MABUCHI GEOMETRY OF BIG COHOMOLOGY CLASSES WITH PRESCRIBED SINGULARITIES

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Abstract. Let X be a compact Kähler unibranch complex analytic space of pure dimension. Fix a big class α with smooth representative θ and a model potential ϕ with positive mass. We define and study the non-pluripolar products of quasi-plurisubharmonic functions on X. We study the spaces $\mathcal{E}^p(X,\theta;[\phi])$ of finite energy Kähler potentials with prescribed singularities for each $p \geq 1$. We define a metric d_p without solving Monge–Ampère equations. This construction generalizes the usual d_p-metric defined for an ample class.

Contents

1. Introduction

1.1. **Motivation.** Let X be a compact Kähler manifold. Let ω be a Kähler form on X. It is well-known that the space $\mathcal{H}(X,\omega)$ of smooth strictly ω-plurisubharmonic functions admits a natural Riemannian structure:

$$\langle f,g\rangle_\varphi := \int_X fg\,\omega_\varphi^n\,,\quad \varphi\in\mathcal{H}(X,\omega)\,, f,g\in C^\infty(X)=\mathrm{T}_\varphi\mathcal{H}(X,\omega)\,.$$

It is shown by Chen ([Che00]) that the Riemannian structure endows $\mathcal{H}(X,\omega)$ with a *bona fide* metric d_2. Darvas ([Dar17], [Dar15]) proved that the metric completion of $\mathcal{H}(X,\omega)$ with respect to the Riemannian metric can be naturally identified with the space $\mathcal{E}^2(X,\omega)$ of ω-psh functions with finite energy, confirming a conjecture of Guedj. In fact, the results of Darvas show that the metric completion of $\mathcal{H}(X,\omega)$ with respect to the natural p-Finsler metric ($p\geq 1$) is given by $\mathcal{E}^p(X,\omega)$, namely, the space of ω-psh functions with finite p-energy. The space $\mathcal{E}^p(X,\omega)$ is a geodesic metric space. In the non-Archimedean world, one can similarly define a space $\mathcal{E}^1$, which enjoys similar properties as the Archimedean counterpart, see [BJ18b].

The spaces $\mathcal{E}^1$ and $\mathcal{E}^2$ have found numerous applications in the study of Kähler geometry and in non-Archimedean geometry, especially in the problem of canonical metrics. See [CC17], [BDL16], [LTW21], [BJ18a] for example.

The theory of $\mathcal{E}^p$ together with the metric d_p is, however, not completely satisfactory for the following reasons:

(1) The spaces $\mathcal{E}^p$ accounts for only a very small portion of $\mathrm{PSH}(X,\omega)$. In fact, $\mathcal{E}^p(X,\omega)\subseteq\mathcal{E}(X,\omega)$, where the latter space is the space of ω-psh functions with full (non-pluripolar) mass. In particular, we cannot handle general singularities.

(2) Although $\mathcal{E}^p$ can be defined for a general big cohomology class, it is not clear how to define a metric on $\mathcal{E}^p$. In particular, this makes it hard to apply them in the study of birational geometry.

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(3) The spaces $\mathcal{E}^p$ ($p>1$) are only defined when X is smooth. In particular, it is not powerful enough to deal with Kähler–Einstein metrics on normal varieties directly.

The problem (1) is effectively solved by Darvas–Di Nezza–Lu in [DDL18b] by introducing the so-called potentials with *prescribed singularities*. Roughly speaking, this means that we prescribe a *nice* singularity type ϕ, then we can define a space $\mathcal{E}(X,\omega;[\phi])$ of potentials with singularities controlled by ϕ. It can be shown that these spaces for various ϕ give a partition of $\mathrm{PSH}(X,\omega)$. Moreover, each space $\mathcal{E}(X,\omega;[\phi])$ behaves exactly like $\mathcal{E}(X,\omega)$. So we have turned the study of general potentials into the study of potentials with *relatively* full mass.

As for problem (2), so far only the following partial generalizations are known:

(1) When $p=1$, the metric on $\mathcal{E}^1$ is defined for a general big class in [DDL18a]. A further generalization to potentials with prescribed singularities is established in [Tru22].
(2) When the class is big and nef, the metric on $\mathcal{E}^p$ is defined in [DNL20] by perturbation.
(3) On a general normal compact complex analytic space, in [BBEGZ16], the space $\mathcal{E}^1$ with respect to a Kähler class is studied.

In particular, there are no satisfactory theory of $\mathcal{E}^p$ for a general big cohomology class yet.

As for problem (3), this is in fact just technical. The pluripotential theory has not been fully developed on a general normal variety, except in the case of bounded potential ([Dem85]). In particular, the theory of non-pluripolar products ([BEGZ10]) has never been fully developed on singular spaces. It seems that this theory is well-known to experts. And in fact, it has been widely applied in various circumstances without rigorous justifications.

The goal of this paper is to solve all of these problems simultaneously.

1.2. Main results.

1.2.1. *Non-pluripolar products.* The first part of this paper is devoted to develop the theory of non-pluripolar products on a singular space.

Let X be a unibranch (see Definition 2.3) complex analytic space of pure dimension n. Let X^{red} be the reduced space underlying X. Recall that there is a well-defined notion of psh functions on X ([FN80]). Let $\pi: Y\to X^{\mathrm{red}}$ be a proper resolution of singularities. As in the algebraic setting, the unibranchness assumption ensures that the fibres of π are connected, a result classically known as *Zariski's main theorem*, see Theorem 2.12. In particular, we prove that π^* gives a bijection between the spaces $\mathrm{PSH}(X)=\mathrm{PSH}(X^{\mathrm{red}})$ and $\mathrm{PSH}(Y)$. Therefore a large part the study of $\mathrm{PSH}(X)$ can be essentially reduced to the known results in the smooth setting.

We prove in Theorem 3.24 the following generalization of Lelong's conjecture:

Theorem 1.1. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $E\subseteq X$ be a subset. Then E is negligible if and only if E is pluripolar.*

This result is in fact equivalent to the so-called continuity of envelope property. To the best of the author's knowledge this is the first written proof in the singular setting, although this theorem has been applied implicitly in the literature for a long time.

1.2.2. *Pluripotential theory on singular spaces.* In the second part, we study the space $\mathcal{E}^p$. Let X be a compact unibranch Kähler space of pure dimension n. Let α be a big class on X represented by a smooth form θ. Let ϕ be a model potential (Definition 3.48) in $\mathrm{PSH}(X,\theta)$ with positive mass. Then we can define the space $\mathcal{E}^p(X,\theta)$ exactly as in the smooth case,

$$\mathcal{E}^p(X,\theta):=\left\{\varphi\in\mathrm{PSH}(X,\theta):[\varphi]\preceq[\phi],\int_X\theta_\varphi^n=\int_X\theta_\phi^n,\int_X|\phi-\varphi|^p\theta_\varphi^n<\infty\right\},$$

see Definition 4.15 for details. We generalize various known results in the smooth setting to the singular setting. In particular we establish a general comparison principle (Lemma 4.6) and several energy estimates. We show that the rooftop operator is well-defined and has the expected support:

Theorem 1.2 (=Theorem 4.18+Corollary 4.21)**.** *Let $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, then*

$$\varphi\wedge\psi:=\sup{}^*\{\eta\in\mathrm{PSH}(X,\theta):\eta\leq\varphi,\eta\leq\psi\}\in\mathcal{E}^p(X,\theta)\,.$$

Moreover, $\theta^n_{\varphi\wedge\psi}$ is supported on the set $\{\varphi\wedge\psi=\varphi\}\cup\{\varphi\wedge\psi=\psi\}$.

1.2.3. *Metrics on $\mathcal{E}^p$.* The main result in this part is the definition of a metric d_p. This part is the main innovation in this paper. To demonstrate the idea, let us consider the case of a compact Kähler manifold X. Let ω be a Kähler form on X. Let us see how the usual d_p-metric can be defined without solving Monge–Ampère equations. Let $\varphi_0, \varphi_1 \in \mathcal{H}(X,\omega)$. Recall that solving a suitable Monge–Ampère equation gives us a unique geodesic $(\varphi_t)_{t\in[0,1]}$, see [Che00]. The geodesic has $C^{1,1}$-regularity, see [CTW17]. In this case, we have

$$d_p(\varphi_0,\varphi_1)^p = \int_X |\dot{\varphi}_0|^p\, \omega_{\varphi_0}^n\,.$$

Recall that d_p satisfies a so-called Pythagorean equality, which allows us to assume $\varphi_0 \leq \varphi_1$. It is shown in [Dar15] that $d_p(\varphi_0,\varphi_1)$ is of the order $\left(\int_X (\varphi_1-\varphi_0)^p\,\theta_{\varphi_0}^n\right)^{1/p}$. We could reverse this machinery by *defining*

$$d_p(\varphi_0,\varphi_1) \approx \left(\int_X (\varphi_1-\varphi_0)^p\,\theta_{\varphi_0}^n\right)^{1/p}\,,$$

when $\varphi_1 \geq \varphi_0$ is sufficiently close to φ_0. This turns out to work and the resulting metric is equal to the original d_p, see Section 6.5.1. Note that this definition is purely pluripotential-theoretic, hence does not require to solve any PDE. The same definition works in general. This gives our definition of d_p.

The second main result in this part is that the space $\mathcal{E}^p(X,\theta)$ is locally complete with respect to d_p. Here locally completeness means that subspaces of the form $\{\psi \in \mathcal{E}^p(X,\theta) : \psi \geq \varphi\}$ are complete where $\varphi \in \mathcal{E}^p(X,\theta)$. In our proof, we found that an algebraic structure, which we call the *rooftop structure*, plays a key role. So we develop this algebraic structure in detail. Roughly speaking, a rooftop structure on a metric space is an associative, commutative, idempotent binary structure $\wedge$, such that $\bullet\wedge\bullet$ is distance-decreasing in both variables. We prove a general criterion for the completeness of a rooftop metric space (Proposition 5.4). Back to $\mathcal{E}^p(X,\theta)$, the rooftop structure $\wedge$ is defined as follows: given $\varphi,\psi \in \mathcal{E}^p(X,\theta)$, we let $\varphi\wedge\psi$ be the maximal θ-psh function lying below both φ and ψ. We show that this rooftop structure verifies the conditions in Proposition 5.4 locally. Thus d_p is a locally complete metric on $\mathcal{E}^p(X,\theta)$. In other words,

Theorem 1.3 (=Theorem 6.32)**.** *The space $(\mathcal{E}^p(X,\theta), d_p, \wedge)$ is a p-strict locally complete rooftop metric space.*

See Definition 5.2 for the definition of p-strictness. We will prove in Section 6.5 that our definition generalizes all known definitions in the literature.

We have developed our theory in an axiomatic manner. The reason is that these results can be formally generalized to the non-Archimedean setting with essentially the same proofs as long as the conjecture of continuity of envelopes holds ([BJ18b, Conjecture 4.51]). The details will appear in a separate paper.

1.2.4. *The space of geodesic rays.* In the last part, based on our previous results, we study the space $\mathcal{R}^1(X,\theta)$ of finite energy geodesic rays. The study of such spaces as metric spaces is initiated by Chen–Cheng ([CC18]) and Darvas–Lu ([DL20]). In the case of a big class, $\mathcal{R}^1$ is studied in [DDL21b]. We generalize these results to the singular setting as well. Similar to the case of potentials, we construct a natural metric d_1 and a rooftop structure $\wedge$ on $\mathcal{R}^1$. We prove the following theorem

Theorem 1.4 (=Theorem 7.6)**.** *The space $(\mathcal{R}^1(X,\theta), d_1, \wedge)$ is a 1-strict complete rooftop metric space.*

The importance of this theorem lies in its relation to the conjecture of continuity of envelopes ([BJ18b, Conjecture 4.51]) in non-Archimedean geometry. Let X be a unibranch projective complex variety. Let L be an ample $\mathbb{Q}$-line bundle on X. Take $k>0$ so that $L^{\otimes k}$ is a very ample line bundle and embeds X into $\mathbb{P}^N$. Let ω be the pull-back of the Fubini–Study metric on $\mathbb{P}^N$, normalized by k^{-1}. Recall that continuity of envelopes is equivalent to the completeness of $\mathcal{E}^{1,\mathrm{NA}}(L)$ ([BJ18b, Theorem 9.8]).

1.3. **Structure of the paper.** In Section 2, we recall some basic properties of complex analytic spaces. These results are certainly well-known, but the author cannot find a good reference to part of them, so we provide detailed proofs in that case.

In Section 3, we develop the theory of non-pluripolar products on unibranch spaces. This section is an abridgement of the author's notes [Xia21].

In Section 4, we develop the theory of potentials with prescribed singularities in detail, generalizing a number of results in the literature. In particular, we carry out a detailed study of some energy functionals.

In Section 5, we study abstract rooftop structures.

In Section 6, we define the d_p metric on $\mathcal{E}^p$ and prove its completeness.

In Section 7, we study the space $\mathcal{R}^1$ of geodesic rays.

1.4. **Conventions.** All Monge–Ampère operators are taken in the non-pluripolar sense. We use the terms *increasing*, *decreasing* in the French sense, namely, in the non-strict sense. We write $\wedge$ for the rooftop operator instead of the more common $P(\cdot,\cdot)$. For $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, we write $[\varphi]\wedge\psi$ for $P[\varphi](\psi)$. Given two qpsh functions φ,ψ, we write $\varphi\vee\psi=\max\{\varphi,\psi\}$. We follow the convention

$$\mathrm{dd^c}=\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\,.$$

We use C for a positive constant, whose value may change from line to line.

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2. Preliminaries on complex analytic spaces

For each $N\geq 0$, we endow an open subset $\Omega\subseteq\mathbb{C}^N$ with the sheaf of holomorphic functions, so that Ω can be regarded as a locally $\mathbb{C}$-ringed space.

2.1. Complex analytic spaces.

In this section we recall some basic facts about complex analytic spaces. This section is by no means intended to be complete. For details, we refer to [Fis06], [CAS].

Definition 2.1. A *local model* of a complex analytic space is a locally $\mathbb{C}$-ringed space $(V,\mathcal{O}_V)$, such that there is an analytic closed immersion $V\hookrightarrow\Omega$ into a bounded pseudo-convex domain Ω in some $\mathbb{C}^N$. To be more precise, this means that there are $f_1,\dots,f_M\in H^0(\Omega,\mathcal{O}_\Omega)$, such that V is closed subset of Ω defined as the common zero locus of all f_i's, $\mathcal{O}_V$ is the quotient of $\mathcal{O}_\Omega$ by the ideal $(f_1,\dots,f_M)$, regarded as a sheaf on V.

The category is local models is a full subcategory of the category of locally $\mathbb{C}$-ringed spaces.

Here requiring pseudo-convexity of Ω is just for convenience. The notion does not change if we remove it.

Definition 2.2. A *complex analytic space* is a locally $\mathbb{C}$-ringed space $(X,\mathcal{O}_X)$ such that

(1) X is a para-compact, Hausdorff space.
(2) For any $x\in X$, there is an open neighbourhood $U\subseteq X$ of x, such that $(U,\mathcal{O}_U)$ (with $\mathcal{O}_U$ being the restriction of $\mathcal{O}_X$ to U) is isomorphic to a local model as locally $\mathbb{C}$-ringed spaces.

The category is complex analytic spaces is a full subcategory of the category of locally $\mathbb{C}$-ringed spaces.

By abuse of language, we say that X is a complex analytic space as well.

Definition 2.3. A complex analytic space $(X,\mathcal{O}_X)$ is

(1) *reduced* (resp. *normal*) at $x\in X$ if $\mathcal{O}_{X,x}$ is reduced (resp. normal). We say X is *reduced* (resp. *normal*) if it is reduced (resp. normal) at all points.
(2) *unibranch* at $x\in X$ if $\mathcal{O}_{X,x}$ is unibranch. We say X is *unibranch* if it is unibranch at all points.

Recall that the notion of unibranchness is defined in [EGA IV_1, Section 0.23.2.1]. A local ring is unibranch is A^{red} is integral and the integral closure of A^{red} in its fraction field is local.

Remark 2.4. For us, a ring is normal if it is an integral domain and integrally closed. So in particular, a normal analytic space is reduced. Also recall that a normal ring is always unibranch.

Proposition 2.5. *Let X be a complex analytic space and $x\in X$. Then the following are equivalent:*

(1) X is unibranch at x;
(2) X^{red} is unibranch at x;
(3) $\mathcal{O}_{X,x}$ is geometrically unibranch;
(4) $\mathcal{O}_{X,x}^{\mathrm{red}}$ is geometrically unibranch;
(5) $\mathcal{O}_{X,x}$ has a unique minimal prime ideal.

Proof. (1) ⇔ (3): As $\mathcal{O}_{X,x}$ is excellent [DG67], the integral closure $\overline{\mathcal{O}_{X,x}^{\mathrm{red}}}$ is a finite $\mathcal{O}_{X,x}^{\mathrm{red}}$-algebra, so the residue field extension is finite. But the residue field of $\mathcal{O}_{X,x}$ is $\mathbb{C}$, so the residue field extension is the trivial extension.

(1) ⇔ (5): This follows from [Stacks, Tag 0BQ0] and the fact that $\mathcal{O}_{X,x}$ is Henselian ([CAS, Page 45]).

(1) ⇔ (2): This follows from the observation that (5) holds for $\mathcal{O}_{X,x}$ if and only if (5) holds for $\mathcal{O}_{X,x}^{\mathrm{red}}$.

(3) ⇔ (4): This follows from the same argument as (1) ⇔ (2). □

Remark 2.6. Note that our definition of unibranch space is different from the notion of locally irreducible space in [CAS, Page 8]. More precisely, a complex analytic space X is unibranch in our sense if and only if X^{red} is locally irreducible in the sense of [CAS].

Remark 2.7. For complex varieties, one could also define unibranchness using Zariski topology. This definition is equivalent to ours.

To be more precise, let X be a scheme of finite type over $\mathbb{C}$. Let X^{an} be the complex analytification of X. Let $x \in X(\mathbb{C})$. Then we claim that X is unibranch at x if and only if X^{an} is unibranch at x. By GAGA, there is a natural morphism of ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}) \to (X, \mathcal{O}_X)$. We may assume that X is reduced. Then X^{an} is also reduced. Then since $\mathcal{O}_{X,x}$ is excellent, X is unibranch at x if and only if $\widehat{\mathcal{O}_{X,x}} = \widehat{\mathcal{O}_{X^{\mathrm{an}},x}}$ is integral ([EGA IV$_4$, Théorème 18.9.1]). The latter implies that $\mathcal{O}_{X^{\mathrm{an}},x}$ is unibranch by [AC, Chapitre 5, Exercise 2.8]. Running the same argument the other way round, we conclude that $\mathcal{O}_{X,x}$ is unibranch if and only if $\mathcal{O}_{X^{\mathrm{an}},x}$ is.

Definition 2.8. A complex analytic space $(X, \mathcal{O}_X)$ is

(1) *Holomorphically separable* if for any $x, y \in X$, $x \neq y$, there is $f \in H^0(X, \mathcal{O}_X)$, such that $f(x) \neq f(y)$.

(2) *Holomorphically convex* if for any compact set $K \subseteq X$, the set
$$\hat{K} := \left\{ x \in X : |f(x)| \leq \sup_{k \in K} |f(k)| , \quad \forall f \in H^0(X, \mathcal{O}_X) \right\}$$
is compact.

(3) *Stein* if it is both holomorphically separable and holomorphically convex.

Note that all three conditions are stable under passing to closed subspaces.

Theorem 2.9. *Let $(X, \mathcal{O}_X)$ be a complex analytic space. Then X is Stein if and only if X^{red} is.*

This highly non-trivial result is proved in [Gra60]. It is based on a cohomological characterization of Stein spaces. See [GR77] for a simplified proof.

Definition 2.10. Let $f : X \to S$ be a morphism of complex analytic spaces. We say the morphism f is *formally smooth* if given any solid commutative diagram
$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd},$$
where $i : T \to T'$ is a first order thickening of Stein spaces, there is a dotted morphism $T' \to X$ making the whole diagram commutative.

A complex analytic space X is *formally smooth* if the morphism to the final object $\mathbb{C}^0$ is formally smooth.

Theorem 2.11. *Let $f : X \to S$ be a morphism of complex analytic spaces. Assume that f is smooth, then f is formally smooth.*

The proof is a simple extension of its algebraic analogue [Stacks, Tag 02H6]. We refer to Grothendieck's exposé [Gro60b] for the notion of $\Omega^1_{X/S}$.

Proof. Suppose that we are given a diagram as in Definition 2.10.

Consider the sheaf $\mathcal{F}$ of sets on T':
$$H^0(U', \mathcal{F}) := \{ a' : U' \to X : a'|_U = a|_U \} , \quad U = U' \cap T$$
for any open set $U' \subseteq T'$. We want to show that $\mathcal{F}$ admits a global section on T'. Let
$$\mathcal{H} := \mathcal{H}\mathrm{om}_{\mathcal{O}_T}(a^* \Omega^1_{X/S}, \mathcal{C}_{T/T'}) ,$$

where $\mathcal{C}_{T/T'}$ is the conormal sheaf of T in T', namely, if T is defined by a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{T'}$, then $\mathcal{C}_{T/T'}$ is $\mathcal{I}/\mathcal{I}^2$ regarded as a sheaf of $\mathcal{O}_T$-modules. There is an obvious action of $\mathcal{H}$ on $\mathcal{F}$, making $\mathcal{F}$ a pseudo-$\mathcal{H}$-torsor. We will show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor.

First of all, let us show that $\mathcal{F}$ has non-trivial fibres. Let $t \in T$. Let $x = a(t)$, $s = f(x)$, $t' = i(t)$. We know that $f^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is smooth, hence formally smooth. Thus we can find a local homomorphism $g^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_{S,s} \arrow[r, "a^\sharp"] \arrow[d, "f^\sharp"] & \mathcal{O}_{T',t'} \arrow[d, "i^\sharp"] \\ \mathcal{O}_{X,x} \arrow[r] \arrow[ur, "g^\sharp"] & \mathcal{O}_{T,t} \end{tikzcd} .$$

Now by [Gro60a, Théorème 1.3] or [Fis06, Section 0.21], the homomorphism $g^\sharp$ induces a morphism of germs $g : (T', t') \to (X, x)$. This shows that $\mathcal{F}_t \neq \emptyset$.

Now we prove $H^1(T, \mathcal{H}) = 0$. In fact, $\mathcal{H}$ is coherent, so we could apply Cartan's Theorem B ([Fis06, Page 33]). □

We say an analytic space X is irreducible if X^{red} is irreducible ([CAS, Section 9.1]).

Theorem 2.12 (Zariski's main theorem)**.** *Let $f : Y \to X$ be a proper bimeromorphic morphism of complex analytic spaces. Assume that X is unibranch at $x \in X$. Then the fibre $f^{-1}(x)$ is connected.*

Proof. Note that f induces a morphism $f^{\mathrm{red}} : Y^{\mathrm{red}} \to X^{\mathrm{red}}$ satisfying all assumptions of the theorem, so we may assume that X and Y are reduced. Then the result is proved in [Dem85, Proof of Théorème 1.7]. □

2.2. **Kähler spaces.** Let X be a complex analytic space. See [HL71] for the notion of differential forms and currents on a reduced complex analytic space. A differential form/current on X is defined as a differential form/current on X^{red}.

Definition 2.13. A *Kähler form* on X is a smooth $(1,1)$-form ω on X, such that at any point $x \in X$, there is a neighbourhood $V \subseteq X$ of x, a closed immersion $V^{\mathrm{red}} \hookrightarrow \Omega$ into some bounded pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, a Kähler form ω_Ω on Ω, such that $\omega = \omega_\Omega|_V$.

