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Analytic sets

1. Introduction

2. Remmert–Stein theorem

Lemma 2.1. Let $n \in \mathbb{N}$ and U be a relatively compact open neighbourhood of 0 in $\mathbb{C}^n$. Let $k \in \{0, 1, \ldots, n-1\}$. We write L^k for the intersection of $z_1 = \cdots = z_{n-k} = 0$ with U, where $z_1, \ldots, z_n$ are the coordinates on $\mathbb{C}^n$. Let A be an analytic set in $U \setminus L^k$ of dimension $\leq k$. Then for $i = 0, \ldots, k$, we can find a linear subspace L' of $\mathbb{C}^n$ of dimension $n - k + i$ such that

$$\dim L' \cap A \leq i, \quad \dim L' \cap L^k \leq i.$$

Proof. We make an induction on n. When $n = 0, 1$, there is nothing to prove. Let $n > 1$. If $i = k$, we just take $L' = \mathbb{C}^n$. Assume $0 \leq i < k$.

Let $M_1, \ldots, M_N$ be the irreducible components of A. We may assume that no components are single points. Take a non-zero base point $p_j \in M_j$ for $j = 1, \ldots, N$. Let H be an $(n-1)$-dimensional linear subspace of $\mathbb{C}^n$ which does not contain L^k or any of the points $p_1, \ldots, p_N$. Without loss of generality, we may guarantee that H is given by $z_n = 0$.

Let k_j denote the dimension of M_j for $j = 1, \ldots, N$. Let $M'_j = M_j \cap H$ for $j = 1, \ldots, N$. Observe that the dimension of M'_j is either k_j or $k_j - 1$ for $j = 1, \ldots, N$. Let

$$M' := \bigcup_{i=1}^{N} M'_i.$$

Then $\dim M' \leq k - 1$. By the inductive hypothesis, we can find a linear subspace L' of $\mathbb{C}^n$ of dimension $n - k + 1$ with the desired properties. □

Lemma 2.2. Let $k \leq n$ be two elements in $\mathbb{N}$ and $D = \Delta^k \times \Delta^{n-k}$ be the product of two unit polycylinders. Write L for $\Delta^k \times \{0\}$. Consider a non-empty analytic subset M of $D \setminus L$ of dimension k everywhere. Assume that M does not intersect a neighbourhood of $\Delta^k \times \{y \in \mathbb{C}^{n-k} : \|y\|_{L^\infty} = 1\}$. Then for any $\epsilon > 0$, M meets the polycylinder $\{(x, y) \in D : \|x\|_{L^\infty} < \epsilon, |y|_{L^\infty} \in (0, 1)\}$.

Proof. **Step 1**. We observe that for each $a \in \Delta^k$, the intersection

$$\{(x, y) \in D : x = a\} \cap M$$

is discrete. In fact, by our assumption, the absolute values the coordinate functions of Δ^{n-k} obtain their maxima on each irreducible component of the intersection. By Corollary 4.23 in Morphisms between complex analytic spaces, these coordinates are all constant.

Step 2. Let $(x^1, y^1) \in M$. Then $y^1 \neq 0$ by assumption. We may assume that $x^1 \neq 0$ as otherwise there is nothing to prove. Let us write $x^1 = (x^1_1, \dots, x^1_k)$, $y^1 = (y^1_1, \dots, y^1_{n-k})$ with $x^1_1 \neq 0$ and $y^1_1 \neq 0$.

Let $b = (x^1_2, \dots, x^1_k)$. Let N be the intersection of M with $\Delta \times \{b\} \times \Delta^{n-k}$. Then N is non-empty and has dimension 1 everywhere. In fact, by Krulls Hauptidealsatz, the dimension of N at each point is at least 1. By Step 1, the dimension is at most 1.

