

## Commutative algebra



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## 1. Introduction

## 2. Graded commutative algebra

Let  $G$  be an Abelian group. We write the group operation of  $G$  multiplicatively and denote the identity of  $G$  as 1.

**Definition 2.1.** Let  $A$  be an Abelian group. A  $G$ -grading on  $A$  is a coproduct decomposition

$$A = \bigoplus_{g \in G} A_g$$

of Abelian groups such that  $A_g \subseteq A$ . An Abelian group with a  $G$ -grading is called a  $G$ -graded Abelian group.

An element  $a \in A$  is said to be *homogeneous* if there is  $g \in G$  such that  $a \in A_g$ . If  $a$  is furthermore non-zero, we write  $g = \rho(a)$ . We set  $\rho(0) = 0$ . We will write  $\rho(A)$  for the set of  $\rho(a)$  when  $a$  runs over all homogeneous elements in  $A$ .

A  $G$ -graded homomorphism between  $G$ -graded Abelian groups  $A$  and  $B$  is a homogeneous of the underlying Abelian groups  $f : A \rightarrow B$  such that  $f(A_g) \subseteq B_g$  for any  $g \in G$ .

The category of  $G$ -graded Abelian groups is denoted by  $\mathcal{Ab}^G$ .

**Remark 2.2.** A usual Abelian group  $A$  can be given the *trivial  $G$ -grading*:  $A_0 = A$  and  $A_g = 0$  for  $g \in G, g \neq 0$ . In this way, we find a fully faithful embedding

$$\mathcal{Ab} \rightarrow \mathcal{Ab}^G.$$

When we regard an Abelian group as a  $G$ -graded Abelian group and there are no natural gradings, we always understand that we are taking the trivial  $G$ -grading.

More generally, let  $G'$  be a subring of  $G$ . Then any  $G'$ -graded Abelian group can be canonically identified with a  $G$ -graded Abelian group: for the extra pieces in the grading, we simply put 0.

The same remark applies to all the other constructions in this section, which we will not repeat.

**Definition 2.3.** A  $G$ -graded ring is a commutative ring  $A$  endowed with a  $G$ -grading:

$$A = \bigoplus_{g \in G} A_g$$

as Abelian groups and such that

- (1)  $A_g A_h \subseteq A_{gh}$  for any  $g, h \in G$ ;
- (2)  $1 \in A_1$ .

A  $G$ -graded homomorphism of  $G$ -graded rings  $A$  and  $B$  is a ring homomorphism  $f : A \rightarrow B$  such that  $f(A_g) \subseteq B_g$  for each  $g \in G$ . A  $G$ -graded subring of a  $G$ -graded ring  $B$  is a subring  $A$  of  $B$  such that the grading on  $B$  restricts to a grading on  $A$ .

The category of  $G$ -graded rings is denoted by  $\mathcal{Ring}^G$ .

**Example 2.4.** Let  $A$  be a  $G$ -graded ring,  $n \in \mathbb{N}$  and  $g = (g_1, \dots, g_n) \in G^n$ . Then there is a unique  $G$ -grading on  $A[T_1, \dots, T_n]$  extending the grading on  $A$  and such that  $\rho(T_i) = g_i$  for  $i = 1, \dots, n$ . We will denote  $A[T_1, \dots, T_n]$  with this grading as  $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n]$  or simply  $A[g^{-1}T]$ .

**Example 2.5.** Let  $A$  be a  $G$ -graded ring and  $S$  be a multiplicative subset of  $A$  consisting of homogeneous elements, then  $S^{-1}A$  has a natural  $G$ -grading. To see this,

recall the construction of  $S^{-1}A$  in [Stacks, Tag 00CM]. One defines an equivalence relation on  $A \times S$ :  $(x, s) \sim (y, t)$  if there is  $u \in S$  such that  $(xt - ys)u = 0$ . For each  $g \in G$ , we define  $(S^{-1}A)_g$  as the set of  $(x, s)$  for all  $s \in S$  and  $x \in A_{g\rho(s)}$ . It is easy to verify that this is a well-defined  $G$ -grading on  $S^{-1}A$ . **Add details.**

In particular, if  $f \in A$  is a non-zero homogeneous element, then we define  $A_f$  as  $S^{-1}f$  with  $S = \{f^n : n \in \mathbb{N}\}$ .

**Definition 2.6.** Let  $A$  be a  $G$ -graded ring. A  $G$ -homogeneous ideal in  $A$  is an ideal  $I$  in  $G$  such that if  $a \in A$  can be written as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all  $a_g = 0$ , then  $a_g \in I$ .

