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Contents

Affinoid algebras

1. Introduction

Our references for this chapter include [BGR84], [Ber12].

2. Tate algebras

Let $(k,|\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n\in\mathbb{N}$ and $r=(r_1,\ldots,r_n)\in\mathbb{R}^n_{>0}$. We set

$$\begin{aligned}k\{r^{-1}T\}=&k\{r_1^{-1}T_1,\ldots,r_nT_n^{-1}\}\\ &:=\left\{f=\sum_{\alpha\in\mathbb{N}^n}a_\alpha T^\alpha\in k[[T_1,\ldots,T_n]]:a_\alpha\in k,|a_\alpha|r^\alpha\to 0\text{ as }|\alpha|\to\infty\right\}.\end{aligned}$$

For any $f=\sum_{\alpha\in\mathbb{N}^n}a_\alpha T^\alpha\in k\{r^{-1}T\}$, we set

$$\|f\|_r=\max_\alpha|a_\alpha|r^\alpha.$$

We call $(k\{r^{-1}T\},\|\bullet\|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\|\bullet\|_r$ is called the *Gauss norm.*

We omit r from the notation if $r=(1,\ldots,1)$.

This is a special case of Example 4.15 in Banach rings.

Proposition 2.2. Let $n\in\mathbb{N}$ and $r=(r_1,\ldots,r_n)\in\mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\},\|\bullet\|_r)$ is a Banach k-algebra and $\|\bullet\|_r$ is a valuation.

PROOF. This is a special case of Proposition 4.16 in Banach rings. □

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T\rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T\rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n\in\mathbb{N}$ and $r\in\mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\}\cong k[T_1,\ldots,T_n]$ if $r_i\geq 1$ for all i and $k\{r^{-1}T\}\cong k[[T_1,\ldots,T_n]]$ otherwise.

Lemma 2.5. Let A be a Banach k-algebra. For each $n\in\mathbb{N}$ and $a_1,\ldots,a_n\in\mathring{A}$, there is a unique continuous homomorphism $k\{T_1,\ldots,T_n\}\to A$ sending T_i to a_i.

PROOF. This is a special case of Proposition 4.18 in Banach rings. □

3. Affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}_{>0}^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

More generally, a Banach k-algebra A is *k_H-affinoid* if there are $n \in \mathbb{N}$, $r \in H^n$ and an admissible epimorphism $k\{r^{-1}T\} \to A$.

A morphism between k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is a bounded k-algebra homomorphism.

The category of k-affinoid (resp. strictly k-affinoid, resp. k_H-affinoid) algebras is denoted by k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$ (resp. st-k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$, resp. k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$).

For the notion of admissible morphisms, we refer to Definition 2.5 in Banach rings.

Although we have defined strictly k-affinoid algebra when k is trivially valued, we will deliberately avoid talking about it in order to avoid pathologies.

Remark 3.2. Berkovich also introduced the notion of *affinoid k-algebras*: it is a K-affinoid algebra for some complete non-Archimedean field extension K/k. We will not use this notion.

Definition 3.3. The category of *k-affinoid spectra* k-$\mathcal{A}\mathrm{ff}$ (resp. *strictly k-affinoid spectra* st-k-$\mathcal{A}\mathrm{ff}$, resp. *k_H-affinoid spectra* k_H-$\mathcal{A}\mathrm{ff}$) is the opposite category of k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$ (resp. st-k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$, resp. k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$). An object in these categories are called a *k-affinoid spectrum*, *strictly k-affinoid spectrum* and *k_H-affinoid spectrum* respectively.

Given an object A of k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$ (resp. st-k-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$, resp. k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$), we denote the corresponding object in k-$\mathcal{A}\mathrm{ff}$ (resp. st-k-$\mathcal{A}\mathrm{ff}$, resp. k_H-$\mathcal{A}\mathrm{ff}$) by $\operatorname{Sp} A$. We call $\operatorname{Sp} A$ the *affinoid spectrum* of A.

In Definition 6.1 in Banach rings., we defined functors $\operatorname{Sp} : k\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$, $\operatorname{Sp} : \text{st-}k\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$ and $\operatorname{Sp} : k_H\text{-}\mathcal{A}\mathrm{ff} \to \mathcal{T}\mathrm{op}$. This motivates our notation. We will freely view $\operatorname{Sp} A$ as an object in these categories or as a topological space.

Proposition 3.4. Finite limits exist in k_H-$\mathcal{A}\mathrm{ff}$. Moreover, fiber products in k_H-$\mathcal{A}\mathrm{ff}$ corresponds to completed tensor product in k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$.

Proof. It suffices to prove that finite fibered products exsit.

We prove the equivalent statement, finite fibered coproducts exist in k_H-$\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}$. Given k_H-affinoid algebras A, B, C and morphisms $A \to B$, $A \to C$, we claim that $B \hat{\otimes}_A C$ represents the fibered coproduct of B and C over A. By general abstract nonsense, we are reduced to handle the following cases: $A = k$ and $A \to C$ is the codiagonal $C \hat{\otimes}_k C \to C$. In both cases, the proposition is clear.

□

Example 3.5. Let $r \in \mathbb{R}_{>0}$. We let k_r denote the subring of $k[[T]]$ consisting of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ satisfying $|a_i| r^i \to 0$ for $i \to \infty$ and $i \to -\infty$. We define a norm $\|\bullet\|_r$ on k_r as follows:

$$\|f\|_r := \max_{i \in \mathbb{Z}} |a_i| r^i.$$

We will show in Proposition 3.6 that k_r is k-affinoid.

Proposition 3.6. Let $r \in \mathbb{R}_{>0}$, then $(k_r, \|\bullet\|_r)$ defined in Example 3.5 is a k-affinoid algebra. Moreover, $\|\bullet\|_r$ is a valuation.

Proof. Observe that we have an admissible epimorphism

$$\iota : k\{r^{-1}T_1, rT_2\} \to k_r, \quad T_1 \mapsto T, T_2 \mapsto T^{-1}.$$

As we do not have the universal property at our disposal yet, let us verify by hand that this defines a ring homomorphism: consider a series

$$f = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\},$$

namely,

$$|a_{i,j}| r^{i-j} \to 0 \tag{3.1}$$

as $i + j \to \infty$. Observe that for each $k \in \mathbb{Z}$, the series

$$c_k := \sum_{i-j=k, i,j\in\mathbb{N}} a_{i,j}$$

is convergent.

Then by definition, the image $\iota(f)$ is given by

$$\sum_{k=-\infty}^{\infty} c_k T^k.$$

We need to verify that $\iota(f) \in k_r$. That is

$$|c_k| r^k \to 0$$

as $k \to \pm\infty$. When $k \geq 0$, we have $|c_k| \leq |a_{k0}|$ by definition of c_k. So $|c_k| r^k \to 0$ as $k \to \infty$ by (3.1). The case $k \to -\infty$ is similar.

We conclude that we have a well-defined map of sets ι. It is straightforward to verify that ι is a ring homomorphism. Next we show that ι is surjective. Take $g = \sum_{i=-\infty}^{\infty} c_i T^i \in k_r$. We want to show that g lies in the image of ι. As ι is a ring homomorphism, it suffices to treat two cases separately: $g = \sum_{i=0}^{\infty} c_i T^i$ and $g = \sum_{i=-\infty}^{0} c_i T^i$. We handle the first case only, as the second case is similar. In this case, it suffices to consider $f = \sum_{i=0}^{\infty} c_i T_1^i \in k\{r^{-1}T_1, rT_2\}$. It is immediate that $\iota(f) = g$.

Next we show that ι is admissible. We first identify the kernel of ι. We claim that the kenrel is the ideal I generated by $T_1T_2 - 1$. It is obvious that $I \subseteq \ker \iota$. Conversely, consider an element

$$f = \sum_{(i,j)\in\mathbb{N}^2} a_{i,j} T_1^i T_2^j \in k\{r^{-1}T_1, rT_2\}$$

lying in the kenrel of ι. Observe that

$$f = \sum_{k=-\infty}^{\infty} f_k, \quad f_k = \sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} T_1^i T_2^j.$$

If $f \in \ker \iota$, then so is each f_k by our construction.

We first show that each f_k lies in the ideal generated by $T_1T_2 - 1$. The condition that $f_k \in \ker \iota$ means

$$\sum_{(i,j)\in\mathbb{N}^2, i-j=k} a_{i,j} = 0.$$

It is elementary to find $b_{i,j} \in k$ for $i,j \in \mathbb{N}$, $i-j=k$ such that

$$a_{i,j} = b_{i-1,j-1} - b_{i,j}.$$

Then

$$f_k = (T_1T_2 - 1) \sum_{i,j\in\mathbb{N}, i-j=k} b_{i,j} T_1^i T_2^j.$$

Observe that we can make sure that $|b_{i,j}| \leq \max\{|a_{i',j'}| : i-j = i'-j'\}$. In particular, the sum of $\sum_{i,j\in\mathbb{N},i-j=k} b_{i,j}T_1^iT_2^j$ for various k converges to some $g \in k\{r^{-1}T_1, rT_2\}$ and hence $f_k = (T_1T_2-1)g$. Therefore, we have proved that $\ker \iota$ is generated by $T_1T_2 - 1$.

It remains to show that ι is admissible. In fact, we will prove a stronger result: ι induces an isometric isomorphism

$$k\{r^{-1}T_1, rT_2\}/I \to k_r.$$

To see this, take $f = \sum_{k=-\infty}^{\infty} c_k T^k \in k_r$ and we need to show that

$$\|f\|_r = \inf\{\|g\|_{(r,r^{-1})} : \iota(g) = f\}.$$

Observe that if we set $g = \sum_{k=0}^{\infty} c_k T_1^k + \sum_{k=1}^{\infty} c_{-k} T_2^k$, then $\iota(g) = f$ and $\|g\|_{(r,r^{-1})} = \|f\|$. So it suffices to show that for any $h = \sum_{(i,j)\in\mathbb{N}^2} d_{i,j}T_1^iT_2^j \in k\{r^{-1}T_1, rT_2\}$, we have

$$\|f\|_r \leq \|g + h(T_1T_2 - 1)\|_{r,r^{-1}}. \tag{3.2}$$

We compute

$$g+h(T_1T_2-1) = \sum_{k=1}^{\infty}(c_k - d_{k,0})T_1^k + \sum_{k=1}^{\infty}(c_{-k} - d_{0,k})T_2^k + (c_0 - d_0) + \sum_{i,j\geq 1}(d_{i-1,j-1} - d_{i,j})T_1^iT_2^j.$$

So

$$\|g + h(T_1T_2 - 1)\|_{r,r^{-1}} = \max\left\{\max_{k\geq 0} C_{1,k}, \max_{k\geq 1} C_{2,k}\right\},$$

where

$$C_{1,k} = \max\left\{|c_k - d_{k,0}|, \left|\sum_{i-j=k, i,j\geq 1} d_{i-1,j-1} - d_{i,j}\right|\right\}$$

for $k \geq 0$ and

$$C_{2,k} = \max\left\{|c_{-k} - d_{0,k}|, \left|\sum_{i-j=-k, i,j\geq 1} d_{i-1,j-1} - d_{i,j}\right|\right\}$$

for $k \geq 1$. It follows from the strong triangle inequality that $|c_k| \leq C_{1,k}$ for $k \geq 0$ and $c_{-k} \leq C_{2,k}$ for $k \geq 1$. So (3.2) follows. $\square$

Proposition 3.7. Let $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$, then $\|\bullet\|_r$ defined in Example 3.5 is a valuation on k_r.

PROOF. Take $f, g \in k_r$, we need to show that

$$\|fg\|_r \geq \|f\|_r \|g\|_r.$$

Let us expand

$$f = \sum_{i=-\infty}^{\infty} a_i T^i, \quad g = \sum_{i=-\infty}^{\infty} b_i T^i.$$

Take i and j so that

$$|a_i|r^i = \|f\|_r, \quad |b_j|r^j = \|g\|_r. \tag{3.3}$$

By our assumption on r, i, j are unique. Then

$$\|fg\|_r = \max_{k\in\mathbb{Z}}\{|c_k|r^k\},$$

where

$$c_k := \sum_{u,v\in\mathbb{Z}, u+v=k} a_u b_v.$$

It suffices to show that

$$|c_k|r^k = \|f\|_r\|g\|_r. \tag{3.4}$$

for $k = i + j$. Of course, we may assume that $a_i \neq 0$ and $b_j \neq 0$ as otherwise there is nothing to prove. For $u, v \in \mathbb{Z}$, $u + v = i + j$ while $(u, v) \neq (i, j)$, we may assume that $u \neq i$. Then $|a_u|r^u < |a_i|r^i$ and $|b_v|r^v \leq |b_j|r^j$. So $|a_u b_v| < |a_i b_j|$ and we conclude (3.4). □

Remark 3.8. The argument of Proposition 4.16 in Banach rings does not work here if $r \in \sqrt{|k^\times|}$, as in general one can not take minimal i, j so that (3.3) is satisfied.

Proposition 3.9. Assume that $r \in \mathbb{R}_{>0} \setminus \sqrt{|k^\times|}$. Then k_r is a valuation field and $\|\bullet\|_r$ is non-trivial.

Proof. We first show that $\operatorname{Sp} k_r$ consists of a single point: $\|\bullet\|_r$. Assume that $|\bullet| \in \operatorname{Sp} k_r$. As $\|\bullet\|_r$ is a valuation, we find

$$|\bullet| \leq \|\bullet\|_r. \tag{3.5}$$

In particular, $|\bullet|$ restricted to k is the given valuation on k. It suffices to show that $|T| = r$. This follows from (3.5) applied to T and T^{-1}.

It follows that k_r does not have any non-zero proper closed ideals: if I is such an ideal, k_r/I is a Banach k-algebra. By Proposition 6.10 in Banach rings., $\operatorname{Sp} k_r$ is non-empty. So k_r has to admit bounded semi-valuation with non-trivial kernel.

In particular, by Corollary 4.7 in Banach rings., the only maximal ideal of k_r is 0. It follows that k_r is a field.

The valuation $\|\bullet\|_r$ is non-trivial as $\|T\|_r = r$. □

Definition 3.10. An element $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ for some $n \in \mathbb{N}$ is called a *k-free polyray* if $r_1, \ldots, r_n$ are linearly independent in the $\mathbb{Q}$-linear space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R}_{>0}/\sqrt{|k^\times|}$.

Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. Assume that r is a k-free polyray. We define

$$k_r = k_{r_1}\hat{\otimes}_k \cdots \hat{\otimes}_k k_{r_n}.$$

By an interated application of Proposition 3.9, k_r is a complete valuation field.

As a general explanation of why k_r is useful, we prove the following proposition:

Proposition 3.11. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n)$ be a k-free polyray.

(1) For any k-Banach space X, the natural map

$$X \to X\hat{\otimes}_k k_r$$

is an isometric embedding.

(2) Consider a sequence of bounded homomorphisms of k-Banch spaces $X \to Y \to Z$. Then the sequence is admissible and exact (resp. coexact) if and only if $X\hat{\otimes}_k k_r \to Y\hat{\otimes}_k k_r \to Z\hat{\otimes}_k k_r$ is admissible and exact (resp. coexact).

Proof. We may assume that $n = 1$.

(1) We have a more explicit description of $X\hat{\otimes}_k k_r$: as a vector space, it is the space of $f = \sum_{i=-\infty}^{\infty} a_i T^i$ with $a_i \in X$ and $\|a_i\| r^i \to 0$ when $|i| \to \infty$. The norm is given by $\max_i \|a_i\| r^i$. From this description, the embedding is obvious.

(2) This follows easily from the explicit description in (1). □

When X is a Banach k-algebra, $X\hat{\otimes}_k k_r$ is a Banach k_r-algebra.

Example 3.12. For any $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$, not necessarily k-free. We define k_r as the completed fraction field of $k\{r^{-1}T\}$ provided with the extended valuation $|\bullet|_r$. Then k_r is still a valuation field extending k.

When r is a k-free polyray, we claim that k_r coincides with k_r defined in Definition 3.10. To see this, let us temporarily denote the k_r defined in this example as k'_r consider the extension of field:

$$\operatorname{Frac} k\{r^{-1}T\} \to k_r = k\{r^{-1}T, rS\}/(T_1S_1 - 1, \ldots, T_nS_n - 1)$$

sending T_i to T_i for $i = 1, \ldots, n$. Observe that this is an extension of valuation field as well by the same arguments as in Proposition 3.6. In particular, it induces an extension of complete valuation fields $k'_r \to k_r$. But the image clearly contains the classes of all polynomials in $k[T, S]$, so $k'_r \to k_r$ is an isometric isomorphism.

Proposition 3.13. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite bounded k-algebra homomorphism into a k-Banach algebra A. Then A is also strictly k-affinoid.

Proof. We may assume that $B = k\{T_1, \ldots, T_n\}$ for some $n \in \mathbb{N}$. By assumption, we can find finitely many $a_1, \ldots, a_m \in A$ such that $A = \sum_{i=1}^m \varphi(B)a_i$.

We may assume that $a_i \in \mathring{A}$ as k is non-trivially valued. By Proposition 4.18 in Banach rings., φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi : k\{T_1, \ldots, T_n, S_1, \ldots, S_m\} \to A$$

sending S_i to a_i. By Corollary 7.5 in Banach rings., Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is strictly k-affinoid. □

Proposition 3.14. Assume that k is non-trivially valued. Let B be a strict k-affinoid algebra and $\varphi : B \to A$ be a finite k-algebra homomorphism into a k-algebra A. Then there is a norm on A such that the morphism is bounded and A is strictly k-affinoid.

Proof. By Proposition 8.4 in Banach rings., we can endow A with a Banach norm such that φ is admissible. Then we can apply Proposition 3.13. □

Lemma 3.15. Assume that k is non-trivially valued. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$. The algebra $k\{r^{-1}T\}$ is strictly k-affinoid if $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$.

Remark 3.16. The converse is also true.

PROOF. Assume that $r_i \in \sqrt{|k^\times|}$ for all $i = 1, \ldots, n$. Take $s_i \in \mathbb{N}$ and $c_i \in k^\times$ such that

$$r_i^{s_i} = |c_i^{-1}|$$

for $i = 1, \ldots, n$. We deifne a bounded k-algebra homomorphism $\varphi : k\{T_1, \ldots, T_n\} \to k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ by sending T_i to $c_i T_i^{s_i}$. This is possible by Proposition 4.18 in Banach rings.

We claim that φ is finite. To see this, it suffices to observe that if we expand $f \in k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha,$$

we can regroup

$$f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} T^\beta \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (cT^s)^\gamma,$$

where the product γs is taken component-wise. For each $\beta \in \mathbb{N}^n, \beta_i < s_i$, we set

$$g_\beta := \sum_{\gamma \in \mathbb{N}^n} a_{\gamma s + \beta} c^{-\gamma} (T)^\gamma \in k\{T_1, \ldots, T_n\}.$$

While $f = \sum_{\beta \in \mathbb{N}^n, \beta_i < s_i} \varphi(g_\beta) T^\beta$. So We have shown that φ is finite. Hence, $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is k-affinoid by Proposition 3.13. □

Proposition 3.17. Let A be a k-affinoid algebra, then there is $n \in \mathbb{N}$ and a k-free polyray $r = (r_1, \ldots, r_n)$ such that $A \hat{\otimes}_k k_r$ is strictly k_r-affinoid. Moreover, we can guarantee that k_r is non-trivially valued.

PROOF. By Proposition 3.11, we may assume that $A = k\{t^{-1}T\}$ for some $t \in \mathbb{R}_{>0}^m$. By Lemma 3.15, it suffices to take r so that the linear subspace of $\mathbb{R}_{>0}/\sqrt{|k^\times|}$ generated by $r_1, \ldots, r_n$ contains all components of t. By taking $n \geq 1$, we can guarantee that k_r is non-trivially valued. □

Proposition 3.18. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a monomorphism in k_H-$\mathcal{A}$ff. Then for any $y \in \operatorname{Sp} B$ with $x = \varphi(y)$, one has $\varphi^{-1}(x) = \{y\}$ and the natural map $\mathscr{H}(x) \to \mathscr{H}(y)$ is an isomorphism of complete valuation rings.

PROOF. It suffices to show that $\mathscr{H}(x) \to B \hat{\otimes}_A \mathscr{H}(y)$ is an isomorphism as Banach k-algebras. Include details about cofiber products in affalg. By assumption, the codiagonal map $B \hat{\otimes}_A B \to B$ is an isomorphism. It follows that the base change with respect to $A \to \mathscr{H}(x)$ is also an isomorphism: $B' \hat{\otimes}_{\mathscr{H}(x)} B' \to B'$, where $B' = B \hat{\otimes}_A \mathscr{H}(x)$.

Include the fact that the first map is injective. It follows that the composition $B' \otimes_{\mathscr{H}(x)} B \to B' \hat{\otimes}_{\mathscr{H}(x)} B' \to B'$ is injective. Therefore, $\mathscr{H}(x) \to B'$ is an isomorphism of rings. We also know that this map is bounded. But we already know that $\mathscr{H}(x)$ is a complete valuation ring, so the map $\mathscr{H}(x) \to B'$ is an isomorphism of complete valuation rings. □

4. Weierstrass theory

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Proposition 4.1. We have canonical identifications

$$(k\{T_1,\ldots,T_n\})^\circ \cong \mathring{k}\{T_1,\ldots,T_n\},$$
$$(k\{T_1,\ldots,T_n\})^{\check{}} \cong \check{k}\{T_1,\ldots,T_n\},$$
$$\widetilde{k\{T_1,\ldots,T_n\}} \cong \tilde{k}[T_1,\ldots,T_n].$$

The last identification extends $\mathring{k} \to \tilde{k}$ and T_i is mapped to T_i.

Proof. This follows from Corollary 4.20 from the chapter Banach rings. □

We will denote the reduction map $\mathring{k}\{T_1,\ldots,T_n\} \to \tilde{k}[T_1,\ldots,T_n]$ by $\tilde{\bullet}$.

Definition 4.2. Let $n \in \mathbb{N}$. A system $f_1,\ldots,f_n \in k\{T_1,\ldots,T_n\}$ is called an *affinoid chart* of $k\{T_1,\ldots,T_n\}$ if $f_i \in \mathring{k}\{T_1,\ldots,T_n\}$ for each $i = 1,\ldots,n$ and the continuous k-algebra homomorphism $k\{T_1,\ldots,T_n\} \to k\{T_1,\ldots,T_n\}$ sending T_i to f_i is an isomorphism.

The map $k\{T_1,\ldots,T_n\} \to k\{T_1,\ldots,T_n\}$ is well-defined by Proposition 4.1 and Lemma 2.5.

Lemma 4.3. Let $n \in \mathbb{N}$ and $f \in k\{T_1,\ldots,T_n\}$. Assume that $\|f\|_1 = 1$. Then the following are equivalent:

(1) f is a unit $k\{T_1,\ldots,T_n\}$.
(2) $\tilde{f}$ is a unit in $\tilde{k}[T_1,\ldots,T_n]$.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 3.6, f is a unit in $k\{T_1,\ldots,T_n\}$ if and only if it is a unit in $(k\{T_1,\ldots,T_n\})^\circ$, which is identified with $\mathring{k}\{T_1,\ldots,T_n\}$ by Proposition 4.1. This result then follows from Corollary 4.21 in Banach rings. □

Definition 4.4. Let $n \in \mathbb{N}$. Consider $g \in k\{T_1,\ldots,T_n\}$. We expand g as

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1,\ldots,T_{n-1}\}.$$

For $s \in \mathbb{N}$, we say g is *X_n-distinguished of degree s* if g_s is a unit in $k\{T_1,\ldots,T_{n-1}\}$, $\|g_s\|_1 = \|g\|_1$ and $\|g_s\|_1 > \|g_t\|_1$ for all $t > s$.

Theorem 4.5 (Weierstrass division theorem)**.** Let $n, s \in \mathbb{N}$ and $g \in k\{T_1,\ldots,T_n\}$ be X_n-distinguished of degree s. Then for each $f \in k\{T_1,\ldots,T_n\}$, there exist $q \in k\{T_1,\ldots,T_n\}$ and $r \in k\{T_1,\ldots,T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$ such that

$$f = qg + r.$$

Moreover, q and r are uniquely determined. We have the following estimates

$$\|q\|_1 \leq \|g\|_1^{-1}\|f\|_1, \quad \|r\|_1 \leq \|f\|_1. \tag{4.1}$$

If in addition, $f, g \in k\{T_1,\ldots,T_{n-1}\}[T_n]$, then $q \in k\{T_1,\ldots,T_{n-1}\}[T_n]$ as well.

