# Lectures on Vertex Operator Algebras and Conformal Blocks

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## Contents

# 0 Notations

- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$.
- $\mathbf{i} = \sqrt{-1}$, $\mathbb{S}^1$ =unit circle, $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$, $\mathbb{D}_r^{\mathrm{cl}} = \{z \in \mathbb{C} : |z| \leqslant r\}$
- $\mathscr{O}(X)$ is the space of holomorphic functions on a complex manifold $X$.
- Configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}$.
- $z$ and $\zeta$ could mean either points, or the standard coordinate of $\mathbb{C}$, or formal variables. We will give their meanings when the context is unclear.
- All vector spaces are over $\mathbb{C}$, unless otherwise stated. If $W$ is a vector space equipped with a Hermitian form $\langle\cdot|\cdot\rangle$, we let $|\cdot\rangle$ be the linear variable and $\langle\cdot|$ be the antilinear (i.e. conjugate linear) one, following physicists' convention.
- If $W, W'$ are vector spaces, then $\mathrm{Hom}(W, W')$ denote the space of linear operators from $W$ to $W'$. We let $\mathrm{End}(W) = \mathrm{Hom}(W, W)$.
- We use symbols $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$ to denote bilinear forms (i.e., linear on both variables).
- Given a vector space $W$ and a formal variable $z$,

$$W[z] = \{\text{polynomials of } z \text{ whose coefficients are elements of } W\}$$
$$W[[z]] = \Big\{\sum_{n\in\mathbb{N}} w_n z^n : w_n \in W\Big\}$$
$$W((z)) = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W, \text{ and } w_n = 0 \text{ when } n \text{ is sufficiently negative}\Big\}$$
$$W[[z^{\pm 1}]] = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W\Big\}.$$

  Each line is a subspace of the subsequent line. In case there are several formal variables, the spaces are defined in a similar way, expect $W((\cdots))$. For instance,

$$W[[z, \zeta^{\pm 1}]] := W[[z]][[\zeta^{\pm 1}]] = W[[\zeta^{\pm 1}]][[z]]$$

  consists of $\sum_{m\in\mathbb{N}, n\in\mathbb{Z}} w_{m,n} z^m \zeta^n$ where each $w_{m,n} \in W$. However, note that $W((z))((\zeta))$ and $W((\zeta))((z))$ are not equal. (For instance, $\sum_{m\geqslant -n}\sum_{n\geqslant -1} z^m\zeta^n$ belongs to $\mathbb{C}((z))((\zeta))$ but not $\mathbb{C}((\zeta))((z))$.)

  Elements in $W[[z^{\pm 1}]]$ are called **formal Laurent series** of $z$.

- We let

$$W((z_1, \dots, z_N)) = \Big\{\sum_{n_1,\dots,n_N \geqslant L} w_{n_1,\dots,n_N} z_1^{n_1} \cdots z_N^{n_N} \text{ for some } L \in \mathbb{Z}\Big\}.$$

  Then $W((z_1, z_2))$ is a proper subspace of both $W((z_1))((z_2))$ and $W((z_2))((z_1))$.

- We set
$$\mathrm{Res}_{z=0} \sum_{n\in\mathbb{Z}} w_n z^n dz = w_{-1}. \tag{0.2}$$
This is in line with the complex analytic residue.
- A vector of $W_1 \otimes \cdots \otimes W_N$ writen as $w_\bullet$ means that it is of the form $w_1 \otimes \cdots \otimes w_N$ where each $w_i \in W_i$. Depending on the context, $w_\bullet$ will also mean a tuple $(w_1, \dots, w_N)$.
- Unless otherwise stated, by a manifold, we mean one *without* boundaries. Also, "with boundaries" means "possibly with boundaries".

# 1 Segal's picture of 2d CFT; motivations of VOAs and conformal blocks

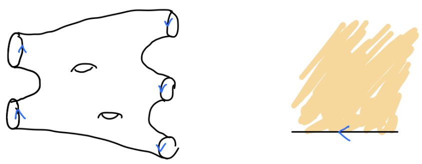
## 1.1

Vertex operator algebras (VOAs) are mathematical objects defined to understand and construct 2-dimensional conformal field theory (CFT for short). A CFT describes propagations and interactions of strings. The are two types of strings: closed strings $\simeq \mathbb{S}^1$ and open strings $\simeq [0,1]$. In this course, we will mainly focus on closed strings.

Let me explain how mathematicians understand CFT. Just like any quantum field theory (QFT), in CFT we must have a Hilbert space $\mathcal{H}$. The vectors in $\mathcal{H}$ are called "states", but unlike ordinary QFT, a vector $\xi \in \mathcal{H}$ is not a state of a particle, but a state of a closed string $\mathbb{S}^1$.

The most important and non-trivial part in CFT is to define/understand string interactions. According to Segal's picture [Seg88], an interaction is uniquely determined by a compact Riemann surface $\Sigma$ with boundaries $\partial\Sigma$, where $\partial\Sigma$ is a disjoint union of some circles (strings). Each string is called either an incoming string or an outcomming one. Suppose $\partial\Sigma$ has $N$ incoming strings and $M$ outgoing ones, then this picture describes an interaction where $N$ strings are going inside, and $M$ strings are going outside.

Moreover, the boundary $\partial\Sigma$ must be **parametrized**. This means that to each connected component $\partial\Sigma_i$ a diffeomorphism $\eta_i : \partial\Sigma_i \xrightarrow{\simeq} \mathbb{S}^1$ is associated. The orientation on $\partial\Sigma_i$ defined by pulling back the one of $\mathbb{S}^1$ along $\eta_i$ is assumed to be the opposite of the one defined in Stokes' theorem, shown as follows



(1.1)

## 1.2

Unless otherwise stated, we assume that the boundary parametrization is also **analytic**. Roughly speaking, this means that $\Sigma$ can be obtained by removing some open discs from a compact Riemann surface $C$ (without boundary) such that the parametrizations of $\partial\Sigma$ are given by local holomorphic functions of $C$.