**Example 2.7.** Let  $A$  be a  $G$ -graded ring and  $n \in \mathbb{N}$  and  $a_1, \dots, a_n$  be homogeneous elements in  $A$ . Then  $a_1, \dots, a_n$  generate a  $G$ -homogeneous ideal  $(a_1, \dots, a_n)$  as follows:

$$(a_1, \dots, a_n)_g = \sum_{i=1}^n A_{g\rho(a_i)^{-1}} a_i$$

for any  $g \in G$ .

**Lemma 2.8.** Let  $f : A \rightarrow B$  be a  $G$ -homomorphism of  $G$ -graded rings. Then  $\ker f$  is a  $G$ -homogeneous ideal in  $A$ .

PROOF. We need to show that

$$\ker f = \sum_{g \in G} (\ker f) \cap A_g.$$

Take  $x \in \ker f$ , we can write  $x$  as

$$\sum_{g \in G} a_g, \quad a_g \in A_g$$

and almost all  $a_g$ 's are 0. Then

$$f(x) = \sum_{g \in G} f(a_g), \quad f(a_g) \in B_g.$$

It follows that  $f(a_g) = 0$  for each  $g \in G$  and hence  $a_g \in (\ker f) \cap A_g$ .  $\square$

**Definition 2.9.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then we define a  $G$ -grading on  $A/I$  as follows: for any  $g \in G$

$$(A/I)_g := (A_g + I)/I.$$

**Proposition 2.10.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then the construction in Definition 2.9 defines a grading on  $A/I$ . The natural map  $\pi : A \rightarrow A/I$  is a  $G$ -homomorphism.

For any  $G$ -graded ring  $B$  and any  $G$ -homomorphism  $f : A \rightarrow B$  such that  $I \subseteq \ker A$ , there is a unique  $G$ -homomorphism  $f' : A/I \rightarrow B$  such that  $f' \circ \pi = f$ .

PROOF. We first argue that for different  $g, h \in G$ ,  $(A/I)_g \cap (A/I)_h = 0$ . Suppose  $x \in (A/I)_g \cap (A/I)_h$ , we can lift  $x$  to both  $y_g + i_g \in A$  and  $y_h + i_h \in A$  with  $y_g, y_h \in A$  and  $i_g, i_h \in I$ . It follows that  $y_g - y_h \in I$ . But  $I$  is a  $G$ -homogeneous ideal, so it follows that  $y_g, y_h \in I$  and hence  $x = 0$ .

Next we argue that

$$A/I = \sum_{g \in G} (A/I)_g.$$

Lift an element  $x \in A/I$  by  $a \in A$ , we represent  $a$  as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all  $a_g$ 's equal to 0. Then  $x$  can be represented as

$$x = \sum_{g \in G} \pi(a_g).$$

We have shown that the construction in [Definition 2.9](#) gives a  $G$ -grading on  $A$ . It is clear from the definition that  $\pi$  is a  $G$ -homomorphism.

Next assume that  $B$  and  $f$  are given as in the proposition. Then there is a ring homomorphism  $f' : A/I \rightarrow B$  such that  $f = f' \circ \pi$ . We need to argue that  $f'$  is a  $G$ -homomorphism. For this purpose, take  $g \in G$ ,  $x \in (A/I)_g$ , we need to show that  $f'(x) \in B_g$ . Lift  $x$  to  $y + i$  with  $y \in A_g$  and  $i \in I$ , then we know that  $f'(x) = \pi(y + i) = \pi(y) \in B_g$ .  $\square$

**Definition 2.11.** Let  $A$  be a  $G$ -graded ring.

Let  $M$  an  $A$ -module which is also a  $G$ -graded Abelian group. We say  $M$  is a  $G$ -graded  $A$ -module if for each  $g, h \in G$ , we have

$$A_g M_h \subseteq M_{gh}.$$

A  $G$ -graded homomorphism of  $G$ -graded  $A$ -modules  $M$  and  $N$  is an  $A$ -module homomorphism  $f : M \rightarrow N$  which is at the same time a homomorphism of the underlying  $G$ -graded Abelian groups.

The category of  $G$ -graded  $A$ -modules is denoted by  $\text{Mod}_A^G$ .

A  $G$ -graded  $A$ -algebra is a  $G$ -graded ring  $B$  together with a  $G$ -graded ring homomorphism  $A \rightarrow B$  such that  $B$  is also a  $G$ -graded  $A$ -module.

A  $G$ -graded homomorphism between  $G$ -graded  $A$ -algebras  $B$  and  $C$  is a  $G$ -graded homomorphism between the underlying  $G$ -graded rings that is at the same time a  $G$ -graded homomorphism of  $G$ -graded  $A$ -modules.

Observe that  $G$ -homogeneous ideals of  $A$  are  $G$ -graded submodules of  $A$ . Also observe that  $\text{Mod}_{\mathbb{Z}}^G$  is isomorphic to  $\text{Ab}^G$ .