Proof. We may assume that $\|g\|_1 = 1$.

Step 1. Assuming the existence of the division. Let us prove (4.1). We may assume that $f \neq 0$, so that one of q, r is non-zero. Up to replacing q, r by a scalar multiple, we may assume that $\max\{\|q\|_1, \|r\|_1\} = 1$. So $\|f\|_1 \leq 1$ as well. We need to show that $\|f\|_1 = 1$. Assume the contrary, then

$$0 = \tilde{f} = \tilde{q}\tilde{g} + \tilde{r}.$$

Here $\tilde{\bullet}$ denotes the reduction map. By our assumption, $\deg_{T_n} = s > \deg_{T_n} r \geq \deg_{T_n} \tilde{r}$. From Proposition 4.1, the equality is in $\tilde{k}[T_1, \ldots, T_n]$. From the usual Euclidean division, we have $\tilde{q} = \tilde{r} = 0$. This is a contradiction to our assumption.

Step 2. Next we verify the uniqueness of the division. Suppose that

$$0 = qg + r$$

with q and r as in the theorem. The estimate in Step 1 shows that $q = r = 0$.

Step 3. We prove the existence of the division.

We define

$$B := \left\{ qg + r : r \in k\{T_1, \ldots, T_{n-1}\}[T_n], \deg_{T_n} r < s, q \in k\{T_1, \ldots, T_n\} \right\}.$$

From Step 1, B is a closed subgroup of $k\{T_1, \ldots, T_n\}$. In fact, suppose $f_i \in B$ is a sequence converging to $f \in k\{T_1, \ldots, T_n\}$. From Step 1, we can represent $f_i = q_i g + r_i$, then from Step 1, q_i and r_i are both Cauchy sequences, we may assume that $q_i \to q \in k\{T_1, \ldots, T_n\}$ and $r_i \to r$. As $\deg_{T_n} r_i < s$, it follows that $r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ and $\deg_{T_n} r < s$. So $f = qg + r$ and hence B is closed.

It suffices to show that B is dense $k\{T_1, \ldots, T_n\}$. We write

$$g = \sum_{i=0}^{\infty} g_i T_n^i, \quad g_i \in k\{T_1, \ldots, T_{n-1}\}.$$

We may assume that $\|g\|_1 = 1$. Define $\epsilon := \max_{j \geq s} \|g_j\|$. Then $\epsilon < 1$ by our assumption. Let $k_\epsilon = \{x \in k : |x| \leq \epsilon\}$ for the moment. There is a natural surjective ring homomorphism

$$\tau_\epsilon : (k\{T_1, \ldots, T_n\})^\circ \to (\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$$

with kernel $\{f \in k\{T_1, \ldots, T_n\} : \|f\|_1 \leq \epsilon\}$. We now apply Euclidean division in the ring $(\mathring{k}/k_\epsilon)[T_1, \ldots, T_n]$ to write

$$\tau_\epsilon(f) = \tau_\epsilon(q)\tau_\epsilon(g) + \tau_\epsilon(r)$$

for some $q \in (k\{T_1, \ldots, T_n\})^\circ$ and $r \in (k\{T_1, \ldots, T_{n-1}\})^\circ[T_n]$ with $\deg_{T_n} r < s$. So

$$\|f - qg - r\|_1 \leq \epsilon.$$

This proves that B is dense in $k\{T_1, \ldots, T_n\}$ by Proposition 2.8 in Banach rings.

Step 4. It remains to prove the last assertion. But this is a consequence of the usual Euclidean division theorem for the ring $k\{T_1, \ldots, T_{n-1}\}[T_n]$ and the uniqueness proved in Step 2. □

Lemma 4.6. Let $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ be a Weierstrass polynomial and $g \in k\{T_1, \ldots, T_n\}$. Assume that $\omega g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, then $g \in k\{T_1, \ldots, T_{n-1}\}[T_n]$.

Proof. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in k\{T_1, \ldots, T_{n-1}\}[T_n]$, $\deg_{T_n} r < \deg_{T_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of Theorem 4.5, we know that $q = g$, so g is a polynomial in T_n. □

As a consequence, we deduce Weierstrass preparation theorem.

Definition 4.7. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n-variables is a monic polynomial $\omega \in k\{T_1, \ldots, T_{n-1}\}[T_n]$ with $\|\omega\|_1 = 1$.

Lemma 4.8. Let $n \in \mathbb{Z}_{>0}$ and $\omega_1, \omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ be two monic polynomials. If $\omega_1\omega_2$ is a Weierstrass polynomial then so are ω_1 and ω_2.

Proof. As ω_1 and ω_2 are monic, $\|\omega_i\|_1 \geq 1$ for $i = 1, 2$. On the other hand, $\|\omega_1\|_1 \cdot \|\omega_2\|_1 = \|\omega_1\omega_2\|_1 = 1$, so $\|\omega_i\|_1 = 1$ for $i = 1, 2$. □

Theorem 4.9 (Weierstrass preparation theorem)**.** Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished of degree s. Then there are a Weierstrass polynomial $\omega \in k\{T_1, \dots, T_{n-1}\}[T_n]$ of degree s and a unit $e \in k\{T_1, \dots, T_n\}$ such that

$$g = e\omega.$$

Moreover, e and ω are unique. If $g \in k\{T_1, \dots, T_{n-1}\}[T_n]$, then so is e.

Proof. We first prove the uniqueness. Assume that a decomposition as in the theorem is given. Let $r = T_n^s - \omega$. Then $T_n^s = e^{-1}g + r$. The uniqueness part of Theorem 4.5 implies that e and r are uniquely determined, hence so is ω.

Next we prove the existence. By Weierstrass division theorem Theorem 4.5, we can write

$$T_n^s = qg + r$$

for some $q \in k\{T_1, \dots, T_n\}$ and $r \in k\{T_1, \dots, T_{n-1}\}[T_n]$ with $\deg_{T_n} r < s$. Let $\omega = T_n^s - r$. From the estimates in Theorem 4.5, $\|r\|_1 \leq 1$. So $\|\omega\|_1 = 1$. Then ω is a Weierstrass polynomial of degree s and $\omega = qg$. It suffices to argue that q is a unit.

We may assume that $\|g\|_1 = 1$. By taking reductions, we find

$$\tilde{\omega} = \tilde{q}\tilde{g}.$$

As $\deg_{T_n} \tilde{g} = \deg_{T_n} \tilde{\omega}$ and the leading coefficients of both polynomials are units in $\tilde{k}[T_1, \dots, T_{n-1}]$, it follows that $\tilde{q}$ is a unit in $\tilde{k}[T_1, \dots, T_{n-1}]$. It follows that $\tilde{q}$ is also a unit in $\tilde{k}[T_1, \dots, T_n]$. By Lemma 4.3, q is a unit in $k\{T_1, \dots, T_n\}$.

The lsat assertion is already proved in Theorem 4.5.

□

Definition 4.10. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished. Then the Weierstrass polynomial ω constructed in Theorem 4.9 is called the *Weierstrass polynomial* defined by g.

Corollary 4.11. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ be X_n-distinguished. Let ω be the Weierstrass polynomial of g. Then the injection

$$k\{T_1, \dots, T_{n-1}\}[T_n] \to k\{T_1, \dots, T_n\}$$

induces an isomorphism of k-algebras

$$k\{T_1, \dots, T_{n-1}\}[T_n]/(\omega) \to k\{T_1, \dots, T_n\}/(g).$$

Proof. The surjectivity follows from Theorem 4.5 and the injectivity follows from Lemma 4.6. □

In the complex setting, we can perturb a convergent power series so that it has finite degree along a fixed axis, the corresponding result in the current setting is:

Lemma 4.12. Let $n \in \mathbb{Z}_{>0}$ and $g \in k\{T_1, \dots, T_n\}$ is non-zero. Then there is a k-algebra automorphism σ of $k\{T_1, \dots, T_n\}$ so that $\sigma(g)$ is T_n-distinguished.

PROOF. We may assume that $\|g\|_1 = 1$. We expand g as

$$g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha.$$

Endow $\mathbb{N}^n$ with the lexicographic order. Take the maximal $\beta \in \mathbb{N}^n$ so that $|a_\beta| = 1$. Take $t \in \mathbb{Z}_{>0}$ so that $t \geq \max_{i=1,\ldots,n} \alpha_i$ for all $\alpha \in \mathbb{N}^n$ with $\tilde{a}_\alpha \neq 0$.

We will define σ by sending T_i to $T_i + T_n^{c_i}$ for all $i = 1, \ldots, n-1$. The c_i's are to be defined. We begin with $c_n = 1$ and define the other c_i's inductively:

$$c_{n-j} = 1 + t \sum_{d=0}^{j-1} c_{n-d}$$

for $j = 1, \ldots, n-1$. We claim that $\sigma(f)$ is T_n-distinguished of order $s = \sum_{i=1}^n c_i \beta_i$.

A straightforward computation shows that

$$\widetilde{\sigma(g)} = \sum_{i=1}^{s} p_i T_n^i$$

for some $p_i \in \tilde{k}[T_1, \ldots, T_{n-1}]$ and $p_s = \tilde{a_\beta}$. Our claim follows. □

Proposition 4.13. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Noetherian.

PROOF. We make induction on n. The case $n = 0$ is trivial. Assume that $n > 0$. It suffices to show that for any non-zero $g \in k\{T_1, \ldots, T_n\}$, $k\{T_1, \ldots, T_n\}/(g)$ is Noetherian. By Lemma 4.12, we may assume that g is T_n-distinguished. By Theorem 4.5, $k\{T_1, \ldots, T_n\}/(g)$ is a finite free $k\{T_1, \ldots, T_{n-1}\}$-module. By the inductive hypothesis and Hilbert basis theorem, $k\{T_1, \ldots, T_n\}/(g)$ is indeed Noetherian. □

Proposition 4.14. Let $n \in \mathbb{N}$. Then $k\{T_1, \ldots, T_n\}$ is Jacobson.

PROOF. When $n = 0$, there is nothing to prove. We make induction on n and assume that $n > 0$. Let $\mathfrak{p}$ be a prime ideal in $k\{T_1, \ldots, T_n\}$, we want to show that the Jacobson radical of $\mathfrak{p}$ is equal to $\mathfrak{p}$.

We distinguish two cases. First we assume that $\mathfrak{p} \neq 0$. Let $\mathfrak{p}' = \mathfrak{p} \cap k\{T_1, \ldots, T_{n-1}\}$. By Lemma 4.12, we may assume that $\mathfrak{p}$ contains a Weierstrass polynomial ω. Observe that

$$k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' \to k\{T_1, \ldots, T_n\}/\mathfrak{p}$$

is finite by Theorem 4.5. For any $b \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p})$ (where J denotes the Jacobson radical), we consider a monic integral equation of minimal degree over $k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'$:

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0, \quad a_i \in k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}'.$$

Then

$$a_n \in J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) \cap k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}' = J(k\{T_1, \ldots, T_{n-1}\}/\mathfrak{p}') = 0$$

by our inductive hypothesis. It follows that $n = 1$ and so $b = 0$. This proves $J(k\{T_1, \ldots, T_n\}/\mathfrak{p}) = 0$.

On the other hand, let us consider the case $\mathfrak{p} = 0$. As $k\{T_1, \ldots, T_n\}$ is a valuation ring, it is an integral domain, so the nilradical is 0. We need to show that

$$J(k\{T_1, \ldots, T_n\}) = 0.$$

Assume that there is a non-zero element f in $J(k\{T_1,\dots,T_n\})$. We may assume that $\|f\|_1=1$.

We claim that there is $c\in k$ with $|c|=1$ such that $c+f$ is not a unit in $k\{T_1,\dots,T_n\}$. Assuming this claim for the moment, we can find a maximal ideal $\mathfrak{m}$ of $k\{T_1,\dots,T_n\}$ such that $c+f\in\mathfrak{m}$. But $f\in\mathfrak{m}$ by our assumption, so $c\in\mathfrak{m}$ as well. This contradicts the fact that $c\in k^\times$.

It remains to prove the claim. We treat two cases separately. When $|f(0)|<1$, we simply take $c=1$, which works thanks to Lemma 4.3. If $|f(0)|=1$, we just take $c=-f(0)$. □

Proposition 4.15. Let $n\in\mathbb{N}$. Then $k\{T_1,\dots,T_n\}$ is UFD. In particular, $k\{T_1,\dots,T_n\}$ is normal.

Proof. As $\|\bullet\|_1$ is a valuation by Proposition 2.2, $k\{T_1,\dots,T_n\}$ is an integral domain. In order to see that $k\{T_1,\dots,T_n\}$ has the unique factorization property, we make induction on $n\geq 0$. When $n=0$, there is nothing to prove. Assume that $n>0$. Take a non-unit element $f\in k\{T_1,\dots,T_n\}$. By Theorem 4.9 and Lemma 4.12, we may assume that f is a Weierstrass polynomial. By inductive hypothesis, $k\{T_1,\dots,T_{n-1}\}$ is a UFD, hence so is $k\{T_1,\dots,T_{n-1}\}[T_n]$ by [Stacks, Tag 0BC1]. It follows that f can be decomposed into the products of monic prime elements $f_1,\dots,f_r\in k\{T_1,\dots,T_{n-1}\}[T_n]$, which are all Weierstrass polynomials by Lemma 4.8. Then by Corollary 4.11, we see that each f_i is prime in $k\{T_1,\dots,T_n\}$.

Any UFD is normal by [Stacks, Tag 0AFV]. □

Corollary 4.16. Let A be a strictly k-affinoid algebra, $d\in\mathbb{N}$ and $\varphi: k\{T_1,\dots,T_d\}\to A$ be an integral torsion-free injective homomorphism of k-algebras. Then ρ is a faithful $k\{T_1,\dots,T_d\}$-algebra norm on A. If $f^n+\varphi(t_1)f^{n-1}+\cdots+\varphi(t_n)=0$ is the minimal integral equation of f over $k\{T_1,\dots,T_d\}$, then

$$|f|_{\sup}=\max_{i=1,\dots,n}|t_i|^{1/i}.$$

Proof. This follows from Proposition 9.11 in Banach rings. and Proposition 4.15. □

5. Noetherian normalization and maximal modulus principle

Let $(k,|\bullet|)$ be a complete non-trivially valued non-Archimedean valued-field.

Theorem 5.1. Let A be a non-zero strictly k-affinoid algebra, $n\in\mathbb{N}$ and $\alpha: k\{T_1,\dots,T_n\}\to A$ be a finite (resp. integral) k-algebra homomorphism. Then up to replacing $T_1,\dots,T_n$ by an affinoid chart, we can guarantee that there exists $d\in\mathbb{N}$, $d\leq n$ such that α when restricted to $k\{T_1,\dots,T_d\}$ is finite (resp. integral) and injective.

Proof. We make an induction on n. The case $n=0$ is trivial. Assume that $n>0$. If $\ker\alpha=0$, there is nothing to prove, so we may assume that $\ker\alpha\neq 0$. By Lemma 4.12 and Theorem 4.9, we may assume that there is a Weierstrass polynomial $\omega\in k\{T_1,\dots,T_{n-1}\}[T_n]$ in $\ker\alpha$. Then α induces a finite (resp. integral) homomorphism $\beta: k\{T_1,\dots,T_n\}/(\omega)\to A$. By Theorem 4.5, $k\{T_1,\dots,T_{n-1}\}\to k\{T_1,\dots,T_n\}/(\omega)$ is a finite homomorphism. So their composition is a finite (resp. integral) homomorphism $k\{T_1,\dots,T_{n-1}\}\to A$. We can apply the inductive hypothesis to conclude. □

Corollary 5.2. Let A be a non-zero strictly k-affinoid algebra, then there is $d \in \mathbb{N}$ and a finite injective k-algebra homomorphism: $k\{T_1, \ldots, T_d\} \to A$.

PROOF. Take some $n \in \mathbb{N}$ and a surjective k-algebra homomorphism $k\{T_1, \ldots, T_n\} \to A$ and apply Theorem 5.1, we conclude. □

Corollary 5.3. Let A be a strictly k-affinoid algebra and I be an ideal in A such that $\sqrt{I}$ is a maximal ideal in A, then A/I is finite-dimensional over k.

In particular, $\operatorname{Spm} A = \operatorname{Spm}_k A$.

PROOF. By Corollary 5.2, there is $d \in \mathbb{N}$ and a finite monomorphism $f : k\{T_1, \ldots, T_d\} \to A/I$. It suffices to show that $d = 0$. Observe that the composition

$$k\{T_1, \ldots, T_d\} \xrightarrow{f} A/I \to A/\sqrt{I}$$

is finite and injective as $k\{T_1, \ldots, T_d\}$ is an integral domain, so $k\{T_1, \ldots, T_d\}$ is a field. This is possible only when $d = 0$. □

Corollary 5.4. Let B be a strictly k-affinoid algebra and A be a Noetherian Banach k-algebra. Let $f : A \to B$ a k-algebra homomorphism. Then f is bounded.

PROOF. This follows from Proposition 8.1 in Banach rings. and Proposition 4.13. □

In particular, we see that the topology of a k-affinoid algebra is uniquely determined by the algebraic structure.

Corollary 5.5. Let A, B be strictly k-affinoid algebras. Let f be a finite k-algebra homomorphism, then f is admissible.

PROOF. This follows from Proposition 3.14 and Corollary 5.4, □

Definition 5.6. For any non-Archimedean valuation field $(K, |\bullet|)$ and $n \in \mathbb{N}$, we define the *n-dimensional polydisk* with value in K:

$$B^n(K) := \left\{ (x_1, \ldots, x_n) \in K^n : \max_{i=1,\ldots,n} |x_i| \leq 1 \right\}.$$

Definition 5.7. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, say with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha, \quad a_\alpha \in k.$$

We define the associated function $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ as sending $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ to

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha.$$

Lemma 5.8. Let $n \in \mathbb{N}$ and $f \in k\{T_1, \ldots, T_n\}$, then $f : B^n(k^{\mathrm{alg}}) \to k^{\mathrm{alg}}$ is continuous and for any $x \in B^n(k^{\mathrm{alg}})$,

$$|f(x)| \leq \|f\|_1.$$

There is $x = (x_1, \ldots, x_n) \in B^n(k^{\mathrm{alg}})$ such that $|f(x)| = \|f\|_1$.

PROOF. To see that f is continuous, it suffices to observe that f is a uniform limit of polynomials. For any $x=(x_1,\dots,x_n)\in B^n(k^{\mathrm{alg}})$, we have

$$|f(x)|=\left|\sum_{\alpha\in\mathbb{N}^n}a_\alpha x^\alpha\right|\leq\max_{\alpha\in\mathbb{N}^n}|a_\alpha x^\alpha|\leq\|f\|_1.$$

To prove the last assertion, we may assume that $\|f\|_1=1$. As the residue field of k^{alg} is equal to $\tilde{k}^{\mathrm{alg}}$, it has infinitely many elements, so there is a point $x\in B^n(k^{\mathrm{alg}})$ such that $\widetilde{f(x)}=\tilde{f}(\tilde{x})\neq 0$. In other words, $\|f(x)\|_1=1$. □

Proposition 5.9. Let $n\in\mathbb{N}$, then the maximal modulus principle holds for $k\{T_1,\dots,T_n\}$. Moreover, for any $f\in k\{T_1,\dots,T_n\}$, $\|f\|_1=|f|_{\mathrm{sup}}$.

PROOF. By Lemma 6.3 in Banach rings., we have

$$\|f\|_1\geq|f|_{\mathrm{sup}}$$

for any $f\in A$. We only have to show that for any $f\in k\{T_1,\dots,T_n\}$ there is a maximal ideal $\mathfrak{m}\subseteq k\{T_1,\dots,T_n\}$ such that $|f(\mathfrak{m})|=\|f\|_1$.

By Lemma 5.8 we can take $x=(x_1,\dots,x_n)\in B^n(k^{\mathrm{alg}})$ such that $|f(x)|=\|f\|_1$. Let L be the field extension of k generated by $x_1,\dots,x_n$, then L/k is finite. Then we can define a homomorphism

$$\mathrm{ev}_x:k\{T_1,\dots,T_n\}\to L$$

sending $g\in k\{T_1,\dots,T_n\}$ to $g(x)$. Observe that the image is indeed in L. Clearly ev_x is surjective. So $\mathfrak{m}_x:=\ker\mathrm{ev}_x$ is a k-algebraic maximal ideal in $k\{T_1,\dots,T_n\}$. Then

$$|f(\mathfrak{m}_x)|=|f(x)|=\|f\|_1.$$

□

Corollary 5.10. Let A be a strictly k-affinoid algebra. Then for any $f\in A$,

$$|f|_{\mathrm{sup}}\subseteq\sqrt{|k^\times|}\cup\{0\}.$$

PROOF. We may assume that $A\neq 0$. By Corollary 5.2 and Proposition 9.11 in Banach rings., we may assume that $A=k\{T_1,\dots,T_n\}$ for some $n\in\mathbb{N}$. The result then follows from Proposition 5.9. □

Corollary 5.11. Maximal modulus principle holds for any strictly k-affinoid algebras.

PROOF. This follows from Corollary 5.2, Proposition 9.11 in Banach rings. and Proposition 5.9. □

Proposition 5.12. Let $\varphi:B\to A$ be an integral k-algebra homomorphism of strictly k-affinoid algebras. Then for each non-zero $f\in A$, there is a moinc polynomial $q(f)=f^n+\varphi(b_1)f^{n-1}+\cdots+\varphi(b_n)$ of f over B. Then

$$|f|_{\mathrm{sup}}=\max_{i=1,\dots,n}|b_i|_{\mathrm{sup}}^{1/i}.$$

PROOF. One side is simple: choose $j=1,\dots,n$ that maximizes $|\varphi(b_j)f^{n-j}|_{\mathrm{sup}}$, then

$$|f|_{\mathrm{sup}}^n=|f^n|_{\mathrm{sup}}\leq|\varphi(b_j)f^{n-j}|_{\mathrm{sup}}\leq|b_j|_{\mathrm{sup}}\cdot|f|_{\mathrm{sup}}^{n-j}.$$

So

$$|f|_{\mathrm{sup}}\leq|b_j|_{\mathrm{sup}}^{1/j}.$$

To prove the reverse inequality, let us begin with the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k-algebra homomorphism $\psi : k\{T_1, \dots, T_d\} \to B$ such that $\varphi \circ \psi$ is integral and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \dots, T_d\} \to A$, we can apply Theorem 5.1 to find $\phi \circ \alpha$ to conclude.

By Corollary 4.16, if p denotes the minimal polynomial of f over $k\{T_1, \dots, T_d\}$, we have $|f|_{\sup} = \sigma(p)$. In particular, $p(f) = 0$. Let $q \in B[X]$ be the polynomial obtained from p by replacing all coefficients by their ψ-images in B. Then clearly, $|f|_{\sup} = \sigma(q)$.

