $(A, \|\bullet\|)$.

3. Semi-normed rings

Definition 3.1. Let A be a ring. A *semi-norm* $\|\bullet\|$ on A is a semi-norm $\|\bullet\|$ on the underlying additive group satisfying the following extra properties:

(3) $\|1\| = 1$;
(4) for any $f, g \in A$, $\|fg\| \leq \|f\| \cdot \|g\|$.

A semi-norm $\|\bullet\|$ on A is called *power-multiplicative* if $\|f\|^n = \|f^n\|$ for all $f \in A$ and $n \in \mathbb{N}$.

A semi-norm $\|\bullet\|$ on A is called *multiplicative* if $\|fg\| = \|f\|\|g\|$ for all $f, g \in A$.

Definition 3.2. A *semi-normed ring* (resp. *normed ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring A and a semi-norm (resp. norm) $\|\bullet\|$ on A. When $\|\bullet\|$ is clear from the context, we also say A is a semi-normed ring (resp. normed ring).

Definition 3.3. Let $(A, \|\bullet\|)$ be a semi-normed ring. An element $a \in A$ is *multiplicative* if $a \notin \ker \|\bullet\|$ and for any $x \in A$,

$$\|ax\| = \|a\| \cdot \|x\|.$$

Definition 3.4. Let $(A, \|\bullet\|)$ be a normed ring. An element $a \in A$ is *power-bounded* if $\{|a^n| : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. The set of power-bounded elements in A is denoted by $\mathring{A}$.

An element $a \in A$ is called *topologically nilpotent* if $a^n \to 0$ as $n \to \infty$. The set of topologically nilpotent elements in A is denoted by $\check{A}$.

Proposition 3.5. Let $(A, \|\bullet\|)$ be a non-Archimedean normed ring. Then $\mathring{A}$ is a subring of A and $\check{A}$ is an ideal in $\mathring{A}$. Moreover, $\mathring{A}$, $\check{A}$ are open and closed in A.

Proof. Choose $a, b \in \mathring{A}$, by definition, there is a constant $C > 0$ so that for any $n \in \mathbb{N}$,

$$\|a^n\| \leq C, \quad \|b^n\| \leq C.$$

It follows that

$$\|(ab)^n\| \leq \|a^n\| \cdot \|b^n\| \leq C^2$$

and

$$\|(a-b)^n\| \leq \max_{i=0,\ldots,n} \|a^i b^{n-i}\| \leq C^2.$$

So $\mathring{A}$ is a subring.

Next we show that $\check{A}$ is an ideal in $\mathring{A}$. On the other hand, take $c \in \check{A}$, then

$$\|(ac)^n\| \leq \|a^n\| \cdot \|c^n\| \leq C\|c^n\|$$

But $\|c^n\| \to 0$ as $n \to \infty$, hence $ac \in \check{A}$.

On the other hand, consider $c, d \in \check{A}$, we need to show $c - d \in \check{A}$. Choose $C > 0$ so that

$$\|a^n\| \leq C, \quad \|b^n\| \leq C$$

for all $n \in \mathbb{N}$. Fix $\epsilon > 0$, then there is $m \in \mathbb{N}$ so that for any $k \geq m$,

$$\|a^k\| \leq \epsilon C^{-1}, \quad \|b^k\| \leq \epsilon C^{-1}.$$

In particular, for $k \geq 2m$, we have

$$\|(a-b)^k\| \leq \max_{i=0,\ldots,k} \|a^i\| \cdot \|b^{k-i}\| \leq \epsilon.$$

It follows that $a - b \in \check{A}$. This proves that $\check{A}$ is an ideal in $\mathring{A}$.

In order to see $\check{A}$ is open and closed in A, observe that it is a subgroup of A, so it suffices to show that $\check{A}$ is open in A. It suffices to show that

$$\{a \in A : \|a\| < 1\} \subseteq \check{A}.$$

But this is obvious, if $\|a\| < 1$, then $\|a^n\| \leq \|a\|^n$ for all $n \in \mathbb{N}$, it follows that $a^n \to 0$ as $n \to \infty$, namely, $a \in \check{A}$.

As $\check{A}$ is a subgroup of $\mathring{A}$, it follows that $\mathring{A}$ is both open and closed. $\square$

Definition 3.6. Let $(A, \|\bullet\|)$ be a non-Archimedean normed ring. We define the *reduction* of A as $\tilde{A} = \mathring{A}/\check{A}$. The map $\mathring{A} \to \tilde{A}$ is called the *reduction map*. We usually denote the reduction map by $a \mapsto \tilde{a}$.

This definition makes sense thanks to Proposition 3.5.

Definition 3.7. Let A be a ring. A *semi-valuation* on A is a multiplicative semi-norm on A. A semi-valuation on A is a *valuation* on A if its underlying semi-norm of Abelian groups is a norm.

Definition 3.8. A *semi-valued ring* (resp. *valued ring*) is a pair $(A, \|\bullet\|)$ consisting of a ring A and a semi-valuation (resp. valuation) $\|\bullet\|$ on A. When $\|\bullet\|$ is clear from the context, we also say A is a semi-valued ring (resp. valued ring).

A semi-valued ring (resp. valued ring) $(A, \|\bullet\|)$ is called a *semi-valued field* (resp. *valued field*) if A is a field.

4. Banach rings

Definition 4.1. A *Banach ring* is a normed ring that is complete with respect to the metric defined in Lemma 2.6.

Definition 4.2. A Banach ring $(A, \|\bullet\|_A)$ is *uniform* if $\|\bullet\|_A$ is power-multiplicative.

Definition 4.3. Let A be a semi-normed ring. There is an obvious ring structure on the completion $\hat{A}$ of A defined in Definition 2.9. We call the resulting Banach ring the *completion* of A.

Proposition 4.4. Let $(A, \|\bullet\|)$ be a Banach ring and $f \in A$. Assume that $\|f\| < 1$, then $1 - f$ is invertible.

Proof. Define
$$g = \sum_{i=0}^{\infty} f^i.$$
From our assumption, the series converges and $g \in A$. It is elementary to check that g is the inverse of $1 - f$. $\square$

In the non-Archimedean case, we have a stronger result:

Proposition 4.5. Let $(A, \|\bullet\|)$ be a non-Archimedean Banach ring and $f \in \check{A}$. Then $1 - f$ is invertible. Moreover, $(1 - f)^{-1}$ can be written as $1 + z$ for some $z \in \check{A}$.

Proof. Define
$$g = \sum_{i=0}^{\infty} f^i.$$
From our assumption, the series converges and $g \in A$. It is elementary to check that g is the inverse of $1 - f$. Moreover, in view of Proposition 3.5 as for any $i \geq 1$, $f^i \in \check{A}$, the same holds for their sum, we conclude the final assertion. $\square$

Corollary 4.6. Let $(A, \|\bullet\|)$ be a Banach ring. Then the set of invertible elements in A is open.

Proof. Let $x \in A$ be an invertible element. It suffices to show that for any $y \in A$, $|y| < 1/(\|x^{-1}\|)$, $y + x$ is invertible. For this purpose, it suffices to show that $1 + x^{-1}y$ is invertible. But this follows from Proposition 4.4. $\square$

Corollary 4.7. Let A be a Banach ring and $\mathfrak{m}$ be a maximal ideal in A. Then $\mathfrak{m}$ is closed.

Proof. The closure $\bar{\mathfrak{m}}$ is obviously an ideal in A. We need to show that $\mathfrak{m} \neq A$. Namely, 1 is not in the closure of $\mathfrak{m}$. But clearly, $\mathfrak{m}$ is contained in the set of non-invertible elements, the latter being closed by Corollary 4.6. So we conclude. $\square$

Lemma 4.8. Let A be a non-Archimedean Banach ring. An element $a \in \mathring{A}$ is a unit in $\mathring{A}$ if and only if $\tilde{a}$ is a unit in $\tilde{A}$.

