Commutative algebra

Contents

1. Introduction

In this chapter, we study the notion of complex analytic local algberas (*Analytische Stellenalgebren* in German) in the sense of [GR71]. Most of the materials in this chapter are standard, but the proofs are scattered in tons of papers and books.

Results regarding the completion of the analytic tensor products are certainly known, though the author does not know any written references.

2. Ring of convergent power series

Definition 2.1. For any $n \in \mathbb{N}$, let $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ denote the subring of $\mathbb{C}[[z_1, \ldots, z_n]]$ consisting of

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},$$

which is convergent in a neighbourhood of 0: there is $\epsilon > 0$ such that for any $x_1, \ldots, x_n \in \mathbb{C}$ with $|x_i| < \epsilon$, $\sum_\alpha a_\alpha x^\alpha$ is a convergent power series. We will write $f(0)$ for $a_{0,\ldots,0}$.

Definition 2.2. Fix $n \in \mathbb{N}$ and $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$. For any $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ with an expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},$$

we define

$$\|f\|_t := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| t^\alpha \in [0, \infty].$$

We define

$$\mathbb{C}\langle z_1, \ldots, z_n\rangle_t := \{f \in \mathbb{C}[[z_1, \ldots, z_n]] : \|f\|_t < \infty\}.$$

Observe that $(\mathbb{C}\langle z_1, \ldots, z_n\rangle_t, \|\bullet\|_t)$ is a normed $\mathbb{C}$-vector space and

$$\mathbb{C}\langle z_1, \ldots, z_n\rangle = \bigcup_{t \in \mathbb{R}^n_{>0}} \mathbb{C}\langle z_1, \ldots, z_n\rangle_t. \tag{2.1}$$

Proposition 2.3 (Cauchy coefficients estimate)**.** Let

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in \mathbb{C}\langle z_1, \ldots, z_n\rangle_t.$$

Then for any $\alpha \in \mathbb{N}^n$,

$$|a_\alpha| \leq \frac{\|f\|_t}{t^\alpha}.$$

PROOF. This follows from the definition. □

Proposition 2.4. $(\mathbb{C}\langle z_1, \ldots, z_n\rangle_t, \|\bullet\|_t)$ is a $\mathbb{C}$-Banach algebra.

There has to be a section about Banach algebras.

Lemma 2.5. Let A be a $\mathbb{C}$-Banach algebra. An element $f \in A$ with $\|1 - f\| < 1$ is invertible.

Lemma 2.6. For any $n \in \mathbb{N}$, an element $f \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$ is a unit if and only if $f(0) \neq 0$.

PROOF. In fact, as $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is a subring of $\mathbb{C}[[z_1, \ldots, z_n]]$, a unit f in $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is also a unit in $\mathbb{C}[[z_1, \ldots, z_n]]$, hence $f(0) \neq 0$.

Conversely, assume that $f \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$ and $f(0) \neq 0$, then

$$\lim_{t\to 0+} \|1 - f(0)^{-1} f\|_t = 0.$$

So by Lemma 2.5, $a^{-1}f$ is a unit in $\mathbb{C}\langle z_1, \ldots, z_n\rangle_t$ when t is small enough. As $\mathbb{C}\langle z_1, \ldots, z_n\rangle_t$ is a subring of $\mathbb{C}\langle z_1, \ldots, z_n\rangle$, it follows that $f(0)^{-1}f$ is invertible in $\mathbb{C}\langle z_1, \ldots, z_n\rangle$, hence so is f. □

Theorem 2.7. Let $m, n \in \mathbb{N}$.

(1) The ring $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is local, and the maximal ideal is given by convergent power series without constant terms.
(2) Any homomorphism of $\mathbb{C}$-algebras $F : \mathbb{C}\langle z_1, \ldots, z_n\rangle \to \mathbb{C}\langle w_1, \ldots, w_m\rangle$ is local.

PROOF. (1) This follows from Lemma 2.6 and [Stacks, Tag 00E9].

