

Affinoid algebras

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1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. We set

$$\begin{aligned} k\{r^{-1}T\} &= k\{r_1^{-1}T_1, \dots, r_n T_n^{-1}\} \\ &:= \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \right\}. \end{aligned}$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$\|f\|_r = \max_{\alpha} |a_\alpha| r^\alpha.$$

We call $(k\{r^{-1}T\}, \|\bullet\|_r)$ the *Tate algebra* in n -variables with radii r . The norm $\|\bullet\|_r$ is called the *Gauss norm*.

We omit r from the notation if $r = (1, \dots, 1)$.

This is a special case of [Example 4.15](#) in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. Then the Tate algebra $(k\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach k -algebra and $\|\bullet\|_r$ is a valuation.

PROOF. This is a special case of [Proposition 4.16](#) in the chapter Banach Rings. \square

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T \rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T \rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}^n$. Then $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$ otherwise.

Lemma 2.5. Let A be a Banach k -algebra. For each $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathring{A}$, there is a unique continuous homomorphism $k\{T_1, \dots, T_n\} \rightarrow A$ sending T_i to a_i .

PROOF. This is a special case of [Proposition 4.18](#) in the chapter Banach Rings. \square

3. Affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. A Banach k -algebra A is *k -affinoid* (resp. *strictly k -affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism $k\{r^{-1}T\} \rightarrow A$ (resp. an admissible epimorphism $k\{T\} \rightarrow A$).

More generally, a Banach k -algebra A is *k_H -affinoid* if there are $n \in \mathbb{N}$, $r \in H^n$ and an admissible epimorphism $k\{r^{-1}T\} \rightarrow A$.

A morphism between k -affinoid (resp. strictly k -affinoid, resp. k_H -affinoid) algebras is a bounded k -algebra homomorphism.

The category of k -affinoid (resp. strictly k -affinoid, resp. k_H -affinoid) algebras is denoted by $k\text{-AffAlg}$ (resp. $\text{st-}k\text{-AffAlg}$, resp. $k_H\text{-AffAlg}$).

For the notion of admissible morphisms, we refer to [Definition 2.5](#) in the chapter Banach rings.

Although we have defined strictly k -affinoid algebra when k is trivially valued, we will deliberately avoid talking about it in order to avoid pathologies.

Remark 3.2. Berkovich also introduced the notion of *affinoid k -algebras*: it is a K -affinoid algebra for some complete non-Archimedean field extension K/k . We will not use this notion.

Definition 3.3. The category of *k -affinoid spectra* $k\text{-Aff}$ (resp. *strictly k -affinoid spectra* $\text{st-}k\text{-Aff}$, resp. *k_H -affinoid spectra* $k_H\text{-Aff}$) is the opposite category of $k\text{-AffAlg}$ (resp. $\text{st-}k\text{-AffAlg}$, resp. $k_H\text{-AffAlg}$). An object in these categories are called a *k -affinoid spectrum*, *strictly k -affinoid spectrum* and *k_H -affinoid spectrum* respectively.

Given an object A of $k\text{-AffAlg}$ (resp. $\text{st-}k\text{-AffAlg}$, resp. $k_H\text{-AffAlg}$), we denote the corresponding object in $k\text{-Aff}$ (resp. $\text{st-}k\text{-Aff}$, resp. $k_H\text{-Aff}$) by $\text{Sp } A$. We call $\text{Sp } A$ the *affinoid spectrum* of A .

In [Definition 6.1](#) in the chapter Banach Rings, we defined functors $\text{Sp} : k\text{-Aff} \rightarrow \text{Top}$, $\text{Sp} : \text{st-}k\text{-Aff} \rightarrow \text{Top}$ and $\text{Sp} : k_H\text{-Aff} \rightarrow \text{Top}$. This motivates our notation. We will freely view $\text{Sp } A$ as an object in these categories or as a topological space.

Proposition 3.4. Finite limits exist in $k_H\text{-Aff}$.

PROOF. It suffices to prove that finite fibered products exist.

We prove the equivalent statement, finite fibered coproducts exist in $k_H\text{-AffAlg}$. Given k_H -affinoid algebras A, B, C and morphisms $A \rightarrow B$, $A \rightarrow C$, we claim that $B \hat{\otimes}_A C$ represents the fibered coproduct of B and C over A . By general abstract nonsense, we are reduced to handle the following cases: $A = k$ and $A \rightarrow C$ is the codiagonal $C \hat{\otimes}_k C \rightarrow C$. In both cases, the proposition is clear. \square

Example 3.5. Let $r \in \mathbb{R}_{>0}$. We let k_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \rightarrow 0$ for $i \rightarrow \infty$ and $i \rightarrow -\infty$. We define a norm $\|\bullet\|_r$ on k_r as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in [Proposition 3.6](#) that k_r is k -affinoid.

Proposition 3.6. Let $r \in \mathbb{R}_{>0}$, then $(k_r, \|\bullet\|_r)$ defined in [Example 3.5](#) is a k -affinoid algebra. Moreover, $\|\bullet\|_r$ is a valuation.

PROOF. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \rightarrow k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$(3.1) \quad |a_{i,j}|r^{i-j} \rightarrow 0$$

as $i + j \rightarrow \infty$. Observe that for each $k \in \mathbb{Z}$, the series

$$c_k := \sum_{i-j=k, i,j \in \mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that $\iota(f) \in k_r$. That is

$$|c_k|r^k \rightarrow 0$$

as $k \rightarrow \pm\infty$. When $k \geq 0$, we have $|c_k| \leq |a_{k,0}|$ by definition of c_k . So $|c_k|r^k \rightarrow 0$ as $k \rightarrow \infty$ by (3.1). The case $k \rightarrow -\infty$ is similar.

We conclude that we have a well-defined map of sets ι . It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g = \sum_{i=-\infty}^{\infty} c_i T^i \in k_r$. We want to show that g lies in the image of ι . As ι is a ring homomorphism, it suffices to treat two cases separately: $g = \sum_{i=0}^{\infty} c_i T^i$ and $g = \sum_{i=-\infty}^0 c_i T^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$. It is immediate that $\iota(f) = g$.

Next we show that ι is admissible. We first identify the kernel of ι . We claim that the kernel is the ideal I generated by $T_1 T_2 - 1$. It is obvious that $I \subseteq \ker \iota$. Conversely, consider an element

$$f = \sum_{(i,j) \in \mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kernel of ι . Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If $f \in \ker \iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by $T_1 T_2 - 1$. The condition that $f_k \in \ker \iota$ means

$$\sum_{(i,j) \in \mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find $b_{i,j} \in k$ for $i, j \in \mathbb{N}$, $i - j = k$ such that

$$a_{i,j} = b_{i-1,j-1} - b_{i,j}.$$

Then

$$f_k = (T_1 T_2 - 1) \sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$. In particular, the sum of $\sum_{i,j \in \mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j$ for various k converges to some $g \in k\{r^{-1}T_1, rT_2\}$ and hence $f_k = (T_1 T_2 - 1)g$. Therefore, we have proved that $\ker \iota$ is generated by $T_1 T_2 - 1$.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \rightarrow k_r.$$

To see this, take $f = \sum_{k=-\infty}^{\infty} c_k T^k \in k_r$ and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$, then $\iota(g) = f$ and $\|g\|_{(r,r^{-1})} = \|f\|_r$. So it suffices to show that for any $h = \sum_{(i,j) \in \mathbb{N}^2} d_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$, we have

$$(3.2) \quad \|f\|_r \leq \|g + h(T_1 T_2 - 1)\|_{r,r^{-1}}.$$

We compute

$$g + h(T_1 T_2 - 1) = \sum_{k=1}^{\infty} (c_k - d_{k,0}) T_1^k + \sum_{k=1}^{\infty} (c_{-k} - d_{0,k}) T_2^k + (c_0 - d_0) + \sum_{i,j \geq 1} (d_{i-1,j-1} - d_{i,j}) T_1^i T_2^j.$$

So

$$\|g + h(T_1 T_2 - 1)\|_{r,r^{-1}} = \max \left\{ \max_{k \geq 0} C_{1,k}, \max_{k \geq 1} C_{2,k} \right\},$$

where

$$C_{1,k} = \max \left\{ |c_k - d_{k,0}|, \left| \sum_{i-j=k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for $k \geq 0$ and

$$C_{2,k} = \max \left\{ |c_{-k} - d_{0,k}|, \left| \sum_{i-j=-k, i,j \geq 1} d_{i-1,j-1} - d_{i,j} \right| \right\}$$

for $k \geq 1$. It follows from the strong triangle inequality that $|c_k| \leq C_{1,k}$ for $k \geq 0$ and $c_{-k} \leq C_{2,k}$ for $k \geq 1$. So (3.2) follows. \square

Proposition 3.7. Let $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$, then $\|\bullet\|_r$ defined in Example 3.5 is a valuation on k_r .

PROOF. Take $f, g \in k_r$, we need to show that

$$\|fg\|_r \geq \|f\|_r \|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_i T^i, \quad g = \sum_{i=-\infty}^{\infty} b_i T^i.$$

Take i and j so that

$$(3.3) \quad |a_i| r^i = \|f\|_r, \quad |b_j| r^j = \|g\|_r.$$

By our assumption on r , i, j are unique. Then

$$\|fg\|_r = \max_{k \in \mathbb{Z}} \{|c_k| r^k\},$$

where

$$c_k := \sum_{u, v \in \mathbb{Z}, u+v=k} a_u b_v.$$

It suffices to show that

$$(3.4) \quad |c_k| r^k = \|f\|_r \|g\|_r.$$

for $k = i + j$. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. For $u, v \in \mathbb{Z}$, $u + v = i + j$ while $(u, v) \neq (i, j)$, we may assume that $u \neq i$. Then $|a_u| r^u < |a_i| r^i$ and $|b_v| r^v \leq |b_j| r^j$. So $|a_u b_v| < |a_i b_j|$ and we conclude (3.4). \square

Remark 3.8. The argument of [Proposition 4.16](#) in the chapter Banach Rings does not work here if $r \in \sqrt{|k^\times|}$, as in general one can not take minimal i, j so that (3.3) is satisfied.

Proposition 3.9. Assume that $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$. Then k_r is a valuation field and $\|\bullet\|_r$ is non-trivial.

PROOF. We first show that $\text{Sp } k_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet| \in \text{Sp } k_r$. As $\|\bullet\|_r$ is a valuation, we find

$$(3.5) \quad |\bullet| \leq \|\bullet\|_r.$$

In particular, $|\bullet|$ restricted to k is the given valuation on k . It suffices to show that $|T| = r$. This follows from (3.5) applied to T and T^{-1} .

It follows that k_r does not have any non-zero proper closed ideals: if I is such an ideal, k_r/I is a Banach k -algebra. By [Proposition 6.10](#) in the chapter Banach rings, $\text{Sp } k_r$ is non-empty. So k_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by [Corollary 4.7](#) in the chapter Banach rings, the only maximal ideal of k_r is 0. It follows that k_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r = r$. \square

Definition 3.10. An element $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$ for some $n \in \mathbb{N}$ is called a *k-free polyray* if r_1, \dots, r_n are linearly independent in the \mathbb{Q} -linear space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_{>0} / \sqrt{|k^\times|}$.

Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. Assume that r is a k -free polyray. We define

$$k_r = k_{r_1} \hat{\otimes}_k \cdots \hat{\otimes}_k k_{r_n}.$$

By an iterated application of [Proposition 3.9](#), k_r is a complete valuation field. As a general explanation of why k_r is useful, we prove the following proposition:

Proposition 3.11. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k -free polyray.

- (1) For any k -Banach space X , the natural map

$$X \rightarrow X \hat{\otimes}_k k_r$$

is an isometric embedding.

- (2) Consider a sequence of bounded homomorphisms of k -Banach spaces $X \rightarrow Y \rightarrow Z$. Then the sequence is admissible and exact (resp. coexact) if and only if $X \hat{\otimes}_k k_r \rightarrow Y \hat{\otimes}_k k_r \rightarrow Z \hat{\otimes}_k k_r$ is admissible and exact (resp. coexact).

PROOF. We may assume that $n = 1$.

(1) We have a more explicit description of $X \hat{\otimes}_k k_r$: as a vector space, it is the space of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ with $a_i \in X$ and $\|a_i\| r^i \rightarrow 0$ when $|i| \rightarrow \infty$. The norm is given by $\max_i \|a_i\| r^i$. From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1). \square

When X is a Banach k -algebra, $X \hat{\otimes}_k k_r$ is a Banach k_r -algebra.

Example 3.12. For any $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$, not necessarily k -free. We define k_r as the completed fraction field of $k\{r^{-1}T\}$ provided with the extended valuation $|\bullet|_r$. Then k_r is still a valuation field extending k .

When r is a k -free polyray, we claim that k_r coincides with k_r defined in [Definition 3.10](#). To see this, let us temporarily denote the k_r defined in this example as k'_r consider the extension of field:

$$\text{Frac } k\{r^{-1}T\} \rightarrow k_r = k\{r^{-1}T, rS\} / (T_1 S_1 - 1, \dots, T_n S_n - 1)$$

sending T_i to T_i for $i = 1, \dots, n$. Observe that this is an extension of valuation field as well by the same arguments as in [Proposition 3.6](#). In particular, it induces an extension of complete valuation fields $k'_r \rightarrow k_r$. But the image clearly contains the classes of all polynomials in $k[T, S]$, so $k'_r \rightarrow k_r$ is an isometric isomorphism.

Proposition 3.13. Assume that k is non-trivially valued. Let B be a strict k -affinoid algebra and $\varphi : B \rightarrow A$ be a finite bounded k -algebra homomorphism into a k -Banach algebra A . Then A is also strictly k -affinoid.

PROOF. We may assume that $B = k\{T_1, \dots, T_n\}$ for some $n \in \mathbb{N}$. By assumption, we can find finitely many $a_1, \dots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(B) a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings, φ admits a unique extension to a bounded k -algebra epimorphism

$$\Phi : k\{T_1, \dots, T_n, S_1, \dots, S_m\} \rightarrow A$$

sending S_i to a_i . By [Corollary 7.5](#) in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is strictly k -affinoid. \square

Proposition 3.14. Assume that k is non-trivially valued. Let B be a strict k -affinoid algebra and $\varphi : B \rightarrow A$ be a finite k -algebra homomorphism into a k -algebra A . Then there is a norm on A such that the morphism is bounded and A is strictly k -affinoid.

PROOF. By [Proposition 8.4](#) in the chapter Banach Rings, we can endow A with a Banach norm such that φ is admissible. Then we can apply [Proposition 3.13](#). \square

Lemma 3.15. Assume that k is non-trivially valued. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$. The algebra $k\{r^{-1}T\}$ is strictly k -affinoid if $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \dots, n$.

Remark 3.16. The converse is also true.