In general, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes in A. The number is finite by Proposition 4.13. For each $i = 1, \dots, r$, let $\pi_i : A \to A/\mathfrak{p}_i$ denote the natural homomorphism. We know that there are monic polynomials $q_i \in B[X]$ such that $q_i(\pi(f)) = 0$ and $|\pi_i(f)|_{\sup} = \sigma(q_i)$ for $i = 1, \dots, r$. We let $q' = q_1 \cdots q_r$. Then

$$q'(f) \in \bigcap_{i=1}^{r} \mathfrak{p}_i.$$

So there is $e \in \mathbb{Z}_{>0}$ such that $q'(f)^e = 0$. Let $q = q'^e$. By Proposition 9.5 in Banach rings.,

$$\sigma(q) \leq \max_{i=1,\dots,r} \sigma(q_i) = \max_{i=1,\dots,r} |\pi_i(f)|_{\sup} = |f|_{\sup}.$$

The last equality follows from Proposition 9.9 in Banach rings. □

Lemma 5.13. Let $\varphi : B \to A$ be an admissible k-algebra homomorphism between strictly k-affinoid algebras. Let $\tau : \mathring{B} \to \tilde{B}$ be the reduction map, then

$$\tau^{-1}(\ker \tilde{\varphi}) = \sqrt{\check{B} + \ker \mathring{\varphi}}, \quad \ker \tilde{\varphi} = \sqrt{\tau(\ker \mathring{\varphi})}.$$

Proof. The second equation follows from the first one by applying τ. Let us prove the first equation. By assumption, $\varphi(\check{B})$ is open in $\varphi(B)$. Consider $g \in \tau^{-1}(\ker \tilde{\varphi})$, we know that

$$\lim_{n \to \infty} \varphi(g)^n = 0.$$

So $\varphi(g)^n \in \varphi(\check{B})$ for n large enough, and hence $g^n \in \check{B} + \ker \mathring{\varphi}$. □

Lemma 5.14. Let $m \in \mathbb{N}$ and $T = k\{T_1, \dots, T_m\}$. Let A be a k-affinoid algebra and $\varphi : T\{S_1, \dots, S_n\} \to A$ be a finite morphism such that $\tilde{\varphi}(S_i)$ is integral over $\tilde{T}$. Then $\varphi|_T : T \to A$ is finite.

Proof. We make an induction on n. When $n = 0$, there is nothing to prove. So assume $n > 0$ and the lemma has been proved for smaller values of n.

Let $T' = T\{S_1, \dots, S_n\}$. By assumption, there is a Weierstrass polynomial

$$\omega = S_n^k + a_1 S_n^{k-1} + \dots + a_k \in \mathring{T}[S_n]$$

such that $\tilde{\omega} \in \ker \tilde{\varphi}$. As φ is admissible by Corollary 5.5, we have $\omega^q \in \check{T}' + \ker \mathring{\varphi}$ for some $q \in \mathbb{Z}$ by Lemma 5.13.

In particular, we can find $r \in (T')\check{}$ such that $g := \omega^q - r \in \ker \mathring{\varphi}$. Observe that g is S_n distinguished of order mq as $\tilde{g} = \tilde{\omega}^q$. By Corollary 4.11, the restriction of φ to $T\{S_1, \dots, S_{n-1}\}$ is finite. We can apply the inductive hypothesis to conclude. □

Lemma 5.15. Let $\varphi : B \to A$ be a k-algebra homomorphism of strictly k-affinoid algebras. Assume that there exist affinoid generators $f_1, \dots, f_n \in \mathring{A}$ of A such that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over $\tilde{B}$, then φ is finite.

Proof. By assumption, we can find $s_i \in \mathbb{Z}_{>0}$, $b_{ij} \in \mathring{B}$ for $i = 1, \dots, n$, $j = 1, \dots, s_i$ such that

$$\tilde{f}_i^{s_i} + \tilde{\varphi}(\tilde{b}_{i1})\tilde{f}_i^{s_i-1} + \cdots + \tilde{\varphi}(\tilde{b}_{is_i}) = 0$$

for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$ and define a bounded k-algebra homomorphism $\psi : D := k\{T_{ij}\} \to B$ sending T_{ij} to b_{ij}, for $i = 1, \dots, n$ and $j = 1, \dots, s_i$. Observe that $\tilde{f}_1, \dots, \tilde{f}_n$ are all integral over $\tilde{D}$. So it suffices to prove the theorem when $B = k\{T_1, \dots, T_m\}$. We extend φ to a bounded k-algebra epimorphism $\varphi' : T\{S_1, \dots, S_n\} \to A$ sending S_i to f_i for $i = 1, \dots, n$. Then $\varphi'(\tilde{S}_i)$ is integral over $\tilde{B}$. It suffices to apply Lemma 5.14. □

6. Properties of affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^{\times}| \cdot H \neq \{1\}$.

Proposition 6.1. Assume that k is non-trivially valued. Let A be a strictly k-afifnoid algebra. Then

$$\mathring{A} = \{f \in A : \rho(f) \leq 1\} = \{f \in A : |f|_{\sup} \leq 1\}.$$

Proof. By Lemma 6.3, we have

$$\mathring{A} \subseteq \{f \in A : \rho(f) \leq 1\} \subseteq \{f \in A : |f|_{\sup} \leq 1\}.$$

Conversely, let $f \in A$, $|f|_{\sup} \leq 1$. Choose $d \in \mathbb{N}$ and a surjective k-algebra homomorphism

$$\varphi : k\{T_1, \dots, T_d\} \to A.$$

Let $f^n + t_1 f^{n-1} + \cdots + t_n = 0$ be the minimal equation of f over $k\{T_1, \dots, T_d\}$. Then $t_i \in (k\{T_1, \dots, T_d\})^{\circ}$ by Proposition 9.11 in Banach rings. An induction on $i \geq 0$ shows that

$$f^{n+i} \in \sum_{j=0}^{n-1} \varphi\left((k\{T_1, \dots, T_d\})^{\circ}\right) f^j.$$

The right-hand side is clearly bounded. □

Corollary 6.2. Assume that k is non-trivially valued. Let $(A, \|\bullet\|)$ be a strictly k-affinoid algebra. For any $f \in A$,

$$\rho(f) = |f|_{\sup}.$$

Proof. We have shown that $\rho(f) \geq |f|_{\sup}$ in Lemma 6.3 from the chapter Banach Rings. Assume tha tthe inverse inequality fails: for some $f \in A$,

$$\rho(f) > |f|_{\sup}.$$

If $|f|_{\sup} = 0$, then f lies in the Jacobson radical of A, which is equal to the nilradial of A by Proposition 4.14. But then $\rho(f) = 0$ as well. We may therefore assume that $|f|_{\sup} \neq 0$. By Corollary 5.10, we may assume that $|f|_{\sup} = 1$ as ρ is power-multiplicative. Then $\rho(f) > 1$. This contradicts Proposition 6.1. □

Theorem 6.3. A k-affinoid algebra A is Noetherian and all ideals of A are closed.

Proof. Let I be an ideal in A. By Proposition 3.17, we can take a suitable $r \in \mathbb{R}^m_{>0}$ so that $A\hat{\otimes}k_r$ is strictly k_r-affinoid. Then $I(A\hat{\otimes}k_r)$ is an ideal in $A\hat{\otimes}k_r$. By Proposition 4.13, the latter ring is Noetherian. So we may take finitely many generators $f_1, \ldots, f_k \in I$. Each $f \in I$ can be written as

$$f = \sum_{i=1}^{k} f_i g_i$$

with $g_i = \sum_{j=-\infty}^{\infty} g_{i,j} T^j \in A\hat{\otimes}k_r$. But then

$$f = \sum_{i=1}^{k} f_i g_{i,0}.$$

So I is finitely generated.

As $I = A \cap (I(A\hat{\otimes}k_r))$, by Corollary 7.4 in Banach rings., we see that I is closed in $A\hat{\otimes}k_r$ and hence closed in A. $\square$

Proposition 6.4. Let $(A, \|\bullet\|)$ be a k-affinoid algebra and $f \in A$. Then there is $C > 0$ and $N \geq 1$ such that for any $n \geq N$, we have

$$\|f^n\| \leq C\rho(f)^n.$$

Recall that ρ is the spectral radius map defined in Definition 4.9 in Banach rings.

Proof. By Proposition 3.11, we may assume that k is non-trivially valued and k is non-trivially valued.

If $\rho(f) = 0$, then f lies in each maximal ideal of A. To see this, we may assume that A is a field, then by Proposition 6.10 in Banach rings., there is a bounded valuation $\|\bullet\|'$ on A. But then $\rho(f) = 0$ implies that $\|f\|' = 0$ and hence $f = 0$.

It follows that if $\rho(f) = 0$ then f lies in $J(A)$, the Jacobson radical of A. By Proposition 4.14, A is a Jacobson ring. So f is nilpotent. The assertion follows.

So we can assume that $\rho(f) > 0$. In this case, by Corollary 5.2 and Proposition 9.11 in Banach rings., we have $\rho(f) \in \sqrt{|k^\times|}$. Take $a \in k^\times$ and $d \in \mathbb{Z}_{>0}$ so that $\rho(f)^d = |a|$. Then $\rho(f^d/a) = 1$ and hence it is powerly-bounded by Proposition 6.1. It follows that there is $C > 0$ so that for $n \geq 1$,

$$\|f^{nd}\| \leq C|a|^n = C\rho(f)^{nd}.$$

It follows that $\|f^n\| \leq C\rho(f)$ for $n \geq d$ as long as we enlarge C. $\square$

Corollary 6.5. Let $\varphi : A \to B$ be a bounded homomorphism of k-affinoid algebras. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in B$ and $r_1, \ldots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$ for $i = 1, \ldots, n$. Write $r = (r_1, \ldots, r_n)$, then there is a unique bounded homomorphism $\Phi : A\{r^{-1}T\} \to B$ extending φ and sending T_i to f_i.

Proof. The uniqueness is clear. Let us consider the existence. Given

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in A\{r^{-1}T\},$$

we define

$$\Phi(h) = \sum_{\alpha \in \mathbb{N}^n} \varphi(a_\alpha) f^\alpha.$$

It follows from Proposition 6.4 that the right-hand side the series converges. The boundedness of Φ is obvious. $\square$

Proposition 6.6. Let $(A,\|\bullet\|_A),(B,\|\bullet\|_B)$ be k-affinoid algebras, $r\in\mathbb{R}^n_{>0}$ and $\varphi: A\{r^{-1}T\}\to B$ be an admissible epimorphism. Write $f_i=\varphi(T_i)$ for $i=1,\ldots,n$. Then there is $\epsilon>0$ such that for any $g=(g_1,\ldots,g_n)\in B^n$ with $\|f_i-g_i\|_B<\epsilon$ for all $i=1,\ldots,n$, there exists a unique bounded k-algebra homomorphism $\psi: A\{r^{-1}T\}\to B$ that coincides with φ on A and sends T_i to g_i. Moreover, ψ is also an admissible epimorphism.

PROOF. The uniqueness of ψ is obvious. We prove the remaining assertions. Taking $\epsilon>0$ small enough, we could further guarantee that $\rho(g_i)\leq r_i$. It follows from Corollary 6.5 that there exists a bounded homomorphism ψ as in the statement of the proposition.

As φ is an admissible epimorphism, we may assume that $\|\bullet\|_B$ is the residue induced by $\|\bullet\|_r$ on $A\{r^{-1}T\}$.

By definition of the residue norm, for any $\delta>0$ and any $h\in B$, we can find

$$k_0=\sum_{\alpha\in\mathbb{N}^n}a_\alpha T^\alpha\in A\{r^{-1}T\}$$

with

$$\|a_\alpha\|_A r^\alpha\leq(1+\delta)\|h\|_B$$

for any $\alpha\in\mathbb{N}^n$. Choose $\epsilon\in(0,(1+\delta)^{-1})$. Now for $g_1,\ldots,g_n$ as in the statement of the proposition, we can write

$$h=\sum_{\alpha\in\mathbb{N}^n}a_\alpha f^\alpha=\sum_{\alpha\in\mathbb{N}^n}a_\alpha g^\alpha+h_1=\psi(k_0)+h_1.$$

It follows that

$$\|h_1\|_B=\left\|\sum_{\alpha\in\mathbb{N}^n}a_\alpha(f^\alpha-g^\alpha)\right\|_B\leq(1+\delta)\epsilon\|h\|_B.$$

Repeating this procedure, we can construct $k_i\in A\{r^{-1}T\}$ for $i\in\mathbb{N}$ and $h_j\in B$ for $j\in\mathbb{Z}_{>0}$ such that for any $i\in\mathbb{Z}_{>0}$, we have

$$\begin{aligned}h=&\psi(k_0+\cdots+k_{i-1})+h_i,\\ \|k_i\|_r\leq&((1+\delta)\epsilon)^i(1+\delta)\|h\|_B,\\ \|h_i\|_B\leq&((1+\delta)\epsilon)^i\|h\|_B.\end{aligned}$$

In particular, $k:=\sum_{i=0}^\infty k_i$ converges in $A\{r^{-1}T\}$ and

$$\|k\|_r\leq(1+\delta)\|h\|_B.$$

It follows that ψ is an admissible epimorphism. □

Corollary 6.7. Let A be a Banach k-algebra, $n\in\mathbb{N}$ and $r=(r_1,\ldots,r_n)$ be a k-free polyray. Assume that $A\hat{\otimes}_k k_r$ is k_r-affinoid, then A is k-affinoid.

If $A\hat{\otimes}_k k_r$ is k_H-affinoid and $r\in H$, then A is also k_H-affinoid.

PROOF. We may assume that r has only one component.

Take $m\in\mathbb{N}$, $p_1,\ldots,p_m\in\mathbb{R}_{>0}$ and an admissible epimorphism

$$\pi: k_r\{p_1^{-1}S_1,\ldots,p_m^{-1}S_m\}\to A\hat{\otimes}_k k_r.$$

Let

$$\pi(S_i)=\sum_{j=-\infty}^{\infty}a_{i,j}T^j,\quad a_{i,j}\in A$$

for $i=1,\ldots,m$. By Proposition 6.6, we may assume that there is a large integer l such that $a_{i,j}=0$ for $|j|>l$ and for any $i=1,\ldots,m$. We define $B=k\{p_i^{-1}r^jT_{i,j}\}$, $i=1,\ldots,n$ and $j=-l,-l+1,\ldots,l$. Let $\varphi:B\to A$ be the bounded k-algebra homomorphism sending $T_{i,j}$ to $a_{i,j}$. The existence of φ is guaranteed by Corollary 6.5.

We claim that φ is an admissible epimorphism. It is clearly an epimorphism. Let us show that φ is admissible. Let $\eta:k_r\{p_1^{-1}S_1,\ldots,p_m^{-1}S_m\}\to B\hat{\otimes}_k k_r$ be the bounded homomorphism sending S_i to $\sum_{j=-l}^{l}T_{i,j}T^j$, then we have the following commutative diagram

$$\begin{tikzcd} k_r\{p^{-1}S\} \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\ B\hat{\otimes}_k k_r \arrow[r, "\varphi\hat{\otimes}_k k_r"] & A\hat{\otimes}_k k_r \end{tikzcd}$$

It follows that $\varphi\hat{\otimes}_k k_r$ is also an admissible epimorphism. By Proposition 3.11, φ is also admissible. $\square$

7. Examples of the Berkovich spectra of affinoid algebras

Let $(k,|\bullet|)$ be a complete non-Archimedean valued field.

Example 7.1. Take $r>0$. We will study the Berkovich spectrum $\operatorname{Sp}k\{r^{-1}T\}$.

We first assume that k is non-trivially valued and k is algebraically closed.

For $a\in k$ with $|a|\le r$ and $\rho\in(0,r]$, we set

$$E(a,\rho)=\left\{x\in\operatorname{Sp}k\{r^{-1}T\}:|(T-a)(x)|\le\rho\right\},$$
$$D(a,\rho)=\left\{x\in\operatorname{Sp}k\{r^{-1}T\}:|(T-a)(x)|<\rho\right\}.$$

We give a list of points on $\operatorname{Sp}k\{r^{-1}T\}$. The two classes are called *closed disks* and *open disks* with center a and with radius r.

(1) Any element $a\in k$ with $|a|\le r$ determines a bounded semi-valuation on $k\{r^{-1}T\}$ sending f to $|f(a)|$. Such points are called *points of type (1)*.
(2) For any $a\in k$ with $|a|\le r$ and $\rho\in|k|\cap(0,r]$, we define a bounded semi-valuation on $k\{r^{-1}T\}$ sending $f=\sum_{n=0}^{\infty}a_n(T-a)^n$ to
$$|f|_{E(a,\rho)}:=\max_{n\in\mathbb{N}}|a_n|\rho^n.$$
Such points are called *points of type (2)*.
(3) For any $a\in k$ with $|a|\le r$ and $\rho\in(0,r]\setminus|k|$, we define a bounded semi-valuation on $k\{r^{-1}T\}$ sending $f=\sum_{n=0}^{\infty}a_n(T-a)^n$ to
$$|f|_{E(a,\rho)}:=\max_{n\in\mathbb{N}}|a_n|\rho^n.$$
Such points are called *points of type (3)*.
(4) Let $\mathcal{E}=\{E^\rho\}_{\rho\in I}$ be a family of closed disks with radii ρ and such that $E^\rho\supseteq E^{\rho'}$ when $\rho>\rho'$, where I is a non-empty subset of $\mathbb{R}_{>0}$. We define a bounded semi-valuation on $k\{r^{-1}T\}$ sending f to
$$|f|_{\mathcal{E}}:=\inf_{\rho\in I}|f|_{E^\rho}.$$
If $\bigcap_{\rho\in I}E^\rho\cap k=\emptyset$, we call the point $|\bullet|_{\mathcal{E}}$ a *point of type (4)*.

We verify that points of type (1) are indeed points in $\mathrm{Sp}\, k\{r^{-1}T\}$: $f \mapsto |f(a)|$ is a bounded semi-valuation. It is clearly a semi-valuation. It is bounded by Lemma 6.3 in Banach rings.

We verify that points of type (2) and type (3) are indeed points in $\mathrm{Sp}\, k\{r^{-1}T\}$. We first need to make sense of the expansion

$$f = \sum_{n=0}^{\infty} a_n (T-a)^n. \tag{7.1}$$

In fact, by Corollary 6.5, there is an isomorphism of k-affinoid algebras $\iota : A\{r^{-1}T\} \to A\{r^{-1}S\}$ sending T to $S+a$, as $\|(S+a)^n\|_r = r^n$ and hence $\rho(S+a) = r$. We expand the image of $\sum_{n=0}^{\infty} a_n S^n$ and then (7.1) is just formally expressing this expansion. Now in order to show that $|\bullet|_{E(a,\rho)}$ is a bounded semi-valuation, we may assume that $a = 0$ after applying ι. It is a semi-valuation as $|\bullet|_\rho$ is a valuation on the larger ring $k\{\rho^{-1}T\}$. Again, the boundedness is a consequence of Lemma 6.3 in Banach rings.

We verify that points of type (4) are bounded semi-valuations. Take $\mathcal{E} = \{E^\rho\}_{\rho \in I}$ as above. It is a semi-valuation as the infimum of bounded semi-valuations. It is bounded as E^ρ is for any $\rho \in I$.

Proposition 7.2. Assume that k is non-trivially valued and algebraically closed. For any $r > 0$, a point in $\mathrm{Sp}\, k\{r^{-1}T\}$ belongs to one of the following classes: type (1), type (2), type (3), type (4).

Proof. Let $\|\bullet\|$ be a bounded semi-valuation on $k\{r^{-1}T\}$. Consider the family

$$\mathcal{E} := \{E(a, \|T-a\|) : a \in k, |a| \leq r\}.$$

We claim that if $a, b \in k$, $|a|, |b| \leq r$ and $\|T-a\| \leq \|T-b\|$, then

$$E(a, \|T-a\|) \subseteq E(b, \|T-b\|).$$

In fact, if $x \in E(a, \|T-a\|)$, then

$$|(T-a)(x)| \leq \|T-a\|.$$

Observe that $|a-b| \leq \max\{\|T-a\|, \|T-b\|\} = \|T-b\|$, so

$$|(T-b)(x)| \leq \max\{|(T-a)(x)|, |a-b|\} \leq \|T-b\|.$$

So $x \in E(b, \|T-b\|)$ proving our claim.

Now we claim that for any $a \in k$,

$$\|T-a\| = |T-a|_{\mathcal{E}}.$$

From this, it follows that the bounded semi-valuation $\|\bullet\|$ is necessarily of the form $|\bullet|_{\mathcal{E}}$, hence of type (1), type (2), type (3) or type (4).

In order to prove the claim, we observe that

$$|T-a|_{\mathcal{E}} = \inf_{b \in k, |b| \leq r} |T-a|_{E(b, \|T-b\|)}.$$

We write $T - a = T - b + b - a$, then

$$|T-a|_{E(b,\|T-b\|)} = \max\{\|T-b\|, |b-a|\} \geq \|T-a\|.$$

In particular $\|T-a\| \leq |T-a|_{\mathcal{E}}$. On the other hand, the computation shows that

$$|T-a|_{\mathcal{E}} = \inf_{b \in k, |b| \leq r} \max\{\|T-a\|, |b-a|\}.$$

In order to show that $\|T-a\| \geq |T-a|_{\mathcal{E}}$, it suffices to show that

$$\inf_{b\in k,|b|\leq r} |b-a| \leq \|T-a\|$$

when $|a|>r$. In this case, $1-a^{-1}T$ is invertible by Proposition 4.4 in Banach rings., so

$$\|1-a^{-1}T\| = \|1-a^{-1}T\|_r = 1+|a|^{-1}r.$$

We need to show

$$\inf_{b\in k,|b|\leq r} |b-a| \leq |a|+r,$$

which is obvious. This proves our claim.

□

Proposition 7.3. Assume that k is non-trivially valued and algebraically closed. Let $r>0$, and $x\in \operatorname{Sp} k\{r^{-1}T\}$.

(1) If x is of type (1), then $\mathscr{H}(x)=k$.

(2) If x is of type (2), then $\mathscr{H}(x)=k_\rho$, $\widetilde{\mathscr{H}(x)}=\tilde{k}(T)$ and $|\mathscr{H}(x)|=|k|$.

(3) If x is of type (3), then $\mathscr{H}(x)=k_\rho$, $\widetilde{\mathscr{H}(x)}=\tilde{k}$ and $|\mathscr{H}(x)^\times|$ is generated by ρ and $|k^\times|$.

(4) If x is of type (4), then $\widetilde{\mathscr{H}(x)}=\tilde{k}$ and $|\mathscr{H}(x)|=|k|$. Moreover, $\mathscr{H}(x)\neq k$. In other words, $\mathscr{H}(x)\supsetneq k$ is a non-trivial immediate extension.

In particular, the four types do no overlap.

Proof. (1) Assume that x is defined by $a\in k$ with $|a|\leq r$. Observe that the valuation factorizes through $k\{r^{-1}T\}\to k$, so $\mathscr{H}(x)$ is a subfield of k. But for $b\in k$, $b(x)=b$, so $\mathscr{H}(x)=k$.

(2) Assume that x is defined by $E(a,\rho)$ with $a\in k$, $|a|\leq r$ and $\rho\in(0,r]\cap|k|$. We may assume that $a=0$. Observe that $|\bullet|_{E(a,\rho)}$ is a valuation. So $\mathscr{H}(x)$ is the completion of the fraction field of $k\{r^{-1}T\}$, namely $\mathscr{H}(x)=k_\rho$. Observe that for any $f\in k\{r^{-1}T\}$, $|f|_{E(a,\rho)}$ is of the form $|a_n|\rho^n$ for some $a_n\in k$, $n\in\mathbb{N}$, so $|f|_{E(a,\rho)}\in|k|$ and hence $|\mathscr{H}(x)|\subseteq|k|$. The reverse inequality is trivial. The residue field is computed as in Corollary 4.20 from the chapter Banach rings.

(3) It follows from the same argument in (2) that $\mathscr{H}(x)=k_\rho$. On the other hand, an element

$$f=\sum_{i=-\infty}^{\infty} a_iT^i\in k_\rho$$

satisfies $|f|\leq 1$ (resp. $|f|<1$) if and only if $a_0\in\mathring{k}$ (resp. $a_0\in\check{k}$) and $|a_i|\rho^i<1$ for $i\neq 0$. It follows that $\widetilde{H(x)}=\tilde{k}$.

(4) To be finished

□

8. H-strict affinoid algebras

Let $(k,|\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $R_{>0}$ such that $|k^\times|\cdot H\neq\{1\}$.

We next give a non-strict extension of Proposition 3.13.

Proposition 8.1. Let B be a k_H-affinoid algebra and $\varphi: B\to A$ be a finite bounded homomorphism into a k-Banach algebra A. Then A is also k_H-affinoid.

Proof. We first assume that k is non-trivially valued.

We may assume that $B=k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}$ for some $n\in\mathbb{N}$ and $r_1,\dots,r_n\in H$. By assumption, we can find finitely many $a_1,\dots,a_m\in A$ such that $A=\sum_{i=1}^m\varphi(B)a_i$.