(2) Suppose it is not the case. As $z_1, \ldots, z_n$ generate the maximal ideal in $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ by (1), at least one of $F(z_1), \ldots, F(z_n)$ is not in the maximal ideal of $\mathbb{C}\langle w_1, \ldots, w_m\rangle$. We may assume that it is $F(z_1)$, namely

$$F(z_1) = a + \mathcal{O}(w)$$

with $a \in \mathbb{C}^\times$. As F is a $\mathbb{C}$-algebra homomorphism, $F(z_1 - a) = F(z_1) - a$ is a unit. This contradicts the fact that $z_1 - a$ is a unit. □

Lemma 2.8. Let $m, n \in \mathbb{N}$ and $g_1, \ldots, g_m \in \mathbb{C}\langle w_1, \ldots, w_n\rangle$. Assume that $g_i(0) = 0$ for all $i = 1, \ldots, m$. Then the formal substitution

$$F : \mathbb{C}[[z_1, \ldots, z_m]] \to \mathbb{C}[[w_1, \ldots, w_n]]$$

sending z_i to g_i restricts to a homomorphism of $\mathbb{C}$-algebras

$$F : \mathbb{C}\langle z_1, \ldots, z_m\rangle \to \mathbb{C}\langle w_1, \ldots, w_n\rangle.$$

PROOF. Fix $t \in \mathbb{R}^m_{>0}$. Take $s \in \mathbb{R}^n_{>0}$ so that $\|g_i\|_s \leq t_i$ for $i = 1, \ldots, m$. This is possible as $g_i(0) = 0$. Then we claim that F sends $\mathbb{C}\langle z_1, \ldots, z_m\rangle_t$ to $\mathbb{C}\langle w_1, \ldots, w_n\rangle_s$. This implies our lemma.

To prove the assertion, let $f \in \mathbb{C}\langle z_1, \ldots, z_m\rangle_t$, which we expand as

$$f = \sum_{\alpha\in\mathbb{N}^m} a_\alpha z^\alpha.$$

Then

$$\|F(f)\|_s = \sum_{j=0}^{\infty} \|\sum_{|\alpha|=j} a_\alpha g^\alpha\|_s \leq \sum_\alpha |a_\alpha| \|g\|_s^\alpha \leq \|f\|_t.$$

Here the first inequality follows from Proposition 2.4. □

Conversely, we have

Lemma 2.9. Let $m, n \in \mathbb{N}$. Then any homomorphism of $\mathbb{C}$-algebras

$$F : \mathbb{C}\langle z_1, \ldots, z_m\rangle \to \mathbb{C}\langle w_1, \ldots, w_n\rangle$$

is the restriction of a substitution homomorphism

$$\mathbb{C}[[z_1, \ldots, z_m]] \to \mathbb{C}[[w_1, \ldots, w_n]].$$

In particular, F is uniquely determined by $F(z_1), \ldots, F(z_m)$.

PROOF. Let $g_i = F(z_i)$. It follows from Theorem 2.7 that $g_i \in \mathfrak{m}$, the maximal ideal in $\mathbb{C}[[w_1, \dots, w_n]]$. Let $G : \mathbb{C}\langle z_1, \dots, z_m\rangle \to \mathbb{C}\langle w_1, \dots, w_n\rangle$ be the substitution homomorphism sending z_i to g_i. Then F and G agrees on $\mathbb{C}[z_1, \dots, z_m]$. In particular, for any $f \in \mathbb{C}\langle z_1, \dots, z_m\rangle$ and $a \in \mathbb{N}$ if we write f_1 the sum of the homogeneous parts of f of degree no more than a and $f_2 = f - f_1$, we see that $f_2 \in \mathfrak{m}_1^{a+1}$, where $\mathfrak{m}_1$ is the maximal ideal of $\mathbb{C}\langle z_1, \dots, z_m\rangle$. It follows that $(F - G)(f_1) = 0$ and $(F - G)(f_2) \in \mathfrak{m}^{a+1}$, the latter is a consequence of Theorem 2.7. As a is arbitrary, we find that

$$(F - G)(f) \in \bigcap_{a=1}^{\infty} \mathfrak{m}^a \subseteq \bigcap_{a=1}^{\infty} \mathfrak{m}_2^a,$$

where $\mathfrak{m}_2$ is the maximal ideal in $\mathbb{C}[[w_1, \dots, w_n]]$. As $\mathbb{C}[[w_1, \dots, w_n]]$ is Noetherian, it follows from Krull's intersection theorem [Stacks, Tag 00IP] that $\bigcap_{a=1}^{\infty} \mathfrak{m}_2^a = 0$, so $F = G$. □

We prove a few elementary results about the structure of the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$. In the following sections, we will develop deeper structures after developing the Weierstrass theory.

Proposition 2.10. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is an integral domain.

PROOF. It suffices to observe that $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is a subring of $\mathbb{C}[[z_1, \dots, z_n]]$. □

Theorem 2.11. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is strictly Henselian.