Here is a more rigorous explanation. By a **local coordinate** $\eta$ of $C$ at $x \in C$, we mean $\eta$ is a holomorphic injective function on a neighborhood $U$ of $x$ such that $\eta(x) = 0$. So $\eta$ is a biholomorphism between $U$ and a neighborhood $\eta(U)$ of $0$. Now, suppose we have local coordinates $\eta_1, \dots, \eta_N$ at distinct points $x_1, \dots, x_N \in C$. The data

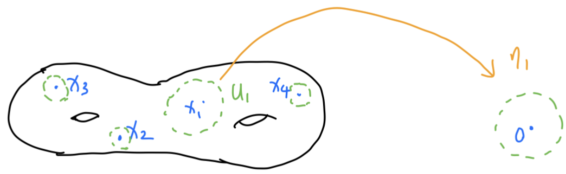
$$\mathfrak{X} := (C; x_\bullet; \eta_\bullet) = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N) \tag{1.2}$$

is called an $N$**-pointed compact Riemann surface with local coordinates**.

Let each $\eta_i$ be defined on a neighborhood $U_i \ni x_i$. We assume moreover the following

**Assumption 1.1.** $U_i \cap U_j = \varnothing$ if $i \neq j$ (indeed, $\eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}}) \cap \eta_j^{-1}(\mathbb{D}_1^{\mathrm{cl}}) = \varnothing$ is sufficient), and $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$ for each $i$. Here $\mathbb{D}_1^{\mathrm{cl}}$ is the closed unit disc.

By removing all $\eta_i^{-1}(\mathbb{D}_1)$, we get $\Sigma$ with boundary strings $\eta_i^{-1}(\partial\mathbb{D}_1^{\mathrm{cl}}) = \eta_i^{-1}(\mathbb{S}^1)$ whose parametrizations are $\eta_i$.




## 1.3

Any $\Sigma$ as above determines uniquely an interaction of strings. Suppose it has $N$ incoming strings and $M$ outgoing ones. Then mathematically, such an interaction is described by a bounded linear map $T = T_\Sigma : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes M}$. (The boundedness is automatic thanks to the uniform boundedness principle. But this is not an important point in this course.) Given $\xi_\bullet = \xi_1 \otimes \cdots \otimes \xi_N \in \mathcal{H}^{\otimes N}$ and $\eta_\bullet = \eta_1 \otimes \cdots \otimes \eta_M \in \mathcal{H}^{\otimes M}$, the value

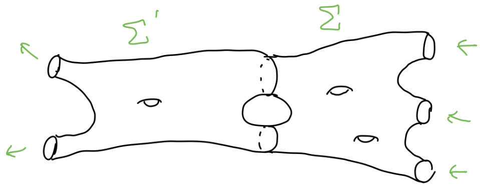
$$\langle \eta_\bullet | T\xi_\bullet \rangle \tag{1.3}$$

describes the probability amplitude that the $N$ incoming closed strings with states $\xi_1, \dots, \xi_N$ become $\eta_1, \dots, \eta_M$ after interaction.

The word "conformal" in conformal field theory reflects the fact that $T$ depends only on the complex structure of $\Sigma$ and its parametrization, but not on the metric for instance. Thus, a CFT is more rigid than a topological quantum field theory (TQFT): in the latter case, $T$ depends only on the topological structures of the manifolds.

## 1.4

Suppose we have another interaction $S : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes L}$ corresponding to the parametrized surface $\Sigma'$, then the composition of them $S \circ T : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes L}$ corresponds to the **sewing** $\Sigma \# \Sigma'$ of $\Sigma$ and $\Sigma'$, where the $j$-th outgoing string $\partial_+\Sigma_j$ of $\Sigma$ is sewn with the $j$-th incoming one $\partial_-\Sigma'_j$ of $\Sigma'$.




It is important to specify how $\partial_+\Sigma_j$ (with parametrization $\eta_j$) is identified with $\partial_-\Sigma'_j$ (with parametrization $\eta'_j$). Pick $x \in \partial_+\Sigma_j$ and $y \in \partial_-\Sigma'_j$. Then

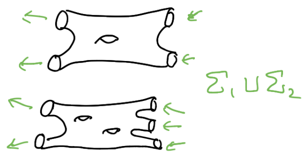
$$x = y \quad \Longleftrightarrow \quad \eta_j(x)\eta'_j(y) = 1. \tag{1.4}$$

It is clear from the picture that the orientations of $\partial_+\Sigma_j$ and $\partial_-\Sigma_j$ are opposite to each other. This is related to the fact that our rule for sewing is $\eta_j(x) = 1/\eta'_j(y)$ but not (say) $\eta_j(x) = \eta'_j(y)$.

Recall we assume that the parametrizations are analytic. We leave it to the readers to check that the sewing of $\Sigma$ and $\Sigma'$, a priori only a topological surface, has a natural complex analytic structure.

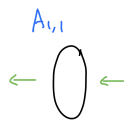
## 1.5

Suppose $T_1 : \mathcal{H}^{\otimes N_1} \to \mathcal{H}^{\otimes M_1}$ corresponds to $\Sigma_1$ and $T_2 : \mathcal{H}^{\otimes N_2} \to \mathcal{H}^{\otimes M_2}$ to $\Sigma_2$, then $T_1 \otimes T_2 : \mathcal{H}^{\otimes(N_1+N_2)} \to \mathcal{H}^{\otimes(M_1+M_2)}$ corresponds to the disjoint union $\Sigma_1 \sqcup \Sigma_2$.




