

## Topology and bornology



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## 1. Introduction

In the whole project, a neighbourhood in a topology space is taken in Bourbaki's sense. In particular, a neighbourhood is not necessarily open.

We follow Bourbaki's convention about compact space. A compact space is always Hausdorff.

On the other hand, we do not require locally compact spaces and paracompact spaces be Hausdorff.

References to this chapter include [Ber93].

## 2. Nets

Let  $X$  be a set,  $Y \subseteq X$  be a subset. Consider a collection  $\tau$  of subsets of  $X$ , we write

$$\tau|_Y := \{V \in \tau : V \subseteq Y\}.$$

**Definition 2.1.** Let  $X$  be a topology space and  $\tau$  be a collection of subsets of  $X$ .

- (1) We say  $\tau$  is *dense* if for any  $V \in \tau$  and any  $x \in V$ , there is a fundamental system of neighbourhoods of  $x$  in  $V$  consisting of sets from  $\tau|_V$ .
- (2) We say  $\tau$  is a *quasi-net* on  $X$  if for each  $x \in X$ , there exist  $n \in \mathbb{Z}_{>0}$ ,  $V_1, \dots, V_n \in \tau$  such that  $x \in V_1 \cap \dots \cap V_n$  and that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ .
- (3) We say  $\tau$  is a *net* on  $X$  if it is a quasi-net and if for any  $U, V \in \tau$ ,  $\tau|_{U \cap V}$  is a quasi-net on  $U \cap V$ .

We observe that if  $\tau$  is a net,  $\tau|_{U \cap V}$  is in fact a net.

**Lemma 2.2.** Let  $X$  be a topological space and  $\tau$  be a quasi-net on  $X$ .

- (1) A subset  $U \subseteq X$  is open if and only if for each  $V \in \tau$ ,  $U \cap V$  is open in  $V$ .
- (2) Suppose that  $\tau$  consists of compact sets. Then  $X$  is Hausdorff if and only if for any  $U, V \in \tau$ ,  $U \cap V$  is compact.

We remind the readers that a compact space is Hausdorff by our convention.

**PROOF.** (1) The direct implication is trivial. Suppose that  $U \cap V$  is open in  $V$  for all  $V \in \tau$ . We want to show that  $U$  is open. Take  $x \in U$ , we can find  $n \in \mathbb{Z}_{>0}$ ,  $V_1, \dots, V_n \in \tau$  all containing  $x$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ . By our hypothesis, we can find open sets  $W_1, \dots, W_n$  in  $W$  such that  $W \cap V_i = U \cap V_i$  for  $i = 1, \dots, n$ . Then  $W = W_1 \cap \dots \cap W_n$  is an open neighbourhood of  $x$  in  $X$ . But then

$$U \cap (V_1 \cup \dots \cup V_n) \supseteq W \cap (V_1 \cup \dots \cup V_n),$$

the latter is a neighbourhood of  $x$  hence so is the former. It follows that  $U$  is open.

(2) The direct implication is trivial. Consider the quasi-net  $\tau \times \tau := \{U \times V : U, V \in \tau\}$  on  $X \times X$ . By (1), it suffices to verify that the intersection of the diagonal with  $U \times V$  is closed in  $U \times V$  for any  $U, V \in \tau$ . But this intersection is homeomorphic to  $U \cap V$ , which is compact by our assumption and hence closed as  $U, V$  are both Hausdorff.  $\square$

**Lemma 2.3.** Let  $X$  be a Hausdorff space. Assume that  $X$  admits a quasi-net  $\tau$  consisting of compact sets. Then  $X$  is locally compact.

PROOF. Take  $x \in X$ . By assumption, we can find  $n \in \mathbb{N}$  and  $V_1, \dots, V_n \in \tau$  all containing  $x$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$ . This neighbourhood is clearly compact.  $\square$

**Lemma 2.4.** Let  $X$  be a Hausdorff space and  $\tau$  be a collection of compact subsets of  $X$ . Then the following are equivalent:

- (1)  $\tau$  is a quasi-net;
- (2) For each  $x \in X$ , there are  $n \in \mathbb{N}$  and  $V_1, \dots, V_n \in \tau$  such that  $V_1 \cup \dots \cup V_n$  is a neighbourhood of  $x$  in  $X$ .

