Commutative algebra

Contents

1. Power ring series

Definition 1.1. For any $n \in \mathbb{N}$, let $\mathbb{C}\langle z_1, \dots, z_n\rangle$ denote the subring of $\mathbb{C}[[z_1, \dots, z_n]]$:

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},$$

which is convergent in a neighbourhood of 0: there is $\epsilon > 0$ such that for any $x_1, \dots, x_n \in \mathbb{C}$ with $|x_i| < \epsilon$, $\sum_\alpha a_\alpha x^\alpha$ is a convergent power series.

Theorem 1.2. *For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is local, and the maximal ideal is given by convergent power series without constant terms.*

Theorem 1.3. *For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is noetherian and factorial.*

Theorem 1.4. *For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is strictly Henselian.*

2. Complex analytic local algebras

I do not know the "official" translation of Analytische Stellenalgebren, so I choose the term complex analytic local algebra. Please contact me if you have a better name.

Definition 2.1. A *complex analytic local algebra* is a $\mathbb{C}$-algebra A such that $A \neq 0$ and there exists some $n \in \mathbb{N}$ and an ideal I in $\mathbb{C}\langle z_1, \dots, z_n\rangle$ such that

$$A \cong \mathbb{C}\langle z_1, \dots, z_n\rangle / I$$

as $\mathbb{C}$-algebras.

A morphism between complex analytic local algebras A and B is a $\mathbb{C}$-algebra homomorphism $A \to B$.

The category of complex analytic local algebras is denoted by $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$.

Observe that a complex analytic local algebra is always local with residue field $\mathbb{C}$ and a morphism in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ is always a local homomorphism. We will write $\mathfrak{m}_A$ for the maximal ideal in A.

Lemma 2.2. *Let A be a complex analytic local algebra and $n \in \mathbb{N}$, then there is a natural bijection*

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\langle z_1, \dots, z_n\rangle, A) \cong \mathfrak{m}_A^n$$

sending a morphism f to $(f(z_1), \dots, f(z_n))$.

Proof. As a morphism $f : \mathbb{C}\langle z_1, \dots, z_n\rangle \to A$ is necessarily local, we see that $f(z_i) \in A$ for all $i = 1, \dots, n$. So the map $\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\langle z_1, \dots, z_n\rangle, A) \to \mathfrak{m}_A^n$ is well-defined. Conversely, given $w_1, \dots, w_n \in \mathfrak{m}_A$, we claim that there is a unique morphism $f : \mathbb{C}\langle z_1, \dots, z_n\rangle \to A$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ sending z_i to w_i.

The uniqueness is easy Add details, so let us consider only the existence. Let $\mathbb{C}\langle z_1, \dots, z_m\rangle \to A$ be a surjective morphism. Lift w_i to $w_i' \in \mathbb{C}\langle z_1, \dots, z_m\rangle$, it suffices to construct a morphism $\mathbb{C}\langle z_1, \dots, z_n\rangle \to \mathbb{C}\langle z_1, \dots, z_m\rangle$ sending z_i to w_i'. So we may assume that $A = \mathbb{C}\langle z_1, \dots, z_m\rangle$. □

Definition 2.3. Let A_1, A_2 be complex analytic local algebras, *an analytic tensor product* of A_1 and A_2 is a complex analytic local algebra A together with morphisms $A_1 \to A$ and $A_2 \to A$ such that for any complex analytic local algebra C, the induced map

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A, C) \to \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_1, C) \times \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_2, C)$$

is bijective.

As analytic tensor product is unique up to unique isomorphism, so we can choose a specific analytic tensor product $A_1 \overline{\otimes} A_2$ and call it *the analytic tensor product* of A_1 and A_2.

By definition, there are natural morphisms

$$A_1 \otimes A_2 \to A_1 \overline{\otimes} A_2.$$

The simplest example is

Lemma 2.4. *For any $m, n \in \mathbb{N}$, we have*

$$\mathbb{C}\langle z_1, \ldots, z_m \rangle \overline{\otimes} \mathbb{C}\langle z_1, \ldots, z_n \rangle \cong \mathbb{C}\langle z_1, \ldots, z_{m+n} \rangle$$

as complex analytic local algebras.

Proof. $\square$

Lemma 2.5. *Assume that $f_i : A_i \to B_i$ are surjective (i.e. the underlying homomorphisms of algebras are surjective) morphisms in $\mathbb{C}$-$\mathcal{L}$A for $i = 1, 2$. Let I_i be the kernel of f_i as homomorphisms of algebras. If $A_1 \overline{\otimes} A_2$ exists, then so is $B_1 \overline{\otimes} B_2$ and*

$$B_1 \overline{\otimes} B_2 \cong A_1 \overline{\otimes} A_2 / (I_1 \otimes 1 + 1 \otimes I_2)(A_1 \overline{\otimes} A_2).$$

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Bibliography

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.