PROOF. The direct implication is trivial. Conversely, assume that $a \in \mathring{A}$ and there is an element $b \in \mathring{A}$ such that

$$\tilde{a}\tilde{b} = 1.$$

Then $1 - ab \in \check{A}$. It follows from Proposition 4.5 that ab is a unit in $\mathring{A}$ and hence a is a unit in $\mathring{A}$. $\square$

Definition 4.9. Let $(A, \|\bullet\|)$ be a Banach ring. We define the *spectral radius* $\rho = \rho_A : A \to [0, \infty)$ as follows:

$$\rho(f) = \inf_{n \geq 1} \|f^n\|^{1/n}, \quad f \in A.$$

Lemma 4.10. Let $(A, \|\bullet\|)$ be a Banach ring. Then for any $f \in A$, we have

$$\rho(f) = \lim_{n \to \infty} \|f^n\|^{1/n}.$$

PROOF. This follows from the multiplicative version of Fekete's lemma. $\square$

Example 4.11. The ring $\mathbb{C}$ with its usual norm $|\bullet|$ is a Banach ring. In fact, $(\mathbb{C}, |\bullet|)$ is a complete valued field.

Example 4.12. Let $\{(A_i, \|\bullet\|_i\}_{i \in I}$ be a family of Banach rings. We define their *product* $\prod_{i \in I} A_i$ as the following Banach ring: as a set it consists of all elements $f = (f_i)_{i \in I}$ with

$$\|f\| := \sup_{i \in I} \|f_i\|_i < \infty.$$

The norm is given by $\|\bullet\|$. It is easy to verify that $\prod_{i \in I} A_i$ is indeed a Banach ring.

Example 4.13. For any Banach ring $(A, \|\bullet\|)$, any $n \in \mathbb{N}$ and any $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$, we define $A\langle r^{-1}z\rangle = A\langle r_1^{-1}z_1, \ldots, r_n^{-1}z_n\rangle$ as the subring of $A[[z_1, \ldots, z_n]]$ consisting of formal power series

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in A$$

such that

$$\|f\|_r := \sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha < \infty.$$

We will verify in Proposition 4.14 that $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

When $r = (1, \ldots, 1)$, we omit r^{-1} from our notations.

Proposition 4.14. In the setting of Example 4.13, $(A\langle r^{-1}z\rangle, \|\bullet\|_r)$ is a Banach ring.

PROOF. By induction, we may assume that $n = 1$.

It is obvious that $\|\bullet\|_r$ is a norm on the undelrying Abelian group. To see that $\|\bullet\|_r$ is a norm on the ring $A\langle r^{-1}z\rangle$, we need to verify the condition in Definition 3.1. Condition (3) in Definition 3.1 is obvious. Let us consider Condition (4). Let

$$f = \sum_{i=0}^{\infty} a_i z^i, \quad g = \sum_{j=0}^{\infty} b_j z^j$$

be two elements in $A\langle r^{-1}z\rangle$. Then

$$fg=\sum_{k=0}^{\infty}\left(\sum_{i+j=k}a_ib_j\right)z^k.$$

We compute

$$\|fg\|_r=\sum_{k=0}^{\infty}\left\|\sum_{i+j=k}a_ib_j\right\|r^k\leq\sum_{k=0}^{\infty}\left(\sum_{i+j=k}\|a_i\|\cdot\|b_j\|\right)r^k=\|f\|_r\cdot\|g\|_r.$$

It remains to verify that $A\langle r^{-1}z\rangle$ is complete.

For this purpose, take a Cauchy sequence

$$f^b=\sum_{i=0}^{\infty}a_i^bz^i\in A\langle r^{-1}z\rangle$$

for $b\in\mathbb{N}$. Then for each i, the coefficients $(a_i^b)_b$ is a Cauchy sequence in A. Let a_i be the limit of a_i^b as $b\to\infty$ and set

$$f=\sum_{i=0}^{\infty}a_iz^i\in A[[z]].$$

We need to show that $f\in A\langle r^{-1}z\rangle$ and $f^b\to f$.

Fix a constant $\epsilon>0$. There is $m=m(\epsilon)>0$ such that for all $j\geq m$ and all $k\geq 0$, we have

$$\sum_{i=0}^{\infty}\|a_i^{j+k}-a_i^j\|r^i<\epsilon/2.$$

In particular, for any $s>0$, we have

$$\sum_{i=0}^{s}\|a_i-a_i^j\|r^i\leq\sum_{i=0}^{s}\|a_i-a_i^{j+k}\|r^i+\sum_{i=0}^{s}\|a_i^j-a_i^{j+k}\|r^i\leq\sum_{i=0}^{s}\|a_i-a_i^{j+k}\|r^i+\epsilon/2.$$

When k is large enough, we can guarantee that

$$\sum_{i=0}^{s}\|a_i-a_i^{j+k}\|r^i<\epsilon/2.$$

So

$$\sum_{i=0}^{s}\|a_i-a_i^j\|r^i\leq\epsilon.$$

Let $s\to\infty$, we find

$$\|f-f^j\|_r\leq\sum_{i=0}^{\infty}\|a_i-a_i^j\|r^i\leq\epsilon.$$

In particular, $\|f\|_r<\infty$ and $f^j\to f$ as $j\to\infty$. □

Example 4.15. For any non-Archimedean Banach ring $(A,\|\bullet\|)$, any $n\in\mathbb{N}$ and any $r=(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$, we define $A\{r^{-1}T\}=A\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}$ as the subring of $A[[T_1,\dots,T_n]]$ consisting of formal power series

$$f=\sum_{\alpha\in\mathbb{N}^n}a_\alpha T^\alpha,\quad a_\alpha\in A$$

such that $\|a_\alpha\| r^\alpha \to 0$ as $|\alpha| \to \infty$. We set

$$\|f\|_r := \max_{\alpha \in \mathbb{N}^n} \|a_\alpha\| r^\alpha.$$

We will verify in Proposition 4.16 that $(A\langle r^{-1}T\rangle, \|\bullet\|_r)$ is a Banach ring.

The semi-norm $\|\bullet\|_r$ is called the *Gauss norm*.

Proposition 4.16. In the setting of Example 4.15, $(A\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach ring.

Moreover, if the norm $\|\bullet\|$ on A is a valuation, so is $\|\bullet\|_r$.

The second part is usually known as the *Gauss lemma*.

Proof. By induction on n, we may assume that $n = 1$.

The proof of the fact that $\|\bullet\|_r$ is a norm is similar to that of Proposition 4.14. We leave the details to the readers.

Next we argue that $(A\{r^{-1}T\}, \|\bullet\|_r)$ is complete. Take a Cauchy sequence

$$f^b = \sum_{i=0}^{\infty} a_i^b T^i \in A\{r^{-1}T\}$$

for $b \in \mathbb{N}$. As

$$\|a_i^b - a_i^{b'}\| r^i \leq \|f^b - f^{b'}\|_r$$

for any $i, b, b' \geq 0$, it follows that for any $i \geq 0$, $\{a_i^b\}_b$ is a Cauchy sequence. Let $a_i \in A$ be its limit and set

$$f = \sum_{i=0}^{\infty} a_i T^i \in A[[T]].$$

We need to show that $f \in A\{r^{-1}T\}$ and $f^b \to f$.

Fix $\epsilon > 0$. We can find $m = m(\epsilon) > 0$ such that for all $j \geq m$ and all $k \geq 0$,

$$\|f^j - f^{j+k}\|_r \leq \epsilon.$$

It follows that $\|a_i^j - a_i^{j+k}\| r^i \leq \epsilon$ for all $i \geq 0$. Let $k \to \infty$, we find

$$\|a_i^j - a_i\| r^i \leq \epsilon$$

for all $i \geq 0$. Fix $j \geq 0$, take i large enough so that $|a_i^j| r^i < \epsilon$. Then $\|a_i\| r^i \leq \epsilon$. So we find $f \in A\{r^{-1}T\}$. On the other hand,

$$\|f - f^j\|_r = \max_i \|a_i^j - a_i\| r^i \leq \epsilon.$$

This proves that $f^j \to f$.