## 1.6

Consider an annulus $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$ obtained by removing two open discs from the compact Riemann sphere $\mathbb{P}^1$ via the local coordinate $\eta_1(z) = z/r$ at $0$ and $\eta_2(z) = R/z$ at $\infty$. We call such $A_{r,R}$ (with the given boundary parametrization) a **standard annlus**. Let $r \nearrow 1, R \searrow 1$. The limit of this annulus is a "degenerate" Riemann surface with 1 incoming boundary circle and 1 outing one. Both circles are $\mathbb{S}^1$. The incoming one has parametrization $z \mapsto z$ and the outgoing one $z \mapsto z^{-1}$. We call this annulus the **standard thin annulus** and denote it by $A_{1,1}$. *The map $T : \mathcal{H} \to \mathcal{H}$ associated to $A_{1,1}$ is the identity map.* This reflects the fact that sewing any $\Sigma$ with a disjoint union of $A_{1,1}$ gives $\Sigma$.



## 1.7

We give a fancy way to summarize what we have so far: Let $\mathscr{C}$ be the monoidal category of compact 1-dimensional smooth manifolds such that a morphism from an object $S_1$ to another $S_2$ is a compact Riemann surface with incomming parametrized boundary $\simeq S_1$ and outgoing one $\simeq S_2$, that the identity morphism for a union of $N$ circles is a disjoint union of $N$ pieces of $A_{1,1}$, that the unit object is the empty set, and that the tensor product of objects and morphisms are respectively the disjoint unions of strings and Riemann surfaces. Then a CFT is a monoidal functor from $\mathscr{C}$ to the monoidal category of Hilbert spaces. So, roughly speaking, a CFT is a representation of $\mathscr{C}$.

Since we choose Hilbert spaces as our underlying spaces, we should expect that the representation of $\mathscr{C}$ is unitary. Technically, the functor mentioned above should be a $*$-functor: this means that for each morphism $\Sigma$ from $N$ strings to $M$ strings, we should define its adjoint morphism $\Sigma^*$ from $M$ strings to $N$ ones whose corresponding map is the adjoint $T^* : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes N}$ of $T$. $\Sigma^*$ is defined simply to be the **complex conjugate** $\overline{\Sigma}$ of $\Sigma$:

**Definition 1.2.** $\overline{\Sigma}$ consists of points $\overline{x}$ where $x \in \Sigma$; the local holomorphic functions on $\overline{\Sigma}$ are $\eta^*$ where $\eta$ is a locally defined holomorphic function on $\Sigma$ and

$$\eta^*(\overline{x}) = \overline{\eta(x)} \tag{1.5}$$

whenever $\eta$ is defined on $x \in \Sigma$; similarly, boundary parametrizations are given by $\eta_j^*$. Note that if $\Sigma$ is obtained by removing open discs from an $N$ pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, then $\overline{\Sigma}$ is obtained by removing discs from

$$\overline{\mathfrak{X}} := (\overline{C}; \overline{x}_1, \ldots, \overline{x}_N; \eta_1^*, \ldots, \eta_N^*) \tag{1.6}$$

$\eta^*$ should not be confused with $\overline{\eta}$ defined on $\Sigma$ by

$$\overline{\eta}(x) = \overline{\eta(x)}.$$

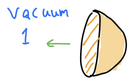
In the present context, we should assume that an incoming (resp. outgoing) string of $\Sigma$ becomes an outgoing (resp. incoming) one of $\overline{\Sigma}$ via the conjugate map $\complement : x \in \Sigma \mapsto \overline{x} \in \overline{\Sigma}$. In the future, we will often consider all strings as incoming ones if necessary (cf. 1.9). In that case, we shall also assume all the boundary strings of $\overline{\Sigma}$ as incoming.

We should point out that although unitarity is a very important condition, there are important non-unitary CFTs, for instance, the logarithmic CFTs. (In such cases, $\mathcal{H}$ is a vector space without inner products.) Also, many VOA results and techniques do not rely on the unitarity. Nevertheless, assuming unitarity will often reasonably simply discussions or give motivations.

**Example 1.3.** Let $\mathfrak{X} = (\mathbb{P}^1; 0; \lambda\zeta)$ where $\zeta$ is the standard coordinate of $\mathbb{C}$ and $\lambda \in \mathbb{C}^\times$. We can identify the conjugate of $\mathbb{P}^1$ with $\mathbb{P}^1$ by letting $x \in \mathbb{P}^1 \mapsto \overline{x}$ be the standard conjugate of $\mathbb{C}$: $z \mapsto \overline{z}$. Then $(\lambda\zeta)^*(\overline{z}) = \overline{\lambda\zeta(z)} = \overline{\lambda} \cdot \overline{z} = \overline{\lambda}\zeta(\overline{z})$. So the conjugate of $\mathfrak{X}$ is isomorphic to $\overline{\mathfrak{X}} = (\mathbb{P}^1; 0; \overline{\lambda}\zeta)$.

## 1.8

An interaction process could have no incoming or outgoing strings. *The Hilbert space for the empty string $\varnothing$ is $\mathbb{C}$.* The most elementary and important example with no incoming boundary is the closed unit disc $\mathbb{D}_1^{\mathrm{cl}}$ with 1 outgoing boundary parametrized by $z \mapsto z^{-1}$. The corresponding map $\mathbb{C} \to \mathcal{H}$ can be identified with its value at $1$. This element in $\mathcal{H}$ is denoted by $\mathbf{1}$ and called the **vacuum vector**.