PROOF. Assume that $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \dots, n$. Take $s_i \in \mathbb{N}$ and $c_i \in k^\times$ such that

$$r_i^{s_i} = |c_i^{-1}|$$

for $i = 1, \dots, n$. We define a bounded k -algebra homomorphism $\varphi : k\{T_1, \dots, T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ by sending T_i to $c_i T_i^{s_i}$. This is possible by [Proposition 4.18](#) in the chapter Banach Rings.

We claim that φ is finite. To see this, it suffices to observe that if we expand $f \in k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (c T^s)^\gamma,$$

where the product γs is taken component-wise. For each $\beta \in \mathbb{N}^n, \beta_i < s_i$, we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \dots, T_n\}.$$

While $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$. So We have shown that φ is finite. Hence, $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is k -affinoid by [Proposition 3.13](#). \square

Proposition 3.17. Let A be a k -affinoid algebra, then there is $n \in \mathbb{N}$ and a k -free polyray $r = (r_1, \dots, r_n)$ such that $A \hat{\otimes}_k k_r$ is strictly k_r -affinoid. Moreover, we can guarantee that k_r is non-trivially valued.

PROOF. By [Proposition 3.11](#), we may assume that $A = k\{t^{-1}T\}$ for some $t \in \mathbb{R}_{>0}^m$. By [Lemma 3.15](#), it suffices to take r so that the linear subspace of $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ generated by r_1, \dots, r_n contains all components of t . By taking $n \geq 1$, we can guarantee that k_r is non-trivially valued. \square

Proposition 3.18. Let $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a monomorphism in $k_H\text{-Aff}$. Then for any $y \in \mathrm{Sp} B$ with $x = \varphi(y)$, one has $\varphi^{-1}(x) = \{y\}$ and the natural map $\mathcal{H}(x) \rightarrow \mathcal{H}(y)$ is an isomorphism of complete valuation rings.

PROOF. It suffices to show that $\mathcal{H}(x) \rightarrow B \hat{\otimes}_A \mathcal{H}(y)$ is an isomorphism as Banach k -algebras. [Include details about cofiber products in affalg](#). By assumption, the codiagonal map $B \hat{\otimes}_A B \rightarrow B$ is an isomorphism. It follows that the base change with respect to $A \rightarrow \mathcal{H}(x)$ is also an isomorphism: $B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$, where $B' = B \hat{\otimes}_A \mathcal{H}(x)$.

[Include the fact that the first map is injective](#). It follows that the composition $B' \otimes_{\mathcal{H}(x)} B \rightarrow B' \hat{\otimes}_{\mathcal{H}(x)} B' \rightarrow B'$ is injective. Therefore, $\mathcal{H}(x) \rightarrow B'$ is an isomorphism of rings. We also know that this map is bounded. But we already know that $\mathcal{H}(x)$ is a complete valuation ring, so the map $\mathcal{H}(x) \rightarrow B'$ is an isomorphism of complete valuation rings. \square

4. Weierstrass theory

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 4.1. We have canonical identifications

$$\begin{aligned} (k\{T_1, \dots, T_n\})^\circ &\cong \mathring{k}\{T_1, \dots, T_n\}, \\ (k\{T_1, \dots, T_n\})^\vee &\cong \check{k}\{T_1, \dots, T_n\}, \\ k\{\widetilde{T_1, \dots, T_n}\} &\cong \tilde{k}[T_1, \dots, T_n]. \end{aligned}$$

The last identification extends $\mathring{k} \rightarrow \tilde{k}$ and T_i is mapped to T_i .

PROOF. This follows from [Corollary 4.20](#) from the chapter Banach rings. \square

We will denote the reduction map $\mathring{k}\{T_1, \dots, T_n\} \rightarrow \tilde{k}[T_1, \dots, T_n]$ by $\tilde{\bullet}$.

Definition 4.2. Let $n \in \mathbb{N}$. A system $f_1, \dots, f_n \in k\{T_1, \dots, T_n\}$ is called an *affinoid chart* of $k\{T_1, \dots, T_n\}$ if $f_i \in \mathring{k}\{T_1, \dots, T_n\}$ for each $i = 1, \dots, n$ and the continuous k -algebra homomorphism $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$ sending T_i to f_i is an isomorphism.

The map $k\{T_1, \dots, T_n\} \rightarrow k\{T_1, \dots, T_n\}$ is well-defined by [Proposition 4.1](#) and [Lemma 2.5](#).

Lemma 4.3. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \dots, T_n\}$. Assume that $\|f\|_1 = 1$. Then the following are equivalent:

- (1) f is a unit in $k\{T_1, \dots, T_n\}$.
- (2) \tilde{f} is a unit in $\tilde{k}[T_1, \dots, T_n]$.

PROOF. As $\|\bullet\|_1$ is a valuation by [Proposition 3.6](#), f is a unit in $k\{T_1, \dots, T_n\}$ if and only if it is a unit in $(k\{T_1, \dots, T_n\})^\circ$, which is identified with $\mathring{k}\{T_1, \dots, T_n\}$ by [Proposition 4.1](#). This result then follows from [Corollary 4.21](#) in the chapter Banach Rings. \square

Definition 4.4. Let $n \in \mathbb{N}$. Consider $g \in k\{T_1, \dots, T_n\}$. We expand g as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

For $s \in \mathbb{N}$, we say g is *X_n -distinguished of degree s* if g_s is a unit in $k\{T_1, \dots, T_{n-1}\}$, $\|g_s\|_1 = \|g\|_1$ and $\|g_s\|_1 > \|g_t\|_1$ for all $t > s$.

Theorem 4.5 (Weierstrass division theorem). Let $n, s \in \mathbb{N}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n -distinguished of degree s . Then for each $f \in k\{T_1, \dots, T_n\}$, there exist $q \in k\{T_1, \dots, T_n\}$ and $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$ such that

$$f = qg + r.$$

Moreover, q and r are uniquely determined. We have the following estimates

$$(4.1) \quad \|q\|_1 \leq \|g\|_1^{-1} \|f\|_1, \quad \|r\|_1 \leq \|f\|_1.$$

If in addition, $f, g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then $q \in k\{T_1, \dots, T_{n-1}\}[T_n]$ as well.

PROOF. We may assume that $\|g\|_1 = 1$.

Step 1. Assuming the existence of the division. Let us prove (4.1). We may assume that $f \neq 0$, so that one of q, r is non-zero. Up to replacing q, r by a scalar multiple, we may assume that $\max\{\|q\|_1, \|r\|_1\} = 1$. So $\|f\|_1 \leq 1$ as well. We need to show that $\|f\|_1 = 1$. Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here $\tilde{\bullet}$ denotes the reduction map. By our assumption, $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$. From [Proposition 4.1](#), the equality is in $\tilde{k}[T_1, \dots, T_n]$. From the usual Euclidean division, we have $\tilde{q} = \tilde{r} = 0$. This is a contradiction to our assumption.

Step 2. Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with q and r as in the theorem. The estimate in Step 1 shows that $q = r = 0$.

Step 3. We prove the existence of the division.

We define

$$B := \{qg + r : r \in k\{T_1, \dots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \dots, T_n\}\}.$$

From Step 1, B is a closed subgroup of $k\{T_1, \dots, T_n\}$. In fact, suppose $f_i \in B$ is a sequence converging to $f \in k\{T_1, \dots, T_n\}$. From Step 1, we can represent $f_i = q_i g + r_i$, then from Step 1, q_i and r_i are both Cauchy sequences, we may assume that $q_i \rightarrow q \in k\{T_1, \dots, T_n\}$ and $r_i \rightarrow r$. As $\deg_{T_n} r_i < s$, it follows that $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ and $\deg_{T_n} r < s$. So $f = qg + r$ and hence B is closed.

It suffices to show that B is dense $k\{T_1, \dots, T_n\}$. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \dots, T_{n-1}\}.$$

We may assume that $\|g\|_1 = 1$. Define $\epsilon := \max_{j \geq s} \|g_j\|$. Then $\epsilon < 1$ by our assumption. Let $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$ for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \dots, T_n\})^\circ \rightarrow (\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$$

with kernel $\{f \in k\{T_1, \dots, T_n\} : \|f\|_1 \leq \epsilon\}$. We now apply Euclidean division in the ring $(\mathring{k}/k_\epsilon)[T_1, \dots, T_n]$ to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some $q \in (k\{T_1, \dots, T_n\})^\circ$ and $r \in (k\{T_1, \dots, T_{n-1}\})^\circ[T_n]$ with $\deg_{T_n} r < s$. So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that B is dense in $k\{T_1, \dots, T_n\}$ by [Proposition 2.8](#) in the chapter Banach rings.

Step 4. It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring $k\{T_1, \dots, T_{n-1}\}[T_n]$ and the uniqueness proved in Step 2. \square

Lemma 4.6. Let $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ be a Weierstrass polynomial and $g \in k\{T_1, \dots, T_n\}$. Assume that $\omega g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$.

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in k\{T_1, \dots, T_{n-1}\}[T_n]$, $\deg_{T_n} r < \deg_{T_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of [Theorem 4.5](#), we know that $q = g$, so g is a polynomial in T_n . \square

As a consequence, we deduce Weierstrass preparation theorem.

Definition 4.7. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n -variables is a monic polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\|\omega\|_1 = 1$.

Lemma 4.8. Let $n \in \mathbb{Z}_{>0}$ and $\omega_1, \omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ be two monic polynomials. If $\omega_1\omega_2$ is a Weierstrass polynomial then so are ω_1 and ω_2 .

PROOF. As ω_1 and ω_2 are monic, $\|\omega_i\|_1 \geq 1$ for $i = 1, 2$. On the other hand, $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1\omega_2\|_1 = 1$, so $\|\omega_i\|_1 = 1$ for $i = 1, 2$. \square

Theorem 4.9 (Weierstrass preparation theorem). Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n -distinguished of degree s . Then there are a Weierstrass polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ of degree s and a unit $e \in k\{T_1, \dots, T_n\}$ such that

$$g = e\omega.$$

Moreover, e and ω are unique. If $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then so is e .

PROOF. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let $r = T_n^s - \omega$. Then $T_n^s = e^{-1}g + r$. The uniqueness part of [Theorem 4.5](#) implies that e and r are uniquely determined, hence so is ω .

Next we prove the existence. By Weierstrass division theorem [Theorem 4.5](#), we can write

$$T_n^s = qg + r$$

for some $q \in k\{T_1, \dots, T_n\}$ and $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$. Let $\omega = T_n^s - r$. From the estimates in [Theorem 4.5](#), $\|r\|_1 \leq 1$. So $\|\omega\|_1 = 1$. Then ω is a Weierstrass polynomial of degree s and $\omega = qg$. It suffices to argue that q is a unit.

We may assume that $\|g\|_1 = 1$. By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$ and the leading coefficients of both polynomials are units in $\tilde{k}[T_1, \dots, T_{n-1}]$, it follows that \tilde{q} is a unit in $\tilde{k}[T_1, \dots, T_{n-1}]$. It follows that \tilde{q} is also a unit in $\tilde{k}[T_1, \dots, T_n]$. By [Lemma 4.3](#), q is a unit in $k\{T_1, \dots, T_n\}$.

The last assertion is already proved in [Theorem 4.5](#). \square

Definition 4.10. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n -distinguished. Then the Weierstrass polynomial ω constructed in [Theorem 4.9](#) is called the *Weierstrass polynomial* defined by g .

Corollary 4.11. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n -distinguished. Let ω be the Weierstrass polynomial of g . Then the injection

$$k\{T_1, \dots, T_{n-1}\}[T_n] \rightarrow k\{T_1, \dots, T_n\}$$

induces an isomorphism of k -algebras

$$k\{T_1, \dots, T_{n-1}\}[T_n]/(\omega) \rightarrow k\{T_1, \dots, T_n\}/(g).$$

PROOF. The surjectivity follows from [Theorem 4.5](#) and the injectivity follows from [Lemma 4.6](#). \square

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

Lemma 4.12. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ is non-zero. Then there is a k -algebra automorphism σ of $k\{T_1, \dots, T_n\}$ so that $\sigma(g)$ is T_n -distinguished.

PROOF. We may assume that $\|g\|_1 = 1$. We expand g as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow \mathbb{N}^n with the lexicographic order. Take the maximal $\beta \in \mathbb{N}^n$ so that $|a_\beta| = 1$. Take $t \in \mathbb{Z}_{>0}$ so that $t \geq \max_{i=1, \dots, n} \alpha_i$ for all $\alpha \in \mathbb{N}^n$ with $\tilde{a}_\alpha \neq 0$.

We will define σ by sending T_i to $T_i + T_n^{c_i}$ for all $i = 1, \dots, n-1$. The c_i 's are to be defined. We begin with $c_n = 1$ and define the other c_i 's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for $j = 1, \dots, n-1$. We claim that $\sigma(f)$ is T_n -distinguished of order $s = \sum_{i=1}^n c_i \beta_i$.

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^s p_i T_n^i$$

for some $p_i \in \tilde{k}[T_1, \dots, T_{n-1}]$ and $p_s = \tilde{a}_\beta$. Our claim follows. \square

Proposition 4.13. Let $n \in \mathbb{N}$. Then $k\{T_1, \dots, T_n\}$ is Noetherian.

PROOF. We make induction on n . The case $n = 0$ is trivial. Assume that $n > 0$. It suffices to show that for any non-zero $g \in k\{T_1, \dots, T_n\}$, $k\{T_1, \dots, T_n\}/(g)$ is Noetherian. By Lemma 4.12, we may assume that g is T_n -distinguished. By Theorem 4.5, $k\{T_1, \dots, T_n\}/(g)$ is a finite free $k\{T_1, \dots, T_{n-1}\}$ -module. By the inductive hypothesis and Hilbert basis theorem, $k\{T_1, \dots, T_n\}/(g)$ is indeed Noetherian. \square

Proposition 4.14. Let $n \in \mathbb{N}$. Then $k\{T_1, \dots, T_n\}$ is Jacobson.

PROOF. When $n = 0$, there is nothing to prove. We make induction on n and assume that $n > 0$. Let \mathfrak{p} be a prime ideal in $k\{T_1, \dots, T_n\}$, we want to show that the Jacobson radical of \mathfrak{p} is equal to \mathfrak{p} .

We distinguish two cases. First we assume that $\mathfrak{p} \neq 0$. Let $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \dots, T_{n-1}\}$. By Lemma 4.12, we may assume that \mathfrak{p} contains a Weierstrass polynomial ω . Observe that

$$k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' \rightarrow k\{T_1, \dots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any $b \in J(k\{T_1, \dots, T_n\}/\mathfrak{p})$ (where J denotes the Jacobson radical), we consider a monic integral equation of minimal degree over $k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'$:

$$b^n + a_1 b^{n-1} + \dots + a_n = 0, \quad a_i \in k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \dots, T_n\}/\mathfrak{p}) \cap k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \dots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that $n = 1$ and so $b = 0$. This proves $J(k\{T_1, \dots, T_n\}/\mathfrak{p}) = 0$.

