

Commutative algebra

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1. Introduction

2. Graded commutative algebra

Let G be an Abelian group. We write the group operation of G multiplicatively and denote the identity of G as 1.

Definition 2.1. Let A be an Abelian group. A G -grading on A is a coproduct decomposition

$$A = \bigoplus_{g \in G} A_g$$

of Abelian groups such that $A_g \subseteq A$. An Abelian group with a G -grading is called a G -graded Abelian group.

An element $a \in A$ is said to be *homogeneous* if there is $g \in G$ such that $a \in A_g$. If a is furthermore non-zero, we write $g = \rho(a)$. We set $\rho(0) = 0$. We will write $\rho(A)$ for the set of $\rho(a)$ when a runs over all homogeneous elements in A .

A G -graded homomorphism between G -graded Abelian groups A and B is a homogeneous of the underlying Abelian groups $f : A \rightarrow B$ such that $f(A_g) \subseteq B_g$ for any $g \in G$.

The category of G -graded Abelian groups is denoted by \mathcal{Ab}^G .

Remark 2.2. A usual Abelian group A can be given the *trivial G -grading*: $A_0 = A$ and $A_g = 0$ for $g \in G$, $g \neq 0$. In this way, we find a fully faithful embedding

$$\mathcal{Ab} \rightarrow \mathcal{Ab}^G.$$

When we regard an Abelian group as a G -graded Abelian group and there are no natural gradings, we always understand that we are taking the trivial G -grading.

More generally, let G' be a subring of G . Then any G' -graded Abelian group can be canonically identified with a G -graded Abelian group: for the extra pieces in the grading, we simply put 0.

The same remark applies to all the other constructions in this section, which we will not repeat.

Definition 2.3. A G -graded ring is a commutative ring A endowed with a G -grading:

$$A = \bigoplus_{g \in G} A_g$$

as Abelian groups and such that

- (1) $A_g A_h \subseteq A_{gh}$ for any $g, h \in G$;
- (2) $1 \in A_1$.

A G -graded homomorphism of G -graded rings A and B is a ring homomorphism $f : A \rightarrow B$ such that $f(A_g) \subseteq B_g$ for each $g \in G$. A G -graded subring of a G -graded ring B is a subring A of B such that the grading on B restricts to a grading on A .

The category of G -graded rings is denoted by \mathcal{Ring}^G .

Example 2.4. Let A be a G -graded ring, $n \in \mathbb{N}$ and $g = (g_1, \dots, g_n) \in G^n$. Then there is a unique G -grading on $A[T_1, \dots, T_n]$ extending the grading on A and such that $\rho(T_i) = g_i$ for $i = 1, \dots, n$. We will denote $A[T_1, \dots, T_n]$ with this grading as $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n]$ or simply $A[g^{-1}T]$.

Example 2.5. Let A be a G -graded ring and S be a multiplicative subset of A consisting of homogeneous elements, then $S^{-1}A$ has a natural G -grading. To see this,

recall the construction of $S^{-1}A$ in [Stacks, Tag 00CM]. One defines an equivalence relation on $A \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}A)_g$ as the set of (x, s) for all $s \in S$ and $x \in A_{g\rho(s)}$. It is easy to verify that this is a well-defined G -grading on $S^{-1}A$. **Add details.**

In particular, if $f \in A$ is a non-zero homogeneous element, then we define A_f as $S^{-1}f$ with $S = \{f^n : n \in \mathbb{N}\}$.

Definition 2.6. Let A be a G -graded ring. A G -homogeneous ideal in A is an ideal I in G such that if $a \in A$ can be written as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all $a_g = 0$, then $a_g \in I$.

Example 2.7. Let A be a G -graded ring and $n \in \mathbb{N}$ and a_1, \dots, a_n be homogeneous elements in A . Then a_1, \dots, a_n generate a G -homogeneous ideal (a_1, \dots, a_n) as follows:

$$(a_1, \dots, a_n)_g = \sum_{i=1}^n A_{g\rho(a_i)^{-1}} a_i$$

for any $g \in G$.