**Proposition 2.12.** Let  $A$  be a  $G$ -graded ring. Then  $\text{Mod}_A^G$  is an Abelian category satisfying AB5.

PROOF. We first show that  $\text{Mod}_A^G$  is preadditive. Given  $M, N \in \text{Mod}_A^G$ , we can regard  $\text{Hom}_{\text{Mod}_A^G}(M, N)$  as a subgroup of  $\text{Hom}_A(M, N)$ . It is easy to see that this gives  $\text{Mod}_A^G$  an enrichment over  $\text{Ab}$ .

Next we show that  $\text{Mod}_A^G$  is additive. The zero object is clearly given by 0 with the trivial grading. Given  $M, N \in \text{Mod}_A^G$ , we define

$$(M \oplus N)_g := M_g \oplus N_g, \quad g \in G.$$

This construction makes  $M \oplus N$  a  $G$ -graded  $A$ -module. It is easy to verify that  $M \oplus N$  is the biproduct of  $M$  and  $N$ .

Next we show that  $\text{Mod}_A^G$  is pre-Abelian. Given an arrow  $f : M \rightarrow N$  in  $\text{Mod}_A^G$ , we need to define its kernel and cokernel. We define

$$(\ker f)_g := (\ker f) \cap M_g$$

and  $(\text{coker } f)_g$  as the image of  $N_g$  for any  $g \in G$ . It is straightforward to verify that these are kernels and cokernels.

Next, given a monomorphism  $f : M \rightarrow N$ , it is obvious that the map  $f$  is injective and  $f$  can be identified with the kernel of the natural map  $N/\text{Im } f$ . A dual argument shows that an epimorphism is the cokernel of some morphism as well.

The AB5 condition is easily verified. **Expand the details of this argument!**  $\square$

Next we define the tensor product of  $G$ -graded modules.

**Definition 2.13.** Let  $A$  be a  $G$ -graded ring and  $M, N$  be  $G$ -graded  $A$ -modules. We define a  $G$ -grading on  $M \otimes_A N$  as follows: for any  $g \in G$ ,  $(M \otimes_A N)_g$  is defined as the image of  $\sum_{h \in G} M_h \times N_{gh^{-1}}$  in  $M \otimes_A N$ . We always endow  $M \otimes_A N$  with this  $G$ -grading.

**verify the universal property; show that this is indeed a grading**

**Example 2.14.** This is a continuation of **Example 2.5**. Let  $A$  be a  $G$ -graded ring and  $S$  be a multiplicative subset of  $A$  consisting of homogeneous elements. Consider a  $G$ -graded  $A$ -module  $M$ . We define a  $G$ -grading on  $S^{-1}M$ . Recall that  $S^{-1}M$  can be realized as follows: one defines an equivalence relation on  $M \times S$ :  $(x, s) \sim (y, t)$  if there is  $u \in S$  such that  $(xt - ys)u = 0$ . For each  $g \in G$ , we define  $(S^{-1}M)_g$  as the set of  $(x, s)$  for all  $s \in S$  and  $x \in M_{gp(s)}$ . It is easy to verify that this is a well-defined  $G$ -grading on  $S^{-1}M$  and  $S^{-1}M$  is a  $G$ -graded  $S^{-1}A$ -module. **Add details.**

**Example 2.15.** Let  $A$  be a  $G$ -graded ring and  $g \in G$ . We define  $g^{-1}A$  as the  $G$ -graded  $A$ -module:

$$(g^{-1}A)_h = A_{g^{-1}h}$$

for any  $h \in G$ . Observe that  $1 \in (g^{-1}A)_g$ .

**Definition 2.16.** Let  $A$  be a  $G$ -graded ring and  $M$  be a  $G$ -graded  $A$ -module. We say  $M$  is *free* if there exists a family  $\{g_i\}_{i \in I}$  in  $G$  such that

$$M = \coprod_{i \in I} g_i^{-1}A.$$

**Definition 2.17.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. We say  $f$  is *finite* (resp. *finitely generated*, resp. *integral*) if it is finite (resp. finitely generated, resp. integral) as a usual ring map.

**Proposition 2.18.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. Then

- (1)  $f$  is finite if and only if there are  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G$  and a surjective  $G$ -graded homomorphism

$$\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$$

of graded  $A$ -modules.