We may assume that $a_i\in \mathring{A}$ as k is non-trivially valued. By Proposition 4.18 in Banach rings., φ admits a unique extension to a bounded k-algebra epimorphism

$$\Phi: k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n,S_1,\dots,S_m\}\to A$$

sending S_i to a_i. By Corollary 7.5 in Banach rings., Φ is admissible. Moreover, the homomorphism Φ is surjective by our assumption. It follows that A is k_H-affinoid.

If k is trivially valued, then H is non-trivial. Take $s\in H\setminus\{1\}$. It follows from the previous case applied to $\varphi\hat{\otimes}k_s: B\hat{\otimes}k_s\to A\hat{\otimes}k_s$ that $A\hat{\otimes}k_s$ is k_H-affinoid. By Corollary 6.7, A is also k_H-affinoid. $\square$

Proposition 8.2. Let A be a Banach k-algebra. Then the following are equivalent:

(1) A is k_H-affinoid;
(2) there are $n\in\mathbb{N}$, $r\in\sqrt{|k^\times|\cdot H}$ and an admissible epimorphism $k\{r^{-1}T\}\to A$.

Proof. The non-trivial direction is (2). Assume (2). Take $s_1,\dots,s_n\in\mathbb{Z}_{>0}$, $c_1,\dots,c_n\in k^\times$ and $h_1,\dots,h_n\in H$ such that

$$r_i^{s_i}=|c_i^{-1}|h_i$$

for $i=1,\dots,n$. We define a bounded k-algebra homomorphism

$$\varphi: k\{h_1^{-1}T_1,\dots,h_n^{-1}T_n\}\to k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}$$

by sending T_i to $c_iT_i^{s_i}$. The existence of such a homomorphism is guaranteed by Corollary 6.5. The same proof of Lemma 3.15 shows that φ is finite. By Proposition 8.1, $k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\}$ is k_H-affinoid. $\square$

Lemma 8.3. Assume that k is non-trivially valued. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is strictly k-affinoid;
(2) for any $a\in A$, $\rho(a)\in\sqrt{|k^\times|}\cup\{0\}$.

Proof. (1) $\implies$ (2) by Corollary 5.10 and Corollary 6.2.

(2) $\implies$ (1): Take $n\in\mathbb{N}$, $r\in\mathbb{R}^n_{>0}$ and an admissible epimorphism

$$\varphi: k\{r^{-1}T\}\to A.$$

Let $f_i=\varphi(T_i)$ for $i=1,\dots,n$. Suppose $r_1,\dots,r_m\notin\sqrt{|k^\times|}$ and $r_{m+1},\dots,r_n\in\sqrt{|k^\times|}$. Then $\rho(f_i)<r_i$ for $i=1,\dots,m$ and we can choose $r_1',\dots,r_m'\in\sqrt{|k^\times|}$ such that

$$\rho(f_i)\leq r_i'<r_i$$

for $i=1,\dots,m$. Set $r_i'=r_i$ when $i=m+1,\dots,n$. We can then define a bounded k-algebra homomorphism $\psi: k\{r'^{-1}T\}\to A$ sending T_i to f_i for $i=1,\dots,n$. The existence of ψ is guaranteed by Corollary 6.5. Observe that ψ is surjective and admissible. It follows that A is strictly k-affinoid. $\square$

Theorem 8.4. Let A be a k-affinoid algebra. Then the following are equivalent:

(1) A is k_H-affinoid;
(2) A is $k_{\sqrt{|k^\times|\cdot H}}$-affinoid;

(3) For any non-zero $a \in A$, $\rho(a) \in \sqrt{|k^\times| \cdot H} \cup \{0\}$.

PROOF. The equivalence between (1) and (2) follows from Proposition 8.2.

(1) $\implies$ (3): we may assume that $H \supseteq |k^\times|$. Take $n \in \mathbb{N}$, $r = (r_1, \dots, r_n) \in H^n$ and an admissible epimorphism

$$\varphi : k\{r^{-1}T\} \to A.$$

Take a k-free polyray s with at least one component so that $|k_s| \supseteq \{r_1, \dots, r_n\}$. We can apply Lemma 8.3 to $\varphi \hat{\otimes}_k k_s$, it follows that $\rho(A) \subseteq \sqrt{|k_s^\times|} \cup \{0\}$.

(3) $\implies$ (2): we may assume that $H \supseteq |k^\times|$. It suffices to apply the same argument as (2) $\implies$ (1) in the proof of Lemma 8.3.

□

9. Finite modules over affinoid algebras

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

For any k-affinoid algebra A, we have defined the category $\mathcal{B}\mathrm{an}^f_A$ of finite Banach A-modules in Definition 5.3 in Banach rings. We write $\mathcal{M}\mathrm{od}^f_A$ for the category of finite A-modules.

Lemma 9.1. Let A be a k-affinoid algebra, $(M, \|\bullet\|_M)$ be a finite Banach A-module and $(N, \|\bullet\|_N)$ be a Banach A-module N. Let $\varphi : M \to N$ be an A-linear homomorphism. Then φ is bounded.

PROOF. Take $n \in \mathbb{N}$ such that there is an admissible epimorphism

$$\pi : A^n \to M.$$

It suffices to show that $\varphi \circ \pi$ is bounded. So we may assume that $M = A^n$. For $i = 1, \dots, n$, let e_i be the vector with $(0, \dots, 0, 1, 0, \dots, 0)$ of A^n with 1 placed at the i-th place. Set $C = \max_{i=1,\dots,n} \|\varphi(e_i)\|_N$. For a general $f = \sum_{i=1}^n a_i e_i$ with $a_i \in A$, we have

$$\|\varphi(f)\|_N \leq C\|f\|_M.$$

So φ is bounded. □

Proposition 9.2. Let A be a k-affinoid algebra. The forgetful functor $\mathcal{B}\mathrm{an}^f_A \to \mathcal{M}\mathrm{od}^f_A$ is an equivalence of categories.

PROOF. It suffices to construct the inverse functor. Let M be a finite A-module. Choose $n \in \mathbb{N}$ and an A-linear epimorphism $\pi : A^n \to M$. By Theorem 6.3, $\ker \pi$ is closed in A^n. We can endow M with the residue norm. By Lemma 9.1, the equivalence class of the norm does not depend on the choice of π.

For any A-linear homomorphism $f : M \to N$ of finite A-modules, we endow M and N with the Banach structures as above. It follows from Lemma 9.1 that f is bounded. We have defined the inverse functor of the forgetful functor $\mathcal{B}\mathrm{an}^f_A \to \mathcal{M}\mathrm{od}^f_A$. □

Remark 9.3. Let A be a k-affinoid algebra. It is not true that a Banach A-module which is finite as A-module is finite as Banach A-module.

As an example, take $0 < p < q < 1$ and $A = k\{q^{-1}T\}$, $B = k\{p^{-1}T\}$. Then B is a Banach A-module. By Example 2.4, the underlying rings of A and B are both $k[[T]]$. So the canonical map $A \to B$ is bijective. But B is not a finite A-module.

As otherwise, the inverse map $B \to A$ is bounded by Lemma 9.1, which is not the case.

The correct statement is the following: consider a Banach A-module $(M, \|\bullet\|_M)$ which is finite as A-module, then there is a norm on M such that M becomes a finite Banach A-module. The new norm is not necessarily equivalent to the given norm $\|\bullet\|_M$.

Proposition 9.4. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Then the natural map

$$M \otimes_A N \to M \hat{\otimes}_A N$$

is an isomorphism of Banach A-modules and $M\hat{\otimes}_A N$ is a finite Banach A-module.

Here the Banach A-module structure on $M \otimes_A N$ is given by Proposition 9.2.

PROOF. Choose $m, m' \in \mathbb{N}$ an admissibly coexact sequence

$$A^{m'} \to A^m \to M \to 0$$

of Banach A-modules. Then we have a commutative diagram of A-modules:

$$\begin{tikzcd}
A^{m'} \otimes_A N \arrow[r] \arrow[d] & A^m \otimes_A N \arrow[r] \arrow[d] & M \otimes_A N \arrow[r] \arrow[d] & 0 \\
A^{m'} \hat{\otimes}_A N \arrow[r] & A^m \hat{\otimes}_A N \arrow[r] & M \hat{\otimes}_A N \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By 5-lemma, in order to prove $M \otimes_A N \xrightarrow{\sim} M\hat{\otimes}_A N$ and $M\hat{\otimes}_A N$ is a finite Banach A-module, we may assume that $M = A^m$ for some $m \in \mathbb{N}$. Similarly, we can assume $N = A^n$ for some $n \in \mathbb{N}$. In this case, the isomorphism is immediate and $M\hat{\otimes}_A N$ is clearly a finite Banach A-module. By Lemma 9.1, the Banach A-module structure on $M\hat{\otimes}_A N$ coincides with the Banach A-module strucutre on $M \otimes_A N$ induced by Proposition 9.2. □

Proposition 9.5. Let A, B be a k-affinoid algebra and $A \to B$ be a bounded k-algebra homomorphism. Let M be a finite Banach A-module, then the natural map

$$M \otimes_A B \to M\hat{\otimes}_A B$$

is an isomorphism of Banach B-modules and $M\hat{\otimes}_A B$ is a finite Banach B-module.

PROOF. By the same argument as Proposition 9.4, we may assume that $M = A^n$ for some $n \in \mathbb{N}$. In this case, the assertions are trivial. □

Proposition 9.6. Let A be a k-affinoid algebra and M, N be finite Banach A-modules. Let $\varphi : M \to N$ be an A-linear map. Then φ is admissible.

PROOF. By Lemma 9.1, φ is always bounded. By Proposition 9.5 and Proposition 3.11, we may assume that k is non-trivially valued. By Theorem 6.3, N is a Noetherian A-module. It follows from Corollary 7.4 in Banach rings. that $\operatorname{Im} \varphi$ is closed in N and is finite as an A module. In particular, the norm induced from N and from M are equivalent by Lemma 9.1. It follows that φ is admissible. □

Proposition 9.7. Let A be a k-affinoid algebra. Let $n \in \mathbb{N}$ and $r = (r_1, \ldots, r_n)$ be a k-free polyray. Then M is a finite Banach A-module if and only if $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module.

PROOF. We may assume that r has only one component and write $r_1 = r$. The direct implication is trivial. Let us assume that $M\hat{\otimes}_k k_r$ is a finite Banach $A\hat{\otimes}_k k_r$-module. Take $n \in \mathbb{N}$ and an admissible epimorphism of $A\hat{\otimes}_k k_r$-modules

$$\varphi : (A\hat{\otimes}_k k_r)^n \to M\hat{\otimes}_k k_r.$$

Let $e_1, \dots, e_n$ denotes the standard basis of $(A\hat{\otimes}_k k_r)^n$. We expand

$$\varphi(e_i) = \sum_{j=-\infty}^{\infty} m_{i,j} T^j.$$

By Proposition 6.6, we can assume that there is $l > 0$ such that $m_{i,j} = 0$ for all $i = 1, \dots, n$ and $|j| > l$. It follows that

$$A^{n(2l+1)} \to M$$

sending the standard basis to $m_{i,j}$ with $i = 1, \dots, n$ and $j = -l, -l+1, \dots, l$ is an admissible epimorphism. □

Proposition 9.8. Let $\phi : A \to B$ be a morphism of k-affinoid algebras, $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then the following are equivalent:

(1) ϕ is finite and admissible.
(2) $\phi\hat{\otimes}_k k_r$ is finite and admissible.

This is [Tem04, Lemma 3.2]. I do not understand Temkin's argument. The following proof is a modification of the argument of Temkin's.

PROOF. (1) $\implies$ (2): This is straightforward.

(2) $\implies$ (1): The admissible part is straightforward. Let us prove that ϕ is finite. We may assume that $n = 1$. When r is not in $\sqrt{|k^\times|}$, we just apply Proposition 9.7. Now suppose $r \in \sqrt{|k^\times|}$. Let us take $m \in \mathbb{Z}_{>0}$ such that $r^m = |c^{-1}|$ for some $c \in k^\times$. Define a bounded k-algebra homomorphism

$$\varphi : k\{T\} \to k\{r^{-1}T\}$$

sending T to cT^m. Observe that φ is injective. We have argued in the proof of Lemma 3.15 that this homomorphism is finite.

Then φ induces a finite extension of ring $\operatorname{Frac} k\{r^{-1}T\}/\operatorname{Frac} k\{T\}$. In particular, the closure of $\operatorname{Frac} k\{T\}$ in k_r is a subfield over which k_r is finite. But this valuation field is isomorphic to $k\{T\}$. By Proposition 9.5 and fpqc descent [Stacks, Tag 02LA], we may assume that $r = 1$.

Recall that k_1 is the completion of $\operatorname{Frac} k\{T\}$. Let $\{\tilde{f}_i\}_{i\in I}$ be the set of irreducible monic polynomials in $\tilde{k}[T]$. Lift each $\tilde{f}_i$ to $f_i \in \mathring{k}[T]$. Let $a \in A\hat{\otimes}_k k_1$, we represent a as

$$a = \sum_{l=0}^{\infty} a_l T^l + \sum_{i\in I, j\geq 1, 0\leq k<\deg f_i} a_{ijk} T^k / f_i^j.$$

A similar expression exists for elements in $B\hat{\otimes}_k k_1$ as well. Moreover, the representation is unique.

As $B\hat{\otimes}_k k_1$ is finite over $A\hat{\otimes}_k k_1$, we can find $b_1, \dots, b_m$ such that any $b \in B$ can be written as

$$b = \sum_{j=1}^{m} \phi\hat{\otimes}_k k_1(a_j) b_j,$$

where $a_j \in A\hat{\otimes}_k k'$. We can replace b_j by $b_{j,0}$ and a_j by $a_{j,0}$. It follows that B is generated $b_{1,0}, \dots, b_{m,0}$ over A. $\square$

For any ring A, $\mathcal{A}\mathrm{lg}^f_A$ denotes the category of finitely generated A-algebras.

Proposition 9.9. Let A be a k-affinoid algebra. Then the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A \to \mathcal{A}\mathrm{lg}^f_A$ is an equivalence of categories.

Recall that $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A$ is defined in Definition 5.9 in Banach rings.

Proof. It suffices to construct an inverse functor. Let B be a finite A-algebra. We endow B with the norm $\|\bullet\|_B$ as in Proposition 9.2. We claim that B is a Banach A-algebra.

Let us recall the definition of the norm. Take $n \in \mathbb{N}$, an epimorphism $\varphi : A^n \to B$ of A-modules. Then $\|\bullet\|_B$ is the residue norm induced by φ.

Consider the A-linear epimorphism $\psi : A^n \otimes_A A^n \to B \otimes_A B$. By Proposition 9.6, when both sides are endowed with the norms $\|\bullet\|_{A^n\otimes_A A^n}$ and $\|\bullet\|_{B\otimes_A B}$ as in Proposition 9.2, ψ is admissible. It follows that there is $C > 0$ such that for any $f, g \in B$,

$$\|f \otimes g\|_{B\otimes B} \leq C\|f\|_B \cdot \|g\|_B.$$

On the other hand, by Proposition 9.2, the natural map $B \otimes_A B \to B$ is bounded. It follows that there is a constant $C' > 0$ such that

$$\|fg\|_B \leq C'\|f \otimes g\|_{B\otimes B}.$$

It follows that the multiplication in B is bounded and hence B is a finite Banach algebra. Given any morphism $B \to B'$ in $\mathcal{A}\mathrm{lg}^f_A$, we endow B and B' with the norms given by Proposition 9.2. It follows from Lemma 9.1 that $B \to B'$ is a bounded homomorphism of finite Banach A-algebras. So we have defined an inverse functor to the forgetful functor $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}^f_A \to \mathcal{A}\mathrm{lg}^f_A$. $\square$

Remark 9.10. It is not true that any homomorphism of k-affinoid algebras is bounded. For example, if the valuation on k is trivial. Take $0 < p < q < 1$ and consider the natural homomorphism $k_p \to k_q$. This homomorphism is bijective but not bounded.

10. Affinoid domains

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 10.1. Let A be a k_H-affinoid algebra. A closed subset $V \subseteq \mathrm{Sp}\,A$ is said to be a *k_H-affinoid domain* in X if there is an object $\mathrm{Sp}\,A_V \in k_H$-$\mathcal{A}\mathrm{ff}$ and a morphism $\phi : \mathrm{Sp}\,A_V \to \mathrm{Sp}\,A$ in k_H-$\mathcal{A}\mathrm{ff}$ such that

(1) the image of ϕ in $\mathrm{Sp}\,A$ is V;
(2) given any object $\mathrm{Sp}\,B \in k_H$-$\mathcal{A}\mathrm{ff}$ and a morphism $\mathrm{Sp}\,B \to \mathrm{Sp}\,A$ whose image lies in V, there is a unique morphism $\mathrm{Sp}\,B \to \mathrm{Sp}\,A$ in k_H-$\mathcal{A}\mathrm{ff}$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Sp}\,B \arrow[d, dotted, "!"] \arrow[dr] & \\ \mathrm{Sp}\,A_V \arrow[r, "\phi"] & \mathrm{Sp}\,A \end{tikzcd}$$

We say V is *represented by* the morphism ϕ or by the corresponding morphism $A \to A_V$.

When $H = \mathbb{R}_{>0}$, we say V is a *k-affinoid domain* in X. When $H = |k^\times|$, we say V is a *strict k-affinoid domain* in X.

We observe that A_V is canonically determined by the universal property.

Remark 10.2. This definition differs from the original definition of [Ber12], we follow the approach of Temkin instead. It can be shown that this definition is equivalent to the orignal definition of Berkovich when $H = \mathbb{R}_{>0}$.

A priori, this does not seem to be a good definition, as it is not easy to see that it is preserved by base field extension. But we will prove that it is the case after establishing the Gerritzen–Grauert theorem.

We begin with a few examples.

Example 10.3. Let A be a k_H-affinoid domain. Let $n, m \in \mathbb{N}$ and $f = (f_1, \dots, f_n) \in A^n$, $g = (g_1, \dots, g_m) \in A^m$. Let $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ and $s = (s_1, \dots, s_m) \in \sqrt{|k^\times| \cdot H}^m$. We define

$$(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\} := \{x \in \operatorname{Sp} A : |f_i(x)| \leq r_i, |g_j(x)| \geq s_j, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

We claim that $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ is a k_H-affinoid domain in $\operatorname{Sp} A$. These domains are called *k_H-Laurent domains* in $\operatorname{Sp} A$. When $m = 0$, the domains $\operatorname{Sp} A\{r^{-1}f\}$ are called *k_H-Weierstrass domains* in $\operatorname{Sp} A$.

To see this, we define

$$A\left\{r^{-1}f, sg^{-1}\right\} := A\{r^{-1}T, sS\}/(T_1 - f_1, \dots, T_n - f_n, g_1S_1 - 1, \dots, g_mS_m - 1).$$

By Theorem 6.3, this defines a Banach k-algebra structure. We write $\|\bullet\|'$ for the quotient norm. By definition, $A\left\{r^{-1}f, sg^{-1}\right\}$ is a k_H-affinoid algebra and there is a natural morphism $A \to A\left\{r^{-1}f, sg^{-1}\right\}$. We claim that this morphism represents $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$.

For this purpose, we first compute $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$. We observe that $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\} \to \operatorname{Sp} A$ is injective since $A[f, g^{-1}]$ is dense in $A\left\{r^{-1}f, sg^{-1}\right\}$. We will therefore identify $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ with a subset of $\operatorname{Sp} A$.

Next we show that the image of $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$ in $\operatorname{Sp} A$ is contained in $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$. Take $\|\bullet\| \in \operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$. Then there is a constant $C > 0$ such that

$$\|\bullet\| \leq C\|\bullet\|'.$$

Applying this to f_i^k for some $k \in \mathbb{Z}_{>0}$ and $i = 1, \dots, n$, we find that

$$\|f_i\|^k = \|f_i^k\| \leq C\|f_i^k\|' \leq C\|T_i^i\|_{r,s^{-1}} = Cr_i^k.$$

It follows that

$$\|f_i\| \leq r_i.$$

Similarly, we deduce $|g_j| \geq s_j$ for $j = 1, \dots, m$. Namely, $\|\bullet\| \in (\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$.

Next we verify the universal property: let $\operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid domains that factorizes through $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$. We write $\psi : A \to B$ for the corresponding morphism of k_H-affinoid algebras. By Corollary 6.12 in Banach rings., we have

$$\rho_B(f_i) = \sup_{x \in \operatorname{Sp} B} |f_i(x)| \leq \sup_{y \in (\operatorname{Sp} A)\{r^{-1}f, sg^{-1}\}} |f_i(y)| \leq r_i$$

for $i = 1, \dots, n$. Similarly, one deduces that $\rho(g_j) \le s_j^{-1}$ for $j = 1, \dots, m$.

We will construct the dotted arrows:

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d] & B \\
A\{r^{-1}T, sS\}^{\tau} \arrow[ur, dotted, "\eta"] \arrow[d] & \\
A\{r^{-1}f, sg\} \arrow[uur, dotted] &
\end{tikzcd}$$

so that this diagram commutes. We define η as the unique morphism sending T_i to f_i and S_j to g_j for $i = 1, \dots, n$, $j = 1, \dots, m$. The existence of such a morphism is guaranteed by Corollary 6.5. In order to descend this morphism to η', it suffices to show that $T_i - f_i$ and $g_j S_j - 1$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ lie in the kernel of η. But this is immediate from our definition. Moreover, it is clear that η' is necessarily unique.

It remains to show that each point in $(\operatorname{Sp} A)\left\{r^{-1}f, sg^{-1}\right\}$ lies in $\operatorname{Sp} A\left\{r^{-1}f, sg^{-1}\right\}$.

It suffices to treat the cases $(n, m) = (1, 0)$ and $(n, m) = (0, 1)$. We will only handle the former case, as the latter is similar. In concrete terms, we need to show that for any $x \in \operatorname{Sp} A$ corresponding to a bounded semi-valuation $|\bullet|_x$ on A satisfying $|f(x)| \le r$, we can always extend $|\bullet|_x$ to a bounded semi-valuation $\|\bullet\|$ on $A\{r^{-1}f\}$. Replacing A by $A/\ker|\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A. We endow $A\{r^{-1}T\}$ with the Gauss norm $\|\bullet\|_{x,r}$ induced by $|\bullet|_x$ and $A\{r^{-1}T\}$ with the quotient norm $\|\bullet\|$. This norm is bounded by construction. It suffices to show that it is a valuation and it extends the given valuation on A. The former is a consequence of the latter, as A is dense in $A\{r^{-1}f\}$. Now suppose $a \in A$. A general preimage of a in $A\{r^{-1}T\}$ is

$$a + (T - f)\sum_{j=0}^{\infty} b_j T^j = a - fb_0 + \sum_{j=1}^{\infty}(b_{j-1} - fb_j)T^j$$

with $\|b_j\|_A r^j \to 0$ as $j \to \infty$. Now we compute

$$\begin{aligned}
\|a - fb_j + \sum_{j=1}^{\infty}(b_{j-1} - fb_j)\|_{x,r} &= \max\left\{|a - fb_0|_x, \max_{j\ge1}|b_{j-1} - fb_j|_x r^j\right\} \\
&\ge \max\left\{|a - fb_0|_x, \max_{j\ge1}|b_{j-1} - fb_j|_x |f|_x^j\right\} \\
&= \max\left\{|a - fb_0|_x, \max_{j\ge1}|f^j b_{j-1} - f^{j+1}b_j|_x\right\} \ge \quad |a|_x.
\end{aligned}$$

So $\|a\| \ge |a|_x$. The reverse inequality is trivial. We conclude.