PROOF. As the residue field of $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is $\mathbb{C}$ by Theorem 2.7, it suffices to show that $\mathbb{C}\langle z_1, \dots, z_n\rangle$ is Henselian. Take a monic polynomial $f \in \mathbb{C}\langle z_1, \dots, z_n\rangle[w]$, say

$$f = w^b + f_1 w^{b-1} + \dots + f_b, \quad f_i \in \mathbb{C}\langle z_1, \dots, z_n\rangle.$$

Suppose that $\bar{a} \in \mathbb{C}$ is a simple root of $\bar{f} = w^b + f_1(0)w^{b-1} + \dots + f_b(0)$. We want to lift find $a \in \mathbb{C}\langle z_1, \dots, z_n\rangle$ such that $f(a) = 0$ and $a(0) = \bar{a}$. As $\mathbb{C}$ is algebraically closed, we can prove a stronger result: suppose that

$$\bar{f} = (w - c_1)^{b_1} \cdots (w - c_t)^{b_t}$$

for some $c_i \in \mathbb{C}$ and $b_i \in \mathbb{Z}_{>0}$. Then we claim that there are monic polynomials $g_i \in \mathbb{C}\langle z_1, \dots, z_n\rangle[w]$ such that $\bar{g}_i = (w - c_i)^{b_i}$ for all $i = 1, \dots, t$ and $f = g_1 \cdots g_t$.

We make an induction on t. When $t = 1$, there is nothing to prove, so assume that $t > 1$ and the theorem has been proved for $t - 1$. We may assume that $c_1 = 0$. By Weierstrass preparation theorem, we can find a Weierstrass polynomial $h \in \mathbb{C}\langle z_1, \dots, z_n\rangle[w]$ of degree b_1 and a monic polynomial $k \in \mathbb{C}\langle z_1, \dots, z_n\rangle[w]$ such that $f = hk$. By the inductive hypothesis, we can find monic polynomials $g_2, \dots, g_t \in \mathbb{C}\langle z_1, \dots, z_n\rangle[w]$ such that $\bar{g}_i = (w - c_i)^{b_i}$ for all $i = 2, \dots, t$ and $k = g_2 \cdots g_t$. It suffices to take $g_1 = h$. □

3. Weierstrass division and excellence of the ring of formal power series

Definition 3.1. Let $n \in \mathbb{N}$ and $f \in \mathbb{C}\langle z_1, \dots, z_n\rangle$, we say f *has order e in z_n* for some $e \in \mathbb{N}$ if we expand f as

$$f = \sum_{i=0}^{\infty} f_i z_n^i$$

with $f_i \in \mathbb{C}\langle z_1, \dots, z_{n-1}\rangle$, then $f_0(0) = \dots = f_{e-1}(0) = 0$ while $f_e(0) \neq 0$.

If $f_i(0)=0$ for all i, we say f *has order* ∞ *in* z_n. We will write $\mathrm{ord}_{z_n} f$ for the order of f in z_n.

Theorem 3.2 (Weierstrass division theorem)**.** Let $n\in\mathbb{N}$ and $g\in\mathbb{C}\langle z_1,\dots,z_n\rangle$. Assume that $\mathrm{ord}_{z_n} g<\infty$. Then for any $f\in\mathbb{C}\langle z_1,\dots,z_n\rangle$ there is $q\in\mathbb{C}\langle z_1,\dots,z_n\rangle$ and $r\in\mathbb{C}\langle z_1,\dots,z_{n-1}\rangle[w]$ with $\deg_{z_n} r<\mathrm{ord}_{z_n} g$ such that

$$f=qg+r.$$

Moreoer, q and r are uniquely determined.

Remark 3.3. The restriction $\mathrm{ord}_{z_n} g<\infty$ is not too severe. In fact, given any non-zero g, we can always find an invertible $n\times n$ matrix A, so that if we consider gA^{-1} defined in the obvious way, we have $\mathrm{ord}_{z_n} gA^{-1}<\infty$.

Proof. Fix $\epsilon\in(0,1)$.

Choose a small enough $t=(t_1,\dots,t_n)\in\mathbb{R}^n_{>0}$ so that $f,g\in\mathbb{C}\langle z_1,\dots,z_n\rangle_t$. This is possible by (2.1). We expand g as

$$g=\sum_{i=0}^{\infty}a_iz_n^i \tag{3.1}$$

with $a_i\in\mathbb{C}\langle z_1,\dots,z_{n-1}\rangle$. We decompose g into

$$g=g_1+z_n^{\mathrm{ord}_{z_n} g}g_2,\quad g_1=\sum_{i=0}^{\mathrm{ord}_{z_n} g-1}a_iz_n^i,\quad g_2=\sum_{i=\mathrm{ord}_{z_n} g}^{\infty}a_iz_n^{i-\mathrm{ord}_{z_n} g}.$$

