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Contents

Global properties of complex analytic spaces

1. Introduction

2. Holomorphically convex hulls

Definition 2.1. Let X be a complex analytic space and M be a subset of X, we define the *holomorphically convex hull* of M in X as

$$\hat{M}^X := \left\{ x \in X : |f(x)| \leq \sup_{y \in M} |f(y)| \text{ for all } f \in \mathcal{O}_X(X) \right\}.$$

Proposition 2.2. Let X be a complex analytic space and M be a subset of X. Then the following properties hold:

(1) $\hat{M}^X$ is closed in X;

(2) $M \subseteq \hat{M}^X$ and $\widehat{\hat{M}^X}^X = \hat{M}^X$;

(3) If M' is another subset of X containing M, then $\hat{M}^X \subseteq \hat{M'}^X$;

(4) If $f : Y \to X$ is a morphism of complex analytic spaces, then

$$\widehat{f^{-1}(M)}^Y \subseteq f^{-1}(\hat{M}^X);$$

(5) If X' is another complex analytic space and M' is a subset of X', then

$$\widehat{M \times M'}^{X \times X'} \subseteq \hat{M}^X \times \hat{M'}^{X'};$$

(6) If M' is another subset of X and $\hat{M}^X = M$, $\hat{M'}^X = M'$, then

$$\widehat{M \cap M'}^X = M \cap M'.$$

PROOF. (1), (2), (3), (4), (5) are obvious by definition.
(6) is a consequence of (3). $\square$

Example 2.3. Let Q be a compact cube in $\mathbb{C}^n$ for some $n \in \mathbb{N}$, then $\hat{Q}^{\mathbb{C}^n} = Q$.

In fact, by Proposition 2.2(5), we may assume that $n = 1$. Given $p \in \mathbb{C} \setminus Q$, we can take a closed disk $T \subseteq \mathbb{C}$ centered at $a \in \mathbb{C}$ such that $Q \subseteq T$ while $p \notin T$. Consider $z - a \in \mathcal{O}_{\mathbb{C}}(\mathbb{C})$, then

$$|f(p)| > \sup_{q \in Q} |f(q)|.$$

So $p \notin \hat{Q}^{\mathbb{C}}$.

3. Stones

Definition 3.1. Let X be a complex analytic space. A *stone* in X is a pair (P, π) consisting of

(1) a non-empty compact set P in X and

(2) a morphism $\pi : X \to \mathbb{C}^n$ for some $n \in \mathbb{N}$

such that there is a compact tube Q in $\mathbb{C}^n$ and an open set W in X such that $P = \pi^{-1}(Q) \cap W$.

We call $P^0 := \pi^{-1}(\operatorname{Int} Q) \cap W$ the *analytic interior* of the stone (P, π). It clearly does not depend on the choice of W.

We observe that $\hat{P}^X \cap W = P$. In fact, $P \subseteq \pi^{-1}(Q)$, so

$$\hat{P}^X \subseteq \pi^{-1}(\hat{Q}^{\mathbb{C}^n}) = \pi^{-1}(Q) = P \cap W = P.$$

Here we applied Proposition 2.2 and Example 2.3.

In general, $P^0 \subseteq \operatorname{Int} P$, but they can be different.

Theorem 3.2. Let X be a Hausdorff complex analytic space and $K \subseteq X$ be a compact subset. Then the following are equivalent:

(1) There is an open neighbourhood W of K in X such that $\hat{K}^X \cap W$ is compact;
(2) There is an open relative compact neighbourhood W of K in X such that $\partial W \cap \hat{K} = \emptyset$;
(3) There is a stone (P, π) in X with $K \subseteq P^0$.

PROOF. (1) $\implies$ (2): This is trivial, in fact, we may assume that W in (1) is relatively compact in X.

(2) $\implies$ (3): As $\hat{K}^X$ is closed by Proposition 2.2(1) and $\partial W \cap \hat{K}^X = \emptyset$, given $p \in \partial W$, we can find $h \in \mathcal{O}_X(X)$ such that

$$\sup_{x \in K} |h(x)| < 1 < |h(p)|.$$

We will denote the left-hand side by $|h|_K$. Up to raising h to a power, we may assume that

$$\max\{|\operatorname{Re} h(p)|, |\operatorname{Im} h(p)|\} > 1.$$

As ∂W is compact, we can find finitely many sections $h_1, \ldots, h_m \in \mathcal{O}_X(X)$ so that

$$\max_{j=1,\ldots,m}\{|\operatorname{Re} h_j|_K, |\operatorname{Im} h_j|_K\} < 1, \quad \max_{j=1,\ldots,m}\{|\operatorname{Re} h_j(p)|, |\operatorname{Im} h_j(p)|\} > 1.$$

Let

$$Q := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |\operatorname{Re} z_i| \le 1, |\operatorname{Im} z_i| \le 1 \text{ for all } i = 1, \ldots, m\}.$$

The sections $h_1, \ldots, h_m$ defines a homomorphism $\pi : X \to \mathbb{C}^m$ by **??** in **??**. Obviously, $P = \pi^{-1}(Q) \cap W$ satisfies our assumptions.

(3) $\implies$ (1): Let W be the open set as in Definition 3.1. As $\hat{P}^X \cap W = P$ and $K \subseteq P$, we have

$$\hat{K} \cap W \subseteq P \cap W = P.$$

As P is compact, so is $\hat{K} \cap W$. $\qquad\square$

Theorem 3.3. Let X be a Hausdorff complex analytic space and $(P, \pi : X \to \mathbb{C}^n)$ be a stone in X. Let Q be the tube in $\mathbb{C}^m$ as in Definition 3.1. Then there are open neighbourhoods U and V of P and Q in X and $\mathbb{C}^n$ respectively with $\pi(U) \subseteq V$ and $P = \pi^{-1}(Q) \cap U$ such that $\pi|_U : U \to V$ is proper.

PROOF. Let $W \subseteq X$ be the open set as in Definition 3.1. We may assume that W is relatively compact. Then ∂W and $\pi(\partial W)$ are also compact. As $\partial W \cap \pi^{-1}(Q)$ is empty, we know that $V := \mathbb{C}^n \setminus \pi(\partial W)$ is an open neighbourhood of Q. The set $U := W \cap \pi^{-1}(V) = W \setminus \pi^{-1}(\pi(\partial W))$ is open in X and $\pi(U) \subseteq V$. Observe that $\pi|_U : U \to V$ is proper by Lemma 4.6 in Topology and bornology.