We argue that we can take $|z_1|$ on M as small as wel wish. Suppose otherwise,

$$\sup_{z \in M} |z_1| > 0.$$

Tkae $q \in \mathbb{Z}_{>0}$ with

$$|x^1_1|^q < |y^1_1|.$$

Consider the function $f : N \to \mathbb{C}$ sending (x, y) to y_1/x^q_1. Then f is a morphism of complex analytic spaces and is bounded, say

$$\sup_{(x,y) \in N} |f(x, y)| = C_0.$$

Then $C_0 > 1$ by our choice of q. But at the boundary of D, $|z_1| = 1$, so we find that $|f(x, y)|$ obtains its maximum on each irreducible component of N. So in particular, $|z_1|$ obtains its infimum on each irreducible component of N. This contradicts the fact that N has dimension 1 everywhere.

We can now assume that $|x^1_1| < \epsilon$. Now we can replace M by $\{x^1_1\} \times \Delta^{k-1} \times \Delta^{n-k}$ and reduce the value of k by 1. By induction, we conclude. $\square$

Lemma 2.3 (Fundamental lemma)**.** Let X be a complex manifold and F be a nowhere dense analytic set of dimension $\leq k$, where $k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ such that for any $x \in E$,

$$\dim_x E = k.$$

Then

$$\left\{x \in F : \bar{E} \text{ is analytic at } x\right\}$$

is clopen in X.

Proof. The given set is clearly open. It suffices to show that it is closed.

Let $p \in F$ be a point in the closure of the given set. We need to show that $\bar{E}$ is analytic at p. The problem is local on X, we may assume that X is a complex model space. Then it is immediate that we can reduce to the case where X is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. By enlarging F, we may assume that F is defined by $y = 0$, where x, y denote the first k and the last $n - k$ coordinates on $X \subseteq \mathbb{C}^n$. Finally, we may assume that $p = 0$.

By Lemma 2.1, we can take a linear subspace L of $\mathbb{C}^n$ which meets F and E only at discrete points. We may arrange that L is defined by the condition $x = 0$.

Take $\epsilon, \delta > 0$ so that

(1)
$$S := \left\{(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \|x\|_{L^\infty} < \epsilon, \|y\|_{L^\infty} < \delta\right\} \subseteq D;$$

(2)
$$\left\{(x, y) \in \mathbb{C}^k \times \mathbb{C}^{n-k} : \|x\|_{L^\infty} < \epsilon, \|y\|_{L^\infty} = \delta\right\} \cap E = \emptyset.$$

Observe that for all $a \in \mathbb{C}^k$, $\|a\|_{L^\infty} < \epsilon$, the intersection

$$(\{a\} \times \mathbb{C}^{n-k}) \cap E \cap \operatorname{Int} S$$

is discrete. In fact, the intersection is an analytic set in $S \setminus F$ and the absolute values of $y_1, \dots, y_{n-k}$ take their maxima on each irreducible components by (2). So they are in fact constant.

By our assumption, there are points at which $\bar{E}$ is analytic on $Z := \{|x| < \epsilon, y = 0\}$. Let B_0 be a connected component of the set of such points. We can equivalently view B_0 as an open subset of $\{|x| < \epsilon\}$. Then for any $a \in B_0$, the set

$$F_a := \{(x,y) \in \mathbb{C}^n : x = a\} \cap \bar{E} \cap \operatorname{Int} S$$

is discrete. Let (x^1, y^1) be a point in this set, then $\bar{E}$ is equidimensional of dimension k at this point. Each irreducible component K_j at (x^1, y^1) is a ramified covering of order m_j. We define the order $m(x^1, y^1)$ as this sum.

For each $a \in B_0$, we define $s(a)$ as the sum of multiplicities of points of F_a. Then $s(a)$ is locally constant on B_0 and by (2), $s(a)$ is actually constant. Let s be this common value.

Assume that $\bar{E}$ is not analytic at 0. Then B_0 meets $|x| = \epsilon$, say at x'. Let s' be the number of intersection points of $\{x = x'\} \cap E$ counting mulitiplicity.

Observe that $s' \leq s$, as otherwise, there will be more than s points of E over points of B_0 close to x'. But $s' \neq s$ as otherwise, we contradict Lemma 2.2.