- (2)  $f$  is finitely generated if and only if there are  $n \in \mathbb{N}$ ,  $g_1, \dots, g_n \in G$  and a surjective  $G$ -graded  $A$ -algebra homomorphism

$$A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B.$$

- (3)  $f$  is integral if and only if for any non-zero homogeneous element  $b \in B$ , there is  $n \in \mathbb{N}$  and homogeneous elements  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

- (4) A non-zero homogeneous element  $b \in B$  is integral over  $A$  if there is  $n \in \mathbb{N}$  and homogeneous elements  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

PROOF. (1) The non-trivial direction is the direct implication. Assume that  $f$  is finite. Take  $b_1, \dots, b_n \in B$  so that  $\sum_{i=1}^n f(A)b_i = B$ . Up to replacing the collection  $\{b_i\}_i$  by the finite collection of non-zero homogeneous components of the  $b_i$ 's, we may assume that each  $b_i$  is homogeneous. We define  $g_i = \rho(b_i)$  and the map  $\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$  sends 1 at the  $i$ -th place to  $b_i$ .

(2) The non-trivial direction is the direct implication. Suppose  $f$  is finitely generated, say by  $b_1, \dots, b_n$ . Up to replacing the collection  $\{b_i\}_i$  by the finite collection of non-zero homogeneous components of the  $b_i$ 's, we may assume that each  $b_i$  is homogeneous. Then we define  $g_i = \rho(b_i)$  for  $i = 1, \dots, n$  and the  $A$ -algebra homomorphism  $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B$  sends  $T_i$  to  $b_i$  for  $i = 1, \dots, n$ .

(3) Assume that  $f$  is integral, then for any non-zero homogeneous element  $b \in B$ , we can find  $a_1, \dots, a_n \in A$  such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

Obviously, we can replace  $a_i$  by its component in  $\rho(b)^i$  for  $i = 1, \dots, n$  and the equation remains true.

The reverse direction follows from [\[Stacks, Tag 00GO\]](#).

(4) This is argued in the same way as (3). □

**Definition 2.19.** A  $G$ -graded ring  $A$  is a  *$G$ -graded field* if

- (1)  $A \neq 0$ .
- (2)  $A$  does not admit any non-zero proper  $G$ -homogeneous ideals.

**Proposition 2.20.** Let  $A$  be a non-zero  $G$ -graded ring. Then the following conditions are equivalent:

- (1)  $A$  is a  $G$ -graded field.
- (2) Any non-zero homogeneous element in  $A$  is invertible.

PROOF. Assume that  $A$  is a  $G$ -graded field. Let  $a \in A$  be a non-zero homogeneous element. Consider the  $G$ -homogeneous ideal  $(a)$  generated by  $a$  as in [Example 2.7](#). As  $a \neq 0$ , it follows that  $(a) = 1$ . Hence,  $a$  is invertible.

Conversely, suppose that any non-zero homogeneous element in  $A$  is invertible. If  $I$  is a non-zero  $G$ -homogeneous ideal in  $A$ . There is a non-zero homogeneous element  $a \in I$ . But we know that  $a$  is invertible and hence  $I = A$ . □

**Definition 2.21.** A  $G$ -graded ring  $A$  is an *integral domain* if for any non-zero homogeneous elements  $a, b \in A$ ,  $ab \neq 0$ .



**Lemma 2.22.** Let  $A$  be a  $G$ -graded integral domain. Let  $S$  denote the set of non-zero homogeneous elements in  $A$ . Then  $S^{-1}A$  is a graded field. The natural map  $A \rightarrow S^{-1}A$  is injective.

Recall that  $S^{-1}A$  is defined in [Example 2.5](#).

PROOF. By [Proposition 2.20](#), it suffices to show that each non-zero homogeneous element in  $S^{-1}A$  is invertible. Such an element has the form  $a/s$  for some homogeneous element  $a \in A$  and  $s \in S$ . As  $A$  is a  $G$ -graded integral domain,  $a$  is invertible and hence  $s/a \in S^{-1}A$ .

In general, the kernel of the localization map is given by  $\{a \in A : \text{there is } s \in S \text{ such that } sa = 0\}$ . As  $A \rightarrow S^{-1}A$  is a  $G$ -graded homomorphism, the kernel is in addition a  $G$ -homogeneous ideal in  $A$  by [Lemma 2.8](#). So it suffices to show that each homogeneous element in the kernel vanishes: if  $a \in A$  is a homogeneous element and there is  $s \in S$  such that  $sa = 0$ , then  $a = 0$ . Otherwise,  $a$  is invertible by [Proposition 2.20](#), which is a contradiction.  $\square$

**Definition 2.23.** Let  $A$  be a  $G$ -graded integral domain. We call the graded field defined in [Lemma 2.22](#) the *fraction  $G$ -graded field* of  $A$  and denote it by  $\text{Frac}^G A$ .

**Definition 2.24.** Let  $A$  be a  $G$ -graded ring. A proper  $G$ -homogeneous ideal  $I$  in  $A$  is called *prime* if the  $G$ -graded ring  $A/I$  is a  $G$ -graded integral domain.