Furthermore,

$$\pi^{-1}(Q) \cap U = \pi^{-1}(Q) \cap \left(W \setminus \left(\pi^{-1}(Q) \cap \pi^{-1}\pi(\partial W) \right) \right).$$

But $\pi^{-1}Q \cap \pi^{-1}\pi(\partial W)$ is empty as $Q \cap \pi(\partial W)$ is. It follows that $\pi^{-1}(Q) \cap U = P$ and hence U is a neighbourhood of P. $\qquad\square$

Definition 3.4. Let X be a complex analytic space. Let $(P, \pi : X \to \mathbb{C}^n)$, $(P', \pi' : X \to \mathbb{C}^{n'})$ be two stones on X. We say (P, π) *is contained in* (P', π') if the following conditions are satisfied:

(1) P lies in the analytic interior of P';
(2) $n' \geq n$ and there is $q \in \mathbb{C}^{n'-n}$ such that if $Q \subseteq \mathbb{C}^n$, $Q' \subseteq \mathbb{C}^{n'}$ be the tubes as in Definition 3.1, then

$$Q \times \{q\} \subseteq Q'.$$

(3) There is a morphism $\varphi : X \to \mathbb{C}^{n'-n}$ such that

$$\pi' = (\pi, \varphi).$$

We formally write $(P, \pi) \subseteq (P', \pi')$ in this case. Clearly, this defines a partial order on the set of stones on X.

Definition 3.5. Let X be a complex analytic space. An *exhaustion of X by stones* is a sequence $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ of stones such that

(1) $(P_i, \pi_i) \subseteq (P_{i+1}, \pi_{i+1})$ for all $i \in \mathbb{Z}_{>0}$;
(2)

$$X = \bigcup_{i=1}^{\infty} P_i^0.$$

We say X is *weakly holomorphically convex* if it there is an exhaustion of X by stones.

Theorem 3.6. Let X be a Hausdorff complex analytic space. Consider the following conditions:

(1) X is weakly holomorphically convex;
(2) For any compact subset $K \subseteq X$, there is an open set $W \subseteq X$ such that $\hat{K}^X \cap W$ is compact.

Then (1) $\implies$ (2). If X is Lindelöf, then (2) $\implies$ (1).

PROOF. (1) $\implies$ (2): It suffices to observe that $K \subseteq P_j^0$ when j is large enough and apply Theorem 3.2.

Assume that X is Lindelöf. (2) $\implies$ (1): Let (K_i) a compact exhaustion of X. We construcct the stones $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ so that

$$K_i \subseteq P_i^0$$

for all $i \in \mathbb{Z}_{>0}$ inductively. Let P_1 be an arbitrary stone in X such that $K_1 \subseteq P_1^0$. The existence of P_1 is guaranteed by Theorem 3.2.

Assume that we have constructed $(P_{i-1}, \pi_{i-1} : X \to \mathbb{C}^{n_{i-1}})$ for $i \geq 2$. Let $Q_{i-1} \subseteq \mathbb{C}^{n_{i-1}}$ be the associated tube. By Theorem 3.2 again, take a stone $(P_i, \pi_i^* : X \to \mathbb{C}^n)$ with $K_i \cup P_{i-1} \subseteq P_i^0$. Let $Q_i^* \subseteq \mathbb{C}^n$ be the associated tube. Let W be an open subset of X with

$$P_i = \pi_i^{*,-1}(Q_i^*) \cap W.$$

Choose a tube $Q_i' \subseteq \mathbb{C}^{n_{i-1}}$ with $Q_{i-1} \subseteq \operatorname{Int} Q_i'$ so that

$$\pi_{i-1}(P_i) \subseteq \operatorname{Int} Q_i'.$$

Let $\pi_i := (\pi_{i-1}, \pi_i^*) : X \to \mathbb{C}^{n_{i-1}+n}$ and $Q_i := Q_i' \times Q_i^*$. Then (P_i, π_i) is a stone and $(P_{i-1}, \pi_{i-1}) \subseteq (P_i, \pi_i)$. $\qquad\square$

4. Holomorphical separable spaces

Definition 4.1. Let X be a complex analytic space. We say X is *holomorphically separable* if for any $x, y \in X$ with $x \neq y$, there is $f \in \mathcal{O}_X(X)$ with $f(x) \neq f(y)$.

Here we regard f as a continuous function $X \to \mathbb{C}$. In particular, a holomorphically separable space is Hausdorff.

Definition 4.2. Let X be a complex analytic space. We say X is *holomorphically convex* if $|X|$ is Hausdorff and for any compact set $K \subseteq X$, $\hat{K}^X$.

We say X is *weakly holomorphically convex* if for any quasi-compact set $K \subseteq X$, the connected components of $\hat{K}^X$ are all quasi-compact.

Proposition 4.3. Let X be a holomorphically convex complex analytic space. Then X^{red} is holomorphically convex.

PROOF. This follows immediately from the definition. $\qquad\square$

Proposition 4.4. Let X be a Hausdorff complex analytic space. Consider the following conditions:

  (1) X is holomorphically convex;
  (2) For any sequence $x_i \in X$ ($i \in \mathbb{Z}_{>0}$) without accumulation points, there is $f \in \mathcal{O}_X(X)$ such that $|f(x_i)|$ is unbounded.

Then (2) $\implies$ (1) if X is Lindelöf.

PROOF. (2) $\implies$ (1): For a Lindelöf Hausdorff space, sequential compactness implies compactness.

$\qquad\square$

Corollary 4.5. Let $n \in \mathbb{N}$ and Ω be a domain in $\mathbb{C}^n$. Assume that for each $p \in \partial\Omega$, there is a holomorphic function f on an open neighbourhood U of $\bar{\Omega}$ such that $f(p) = 0$ and f is non-zero on Ω. Then Ω is holomorphically convex.

PROOF. Let $x_i \in \Omega$ ($i \in \mathbb{Z}_{>0}$) be a sequence without accumulation points in Ω. We need to construct $f \in \mathcal{O}_\Omega(\Omega)$ such that $(|f(x_i)|)_{i \in \mathbb{Z}_{>0}}$ is unbounded. This is clear if x_i itself is unbounded. Assume that x_i is bounded. Then up to passing to a subsequence, we may assume that $x_i \to p \in \partial\Omega$ as $i \to \infty$. The inverse of the function f in our assumption of the corollary works. $\qquad\square$

5. Stein sets

Definition 5.1. Let X be a complex analytic space and P be a closed subset of X. We say P is a *Stein set* in X if for any coherent $\mathcal{O}_U$-module $\mathcal{F}$ for some open neighbourhood U of P in X, we have

$$H^i(P, \mathcal{F}) = 0 \quad \text{for all } i \in \mathbb{Z}_{>0}.$$

A *coherent $\mathcal{O}_P$-module* is a coherent $\mathcal{O}_U$-module for some open neighbourhood U of P in X. Two coherent $\mathcal{O}_P$-modules are isomorphic if there is a small enough open neighbourhood V of P in X such that they are isomorphic when restricted to V. In particular, $\mathcal{O}_P$ denotes the coherent $\mathcal{O}_P$-module defined by $\mathcal{O}_X$ on X.