Now assume that $\|\bullet\|$ is a valuation, we verify that $\|\bullet\|_r$ is also a valuation. Again, we may assume that $n = 1$. Take two elements $f, g \in A\{r^{-1}T\}$:

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad g = \sum_{j=0}^{\infty} b_j T^j.$$

As we have already shown $|fg|_r \leq |f|_r |g|_r$, it suffices to check the reverse inequality. For this purpose, choose the minimal indices i, j so that

$$\|f\|_r = \|a_i\| r^i, \quad \|g\|_r = \|b_j\| r^j.$$

Write

$$fg = \sum_{k=0}^{\infty} \left(\sum_{p+q=k} a_p b_q \right) T^k.$$

Then we claim that

$$\left\| \sum_{p+q=k} a_p b_q \right\| r^k = \|f\|_r \|g\|_r$$

when $k = i + j$. This implies the desired inequality. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. To verify our claim, it suffices to observe that for $(p, q) \neq (i, j)$, $r + s = i + j$, say $p < i$ and $q > j$, we have

$$\|a_p b_q\| r^k = \|a_p\| r^p \cdot \|b_q\| r^q < \|a_i\| r^i \cdot \|b_j\| r^j.$$

So

$$\|a_p b_q\| < \|a_i b_j\|.$$

Since the valuation on A is non-Archimedean, it follows that

$$\| \sum_{p+q=k} a_p b_q \| = \|a_i b_j\|.$$

Our claim follows. $\square$

Proposition 4.17. Let A, B be a non-Archimedean Banach ring and $f : A \to B$ be a continuous homomorphism. Then for any $b \in \mathring{B}$, there is a unique continuous homomorphism $F : A\{T\} \to B$ extending f and sending T to b.

Proof. From the continuity and the fact that $A[T]$ is dense in $A\{T\}$, F is clearly unique. To prove the existence, we define F directly: consider $g = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}$, we define

$$F(g) := \sum_{i=0}^{\infty} f(a_i) f^i.$$

As $f_i \in \mathring{A}$ and $a_i \to 0$, the right-hand side is well-defined. It is straightforward to check that F is a continuous homomorphism. $\square$

Proposition 4.18. For any non-Archimedean Banach ring $(A, \| \bullet \|)$, we have

$$(A\{T\})^{\circ} = \mathring{A}\{T\}, \quad (A\{T\})^{\vee} = \check{A}\{T\}.$$

For the definitions of $\mathring{\bullet}$ and $\check{\bullet}$, we refer to Definition 3.4.

Proof. We first show that

$$\mathring{A}\{T\} \subseteq (A\{T\})^{\circ}.$$

Let $f \in \mathring{A}\{T\}$. We expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \mathring{A}.$$

Then for each $i, j \in \mathbb{N}$, $\|a_i T^i\|_1^j = \|a_i\|^j$. So for each $i \in \mathbb{N}$, $a_i T^i \in (A\{T\})^{\circ}$. By Proposition 3.5, it follows that $f \in (A\{T\})^{\circ}$.

Next we prove the reverse inclusion. Take $f \in (A\{T\})^\circ$, suppose by contrary that $f \notin \mathring{A}\{T\}$. Expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal $m \in \mathbb{N}$ so that $a_m \notin \mathring{A}$. Then $\sum_{i=0}^{m-1} a_i T^i \in \mathring{A}\{T\} \subseteq (A\{T\})^\circ$ by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})^\circ.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any $j \in \mathbb{N}$. It follows that $a_m \in \mathring{A}$, which is a contradiction.

Next we show that

$$\check{A}\{T\} \subseteq (A\{T\})\check{}.$$

Let $f \in \check{A}\{T\}$. We expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in \check{A}.$$

Then for each $i, j \in \mathbb{N}$, $\|a_i T^i\|_1^j = \|a_i\|^j$. So for each $i \in \mathbb{N}$, $a_i T^i \in (A\{T\})\check{}$. By Proposition 3.5, it follows that $f \in (A\{T\})\check{}$.

Conversely, take $f \in (A\{T\})\check{}$, suppose by contrary that $f \notin \check{A}\{T\}$. Expand f as

$$f = \sum_{i=0}^{\infty} a_i T^i, \quad a_i \in A.$$

We can take a minimal $m \in \mathbb{N}$ so that $a_m \notin \check{A}$. Then $\sum_{i=0}^{m-1} a_i T^i \in \check{A}\{T\} \subseteq (A\{T\})\check{}$ by what we have proved. It follows that

$$g := f - \sum_{i=0}^{m-1} a_i T^i = \sum_{i=m}^{\infty} a_i T^i \in (A\{T\})\check{}.$$

Then it follows that

$$\|g^j\| \geq \|a_m^j\|$$

for any $j \in \mathbb{N}$. It follows that $a_m \in \check{A}$, which is a contradiction. □

Corollary 4.19. For any non-Archimedean Banach ring $(A, \|\bullet\|)$, we have a canonical isomorphism

$$\widetilde{A\{T\}} \cong \tilde{A}[T].$$

The natural map $A\{T\}^\circ \to \widetilde{A\{T\}}$ corresponds to a homomorphism $\mathring{A}\{T\} \to \tilde{A}[T]$ extending the homomorphism $\mathring{A} \to \tilde{A}$ and sending T to T.

Proof. Let $f = \sum_{i=0}^{\infty} a_i T^i \in A\{T\}^\circ$. Then $a_i \in \mathring{A}$ by Proposition 4.18. But $\|a_i\| \to 0$ as $i \to \infty$, so $a_i \in \check{A}$ for almost all i. It follows that the image of f in $\widetilde{A\{T\}}$ is the same as the image of an element from $\mathring{A}[T]$. On the other hand, for each $f \in \tilde{A}[T]$, we can expand $f = a_N T^N + \cdots + a_1 T^1 + a_0$ with $a_N \in \tilde{A}$. Lift each a_i to $b_i \in \mathring{A}$. Then the image of $b_N T^N + \cdots + b_1 T^1 + b_0$ under the reduction corresponds to f. The assertions follow. □

Corollary 4.20. Let $(A, \|\bullet\|)$ be a non-Archimedean Banach ring. An element $f = \sum_{i=0}^{\infty} a_i T^i \in \mathring{A}\{T\}$ is a unit in $\mathring{A}\{T\}$ if and only if a_0 is a unit in $\mathring{A}$ and $a_i \in \check{A}$ for all $i > 0$.

Proof. By Proposition 4.16, we know that $A\{T\}$ is complete. By Lemma 4.8 and Proposition 4.18, f is a unit in $\mathring{A}\{T\}$ if and only if $\sum_{i=0}^{\infty} \tilde{a}_i T^i$ is a unit in $\tilde{A}[T]$. By Lemma 4.8 again, a_0 is a unit in A if and only if $\tilde{a_0}$ is a unit in $\tilde{A}$. So we are reduced to argue that units in $\tilde{A}[T]$ are exactly units in $\tilde{A}$. This follows from the general fact about units in polynomial rings over a reduced ring. □

The lemma needs to be places elsewhere.

Lemma 4.21. Let R be a commutative ring. A polynomial $a_0 + a_1 X + \cdots + a_n X^n \in R[X]$ is a unit if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotents.

5. Semi-normed modules

Definition 5.1. Let $(A, \|\bullet\|_A)$ be a normed ring. A *semi-normed A-module* (resp. *normed A-module*) is a pair $(M, \|\bullet\|_M)$ consisting of a A-module M and a semi-norm (resp. norm) on the underlying Abelian group of M such that there is a constant $C > 0$ such that

$$\|fm\|_M \leq C\|f\|_A\|m\|_M$$

for all $f \in A$ and $m \in M$. In case $\|\bullet\|_A$ is non-Archimedean, we require that $\|\bullet\|_M$ is also non-Archimedean.

We say the semi-normed A-module (resp. normed A-module) M is faithful if we can take $C = 1$.

When $\|\bullet\|_M$ is clear from the context, we say M is a semi-normed A-module (resp. normed A-module).

An A-module homomorphism $\varphi : M \to N$ between two semi-normed A-modules M and N is *bounded* if the homomorphism of the underlying semi-normed Abelian groups is bounded in the sense of Definition 2.5.

A *Banach A-module* is a normed A-module which is complete with respect to the metric Lemma 2.6.

We denote by $\mathcal{B}\mathrm{an}_A$ the category of Banach A-modules with bounded A-module homomorphisms as morphisms.

Definition 5.2. Let A be a Banach ring and $(M, \|\bullet\|_M), (N, \|bullet\|_N)$ be two Banach A-modules. Define their *direct sum* as the Banach A-module $(M \oplus N, \|\bullet\|_{M\oplus N})$, where for $m \in M, n \in N$, we set

$$\|(m,n)\|_{M\oplus N} := \max\{\|m\|_M, \|n\|_N\}.$$

This definition extends immediately to finite direct sums of Banach A-modules.

Definition 5.3. Let A be a Banach ring. A Banach A-module M is said to be *finite* if there is $n \in \mathbb{N}$ and an admissible epimorphism $A^n \to M$.

