Commutative algebra

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Let G be an Abelian group. We write the group operation of G multiplicatively and denote the identity of G as 1.

Definition 2.1. Let A be an Abelian group. A *G-grading* on A is a coproduct decomposition

$$A = \bigoplus_{g \in G} A_g$$

of Abelian groups such that $A_g \subseteq A$. An Abelian group with a G-grading is called a *G-graded Abelian group*.

An element $a \in A$ is said to be *homogeneous* if there is $g \in G$ such that $a \in A_g$. If a is furthermore non-zero, we write $g = \rho(a)$. We set $\rho(0) = 0$. We will write $\rho(A)$ for the set of $\rho(a)$ when a runs over all homogeneous elements in A.

A *G-graded homomorphism* between G-graded Abelian groups A and B is a homogeneous of the underlying Abelian groups $f : A \to B$ such that $f(A_g) \subseteq B_g$ for any $g \in G$.

The category of G-graded Abelian groups is denoted by $\mathcal{A}\mathrm{b}^G$.

Remark 2.2. A usual Abelian group A can be given the *trivial G-grading*: $A_0 = A$ and $A_g = 0$ for $g \in G$, $g \neq 0$. In this way, we find a fully faithful embedding

$$\mathcal{A}\mathrm{b} \to \mathcal{A}\mathrm{b}^G.$$

When we regard an Abelian group as a G-graded Abelian group and there are no natural gradings, we always understand that we are taking the trivial G-grading.

More generally, let G' be a subring of G. Then any G'-graded Abelian group can be canonically identified with a G-graded Abelian group: for the extra pieces in the grading, we simply put 0.

The same remark applies to all the other constructions in this section, which we will not repeat.

Definition 2.3. A *G-graded ring* is a commutative ring A endowed with a G-grading:

$$A = \bigoplus_{g \in G} A_g$$

as Abelian groups and such that

(1) $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$;
(2) $1 \in A_1$.

A *G-graded homomorphism* of G-graded rings A and B is a ring homomorphism $f : A \to B$ such that $f(A_g) \subseteq B_g$ for each $g \in G$. A *G-graded subring* of a G-graded ring B is a subring A of B such that the grading on B restricts to a grading on A.

The category of G-graded rings is denoted by $\mathcal{R}\mathrm{ing}^G$.

Example 2.4. Let A be a G-graded ring, $n \in \mathbb{N}$ and $g = (g_1, \ldots, g_n) \in G^n$. Then there is a unique G-grading on $A[T_1, \ldots, T_n]$ extending the grading on A and such that $\rho(T_i) = g_i$ for $i = 1, \ldots, n$. We will denote $A[T_1, \ldots, T_n]$ with this grading as $A[g_1^{-1}T_1, \ldots, g_n^{-1}T_n]$ or simply $A[g^{-1}T]$.

Example 2.5. Let A be a G-graded ring and S be a multiplicative subset of A consisting of homogeneous elements, then $S^{-1}A$ has a natural G-grading. To see this, recall the construction of $S^{-1}A$ in [Stacks, Tag 00CM]. One defines an equivalence relation on $A \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}A)_g$ as the set of (x, s) for all $s \in S$ and $x \in A_{g\rho(s)}$. It is easy to verify that this is a well-defined G-grading on $S^{-1}A$. Add details.

In particular, if $f \in A$ is a non-zero homogeneous element, then we define A_f as $S^{-1}f$ with $S = \{f^n : n \in \mathbb{N}\}$.

Definition 2.6. Let A be a G-graded ring. A *G-homogeneous ideal* in A is an ideal I in G such that if $a \in A$ can be written as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all $a_g = 0$, then $a_g \in I$.

Example 2.7. Let A be a G-graded ring and $n \in \mathbb{N}$ and $a_1, \ldots, a_n$ be homogeneous elements in A. Then $a_1, \ldots, a_n$ generate a G-homogeneous ideal $(a_1, \ldots, a_n)$ as follows:

$$(a_1, \ldots, a_n)_g = \sum_{i=1}^{n} A_{g\rho(a_i)^{-1}} a_i$$

for any $g \in G$.

Lemma 2.8. Let $f : A \to B$ be a G-homomorphism of G-graded rings. Then $\ker f$ is a G-homogeneous ideal in A.