Lemma 2.8. Let $f : A \rightarrow B$ be a G -homomorphism of G -graded rings. Then $\ker f$ is a G -homogeneous ideal in A .

PROOF. We need to show that

$$\ker f = \sum_{g \in G} (\ker f) \cap A_g.$$

Take $x \in \ker f$, we can write x as

$$\sum_{g \in G} a_g, \quad a_g \in A_g$$

and almost all a_g 's are 0. Then

$$f(x) = \sum_{g \in G} f(a_g), \quad f(a_g) \in B_g.$$

It follows that $f(a_g) = 0$ for each $g \in G$ and hence $a_g \in (\ker f) \cap A_g$. \square

Definition 2.9. Let A be a G -graded ring and I be a G -homogeneous ideal in A . Then we define a G -grading on A/I as follows: for any $g \in G$

$$(A/I)_g := (A_g + I)/I.$$

Proposition 2.10. Let A be a G -graded ring and I be a G -homogeneous ideal in A . Then the construction in Definition 2.9 defines a grading on A/I . The natural map $\pi : A \rightarrow A/I$ is a G -homomorphism.

For any G -graded ring B and any G -homomorphism $f : A \rightarrow B$ such that $I \subseteq \ker A$, there is a unique G -homomorphism $f' : A/I \rightarrow B$ such that $f' \circ \pi = f$.

PROOF. We first argue that for different $g, h \in G$, $(A/I)_g \cap (A/I)_h = 0$. Suppose $x \in (A/I)_g \cap (A/I)_h$, we can lift x to both $y_g + i_g \in A$ and $y_h + i_h \in A$ with $y_g, y_h \in A$ and $i_g, i_h \in I$. It follows that $y_g - y_h \in I$. But I is a G -homogeneous ideal, so it follows that $y_g, y_h \in I$ and hence $x = 0$.

Next we argue that

$$A/I = \sum_{g \in G} (A/I)_g.$$

Lift an element $x \in A/I$ by $a \in A$, we represent a as

$$a = \sum_{g \in G} a_g, \quad a_g \in A_g$$

with almost all a_g 's equal to 0. Then x can be represented as

$$x = \sum_{g \in G} \pi(a_g).$$

We have shown that the construction in [Definition 2.9](#) gives a G -grading on A . It is clear from the definition that π is a G -homomorphism.

Next assume that B and f are given as in the proposition. Then there is a ring homomorphism $f' : A/I \rightarrow B$ such that $f = f' \circ \pi$. We need to argue that f' is a G -homomorphism. For this purpose, take $g \in G$, $x \in (A/I)_g$, we need to show that $f'(x) \in B_g$. Lift x to $y + i$ with $y \in A_g$ and $i \in I$, then we know that $f'(x) = \pi(y + i) = \pi(y) \in B_g$. \square

Definition 2.11. Let A be a G -graded ring.

Let M an A -module which is also a G -graded Abelian group. We say M is a G -graded A -module if for each $g, h \in G$, we have

$$A_g M_h \subseteq M_{gh}.$$

A G -graded homomorphism of G -graded A -modules M and N is an A -module homomorphism $f : M \rightarrow N$ which is at the same time a homomorphism of the underlying G -graded Abelian groups.

The category of G -graded A -modules is denoted by Mod_A^G .

A G -graded A -algebra is a G -graded ring B together with a G -graded ring homomorphism $A \rightarrow B$ such that B is also a G -graded A -module.

A G -graded homomorphism between G -graded A -algebras B and C is a G -graded homomorphism between the underlying G -graded rings that is at the same time a G -graded homomorphism of G -graded A -modules.

Observe that G -homogeneous ideals of A are G -graded submodules of A . Also observe that $\text{Mod}_{\mathbb{Z}}^G$ is isomorphic to Ab^G .

Proposition 2.12. Let A be a G -graded ring. Then Mod_A^G is an Abelian category satisfying AB5.

PROOF. We first show that Mod_A^G is preadditive. Given $M, N \in \text{Mod}_A^G$, we can regard $\text{Hom}_{\text{Mod}_A^G}(M, N)$ as a subgroup of $\text{Hom}_A(M, N)$. It is easy to see that this gives Mod_A^G an enrichment over Ab .