Example 10.4. Let A be a k_H-affinoid domain. Let $n \in \mathbb{N}$, $g \in A$, $f = (f_1, \dots, f_n) \in A^n$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$. Assume that $g, f_1, \dots, f_n$ generates the unit ideal. Define

$$(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\} = \{x \in \operatorname{Sp} A : |f_i(x)| \le r_i|g(x)| \text{ for } i = 1, \dots, n\}.$$

Then we claim that $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$ is a k_H-affinoid domain in $\operatorname{Sp} A$. Domains of this form are called *k_H-rational domains.*

To see this, we define

$$A\left\{r^{-1}\frac{f}{g}\right\} := A\{r^{-1}T\}/(gT_1 - f_1, \ldots, gT_n - f_n).$$

By Theorem 5.1, this is indeed a k_H-affinoid domain. We will denote by $\|\bullet\|'$ the residue norm. We will prove that the natural map $A \to A\left\{r^{-1}\frac{f}{g}\right\}$ represents the affinoid domain $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. Observe that

$$\operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$$

is injective as elemnts of the form a/g with $a \in A$ is dense in $A\left\{r^{-1}\frac{f}{g}\right\}$. Next we show that

$$(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\} \supseteq \operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}.$$

Let $x \in \operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$, take $|\bullet|_x$ as the correspoding bounded semi-valuation on $A\left\{r^{-1}\frac{f}{g}\right\}$. Then there is a constant $C > 0$ such that for any $k \in \mathbb{Z}_{>0}$,

$$|f_i|_x^k = |f_i^k|_x = |g|_x^k \cdot |T_i^k|_x \le C|g|_x^k r_i^k.$$

for all $i = 1, \ldots, n$. In particular,

$$|f_i|_x \le r_i |g|_x.$$

Hence, $x \in (\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$.

Next we verify the universal property. Let $\operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spectra factorizing through $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. Observe that $g(x) \neq 0$ for all $x \in (\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. As otherwise, $f_i(x) = 0$ for all $i = 1, \ldots, n$. This contradicts our assumption on $g, f_1, \ldots, f_n$. It follows that $\psi(g)$ is invertible by Corollary 6.11 int the chapter Banach Rings. From the definition of $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$, it is clear that $\rho(\psi(f_i)) \le r\rho(\psi(g))$ for $i = 1, \ldots, n$.

We construct

$$\begin{tikzcd}
A \arrow[r, "\psi"] \arrow[d] & B \\
A\{r^{-1}T\} \arrow[ur, dotted, "\eta"] \arrow[d] & \\
A\left\{r^{-1}\frac{f}{g}\right\} \arrow[uur, dotted, "\tau"'] &
\end{tikzcd}$$

successively. The morphism η sends T_i to $\psi(f_i)/\psi(g)$ for $i = 1, \ldots, n$. The existence of such a morphism is guaranteed by Corollary 6.5. Clearly $gT_i - f_i$ is contained in $\ker \eta$, so η descends to τ. The morphism τ is clearly unique.

It remains to verify that the image of $\operatorname{Sp} A\left\{r^{-1}\frac{f}{g}\right\}$ in $\operatorname{Sp} A$ is exactly $(\operatorname{Sp} A)\left\{r^{-1}\frac{f}{g}\right\}$. In other words, we need to verify that if $|\bullet|_x$ is a bounded semi-valuation on A satisfying $|f_i|_x \le r_i |g|_x$, then $|\bullet|_x$ extends to a bounded semi-valuation on $A\left\{r^{-1}\frac{f}{g}\right\}$. Replacing A by $A/\ker|\bullet|_x$, we may assume that $|\bullet|_x$ is a valuation on A. Consider the Gauss valuation $|\bullet|_{x,r}$ on $A\{r^{-1}T\}$ and the residue norm $\|\bullet\|$ on $A\left\{r^{-1}\frac{f}{g}\right\}$. It suffices to show that $\|\bullet\|$ is a valuation extending the valuation $|\bullet|_x$ on A. The former is a consequence of the latter. Take $a\in A$, we need to show that $|a|_x=\|a\|$.

A general preimage of a in $A\{r^{-1}T\}$ has the form

$$a+\sum_{i=1}^{n}(gT_i-f_i)\sum_{\alpha\in\mathbb{N}^n}^{\infty}b_{i,\alpha}T^{\alpha}$$

with $\|b_{i,\alpha}\|_A r^{\alpha}$, where $\|\bullet\|_A$ denotes the initial norm on A. The same argument as in Example 10.3 shows that

$$\|a+\sum_{i=1}^{n}(gT_i-f_i)\sum_{\alpha\in\mathbb{N}^n}^{\infty}b_{i,\alpha}T^{\alpha}\|_{x,r}\geq|a|_x.$$

So $\|a\|_x\geq|a_x|$, the reverse inequality is trivial.

Proposition 10.5. Let A be a k_H-affinoid algebra and $V\subseteq\operatorname{Sp}A$ be a k_H-affinoid domain represented by $\varphi:A\to A_V$. Then $\operatorname{Sp}\varphi$ induces a homeomorphism $\operatorname{Sp}A_V\to V$.

In particular, we will identify V with $\operatorname{Sp}A_V$ and say $\operatorname{Sp}A_V$ is a k_H-affinoid domain in $\operatorname{Sp}A$.

Proof. We observe that $\operatorname{Sp}A_V\to\operatorname{Sp}A$ is a monomorphism in the category k_H-$\mathcal{A}$ff. In other words, $A\to A_V$ is an epimorphism in the category k_H-$\mathcal{A}$ff$\mathcal{A}$lg. To see this, let $\eta_1,\eta_2:A_V\to B$ be two arrows in k_H-$\mathcal{A}$ff$\mathcal{A}$lg such that $\eta_1\circ\varphi=\eta_2\circ\varphi$. It follows from the universal property in Definition 10.1 that $\eta_1=\eta_2$. By Proposition 3.18, $\operatorname{Sp}A_V\to\operatorname{Sp}A$ is a bijection. But $\operatorname{Sp}A_V$ and Sp_A are both compact and Hausdorff by Theorem 6.13 in Banach rings., so $\operatorname{Sp}A_V\to V$ is a homeomorphism. □

Corollary 10.6. Let A be a k_H-affinoid algebra. Let $\operatorname{Sp}B$ be a k_H-affinoid domain in $\operatorname{Sp}A$ and $\operatorname{Sp}C$ is a k_H-affinoid domain in $\operatorname{Sp}A$, then $\operatorname{Sp}C$ is a k_H-affinoid domain in $\operatorname{Sp}A$.

Proof. This follows immediately from Proposition 10.5. □

Proposition 10.7. Let A be a k_H-affinoid algebra and V,W be k_H-Weierstrass domains (resp. k_H-Laurent domains, resp. k_H-rational domains) in $\operatorname{Sp}A$. Then $V\cap W$ is also a k_H-Weierstrass domain (resp. k_H-Laurent domain, resp. k_H-rational domain).

Proof. This is clear in the Weierstrass and Laurent cases. We will prove therefore assume that V and W are k_H-rational.

We take $f_1,\ldots,f_n\in A$, $g_1,\ldots,g_m\in A$ both generating the unit ideal and $r=(r_1,\ldots,r_n)\in\sqrt{|k^{\times}|\cdot H}^n$, $s=(s_1,\ldots,s_m)\in\sqrt{|k^{\times}|\cdot H}^m$ such that

$$V=\operatorname{Sp}A\left\{r^{-1}\frac{f}{f_m}\right\},\quad W=\operatorname{Sp}A\left\{s^{-1}\frac{g}{g_n}\right\}.$$

We may assume that $r_n = s_m = 1$. Now let $R = (R_{i,j}) \in \sqrt{|k^\times| \cdot H}^{mn}$ where $R_{i,j} = r_i s_j$ and $F = (F_{i,j})$ with $F_{i,j} = f_i g_j$ for $i = 1, \dots, n$, $j = 1, \dots, m$. Observe that the $F_{i,j}$'s generate the unit ideal. We consider the k_H-rational domain

$$Z = \operatorname{Sp} A \left\{ R^{-1} \frac{F}{f_n g_m} \right\}.$$

We clearly have $V \cap W \subseteq Z$. We need to prove the reverse inequality. Let $x \in Z$, so we have

$$|f_i g_j(x)| \leq r_i s_j |f_n g_m(x)|$$

for any $i = 1, \dots, n$, $j = 1, \dots, m$. In particular, when $j = m$, we have

$$|f_i g_m(x)| \leq r_i |f_n g_m(x)|$$

for any $i = 1, \dots, n$. But $f_n g_m$ is invertible, so we can cancel $g_m(x)$ to find

$$|f_i(x)| \leq r_i |f_n(x)|.$$

So $x \in V$. Similarly, we have $x \in W$. □

Corollary 10.8. Let A be a k_H-affinoid algebra and V be a k_H-Laurent domain in $\operatorname{Sp} A$. Then V is also a k_H-rational domain.

Proof. By Proposition 10.7, it suffices to show consider k_H-Laurent domains of the following form:

$$\operatorname{Sp} A\{r^{-1} f\}, \quad \operatorname{Sp} A\{s g^{-1}\}$$

where $r, s \in \sqrt{|k^\times| \cdot H}$ and $f, g \in A$. Both domains are k_H-rational by definition. □

Proposition 10.9. Let A be a k_H-affinoid algebra and $\operatorname{Sp} B$ be a k_H-rational domain in $\operatorname{Sp} A$. Then there is a k_H-Laurent domain $\operatorname{Sp} C$ in $\operatorname{Sp} A$ such that $\operatorname{Sp} B \subseteq \operatorname{Sp} C$ and $\operatorname{Sp} B$ is a k_H-Weierstrass domain in $\operatorname{Sp} C$.

Proof. We write

$$B = A \left\{ r^{-1} \frac{f}{g} \right\}$$

for some $n \in \mathbb{N}$, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$, $f = (f_1, \dots, f_n) \in A^n$ and $g \in A$ such that $f_1, \dots, f_n, g$ generate the unit ideal. Let g'' be the image of g in B, which is a unit. Choose $c \in \sqrt{|k^\times| \cdot H}$ such that $\rho_B(g^{-1}) < c^{-1}$. Set $C = A\{c g^{-1}\}$, then $\operatorname{Sp} B \subseteq \operatorname{Sp} C$. Moreover,

$$\operatorname{Sp} B \cap \operatorname{Sp} C = \emptyset.$$

Let $f_1', \dots, f_n', g'$ be the images of $f_1, \dots, f_n, g$ in C. Write $f' = (f_1', \dots, f_n')$. Then by Corollary 6.11 in Banach rings., g' is a unit and

$$\operatorname{Sp} B = \operatorname{Sp} C\{r^{-1} g'^{-1} f'\}.$$

□

Proposition 10.10. Let A be a k_H-affinoid algebra, $\operatorname{Sp} B$ be a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} A$ and $\operatorname{Sp} C$ be a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} B$. Then $\operatorname{Sp} C$ is a k_H-Weierstrass domain (resp. k_H-rational domain) in $\operatorname{Sp} A$.

Proof. We first handle the Weierstrass case. Write

$$B = \operatorname{Sp} A\{r^{-1}f\}, C = \operatorname{Sp} B\{s^{-1}g\}$$

for some $n, m \in \mathbb{N}$, $r \in \sqrt{|k^\times| \cdot H}^n$, $s \in \sqrt{|k^\times| \cdot H}^m$ and $f = (f_1, \dots, f_n) \in A^n$, $g = (g_1, \dots, g_m) \in B^m$. Observe that if we replace g with a small perturbation, the domain $\operatorname{Sp} C$ in $\operatorname{Sp} B$ remains the same, so we may assume that $g_1, \dots, g_m \in A$. Then

$$\operatorname{Sp} C = \operatorname{Sp} A\{r^{-1}f\} \cap \operatorname{Sp} A\{s^{-1}g\}$$

is a k_H-Weierstrass domain by Proposition 10.7.

Next we handle the rational case. Write

$$B = A\left\{s^{-1}\frac{f}{g}\right\}$$

for some $m \in \mathbb{N}$, $f = (f_1, \dots, f_m) \in A^m$, $r = (r_1, \dots, r_m) \in \sqrt{|k^\times| \cdot H}^m$ and $g \in A$ such that $f_1, \dots, f_m, g$ generate the unit ideal.

By Proposition 10.9 and Proposition 10.7, it suffices to handle the special cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ for some $r \in \sqrt{|k^\times| \cdot H}$ and $h \in B$. Observe that making a small perturbation on h does not change the domain. As $A[g^{-1}]$ is dense in B, we may assume that there is $n \in \mathbb{Z}_{>0}$ such that $h' = g^n h \in A$. As g is invertible on $\operatorname{Sp} B$, we can find $c \in \sqrt{|k^\times| \cdot H}$ so that

$$|g(x)|^n > c^{-1}$$

for $x \in \operatorname{Sp} B$.

We need to treat the cases $C = B\{r^{-1}h\}$ and $C = B\{rh^{-1}\}$ separately. In the first case, we write

$$\operatorname{Sp} C = \operatorname{Sp} B \cap \operatorname{Sp} A\left\{(r, c)^{-1}\frac{(h', 1)}{g^n}\right\}.$$

In the second case,

$$\operatorname{Sp} C = \operatorname{Sp} B \cap \operatorname{Sp} A\left\{(r, c)^{-1}\frac{(g^n, 1)}{h'}\right\}.$$

□

Lemma 10.11. Let A be a k_H-affinoid algebra and $\operatorname{Sp} B$ be a k_H-affinoid domain in $\operatorname{Sp} A$. Let $\operatorname{Sp} C$ be a rational domain in $\operatorname{Sp} A$, then $(\operatorname{Sp} C) \cap (\operatorname{Sp} B)$ is a k_H-affinoid domain in $\operatorname{Sp} A$ represented by $A \to B\hat{\otimes}_A C$.

Proof. We first recall that $B\hat{\otimes}_A C$ is k_H-affinoid by Proposition 3.4.

We may assume that

$$C = A\left\{s\frac{f}{g}\right\}$$

for some $m \in \mathbb{N}$, $f = (f_1, \dots, f_m) \in A^m$, $r = (r_1, \dots, r_m) \in \sqrt{|k^\times| \cdot H}^m$ and $g \in A$ such that $f_1, \dots, f_m, g$ generate the unit ideal.

We observe that there is a natural isomorphism

$$B\hat{\otimes}_A C \cong B\left\{s^{-1}\frac{f}{g}\right\}.$$

Hence,

$$\operatorname{Sp} B\hat{\otimes}_A C = \{x \in \operatorname{Sp} B : |f_i(x)| \leq s|g(x)| \text{ for } i = 1, \dots, m\}.$$

On the other hand,

$$\operatorname{Sp} C = \{x \in \operatorname{Sp} A : |f_i(x)| \leq s|g(x)| \text{ for } i = 1, \ldots, m\}\,.$$

So $\operatorname{Sp} B\hat{\otimes}_A C = B\hat{\otimes}_A C$. By Proposition 3.4, we have the Cartesian square in the diagram below:

$$\begin{tikzcd}
\operatorname{Sp} D \arrow[dr, dotted] \arrow[drr, bend left=10] \arrow[ddr, dotted] & & \\
& \operatorname{Sp} B\hat{\otimes}_A C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Sp} C \arrow[d] \\
& \operatorname{Sp} B \arrow[r] & \operatorname{Sp} A
\end{tikzcd}$$

It remains to verify the universal property. Let $\operatorname{Sp} D \to \operatorname{Sp} C$ be a morphism of k_H-affinoid spectra that factorizes through $(\operatorname{Sp} C) \cap (\operatorname{Sp} B)$. Then by the universal property of $\operatorname{Sp} B$ in $\operatorname{Sp} A$, we find the dotted morphism $\operatorname{Sp} D \to \operatorname{Sp} B$ making the diagram commutes. Then as the square is Cartesian, we get the desired morphism $\operatorname{Sp} D \to \operatorname{Sp} B\hat{\otimes}_A C$. This morphism is clearly unique. □

11. Graded reduction

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 11.1. Let A be a Banach k-algebra, we define the *graded reduction* of A as

$$\tilde{A} := \bigoplus_{h \in \mathbb{R}_{>0}} \{x \in A : \rho(x) \leq h\} \,/\, \{x \in A : \rho(x) < h\}\,.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define $\tilde{f}$ as the image of f in the $\rho(f)$-graded piece of $\tilde{A}$.

Definition 11.2. Let A be a k_H-affinoid algebra. We define the *k_H-graded reduction* of A as the $\sqrt{|k^\times| \cdot H}$-graded ring

$$\tilde{A}^H := \bigoplus_{h \in \sqrt{|k^\times| \cdot H}} \{x \in A : \rho(x) \leq h\} \,/\, \{x \in A : \rho(x) < h\}\,.$$

For any $f \in A$ with $\rho(f) \neq 0$, we define $\tilde{f}$ as the image of f in the $\rho(f)$-graded piece of $\tilde{A}^H$.

For any morphism $f : A \to B$ of k_H-affinoid algebras, we define

$$\tilde{f}^H : \tilde{A}^H \to \tilde{B}^H$$

as the map induced by sending the class of $x \in A$ with $\rho(x) \leq h$ for any $h \in \sqrt{|k^\times| \cdot H}$ to the class of $f(x) \in B$.

Recall that $\rho(A) = \sqrt{|k^\times| \cdot H} \cup \{0\}$ by Theorem 8.4, so $\tilde{f}$ is well-defined. This definition is compatible with Definition 11.1 in the sense that if we regard a $\sqrt{|k^\times| \cdot H}$-graded ring as a $\mathbb{R}_{>0}$-graded ring, the two definitions give the same object.

Example 11.3. If K is a k_H-affinoid algebra which is a field as well, then $\tilde{K}^H$ is a $\sqrt{|k^\times| \cdot H}$-graded field. This is immediate from the definition.

Lemma 11.4. Let $(A, \|\bullet\|)$ be a k-affinoid algebra, $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Let $f \in k\{r^{-1}T\}$. Expand f as

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha .$$

Then

$$\rho(f) = \max_{\alpha \in \mathbb{N}^n} \rho(a_\alpha) r^\alpha .$$

PROOF. By induction, we may assume that $n = 1$ and write $r = r_1$. As ρ is a bounded powerly bounded semi-norm, we have

$$\rho(f) \leq \max_{j \in \mathbb{N}} \rho(a_j T^j) \leq \max_{j \in \mathbb{N}} \rho(a_j)\rho(T^j) = \max_{j \in \mathbb{N}} \rho(a_j) r^j .$$

Observe that $\rho(a_j)$ is not ambiguous: when intepreted as in A and in $A\{r^{-1}T\}$, it has the same value.

Conversely, we need to show that for any $j \in \mathbb{N}$,

$$\rho(f) \geq \rho(a_j) r^j .$$

Equivalently, this means for any $k \in \mathbb{Z}_{>0}$ and any $j \in \mathbb{N}$, we need to show that

$$\|f^k\|_r \geq \rho(a_j)^k r^{jk} .$$

Fix j and k as above. We compute the left-hand side:

$$f^k = \sum_{\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k} b_\beta T^{|\beta|}, \quad b_\beta = \prod_{l=1}^{k} a_{\beta_l} .$$

It follows that

$$\|f^k\|_r = \max_{\beta \in \mathbb{N}^k} \|b_\beta\| T^{|\beta|} .$$

Take $\beta = (j, j, \ldots, j)$, we find

$$\|f^k\|_r \geq \|a_j^k\| r^{jk} \geq \rho(a_j)^k r^{jk} .$$

□

Lemma 11.5. Assume that k is non-trivially valued. Let A be a strictly k-affinoid algebra. Then for any $a, f \in A$, the set of non-zero values $\rho(f^n a)$ for $n \in \mathbb{N}$ is a discrete subset of $\mathbb{R}_{>0}$.

PROOF. As A is noetherian Theorem 6.3, it has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. It follows that

$$\operatorname{Sp} A = \bigcup_{i=1}^{m} \operatorname{Sp} A/\mathfrak{p}_i .$$

Here we make the obvious identification by identifying $\operatorname{Sp} A/\mathfrak{p}_i$ with a subset of $\operatorname{Sp} A$.

By Corollary 6.12 in Banach rings., it suffices to consider each of $\operatorname{Sp} A/\mathfrak{p}_i$ separately, so we may assume that A is an integral domain.

By Corollary 5.2, we can take $d \in \mathbb{N}$ and a finite injective homomorphism of k-algebras $\iota : k\{T_1, \ldots, T_d\} \to A$. According to Proposition 9.11 in Banach rings., ρ_A is the restriction of the norm $\|\bullet\|_{\operatorname{Frac} A}$ on $\operatorname{Frac} A$ induced by the finite extension $\operatorname{Frac} A / \operatorname{Frac} k\{T_1, \ldots, T_d\}$ from the Gauss valuation. But it is well-known that $\|\bullet\|_{\operatorname{Frac} A}$ is the maximum of finitely many valuations on $\operatorname{Frac} A$. Reproduce BGR3.3.3.1 somewhere. The assertion is by now obvious. □

Lemma 11.6. Let $(A,\|\bullet\|)$ be a k-affinoid algebra, $f\in A$ with $r=\rho(f)>0$. Let $B=A\{r^{-1}f\}$. Then for any $a\in A$, we have

$$\rho_B(a)=\lim_{n\to\infty} r^{-n}\rho_A(f^na).$$

If moreover, $\rho_B(a)>0$, then there is $n_0>0$ such that for $n\geq n_0$,

$$\rho_B(a)=r^{-n}\rho_A(f^na),\quad \rho_B(f^na)=r^{-n}\rho_A(a).$$

Proof. We observe that for any $a\in A$, $n\in\mathbb{Z}_{>0}$, we have

$$\rho_B(f^na)=r^n\rho_B(a).$$

So the last two assertions are equivalent.

Take a k-free polyray s such that $A\hat{\otimes}_k k_s$ and $B\hat{\otimes}_k k_s$ are both strictly k_s-affinoid. By Proposition 3.11, $A\hat{\otimes}_k k_s\{r^{-1}f\}\xrightarrow{\sim} B\hat{\otimes}_k k_s$. Moreover, ρ_A and ρ_B are both preserved after base change to k_s. So we may assume that k is non-trivially valued and A and B are strictly k-affinoid.

Observe that for $n\in\mathbb{Z}_{>0}$,

$$\rho_A(f^{n+1}a)\leq\rho_A(f)\rho_A(f^na)=r\rho_A(f^na).$$

So $r^{-n}\rho_A(f^na)$ is decreasing in n. Moreover, for any $x\in\operatorname{Sp}A\{r^{-1}f\}$, by Example 10.3, we have

$$|f(x)|\geq r.$$

By Corollary 6.12 in Banach rings., we have

$$|f(x)|=r$$

for any $x\in\operatorname{Sp}A\{r^{-1}f\}$. It follows from Corollary 6.12 in Banach rings. that for any $n\in\mathbb{Z}_{>0}$,

$$\rho_A(f^na)=\sup_{x\in\operatorname{Sp}A}|f^na(x)|\geq r^n\sup_{x\in\operatorname{Sp}A\{rf^{-1}\}}|a(x)|=r^n\rho_B(a).$$

By Lemma 11.5, the decreasing sequence $\{r^{-n}\rho_A(f^na)\}_n$ either tends to 0 or is eventually constant. It converges to 0, there is nothing else to prove. So let us assume that there is $\alpha\in\mathbb{R}_{>0}$ and $n_0>0$ such that for $n\geq n_0$, we have

$$r^{-n}\rho_A(f^na)=\alpha.$$

We have to show that $\alpha\leq\rho_B(a)$. Assume the contrary $\alpha>\rho_B(a)$. Then for all $x\in\operatorname{Sp}A$, we have

$$|f^na(x)|\leq r^n|a(x)|.$$

So f^na must obtain its maximum on $U:=\{x\in\operatorname{Sp}A:|a(x)|\geq\alpha\}$. But U is disjoint from $\operatorname{Sp}A\{r^{-1}f\}$ as

$$\alpha>\rho_B(a).$$

It follows from Example 10.3 that

$$\beta:=\sup_{x\in U}|f(x)|=\max_{x\in U}|f(x)|<r.$$

So

$$\rho(f^na)=\sup_{x\in\operatorname{Sp}A}|f^na(x)|=\sup_{x\in U}|f^na(x)|\leq\beta^n\sup_{x\in U}|a(x)|.$$

This contradicts the fact that $\alpha>0$. ∎

Proposition 11.7. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\widetilde{A\{r^{-1}T\}}^H \xrightarrow{\sim} \tilde{A}^H[r^{-1}T]$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that k_r is defined in Example 3.12.