The germ-wise notions obviously make sense for coherent $\mathcal{O}_P$-modules.

The given condition is usually known as *Cartan's Theorem B*. It implies *Cartan's Theorem A*:

Theorem 5.2 (Cartan's Theorem A)**.** Let X be a complex analytic space and P be a Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module for some open neighbourhood U of P in X. Then $H^0(P, \mathcal{F})$ generates $\mathcal{F}_x$ for each $x \in P$.

PROOF. Fix $x \in P$. Let $\mathcal{M}$ be the coherent ideal sheaf on U consisting of holomorphic functions vanishing at x. Then $\mathcal{F}\mathcal{M}$ is a coherent $\mathcal{O}_U$-module. It follows from Theorem B that

$$H^0(P, \mathcal{F}) \to H^0(P, \mathcal{F}/\mathcal{F}\mathcal{M})$$

is surjective. Note that we can identify this map with the natural map

$$H^0(P, \mathcal{F}) \to \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x.$$

Let $e_1, \ldots, e_m$ be a basis of $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Lift them to $s_1, \ldots, s_m \in H^0(P, \mathcal{F})$. By Nakayama's lemma, $s_{1x}, \ldots, s_{mx}$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$. $\qquad\square$

Corollary 5.3. Let X be a complex analytic space and P be a quasi-compact Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-module. Then there is $n \in \mathbb{Z}_{>0}$ and an epimorphism

$$\mathcal{O}_P^n \to \mathcal{F}.$$

PROOF. By Theorem 5.2, we can find an open covering $\{U_i\}_{i \in I}$ of P such that there are homomorphisms

$$h_i : \mathcal{O}_P^{n_i} \to \mathcal{F}$$

for some $n_i \in \mathbb{Z}_{>0}$, which is surjective on U_i for each $i \in I$. By the quasi-compactness of P, we may assume that I is a finite set. Then it suffices to set $n = \sum_{i \in I} n_i$ and consider the epimorphism $\mathcal{O}_P^n \to \mathcal{F}$ induced by the h_i's. $\qquad\square$

Theorem 5.4. Let X be a complex analytic space and $P \subseteq X$ be a set with the following properties:

  (1) there is an open neighbourhood U of P in X, a domain V in $\mathbb{C}^m$ for some $m \in \mathbb{N}$ and a finite holomorphic morphism $\tau : U \to V$;
  (2) There exists a compact tube in $\mathbb{C}^m$ contained in V such that $P = \tau^{-1}(Q)$.

Then P is a compact Stein set in X.

PROOF. As $P = \tau^{-1}(Q)$ and τ is proper, we see that P is compact.

It remains to show that P is a Stein set in X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_P$-module.

Step 1. We first reduce to the case where $\mathcal{F}$ is defined by a coherent $\mathcal{O}_U$-module.

Take an open neighbourhood U' of P in X contained in U such that $\mathcal{F}$ is defined by a coherent $\mathcal{O}_{U'}$-module. By Lemma 4.2 in Topology and bornology, we can take an open neighbourhood V' of Q in V such that $\tau^{-1}(V') \subseteq U'$. The restriction of τ to $\tau^{-1}(V') \to V'$ is again finite.

Step 2. By Leray spectral sequence,

$$H^i(P, \mathcal{F}) \cong H^i(Q, (\tau|_P)_* \mathcal{F})$$

for all $i \geq 0$. By **??** in **??**, $(\tau|_P)_* \mathcal{F}$ is a coherent $\mathcal{O}_Q$-module, so we are reduced to show that Q is a Stein set in $\mathbb{C}^m$, which is well-known.   □

Definition 5.5. Let X be a Hausdorff complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. A *Stein exhaustion of X relative to $\mathcal{F}$* is a compact exhaustion $(P_i)_{i \in \mathbb{Z}_{>0}}$ such that the following conditions are satisfied:

(1) P_i is a Stein set in X for each $i \in \mathbb{Z}_{>0}$;
(2) the $\mathbb{C}$-vector space $H^0(P_i, \mathcal{F})$ admits a semi-norm $| \bullet |_i$ such that the restriction map

$$H^0(X, \mathcal{F}) \to H^0(P_i, \mathcal{F})$$

has dense image with respect to the topological defined by $| \bullet |_i$ for each $i \in \mathbb{Z}_{>0}$;
(3) The restriction map

$$H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$$

is bounded for each $i \in \mathbb{Z}_{>0}$;
(4) Let $i \in \mathbb{Z}_{\geq 2}$. Suppose that $(s_j)_{j \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in $H^0(P_i, \mathcal{F})$, then the restricted sequence $s_j|_{P_{i-1}}$ has a limit in $H^0(P_{i-1}, \mathcal{F})$;
(5) Let $i \in \mathbb{Z}_{\geq 2}$. If $s \in H^0(P_i, \mathcal{F})$ and $|s|_i = 0$, then $s|_{P_{i-1}} = 0$.

A *Stein exhaustion of X* is a compact exhaustion of X that is a Stein exhaustion of X relative to any coherent $\mathcal{O}_X$-module.

Theorem 5.6. Let X be a Hausdorff complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume that $(P_i)_{i \in \mathbb{Z}_{>0}}$ is a Stein exhaustion of X relative to $\mathcal{F}$. Then

$$H^q(X, \mathcal{F}) = 0 \quad \text{for any } q \in \mathbb{Z}_{>0}.$$

PROOF. When $q \geq 2$, this follows from the general facts proved in Lemma 5.3 in Topology and bornology. We will assume that $q = 1$.

We may assume that X is connected. First observe that X is necessarily paracompact. This follows from Proposition 3.2 in Topology and bornology. In particular, we can take a flabby resolution

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \cdots.$$

Taking global sections, we get a complex

$$0 \to H^0(X, \mathcal{F}) \xrightarrow{i} H^0(X, \mathcal{G}^0) \xrightarrow{d_0} H^0(X, \mathcal{G}^1) \xrightarrow{d_1} H^0(X, \mathcal{G}^2) \xrightarrow{d_2} \cdots.$$

We need to show that $\ker d_1 = \operatorname{Im} d_0$. Let $\alpha \in \ker d_1$. We need to construct $\beta \in H^0(X, \mathcal{G}^0)$ with $d_0 \beta = \alpha$.

We take semi-norms $|\bullet|_i$ on $H^0(P_i, \mathcal{F})$ for each $i \in \mathbb{Z}_{>0}$ satisfying the conditions in Definition 5.5. We may furthermore assume that the restriction $H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$ is a contraction for each $i \in \mathbb{Z}_{>0}$.

For each $j \in \mathbb{Z}_{\geq 2}$, we will construct $\beta_j \in H^0(P_j, \mathcal{G}^0)$ and $\delta_j \in H^0(P_{j-1}, \mathcal{F})$ such that

(1) $(d_0|_{P_j})\beta_j = \alpha|_{P_j}$;
(2) $(\beta_{j+1} + \delta_{j+1})|_{P_{j-1}} = (\beta_j + \delta_j)|_{P_{j-1}}$.