Next we show that Mod_A^G is additive. The zero object is clearly given by 0 with the trivial grading. Given $M, N \in \text{Mod}_A^G$, we define

$$(M \oplus N)_g := M_g \oplus N_g, \quad g \in G.$$

This construction makes $M \oplus N$ a G -graded A -module. It is easy to verify that $M \oplus N$ is the biproduct of M and N .

Next we show that Mod_A^G is pre-Abelian. Given an arrow $f : M \rightarrow N$ in Mod_A^G , we need to define its kernel and cokernel. We define

$$(\ker f)_g := (\ker f) \cap M_g$$

and $(\text{coker } f)_g$ as the image of N_g for any $g \in G$. It is straightforward to verify that these are kernels and cokernels.

Next, given a monomorphism $f : M \rightarrow N$, it is obvious that the map f is injective and f can be identified with the kernel of the natural map $N/\text{Im } f$. A dual argument shows that an epimorphism is the cokernel of some morphism as well.

The AB5 condition is easily verified. **Expand the details of this argument!** \square

Next we define the tensor product of G -graded modules.

Definition 2.13. Let A be a G -graded ring and M, N be G -graded A -modules. We define a G -grading on $M \otimes_A N$ as follows: for any $g \in G$, $(M \otimes_A N)_g$ is defined as the image of $\sum_{h \in G} M_h \times N_{gh^{-1}}$ in $M \otimes_A N$. We always endow $M \otimes_A N$ with this G -grading.

verify the universal property; show that this is indeed a grading

Example 2.14. This is a continuation of **Example 2.5**. Let A be a G -graded ring and S be a multiplicative subset of A consisting of homogeneous elements. Consider a G -graded A -module M . We define a G -grading on $S^{-1}M$. Recall that $S^{-1}M$ can be realized as follows: one defines an equivalence relation on $M \times S$: $(x, s) \sim (y, t)$ if there is $u \in S$ such that $(xt - ys)u = 0$. For each $g \in G$, we define $(S^{-1}M)_g$ as the set of (x, s) for all $s \in S$ and $x \in M_{gp(s)}$. It is easy to verify that this is a well-defined G -grading on $S^{-1}M$ and $S^{-1}M$ is a G -graded $S^{-1}A$ -module. **Add details.**

Example 2.15. Let A be a G -graded ring and $g \in G$. We define $g^{-1}A$ as the G -graded A -module:

$$(g^{-1}A)_h = A_{g^{-1}h}$$

for any $h \in G$. Observe that $1 \in (g^{-1}A)_g$.

Definition 2.16. Let A be a G -graded ring and M be a G -graded A -module. We say M is *free* if there exists a family $\{g_i\}_{i \in I}$ in G such that

$$M = \coprod_{i \in I} g_i^{-1}A.$$

Definition 2.17. Let $f : A \rightarrow B$ be a G -graded homomorphism of G -graded rings. We say f is *finite* (resp. *finitely generated*, resp. *integral*) if it is finite (resp. finitely generated, resp. integral) as a usual ring map.

Proposition 2.18. Let $f : A \rightarrow B$ be a G -graded homomorphism of G -graded rings. Then

- (1) f is finite if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G -graded homomorphism

$$\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$$

of graded A -modules.

- (2) f is finitely generated if and only if there are $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and a surjective G -graded A -algebra homomorphism

$$A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B.$$

- (3) f is integral if and only if for any non-zero homogeneous element $b \in B$, there is $n \in \mathbb{N}$ and homogeneous elements $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

- (4) A non-zero homogeneous element $b \in B$ is integral over A if there is $n \in \mathbb{N}$ and homogeneous elements $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

PROOF. (1) The non-trivial direction is the direct implication. Assume that f is finite. Take $b_1, \dots, b_n \in B$ so that $\sum_{i=1}^n f(A)b_i = B$. Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i 's, we may assume that each b_i is homogeneous. We define $g_i = \rho(b_i)$ and the map $\bigoplus_{i=1}^n (g_i^{-1}A)^n \rightarrow B$ sends 1 at the i -th place to b_i .