A morphism between finite A modules M and N is a morphism $M \to N$ in $\mathcal{B}\mathrm{an}_A$. We write $\mathcal{B}\mathrm{an}_A^f$ for the category of finite Banach A-modules.

Definition 5.4. Let A be a semi-normed ring and M be a semi-normed A-module. There is an obvious $\hat{A}$-module structure on the completion $\hat{M}$ of A defined in Definition 2.9. We call the resulting Banach module the *completion* of M.

Definition 5.5. Let A be a non-Archimedean semi-normed ring. Consider semi-normed A-modules $(M, \|\bullet\|_M)$ and $(N, \|\bullet\|_N)$. We define the *tensor product* of $(M, \|\bullet\|_M)$ and $(N, \|\bullet\|_N)$ as the semi-normed A-module $(M \otimes N, \|\bullet\|_{M\otimes N})$, where

$$\|x\|_{M\otimes N} = \inf \max_i (\|m_i\|_M \cdot \|n_i\|_N),$$

where the infimum is taken over all decompositions $x = \sum_i m_i \otimes n_i$.

Definition 5.6. Let A be a non-Archimedean Banach ring. Consider semi-normed A-modules M and M, we define the *complete tensor product* of M and N as the metric completion $M\hat{\otimes}_A N$ of the tensor product of M and N defined in Definition 5.5.

Theorem 5.7. Let $(A, \|\bullet\|_A)$ be a normed ring. Then $\mathcal{B}\mathrm{an}_A$ is a quasi-Abelian category.

Proof. We first observe that $\mathcal{B}\mathrm{an}_A$ is preadditive, as for any $M, N \in \mathcal{B}\mathrm{an}_A$, $\mathrm{Hom}_{\mathcal{B}\mathrm{an}_A}(M, N)$ can be given the group structure inherited from the Abelian group $\mathrm{Hom}_A(M, N)$. It is obvious that $\mathcal{B}\mathrm{an}_A$ is preadditive.

Next we show that finite biproducts exist in $\mathcal{B}\mathrm{an}_A$. Given $(M, \|\bullet\|_M), (N, \|\bullet\|_N) \in \mathcal{B}\mathrm{an}_A$, we set

$$(M, \|\bullet\|_M) \oplus (N, \|\bullet\|_N) := (M \oplus N, \|\bullet\|_{M\oplus N}), \tag{5.1}$$

where $\|(m, n)\|_{M\oplus N} := \|m\|_M + \|n\|_N$ for $m \in M$ and $n \in N$. It is easy to verify that this gives the biproduct in $\mathcal{B}\mathrm{an}_A$.

We have shown that $\mathcal{B}\mathrm{an}_A$ is an additive category.

Next given a morphism $\varphi : (M, \|\bullet\|_M) \to (N, \|\bullet\|_N)$ in $\mathcal{B}\mathrm{an}_A$, we construct its kernel $(\ker\varphi, \|\bullet\|_{\ker\varphi})$ as the kernel of the underlying homomorphism of A-modules of φ endowed with the subgroup semi-norm induced from $\|\bullet\|_M$ as in Definition 2.3. It is easy to verify that $(\ker\varphi, \|\bullet\|_{\ker\varphi})$ is the kernel of φ in $\mathcal{B}\mathrm{an}_A$.

We can similarly construct the cokernels. To be more precise, let $\varphi : (M, \|\bullet\|_M) \to (N, \|\bullet\|_N)$ be a morphism in $\mathcal{B}\mathrm{an}_A$, then the $\operatorname{coker}\varphi = \{N/\overline{\varphi(M)}\}$ with quotient norm.

We have shown that $\mathcal{B}\mathrm{an}_A$ is a pre-Abelian category.

Observe that given a morphism $\varphi : (M, \|\bullet\|_M) \to (N, \|\bullet\|_N)$ in $\mathcal{B}\mathrm{an}_A$, its image is given by $\mathrm{Im}\,\varphi = \overline{\varphi(M)}$ with the subspace norm induced from N; its coimage is $M/\ker f$ with the residue norm. The morphism φ is admissible if the natural map

$$M/\ker f \to \overline{\varphi(M)}$$

is an isomorphism in $\mathcal{B}\mathrm{an}_A$.

It remains to show that pull-backs preserve admissible epimorphisms and pushouts preserve admissible monomorphisms. We first handle the case of admissible epimorphisms. Consider a Cartesian square in $\mathcal{B}\mathrm{an}_A$:

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\ V \arrow[r, "g"] & W \end{tikzcd}$$

with g being an admissible epimorphism. We need to show that p is also an admissible epimorphism, namely $U \cong M/\ker p$.

We define $\alpha : U \oplus V \to W$, $\alpha = (f, -g)$, then there is a natural isomorphism $j : M \to \ker\alpha$. Let us write $i : \ker\alpha \to U \oplus V$ the natural morphism. Then

$$q = \pi_V \circ i \circ j, \quad p = \pi_U \circ i \circ j,$$

where $\pi_U : U \oplus V \to U, \pi_V : U \oplus V \to V$ are the natural morphisms. We may assume that $M = \ker \alpha$ and j is the identity. Then it is obvious that p is surjective on the underlying sets. In order to compute the quotient norm on $M/\ker p$, we need a more explicit description of $\ker p \subseteq \ker \alpha$. We know that

$$\ker \alpha = \{(u, v) \in U \oplus V : f(u) = g(v)\}$$

with the subspace norm induced from the product norm on $U \oplus V$ defined in (5.1). Then

$$\ker p = \{(u, v) \in U \oplus V : u = 0, g(v) = 0\}.$$

It follows that for $(u, v) \in \ker \alpha$,

$$\inf_{(u',v') \in \ker p} \|(u, v) + (u', v')\|_{U \oplus V} = \inf_{v' \in \ker g} (\|v + v'\|_V) + \|x\|_U,$$

where $\|\bullet\|_U$ and $\|\bullet\|_V$ denote the norms on U and V respectively. By our assumption that g is an admissible epimorphism, there is a constant $C > 0$ so that

$$\inf_{v' \in \ker g} (\|v + v'\|_V) \leq C\|g(v)\|_W$$

for any $v \in V$. As f is bounded, we can also find a constant $C' > 0$ so that for any $(u, v) \in \ker \alpha$,

$$\|g(v)\|_W = \|f(u)\|_W \leq C'\|u\|_U.$$

It follows that p is admissible epimorphism.

It remains to check that the pushforwards preserve admissible monomorphisms. Consider a co-Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q} \\ V & \xrightarrow{p} & M \end{array}$$

with g being an admissible monomorphism. We need to show that p is an adimissible monomorphism. This boils down to the following: p is injective with closed image and the norms on $p(V)$ obtained in the obvious ways are equivalent. As in the case of pull-backs, we may let $\alpha : W \to U \oplus V$ be the morphism $(g, -f)$ and assume that $M = \operatorname{coker} \alpha$. It is then easy to see that p is injective. The proof that the two norms on $p(V)$ are equivalent is parallel to the argument in the pull-back case, and we omit it.

It remains to verify that $p(V)$ is closed in W. Consider the admissibly coexact sequence in $\mathcal{B}\mathrm{an}_A$:

$$W \xrightarrow{\alpha} U \oplus V \xrightarrow{\pi} M \to 0.$$

It is also admissibly coexact in the category of semi-normed A-modules. Include details later. Let $x_n \in V$ be a sequence so that $p(x_n) \to y \in M$. We may write $y = \pi(u, v)$ for some $(u, v) \in U \oplus V$. Then

$$\pi(-u, x_n - v) \to 0$$

as $n \to \infty$. From the strict coexact sequence, we can find a sequence $w_n \in W$ so that

$$(-u - g(w_n), x_n - v + f(w_n)) \to 0$$

as $n \to \infty$. Then $g(w_n) \to -u$ in U and hence there is $w \in W$ so that $w_n \to w \in W$ and $g(w) = -u$. But then $x_n \to x$ and $p(x) = y$. $\square$

Definition 5.8. Let $(A, \|\bullet\|_A)$ be a normed ring. A *Banach A-algebra* is a pair $(B, \|\bullet\|_B)$ such that $(B, \|\bullet\|_B)$ is a Banach A-module and $(B, \|\bullet\|_B)$ is a Banach ring.

A morphism of Banach A-algebras is a bounded A-algebra homomorphism. The category of Banach A-algebras is denoted by $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_A$.