Proof. By Lemma 11.4, we have a natural isomorphism

$$\widetilde{A\{r^{-1}T\}}^H_s \xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{N}^n} \tilde{A}^H_{sr^{-\alpha}}$$

for any $s \in \sqrt{|k^\times| \cdot H}$. This establishes the desired isomorphism. □

Proposition 11.8. Let A be a k_H-affinoid algebra and $f \in A$ with $r = \rho(f) > 0$. Then there is a natural isomorphism

$$\tilde{A}^H_{\tilde{f}} \xrightarrow{\sim} \widetilde{A\{rf^{-1}\}}^H$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

Recall that $A\{rf^{-1}\}$ is defined in Example 10.3, by Theorem 8.4, it is k_H-affinoid.

Proof. Let $B = A\{rf^{-1}\}$ and denote by $\phi : \tilde{A}^H \to \tilde{A}^H_{\tilde{f}}$ the natural $\sqrt{|k^\times| \cdot H}$-graded homomorphism. From the universal property add details, we can factor the natural map $\tilde{A}^H \to \tilde{B}^H$ as $\psi : \tilde{A}^H_{\tilde{f}} \to \tilde{B}^H$. We have a commutative diagram:

$$\begin{tikzcd} \tilde{A}^H \arrow[r] \arrow[d, "\phi"'] & \tilde{B}^H \\ \tilde{A}^H_{\tilde{f}} \arrow[ur, "\psi"] & \end{tikzcd}$$

We claim that ψ is bijective. Let $\tilde{a}/\tilde{f}^m$ be an element in $\ker\psi$, where $\tilde{a} \in \tilde{A}^H$ is homogeneous. Lift $\tilde{a}$ to $a \in A$. Then $\rho_B(a) < \rho_A(a)$. By Lemma 11.6, $\rho_A(f^n a) < r^n \rho_A(a)$ when n is large enough, so

$$\tilde{f}^n \tilde{a} = 0$$

in $\tilde{A}$. Therefore, $\tilde{a}/f^m = 0$ in $\tilde{A}^H_{\tilde{f}}$. We have shown that ψ is injective.

It remains to show that ψ is surjective. Let $\tilde{b} \in \tilde{B}^H$ be a non-zero homogeneous element. Lift $\tilde{b}$ to $b \in B$ of the form $f^{-n}a$ for some $a \in A$. By Lemma 11.6 again, up to enlarging n, we can assume that $\rho_B(a) = \rho_A(a)$. Then $\tilde{a} = \tilde{f}^n \tilde{b}$ has a preimage in $\tilde{A}$.

□

Corollary 11.9. Let A be a k_H-affinoid algebra and $r \in \mathbb{R}^n_{>0}$, then there is a functorial isomorphism

$$\tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H \cong \widetilde{A \hat{\otimes}_k k_r}^H$$

of $\sqrt{|k^\times| \cdot H}$-graded rings.

PROOF. We can write

$$A\hat{\otimes}_k k_r = \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} A\{r^{-1}T\}\{\rho(g)g^{-1}\}.$$

Taking graded reduction, we find

$$\begin{aligned}\widetilde{A\hat{\otimes}_k k_r}^H &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \widetilde{A\{r^{-1}T\}\{\rho(g)g^{-1}\}}^H\\ &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \widetilde{A\{r^{-1}T\}}^H_{\tilde{g}}\\ &= \varinjlim_{g\in k\{r^{-1}T\}, g\neq 0} \tilde{A}^H[r^{-1}T]_{\tilde{g}}\\ &= \tilde{A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H.\end{aligned}$$

Here we have applied Proposition 11.8 in the second equality and Proposition 11.7 in the third equality. The first equality follows from the simple observation that graded reduction commutes with filtered colimits. □

Theorem 11.10. Let $\phi : A \to B$ be a morphism of k_H-affinoid algebras. Then the following are equivalent:

(1) ϕ is finite and admissible.
(2) $\tilde{\phi} : \tilde{A}^H \to \tilde{B}^H$ is finite.

PROOF. Take $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$ so that

$$\rho(A\hat{\otimes}_k k_r) = \rho(B\hat{\otimes}_k k_r) = |k_r|$$

and k_r is non-trivially valued. Proof that this is possible.

By Corollary 2.36 in Commutative algebras and Proposition 9.8, we may assume that k is non-trivially valued and $\rho(A) = \rho(B) = |k|$. By Lemma 2.33 in the chapter Commutative Algebra, we have $\tilde{A} = \tilde{A}_1 \otimes_{\tilde{k}_1} \tilde{k}$. By Corollary 5.5, ϕ is automatically admissible if it is finite.

So it suffices to argue that ϕ is finite if and only if $\tilde{\phi} : \tilde{A} \to \tilde{B}$ is finite.

Assume that φ is finite. We show that $\tilde{\varphi}$ is finite.

First consider the case where A is an integral domain.

We claim that there is $d \in \mathbb{N}$ and a k-algebra homomorphism $\psi : k\{T_1, \ldots, T_d\} \to A$ such that $\phi \circ \psi$ is finite and injective. In fact, choosing an epimorphism $\alpha : k\{T_1, \ldots, T_d\} \to A$, we can apply Theorem 5.1 to find $\phi \circ \alpha$ to conclude.

It suffices to show that $\widetilde{\phi \circ \psi}$ is finite in order to conclude that $\tilde{\phi}$ is finite. So we are reduced to the case $A = k\{T_1, \ldots, T_d\}$ and $\ker \phi = 0$.

We will show that the conditions of Lemma 10.1 in Banach rings. is satisfied with ρ_B as the norm B. We have shown that ρ_B is a faithful $k\{T_1, \ldots, T_d\}$-algebra nrom in Corollary 4.16. As B is of finite over $k\{T_1, \ldots, T_d\}$, the rank condition is clearly satisfied. It remains to establish that $\mathring{\phi}$ is integral.

By Proposition 5.12, for $f \in B$, there is an integral equation

$$f^n + \phi(a_1)f^{n-1} + \cdots + \phi(a_n) = 0$$

over A such that $\rho_B(f) = \max_{i=1,\ldots,n} |b_i|^{1/i}_{\sup}$. If $f \in \mathring{B}$, then $|b_i|_{\sup} \leq 1$, hence $b_i \in \mathring{B}$. Add a ref

Conversely, assume that $\tilde{\phi}$ is finite. It suffices to apply Lemma 5.15 to conclude that ϕ is finite. □

Corollary 11.11. Let A be a k_H-affinoid algebra, then $\tilde{A}^H$ is finitely generated over $\tilde{k}^H$.

PROOF. Take $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism

$$\pi : k\{r^{-1}T\} \to A.$$

Applying Theorem 11.10, we find that it suffices to prove that $\widetilde{k\{r^{-1}T\}}^H$ is finitely generated over $\tilde{k}^H$. But this follows from Proposition 11.7. □

Definition 11.12. Let A be a k_H-affinoid algebra, we define the *reduction map*

$$\tilde{\mathrm{Sp}}\, A^H := \mathrm{Spec}^{\sqrt{|k^\times|\cdot H}}\,\tilde{A}^H.$$

We have a natural map $\pi^H : \mathrm{Sp}\, A \to \tilde{\mathrm{Sp}}\, A^H$: given $x \in \mathrm{Sp}\, A$, it defines a character $\chi_x : A \to \mathscr{H}(x)$, which in turn induces $\widetilde{\chi_x} : \tilde{A}^H \to \widetilde{\mathscr{H}(x)}$. We define $\pi^H(x) = \ker \widetilde{\chi_x}$.

Lemma 11.13. Assume that k is non-trivially valued and A is a strictly k-affinoid algebra. Then the reduction map

$$\pi : \mathrm{Sp}\, A \to \mathrm{Spec}\, \tilde{A}$$

is surjective.

The reduction map is defined as follows: a point $x \in \mathrm{Sp}\, A$ defines a character $\chi_x : A \to \mathscr{H}(x)$. By reduction, we get $\tilde{\chi_x} : \tilde{A} \to \widetilde{\mathscr{H}(x)}$. The kernel is the image of x.

PROOF. **Step 1**. We assume that $A = k\{T_1, \dots, T_n\}$ for some $n \in \mathbb{N}$.

We make induction on n. The case $n = 0$ is trivial. We first handle the case $n = 1$. In this case, we have an explicit description of the Berkovich disk Example 7.1 when k is algebraically closed.

By Corollary 8.6 in Banach rings, we have a natural identification

$$\mathrm{Sp}\, k\{T\} = \mathrm{Sp}\, \widehat{k^{\mathrm{alg}}\{T\}} / \mathrm{Gal}(k^{\mathrm{sep}}/k).$$

By Proposition 4.1, we have an identification $\widetilde{k\{T\}} = \tilde{k}[T]$. The prime ideals are of two types: $(T - a)$ for some $a \in k$ and 0. In the former case, the type (1) point defined by a lies in the inverse image of $(T - a)$ by definition. In the second case, we take the Gauss point $\|\bullet\|_1$.

Consider the case $n > 1$. Assume that the assertion has been proved for lower n. Let $p : \mathrm{Sp}\, k\{T_1, \dots, T_n\} \to \mathrm{Sp}\, k\{T_1\}$ be the projection induced by $k\{T_1\} \to k\{T_1, \dots, T_n\}$ sending T_1 to T_1. We have a comutative diagram

$$\begin{tikzcd}
\mathrm{Sp}\, k\{T_1, \dots, T_n\} \arrow[r, "p"] \arrow[d, "\pi"] & \mathrm{Sp}\, k\{T_1\} \arrow[d, "\pi"] \\
\mathrm{Spec}\, \tilde{k}[T_1, \dots, T_n] \arrow[r] & \mathrm{Spec}\, \tilde{k}[T_1]
\end{tikzcd}.$$

Let $\tilde{x} \in \operatorname{Spec} \tilde{k}[T_1, \ldots, T_n]$ and $\tilde{y}$ be its image in $\operatorname{Spec} \tilde{k}[T_1]$. By the case $n = 1$, we can find $y \in \operatorname{Sp} k\{T_1\}$ with $\pi(y) = \tilde{y}$. There is a bijection $p^{-1}(y)$ with $\operatorname{Sp} \mathscr{H}(y)\{T_2, \ldots, T_n\}$. So it suffices to show that

$$\operatorname{Sp} \mathscr{H}(y)\{T_2, \ldots, T_n\} \to \operatorname{Spec} \kappa(\tilde{y})[T_2, \ldots, T_n] \tag{11.1}$$

is surjective. By construction, we have an embedding $\kappa(\tilde{y}) \to \widetilde{\mathscr{H}(y)}$, so we can factorize (11.1) as

$$\operatorname{Sp} \mathscr{H}(y)\{T_2, \ldots, T_n\} \to \operatorname{Spec} \widetilde{\mathscr{H}(y)}[T_2, \ldots, T_n] \to \operatorname{Spec} \kappa(\tilde{y})[T_2, \ldots, T_n].$$

By induction, the first map is surjective. The second map is obviously surjective. It follows that the map (11.1) is also surjective.

Step 2. We handle the case where A is an integral domain. By Corollary 5.2, we can find $d \in \mathbb{N}$ and a finite injective morphism

$$k\{T_1, \ldots, T_d\} \to A.$$

Then $\operatorname{Frac} A$ is a finite extension of $\operatorname{Frac} k\{T_1, \ldots, T_d\}$. Fix an algebraic closure of $\operatorname{Frac} k\{T_1, \ldots, T_d\}$. Let K be the smallest extension of $\operatorname{Frac} k\{T_1, \ldots, T_d\}$ inside this algebraic closure which is norm over $\operatorname{Frac} k\{T_1, \ldots, T_d\}$ and which contains A. Let $G = \operatorname{Gal}(K/\operatorname{Frac} k\{T_1, \ldots, T_d\})$. Let B be the smallest k-subalgebra of K containing all $\gamma(A)$ for $\gamma \in G$. Then B is finite over $k\{T_1, \ldots, T_d\}$ and hence strictly k-affinoid by Proposition 8.1. We therefore have a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Sp} B & \longrightarrow & \operatorname{Sp} A & \longrightarrow & \operatorname{Sp} k\{T_1, \ldots, T_d\} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} \tilde{B} & \longrightarrow & \operatorname{Sp} \tilde{A} & \longrightarrow & \operatorname{Spec} k[T_1, \ldots, T_d]
\end{array}.$$

By going up theorem, all horizonal maps are surjective. So we only have to show that π_B is surjective by diagram chasing.

The group G acts on K and hence on B. For any $\gamma \in G$, we write the corresponding automorphism $B \to B$ as γ. The induced map on the reduction $\tilde{B} \to \tilde{B}$ is denoted by $\tilde{\gamma}$. In this way, we see that the G-action is compatible with the big square. All maps but the left vertical map are surjective. So it suffices to show that G acts transitively on each fiber of $\operatorname{Spec} \tilde{B} \to \operatorname{Spec} \tilde{k}[T_1, \ldots, T_d]$.

Let $\tilde{x} \in \operatorname{Spec} \tilde{k}[T_1, \ldots, T_d]$ and $\tilde{y}, \tilde{y}' \in \operatorname{Spec} \operatorname{Spec} \tilde{B}$ lying over $\tilde{x}$. If no elements in $\gamma \in G$ transforms $\tilde{y}$ to $\tilde{y}'$, we have

$$\mathfrak{p}_{\tilde{y}'} \not\subseteq \mathfrak{p}_{\tilde{\gamma}(\tilde{y})}$$

as $\tilde{B}$ is finite over $\tilde{k}[T_1, \ldots, T_d]$. Here $\mathfrak{p}_\bullet$ denotes the prime ideal corresponding to $\bullet$. By prime avoidance [Stacks, Tag 00DS], we can find $f \in \mathring{B}$ such that $\tilde{f} \in \mathfrak{p}_{\tilde{y}'}$ by $\tilde{\gamma}(\tilde{f}) \notin \mathfrak{p}_{\tilde{y}}$ for any $\gamma \in G$.

Take the minimal equation of f over $\operatorname{Frac} k\{T_1, \ldots, T_d\}$:

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0.$$

Up to sign, a_r is a power of the product of all conjugates of f. So

$$\widetilde{a_r} \in \mathfrak{p}_{\tilde{y}'} \setminus \mathfrak{p}_{\tilde{y}}.$$

By $a_r \in T_n$ as it is integral over T_n by Proposition 4.15. While $f \in \mathring{B}$ implies that $a_r \in (k\{T_1, \dots, T_d\})^\circ$ by Corollary 4.16. Thus,

$$\widetilde{a_r} \in \mathfrak{p}_{\tilde{y}'} \cap \widetilde{k\{T_1, \dots, T_d\}} = \mathfrak{p}_{\tilde{x}},$$

which contradicts the fact that $\tilde{a}_r \notin \mathfrak{p}_{\tilde{y}}$.

Step 3. We handle the general case. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes of A. The number is finite by Theorem 6.3. We then have a map

$$A \to \prod_{i=1}^r A/\mathfrak{p}_i.$$

We have a commutative diagram

$$\begin{tikzcd} \coprod_{i=1}^r \operatorname{Sp} A/\mathfrak{p}_i \arrow[r] \arrow[d] & \operatorname{Sp} A \arrow[d] \\ \coprod_{i=1}^r \operatorname{Spec} \widetilde{A/\mathfrak{p}_i} \arrow[r] & \operatorname{Spec} \tilde{A} \end{tikzcd}.$$

All maps but the right vertical one are surjective. Hence the right vertical map is surjective as well. □

Remark 11.14. Berkovich [Ber12] claimed that this follows from the proofs in [BGR84]. The author does not understand how this works. The current proof is due to Mattias Jonsson.

Theorem 11.15. Let A be a k_H-affinoid algebra. Then the reduction $\pi^H : \operatorname{Sp} A \to \widetilde{\operatorname{Sp} A}^H$ is surjective.

PROOF. **Step 1**. We reduce to the case where $\rho(A) = |k|$.

Take $n \in \mathbb{Z}_{>0}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$ such that $\rho(A \hat{\otimes}_k k_r) = |k_r|$ such that r_1 is k-free. Let $B = A \hat{\otimes}_k k_r$. Then we have a commutative diagram

$$\begin{tikzcd} \operatorname{Sp} B \arrow[r] \arrow[d, "\pi^H"] & \operatorname{Sp} A \arrow[d, "\pi^H"] \\ \widetilde{\operatorname{Sp} B}^H \arrow[r] & \widetilde{\operatorname{Sp} A}^H \end{tikzcd}.$$

It suffices to show that the left vertical map is surjective and the bottom map is surjective.

We begin with the bottom map. By Corollary 11.9, we can identify

$$\widetilde{\operatorname{Sp} B}^H \xrightarrow{\sim} \widetilde{\operatorname{Sp} A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H.$$

It suffices to show that

$$\widetilde{\operatorname{Sp} A}^H \otimes_{\tilde{k}^H} \tilde{k_r}^H \to \widetilde{\operatorname{Sp} A}^H$$

is surjective, which is trivial.

Step 2. We may assume that k is non-trivially valued, A is strictly k-affinoid and $\rho(A) = |k|$. By Lemma 2.34 in Commutative algebras, it suffices to show that the usual reduction $\pi : A \to \operatorname{Spec} \tilde{A}$ is surjective, which is exactly Lemma 11.13. □

Proposition 11.16. Let A be a k_H-affinoid algebra. Then for any generic point $\tilde{x}$ of an irreducible component of $\widetilde{\operatorname{Sp} A}^H$, $\pi^{H,-1}(\tilde{x})$ is a single point.

PROOF. We first suppose that $\operatorname{Sp}\tilde{A}^H$ is irreducible. Note that the character

$$\tilde{A}^H \to \kappa(\tilde{x})$$

corresponding to $\tilde{x}$ is injective, since $\tilde{A}^H$ does not have non-trival homogeneous nilpotents. By Theorem 11.15, we can find $x \in \operatorname{Sp} A$ whose reduction is $\tilde{x}$, we have

$$\rho_A(f) \leq |f(x)|.$$

So equality holds by Corollary 6.12 in **??** . In other words, $\pi^{H,-1}(\tilde{x}) = \{\rho_A\}$.

In general, by Lemma 3.2 in **??** , we can find $\tilde{f} \in \tilde{A}^H$ that is not contained on all generic points of irreducible components by x. Include graded version of prime avoidance somewhere. Lift $\tilde{f}$ to $f \in A$ and $r = \rho_A(f)$. Let $B = A\{r^{-1}f\}$, then

$$\pi^{H,-1}\{x\} \subseteq \operatorname{Sp} A\{r^{-1}f\} = \operatorname{Sp} B.$$

By Proposition 11.8, we have an identification

$$\tilde{B}^H = \tilde{A}^H_{\tilde{f}}.$$

It suffices to apply the special case to B. □

Proposition 11.17. Let A be a k_H-affinoid algebra. Let Z be the set of generic points of irreducible components of $\operatorname{Sp}\tilde{A}^H$. Then $\pi^{H,-1}(Z)$ is the Shilov boundary of A.

In particular, A admits a Shilov boundary.

Recall that the Shilov boundary is defined in Definition 8.7 in **??** .

PROOF. Let $f \in A$ be an element with $\rho(f) = r > 0$. Assume that $\tilde{f} \in \tilde{A}$ is not contained in some $\tilde{x} \in Z$, take the unique lift $x \in A$ of $\tilde{x}$ by Proposition 11.16. Then $|f(x)| = r$. In particular, $\pi^{H,-1}(Z)$ is a boundary.

To show that $\pi^{H,-1}(Z)$ is a minimal boundary, let $x \in \pi^{H,-1}(Z)$ and U be an open neighbourhood of x. As

$$x = \bigcup_{\tilde{f}(\tilde{x})} \pi_X^{-1}(D(\tilde{f})),$$

we can find $f \in A$ with $\tilde{f}(\tilde{x}) \neq 0$ and $\operatorname{Sp} A\{rf^{-1}\} \subseteq U$, where $r = \rho(f)$. As U is open, we can find $\epsilon > 0$ such that

$$\operatorname{Sp} A\{(r-\epsilon)f^{-1}\} \subseteq U.$$

So x belongs to any boundary of A.

□

12. Gerritzen–Grauert theorem

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 12.1. Let A be a k_H-affinoid algebra. A morphism $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ in k_H-$\mathcal{A}$ff is a *closed immersion* if the corresponding morphism $A \to B$ in k_H-$\mathcal{A}$ff$\mathcal{A}$lg is an admissible epimorphism.

Example 12.2. Let A be a k_H-affinoid algebra. Consider the diagonal morphism $\Delta : \operatorname{Sp} A \to \operatorname{Sp} A \times \operatorname{Sp} A$, defined by the codiagonal $A\hat{\otimes}_k A \to A$. We claim that Δ is a closed immersion.

We first observe that we have a factorization

$$A \otimes_k A \to A\hat{\otimes}_k A \to A$$

of the usual codiagonal, but $A \otimes_k A \to A$ is clearly surjective. Hence, so is $A\hat{\otimes}_k A \to A$.

In order to see that the codiagonal is admissible, we first observe that it is bounded by definition. Take a k-free polyray r with at least one component, then by Proposition 3.11, we may reduce to the case where k is non-trivially valued. Then it suffices to apply the open mapping theorem Theorem 7.2 in Banach rings.

Proposition 12.3. Let A, C be a k_H-affinoid algebra. Let $\operatorname{Sp} B \to \operatorname{Sp} A$ be a closed immersion. Consider the Cartesian diagram:

$$\begin{tikzcd}
\operatorname{Sp} B\hat{\otimes}_A C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Sp} B \arrow[d] \\
\operatorname{Sp} C \arrow[r] & \operatorname{Sp} A
\end{tikzcd}$$

Then $\operatorname{Sp} B\hat{\otimes}_A C \to \operatorname{Sp} C$ is also a closed immersion.

Proof. This follows from the right-exactness of completed tensor products. □

Definition 12.4. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism in k_H-$\mathcal{A}\mathrm{ff}$. We call φ a *k_H-Runge immersion* if there is a factorization in k_H-$\mathcal{A}\mathrm{ff}$ of φ:

$$\operatorname{Sp} B \to \operatorname{Sp} C \to \operatorname{Sp} A,$$

such that $\operatorname{Sp} B \to \operatorname{Sp} C$ is a closd immersion and $\operatorname{Sp} C \to \operatorname{Sp} A$ is a k_H-Weierstrass domain.

Add a prop rational domains form basis

Lemma 12.5. Let A be a k_H-affinoid algebra and V be a k_H-Laurent domain in $\operatorname{Sp} A$ represented by $A \to B = A\{r^{-1}f, sg\}$ for some $n, m \in \mathbb{N}$, $f = (f_1, \ldots, f_n) \in A^n$ and $g = (g_1, \ldots, g_m) \in A^m$, $r = (r_1, \ldots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ and $s = (s_1, \ldots, s_m) \in \sqrt{|k^\times| \cdot H}^m$. Then

(1) $\tilde{B}^H$ is finite over the subalgebra generated by $\tilde{A}^H$ and $\tilde{f}_1, \ldots, \tilde{f}_n, \tilde{g}_1^{-1}, \ldots, \tilde{g}_m^{-1}$;
(2) if V is a neighbourhood of a point $x \in \operatorname{Sp} A$, then $\tilde{\chi_x}(\tilde{B}^H)$ is finite over $\tilde{\chi_x}(\tilde{A}^H)$.

Proof. (1) Consider the admissible epimorphism

$$A\{r^{-1}T, sS\} \to B.$$

By Theorem 11.10, it induces a finite homomorphism

$$A\{\widetilde{r^{-1}T, sS}\}^H \to \tilde{B}^H.$$

The former is computed in Proposition 11.7 and our assertion follows.

(2) This is a special case of (1). □

Theorem 12.6 (Gerritzen–Grauert, Temkin)**.** Let $\varphi : \operatorname{Sp} A \to \operatorname{Sp} B$ be a monomorphism in k_H-$\mathcal{A}\mathrm{ff}$. Then there is a finite cover of X by k_H-rational domains $W_1, \dots, W_k$ such that the restrictions $\varphi_i : \varphi^{-1}(W_i) \to W_i$ are k_H-Runge immersions for $i = 1, \dots, k$.

PROOF. **Step 1**. We reduce to the following claim: for each $x \in \operatorname{Sp} A$, there is a k_H-rational domain U in $\operatorname{Sp} B$ containing $y = \varphi(x)$ such that $V = \varphi^{-1} U$ is a neighbourhood of x in $\operatorname{Sp} A$ and the induced map $V \to U$ is a closed immersion.