PROOF. (1)  $\implies$  (2): This is trivial.

(2)  $\implies$  (1): Given  $x \in X$ , take  $V_1, \dots, V_n$  as in (2). We may assume that  $x \in V_1, \dots, V_m$  and  $x \notin V_{m+1}, \dots, V_n$  for some  $1 \leq m \leq n$ . Then  $V_1 \cup \dots \cup V_m$  is a neighbourhood of  $x$  in  $X$ : if  $U$  is an open neighbourhood of  $x$  in  $X$  contained in  $V_1 \cup \dots \cup V_n$ , then  $U \setminus (V_{m+1} \cup \dots \cup V_n)$  is an open neighbourhood of  $x$  in  $X$  contained in  $V_1 \cup \dots \cup V_m$ .  $\square$

**Lemma 2.5.** Let  $X$  be a topological space and  $\tau$  be a net on  $X$  consisting of compact sets. Then

- (1) for any pair  $U, V \in \tau$ , the intersection  $U \cap V$  is locally closed in  $U$  and in  $V$ ;
- (2) If  $n \in \mathbb{Z}_{>0}$ ,  $V, V_1, \dots, V_n \in \tau$  are such that

$$V \subseteq V_1 \cup \dots \cup V_n,$$

then there are  $m \in \mathbb{Z}_{>0}$  and  $U_1, \dots, U_m \in \tau$  such that

$$V = U_1 \cup \dots \cup U_m$$

and each  $U_j$  is contained in some  $V_i$ .

PROOF. (1) It suffices to show that  $U \cap V$  is locally compact in the induced topology. This follows from [Lemma 2.3](#).

(2) For each  $x \in V$  and each  $i = 1, \dots, n$  such that  $x \in V_i$ , we take a neighbourhood of  $x$  in  $V \cap V_i$  of the form  $W_i V_{i1} \cup \dots \cup V_{im_i}$  for some  $m_i \in \mathbb{Z}_{>0}$  and  $V_{ij} \in \tau$  for  $j = 1, \dots, m_i$ . Then the union of all  $W_i$ 's is a neighbourhood of  $x$  of the form  $U_1 \cup \dots \cup U_m$ , where  $U_j$  belongs to  $\tau$  and is contained in some  $V_i$ . Using the compactness of  $V$ , we conclude.  $\square$

### 3. Bornology

**Definition 3.1.** Let  $X$  be a set. A *bornology* on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  such that

- (1) For any  $x \in X$ , there is  $B \in \mathcal{B}$  such that  $x \in B$ ;
- (2) For any  $B \in \mathcal{B}$  and any subset  $A \subseteq B$ ,  $A \in \mathcal{B}$ ;
- (3)  $\mathcal{B}$  is stable under finite union.

The pair  $(X, \mathcal{B})$  is called a *bornological set*. The elements of  $\mathcal{B}$  are called the *bounded subsets* of  $(X, \mathcal{B})$ . When  $\mathcal{B}$  is obvious from the context, we omit it from the notations.

A morphism between bornological sets  $(X, \mathcal{B}_X)$  and  $(Y, \mathcal{B}_Y)$  is a map of sets  $f : X \rightarrow Y$  such that for any  $A \in \mathcal{B}_X$ ,  $f(A) \in \mathcal{B}_Y$ . Such a map is called a *bounded map*.

**Definition 3.2.** Let  $(X, \mathcal{B})$  be a bornological set. A *basis* for  $\mathcal{B}$  is a subset  $\mathcal{A} \subseteq \mathcal{B}$  such that for any  $B \in \mathcal{B}$ , there are  $A_1, \dots, A_n \in \mathcal{A}$  such that  $B \subseteq A_1 \cup \dots \cup A_n$ .

## Bibliography

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