Proof. We need to show that

$$\ker f = \sum_{g \in G} (\ker f) \cap A_g.$$

Take $x \in \ker f$, we can write x as

$$\sum_{g \in G} a_g, \quad a_g \in A_g$$

and almost all a_g's are 0. Then

$$f(x) = \sum_{g \in G} f(a_g), \quad f(a_g) \in B_g.$$

It follows that $f(a_g) = 0$ for each $g \in G$ and hence $a_g \in (\ker f) \cap A_g$. ☐

Definition 2.9. Let A be a G-graded ring and I be a G-homogeneous ideal in A. Then we define a G-grading on A/I as follows: for any $g \in G$

$$(A/I)_g := (A_g + I)/I.$$

Proposition 2.10. Let A be a G-graded ring and I be a G-homogeneous ideal in A. Then the construction in Definition 2.9 defines a grading on A/I. The natural map $\pi : A \to A/I$ is a G-homomorphism.

For any G-graded ring B and any G-homomorphism $f : A \to B$ such that $I \subseteq \ker A$, there is a unique G-homomorphism $f' : A/I \to B$ such that $f' \circ \pi = f$.

Proof. We first argue that for different $g, h \in G$, $(A/I)_g \cap (A/I)_h = 0$. Suppose $x \in (A/I)_g \cap (A/I)_h$, we can lift x to both $y_g + i_g \in A$ and $y_h + i_h \in A$ with $y_g, y_h \in A$ and $i_g, i_h \in I$. It follows that $y_g - y_h \in I$. But I is a G-homogeneous ideal, so it follows that $y_g, y_h \in I$ and hence $x = 0$.

Next we argue that

$$A/I = \sum_{g \in G} (A/I)_g.$$

Lift an element $x \in A/I$ by $a \in A$, we represent a as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all a_g's equal to 0. Then x can be represented as

$$x = \sum_{g \in G} \pi(a_g).$$

We have shown that the construction in Definition 2.9 gives a G-grading on A. It is clear from the definition that π is a G-homomorphism.

Next assume that B and f are given as in the proposition. Then there is a ring homomorphism $f' : A/I \to B$ such that $f = f' \circ \pi$. We need to argue that f' is a G-homomorphism. For this purpose, take $g \in G$, $x \in (A/I)_g$, we need to show that $f'(x) \in B_g$. Lift x to $y + i$ with $y \in A_g$ and $i \in I$, then we know that $f'(x) = \pi(y + i) = \pi(y) \in B_g$. □

Definition 2.11. Let A be a G-graded ring.

Let M an A-module which is also a G-graded Abelian group. We say M is a *G-graded A-module* if for each $g, h \in G$, we have

$$A_g M_h \subseteq M_{gh}.$$

A *G-graded homomorphism* of G-graded A-modules M and N is an A-module homomorphism $f : M \to N$ which is at the same time a homomorphism of the underlying G-graded Abelian groups.

The category of G-graded A-modules is denoted by $\mathcal{M}\mathrm{od}_A^G$.

A *G-graded A-algebra* is a G-graded ring B together with a G-graded ring homomorphism $A \to B$ such that B is also a G-graded A-module.

A *G-graded homomorphism* between G-graded A-algebras B and C is a G-graded homomorphism between the underlying G-graded rings that is at the same time a G-graded homomorphism of G-graded A-modules.

Observe that G-homogeneous ideals of A are G-graded submodules of A. Also observe that $\mathcal{M}\mathrm{od}_{\mathbb{Z}}^G$ is isomorphic to $\mathcal{A}\mathrm{b}^G$.

Proposition 2.12. Let A be a G-graded ring. Then $\mathcal{M}\mathrm{od}_A^G$ is an Abelian category satisfying AB5.

PROOF. We first show that $\mathcal{M}\mathrm{od}_A^G$ is preadditive. Given $M, N \in \mathcal{M}\mathrm{od}_A^G$, we can regard $\mathrm{Hom}_{\mathcal{M}\mathrm{od}_A^G}(M, N)$ as a subgroup of $\mathrm{Hom}_A(M, N)$. It is easy to see that this gives $\mathcal{M}\mathrm{od}_A^G$ an enrichment over $\mathcal{A}\mathrm{b}$.