It suffices to take $\beta \in H^0(X, \mathcal{G}^0)$ as the section defined by the $\beta_j + \delta_j$'s.

We first construct β_j. Choose a sequence $\beta'_j \in H^0(P_j, \mathcal{G}^0)$ with

$$(d_0|_{P_j})\beta'_j = \alpha|_{P_j}$$

for each $j \in \mathbb{Z}_{>0}$. This is possible because P_j is Stein. We define β_j satisfying Condition (1) for $j \in \mathbb{Z}_{>0}$ inductively. We begin with $\beta_1 = \beta'_1$. Assume that $\beta_1, \ldots, \beta_j$ have been constructed. Let

$$\gamma'_j := \beta'_{j+1}|_{P_j} - \beta_j.$$

Then

$$(d_0|_{P_j})\gamma'_j = 0.$$

It follows that $\gamma'_j \in H^0(P_j, \mathcal{F})$. Take $\gamma_j \in H^0(X, \mathcal{F})$ with

$$|\gamma'_j - \gamma_j|_{P_j}|_j \leq 2^{-j}.$$

Define

$$\beta_{j+1} = \beta'_{j+1} - \gamma_i|_{P_{j+1}}.$$

Then clearly β_{j+1} satisfies (1).

Next we construct the sequence δ_j.

We observe that for each $j \in \mathbb{Z}_{>0}$,

$$\left|\beta_{j+1}|_{P_j} - \beta_j\right|_j \leq 2^{-j}.$$

Let

$$s_k^j := \beta_{j+k}|_{P_j} - \beta_j \in H^0(P_j, \mathcal{F})$$

for all $j \in \mathbb{Z}_{>0}$ and $k \in \mathbb{N}$. By definition,

$$s_k^j - s_{k-1}^{j+1}|_{P_j} = \beta_{j+1}|_{P_j} - \beta_j$$

for all $j \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}_{>0}$.

We claim that $(s_k^j|_{P_{j-1}})_k$ converges in $H^0(P_{j-1}, \mathcal{F})$ as $k \to \infty$. By our assumption, it suffices to show that $(s_k^j)_k$ is a Cauchy sequence in $H^0(P_j, \mathcal{F})$ for each $j \in \mathbb{Z}_{>1}$. We first compute

$$\left|\beta_{j+l}|_{P_j} - \beta_{j+l-1}|_{P_j}\right|_j \leq \left|\beta_{j+l}|_{P_{j+l-1}} - \beta_{j+l-1}\right|_{j+l-1} \leq 2^{1-j-l}$$

for all $l \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{>0}$. As a consequence for $k' > k \geq 1$, we have

$$|s_k^j - s_{k'}^j|_j \leq \sum_{l=k+1}^{k} 2^{1-j-l} \leq 2^{1-j+k}.$$

So we conclude our claim.

Let δ_j be the limit of $s_k^j|_{P_{j-1}}$ as $k \to \infty$ for each $j \in \mathbb{Z}_{\geq 2}$. Then

$$\lim_{k \to \infty} \left(s_k^j - s_{k-1}^{j+1}\right)|_{P_{j-1}} = (\delta_j - \delta_{j+1})|_{P_{j-1}}$$

for each $j \in \mathbb{Z}_{\geq 2}$. The desired identity is clear. $\square$

6. Analytic blocks

Definition 6.1. Let X be a Hausdorff complex analytic space. A stone $(P, \pi : X \to \mathbb{C}^n)$ on X is an *analytic block* in X if there are open neighbourhoods U and V of P and Q in X and Y respectively, where $Q \subseteq \mathbb{C}^n$ denotes the tube associated with the stone, such that

(1) $\pi(U) \subseteq V$;
(2) $P = \pi^{-1}(Q) \cap U$;
(3) $U \to V$ induced by π is a finite morphism.

Recall that by Theorem 3.3, we can always assume that $U \to V$ is proper.

Proposition 6.2. Let X be a Hausdorff complex analytic space and (P, π) be an analytic block in X. Then P is a compact Stein set in X.

PROOF. This follows from Theorem 5.4 applied to $U \to V$ in Definition 6.1. $\square$

Proposition 6.3. Let X be a complex analytic space such that each compact analytic set in X is finite, then every stone in X is an analytic block in X.

PROOF. Let $(P, \pi : X \to \mathbb{C}^n)$ be a stone in X. We consider the proper morphism $\tau : U \to V$ as in Theorem 3.3. Each fiber of τ is a compact subset of U and hence a compact subset of X. By our assumption, it is finite. It suffices to apply Proposition 4.5 in Topology and bornology to conclude that τ is finite. $\square$

7. Holomorphically spreadable spaces

Definition 7.1. Let X be a complex analytic space. We say X is *holomorphically spreadable* if $|X|$ is Hausdorff and for any $x \in X$, we can find an open neighbourhood U of x in X such that

$$\{y \in U : f(x) = f(y) \text{ for all } f \in \mathcal{O}_X(X)\} = \{x\}.$$

A holomorphically separable space is clearly holomorphically spreadable.

Proposition 7.2. Let X be a holomorphically spreadable complex analytic space and $x \in X$. Then there exist finitely many $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ such that x is an isolated point of $W(f_1, \ldots, f_n)$.

PROOF. By induction on $\dim_x X$, it suffices to prove the following claim: if A is an analytic set in X and $a \in A$ such that $\dim_a A \geq 1$. Then there is $f \in \mathcal{O}_X(X)$ such that $\dim_a(A \cap W(f)) = \dim_a A - 1$.

To prove the claim, let $A_1, \ldots, A_k$ be the irreducible components of A. We may assume that all of them contain a. Choose $a_j \in A_j$ for each $j = 1, \ldots, k$ so that $a, a_1, \ldots, a_k$ are pairwise different. Then there is a function $f \in \mathcal{O}_X(X)$ with $f(a) = 0$ while $f(a_j) \neq 0$ for $j = 1, \ldots, k$. Then $a \in W(f)$ while $f|_{A_j}$ is not identically 0. By Krulls Hauptidealsatz, $\dim_a(A_j \cap W(f)) = \dim_a A_j - 1$ for all $j = 1, \ldots, k$. Observe that $A \cap W(f)$ and $\bigcup_{j=1}^k (A_j \cap W(f))$ coincide near a, so

$$\dim_a(A \cap W(f)) = \max_{j=1,\ldots,k} \dim_a(A_j \cap W(f)) = \max_{j=1,\ldots,k} (\dim_a A_j - 1) = \dim_a A - 1.$$

$\square$

Proposition 7.3. Let X be an irreducible holomorphically spreadable complex analytic space. Then X has countable basis.