Then our assumption implies that g_2 is a unit in $\mathbb{C}\langle z_1,\dots,z_n\rangle$ and by (2.1) again, choosing t small enough, we may assume that $g_2^{-1}\in\mathbb{C}\langle z_1,\dots,z_n\rangle_t$. Then

$$\|gg_2^{-1}-z_n^{\mathrm{ord}_{z_n} g}\|_t=\|g_1g_2^{-1}\|_t\leq\|g_1\|_t\cdot\|g_2^{-1}\|_t.$$

As $\lim_{t\to0+}\|g_1\|_t=0$ by our assumption, taking t small enough, we can also guarantee that

$$\|gg_2^{-1}-z_n^{\mathrm{ord}_{z_n} g}\|_t\leq t_n^{\mathrm{ord}_{z_n} g}\epsilon.$$

We define $v_j\in\mathbb{C}\langle z_1,\dots,z_n\rangle_t$ for $j\in\mathbb{N}$ as follows: $v_0=f$ and

$$v_{j+1}=(z_n^{\mathrm{ord}_{z_n} g}-gg_2^{-1})v_{j,1},$$

where $v_{j,1}$ is defined from v_j in the same way g_1 is defined from g.

Observe that

$$\|v_{j,1}\|_t\leq t_n^{-\mathrm{ord}_{z_n} g}\|v_j\|_t.$$

It follows that

$$\|v_{j+1}\|_t\leq\epsilon\|v_j\|_t.$$

In particular,

$$w=\sum_{j=0}^{\infty}v_j$$

converges in $\mathbb{C}\langle z_1,\dots,z_n\rangle_t$ by Proposition 2.4. Now we can define $q=g_2^{-1}w_2$ and $r=w_1$. Again w_1,w_2 are defined from w using the same way g_1,g_2 are defined from g. It follows that

$$f=\sum_{j=0}^{\infty}(v_j-v_{j+1})=\sum_{j=0}^{\infty}(gg_2^{-1}w_2+w_1)=qg+r.$$

We conclude the existence part.

As for the uniqueness, suppose that $qg+r=0$ for some $q\in\mathbb{C}\langle z_1,\dots,z_n\rangle$ and $r\in\mathbb{C}\langle z_1,\dots,z_{n-1}\rangle[z_n]$ with $\deg r<\mathrm{ord}_{z_n} g$. We want to deduce $q=r=0$. Take $t\in\mathbb{R}^n_{>0}$ small enough, we may assume that $q,g,r\in\mathbb{C}\langle z_1,\dots,z_n\rangle_t$. Expand g as in (3.1), we may assume that $a^{-1}_{\mathrm{ord}_{z_n} g}\in\mathbb{C}\langle z_1,\dots,z_{n-1}\rangle_t$. We can then write

$$a^{-1}_{\mathrm{ord}_{z_n} g}g=z_n^{\mathrm{ord}_{z_n} g}+h$$

for some $h\in\mathbb{C}\langle z_1,\dots,z_n\rangle_t$, $h(0)=0$. Fix $\epsilon\in(0,1)$. Choose t small enough, we can then guarantee that

$$\|h\|_t\leq t_n^b\epsilon.$$

Now

$$qa_{\mathrm{ord}_{z_n} g}z_n^{\mathrm{ord}_{z_n} g}+r=-qha_{\mathrm{ord}_{z_n} g}.$$

If we set $M=\|qa_{\mathrm{ord}_{z_n} g}\|_t t_n^{\mathrm{ord}_{z_n} g}$, then we see immediately

$$M=\|qa_{\mathrm{ord}_{z_n} g}z_n^{\mathrm{ord}_{z_n} g}\|_t\leq\|qa_{\mathrm{ord}_{z_n} g}z_n^{\mathrm{ord}_{z_n} g}+r\|_t=\|qha_{\mathrm{ord}_{z_n} g}\|_t\leq M\epsilon.$$

It follows that $M=0$ and hence $qa_{\mathrm{ord}_{z_n} g}=0$. It follows that $q=0$ by Proposition 2.10. Therefore, $r=0$ as well. □

Proposition 3.4. Let $n\in\mathbb{N}$ and $g\in\mathbb{C}\langle z_1,\dots,z_n\rangle$. Assume that $\mathrm{ord}_{z_n} g<\infty$. Then there is a canonical isomorphism

$$\mathbb{C}\langle z_1,\dots,z_n\rangle/g\mathbb{C}\langle z_1,\dots,z_n\rangle\xrightarrow{\sim}\mathbb{C}\langle z_1,\dots,z_{n-1}\rangle^{\mathrm{ord}_{z_n} g}.$$

Proof. Given any $f\in\mathbb{C}\langle z_1,\dots,z_n\rangle$, we consider the Weierstrass division $f=qg+r$ as in Theorem 3.2. Write

$$r=r_0+r_1z_n+\cdots+r_{\mathrm{ord}_{z_n} g-1}z_n^{\mathrm{ord}_{z_n} g-1}.$$

Then we map f to $(r_0,\dots,r_{\mathrm{ord}_{z_n} g-1})$. Clearly, this defines the isomorphism as in the proposition. □

As an application of Weierstrass theory, we prove a few results about the structure of the ring $\mathbb{C}\langle z_1,\dots,z_n\rangle$.