A *Kähler space* is a complex analytic space which admits a Kähler form.

Example 2.14. *Any Kähler manifold is a Kähler space. In fact, when X is smooth, the notion of Kähler form in Definition 2.13 coincides with the usual one.*

Example 2.15. *Let X be a Kähler space. Let Y be a closed analytic subspace, then Y is a Kähler space.*

As a consequence, any projective analytic space is Kähler. More generally,

Lemma 2.16. *Let X be a compact Kähler space. Let $f : Y \to X$ be a projective morphism. Then Y is a Kähler space.*

For the definition of projective morphism, see [GPR94, Section V.4].

Proof. We can embed Y as a closed subspace of $X \times \mathbb{P}^N$ for some $N \geq 0$ preserving the morphism to X. Let p_1, p_2 be the projection from $X \times \mathbb{P}^N$ to two factors. Take a Kähler form ω on X. Let ω_{FS} be the Fubini–Study metric on $\mathbb{P}^N$. Then we claim that $p_1^*\omega + p_2^*\omega_{\mathrm{FS}}$ defines a Kähler form on Y. By Example 2.15, it suffices to show that $p_1^*\omega + p_2^*\omega_{\mathrm{FS}}$ is a Kähler form on $X \times \mathbb{P}^N$. We may assume that X is reduced. The problem is also local in $\mathbb{P}^N$, we could replace $\mathbb{P}^N$ by a polydisk Δ in it.

The problem is local on X, so we may assume that X is a closed subspace of a pseudo-convex domain Ω in $\mathbb{C}^M$ and there is a Kähler form ω_Ω such that $\omega = \omega_\Omega|_X$. Now $X \times \Delta \hookrightarrow \Omega \times \Delta$ and the form

$$p_1^*\omega + p_2^*\omega_{\mathrm{FS}} = (\pi_1^*\omega_\Omega + \pi_2^*\omega_{\mathrm{FS}})\,|_{X \times \Delta} ,$$

where π_1, π_2 are the two projection from $\Omega \times \mathbb{P}^N$ to the two factors. □

Corollary 2.17. *Let X be a compact Käher space. Let Y be a closed subspace. Then $\mathrm{Bl}_Y X$ is a Kähler space.*

Here $\mathrm{Bl}_Y X$ is the blowing-up of X with center Y. For its precise definition, we refer to [GPR94, Section VII.2].

Recall that we can always resolve singularities of a complex analytic space.

Theorem 2.18. *Let X be a reduced complex analytic space. Then there is a (proper) resolution of singularity $\pi : Y \to X$ of X. Moreover, we may assume that π is an isomorphism over the non-singular part of X.*

This theorem was first proved by Aroca–Hironaka–Vicente, see the book [AHV18]. Later simplifications are due to Bierstone–Milman, Villamayor and Włodarczyk. See [Wło09] for details and further references.

Corollary 2.19. *Let X be a reduced compact complex Kähler space. Then there is a resolution of singularity $\pi : Y \to X$ such that Y is a compact Kähler manifold.*

Proof. Let $p : Z \to X$ be a resolution. By Hironaka's Chow lemma ([Hir75]), Z is dominated (over X) by a complex analytic space W, which is a sequence of blowing-ups of X with smooth centers. By Corollary 2.17, W admits a Kähler form. Hence Z admits a Kähler current. Hence Z is of Fujiki's class $\mathcal{C}$ ([DP04]). In particular, there is a proper bimeromorphic morphism $Y \to Z$, such that Y is a compact Kähler manifold. □

In the remaining of this paper, by a resolution of singularity $f : Y \to X$ of a reduced Kähler space X, we always assume that Y is Kähler.

3. Non-pluripolar products on unibranch spaces

Let X be a complex analytic space.

3.1. Plurisubharmonic functions.

Definition 3.1. Let $U \subseteq X$ be an open immersion. A function $\varphi : U \to [-\infty, \infty)$ is *pluri-subharmonic* if

(1) φ is not identically $-\infty$ on any irreducible component of U.

(2) For any $x \in U$, there is an open neighbourhood V of x in U, a bounded pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$, an open set $\tilde{V} \subseteq \Omega$ with $x \in \tilde{V}$, a pluri-subharmonic function $\tilde{\varphi}$ on $\tilde{V}$, such that $\varphi|_{\tilde{V}\cap V} = \tilde{\varphi}|_{\tilde{V}\cap V}$.

The set of pluri-subharmonic functions on U is denoted by $\mathrm{PSH}(U)$.

Proposition 3.2. *Let X be a complex manifold. Then with the canonical complex analytic space structure on X, the definition of $\mathrm{PSH}(X)$ coincides with the usual one.*

Proof. It suffices to recall that a psh function on a domain restricts to a psh function on a closed analytic submanifold. □

Proposition 3.3. *Let $f : X \to Y$ be a morphism between complex analytic spaces. Let $\varphi \in \mathrm{PSH}(Y)$. Assume that $f^*\varphi$ is not identically equal to $-\infty$ on each irreducible component of X, then $f^*\varphi \in \mathrm{PSH}(X)$.*

Proof. Let $x \in X$, $y = f(x)$. We need to verify Condition (2). The problem is local, so we may assume that there is a closed immersion $X \hookrightarrow \Sigma$ and that there is an open neighbourhood $V \subseteq Y$ of y, a closed immersion $V \hookrightarrow \Omega$, an open set $\tilde{V} \subseteq \Omega$ containing y, a psh function $\tilde{\varphi}$ on $\tilde{V}$, such that $\tilde{\varphi}|_{\tilde{V}\cap V} = \varphi|_{\tilde{V}\cap V}$, where Σ and Ω are bounded pseudoconvex domains in $\mathbb{C}^N$ and $\mathbb{C}^M$ respectively. Shrinking X, we may assume that $f(X) \subseteq \tilde{V} \cap V$. We the get a closed immersion:

$$X \hookrightarrow X \times V \hookrightarrow \Sigma \times \Omega\,,$$

where the first morphism is the base change of $\Delta_V : V \to V \times V$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X \times V \arrow[r, hook] \arrow[d] & \Sigma \times \Omega \\ V \arrow[r, hook, "\Delta_V"] & V \times V & \end{tikzcd}.$$

Define $\tilde{X} := \Sigma \times \tilde{V}$. Define a psh function ψ on $\tilde{X}$ as the pull-back of $\tilde{\varphi}$ from the projection onto the second variable. Then $f^*\varphi|_{\tilde{X}\cap X} = \psi|_{\tilde{X}\cap X}$. □

Proposition 3.4. *There is a canonical bijection*

$$\mathrm{PSH}(X) \xrightarrow{\sim} \mathrm{PSH}(X^{\mathrm{red}})\,. \tag{3.1}$$

Proof. Let $\varphi \in \mathrm{PSH}(X)$. We claim that $\varphi \in \mathrm{PSH}(X^{\mathrm{red}})$ as well. Condition (1) is trivially satisfied. Let us prove Condition (2). The problem is local. Fix $x \in X$. We may assume that there is a closed immersion $X \hookrightarrow \Omega$, where Ω is a bounded pseudo-convex domain in $\mathbb{C}^N$, an open set $\tilde{X} \subseteq \Omega$ with $x \in \tilde{X}$, a psh function $\tilde{\varphi}$ on $\tilde{X}$, such that $\tilde{\varphi}|_{\tilde{X}\cap X} = \varphi|_{\tilde{X}\cap X}$. Note that $X^{\mathrm{red}} \hookrightarrow X$ induces a closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$. Now we can take the same $\tilde{\varphi}$ to conclude.

Now let $\varphi \in \mathrm{PSH}(X^{\mathrm{red}})$. We want to show $\varphi \in \mathrm{PSH}(X)$. Again, it suffices to prove Condition (2). The problem is local. We may assume that X is Stein. Take $x \in X$. We may assume that there is a closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$, where Ω is a bounded pseudo-convex domain in $\mathbb{C}^N$, an open set $\tilde{X} \subseteq \Omega$ with $x \in \tilde{X}$, a psh function $\tilde{\varphi}$ on $\tilde{X}$, such that $\tilde{\varphi}|_{\tilde{X}\cap X} = \varphi|_{\tilde{X}\cap X}$.

By Theorem 2.11, after possibly shrinking X, we can lift the closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$ to a morphism $j : X \to \Omega$:

$$\begin{tikzcd} X^{\mathrm{red}} \arrow[r, hook] \arrow[d, hook] & \Omega \\ X \arrow[ur, "j"] & \end{tikzcd} .$$

By Proposition 3.3, $j^*\varphi$ is psh. Now φ is the image of $j^*\varphi$ under (3.1). □

By this proposition, we could always restrict our attention to reduced analytic spaces.

Theorem 3.5 (Fornaess–Narasimhan)**.** *Let $\varphi : X \to [-\infty, \infty)$ be a function. Assume that φ is not identically $-\infty$ on any irreducible component of X, then the following are equivalent:*

(1) φ is psh.

(2) φ is usc and for any morphism $f : \Delta \to X$ from the open unit disk Δ in $\mathbb{C}$ to X such that $f^*\varphi$ is not identically $-\infty$, then $f^*\varphi$ is psh.

Moreover, assume that X is unibranch, then the conditions are equivalent to

(3) $\varphi|_{X\setminus \operatorname{Sing} X^{\mathrm{red}}}$ is psh, φ is locally bounded from above near $\operatorname{Sing} X^{\mathrm{red}}$ and φ is strongly usc in the following sense:

$$\varphi(x) = \varlimsup_{y\to x, y\in X\setminus \operatorname{Sing} X^{\mathrm{red}}} \varphi(y)\,, \quad x \in X\,. \tag{3.2}$$

(4) φ is locally integrable, locally bounded from above, strongly usc and $\mathrm{dd^c}\varphi \geq 0$.

Proof. Note that we can assume that X is reduced. The equivalence between (1) and (2) is the classical Fornaess–Narasimhan theorem. See [FN80]. For equivalence with the other conditions, see [Dem85, Section 1.8]. □

Corollary 3.6. *Assume that X is unibranch. Let $\varphi \in \mathrm{PSH}(X \setminus \operatorname{Sing} X^{\mathrm{red}})$. Assume that φ is locally bounded from above near $x \in \operatorname{Sing} X^{\mathrm{red}}$, then there is a unique extension $\varphi \in \mathrm{PSH}(X)$.*

Proposition 3.7.

(1) Assume that X is unibranch. Let φ_θ be a family in $\mathrm{PSH}(X)$, locally uniformly bounded from above. Then $\sup^*_\theta \varphi_\theta$ is also psh.

(2) Let φ_θ be a decreasing net in $\mathrm{PSH}(X)$, such that $\inf_\theta \varphi_\theta$ is not identically $-\infty$ on each irreducible component of X, then $\inf_\theta \varphi_\theta$ is psh.

Here

$$\sup{}^* f_\theta := (\sup_\theta f_\theta)^*$$

and

$$f^*(x) := \varlimsup_{y\to x, y\in X\setminus \operatorname{Sing} X^{\mathrm{red}}} f(y)\,.$$

Proof. (1) Observe that

$$(\sup_\theta{}^* \varphi_\theta)|_{X\setminus \operatorname{Sing} X^{\mathrm{red}}} = \sup_\theta{}^* \varphi_\theta|_{X\setminus \operatorname{Sing} X^{\mathrm{red}}}\,,$$

where the right-hand side is psh by the classical theory. Now $\sup^*_\theta \varphi_\theta$ is clearly locally bounded from above, (3.2) also clearly holds.

(2) Note that $\inf_\theta \varphi_\theta$ is usc. Condition (2) of Theorem 3.5 clearly holds. □

3.2. Extension theorem.

Theorem 3.8 (Extension theorem)**.** *Let M be a Stein manifold. Let $N \subseteq M$ be a closed reduced complex analytic subvariety. Let φ be a psh function on N. Assume that ψ is a continuous psh exhausion function on M, such that $\varphi \leq \psi|_N$. Let $c > 1$. Then there is a psh extension of φ to M, such that $\varphi \leq c \max\{\psi, 0\}$.*

For the proof, we refer to [CGZ13].

3.3. **Local regularization.** For each $N \geq 0$, we fix a Friedrichs kernel $\rho = \rho_N : [0, \infty) \to [0, \infty)$, such that ρ is smooth, $\rho(r) = 0$ for $r \geq 1$ and

$$\int_{\mathbb{C}^N} \rho(|z|) \, \mathrm{d}\lambda(z) = 1 \,.$$

Let $U \subseteq \mathbb{C}^N$ be an open subset. For any locally integrable function $u : U \to [-\infty, \infty)$ and any $\delta > 0$, define

$$\varphi_\delta(x) := \int_{\mathbb{C}^N} u(x - \delta y)\rho(|y|) \, \mathrm{d}\lambda(y) \,, \quad x \in U_\delta \,,$$

where

$$U_\delta := \{ \, x \in U : B(x, \delta) \subseteq U \, \} \,.$$

Lemma 3.9. *Let $\Omega \subseteq \mathbb{C}^N$ be a bounded pseudo-convex domain. Let $V \hookrightarrow \Omega$ be a closed analytic subspace. Let $W \Subset V$ be an open subset. Let $\varphi \in \mathrm{PSH}(V) \cap L^\infty(V)$. There there exists a decreasing sequence φ_i of smooth psh functions on W, converging pointwisely to φ on W.*

Proof. By Theorem 3.8, φ can be extended to a psh function on Ω. Define $\varphi_i = \varphi_{1/i}|_W$. □

3.4. **Bedford–Taylor product.** Proofs in this section are mostly taken from [Dem85] and the book [GZ17].
Fix a complex analytic space X.

Definition 3.10 (Bedford–Taylor)**.** Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1, \dots, k)$. Let T be a closed positive current of bidimension (m, m) on X. We define

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T := \mathrm{dd^c} \left(\varphi_1 \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \right) . \tag{3.3}$$

Remark 3.11. Unless X is equi-dimensional, the bidegree of a current is not well-defined. So we only talk about bidimensions.

Proposition 3.12. *Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1, \dots, k)$. Let T be a closed positive current of bidimension (m, m) on X. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T$ is a closed positive current of bidimension $(m - k, m - k)$.*

Proof. We prove by induction. When $k = 0$, there is nothing to prove. Assume that the result is known for $k - 1$, namely, assume that $\mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T$ is closed positive. Then for any smooth psh function φ,

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \mathrm{dd^c}\varphi_k \wedge T$$

is clearly closed positive. As our problem is local, we may assume that there is a decreasing sequence of smooth psh functions φ^i converging pointwisely to φ_1, then

$$\varphi^i \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \rightharpoonup \varphi_1 \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \,,$$

so

$$\mathrm{dd^c}\varphi^i \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \,,$$

this proves our result. □

Now we prove the functoriality of this product.

Proposition 3.13 (Projection formula)**.** *Let $\pi : Y \to X$ be a proper morphism of complex analytic spaces. Let $\varphi_0, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Let T be a closed positive current of bidimension (m, m) on Y. Then*

$$\pi_*(\pi^*\varphi_0 \mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k \wedge T) = \varphi_0 \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \pi_*T \,.$$

Proof. By induction on k, we may assume that $k = 1$. The result simply follows from the fact that d and $\mathrm{d^c}$ are functorial and the classical projection formula. □

In particular, let Y be an irreducible component of X of dimension n, let $i : Y \to X$ be the inclusion, $T = [Y]$ is a closed positive current of bidimension (n, n) (see [GH78, Section 0.2] for example). Then we conclude

Corollary 3.14. *Let Y be an irreducible component of X of dimension n, let $i : Y \to X$ be the inclusion. Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then*

$$i_*(\mathrm{dd^c}\varphi_1|_Y \wedge \cdots \wedge \mathrm{dd^c}\varphi_k|_Y) = \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge [Y]\,.$$

Corollary 3.15. *Let $\pi : Y \to X$ be a proper bimeromorphic morphism between complex analytic spaces. Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then*

$$\pi_*(\pi^*\varphi_0 \mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k) = \varphi_0 \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k\,.$$

Corollary 3.16. *Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Let $i : X^{\mathrm{red}} \to X$ be the canonical inclusion. Then*

$$i_*(\mathrm{dd^c}\varphi_1|_{X^{\mathrm{red}}} \wedge \cdots \wedge \mathrm{dd^c}\varphi_k|_{X^{\mathrm{red}}}) = \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k\,.$$

Definition 3.17. Let $\varphi \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$, let T be a closed positive current of bidimension (p, p) on X, then on each open set $U \Subset X$,

$$\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge T := \frac{1}{2}\mathrm{dd^c}(\varphi - \inf_U \varphi)^2 \wedge T - (\varphi - \inf_U \varphi)\mathrm{dd^c}\varphi \wedge T\,.$$

This gives a well-defined $(k+1, k+1)$-current on X.

Theorem 3.18 (Chern–Levine–Nirenberg)**.** *Let $U \subseteq X$ be an open subset. Let $\varphi_i \in \mathrm{PSH}(U) \cap L^\infty_{\mathrm{loc}}(U)$ $(i = 1, \dots, k)$. Let T be a closed positive current of bidimension (k, k) on U. Let $W \Subset V \Subset U$ be two open sets, then there is a constant $C = C(W, V) > 0$, such that for any compact set $K \subset W$,*

$$\int_K \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \leq C \left\|\varphi_1\right\|_{L^\infty(E)} \cdots \left\|\varphi_k\right\|_{L^\infty(E)} \left\|T\right\|_E \tag{3.4}$$

and

$$\int_K \mathrm{d}\varphi_1 \wedge \mathrm{d^c}\varphi_1 \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \leq C \left\|\varphi_1\right\|^2_{L^\infty(E)} \left\|\varphi_2\right\|_{L^\infty(E)} \cdots \left\|\varphi_k\right\|_{L^\infty(E)} \left\|T\right\|_E\,,$$

where $E = \operatorname{Supp} T \cap (V \setminus W)$.

Here $\|T\|_E$ is a semi-norm, defined as follows: take finitely many open sets $U_i \subseteq U$, open subsets $V_i \Subset U_i$, such that $\{V_i\}$ covers U and such that there are Kähler forms ω_i on each U_i. Then

$$\|T\|_E := \sum_i \int_{V_i} \omega_i^k \wedge T\,.$$

Proof. We first prove (3.4). By induction, it suffices to treat the case where $k = 1$. We omit the index and write $\varphi = \varphi_1$. We may assume that $\varphi|_V \leq 0$. Let χ be a compactly supported smooth function on V, equal to 1 on W. Then

$$\int_W \mathrm{dd^c}\varphi_1 \wedge T = \int_V \chi \mathrm{dd^c}\varphi_1 \wedge T = \int_V \varphi_1 \mathrm{dd^c}\chi \wedge T = \int_{V \setminus W} \varphi_1 \mathrm{dd^c}\chi \wedge T\,,$$

where we have applied Lemma 3.19. Now (3.4) is obvious.

The other part is similar. □

Lemma 3.19. *Let $U \subseteq X$ be an open subset. Let T be a closed positive current of bidimension $(1, 1)$ on U. Let $\varphi, \psi \in \mathrm{PSH}(U) \cap L^\infty_{\mathrm{loc}}(U)$, $\varphi, \psi \leq 0$. Assume $\lim_{x \to \partial U} \varphi(x) = 0$ and $\int_U \mathrm{dd^c}\psi \wedge T < \infty$, then*

$$\int_U \psi \, \mathrm{dd^c}\varphi \wedge T \leq \int_U \varphi \, \mathrm{dd^c}\psi \wedge T\,.$$

Proof. For $\epsilon > 0$, let $\varphi_\epsilon := \varphi \vee (-\epsilon)$. By monotone convergence theorem,

$$\int_U \varphi \, \mathrm{dd^c}\psi \wedge T = \lim_{\epsilon \to 0+} \int_U (\varphi - \varphi_\epsilon) \, \mathrm{dd^c}\psi \wedge T\,.$$

Let K_ϵ be the integral closure of $\{\varphi < -\epsilon\}$. Fix $\epsilon > 0$. Let $D_1 \Subset U$ be a domain close to U and such that $K \subseteq D_1$. Consider a standard mollifier ρ_η. Then

$$\int_{D_1} (\varphi - \varphi_\epsilon)\,\mathrm{dd^c}\psi \wedge T = \lim_{\eta\to 0+} \int_{D_1} (\varphi - \varphi_\epsilon) * \rho_\eta \,\mathrm{dd^c}\psi \wedge T$$

$$= \lim_{\eta\to 0+} \int_{D_1} \psi\,\mathrm{dd^c}((\varphi - \varphi_\epsilon) * \rho_\eta) \wedge T \geq \lim_{\eta\to 0+} \int_{D_1} \psi\,\mathrm{dd^c}(\varphi * \rho_\eta) \wedge T\,.$$

When ψ is continuous in a neighbourhood of $\bar{D}_1$. We could assume that ∂D_1 is null with respect to $-\psi\mathrm{dd^c}\varphi \wedge T$. Thus the right-hand side is equal to $\int_{D_1} \psi\mathrm{dd^c}\varphi \wedge T$. Hence

$$\int_{D_1} \varphi\,\mathrm{dd^c}\psi \wedge T \geq \int_{D_1} \psi\,\mathrm{dd^c}\varphi \wedge T - \epsilon \int_{D_1} \mathrm{dd^c}\psi \wedge T\,.$$

In general, take a continuous non-positive psh functions ψ_j on Ω converging to ψ on a neighbourhood of $\bar{D}_1$. Choose a domain $D_2 \Subset D_1$, let L be the closure of D_2. We get

$$\int_L \varphi\,\mathrm{dd^c}\psi \wedge T \geq \varlimsup_{j\to\infty} \int_L \varphi\,\mathrm{dd^c}\psi_j \wedge T \geq \lim_{j\to\infty} \int_{D_1} \psi_j\,\mathrm{dd^c}\varphi \wedge T - \epsilon \varliminf_{j\to\infty} \int_{D_1} \mathrm{dd^c}\psi_j \wedge T$$

$$\geq \int_{D_1} \psi\,\mathrm{dd^c}\varphi \wedge T - \epsilon \int_{D_1} \mathrm{dd^c}\psi \wedge T\,.$$

Take limit in D_1, D_2 and ϵ, we conclude. □

Definition 3.20. A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighbourhood $V \subseteq X$ of x, a psh function φ on V, such that $E \cap V \subseteq \{x \in V : \varphi(x) = -\infty\}$.

Definition 3.21. The σ-algebra of *quasi-Borel sets* is the σ-algebra generated by all Borel sets and all pluripolar set. A set in this σ-algebra is called a *quasi-Borel set*.

Definition 3.22. Assume that X is unibranch. A set $E \subseteq X$ is *negligible* if for any $x \in X$, there is an open neighbourhood $V \subseteq X$ of x, a bounded from above family $\{\psi_\theta\}$ of psh functions on V, such that

$$E \cap V \subseteq \{\, x \in V : \sup{}^*\{\psi_\theta\}(x) > \sup\{\psi_\theta\}(x) \,\}\ .$$

Remark 3.23. There are different definitions in the literature about both pluripolar sets and negligible sets. In some literature, our notion of pluri-polar sets is called locally pluripolar. However, we want to emphasis that what is actually proved in [BT82] is that pluri-polarity is equivalent to negligiblility *in our sense*.

Now we study the pluri-fine topology on X.

Classical analogues of the remaining results in this section are proved in [BT82] and [BT87].