The statement of this proposition in [Fis76, Proposition 0.37] is clearly wrong. I do not understand the argument of either [Jur59] or [Gra55], where they claim that this result holds for connected holomorphically spreadable complex analytic spaces.

PROOF. We may assume that X is connected. Recall that by **??** in **??**, X is locally connected. Let $F : X \to \mathbb{C}^{\mathcal{O}_X(X)}$ be the map sending $x \in X$ to $(f(x))_{f \in \mathcal{O}_X(X)}$. By our assumption, F is continuous and has discrete fibers. In particular, for each $x \in X$, we may assume take finitely many $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ so that the induced morphism $F' : X \to \mathbb{C}^n$ is quasi-finite at x. By **??** in **??**, we can find a nowhere dense analytic set A in X such that the map $X \setminus A \to \mathbb{C}^n$ induced by F' is quasi-finite. Now we endow $\mathcal{O}_X(X)$ with the compact-open topology. It is a metric space. By Proposition 6.2 in Topology and bornology, $X \setminus A$ has countable basis. It follows that $\mathcal{O}_X(X \setminus A)$ is a separable metric space. Hence, so it $\mathcal{O}_X(X)$. In particular, there is a continous map with discrete fibers

$$X \to \mathbb{C}^\omega.$$

It follows again from Proposition 6.2 in Topology and bornology that X has countable basis. $\square$

Proposition 7.4. Let X be a holomorphically spreadable complex analytic space. Then any compact analytic set A in X is finite.

PROOF. Let B be a connected component of A and $p \in B$. We need to show that $B = \{p\}$. Take finitely many $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ so that p is an isolated point of $W(f_1, \ldots, f_n)$. This is possible by Proposition 7.2. As f_i vanishes on B for each $i = 1, \ldots, n$, we have $B = \{p\}$. $\square$

Corollary 7.5. Let X be a complex analytic space and A be a compact analytic subset of X. Suppose that there exists an analytic block $(P, \pi : X \to \mathbb{C}^n)$ in X with $A \subseteq P$, then A is finite.

PROOF. Take $U \subseteq X, V \subseteq \mathbb{C}^n$ as in Definition 6.1 so that $U \to V$ is finite. Then U is clearly holomorphically spreadable. By Proposition 7.4, A is finite. $\square$

8. Holomorphically complete spacs

Definition 8.1. Let X be a complex analytic space. An *exhaustion of X by analytic blocks* is an exhaustion of X by stones $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ such that (P_i, π_i) is an analytic block for each $i \in \mathbb{Z}_{>0}$.

We say X is *holomorphically complete* if X is Hausdorff and there is an exhaustion of X by analytic stones.

Theorem 8.2. Let X be a Hausdorff complex analytic space. Then the following are equivalent:

(1) X is holomorphically complete;
(2) X is weakly holomorphically convex and every compact analytic subset of X is finite.

PROOF. (1) $\implies$ (2): X is weakly holomorphically convex by definition. Each compact analytic subset A of X is contained in some analytic block, hence finite by Corollary 7.5.

(2) $\implies$ (1): This follows from Proposition 6.3. $\square$

Lemma 8.3. Let X be a complex manifold and $\mathcal{I}$ be a coherent subsheaf of $\mathcal{O}_X^l$ for some $l \in \mathbb{Z}_{>0}$. Then $\mathcal{I}(X)$ is a closed subspace of $\mathcal{O}_X(X)^l$ endowed with the compact-open topology.

PROOF. Let $(f_j \in \mathcal{I}(X))_{j \in \mathbb{Z}_{>0}}$ be a sequence with a limit $f \in \mathcal{O}_X^l(X)$. Let $x \in X$. It suffices to show that $f_x \in \mathcal{I}_x$. Observe that f_x is the limit of f_{jx} as $j \to \infty$. As $\mathcal{O}_{X,x}$ is noetherian, the submodule $\mathcal{I}_x$ of $\mathcal{O}_x^l$ is closed by Corollary 7.4 in **??**. We conclude. $\qquad\square$

Definition 8.4. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$.

Choose $U \subseteq X, V \subseteq \mathbb{C}^n$ as in Definition 6.1 so that $\tau : U \to V$ induced by π is finite. Then $\mathcal{G} := \tau_*(\mathcal{F}|_U)$ is a coherent $\mathcal{O}_V$-module. By Corollary 5.3, we can find $l \in \mathbb{Z}_{>0}$ and an epimorphism $\mathcal{O}_Q^l \to \mathcal{G}|_Q$. It induces an epimorphism $\epsilon : H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l \to H^0(Q, \mathcal{G}) \xrightarrow{\sim} H^0(P, \mathcal{F})$. We define a semi-norm $|\bullet|$ on $H^0(P, \mathcal{F})$ as the quotient semi-norm induced by the sup seminorm on $H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$.

A seminorm on $H^0(P, \mathcal{F})$ defined in this way is called a *good semi-norm* on $H^0(P, \mathcal{F})$ with respect to (P, π).

Lemma 8.5. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let (P, π) be an analytic block on X. A good semi-norm on $H^0(P, \mathcal{F})$ induces a metric on $H^0(P^0, \mathcal{F})$.

PROOF. We need to show that if $|s| = 0$ for some $s \in H^0(P, \mathcal{F})$, then $s|_{P^0} = 0$, where P^0 is the analytic interior of P.

We use the same notations as in Definition 8.4. We can take $h \in H^0(Q, \mathcal{O}_{\mathbb{C}^n})^l$ and $h_j \in \ker \epsilon$ for each $j \in \mathbb{Z}_{>0}$ so that $\epsilon(h) = s$ and $\|h_j - h\|_{L^\infty} \to 0$. So $h_j|_Q \to h|_Q$ with respect to the compact-open topology. From Lemma 8.3, we conclude that the image of $h|_{\mathrm{Int}\, Q}$ is 0. Namely, s vanishes on $P^0 = \tau^{-1}(\mathrm{Int}\, Q)$. $\qquad\square$

Lemma 8.6. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n)$ be an analytic block on X with a non-zero associated tube $Q \subseteq \mathbb{C}^n$. Consider the epimorphism of sheaves

$$\mathcal{O}_Q^l \to \pi_*(\mathcal{F}|_P)$$

as in Definition 8.4 and endow $H^0(P^0, \mathcal{F})$ with the metric induced by the corresponding good semi-norm. Let

$$Q_1 \subseteq Q_2 \subseteq \cdots$$

be a compact exhaustion of $\mathrm{Int}\, Q$ by tubes with the same centers in $\mathbb{C}^n$. We get an induced map

$$\epsilon_j : H^0(Q_j, \mathcal{O}_{\mathbb{C}^n}^l) \to \pi_*(\mathcal{F}|_P)(Q_j)$$

for each $j \in \mathbb{Q}_{>0}$. We therefore get good semi-norms $|\bullet|_j$ on $H^0(P^0, \mathcal{F})$ for each $j \in \mathbb{Z}_{>0}$. Let

$$d(s_1, s_2) := \sum_{j=1}^\infty 2^{-j} \frac{|s_1 - s_2|_j}{1 + |s_1 - s_2|_j}$$

for each $s_1, s_2 \in H^0(P^0, \mathcal{F})$. Then d is a metric on $H^0(P^0, \mathcal{F})$ and $H^0(P^0, \mathcal{F})$ is a Fréchet space with respect to this topology.