Theorem 3.5. For any $n\in\mathbb{N}$, the ring $\mathbb{C}\langle z_1,\dots,z_n\rangle$ is a noetherian integral domain.

Proof. We make an induction on n to prove that $\mathbb{C}\langle z_1,\dots,z_n\rangle$ is noetherian. The case $n=0$ is trivial. So assume $n>0$ and that the theorem has been proved for all smaller values of n. Let $f\in\mathbb{C}\langle z_1,\dots,z_n\rangle$ be a non-zero element. It suffices to show that $\mathbb{C}\langle z_1,\dots,z_n\rangle/f\mathbb{C}\langle z_1,\dots,z_n\rangle$ is noetherian. By Remark 3.3, we may assume that $\mathrm{ord}_{z_n} f<\infty$. By Proposition 3.4, we know that

$$\mathbb{C}\langle z_1,\dots,z_n\rangle/f\mathbb{C}\langle z_1,\dots,z_n\rangle\cong\mathbb{C}\langle z_1,\dots,z_n\rangle^c$$

for some $c\in\mathbb{N}$. By the inductive hypothesis, the latter ring is noetherian. □

Proposition 3.6. Let $n\in\mathbb{N}$. Then the $\mathfrak{m}$-adic completion of the ring $\mathbb{C}\langle z_1,\dots,z_n\rangle$ is canonically isomorphic to $\mathbb{C}[[z_1,\dots,z_n]]$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{C}\langle z_1,\dots,z_n\rangle$.

PROOF. Let $\mathfrak{m}_1$ be the maximal ideal in $\mathbb{C}[z_1, \ldots, z_n]$. It suffices to observe that we have canonical identifications

$$\mathbb{C}[z_1, \ldots, z_n]/\mathfrak{m}_1^n \xrightarrow{\sim} \mathbb{C}\langle z_1, \ldots, z_n\rangle/\mathfrak{m}^n$$

for any $n \in \mathbb{N}$. So in particular, the $\mathfrak{m}$-adic completion of $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is isomorphic to the $\mathfrak{m}_1$-adic completion of $\mathbb{C}[z_1, \ldots, z_n]$, which is $\mathbb{C}\langle z_1, \ldots, z_n\rangle$. $\square$

Corollary 3.7. Let $n \in \mathbb{N}$. Then the Krull dimension of $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is n.

PROOF. This follows from Proposition 3.6 and [Stacks, Tag 07NV]. $\square$

Theorem 3.8. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is regular. In particular, it is a UFD.

PROOF. We have computed that the completion of $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is $\mathbb{C}[[z_1, \ldots, z_n]]$ in Proposition 3.6. The regularity of $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ follows from the regularity of the completion by descent [Stacks, Tag 07NY]. It is a UFD by Auslander–Buchsbaum theorem. $\square$

Proposition 3.9. Let A be a complex analytic local algebra. Then A is a excellent, strictly Henselian.

PROOF. This follows from the corresponding results in Theorem 3.11 and Theorem 2.11. $\square$

We recall the following criterion.

Theorem 3.10. Let k be a field of characteristic 0 and R be a regular ring containing k. Suppose that there is $n \in \mathbb{N}$ such that

(1) for any maixmal ideal $\mathfrak{m}$ of R, the residue $R/\mathfrak{m}$ is algebraic over k and the height of $\mathfrak{m}$ is n;
(2) there exists $D_1, \ldots, D_n \in \mathrm{Der}_k(R)$ and $x_1, \ldots, x_n \in R$ such that $D_i x_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

Then R is excellent.

PROOF. We refer to [Mat80, Theorem 102]. $\square$

Theorem 3.11. For any $n \in \mathbb{N}$, the ring $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ is excellent.

PROOF. This is an immediate consequence of Theorem 3.10. $\square$

4. Weierstrass preparation theorem

Definition 4.1. Let $n \in \mathbb{Z}_{>0}$. A *Weierstrass polynomial* in n variables is a monic polynomial

$$\omega = z_n^b + a_1 z_n^{b-1} + \cdots + a_b \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$$

such that $a_i(0) = 0$ for all $i = 0, \ldots, b$.