Assume this holds. Write $U = \operatorname{Sp} B \left\{ r \frac{f}{g} \right\}$ for some $n \in \mathbb{N}$, $f = (f_1, \dots, f_n) \in B^n$ and $g \in B$ such that $f_1, \dots, f_n, g$ generates the unit ideal and $r \in \sqrt{|k^\times| \cdot H}^n$. As g is invertible on U, we can find a small k_H-rational domain W in $\operatorname{Sp} B$ containing y such that

(1) g is invertible on W;
(2) $\varphi^{-1} W \subseteq \varphi^{-1} U$.

Then $U \cap W$ is a k_H-Weierstrass domain in W and $\varphi^{-1} W \to W$ is therefore a k_H-Runge immersion. From the compactness of $\operatorname{Sp} A$, this implies that we can find k_H-rational domains $W_1, \dots, W_m$ of $\operatorname{Sp} B$ such that $\varphi^{-1}(W_i) \to W_i$ is a k_H-Runge immersion for $i = 1, \dots, m$ and $X_1 \cup \dots \cup X_m$ contains an open neighbourhood U of $\varphi(\operatorname{Sp} A)$. As $\operatorname{Sp} B$ is compact, we can find finitely many k_H-rational domains $W_{m+1}, \dots, W_k$ which do not intersection $\varphi(\operatorname{Sp} A)$ that covers $\operatorname{Sp} B \setminus U$. Then the covering $W_1, \dots, W_k$ satisfies all the requirements.

We have reduced the problem to a local one on $\operatorname{Sp} B$.

Step 2. We show that we may assume that $\widetilde{\chi_x}(\tilde{A}^H)$ is finite over $\widetilde{\chi_y}(\tilde{B}^H)$. Here the notation χ_y is defined in Definition 6.7 in Banach rings.

By Corollary 11.11, $\widetilde{\chi_x}(\tilde{A}^H)$ is finitely generated over $\widetilde{\chi_y}(\tilde{B}^H)$. Take generators $h_1, \dots, h_l \in A$. By Proposition 3.18, $\mathscr{H}(x) \xrightarrow{\sim} \mathscr{H}(y)$, so we can find $f_1, \dots, f_l, g \in B$ with $|g(y)| = 1$ such that

$$\left| \left(\frac{f_i}{g} - h_i \right)(x) \right| < \rho(h_i)$$

for all $i = 1, \dots, l$.

In fact, we can take $g = 1$. This can be seen as follows. Let $B' = B\{a g^{-1}\}$ for some $a \in \sqrt{|k^\times| \cdot H}$ with $a < 1$. Then by Lemma 12.5, $\tilde{\chi}_y(\tilde{B'}^H)$ is finite over $\tilde{\chi}_y(\tilde{B}^H)$. So up to replacing B by the B' and $\operatorname{Sp} A$ by the inverse image of $\operatorname{Sp} B'$, we may assume that g is invertible. Replacing f_i by f_i/g, we could then assume that $g = 1$.

Up to replacing $\operatorname{Sp} B$ by $\operatorname{Sp} B\{\rho(h_1)^{-1} f_1, \dots, \rho(h_l)^{-1} f_l\}$, we can guarantee that $\tilde{f}_i = \tilde{h}_i$ for $i = 1, \dots, l$. So our assertion follows.

Step 3. We may assume that $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$ for any $x' \in \operatorname{Sp} A$ and $y' = \varphi(x')$.

Let $\pi : \operatorname{Sp} A \to \widetilde{\operatorname{Sp} A}^H$ be the reduction map. Let $\mathcal{X}$ denote the Zariski closure of $\pi(x)$. Then for any $x' \in \operatorname{Sp} A$ with $\pi(x') \in \mathcal{X}$, we have

$$\ker \widetilde{\chi_x} \subseteq \ker \widetilde{\chi_{x'}}.$$

It follows that $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$.

Since $\pi^{-1}\mathcal{X}$ is open in $\operatorname{Sp} A$ Include the proof, we can find a k_H-Laurent neighbourhood $\operatorname{Sp} B\{rf, sg^{-1}\}$ for soem suitable tuples r, f, s, g of y such that $\varphi^{-1} \operatorname{Sp} B\{rf, sg^{-1}\} \subseteq \pi^{-1}\mathcal{X}$. Observe that for each $x' \in \operatorname{Sp} A$, $\widetilde{\chi_{x'}}(\tilde{A}^H)$ is finite over $\widetilde{\chi_{y'}}(\tilde{B}^H)$. This follows simply from Lemma 12.5. So up to replacing B with $B\{rf, sg^{-1}\}$, we conclude.

Step 4. We claim that after all of these reductions, φ becomes a closed immersion. By our assumptions, for any minimal homogeneous prime ideal $\mathfrak{p}$ of $\tilde{A}^H$, there is a point $x \in \operatorname{Sp} A$ with $\ker \widetilde{\chi_y} = \mathfrak{p}$ and $\tilde{A}^H/\mathfrak{p}$ is finite over $\tilde{A}^H$.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the list of minimal homogeneous prime ideals of $\tilde{A}^H$ prove finiteness, then

$$\tilde{A}^H \to \bigoplus_{i=1}^{k} \tilde{A}^H/\mathfrak{p}_i$$

is injective. Since $\tilde{B}^H$ is graded noetherian Introduce this notion, we find that $\tilde{A}^H$ is finite over $\tilde{B}^H$. So $B \to A$ is finite by Theorem 11.10. It follows that the natural map $A \otimes_B A \to A \hat{\otimes}_B A$ is an isomorphism by Proposition 9.4. As φ is a monomorphism, from general abstract nonsense, the codiagonal $A \hat{\otimes}_B A \xrightarrow{\sim} A$ is an isomorphism. In particular, the codiagonal $A \otimes_B A \to A$ is an isomorphism. This implies that $A \to B$ is surjective. $\square$

Lemma 12.7. Let A be a k_H-affinoid domain and V be a k_H-affinoid domain in A represented by $A \to A_V$. Assume that $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a closed immersion, then V is a k_H-Weierstrass domain.

Proof. As $\operatorname{Sp} A_V \to \operatorname{Sp} A$ is a closed immersion, we can find an ideal $I \subseteq A$ and assume that $A_V = A/I$. Consider the morphism of k_H-affinoid spectra $\psi : \operatorname{Sp} A/I^2 \to \operatorname{Sp} A$ induced by the natural map A/I^2. By the universal property of V, we have a commutative diagram:

$$\begin{tikzcd} \operatorname{Sp} A/I^2 \arrow[d] \arrow[dr] & \\ \operatorname{Sp} A/I \arrow[r] & \operatorname{Sp} A \end{tikzcd}$$

On the other hand, the natural map $A/I^2 \to A/I$ induces a morphism of k_H-affinoid spectra $\varphi : \operatorname{Sp} A/I \to \operatorname{Sp} A/I^2$. From the universal property again, the composition $\psi \circ \varphi$ is the identity. In particular, $A/I^2 \to A/I$ is injective and hence $I = I^2$. It follows that I is the principal ideal generated by an idempotent element e. We may assume that $e \neq 0$, $e \neq 1$. Take $c \in \sqrt{|k^\times| \cdot H}$ such that $0 < c < 1$, then $V = (\operatorname{Sp} A)\{c^{-1}e\}$. $\square$

Corollary 12.8. Let A be a k_H-affinoid algebra and V be a k_H-affinoid domain in $\operatorname{Sp} A$. Then there are finitely many k_H-affinoid domains $W_1, \dots, W_n$ in $\operatorname{Sp} A$ such that

$$V = \bigcup_{i=1}^{n} W_i.$$

Proof. By Theorem 12.6, we can find finitely many k_H-rational domains $U_1, \dots, U_m$ in $\operatorname{Sp} A$ such that $V \cap U_i \to U_i$ is a k_H-Runge immersion for each $i = 1, \dots, m$. By Proposition 10.10, it suffices to prove that $V \cap U_i$ is a k_H-rational domain in U_i. Observe that $V \cap U_i$ is a k_H-affinoid domain in U_i by Lemma 10.11. So we are reduced to the case where $V \to \operatorname{Sp} A$ is also a Runge immersion.

By Lemma 10.11 and Proposition 10.10 again, we may assume that $V \to \operatorname{Sp} A$ is a Runge immersion.

In this case, the result follows from Lemma 12.7. □

13. Tate acyclicity theorem

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field and H be a subgroup of $\mathbb{R}_{>0}$ such that $|k^\times| \cdot H \neq \{1\}$.

Definition 13.1. Let A be a k_H-affinoid algebra. Let $\mathcal{V} = \{V_i\}_{i \in 1,\dots,n}$ be a finite covering of $\mathrm{Sp}\, A$ by k_H-affinoid domains. Let M be an A-module. We define the *augmented Čech complex* $\check{C}(\mathcal{V}, M)$ as the following cochain complex with M placed at the place 0:

$$\check{C}(\mathcal{V}, M) = 0 \to M \to \prod_{i=1}^{n} M \otimes_A A_{V_i} \to \prod_{1 \leq i < j \leq n} M \otimes_A A_{V_i} \hat{\otimes}_A A_{V_j} \to \cdots .$$

Definition 13.2. Let A be a k_H-affinoid algebra. A *finite k_H-affinoid covering* of $\mathrm{Sp}\, A$ is a finite covering of A by k_H-affinoid domains.

A finite k_H-affinoid covering $\mathcal{U}$ is a

(1) *k_H-Laurent covering* if there are $n \in \mathbb{N}$, $f_1, \dots, f_n \in A$ and $r_1, \dots, r_n \in \sqrt{|k^\times| \cdot H}$ such that $\mathcal{U}$ consists of
$$\mathrm{Sp}\, A \left\{ r_1^{-\epsilon_1} f_1^{\epsilon_1}, \dots, r_1^{-\epsilon_n} f_1^{\epsilon_n} \right\}$$
for all $\epsilon_i = \pm 1$, $i = 1, \dots, n$. In this case, we say that $\mathcal{U}$ is the k_H-Laurent covering *generated by* $r_1^{-1} f_1, \dots, r_n^{-1} f_n$.

(2) *k_H-rational covering* if there are $n \in \mathbb{N}$, $f_1, \dots, f_n \in A$ generating the unit ideal, $r = (r_1, \dots, r_n) \in \sqrt{|k^\times| \cdot H}^n$ such that $\mathcal{U}$ consists of
$$\mathrm{Sp}\, A \left\{ (r/r_j)^{-1} \frac{f}{f_j} \right\}$$
for $j = 1, \dots, n$. In this case, we say that $\mathcal{U}$ is the k_H-rational covering *generated by* $r_1^{-1} f_1, \dots, r_n^{-1} f_n$.

In both cases, if $f_1, \dots, f_n$ are all units in A, we say the covering is *generated by units in A*.

Lemma 13.3. Let A be a k_H-affinoid algebra and $\mathcal{V} = \{V_i\}_{i \in 1,\dots,m}$ be a finite k_H-affinoid covering of $\mathrm{Sp}\, A$. Then there is a k_H-rational covering refining $\mathcal{V}$.

Proof. By Corollary 12.8, we may assume that all V_i's are k_H-rational domains in $\mathrm{Sp}\, A$. Take $n_i \in \mathbb{N}$, $g_1^{(i)}, \dots, g_{n_i}^{(i)} \in A$ generating the unit ideal, $r^{(i)} = (r_1^{(i)}, \dots, r_{n_i-1}^{(i)}, r_{n_i}^{(i)}) \in \sqrt{|k^\times| \cdot H}^{n_i}$ for each $i = 1, \dots, m$ such that if we write $g^{(i)} = (g_1^{(i)}, \dots, g_{n_i}^{(i)})$, then

$$V_i = \mathrm{Sp}\, A \left\{ \left(r^{(i)} / r_{n_i}^{(i)} \right)^{-1} \frac{g^{(i)}}{g_{n_i}^{(i)}} \right\}$$

for $i = 1, \dots, m$. Let $\mathcal{B}^i$ be the k_H-rational covering generated by

$$(r^{(i)})^{-1} f_1^{(i)}, \dots, (r^{(i)})^{-1} f_{n_i}^{(i)}$$

for $i = 1, \dots, m$. We denote the elements in $\mathcal{B}^i$ by V_j^i, $j = 1, \dots, n_i$:

$$V_j^i := \mathrm{Sp}\, A \left\{ \left(r^{(i)} / r_j^{(i)} \right)^{-1} \frac{g^{(i)}}{g_j^{(i)}} \right\}.$$

Let

$$I := \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m : 1 \leq \alpha_i \leq n_i \text{ for } i = 1, \dots, m\}$$

and

$$I' := \{\alpha = (\alpha_1, \dots, \alpha_m) \in I : \alpha_i = n_i \text{ for some } i = 1, \dots, n\}.$$

Next for $\beta = (\beta_1, \dots, \beta_m) \in I$, we let

$$g_\beta = g^{(1)}_{\beta_1} \cdots g^{(m)}_{\beta_m}, \quad r_\beta = r^{(1)}_{\beta_1} \cdots r^{(m)}_{\beta_m}$$

and we have

$$V_\beta := V^1_{\beta_1} \cap \dots \cap V^m_{\beta_m} = \operatorname{Sp} A \left\{ ((r_\alpha)_{\alpha \in I}/r_\beta)^{-1} \frac{(g_\alpha)_{\alpha \in I}}{g_\beta} \right\}$$

as in the proof of Proposition 10.7.

When $\beta \in I'$, we claim that

$$V_\beta = \operatorname{Sp} A \left\{ ((r_\alpha)_{\alpha \in I'}/r_\beta)^{-1} \frac{(g_\alpha)_{\alpha \in I'}}{g_\beta} \right\}.$$

It is clear that the left-hand side is contained in the right-hand side. Conversely, x in the right-hand side. By rearranging $U_1, \dots, U_m$, we may assume that $x \in U_1$. Let $\gamma = (\gamma_1, \dots, \gamma_m) \in I \setminus I'$. Then

$$r_\gamma^{-1} |g_\gamma(x)| \leq (r^{(1)}_{n_1})^{-1} (r^{(2)}_{\gamma_2})^{-1} \cdots (r^{(m)}_{\gamma_m})^{-1} \left| g^{(1)}_{n_1} g^{(2)}_{\gamma_2} \cdots g^{(m)}_{\gamma_m} \right| \leq r_\beta^{-1} |g_\beta(x)|.$$

The claim follows. Now $\{V_\beta\}_{\beta \in I'}$ is the k_H-rational covering generated by $r_\beta^{-1} g_\beta$ for $\beta \in I'$. It is clear that this covering refines $\mathcal{V}$. □

Lemma 13.4. Let A be a k_H-affinoid algebra and $\mathcal{U}$ be a k_H-rational covering of $\operatorname{Sp} A$. Then there is a k_H-Laurent covering $\mathcal{V}$ of $\operatorname{Sp} A$ such that for each $\operatorname{Sp} C \in \mathcal{V}$, the restriction $\mathcal{U}|_{\operatorname{Sp} C}$ is a k_H-rational covering of $\operatorname{Sp} C$ generated by units in C.

Proof. We take $n \in \mathbb{N}$, $f_1, \dots, f_n \in A$ generating the unit ideal and $r_1, \dots, r_n \in \sqrt{|k^\times| \cdot H}$ such that $\mathcal{U}$ is generated by $r_1^{-1} f_1, \dots, r_n^{-1} f_n$. Choose $c \in \sqrt{|k^\times| \cdot H}$ such that

$$c < \inf_{x \in \operatorname{Sp} A} \max_{i=1,\dots,n} r_i^{-1} |f_i(x)|.$$

Let $\mathcal{V}$ be the k_H-Laurent covering of $\operatorname{Sp} A$ generated by $(cr_1)^{-1} f_1, \dots, (cr_n)^{-1} f_n$. We claim that $\mathcal{V}$ satisfies our requirements.

Take

$$V = \operatorname{Sp} A \left\{ (cr_1)^{-\epsilon_1} f_1^{\epsilon_1}, \dots, (cr_n)^{-\epsilon_n} f_n^{\epsilon_n} \right\}$$

be an element in $\mathcal{V}$, $\epsilon_i = \pm 1$ for $i = 1, \dots, n$. We may assume that there is $s \in [0, n]$ such that $\epsilon_1 = \dots = \epsilon_s = 1$ and $\epsilon_{s+1} = \dots = \epsilon_n = -1$. We claim that $\mathcal{U}|_V$ is teh k_H-rational covering generated by the images of $r_{s+1}^{-1} f_{s+1}, \dots, r_n^{-1} f_n$ in

$$A \left\{ (cr_1)^{-1} f_1, \dots, (cr_s)^{-1} f_s, (cr_{s+1}) f_{s+1}^{-1}, \dots, (cr_n) f_n^{-1} \right\}$$

and these elements are units.

In fact, by our assumption, for $x \in V$,

$$\begin{aligned} |f_i(x)| &\leq cr_i, \quad \text{for } i = 1, \dots, s; \\ |f_i(x)| &\geq cr_i, \quad \text{for } i = s+1, \dots, n. \end{aligned}$$

In particular,

$$\max_{i=1,\dots,s} r_i^{-1} |f_i(x)| \leq c < \max_{i=1,\dots,n} r_i^{-1} |f_i(x)|.$$

Hence,
$$\max_{i=1,\ldots,s} r_i^{-1}|f_i(x)| = \max_{i=s+1,\ldots,n} r_i^{-1}|f_i(x)|.$$
Our claim follows. □

Lemma 13.5. Let A be a k_H-affinoid algebra and $\mathcal{U}$ be a k_H-rational covering of $\operatorname{Sp} A$ generated by units in A. Then there is a k_H-Laurent convering $\mathcal{V}$ of $\operatorname{Sp} A$ refining $\mathcal{U}$.

Proof. We take $n \in \mathbb{N}$, units $f_1, \ldots, f_n \in A$ and $r_1, \ldots, r_n \in \sqrt{|k^\times| \cdot H}$ such that $\mathcal{U}$ is generated by $r_1^{-1}f_1, \ldots, r_n^{-1}f_n$.

We take $\mathcal{V}$ as the Laurent covering generated by $(r_ir_j^{-1})^{-1}f_if_j^{-1}$ for $1 \leq i < j \leq n$. We claim that $\mathcal{V}$ refines $\mathcal{U}$. Write $I = \{(i,j) \in \mathbb{N}^2 : 1 \leq i < j \leq n\}$. To see this, consider $V \in \mathcal{V}$, say
$$V = \bigcap_{(i,j)\in I_1} \operatorname{Sp} A\{(r_ir_j^{-1})^{-1}f_if_j^{-1}\} \cap \bigcap_{(i,j)\in I_2} \operatorname{Sp} A\{(r_ir_j^{-1})^{+1}f_i^{-1}f_j\},$$
where I_1, I_2 is a partition of I. For $i, j \in \{1, \ldots, n\}$, we write $i \preceq j$ if $(i,j) \in I_1$ and $j \preceq i$ if $(i,j) \in I_2$. Consider a maximal chain
$$i_1 \preceq i_2 \preceq \cdots \preceq i_s$$
on the set $\{1, \ldots, n\}$. Then $i \preceq i_s$ for each $i = 1, \ldots, n$. In other words, for $x \in X$, we have
$$|f_if_{i_s}^{-1}(x)| \leq r_ir_{i_s}^{-1}.$$
The right-hand side defines an element in $\mathcal{U}$. □

We first prove Tate acyclicity theorem in a special case.

Lemma 13.6. Let A be a k_H-affinoid algebra. Let $\mathcal{V} = \{V_i\}_{i\in 1,\ldots,n}$ be a finite k_H-affinoid covering of $\operatorname{Sp} A$. Assume that each V_i is a k_H-rational domain. Then $\check{C}(\mathcal{V}, A)$ is exact and admissible.

Proof. **Step 1**. We reduce to the case where
$$\mathcal{V} = \big\{\{\operatorname{Sp} A\{r^{-1}f\}\}, \{\operatorname{Sp} A\{rf^{-1}\}\}\big\}$$
for some $r \in \sqrt{|k^\times| \cdot H}$ and $f \in A$.

Take a k-free polyray s with at least one component. By Proposition 3.11, we can make the base change to k_s and assume that k is non-trivially valued. In this case, by open mapping theorem Theorem 7.2 in Banach rings., the admissibility is automatic. It suffices to prove the exactness.

In this case, we can define a presheaf $\mathcal{O}_X$ on X on the family of k_H-rational domains in $\operatorname{Sp} A$: $\mathcal{O}_X(\operatorname{Sp} C) = C$. From the general comparison theorem of Čech cohomology BGR P327 reproduce in the topology part and Lemma 13.3, we may assume that the covering $\mathcal{V}$ is k_H-rational covering. But then we need to show that for each k_H-rational domain W in $\operatorname{Sp} A$, $\check{C}(\mathcal{V}|_W, A)$ is exact. Similarly, by Lemma 13.4, we may assume that the k_H-rational covering is generated by units. Again, by Lemma 13.5, we can reduce to the case where $\mathcal{V}$ is a k_H-Laurent covering.

We need to show that for each k_H-affinoid domain $\operatorname{Sp} C$ in $\operatorname{Sp} A$, $\check{C}(\mathcal{V}|_W, A)$ is exact. But $\mathcal{V}|_W$ is also a k_H-Laurent covering. In particular, it suffices to show that $\check{C}(\mathcal{V}, A)$ is exact. By induction on the number of generators of $\mathcal{V}$, we can reduce the case stated in the beginning.

Step 2. After the reduction, we need to show that the following sequence is exact:

$$0 \to A \xrightarrow{i} A\{r^{-1}f\} \times A\{rf^{-1}\} \xrightarrow{d^0} A\{r^{-1}f, rf^{-1}\} \to 0,$$

where $i(a) = (a, a)$ and $d^0(f, g) = f - g$. We extend the sequence to the following commutative diagram in k_H-$\mathcal{A}$ff$\mathcal{A}$lg:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & (\zeta - f)A\{r^{-1}\zeta\} \times (1 - f\eta)A\{r\eta\} \arrow[r, "\lambda'"] \arrow[d] & (\zeta - f)A\{r^{-1}\zeta, r\zeta^{-1}\} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, Rightarrow] & A\{r^{-1}\zeta\} \times A\{r\eta\} \arrow[r, "\lambda"] \arrow[d] & A\{r^{-1}\zeta, r\eta\}/(\zeta\eta - 1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\epsilon"] & A\{r^{-1}f\} \times A\{rf^{-1}\} \arrow[r, "d^0"] \arrow[d] & A\{r^{-1}f, rf^{-1}\} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd},$$

where $\iota(a) = (a, a)$ and λ sends ζ to ζ and η to η. The two colomns are clearly exact. It is straightforward to see that everywhere the first non-zero row is exact. The second non-zero row is also exact. The non-trivial part is to show that if $\sum_{i=0}^{\infty} a_i \zeta^i \in A\{r^{-1}\zeta\} \in A\{r^{-1}\zeta\}$ and $\sum_{i=0}^{\infty} b_i \zeta^i \in A\{r^{-1}\eta\} \in A\{r\eta\}$ are such that their pair lies in the kernel of λ, then

$$0 = \sum_{i=0}^{\infty} a_i \zeta^i - \sum_{i=0}^{\infty} b_i \zeta^{-i}.$$

It follows that $a_i = 0 = b_i$ for $i > 0$ and $a_i = b_i$. So we find that the second row is also exact. By diagram chasing, the third row is also exact. $\square$

Corollary 13.7. Let A be a k_H-affinoid algebra and $\operatorname{Sp} B$ be a k-affinoid domain in $\operatorname{Sp} A$. Then for any complete non-Archimedean field extension K/k, any K-affinoid algebra C and any bounded ring homomorphism $A \to C$ such that $\operatorname{Sp} C \to \operatorname{Sp} A$ factorizes through $\operatorname{Sp} B$, there is a unique bounded ring homomorphism $B \to C$ making the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Sp} C \arrow[d, dotted] \arrow[dr] & \\
\operatorname{Sp} B \arrow[r] & \operatorname{Sp} A
\end{tikzcd}$$

Proof. The proof is the same as in Example 10.4 when $\operatorname{Sp} B$ is an affinoid domain in $\operatorname{Sp} A$.