Next we show that $\mathcal{M}\mathrm{od}_A^G$ is additive. The zero object is clearly given by 0 with the trivial grading. Given $M, N \in \mathcal{M}\mathrm{od}_A^G$, we define

$$(M \oplus N)_g := M_g \oplus N_g, \quad g \in G.$$

This construction makes $M \oplus N$ a G-graded A-module. It is easy to verify that $M \oplus N$ is the biproduct of M and N.

Next we show that $\mathcal{M}\mathrm{od}_A^G$ is pre-Abelian. Given an arrow $f : M \to N$ in $\mathcal{M}\mathrm{od}_A^G$, we need to define its kernel and cokernel. We define

$$(\ker f)_g := (\ker f) \cap M_g$$

and $(\operatorname{coker} f)_g$ as the image of N_g for any $g \in G$. It is straightforward to verify that these are kernels and cokernels.

Next, given a monomorphism $f : M \to N$, it is obvious that the map f is injective and f can be identified with the kenrel of the natural map $N/\operatorname{Im} f$. A dual argument shows that an epimorphism is the cokernel of some morphism as well.

The AB5 condition is easily verified. Expand the details of this argument! □

Example 2.13. This is a continuition of Example 2.5. Let A be a G-graded ring and S be a multiplicative subset of A consisting of homogeneous elements. Consider a G-graded A-module M. We define a G-grading on $S^{-1}M$. Recall that $S^{-1}M$ can be realized as follows: one defines an equivalence relation on $M \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}M)_g$ as the set of (x, s) for all $s \in S$ and $x \in M_{g\rho(s)}$. It is easy to verify that this is a well-defined G-grading on $S^{-1}M$ and $S^{-1}M$ is a G-graded $S^{-1}A$-module. Add details.

Example 2.14. Let A be a G-graded ring and $g \in G$. We define $g^{-1}A$ as the G-graded A-module:

$$(g^{-1}A)_h = A_{g^{-1}h}$$

for any $h \in G$. Observe that $1 \in (g^{-1}A)_g$.

Definition 2.15. Let A be a G-graded ring and M be a G-graded A-module. We say M is *free* if there exists a family $\{g_i\}_{i \in I}$ in G such that

$$M = \coprod_{i \in I} g_i^{-1}A.$$

Definition 2.16. Let $f : A \to B$ be a G-graded homomorphism of G-graded rings. We say f is *finite* (resp. *finitely generated*, resp. *integral*) if it is finite (resp. finitely generated, resp. integral) as a usual ring map.

Proposition 2.17. Let $f : A \to B$ be a G-graded homomorphism of G-graded rings. Then

(1) f is finite if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G-graded homomorphism

$$\bigoplus_{i=1}^{n} (g_i^{-1}A)^n \to B$$

of graded A-modules.

(2) f is finitely generated if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G-graded A-algebra homomorphism

$$A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \to B.$$

(3) f is integral if and only if for any non-zero homogeneous element $b \in B$, there is $n \in \mathbb{N}$ and homogeneous elements $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

(4) A non-zero homogeneous element $b \in B$ is integral over A if there is $n \in \mathbb{N}$ and homogeneous elements $a_1, \ldots, a_n \in A$ such that
$$b^n + f(a_1)b^{n-1} + \cdots + f(a_n) = 0.$$

Proof. (1) The non-trivial direction is the direct implication. Assume that f is finite. Take $b_1, \ldots, b_n \in B$ so that $\sum_{i=1}^n f(A)b_i = B$. Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i's, we may assume that each b_i is homogeneous. We define $g_i = \rho(b_i)$ and the map $\bigoplus_{i=1}^n (g_i^{-1}A)^n \to B$ sends 1 at the i-th place to b_i.

(2) The non-trivial direction is the direct implication. Suppose f is finitely generated, say by $b_1, \ldots, b_n$. Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i's, we may assume that each b_i is homogeneous. Then we define $g_i = \rho(b_i)$ for $i = 1, \ldots, n$ and the A-algebra homomorphism $A[g_1^{-1}T_1, \ldots, g_n^{-1}T_n] \to B$ sends T_i to b_i for $i = 1, \ldots, n$.