Observe that by definition, $\mathrm{ord}_{z_n} \omega = b$.

Lemma 4.2. Let $\omega \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$ be a Weierstrass polynomial and $g \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$. Assume that $\omega g \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$, then $g \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$.

PROOF. By the division theorem of polynomial rings, we can write

$$\omega g = q\omega + r$$

for some $q, r \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$, $\deg_{z_n} r < \deg_{z_n} \omega g$. But we can write $\omega g = \omega \cdot g$. From the uniqueness part of Theorem 3.2, we know that $q = g$, so g is a polynomial in z_n. □

THEOREM 4.3 (Weierstrass preparation theorem). Let $g \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$. Assume that $\operatorname{ord}_{z_n} g < \infty$, then there is a unique Weierstrass polynomial $\omega \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$ of degree $\operatorname{ord}_{z_n} g$ and a unit $e \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$ such that $g = e\omega$.

Moreover, if $g \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$, then so is e.

PROOF. By Theorem 3.2, we can write

$$z_n^{\operatorname{ord}_{z_n} g} = qg + r$$

for $q \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$ and $r \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$ with $\deg_{z_n} r < \operatorname{ord}_{z_n} g$. Write $g = \sum_{i=0}^{\infty} a_i z_n^i$ for some $a_i \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle$. Define $\bar{g} = \sum_{i=0}^{\infty} a_i(0) z_n^i$, set $e' := z_n^{-\operatorname{ord}_{z_n} g}\bar{g} \in \mathbb{C}[z_n]$. Then $e'(0) \neq 0$. Similarly define $\bar{q}$ and $\bar{r}$, then we have

$$z_n^{\operatorname{ord}_{z_n} g} = \bar{q}\bar{g} + \bar{r}.$$

Or

$$1 = \bar{q}e' + \bar{r}.$$

From the uniqueness part of Theorem 3.2, we conclude that $\bar{q} = e'^{-1}$, namely q is a unit. Now

$$g = q^{-1}(z_n^{\operatorname{ord}_{z_n} g} - r)$$

is the desired decomposition. When $g \in \mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]$, so is e, as can be seen from Lemma 4.2.

It remains to prove the uniqueness: if e is a unit in $\mathbb{C}\langle z_1, \ldots, z_n\rangle$, ω, ω' are two Weierstrass polynomials and $e\omega = \omega'$, then we need to show that $e = 1$. It follows from Lemma 4.2 that e is a polynomial in z_n. Setting $z_1 = \cdots = z_{n-1} = 0$, we find that $\bar{e}$ is a power of z_n. As e is a unit, it follows that $\bar{e} = 1$. On the other hand, clearly e is a monic polynomial, it follows that $e = 1$. □

Definition 4.4. Let $g \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$. Assume that $\operatorname{ord}_{z_n} g < \infty$. The Weierstrass polynomial constructed in Theorem 4.3 is called the *Weierstrass polynomial* of g.

Corollary 4.5. Let $g \in \mathbb{C}\langle z_1, \ldots, z_n\rangle$. Assume that $\operatorname{ord}_{z_n} g < \infty$. Let ω be the Weierstrass polynomial of g. Then the injection

$$\mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n] \to \mathbb{C}\langle z_1, \ldots, z_n\rangle$$

induces an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n]/\omega\mathbb{C}\langle z_1, \ldots, z_{n-1}\rangle[z_n] \to \mathbb{C}\langle z_1, \ldots, z_n\rangle/g\mathbb{C}\langle z_1, \ldots, z_n\rangle.$$

PROOF. The morphism is surjective by Theorem 3.2 and injective by Lemma 4.2. □

5. Complex analytic local algebras

Definition 5.1. A *complex analytic local algebra* is a $\mathbb{C}$-algebra A such that $A \neq 0$ and there exists some $n \in \mathbb{N}$ and an ideal I in $\mathbb{C}\langle z_1, \ldots, z_n\rangle$ such that

$$A \cong \mathbb{C}\langle z_1, \ldots, z_n\rangle / I$$

as $\mathbb{C}$-algebras.

A morphism between complex analytic local algebras A and B is a $\mathbb{C}$-algebra homomorphism $A \to B$.

The category of complex analytic local algebras is denoted by $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$.

Proposition 5.2. Let A be an object in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ and $f : B \to C$ be a morphism in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$. Then

(1) A is local with residue field $\mathbb{C}$.
(2) f is a local homomorphism.