In general, by Corollary 12.8, we can cover $\operatorname{Sp} B$ by finitely many affinoid domains $\operatorname{Sp} B_1, \ldots, \operatorname{Sp} B_n$ in $\operatorname{Sp} A$. Let $\operatorname{Sp} C_i$ be the rational domain in $\operatorname{Sp} C$ defined by the preimage of $\operatorname{Sp} B_i$ for $i = 1, \ldots, n$. In other words, we have Cartesian diagrams for $i = 1, \dots, n$:

$$\begin{tikzcd} \operatorname{Sp} C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Sp} C \arrow[d] \\ \operatorname{Sp} B_i \arrow[r] & \operatorname{Sp} A \end{tikzcd}$$

It follows from Lemma 13.6 that we have an admissible exact sequence

$$0 \to C \to \prod_{i=1}^{n} C_i \to \prod_{1 \le i < j \le n}^{n} C_i \hat{\otimes}_C C_j.$$

From general abstract nonsense, to construct bounded A-homomorphisms $\varphi : B \to C$ is the same as to construct bounded homomorphisms $\varphi_i : B \to C_i$ over A such that the induced maps $B \to C_i \hat{\otimes}_C C_j$ are compatible. On the other hand, by our definition of B_i, in order to construct the morphisms φ_i, it suffices to construct $\psi_i : B_i \to C_i$ over A. This reduces to the known case. $\square$

Corollary 13.8. Let A be a k_H-affinoid algebra and $H' \supseteq H$ is a subgroup of $\mathbb{R}_{>0}$. Let $V = \operatorname{Sp} B$ be a k_H-affinoid domain in $\operatorname{Sp} A$, then $\operatorname{Sp} B$ is a $k_{H'}$-affinoid domain in $\operatorname{Sp} A$.

Proof. This follows immediately from Corollary 13.7. $\square$

Introduce the Shilov point

Proposition 13.9. Let A be a k-affinoid algebra and $V \subseteq X$ is a closed subset. Let $f : A \to B$ be a morphism of k-affinoid algebras. Assume that for any complete non-Archimedean field extension K/k, any K-affinoid algebra C and any bounded ring homomorphism $A \to C$ such that $\operatorname{Sp} C \to \operatorname{Sp} A$ factorizes through V, there is a unique bounded ring homomorphism $B \to C$ making the following diagram commutes:

$$\begin{tikzcd} \operatorname{Sp} C \arrow[d, dotted] \arrow[dr] & \\ \operatorname{Sp} B \arrow[r] & \operatorname{Sp} A \end{tikzcd}.$$

Then V is an affinoid domain represented by the given $A \to B$.

Proof. The only non-trival thing is to show that the image of $\operatorname{Sp} B \to \operatorname{Sp} A$ is V.

Step 1. We reduce to the case where k is non-trivially valued and A, B are both strictly k-affinoid.

Let r be a k-free polyray with at least one component such that $A \hat{\otimes}_k k_r$ and $B \hat{\otimes}_k k_r$ are both strictly k_r-affinoid. Let V' be the inverse image of V in $\operatorname{Sp} A \hat{\otimes}_k k_r$. Then clearly, V' has the same universal property. Assume that we have already shown that the image of

$$\operatorname{Sp} B \hat{\otimes}_k k_r \to A \hat{\otimes}_k k_r$$

is exactly V'. We have a commutative diagram:

$$\begin{tikzcd} \operatorname{Sp} B \hat{\otimes}_k k_r \arrow[r] \arrow[d] & \operatorname{Sp} A \hat{\otimes}_k k_r \arrow[d] \\ \operatorname{Sp} B \arrow[r] & \operatorname{Sp} A \end{tikzcd}$$

From the existence of the Shilov points, both vertical sections are surjective. Hence, the image of $\operatorname{Sp} B$ in $\operatorname{Sp} A$ is exactly V.

Step 2. After the reduction, it suffices to argue that each point in $V \cap \operatorname{Spm} A$ lies in the image. Let y be such a point corresponding to a maximal ideal $\mathfrak{m}_y$ of A. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi"] & B \arrow[d, "\pi'"] \arrow[dl, dotted, "\alpha"'] \\ A/\mathfrak{m}_y \arrow[r, "\sigma"] & B/\mathfrak{m}_y B \end{tikzcd} .$$

The two vertical maps are the natural projections and σ is the map induced by f. The existence of α and the commutativity of the diagram follow from the universal property. Observe that σ is surjective as π' is. Similarly, α is surjective as π is. Moreover, $\mathfrak{m}_y B = \ker \pi' \subseteq \ker \alpha$. In particular, σ is bijection. So $\mathfrak{m}_y B$ is a maximal ideal in B and the corresponding point $x \in \operatorname{Spm} B$ sends x to y. $\square$

Remark 13.10. In fact, the proof proves the following result: assume that the valuation on k is non-trivial and A is a strictly k-affinoid algebra. Let $\operatorname{Sp} B$ be a strictly k-affinoid domain. Then for each $x \in \operatorname{Spm} B$ corresponding to a maximal ideal $\mathfrak{m}_x$ in B and any $n \in \mathbb{Z}_{>0}$, we have a natural isomorphism

$$A/\mathfrak{m}_y^n \xrightarrow{\sim} B/\mathfrak{m}_x^n,$$

where y is the image of x in $\operatorname{Sp} A$ and $\mathfrak{m}_y$ is the corresponding maximal ideal in A. Moreover, $\mathfrak{m}_x = \mathfrak{m}_y B$.

In particular, the natural map $\hat{A}_{\mathfrak{m}_y} \to \hat{B}_{\mathfrak{m}_x}$ is an isomorphism.

Corollary 13.11. Let A be a k-affinoid algebra and $\operatorname{Sp} B$ be a k-affinoid domain in $\operatorname{Sp} A$. Assume that K/k is an extension of complete valued field. Then $\operatorname{Sp} B\hat{\otimes}_k K$ is a K-affinoid domain in $\operatorname{Sp} A\hat{\otimes}_k K$. Moreover, the image of $\operatorname{Sp} B\hat{\otimes}_k K$ in $\operatorname{Sp} A\hat{\otimes}_k K$ is the inverse image of the image of $\operatorname{Sp} B$ in $\operatorname{Sp} A$.

Proof. This is an immediate consequence of Proposition 13.9 and Corollary 13.7. $\square$

Corollary 13.12. Let $\varphi : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism of k_H-affinoid spectra. Let $V \subseteq \operatorname{Sp} A$ be a k_H-affinoid domain in $\operatorname{Sp} A$, then $\varphi^{-1}(V)$ is a k_H-affinoid domain in $\operatorname{Sp} B$.

In fact, suppose that V is represented by $A \to A_V$, then $B \to B\hat{\otimes}_A A_V$ represents $\varphi^{-1}V$.

Proof. It is an immediate consequence of Proposition 13.9 and Corollary 13.7 that $\varphi^{-1}(V)$ is a k-affinoid domain. As $B\hat{\otimes}_A A_V$ is k_H-affioid, we find that it is also a k_H-affinoid domain. $\square$

Corollary 13.13. Let A be a k_H-affinoid algbera and $\operatorname{Sp} B$, $\operatorname{Sp} C$ be k_H-affinoid domains in $\operatorname{Sp} A$. Then $\operatorname{Sp} B \cap \operatorname{Sp} C$ is a k_H-affinoid domain represented by the natural morphism $A \to B\hat{\otimes}_A C$.

Proof. This is an immediate consequence of Corollary 13.12. $\square$

Corollary 13.14. Let A be a k_H-affinoid algbera and $\operatorname{Sp} B$, $\operatorname{Sp} C$ be k_H-affinoid domains in $\operatorname{Sp} A$. Then the natural morphism

$$\operatorname{Sp} B \cap \operatorname{Sp} C \to \operatorname{Sp} B \times \operatorname{Sp} C$$

is a closed immersion.

Proof. By Corollary 13.13, we need to show that the natural map

$$B\hat{\otimes}_k C \to B\hat{\otimes}_A C$$

is an admissible epimorphism. From general abstract nonsense and Proposition 12.3, it suffices to show that the codiagonal

$$A\hat{\otimes}_k A \to A$$

is an admissible epimorphism. This follows from Example 12.2. □

Corollary 13.15. Let A be a k_H-affinoid algebra. Let V, W be k_H-affinoid domains in $\mathrm{Sp}\, A$ represented by $A \to A_V$ and $A \to A_W$ respectively. Then $V \cap W$ is a k_H-affinoid domain represented by $A \to A_V\hat{\otimes}_A A_W$.

Proof. This is an immediate consequence of Corollary 13.12. □

Definition 13.16. Let $X = \mathrm{Sp}\, A$ be a k-affinoid spectra, we define a presheaf $\mathcal{O}_X$ of Banach rings on the family of k-affinoid domains in X as follows: for any k-affinoid domain $\mathrm{Sp}\, B$, we set

$$\mathcal{O}_X(\mathrm{Sp}\, B) = B.$$

Given an inclusion of affinoid domains, $\mathrm{Sp}\, C \to \mathrm{Sp}\, B$, we define the corresponding restriction map as the given morphism $B \to C$.

Theorem 13.17. Let A be a k-affinoid algebra and $V' = \mathrm{Sp}\, B$ be a k-affinoid domain in $\mathrm{Sp}\, A$. Then B is a flat A-algebra.

Proof. **Step 1**. We reduce to the case where k is non-trivially valued and A is strictly k-affinoid.

Let r be a k-free polyray with at least one component. Let $\varphi : M \to N$ be an injective A-module homomorphism. We endow M and N with the structures of finite Banach A-modules by Proposition 9.2 and then φ is admissible by Proposition 9.6. By Proposition 3.11, the induced homomorphism

$$M\hat{\otimes}_k k_r \to N\hat{\otimes}_k k_r$$

is injective and admissible. Let V' be the inverse image of V in $\mathrm{Sp}\, A\hat{\otimes}_k k_r$. By Corollary 13.11, V' is a k_r-affinoid domain represented by $A\hat{\otimes}_k k_r \to B\hat{\otimes}_k k_r$.

If we have shown the result in the special case, we know that

$$(M\hat{\otimes}_k k_r) \otimes_{A\hat{\otimes}_k k_r} (B\hat{\otimes}_k k_r) \to (N\hat{\otimes}_k k_r) \otimes_{A\hat{\otimes}_k k_r} (B\hat{\otimes}_k k_r)$$

is injective. By Proposition 9.5, this map can be identified with

$$(M\hat{\otimes}_k k_r)\hat{\otimes}_{A\hat{\otimes}_k k_r}(B\hat{\otimes}_k k_r) \to (N\hat{\otimes}_k k_r)\hat{\otimes}_{A\hat{\otimes}_k k_r}(B\hat{\otimes}_k k_r).$$

The latter map is easily identified with

$$M\hat{\otimes}_A B \to N\hat{\otimes}_A B.$$

By Proposition 9.5 again, the latter map is identified with

$$M \otimes_A B \to N \otimes_A B.$$

We conclude that $A \to B$ is flat.

Step 2. After the reduction, we take a maximal ideal $\mathfrak{m}_x$ in B corresponding to a point $x \in \mathrm{Sp}\, B$. Let y be the image of y in $\mathrm{Sp}\, A$ and $\mathfrak{m}_y$ denotes the corresponding maximal ideal. Then by Remark 13.10, $\hat{A}_{\mathfrak{m}_y} \to \hat{B}_{\mathfrak{m}_y}$ is an isomorphism. By [Stacks, Tag 0C4G] and [Stacks, Tag 0399], we conclude that $A \to B$ is flat. □

Theorem 13.18 (Tate acyclicity theorem)**.** Let A be a k-affinoid algebra. Let $\mathcal{V} = \{V_i\}_{i \in 1,\ldots,n}$ be a finite k-affinoid covering of $\operatorname{Sp} A$. Let M be an A-module. Then the complex $\check{C}(\mathcal{V}, A)$ is exact. It is exact and admissible if M is finite as A-module.

Proof. We first observe that teh admissibility follows from the same argument as in Lemma 13.6. We will only concentrate on the exactness.

Step 1. We first reduce to the case $M = A$.

As the covering $\mathcal{V}$ is finite, we can find $N \in \mathbb{N}$ such that $\check{H}^j(\mathcal{V}, M'') = 0$ for all $j \geq N$ and all A-module M''. We take the minimum of such N. Assume that $N > 0$.

Assume we have proved the theorem in this case, then the case where M is free is immediate. In general, choose an exact sequence of A-modules:

$$0 \to M' \to F \to M \to 0$$

with F free. In this case, we have a short exact sequence

$$0 \to \check{C}(\mathcal{V}, M') \to \check{C}(\mathcal{V}, F) \to \check{C}(\mathcal{V}, M) \to 0.$$

The exactness follows from Theorem 13.17.

From the long exact sequence, we find that

$$H^{q-1}(\mathcal{V}, M) \cong H^q(\mathcal{V}, M').$$

for all $q \in \mathbb{Z}$. It follows that $H^q(\mathcal{V}, M) = 0$ for all $q \geq N - 1$. This argument works for any A-module M and we get a contradiction with our choice of N.

Step 2. After the reduction in Step 1 and the successful defition of $\mathcal{O}_X$ in Definition 13.16, the remaining of the argument is exactly the same as Lemma 13.6. □

Corollary 13.19. Let A be a k-affinoid algebra and $\{\operatorname{Sp} B_i\}$ be a finite k_H-affinoid covering of $\operatorname{Sp} A$. Then A is k_H-affinoid.

Proof. By Theorem 13.18, we have an admissible injective morphism

$$A \to \prod_{i \in I} B_i$$

of Banach k-algebras. Then for any $a \in A$,

$$\rho_A(a) = \max_{i \in I} \rho_{B_i}(a).$$

We conclude using Theorem 8.4. □

14. Kiehl's theorem

Let $(k, |\bullet|)$ be a complete non-Archimedean valued field.

Theorem 14.1. Let A be a k-affinoid algebra and $\mathcal{U} = \{\operatorname{Sp} B_i\}_{i \in I}$ a finite k-affinoid covering of $\operatorname{Sp} A$. Suppose that we are given

(1) for each $i \in I$ a finite B_i-module M_i;

(2) for each $i, j \in I$, an isomorphism

$$\alpha_{ij} : M_i \otimes_{B_i} B_{ij} \to M_j \otimes_{B_j} B_{ji}$$

of B_{ij}-modules, where $B_{ij} = B_i \hat{\otimes}_A B_j$ such that
(a) α_{ii} is identity for all $i \in I$;
(b) $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ on $\operatorname{Sp} B_i \cap \operatorname{Sp} B_j \cap \operatorname{Sp} B_k$ for $i, j, k \in I$.
Then there is a finite A-module M and isomorphisms

$$\beta_i : M \otimes_A B_i \to M_i$$

of B_i-modules for each $i \in I$ and such that the following diagram is commutative:

$$\begin{tikzcd}
M \otimes_A B_i \otimes_{B_i} B_{ij} \arrow[r, "\beta_i \otimes_{B_i} B_{ij}"] \arrow[d, Rightarrow, no head] & M_i \otimes_{B_i} B_{ij} \arrow[d, "\alpha_{ij}"] \\
M \otimes_A B_j \otimes_{B_j} B_{ji} \arrow[r, "\beta_i \otimes_{B_j} B_{ji}"] & M_i \otimes_{B_j} B_{ji}
\end{tikzcd}.$$

If moreover each M_i is an A_i-algebra for $i \in I$ and the maps α_{ij} are B_{ij}-algebra homomorphisms for $i, j \in I$, then we can endow M with the structure of an A-algebra and β_i is a B_i-algebra homomorphism for $i \in I$.

Proof. By the same reduction as in our proof of Lemma 13.6, it suffices to handle the case where $\mathcal{U}$ is a Laurent covering generated by a single element:

$$\mathcal{U} = \left\{ \operatorname{Sp} A\{r^{-1}f\}, \operatorname{Sp} A\{rf^{-1}\} \right\}$$

for some $r > 0$ and $f \in A$. We write $B_1 = A\{r^{-1}f\}$ and $B_2 = A\{rf^{-1}\}$. Then $B_{12} = A\{r^{-1}f, rf^{-1}\}$. Let $M_{12} = M_1 \otimes_{B_1} B_{12}$. We endow M_1 (resp. M_2, resp. M_{12}) with the structure of finite Banach B_1-(resp. B_2-, resp. B_{12}-)module by Proposition 9.2. We will denote the Banach norms on these modules by $\|\bullet\|$ without specifying the index. Let $\|\bullet\|_A$, $\|\bullet\|_1$, $\|\bullet\|_2$, $\|\bullet\|_{12}$ denote the norms on A, B_1, B_2, B_{12} respectively.

Step 1. We show that

$$d^0 : M_1 \times M_2 \to M_{12}$$

is surjective, where $d^0(m_1, m_2) = m_1 - m_2$. We have omitted the obvious map $M_1 \to M_{12}$ and $M_2 \to M_{12}$.

We will prove the following claim: let $\epsilon > 0$ be a constant. Then there is a constant $\alpha > 0$ such that for each $u \in M_{12}$, there exist $u^+ \in M_1$ and $u^- \in M_2$ with

$$\|u^\pm\| \leq \alpha \|u\|, \quad \|u - u^+ - u^-\| \leq \epsilon \|u\|.$$

This implies that d^0 is surjective.

Let $v_1, \ldots, v_n$ be generators of the B_1-module M_1 and $w_1, \ldots, w_m$ be generators of the B_2-module M_2. We write the images of $v_1, \ldots, v_n$ in M_{12} as $v'_1, \ldots, v'_n$ and the images of $w_1, \ldots, w_m$ in M_{12} as $w'_1, \ldots, w'_m$. We could assume that the norms $\|\bullet\|$ on M_1, M_2, M_{12} are the residue norms induced from B_1^n, B_2^m, B_{12}^n by the basis $\{v_i\}$, $\{w_j\}$, $\{v'_i\}$ respectively. Then we can find an $n \times m$-matrix $C = (c_{ij})$ with value in B_{12} and an $m \times n$-matrix $D = (D_{ji})$ with value in B_{12} such that

$$v'_i = \sum_{j=1}^{m} c_{ij} w'_j, \quad i = 1, \dots, n;$$

$$w'_j = \sum_{i=1}^{n} d_{ji} v'_i, \quad i = 1, \dots, n.$$

Fix $\beta > 1$. As B_2 is dense in B_{12}, we can find $c'_{ij} \in B_2$ for $i = 1, \dots, n$, $j = 1, \dots, m$ such that

$$\max_{i,l=1,\dots,n} \max_{j=1,\dots,m} \|c_{ij} - c'_{ij}\|_2 \cdot \|d_{jl}\|_2 \le \beta^{-2}\epsilon.$$

We write

$$u = \sum_{i=1}^{n} a_i \|v'_i\|$$

with $a_1, \dots, a_n \in B_{12}$ with $\|a_i\|_{12} \le \beta\|u\|$. For each a_i with $i = 1, \dots, n$, we can expand lift them into series

$$a_i = \sum_{j,k=0}^{\infty} c^i_{jk} T^j S^k \in A\{r^{-1}T, rS\}$$

with

$$\|c^i_{jk}\|_A r^{j-k} \le \beta\|a_i\|_{12}.$$

In particular, we can find $a_i^+ \in B_1$ and $a_i^- \in B_2$ with

$$\|a_i^+\|_1 \le \beta\|a_i\|_{12}, \quad \|a_i^-\|_2 \le \beta\|a_i\|_{12}.$$

Take

$$u^+ = \sum_{i=1^n} a_i^+ v_i \in M_1, \quad u^- = \sum_{i=1^n} \sum_{j=1}^{m} a_i^- c'_{ij} w_j \in M_2.$$

Then $u^\pm$ satisfies all the requirements.

Step 2. We define $M = \ker d^0$. We will see that M satisfies the desired requirement. To prove this assertion, it suffices to know that M generates M_i as A_i-modules for $i = 1, 2$.

In fact, assuming that this holds, we can choose $f_1, \dots, f_s \in M$ so that they generate M_i as A_i-module for $i = 1, 2$. In this way we get a surjective homomorphism $A^s \to M$. Similarly, we apply the same construction to the kernel of this map, we get a presentation

$$A^r \to A^s \to M \to 0,$$

which can be embedded in the large commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^r \arrow[r] \arrow[d] & A_1^r \times A_2^r \arrow[r] \arrow[d] & A_{12}^r \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^s \arrow[r] \arrow[d] & A_1^s \times A_2^s \arrow[r] \arrow[d] & A_{12}^s \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & M_1 \times M_2 \arrow[r, "d^0"] \arrow[d] & M_{12} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

All colomns are exact by our assumptions. All rows are exact: the third row is Step 1 and our construction of M; the first two rows are trivial. The desired result follows from the right-exactness of tensor products.

In order to prove that M generates M_i as A_i-module for $i = 1, 2$ is the same as verifying

$$M \otimes_A A_i \to M_i$$

is surjective for $i = 1, 2$. Endow M and M_i with the structure of finite Banach A-module and finite Banach A_i-module respectively by Proposition 9.2. By Proposition 9.5, we can identify $M \otimes_A A_i$ with $M \hat{\otimes}_A A_i$. Now take a k-free polyray r with at least one component such that $A\hat{\otimes}_k k_r$, $A_1\hat{\otimes}_k k_r$, $A_2\hat{\otimes}_k k_r$ and $A_{12}\hat{\otimes}_k k_r$ are all strictly k_r-affinoid. By Proposition 3.11, we can then reduce to the strictly affinoid case.

Step 3. After the reductions, we can assume that k is non-trivially valued and A, A_1, A_2, A_{12} are all strictly k-affinoid. We need to show that M generates M_1 and M_2 as A_1-module and A_2-module respectively.

For each $x \in \operatorname{Spm} A$ with kernel $\mathfrak{m}$, we claim that teh natural map $M \to M/\mathfrak{m}M_i$ is surjective for $i = 1, 2$.

Assuming this claim, by Nakayama's lemma, we see that M generates M_i as A-module for $i = 1, 2$.

It remains to prove the claim. We have a short exact sequences

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0.$$

By [Stacks, Tag 03OM], we have a short exact sequence of Čech complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker\eta \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \ker\iota \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{m}M_1 \times \mathfrak{m}M_2 \arrow[r] \arrow[d, "\eta"] & M_1 \times M_2 \arrow[r] \arrow[d] & M_1/\mathfrak{m}M_1 \times M_2/\mathfrak{m}M_2 \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \mathfrak{m}M_{12} \arrow[r] \arrow[d] & M_{12} \arrow[r] \arrow[d] & M_{12}/\mathfrak{m}M_{12} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows are exact and the colomns are complexes. It follows from Step 1 and the snake lemma that we have an exact sequence

$$0 \to \ker\eta \to M \to \ker\iota \to 0.$$

In particular, the map $M \to \ker\iota$ is surjective.

Next assume that $x \in \operatorname{Sp} B_1$, we will prove that $\ker\iota \to M_1/\mathfrak{m}M_1$ is bijective. A dual arguement applies in the case $x \in \operatorname{Sp} B_2$. Note that this assertion readily implies our claim.

By Remark 13.10, we have the natural map is a bijection

$$B_2/\mathfrak{m}B_2 \to B_{12}/\mathfrak{m}B_{12}.$$

It follows that the following natural map is a bijection

$$M_2/\mathfrak{m}M_2 \to M_{12}/\mathfrak{m}M_{12}.$$

In particular, we find that $\ker \iota = M_1/\mathfrak{m}M_1$. This proves our assertion.

Finally, the last assertion is clear as M is constructed as an equalizer. $\square$

Bibliography

[Ber12] V. G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields. 33. American Mathematical Soc., 2012.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. Non-Archimedean analysis. Vol. 261. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. A systematic approach to rigid analytic geometry. Springer-Verlag, Berlin, 1984, pp. xii+436. URL: https://doi.org/10.1007/978-3-642-52229-1.

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.

[Tem04] M. Temkin. On local properties of non-Archimedean analytic spaces. II. *Israel J. Math.* 140 (2004), pp. 1–27. URL: https://doi.org/10.1007/BF02786625.