(3) Assume that f is integral, then for any non-zero homogeneous element $b \in B$, we can find $a_1, \ldots, a_n \in A$ such that
$$b^n + f(a_1)b^{n-1} + \cdots + f(a_n) = 0.$$
Obviously, we can replace a_i by its component in $\rho(b)^i$ for $i = 1, \ldots, n$ and the equation remains true.

The reverse direction follows from [Stacks, Tag 00GO].

(4) This is argued in the same way as (3). □

Definition 2.18. A G-graded ring A is a *G-graded field* if

(1) $A \neq 0$.
(2) A does not admit any non-zero proper G-homogeneous ideals.

Proposition 2.19. Let A be a non-zero G-graded ring. Then the following conditions are equivalent:

(1) A is a G-graded field.
(2) Any non-zero homogeneous element in A is invertible.

Proof. Assume that A is a G-graded field. Let $a \in A$ be a non-zero homogeneous element. Consider the G-homogeneous ideal (a) generated by a as in Example 2.7. As $a \neq 0$, it follows that $(a) = 1$. Hence, a is invertible.

Conversely, suppose that any non-zero homogeneous element in A is invertible. If I is a non-zero G-homogeneous ideal in A. There is a non-zero homogeneous element $a \in I$. But we know that a is invertible and hence $I = A$. □

Definition 2.20. A G-graded ring A is an *integral domain* if for any non-zero homogeneous elements $a, b \in A$, $ab \neq 0$.

Lemma 2.21. Let A be a G-graded integral domain. Let S denote the set of non-zero homogeneous elemnts in A. Then $S^{-1}A$ is a graded field. The natural map $A \to S^{-1}A$ is injective.

Recall that $S^{-1}A$ is defined in Example 2.5.

Proof. By Proposition 2.19, it suffices to show that each non-zero homogeneous element in $S^{-1}A$ is invertible. Such an element has the form a/s for some homogeneous element $a \in A$ and $s \in S$. As A is a G-graded integral domain, a is invertible and hence $s/a \in S^{-1}A$.

In general, the kernel of the localzation map is given by $\{a \in A :$ there is $s \in S$ such that $sa = 0\}$. As $A \to S^{-1}A$ is a G-graded homomorphism, the kernel is in addition a G-homogeneous ideal in A by Lemma 2.8. So it suffices to show that each homogeneous element in the kenrel vanishes: if $a \in A$ is a homogeneous element and there is $s \in S$ such that $sa = 0$, then $a = 0$. Otherwise, a is invertible by Proposition 2.19, which is a contradiction. $\square$

Definition 2.22. Let A be a G-graded integral domain. We call the graded field defined in Lemma 2.21 the *fraction G-graded field* of A and denote it by $\operatorname{Frac}^G A$.

Definition 2.23. Let A be a G-graded ring. A proper G-homogeneous ideal I in A is called *prime* if the G-graded ring A/I is a G-graded integral domain.

Proposition 2.24. Let A be a G-graded ring and I be a proper homogeneous ideal in A. Then the following are equivalent:

(1) I is a G-graded prime ideal in A.
(2) For any homogeneous elements $a, b \in A$ satisfying $ab \in I$, at least one of a and b lies in I.

Proof. Assume that I is a G-graded prime ideal in A. Let $a, b \in A$ be homogeneous elements satisfying $ab \in I$. Let $\bar{a}, \bar{b}$ be the images of a, b in A/I. Then $\bar{a}, \bar{b}$ are homogeneous and $\bar{a}\bar{b} = 0$. So at least one of $\bar{a}$ and $\bar{b}$ is zero. That is, a or b lies in I.

Conversely, assume that the conditon in (2) is satisfied. Take $x, y \in A/I$ with $xy = 0$. We need to show that at least one of x and y is 0. Lift x and y to $a + i$ and $b + i'$ in A with a, b being homogeneous and $i, i' \in I$. Then $ab \in I$ and hence $a \in I$ or $b \in I$. It follows that $x = 0$ or $y = 0$. $\square$

Definition 2.25. Let A be a G-graded ring and $\mathfrak{p}$ be a G-homogeneous prime ideal in A. Then we define the *G-graded localization* $A^G_{\mathfrak{p}}$ of A at $\mathfrak{p}$ as $S^{-1}A$, where S is the set of homogeneous elements in $A \setminus \mathfrak{p}$.