Assume that X has pure dimension n. Let $\varphi_1, \dots, \varphi_n \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_n$ is a Borel measure. We extend this measure by taking its completion. Then all quasi-Borel sets are measurable.

Theorem 3.24. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $E \subseteq X$ be a subset. Then E is negligible if and only if E is pluripolar.*

Proof. We may assume that X is reduced.

Assume that E is pluripolar. We show that E is negligible. The problem is local, so we may assume that there is a psh function $\varphi \leq 0$ on X, such that $E \subseteq \{\varphi = -\infty\}$. We may assume that equality actually holds. For each $i \geq 1$, let $\varphi_i := i^{-1}\varphi$. Observe that E has empty interior by the classical theory on $E \cap (X \setminus \operatorname{Sing} X)$. Now $\sup_i \varphi_i = 0$ on $X \setminus E$, while $\sup_i \varphi_i|_E = -\infty$, so $\sup^* \varphi_i = 0$. Thus

$$E = \left\{ \sup_i \varphi_i < \sup_i{}^* \varphi_i \right\}\,.$$

Now assume that E is negligible. We show that E is pluripolar. We may assume that X is a reduced, unibranch Kähler space with a Kähler form ω.

Let $\pi : Y \to X$ be a resolution of singularity. Observe that $\pi^{-1}(E)$ is negligible, as the union of two negligible sets. Let

$$u^*_{E,\omega} := \sup{}^* \{\, \varphi \in \mathrm{PSH}(X, \omega) : \varphi \leq 0, \varphi|_E \leq -1 \,\} \in \mathrm{PSH}(X, \omega)\,.$$

Observe that

$$\pi^* u^*_{E,\omega} = u^*_{\pi^{-1}E,\pi^*\omega} = 0\,,$$

where the second equality follows from the classical theory (see [Lu21, Lemma 2.3] for example). In particular, $u^*_{E,\omega} = 0$. So by Choquet's lemma, we may take an increasing sequence of ω-psh functions ψ_i, $\psi_i \leq 0$, $\psi_i|_E \leq -1$ such that the L^1-norm of ψ_i is bounded from above by 2^{-i}. Then take $\psi = \sum_i \psi_i$. We find $E \subseteq \{\psi = -\infty\}$. □

Theorem 3.25. *Assume that X is an open subspace of a compact unibranch Kähler space. An arbitrary union of pluri-fine open subsets differs from a countable subunion by at most a pluripolar set.*

Proof. Let $U_\theta \subseteq X$ ($\theta \in I$) be a family of pluri-fine open subsets in X. Let $U = \cup_\theta U_\theta$. Take a countable basis B_i of the topology of X. We may assume that each U_θ is of the form $B_i \cap \{\varphi_\theta > -1\}$ for some B_i, where $\varphi_\theta \leq 0$ is a bounded psh function on B_i. It suffices to prove that for each fixed i,

$$\bigcup_\theta B_i \cap \{\varphi_\theta > -1\}$$

with θ running through the subset set J_i of I consisting of all θ such that U_θ is of the form $B_i \cap \{\varphi_\theta > -1\}$. So we may assume that there are bounded psh functions φ_θ defined on X, such that $-1 \leq \varphi_\theta \leq -1$, $U_\theta = \{\varphi_\theta > 0\}$. Now by Choquet's lemma, there is a countable subset $J \subseteq I$, such that

$$\sup_{\theta\in I}{}^* \varphi_\theta = \sup_{\theta\in J}{}^* \varphi_\theta\,.$$

So by Theorem 3.24,

$$\sup_{\theta\in I} \{\varphi_\theta > 0\} = \bigcup_{\theta\in I} U_\theta$$

differs from

$$\sup_{\theta\in J} \{\varphi_\theta > 0\} = \bigcup_{\theta\in J} U_\theta$$

by at most a pluripolar set. This proves our theorem. □

This is a corollary of the previous theorem.

Corollary 3.26. *Assume that X is an open subspace of a compact unibranch Kähler space, then all pluri-fine Borel sets are quasi-Borel.*

Theorem 3.27. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_0^j, \dots, \varphi_k^j$ be $(n+1)$-sequences of psh functions on X. Assume that the sequences are all uniformly bounded and converging a.e. monotonically (either increasing or decreasing) to psh functions $\varphi_0, \dots, \varphi_k$ on X. Then*

(1)

$$\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k\,.$$

(2)

$$\varphi_0^j\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \varphi_0\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k\,.$$

(3)

$$\mathrm{d}\varphi_0^j \wedge \mathrm{d^c}\varphi_0^j \wedge \mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \mathrm{d}\varphi_0 \wedge \mathrm{d^c}\varphi_0 \wedge \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k\,.$$

Here p.f. *means that the weak convergence is with respect to the pluri-fine topology.*

Proof. It suffices to prove (2).

We may assume that X is reduced. Let $\pi : Y \to X$ be a resolution of singularity. From the classical theory, we know that

$$\pi^*\varphi_1^j\mathrm{dd^c}\pi^*\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k^j \xrightarrow{\text{p.f.}} \pi^*\varphi_1\mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k\,.$$

We conclude by Proposition 3.13. □

Corollary 3.28. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1, \dots, k)$. Then for any $\sigma \in \mathcal{S}_k$,*

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k = \mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(k)}\,.$$

This is a standard consequence of Theorem 3.27, we omit the details.

In particular, we may use the following notation for any of these products: $\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j$.

Lemma 3.29. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X)$. Let E be a pluripolar set. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k$ does not charge E in the following sense: For each irreducible component Y of X, $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge [Y]$ does not charge $E \cap Y$.*

Proof. We may assume that X is irreducible of dimension n. The problem is local, by Theorem 3.24, we may assume that there is a bounded from above sequence $\psi_j \in \mathrm{PSH}(X)$, such that $E \subseteq N$, where $N := \{\sup_j \psi_j < \sup^*{}_j \psi_j\}$. We want to show that N is null with respect to $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k$. By taking a cut-off, we may assume that ψ_j are uniformly bounded. Take a compact set $K \subseteq X$, a smooth positive $(n-k, n-k)$-form θ on X. Then it suffices to show that

$$\int_{K \cap N} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \theta = 0 \,.$$

Pick a smooth function $\chi : X \to [0, 1]$ with compact support, such that χ is equal to 1 near K. Then by Theorem 3.27,

$$\begin{aligned}\int_X \chi \sup_j \psi_j \, \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \theta &= \lim_{j\to\infty} \int_X \chi\psi_j \, \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \theta \\ &= \int_X \chi \sup_j{}^* \psi_j \, \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \theta \,.\end{aligned}$$

We conclude. □

Lemma 3.30. *Assume that X is an open subspace of a compact unibranch Kähler space. Assume that X is of pure dimension n. Let $\varphi, \psi \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Let $O := \{\varphi > \psi\}$. Then*

$$(\mathrm{dd^c}(\varphi \vee \psi))^n|_O = (\mathrm{dd^c}\varphi)^n|_O \,. \tag{3.5}$$

Proof. The problem is local. We may assume that X is a closed analytic subspace of a bounded pseudo-convex domain Ω in some $\mathbb{C}^N$. By Lemma 3.9, we may assume that there is a decreasing sequence of smooth psh functions φ_k on X converging pointwisely to φ. It is obvious that (3.5) holds for φ_k in place of φ. Hence we conclude by Lemma 3.31. □

Lemma 3.31. *Let $\Omega \subseteq \mathbb{C}^N$ be a bounded pseudo-convex domain. Let X be a closed subspace of Ω. Assume that X can be realized as an open subspace of a compact unibranch Kähler space. Assume that X is of pure dimension n. Let $O \subseteq X$ be a pluri-fine open subset. Let φ be a bounded psh function on X. Let φ_k be a decreasing sequence of bounded psh functions on X converging pointwisely to φ. Let $\psi \in \mathrm{PSH}(X) \cap L^\infty(X)$. Assume that for any k,*

$$(\mathrm{dd^c}\varphi_k)^n|_O = (\mathrm{dd^c}\psi)^n|_O \,,$$

then

$$(\mathrm{dd^c}\varphi)^n|_O = (\mathrm{dd^c}\psi)^n|_O \,,$$

Proof. By Theorem 3.25, we may assume that $O = B \cap \{\eta > 0\}$, where $B \cap X$ is an open ball in $\mathbb{C}^n$ and η is a bounded psh function on B. Replace η with a non-negative pluri-fine continuous function with compact support in X such that $O = \{\eta > 0\}$. Then by Theorem 3.27, for any continuous function f on X,

$$\int_X f\eta \, (\mathrm{dd^c}\psi)^n = \lim_{k\to\infty} \int_X f\eta \, (\mathrm{dd^c}\varphi_k)^n = \int_X f\eta \, (\mathrm{dd^c}\varphi)^n \,.$$

We conclude. □

Theorem 3.32 (Plurilocality)**.** *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi^i, \psi^i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1, \dots, n)$. Let $W \subseteq X$ be an open set with respect to the plurifine topology. Assume that $\varphi^i|_W = \psi^i|_W$ for all i, then*

$$\mathrm{dd^c}\varphi^1 \wedge \cdots \wedge \mathrm{dd^c}\varphi^n|_W = \mathrm{dd^c}\psi^1 \wedge \cdots \wedge \mathrm{dd^c}\psi^n|_W \,.$$

Proof. By polarization, we may assume that all φ^i are equal and all ψ^i are equal. We omit the superindex. Then we want to show $(\mathrm{dd^c}\varphi)^n = (\mathrm{dd^c}\psi)^n$. Note that $\varphi = \varphi \vee (\psi - \epsilon)$ for any $\epsilon > 0$. It suffices to apply Lemma 3.30. □

Finally, we introduce the notion of quasi-psh functions.

Definition 3.33. Let θ be a smooth strongly closed real $(1,1)$-form on X (in the sense that locally it is the pull-back of a smooth closed real $(1,1)$-form $\tilde{\theta}$ on pseudo-convex domains). Then the set $\mathrm{PSH}(X,\theta)$ consists of all functions $\varphi : X \to [-\infty,\infty)$, such that on each open set $V \subseteq X$, embedded as a closed analytic subspace in a pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, such that for any smooth function a on U with $\mathrm{dd^c} a = \theta$, $a|_V + \varphi$ is psh.

3.5. **Non-pluripolar product.** Assume that X is an open subset of a compact unibranch Kähler space of pure dimension n.

Definition 3.34. Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Let

$$O_k = \bigcap_{j=1}^{p} \{\varphi_j > -k\} \;.$$

We say that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is *well-defined* if for each open subset $U \subseteq X$ such that there is a Kähler form ω on U, each compact subset $K \subseteq U$, we have

$$\sup_{k\geq 0} \int_{K\cap O_k} \left(\bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j,-k\} \right)\Bigg|_U \wedge \omega^{n-p} < \infty. \tag{3.6}$$

In this case, we define $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ by

$$\mathbb{1}_{O_k} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j,-k\}. \tag{3.7}$$

on $\bigcup_{k\geq 0} O_k$ and make a zero-extension to X.

Remark 3.35. The condition (3.6) is clearly independent of the choice of U and ω.

Proposition 3.36. *Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Let $\sigma \in \mathcal{S}_p$. By definition, $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_p$ is well-defined if and only if $\mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(p)}$ is. Moreover, in this case,*

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(p)} \,.$$

In particular, we may use the following notation for any of these products: $\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j$.

Proof. This is a direct consequence of Corollary 3.28. □

Lemma 3.37. *Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined. Let $E_k \subseteq O_k$ ($k \geq 0$) be Borel sets such that $X - \cup_k E_k$ is pluripolar. Let Ω be a $(n-p,n-p)$-form with measurable coefficients. Assume that the following conditions are satisfied:*

(1) $\operatorname{Supp}\Omega$ *is compact.*
(2) For each open subset $U \subseteq X$, each Kähler form ω on U, there is a constant $C > 0$ such that $-C\omega^{n-p} \leq \Omega \leq C\omega^{n-p}$ holds on $\operatorname{Supp}\Omega \cap U$.

Then

$$\lim_{k\to\infty} \int_X \mathbb{1}_{E_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j,-k\} \wedge \Omega = \int_X \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \wedge \Omega \,.$$

In particular,

$$\mathbb{1}_{E_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j,-k\} \rightharpoonup \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \,, \quad k \to \infty$$

as currents and the convergence is strong on each compact subset of X.

Proof. Since the problem is local, we may assume that $\operatorname{Supp}\Omega \subseteq U$, where $U \subseteq X$ is an open subset with a Kähler form ω. Take $C > 0$ so that $-C\omega^{n-p} \leq \Omega \leq C\omega^{n-p}$. Then observe that

$$
\begin{aligned}
0 \leq \int_X \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\} \wedge \Omega - \int_X \mathbb{1}_{E_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\} \wedge \Omega \\
\leq \int_{\operatorname{Supp}\Omega} (1 - \mathbb{1}_{E_k}) \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \wedge \Omega \,.
\end{aligned}
$$

The right-hand side tends to 0 by dominated convergence theorem. So it suffices to prove the theorem for $E_k = O_k$. In this case, the theorem again follows from dominated convergence theorem. □

Proposition 3.38. *Let $\varphi_1, \dots, \varphi_m$ be psh functions on X. Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity. Then $\varphi_1 \wedge \dots \wedge \varphi_m$ is well-defined if and only if $\pi^*\varphi_1 \wedge \dots \wedge \pi^*\varphi_m$ is. In this case,*

$$
\pi_* \left(\mathrm{dd^c}\pi^*\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\pi^*\varphi_m\right) = \mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_m \,.
$$

Proof. This follows directly from Proposition 3.13. □

This proposition allows us to generalize directly known facts about non-pluripolar products in the smooth setting to the current setting.

Proposition 3.39. *Let $\varphi_1, \dots, \varphi_p \in \mathrm{PSH}(X)$.*

(1) The product $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ is local in pluri-fine topology. In the following sense: let $O \subseteq X$ be a pluri-fine open subset, let $v_1, \dots, v_p \in \mathrm{PSH}(X)$, assume that $\varphi_j|_O = v_j|_O$, for $j = 1, \dots, p$. Assume that $\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j$, $\bigwedge_{j=1}^p \mathrm{dd^c}v_j$ are both well-defined, then

$$
\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j\right|_O = \left.\bigwedge_{j=1}^{p} \mathrm{dd^c}v_j\right|_O . \tag{3.8}
$$

If O is open in the usual topology, then the product $\bigwedge_{j=1}^p \mathrm{dd^c}(\varphi_j|_O)$ on O is well-defined and

$$
\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j\right|_O = \bigwedge_{j=1}^{p} \mathrm{dd^c}(\varphi_j|_O) \,. \tag{3.9}
$$

Let $\mathcal{U}$ be an open covering of X. Then $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ is well-defined if and only if each of the following product is well-defined

$$
\bigwedge_{j=1}^{p} \mathrm{dd^c}(\varphi_j|_U)\,, \quad U \in \mathcal{U}\,.
$$

(2) The current $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ and the fact that it is well-defined depend only on the currents $\mathrm{dd^c}\varphi_j$, not on specific φ_j.
(3) When $\varphi_1, \dots, \varphi_p \in L^\infty_{\mathrm{loc}}(X)$, $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ is well-defined and is equal to the Bedford–Taylor product.
(4) Assume that $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ puts not mass on pluripolar sets.
(5) Assume that $\mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j$ is a closed positive current of bidegree (p,p) on X.
(6) The product is multi-linear: let $v_1 \in \mathrm{PSH}(X)$, then

$$
\mathrm{dd^c}(\varphi_1 + v_1) \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j = \mathrm{dd^c}\varphi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j + \mathrm{dd^c}v_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j \tag{3.10}
$$

in the sense that the left-hand side is well-defined if and only if both terms on the right-hand side are well-defined, and the equality holds in that case.

Proof. The only non-trivial part is (2).

(2) By (1), we may assume that there is a Kähler form ω on X. Let w_j $(j=1,\dots,p)$ be pluri-harmonic functions on X. Assume that $\langle \mathrm{dd^c}\varphi_1\wedge\dots\wedge\mathrm{dd^c}\varphi_p\rangle$ is well-defined. We want to prove that $\langle \mathrm{dd^c}(w_1+\varphi_1)\wedge\dots\wedge\mathrm{dd^c}(w_p+\varphi_p)\rangle$ is also well-defined and

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}\varphi_j=\bigwedge_{j=1}^{p}\mathrm{dd^c}(w_j+\varphi_j)\,. \tag{3.11}$$

By further shrinking X, we may assume that w_j are bounded from above on X, say $w_j\leq C$, for $j=1,\dots,p$ Then for any $k\geq 0$, on the pluri-fine open set

$$V_k:=\bigcap_{j=1}^{p}\{\varphi_j+w_j>-k\}\,,$$

we have $\varphi_j>-k-C$, so by (1),

$$\max\{\varphi_j+w_j,-k\}=\max\{\varphi_j,-k-C\}+w_j\,.$$

Let $K\subseteq X$ be a compact subset, then

$$\int_K \mathbb{1}_{O_k}\bigwedge_{j=1}^{p}\mathrm{dd^c}\max\{\varphi_j+w_j,-k\}\wedge\omega^{n-p}=\int_K \mathbb{1}_{O_k}\bigwedge_{j=1}^{p}\mathrm{dd^c}\max\{\varphi_j,-k-C\}\wedge\omega^{n-p}\,.$$

The right-hand side is bounded by assumption. So the right-hand side of (3.11) is well-defined and (3.11) follows. □

Let $\theta_1,\dots,\theta_p$ be smooth strongly closed $(1,1)$-forms on X. Let $\varphi_i\in\mathrm{PSH}(X,\theta_i)$. Let $T_i=\theta_i+\mathrm{dd^c}\varphi_i$. Then $T_1\wedge\dots\wedge T_p$ can be defined in the obvious way.

Definition 3.40. We say a closed positive $(1,1)$-current T on X is *good* if for any $x\in X$, there is a neighbourhood $V\subseteq X$ of x, such that there exists a smooth strongly closed $(1,1)$-form θ on V, a function $\varphi\in\mathrm{PSH}(V,\theta)$, such that $T=\theta+\mathrm{dd^c}\varphi$ on V.

Let $T_1,\dots,T_p$ be good closed positive $(1,1)$-currents on X, we can define $T_1\wedge\dots\wedge T_p$ in the obvious way. Namely,

Definition 3.41. Let $T_1,\dots,T_p$ be good closed positive $(1,1)$-currents on X. Locally on a small enough open set $V\subseteq X$, write $T_i=\theta_i+\mathrm{dd^c}\varphi_i$, where there are closed immersions $X\hookrightarrow\Omega_i$ with $\Omega_i\subseteq\mathbb{C}^{N_i}$ being bounded pseudo-convex domains, smooth closed real forms $\tilde{\theta}_i$ on Ω_i, such that $\theta_i|_V=\tilde{\theta}_i|_V$, $\varphi_i\in\mathrm{PSH}(V,\theta_i)$. Then writing $\tilde{\theta}_i=\mathrm{dd^c}a_i$ for some smooth functions a_i on Ω_i, we define

$$T_1\wedge\dots\wedge T_p:=\bigwedge_{i=1}^{p}\mathrm{dd^c}(a_i+\varphi_i)\,.$$

Proposition 3.42. *Let $T_1,\dots,T_p$ be closed positive currents of bidegree $(1,1)$ on X. Assume that all T_i's are good.*

(1) The product $T_1\wedge\dots\wedge T_p$ is local in pluri-fine topology in the following sense: let $O\subseteq X$ be a pluri-fine open subset, let $S_1,\dots,S_p$ be closed positive currents of bidegree $(1,1)$ on X. Assume that all S_i's are good. Assume that

$$T_j|_O=S_j|_O,\quad j=1,\dots,p.$$

Assume that $T_1\wedge\dots\wedge T_p$, $S_1\wedge\dots\wedge S_p$ are both well-defined, then

$$T_1\wedge\dots\wedge T_p|_O=S_1\wedge\dots\wedge S_p|_O\,. \tag{3.12}$$

If O is open in the usual topology, then the product $T_1\wedge\dots\wedge T_p|_O$ on O is well-defined and

$$T_1\wedge\dots\wedge T_p|_O=T_1\wedge\dots\wedge T_p|_O. \tag{3.13}$$

Let $\mathcal{U}$ be an open covering of X. Then $T_1\wedge\dots\wedge T_p$ is well-defined if and only if each of the following product is well-defined

$$T_1\wedge\dots\wedge T_p|_U,\quad U\in\mathcal{U}.$$

(2) *Assume that $T_1 \wedge \cdots \wedge T_p$ is well-defined, then the product $T_1 \wedge \cdots \wedge T_p$ puts not mass on pluripolar sets.*
(3) *Assume that $T_1 \wedge \cdots \wedge T_p$ is well-defined, then $T_1 \wedge \cdots \wedge T_p$ is a closed positive current of bidegree (p,p).*
(4) *The product $T_1 \wedge \cdots \wedge T_p$ is symmetric.*
(5) *The product is multi-linear: let T_1' be a good closed positive current of bidegree $(1,1)$, then*

$$(T_1 + T_1') \wedge T_2 \wedge \cdots \wedge T_p = T_1 \wedge T_2 \wedge \cdots \wedge T_p + T_1' \wedge T_2 \wedge \cdots \wedge T_p$$

in the sense that left-hand side is well-defined if and only if both terms on right-hand side are well-defined, and the equality holds in that case.

Proposition 3.43. *Let X be a compact unibranch Kähler space of pure dimension n. Let $T_1, \dots, T_p$ be good closed positive currents of bidegree $(1,1)$ on X. Then $T_1 \wedge \cdots \wedge T_p$ is well-defined.*

Proof. We may assume that X is reduced.

Fix a Kähler form ω on X. In this case, write $T_j = (T_j + C\omega) - C\omega$ for $C > 0$ large enough and apply Proposition 3.42 (5), we may assume that T_j is in a Kähler class. So we can write $T_j = \omega_j + \mathrm{dd^c}\varphi_j$, where ω_j is a Kähler form and φ_j is ω_j-psh. Let U be an open subset on which we can write $\omega_j = \mathrm{dd^c}\psi_j$ with psh functions $\psi_j \leq 0$ on U. Now on U, for each $k \geq 0$, $\{\psi_j + \varphi_j > -k\} \subseteq \{\varphi_j > -k\}$ so for each compact subset $K \subseteq U$,

$$\begin{aligned}
&\int_K \mathbb{1}_{\cap_{j=1}^p \{\psi_j + \varphi_j > -k\}} \bigwedge_{j=1}^p \mathrm{dd^c} \max\{\psi_j + \varphi_j, -k\} \wedge \omega^{n-p} \\
=&\int_K \mathbb{1}_{\cap_{j=1}^p \{\psi_j + \varphi_j > -k\}} \bigwedge_{j=1}^p (\omega_j + \mathrm{dd^c} \max\{\varphi_j, -k\}) \wedge \omega^{n-p} \\
\leq &\int_X \bigwedge_{j=1}^p (\omega_j + \mathrm{dd^c} \max\{\varphi_j, -k\}) \wedge \omega^{n-p} \\
=&\int_X \bigwedge_{j=1}^p \omega_j \wedge \omega^{n-p}.
\end{aligned}$$

The last step follows from the corresponding result on a resolution. □

3.6. **Bimeromorphic behaviour.** Let X be a compact unibranch Kähler complex analytic space of pure dimension n.

Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity that is an isomorphism over $X \setminus \operatorname{Sing} X^{\mathrm{red}}$.

Let θ be a strongly closed smooth $(1,1)$-form on X. Assume that $[\theta]$ is big: for all proper bimeromorphic morphism $f : Y \to X$ from a smooth manifold Y, $f^*\theta$ is big. In this case, define $\operatorname{vol}\theta = \operatorname{vol} f^*\theta$.

