Affinoid algebras

Contents

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 2.1. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. We set

$$k\{r^{-1}T\} = k\{r_1^{-1}T_1, \dots, r_nT_n^{-1}\} := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] : a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ as } |\alpha| \to \infty \right\}.$$

For any $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha T^\alpha \in k\{r^{-1}T\}$, we set

$$\|f\|_r = \max_\alpha |a_\alpha| r^\alpha.$$

We call $(k\{r^{-1}T\}, \|\bullet\|_r)$ the *Tate algebra* in n-variables with radii r. The norm $\|\bullet\|_r$ is called the *Gauss norm*.

We omit r from the notation if $r = (1, \dots, 1)$.

This is a special case of Example 4.8 in the chapter Banach Rings.

Proposition 2.2. Let $n \in \mathbb{N}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. Then the Tate algebra $(k\{r^{-1}T\}, \|\bullet\|_r)$ is a Banach k-algebra and $\|\bullet\|_r$ is a valuation.

Proof. This is a special case of Proposition 4.9 in the chapter Banach Rings. □

Remark 2.3. One should think of $k\{r^{-1}T\}$ as analogues of $\mathbb{C}\langle r^{-1}T\rangle$ in the theory of complex analytic spaces. We could have studied complex analytic spaces directly from the Banach rings $\mathbb{C}\langle r^{-1}T\rangle$, as we will do in the rigid world. But in the complex world, the miracle is that we have *a priori* a good theory of functions on all open subsets of the unit polydisk, so things are greatly simplified. The unit polydisk is a ringed space for free.

As we will see, constructing a good function theory, or more precisely, enhancing the unit disk to a ringed site is the main difficulty in the theory of rigid spaces. And Tate's innovation comes in at this point.

Example 2.4. Assume that the valuation on k is trivial.

Let $n \in \mathbb{N}$ and $r \in \mathbb{R}^n_{>0}$. Then $k\{r^{-1}T\} \cong k[T_1, \dots, T_n]$ if $r_i \geq 1$ for all i and $k\{r^{-1}T\} \cong k[[T_1, \dots, T_n]]$ otherwise.

Let $(k, |\bullet|)$ be a complete non-Archimedean valued-field.

Definition 3.1. A Banach k-algebra A is *k-affinoid* (resp. *strictly k-affinoid*) if there are $n \in \mathbb{N}$, $r \in \mathbb{R}^n_{>0}$ and an admissible epimorphism $k\{r^{-1}T\} \to A$ (resp. an admissible epimorphism $k\{T\} \to A$).

An affinoid k-algebra is a K-affinoid algebra for some complete non-Archimedean field extension K/k.

For the notion of admissible morphisms, we refer to Definition 2.5 in the chapter Banach rings.

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Bibliography

[Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.