(2) The non-trivial direction is the direct implication. Suppose f is finitely generated, say by b_1, \dots, b_n . Up to replacing the collection $\{b_i\}_i$ by the finite collection of non-zero homogeneous components of the b_i 's, we may assume that each b_i is homogeneous. Then we define $g_i = \rho(b_i)$ for $i = 1, \dots, n$ and the A -algebra homomorphism $A[g_1^{-1}T_1, \dots, g_n^{-1}T_n] \rightarrow B$ sends T_i to b_i for $i = 1, \dots, n$.

(3) Assume that f is integral, then for any non-zero homogeneous element $b \in B$, we can find $a_1, \dots, a_n \in A$ such that

$$b^n + f(a_1)b^{n-1} + \dots + f(a_n) = 0.$$

Obviously, we can replace a_i by its component in $\rho(b)^i$ for $i = 1, \dots, n$ and the equation remains true.

The reverse direction follows from [\[Stacks, Tag 00GO\]](#).

(4) This is argued in the same way as (3). □

Definition 2.19. A G -graded ring A is a *G -graded field* if

- (1) $A \neq 0$.
- (2) A does not admit any non-zero proper G -homogeneous ideals.

Proposition 2.20. Let A be a non-zero G -graded ring. Then the following conditions are equivalent:

- (1) A is a G -graded field.
- (2) Any non-zero homogeneous element in A is invertible.

PROOF. Assume that A is a G -graded field. Let $a \in A$ be a non-zero homogeneous element. Consider the G -homogeneous ideal (a) generated by a as in [Example 2.7](#). As $a \neq 0$, it follows that $(a) = 1$. Hence, a is invertible.

Conversely, suppose that any non-zero homogeneous element in A is invertible. If I is a non-zero G -homogeneous ideal in A . There is a non-zero homogeneous element $a \in I$. But we know that a is invertible and hence $I = A$. □

Definition 2.21. A G -graded ring A is an *integral domain* if for any non-zero homogeneous elements $a, b \in A$, $ab \neq 0$.

Lemma 2.22. Let A be a G -graded integral domain. Let S denote the set of non-zero homogeneous elements in A . Then $S^{-1}A$ is a graded field. The natural map $A \rightarrow S^{-1}A$ is injective.

Recall that $S^{-1}A$ is defined in [Example 2.5](#).

PROOF. By [Proposition 2.20](#), it suffices to show that each non-zero homogeneous element in $S^{-1}A$ is invertible. Such an element has the form a/s for some homogeneous element $a \in A$ and $s \in S$. As A is a G -graded integral domain, a is invertible and hence $s/a \in S^{-1}A$.

In general, the kernel of the localization map is given by $\{a \in A : \text{there is } s \in S \text{ such that } sa = 0\}$. As $A \rightarrow S^{-1}A$ is a G -graded homomorphism, the kernel is in addition a G -homogeneous ideal in A by [Lemma 2.8](#). So it suffices to show that each homogeneous element in the kernel vanishes: if $a \in A$ is a homogeneous element and there is $s \in S$ such that $sa = 0$, then $a = 0$. Otherwise, a is invertible by [Proposition 2.20](#), which is a contradiction. \square

Definition 2.23. Let A be a G -graded integral domain. We call the graded field defined in [Lemma 2.22](#) the *fraction G -graded field* of A and denote it by $\text{Frac}^G A$.

Definition 2.24. Let A be a G -graded ring. A proper G -homogeneous ideal I in A is called *prime* if the G -graded ring A/I is a G -graded integral domain.

Proposition 2.25. Let A be a G -graded ring and I be a proper homogeneous ideal in A . Then the following are equivalent:

- (1) I is a G -graded prime ideal in A .
- (2) For any homogeneous elements $a, b \in A$ satisfying $ab \in I$, at least one of a and b lies in I .

PROOF. Assume that I is a G -graded prime ideal in A . Let $a, b \in A$ be homogeneous elements satisfying $ab \in I$. Let \bar{a}, \bar{b} be the images of a, b in A/I . Then \bar{a}, \bar{b} are homogeneous and $\bar{a}\bar{b} = 0$. So at least one of \bar{a} and \bar{b} is zero. That is, a or b lies in I .