We set

$$V_\theta := \sup{}^* \{ \varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0 \} . \tag{3.14}$$

Definition 3.44. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Define

$$\varphi \wedge \psi := \sup{}^* \{ \eta \in \mathrm{PSH}(X,\theta) : \eta \leq \varphi, \eta \leq \psi \} .$$

When the set is empty, we just define $\sup^* \emptyset = -\infty$.

Definition 3.45. Let $U \subseteq X$ be an open subset. Let $\varphi \in \mathrm{PSH}(\pi^{-1}U)$, define $\pi_*\varphi \in \mathrm{PSH}(U)$ as the unique psh extension of $\varphi|_{\pi^{-1}(U \setminus \operatorname{Sing} X)}$. We call $\pi_*\varphi$ the psh pushforward of φ.

Definition 3.46. Let

$$\mathcal{E}(X,\theta) := \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n = \operatorname{vol}\theta \right\} .$$

For any $p \geq 1$,

$$\mathcal{E}^p(X,\theta) := \left\{ \varphi \in \mathcal{E}(X,\theta) : \int_X |\varphi|^p \theta_\varphi^n < \infty \right\} .$$

Let

$$\mathcal{E}^\infty(X,\theta) := \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \sup_X |\varphi - V_\theta| < \infty \right\} .$$

Definition 3.47. Let $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a map $\varphi : (a,b) \to \mathcal{E}^1(X,\theta)$, such that

(1) The potential Φ on $X \times \{z \in \mathbb{C} : e^{-b} < |z| < e^{-a}\}$ defined by $\Phi(x,z) := \varphi_{-\log|z|}(x)$ is $\pi_1^*\theta$-psh, where $\pi_1 : X \times \{z \in \mathbb{C} : e^{-b} < |z| < e^{-a}\} \to X$ is the natural projection.

(2)
$$\lim_{t\to a+} \varphi_t = \varphi_a \,, \quad \lim_{t\to b-} \varphi_t = \varphi_b \,.$$

Definition 3.48. A potential $\varphi \in \mathrm{PSH}(X,\theta)$ is *model* if

$$\varphi = \sup_{C>0}{}^{*} (\varphi + C) \wedge V_\theta \,.$$

Definition 3.49. For any $\varphi \in \mathcal{E}^1(X,\theta)$, define $\mathcal{R}^1(X,\theta)$ as the set of all geodesic rays in $\mathcal{E}^1$ emanating from φ.

Definition 3.50. The *geodesic* between $\varphi_0, \varphi_1 \in \mathcal{E}^1$ is the maximal subgeodesic between them.

Proposition 3.51. *The functions π^* and π_* are inverse to each other. Under this correspondence, we get a bijection between*

(1) $\mathrm{PSH}(X,\theta)$ *and* $\mathrm{PSH}(Y,\pi^*\theta)$*. This bijection preserves the pre-rooftop structures.*
(2) $\mathcal{E}(X,\theta)$ *and* $\mathcal{E}(Y,\pi^*\theta)$*.*
(3) $\mathcal{E}^p(X,\theta)$ *and* $\mathcal{E}^p(Y,\pi^*\theta)$*.*
(4) Subgeodesics in $\mathcal{E}^1(X,\theta)$ *and subgeodesics in* $\mathcal{E}^1(Y,\pi^*\theta)$*.*
(5) Geodesics in $\mathcal{E}^1(X,\theta)$ *and geodesics in* $\mathcal{E}^1(Y,\pi^*\theta)$*.*
(6) $\mathcal{R}^1(X,\theta)$ *and* $\mathcal{R}^1(X,\pi^*\theta)$*.*
(7) Model potentials in $\mathrm{PSH}(X,\theta)$ *and in* $\mathrm{PSH}(Y,\pi^*\theta)$*.*

Proof. Only (4) needs a proof. Let Φ be a subgeodesic in $\mathcal{E}^1(Y,\pi^*\theta)$, regarded as a potential on $Y \times A$, where $A = \{z \in \mathbb{C} : e^{-1} < |z| < 1\}$. It is easy to verify then that the psh pushforward of Φ is the same as the ensemble of all psh pushforwards for fixed $z \in A$. □

In particular, d_p metric is defined on $\mathcal{E}^p(X,\theta)$. This result partially generalizes [Dar17].

Let $\varphi, \phi \in \mathrm{PSH}(X,\theta)$. We define

$$[\varphi] \wedge \psi := \sup_{C>0}{}^{*} \left((\varphi + C) \wedge \psi \right) .$$

Lemma 3.52. *Let $\varphi, \phi \in \mathrm{PSH}(X,\theta)$. Then $[\pi^*\varphi] \wedge \pi^*\psi = \pi^*\left([\varphi] \wedge \psi\right)$.*

Proof. By Proposition 3.51, for each $C > 0$,

$$(\pi^*\varphi + C) \wedge \pi^*\psi = \pi^*\left((\varphi + C) \wedge \psi\right) .$$

As $[\varphi] \wedge \psi$ is by definition the minimal θ-psh function lying above all $(\varphi + C) \wedge \psi$, by Proposition 3.51 again, $\pi^*([\varphi] \wedge \psi)$ is the minimal $\pi^*\theta$-psh function lying above all $\pi^*((\varphi + C) \wedge \psi) = (\pi^*\varphi + C) \wedge \pi^*\psi$. Hence we conclude. □

3.7. Basic properties.

Let X be a compact Kähler unibranch complex analytic space of pure dimension n.

Theorem 3.53 (Integration by parts)**.** *Let θ_j $(j = 0, \dots, n)$ be big cohomology classes on X. Let θ_j $(j = 0, \dots, n)$ be smooth representatives in θ_j. Let $\gamma_j \in \mathrm{PSH}(X,\theta_j)$ $(j = 2, \dots, n)$. Let $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta_0)$, $\psi_1, \psi_2 \in \mathrm{PSH}(X,\theta_1)$. Let $u = \varphi_1 - \varphi_2$, $v = \psi_1 - \psi_2$. Assume that*

$$[\varphi_1] = [\varphi_2], \quad [\psi_1] = [\psi_2].$$

Then

$$\int_X u \,\mathrm{dd^c} v \wedge \theta_{2,\gamma_2} \wedge \cdots \wedge \cdots \wedge \theta_{n,\gamma_n} = \int_X v \,\mathrm{dd^c} u \wedge \theta_{2,\gamma_2} \wedge \cdots \wedge \cdots \wedge \theta_{n,\gamma_n}. \tag{3.15}$$

Proof. In the smooth setting, this is proved in [Xia19] and [Lu21]. The general case follows immediately. □

Theorem 3.54 (Semi-continuity)**.** *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ ($k \in \mathbb{Z}_{>0}$, $j = 1, \dots, n$). Let $\chi \geq 0$ be a bounded quasi-continuous function on X. Assume that for any $j = 1, \dots, n$, $i = 1, \dots, m$, as $k \to \infty$, φ_j^k converges to φ_j monotonely. Then for any pluri-fine open set $U \subseteq X$, we have*

$$\varliminf_{k\to\infty} \int_U \chi\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_U \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \,. \tag{3.16}$$

Proof. In the smooth setting, this is due to [DDL18b, Theorem 2.3]. The general case follows immediately. □

Theorem 3.55 (Monotonicity)**.** *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X, \theta_j)$. Assume that $[\varphi_j] \succeq [\psi_j]$ for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n} \,.$$

Proof. In the smooth setting, this is proved in [Wit19] and [DDL18b]. The general case follows immediately. □

4. Potentials with prescribed singularities

Let X be a compact unibranch Kähler space of pure dimension n. Let α be a big $(1,1)$-cohomology class. Let θ be a strongly closed smooth $(1,1)$-form in the class α. Let V be the volume of α. Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity.

For $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, define

$$[\varphi] \wedge \psi := \sup_{C\in\mathbb{R}}{}^* \left((\varphi + C) \wedge \psi\right).$$

This is usually denoted by $P[\varphi](\psi)$.

4.1. Potentials with prescribed singularities.

Let $\phi \in \mathrm{PSH}(X, \theta)$ be a model potential with mass $V_\phi > 0$. Fix a resolution of singularities $\pi : Y \to X^{\mathrm{red}}$. Recall that we may always assume that Y is Kähler.

Proposition 4.1. *The following set is relatively compact in L^1-topology.*

$$\left\{ \varphi \in \mathrm{PSH}(X, \theta) : \sup_X (\varphi - \phi) = 0 \right\} \subseteq \mathrm{PSH}(X, \theta) \,.$$

Remark 4.2. By the L^1-topology on $\mathrm{PSH}(X, \theta)$, we mean the L^1-topology with respect to the measure defined by ω^n, where ω is a Kähler form on X. It is easy to see that this topology does not depend on the choice of ω. Moreover, this topology coincides with the topology defined by *la mesure aire* in the sense of [Dem85, Proposition 1.8].

Proof. We may assume that X is reduced. Let $\varphi_j \in \mathrm{PSH}(X, \theta)$ be a net such that $\sup_X(\varphi_j - \phi) = 0$. Then

$$\pi^*\varphi_j \in \left\{ \psi \in \mathrm{PSH}(Y, \pi^*\theta) : \sup_Y (\psi - \pi^*\phi) = 0 \right\} .$$

By [DDL21a, Lemma 2.2], up to subtracting a subnet, we may assume that $\pi^*\varphi_j$ converges to some $\pi^*\varphi$, $\varphi \in \mathrm{PSH}(X, \theta)$. Clearly, $\sup_X(\varphi - \phi) = 0$. Note that $\pi^*\varphi_j \to \pi^*\varphi$ in L^1 implies that $\varphi_j \to \varphi$ in L^1, we conclude. □

Definition 4.3. Define the relative full mass class as

$$\mathcal{E}(X, \theta; [\phi]) := \left\{ \varphi \in \mathrm{PSH}(X, \theta) : [\varphi] \preceq [\phi], \int_X \theta_\varphi^n = \int_X \theta_\phi^n \right\} .$$

Remark 4.4. It is easy to see that under Proposition 3.4,

$$\mathcal{E}(X, \theta; [\phi]) = \mathcal{E}(X^{\mathrm{red}}, \theta; [\phi]) \,.$$

Pull-back induces a bijection

$$\pi^* : \mathcal{E}(X, \theta; [\phi]) \xrightarrow{\sim} \mathcal{E}(Y, \pi^*\theta; [\pi^*\phi]) \,.$$

We will use these identifications without explicit mentioning.

Proposition 4.5. *Let $\varphi \in \mathrm{PSH}(X, \theta)$. Then $\varphi \in \mathcal{E}(X, \theta; [\phi])$ if and only if $[\varphi] \wedge V_\theta = \phi$.*

Proof. We may assume that X is reduced. By Proposition 3.51, Lemma 3.52, it suffices to prove the corresponding result on Y, in which case, this is exactly [DDL21a, Theorem 2.1]. ☐

Lemma 4.6. *Assume that $\gamma_j \in \mathcal{E}(X,\theta;[\phi])$ $(j=1,\dots,j_0 \leq n)$. Let $\varphi,\psi \in \mathcal{E}(X,\theta;[\phi])$. Then*

$$\int_{\{\varphi<\psi\}} \theta_\psi^{n-j_0} \wedge \theta_{\gamma_1} \wedge \cdots \wedge \theta_{\gamma_{j_0}} \leq \int_{\{\varphi<\psi\}} \theta_\varphi^{n-j_0} \wedge \theta_{\gamma_1} \wedge \cdots \wedge \theta_{\gamma_{j_0}}.$$

Proof. We may assume that X is reduced. By Proposition 3.51, it suffices to prove the corresponding result on Y, in which case, this is [DDL18b, Corollary 3.16]. ☐

Proposition 4.7. *Let $\varphi,\psi,\gamma \in \mathcal{E}(X,\theta;[\phi])$. Assume that $\varphi \leq \psi \leq \gamma$. Then*

$$\int_X (\gamma-\psi)^p \, \theta_\psi^n \leq 2^{n+p} \int_X (\gamma-\varphi)^p \, \theta_\varphi^n .$$

Remark 4.8. Note that our measure θ_ψ^n does not charge any pluripolar set, as this is true on a resolution. The function $\gamma-\psi$ is well-defined outside a pluripolar set, so the integral $\int_X (\gamma-\psi)^p \, \theta_\psi^n$ makes sense. We will omit this kind of arguments from now on.

Proof. We may assume that X is a Kähler manifold.

Let $S=\varphi^{-1}(-\infty)$. Observe that for any $t \geq 0$,

$$\{\gamma > \psi+2t\} \setminus S \subseteq \{(\gamma+\psi)/2 > \psi+t\} \setminus S \subseteq \{(\gamma+\psi)/2 > \varphi+t\} \setminus S \subseteq \{\gamma > \varphi+t\} \setminus S . \tag{4.1}$$

So

$$\begin{aligned}
\int_X (\gamma-\psi)^p \, \theta_\psi^n =& 2^p \int_X p t^{p-1} \int_{\{\gamma-\psi>2t\}} \theta_\psi^n \, \mathrm{d}t \\
\leq& 2^p \int_X p t^{p-1} \int_{\{(\gamma+\psi)/2>\psi+t\}} \theta_\psi^n \, \mathrm{d}t && \text{By (4.1)} \\
\leq& 2^{n+p} \int_X p t^{p-1} \int_{\{(\gamma+\psi)/2>\varphi+t\}} \theta_{(\gamma+\psi)/2}^n \, \mathrm{d}t \\
\leq& 2^{n+p} \int_X p t^{p-1} \int_{\{(\gamma+\psi)/2>\varphi+t\}} \theta_\varphi^n \, \mathrm{d}t && \text{By Lemma 4.6} \\
\leq& 2^{n+p} \int_X p t^{p-1} \int_{\{\gamma>\varphi+t\}} \theta_\varphi^n \, \mathrm{d}t && \text{By (4.1)} \\
=& 2^{n+p} \int_X (\gamma-\varphi)^p \, \theta_\varphi^n && \text{By Lemma 4.6} .
\end{aligned}$$

☐

Remark 4.9. Proposition 4.7 is a direct generalization of the fundamental inequality of Guedj–Zeriahi ([GZ07, Lemma 2.3]). See also [DDL21a, Lemma 2.4]. The same proof applies to a general weight function in $\mathcal{W}_M^+$ (See [Dar15] for the precise definition).

Proposition 4.10. *Let $\varphi,\psi,\gamma \in \mathcal{E}(X,\theta;[\phi])$. Assume that $\gamma \geq \varphi \vee \psi$. Then*

$$\int_X (\gamma-\varphi)^p \, \theta_\psi^n \leq 2^p \int_X (\gamma-\varphi)^p \, \theta_\varphi^n + 2^p \int_X (\gamma-\psi)^p \, \theta_\psi^n .$$

Proof. We may assume that X is a Kähler manifold.

Observe that

$$\{\gamma > \varphi+2t\} \subseteq \{\gamma > \psi+t\} \cup \{\psi > \varphi+t\} . \tag{4.2}$$

So

$$\begin{aligned}\int_X(\gamma-\varphi)^p\,\theta_\psi^n =&2^p\int_0^\infty pt^{p-1}\int_{\{\gamma>\varphi+2t\}}\theta_\psi^n\,\mathrm{d}t\\ \leq&2^p\int_0^\infty pt^{p-1}\int_{\{\gamma>\psi+t\}}\theta_\psi^n\,\mathrm{d}t+2^p\int_0^\infty pt^{p-1}\int_{\{\psi>\varphi+t\}}\theta_\psi^n\,\mathrm{d}t \quad \text{By (4.2)}\\ \leq&2^p\int_X(\gamma-\psi)^p\,\theta_\psi^n+2^p\int_0^\infty pt^{p-1}\int_{\{\psi>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t \quad \text{By Lemma 4.6}\\ \leq&2^p\int_X(\gamma-\psi)^p\,\theta_\psi^n+2^p\int_0^\infty pt^{p-1}\int_{\{\gamma>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t\\ =&2^p\int_X(\gamma-\psi)^p\,\theta_\psi^n+2^p\int_X(\gamma-\varphi)^p\,\theta_\varphi^n\,.\end{aligned}$$

□

4.2. **Relative $\mathcal{E}^\infty$ spaces.**

Definition 4.11.

$$\mathcal{E}^\infty(X,\theta;[\phi])=\{\,\varphi\in\mathrm{PSH}(X,\theta):\varphi-\phi\in L^\infty(X)\,\}\,.$$

We say that a potential $\varphi\in\mathcal{E}^\infty(X,\theta;[\phi])$ has *relative minimal singularity* with respect to $[\phi]$.

Remark 4.12. It is easy to see that under Proposition 3.4,

$$\mathcal{E}^\infty(X,\theta;[\phi])=\mathcal{E}^\infty(X^{\mathrm{red}},\theta;[\phi])\,.$$

Pull-back induces a bijection

$$\pi^*:\mathcal{E}^\infty(X,\theta;[\phi])\xrightarrow{\sim}\mathcal{E}^\infty(Y,\pi^*\theta;[\pi^*\phi])\,.$$

We will use these identifications without explicit mentioning.

Remark 4.13. Note that $\mathcal{E}^\infty(X,\theta;[\phi])\subseteq\mathcal{E}(X,\theta;[\phi])$ by Theorem 3.55.

We have the following easy observation.

Lemma 4.14. *Each $\varphi\in\mathcal{E}(X,\theta;[\phi])$ is a decreasing limit of $\varphi^j\in\mathcal{E}^\infty(X,\theta;[\phi])$.*

Proof. It suffices to take $\varphi^j=\varphi\vee(\phi-j)$. □

We call φ^j constructed in this way the *canonical approximations* of φ.

4.3. **Relative $\mathcal{E}^p$ spaces.** Fix $p\in[1,\infty)$.

Definition 4.15. Define

$$\mathcal{E}^p(X,\theta;[\phi]):=\left\{\varphi\in\mathcal{E}(X,\theta;[\phi]):\int_X|\phi-\varphi|^p\,\theta_\varphi^n<\infty\right\}\,.$$

Remark 4.16. It is easy to see that under Proposition 3.4,

$$\mathcal{E}^p(X,\theta;[\phi])=\mathcal{E}^p(X^{\mathrm{red}},\theta;[\phi])\,.$$

Pull-back induces a bijection

$$\pi^*:\mathcal{E}^p(X,\theta;[\phi])\xrightarrow{\sim}\mathcal{E}^p(Y,\pi^*\theta;[\pi^*\phi])\,.$$

We will use these identifications without explicit mentioning.

Proposition 4.17. *Let $\varphi_j,\gamma\in\mathcal{E}^p(X,\theta)$ ($j\in\mathbb{Z}_{>0}$). Assume that $\varphi_j\leq\gamma$ for each j and that $\varphi_j\to\varphi\in\mathrm{PSH}(X,\theta)$ in L^1-topology. Assume that there is a constant $A>0$ such that*

$$\int_X(\gamma-\varphi_j)^p\,\theta_{\varphi_j}^n\leq A\,.$$

Then $\varphi\in\mathcal{E}^p(X,\theta)$ and

$$\int_X(\gamma-\varphi)^p\,\theta_\varphi^n\leq 2^{n+2p+1}A\,.$$

Proof. We may assume that X is reduced.

Step 1. Assume that φ_j is decreasing. In this case, we prove

$$\int_X (\gamma-\varphi)^p\,\theta_\varphi^n \leq 2^{p+1}A\,.$$

By Proposition 4.10, for any j,k,

$$\int_X (\gamma-\varphi_j)^p\,\theta_{\varphi_k}^n \leq 2^{p+1}A\,.$$

For any $C>0$,

$$\int_X (\gamma-\varphi_j\vee(\gamma-C))^p\ \theta_{\varphi_k}^n \leq \int_X (\gamma-\varphi_j)^p\,\theta_{\varphi_k}^n \leq 2^{p+1}A\,.$$

Let $k\to\infty$, by Theorem 3.54, we find

$$\int_X (\gamma-\varphi_j\vee(\gamma-C))^p\ \theta_{\varphi}^n \leq 2^{p+1}A\,.$$

Let $j\to\infty$, by the monotone convergence theorem,

$$\int_X (\gamma-\varphi\vee(\gamma-C))^p\ \theta_{\varphi}^n \leq 2^{p+1}A\,.$$

Let $C\to\infty$, again by the monotone convergence theorem,

$$\int_X (\gamma-\varphi)^p\ \theta_{\varphi}^n \leq 2^{p+1}A\,.$$

Step 2. In general, let $\psi^j=\sup^*_{k\geq j}\varphi_k$. For each $C>0$, let

$$\psi^{j,C}=\psi^j\vee(\gamma-C),\quad \varphi^C=\varphi\vee(\gamma-C)\,.$$

Observe that $\psi^{j,C}$ decreases to φ^C as $j\to\infty$. Moreover, $\gamma\geq\psi^{j,C}\geq\psi^j\geq\varphi_j$. By Proposition 4.7,

$$\int_X (\gamma-\psi^{j,C})^p\,\theta_{\psi^{j,C}} \leq 2^{n+p}A\,.$$

By Step 1,

$$\int_X (\gamma-\varphi^C)^p\,\theta_{\varphi^C}^n \leq 2^{n+2p+1}A\,. \tag{4.3}$$

In particular,

$$\int_{\{\varphi>\gamma-C\}} (\gamma-\varphi^C)^p\,\theta_{\varphi}^n \leq 2^{n+2p+1}A\,.$$

Let $C\to\infty$, by the monotone convergence theorem,

$$\int_X (\gamma-\varphi)^p\,\theta_{\varphi}^n \leq 2^{n+2p+1}A\,.$$

In order to conclude that $\varphi\in\mathcal{E}^p(X,\theta)$, we still have to prove that $\varphi\in\mathcal{E}(X,\theta;[\phi])$. In fact, by (4.3),

$$\int_{\{\varphi\leq\gamma-C\}} \theta_{\varphi^C}^n \leq \frac{1}{C^p}\int_X (\gamma-\varphi^C)^p\,\theta_{\varphi^C}^n \leq 2^{n+2p+1}C^{-p}A\,.$$

Then $\pi^*\varphi\in\mathcal{E}(Y,\pi^*\theta;[\pi^*\phi])$ by [DDL18b, Lemma 3.4]. Thus $\varphi\in\mathcal{E}(X,\theta;[\phi])$. □

Theorem 4.18. *Let $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, then $\varphi\wedge\psi\in\mathcal{E}^p(X,\theta)$.*

The proof of the theorem follows from [DDL18c, Theorem 2.13]. We reproduce the proof for the convenience of the readers. We prove some preliminary results at first.

Lemma 4.19. *Assume that X is a Kähler manifold. Let $\varphi,\psi\in\mathcal{E}^\infty(X,\theta;[\phi])$. Then there is $\gamma\in\mathcal{E}^\infty(X,\theta;[\phi])$, such that*

$$\theta_\gamma^n = e^{\gamma-\varphi}\theta_\varphi^n + e^{\gamma-\psi}\theta_\psi^n\,. \tag{4.4}$$

The proof is similar to that of [DDL18c, Lemma 2.14].