Definition 5.9. Let A be a normed ring. A Banach A-algebra B is said to be *finite* if B is finite as a Banach A-module. A morphism of finite Banach A-algebras is a morphism in $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_A$. The category of finite Banach A-algebras is denoted by $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A$.

6. Berkovich spectra

Definition 6.1. Let $(A, \|\bullet\|_A)$ be a Banach ring. A semi-norm $|\bullet|$ on A is *bounded* if there is a constant $C > 0$ such that for any $f \in A$, $|f| \leq C\|f\|_A$.

We write $\mathrm{Sp}\, A$ for the set of bounded semi-valuations on A. We call $\mathrm{Sp}\, A$ the *Berkovich spectrum* of A.

We endow $\mathrm{Sp}\, A$ with the weakest topology such that for each $f \in A$, the map $\mathrm{Sp}\, A \to \mathbb{R}_{\geq 0}$ sending $\|\bullet\|$ to $\|f\|$ is continuous.

It is sometimes preferable to denote an element $\|\bullet\|$ in $\mathrm{Sp}\, A$ by a single letter x. In this case, we write $|f(x)| = \|f\|$ for any $f \in A$.

Given a bounded homomorphism $\varphi : A \to B$ of Banach rings, we define $\mathrm{Sp}\,\varphi : \mathrm{Sp}\, B \to \mathrm{Sp}\, A$ as follows: given a bounded semi-valuation $\|\bullet\|$ on B, we define $\mathrm{Sp}\,\varphi(\|\bullet\|)$ as the bounded semi-valuation on A sending $f \in A$ to $\|\varphi(f)\|$.

Observe that there is a natural map of sets:

$$\mathrm{Sp}\, A \to \{\mathfrak{p} \in \mathrm{Spec}\, A : \mathfrak{p} \text{ is closed.}\} \tag{6.1}$$

sending each bounded semi-valuation to its kernel. The fiber over a closed ideal $\mathfrak{p} \in \mathrm{Spec}\, A$ is identified with the set of bounded valuations on $A/\mathfrak{p}$. Here boundedness is with respect to the residue norm.

Remark 6.2. In the literature, it is more common to denote $\mathrm{Sp}\, A$ by $\mathcal{M}(A)$.

Lemma 6.3. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then for any $x \in \mathrm{Sp}\, A$, we have

$$|f(x)| \leq \rho(f) \leq \|f\|_A.$$

Proof. Let $\|\bullet\|_x$ be the bounded semi-valuation corresponding to x. Then there is a constant $C > 0$ such that

$$\|\bullet\|_x \leq C\|\bullet\|_A.$$

It follows that for any $n \in \mathbb{N}$,

$$\|f\|_x^n = \|f^n\|_x \leq C\|f^n\|_A.$$

Taking n-th root and letting $n \to \infty$, we find

$$\|f\|_x \leq \rho(f).$$

The inequality $\rho(f) \leq \|f\|_A$ follows from the definition of ρ. □

Example 6.4. If $(k, |\bullet|)$ is a complete valuation field, then $\mathrm{Sp}\, k$ is a single point $|\bullet|$.

To see this, let $\|\bullet\| \in \mathrm{Sp}\, k$, then by Lemma 6.3,

$$\|f\| \leq |f|$$

for any $f \in k$. If $f \neq 0$, the same inequality applied to f^{-1} implies that $\|f\| = |f|$. When $f = 0$, the equality is trivial.

Example 6.5. Let $\{K_i\}_{i\in I}$ be a family of complete valuation fields. Recall that $\prod_{i\in I} K_i$ is defined in Example 4.12. Then $\mathrm{Sp}\prod_{i\in I} K_i$ is homeomorphic to the Stone–Čech compactification of the discrete set I.

To see this, we first identify the set of proper closed ideals in $\prod_{i\in I} K_i$ with the set of filters on I.

We first introduce a notation: for each $J \subseteq I$, we write $a_J \in \prod_{i\in I} K_i$ for the element

$$a_{J,i} = \begin{cases} 0, & \text{if } i \in J; \\ 1, & \text{if } i \notin J. \end{cases}$$

Givena proper closed ideal $\mathfrak{a} \subseteq \prod_{i\in I} K_i$, we define a filter $\Phi_{\mathfrak{a}} = \{J \subseteq I : a_J \in \mathfrak{a}\}$. Conversely, given a filter Φ on I, we denote by $\mathfrak{a}_\Phi$ the closed ideal of $\prod_{i\in I} K_i$ generated by a_J for all $J \in \Phi$. These maps are inverse to each other and order preserving. In particular, the maximal ideals of $\prod_{i\in I} K_i$ are identified with ultrafilters of I by Corollary 4.7.

Next we show that all prime ideals of $\prod_{i\in I} K_i$ are maximal. In fact, take $\mathfrak{p} \in \mathrm{Spec}\prod_{i\in I} K_i$ and suppose that there is a maximal ideal $\mathfrak{m}$ properly containing $\mathfrak{p}$. Let $J \in \Phi_{\mathfrak{m}} \setminus \Phi_{\mathfrak{p}}$ so that $a_J \in \mathfrak{m} \setminus \mathfrak{p}$. As $I \setminus J \notin \Phi_{\mathfrak{m}}$, we have $a_{I\setminus J} \notin \mathfrak{m}$. But $a_J \cdot a_{I\setminus J} = 0$. This contradicts the fact that $a_J \notin \mathfrak{p}$ and $a_{I\setminus J} \notin \mathfrak{p}$.

So we have shown that as a set $\mathrm{Spec}\prod_{i\in I} K_i$ is identified with the Stone–Čech compactification of I.

Next we show taht if $\mathfrak{m} \in \mathrm{Spec}\prod_{i\in I} K_i$, then the residue norm on $\prod_{i\in I} K_i/\mathfrak{m}$ is multiplicative. In fact, for each $f \in \prod_{i\in I} K_i$, we have

$$\|\pi(f)\|_{\prod_{i\in I} K_i/\mathfrak{m}} = \inf_{J\in\Phi_{\mathfrak{m}}} \sup_{i\in J} \|f\|.$$

Here $\pi : \prod_{i\in I} K_i \to \prod_{i\in I} K_i/\mathfrak{m}$ is the natural map and $\|\bullet\|$ denotes the norm on $\prod_{i\in I} K_i$ defined in Example 4.12. It follows immediately that the residue norm on $\prod_{i\in I} K_i/\mathfrak{m}$ is multiplicative. In particular, by Example 6.4, $\mathrm{Sp}\prod_{i\in I} K_i$ and $\mathrm{Spec}\prod_{i\in I} K_i$ are identified as sets under the natural map (6.1).

It remains to identify the topologies. But this is easy: for any ultrafilter Φ on I, let $\mathfrak{m} = \mathfrak{m}_\Phi$, then $\|\pi(a_J)\| = 0$ for $J \in \Phi$ and $\|\pi(a_J)\| = 1$ otherwise.

Proposition 6.6. Let $\varphi : A \to B$ be a bounded homomorphism of Banach rings, then $\mathrm{Sp}\,\varphi : \mathrm{Sp}\,B \to \mathrm{Sp}\,A$ is continuous.

Proof. For each $f \in A$, we define $\mathrm{ev}_f : \mathrm{Sp}\,A \to \mathbb{R}$ by sending $\|\bullet\|$ to $\|f\|$. It suffices to show that for any $f \in A$, the map $\mathrm{Sp}\,\varphi \circ \mathrm{ev}_f$ is continuous. But the composition is just the map sending $\|\bullet\| \in \mathrm{Sp}\,B$ to $\|\varphi(f)\|$. It is continuous by definition of the topology on $\mathrm{Sp}\,B$ as φ is bounded. □

Definition 6.7. Let $(A, \|\bullet\|_A)$ be a Banach ring. For each $x \in \mathrm{Sp}\,A$ corresponding to a bounded semi-valuation $\|\bullet\|_x$ on A, there is a natural induced valuation on $\mathrm{Frac}\ker\|\bullet\|_x$. We write $\mathscr{H}(x)$ for the completion of $\mathrm{Frac}\ker\|\bullet\|_x$ with the induced valuation. The complete valuation field $\mathscr{H}(x)$ is called the *complete residue field* of A at x.

We will write $f(x)$ for the residue class of f in $\mathscr{H}(x)$.