(1.7)

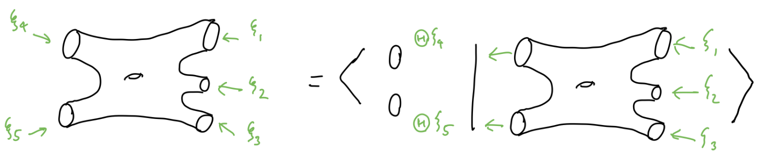
Assume as before that out theory is unitary. Then conjugate of the above disk is the same disk and boundary parametrization, but the original outgoing string is now the incoming one. The corresponding map $\mathcal{H} \to \mathbb{C}$ is, according to 1.7, the linear functional $\langle \mathbf{1} | \cdot \rangle$.

## 1.9

In general, one may wonder what the interaction $T : \mathcal{H}^{\otimes N} \to \mathbb{C}$ means physically for a surface $\Sigma$ with $N$ incoming strings but no outgoing ones. Choose $0 < M < N$, and make $M$ of the $N$ strings of $\partial\Sigma$ be outgoing strings. Then the corresponding interaction is a map $\widetilde{T} : \mathcal{H}^{\otimes(N-M)} \to \mathcal{H}^{\otimes M}$. In unitary CFT, $T$ can be related to $\widetilde{T}$ by a anti-unitary (i.e. conjugate-unitary) map $\Theta$ on $\mathcal{H}$, called the **CPT operator**, such that for $\xi_1, \dots, \xi_N \in \mathcal{H}$ (where the last $M$ vectors are associated to the outgoing strings), we have

$$T(\xi_1 \otimes \cdots \otimes \xi_N) = \langle \Theta\xi_{N-M+1} \otimes \cdots \otimes \Theta\xi_N | \widetilde{T}(\xi_1 \otimes \cdots \otimes \xi_{N-M}) \rangle, \tag{1.8}$$

interpreted pictorially as



The operator $\Theta$ is an involution, i.e., $\Theta^2 = \mathbf{1}_{\mathcal{H}}$.

Such a linear functional $T$ corresponding to an interaction with no outgoing strings is called a **correlation function** (or an **$N$-point function**). These functions are the central objects in CFT (and indeed, in any quantum field theory). Relation (1.8) teaches us that: (1) correlation functions can be interpreted as probability amplitudes in string interactions with the help of $\Theta$, and (2) to study arbitrary interactions, it suffices to study those with no outgoing strings.

Let me close this subsection by mentioning an important fact: suppose the complex structure of $\Sigma$ and the (assumed analtytic) boundary parametrizations are parametrized holomorphically by some complex variables $\tau_\bullet = (\tau_1, \dots, \tau_k)$, then the value of $T(\xi_\bullet)$ is now a *real analytic function* of $\tau_\bullet$, i.e., it is locally a power series of $\tau_1, \dots, \tau_k$ and their conjugates. Actually, the word "function" in "correlation function" means a function of $\tau_\bullet$, but not of $\xi_\bullet$.

## 1.10

You must be curious what CPT means. Indeed, $\Theta$ is responsible for the simultaneous symmetry of charge conjugation (C), parity transformation (P), and time reversal (T). P+T together means an *anti-biholomorphism* $\Sigma \to \Sigma'$. Now we have arrived at a point that we missed previously: since anti-holomorphic maps are also conformal maps, should we expect that the interaction maps (or the correlation functions) for anti-biholomorphic surfaces are equal? The answer is no. (Namely, P+T are not preserved.) Indeed, if we let $\Sigma$ have $N$ incomes and no outcomes, let $\overline{\Sigma}$ be its complex conjugate (cf. 1.7) but still with $N$ incomes, and let $T_\Sigma, T_{\overline{\Sigma}}$ be the correlation functions associated to them. Then from 1.7 and relation (1.8), we have

$$T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N) = \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \tag{1.9}$$

*Proof.* By the description in Subsec. 1.7, the interaction map $\widetilde{T}_{\overline{\Sigma}}$ associated $\overline{\Sigma}$ with no input and $N$ outputs is $T_\Sigma^* : \mathbb{C} \to \mathcal{H}^{\otimes N}$, the adjoint of $T_\Sigma$. By $\Theta^2 = 1$, we have

$$\begin{aligned} &T_\Sigma(\xi_1 \otimes \cdots \xi_N) = \langle 1 | T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N)\rangle = \langle T_\Sigma^* 1 | \xi_1 \otimes \cdots \otimes \xi_N\rangle \\ =&\overline{\langle \xi_1 \otimes \cdots \otimes \xi_N | \widetilde{T}_{\overline{\Sigma}} 1\rangle} \xlongequal{(1.8)} \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \end{aligned}$$

Note that mathematically, the point of formula (1.9) is to translate (using (1.8)) the relation $\widetilde{T}_{\overline{\Sigma}} = T_\Sigma^*$ (regarding all the strings of $\overline{\Sigma}$ as outgoing) to the case that all the strings of $\overline{\Sigma}$ are incoming. □

Formula (1.9) explains CPT symmetry: the symmetries of charge (taking complex conjugate of the values of correlation functions) and parity+time (the conjugate biholomophism $\complement : \Sigma \to \overline{\Sigma}$) are preserved, and the operator realizing this simultaneous symmetry is $\Theta$.

Note that mathematically, charge conjugation $C$ is related to taking complex conjugate of numbers (but not of $\Sigma$). Physically, it means making a string into its "anti-string", or (in general QFT) making a particle (e.g. an election with negative charge) to its anti-particle (e.g. an antielectron with positive charge).