Proof. For each $j \geq 1$, let $\varphi_j := \varphi \vee (-j)$, $\psi_j := \psi \vee (-j)$. Let

$$\mu_j = e^{-\varphi_j}\theta^n_\varphi + e^{-\psi_j}\theta^n_\psi\,.$$

By [DDL21a, Theorem 5.3], we can find $\gamma_j \in \mathcal{E}^\infty(X,\theta;[\phi])$, such that

$$\theta^n_{\gamma_j} = e^{\gamma_j}\mu_j\,. \tag{4.5}$$

Take a constant $C > 0$ so that $|\varphi - \psi| \leq 2C$. Let

$$\eta = \frac{\varphi+\psi}{2} - C - n\log 2\,.$$

Then $\eta \in \mathcal{E}^\infty(X,\theta;[\phi])$ and $\theta^n_\eta \geq e^\eta\mu_j$. Hence $\gamma_j \geq \eta$ by [DDL21a, Lemma 5.4]. By the same lemma, γ_j is decreasing in j, let $\gamma = \lim_{j\to\infty}\gamma_j$ in the pointwise sense. Then $\gamma \geq \eta$, hence $\gamma \in \mathcal{E}^\infty(X,\theta;[\phi])$. Now (4.4) follows from (4.5) by letting $j \to \infty$ using [DDL18b, Theorem 2.3]. □

Proof of Theorem 4.18. We may assume that X is a Kähler manifold. We may assume that $\varphi,\psi \leq \phi$. For each $j \geq 1$, consider the canonical approximations:

$$\varphi_j := \varphi \vee (\phi - j)\,,\quad \psi_j := \psi \vee (\phi - j)\,.$$

By Lemma 4.19, we can take $\gamma_j \in \mathcal{E}^\infty(X,\theta;[\phi])$ solving the following equation:

$$\theta^n_{\gamma_j} = e^{\gamma_j-\varphi_j}\theta^n_{\varphi_j} + e^{\gamma_j-\psi_j}\theta^n_{\psi_j}\,.$$

It follows from [DDL21a, Lemma 5.4] that $\gamma_j \leq \varphi_j \wedge \psi_j$. We claim that

$$\int_X (\phi-\gamma_j)^p\,\theta^n_{\gamma_j} \leq C\,. \tag{4.6}$$

Assume the claim is true for now. We get immediately that $\sup_X(\gamma_j - \phi) \geq -C$. Hence, according to [DDL21a, Lemma 2.2], after possibly subtracting a subsequence, we may assume that $\gamma_j \to \gamma \in \mathrm{PSH}(X,\theta)$ in L^1-topology. Then $\gamma \in \mathcal{E}^p(X,\theta)$ by Proposition 4.17. Moreover, since $\gamma_j \leq \varphi_j \wedge \psi_j$, we know that $\gamma \leq \varphi\wedge\psi$. In particular, $\varphi\wedge\psi \in \mathrm{PSH}(X,\theta)$. Now by Proposition 4.7, $\varphi\wedge\psi \in \mathcal{E}^p(X,\theta)$.

Now we prove the claim. By symmetry, it suffices to prove

$$\int_X (\phi-\gamma_j)^p e^{\gamma_j-\varphi_j}\,\theta^n_{\varphi_j} \leq C\,.$$

But note that

$$\int_X (\phi-\gamma_j)^p e^{\gamma_j-\varphi_j}\,\theta^n_{\varphi_j} \leq C\int_X (\phi-\varphi_j)^p e^{\gamma_j-\varphi_j}\,\theta^n_{\varphi_j} + C\int_X (\varphi_j-\gamma_j)^p e^{\gamma_j-\varphi_j}\,\theta^n_{\varphi_j}\,.$$

But $x^p e^{-x} \leq C$ when $x \geq 0$, so it suffices to prove

$$\int_X (\phi-\varphi_j)^p e^{\gamma_j-\varphi_j}\,\theta^n_{\varphi_j} \leq C\,.$$

As $\gamma_j \leq \varphi_j$, it suffices to prove

$$\int_X (\phi-\varphi_j)^p\,\theta^n_{\varphi_j} \leq C\,. \tag{4.7}$$

It follows from the argument of [BEGZ10, Proposition 2.10] that

$$\int_X (\phi-\varphi_j)^p\,\theta^n_{\varphi_j} \leq \int_X (\phi-\varphi)^p\,\theta^n_\varphi\,.$$

Thus (4.7) follows. □

Corollary 4.20. *The space $\mathcal{E}^p(X,\theta)$ is convex.*

Proof. Let $\varphi_0,\varphi_1 \in \mathcal{E}^p(X,\theta)$, for $t \in [0,1]$, let $\varphi_t = t\varphi_1 + (1-t)\varphi_0$. Note that $\varphi_0\wedge\varphi_1 \leq \varphi_t$. Since $\varphi_0\wedge\varphi_1 \in \mathcal{E}^p(X,\theta)$ by Theorem 4.18. So $\varphi_t \in \mathcal{E}^p(X,\theta)$ by Proposition 4.7. □

Corollary 4.21. *Let $\varphi,\psi \in \mathcal{E}^p(X,\theta;[\phi])$, then*

$$\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\theta^n_\varphi + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\theta^n_\psi.$$

In particular, $\theta^n_{\varphi\wedge\psi}$ is supported on $\{\varphi\wedge\psi=\varphi\}\cup\{\varphi\wedge\psi=\psi\}$.

Proof. We may assume that X is a Kähler manifold. In this case, the assertion follows from Theorem 4.18 and [DDL18b, Lemma 3.7]. $\square$

4.4. Energy functionals.

Definition 4.22. Let $\varphi,\psi\in\mathcal{E}^p(X,\theta)$.

(1) Define $F_p(\varphi,\psi)\coloneqq\int_X|\varphi-\psi|^p\,\theta^n_{\varphi\wedge\psi}$.
(2) Assume $\varphi\leq\psi$, define $G_p(\varphi,\psi)\coloneqq\int_X(\psi-\varphi)^p\,\theta^n_\psi$. In general, define

$$G_p(\varphi,\psi)=G_p(\varphi\wedge\psi,\varphi)+G_p(\varphi\wedge\psi,\psi).$$

(3) Assume $\varphi\leq\psi$, $[\varphi]=[\psi]$, define

$$E_p(\varphi,\psi)\coloneqq\frac{1}{n+1}\sum_{j=0}^n\int_X(\psi-\varphi)^p\,\theta^j_\psi\wedge\theta^{n-j}_\varphi. \tag{4.8}$$

Assume $\varphi\leq\psi$, define $E_p(\varphi,\psi)\coloneqq\sup_\eta E_p(\eta,\psi)\in(-\infty,\infty]$, where the sup is taken over $\eta\in\mathcal{E}^p(X,\theta)$, such that $[\eta]=[\psi]$, $\varphi\leq\eta\leq\psi$.
In general, define

$$E_p(\varphi,\psi)\coloneqq E_p(\varphi\wedge\psi,\varphi)+E_p(\varphi\wedge\psi,\psi).$$

The functional E_p is similar to the energy functional studied in [BEGZ10, Section 2.2]. As we will see in the next proposition, E_p always takes finite values.

Proposition 4.23. *Let $\varphi,\psi,\gamma\in\mathcal{E}^p(X,\theta)$, $\varphi\leq\psi\leq\gamma$.*

(1) $j\mapsto\int_X(\psi-\varphi)^p\,\theta^j_\psi\wedge\theta^{n-j}_\varphi$ is decreasing in j.
(2) Let $\varphi',\psi'\in\mathcal{E}^p(X,\theta)$ with $\varphi\leq\varphi'\leq\psi\leq\psi'$. Then

$$E_p(\varphi',\psi)\leq E_p(\varphi,\psi)\leq E_p(\varphi,\psi').$$

(3) We have

$$G_p(\varphi,\psi)\leq E_p(\varphi,\psi)\leq F_p(\varphi,\psi)\leq(n+1)E_p(\varphi,\psi)<\infty. \tag{4.9}$$

(4) When $p>1$, for any $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, $\varphi\leq\psi$, then there is a constant $C=C(p,n,V_\theta)>0$ such that, $E_p(\varphi,\psi)^{1/p}\geq C^{-1}E_1(\varphi,\psi)$.
(5) When $p=1$, we have

$$E_1(\varphi,\gamma)=E_1(\varphi,\psi)+E_1(\psi,\gamma).$$

Proof. (i) Write

$$\int_X(\psi-\varphi)^p\,\theta^j_\varphi\wedge\theta^{n-j}_\psi=p\int_0^\infty t^{p-1}\int_{\{\psi>\varphi+t\}}\theta^j_\varphi\wedge\theta^{n-j}_\psi\,\mathrm{d}t.$$

By Lemma 4.6, $j\mapsto\int_{\{\psi>\varphi+t\}}\theta^j_\varphi\wedge\theta^{n-j}_\psi$ is decreasing in j.

(ii) The proof follows from the same argument as that of [BEGZ10, Proposition 2.8(ii)].

(iii) When $[\varphi]=[\psi]$, this is a direct consequence of (i). In general, for each $j\geq1$, let $\varphi_j=(\psi-j)\vee\varphi$. For any $j\geq1$,

$$\begin{aligned}\int_X(\psi-\varphi_j)^p\,\theta^n_\varphi&=\lim_{k\to\infty}\int_{\{\varphi>\psi-k\}}(\psi-\varphi_j)^p\,\theta^n_{\varphi_k}\\&\leq\lim_{k\to\infty}\int_{\{\varphi>\psi-k\}}(\psi-\varphi_k)^p\,\theta^n_{\varphi_k}\\&\leq(n+1)\lim_{k\to\infty}E_p(\varphi_k,\psi)\\&\leq(n+1)E_p(\varphi,\psi).\end{aligned}$$

Let $j \to \infty$, by monotone convergence theorem, $F_p(\varphi,\psi) \leq (n+1)E_p(\varphi,\psi)$. Now observe that $F_p(\varphi_k,\psi) \leq F_p(\varphi,\psi)$. In fact, for any $\epsilon > 0$,

$$\begin{aligned}
\int_X (\psi-\varphi_k)^p\,\theta_{\varphi_k}^n \leq & \int_{\{\varphi<\psi-k\}} (\psi-\varphi_k)^p\,\theta_{\varphi_k}^n + \int_{\{\varphi>\psi-k-\epsilon\}} (\psi-\varphi_k)^p\,\theta_{\varphi}^n \\
= & k^p \int_{\{\varphi<\varphi_k\}} \theta_{\varphi_k}^n + \int_{\{\varphi>\psi-k-\epsilon\}} (\psi-\varphi_k)^p\,\theta_{\varphi}^n \\
\leq & k^p \int_{\{\varphi<\psi-k\}} \theta_{\varphi}^n + \int_{\{\varphi>\psi-k-\epsilon\}} (\psi-\varphi)^p\,\theta_{\varphi}^n \qquad \text{By Lemma 4.6} \\
\leq & \int_X (\psi-\varphi)^p\,\theta_{\varphi}^n + (k+\epsilon)^p \int_{\{\psi-k<\varphi<\psi-k+\epsilon\}} \theta_{\varphi}^n,
\end{aligned}$$

where the second term tends to 0 as $\epsilon \to 0+$ by dominated convergence theorem. By the arguments of [BEGZ10, Proposition 2.10(ii)], $E_p(\varphi,\psi) = \lim_{k\to\infty} E_p(\varphi_k,\psi)$. So

$$E_p(\varphi,\psi) = \lim_{k\to\infty} E_p(\varphi_k,\psi) \leq \varlimsup_{k\to\infty} F_p(\varphi_k,\psi) \leq F_p(\varphi,\psi).$$

By Fatou's lemma,

$$G_p(\varphi,\psi) \leq \varliminf_{k\to\infty} G_p(\varphi_k,\psi) \leq \lim_{k\to\infty} E_p(\varphi_k,\psi) = E_p(\varphi,\psi).$$

Finally, let us prove that $F_p(\varphi,\psi) < \infty$. Take a constant $C_1 > 0$ such that $\psi < V_\theta + C_1$, then

$$\int_X (\psi-\varphi)^p\,\theta_{\varphi}^n \leq \int_X (C_1+V_\theta-\varphi)^p\,\theta_{\varphi}^n \leq C + C\int_X (V_\theta-\varphi)^p\,\theta_{\varphi}^n < \infty.$$

(iv) We may assume that $\varphi,\psi \in \mathcal{E}^\infty(X,\theta)$. This is a consequence of Hölder's inequality.

(v) Assume that in addition, $[\varphi]=[\psi]=[\gamma]$, then this is a direct generalization of [DDL18b, Theorem 4.10]. One just needs the integration by parts formula Theorem 3.53. In general, one concludes by canonical approximations and a generalization of [GZ17, Theorem 10.37]. □

Recall that as in [DDL18b, Section 4.2], one can define a functional $E^\phi : \mathrm{PSH}(X,\theta;[\phi]) \to [-\infty,\infty)$ as follows:

$$E^\phi(\varphi) := \frac{1}{n+1}\sum_{j=0}^{n} \int_X (\varphi-\phi)\,\theta_{\varphi}^j \wedge \theta_{\phi}^{n-j}$$

when $\varphi \in \mathcal{E}^\infty(X,\theta)$ and $E^\phi(\varphi) := \inf_\psi E^\phi(\psi)$ in general, where the inf is taken over $\psi \in \mathcal{E}^\infty(X,\theta)$ such that $\varphi \leq \psi$.

Note that

$$E_1(\varphi,\psi) = E^\phi(\psi) - E^\phi(\varphi) \tag{4.10}$$

when $\varphi \leq \psi$ and $\varphi,\psi \in \mathcal{E}^1(X,\theta;[\phi])$.

Remark 4.24. In [DDL18b], the authors assumed in addition that ϕ has small unbounded locus to make sure that one can perform the integration by parts. Since the general integration by parts formula is proved in Theorem 3.53, we no longer need this assumption. One can easily check that all results in [DDL18b] still hold in general by the same proof even in the unibranch setting. We will apply this remark without explicitly mentioning.

Proposition 4.25. *Let $\varphi,\psi,\gamma,\varphi_j \in \mathcal{E}^p(X,\theta)$ ($j \in \mathbb{Z}_{>0}$).*

(1) Assume that $\gamma \geq \varphi \vee \psi$. Then

$$F_p(\gamma,\varphi) + F_p(\gamma,\psi) \geq F_p(\varphi,\psi) = F_p(\varphi,\varphi\wedge\psi) + F_p(\psi,\varphi\wedge\psi)\,. \tag{4.11}$$

(2) $F_p(\gamma\wedge\varphi,\gamma\wedge\psi) \leq F_p(\varphi,\psi)$.

(3) Assume that $\varphi \leq \psi$. Let $\varphi_t = (1-t)\varphi + t\psi$. Then for any $N > 0$,

$$\sum_{j=0}^{N-1} F_p(\varphi_{j/N},\varphi_{(j+1)/N})^{1/p} \leq F_p(\varphi,\psi)^{1/p}\,.$$

(4) Assume that φ_j increases a.e. to φ, then $\lim_{j\to\infty} F_p(\varphi_j,\varphi) = 0$,

Proof. We may assume that X is a Kähler manifold throughout the proof.

(1) By Corollary 4.21,

$$\mathbb{1}_{\{\varphi\neq\psi\}}\theta^n_{\varphi\wedge\psi}\leq \mathbb{1}_A\theta^n_\varphi+\mathbb{1}_B\theta^n_\psi\,,$$

where $A=\{\varphi\wedge\psi=\varphi<\psi\}$, $B=\{\varphi\wedge\psi=\psi<\varphi\}$. Note that A and B are disjoint sets.

In particular,

$$F_p(\varphi,\psi)\leq\int_X|\varphi-\psi|^p\mathbb{1}_A\,\theta^n_\varphi+\int_X|\varphi-\psi|^p\mathbb{1}_B\,\theta^n_\psi\,.$$

Observe that on the set A, we have $\varphi<\psi\leq\gamma$. So

$$\int_X|\varphi-\psi|^p\mathbb{1}_A\,\theta^n_\varphi\leq\int_X|\varphi-\gamma|^p\,\theta^n_\varphi\,.$$

Similarly,

$$\int_X|\varphi-\psi|^p\mathbb{1}_B\,\theta^n_\varphi\leq\int_X|\psi-\gamma|^p\,\theta^n_\psi\,.$$

The first inequality in (4.11) follows. For the second, we observe that

$$\begin{aligned}
F_p(\varphi,\varphi\wedge\psi)+F_p(\psi,\varphi\wedge\psi)=&\int_X\left((\varphi-\varphi\wedge\psi)^p+(\psi-\varphi\wedge\psi)^p\right)\,\theta^n_{\varphi\wedge\psi}\\
=&\int_X\mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\left((\varphi-\varphi\wedge\psi)^p+(\psi-\varphi\wedge\psi)^p\right)\,\theta^n_{\varphi\wedge\psi}\\
&+\int_X\mathbb{1}_{\{\varphi\wedge\psi=\psi<\varphi\}}\left((\varphi-\varphi\wedge\psi)^p+(\psi-\varphi\wedge\psi)^p\right)\,\theta^n_{\varphi\wedge\psi} && \text{By Corollary 4.21}\\
=&\int_X\mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}|\psi-\varphi|^p\theta^n_{\varphi\wedge\psi}+\int_X\mathbb{1}_{\{\varphi\wedge\psi=\psi<\varphi\}}|\psi-\varphi|^p\theta^n_{\varphi\wedge\psi}\\
=&\int_X|\psi-\varphi|^p\theta^n_{\varphi\wedge\psi} && \text{By Corollary 4.21}\\
=&F_p(\varphi,\psi)\,.
\end{aligned}$$

(2) By (1), we may assume that $\varphi\leq\psi$. Then

$$\begin{aligned}
F_p(\gamma\wedge\varphi,\gamma\wedge\psi)=&\int_X(\gamma\wedge\psi-\gamma\wedge\varphi)^p\,\theta^n_{\gamma\wedge\varphi}\\
\leq&\int_{\{\gamma\wedge\varphi=\varphi\}}(\gamma\wedge\psi-\varphi)^p\,\theta^n_\varphi\quad\text{By Corollary 4.21}\\
\leq&\int_{\{\gamma\wedge\varphi=\varphi\}}(\psi-\varphi)^p\,\theta^n_\varphi\\
\leq&\int_X(\psi-\varphi)^p\,\theta^n_\varphi\\
=&F_p(\varphi,\psi)\,.
\end{aligned}$$

(3) We observe that

$$\sum_{j=0}^{N-1}\left(\int_X(\varphi_{(j+1)/N}-\varphi_{j/N})^p\,\theta^n_{\varphi_{j/N}}\right)^{1/p}=\frac{1}{N}\sum_{j=0}^{N-1}\left(\int_X(\psi-\varphi)^p\,\theta^n_{\varphi_{j/N}}\right)^{1/p}\,.$$

So it suffices to find a uniform upper bound of the summand.

In fact,

$$\begin{aligned}\int_X (\psi-\varphi)^p\,\theta_{\varphi_t}^n &= \int_X (\psi-\varphi)^p\,\left(t\theta_\psi+(1-t)\theta_\varphi\right)^n\\ &= \sum_{j=0}^n \binom{n}{j}\int_X t^j(1-t)^{n-j}(\psi-\varphi)^p\,\theta_\psi^j\wedge\theta_\varphi^{n-j}\\ &\leq \sum_{j=0}^n \binom{n}{j}\int_X t^j(1-t)^{n-j}(\psi-\varphi)^p\,\theta_\varphi^n \qquad \text{By Proposition 4.23}\\ &= \int_X (\psi-\varphi)^p\,\theta_\varphi^n\,.\end{aligned}$$

(4) We may assume that $\varphi\leq 0$. For each $C\geq 0$, let

$$\varphi_j^C=\varphi_j\vee(\phi-C)\,,\quad \varphi^C=\varphi\vee(\phi-C)\,.$$

By [DDL18b, Theorem 2.3, Remark 2.5], $\lim_{j\to\infty}F_p(\varphi_j^C,\varphi^C)=0$. The remaining of the proof is exactly the same as [BEGZ10, Theorem 2.17]. We reproduce the proof for the convenience of readers.

Take a function $\chi:(-\infty,\infty]\to\mathbb{R}$ satisfying

(1) χ is concave, continuous, decreasing, $\chi(0)=0$, $\chi(\infty)=\infty$.
(2) There is a constant $M>1$, such that $|t\chi'(t)|\leq M|\chi(t)|$ for all $t\geq 0$.
(3) $\frac{t^p}{\chi(t)}$ decreases to 0 as $t\to\infty$.
(4)
$$\int_X \chi\circ(\phi-\varphi_k)\theta_{\varphi_k}^n<\infty\,.$$

The existence of such weight follows from the standard analysis, see [GZ07] for example.

Now we estimate

$$\begin{aligned}\left|F_p(\varphi_j,\varphi)-F_p(\varphi_j^C,\varphi^C)\right| &\leq \left|\int_{\{\varphi_k\leq\phi-C\}}(\varphi-\varphi_k)^p\theta_{\varphi_k}^n-\int_{\{\varphi_k\leq\phi-C\}}(\varphi^C-\varphi_k^C)^p\theta_{\varphi_k^C}^n\right|\\ &\leq \int_{\{\varphi_k\leq\phi-C\}}(\phi-\varphi_k)^p\theta_{\varphi_k}^n-\int_{\{\varphi_k\leq\phi-C\}}(\phi-\varphi_k^C)^p\theta_{\varphi_k^C}^n\\ &\leq \frac{C^p}{\chi(C)}\left(\int_{\{\varphi_k\leq\phi-C\}}\chi\circ(\phi-\varphi_k)\theta_{\varphi_k}^n+\int_{\{\varphi_k\leq\phi-C\}}\chi\circ(\phi-\varphi_k^C)\theta_{\varphi_k^C}^n\right)\\ &\leq C_0\frac{C^p}{\chi(C)}\quad \text{By Proposition 4.7 and Remark 4.9}\,.\end{aligned}$$

Hence we conclude. ☐

Proposition 4.26. *Let $\varphi,\psi,\gamma\in\mathcal{E}^p(X,\theta)$, $\varphi\leq\psi$. Then*

(1) $G_p(\gamma\vee\varphi,\gamma\vee\psi)\leq G_p(\varphi,\psi)$.
(2) Let $\varphi_t=(1-t)\psi+t\varphi$. Then for any $N>0$,

$$\sum_{j=0}^{N-1}G_p(\varphi_{j/N},\varphi_{(j+1)/N})^{1/p}\geq G_p(\varphi,\psi)^{1/p}\,.$$

Proof. (1) We calculate

$$\begin{aligned}G_p(\gamma\vee\varphi,\gamma\vee\psi) &= \int_X (\gamma\vee\psi-\gamma\vee\varphi)^p\,\theta_{\gamma\vee\psi}^n\\ &= \int_{\{\gamma<\psi\}}(\gamma\vee\psi-\gamma\vee\varphi)^p\,\theta_\psi^n\\ &\leq \int_X(\psi-\varphi)^p\,\theta_\psi^n\\ &= G_p(\varphi,\psi)\,.\end{aligned}$$

(2) The proof is similar to that of Proposition 4.25, we omit it. ☐

Lemma 4.27. *Let $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, assume that $\varphi\leq\psi$. Let $\varphi_t=t\psi+(1-t)\varphi$. Then for $N\geq 1$,*

$$\sum_{j=0}^{N-1}\left(F_p(\varphi_{j/N},\varphi_{(j+1)/N})^{1/p}-G_p(\varphi_{j/N},\varphi_{(j+1)/N})^{1/p}\right)\leq CN^{-1/p}\,,$$

where $C>0$ depends on φ and ψ.