Observe that for any $f \in A$, $|f(x)|$ is exactly the valuation of $f(x)$ with respect to the valuation on $\mathscr{H}(x)$.

Definition 6.8. Let A be a Banach ring. The *Gelfand transform* of A is the homomorphism

$$A \to \prod_{x \in \operatorname{Sp} A} \mathscr{H}(x).$$

Here the product is defined in Example 4.12.

We will denote the Gelfand transform as $f \mapsto \hat{f} = (f(x))_{x \in \operatorname{Sp} A}$.

By Lemma 6.3, the Gelfand transform is well-defined.

Proposition 6.9. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then the Gelfand transform

$$A \to \prod_{x \in \operatorname{Sp} A} \mathscr{H}(x).$$

is bounded. In fact, the Gelfand transform is contractive.

PROOF. This follows simply from Lemma 6.3. □

Proposition 6.10. Let $(A, \|\bullet\|)$ be a Banach ring. Then $\operatorname{Sp} A$ is empty if and only if $A = 0$.

PROOF. If $A = 0$, $\operatorname{Sp} A$ is clearly empty. Conversely, suppose that $\operatorname{Sp} A$ is empty. Assume that $A \neq 0$. For any maximal ideal $\mathfrak{m}$, by Corollary 4.7, $A/\mathfrak{m}$ is a Banach ring and $\operatorname{Sp} A/\mathfrak{m}$ is a subset of $\operatorname{Sp} A$. So we may assume that A is a field. Let S be the set of bounded semi-norms on A. Then S is non-empty as $\|\bullet\| \in S$. By Zorn's lemma, we can take a minimal element $|\bullet| \in S$. Up to replacing A by the completion with respect to $|\bullet|$, we may assume that $|\bullet|$ is a norm on A. As A is a field, we may further assume that $|\bullet| = \|\bullet\|$.

We claim that $\|\bullet\|$ is multiplicative. As A is a field, it suffices to show that $\|f^{-1}\| = \|f\|^{-1}$ for any non-zero $f \in A$. We may assume that $\|f\|^{-1} < \|f^{-1}\|$.

Let r be a positive real number. Let $\varphi : A \to A\{r^{-1}T\}/(T - f)$ be the natural map. The map is injective as A is a field. We endow $A\{r^{-1}T\}/(T - f)$ with the quotient semi-norm induced by $\|\bullet\|_r$. We still denote this semi-norm by $\|\bullet\|_r$.

We claim that $f - T$ is not invertible in $A\{r^{-1}T\}$ for the choice $r = \|f^{-1}\|^{-1}$. From this, it follows that

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

The last step is our assumption. This contradicts our choice of $\|\bullet\|$.

In order to prove the claim, we need to show that $\|\bullet\|$ is power multiplicative first. Assuming this, it is obvious that

$$\sum_{i=0}^{\infty} |f^{-i}| r^i = \sum_{i=0}^{\infty} |f^{-1}|^i |f^{-1}|^{-i}$$

diverges.

It remains to show that $\|\bullet\|$ is power multiplicative. Suppose that is $f \in A$ so that $\|f^n\| < \|f\|^n$ for some $n > 1$. We claim that $f - T$ is not invertible in $A\{r^{-1}T\}$ for the choice $r = \|f^n\|^{1/n}$. From this,

$$\|\varphi(f)\|_r = \|T\|_r \leq r < \|f\|.$$

This contradicts our choice of $\|\bullet\|$. The claim amounts to the divergence of

$$\sum_{i=0}^{\infty} \|f^{-i}\| r^i.$$

For a general $i \geq 0$, we write $i = pn + q$ for $p, q \in \mathbb{N}$ and $q \leq n-1$. Then $\|f^i\| \leq \|f^n\|^p \|f^q\|$. So

$$\|f^{-i}\| r^i \geq \|f^i\|^{-1} \|f^n\|^{p+n^{-1}q} \geq \|f^n\|^{n^{-1}q} \|f^q\|^{-1}.$$

It therefore follows that $|f^{-i}| r^i$ admits a positive lower bound, and we conclude. $\square$

Corollary 6.11. Let A be a Banach ring. Then an element $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in \operatorname{Sp} A$.

Proof. The direct implication is trivial. Assume that $f(x) \neq 0$ for all $x \in \operatorname{Sp} A$. We claim that $f \notin \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ in A. From this, it follows that f is invertible in A.

By Corollary 4.7, $A/\mathfrak{m}$ is a Banach ring. It follows from Proposition 6.10 that there is a non-trival bounded semi-valuation on $A/\mathfrak{m}$, which lifts to a bounded semi-valuation on A. $\square$

Corollary 6.12. Let $(A, \|\bullet\|_A)$ be a Banach ring. Then for any $f \in A$, we have

$$\rho(f) = \sup_{x \in \operatorname{Sp} A} |f(x)|.$$

Proof. We have already shown $\rho(f) \geq \sup_{x \in \operatorname{Sp} A} |f(x)|$ in Lemma 6.3. To verify the reverse inequality, take $f \in A$ and $r \in \mathbb{R}_{>0}$, it suffices to show that if $|f(x)| < r$ for all $x \in \operatorname{Sp} A$, then $\rho(f) \leq r$.

Consider the Banach ring $B = A\{rT\}$. By Lemma 6.3 again, $|T(x)| \leq \|T\|_{r^{-1}} = r^{-1}$ for all $x \in \operatorname{Sp} B$. Therefore, for any $x \in \operatorname{Sp} B$, $|(fT)(x)| < 1$. Hence, $(1 - fT)(x) \neq 0$ for all $x \in \operatorname{Sp} B$. By Corollary 6.11, $1 - fT$ is invertible in B. But this happens exactly when

$$\sum_{i=0}^{\infty} \|f^i\|_A r^{-i}$$

is convergent. It follows that $\rho(f) \leq r$. $\square$

Theorem 6.13. Let $(A, \|\bullet\|)$ be a Banach ring. Then $\operatorname{Sp} A$ is a compact Hausdorff space.

Proof. We first show that $\operatorname{Sp} A$ is Hausdorff. Take $x_1, x_2 \in A$, $x_1 \neq x_2$. In other words, we can find $f \in A$ so that $|f(x_1)| \neq |f(x_2)|$. We may assume that $|f(x_1)| < |f(x_2)|$. Take a real number $r > 0$ so that

$$|f(x_1)| < r < |f(x_2)|.$$

Then $\{x \in \operatorname{Sp} A : |f(x)| < r\}$ and $\{x \in \operatorname{Sp} A : |f(x)| > r\}$ are disjoint neighbourhoods of x_1 and x_2.

Next we show that $\operatorname{Sp} A$ is compact. By Proposition 6.9 and Proposition 6.6, we can define a continuous map

$$\operatorname{Sp} \prod_{x \in \operatorname{Sp} A} \mathscr{H}(x) \to \operatorname{Sp} A.$$

The map is clearly surjective: for any $x \in \operatorname{Sp} A$, the valuation on $\mathscr{H}(x)$ induces a semi-valuation on $\prod_{x \in \operatorname{Sp} A} \mathscr{H}(x)$, which is clearly bounded. The image of this semi-valuation in $\operatorname{Sp} A$ is just x.

So it suffices to show that $\operatorname{Sp} \prod_{x \in \operatorname{Sp} A} \mathscr{H}(x)$ is compact. This follows from Example 6.5. $\square$

7. Open mapping theorem

Let $(k, |\bullet|)$ be a complete non-trivially valued field. All results in this section fail when k is trivially valued.

Proposition 7.1. Let A be a normed k-algebra and $f : (M, \|\bullet\|_M) \to (N, \|\bullet\|_N)$ be an A-homomorphism of normed A-modules. Then f is bounded if and only if f is continuous.

Proof. The direct implication follows from Proposition 2.7. Assume that f is continuous. We may assume that $A = k$.

Assume that f is not bounded. Fix $a \in k$ with $|a| \in (0, 1)$. This is possible as k is non-trivially valued. Then we can find a sequence $m_i \in M$ such that $\|f(m_i)\|_N > |a|^{-i}\|m_i\|_M$. Up to replace m_i by a scalar multiple, we may assume that $\|m_i\|_M \in [1, |a|^{-1})$: if $\|m_i\|_M \geq 1$, choose $n \in \mathbb{N}$ such that $|a|^{-n} \leq \|m_i\|_M < |a|^{-n-1}$, then replace m_i with $a^n m_i$. The case $|x| < 1$ is similar. Then $\|f(a^i m_i)\|_N > \|m_i\|_M \geq 1$ while $\|a^i m_i\|_M < |a|^n |a|^{-1} \to 0$. This is a contradiction. □

Theorem 7.2 (Open mapping theorem)**.** Let $(V, \|\bullet\|_V), (W, \|\bullet\|_W)$ be k-Banach spaces and $L : V \to W$ be a bounded and surjective k-homomorphism. Then L is open.