On the other hand, let us consider the case $\mathfrak{p} = 0$. As $k\{T_1, \dots, T_n\}$ is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \dots, T_n\}) = 0.$$

Assume that there is a non-zero element f in $J(k\{T_1, \dots, T_n\})$. We may assume that $\|f\|_1 = 1$.

We claim that there is $c \in k$ with $|c| = 1$ such that $c + f$ is not a unit in $k\{T_1, \dots, T_n\}$. Assuming this claim for the moment, we can find a maximal ideal \mathfrak{m} of $k\{T_1, \dots, T_n\}$ such that $c + f \in \mathfrak{m}$. But $f \in \mathfrak{m}$ by our assumption, so $c \in \mathfrak{m}$ as well. This contradicts the fact that $c \in k^\times$.

It remains to prove the claim. We treat two cases separately. When $|f(0)| < 1$, we simply take $c = 1$, which works thanks to [Lemma 4.3](#). If $|f(0)| = 1$, we just take $c = -f(0)$. \square

Proposition 4.15. Let $n \in \mathbb{N}$. Then $k\{T_1, \dots, T_n\}$ is UFD. In particular, $k\{T_1, \dots, T_n\}$ is normal.

PROOF. As $\|\bullet\|_1$ is a valuation by [Proposition 2.2](#), $k\{T_1, \dots, T_n\}$ is an integral domain. In order to see that $k\{T_1, \dots, T_n\}$ has the unique factorization property, we make induction on $n \geq 0$. When $n = 0$, there is nothing to prove. Assume that $n > 0$. Take a non-unit element $f \in k\{T_1, \dots, T_n\}$. By [Theorem 4.9](#) and [Lemma 4.12](#), we may assume that f is a Weierstrass polynomial. By inductive hypothesis, $k\{T_1, \dots, T_{n-1}\}$ is a UFD, hence so is $k\{T_1, \dots, T_{n-1}\}[T_n]$ by [[Stacks, Tag 0BC1](#)]. It follows that f can be decomposed into the products of monic prime elements $f_1, \dots, f_r \in k\{T_1, \dots, T_{n-1}\}[T_n]$, which are all Weierstrass polynomials by [Lemma 4.8](#). Then by [Corollary 4.11](#), we see that each f_i is prime in $k\{T_1, \dots, T_n\}$.

Any UFD is normal by [[Stacks, Tag 0AFV](#)]. \square

Corollary 4.16. Let A be a strictly k -affinoid algebra, $d \in \mathbb{N}$ and $\varphi : k\{T_1, \dots, T_d\} \rightarrow A$ be an integral torsion-free injective homomorphism of k -algebras. Then ρ is a faithful $k\{T_1, \dots, T_d\}$ -algebra norm on A . If $f^n + \varphi(t_1)f^{n-1} + \dots + \varphi(t_n) = 0$ is the minimal integral equation of f over $k\{T_1, \dots, T_d\}$, then

$$|f|_{\sup} = \max_{i=1, \dots, n} |t_i|^{1/i}.$$

PROOF. This follows from [Proposition 9.11](#) in the chapter Banach Rings and [Proposition 4.15](#). \square

5. Noetherian normalization and maximal modulus principle

Let $(k, |\bullet|)$ be a complete non-trivially valued non-Archimedean valued-field.

Theorem 5.1. Let A be a non-zero strictly k -affinoid algebra, $n \in \mathbb{N}$ and $\alpha : k\{T_1, \dots, T_n\} \rightarrow A$ be a finite (resp. integral) k -algebra homomorphism. Then up to replacing T_1, \dots, T_n by an affinoid chart, we can guarantee that there exists $d \in \mathbb{N}$, $d \leq n$ such that α when restricted to $k\{T_1, \dots, T_d\}$ is finite (resp. integral) and injective.

PROOF. We make an induction on n . The case $n = 0$ is trivial. Assume that $n > 0$. If $\ker \alpha = 0$, there is nothing to prove, so we may assume that $\ker \alpha \neq 0$. By [Lemma 4.12](#) and [Theorem 4.9](#), we may assume that there is a Weierstrass polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ in $\ker \alpha$. Then α induces a finite (resp. integral) homomorphism $\beta : k\{T_1, \dots, T_n\}/(\omega) \rightarrow A$. By [Theorem 4.5](#), $k\{T_1, \dots, T_{n-1}\} \rightarrow k\{T_1, \dots, T_n\}/(\omega)$ is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism $k\{T_1, \dots, T_{n-1}\} \rightarrow A$. We can apply the inductive hypothesis to conclude. \square

Corollary 5.2. Let A be a non-zero strictly k -affinoid algebra, then there is $d \in \mathbb{N}$ and a finite injective k -algebra homomorphism: $k\{T_1, \dots, T_d\} \rightarrow A$.

PROOF. Take some $n \in \mathbb{N}$ and a surjective k -algebra homomorphism $k\{T_1, \dots, T_n\} \rightarrow A$ and apply [Theorem 5.1](#), we conclude. \square

Corollary 5.3. Let A be a strictly k -affinoid algebra and I be an ideal in A such that \sqrt{I} is a maximal ideal in A , then A/I is finite-dimensional over k .

In particular, $\text{Spm } A = \text{Spm}_k A$.

PROOF. By [Corollary 5.2](#), there is $d \in \mathbb{N}$ and a finite monomorphism $f : k\{T_1, \dots, T_d\} \rightarrow A/I$. It suffices to show that $d = 0$. Observe that the composition

$$k\{T_1, \dots, T_d\} \xrightarrow{f} A/I \rightarrow A/\sqrt{I}$$

is finite and injective as $k\{T_1, \dots, T_d\}$ is an integral domain, so $k\{T_1, \dots, T_d\}$ is a field. This is possible only when $d = 0$. \square

Corollary 5.4. Let B be a strictly k -affinoid algebra and A be a Noetherian Banach k -algebra. Let $f : A \rightarrow B$ a k -algebra homomorphism. Then f is bounded.

PROOF. This follows from [Proposition 8.1](#) in the chapter Banach Rings and [Proposition 4.13](#). \square

In particular, we see that the topology of a k -affinoid algebra is uniquely determined by the algebraic structure.

Corollary 5.5. Let A, B be strictly k -affinoid algebras. Let f be a finite k -algebra homomorphism, then f is admissible.

PROOF. This follows from [Proposition 3.14](#) and [Corollary 5.4](#), \square

Definition 5.6. For any non-Archimedean valuation field $(K, |\bullet|)$ and $n \in \mathbb{N}$, we define the n -dimensional polydisk with value in K :

$$B^n(K) := \left\{ (x_1, \dots, x_n) \in K^n : \max_{i=1, \dots, n} |x_i| \leq 1 \right\}.$$

Definition 5.7. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \dots, T_n\}$, say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$ as sending $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$ to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

Lemma 5.8. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \dots, T_n\}$, then $f : B^n(k^{\text{alg}}) \rightarrow k^{\text{alg}}$ is continuous and for any $x \in B^n(k^{\text{alg}})$,

$$|f(x)| \leq \|f\|_1.$$

There is $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$ such that $|f(x)| = \|f\|_1$.

PROOF. To see that f is continuous, it suffices to observe that f is a uniform limit of polynomials. For any $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$, we have

$$|f(x)| = \left| \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \right| \leq \max_{\alpha \in \mathbb{N}^n} |a_\alpha x^\alpha| \leq \|f\|_1.$$

To prove the last assertion, we may assume that $\|f\|_1 = 1$. As the residue field of k^{alg} is equal to \tilde{k}^{alg} , it has infinitely many elements, so there is a point $x \in B^n(k^{\text{alg}})$ such that $\widetilde{f(x)} = \tilde{f}(\tilde{x}) \neq 0$. In other words, $\|f(x)\|_1 = 1$. \square

Proposition 5.9. Let $n \in \mathbb{N}$, then the maximal modulus principle holds for $k\{T_1, \dots, T_n\}$. Moreover, for any $f \in k\{T_1, \dots, T_n\}$, $\|f\|_1 = |f|_{\text{sup}}$.

PROOF. By Lemma 6.3 in the chapter Banach Rings, we have

$$\|f\|_1 \geq |f|_{\text{sup}}$$

for any $f \in A$. We only have to show that for any $f \in k\{T_1, \dots, T_n\}$ there is a maximal ideal $\mathfrak{m} \subseteq k\{T_1, \dots, T_n\}$ such that $|f(\mathfrak{m})| = \|f\|_1$.

By Lemma 5.8 we can take $x = (x_1, \dots, x_n) \in B^n(k^{\text{alg}})$ such that $|f(x)| = \|f\|_1$. Let L be the field extension of k generated by x_1, \dots, x_n , then L/k is finite. Then we can define a homomorphism

$$\text{ev}_x : k\{T_1, \dots, T_n\} \rightarrow L$$

sending $g \in k\{T_1, \dots, T_n\}$ to $g(x)$. Observe that the image is indeed in L . Clearly ev_x is surjective. So $\mathfrak{m}_x := \ker \text{ev}_x$ is a k -algebraic maximal ideal in $k\{T_1, \dots, T_n\}$. Then

$$|f(\mathfrak{m}_x)| = |f(x)| = \|f\|_1.$$

\square

Corollary 5.10. Let A be a strictly k -affinoid algebra. Then for any $f \in A$,

$$|f|_{\text{sup}} \subseteq \sqrt{|k^\times|} \cup \{0\}.$$

PROOF. We may assume that $A \neq 0$. By Corollary 5.2 and Proposition 9.11 in the chapter Banach Rings, we may assume that $A = k\{T_1, \dots, T_n\}$ for some $n \in \mathbb{N}$. The result then follows from Proposition 5.9. \square

Corollary 5.11. Maximal modulus principle holds for any strictly k -affinoid algebras.

PROOF. This follows from Corollary 5.2, Proposition 9.11 in the chapter Banach Rings and Proposition 5.9. \square

Proposition 5.12. Let $\varphi : B \rightarrow A$ be an integral k -algebra homomorphism of strictly k -affinoid algebras. Then for each non-zero $f \in A$, there is a moine polynomial $q(f) = f^n + \varphi(b_1)f^{n-1} + \dots + \varphi(b_n)$ of f over B . Then

$$|f|_{\text{sup}} = \max_{i=1, \dots, n} |b_i|_{\text{sup}}^{1/i}.$$

PROOF. One side is simple: choose $j = 1, \dots, n$ that maximizes $|\varphi(b_j)f^{n-j}|_{\text{sup}}$, then

$$|f|_{\text{sup}}^n = |f^n|_{\text{sup}} \leq |\varphi(b_j)f^{n-j}|_{\text{sup}} \leq |b_j|_{\text{sup}} \cdot |f|_{\text{sup}}^{n-j}.$$

So

$$|f|_{\text{sup}} \leq |b_j|_{\text{sup}}^{1/j}.$$

To prove the reverse inequality, let us begin with the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k -algebra homomorphism $\psi : k\{T_1, \dots, T_d\} \rightarrow B$ such that $\varphi \circ \psi$ is integral and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$, we can apply [Theorem 5.1](#) to find $\phi \circ \alpha$ to conclude.

By [Corollary 4.16](#), if p denotes the minimal polynomial of f over $k\{T_1, \dots, T_d\}$, we have $|f|_{\text{sup}} = \sigma(p)$. In particular, $p(f) = 0$. Let $q \in B[X]$ be the polynomial obtained from p by replacing all coefficients by their ψ -images in B . Then clearly, $|f|_{\text{sup}} = \sigma(q)$.

In general, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes in A . The number is finite by [Proposition 4.13](#). For each $i = 1, \dots, r$, let $\pi_i : A \rightarrow A/\mathfrak{p}_i$ denote the natural homomorphism. We know that there are monic polynomials $q_i \in B[X]$ such that $q_i(\pi(f)) = 0$ and $|\pi_i(f)|_{\text{sup}} = \sigma(q_i)$ for $i = 1, \dots, r$. We let $q' = q_1 \cdots q_r$. Then

$$q'(f) \in \bigcap_{i=1}^r \mathfrak{p}_i.$$

So there is $e \in \mathbb{Z}_{>0}$ such that $q'(f)^e = 0$. Let $q = q'^e$. By [Proposition 9.5](#) in the chapter Banach Rings,

$$\sigma(q) \leq \max_{i=1, \dots, r} \sigma(q_i) = \max_{i=1, \dots, r} |\pi_i(f)|_{\text{sup}} = |f|_{\text{sup}}.$$

The last equality follows from [Proposition 9.9](#) in the chapter Banach Rings. \square

Lemma 5.13. Let $\varphi : B \rightarrow A$ be an admissible k -algebra homomorphism between strictly k -affinoid algebras. Let $\tau : \check{B} \rightarrow \check{B}$ be the reduction map, then

$$\tau^{-1}(\ker \tilde{\varphi}) = \sqrt{\check{B} + \ker \tilde{\varphi}}, \quad \ker \tilde{\varphi} = \sqrt{\tau(\ker \tilde{\varphi})}.$$

PROOF. The second equation follows from the first one by applying τ . Let us prove the first equation. By assumption, $\varphi(\check{B})$ is open in $\varphi(B)$. Consider $g \in \tau^{-1}(\ker \tilde{\varphi})$, we know that

$$\lim_{n \rightarrow \infty} \varphi(g)^n = 0.$$

So $\varphi(g)^n \in \varphi(\check{B})$ for n large enough, and hence $g^n \in \check{B} + \ker \tilde{\varphi}$. \square

Lemma 5.14. Let $m \in \mathbb{N}$ and $T = k\{T_1, \dots, T_m\}$. Let A be a k -affinoid algebra and $\varphi : T\{S_1, \dots, S_n\} \rightarrow A$ be a finite morphism such that $\tilde{\varphi}(S_i)$ is integral over \check{T} . Then $\varphi|_T : T \rightarrow A$ is finite.

PROOF. We make an induction on n . When $n = 0$, there is nothing to prove. So assume $n > 0$ and the lemma has been proved for smaller values of n .

Let $T' = T\{S_1, \dots, S_n\}$. By assumption, there is a Weierstrass polynomial

$$\omega = S_n^k + a_1 S_n^{k-1} + \cdots + a_k \in \check{T}[S_n]$$

such that $\tilde{\omega} \in \ker \tilde{\varphi}$. As φ is admissible by [Corollary 5.5](#), we have $\omega^q \in \check{T}' + \ker \tilde{\varphi}$ for some $q \in \mathbb{Z}$ by [Lemma 5.13](#).