**Proposition 2.25.** Let  $A$  be a  $G$ -graded ring and  $I$  be a proper homogeneous ideal in  $A$ . Then the following are equivalent:

- (1)  $I$  is a  $G$ -graded prime ideal in  $A$ .
- (2) For any homogeneous elements  $a, b \in A$  satisfying  $ab \in I$ , at least one of  $a$  and  $b$  lies in  $I$ .

PROOF. Assume that  $I$  is a  $G$ -graded prime ideal in  $A$ . Let  $a, b \in A$  be homogeneous elements satisfying  $ab \in I$ . Let  $\bar{a}, \bar{b}$  be the images of  $a, b$  in  $A/I$ . Then  $\bar{a}, \bar{b}$  are homogeneous and  $\bar{a}\bar{b} = 0$ . So at least one of  $\bar{a}$  and  $\bar{b}$  is zero. That is,  $a$  or  $b$  lies in  $I$ .

Conversely, assume that the condition in (2) is satisfied. Take  $x, y \in A/I$  with  $xy = 0$ . We need to show that at least one of  $x$  and  $y$  is 0. Lift  $x$  and  $y$  to  $a + i$  and  $b + i'$  in  $A$  with  $a, b$  being homogeneous and  $i, i' \in I$ . Then  $ab \in I$  and hence  $a \in I$  or  $b \in I$ . It follows that  $x = 0$  or  $y = 0$ .  $\square$

**Definition 2.26.** Let  $A$  be a  $G$ -graded ring and  $\mathfrak{p}$  be a  $G$ -homogeneous prime ideal in  $A$ . Then we define the  *$G$ -graded localization*  $A_{\mathfrak{p}}^G$  of  $A$  at  $\mathfrak{p}$  as  $S^{-1}A$ , where  $S$  is the set of homogeneous elements in  $A \setminus \mathfrak{p}$ .

Similarly, let  $M$  be a  $G$ -graded  $A$ -module. We define the  *$G$ -graded localization*  $M_{\mathfrak{p}}^G$  as  $S^{-1}M$ .

Recall that  $S^{-1}A$  and  $S^{-1}M$  are defined in [Example 2.5](#) and [Example 2.14](#).

**Definition 2.27.** Let  $A$  be a  $G$ -graded ring.

A  $G$ -homogeneous ideal  $I$  in  $A$  is said to be *maximal* if it is proper, and it is not contained in any other proper  $G$ -homogeneous ideals.

We call  $A$  a  *$G$ -graded local ring* if it has a unique maximal homogeneous ideal. This ideal is called the *maximal  $G$ -homogeneous ideal* of  $A$ .

**Proposition 2.28.** Let  $A$  be a  $G$ -graded ring and  $I$  be a  $G$ -homogeneous ideal in  $A$ . Then the following are equivalent:

- (1)  $I$  is a maximal  $G$ -homogeneous ideal in  $A$ ;
- (2)  $A/I$  is a  $G$ -graded field.

In particular, a maximal  $G$ -homogeneous ideal is a  $G$ -homogeneous prime ideal.

PROOF. Assume (1). Then  $I$  is a proper ideal, so  $A/I$  is non-zero. Suppose that  $A/I$  has a proper  $G$ -homogeneous ideal  $J$ , it lifts to an ideal  $J'$  of  $A$ . We claim that  $J'$  is  $G$ -homogeneous. In fact, we set  $J'_g := \{x \in A_g : x + I \in J\}$  for  $g \in G$ , we need to show that

$$J' = \sum_{g \in G} J'_g.$$

For any  $j \in J'$ , we can expand  $j + I$  as  $\sum_{g \in G} a_g + I$  with  $a_g \in A_g$  and almost all  $a_g$ 's are 0. We take  $i \in I$  so that

$$j = i + \sum_{g \in G} a_g.$$

The desired equation follows. But then it follows that  $J' = I$  and hence  $J = 0$ .

Assume (2). Then  $I$  is a proper ideal in  $A$ . If  $J$  is a  $G$ -homogeneous proper ideal of  $A$  containing  $I$ , then  $J/I$  is a  $G$ -homogeneous proper ideal of  $A/I$ . It follows that  $J/I = 0$  and hence  $J = I$ .  $\square$

**Corollary 2.29.** Let  $A$  be a non-zero  $G$ -graded ring, then  $A$  admits a  $G$ -homogeneous prime ideal.

PROOF. By our assumption,  $0$  is a proper ideal in  $A$ . By Zorn's lemma,  $A$  admits a maximal  $G$ -homogeneous ideal, which is prime by [Proposition 2.28](#).  $\square$

**Lemma 2.30.** Let  $f : A \rightarrow B$  be a  $G$ -graded homomorphism of  $G$ -graded rings. Let  $b_1, \dots, b_n \in B$  be a finite set of homogeneous elements integral over  $A$ , then there is a  $G$ -graded  $A$ -subalgebra  $B' \subseteq B$  containing  $b_1, \dots, b_n$  such that  $A \rightarrow B'$  is finite.