Proof. We write $V_0 = \{v \in V : \|v\|_V < 1\}$. Similarly define W_0.

Step 1. We claim that there is a constant $C > 0$ such that for all $w' \in W$, there is $v' \in V$ such that

$$\|v'\|_V \leq C\|w'\|_W, \quad \|w' - L(v')\|_W < 1/2.$$

As k is non-trivially valued, we can take $c \in k$ with $|c| \in (0, 1)$, so

$$V = \bigcup_{n\in\mathbb{N}} c^n V_0.$$

As L is surjective, we have

$$W = \bigcup_{n\in\mathbb{N}} c^n L(V_0).$$

By Baire's category theorem, we may assume that $\overline{L(V_0)}$ has non-empty interior. Take $w \in W$ and $r > 0$ so that

$$\{w' \in W : \|w - w'\|_W < r\} \subseteq \overline{L(V_0)}.$$

Take $d \in W_0$ and $c' \in k^\times$ so that $|c'| < r$, then $w + c'd \in \overline{L(V_0)}$. It follows that

$$c'd \in \overline{L(V_0)} + \overline{L(V_0)} \subseteq \overline{L(V_0) + L(V_0)} = \overline{L(V_0)}.$$

So

$$W_0 \subseteq \overline{L(c'^{-1}V_0)}.$$

It suffices to take $C = |c'^{-1}|$.

Step 2. Now given $w \in W_0$, we want to show that $w \in L(\{v \in V : \|v\|_V < C\})$. This will finish the argument: as k is non-trivially valued, this implies that $L(V_0)$ contains an open neighbourhood of 0.

From Step 1, we can construct $v_1 \in V$ with $\|v_1\|_V < C$ and $\|w - L(v_1)\|_W < 1/2$. Repeat this process, we can $v_n \in V$ inductively so that

$$\|v_n\|_V < 2^{1-n}C, \quad \|w - L(v_1 + \cdots + v_n)\|_W < 2^{-n}.$$

We set $v = \sum_{i=1}^{\infty} v_i$. Then $v \in V$ and $Av = w$ by continuity. Moreover,

$$\|v\|_V \leq \max_n \|v_n\|_V < C.$$

□

Corollary 7.3. Let A be a k-Banach algebra and M be a normed A-module. Assume that $\hat{M}$ is a finite A-module, then M is complete.

PROOF. Take $x_1, \dots, x_n \in \hat{M}$ so that $\pi : A^n \to \hat{M}$ sending $(a_1, \dots, a_n)$ to $\sum_{i=1}^n a_i x_i$ is surjective. By open mapping theorem Theorem 7.2, $\sum_{i=1}^n \check{A} x_i$ is a neighbourhood of 0 in $\hat{M}$. So

$$x_j \in M + \sum_{i=1}^{n} \check{A} x_i.$$

It follows from (a version of) Nakayama's lemma that $M = \hat{M}$. □

Corollary 7.4. Let A be a k-Banach algebra and M be a Noetherian Banach A-module. Let N be a submodule of M. Then N is closed in M.

In particular, if A is Noetherian, then all ideals of A are closed.

PROOF. As M is noetherian, $\bar{N}$ is a finite A-module. In particular, N is complete by Corollary 7.3. Hence, N is closed in M. □

Corollary 7.5. A bounded epimorphism of k-Banach algebras $f : A \to B$ is admissible.

PROOF. Replacing A by $A/\ker f$, we may assume that f is bijective. It follows from Theorem 7.2 that f is a homeomorphism. The inverse of f is therefore continuous, and hence bounded by Proposition 7.1. □

8. Maximum spectra

Let $(k, |\bullet|)$ a complete non-Archimedean valued field.

Definition 8.1. For any k-algebra A, we write

$$\operatorname{Spm}_k A := \{\mathfrak{m} \in \operatorname{Spm} A : A/\mathfrak{m} \text{ is algebraic over } k\}.$$

For any $x \in \operatorname{Spm}_k A$ and any $f \in A$, we write $f(x)$ for the residue of f in $A/\mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal corresponding to x. We write $|f(x)|$ for the valuation of $f(x)$ with respect to the extended valuation induced from the given valuation on k.

Definition 8.2. Let A be a k-algebra. For each $f \in A$, we write $|f|_{\sup}$ for the supremum of $|f(x)|$ for all $x \in \operatorname{Spm}_k A$ if $\operatorname{Spm}_k A$ is non-empty and 0 otherwise.

Definition 8.3. Let f be a monic polynomial in $k[X]$, we expand $f = X^n + a_1 X^{n-1} + \cdots + a_n \in k[X]$, then we define $\sigma(f) := \max_{i=1,\dots,n} |a_i|^{1/i}$.

Definition 8.4. Let L be a reduced integral k-algebra. We define the *spectral norm* $|\bullet|_{\mathrm{sp}}$ on L as follows: given a non-zero $x \in L$, take a minimal polynomial $X^n + a_1 X^{n-1} + \cdots + a_n \in k[X]$ of x over k. Then we set

$$|x|_{\mathrm{sp}} := \max_{i=1,\dots,n} |a_i|^{1/i}.$$

Proposition 8.5. Let f, g be monic polynomials in $k[X]$, then

$$\sigma(fg) = \max\{\sigma(f), \sigma(g)\}.$$

PROOF. Replacing k by a finite extension, we may assume that f and g split into linear factors a_i and b_j. Then it is straightforward to show that

$$\sigma(f) = \prod_i a_i, \quad \sigma(g) = \prod_j b_j, \sigma(fg) = \prod_i a_i \cdot \prod_j b_j.$$

The assertion follows. □

Proposition 8.6. Let L be a reduced integral k-algebra. Then $|\bullet|_{\mathrm{sp}}$ is a power-multiplicative norm on L, and it extends the norm on k.

PROOF. It is clear that $|\bullet|_{\mathrm{sp}}$ extends the valuation on k. In order to show that $|\bullet|_{\mathrm{sp}}$ is a power-multiplicative norm on L, we may assume that L is finite dimensional over k. Then we can find finite field extensions $L_1, \dots, L_t$ of k such that $L = \bigoplus_{i=1}^t L_i$. By Proposition 8.5, we can immediately reduce to the case where L/k is a finite field extension. In this case, the result is well-known. Expand. □

Proposition 8.7. Let L be a reduced integral k-algebra. For any $\mathfrak{p} \in \operatorname{Spec} L$, write $\pi_{\mathfrak{p}} : L \to L/\mathfrak{p}$ the residue map. Then for any $y \in L$,

$$|y|_{\mathrm{sp}} = \max_{\mathfrak{p} \in \operatorname{Spec} L} |\pi_{\mathfrak{p}}(y)|_{\mathrm{sp}}.$$

PROOF. Fix $y \in L$. For any $\mathfrak{p} \in \operatorname{Spec} L$, let $q_{\mathfrak{p}} \in k[X]$ be the minimal polynomial of $\pi_{\mathfrak{p}}(y)$ over k. Let $q \in k[X]$ be the minimal polynomial of y over k. Then clearly $q_{\mathfrak{p}}$ divides q for all $\mathfrak{p} \in \operatorname{Spec} L$. In particular, there are only finitely many different polynomials among $q_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} L$), say $q_1, \dots, q_r$. Define $q' = q_1 \cdots q_r \in k[X]$. Then for $f \in k[X]$, $f(y) = 0$ if and only if $\pi_{\mathfrak{p}}(f(y)) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} L$ as L is reduced. The latter condition is equivalent to that $q'|f$. It follows that $q' = q$. Now by Proposition 8.5,

$$|y|_{\mathrm{sp}} = \sigma(q) = \max_{i=1,\dots,r} \sigma(q_i) = \max_{\mathfrak{p} \in \operatorname{Spec} L} |\pi_{\mathfrak{p}}(y)|_{\mathrm{sp}}.$$

□

Proposition 8.8. Let $\varphi : B \to A$ be a homomorphism of commutative k-algebras. Then for any $f \in B$,

$$|\varphi(f)|_{\sup} \leq |f|_{\sup}.$$

PROOF. Of course, we can assume that $\operatorname{Spm}_k A \neq \emptyset$. Let $x \in \operatorname{Spm}_k A$, then $\varphi^{-1}x \in \operatorname{Spm}_k B$. But for any $f \in B$, $|\varphi(f)(x)| = |f(\varphi^{-1}x)|$. We conclude. □

Proposition 8.9. Let A be a k-algebra. Let $\mathfrak{M}$ be the set of minimal prime ideals in A and let $\pi_{\mathfrak{p}} : A \to A/\mathfrak{p}$ be the canonical residue map for all $\mathfrak{p} \in \mathfrak{M}$. Then for any $f \in A$,

$$|f|_{\sup} = \sup_{\mathfrak{p} \in \mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup}. \tag{8.1}$$

In particular, if A be a reduced integral k-algebra. Then $|\bullet|_{\sup} = |\bullet|_{\mathrm{sp}}$ on A.