Conversely, assume that the condition in (2) is satisfied. Take $x, y \in A/I$ with $xy = 0$. We need to show that at least one of x and y is 0. Lift x and y to $a + i$ and $b + i'$ in A with a, b being homogeneous and $i, i' \in I$. Then $ab \in I$ and hence $a \in I$ or $b \in I$. It follows that $x = 0$ or $y = 0$. \square

Definition 2.26. Let A be a G -graded ring and \mathfrak{p} be a G -homogeneous prime ideal in A . Then we define the *G -graded localization* $A_{\mathfrak{p}}^G$ of A at \mathfrak{p} as $S^{-1}A$, where S is the set of homogeneous elements in $A \setminus \mathfrak{p}$.

Similarly, let M be a G -graded A -module. We define the *G -graded localization* $M_{\mathfrak{p}}^G$ as $S^{-1}M$.

Recall that $S^{-1}A$ and $S^{-1}M$ are defined in [Example 2.5](#) and [Example 2.14](#).

Definition 2.27. Let A be a G -graded ring.

A G -homogeneous ideal I in A is said to be *maximal* if it is proper, and it is not contained in any other proper G -homogeneous ideals.

We call A a *G -graded local ring* if it has a unique maximal homogeneous ideal. This ideal is called the *maximal G -homogeneous ideal* of A .

Proposition 2.28. Let A be a G -graded ring and I be a G -homogeneous ideal in A . Then the following are equivalent:

- (1) I is a maximal G -homogeneous ideal in A ;
- (2) A/I is a G -graded field.

In particular, a maximal G -homogeneous ideal is a G -homogeneous prime ideal.

PROOF. Assume (1). Then I is a proper ideal, so A/I is non-zero. Suppose that A/I has a proper G -homogeneous ideal J , it lifts to an ideal J' of A . We claim that J' is G -homogeneous. In fact, we set $J'_g := \{x \in A_g : x + I \in J\}$ for $g \in G$, we need to show that

$$J' = \sum_{g \in G} J'_g.$$

For any $j \in J'$, we can expand $j + I$ as $\sum_{g \in G} a_g + I$ with $a_g \in A_g$ and almost all a_g 's are 0. We take $i \in I$ so that

$$j = i + \sum_{g \in G} a_g.$$

The desired equation follows. But then it follows that $J' = I$ and hence $J = 0$.

Assume (2). Then I is a proper ideal in A . If J is a G -homogeneous proper ideal of A containing I , then J/I is a G -homogeneous proper ideal of A/I . It follows that $J/I = 0$ and hence $J = I$. \square

Corollary 2.29. Let A be a non-zero G -graded ring, then A admits a G -homogeneous prime ideal.

PROOF. By our assumption, 0 is a proper ideal in A . By Zorn's lemma, A admits a maximal G -homogeneous ideal, which is prime by [Proposition 2.28](#). \square

Lemma 2.30. Let $f : A \rightarrow B$ be a G -graded homomorphism of G -graded rings. Let $b_1, \dots, b_n \in B$ be a finite set of homogeneous elements integral over A , then there is a G -graded A -subalgebra $B' \subseteq B$ containing b_1, \dots, b_n such that $A \rightarrow B'$ is finite.

PROOF. We may assume that none of the b_i 's is zero. By [Proposition 2.18](#), we can find $m_1, \dots, m_n \in \mathbb{N}$ and homogeneous elements $a_{i,j} \in A$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$ such that

$$b_i^{m_i} + f(a_{i,1})b_i^{m_i-1} + \dots + f(a_{i,m_i}) = 0$$

for $i = 1, \dots, n$. It suffices to take B' as the A -submodule generated by $a_{i,j}$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$. \square

Proposition 2.31. Let $f : A \rightarrow B$ be an injective integral G -graded homomorphism of G -graded rings. Then for any G -homogeneous prime ideal \mathfrak{p} in A , there is a G -homogeneous prime ideal \mathfrak{p}' in B such that $\mathfrak{p} = f^{-1}\mathfrak{p}'$.