Proof. (1) This follows immediately from Theorem 2.7.
(2) This follows from the same arguments as Theorem 2.7 (2). □

Observe that a complex analytic local algebra is always local with residue field $\mathbb{C}$ and a morphism in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ is always a local homomorphism. We will write $\mathfrak{m}_A$ for the maximal ideal in A.

Lemma 5.3. Let A be a complex analytic local algebra and $n \in \mathbb{N}$, then there is a natural bijection

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\langle z_1, \ldots, z_n\rangle, A) \cong \mathfrak{m}_A^n$$

sending a morphism f to $(f(z_1), \ldots, f(z_n))$.

Proof. As a morphism $f : \mathbb{C}\langle z_1, \ldots, z_n\rangle \to A$ is necessarily local, we see that $f(z_i) \in A$ for all $i = 1, \ldots, n$. So the map $\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(\mathbb{C}\langle z_1, \ldots, z_n\rangle, A) \to \mathfrak{m}_A^n$ is well-defined. Conversely, given $w_1, \ldots, w_n \in \mathfrak{m}_A$, we claim that there is a unique morphism $f : \mathbb{C}\langle z_1, \ldots, z_n\rangle \to A$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ sending z_i to w_i.

The uniqueness follows from Lemma 2.9, so let us consider only the existence. Let $\mathbb{C}\langle z_1, \ldots, z_m\rangle \to A$ be a surjective morphism. Lift w_i to $w_i' \in \mathbb{C}\langle z_1, \ldots, z_m\rangle$, it suffices to construct a morphism $\mathbb{C}\langle z_1, \ldots, z_n\rangle \to \mathbb{C}\langle z_1, \ldots, z_m\rangle$ sending z_i to w_i'. So we may assume that $A = \mathbb{C}\langle z_1, \ldots, z_m\rangle$. In this case, the result follows from Lemma 2.8. □

Definition 5.4. Let A_1, A_2 be complex analytic local algebras, *an analytic tensor product* of A_1 and A_2 is a complex analytic local algebra A together with morphisms $A_1 \to A$ and $A_2 \to A$ such that for any complex analytic local algebra C, the induced map

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A, C) \to \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_1, C) \times \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A_2, C)$$

is bijective.

As analytic tensor product is unique up to unique isomorphism, so we can choose a specific analytic tensor product $A_1 \overline{\otimes} A_2$ and call it *the analytic tensor product* of A_1 and A_2.

More generally, let B be a complex analytic local algebra and $A_1, A_2 \in \mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\backslash B}$ (the under-slice of B). *An analytic tensor product* of A_1 and A_2 over B is a complex analytic local algebra A over B together with morphisms $A_1 \to A$ and $A_2 \to A$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\backslash B}$ such that the induced map

$$\mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}}(A, C) \to \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\backslash B}}(A_1, C) \times \mathrm{Hom}_{\mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\backslash B}}(A_2, C)$$

is bijective.

As analytic tensor product is unique up to unique isomorphism, so we can choose a specific analytic tensor product $A_1 \overline{\otimes}_B A_2$ and call it *the analytic tensor product* of A_1 and A_2 over B.

By definition, there are natural morphisms

$$A_1 \otimes A_2 \to A_1 \overline{\otimes} A_2.$$

The simplest example is

Lemma 5.5. For any $m, n \in \mathbb{N}$, we have

$$\mathbb{C}\langle z_1, \ldots, z_m \rangle \overline{\otimes} \mathbb{C}\langle z_1, \ldots, z_n \rangle \cong \mathbb{C}\langle z_1, \ldots, z_{m+n} \rangle$$

as complex analytic local algebras.

Proof. This is a simple consequence of Lemma 5.3 and Yoneda's lemma. □

Lemma 5.6. Assume that $f_i : A_i \to B_i$ are surjective (i.e. the underlying homomorphisms of algebras are surjective) morphisms in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ for $i = 1, 2$. Let I_i be the kernel of f_i as homomorphisms of algebras. If $A_1 \overline{\otimes} A_2$ exists, then so is $B_1 \overline{\otimes} B_2$ and

$$B_1 \overline{\otimes} B_2 \cong A_1 \overline{\otimes} A_2 / (I_1 \otimes 1 + 1 \otimes I_2)(A_1 \overline{\otimes} A_2).$$

Proof. That $A_1 \overline{\otimes} A_2 / (I_1 \otimes 1 + 1 \otimes I_2)(A_1 \overline{\otimes} A_2)$ is a complex analytic local algebra follows from our assumption. That it represents $B_1 \overline{\otimes} B_2$ follows from general abstract nonsense. Include details □

Corollary 5.7. Let A_1, A_2 be complex analytic local algebras, then $A_1 \overline{\otimes} A_2$ exists.

