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Contents

Properties of complex analytic spaces

1. Introduction

2. Dimension

Definition 2.1. Let X be a complex analytic space and $x \in X$, the *dimension* $\dim_x X$ of X at x is

$$\dim_x X = \dim \mathcal{O}_{X,x}.$$

We also define the *dimension* of the pointed complex analytic space (X, x) and the *dimension* of the complex analytic germ X_x as $\dim_x X$.

Definition 2.2. Let X be a complex analytic space, we say X is *equidimensional* at $x \in X$ if $\mathcal{O}_{X,x}$ is equidimensional.

We also say (X, x) or X_x is *equidimensional*.

We say X is *equidimensional of dimension* n if X is equidimensional of dimension n at each $x \in X$.

Recall that in general, a local ring R is equidimensional if $\dim R/\mathfrak{p} = \dim R$ for all minimal prime $\mathfrak{p}$ of R.

Definition 2.3. Let X be a complex analytic space and $x \in X$, we say X is *integral* at x if $\mathcal{O}_{X,x}$ is integral.

This corresponds to the notion defined in **??** in **??**.

Theorem 2.4. Let X be a complex analytic space and $n \in \mathbb{N}$, then the set of points $x \in X$ such that X_x is equidimensional of dimension n is open.

This is analogous to the result for noetherian cartenary schemes.

Proof. Let $x \in X$ be a point such that X_x is equidimensional of dimension n. We want to construct an open neighbourhood V of x in X such that X is equidimensional of dimension n at any $y \in V$.

Step 1. We reduce to the case where X is integral at x.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal primes of $\mathcal{O}_{X,x}$. The number is finite because $\mathcal{O}_{X,x}$ is noetherian. We have

$$\bigcap_{i=1}^{m} \mathfrak{p}_i = \operatorname{rad} \mathcal{O}_{X,x}.$$

Take an open neighbourhood U of x in X such that there are ideals of finite type $\mathcal{I}_1, \ldots, \mathcal{I}_m$ extending $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Up to shrinking U, we may assume that

$$\bigcap_{i=1}^{m} \mathcal{I}_i$$

is nilpotent. For each $i = 1, \dots, m$, let U_i denote the closed analytic subspace of U defined by $\mathcal{I}_i$. Then

$$|U| = \bigcup_{i=1}^{m} |U_i|$$

by **??** in **??**. As for any $y \in U$,

$$\bigcap_{i=1}^{m} \mathcal{I}_{i,y}$$

is nilpotent, we have

$$|\operatorname{Spec} \mathcal{O}_{X,y}| = |\operatorname{Spec} \mathcal{O}_{X,y}/\bigcap_{i=1}^{m} \mathcal{I}_{i,y}| = \bigcup_{i=1}^{m} |\operatorname{Spec} \mathcal{O}_{X,y}/\mathcal{I}_{i,y}|.$$

In particular, for any $y \in U$,

$$\dim_y X = \dim_y U = \max_{i=1,\dots,m} \dim_y U_i.$$

It suffices to handle each W_i separately.

Step 2. We assume that X_x is integral. By **??** in **??**, we may assume that X has the following structure: there is an open neighbourhood W of 0 in $\mathbb{C}^n$, a morphism $(X, x) \to (W, 0)$ and a finite $\mathcal{O}_W$-algebra $\mathcal{A}$ such that $\operatorname{Spec}^{\mathrm{an}}_W \mathcal{A}$ has a unique point x' over 0 and $(\operatorname{Spec}^{\mathrm{an}}_W \mathcal{A}, x')$ is isomorphic to (X, x) over $(W, 0)$. By **??** in **??**, $\mathcal{O}_{W,0} \to \mathcal{O}_{X,x}$ is injective, hence $\mathcal{O}_{X,x}$ is torsion-free over $\mathcal{O}_{W,0}$. As the torsion sheaf is coherent, up to shrinking X, we may assume that $\mathcal{O}_{X,y}$ is torsion-free over $\mathcal{O}_{W,z}$, where z denotes the image of y in W. It suffices to apply **??** in **??**. □

Corollary 2.5. Let X be a complex analytic space and $n \in \mathbb{N}$. Then the set $\{x \in X : \dim_x X \geq n\}$ is an analytic set in X.

After introducing the analytic Zariski topology, we can reformulate this corollary as follows: the map $x \mapsto \dim_x X$ is upper semi-continuous with respect to the analytic Zariski topology.

Proof. The problem is local on X. Fix $x \in X$ and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of $\mathcal{O}_{X,x}$. Up to shrinking X, we may assume that

$$|X| = \bigcup_{i=1}^{m} |W_i|,$$

where W_i is a closed analytic subspace of X defined by a coherent $\mathcal{I}_i$ spreading $\mathfrak{p}_i$. We can guarantee that

$$\dim_y X = \max_{i=1,\dots,m} \dim_y W_i.$$

This is possible as in the proof of Theorem 2.4. By Theorem 2.4, up to shrinking X, we may assume that W_i is equidimensional of dimension n_i for some $n_i \in \mathbb{N}$ for each $i = 1, \dots, m$. In particular, for each $y \in X$, we have

$$\dim_y X = \sup_{y \in W_i} n_i.$$

So

$$\{x \in X : \dim_x X \geq n\} = \bigcup_{i : n_i \geq n} |W_i|.$$

The corollary follows. □

Definition 2.6. Let X_x be an analytic germ and Y_x be a closed analytic subgerm defined by an ideal $I \subseteq \mathcal{O}_{X,x}$.