PROOF. By Proposition 8.8,

$$\sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup} \leq |f|_{\sup}.$$

In order to show the reverse inequality, let $x \in \operatorname{Spm}_k A$. Take $\mathfrak{p} \in \mathfrak{M}$ such that $x \supseteq \mathfrak{p}$. Clearly, $\pi_{\mathfrak{p}}(x) \in \operatorname{Spm}_k A/\mathfrak{p}$ and

$$|f(x)| = |\pi_{\mathfrak{p}}(f)(\pi_{\mathfrak{p}}(x))|\,.$$

In particular,

$$|f(x)| \leq |\pi_{\mathfrak{p}}(f)|_{\sup} \leq \sup_{\mathfrak{p}\in\mathfrak{M}} |\pi_{\mathfrak{p}}(f)|_{\sup}.$$

Take sup with respect to x, we conclude (8.1).

When A is a reduced and integral k-algebra, all prime ideals of A are minimal. The final assertion follows from Proposition 8.7. $\square$

Definition 8.10. Let A be a k-Banach algebra. We say that *maximal modulus principle* holds for A if for any $f \in A$, there is $x \in \operatorname{Spm}_k A$ such that $|f(x)| = |f|_{\sup}$.

Proposition 8.11. Let $\varphi : B \to A$ be an injective integral homomorphism of Banach k-algebras. Assume that B is a normal integral domain.

(1) Fix $f \in A$. Let $f^n + \varphi(b_1)f^{n-1} + \cdots + \varphi(b_n) = 0$ be the minimal equation of f over A. Then

$$|f|_{\sup} = \max_{i=1,\dots,n} |b_i|_{\sup}^{1/i}.$$

(2) Assume that maximal modulus principle holds for B, then it holds for A as well.

PROOF. (1) We first show the inequality

$$|f|_{\sup} \leq \max_{i=1,\dots,n} |b_i|_{\sup}^{1/i}.$$

Of course, we can assume that $\operatorname{Spm}_k A \neq \emptyset$. For all $x \in \operatorname{Spm}_k A$, we have

$$0 = f(x)^n + \varphi(b_1)f(x)^{n-1} + \cdots + \varphi(b_n) = f(x)^n + b_1(\varphi^{-1}x)f(x)^{n-1} + \cdots + b_n(\varphi^{-1}(x)).$$

Then we in fact have that

$$|f(x)| \leq \max_{i=1,\dots,n} |b_i(\varphi^{-1}x)|_{\sup}^{1/i}.$$

Assume that to the contrary that

$$|f(x)|^i > |b_i(\varphi^{-1}x)|$$

for all $i = 1, \dots, n$. Then

$$|b_i(\varphi^{-1}x)f(x)^{n-i}| < |f(x)|^n = |f(x)^n|.$$

It follows that

$$|b_1(\varphi^{-1}x)f(x)^{n-1} + \cdots + b_n(\varphi^{-1}(x))| < |f(x)^n|.$$

This is a contradiction.

It remains to argue that

$$|f|_{\sup} \geq \max_{i=1,\dots,n} |b_i|_{\sup}^{1/i}. \tag{8.2}$$

Next let $A' = B[f]$. We argue that $A' \to A$ is an isometry with respect to $|\bullet|_{\sup}$. If $\operatorname{Spm}_k A'$ is empty, then the assertion follows from Proposition 8.8. Assume that $\operatorname{Spm}_m A'$ is non-empty. Take $y \in \operatorname{Spm}_k A'$. By [Stacks, Tag 00GQ], there is a maximal ideal $x \in \operatorname{Spm} A$ lying over y. As the induced map $A'/y \to A/x$ is integral, we find $x \in \operatorname{Spm}_k A$. So the map $\operatorname{Spm}_k A \to \operatorname{Spm}_k A'$ is surjective. If follows that $A' \to A$ is an isometry with respect to $|\bullet|_{\sup}$.

In order to argue (8.2), we may assume that $A = B[f]$. Let $q \in B[X]$ denote the minimal polynomial of f over A. Then $A = B[X]/(q)$. Let $y \in \operatorname{Spm}_k B$, we write f_y for the residue class of f in A/yA and write $\bar{f}_y$ for the residue class in $(A/yA)^{\mathrm{red}}$. Similarly, let q_y denote the residue class of q in $B/y[X]$. As y is contained in some $\operatorname{Spm}_k A$, we see that

$$|f|_{\sup} = \sup_{y \in \operatorname{Spm}_k B} |f_y|_{\sup} = \sup_{y \in \operatorname{Spm}_k B} |\bar{f}_y|_{\sup}.$$

For $y \in \operatorname{Spm}_k B$, we decompose q_y into prime factors $q_1^{n_1} \cdots q_r^{n_r}$ in $B/y[X]$. Then

$$A/yA \cong B/y[X]/(q_y)$$

and

$$(A/yA)^{\mathrm{red}} \cong \bigoplus_{i=1}^{r} B/y[X]/(q_i).$$

We endow $\bigoplus_{i=1}^{r} B/y[X]/(q_i)$ with the spectral norm over B/y. If $\bar{f}_i$ denotes the residue class of $\bar{f}_y$ in $B/y[X]/(q_i)$, by Proposition 8.9 and Proposition 8.5,

$$|\bar{f}_y|_{\sup} = \max_{i=1,\dots,r} |\bar{f}_i|_{\mathrm{sp}} = \max_{i=1,\dots,r} \sigma(q_i) = \sigma(q_y).$$

Therefore,

$$|f|_{\sup} = \sup_{y \in \operatorname{Spm}_k B} \sigma(q_y) = \max_{i=1,\dots,n} |b_i|_{\sup}^{1/n}.$$

(2) Take a non-zero $f \in A$. Using the notations in (1), we can find $y \in \operatorname{Spm}_k B$ such that

$$|\bar{f}_y|_{\sup} = \sigma(q_y) = |f|_{\sup}.$$

As A/yA contains only finitely many maximal ideals, there is $x \in \operatorname{Spm}_k A$ such that $|\bar{f}_y|_{\sup} = |f(x)|$. So

$$|f|_{\sup} = |f(x)|.$$

□

9. Bornology

This section may be placed elsewhere.

Definition 9.1. Let X be a set. A *bornology* on X is a collection $\mathcal{B}$ of subsets of X such that

(1) For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(2) For any $B \in \mathcal{B}$ and any subset $A \subseteq B$, $A \in \mathcal{B}$;
(3) $\mathcal{B}$ is stable under finite union.

The pair $(X, \mathcal{B})$ is called a *bornological set*. The elements of $\mathcal{B}$ are called the *bounded subsets* of $(X, \mathcal{B})$. When $\mathcal{B}$ is obvious from the context, we omit it from the notations.

A morphism between bornological sets $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ is a map of sets $f : X \to Y$ such that for any $A \in \mathcal{B}_X$, $f(A) \in \mathcal{B}_Y$. Such a map is called a *bounded map*.

Definition 9.2. Let $(X, \mathcal{B})$ be a bornological set. A *basis* for $\mathcal{B}$ is a subset $\mathcal{A} \subseteq \mathcal{B}$ such that for any $B \in \mathcal{B}$, there are $A_1, \dots, A_n \in \mathcal{A}$ such that $B \subseteq A_1 \cup \dots \cup A_n$.

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