PROOF. We may assume that $A \neq 0$, as otherwise there is nothing to prove.

It suffices to show that $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. [Include a proof](#) We could localize that \mathfrak{p} and assume that \mathfrak{p} is a maximal G -homogeneous ideal. [Include details about localization](#) It suffices then to show that $\mathfrak{p}B \neq B$. Assume by contrary that we can write $1 = \sum_{i=1}^n f_i b_i$ for some homogeneous elements $f_i \in \mathfrak{p}$ and some homogeneous elements $b_i \in B$. Let B' be a G -graded subring of B containing A and b_1, \dots, b_n and such that $A \rightarrow B'$ is finite. The existence of B' is guaranteed by [Lemma 2.30](#). Then we find immediately $B' = \mathfrak{m}_A B'$. Then $B' = 0$ by the graded Nakayama's lemma. [Include details](#) So $A = 0$, which is a contradiction. \square

Lemma 2.32. Let A be a G -graded ring. Then the following are equivalent:

- (1) A is a G -graded local ring;
- (2) There is a proper homogeneous ideal I in A such that any non-invertible homogeneous element in A is contained in I .

PROOF. Assume that (1) holds, let I be the maximal G -homogeneous ideal of A . Let a be a non-invertible homogeneous element in A . Then the image of a in A/I is invertible by [Proposition 2.28](#) and [Proposition 2.20](#).

Assume (2). We show that I is the maximal G -homogeneous ideal in A . By [Proposition 2.28](#), it suffices to show that A/I is a graded field. By [Proposition 2.20](#), we need to show that any non-zero homogeneous element $b \in A/I$ is invertible. Lift b to $a + i \in A$ with $a \in A$ homogeneous and $i \in I$. If a is not invertible, then $a \in I$ by the assumption hence $b = 0$. This is a contradiction. \square

Lemma 2.33. Let k be a G -graded field and A be a graded k -algebra. Suppose that $\rho(A) = \rho(k)$, then

- (1) For any $g \in G$, there is a natural isomorphism

$$A_g \cong A_1 \otimes_{k_1} k_g.$$

- (2) The map $I \mapsto I \cap A_1$ is a bijection between the set of G -homogeneous ideals (resp. G -homogeneous prime ideals) in A and ideals (resp. prime ideals) in A_1 .

PROOF. (1) Take $g \in \rho(A)$. As $\rho(A) = \rho(k)$, we can take a non-zero homogeneous element $b \in k_g$. Then b and b^{-1} induces inverse bijections between A_1 and A_g .

(2) The part about ideals can be proved in the same way as (1). The part about prime ideals follows easily. \square

Proposition 2.34. Let k be a G -graded field and M be a G -graded A -module. Then M is free as G -graded A -module.

PROOF. We may assume that $M \neq 0$. Let $\{m_i\}_{i \in I}$ be a maximal set of non-zero homogeneous elements in M such that the corresponding homomorphism

$$F := \bigoplus_{i \in I} (\rho(f))^{-1} k \rightarrow M$$

is injective. The existence of $\{m_i\}_{i \in I}$ follows from Zorn's lemma.

If $f \in M/F$ is a non-zero homogeneous element, then we get a homomorphism $(\rho(f))^{-1} k \rightarrow M/F$. This map is necessarily injective as $(\rho(f))^{-1} k$ does not have non-zero proper graded submodules. This contradicts the definition of F . \square

Corollary 2.35. Let k be a G -graded field, C be a G -graded k -algebra. Consider a G -graded homomorphism of G -graded k -algebras $f : A \rightarrow B$. Then the following are equivalent:

- (1) f is finite (resp. finitely generated);
- (2) $f \otimes_k C$ is finite (resp. finitely generated).

PROOF. (1) \implies (2): This implication is trivial.