In particular, we can find $r \in (\check{T}')^\times$ such that $g := \omega^q - r \in \ker \tilde{\varphi}$. Observe that g is S_n distinguished of order mq as $\tilde{g} = \tilde{\omega}^q$. By [Corollary 4.11](#), the restriction of φ to $T\{S_1, \dots, S_{n-1}\}$ is finite. We can apply the inductive hypothesis to conclude. \square

Lemma 5.15. Let $\varphi : B \rightarrow A$ be a k -algebra homomorphism of strictly k -affinoid algebras. Assume that there exist affinoid generators $f_1, \dots, f_n \in \mathring{A}$ of A such that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over \tilde{B} , then φ is finite.

PROOF. By assumption, we can find $s_i \in \mathbb{Z}_{>0}$, $b_{ij} \in \mathring{B}$ for $i = 1, \dots, n$, $j = 1, \dots, s_i$ such that

$$\tilde{f}_i^{s_i} + \tilde{\varphi}(\tilde{b}_{i1})\tilde{f}_i^{s_i-1} + \dots + \tilde{\varphi}(\tilde{b}_{is_i}) = 0$$

for $i = 1, \dots, n$. Let $s = s_1 + \dots + s_n$ and define a bounded k -algebra homomorphism $\psi : D := k\{T_{ij}\} \rightarrow B$ sending T_{ij} to b_{ij} , for $i = 1, \dots, n$ and $j = 1, \dots, s_i$. Observe that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over \tilde{D} . So it suffices to prove the theorem when $B = k\{T_1, \dots, T_m\}$. We extend φ to a bounded k -algebra epimorphism $\varphi' : T\{S_1, \dots, S_n\} \rightarrow A$ sending S_i to f_i for $i = 1, \dots, n$. Then $\varphi'(\tilde{S}_i)$ is integral over \tilde{B} . It suffices to apply [Lemma 5.14](#). \square

6. Properties of affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Proposition 6.1. Assume that k is non-trivially valued. Let A be a strictly k -affinoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\sup} \leq 1\}.$$

PROOF. By [Lemma 6.3](#), we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\sup} \leq 1\}.$$

Conversely, let $f \in A$, $|f|_{\sup} \leq 1$. Choose $d \in \mathbb{N}$ and a surjective k -algebra homomorphism

$$\varphi : k\{T_1, \dots, T_d\} \rightarrow A.$$

Let $f^n + t_1 f^{n-1} + \dots + t_n = 0$ be the minimal equation of f over $k\{T_1, \dots, T_d\}$. Then $t_i \in (k\{T_1, \dots, T_d\})^\circ$ by [Proposition 9.11](#) in the chapter Banach Rings. An induction on $i \geq 0$ shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi((k\{T_1, \dots, T_d\})^\circ) f^j.$$

The right-hand side is clearly bounded. \square

Corollary 6.2. Assume that k is non-trivially valued. Let $(A, \|\bullet\|)$ be a strictly k -affinoid algebra. For any $f \in A$,

$$\rho(f) = |f|_{\sup}.$$

PROOF. We have shown that $\rho(f) \geq |f|_{\sup}$ in [Lemma 6.3](#) from the chapter Banach Rings. Assume that the inverse inequality fails: for some $f \in A$,

$$\rho(f) > |f|_{\sup}.$$

If $|f|_{\sup} = 0$, then f lies in the Jacobson radical of A , which is equal to the nilradical of A by [Proposition 4.14](#). But then $\rho(f) = 0$ as well. We may therefore assume that $|f|_{\sup} \neq 0$. By [Corollary 5.10](#), we may assume that $|f|_{\sup} = 1$ as ρ is power-multiplicative. Then $\rho(f) > 1$. This contradicts [Proposition 6.1](#). \square

Theorem 6.3. A k -affinoid algebra A is Noetherian and all ideals of A are closed.

PROOF. Let I be an ideal in A . By [Proposition 3.17](#), we can take a suitable $r \in \mathbb{R}_{>0}^m$ so that $A \hat{\otimes} k_r$ is strictly k_r -affinoid. Then $I(A \hat{\otimes} k_r)$ is an ideal in $A \hat{\otimes} k_r$. By [Proposition 4.13](#), the latter ring is Noetherian. So we may take finitely many generators $f_1, \dots, f_k \in I$. Each $f \in I$ can be written as

$$f = \sum_{i=1}^k f_i g_i$$

with $g_i = \sum_{j=-\infty}^{\infty} g_{i,j} T^j \in A \hat{\otimes} k_r$. But then

$$f = \sum_{i=1}^k f_i g_{i,0}.$$

So I is finitely generated.

As $I = A \cap (I(A \hat{\otimes} k_r))$, by [Corollary 7.4](#) in the chapter Banach Rings, we see that I is closed in $A \hat{\otimes} k_r$ and hence closed in A . \square

Proposition 6.4. Let $(A, \|\bullet\|)$ be a k -affinoid algebra and $f \in A$. Then there is $C > 0$ and $N \geq 1$ such that for any $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Recall that ρ is the spectral radius map defined in [Definition 4.9](#) in the chapter Banach Rings.

PROOF. By [Proposition 3.11](#), we may assume that k is non-trivially valued and k is non-trivially valued.

If $\rho(f) = 0$, then f lies in each maximal ideal of A . To see this, we may assume that A is a field, then by [Proposition 6.10](#) in the chapter Banach Rings, there is a bounded valuation $\|\bullet\|'$ on A . But then $\rho(f) = 0$ implies that $\|f\|' = 0$ and hence $f = 0$.

It follows that if $\rho(f) = 0$ then f lies in $J(A)$, the Jacobson radical of A . By [Proposition 4.14](#), A is a Jacobson ring. So f is nilpotent. The assertion follows.

So we can assume that $\rho(f) > 0$. In this case, by [Corollary 5.2](#) and [Proposition 9.11](#) in the chapter Banach Rings, we have $\rho(f) \in \sqrt{|k^\times|}$. Take $a \in k^\times$ and $d \in \mathbb{Z}_{>0}$ so that $\rho(f)^d = |a|$. Then $\rho(f^d/a) = 1$ and hence it is powerly-bounded by [Proposition 6.1](#). It follows that there is $C > 0$ so that for $n \geq 1$,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that $\|f^n\| \leq C\rho(f)$ for $n \geq d$ as long as we enlarge C . \square

Corollary 6.5. Let $\varphi : A \rightarrow B$ be a bounded homomorphism of k -affinoid algebras. Let $n \in \mathbb{N}$ and $f_1, \dots, f_n \in B$ and $r_1, \dots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$ for $i = 1, \dots, n$. Write $r = (r_1, \dots, r_n)$, then there is a unique bounded homomorphism $\Phi : A\{r^{-1}T\} \rightarrow B$ extending φ and sending T_i to f_i .

PROOF. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from [Proposition 6.4](#) that the right-hand side the series converges. The boundedness of Φ is obvious. \square

Proposition 6.6. Let $(A, \|\bullet\|_A), (B, \|\bullet\|_B)$ be k -affinoid algebras, $r \in \mathbb{R}_{>0}^n$ and $\varphi : A\{r^{-1}T\} \rightarrow B$ be an admissible epimorphism. Write $f_i = \varphi(T_i)$ for $i = 1, \dots, n$. Then there is $\epsilon > 0$ such that for any $g = (g_1, \dots, g_n) \in B^n$ with $\|f_i - g_i\|_B < \epsilon$ for all $i = 1, \dots, n$, there exists a unique bounded k -algebra homomorphism $\psi : A\{r^{-1}T\} \rightarrow B$ that coincides with φ on A and sends T_i to g_i . Moreover, ψ is also an admissible epimorphism.

PROOF. The uniqueness of ψ is obvious. We prove the remaining assertions. Taking $\epsilon > 0$ small enough, we could further guarantee that $\rho(g_i) \leq r_i$. It follows from [Corollary 6.5](#) that there exists a bounded homomorphism ψ as in the statement of the proposition.

As φ is an admissible epimorphism, we may assume that $\|\bullet\|_B$ is the residue induced by $\|\bullet\|_r$ on $A\{r^{-1}T\}$.

By definition of the residue norm, for any $\delta > 0$ and any $h \in B$, we can find

$$k_0 = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha \leq (1 + \delta) \|h\|_B$$

for any $\alpha \in \mathbb{N}^n$. Choose $\epsilon \in (0, (1 + \delta)^{-1})$. Now for g_1, \dots, g_n as in the statement of the proposition, we can write

$$h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha f^\alpha = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g^\alpha + h_1 = \psi(k_0) + h_1.$$

It follows that

$$\|h_1\|_B = \left\| \sum_{\alpha \in \mathbb{N}^n} a_\alpha (f^\alpha - g^\alpha) \right\|_B \leq (1 + \delta) \epsilon \|h\|_B.$$

Repeating this procedure, we can construct $k_i \in A\{r^{-1}T\}$ for $i \in \mathbb{N}$ and $h_j \in B$ for $j \in \mathbb{Z}_{>0}$ such that for any $i \in \mathbb{Z}_{>0}$, we have

$$h = \psi(k_0 + \dots + k_{i-1}) + h_i,$$

$$\|k_i\|_r \leq ((1 + \delta)\epsilon)^i (1 + \delta) \|h\|_B,$$

$$\|h_i\|_B \leq ((1 + \delta)\epsilon)^i \|h\|_B.$$

In particular, $k := \sum_{i=0}^{\infty} k_i$ converges in $A\{r^{-1}T\}$ and

$$\|k\|_r \leq (1 + \delta) \|h\|_B.$$

It follows that ψ is an admissible epimorphism. \square

Corollary 6.7. Let A be a Banach k -algebra, $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k -free polyray. Assume that $A \hat{\otimes}_k k_r$ is k_r -affinoid, then A is k -affinoid.

If $A \hat{\otimes}_k k_r$ is k_H -affinoid and $r \in H$, then A is also k_H -affinoid.

PROOF. We may assume that r has only one component.

Take $m \in \mathbb{N}$, $p_1, \dots, p_m \in \mathbb{R}_{>0}$ and an admissible epimorphism

$$\pi : k_r\{p_1^{-1}S_1, \dots, p_m^{-1}S_m\} \rightarrow A \hat{\otimes}_k k_r.$$

Let

$$\pi(S_i) = \sum_{j=-\infty}^{\infty} a_{i,j} T^j, \quad a_{i,j} \in A$$

for $i = 1, \dots, m$. By [Proposition 6.6](#), we may assume that there is a large integer l such that $a_{i,j} = 0$ for $|j| > l$ and for any $i = 1, \dots, m$. We define $B = k\{p_i^{-1} r^j T_{i,j}\}$, $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$. Let $\varphi : B \rightarrow A$ be the bounded k -algebra homomorphism sending $T_{i,j}$ to $a_{i,j}$. The existence of φ is guaranteed by [Corollary 6.5](#).

We claim that φ is an admissible epimorphism. It is clearly an epimorphism. Let us show that φ is admissible. Let $\eta : k_r\{p_1^{-1} S_1, \dots, p_m^{-1} S_m\} \rightarrow B \hat{\otimes}_k k_r$ be the bounded homomorphism sending S_i to $\sum_{j=-l}^l T_{i,j} T^j$, then we have the following commutative diagram

$$\begin{array}{ccc} k_r\{p^{-1} S\} & & \\ \downarrow \eta & \searrow \pi & \\ B \hat{\otimes}_k k_r & \xrightarrow{\varphi \hat{\otimes}_k k_r} & A \hat{\otimes}_k k_r \end{array}$$

It follows that $\varphi \hat{\otimes}_k k_r$ is also an admissible epimorphism. By [Proposition 3.11](#), φ is also admissible. \square

7. H -strict affinoid algebras

Let $(k, |\cdot|)$ be a complete non-Archimedean valued field and H be a subgroup of $R_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

We next give a non-strict extension of [Proposition 3.13](#).

Proposition 7.1. Let B be a k_H -affinoid algebra and $\varphi : B \rightarrow A$ be a finite bounded homomorphism into a k -Banach algebra A . Then A is also k_H -affinoid.

PROOF. We first assume that k is non-trivially valued.

We may assume that $B = k\{r_1^{-1} T_1, \dots, r_n^{-1} T_n\}$ for some $n \in \mathbb{N}$ and $r_1, \dots, r_n \in H$. By assumption, we can find finitely many $a_1, \dots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(B) a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By [Proposition 4.18](#) in the chapter Banach Rings, φ admits a unique extension to a bounded k -algebra epimorphism

$$\Phi : k\{r_1^{-1} T_1, \dots, r_n^{-1} T_n, S_1, \dots, S_m\} \rightarrow A$$

sending S_i to a_i . By [Corollary 7.5](#) in the chapter Banach Rings, Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is k_H -affinoid.

If k is trivially valued, then H is non-trivial. Take $s \in H \setminus \{1\}$. It follows from the previous case applied to $\varphi \hat{\otimes}_k k_s : B \hat{\otimes}_k k_s \rightarrow A \hat{\otimes}_k k_s$ that $A \hat{\otimes}_k k_s$ is k_H -affinoid. By [Corollary 6.7](#), A is also k_H -affinoid. \square

Proposition 7.2. Let A be a Banach k -algebra. Then the following are equivalent:

- (1) A is k_H -affinoid;
- (2) there are $n \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H}$ and an admissible epimorphism $k\{r^{-1} T\} \rightarrow A$.

PROOF. The non-trivial direction is (2). Assume (2). Take $s_1, \dots, s_n \in \mathbb{Z}_{>0}$, $c_1, \dots, c_n \in k^\times$ and $h_1, \dots, h_n \in H$ such that

$$r_i^{s_i} = |c_i^{-1}| h_i$$

for $i = 1, \dots, n$. We define a bounded k -algebra homomorphism

$$\varphi : k\{h_1^{-1}T_1, \dots, h_n^{-1}T_n\} \rightarrow k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$$

by sending T_i to $c_i T_i^{s_i}$. The existence of such a homomorphism is guaranteed by [Corollary 6.5](#). The same proof of [Lemma 3.15](#) shows that φ is finite. By [Proposition 7.1](#), $k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ is k_H -affinoid. \square

Lemma 7.3. Assume that k is non-trivially valued. Let A be a k -affinoid algebra. Then the following are equivalent:

- (1) A is strictly k -affinoid;
- (2) for any $a \in A$, $\rho(a) \in \sqrt{|k^\times|} \cup \{0\}$.

PROOF. (1) \implies (2) by [Corollary 5.10](#) and [Corollary 6.2](#).