PROOF. We may assume that none of the  $b_i$ 's is zero. By [Proposition 2.18](#), we can find  $m_1, \dots, m_n \in \mathbb{N}$  and homogeneous elements  $a_{i,j} \in A$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$  such that

$$b_i^{m_i} + f(a_{i,1})b_i^{m_i-1} + \dots + f(a_{i,m_i}) = 0$$

for  $i = 1, \dots, n$ . It suffices to take  $B'$  as the  $A$ -submodule generated by  $a_{i,j}$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$ .  $\square$

**Proposition 2.31.** Let  $f : A \rightarrow B$  be an injective integral  $G$ -graded homomorphism of  $G$ -graded rings. Then for any  $G$ -homogeneous prime ideal  $\mathfrak{p}$  in  $A$ , there is a  $G$ -homogeneous prime ideal  $\mathfrak{p}'$  in  $B$  such that  $\mathfrak{p} = f^{-1}\mathfrak{p}'$ .

PROOF. We may assume that  $A \neq 0$ , as otherwise there is nothing to prove.

It suffices to show that  $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$ . [Include a proof](#) We could localize that  $\mathfrak{p}$  and assume that  $\mathfrak{p}$  is a maximal  $G$ -homogeneous ideal. [Include details about localization](#) It suffices then to show that  $\mathfrak{p}B \neq B$ . Assume by contrary that we can write  $1 = \sum_{i=1}^n f_i b_i$  for some homogeneous elements  $f_i \in \mathfrak{p}$  and some homogeneous elements  $b_i \in B$ . Let  $B'$  be a  $G$ -graded subring of  $B$  containing  $A$  and  $b_1, \dots, b_n$  and such that  $A \rightarrow B'$  is finite. The existence of  $B'$  is guaranteed by [Lemma 2.30](#). Then we find immediately  $B' = \mathfrak{m}_A B'$ . Then  $B' = 0$  by the graded Nakayama's lemma. [Include details](#) So  $A = 0$ , which is a contradiction.  $\square$

**Lemma 2.32.** Let  $A$  be a  $G$ -graded ring. Then the following are equivalent:

- (1)  $A$  is a  $G$ -graded local ring;
- (2) There is a proper homogeneous ideal  $I$  in  $A$  such that any non-invertible homogeneous element in  $A$  is contained in  $I$ .

PROOF. Assume that (1) holds, let  $I$  be the maximal  $G$ -homogeneous ideal of  $A$ . Let  $a$  be a non-invertible homogeneous element in  $A$ . Then the image of  $a$  in  $A/I$  is invertible by [Proposition 2.28](#) and [Proposition 2.20](#).

Assume (2). We show that  $I$  is the maximal  $G$ -homogeneous ideal in  $A$ . By [Proposition 2.28](#), it suffices to show that  $A/I$  is a graded field. By [Proposition 2.20](#), we need to show that any non-zero homogeneous element  $b \in A/I$  is invertible. Lift  $b$  to  $a + i \in A$  with  $a \in A$  homogeneous and  $i \in I$ . If  $a$  is not invertible, then  $a \in I$  by the assumption hence  $b = 0$ . This is a contradiction.  $\square$

**Lemma 2.33.** Let  $k$  be a  $G$ -graded field and  $A$  be a graded  $k$ -algebra. Suppose that  $\rho(A) = \rho(k)$ , then

- (1) For any  $g \in G$ , there is a natural isomorphism

$$A_g \cong A_1 \otimes_{k_1} k_g.$$

- (2) The map  $I \mapsto I \cap A_1$  is a bijection between the set of  $G$ -homogeneous ideals (resp.  $G$ -homogeneous prime ideals) in  $A$  and ideals (resp. prime ideals) in  $A_1$ .

PROOF. (1) Take  $g \in \rho(A)$ . As  $\rho(A) = \rho(k)$ , we can take a non-zero homogeneous element  $b \in k_g$ . Then  $b$  and  $b^{-1}$  induces inverse bijections between  $A_1$  and  $A_g$ .

(2) The part about ideals can be proved in the same way as (1). The part about prime ideals follows easily.  $\square$

**Proposition 2.34.** Let  $k$  be a  $G$ -graded field and  $M$  be a  $G$ -graded  $A$ -module. Then  $M$  is free as  $G$ -graded  $A$ -module.

PROOF. We may assume that  $M \neq 0$ . Let  $\{m_i\}_{i \in I}$  be a maximal set of non-zero homogeneous elements in  $M$  such that the corresponding homomorphism

$$F := \bigoplus_{i \in I} (\rho(f))^{-1} k \rightarrow M$$

is injective. The existence of  $\{m_i\}_{i \in I}$  follows from Zorn's lemma.