## 1.11

The CFT we have described so far is actually very special: it has no conformal anomaly. There are indeed no nontrivial CFTs which are both unitary and without anomaly. In this course, we will be mainly interested in CFTs with conformal anomaly. Technically, the conformal anomaly is determined by a complex number $c$ (positive for unitary CFT), called **central charge**. To describe such CFT, we modify the previous descriptions as follows: The map (or the correlation function) $T_\Sigma$ for $\Sigma$ is only up to a positive scalar multiplication depending on $\Sigma$. $T_{\Sigma_1} \circ T_{\Sigma_2} = \lambda T_{\Sigma_1 \# \Sigma_2}$ where $\lambda > 0$. (The constants are not necessarily positive in non-unitary CFT.) If $\Sigma$ is parametrized holomorphically by some complex variables $\tau_\bullet$, then by shrinking the domain of $\tau_\bullet$, we can choose $T_\Sigma$ depending real analytically on $\tau_\bullet$.

There are many important cases where a real analytic (or even a holomorphic) $T_\Sigma$ can be chosen globally for $\tau_\bullet$. This will be studied later in details.

Unless otherwise stated, a CFT always means one with (possible) conformal anomaly. Using the fancy language of 1.7, one can say that a unitary CFT is a *projective* monoidal $*$-functor from the category $\mathscr{C}$ in 1.7 to the category of Hilbert spaces. Namely, it is a projective unitary representation of $\mathscr{C}$.

## 1.12

To study the representations of a topological group $G$, one must first understand very well the topological and the algebraic structures of $G$. Similarly, the study of CFTs relies heavily on the geometric and analytic structures of compact Riemann surfaces. However, from what we have discussed, there is a huge obstacle for studying CFTs: the correlation functions are real analytic, but not complex analytic (i.e. holomorphic) functions of the parameters $\tau_\bullet$. Thus, in order to study CFTs using the powerful tools of complex analysis (residue theorem, for instance), we make the following Ansatz: A correlation function $T$ is a sum : $T_\Sigma = \sum_j \Phi^j_\Sigma \cdot \Psi^j_{\overline{\Sigma}}$, where each $\Phi^j$ and $\Psi^j$ relies holomorphically on $\Sigma$ and $\overline{\Sigma}$ respectively (so $\Psi^j_{\overline{\Sigma}}$ relies anti-holomorphically on $\Sigma$).

This Ansatz is very vague. Let me explain it in more details. Consider the annulus $A_{r,R}$ with boundary parametrization as in 1.6. We move the inside circle to another one centered at $z$ (where $z \in A_{r,R}$ is reasonably small), still with radius $r$. The new eccentric annulus $A_{z,r,R}$ has larger outgoing string parametrized by $R/\zeta$ and the smaller incoming one parametrized by $(\zeta - z)/r$, where $\zeta$ is the standard coordinate of $\mathbb{P}^1$. Namely, it is determined by the data

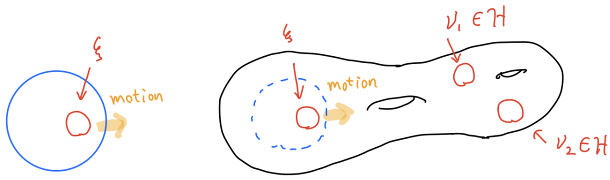
$$(\mathbb{P}^1; z, \infty; (\zeta - z)/r, R/\zeta). \tag{1.10}$$

Let $T_z : \mathcal{H} \to \mathcal{H}$ be the corresponding map. As we have said, for general vectors $\xi, \eta \in \mathcal{H}$, the expression $\langle \eta | T_z \xi \rangle = \langle \Theta\eta, T_z \xi \rangle$ can be chosen to be real analytic with respect to $z$. We now let

$$\begin{aligned} \mathbb{V} = \{\xi \in \mathcal{H} : \ & \text{For all } r, R, \text{ the map } T \text{ can be chosen such that} \\ & z \mapsto \langle \nu | T_z \xi \rangle \text{ is holomorphic for all } \nu \in \mathcal{H}, \text{ and} \\ & \xi \text{ has "finite energy"}\} \end{aligned} \tag{1.11}$$

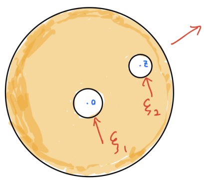
"Finite energy" is a minor condition to be explained later. (See 2.8.)

We can sew $A_{z,r,R}$ with any $\Sigma$, and the motion of the smaller string inside the annulus becomes, after sewing, the motion of a boundary string of $\Sigma$:




Therefore, if a vector $\xi \in \mathbb{V}$ is assigned to an incoming string of $\Sigma$ with (analytic) boundary parametrization $\eta_i$, then, when translating this parametrized string with respect to $\eta_i$, the correlation function $T_\Sigma(\xi \otimes \cdots)$ should be holomorphic with respect to the motion, whatever states we assign to the other strings. We can therefore study $\mathbb{V}$ with the help of complex analysis. $\mathbb{V}$ is called a **vertex operator algebra** (VOA).

We have only described $\mathbb{V}$ as a vector space. But in which sense is $\mathbb{V}$ an algebra? An obvious candidate is as follows: consider $\mathbb{P}^1$ with three marked points $0, z, \infty$ and usual coordinates, e.g. $\eta_0 = \zeta/r_1, \eta_z = (\zeta - z)/r_2, \eta_\infty = R/\zeta$ at $0, z, \infty$ where $r_1, r_2 > 0$ are small and $R > 0$ is large, and $\zeta$ is again the standard coordinate of $\mathbb{C}$. We assume the strings around $0$ and $z$ are ingoing and that around $\infty$ outgoing. If we assign $\xi_1, \xi_2 \in \mathbb{V}$ to the incoming strings, then the outcome can be viewed as a product of $\xi_1$ and $\xi_2$.