(2) \implies (1): Take $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Let $f_i = \varphi(T_i)$ for $i = 1, \dots, n$. Suppose $r_1, \dots, r_m \notin \sqrt{|k^\times|}$ and $r_{m+1}, \dots, r_n \in \sqrt{|k^\times|}$. Then $\rho(f_i) < r_i$ for $i = 1, \dots, m$ and we can choose $r'_1, \dots, r'_m \in \sqrt{|k^\times|}$ such that

$$\rho(f_i) \leq r'_i < r_i$$

for $i = 1, \dots, m$. Set $r'_i = r_i$ when $i = m+1, \dots, n$. We can then define a bounded k -algebra homomorphism $\psi : k\{r'^{-1}T\} \rightarrow A$ sending T_i to f_i for $i = 1, \dots, n$. The existence of ψ is guaranteed by [Corollary 6.5](#). Observe that ψ is surjective and admissible. It follows that A is strictly k -affinoid. \square

Theorem 7.4. Let A be a k -affinoid algebra. Then the following are equivalent:

- (1) A is k_H -affinoid;
- (2) A is $k_{\sqrt{|k^\times|} \cdot H}$ -affinoid;
- (3) For any non-zero $a \in A$, $\rho(a) \in \sqrt{|k^\times| \cdot H} \cup \{0\}$.

PROOF. The equivalence between (1) and (2) follows from [Proposition 7.2](#).

(1) \implies (3): we may assume that $H \supseteq |k^\times|$. Take $n \in \mathbb{N}$, $r = (r_1, \dots, r_n) \in H^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \rightarrow A.$$

Take a k -free polyray s with at least one component so that $|k_s| \supseteq \{r_1, \dots, r_n\}$. We can apply [Lemma 7.3](#) to $\varphi \hat{\otimes}_k k_s$, it follows that $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$.

(3) \implies (2): we may assume that $H \supseteq |k^\times|$. It suffices to apply the same argument as (2) \implies (1) in the proof of [Lemma 7.3](#). \square

8. Finite modules over affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

For any k -affinoid algebra A , we have defined the category $\mathcal{B}\text{an}_A^f$ of finite Banach A -modules in [Definition 5.3](#) in the chapter Banach Rings. We write $\mathcal{M}\text{od}_A^f$ for the category of finite A -modules.

Lemma 8.1. Let A be a k -affinoid algebra, $(M, \|\bullet\|_M)$ be a finite Banach A -module and $(N, \|\bullet\|_N)$ be a Banach A -module N . Let $\varphi : M \rightarrow N$ be an A -linear homomorphism. Then φ is bounded.

PROOF. Take $n \in \mathbb{N}$ such that there is an admissible epimorphism

$$\pi : A^n \rightarrow M.$$

It suffices to show that $\varphi \circ \pi$ is bounded. So we may assume that $M = A^n$. For $i = 1, \dots, n$, let e_i be the vector with $(0, \dots, 0, 1, 0, \dots, 0)$ of A^n with 1 placed at the i -th place. Set $C = \max_{i=1, \dots, n} \|\varphi(e_i)\|_N$. For a general $f = \sum_{i=1}^n a_i e_i$ with $a_i \in A$, we have

$$\|\varphi(f)\|_N \leq C \|f\|_M.$$

So φ is bounded. \square

Proposition 8.2. Let A be a k -affinoid algebra. The forgetful functor $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$ is an equivalence of categories.

PROOF. It suffices to construct the inverse functor. Let M be a finite A -module. Choose $n \in \mathbb{N}$ and an A -linear epimorphism $\pi : A^n \rightarrow M$. By [Theorem 6.3](#), $\ker \pi$ is closed in A^n . We can endow M with the residue norm. By [Lemma 8.1](#), the equivalence class of the norm does not depend on the choice of π .

For any A -linear homomorphism $f : M \rightarrow N$ of finite A -modules, we endow M and N with the Banach structures as above. It follows from [Lemma 8.1](#) that f is bounded. We have defined the inverse functor of the forgetful functor $\mathcal{B}\text{an}_A^f \rightarrow \mathcal{M}\text{od}_A^f$. \square

Remark 8.3. Let A be a k -affinoid algebra. It is not true that a Banach A -module which is finite as A -module is finite as Banach A -module.

As an example, take $0 < p < q < 1$ and $A = k\{q^{-1}T\}$, $B = k\{p^{-1}T\}$. Then B is a Banach A -module. By [Example 2.4](#), the underlying rings of A and B are both $k[[T]]$. So the canonical map $A \rightarrow B$ is bijective. But B is not a finite A -module. As otherwise, the inverse map $B \rightarrow A$ is bounded by [Lemma 8.1](#), which is not the case.

The correct statement is the following: consider a Banach A -module $(M, \|\bullet\|_M)$ which is finite as A -module, then there is a norm on M such that M becomes a finite Banach A -module. The new norm is not necessarily equivalent to the given norm $\|\bullet\|_M$.

Proposition 8.4. Let A be a k -affinoid algebra and M, N be finite Banach A -modules. Then the natural map

$$M \otimes_A N \rightarrow M \hat{\otimes}_A N$$

is an isomorphism of Banach A -modules and $M \hat{\otimes}_A N$ is a finite Banach A -module.

Here the Banach A -module structure on $M \otimes_A N$ is given by [Proposition 8.2](#).

PROOF. Choose $m, m' \in \mathbb{N}$ an admissibly coexact sequence

$$A^{m'} \rightarrow A^m \rightarrow M \rightarrow 0$$

of Banach A -modules. Then we have a commutative diagram of A -modules:

$$\begin{array}{ccccccc} A^{m'} \otimes_A N & \longrightarrow & A^m \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{m'} \hat{\otimes}_A N & \longrightarrow & A^m \hat{\otimes}_A N & \longrightarrow & M \hat{\otimes}_A N & \longrightarrow & 0 \end{array}$$

with exact rows. By 5-lemma, in order to prove $M \otimes_A N \xrightarrow{\sim} M \hat{\otimes}_A N$ and $M \hat{\otimes}_A N$ is a finite Banach A -module, we may assume that $M = A^m$ for some $m \in \mathbb{N}$. Similarly, we can assume $N = A^n$ for some $n \in \mathbb{N}$. In this case, the isomorphism is immediate and $M \hat{\otimes}_A N$ is clearly a finite Banach A -module. By [Lemma 8.1](#), the Banach A -module structure on $M \hat{\otimes}_A N$ coincides with the Banach A -module structure on $M \otimes_A N$ induced by [Proposition 8.2](#). \square

Proposition 8.5. Let A, B be a k -affinoid algebra and $A \rightarrow B$ be a bounded k -algebra homomorphism. Let M be a finite Banach A -module, then the natural map

$$M \otimes_A B \rightarrow M \hat{\otimes}_A B$$

is an isomorphism of Banach B -modules and $M \hat{\otimes}_A B$ is a finite Banach B -module.

PROOF. By the same argument as [Proposition 8.4](#), we may assume that $M = A^n$ for some $n \in \mathbb{N}$. In this case, the assertions are trivial. \square

Proposition 8.6. Let A be a k -affinoid algebra and M, N be finite Banach A -modules. Let $\varphi : M \rightarrow N$ be an A -linear map. Then φ is admissible.

PROOF. By [Lemma 8.1](#), φ is always bounded. By [Proposition 8.5](#) and [Proposition 3.11](#), we may assume that k is non-trivially valued. By [Theorem 6.3](#), N is a Noetherian A -module. It follows from [Corollary 7.4](#) in the chapter Banach Rings that $\text{Im } \varphi$ is closed in N and is finite as an A module. In particular, the norm induced from N and from M are equivalent by [Lemma 8.1](#). It follows that φ is admissible. \square

Proposition 8.7. Let A be a k -affinoid algebra. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n)$ be a k -free polyray. Then M is a finite Banach A -module if and only if $M \hat{\otimes}_k k_r$ is a finite Banach $A \hat{\otimes}_k k_r$ -module.

PROOF. We may assume that r has only one component and write $r_1 = r$. The direct implication is trivial. Let us assume that $M \hat{\otimes}_k k_r$ is a finite Banach $A \hat{\otimes}_k k_r$ -module. Take $n \in \mathbb{N}$ and an admissible epimorphism of $A \hat{\otimes}_k k_r$ -modules

$$\varphi : (A \hat{\otimes}_k k_r)^n \rightarrow M \hat{\otimes}_k k_r.$$

Let e_1, \dots, e_n denotes the standard basis of $(A \hat{\otimes}_k k_r)^n$. We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By [Proposition 6.6](#), we can assume that there is $l > 0$ such that $m_{i,j} = 0$ for all $i = 1, \dots, n$ and $|j| > l$. It follows that

$$A^{n(2l+1)} \rightarrow M$$

sending the standard basis to $m_{i,j}$ with $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$ is an admissible epimorphism. \square

Proposition 8.8. Let $\phi : A \rightarrow B$ be a morphism of k -affinoid algebras, $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}^n$. Then the following are equivalent:

- (1) ϕ is finite and admissible.
- (2) $\phi \hat{\otimes}_k k_r$ is finite and admissible.

This is [Tem04, Lemma 3.2]. I do not understand Temkin's argument. The following proof is a modification of the argument of Temkin's.

PROOF. (1) \implies (2): This is straightforward.

(2) \implies (1): The admissible part is straightforward. Let us prove that ϕ is finite. We may assume that $n = 1$. When r is not in $\sqrt{|k^\times|}$, we just apply [Proposition 8.7](#). Now suppose $r \in \sqrt{|k^\times|}$. Let us take $m \in \mathbb{Z}_{>0}$ such that $r^m = |c^{-1}|$ for some $c \in k^\times$. Define a bounded k -algebra homomorphism

$$\varphi : k\{T\} \rightarrow k\{r^{-1}T\}$$

sending T to cT^m . Observe that φ is injective. We have argued in the proof of [Lemma 3.15](#) that this homomorphism is finite.

Then φ induces a finite extension of ring $\text{Frac } k\{r^{-1}T\} / \text{Frac } k\{T\}$. In particular, the closure of $\text{Frac } k\{T\}$ in k_r is a subfield over which k_r is finite. But this valuation field is isomorphic to $k\{T\}$. By [Proposition 8.5](#) and fpqc descent [[Stacks](#), [Tag 02LA](#)], we may assume that $r = 1$.

Recall that k_1 is the completion of $\text{Frac } k\{T\}$. Let $\{\tilde{f}_i\}_{i \in I}$ be the set of irreducible monic polynomials in $\tilde{k}[T]$. Lift each \tilde{f}_i to $f_i \in k[T]$. Let $a \in A \hat{\otimes}_k k_1$, we represent a as

$$a = \sum_{l=0}^{\infty} a_l T^l + \sum_{i \in I, j \geq 1, 0 \leq k < \deg f_i} a_{ijk} T^k / f_i^j.$$

A similar expression exists for elements in $B \hat{\otimes}_k k_1$ as well. Moreover, the representation is unique.

As $B \hat{\otimes}_k k_1$ is finite over $A \hat{\otimes}_k k_1$, we can find b_1, \dots, b_m such that any $b \in B$ can be written as

$$b = \sum_{j=1}^m \phi \hat{\otimes}_k k_1(a_j) b_j,$$

where $a_j \in A \hat{\otimes}_k k'$. We can replace b_j by $b_{j,0}$ and a_j by $a_{j,0}$. It follows that B is generated $b_{1,0}, \dots, b_{m,0}$ over A . \square

For any ring A , Alg_A^f denotes the category of finitely generated A -algebras.

Proposition 8.9. Let A be a k -affinoid algebra. Then the forgetful functor $\text{BanAlg}_A^f \rightarrow \text{Alg}_A^f$ is an equivalence of categories.

Recall that BanAlg_A^f is defined in [Definition 5.9](#) in the chapter Banach Rings.

PROOF. It suffices to construct an inverse functor. Let B be a finite A -algebra. We endow B with the norm $\|\bullet\|_B$ as in [Proposition 8.2](#). We claim that B is a Banach A -algebra.

Let us recall the definition of the norm. Take $n \in \mathbb{N}$, an epimorphism $\varphi : A^n \rightarrow B$ of A -modules. Then $\|\bullet\|_B$ is the residue norm induced by φ .

Consider the A -linear epimorphism $\psi : A^n \otimes_A A^n \rightarrow B \otimes_A B$. By [Proposition 8.6](#), when both sides are endowed with the norms $\|\bullet\|_{A^n \otimes_A A^n}$ and $\|\bullet\|_{B \otimes_A B}$ as in [Proposition 8.2](#), ψ is admissible. It follows that there is $C > 0$ such that for any $f, g \in B$,

$$\|f \otimes g\|_{B \otimes B} \leq C \|f\|_B \cdot \|g\|_B.$$

On the other hand, by [Proposition 8.2](#), the natural map $B \otimes_A B \rightarrow B$ is bounded. It follows that there is a constant $C' > 0$ such that

$$\|fg\|_B \leq C' \|f \otimes g\|_{B \otimes B}.$$

It follows that the multiplication in B is bounded and hence B is a finite Banach algebra. Given any morphism $B \rightarrow B'$ in Alg_A^f , we endow B and B' with the norms given by [Proposition 8.2](#). It follows from [Lemma 8.1](#) that $B \rightarrow B'$ is a bounded homomorphism of finite Banach A -algebras. So we have defined an inverse functor to the forgetful functor $\text{BanAlg}_A^f \rightarrow \text{Alg}_A^f$. \square

Remark 8.10. It is not true that any homomorphism of k -affinoid algebras is bounded. For example, if the valuation on k is trivial. Take $0 < p < q < 1$ and consider the natural homomorphism $k_p \rightarrow k_q$. This homomorphism is bijective but not bounded.

9. Affinoid domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 9.1. Let A be a k_H -affinoid algebra. A closed subset $V \subseteq \text{Sp } A$ is said to be a k_H -affinoid domain in X if there is an object $\text{Sp } A_V \in k_H\text{-Aff}$ and a morphism $\phi : \text{Sp } A_V \rightarrow \text{Sp } A$ in $k_H\text{-Aff}$ such that

- (1) the image of ϕ in $\text{Sp } A$ is V ;
- (2) given any object $\text{Sp } B \in k_H\text{-Aff}$ and a morphism $\text{Sp } B \rightarrow \text{Sp } A$ whose image lies in V , there is a unique morphism $\text{Sp } B \rightarrow \text{Sp } A_V$ in $k_H\text{-Aff}$ such that the following diagram commutes

$$\begin{array}{ccc} \text{Sp } B & & \\ \downarrow \text{!} & \searrow & \\ \text{Sp } A_V & \xrightarrow{\phi} & \text{Sp } A \end{array}$$

We say V is *represented by* the morphism ϕ or by the corresponding morphism $A \rightarrow A_V$.

When $H = \mathbb{R}_{>0}$, we say V is a k -affinoid domain in X . When $H = |k^\times|$, we say V is a *strict k -affinoid domain* in X .

We observe that A_V is canonically determined by the universal property.

Remark 9.2. This definition differs from the original definition of [\[Ber12\]](#), we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the original definition of Berkovich when $H = \mathbb{R}_{>0}$.

We begin with a few examples.