If  $f \in M/F$  is a non-zero homogeneous element, then we get a homomorphism  $(\rho(f))^{-1} k \rightarrow M/F$ . This map is necessarily injective as  $(\rho(f))^{-1} k$  does not have non-zero proper graded submodules. This contradicts the definition of  $F$ .  $\square$

**Corollary 2.35.** Let  $k$  be a  $G$ -graded field,  $C$  be a  $G$ -graded  $k$ -algebra. Consider a  $G$ -graded homomorphism of  $G$ -graded  $k$ -algebras  $f : A \rightarrow B$ . Then the following are equivalent:

- (1)  $f$  is finite (resp. finitely generated);
- (2)  $f \otimes_k C$  is finite (resp. finitely generated).

PROOF. (1)  $\implies$  (2): This implication is trivial.

(2)  $\implies$  (1): By [Proposition 2.34](#), this implication follows from fpqc descent [[Stacks](#), [Tag 02YJ](#)].  $\square$

**Definition 2.36.** Let  $K$  be a  $G$ -graded field. A  $G$ -graded subring  $A \subseteq K$  is a  *$G$ -graded valuation ring* in  $K$  if

- (1)  $A$  is a local  $G$ -graded ring;
- (2) the natural map  $\text{Frac}^G A \rightarrow K$  is an isomorphism;
- (3) For any non-zero homogeneous element  $f \in K$ , either  $f \in A$  or  $f^{-1} \in A$ .

**Definition 2.37.** Let  $K$  be a  $G$ -graded field and  $A, B$  be  $G$ -graded local subrings of  $K$ . We say  $B$  *dominates*  $A$  if  $A \subseteq B$  and  $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ , where  $\mathfrak{m}_A$  and  $\mathfrak{m}_B$  are the maximal  $G$ -homogeneous ideals in  $A$  and  $B$ .

**Proposition 2.38.** Let  $K$  be a  $G$ -graded field and  $A \subseteq K$  be a  $G$ -graded local subring. Then the following are equivalent:

- (1)  $A$  is a  $G$ -graded valuation ring in  $K$ .
- (2)  $A$  is maximal among the  $G$ -graded local subrings of  $K$  with respect to the order of domination.

PROOF. Assume (1). We may assume that  $A \neq K$ . Then  $A$  is not a  $G$ -graded field as  $\text{Frac}^G A = K$ . Let  $\mathfrak{m}$  be a maximal  $G$ -homogeneous ideal in  $A$ . Then  $\mathfrak{m} \neq 0$ .

We argue first that  $A$  is a  $G$ -graded local ring. Assume the contrary. Let  $\mathfrak{m}' \neq \mathfrak{m}$  be maximal  $G$ -homogeneous ideal in  $A$ . Choose non-zero homogeneous elements  $x, y \in A$  with  $x \in \mathfrak{m}' \setminus \mathfrak{m}$ ,  $y \in \mathfrak{m} \setminus \mathfrak{m}'$ . Then  $x/y \notin A$  as otherwise  $x = (x/y)y \in \mathfrak{m}$ . Similarly,  $y/x \notin A$ . This is a contradiction.

Next suppose that  $A'$  is a  $G$ -graded local subring of  $K$  dominating  $A$ . Let  $x \in A'$  be a non-zero homogeneous element, we need to show that  $x \in A$ . If not, we have  $x^{-1} \in A$  and as  $x^{-1}$  is not a unit,  $x^{-1} \in \mathfrak{m}_A$ . But then  $x^{-1} \in \mathfrak{m}_{A'}$ , the maximal  $G$ -homogeneous ideal in  $A'$ . This contradicts the fact that  $x \in A'$ .

Assume (2). Take a homogeneous element  $x \in K \setminus A$ , we need to argue that  $x^{-1} \in A$ . Let  $A'$  denote the minimal  $G$ -homogeneous subring of  $K$  containing  $A$  and  $x$ . It is easy to see that  $A'$  is the usual subring generated by  $A$  and  $x$ .

By our assumption, there is no  $G$ -graded prime ideal of  $A'$  lying over  $\mathfrak{m}_A$ , as otherwise, if  $\mathfrak{p}$  is such an ideal, the  $G$ -graded local subring  $A'_{\mathfrak{p}}^G$  of  $K$  dominates  $A$ .

In other words, the  $G$ -graded ring  $A'/\mathfrak{m}_A A'$  does not have any homogeneous prime ideals and hence  $A' = \mathfrak{m}_A A'$  by [Corollary 2.29](#).