Similarly, let M be a G-graded A-module. We define the *G-graded localization* $M^G_{\mathfrak{p}}$ as $S^{-1}M$.

Recall that $S^{-1}A$ and $S^{-1}M$ are defined in Example 2.5 and Example 2.13.

Definition 2.26. Let A be a G-graded ring.

A G-homogeneous ideal I in A is said to be *maximal* if it is proper, and it is not contained in any other proper G-homogeneous ideals.

We call A a *G-graded local ring* if it has a unique maximal homogeneous ideal. This ideal is called the *maximal G-homogeneous ideal* of A.

Proposition 2.27. Let A be a G-graded ring and I be a G-homogeneous ideal in A. Then the following are equivalent:

(1) I is a maximal G-homogeneous ideal in A;
(2) A/I is a G-graded field.

In particular, a maximal G-homogeneous ideal is a G-homogeneous prime ideal.

Proof. Assume (1). Then I is a proper ideal, so A/I is non-zero. Suppose that A/I has a proper G-homogeneous ideal J, it lifts to an ideal J' of A. We claim that J' is G-homogeneous. In fact, we set $J'_g := \{x \in A_g : x + I \in J\}$ for $g \in G$, we need to show that

$$J' = \sum_{g \in G} J'_g.$$

For any $j \in J'$, we can expand $j + I$ as $\sum_{g\in G} a_g + I$ with $a_g \in A_g$ and almost all a_g's are 0. We take $i \in I$ so that

$$j = i + \sum_{g\in G} a_g.$$

The desired equation follows. But then it follows that $J' = I$ and hence $J = 0$.

Assume (2). Then I is a proper ideal in A. If J is a G-homogeneous proper ideal of A containing I, then J/I is a G-homogeneous proper ideal of A/I. It follows that $J/I = 0$ and hence $J = I$. □

Corollary 2.28. Let A be a non-zero G-graded ring, then A admits a G-homogeneous prime ideal.

Proof. By our assumption, 0 is a proper ideal in A. By Zorn's lemma, A admits a maximal G-homogeneous ideal, which is prime by Proposition 2.27. □

Lemma 2.29. Let $f : A \to B$ be a G-graded homomorphism of G-graded rings. Let $b_1, \ldots, b_n \in B$ be a finite set of homogeneous elements integral over A, then there is a G-graded A-subalgebra $B' \subseteq B$ containing $b_1, \ldots, b_n$ such that $A \to B'$ is finite.

Proof. We may assume that none of the b_i's is zero. By Proposition 2.17, we can find $m_1, \ldots, m_n \in \mathbb{N}$ and homogeneous elements $a_{i,j} \in A$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$ such that

$$b_i^{m_i} + f(a_{i,1})b_i^{m_i-1} + \cdots + f(a_{i,m_i}) = 0$$

for $i = 1, \ldots, n$. It suffices to take B' as the A-submodule generated by $a_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. □

Proposition 2.30. Let $f : A \to B$ be an injective integral G-graded homomorphism of G-graded rings. Then for any G-homogeneous prime ideal $\mathfrak{p}$ in A, there is a G-homogeneous prime ideal $\mathfrak{p}'$ in B such that $\mathfrak{p} = f^{-1}\mathfrak{p}'$.

Proof. We may assume that $A \neq 0$, as otherwise there is nothing to prove.

It suffices to show that $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. Include a proof We could localize that $\mathfrak{p}$ and assume that $\mathfrak{p}$ is a maximal G-homogeneous ideal. Include details about localization It suffices then to show that $\mathfrak{p}B \neq B$. Assume by contrary that we can write $1 = \sum_{i=1}^n f_i b_i$ for some homogeneous elements $f_i \in \mathfrak{p}$ and some homogeneous elements $b_i \in B$. Let B' be a G-graded subring of B containg A and $b_1, \ldots, b_n$ and such that $A \to B'$ is finite. The existence of B' is guaranteed by Lemma 2.29. Then we find immediately $B' = \mathfrak{m}_A B'$. Then $B' = 0$ by the graded Nakayama's lemma. Include details So $A = 0$, which is a contradiction. □

Lemma 2.31. Let A be a G-graded ring. Then the following are equivalent:

(1) A is a G-graded local ring;
(2) There is a proper homogeneous ideal I in A such that any non-invertible homogeneous element in A is contained in I.