Proof. This is a consequence of Lemma 5.6 and Lemma 5.5. □

Theorem 5.8. Let A_1, A_2 be complex analytic local algebras, then there are natural isomorphisms

$$A_1 \hat{\otimes}_{\mathbb{C}} A_2 \xrightarrow{\sim} (A_1 \overline{\otimes} A_2)\hat{}.$$

Here on the right-hand side, we take the adic completion with respect to the maximal ideal in $A_1 \overline{\otimes} A_2$.

Proof. Observe that the existence of a morphism $A_1 \hat{\otimes} A_2 \to (A_1 \overline{\otimes} A_2)\hat{}$ follows from the universal property.

When A_1 and A_2 are both rings of convergent power series, this result follows from Lemma 5.5 and Proposition 3.6.

In general, represent

$$A_1 = \mathbb{C}\langle z_1, \ldots, z_m \rangle / I_1, \quad A_2 = \mathbb{C}\langle w_1, \ldots, w_n \rangle / I_2.$$

Then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}\langle z_1, \ldots, z_m \rangle \hat{\otimes}_{\mathbb{C}} \mathbb{C}\langle w_1, \ldots, w_n \rangle \arrow[r] \arrow[d] & (\mathbb{C}\langle z_1, \ldots, z_m \rangle \overline{\otimes} \mathbb{C}\langle w_1, \ldots, w_n \rangle)\hat{} \arrow[d] \\
A_1 \hat{\otimes}_{\mathbb{C}} A_2 \arrow[r] & (A_1 \overline{\otimes} A_2)\hat{}
\end{tikzcd}.$$

We already know tha the upper arrow is an isomorphism, it suffices to identify the kernels of the two vertical maps. The kernel of the right vertical map before completion is given by Lemma 5.6, namely, the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$. Accordingly, the kernel of the right vertical map is the closure of the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$ by [Stacks, Tag 0ARZ]. On the other hand, it follows from [Stacks, Tag 0ARZ] that the kenrel of the left vertical map is the closure of the ideal generated by $I_1 \otimes 1 + 1 \otimes I_2$. We conclude. □

Theorem 5.9. Let B be a complex analytic local algebra and $A_1, A_2 \in \mathbb{C}\text{-}\mathcal{L}\mathrm{A}_{\backslash B}$. Then $A_1 \overline{\otimes}_B A_2$ exists. Moreover, there is a natural identification of adic rings

$$A_1 \hat{\otimes}_B A_2 \xrightarrow{\sim} (A_1 \overline{\otimes}_B A_2)\hat{}.$$

Here on the right-hand side, we take the adic completion with respect to the maximal ideal in $A_1 \overline{\otimes}_B A_2$.

Proof. Observe that we have a natural map $B \overline{\otimes} B \to B$: at the level of functor of points, we simply define

$$h^B \to h^{B \overline{\otimes} B}$$

by sending an arrow $f : B \to C$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ to $(f, f) \in h^{B \overline{\otimes} B}(C)$. We claim that $B \overline{\otimes} B \to B$ is surjective. In fact, it is easy to construct a section $B \to B \overline{\otimes} B$, which at the level of functor of points, sends a pair of morphisms $(f_1 : B \to C, f_2 : B \to C)$ in $\mathbb{C}\text{-}\mathcal{L}\mathrm{A}$ to f_1.

It follows from general abstract nonsense that the tensor product

$$(A_1 \overline{\otimes} A_2) \overline{\otimes}_{B \overline{\otimes} B} B$$

represents $A_1 \overline{\otimes}_B A_2$. So we are reduced to the case where $B \to A_2$ is surjective. Let I denote the kernel of the map $B \to A_2$. We denote by J the image of I in A_1. It is obvious that $A_1 / J A_1$ is the desired tensor product.

In order to compute the completed local ring, we similarly reduce to the case where $B \to A_2$ is surjective with kernel I. In this case, $A_1 \overline{\otimes}_B A_2$ is the quotient of A_1 by the ideal generated by I. So after taking completion, $(A_1 \overline{\otimes}_B A_2)\hat{}$ is the quotient of $\hat{A}_1$ by the closed ideal generated by I by [Stacks, Tag 0ARZ]. On the other hand, $A_1 \hat{\otimes}_B A_2$ is also the quotient of $\hat{A}_1$ by the closed ideal generated by I by [Stacks, Tag 0AMZ]. We conclude. □

Remark 5.10. One should remark that in general, the completed tensor products of local rings are not local.

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