Moreover, the topology does not depend on the choice of π, ϵ and the exhaustion.

PROOF. By Lemma 8.5, each $|\bullet|_\nu$ is a norm on $H^0(P^0, \mathcal{F})$. It follows that d is a metric. Next we show that $H^0(P^0, \mathcal{F})$ is Fréchet. Let $(s_j)_{j \in \mathbb{Z}_{>0}}$ be a Cauchy sequence in $H^0(P^0, \mathcal{F})$. We can find bounded sequences $(f_{jk} \in H^0(Q_k, \mathcal{O}^l_{\mathbb{C}^n}))_{k \in \mathbb{Z}_{>0}}$ so that $\epsilon_k(f_{jk}) = s_j|_{\pi^{-1}(Q_k) \cap P}$ $(k \in \mathbb{Z}_{>0})$ for each $j\mathbb{Z}_{>0}$. By Montel's theorem, there is a subsequence of $(f_{jk})_j$ which converges uniformly on Q_{k-1} to $f_k \in H^0(Q_{k-1}, \mathcal{O}^l_{\mathbb{C}^n})$. Then $\epsilon_{k-1}(f_{k+1})|_{\text{Int } Q_{k-1}} = \epsilon_{k-1}(f_k)|_{\text{Int } Q_{k-1}}$ for each $k \in \mathbb{Z}_{\geq 2}$. So we can glue the f_k's to $s \in H^0(P^0, \mathcal{F})$. Clearly, $s_k \to s$ as $k \to \infty$.

Next we show that the topology is independent of the choice of π, ϵ and the exhaustion. The independence of the exhaustion is obvious. We prove the other two independence. Let $(P, \pi' : X \to \mathbb{C}^{n'})$ be another analytic block with $\pi' = (\pi, \varphi) : X \to \mathbb{C}^n \times \mathbb{C}^m$, $n' = n + m$. Let $Q^* \subseteq \mathbb{C}^m$ be a tube such that $\varphi(P) \subseteq Q^*$. Then $P = \pi'^{-1}(Q \times Q^*) \cap U$. We can find an open neighbourhood U' of P in X and V' of $Q \times Q^*$ in $\mathbb{C}^{n'}$ for which the induced map $\tau' : U' \to V'$ is finite by Definition 6.1. Fix an epimorphism $\mathcal{O}^{l'}_{\mathbb{C}^{n'}}|_{Q \times Q^*} \to \pi'_*(\mathcal{F}|_P)$ for some $l' \in \mathbb{Z}_{>0}$. Construct an exhanstion of $\text{Int } Q \times \text{Int } Q^*$ of the product type: $(Q_j \times Q^*_j)_{j \in \mathbb{Z}_{>0}}$ as in the lemma. Let d' denote the induced metric on $H^0(\text{Int } P, \mathcal{F})$.

We will show that d' and d induce the same topology. Let $e_1, \ldots, e_l \in H^0(Q, \mathcal{O}^l_{\mathbb{C}^n})$ be the standard basis. Let $e'_1, \ldots, e'_l$ be the preimages of $\epsilon(e_1), \ldots, \epsilon(e_l) \in \pi_*(\mathcal{F}|P)(Q) = \pi'_*(\mathcal{F}|_P)(Q \times Q^*)$ in $\mathcal{O}_{\mathbb{C}^{n'}}(Q \times Q^*)^{l'}$ under ϵ'. Further, for $f \in \mathcal{O}_{\mathbb{C}^n}(Q_j)$, we denote by $f' \in \mathcal{O}_{\mathbb{C}^{n'}}(Q_j \times Q^*_j)$ the holomorphic extension of f to $Q_j \times Q^*_j$ constant along $\{q\} \times Q^*_j$ for each $q \in Q_j$ for each $j \in \mathbb{Z}_{>0}$. The norms of

$$\mathcal{O}_{\mathbb{C}^n}(Q_j)^l \to \mathcal{O}_{\mathbb{C}^{n'}}(Q_j \times Q^*_j)^l, \quad \sum_{i=1}^l f_i e_i \mapsto \sum_{i=1}^l f'_i e'_i$$

for $j \in \mathbb{Z}_{>0}$ are bounded by a constant independent of j. Therefore, the identity map

$$(H^0(P^0, \mathcal{F}), d) \to (H^0(P^0, \mathcal{F}), d')$$

is continuous. By open mapping theorem, the map is a homeomorphism. $\square$

Theorem 8.7. Let X be a complex analytic space and $(P, \pi) \subseteq (P', \pi')$ be two analytic blocks on X and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, then the restriction map

$$H^0(P', \mathcal{F}) \to H^0(P, \mathcal{F})$$

with respect to any good semi-norms.

PROOF. We claim that there exists an analytic block (P_1, π) such that

$$(P, \pi) \subseteq (P_1, \pi) \subseteq (P', \pi').$$

Assume this claim, then we have a decomposition of the restriction map

$$H^0(P', \mathcal{F}) \to H^0(P^0_1, \mathcal{F}) \to H^0(P, \mathcal{F}).$$

The first map is continuous if we endow $H^0(P^0_1, \mathcal{F})$ with the topology induced by π', the second is continuous if we endow $H^0(P^0_1, \mathcal{F})$ with the topology induced by π. These topologies are identical by Lemma 8.6. Our assertion follows.

To argue the claim, let us write $\pi : X \to \mathbb{C}^n$ and $\pi' = (\pi, \varphi) : X \to \mathbb{C}^n \times \mathbb{C}^m$. Take $q \in \mathbb{C}^m$ with $Q \times \{q\} \subseteq \text{Int } Q'$. Let $Q'' := Q' \cap (\mathbb{C}^n \times \{q\})$ and identify it with a subset of $\mathbb{C}^n$. Let Q^* be the image of Q' under the projection $\mathbb{C}^{n+m} \to \mathbb{C}^m$.

Choose open neighbourhoods $U \subseteq P'^0$, $V \subseteq Q'$ of P and Q respectively such that $\tau : U \to V$ is finite and $U \cap \pi^{-1}(Q) = P$. Take a tube $Q_1 \subseteq \mathbb{C}^n$ such that

$$Q \subseteq \operatorname{Int} Q_1 \subseteq Q_1 \subseteq \operatorname{Int} Q''.$$

Now it suffices to set $P_1 := \pi^{-1}(Q_1) \cap U$. $\hfill\square$

Corollary 8.8. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi) \subseteq (P', \pi')$ be analytic blocks in X. Then for any Cauchy sequence $(s_j)_{j \in \mathbb{Z}_{>0}}$ in $H^0(P', \mathcal{F})$, the restriction sequence $(s_j|_P)_{j \in \mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$.