Proof. In fact, it suffices to estimate

$$\sum_{j=0}^{N-1}\left(F_p(\varphi_{j/N},\varphi_{(j+1)/N})-G_p(\varphi_{j/N},\varphi_{(j+1)/N})\right)^{1/p}\,.$$

We estimate each term

$$\begin{aligned}F_p(\varphi_{j/N},\varphi_{(j+1)/N})-G_p(\varphi_{j/N},\varphi_{(j+1)/N})=&\frac{1}{N^p}\int_X(\varphi_1-\varphi_0)^p(\theta^n_{\varphi_{j/N}}-\theta^n_{\varphi_{(j+1)/N}})\\=&\frac{1}{N^{p+1}}\sum_{a=0}^{n-1}\int_X(\psi-\varphi)^p\mathrm{dd^c}(\varphi-\psi)\wedge\theta^a_{\varphi_{j/N}}\wedge\theta^{n-1-a}_{\varphi_{(j+1)/N}}\,.\end{aligned}$$

Let C_0 be a common upper bound for terms of the form:

$$\left|\int_X(\psi-\varphi)^p\mathrm{dd^c}(\varphi-\psi)\wedge\theta^a_\varphi\wedge\theta^{n-1-a}_\psi\right|\,.$$

Expand $\theta_{\varphi_{j/N}}$ as a linear combination of θ_φ and θ_ψ, we find immediately

$$F_p(\varphi_{j/N},\varphi_{(j+1)/N})-G_p(\varphi_{j/N},\varphi_{(j+1)/N})\leq CC_0N^{-1-p}\,.$$

□

5. Rooftop structures

Definition 5.1. Let E be a set. A *pre-rooftop structure* on E is a binary operator $\wedge:E\times E\to E$, satisfying the following axioms: for $x,y,z\in E$,

(1) $x\wedge y=y\wedge x$.
(2) $(x\wedge y)\wedge z=x\wedge(y\wedge z)$.
(3) $x\wedge x=x$.

We call $(E,\wedge)$ a pre-rooftop space.

A morphism between rooftop spaces $(E,\wedge)\to(E',\wedge')$ is a map $f:E\to E'$, such that

$$f(x\wedge y)=f(x)\wedge' f(y)\,,\quad x,y\in E\,.$$

A pre-rooftop structure $\wedge$ defines a partial order $\leq$ on E as follows:

$$x\leq y\quad\text{if and only if}\quad x\wedge y=x\,.$$

Here by abuse of notation, we use $\leq$ to denote the partial order.

In particular, it makes sense to talk about an increasing and decreasing sequences in E.

Definition 5.2. Let (E,d) be a metric space. A *pre-rooftop structure* on (E,d) is a pre-rooftop structure $\wedge$ on E. We say $(E,d,\wedge)$ is a pre-rooftop metric space. A morphism between pre-rooftop metric spaces $(E,d,\wedge)\to(E',d',\wedge')$ is a morphism $f:(E,\wedge)\to(E',\wedge')$, which is also distance decreasing. Let $\mathcal{PR}\mathrm{T}$ be the category of pre-rooftop metric spaces.

A *rooftop structure* on (E,d) is a pre-rooftop structure $\wedge$ on E such that

$$d(x\wedge z,y\wedge z)\leq d(x,y),\quad\forall x,y,z\in E. \tag{5.1}$$

We call $(E,d,\wedge)$ a *rooftop metric space*. Morphisms between rooftop metric spaces is the same as morphisms between underlying pre-rooftop metric spaces. Let $\mathcal{R}\mathrm{T}$ be the category of rooftop metric spaces.

We say the rooftop structure $\wedge$ is *p-strict* ($p\in[1,\infty)$) if the following *Pythagorean formula* holds:

$$d(x,y)^p=d(x,x\wedge y)^p+d(y,x\wedge y)^p,\quad\forall x,y\in E. \tag{5.2}$$

In this case, we also say $(E,d,\wedge)$ is a p-strict rooftop metric space.

The name rooftop comes from the Kähler setting, where $\varphi \wedge \psi$ is known as the rooftop envelop of the quasi-psh functions φ, ψ in the literature.

Lemma 5.3. *Let $(E, d, \wedge) \in \mathcal{R}\mathrm{T}$. Let $x, y, x', y' \in E$, then*

$$d(x \wedge y, x' \wedge y') \leq d(x, x') + d(y, y'). \tag{5.3}$$

Proof. We compute

$$d(x \wedge y, x' \wedge y') \leq d(x \wedge y, x \wedge y') + d(x \wedge y', x' \wedge y') \leq d(x, x') + d(y, y').$$

□

Proposition 5.4. *Let $(E, d, \wedge) \in \mathcal{R}\mathrm{T}$. Then (E, d) is complete if and only if both of the followings hold:*

(1) Each increasing Cauchy sequence converges.
(2) Each decreasing Cauchy sequence converges.

This is essentially an abstract version of [Dar17, Theorem 9.2].

Proof. The direct implication is trivial.

Conversely, assume that both conditions are true. Let $x_j \in E$ $(j \geq 1)$ be a Cauchy sequence. We want to prove that x_j converges. By passing to a subsequence, we may assume that

$$d(x_j, x_{j+1}) \leq 2^{-j}.$$

For $k, j \geq 1$, let

$$y_j^k \coloneqq x_k \wedge \cdots \wedge x_{k+j}.$$

Then $(y_j^k)_j$ is decreasing, and

$$d(y_j^k, y_{j+1}^k) \leq d(x_{k+j}, x_{k+j+1}) \leq 2^{-k-j}.$$

So $(y_k^j)_j$ is a decreasing Cauchy sequence. Define

$$y^k \coloneqq \lim_{j \to \infty} y_j^k.$$

Then

$$d(y^k, y^{k+1}) = \lim_{j \to \infty} (y_{j+1}^k, y_j^{k+1}) \leq d(x_k, x_{k+1}) \leq 2^{-k}.$$

So y^k is an increasing Cauchy sequence. Let

$$y \coloneqq \lim_{k \to \infty} y^k.$$

Then

$$d(y^k, x_k) = \lim_{j \to \infty} d(y_j^k, x_k) \leq \lim_{j \to \infty} d(y_{j-1}^{k+1}, x_k).$$

Note that

$$d(y_{j-1}^{k+1}, x_k) \leq d(y_{j-1}^{k+1}, x_{k+1}) + d(x_{k+1}, x_k) \leq d(y_{j-1}^{k+2}, x_{k+1}) + 2^{-k}.$$

Hence

$$d(y^k, x_k) \leq 2^{-k} + \lim_{j \to \infty} d(y_{j-1}^{k+2}, x_{k+1}) \leq \lim_{j \to \infty} \sum_{r=k}^{j+k} d(x_r, x_{r+1}) \leq 2^{1-k}.$$

So x_k converges to y. □

Definition 5.5. A rooftop metric space $(E, d, \wedge)$ is *locally complete* if for each $y \in E$, the subspace $\{y \in E : x \geq y\}$ is complete.

Observe that $\{y \in E : x \geq y\}$ is a rooftop metric space with respect to the metric d and the pre-rooftop structure $\wedge$. In particular, we find that

Corollary 5.6. *Let $(E, d, \wedge) \in \mathcal{R}\mathrm{T}$. Then (E, d) is locally complete if and only if both of the followings hold:*

(1) Each increasing Cauchy sequence converges.
(2) Each decreasing Cauchy sequence which admits a lower bound in E converges.

Proposition 5.7. *Let $(E, d, \wedge) \in \mathcal{R}\mathrm{T}$, let $i : (E, d) \to (\bar{E}, \bar{d})$ be the metric completion of (E, d). Then there is a unique rooftop structure $\bar{\wedge}$ on $(\bar{E}, \bar{d})$, so that $i : (E, d, \wedge) \to (\bar{E}, \bar{d}, \bar{\wedge})$ is a morphism in $\mathcal{R}\mathrm{T}$.*

Proof. We first argue the existence. Let $(x_j), (y_j)$ be Cauchy sequences in E. For $j, k > 0$, by Lemma 5.3,

$$d(x_j \wedge y_j, x_k \wedge y_k) \leq d(y_j, y_k) + d(x_j, x_k)\,.$$

Hence $x_j \wedge y_j$ is also Cauchy. Now let $x, y \in \bar{E}$, represented by Cauchy sequences $(x_j), (y_j)$ in E, then we define

$$x \bar{\wedge} y := \lim_{j \to \infty} x_j \wedge y_j\,.$$

We have to show that this is well-defined. Let $(x'_j), (y'_j)$ be two other Cauchy sequences in E representing x, y. By Lemma 5.3,

$$d(x_j \wedge y_j, x'_j \wedge y'_j) \leq d(y_j, y'_j) + d(x_j, x'_j)\,.$$

Hence

$$\lim_{j \to \infty} d(x_j \wedge y_j, x'_j \wedge y'_j) = 0\,.$$

Thus $\bar{\wedge}$ is well-defined. We claim that $\bar{\wedge}$ is a rooftop structure. We only have to verify (5.1). Let $x, y, z \in \bar{E}$, represented by Cauchy sequences $(x_j), (y_j), (z_j)$ in E. Then

$$\bar{d}(x \bar{\wedge} z, y \bar{\wedge} z) = \lim_{j \to \infty} d(x_j \wedge z_j, y_j \wedge z_j) \leq \varliminf_{j \to \infty} d(x_j, y_j) = \bar{d}(x, y)\,.$$

It is clear that i becomes a morphism in $\mathcal{R}\mathrm{T}$.

Now we prove the uniqueness. Assume that we have a rooftop operator $\bar{\wedge}$ on $(\bar{E}, \bar{d})$, such that i becomes a morphism in $\mathcal{R}\mathrm{T}$. Let $x, y \in \bar{E}$, represented by Cauchy sequences $(x_j), (y_j)$ in E. By Lemma 5.3,

$$\bar{d}(x_j \bar{\wedge} y_j, x \bar{\wedge} y) \leq \bar{d}(x_j, x) + \bar{d}(y_j, y)\,.$$

Thus

$$\lim_{j \to \infty} x_j \bar{\wedge} y_j = x \bar{\wedge} y\,.$$

As i is a morphism in $\mathcal{R}\mathrm{T}$, we find

$$\lim_{j \to \infty} x_j \wedge y_j = x \bar{\wedge} y\,.$$

□

Definition 5.8. Let $(E, d, \wedge) \in \mathcal{R}\mathrm{T}$, we call $(\bar{E}, \bar{d}, \bar{\wedge})$ constructed in Proposition 5.7 the *completion* of $(E, d, \wedge)$.

Example 5.9. *$\mathcal{E}^p(X, \theta)$ is a p-strict rooftop metric space.*

This is Theorem 1.3.

Example 5.10. *Let X be a compact Kähler manifold. Let ω be a Kähler form on X. Then $\mathcal{R}^p(X, \omega)$ is a complete p-strict rooftop metric space.*

Proof. The metric d_p is constructed in [DL20]:

$$d_p(\ell^1, \ell^2) := \lim_{t \to \infty} \frac{1}{t} d_p(\ell^1_t, \ell^2_t), \quad \ell^1, \ell^2 \in \mathcal{R}^p\,.$$

It is shown there that $(\mathcal{R}^p, d_p)$ is a complete metric space.

Now we construct the rooftop structure: let $\ell^1, \ell^2 \in \mathcal{R}^p$, define $\ell^1 \wedge \ell^2$ as the maximal geodesic ray in $\mathcal{R}^p$ that lies below both ℓ^1 and ℓ^2. The proof of the existence of ℓ is the same as Lemma 7.4, so we omit the details. In particular, $\ell^1 \wedge \ell^2$ admits the following concrete description: for each $t > 0$, let $(L^t_s)_{s \in [0,t]}$ be the geodesic from 0 to $\ell^1_t \wedge \ell^2_t$. Then $(\ell^1 \wedge \ell^2)_t$ is the limit in $\mathcal{E}^1$ of L^t_s as $s \to \infty$.

It is easy to verify that $\wedge$ is indeed a rooftop structure. That it is p-strict follows from the fact that $\mathcal{E}^p$ is p-strict. □

Example 5.11. *Let X be a compact unibranch Kähler space of pure dimension. Let θ be a strongly closed smooth real $(1,1)$-form on X, representing a big class. Then $\mathcal{R}^1(X, \theta)$ is a complete 1-strict rooftop metric space.*

This is Theorem 1.4.

For the next example, we recall some related results from non-Archimedean geometry. Let L be an ample $\mathbb{Q}$-line bundle on X. Let ω be a Kähler form in $c_1(L)$. Let X^{NA} be the Berkovich analytification of X with respect to the trivial norm on $\mathbb{C}$. There is a natural morphism of ringed spaces $X^{\mathrm{NA}} \to X$. Let L^{NA} be the pull-back of L along this morphism. Let $\mathcal{E}^{1,\mathrm{NA}}(L)$ be the space of non-Archimedean metrics of finite energy on L^{NA}, see [BJ18b, Section 5.2]. Let $E : \mathcal{E}^{1,\mathrm{NA}}(L) \to \mathbb{R}$ be the Monge–Ampère energy functional defined in [BJ18b, Section 5.2]. For $\varphi, \psi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, $\varphi \leq \psi$, define

$$d_1(\varphi, \psi) = E(\psi) - E(\varphi)\,.$$

Recall that from [BBJ21], there is a canonical distance preserving embedding

$$\iota : \mathcal{E}^{1,\mathrm{NA}}(L) \hookrightarrow \mathcal{R}^1(X, \omega)\,,$$

It is not surjective in general ([BBJ21, Example 6.10]). As in [BBJ21], there is a contraction $\Pi : \mathcal{R}^1(X, \omega) \to \mathcal{E}^{1,\mathrm{NA}}(L)$, given by $\Pi(\ell) = \ell^{\mathrm{NA}}$, such that

$$\Pi \circ \iota(\phi) = \phi\,, \quad \phi \in \mathcal{E}^{1,\mathrm{NA}}\,.$$

We may identify $\mathcal{E}^{1,\mathrm{NA}}(L)$ with a subset of $\mathcal{R}^1(X, \omega)$ through ι, known as the set of *maximal geodesic rays*. Finally, recall that ι and Π are both order preserving by definition.

Example 5.12. *Let X be a projective smooth scheme of finite type over $\mathbb{C}$. Let L be an ample $\mathbb{Q}$-line bundle on X. Then $\mathcal{E}^{1,\mathrm{NA}}(L)$ is a complete 1-strict rooftop metric space.*

Proof. **Step 1**. We first show that given $\phi, \psi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, $\ell := \iota(\phi) \wedge \iota(\psi)$ is maximal. Let $\ell' = \iota \circ \Pi(\ell)$. Then $\ell' \geq \ell$ by definition ([BBJ21, Definition 6.5]). Since both ι and Π are order preserving, we have

$$\ell' \leq \iota(\phi)\,, \quad \ell' \leq \iota(\psi)\,.$$

Thus $\ell = \ell'$ and ℓ is maximal.

Note that the result also follows from the characterization of maximal geodesic rays in [DX22].

Step 2. We define the rooftop structure as follows: let $\phi, \psi \in \mathcal{E}^{1,\mathrm{NA}}(L)$, define

$$\phi \wedge \psi := \iota^{-1}\left(\iota(\phi) \wedge \iota(\psi)\right)\,.$$

It is easy to check that $\wedge$ is indeed a 1-strict rooftop structure. It follows from [DX22, Theorem 1.2] that Π is continuous, hence $\mathcal{E}^{1,\mathrm{NA}}(L)$ is identified with a closed subspace of $\mathcal{R}^1(X, \omega)$, hence complete. ☐

6. Metric on $\mathcal{E}^p$ spaces

Let $p \in [1, \infty)$. Let X be a compact unibranch Kähler space of pure dimension n. Let α be a big $(1,1)$-cohomology class. Let θ be a strongly closed smooth $(1,1)$-form in the class α. Let V be the volume of α. Let $\phi \in \mathrm{PSH}(X, \theta)$ be a model potential with mass $V_\phi > 0$. Fix a resolution of singularities $\pi : Y \to X^{\mathrm{red}}$. Recall that we may always assume that Y is Kähler.

6.1. Length elements.

Definition 6.1. A *length element* is a symmetric function $f : \mathcal{E}^p(X, \theta) \times \mathcal{E}^p(X, \theta) \to [0, \infty)$. A length element f is *good* if it satisfies the following conditions:

A.1 For $\varphi \in \mathcal{E}^p(X, \theta)$, $f(\varphi, \varphi) = 0$.
A.2 For $\varphi, \psi \in \mathcal{E}^p(X, \theta)$, $f(\varphi, \psi)^p = f(\varphi \wedge \psi, \varphi)^p + f(\varphi \wedge \psi, \psi)^p$.
A.3 For $\varphi, \psi, \gamma \in \mathcal{E}^p(X, \theta)$, $\gamma \geq \varphi \vee \psi$, $f(\varphi, \gamma)^p + f(\psi, \gamma)^p \geq f(\varphi, \psi)^p$.
A.4 For $\varphi, \psi, \gamma \in \mathcal{E}^p(X, \theta)$, $f(\varphi \wedge \gamma, \psi \wedge \gamma) \leq f(\varphi, \psi)$.
A.5 For $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$, assume that $\varphi_0 \leq \varphi_1$, set $\varphi_t = t\varphi_1 + (1-t)\varphi_0$, then

$$\varlimsup_{N \to \infty} \sum_{j=0}^{N-1} f(\varphi_{j/N}, \varphi_{(j+1)/N}) \leq f(\varphi_0, \varphi_1).$$

A.6 There is a constant $C = C(p) > 0$, so that for any $\varphi, \psi \in \mathcal{E}^p(X, \theta)$, $\varphi \leq \psi$, we have $Cf(\varphi, \psi) \geq E_1(\varphi, \psi)$.

Theorem 6.2. *The function $F_p^{1/p}$ is a good length element.*

$G_p^{1/p}$ is a length element that satisfies the following condition:

A.7 *For $\varphi, \psi, \gamma \in \mathcal{E}^p(X, \theta)$, $f(\varphi \vee \gamma, \psi \vee \gamma) \leq f(\varphi, \psi)$.*

Proof. The first part follows from Proposition 4.25. The second part follows from Proposition 4.26. □

Definition 6.3. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$.

(1) Let $\Gamma(\varphi_0, \varphi_1)$ be the set of $\Psi = (\psi_0, \dots, \psi_N) \in \mathcal{E}^p(X, \theta)^{N+1}$ for various $N \geq 0$ (we call N the *length* of Ψ), such that
  (a) $\psi_0 = \varphi_0$, $\psi_N = \varphi_1$.
  (b) For each $j = 0, \dots, N-1$, either $\varphi_j \leq \varphi_{j+1}$ or $\varphi_j \geq \varphi_{j+1}$.

(2) When $\varphi_0 \leq \varphi_1$, let $\Gamma_+(\varphi_0, \varphi_1)$ be the set of $\Psi = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi_0, \varphi_1)$ such that
$$\psi_0 \leq \psi_1 \leq \dots \leq \psi_N.$$

(3) For $\Psi = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi_0, \varphi_1)$, $\gamma \in \mathcal{E}^p(X, \theta)$, define
$$\gamma \wedge \Psi := (\gamma \wedge \psi_0, \dots, \gamma \wedge \psi_N), \quad \gamma \vee \Psi := (\gamma \vee \psi_0, \dots, \gamma \vee \psi_N).$$

We observe that in (iii) above when $\varphi_0 \leq \varphi_1$ and $\Psi \in \Gamma_+(\varphi_0, \varphi_1)$, we have
$$\gamma \wedge \Psi \in \Gamma_+(\gamma \wedge \varphi_0, \gamma \wedge \varphi_1), \quad \gamma \vee \Psi \in \Gamma_+(\gamma \vee \varphi_0, \gamma \vee \varphi_1).$$

Definition 6.4. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$.

(1) We say that $\Psi = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi_0, \varphi_1)$ is *reduced* if for every $j = 0, \dots, N-1$, $\psi_j \neq \psi_{j+1}$.
(2) For a non-reduced Ψ, we can always delete repeating consecutive elements to get its *reduction* $\tilde{\Psi}$.
(3) Let $\Psi = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi_0, \varphi_1)$. Assume that Ψ is reduced. We say that $j \in [1, N-1]$ is a *turning point* of Ψ is one of the following is true
  (a) $\psi_j \geq \psi_{j-1}$, $\psi_j \geq \psi_{j+1}$.
  (b) $\psi_j \leq \psi_{j-1}$, $\psi_j \leq \psi_{j+1}$.
  We say j is a turning point of *type 1* or *type 2* respectively.

Observe that a turning point cannot be both type 1 and type 2 as Ψ is reduced.

Definition 6.5. Let f be a length element satisfying Condition A.1. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$ and $\Psi = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi_0, \varphi_1)$.

(1) When $\varphi_0 \leq \varphi_1$ and $\Psi \in \Gamma_+(\varphi_0, \varphi_1)$, define
$$\ell(\Psi) := \sum_{j=0}^{N-1} f(\psi_j, \psi_{j+1}). \tag{6.1}$$

(2) When Ψ is reduced, write the turning points of Ψ as $i_1, \dots, i_S$ for some $S \geq 0$. Let $i_0 = 0, i_{S+1} = N$. Then we define
$$\ell(\Psi) := \left(\sum_{j=0}^{S} \ell(\psi_{i_j}, \psi_{i_j+1}, \dots, \psi_{i_{j+1}})^p \right)^{1/p}. \tag{6.2}$$

(3) Define
$$\ell(\Psi) := \ell(\tilde{\Psi}). \tag{6.3}$$

(4) Define
$$|\Psi|_{f^p} := \sup_{j=0,\dots,N-1} f(\psi_j, \psi_{j+1})^p.$$

We write $|\Psi| = |\Psi|_{F_p}$. Note that $|\tilde{\Psi}| = |\Psi|$.
The following lemma is clear.

Lemma 6.6. *Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$, $\varphi_0 \leq \varphi_1$, let $\Psi \in \Gamma_+(\varphi_0, \varphi_1)$, then the definitions of $\ell(\Psi)$ in (6.1) and in (6.3) coincide.*

Fix a length element f satisfying Condition A.1.

Definition 6.7. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$.

(1) When $\varphi_0 \leq \varphi_1$, define

$$d_p(\varphi_0, \varphi_1) \coloneqq \lim_{\delta \to 0+} \inf_{\substack{\Psi \in \Gamma_+(\varphi_0, \varphi_1) \\ |\Psi| < \delta}} \ell(\Psi). \tag{6.4}$$

(2) In general, define

$$d_p(\varphi_0, \varphi_1) \coloneqq \left(d_p(\varphi_0 \wedge \varphi_1, \varphi_0)^p + d_p(\varphi_0 \wedge \varphi_1, \varphi_1)^p\right)^{1/p}. \tag{6.5}$$

Lemma 6.8. *Let $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{E}^p(X, \theta)$ with $\varphi_0 \leq \varphi_1 \leq \varphi_2$. Then*

$$d_p(\varphi_0, \varphi_2) \leq d_p(\varphi_0, \varphi_1) + d_p(\varphi_1, \varphi_2).$$

Proof. This follows immediately from definition. □

Definition 6.9. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$. Assume $\varphi_0 \leq \varphi_1$. Let $\Phi = (\psi_0, \ldots, \psi_N) \in \Gamma_+(\varphi_0, \varphi_1)$. For each $M \geq 1$, define

$$\Phi^{(M)} = (\psi_0^0, \ldots, \psi_0^{M-1}, \psi_1^0, \ldots, \psi_1^{M-1}, \ldots, \psi_{N-1}^0, \ldots, \psi_{N-1}^{M-1}) \in \Gamma_+(\varphi_0, \varphi_1), \tag{6.6}$$

where

$$\psi_j^k = \frac{M-k}{M}\psi_j + \frac{k}{M}\psi_{j+1}, \quad j = 0, \ldots, N-1;\ k = 0, \ldots, M-1.$$

6.2. **d_p-metrics.** From now on, we focus on the length element $F_p^{1/p}$. We write ℓ^F, ℓ^G the function ℓ defined with respect to $F_p^{1/p}$, $G_p^{1/p}$ respectively. When we write ℓ, we refer to ℓ^F.