Example 9.3. Let A be a k_H -affinoid domain. Let $n, m \in \mathbb{N}$ and $f = (f_1, \dots, f_n) \in A^n$, $g = (g_1, \dots, g_m) \in A^m$. Let $r = (r_1, \dots, r_n) \in \sqrt{|k^\times|} \cdot H^n$ and $s = (s_1, \dots, s_m) \in \sqrt{|k^\times|} \cdot H^m$. We define

$$(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\} := \{x \in \mathrm{Sp} A : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

We claim that $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ is a k_H -affinoid domain in $\mathrm{Sp} A$. These domains are called *k_H -Laurent domains* in $\mathrm{Sp} A$. When $m = 0$, the domains $\mathrm{Sp} A \{r^{-1}f\}$ are called *k_H -Weierstrass domains* in $\mathrm{Sp} A$.

To see this, we define

$$A \{r^{-1}f, sg^{-1}\} := A \{r^{-1}T, sS\} / (T_1 - f_1, \dots, T_n - f_n, g_1 S_1 - 1, \dots, g_m S_m - 1).$$

By [Theorem 6.3](#), this defines a Banach k -algebra structure. We write $\|\bullet\|'$ for the quotient norm. By definition, $A \{r^{-1}f, sg^{-1}\}$ is a k_H -affinoid algebra and there is a natural morphism $A \rightarrow A \{r^{-1}f, sg^{-1}\}$. We claim that this morphism represents $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$.

For this purpose, we first compute $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$. We observe that $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\} \rightarrow \mathrm{Sp} A$ is injective since $A[f, g^{-1}]$ is dense in $A \{r^{-1}f, sg^{-1}\}$. We will therefore identify $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ with a subset of $\mathrm{Sp} A$.

Next we show that the image of $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$ in $\mathrm{Sp} A$ is contained in $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$. Take $\|\bullet\| \in \mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$. Then there is a constant $C > 0$ such that

$$\|\bullet\| \leq C \|\bullet\|'.$$

Applying this to f_i^k for some $k \in \mathbb{Z}_{>0}$ and $i = 1, \dots, n$, we find that

$$\|f_i\|^k = \|f_i^k\| \leq C \|f_i^k\|' \leq C \|T_i^k\|_{r, s^{-1}} = C r_i^k.$$

It follows that

$$\|f_i\| \leq r_i.$$

Similarly, we deduce $|g_j| \geq s_j$ for $j = 1, \dots, m$. Namely, $\|\bullet\| \in (\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$.

Next we verify the universal property: let $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism of k_H -affinoid domains that factorizes through $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$. We write $\psi : A \rightarrow B$ for the corresponding morphism of k_H -affinoid algebras. By [Corollary 6.12](#) in the chapter Banach Rings, we have

$$\rho_B(f_i) = \sup_{x \in \mathrm{Sp} B} |f_i(x)| \leq \sup_{y \in (\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}} |f_i(y)| \leq r_i$$

for $i = 1, \dots, n$. Similarly, one deduces that $\rho(g_j) \leq s_j^{-1}$ for $j = 1, \dots, m$.

We will construct the dotted arrows:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow & \searrow \eta & \uparrow \\ A \{r^{-1}T, sS\}^\tau & & \\ \downarrow & \nearrow & \\ A \{r^{-1}f, sg\} & & \end{array}$$

so that this diagram commutes. We define η as the unique morphism sending T_i to f_i and S_j to g_j for $i = 1, \dots, n$, $j = 1, \dots, m$. The existence of such a morphism is guaranteed by [Proposition 6.6](#). In order to descend this morphism to η' , it suffices

to show that $T_i - f_i$ and $g_j S_j - 1$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ lie in the kernel of η . But this is immediate from our definition. Moreover, it is clear that η' is necessarily unique.

It remains to show that each point in $(\mathrm{Sp} A) \{r^{-1}f, sg^{-1}\}$ lies in $\mathrm{Sp} A \{r^{-1}f, sg^{-1}\}$.

It suffices to treat the cases $(n, m) = (1, 0)$ and $(n, m) = (0, 1)$. We will only handle the former case, as the latter is similar. In concrete terms, we need to show that for any $x \in \mathrm{Sp} A$ corresponding to a bounded semi-valuation $|\bullet|_x$ on A satisfying $|f(x)| \leq r$, we can always extend $|\bullet|_x$ to a bounded semi-valuation $\|\bullet\|$ on $A\{r^{-1}f\}$. Replacing A by $A/\ker |\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A . We endow $A\{r^{-1}T\}$ with the Gauss norm $\|\bullet\|_{x,r}$ induced by $|\bullet|_x$ and $A\{r^{-1}T\}$ with the quotient norm $\|\bullet\|$. This norm is bounded by construction. It suffices to show that it is a valuation and it extends the given valuation on A . The former is a consequence of the latter, as A is dense in $A\{r^{-1}T\}$. Now suppose $a \in A$. A general preimage of a in $A\{r^{-1}T\}$ is

$$a + (T - f) \sum_{j=0}^{\infty} b_j T^j = a - fb_0 + \sum_{j=1}^{\infty} (b_{j-1} - fb_j) T^j$$

with $\|b_j\|_A r^j \rightarrow 0$ as $j \rightarrow \infty$. Now we compute

$$\begin{aligned} \|a - fb_j + \sum_{j=1}^{\infty} (b_{j-1} - fb_j) T^j\|_{x,r} &= \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |b_{j-1} - fb_j|_x r^j \right\} \\ &\geq \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |b_{j-1} - fb_j|_x |f|_x^j \right\} \\ &= \max \left\{ |a - fb_0|_x, \max_{j \geq 1} |f^j b_{j-1} - f^{j+1} b_j|_x \right\} \geq |a|_x. \end{aligned}$$

So $\|a\| \geq |a|_x$. The reverse inequality is trivial. We conclude.

Example 9.4. Let A be a k_H -affinoid domain. Let $n \in \mathbb{N}$, $g \in A$, $f = (f_1, \dots, f_n) \in A^n$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times|} \cdot H^n$. Assume that g, f_1, \dots, f_n generates the unit ideal. Define

$$(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\} = \{x \in \mathrm{Sp} A : |f_i(x)| \leq r_i |g(x)| \text{ for } i = 1, \dots, n\}.$$

Then we claim that $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$ is a k_H -affinoid domain in $\mathrm{Sp} A$. Domains of this form are called *k_H -rational domains*.

To see this, we define

$$A \left\{ r^{-1} \frac{f}{g} \right\} := A\{r^{-1}T\} / (gT_1 - f_1, \dots, gT_n - f_n).$$

By [Theorem 5.1](#), this is indeed a k_H -affinoid domain. We will denote by $\|\bullet\|'$ the residue norm. We will prove that the natural map $A \rightarrow A \left\{ r^{-1} \frac{f}{g} \right\}$ represents the affinoid domain $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$. Observe that

$$\mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$$

is injective as elements of the form a/g with $a \in A$ is dense in $A \left\{ r^{-1} \frac{f}{g} \right\}$. Next we show that

$$(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\} \supseteq \mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}.$$

Let $x \in \mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$, take $|\bullet|_x$ as the corresponding bounded semi-valuation on $A \left\{ r^{-1} \frac{f}{g} \right\}$. Then there is a constant $C > 0$ such that for any $k \in \mathbb{Z}_{>0}$,

$$|f_i|_x^k = |f_i^k|_x = |g|_x^k \cdot |T_i^k|_x \leq C |g|_x^k r_i^k.$$

for all $i = 1, \dots, n$. In particular,

$$|f_i|_x \leq r_i |g|_x.$$

Hence, $x \in (\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$.

Next we verify the universal property. Let $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism of k_H -affinoid spectra factorizing through $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$. Observe that $g(x) \neq 0$ for all $x \in (\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$. As otherwise, $f_i(x) = 0$ for all $i = 1, \dots, n$. This contradicts our assumption on g, f_1, \dots, f_n . It follows that $\psi(g)$ is invertible by [Corollary 6.11](#) in the chapter Banach Rings. From the definition of $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$, it is clear that $\rho(\psi(f_i)) \leq r\rho(\psi(g))$ for $i = 1, \dots, n$.

We construct

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow & \searrow \eta & \uparrow \\ A\{r^{-1}T\} & \xrightarrow{\tau} & \\ \downarrow & \nearrow & \\ A\left\{r^{-1}\frac{f}{g}\right\} & & \end{array}$$

successively. The morphism η sends T_i to $\psi(f_i)/\psi(g)$ for $i = 1, \dots, n$. The existence of such a morphism is guaranteed by [Proposition 6.6](#). Clearly $gT_i - f_i$ is contained in $\ker \eta$, so η descends to τ . The morphism τ is clearly unique.

It remains to verify that the image of $\mathrm{Sp} A \left\{ r^{-1} \frac{f}{g} \right\}$ in $\mathrm{Sp} A$ is exactly $(\mathrm{Sp} A) \left\{ r^{-1} \frac{f}{g} \right\}$. In other words, we need to verify that if $|\bullet|_x$ is a bounded semi-valuation on A satisfying $|f_i|_x \leq r_i |g|_x$, then $|\bullet|_x$ extends to a bounded semi-valuation on $A \left\{ r^{-1} \frac{f}{g} \right\}$. Replacing A by $A/\ker |\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A . Consider the Gauss valuation $|\bullet|_{x,r}$ on $A\{r^{-1}T\}$ and the residue norm $\|\bullet\|$ on $A \left\{ r^{-1} \frac{f}{g} \right\}$. It suffices to show that $\|\bullet\|$ is a valuation extending the valuation $|\bullet|_x$ on A . The former is a consequence of the latter. Take $a \in A$, we need to show that $|a|_x = \|a\|$.

A general preimage of a in $A\{r^{-1}T\}$ has the form

$$a + \sum_{i=1}^n (gT_i - f_i) \sum_{\alpha \in \mathbb{N}^n} b_{i,\alpha} T^\alpha$$

with $\|b_{i,\alpha}\|_A r^\alpha$, where $\|\bullet\|_A$ denotes the initial norm on A . The same argument as in [Example 9.3](#) shows that

$$\|a + \sum_{i=1}^n (gT_i - f_i) \sum_{\alpha \in \mathbb{N}^n} b_{i,\alpha} T^\alpha\|_{x,r} \geq |a|_x.$$

So $\|a\|_x \geq |a_x|$, the reverse inequality is trivial.

Proposition 9.5. Let A be a k_H -affinoid algebra and $V \subseteq \mathrm{Sp} A$ be a k_H -affinoid domain represented by $\varphi : A \rightarrow A_V$. Then $\mathrm{Sp} \varphi$ induces a homeomorphism $\mathrm{Sp} A_V \rightarrow V$.

In particular, we will identify V with $\mathrm{Sp} A_V$ and say $\mathrm{Sp} A_V$ is a k_H -affinoid domain in $\mathrm{Sp} A$.

PROOF. We observe that $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ is a monomorphism in the category $k_H\text{-Aff}$. In other words, $A \rightarrow A_V$ is an epimorphism in the category $k_H\text{-AffAlg}$. To see this, let $\eta_1, \eta_2 : A_V \rightarrow B$ be two arrows in $k_H\text{-AffAlg}$ such that $\eta_1 \circ \varphi = \eta_2 \circ \varphi$. It follows from the universal property in [Definition 9.1](#) that $\eta_1 = \eta_2$. By [Proposition 3.18](#), $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ is a bijection. But $\mathrm{Sp} A_V$ and $\mathrm{Sp} A$ are both compact and Hausdorff by [Theorem 6.13](#) in the chapter Banach rings, so $\mathrm{Sp} A_V \rightarrow V$ is a homeomorphism. \square

Proposition 9.6. Let A be a k_H -affinoid algebra and V, W be k_H -Weierstrass domains (resp. k_H -Laurent domains, resp. k_H -rational domains) in $\mathrm{Sp} A$. Then $V \cap W$ is also a k_H -Weierstrass domain (resp. k_H -Laurent domain, resp. k_H -rational domain).

PROOF. This is clear in the Weierstrass and Laurent cases. We will prove therefore assume that V and W are k_H -rational.

We take $f_1, \dots, f_n \in A$, $g_1, \dots, g_m \in A$ both generating the unit ideal and $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H^n}$, $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H^m}$ such that

$$V = \mathrm{Sp} A \left\{ r^{-1} \frac{f}{f_m} \right\}, \quad W = \mathrm{Sp} A \left\{ s^{-1} \frac{g}{g_n} \right\}.$$

We may assume that $r_n = s_m = 1$. Now let $R = (R_{i,j}) \in \sqrt{|k^\times| \cdot H^{mn}}$ where $R_{i,j} = r_i s_j$ and $F = (F_{i,j})$ with $F_{i,j} = f_i g_j$ for $i = 1, \dots, n$, $j = 1, \dots, m$. Observe that the $F_{i,j}$'s generate the unit ideal. We consider the k_H -rational domain

$$Z = \mathrm{Sp} A \left\{ R^{-1} \frac{F}{f_n g_m} \right\}.$$

We clearly have $V \cap W \subseteq Z$. We need to prove the reverse inequality. Let $x \in Z$, so we have

$$|f_i g_j(x)| \leq r_i s_j |f_n g_m(x)|$$

for any $i = 1, \dots, n$, $j = 1, \dots, m$. In particular, when $j = m$, we have

$$|f_i g_m(x)| \leq r_i |f_n g_m(x)|$$

for any $i = 1, \dots, n$. But $f_n g_m$ is invertible, so we can cancel $g_m(x)$ to find

$$|f_i(x)| \leq r_i |f_n(x)|.$$

So $x \in V$. Similarly, we have $x \in W$. \square

Corollary 9.7. Let A be a k_H -affinoid algebra and V be a k_H -Laurent domain in $\mathrm{Sp} A$. Then V is also a k_H -rational domain.

PROOF. By [Proposition 9.6](#), it suffices to show consider k_H -Laurent domains of the following form:

$$\mathrm{Sp} A\{r^{-1}f\}, \quad \mathrm{Sp} A\{sg^{-1}\}$$

where $r, s \in \sqrt{|k^\times| \cdot H}$ and $f, g \in A$. Both domains are k_H -rational by definition. \square

Proposition 9.8. Let A be a k_H -affinoid algebra and $\mathrm{Sp} B$ be a k_H -rational domain in $\mathrm{Sp} A$. Then there is a k_H -Laurent domain $\mathrm{Sp} C$ in $\mathrm{Sp} A$ such that $\mathrm{Sp} B \subseteq \mathrm{Sp} C$ and $\mathrm{Sp} B$ is a k_H -Weierstrass domain in $\mathrm{Sp} C$.