Proof. Assume that (1) holds, let I be the maximal G-homogeneous ideal of A. Let a be a non-invertible homogeneous element in A. Then the image of a in A/I is invertible by Proposition 2.27 and Proposition 2.19.

Assume (2). We show that I is the maximal G-homogeneous ideal in A. By Proposition 2.27, it suffices to show that A/I is a graded field. By Proposition 2.19, we need to show that any non-zero homogeneous element $b \in A/I$ is invertible. Lift b to $a + i \in A$ with $a \in A$ homogeneous and $i \in I$. If a is not invertible, then $a \in I$ by the assumption hence $b = 0$. This is a contradiction. $\square$

Lemma 2.32. Let k be a G-graded field and A be a graded k-algebra. Suppose that $\rho(A) = \rho(k)$, then

(1) For any $g \in G$, there is a natural isomorphism

$$A_g \cong A_1 \otimes_{k_1} k_g.$$

(2) The map $I \mapsto I \cap A_1$ is a bijection between the set of G-homogeneous ideals (resp. G-homogeneous prime ideals) in A and ideals (resp. prime ideals) in A_1.

Proof. (1) Take $g \in \rho(A)$. As $\rho(A) = \rho(k)$, we can take a non-zero homogeneous element $b \in k_g$. Then b and b^{-1} induces inverse bijections between A_1 and A_g.

(2) The part about ideals can be proved in the same way as (1). The part about prime ideals follows easily. $\square$

Proposition 2.33. Let k be a G-graded field and M be a G-graded A-module. Then M is free as G-graded A-module.

Proof. We may assume that $M \neq 0$. Let $\{m_i\}_{i\in I}$ be a maximal set of non-zero homogeneous elements in M such that the corresponding homomorphism

$$F := \bigoplus_{i\in I} (\rho(f))^{-1} k \to M$$

is injective. The existence of $\{m_i\}_{i\in I}$ follows from Zorn's lemma.

If $f \in M/F$ is a non-zero homogeneous element, then we get a homomorphism $(\rho(f))^{-1}k \to M/F$. This map is necessarily injective as $(\rho(f))^{-1}k$ does not have non-zero proper graded submodules. This contradicts the definition of F. $\square$

Corollary 2.34. Let k be a G-graded field, C be a G-graded k-algebra. Consider a G-graded homomorphism of G-graded k-algebras $f : A \to B$. Then the following are equivalent:

(1) f is finite (resp. finitely generated);
(2) $f \otimes_k C$ is finite (resp. finitely generated).

We do not put G-graded structurs on $A \times_k C$, Condition (2) is in the usual sense of commutative algebra.

Proof. (1) $\implies$ (2): This implication is trivial.

(2) $\implies$ (1): By Proposition 2.33, this implication follows from fpqc descent [Stacks, Tag 02YJ]. $\square$

Definition 2.35. Let K be a G-graded field. A G-graded subring $A \subseteq K$ is a *G-graded valuation ring* in K if

(1) A is a local G-graded ring;
(2) the natural map $\mathrm{Frac}^G A \to K$ is an isomorphism;
(3) For any non-zero homogeneous element $f \in K$, either $f \in A$ or $f^{-1} \in A$.

Definition 2.36. Let K be a G-graded field and A, B be G-graded local subrings of K. We say B *dominates* A if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, where $\mathfrak{m}_A$ and $\mathfrak{m}_B$ are the maximal G-homogeneous ideals in A and B.

Proposition 2.37. Let K be a G-graded field and $A \subseteq K$ be a G-graded local subring. Then the following are equivalent:

(1) A is a G-graded valuation ring in K.
(2) A is maximal among the G-graded local subrings of K with respect to the order of domination.

Proof. Assume (1). We may assume that $A \neq K$. Then A is not a G-graded field as $\mathrm{Frac}^G = K$. Let $\mathfrak{m}$ be a maximal G-homogeneous ideal in A. Then $\mathfrak{m} \neq 0$.