Our d_p will be defined relative to $F_p^{1/p}$. However, the length element $G_p^{1/p}$ helps to establish the triangle inequality through the proof of Proposition 6.12 below. It is of interest to know if it is possible to prove Proposition 6.12 only using $F_p^{1/p}$.

Lemma 6.10. *Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$, $\varphi_0 \leq \varphi_1$, then*

$$d_p(\varphi_0, \varphi_1) = \inf_{\Phi \in \Gamma_+(\varphi_0, \varphi_1)} \ell(\Phi).$$

In particular, $d_p(\varphi_0, \varphi_1) \leq F_p(\varphi_0, \varphi_1)^{1/p} < \infty$.

Proof. This follows from Condition A.5. □

Lemma 6.11. *Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \theta)$. Assume $\varphi_0 \leq \varphi_1$. Let $\Phi \in \Gamma_+(\varphi_0, \varphi_1)$.*

(1) For each $M \geq 1$,

$$\ell^G(\Phi) \leq \ell^G(\Phi^{(M)}) \leq \ell^F(\Phi^{(M)}) \leq \ell^F(\Phi).$$

(2) When $M \to \infty$,

$$\ell^F(\Phi^{(M)}) - \ell^G(\Phi^{(M)}) = \mathcal{O}(M^{-1/p}).$$

Proof. (i) This follows from Proposition 4.25, Proposition 4.26.

(ii) This follows from Lemma 4.27. □

Proposition 6.12. *Let $\varphi_0, \varphi_1, \gamma \in \mathcal{E}^p(X, \theta)$, then*

(1) $d_p(\gamma \wedge \varphi_0, \gamma \wedge \varphi_1) \leq d_p(\varphi_0, \varphi_1)$.

(2) Assume that $\varphi_0 \leq \varphi_1$, then $d_p(\gamma \vee \varphi_0, \gamma \vee \varphi_1) \leq d_p(\varphi_0, \varphi_1)$.

As far as the author knows, the second part is new even for Kähler classes.

Proof. (i) We may assume that $\varphi_0 \leq \varphi_1$. Then the statement follows from Condition A.4.

(ii)**Step 1**. We prove this under the additional assumption that $[\varphi_0] = [\varphi_1]$.

In this case, we take $A > 0$ so that

$$\varphi_1 - A \leq \varphi_0. \tag{6.7}$$

Let $\epsilon > 0$. Take $\Phi = (\psi_0, \ldots, \psi_N) \in \Gamma_+(\varphi_0, \varphi_1)$ so that

$$\ell(\Phi) < d_p(\varphi_0, \varphi_1) + \epsilon.$$

For $M \geq 1$, as in Definition 6.9, let $\Phi^{(M)} \in \Gamma_+(\varphi_0, \varphi_1)$ be the refinement of Φ obtained by inserting the following points between ψ_j and ψ_{j+1} ($j = 0, \dots, N-1$):

$$\psi_j^k \coloneqq \frac{k}{M}\psi_{j+1} + \frac{M-k}{M}\psi_j, \quad k = 0, \dots, M-1.$$

We claim that for M large enough,

$$0 \leq \ell^F(\gamma \vee \Phi^{(M)}) - \ell^G(\gamma \vee \Phi^{(M)}) < \epsilon. \tag{6.8}$$

Assume this for the time being, then by Condition A.7,

$$\ell^G(\gamma \vee \Phi^{(M)}) \leq \ell^G(\Phi^{(M)}). \tag{6.9}$$

By (6.8), (6.9) and Lemma 6.11, for M large enough,

$$\ell^F(\gamma \vee \Phi^{(M)}) < \ell^F(\Phi^{(M)}) + \epsilon \leq \ell^F(\Phi) + \epsilon < d_p(\varphi_0, \varphi_1) + 2\epsilon.$$

Let $\epsilon \to 0+$, $d_p(\gamma \vee \varphi_0, \gamma \vee \varphi_1) \leq d_p(\varphi_0, \varphi_1)$.

It remains to prove (6.8), which is refined version of Lemma 6.11(ii). Clearly, it suffices to treat the case $N = 1$. In this case, we write $\gamma \vee \psi_0^k$ as η_k and write ψ^k for ψ_0^k. Let S be the polar locus of ψ^0. By perturbation, we may assume that $\delta \coloneqq \inf_{X \setminus S}(\psi^M - \psi^0) > 0$. Note that for $k = 0, 1, \dots, M-1$,

$$\psi^{k+1} \geq \psi^k + \frac{\delta}{M}. \tag{6.10}$$

Write $\ell^G = \ell^G(\eta_0, \dots, \eta_M)$, and $\ell^F = \ell^F(\eta_0, \dots, \eta_M)$ for simplicity.

Observe that on $\{\gamma \leq \psi^k\}$ ($k = 0, 1, \dots, M-2$),

$$\eta_{k+2} - \eta_{k+1} = \eta_{k+1} - \eta_k. \tag{6.11}$$

This relation allows us to relate ℓ^G and ℓ^F. So

$$\begin{aligned}
&\ell^G - \left(\int_X (\eta_M - \eta_{M-1})^p \, \theta^n_{\eta_M}\right)^{1/p} \\
&\geq \sum_{k=0}^{M-2} \left(\int_{\{\gamma \leq \psi^k\}} (\eta_{k+1} - \eta_k)^p \, \theta^n_{\eta_{k+1}}\right)^{1/p} \\
&= \sum_{k=0}^{M-2} \left(\int_{\{\gamma \leq \psi^k\}} (\eta_{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}}\right)^{1/p} \qquad \text{By (6.11)} \\
&= \sum_{k=0}^{M-2} \left(\int_X (\eta_{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}} - \int_{\{\psi^k < \gamma < \psi^{k+2}\}} (\psi^{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}}\right)^{1/p} \\
&\geq \ell^F - \sum_{k=0}^{M-2} \left(\int_{\{\psi^k < \gamma < \psi^{k+2}\}} (\psi^{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}}\right)^{1/p} - \left(\int_X (\eta_1 - \eta_0)^p \, \theta^n_{\eta_0}\right)^{1/p} \qquad \text{By Lemma 6.13} \\
&\geq \ell^F - (M-1)^{1-1/p} \left(\sum_{k=0}^{M-2} \int_{\{\psi^k < \gamma < \psi^{k+2}\}} (\psi^{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}}\right)^{1/p} - \left(\int_X (\eta_1 - \eta_0)^p \, \theta^n_{\eta_0}\right)^{1/p}.
\end{aligned}$$

The last step follows from the power mean inequality. We estimate the second term, we write

$$\int_{\{\psi^k < \gamma < \psi^{k+2}\}} (\psi^{k+2} - \eta_{k+1})^p \, \theta^n_{\eta_{k+1}} = \int_{\{\psi^k < \gamma < \psi^{k+1}\}} + \int_{\{\psi^{k+1} \leq \gamma < \psi^{k+2}\}} =: J_{1,k} + J_{2,k}.$$

Then

$$\begin{aligned}
J_{1,k} \leq &\int_{\{\psi^k < \gamma < \psi^{k+1}\}} (\psi^{k+2} - \psi^{k+1})^p \, \theta^n_{\psi^{k+1}} \\
\leq &CM^{-p} \int_{\{\psi^k < \gamma < \psi^{k+1}\}} \theta^n_{\psi^{k+1}} \qquad \text{By (6.7)}.
\end{aligned}$$

Now observe that for any $t \geq 0$,

$$\left(\{\psi^{k+1} \leq \gamma < \psi^{k+2}\} \cap \{\psi^{k+2} - \eta_{k+1} > t\}\right) \setminus S \subseteq \left\{ \frac{\psi^{k+2} + \eta_{k+1} + \gamma}{3} > \psi^{k+1} + \frac{t}{3} \right\}. \tag{6.12}$$

Note that outside S,

$$\psi^{k+2} - \eta_{k+1} \leq \psi^{k+2} - \psi^{k+1} \leq \frac{A}{M}.$$

Also observe that

$$\left\{ \frac{A}{3M} \geq \frac{\psi^{k+2} + \eta_{k+1} + \gamma}{3} - \psi^{k+1} > 0 \right\} \setminus S \subseteq \left\{ -\frac{A}{M} < \gamma - \psi^{k+1} \leq 2\frac{A}{M} \right\}. \tag{6.13}$$

In fact, when $\gamma \geq \psi^{k+1}$ and on the complement of S, $\frac{A}{3M} \geq \frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} - \psi^{k+1} > 0$ implies that $\frac{A}{M} \geq 2\gamma - 2\psi^{k+1}$, so $0 \leq \gamma - \psi^{k+1} \leq \frac{A}{2M}$. On the other hand, when $\gamma < \psi^{k+1}$, the same inequality implies $0 < -\psi^{k+1} + \frac{A}{M} + \gamma$, so $-\frac{A}{M} < \gamma - \psi^{k+1} < 0$. This proves (6.13).

So

$$\begin{aligned}
J_{2,k} &= \int_0^\infty pt^{p-1} \int_{\{\psi^{k+2}-\eta_{k+1}>t\}} \mathbb{1}_{\{\psi^{k+1}\leq\gamma<\psi^{k+2}\}}\, \theta^n_{\eta_{k+1}}\, \mathrm{d}t \\
&\leq 3^p \int_0^{\frac{A}{3M}} pt^{p-1} \int_{\left\{\frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} > \psi^{k+1}+t\right\}} \mathbb{1}_{\{\psi^{k+1}\leq\gamma<\psi^{k+2}\}}\, \theta^n_{\eta_{k+1}}\, \mathrm{d}t && \text{By (6.12)} \\
&\leq C \int_0^{\frac{A}{3M}} pt^{p-1} \int_{\left\{\frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} > \psi^{k+1}+t\right\}} \theta^n_{\frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3}}\, \mathrm{d}t \\
&\leq C \int_0^{\frac{A}{3M}} pt^{p-1} \int_{\left\{\frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} > \psi^{k+1}+t\right\}} \theta^n_{\psi^{k+1}}\, \mathrm{d}t && \text{By Lemma 4.6} \\
&= C \int_{\left\{\frac{A}{3M} \geq \frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} - \psi^{k+1} > 0\right\}} \left(\frac{\psi^{k+2}+\eta_{k+1}+\gamma}{3} - \psi^{k+1} \right)^p \theta^n_{\psi^{k+1}} \\
&\leq CM^{-p} \int_{\left\{-\frac{A}{M} < \gamma - \psi^{k+1} \leq 2\frac{A}{M}\right\}} \theta^n_{\psi^{k+1}} && \text{By (6.13).}
\end{aligned}$$

So for $[A/\delta] + 2 \leq k \leq M - [2A/\delta] - 1$,

$$J_{2,k} \leq CM^{-p} \int_{\{\psi^{k-[A/\delta]-2} \leq \gamma \leq \psi^{k+[2A/\delta]+1}\}} \theta^n_{\psi^{k+1}}.$$

Write $\theta_{\psi^{k+1}}$ as a combination of θ_{ψ^0} and θ_{ψ^M}, we find

$$\sum_{k=0}^{M-1} (J_{1,k} + J_{2,k}) \leq CM^{-p}.$$

Putting all estimates together, we find $\ell^G - \ell^F \geq -CM^{-1/p}$. Hence (6.8) holds.

Step 2. For $k \geq 0$. Let $\varphi_1^k = (\varphi_0 + k) \wedge \varphi_1$. Note that φ_1^k increases to φ_1 almost everywhere. In fact, this is equivalent to say $[\varphi_0] \wedge \varphi_1 = \varphi_1$. Obviously, $[\varphi_0] \wedge \varphi_1 \leq \varphi_1$. For the other inequality, by domination principle([DDL18b, Proposition 3.11]), it suffices to prove

$$[\varphi_0] \wedge \varphi_1 \geq \varphi_1, \quad \theta^n_{[\varphi_0]\wedge\varphi_1} - a.e..$$

But by [DDL18b, Theorem 3.8],

$$\theta^n_{[\varphi_0]\wedge\varphi_1} \leq \mathbb{1}_{\{[\varphi_0]\wedge\varphi_1 = \varphi_1\}} \theta^n_{\varphi_1}.$$

The inequality follows.

By Step 1 and (i),

$$d_p(\gamma \vee \varphi_0, \gamma \vee \varphi_1^k) \leq d_p(\varphi_0, \varphi_1^k) \leq d_p(\varphi_0, \varphi_1).$$

Now by Lemma 6.8,

$$d_p(\gamma \vee \varphi_0, \gamma \vee \varphi_1) \leq d_p(\gamma \vee \varphi_0, \gamma \vee \varphi_1^k) + d_p(\gamma \vee \varphi_1^k, \gamma \vee \varphi_1).$$

So it suffices to prove that

$$\lim_{k\to\infty} d_p(\gamma\vee\varphi_1^k,\gamma\vee\varphi_1)=0.$$

By Lemma 6.10,

$$d_p(\gamma\vee\varphi_1^k,\gamma\vee\varphi_1)^p\leq\int_X(\gamma\vee\varphi_1-\gamma\vee\varphi_1^k)^p\,\theta^n_{\gamma\vee\varphi_1^k}.$$

The right-hand side tends to 0 by Proposition 4.25. ☐

Lemma 6.13. *Let $x,y,a,b\in[0,\infty)$. Let $p\in[1,\infty)$. Then*

$$(a^p+b^p+(x+y)^p)^{1/p}\leq(x^p+a^p)^{1/p}+(y^p+b^p)^{1/p}.$$

In particular,

$$(a+b)^{1/p}\leq a^{1/p}+b^{1/p}.$$

Proof. We may assume that $x,y>0$. Let

$$F(a,b)\coloneqq(a^p+b^p+(x+y)^p)^{1/p}-(x^p+a^p)^{1/p}-(y^p+b^p)^{1/p}.$$

Then

$$\partial_aF(a,b)=a^{p-1}\left(a^p+b^p+(x+y)^p\right)^{1/p-1}-a^{p-1}(x^p+a^p)^{1/p-1}\leq 0.$$

Similarly, $\partial_bF(a,b)\leq 0$ So in order to prove that $F(a,b)\leq 0$, it suffices to prove this when $a=0$, $b=0$. But $F(0,0)=0$, so we are done. ☐

It is hard to check the triangle inequality from the definition of d_p directly, so we provide an alternative definition.

Definition 6.14. Let $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\theta)$, define

$$\tilde{d}_p(\varphi_0,\varphi_1)\coloneqq\inf_{\Psi\in\Gamma(\varphi_0,\varphi_1)}\ell(\Psi).\tag{6.14}$$

Proposition 6.15. *For each $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\theta)$, we have $d_p(\varphi_0,\varphi_1)=\tilde{d}_p(\varphi_0,\varphi_1)$.*

Proof. By definition, $d_p(\varphi_0,\varphi_1)\geq\tilde{d}_p(\varphi_0,\varphi_1)$.

For the other direction, fix some $\epsilon>0$, take $\Psi=(\psi_0,\dots,\psi_N)\in\Gamma(\varphi_0,\varphi_1)$ such that

$$\ell(\Psi)<\tilde{d}_p(\varphi_0,\varphi_1)+\epsilon.\tag{6.15}$$

We may take Ψ with smallest N so that (6.15) is satisfied. In particular, Ψ is reduced.

Step 1. We claim that we can always take Ψ with the following additional property: there is $j\in[0,N]$, so that ψ_k is decreasing for $k\leq j$ and increasing for $k\geq j$.

It suffices to show that we may always assume that there are no type 1 turning points. Since then, it suffices to take j to be the first turning point if there is any and take $j=0$ or $j=N$ otherwise.

Assume that j is the smallest type 1 turning point, let i be the previous turning point if there is one, and let $i=0$ otherwise. Similarly, let k be the next turning point if there is one, and $k=N$ otherwise. We claim that we may replace Ψ with the reduction of

$$\Psi'=(\psi_0,\dots,\psi_{j-1},\psi_{j-1}\wedge\psi_{j+1},\psi_{j+1},\dots,\psi_N).$$

Now

$$\begin{aligned}&\left(\sum_{a=i}^{j-2}f(\psi_a,\psi_{a+1})\right)^p+f(\psi_{j-1},\psi_{j-1}\wedge\psi_{j+1})^p+f(\psi_{j-1}\wedge\psi_{j+1},\psi_{j+1})^p+\left(\sum_{b=j+1}^{k-1}f(\psi_b,\psi_{b+1})\right)^p\\ \leq&\left(\sum_{a=i}^{j-2}f(\psi_a,\psi_{a+1})\right)^p+f(\psi_{j-1},\psi_j)^p+f(\psi_j,\psi_{j+1})^p+\left(\sum_{b=j+1}^{k-1}f(\psi_b,\psi_{b+1})\right)^p\quad\text{By Condition A.3,A.2}\\ \leq&\left(\sum_{a=i}^{j-1}f(\psi_a,\psi_{a+1})\right)^p+\left(\sum_{b=j}^{k-1}f(\psi_b,\psi_{b+1})\right)^p.\end{aligned}$$

Here $f=F_p^{1/p}$. It follows that $\ell(\tilde{\Psi}')=\ell(\Psi')\leq\ell(\Psi)$.

Observe that $\psi_{j-1} \wedge \psi_{j+1}$ is not equal to either ψ_{j-1} or ψ_{j+1}, since otherwise, $\tilde{\Psi}'$ has length $N-1$, contrary to our choice of Ψ. So j is a turning point of type 2 of Ψ'. In particular, if we modify the tuple Ψ' further by decreasing some among $\psi_0, \dots, \psi_{j-1}$, the index j will never become a turning point of type 1.

If some $j' \in [1, j-1]$ is a turning point of type 1 of Ψ', we repeat the above procedure replacing j by j', and finally, we arrive at a new chain Ψ'' of length N without any turning point of type 1 in the interval $[1, j]$, satisfying (6.15). If there is no turning points of type 1 in $[j+1, N-1]$, we are done, otherwise, repeat the same procedure.

Step 2. Now we have
$$(\ell(\Psi')^p + \ell(\Psi'')^p)^{1/p} = \ell(\Psi) < \tilde{d}_p(\varphi_0, \varphi_1) + \epsilon,$$
where $\Psi' = (\psi_j, \psi_{j-1}, \dots, \psi_0)$, $\Psi'' = (\psi_j, \psi_{j+1}, \dots, \psi_N)$. Hence
$$d_p(\psi_j, \varphi_0)^p + d_p(\psi_j, \varphi_1)^p \leq \tilde{d}_p(\varphi_0, \varphi_1)^p.$$
By Proposition 6.12,
$$d_p(\varphi_0 \wedge \varphi_1, \varphi_0)^p + d_p(\varphi_0 \wedge \varphi_1, \varphi_1)^p \leq d_p(\psi_j, \varphi_0)^p + d_p(\psi_j, \varphi_1)^p.$$
Hence $d_p(\varphi_0, \varphi_1) \leq \tilde{d}_p(\varphi_0, \varphi_1)$. □

Proposition 6.16. *d_p is a metric on $\mathcal{E}^p(X, \theta)$.*

Proof. It is easy to see that d_p is symmetric. It is finite by Lemma 6.8.

Step 1. The triangle inequality. Let $\varphi, \psi, \gamma \in \mathcal{E}^p(X, \theta)$. We need to prove
$$d_p(\varphi, \gamma) \leq d_p(\varphi, \psi) + d_p(\psi, \gamma).$$
By Proposition 6.15, it suffices to prove
$$\tilde{d}_p(\varphi, \gamma) \leq \tilde{d}_p(\varphi, \psi) + \tilde{d}_p(\psi, \gamma). \tag{6.16}$$
Take $\epsilon > 0$. Take $\Psi_1 = (\psi_0, \dots, \psi_N) \in \Gamma(\varphi, \psi)$, $\Psi_2 = (\psi_N, \dots, \psi_M) \in \Gamma(\psi, \gamma)$. We could assume that $N > 0$, $M > N$. Let
$$\Psi = (\psi_0, \dots, \psi_N, \psi_{N+1}, \dots, \psi_M) \in \Gamma(\varphi, \gamma).$$
Let $i < N$ be last turning point of Ψ_1 if there is one. Let $i = 0$ otherwise. Similarly, let $j > N$ be the first turning point of Ψ_2 if there is one. Let $j = M$ otherwise. Let $A = \ell(\psi_0, \dots, \psi_i)$, $B = \ell(\psi_j, \dots, \psi_M)$.

Then we claim that
$$\ell(\Psi) \leq \ell(\Psi_1) + \ell(\Psi_2). \tag{6.17}$$
Before proving this claim, let us observe that (6.16) follows from this claim.

In order to prove (6.17), we distinguish two cases.

Case 1. ψ_N is not a turning point of Ψ. Then
$$\begin{aligned}
\ell(\Psi) &= \left(A^p + \left(\sum_{k=i}^{N-1} f(\psi_k, \psi_{k+1}) + \sum_{k=N}^{j-1} f(\psi_k, \psi_{k+1}) \right)^p + B^p \right)^{1/p} \\
&\leq \left(A^p + \left(\sum_{k=i}^{N-1} f(\psi_k, \psi_{k+1}) \right)^p \right)^{1/p} + \left(B^p + \left(\sum_{k=N}^{j-1} f(\psi_k, \psi_{k+1}) \right)^p \right)^{1/p} \quad \text{By Lemma 6.13} \\
&= \ell(\Psi_1) + \ell(\Psi_2).
\end{aligned}$$
Here $f = F_p^{1/p}$.

Case 2. Ψ_N is a turning point of Ψ. Then
$$\begin{aligned}
\ell(\Psi) &= \left(A^p + \left(\sum_{k=i}^{N-1} f(\psi_k, \psi_{k+1}) \right)^p + \left(\sum_{k=N}^{j-1} f(\psi_k, \psi_{k+1}) \right)^p + B^p \right)^{1/p} \\
&\leq \left(A^p + \left(\sum_{k=i}^{N-1} f(\psi_k, \psi_{k+1}) \right)^p \right)^{1/p} + \left(\left(\sum_{k=N}^{j-1} f(\psi_k, \psi_{k+1}) \right)^p + B^p \right)^{1/p} \\
&= \ell(\Psi_1) + \ell(\Psi_2).
\end{aligned}$$

Step 2. We prove that d_p is non-degenerate. Let $\varphi,\psi\in\mathcal{E}^p(X,\theta)$, assume that $d_p(\varphi,\psi)=0$. We want to prove that $\varphi=\psi$.

We may assume that $\varphi\leq\psi$. Let $\Psi=(\psi_0,\dots,\psi_N)\in\Gamma_+(\varphi,\psi)$. By Condition A.6,

$$\ell(\Psi)\geq C^{-1}E_1(\varphi,\psi)\geq C^{-1}F_1(\varphi,\psi).$$

Thus $F_1(\varphi,\psi)=0$. Hence $\psi=\varphi$ by domination principle ([DDL18b, Proposition 3.11]). □

Theorem 6.17. *$(\mathcal{E}^p(X,\theta),d_p,\wedge)$ is a p-strict rooftop metric space.*

Proof. By Proposition 6.16, d_p is a metric. The fact that $\wedge$ is a rooftop structure follows from Proposition 6.12. It follows from definition that d_p is p-strict. □

6.3. **Properties of d_1.**

Proposition 6.18. *Let $\varphi,\psi\in\mathcal{E}^1(X,\theta;[\phi])$. Then*

$$d_1(\varphi,\psi)=E^\phi(\varphi)+E^\phi(\psi)-2E^\phi(\varphi\wedge\psi)\,.$$

Proof. We may assume that X is a compact Kähler manifold.