(2) \implies (1): By [Proposition 2.34](#), this implication follows from fpqc descent [[Stacks](#), [Tag 02YJ](#)]. \square

Definition 2.36. Let K be a G -graded field. A G -graded subring $A \subseteq K$ is a *G -graded valuation ring* in K if

- (1) A is a local G -graded ring;
- (2) the natural map $\text{Frac}^G A \rightarrow K$ is an isomorphism;
- (3) For any non-zero homogeneous element $f \in K$, either $f \in A$ or $f^{-1} \in A$.

Definition 2.37. Let K be a G -graded field and A, B be G -graded local subrings of K . We say B *dominates* A if $A \subseteq B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, where \mathfrak{m}_A and \mathfrak{m}_B are the maximal G -homogeneous ideals in A and B .

Proposition 2.38. Let K be a G -graded field and $A \subseteq K$ be a G -graded local subring. Then the following are equivalent:

- (1) A is a G -graded valuation ring in K .
- (2) A is maximal among the G -graded local subrings of K with respect to the order of domination.

PROOF. Assume (1). We may assume that $A \neq K$. Then A is not a G -graded field as $\text{Frac}^G A = K$. Let \mathfrak{m} be a maximal G -homogeneous ideal in A . Then $\mathfrak{m} \neq 0$.

We argue first that A is a G -graded local ring. Assume the contrary. Let $\mathfrak{m}' \neq \mathfrak{m}$ be maximal G -homogeneous ideal in A . Choose non-zero homogeneous elements $x, y \in A$ with $x \in \mathfrak{m}' \setminus \mathfrak{m}$, $y \in \mathfrak{m} \setminus \mathfrak{m}'$. Then $x/y \notin A$ as otherwise $x = (x/y)y \in \mathfrak{m}$. Similarly, $y/x \notin A$. This is a contradiction.

Next suppose that A' is a G -graded local subring of K dominating A . Let $x \in A'$ be a non-zero homogeneous element, we need to show that $x \in A$. If not, we have $x^{-1} \in A$ and as x^{-1} is not a unit, $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$, the maximal G -homogeneous ideal in A' . This contradicts the fact that $x \in A'$.

Assume (2). Take a homogeneous element $x \in K \setminus A$, we need to argue that $x^{-1} \in A$. Let A' denote the minimal G -homogeneous subring of K containing A and x . It is easy to see that A' is the usual subring generated by A and x .

By our assumption, there is no G -graded prime ideal of A' lying over \mathfrak{m}_A , as otherwise, if \mathfrak{p} is such an ideal, the G -graded local subring $A'_{\mathfrak{p}}^G$ of K dominates A .

In other words, the G -graded ring $A'/\mathfrak{m}_A A'$ does not have any homogeneous prime ideals and hence $A' = \mathfrak{m}_A A'$ by [Corollary 2.29](#).

We can therefore write

$$1 = \sum_{i=0}^d t_i x^i$$

with some homogeneous elements $t_i \in \mathfrak{m}_A$. In particular,

$$(1 - t_0)(x^{-1})^d - \sum_{i=1}^d t_i (x^{-1})^{d-i} = 0.$$

So x^{-1} is integral over A . Let A'' be the subring of K generated by A and x^{-1} . Then $A \rightarrow A''$ is finite and there is a G -homogeneous prime ideal \mathfrak{m}'' of A'' lying over \mathfrak{m}_A by [Proposition 2.31](#). By our assumption, $A = A''_{\mathfrak{m}''}^G$ and hence $x^{-1} \in A$.

It remains to verify that $\text{Frac}^G A = K$. Suppose that it is not the case, let $B \subseteq K$ be a G -graded local subring dominating A . Take a homogeneous element $t \in K$ that is not in $\text{Frac}^G A$. Observe that t can not be transcendental over A , as otherwise $A[t] \subseteq K$ is a G -graded subring, and we can localize it at the G -homogeneous prime generated by t and \mathfrak{m}_A . We get a G -graded local ring dominating A that is different from A .