Although this product does not have finite energy, it does satisfy the statement before the last line in (1.11). Thus, this product is almost a vector in $\mathbb{V}$. By modifying this product suitably, we can ensure that the products of vectors in $\mathbb{V}$ are always in $\mathbb{V}$. Details will be give in later sections.

Similarly to (1.11), we define $\widehat{\mathbb{V}} \subset \mathcal{H}$ to be the set of finite energy vectors $\xi$ such that $\langle \nu | T_z \xi \rangle$ is anti-holomorphic over $z$. The vacuum vector $\mathbf{1}$ belongs to $\mathbb{V} \cap \widehat{\mathbb{V}}$: The result of gluing the unit disc into the inside of $A_{z,r,R}$ is just the disc with radius $R$ and parametrization $R/\zeta$, which is independent of $z$. So $T_z \mathbf{1}$ and hence $\langle \nu | T_z \mathbf{1} \rangle$ are constant over $z$, and hence both holomorphic and anti-holomorphic over $z$.

## 1.13

Now we can give a more detailed presentation of our Ansatz. We let $\mathcal{H}^{\mathrm{fin}}$ be the (indeed dense) subspace of vectors in $\mathcal{H}$ with "finite energy", which is acted on by $\mathbb{V} \otimes \widehat{\mathbb{V}}$. Ansatz:

1. $\mathcal{H}^{\mathrm{fin}}$ as a $\mathbb{V} \otimes \widehat{\mathbb{V}}$-module has decomposition

$$\mathcal{H}^{\mathrm{fin}} = \bigoplus_{i \in \mathfrak{I}} \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i \qquad \supset \mathbb{V} \otimes \widehat{\mathbb{V}} \tag{1.12}$$

   where each $\mathbb{W}_i, \widehat{\mathbb{W}}_i$ are respectively irreducible $\mathbb{V}$-modules and $\widehat{\mathbb{V}}$-modules. $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are (according to their definition cf. (1.11)) subspaces of $\mathcal{H}^{\mathrm{fin}}$ by identifying them with $\mathbb{V} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \widehat{\mathbb{V}}$ respectively. The vacuum vector $\mathbf{1}$ of $\mathcal{H}$ is identified with $\mathbf{1} \otimes \mathbf{1}$ (which belongs to $\mathbb{V} \otimes \widehat{\mathbb{V}}$).

2. For some $\Sigma$ without outgoing boundaries, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the corresponding map. Then, corresponding to the above direct sum decomposition, we have

$$T_\Sigma \Big|_{(\mathcal{H}^{\mathrm{fin}})^{\otimes N}} = \sum_{i_1, \dots, i_N \in \mathfrak{I}} \Phi_{\Sigma, i_\bullet} \otimes \Psi_{\overline{\Sigma}, i_\bullet} \tag{1.13}$$

   where

$$\Phi_{\Sigma, i_\bullet} : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \to \mathbb{C},$$

$$\Psi_{\overline{\Sigma},i_\bullet}:\widehat{\mathbb{W}}_{i_1}\otimes\cdots\otimes\widehat{\mathbb{W}}_{i_N}\to\mathbb{C}$$

are linear. Moreover, when the complex structure and boundary parametrization are parametrized analytically by complex variables $\tau_\bullet$, then locally (with respect to the domain of $\tau_\bullet$), $T_\Sigma, \Psi_{\Sigma,i_\bullet}, \Psi_{\overline{\Sigma},i_\bullet}$ can be chosen such that $\Psi_{\Sigma,i_\bullet}$ is holomorphic over $\tau_\bullet$ (for all input vectors), and $\Psi_{\overline{\Sigma},i_\bullet}$ holomorphic over $\overline{\tau}_\bullet$. $\Phi_{\Sigma,i_\bullet}$ and $\Psi_{\overline{\Sigma},i_\bullet}$ are called **conformal blocks** associated to $\Sigma$ (resp. $\overline{\Sigma}$) and $\mathbb{V}$ (resp. $\widehat{\mathbb{V}}$).

In part one, $\bigoplus$ could be finite (our main focus in this course), infinite but discrete, or continuous.

The second part can be summarized by saying that the CFT is separated into the **chiral halves** (those $\Phi$ or $\mathbb{W}_i$) and the **anti-chiral halves** (those $\Psi$ or $\widehat{\mathbb{W}}_i$). Here, "chiral"="holomorphic".

When physicists say a CFT is **rational**, they usually mean that the above direct sum is finite, and each $\mathbb{W}_{i_k}, \widehat{\mathbb{W}}_{i_k}$ are semi-simple (hence, by further decomposition, can be irreducible). So far, the mathematical theory of conformal blocks is complete almost only for rational CFTs. These will be the main examples of this course. For non-rational logarithmic CFTs, even the above Ansatz needs to be modified. (So far, it is not even clear how to do it.)

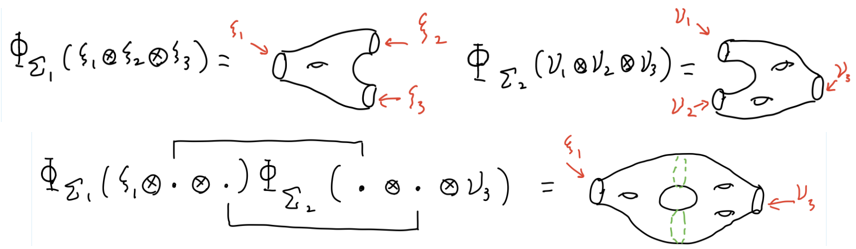
Physicists more or less consider the above description as the definition of conformal blocks. We mathematicians should do the opposite: define conformal blocks in a different way, and use them to *construct* CFTs following the above Ansatz.