PROOF. We write

$$B = A \left\{ r^{-1} \frac{f}{g} \right\}$$

for some $n \in \mathbb{N}$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H^n}$, $f = (f_1, \dots, f_n) \in A^n$ and $g \in A$ such that f_1, \dots, f_n, g generate the unit ideal. Let g'' be the image of g in B , which is a unit. Choose $c \in \sqrt{|k^\times| \cdot H}$ such that $\rho_B(g^{-1}) < c^{-1}$. Set $C = A\{cg^{-1}\}$, then $\mathrm{Sp} B \subseteq \mathrm{Sp} C$. Moreover,

$$\mathrm{Sp} B \cap \mathrm{Sp} C = \emptyset.$$

Let f'_1, \dots, f'_n, g' be the images of f_1, \dots, f_n, g in C . Write $f' = (f'_1, \dots, f'_n)$. Then by [Corollary 6.11](#) in the chapter Banach Rings, g' is a unit and

$$\mathrm{Sp} B = \mathrm{Sp} C\{r^{-1}g'^{-1}f'\}.$$

\square

Proposition 9.9. Let A be a k_H -affinoid algebra, $\mathrm{Sp} B$ be a k_H -Weierstrass domain (resp. k_H -rational domain) in $\mathrm{Sp} A$ and $\mathrm{Sp} C$ be a k_H -Weierstrass domain (resp. k_H -rational domain) in $\mathrm{Sp} B$. Then $\mathrm{Sp} C$ is a k_H -Weierstrass domain (resp. k_H -rational domain) in $\mathrm{Sp} A$.

PROOF. We first handle the Weierstrass case. Write

$$B = \mathrm{Sp} A\{r^{-1}f\}, C = \mathrm{Sp} B\{s^{-1}g\}$$

for some $n, m \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H^n}$, $s \in \sqrt{|k^\times| \cdot H^m}$ and $f = (f_1, \dots, f_n) \in A^n$, $g = (g_1, \dots, g_m) \in B^m$. Observe that if we replace g with a small perturbation, the domain $\mathrm{Sp} C$ in $\mathrm{Sp} B$ remains the same, so we may assume that $g_1, \dots, g_m \in A$. Then

$$\mathrm{Sp} C = \mathrm{Sp} A\{r^{-1}f\} \cap \mathrm{Sp} A\{s^{-1}g\}$$

is a k_H -Weierstrass domain by [Proposition 9.6](#).

Next we handle the rational case. Write

$$B = A \left\{ s \frac{f}{g} \right\}$$

for some $m \in \mathbb{N}$, $f = (f_1, \dots, f_m) \in A^m$, $r = (r_1, \dots, r_m) \in \sqrt{|k^\times| \cdot H^m}$ and $g \in A$ such that f_1, \dots, f_m, g generate the unit ideal.

By [Proposition 9.8](#) and [Proposition 9.6](#), it suffices to handle the special cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ for some $r \in \sqrt{|k^\times| \cdot H}$ and $h \in B$. Observe that making a small perturbation on h does not change the domain. As $A[g^{-1}]$ is dense

in B , we may assume that there is $n \in \mathbb{Z}_{>0}$ such that $g^n h \in A$. As g is invertible on $\mathrm{Sp} B$, we can find $c \in \sqrt{|k^\times| \cdot H}$ so that

$$|g(x)|^n > c$$

for $x \in \mathrm{Sp} B$.

We need to treat the cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ separately. In the first case, we write

$$\mathrm{Sp} C = \mathrm{Sp} B \cap \mathrm{Sp} A \left\{ (r, c)^{-1} \frac{(g^n h, c)}{g^n} \right\}.$$

In the second case,..... **To be added**

□

10. Graded reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 10.1. Let A be a Banach k -algebra, we define the *graded reduction* of A as

$$\tilde{A} := \bigoplus_{h \in \mathbb{R}_{>0}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define \tilde{f} as the image of f in the $\rho(f)$ -graded piece of \tilde{A} .

Definition 10.2. Let A be a k_H -affinoid algebra. We define the k_H -*graded reduction* of A as the $\sqrt{|k^\times| \cdot H}$ -graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times| \cdot H}} \{x \in A : \rho(x) \leq h\} / \{x \in A : \rho(x) < h\}.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define \tilde{f} as the image of f in the $\rho(f)$ -graded piece of \tilde{A}^H .

For any morphism $f : A \rightarrow B$ of k_H -affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \rightarrow \tilde{B}^H$$

as the map induced by sending the class of $x \in A$ with $\rho(x) \leq h$ for any $h \in \sqrt{|k^\times| \cdot H}$ to the class of $f(x) \in B$.

Recall that $\rho(A) = \sqrt{|k^\times| \cdot H} \cup \{0\}$ by **Theorem 7.4**, so \tilde{f} is well-defined. This definition is compatible with **Definition 10.1** in the sense that if we regard a $\sqrt{|k^\times| \cdot H}$ -graded ring as a $\mathbb{R}_{>0}$ -graded ring, the two definitions give the same object.

Example 10.3. If K is a k_H -affinoid algebra which is a field as well, then \tilde{K}^H is a $\sqrt{|k^\times| \cdot H}$ -graded field. This is immediate from the definition.

Lemma 10.4. Let $(A, \|\bullet\|)$ be a k -affinoid algebra, $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}^n$. Let $f \in k\{r^{-1}T\}$. Expand f as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha.$$

PROOF. By induction, we may assume that $n = 1$ and write $r = r_1$. As ρ is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j) \rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j.$$

Observe that $\rho(a_j)$ is not ambiguous: when interpreted as in A and in $A\{r^{-1}T\}$, it has the same value.

Conversely, we need to show that for any $j \in \mathbb{N}$,

$$\rho(f) \geq \rho(a_j) r^j.$$

Equivalently, this means for any $k \in \mathbb{Z}_{>0}$ and any $j \in \mathbb{N}$, we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk}.$$

Fix j and k as above. We compute the left-hand side:

$$f^k = \sum_{\beta=(\beta_1, \dots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^k a_{\beta_l}.$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|}.$$

Take $\beta = (j, j, \dots, j)$, we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk}.$$

□

Lemma 10.5. Assume that k is non-trivially valued. Let A be a strictly k -affinoid algebra. Then for any $a, f \in A$, the set of non-zero values $\rho(f^n a)$ for $n \in \mathbb{N}$ is a discrete subset of $\mathbb{R}_{>0}$.

PROOF. As A is noetherian [Theorem 6.3](#), it has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. It follows that

$$\mathrm{Sp} A = \bigcup_{i=1}^m \mathrm{Sp} A/\mathfrak{p}_i.$$

Here we make the obvious identification by identifying $\mathrm{Sp} A/\mathfrak{p}_i$ with a subset of $\mathrm{Sp} A$.

By [Corollary 6.12](#) in the chapter Banach Rings, it suffices to consider each of $\mathrm{Sp} A/\mathfrak{p}_i$ separately, so we may assume that A is an integral domain.

By [Corollary 5.2](#), we can take $d \in \mathbb{N}$ and a finite injective homomorphism of k -algebras $\iota : k\{T_1, \dots, T_d\} \rightarrow A$. According to [Proposition 9.11](#) in the chapter Banach Rings, ρ_A is the restriction of the norm $\|\bullet\|_{\mathrm{Frac} A}$ on $\mathrm{Frac} A$ induced by the finite extension $\mathrm{Frac} A/\mathrm{Frac} k\{T_1, \dots, T_d\}$ from the Gauss valuation. But it is well-known that $\|\bullet\|_{\mathrm{Frac} A}$ is the maximum of finitely many valuations on $\mathrm{Frac} A$. [Reproduce BGR3.3.3.1 somewhere](#). The assertion is by now obvious. □

Lemma 10.6. Let $(A, \|\bullet\|)$ be a k -affinoid algebra, $f \in A$ with $r = \rho(f) > 0$. Let $B = A\{r^{-1}f\}$. Then for any $a \in A$, we have

$$\rho_B(a) = \lim_{n \rightarrow \infty} r^{-n} \rho_A(f^n a).$$

If moreover, $\rho_B(a) > 0$, then there is $n_0 > 0$ such that for $n \geq n_0$,

$$\rho_B(a) = r^{-n} \rho_A(f^n a), \quad \rho_B(f^n a) = r^{-n} \rho_A(a).$$

PROOF. We observe that for any $a \in A$, $n \in \mathbb{Z}_{>0}$, we have

$$\rho_B(f^n a) = r^n \rho_B(a).$$

So the last two assertions are equivalent.

Take a k -free polyradius s such that $A \hat{\otimes}_k k_s$ and $B \hat{\otimes}_k k_s$ are both strictly k_s -affinoid. By [Proposition 3.11](#), $A \hat{\otimes}_k k_s \{r^{-1}f\} \xrightarrow{\sim} B \hat{\otimes}_k k_s$. Moreover, ρ_A and ρ_B are both preserved after base change to k_s . So we may assume that k is non-trivially valued and A and B are strictly k -affinoid.

Observe that for $n \in \mathbb{Z}_{>0}$,

$$\rho_A(f^{n+1}a) \leq \rho_A(f)\rho_A(f^n a) = r\rho_A(f^n a).$$

So $r^{-n}\rho_A(f^n a)$ is decreasing in n . Moreover, for any $x \in \operatorname{Sp} A\{r^{-1}f\}$, by [Example 9.3](#), we have

$$|f(x)| \geq r.$$

By [Corollary 6.12](#) in the chapter Banach Rings, we have

$$|f(x)| = r$$

for any $x \in \operatorname{Sp} A\{r^{-1}f\}$. It follows from [Corollary 6.12](#) in the chapter Banach Rings that for any $n \in \mathbb{Z}_{>0}$,

$$\rho_A(f^n a) = \sup_{x \in \operatorname{Sp} A} |f^n a(x)| \geq r^n \sup_{x \in \operatorname{Sp} A\{rf^{-1}\}} |a(x)| = r^n \rho_B(a).$$

By [Lemma 10.5](#), the decreasing sequence $\{r^{-n}\rho_A(f^n a)\}_n$ either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is $\alpha \in \mathbb{R}_{>0}$ and $n_0 > 0$ such that for $n \geq n_0$, we have

$$r^{-n}\rho_A(f^n a) = \alpha.$$

We have to show that $\alpha \leq \rho_B(a)$. Assume the contrary $\alpha > \rho_B(a)$. Then for all $x \in \operatorname{Sp} A$, we have

$$|f^n a(x)| \leq r^n |a(x)|.$$

So $f^n a$ must obtain its maximum on $U := \{x \in \operatorname{Sp} A : |a(x)| \geq \alpha\}$. But U is disjoint from $\operatorname{Sp} A\{r^{-1}f\}$ as

$$\alpha > \rho_B(a).$$

It follows from [Example 9.3](#) that

$$\beta := \sup_{x \in U} |f(x)| = \max_{x \in U} |f(x)| < r.$$

So

$$\rho(f^n a) = \sup_{x \in \operatorname{Sp} A} |f^n a(x)| = \sup_{x \in U} |f^n a(x)| \leq \beta^n \sup_{x \in U} |a(x)|.$$

This contradicts the fact that $\alpha > 0$. □

Proposition 10.7. Let A be a k_H -affinoid algebra and $r \in \mathbb{R}_{>0}^n$, then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of $\sqrt{|k^\times|} \cdot H$ -graded rings.

Recall that k_r is defined in [Example 3.12](#).

PROOF. By [Lemma 10.4](#), we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}_s^H \xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{N}^n} \tilde{A}_{sr^{-\alpha}}^H$$

for any $s \in \sqrt{|k^\times|} \cdot H$. This establishes the desired isomorphism. \square

Proposition 10.8. Let A be a k_H -affinoid algebra and $f \in A$ with $r = \rho(f) > 0$. Then there is a natural isomorphism

$$\tilde{A}_f^H \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of $\sqrt{|k^\times|} \cdot H$ -graded rings.

Recall that $A\{rf^{-1}\}$ is defined in [Example 9.3](#), by [Theorem 7.4](#), it is k_H -affinoid.

PROOF. Let $B = A\{rf^{-1}\}$ and denote by $\phi : \tilde{A}^H \rightarrow \tilde{A}_f^H$ the natural $\sqrt{|k^\times|} \cdot H$ -graded homomorphism. From the universal property [add details](#), we can factor the natural map $\tilde{A}^H \rightarrow \tilde{B}^H$ as $\psi : \tilde{A}_f^H \rightarrow \tilde{B}^H$. We have a commutative diagram:

$$\begin{array}{ccc} \tilde{A}^H & \xrightarrow{\quad} & \tilde{B}^H \\ \phi \downarrow & \nearrow \psi & \\ \tilde{A}_f^H & & \end{array}$$

We claim that ψ is bijective. Let \tilde{a}/\tilde{f}^m be an element in $\ker \psi$, where $\tilde{a} \in \tilde{A}^H$ is homogeneous. Lift \tilde{a} to $a \in A$. Then $\rho_B(a) < \rho_A(a)$. By [Lemma 10.6](#), $\rho_A(f^n a) < r^n \rho_A(a)$ when n is large enough, so

$$\tilde{f}^n \tilde{a} = 0$$

in \tilde{A} . Therefore, $\tilde{a}/\tilde{f}^m = 0$ in \tilde{A}_f^H . We have shown that ψ is injective.

It remains to show that ψ is surjective. Let $\tilde{b} \in \tilde{B}^H$ be a non-zero homogeneous element. Lift \tilde{b} to $b \in B$ of the form $f^{-n}a$ for some $a \in A$. By [Lemma 10.6](#) again, up to enlarging n , we can assume that $\rho_B(a) = \rho_A(a)$. Then $\tilde{a} = \tilde{f}^n \tilde{b}$ has a preimage in \tilde{A} . \square

Corollary 10.9. Let A be a k_H -affinoid algebra and $r \in \mathbb{R}_{>0}^n$, then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k}_r^H \cong \widetilde{A \hat{\otimes}_k k_r}^H$$

of $\sqrt{|k^\times|} \cdot H$ -graded rings.

PROOF. We can write

$$A \hat{\otimes}_k k_r = \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}.$$

Taking graded reduction, we find

$$\begin{aligned}
\widetilde{A \hat{\otimes}_k k_r}^H &= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}^H \\
&= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} A\{r^{-1}T\}_{\tilde{g}}^H \\
&= \varinjlim_{g \in k\{r^{-1}T\}, g \neq 0} \tilde{A}^H[r^{-1}T]_{\tilde{g}} \\
&= \tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k}_r^H.
\end{aligned}$$

Here we have applied [Proposition 10.8](#) in the second equality and [Proposition 10.7](#) in the third equality. The first equality follows from the simple observation that graded reduction commutes with filtered colimits. \square

Theorem 10.10. Let $\phi : A \rightarrow B$ be a morphism of k_H -affinoid algebras. Then the following are equivalent:

- (1) ϕ is finite and admissible.
- (2) $\tilde{\phi} : \tilde{A}^H \rightarrow \tilde{B}^H$ is finite.