So $s' < s$. If $x \in B_0$ converges to x', then at lest one of the s points of $\bar{E}$ over x converges to $(x', 0)$ and the coordinates $y_1, \dots, y_{n-k}$ of this point converge to 0. The same holds for all boundary points of B_0 in $\{\|x\|_{L^\infty} < \epsilon\}$.

We introduce $n-k$ unknowns $X_1, \dots, X_{n-k}$ and set

$$z = \sum_{j=1}^{n-k} y_j X_j.$$

If (x, y^i) $(i = 1, \dots, s)$ denotes the s-points of $\bar{E}$ lying over $x \in B_0$, then we set

$$z^i := \sum_{j=1}^{n-k} y^i_j X_j$$

for $i = 1, \dots, s$. Then $z^1 \cdots z^s$ is a homogeneous polynomial of degree s. The coefficients are holomorphic on B_0 by Riemann extension theorem. As B_0 is not contained in $\bar{E}$, the coefficients are not all 0.

If $x \in B_0$ converges to a boundary point of B_0 in $\{\|x\|_{L^\infty} < \epsilon\}$, then all coefficients converge to 0.

By Proposition 4.44 in Morphisms between complex analytic spaces, we conclude that the boundary points of B_0 in the interior of $\{\|x\|_{L^\infty} < \epsilon\}$ lie in an analytic subset of codimension 1.

Let $Q(z) = (z - z^1) \cdots (z - z^s)$. Hen Q is a homogeneous polynomial of degree s with respect to the u_j's. The coefficients are holomorphic on $x \in B_0$ and are polynomials in the y_i's. The vanishing of the coordinates defines exactly the part of $\bar{E}$ over B_0 in the interior of S. But the coefficients are bounded at the boundary, so they extend to holomorphic functions everywhere and in particular on $\{\|x\|_{L^\infty} < \epsilon\}$. The vanishing of the coefficients define an analytic set E' in $B_0 \times \{\|y\|_{L^\infty} < \delta\}$. Each point of E' belongs to the part of $\bar{E}$ lying over B_0. So $\bar{E}$ is analytic at each point of $\{\{(x,0) : \|x\|_{L^\infty} < \epsilon\} < \epsilon\}$. In particular, $B_0 = \{\{\|x\|_{L^\infty} < \epsilon\} < \epsilon\}$. This is a contradiction. □

Theorem 2.4. Let X be a complex manifold and F be a nowhere-dense analytic set in X of dimension $\leq k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ all of whose irreducible components are of dimension $\geq k$ on each point. Consider a point $x \in F$ with $\dim_x F < k$. Then $\bar{E}$ is analytic at x.

Proof. Let $r = \dim_x F$. The problem is local. By Theorem 2.4 in Local properties of complex analytic spaces, we may assume that F is of dimension $\leq r$ everywhere. We need to show that $\bar{E}$ is an analytic set in X. By induction on r, we may clearly assume that F is a complex manifold of equidimension r with respect to the reduced indudced structure.

Again, as the problem is local, we may reduce to the case where X is a complex model space and then to the case where X is an open neighbourhood of 0 $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Let $p \in F$, we want to show that $\bar{E}$ is analytic at p. We may then assume that $p = 0$. We can then rearrange F so that F is a linear subspace of dimension r_0. We can take a closed subspace V of X such that $V \setminus F$ intersects E at an analytic subset of dimension $< k$. Let $E_1 = E \setminus V$. Then

$$\overline{E_1} = \bar{E}.$$

As $\overline{E_1}$ is analytic at all points in $V \setminus F$, it follows from Lemma 2.3 that $\overline{E_1}$ is analytic on all points of V. So $\bar{E}$ is analytic at points in F.

□

Theorem 2.5 (Remmert–Stein)**.** Let X be a complex analytic space and F be a nowhere-dense analytic set in X of dimension $\leq k \in \mathbb{N}$. Let E be an analytic set in $X \setminus F$ all of whose irreducible components are of dimension $\geq k$ on each point. Then

$$\left\{x \in F : \bar{E} \text{ is not analytic at } x\right\}$$

is an analytic set of dimension k at each point.