We can therefore write

$$1 = \sum_{i=0}^d t_i x^i$$

with some homogeneous elements  $t_i \in \mathfrak{m}_A$ . In particular,

$$(1 - t_0)(x^{-1})^d - \sum_{i=1}^d t_i (x^{-1})^{d-i} = 0.$$

So  $x^{-1}$  is integral over  $A$ . Let  $A''$  be the subring of  $K$  generated by  $A$  and  $x^{-1}$ . Then  $A \rightarrow A''$  is finite and there is a  $G$ -homogeneous prime ideal  $\mathfrak{m}''$  of  $A''$  lying over  $\mathfrak{m}_A$  by [Proposition 2.31](#). By our assumption,  $A = A''_{\mathfrak{m}''}^G$  and hence  $x^{-1} \in A$ .

It remains to verify that  $\text{Frac}^G A = K$ . Suppose that it is not the case, let  $B \subseteq K$  be a  $G$ -graded local subring dominating  $A$ . Take a homogeneous element  $t \in K$  that is not in  $\text{Frac}^G A$ . Observe that  $t$  can not be transcendental over  $A$ , as otherwise  $A[t] \subseteq K$  is a  $G$ -graded subring, and we can localize it at the  $G$ -homogeneous prime generated by  $t$  and  $\mathfrak{m}_A$ . We get a  $G$ -graded local ring dominating  $A$  that is different from  $A$ .

So  $t$  is algebraic over  $A$ . We can then take a non-zero homogeneous  $a \in A$  such that  $at$  is integral over  $A$ . The ring  $A' \subseteq K$  generated by  $A$  and  $ta$  is a  $G$ -graded subring and  $A \rightarrow A'$  is finite. By [Proposition 2.31](#), there is a  $G$ -homogeneous prime ideal  $\mathfrak{m}'$  of  $A'$  lifting  $\mathfrak{m}_A$ . But then  $A'_{\mathfrak{m}'}^G$  dominates  $A$  and so  $A = A'_{\mathfrak{m}'}^G$ . It follows that  $t \in \text{Frac}^G A$ , which is a contradiction.  $\square$

**Corollary 2.39.** Let  $K$  be a  $G$ -graded field. Any  $G$ -graded local subring  $B \subseteq K$  is dominated by a  $G$ -graded valuation subring of  $K$ .

PROOF. This follows from [Proposition 2.38](#) and Zorn's lemma.  $\square$

**2.1. Graded algebraic geometry.** Let  $G$  be an Abelian group. We write the group operation of  $G$  multiplicatively and denote the identity of  $G$  as 1.

**Definition 2.40.** Let  $A$  be a  $G$ -graded ring. We define the  $G$ -graded affine spectrum  $\text{Spec}^G(A)$  as follows: as a set  $\text{Spec}^G(A)$  consists of all  $G$ -homogeneous prime ideals of  $A$ ; we endow  $\text{Spec}^G(A)$  with the *Zariski topology*, whose base consists of sets of the form

$$D(f) := \{ \mathfrak{p} \in \text{Spec}^G(A) : f \notin \mathfrak{p} \}$$

for all homogeneous elements  $f \in A$ .

**Lemma 2.41.** Let  $k$  be a  $G$ -graded field and  $A$  be a finitely generated  $G$ -graded  $k$ -algebra. Then  $\text{Spec}^G(A)$  has only finitely many maximal points.

PROOF.  $\square$

### 3. Miscellany

**Proposition 3.1.** Let  $R$  be a noetherian N-2 integral domain. Let  $\psi : R \rightarrow S$  be a ring homomorphism such that  $S$  is reduced, torsion-free as  $R$ -module and has finite rank as  $R$ -module. Then  $\psi$  is finite.

[\[BGR84, Page 122\]](#). Reproduce the argument later.

PROOF. As  $\psi$  is injective by assumption, we may assume that  $R$  is a subring of  $S$  and  $\psi$  is identity. The ring  $S_{R \setminus \{0\}} = \text{Frac } S$  is a finite-dimensional reduced  $\text{Frac } R$ -algebra, hence as a ring,  $\text{Frac } S$  is the product of finitely many finite field extensions of  $\text{Frac } R$ , say  $K_1, \dots, K_t$ . As  $R$  is N-2, the integral closure  $R_i$  of  $R$  in  $K_i$  is finite as  $R$ -module for  $i = 1, \dots, t$ . As  $S$  is integral over  $R$ , we have

$$S \subseteq R_1 \times \dots \times R_t.$$

As  $R$  is noetherian, we conclude that  $S$  is finite as  $R$ -module.  $\square$

**Lemma 3.2.** Let  $R$  be a commutative ring. A polynomial  $a_0 + a_1X + \dots + a_nX^n \in R[X]$  is a unit if and only if  $a_0$  is a unit in  $R$  and  $a_1, \dots, a_n$  are nilpotents.



## Bibliography

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