PROOF. Choose an analytic block (P_1, π) such that

$$(P, \pi) \subseteq (P_1, \pi) \subseteq (P', \pi').$$

The existence of the block (P_1, π) is argued in the proof of Theorem 8.7. We have a decomposition of the restriction map

$$H^0(P', \mathcal{F}) \to H^0(P_1^0, \mathcal{F}) \to H^0(P, \mathcal{F}).$$

The first map is bounded, so the images of $(s_j)_{j \in \mathbb{Z}_{>0}}$ in $H^0(P_1^0, \mathcal{F})$ is a Cauchy sequence. As we have shown that $H^0(P_1^0, \mathcal{F})$ is a Fréchet space in Lemma 8.6, the sequence converges. As the second map is also continuous, it follows that $(s_j|_P)_{j \in \mathbb{Z}_{>0}}$ has a limit in $H^0(P, \mathcal{F})$. $\hfill\square$

Lemma 8.9. Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n) \subseteq (P', \pi' : X \to \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q'. Choose $U' \subseteq X$ and $V' \subseteq \mathbb{C}^{n+m}$ of P' and Q' respectively as in Definition 6.1 such that $U' \to V'$ is finite. Set

$$Q_1 := (Q \times \mathbb{C}^m) \cap Q', \quad P_1 = \pi'^{-1}(Q_1) \cap U'.$$

Then $(P_1. \pi')$ is an analytic block in X with block Q_1 and $H^0(P', \mathcal{F}) \to H^0(P_1, \mathcal{F})$ has dense image. Here we take an epsimorphism

$$\mathcal{O}_{\mathbb{C}^{n+m}}^{l'}|_{Q'} \to (\tau'(\mathcal{F}|_{U'}))_{Q'}$$

and it induces

$$\mathcal{O}_{\mathbb{C}^{n+m}}^{l'}|_{Q_1} \to (\tau'(\mathcal{F}|_{U'}))_{Q_1},$$

which in turn induces a good semi-norm on $H^0(P_1, \mathcal{F})$. This is the semi-norm we are using.

Moreover, there is a compact set $\tilde{P} \subseteq X$ disjoint from P such that

$$P_1 = P \cup \tilde{P}.$$

PROOF. We have a commutative diagram in the category of topological linear spaces:

$$
\begin{array}{ccc}
H^0(Q', \mathcal{O}_{\mathbb{C}^{m+n}}^l) & \longrightarrow & H^0(P', \mathcal{F}) \\
\downarrow & & \downarrow \\
H^0(Q_1, \mathcal{O}_{\mathbb{C}^{m+n}}^l) & \longrightarrow & H^0(P_1, \mathcal{F})
\end{array}.
$$

In order to show that the right vertical map has dense image, it is enough to show that the map on the left-hand side has dense images, which is the Runge approximation.

For the last assertion, as $Q_1 = (Q \times \mathbb{C}^m) \cap Q'$, we have

$$P_1 = \pi^{-1}(Q) \cap P'.$$

As $P \subseteq P'$ and $P \subseteq \pi^{-1}(Q)$, it follows that $P \subseteq P_1$. But there is an open neighbourhood U of P in X so that $P = \pi^{-1}(Q) \cap U$. Hence, $\tilde{P} = P_1 \setminus P$ is compact. $\qquad\square$

Theorem 8.10 (Runge approximation)**.** Let X be a complex analytic space and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $(P, \pi : X \to \mathbb{C}^n) \subseteq (P', \pi' : X \to \mathbb{C}^n \times \mathbb{C}^m)$ be analytic blocks in X with tubes Q and Q'. Then the map

$$H^0(P', \mathcal{F}) \to H^0(P, \mathcal{F})$$

has dense image with respect to a good semi-norm.

PROOF. We use the notations of Lemma 8.9. We extend Q, Q_1, Q' to tubes $\hat{Q}, \hat{Q}_1, \hat{Q}'$ and get $\hat{P}, \hat{P}_1, \hat{P}'$ corresponding to the original P, P_1, P'. The restriction map

$$H^0(\hat{P}_1^{\,0}, \mathcal{F}) \to H^0(\hat{P}^0, \mathcal{F})$$

is a continuous morphism of Fréchet spaces.

Let $s \in H^0(P, \mathcal{F})$ be a section. Lift s to $s_1 \in H^0(P_1, \mathcal{F})$. Up to a suitable modification of the tubes, we can extend s_1 to $\hat{s_1} \in H^0(\hat{P}_1, \mathcal{F})$. Then there is a sequence $(s^j \in H^0(\hat{P}', \mathcal{F}))_{j \in \mathbb{Z}_{>0}}$ such that $s^j|_{\hat{P}_1} \to \hat{s_1}$ as $j \to \infty$ in $H^0(\hat{P}_1, \mathcal{F})$. It follows that $s^j|_{\hat{P}^0} \to \hat{s}_1|_{\hat{P}^0}$ in $H^0(\hat{P}^0, \mathcal{F})$. It follows that $s^j|_P \to s_1|_P = s$ sa $j \to \infty$. $\qquad\square$

Theorem 8.11. Let X be a complex analytic space. Each exhaustion of X by analytic blocks is a Stein exhaustion.

PROOF. Let $(P_i, \pi_i)_{i \in \mathbb{Z}_{>0}}$ be an exhaustion of X by analytic blocks. Take a coherent $\mathcal{O}_X$-module $\mathcal{F}$.

We verify the conditions in Definition 5.5. By Theorem 5.4, P_i is a compact Stein set for each $i \in \mathbb{Z}_{>0}$. So (1) is satisfied.

On $H^0(P_i, \mathcal{F})$, we fix a good semi-norm $|\bullet|_i$ for each $i \in \mathbb{Z}_{>0}$. We may assume that $H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$ is contractive for $i \in \mathbb{Z}_{>0}$.

We have already verified (3), (4) and (5).