Proof. The problem is local on X, so we may assume that X is a complex model space. Then we reduce immediately to the case where X is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In particular, we may assume that X is a complex manifold.

Let F' be set of regular points of F of dimension k and $F'_0 \subseteq F$ be the set of points where $\bar{E}$ is analytic. Then F'_0 is the union of some connected components of F_0 by Lemma 2.3.

Let F'_1 be the union of the other connected components of F'. Observe that $G := \overline{F'_1}$ is an analytic subset of F. Observe that $\bar{E}$ is analytic at no points of G. It suffices to show that $\bar{E}$ is analytic at all points of $F \setminus G$.

Let $p \in F \setminus G$. We show that $\bar{E}$ is If $\dim_p F < k$, we just apply Theorem 2.4. So we may assume that $\dim_p F = k$. By our choice, $p \in \overline{F'_0}$. In a neighbourhood of p, the subset of F consisting of points where $\bar{E}$ is not analytic is contained in $\overline{F'_0} \setminus F'_0$, which is an analytic set of dimension $< k$. We conclude again by Theorem 2.4. □

Corollary 2.6. Let $f : X \to Y$ be a morphism of complex analytic spaces and $n \in \mathbb{N}$. Assume that X is a complex manifold. Then

$$\left\{x \in X : \dim_x f^{-1}(x) \geq n\right\}$$

is closed.

PROOF. Let $x \in X$, $\dim_x f^{-1}(x) = n$. We need to show that the fiber dimension in a neighbourhood of x is at most n.

The problem is local, so we may assume that Y is Hausdorff. Suppose our assertion is false, then we can find a sequence $x_i \in X$ convering to x such that $\dim_{x_i} f^{-1}(x_i) > d$ for all $i \in \mathbb{Z}_{>0}$. Let E_i be the irreducible component of $f^{-1}(x_i)$ containing x_i such that $\dim_{x_i} E_i = \dim_{x_i} f^{-1}(x_i)$ for $i \in \mathbb{Z}_{>0}$.

We may assume that E_i's have the same dimension $d > n$ and x_i and x are all different. Let M be the union of the E_i's, then M is an analytic set in $X \setminus f^{-1}(x)$. By Theorem 2.5, $\bar{M}$ is analytic near x. This is absurd. □

Corollary 2.7 (Remmert)**.** Let $f : X \to Y$ be a morphism of complex analytic spaces and $n \in \mathbb{N}$. Then

$$\left\{x \in X : \dim_x f^{-1}(x) \geq n\right\}$$

is an analytic set in X.

This result is not stated in the correct way in Remmert's paper. In most of Remmert's papers, the notion of codimension is misused.

PROOF. By Corollary 2.6, the given set is closed. It suffices to show that it is analytic along each point on X. In particular, we may assume that X is connected.

Step 1. We reduce to the case where Y is a complex manifold.

The problem is local on Y, so we may assume that Y is a complex model space. Then clearly, we can assume that Y is a domain in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. In particular, Y is a complex manifold.

Step 2. We first handle the case where X is a complex manifold and the rank of $\Omega_{X/Y}$ is constant.

In this case, we simply observe that $\dim_x f^{-1}(x) = \operatorname{rank}_x \Omega_{X/Y}$ and our assertion is obvious.

Step 3. The problem is local on X, so we may assume that $\dim X < \infty$.

Let

$$B = \left\{x \in X^{\mathrm{reg}} : \operatorname{rank}_x \Omega_{X/Y} > \tau\right\},$$

where

$$\tau := \min_{x' \in X} \operatorname{rank}_{x'} \Omega_{X/Y}.$$

Then B is an analytic set in X^{reg} by Step 2. The closure $\bar{B}$ is an analytic set in X, as this can be characterized by the condition that $\operatorname{rank}_x \Omega_{X/Y} > \tau$. Moreover, $\dim \bar{B} < N$.