We argue first that A is a G-graded local ring. Assume the contrary. Let $\mathfrak{m}' \neq \mathfrak{m}$ be maximal G-homogeneous ideal in A. Choose non-zero homogeneous elements $x, y \in A$ with $x \in \mathfrak{m}' \setminus \mathfrak{m}$, $y \in \mathfrak{m} \setminus \mathfrak{m}'$. Then $x/y \notin A$ as otherwise $x = (x/y)y \in \mathfrak{m}$. Similarly, $y/x \notin A$. This is a contradiction.

Next suppose that A' is a G-graded local subring of K dominating A. Let $x \in A'$ be a non-zero homogeneous element, we need to show that $x \in A$. If not, we have $x^{-1} \in A$ and as x^{-1} is not a unit, $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$, the maximal G-homogeneous ideal in A'. This contradicts the fact that $x \in A'$.

Assume (2). Take a homogeneous element $x \in K \setminus A$, we need to argue that $x^{-1} \in A$. Let A' denote the minimal G-homogeneous subring of K containing A and x. It is easy to see that A' is the usual subring generated by A and x.

By our assumption, there is no G-graded prime ideal of A' lying over $\mathfrak{m}_A$, as otherwise, if $\mathfrak{p}$ is such an ideal, the G-graded local subring $A'^G_{\mathfrak{p}}$ of K dominates A.

In other words, the G-graded ring $A'/\mathfrak{m}_A A'$ does not have any homogeneous prime ideals and hence $A' = \mathfrak{m}_A A'$ by Corollary 2.28.

We can therefore write

$$1 = \sum_{i=0}^{d} t_i x^i$$

with some homogeneous elements $t_i \in \mathfrak{m}_A$. In particular,

$$(1 - t_0)(x^{-1})^d - \sum_{i=1}^{d} t_i (x^{-1})^{d-i} = 0.$$

So x^{-1} is integral over A. Let A'' be the subring of K generated by A and x^{-1}. Then $A \to A''$ is finite and there is a G-homgeneous prime ideal $\mathfrak{m}''$ of A'' lying over $\mathfrak{m}_A$ by Proposition 2.30. By our assumption, $A = A''^G_{\mathfrak{m}''}$ and hence $x^{-1} \in A$.

It remains to verify that $\mathrm{Frac}^G A = K$. Suppose that it is not the case, let $B \subseteq K$ be a G-graded local subring dominating A. Take a homogeneous element $t \in K$ that is not in $\mathrm{Frac}^G A$. Observe that t can not be transcendental over A, as otherwise $A[t] \in K$ is a G-graded subring, and we can localize it at the G-homogeneous prime generated by t and $\mathfrak{m}_A$. We get a G-graded local ring dominating A that is different from A.

So t is algebraic over A. We can then take a non-zero homogeneous $a \in A$ such that at is integral over A. The ring $A' \subseteq K$ generated by A and ta is a G-graded subring and $A \to A'$ is finite. By Proposition 2.30, tehre is a G-homogeneous prime ideal $\mathfrak{m}'$ of A' lifting $\mathfrak{m}_A$. But then $A'^G_{\mathfrak{m}'}$ dominates A and so $A = A'^G_{\mathfrak{m}'}$. It follows that $t \in \mathrm{Frac}^G A$, which is a contradiction. $\square$

Corollary 2.38. Let K be a G-graded field. Any G-graded local subring $B \subseteq K$ is dominated by a G-graded valuation subring of K.

Proof. This follows from Proposition 2.37 and Zorn's lemma. $\square$

2.1. Graded algebraic geometry. Let G be an Abelian group. We write the group operation of G multiplicatively and denote the identity of G as 1.

Definition 2.39. Let A be a G-graded ring. We define the *G-graded affine spectrum* $\operatorname{Spec}^G(A)$ as follows: as a set $\operatorname{Spec}^G(A)$ consists of all G-homogeneous prime ideals of A; we endow $\operatorname{Spec}^G(A)$ with the *Zariski topology*, whose base consists of sets of the form

$$D(f) := \left\{ \mathfrak{p} \in \operatorname{Spec}^G(A) : f \notin \mathfrak{p} \right\}$$

for all homogeneous elements $f \in A$.

Lemma 2.40. Let k be a G-graded field and A be a finitely generated G-graded k-algebra. Then $\operatorname{Spec}^G(A)$ has only finitely many maximal points.

Proof. $\square$

Bibliography

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