We verify (2). It suffices to show that

$$H^0(X, \mathcal{F}) \to H^0(P_1, \mathcal{F})$$

has dense image. Let $s \in H^0(P_1, \mathcal{F})$ and $\delta > 0$. By Theorem 8.10, we can find $s_i \in H^0(P_i, \mathcal{F})$ for $i \in \mathbb{Z}_{>0}$ such that $s_1 = s$,

$$|s_{i+1}|_{P_i} - s_i|_i < 2^{-i}\delta$$

for $i \in \mathbb{Z}_{>0}$. By Corollary 8.8, $(s_j|_{P_i})_{j \in \mathbb{Z}_{>0}}$ has a limit $t_i \in H^0(P_i, \mathcal{F})$ for each $i \in \mathbb{Z}_{>0}$. As $H^0(P_{i+1}, \mathcal{F}) \to H^0(P_i, \mathcal{F})$ is continuous for $i \in \mathbb{Z}_{>0}$, the $t_{i+1}|_{P_i}$'s are compatible and defines $t \in H^0(X, \mathcal{F})$. It is easy to see that $|t|_{P_1} - s|_1 < \delta$. Thus condition (2) is satisfied. $\qquad\square$

9. Stein spaces

Definition 9.1. Let X be a complex analytic space. We say that X is a Stein space if X is a Stein set in X and $|X|$ is paracompact and Hausdorff.

Definition 9.2. Let X be a complex analytic space. An *effective formal* 0-*cycle* on X consists of

(1) A disrete set $D \subseteq X$;
(2) An integer n_x for each $x \in D$.

We write the effective formal 0-cycle as $\sum_{x \in D} n_x x$. We define the *ideal sheaf* $\mathcal{O}_X(-\sum_{x \in D} n_x x)$ of an effective formal 0-cycle as $\sum_{x \in D} n_x x$ as

$$\mathcal{O}_X(-\sum_{x \in D} n_x x)(U) = \{ f \in H^0(U, \mathcal{O}_X) : f_x \in \mathfrak{m}_x^{n_x} \text{ for each } x \in D \cap U \}$$

for each open subset $U \subseteq X$.

Observe that $\mathcal{O}_X(-\sum_{x \in D} n_x x)$ is a coherent $\mathcal{O}_X$-module. In fact, the problem is local, so we may assume that D is finite. In this case, D is an effective 0-cycle and the result is clear.

Lemma 9.3. Let X be a complex analytic space and $\sum_{x \in D} n_x x$ be an effective formal 0-cycle on X. Assume that

$$H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X/\mathcal{O}_X(-\sum_{x \in D} n_x x))$$

is surjective. Suppose that for each $x \in D$, we assign $g_x \in \mathcal{O}_{X,x}$. Then there is $f \in H^0(X, \mathcal{O}_X)$ such that

$$f_x - g_x \in \mathfrak{m}_x^{n_x}$$

for all $x \in D$.

PROOF. We define $s \in H^0(X, \mathcal{O}_X/\mathcal{O}_X(-\sum_{x \in D} n_x x))$ by $s_x = g_x$ for each $x \in D$. Lift s to $f \in H^0(X, \mathcal{O}_X)$. Then f clearly satisfies the required properties. $\square$

Proposition 9.4. Let X be a complex analytic space. Assume that $H^1(X, \mathcal{I}) = 0$ for each coherent ideal sheaf $\mathcal{I}$ on X. Let $(x_i \in X)_{i \in \mathbb{Z}_{>0}}$ be a sequence without accumulation points and $(c_i)_{i \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{C}$. Then there is $f \in \mathcal{O}_X(X)$ with $f(x_i) = c_i$ for each $i \in \mathbb{Z}_{>0}$.

PROOF. Consider the formal cycle $\sum_{i=1}^{\infty} x_i$. Apply Lemma 9.3 with $g_{x_i} = c_i$. $\square$

Theorem 9.5. Let X be a complex analytic space. Then the following are equivalent:

(1) X is a Stein space;
(2) For any coherent ideal sheaf $\mathcal{I}$ on X, we have $H^1(X, \mathcal{I}) = 0$;
(3) X is holomorphically separable and holomorphically convex;
(4) X is holomorphically spreadable and weakly holomorphically convex;
(5) X is holomorphically complete.

PROOF. (1) $\implies$ (2): This is trivial.
(2) $\implies$ (3): X is holomorphicaly convex by Proposition 9.4 and Proposition 4.4. X is holomorphically separable by Proposition 9.4.

(3) $\implies$ (4): X is holomorphically spreadable and weakly holomorphically convex by definition.

(4) $\implies$ (5): This follows from Theorem 8.2 and Proposition 7.4.

(5) $\implies$ (1):This follows from Theorem 8.11 and Theorem 5.6. $\qquad\square$

Corollary 9.6. Let X be a complex analytic space. Then the following are equivalent:

(1) X is Stein;

(2) X^{red} is Stein;

(3) The normalization $\overline{X^{\mathrm{red}}}$ is Stein.

The equivalence of (1) and (2) is due to Grauert PIHES. The difficult direction (3) implies (2) is claimed in Grauert–Remmert, where the proof is nonsense. We follow the argument of Narasimhan here.

PROOF. (1) $\implies$ (2): This follows from Leray's spectral sequence.

(2) $\implies$ (1): By Theorem 9.5(3), it suffices to show that the restriction map $H^0(X, \mathcal{O}_X) \to H^0(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$ is surjective.

Let $\mathcal{I}$ be the nilradical of $\mathcal{O}_X$. It is coherent by Cartan–Oka theorem. For each $i \in \mathbb{Z}_{>0}$, we have a short exact sequence

$$0 \to \mathcal{I}^i/\mathcal{I}^{i+1} \to \mathcal{O}_X/\mathcal{I}^{i+1} \to \mathcal{O}_X/\mathcal{I}^i \to 0.$$

As $\mathcal{I}^i/\mathcal{I}^{i+1}$ is a coherent $\mathcal{O}_{X^{\mathrm{red}}}$-module, we conclude that

$$\varphi_i : H^0(X, \mathcal{O}_X/\mathcal{I}^{i+1}) \to H^0(X, \mathcal{O}_X/\mathcal{I}^i)$$

is surjective for each $i \in \mathbb{Z}_{>0}$. Let $h_1 \in H^0(X, \mathcal{O}_X/\mathcal{I}) = H^0(X^{\mathrm{red}}, \mathcal{O}_{X^{\mathrm{red}}})$. We want to lift it to $h \in H^0(X, \mathcal{O}_X)$.

We successively lift h_1 to $h_i \in H^0(X, \mathcal{O}_X/\mathcal{I}^i)$ for each $i \in \mathbb{Z}_{>0}$. Let $X_i = X \setminus \mathrm{Supp}\,\mathcal{I}^i$ of each $i \in \mathbb{Z}_{>0}$. Then clearly

$$X = \bigcup_{i=1}^{\infty} X_i.$$

It is easy to see that

$$h_{i+1}|_{X_i} = h_i|_{X_i}$$

for each $i \in \mathbb{Z}_{>0}$. It follows that we can glue the $h_i|_{X_i}$'s to $h \in H^0(X, \mathcal{O}_X)$ which restricts to h_1.

(2) $\implies$ (3): This follows from Leray's spectral sequence as $\overline{X^{\mathrm{red}}} \to X^{\mathrm{red}}$ is finite by **??** in **??**.

(3) $\implies$ (2): Reproduce Narasimhan's work $\qquad\square$

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