We may assume that $n > \tau$, as there is nothing to prove otherwise. In particular,

$$\left\{x \in X : \dim_x f^{-1}(x) \geq n\right\} \subseteq \bar{B} \cup X^{\mathrm{Sing}}.$$

We write $D = \bar{B} \cup X^{\mathrm{Sing}}$ and endow it with the reduced induced structure.

We make induction on $N := \dim X$. The problem is trivial when $N = 0$. Assume that $N \geq 1$. Then

$$\left\{x \in D_0 : \dim_x f^{-1}(x) \geq n\right\}$$

is an analytic set in D for each connected component D_0 of D.

We observe that

$$\left\{x \in X : \dim_x f^{-1}(x) \geq n\right\} = \bigcup_{D_0} \left\{x \in D_0 : \dim_x f^{-1}(x) \geq n\right\},$$

where D_0 runs over all connected components of D and $N - s_0$ is the dimension of D_0. From this it follows that $\left\{x \in X : \dim_x f^{-1}(x) \geq n\right\}$ is analytic, as the formula union on the right-hand side is locally finite. □

Corollary 2.8. Let $f : X \to Y$ be a morphism of complex analytic spaces and $x \in X$. Assume that X is equidimensional at x, Y is unibranch at $f(x)$ and

$$\dim_x X - \dim_x f^{-1}(f(x)) = \dim_{f(x)} Y.$$

Then there is an open neighbourhood U of x in X such that $U \to Y$ induced by f is open.

Proof. The problem is local on X. By Theorem 2.4 in Local properties of complex analytic spaces, up to shrinking X, we may assume that X is equidimensional of dimension $\dim_x X$. By Corollary 4.19 in Morphisms between complex analytic spaces,

$$\dim_x X - \dim_z f^{-1}(f(z)) \leq \dim_{f(z)} Y$$

for all $z \in X$. But as $\dim_z f^{-1}(f(z))$ is upper semi-continuous by Corollary 2.7, the set where equality holds is open. Our assertion follows from Corollary 4.19 in Morphisms between complex analytic spaces. □

Corollary 2.9. Let X be a complex analytic space, $x \in X$ and $f \in \mathcal{O}_X(X)$. Assume that $f(x) = 0$. Consider the following conditions:

(1) $f : X \to \mathbb{C}$ is open in a neighbourhood of x;
(2) f_x is a non-zero divisor in modulo each minimal prime of $\mathcal{O}_{X,x}$;
(3) $f : X \to \mathbb{C}$ is open at x.

Then (1) implies (2) implies (3). If moreover X is equidimensional at x, then (1) is equivalent to (2).

Proof. (1) $\implies$ (2): If f_x is a zero-divisor modulo some minimal prime of $\mathcal{O}_{X,x}$, then f is identically 0 on some irreducible component up to shrinking X. So f cannot be open in a neighbourhood of x.

(2) $\implies$ (3): By Krulls Hauptidealsatz,

$$\dim_x W(f) = \dim_x X - 1.$$

By Corollary 4.19 in Morphisms between complex analytic spaces, f is open at x.

(2) $\implies$ (1): If X is equidimensional at x, then by Krulls Hauptidealsatz,

$$\dim_x W(f) = \dim_x X - 1.$$

We conclude by Corollary 2.8. □

Corollary 2.10. Let $f : X \to Y$ be an open morphism of complex analytic spaces and $x \in X$. Assume that Y is equidimensional at $f(x)$, then for any $g \in \mathfrak{m}_{Y,f(x)}$ which is a non-zero divisor modulo each minimal prime, then $f_x^{\#}(g) \in \mathfrak{m}_{X,x}$ is also a non-zero divisor modulo each minimal prime.