(1) When Y_x is irreducible, namely when I is a prime ideal, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \operatorname{het}_{\mathcal{O}_{X,x}}(I).$$

(2) In general, we define the *codimension* of Y_x in X_x as

$$\operatorname{codim}_x(Y, X) := \inf_{Z_x \subseteq Y_x} \operatorname{codim}_x(Y, X),$$

where Z_x runs over closed analytic subgerms of X_x contained in Y_x.

We also call $\operatorname{codim}_x(Y, X)$ the codimension of Y in X at x.

Observe that

$$\operatorname{codim}_x(Y, X) \leq \dim_x X - \dim_x Y.$$

When X_x is equidimensional, $\operatorname{codim}_x(Y, X)$ is nothing but $\dim_x X - \dim_x Y$.

3. Serre's condition R_n

Fix $n \in \mathbb{N}$ in this section.

Definition 3.1. Let X be a complex analytic space, we say X *satisfies* R_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* R_n at $x \in X$.

We say X *satisfies* R_n if X satisfies R_n at all points $x \in X$.

Proposition 3.2.

4. Serre's condition S_n

Fix $n \in \mathbb{N}$ in this section.

Definition 4.1. Let X be a complex analytic space, we say X *satisfies* S_n at $x \in X$ if $\mathcal{O}_{X,x}$ satisfies R_n. We also say (X, x) or X_x *satisfies* S_n at $x \in X$.

We say X *satisfies* S_n if X satisfies S_n at all points $x \in X$.

Proposition 4.2.

5. Reducedness

Definition 5.1. Let X be a complex analytic space, we say X is *reduced* at $x \in X$ if $\mathcal{O}_{X,x}$ is reduced. We also say (X, x) or X_x is *reduced* at $x \in X$.

We say X is *reduced* if X is reduced at all points $x \in X$.

Theorem 5.2. Let X be a complex analytic space. Then the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is reduced is the complement of an analytic set.

Proof. $\square$

Corollary 5.3. Let X be a complex analytic space, then the nilradical $\operatorname{rad} \mathcal{O}_X$ is coherent.

PROOF. The problem is local on X. Take $x \in X$. Up to shrinking X, we may assume that $\mathcal{O}_{X,x}/(\operatorname{rad}\mathcal{O}_X)_x$ spreads to a finite $\mathcal{O}_X$-algebra $\mathcal{A}$ by **??** in **??**. Up to further shrinking X, we may assume that $\mathcal{A}$ is the quotient of $\mathcal{O}_X$, say $\mathcal{A} \cong \mathcal{O}_X/\mathcal{I}$ for some coherent ideal $\mathcal{I}$ on X. As $\mathcal{I}_x$ is nilpotent by assumption, up to shrinking X, we may assume that $\mathcal{I}$ is also nilpotent, namely

$$\mathcal{I} \subseteq \operatorname{rad}\mathcal{O}_X.$$

Let Y be the closed analytic subspace of X defined by the ideal $\mathcal{I}$. Then $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}/(\operatorname{rad}\mathcal{O}_X)_x$ is reduced. Up to shrinking X, by Theorem 5.2, we may assume that Y is reduced. But then for any $y \in Y$,

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,y}/\mathcal{I}_y$$

is reduced, so

$$\mathcal{I}_y \supseteq (\operatorname{rad}\mathcal{O}_X)_y.$$

It follows that $\operatorname{rad}\mathcal{O}_X = \mathcal{I}$, hence the nilradical is coherent. □

Corollary 5.4 (Cartan–Oka)**.** Let X be a complex analytic space and A be an analytic subset of X, then the sheaf $\mathcal{J}_A$ is coherent.

Recall that the sheaf $\mathcal{J}_A$ is introduced in **??** in **??**.

PROOF. By **??** in **??**, we may assume that A is a closed analytic subspace of X defined by a coherent ideal $\mathcal{I}$. By **??** in **??**, the sheaf $\mathcal{J}_A$ is nothing but $\sqrt{I}$, which is coherent by Corollary 5.3. □

Corollary 5.5. Let X be a complex analytic space and A be an analytic subset of X, then there is a unique reduced closed analytic space Y of X with underlying set A.

PROOF. The existence follows from Corollary 5.4. The uniqueness follows from **??** in **??**. □

Definition 5.6. Let X be a complex analytic space and A be an analytic subset of X. The analytic space structure on A defined in Corollary 5.5 is called the *reduced induced structure* on A.

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Bibliography

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