So t is algebraic over A . We can then take a non-zero homogeneous $a \in A$ such that at is integral over A . The ring $A' \subseteq K$ generated by A and ta is a G -graded subring and $A \rightarrow A'$ is finite. By [Proposition 2.31](#), there is a G -homogeneous prime ideal \mathfrak{m}' of A' lifting \mathfrak{m}_A . But then $A'_{\mathfrak{m}'}^G$ dominates A and so $A = A'_{\mathfrak{m}'}^G$. It follows that $t \in \text{Frac}^G A$, which is a contradiction. \square

Corollary 2.39. Let K be a G -graded field. Any G -graded local subring $B \subseteq K$ is dominated by a G -graded valuation subring of K .

PROOF. This follows from [Proposition 2.38](#) and Zorn's lemma. \square

3. Graded algebraic geometry

Let G be an Abelian group. We write the group operation of G multiplicatively and denote the identity of G as 1.

Definition 3.1. Let A be a G -graded ring. We define the G -graded affine spectrum $\text{Spec}^G(A)$ as follows: as a set $\text{Spec}^G(A)$ consists of all G -homogeneous prime ideals of A ; we endow $\text{Spec}^G(A)$ with the *Zariski topology*, whose base consists of sets of the form

$$D(f) := \left\{ \mathfrak{p} \in \text{Spec}^G(A) : f \notin \mathfrak{p} \right\}$$

for all homogeneous elements $f \in A$.

Lemma 3.2. Let k be a G -graded field and A be a finitely generated G -graded k -algebra. Then $\text{Spec}^G(A)$ has only finitely many maximal points.

PROOF. \square

4. The birational category

Let G be an Abelian group. Let k be a G -graded field and K/k be a G -graded field extension.

Definition 4.1. We let $\mathbf{P}_{K/k}$ denote the set of G -graded valuation rings \mathcal{O} of K with G -graded fraction field K such that $k \subseteq \mathcal{O}$.

We endow $\mathbf{P}_{K/k}$ with the weakest topology with respect to which $\{\mathcal{O} \in \mathbf{P}_{K/k} : f \in \mathcal{O}\}$ is open for any homogeneous element $f \in K$.

Given $X \subseteq \mathbf{P}_{K/k}$ and $A \subseteq K$ consisting of homogeneous elements, we write

$$\begin{aligned} X\{A\} &:= \{\mathcal{O} \in X : f \in \mathcal{O}\}, \\ X\{\{A\}\} &:= \{\mathcal{O} \in X : f \in \mathfrak{m}_{\mathcal{O}}\}, \end{aligned}$$

where $\mathfrak{m}_{\mathcal{O}}$ is the maximal G -homogeneous ideal of \mathcal{O} .

Lemma 4.2. Let $X \subseteq \mathbf{P}_{K/k}$ and $A \subseteq K$ consisting of homogeneous elements. Then

$$X \setminus X\{A\} = X\{\{A\}\}.$$

Lemma 4.3. Let $A \subseteq K$ be a subset of K consisting of homogeneous elements, then $\mathbb{P}_{K/k}\{A\}$ is quasi-compact.

PROOF. We may replace A by the G -graded subring generated of K generated by A . So we may assume that A is a G -graded subring of K . \square

5. Miscellany

Proposition 5.1. Let R be a noetherian N-2 integral domain. Let $\psi : R \rightarrow S$ be a ring homomorphism such that S is reduced, torsion-free as R -module and has finite rank as R -module. Then ψ is finite.

[BGR84, Page 122]. Reproduce the argument later.

PROOF. As ψ is injective by assumption, we may assume that R is a subring of S and ψ is identity. The ring $S_{R \setminus \{0\}} = \text{Frac } S$ is a finite-dimensional reduced $\text{Frac } R$ -algebra, hence as a ring, $\text{Frac } S$ is the product of finitely many finite field extensions of $\text{Frac } R$, say K_1, \dots, K_t . As R is N-2, the integral closure R_i of R in K_i is finite as R -module for $i = 1, \dots, t$. As S is integral over R , we have

$$S \subseteq R_1 \times \cdots \times R_t.$$

As R is noetherian, we conclude that S is finite as R -module. \square

Lemma 5.2. Let R be a commutative ring. A polynomial $a_0 + a_1X + \cdots + a_nX^n \in R[X]$ is a unit if and only if a_0 is a unit in R and a_1, \dots, a_n are nilpotents.

Bibliography

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