## 1.14

You may notice that to make this Ansatz compatible with 1.4 and 1.5, it is necessarily to assume that

1. The tensor product of conformal blocks $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ associated to $\Sigma_1, \Sigma_2$ respectively should be a conformal block associated to $\Sigma_1 \sqcup \Sigma_2$.

2. The composition of $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ (or more precisely, their contractions) should be conformal blocks associated to the sewings of $\Sigma_1$ and $\Sigma_2$, where the pair of $\mathbb{V}$-modules to be contracted must be dual to each other.




(A side note on linear algebra: If $V^\vee$ is the dual space (or a suitable dense subspace of the dual space) of a vetor space $V$, we choose a basis $\{v_\alpha\}_{\alpha\in\mathfrak{A}}$ labeled by elements of $\mathfrak{A}$, and choose a dual basis $\{v_\alpha^\vee\}_{\alpha\in\mathfrak{A}}$ of $V^\vee$ (i.e. the one determined by $\langle v_\alpha, v_\beta\rangle=\delta_{\alpha,\beta}$), then taking contraction means substituting $\sum_{\alpha\in\mathfrak{A}} v_\alpha\otimes v_\alpha^\vee$ inside the linear functional on a tensor product of vector spaces such that $V, V^\vee$ are tensor components.)

After we define conformal blocks rigorously, we will see that the first point is obvious, while the second one is a non-trivial theorem.

We briefly explain the meaning of "dual", and why the dual modules appear in $\mathcal{H}$. For instance, in the above picture, the unitary $\mathbb{V}$-module containing $\xi_2$ is dual to the one containing $\eta_1$. As vector spaces, they are "graded" dual spaces of each other. (It is a dense subspace of the full dual space, the subspace of "finite energy" linear functionals. We will talk about this in future sections.) In unitary CFTs, all $\mathbb{V}$ and $\widehat{\mathbb{V}}$ modules are unitary, and $\Theta(\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i)$ is equivalent to $\mathbb{W}_i'\otimes\widehat{\mathbb{W}}_i'$ where $\mathbb{W}_i'$ is a $\mathbb{V}$-module dual to $\mathbb{W}_i$, and $\widehat{\mathbb{W}}_i'$ a $\widehat{\mathbb{V}}$-module dual to $\widehat{\mathbb{W}}_i$. The formal name for dual module is **contragredient module**, to be defined rigorously in later sections.

### 1.15

Let us describe the equivalence $\Theta(\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i)\simeq\mathbb{W}_i'\otimes\widehat{\mathbb{W}}_i'$ in more details.

For each $w_i\otimes\widehat{w}_i\in\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$, the vector $\Theta(w_i\otimes\widehat{w}_i)$ is regarded as a linear functional on $\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$ in the following way. Let the (clearly symmetric) bilinear form $\langle\cdot,\cdot\rangle:\mathcal{H}^{\otimes 2}\to\mathbb{C}$ be the correlation function $T_{A_{1,1}}$ for the standard thin annulus $A_{1,1}$ (with two inputs and no outputs). Note that by (1.8), for each $\xi,\nu\in\mathcal{H}$, we have

$$\langle\Theta\xi,\nu\rangle=\langle\xi|\nu\rangle. \tag{1.14}$$

Then $\Theta(w_i\otimes\widehat{w}_i)$ is equivalent to the linear functional

$$\langle\Theta(w_i\otimes\widehat{w}_i),\cdot\rangle=\langle w_i\otimes\widehat{w}_i|\cdot\rangle \tag{1.15}$$

restricted onto $\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$.

A conformal block with $M+N$ inputs $\Phi_\Sigma:\mathbb{W}_{i_1}\otimes\cdots\otimes\mathbb{W}_{i_N}\otimes\mathbb{W}_{j_1}\otimes\cdots\otimes\mathbb{W}_{j_N}\to\mathbb{C}$ can be regarded as one with $N$ inputs and $M$ outputs $\Phi_\Sigma:\mathbb{W}_{j_1}\otimes\cdots\otimes\mathbb{W}_{j_N}\to\mathcal{H}_{i_1}'\otimes\cdots\otimes\mathcal{H}_{i_M}'$ where $\mathcal{H}_{i_k}'$ is the Hilbert space completion of $\mathbb{W}_{i_k}'$ and $\mathbb{W}_{i_k}'$ is the contragredient $\mathbb{V}$-module of $\mathbb{W}_{i_k}$. Using (1.14), it is not hard to show that taking compositions of conformal blocks with outputs is equivalent to taking contractions for conformal blocks without outputs.

## 2 Virasoro relations; change of boundary parametrizations; strings vs. punctures

### 2.1

The goal of this section is to understand conformal blocks associated to 2-pointed Riemann spheres, equivalently, genus-0 surfaces with two boundary strings. We simply call them **annuli**, although their complex structures and boundary parametrizations are not necessarily the standard ones as in 1.6.