PROOF. Take $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}^n$ so that

$$\rho(A \hat{\otimes}_k k_r) = \rho(B \hat{\otimes}_k k_r) = |k_r|$$

and k_r is non-trivially valued. [Proof that this is possible.](#)

By ?? in the chapter Commutative Algebra and [Proposition 8.8](#), we may assume that k is non-trivially valued and $\rho(A) = \rho(B) = |k|$. By ?? in the chapter Commutative Algebra, we have $\tilde{A} = \tilde{A}_1 \otimes_{\tilde{k}_1} \tilde{k}$. By [Corollary 5.5](#), ϕ is automatically admissible if it is finite.

So it suffices to argue that ϕ is finite if and only if $\tilde{\phi} : \tilde{A} \rightarrow \tilde{B}$ is finite.

Assume that ϕ is finite. We show that $\tilde{\phi}$ is finite.

First consider the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k -algebra homomorphism $\psi : k\{T_1, \dots, T_d\} \rightarrow A$ such that $\phi \circ \psi$ is finite and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \dots, T_d\} \rightarrow A$, we can apply [Theorem 5.1](#) to find $\phi \circ \alpha$ to conclude.

It suffices to show that $\phi \circ \psi$ is finite in order to conclude that $\tilde{\phi}$ is finite. So we are reduced to the case $A = k\{T_1, \dots, T_d\}$ and $\ker \phi = 0$.

We will show that the conditions of [Lemma 10.1](#) in the chapter Banach Rings is satisfied with ρ_B as the norm B . We have shown that ρ_B is a faithful $k\{T_1, \dots, T_d\}$ -algebra norm in [Corollary 4.16](#). As B is of finite over $k\{T_1, \dots, T_d\}$, the rank condition is clearly satisfied. It remains to establish that $\tilde{\phi}$ is integral.

By [Proposition 5.12](#), for $f \in B$, there is an integral equation

$$f^n + \phi(a_1)f^{n-1} + \dots + \phi(a_n) = 0$$

over A such that $\rho_B(f) = \max_{i=1, \dots, n} |b_i|_{\sup}^{1/i}$. If $f \in \mathring{B}$, then $|b_i|_{\sup} \leq 1$, hence $b_i \in \mathring{B}$. [Add a ref](#)

Conversely, assume that $\tilde{\phi}$ is finite. It suffices to apply [Lemma 5.15](#) to conclude that ϕ is finite. \square

Corollary 10.11. Let A be a k_H -affinoid algebra, then \tilde{A}^H is finitely generated over \tilde{k}^H .

PROOF. Take $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism

$$\pi : k\{r^{-1}T\} \rightarrow A.$$

Applying [Theorem 10.10](#), we find that it suffices to prove that $\widetilde{k\{r^{-1}T\}}^H$ is finitely generated over \tilde{k}^H . But this follows from [Proposition 10.7](#). \square

Definition 10.12. Let A be a k_H -affinoid algebra, we define the *reduction map*

$$\mathrm{Sp} \tilde{A}^H := \mathrm{Spec} \sqrt{|k^\times| \cdot H} \tilde{A}^H.$$

We have a natural map $\pi^H : \mathrm{Sp} A \rightarrow \mathrm{Sp} \tilde{A}^H$.

11. Gerritzen–Grauert theorem

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 11.1. Let A be a k_H -affinoid algebra. A morphism $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ in $k_H\text{-Aff}$ is a *closed immersion* if the corresponding morphism $A \rightarrow B$ in $k_H\text{-AffAlg}$ is an admissible epimorphism.

Definition 11.2. Let $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism in $k_H\text{-Aff}$. We call φ a *k_H -Runge immersion* if there is a factorization in $k_H\text{-Aff}$ of φ :

$$\mathrm{Sp} B \rightarrow \mathrm{Sp} C \rightarrow \mathrm{Sp} A,$$

such that $\mathrm{Sp} B \rightarrow \mathrm{Sp} C$ is a closed immersion and $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$ is a k_H -Weierstrass domain.

Add a prop rational domains form basis

Lemma 11.3. Let A be a k_H -affinoid algebra and V be a k_H -Laurent domain in $\mathrm{Sp} A$ represented by $A \rightarrow B = A\{r^{-1}f, sg\}$ for some $n, m \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in A^n$ and $g = (g_1, \dots, g_m) \in A^m$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ and $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H}^m$. Then

- (1) \tilde{B}^H is finite over the subalgebra generated by \tilde{A}^H and $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1^{-1}, \dots, \tilde{g}_m^{-1}$;
- (2) if V is a neighbourhood of a point $x \in \mathrm{Sp} A$, then $\tilde{\chi}_x(\tilde{B}^H)$ is finite over $\tilde{\chi}_x(\tilde{A}^H)$.

PROOF. (1) Consider the admissible epimorphism

$$A\{r^{-1}T, sS\} \rightarrow B.$$

By [Theorem 10.10](#), it induces a finite homomorphism

$$A\{\widetilde{r^{-1}T}, sS\}^H \rightarrow \tilde{B}^H.$$

The former is computed in [Proposition 10.7](#) and our assertion follows.

- (2) This is a special case of (1). \square

THEOREM 11.4 (Gerritzen–Grauert, Temkin). Let $\varphi : \mathrm{Sp} A \rightarrow \mathrm{Sp} B$ be a monomorphism in $k_H\text{-Aff}$. Then there is a finite cover of X by k_H -rational domains W_1, \dots, W_k such that the restrictions $\varphi_i : \varphi^{-1}(W_i) \rightarrow W_i$ are k_H -Runge immersions for $i = 1, \dots, k$.

PROOF. Step 1. We reduce to the following claim: for each $x \in \operatorname{Sp} A$, there is a k_H -rational domain U in $\operatorname{Sp} B$ containing $y = \varphi(x)$ such that $V = \varphi^{-1}U$ is a neighbourhood of x in $\operatorname{Sp} A$ and the induced map $V \rightarrow U$ is a closed immersion.

Assume this holds. Write $U = \operatorname{Sp} B \left\{ r \frac{f}{g} \right\}$ for some $n \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in B^n$ and $g \in B$ such that f_1, \dots, f_n, g generates the unit ideal and $r \in \sqrt{|k^\times| \cdot H}^n$. As g is invertible on U , we can find a small k_H -rational domain W in $\operatorname{Sp} B$ containing y such that

- (1) g is invertible on W ;
- (2) $\varphi^{-1}W \subseteq \varphi^{-1}U$.

Then $U \cap W$ is a k_H -Weierstrass domain in W and $\varphi^{-1}W \rightarrow W$ is therefore a k_H -Runge immersion. From the compactness of $\operatorname{Sp} A$, this implies that we can find k_H -rational domains W_1, \dots, W_m of $\operatorname{Sp} B$ such that $\varphi^{-1}(W_i) \rightarrow W_i$ is a k_H -Runge immersion for $i = 1, \dots, m$ and $X_1 \cup \dots \cup X_m$ contains an open neighbourhood U of $\varphi(\operatorname{Sp} A)$. As $\operatorname{Sp} B$ is compact, we can find finitely many k_H -rational domains W_{m+1}, \dots, W_k which do not intersect $\varphi(\operatorname{Sp} A)$ that covers $\operatorname{Sp} B \setminus U$. Then the covering W_1, \dots, W_k satisfies all of the requirements.

We have reduced the problem to a local one on $\operatorname{Sp} B$.

Step 2. We show that we may assume that $\widetilde{\chi}_x(\tilde{A}^H)$ is finite over $\widetilde{\chi}_y(\tilde{B}^H)$. Here the notation χ_y is defined in ?? in the chapter Banach Rings.

By [Corollary 10.11](#), $\widetilde{\chi}_x(\tilde{A}^H)$ is finitely generated over $\widetilde{\chi}_y(\tilde{B}^H)$. Take generators $h_1, \dots, h_l \in A$. By [Proposition 3.18](#), $\mathcal{H}(x) \xrightarrow{\sim} \mathcal{H}(y)$, so we can find $f_1, \dots, f_l, g \in B$ with $|g(y)| = 1$ such that

$$\left| \left(\frac{f_i}{g} - h_i \right) (x) \right| < \rho(h_i)$$

for all $i = 1, \dots, l$.

In fact, we can take $g = 1$. This can be seen as follows. Let $B' = B\{ag^{-1}\}$ for some $a \in \sqrt{|k^\times| \cdot H}$ with $a < 1$. Then by [Lemma 11.3](#), $\tilde{\chi}_y(\tilde{B}'^H)$ is finite over $\tilde{\chi}_y(\tilde{B}^H)$. So up to replacing B by the B' and $\operatorname{Sp} A$ by the inverse image of $\operatorname{Sp} B'$, we may assume that g is invertible. Replacing f_i by f_i/g , we could then assume that $g = 1$.

Up to replacing $\operatorname{Sp} B$ by $\operatorname{Sp} B\{\rho(h_1)^{-1}f_1, \dots, \rho(h_l)^{-1}f_l\}$, we can guarantee that $\tilde{f}_i = \tilde{h}_i$ for $i = 1, \dots, l$. So our assertion follows.

Step 3. We may assume that $\widetilde{\chi}_{x'}(\tilde{A}^H)$ is finite over $\widetilde{\chi}_{y'}(\tilde{B}^H)$ for any $x' \in \operatorname{Sp} A$ and $y' = \varphi(x')$.

Let $\pi : \operatorname{Sp} A \rightarrow \widetilde{\operatorname{Sp} A}^H$ be the reduction map. Let \mathcal{X} denote the Zariski closure of $\pi(x)$. Then for any $x' \in \operatorname{Sp} A$ with $\pi(x') \in \mathcal{X}$, we have

$$\ker \widetilde{\chi}_x \subseteq \ker \widetilde{\chi}_{x'}.$$

It follows that $\widetilde{\chi}_{x'}(\tilde{A}^H)$ is finite over $\widetilde{\chi}_{y'}(\tilde{B}^H)$.

Since $\pi^{-1}\mathcal{X}$ is open in $\operatorname{Sp} A$ [Include the proof](#), we can find a k_H -Laurent neighbourhood $\operatorname{Sp} B\{rf, sg^{-1}\}$ for some suitable tuples r, f, s, g of y such that $\varphi^{-1}\operatorname{Sp} B\{rf, sg^{-1}\} \subseteq \pi^{-1}\mathcal{X}$. Observe that for each $x' \in \operatorname{Sp} A$, $\widetilde{\chi}_{x'}(\tilde{A}^H)$ is finite over $\widetilde{\chi}_{y'}(\tilde{B}^H)$. This follows simply from [Lemma 11.3](#). So up to replacing B with $B\{rf, sg^{-1}\}$, we conclude.

Step 4. We claim that after all of these reductions, φ becomes a closed immersion. By our assumptions, for any minimal homogeneous prime ideal \mathfrak{p} of \tilde{A}^H , there is a point $x \in \mathrm{Sp} A$ with $\ker \widetilde{\chi}_y = \mathfrak{p}$ and \tilde{A}^H/\mathfrak{p} is finite over \tilde{A}^H .

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the list of minimal homogeneous prime ideals of \tilde{A}^H **prove finiteness**, then

$$\tilde{A}^H \rightarrow \bigoplus_{i=1}^k \tilde{A}^H/\mathfrak{p}_i$$

is injective. Since \tilde{B}^H is graded noetherian **Introduce this notion**, we find that \tilde{A}^H is finite over \tilde{B}^H . So $B \rightarrow A$ is finite by **Theorem 10.10**. It follows that the natural map $A \otimes_B A \rightarrow A \hat{\otimes}_B A$ is an isomorphism by **Proposition 8.4**. As φ is a monomorphism, from general abstract nonsense, the codiagonal $A \hat{\otimes}_B A \xrightarrow{\sim} A$ is an isomorphism. In particular, the codiagonal $A \otimes_B A \rightarrow A$ is an isomorphism. This implies that $A \rightarrow B$ is surjective. \square

Lemma 11.5. Let A be a k_H -affinoid domain and V be a k_H -affinoid domain in A represented by $A \rightarrow A_V$. Assume that $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ is a closed immersion, then V is a k_H -affinoid domain.

PROOF. As $\mathrm{Sp} A_V \rightarrow \mathrm{Sp} A$ is a closed immersion, we can find an ideal $I \subseteq A$ and assume that $A_V = A/I$. Consider the morphism of k_H -affinoid spectra $\psi : \mathrm{Sp} A/I^2 \rightarrow \mathrm{Sp} A$ induced by the natural map A/I^2 . By the universal property of V , we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Sp} A/I^2 & & \\ \downarrow & \searrow & \\ \mathrm{Sp} A/I & \longrightarrow & \mathrm{Sp} A \end{array}$$

On the other hand, the natural map $A/I^2 \rightarrow A/I$ induces a morphism of k_H -affinoid spectra $\varphi : \mathrm{Sp} A/I \rightarrow \mathrm{Sp} A/I^2$. From the universal property again, the composition $\psi \circ \varphi$ is the identity. In particular, $A/I^2 \rightarrow A/I$ is injective and hence $I = I^2$. It follows that I is the principal ideal generated by an idempotent element e . We may assume that $e \neq 0$, $e \neq 1$. Take $c \in \sqrt{|k^\times| \cdot H}$ such that $0 < c < 1$, then $V = (\mathrm{Sp} A)\{c^{-1}e\}$. \square

Corollary 11.6.

Corollary 11.7 (?). Let $\mathrm{Sp} A$ be a k -affinoid spectrum and $\mathrm{Sp} B$ be an affinoid domain. Then for any complete non-Archimedean field extension K/k , any K -affinoid algebra C and any bounded algebra homomorphism $A \rightarrow C$ such that $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$ factorizes through $\mathrm{Sp} B$, there is a unique bounded homomorphism $B \rightarrow C$ making the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Sp} C & & \\ \vdots \downarrow & \searrow & \\ \mathrm{Sp} B & \longrightarrow & \mathrm{Sp} A \end{array}$$

PROOF.

\square

The following propositions are a priori not clear with the current definition, we need Gerritzen–Grauert first

Proposition 11.8 (?). Let $\varphi : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism of k_H -affinoid spectra. Let $V \subseteq \mathrm{Sp} A$ be a k_H -affinoid domain in $\mathrm{Sp} A$, then $\varphi^{-1}(V)$ is a k_H -affinoid domain in $\mathrm{Sp} B$.

In fact, suppose that V is represented by $A \rightarrow A_V$, then $B \rightarrow B \hat{\otimes}_A A_V$ represents $\varphi^{-1}V$.

PROOF. □

Proposition 11.9 (?). Let A be a k_H -affinoid algebra. Let V, W be k_H -affinoid domains in $\mathrm{Sp} A$ represented by $A \rightarrow A_V$ and $A \rightarrow A_W$ respectively. Then $V \cap W$ is a k_H -affinoid domain represented by $A \rightarrow A_V \hat{\otimes}_A A_W$.

PROOF. □

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