Proof. The problem is local on X and Y. Up to shrinking X and Y, we may assume that g and f spreads to morphisms $Y \to \mathbb{C}$ and $X \to \mathbb{C}$ such that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & \mathbb{C} \end{tikzcd}.$$

The morphism $Y \to \mathbb{C}$ is open by Corollary 2.9. It follows that α is also open. We conclude again by Corollary 2.9. □

Corollary 2.11. Let $f : X \to Y$ be a morphism of complex analytic spaces. Assume that Y is equidimensional. Consider the following conditions:

(1) f is open;
(2) For any $x \in X$,

$$\dim_x X - \dim_x f^{-1}(f(x)) = \dim_{f(x)} Y.$$

Then (1) implies (2). If moreover, Y is unibranch, then (1) and (2) are equivalent.

Proof. (2) $\implies$ (1): Suppose that Y is unibranch. This is a consequence of Corollary 4.20 in Morphisms between complex analytic spaces.

(1) $\implies$ (2): We may assume that Y is connected and X, Y are reduced. Fix $x \in X$ and write $y = f(x)$. We make an induction on $n = \dim Y$. When $n = 0$, the assertion is trivial. Take $g \in \mathfrak{m}_{Y,y}$ which is a non-zero divisor modulo each minimal prime in $\mathcal{O}_{Y,y}$. By Corollary 2.10, $h := f_x^{\#}(g) \in \mathfrak{m}_{X,x}$ is also a non-zero divisor modulo each minimal prime. Let X' and Y' be the closed analytic spaces of X and Y defined by h and g respectively. Up to shrinking X and Y, we may assume that there is a commutative square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}.$$

By inductive hypothesis,

$$\dim_x X' = \dim_x X'_y + \dim_y Y'.$$

We conclude using Krulls Hauptidealsatz. □

Corollary 2.12. Let $f : X \to Y$ be a flat morphism of complex analytic spaces. Then f is open.

Proof. **Step 1**. If Y is unibranch, then we conclude using Corollary 2.11 and Proposition 5.3 in Morphisms between complex analytic spaces.

Step 2. In general, we may assume that Y is reduced. Let $\bar{Y}$ be the normalization of Y. Then Y has the quotient topology with respect to $\bar{Y} \to Y$. So it suffices to show that the base change $X \times_Y \bar{Y} \to \bar{Y}$ is open. But we know that the latter is flat by Proposition 5.2 in Morphisms between complex analytic spaces. We conclude using Step 1. □

Corollary 2.13. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then

$$\{x \in X : f \text{ is quasi-finite at } x\}$$

is co-analytic.

Proof. This follows immediately from Corollary 2.7. □

As an application of Remmert–Stein theorem, we prove Chow's theorem.

Theorem 2.14. Let $n \in \mathbb{N}$ and X be a closed analytic subspace of $\mathbb{P}^n$. Then X is the analytification of a closed subvariety of $\mathbb{P}^n$.

PROOF. We may assume that X is non-empty. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the projection and $Y = \pi^{-1}(X)$. Then X is analytic in $\mathbb{C}^{n+1} \setminus \{0\}$. By Theorem 2.5, $\bar{X}$ is an analytic set in $\mathbb{C}^{n+1}$.

Choose an open ball U in $\mathbb{C}^{n+1}$ centered at 0 and finitely many holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_{\mathbb{C}^{n+1}}(U)$ such that $\bar{X} \cap U = W(f_1, \dots, f_k)$. Let $\mathcal{P}$ be the collection of homogeneous components of the f_i's. Then

$$X = \bigcap_{f \in \mathcal{P}} W(f).$$

In fact, let us denote the right-hand side by Y for the moment. It is clear that $\bar{X} \cap U$ contains $Y \cap U$ and hence $\bar{X} \supseteq Y$. Conversely, take $x \in \bar{X} \cap U$, from the fact that $\lambda_x \in \bar{X} \cap U$ for all $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we find easily that all homogeneous components of the f_i's vanishes at x. So $x \in Y$. We conclude that $\bar{X} \subseteq Y$.

Now as $\mathbb{C}[X_0, \dots, X_n]$ is noetherian, we may take a finite subcollection $\mathcal{P}'$ of $\mathcal{P}$ such that

$$X = \bigcap_{f \in \mathcal{P}'} W(f).$$

□

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