By Pythagorean formula (5.2), we may assume that $\varphi\leq\psi$. Take $\epsilon>0$. Let $\Psi=(\psi_0,\dots,\psi_N)\in\Gamma_+(\varphi,\psi)$, so that

$$\ell(\Psi)\leq d_1(\varphi,\psi)+\epsilon.$$

By Lemma 6.11, we may assume that $\ell(\Psi)-\ell^G(\Psi)<\epsilon$. Then by (4.10),

$$d_1(\varphi,\psi)+\epsilon\geq\ell(\Psi)\geq\sum_{j=0}^{N-1}E_1(\psi_j,\psi_{j+1})=\sum_{j=0}^{N-1}\left(E^\phi(\psi_{j+1})-E^\phi(\psi_j)\right)=E^\phi(\psi)-E^\phi(\varphi).$$

On the other hand,

$$d_1(\varphi,\psi)-\epsilon\leq\ell^G(\Psi)\leq\sum_{j=0}^{N-1}E_1(\psi_j,\psi_{j+1})=E^\phi(\psi)-E^\phi(\varphi).$$

As $\epsilon>0$ is arbitrary, we conclude that $d_1(\varphi,\psi)=E^\phi(\psi)-E^\phi(\varphi)$. □

Lemma 6.19. *Let $\varphi,\psi\in\mathcal{E}^1(X,\theta;[\phi])$. Then*

$$d_1\left(\varphi,\frac{\varphi+\psi}{2}\right)\leq\frac{3n+3}{2}d_1(\varphi,\psi)\,.$$

The argument is the same as [DDL18a, Lemma 3.8].

Proposition 6.20. *There is a constant $C>0$, such that for any $\varphi,\psi\in\mathcal{E}^1(X,\theta;[\phi])$,*

$$d_1(\varphi,\psi)\leq F_1(\varphi,\psi)\leq Cd_1(\varphi,\psi)\,.$$

Proof. We may assume that X is a compact Kähler manifold.

We prove the left-hand inequality at first.

$$d_1(\varphi,\psi)=\left(E^\phi(\varphi)-E^\phi(\varphi\wedge\psi)\right)+\left(E^\phi(\psi)-E^\phi(\varphi\wedge\psi)\right)\,.$$

By symmetry, it suffices to deal with the first bracket.

$$E^\phi(\varphi)-E^\phi(\varphi\wedge\psi)\leq\int_X(\varphi-\varphi\wedge\psi)\,\theta^n_{\varphi\wedge\psi}\leq\int_{\{\varphi\wedge\psi=\psi\}}(\varphi-\psi)\,\theta^n_\psi\leq\int_X|\varphi-\psi|\,\theta^n_\psi\,.$$

For the right-hand inequality, the argument is exactly the same as in [DDL18a, Theorem 3.7]. □

Proposition 6.21. *There is a constant $C>0$, such that for any $\varphi\in\mathcal{E}^1(X,\theta;[\phi])$,*

$$|\sup_X(\varphi-\phi)|\leq Cd_1(\varphi,\phi)+C\,.$$

Proof. We may assume that X is a compact Kähler manifold.

In fact, we shall prove a stronger result

$$-d_1(\varphi,\phi) \leq \sup_X(\varphi-\phi) \leq C d_1(\varphi,\phi) + C\,.$$

If $\sup_X(\varphi-\phi)\leq 0$, the right-hand inequality is trivial and

$$-d_1(\varphi,\phi) = E^{\phi}(\varphi) \leq \sup_X(\varphi-\phi)\,.$$

So we may assume that $\sup_X(\varphi-\phi)\geq 0$. In this case, the left-hand inequality is trivial. Recall that

$$\theta_\phi^n \leq \mathbb{1}_{\{\phi=V_\theta\}}\theta_{V_\theta}^n \leq \mathbb{1}_{\{\phi=V_\theta=0\}}\theta^n\,.$$

For a proof, see [DDL18b, Theorem 3.8], [DDL18c, Theorem 2.6].

Take a Kähler form ω, such that $\theta\leq\omega$ Then by Proposition 4.1,

$$\int_X \left|\varphi - \sup_X(\varphi-\phi) - \phi\right| \theta_\phi^n \leq C\,,$$

where $C>0$ in independent of the choice of φ.

Then

$$\begin{aligned} d_1(\varphi,\phi) \geq& C^{-1}\int_X |\varphi-\phi|\,\theta_\phi^n \\ \geq& C^{-1}\sup_X(\varphi-\phi) - C^{-1}\int_X \left|\varphi-\sup_X(\varphi-\phi)-\phi\right|\,\theta_\phi^n \\ \geq& C^{-1}\sup_X(\varphi-\phi) - C\,. \end{aligned}$$

□

6.4. **Properties of d_p.**

Proposition 6.22. *Let φ^k be a sequence in $\mathcal{E}^p(X,\theta)$, let $\varphi\in\mathcal{E}^p(X,\theta)$,*

(1) Assume that φ^k is decreasing with pointwise limit φ. Let $\psi\in\mathcal{E}^p(X,\theta)$, $\psi\leq\varphi$, then $\varphi\in\mathcal{E}^p(X,\theta)$, $d_p(\psi,\varphi)=\lim_{k\to\infty}d_p(\psi,\varphi^k)$.

(2) Assume that φ^k increases with $\varphi:=\sup{}^*\varphi^k\in\mathrm{PSH}(X,\theta)$. Let $\psi\in\mathcal{E}^p(X,\theta)$, $\psi\geq\varphi$, then $\varphi\in\mathcal{E}^p(X,\theta)$, $d_p(\psi,\varphi)=\lim_{k\to\infty}d_p(\psi,\varphi^k)$.

Proof. (i) By Proposition 4.7, $\varphi\in\mathcal{E}^p(X,\theta)$. By Proposition 6.12, $d_p(\psi,\varphi^k)$ is decreasing and is greater than $d_p(\psi,\varphi)$. So the limit exists and $d_p(\psi,\varphi)\leq\lim_{k\to\infty}d_p(\psi,\varphi^k)$. On the other hand, $d_p(\psi,\varphi^k)\leq d_p(\psi,\varphi)+d_p(\varphi,\varphi^k)$. So we need to prove $\lim_{k\to\infty}d_p(\varphi,\varphi^k)=0$. By Lemma 6.10, it suffices to show

$$\left(\int_X(\varphi^k-\varphi)^p\,\theta_\varphi^n\right)^{1/p}\to 0,$$

which follows from the dominated convergence theorem.

(ii) As in (i), we only have to prove $\lim_{k\to\infty}d_p(\varphi,\varphi^k)=0$. Again, by Lemma 6.10, it suffices to prove $\lim_{k\to\infty}F_p(\varphi^k,\varphi)=0$. This follows from Proposition 4.25. □

Proposition 6.23. *Let $\varphi_j\in\mathcal{E}^p(X,\theta)$ be a d_p-bounded increasing sequence. Then φ_j converges to some $\varphi\in\mathcal{E}^p(X,\theta)$ with respect to d_p.*

Proof. By Proposition 6.21 and Condition A.6, φ_j converges in L^1-topology to $\varphi\in\mathrm{PSH}(X,\theta)$. By Choquet's lemma and the fact that φ_j is increasing, we conclude that $\varphi_j\to\varphi$ almost everywhere. Hence $\varphi=\sup_j^*\varphi_j$. By Proposition 4.7, $\varphi\in\mathcal{E}^p(X,\theta)$. By Proposition 6.22, $d_p(\varphi_j,\varphi)\to 0$. □

Similarly, for decreasing sequences, we have

Proposition 6.24. *Let $\varphi_j\in\mathcal{E}^p(X,\theta)$ be a d_p-bounded decreasing sequence. Assume that there is $\psi\in\mathcal{E}^p(X,\theta)$ such that $\varphi_j\geq\psi$ for each j. Then φ_j converges to some $\varphi\in\mathcal{E}^p(X,\theta)$ with respect to d_p.*

Corollary 6.25. *$\mathcal{E}^p(X,\theta)$ is a p-strict locally complete rooftop metric space.*

Proof. This follows from Theorem 6.17, Proposition 6.23, Proposition 6.24 and Corollary 5.6. □

Next we consider the problem of completeness. We do not expect $\mathcal{E}^p(X,\theta)$ to be complete in general. However, in some useful cases, we show that $\mathcal{E}^p(X,\theta)$ is indeed complete.

Proposition 6.26. *Let $\varphi_j \in \mathcal{E}^1(X,\theta)$ be a d_1-bounded decreasing sequence. Then φ_j converges to some $\varphi \in \mathcal{E}^1(X,\theta)$ with respect to d_1.*

Proof. Let $\varphi = \inf_j \varphi_j$. By Proposition 6.21, $\varphi \in \mathrm{PSH}(X,\theta)$. Moreover, $\varphi_j \to \varphi$ in L^1-topology. Since φ_j is bounded in $\mathcal{E}^1(X,\theta)$, we know that $E(\varphi_j) \geq -C$ for some constant $C > 0$. Hence $\int_X |\varphi_j - V_\theta|\, \theta^n_{\varphi_j} \leq C$. So by Proposition 4.17, $\varphi \in \mathcal{E}^1(X,\theta)$. Hence by Proposition 6.22, $\varphi_j \to \varphi$ in $\mathcal{E}^1(X,\theta)$. □

Corollary 6.27. *$(\mathcal{E}^1(X,\theta), d_1)$ is complete.*

Proof. This follows from Proposition 5.4, Proposition 6.23 and Proposition 6.26. □

Corollary 6.27 is also proved in [DDL18a] and [Tru22].

In order to proceed, we need

Condition 6.28. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$, $\varphi_0 \leq \varphi_1$. Then

$$G_p(\varphi_0,\varphi_1)^{1/p} \leq d_p(\varphi_0,\varphi_1).$$

Equivalently, given $\psi_0 \leq \psi_1 \leq \cdots \leq \psi_N$ in $\mathcal{E}^p(X,\theta)$, then

$$\left(\int_X (\psi_N - \psi_0)^p\, \theta^n_{\psi_N}\right)^{1/p} \leq \sum_{j=0}^{N-1} \left(\int_X (\psi_{j+1} - \psi_j)^p\, \theta^n_{\psi_j}\right)^{1/p}.$$

This condition holds when $\phi = 0$ and the cohomology class of θ is Kähler, as a consequence of [Dar15, Lemma 4.1]. As an immediate consequence, this condition also holds if θ is semi-positive or when the cohomology class of θ is nef if $\phi = 0$. This condition also holds if $p = 1$ by Proposition 4.23.

Proposition 6.29. *Assume that $\mathcal{E}^p(X,\theta)$ satisfies Condition 6.28. Let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$, $\varphi_0 \leq \varphi_1$. Then*

$$F_p(\varphi_0,\varphi_1)^{1/p} \leq 2^{1+n/p} d_p(\varphi_0,\varphi_1).$$

Proof. Let $\psi = (\varphi_0 + \varphi_1)/2$. Then

$$\begin{aligned}\left(\int_X (\varphi_1 - \varphi_0)^p\, \theta^n_{\varphi_0}\right)^{1/p} \leq 2^{1+n/p} \left(\int_X (\psi - \varphi_0)^p\, \theta^n_{\psi}\right)^{1/p} &\leq 2^{1+n/p} d_p(\varphi_0,\psi) \\ &\leq 2^{1+n/p} d_p(\varphi_0,\varphi_1),\end{aligned}$$

where the last step follows from Proposition 6.12. □

Proposition 6.30. *Assume that $\mathcal{E}^p(X,\theta)$ satisfies Condition 6.28. Let $\varphi_j \in \mathcal{E}^p(X,\theta)$ be a d_p-bounded decreasing sequence. Then φ_j converges to some $\varphi \in \mathcal{E}^p(X,\theta)$ with respect to d_p.*

Proof. We may assume that $\varphi_1 \leq V_\theta$.

By Proposition 6.26, we know that $\varphi := \inf_j \varphi_j \in \mathcal{E}^1(X,\theta)$, such that $\varphi_j \to \varphi$ in $\mathcal{E}^1(X,\theta)$.

Now by Proposition 6.29 and the fact that φ_j is d_p-bounded, we know that $\int_X (V_\theta - \varphi_j)^p\, \theta^n_{\varphi_j} \leq C$. So by Proposition 4.17, $\varphi \in \mathcal{E}^p(X,\theta)$. The result follows from Proposition 6.24. □

Corollary 6.31. *Assume that $\mathcal{E}^p(X,\theta)$ satisfies Condition 6.28, then $(\mathcal{E}^p(X,\theta), d_p)$ is complete.*

Proof. This follows from Proposition 5.4, Proposition 6.23 and Proposition 6.30. □

Hence we have proved the following:

Theorem 6.32. *Assume that $\mathcal{E}^p(X,\theta)$ satisfies Condition 6.28, then the space $(\mathcal{E}^p(X,\theta), d_p, \wedge)$ is a complete p-strict rooftop metric space.*

Proposition 6.33. *Assume that $\mathcal{E}^p(X,\theta)$ satisfies Condition 6.28. There is a constant $C = C(p,n) > 0$, such that for any $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$,*

$$d_p(\varphi_0,\varphi_1) \leq F_p(\varphi_0,\varphi_1)^{1/p} \leq C d_p(\varphi_0,\varphi_1).$$

Proof. By Proposition 4.25 and Condition A.2, we may assume that $\varphi_0 \leq \varphi_1$.

The first inequality follows from Lemma 6.10.

The second inequality follows from Proposition 6.29. □

6.5. **Relation with definitions in the literature.** In this section, we assume that X is a Kähler manifold.

6.5.1. *Ample/big and nef classes without prescribed singularities.* Assume that α is an ample class and X is smooth. In this case, we may always take α to be a Kähler form ω. Then the space $\mathcal{E}^p(X,\theta)$ is usually written as $\mathcal{E}^p(X,\omega)$. The d_p-metric is defined in [Dar15].

We recall the definition now. Let $\varphi_0,\varphi_1\in\mathcal{H}$. There is a unique weak geodesic φ_t from φ_0 to φ_1. The geodesic has $C^{1,1}$-regularity([CTW18]). Then

$$d_p(\varphi_0,\varphi_1)\coloneqq\left(\int_X|\dot{\varphi}_t|^p\,\omega_{\varphi_t}^n\right)^{1/p}, \tag{6.18}$$

for any $t\in[0,1]$. In particular, the right-hand side does not depend on the choice of t.

In general, let $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\omega)$, take decreasing sequences $(\varphi_j^k)_{k=1,2,\dots}$ in $\mathcal{H}(X,\omega)$ with limits φ_j for $j=0,1$. Then

$$d_p(\varphi_0,\varphi_1)\coloneqq\lim_{k\to\infty}d_p(\varphi_0^k,\varphi_1^k). \tag{6.19}$$

The limit exists and is independent of the choice of φ_j^k.

Proposition 6.34. *The definition of d_p in (6.18) and (6.19) coincides with d_p defined in Section 6.2.*

Proof. Let us write the metric defined in Section 6.2 as D_p for the time being.

Let $\varphi_0,\varphi_1\in\mathcal{E}^p(X,\omega)$. We want to prove

$$d_p(\varphi_0,\varphi_1)=D_p(\varphi_0,\varphi_1).$$

Since both sides satisfy the Pythagorean formula (Corollary 6.25, [Dar19, Theorem 3.26]), we may assume that $\varphi_0\leq\varphi_1$. Recall that order preserving simultaneous Demailly approximations of two potentials exist by the explicit construction in [BK07]. Since both sides are continuous along Demailly approximations (Proposition 6.22, [Dar15, Proposition 4.9]), we may assume that $\varphi_0,\varphi_1\in\mathcal{H}$.

Let φ_t be the weak geodesic from φ_0 to φ_1. Now take $\Psi=(\varphi_{0/N},\dots,\varphi_{N/N})\in\Gamma_+(\varphi_0,\varphi_1)$. Then

$$\begin{aligned}\ell(\Psi)=&\sum_{j=0}^{N-1}\left(\int_X(\varphi_{(j+1)/N}-\varphi_{j/N})^p\,\omega_{\varphi_{j/N}}^n\right)^{1/p}\\ \leq&\sum_{j=0}^{N-1}\left(\int_X(\dot{\varphi}_{j/N}N^{-1}+CN^{-2})^p\,\omega_{\varphi_{j/N}}^n\right)^{1/p}\quad\text{By } C^{1,1}\text{ regularity}\\ \leq&\sum_{j=0}^{N-1}N^{-1}\left(\int_X(\dot{\varphi}_{j/N})^p\,\omega_{\varphi_{j/N}}^n\right)^{1/p}+CN^{-1}\\ =&d_p(\varphi_0,\varphi_1)+CN^{-1},\end{aligned}$$

Let $N\to\infty$, we find $D_p(\varphi_0,\varphi_1)\leq d_p(\varphi_0,\varphi_1)$.

For the other inequality, let $\Psi=(\psi_0,\dots,\psi_N)\in\Gamma_+(\varphi_0,\varphi_1)$. We want to show that $d_p(\varphi_0,\varphi_1)\leq\ell(\Psi)$. By the triangle inequality of d_p. It suffices to prove that

$$d_p(\psi_j,\psi_{j+1})\leq F_p(\psi_j,\psi_{j+1})^{1/p},\quad j=0,\dots,N-1.$$

This is [Dar15, Lemma 4.1]. □

The construction of d_p has been extended to big and nef classes in [DNL20]. A similar argument shows that our d_p metric coincides with the definition in [DNL20] as well.

6.5.2. *The case of $\mathcal{E}^1$.* The metric on $\mathcal{E}^1(X,\theta;[\phi])$ is defined in [DDL18a]. We recall the definition. Let $\varphi,\psi\in\mathcal{E}^1(X,\theta;[\phi])$. Then

$$d_1(\varphi,\psi)\coloneqq E^\phi(\varphi)+E^\phi(\psi)-2E^\phi(\varphi\wedge\psi). \tag{6.20}$$

Proposition 6.35. *The definition of d_1 in (6.20) coincides with the definition of d_1 in Section 6.2.*

This is nothing but Proposition 6.18.

7. Spaces of geodesic rays

Let X be a compact unibranch Kähler space of pure dimension n. Let α be a big $(1,1)$-cohomology class. Let θ be a strongly closed smooth $(1,1)$-form in the class α. Let V be the volume of α. Let $\phi \in \mathrm{PSH}(X,\theta)$ be a model potential with mass $V_\phi > 0$. Fix a resolution of singularities $\pi : Y \to X^{\mathrm{red}}$. Recall that we may always assume that Y is Kähler.

7.1. **Definition of the metric.** Let $\mathcal{R}^1(X,\theta)$ be the space of geodesic rays in $\mathcal{E}^1(X,\theta)$ emanating from V_θ.

We begin with a lemma:

Lemma 7.1. *Let $(\ell_t)_{t\in[0,1]}$, $(\ell'_t)_{t\in[0,1]}$ be two geodesics in $\mathcal{E}^1(X,\theta)$. Then $d_1(\ell_t,\ell'_t)$ is a convex function in $t\in[0,1]$.*

Proof. One immediately reduces to the smooth case, where it is proved in [DDL21b, Proposition 2.10]. □

In particular, we can define

Definition 7.2. Let $\ell^1,\ell^2 \in \mathcal{R}^1(X,\theta)$, set

$$d_1(\ell^1,\ell^2) = \lim_{t\to\infty}\frac{1}{t} d_1(\ell^1_t,\ell^2_t)\,.$$

Note that $d_1(\ell^1,\ell^2) = d_1(\pi^*\ell^1,\pi^*\ell^2)$. In particular, d_1 is indeed a metric.

Proposition 7.3. *The pull-back induces an isometric isomorphism:*

$$\pi^* : (\mathcal{R}^1(X,\theta), d_1) \xrightarrow{\sim} (\mathcal{R}^1(Y,\pi^*\theta), d_1)\,.$$

Proof. This follows immediately from Proposition 3.51. □

7.2. **Construction of the rooftop structure.** Now we construct the rooftop structure on $\mathcal{R}^1(X,\theta)$.

Lemma 7.4. *Let $\ell^1,\ell^2 \in \mathcal{R}^1(X,\theta)$, then there is a geodesic ray $\ell \in \mathcal{R}^1(X,\theta)$, such that $\ell \le \ell^1$, $\ell\le\ell^2$ that is maximal among all such rays.*

Proof. We may assume that $\ell^j_t \le 0$ for all t.

Step 1. There is a geodesic ray $L\in\mathcal{R}^1(X,\theta)$, $L\le \ell^1$, $L\le\ell^2$.

For each $t>0$, let L^t_s ($s\in[0,t]$) be the finite energy geodesic from V_θ to $\ell^1_t\wedge\ell^2_t$. Then for $0\le t_1\le t_2$, $L^{t_2}_{t_1}\le L^{t_1}_{t_1}$. By symmetry, it suffices to prove $L^{t_2}_{t_1}\le \ell^1_{t_1}$. This follows from $\ell^1_{t_2}\wedge\ell^2_{t_2}\le \ell^1_{t_2}$. So for each $s\ge 0$, L^t_s is decreasing in $t\ge s$. Now we claim that

$$d_1(V_\theta,\ell^1_s\wedge\ell^2_s)\le Cs,\quad s\ge 0$$

for a constant $C\ge 0$. In fact,

$$d_1(V_\theta,\ell^1_s\wedge\ell^2_s)\le d_1(V_\theta,\ell^1_s)+d_1(\ell^1_s,\ell^1_s\wedge\ell^2_s)\le d_1(V_\theta,\ell^1_s)+d_1(V_\theta,\ell^2_s)\le Cs\,.$$

So for each fixed $s\ge0$, L^t_s ($t\ge s$) is bounded in $\mathcal{E}^1(X,\theta)$. Let $L_s=\inf_{t\ge s} L^t_s$. By [Dar15, Lemma 4.16] (applied to a resolution), $L_s\in\mathcal{E}^1(X,\theta)$ and L_s is the d_1-limit of L^t_s as $t\to\infty$. It is easy to see that $L\in\mathcal{R}^1(X,\theta)$. Then L solves our problem.

Step 2. We show L is maximal. Let $\ell\in\mathcal{R}^1(X,\theta)$. Assume that $\ell\le\ell^1$, $\ell\le\ell^2$. We claim that $\ell_s\le L_s$ for any $s\ge0$. In fact, for each $t\ge s$, we have

$$\ell_s\le \ell^1_s\wedge\ell^2_s = L^s_s\le L^t_s\,.$$

Let $t\to\infty$, we find $\ell_s\le L_s$. □

Definition 7.5. Let $\ell^1,\ell^2\in\mathcal{R}^1(X,\theta)$, we define

$$\ell^1\wedge\ell^2 := \sup\left\{\,\ell\in\mathcal{R}^1(X,\theta) : \ell\le\ell^1,\ell\le\ell^2\,\right\}\,. \tag{7.1}$$

Theorem 7.6. *The space $(\mathcal{R}^1(X,\theta),d_1,\wedge)$ is a 1-strict complete rooftop metric space. Moreover, $\pi^* : (\mathcal{R}^1(X,\theta),d_1)\to(\mathcal{R}^1(Y,\pi^*\theta),d_1)$ is an isomorphism of rooftop metric spaces.*

Proof. It is clear that $\wedge$ is a rooftop structure. It follows from Proposition 7.3 that $(\mathcal{R}^1(X,\theta),d_1,\wedge)$ is a 1-strict complete rooftop metric space. The second claim is obvious. □

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