Let us first consider some degenerate examples whose boundary parametrizations are not necessarily analytic. Let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the topological group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. For each $g\in\mathrm{Diff}^+(\mathbb{S}^1)$, we let $A_{1,1}^g$ be the thin annulus whose incoming and outgoing strings are both $\mathbb{S}^1$ with parametrizations

$$\text{Incoming}: z \mapsto z, \qquad \text{Outgoing}: z \mapsto 1/g(z).$$

**Lemma 2.1.** *If $h \in \mathrm{Diff}^+(\mathbb{S}^1)$, then $A^{gh}_{1,1}$ is obtained by gluing the incoming circle of $A^g_{1,1}$ with the outgoing one of $A^h_{1,1}$.*

*Proof.* By (1.4), a point $z \in A^h_{1,1}$ is glued with $\zeta \in A^g_{1,1}$ iff $\zeta \cdot 1/h(z) = 1$, i.e., $\zeta = h(z)$. Now, a point $z$ of $A^h_{1,1}$ becomes the point $h(z)$ of $A^g_{1,1}$ after gluing, which is sent by the outgoing parametrization of $A^g_{1,1}$ to $1/g(h(z))$. □

This proof is not rigorous since we are considering degenerate annuli. A rigorous one would be approximating $A^g_{1,1}$ and $A^h_{1,1}$ by genuine annuli, identifying the sewn annuli, and then taking the limit. This proof is not easy, unless when $g$ and $h$ are real-analytic (e.g., rotations). Nevertheless, we only need this lemma to motivate our following discussions.

## 2.2

Thus, we may consider $\mathrm{Diff}^+(\mathbb{S}^1)$ as the group of thin annuli whose product is the sewing. The merit of this viewpoint is that it convinces us to *consider the semi-group* **Ann** *of annuli as the complexification of* $\mathrm{Diff}^+(\mathbb{S}^1)$. The multiplication $A_1A_2$ of $A_1, A_2 \in$ **Ann** is the sewing of $A_1, A_2$ defined by gluing the inside of $A_1$ with the outside of $A_2$ using their parametrizations.

As an example, consider $\mathbb{P}^1$ with marked points $0, \infty$ and local coordinates $\eta_0(z) = z, \eta_\infty(z) = e^{-\mathbf{i}\tau}/z$, which gives a thin annulus corresponding to the rotation $z \mapsto e^{\mathbf{i}\tau}z$ when $\tau$ is real. Now consider $\tau$ as a complex variable $\tau = s + \mathbf{i}t$. Then the outgoing circle is the one with radius $e^t$. This gives a genuine annulus whenever $t > 0$.

The Ansatz in 1.13 should be expanded to include the following point: for each annulus $A \in$ **Ann**, the comformal block decomposition of the interaction $T_A : \mathcal{H} \to \mathcal{H}$ (with one income and one outcome) with respect to $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ is of the form

$$T_A = \sum_i \pi_i(A) \otimes \widehat{\pi}_i(\overline{A}) \tag{2.1}$$

where $\pi_i(A)$ is a bounded linear operator on the Hilbert space completion $\mathcal{H}_i$ of $\mathbb{W}_i$, and $\widehat{\pi}_i(\overline{A})$ is one on the completion $\widehat{\mathcal{H}}_i$ of $\widehat{\mathbb{W}}_i$. ($\overline{A}$ is the complex conjugate of $A$; see Def. 1.2. We assume the conjugate of the incomming string of $\overline{A}$ is the incoming of $A$, and similarly for the outcoming strings.) The choice of $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are unique up to scalar multiplications, and if $A$ vary holomorphically over some complex variable $\tau_\bullet$, then locally $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ can be chosen to vary holomorphically with respect to $\tau_\bullet$ and $\overline{\tau}_\bullet$ respectively. Finally, if $A_1, A_2 \in$ **Ann**, then $\pi_i(A_1A_2)$ equals $\pi_i(A_1)\pi_i(A_2)$ up to scalar multiplication, and a similar thing can be said about $\widehat{\pi}_i$.

Namely, each $\pi_i$ is a projective representation of **Ann** on $\mathcal{H}_i$, and so is $\widehat{\pi}_i$ on $\widehat{\mathcal{H}}_i$. They should be the analytic extensions of projective unitary representations of $\mathrm{Diff}^+(\mathbb{S}^1)$.

We emphasize that $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are conformal blocks associated to $A$ and $\overline{A}$ respectively. Roughly speaking, $\pi_i$ describes the conformal symmetries of chiral halves and $\widehat{\pi}_i$ the anti-chiral halves. $A$ and $\overline{A}$ have to act jointly on the full space $\mathcal{H}$.

## 2.3

Thus, the study of CFT interactions for annuli reduces to that of the projective representations of $\mathbf{Ann}$. Our goal is to describe such representations in terms of Lie algebras.

Let $\mathrm{Vec}(\mathbb{S}^1)$ be the Lie algebra of smooth real vector fields of $\mathbb{S}^1$, whose elements are of the form $f\partial_\theta$ where $\partial_\theta$ is the pushforward of the standard unit vector of the real line under the map $\theta \mapsto e^{\mathbf{i}\theta}$, and $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$. The action of $f\partial_\theta$ on $h \in C^\infty(\mathbb{S}^1, \mathbb{R})$ is the negative of the usual one, $-f(e^{\mathbf{i}\theta}) \cdot \frac{\partial}{\partial\theta} h(e^{\mathbf{i}\theta})$. This is because the action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on $h$ should be $h \circ g^{-1}$ in order to respect the order of group multiplication. Therefore, the Lie bracket in $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the usual one:

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta. \tag{2.2}$$

## 2.4

A projective unitary representation $\pi$ of $\mathrm{Vec}(\mathbb{S}^1)$ and the corresponding one $\pi$ of $\mathrm{Diff}^+(\mathbb{S}^1)$ (if exists) are related as follows. (Here unitary means that for each vector field $f\partial_\theta$, we have $\pi(f\partial_\theta)^\dagger = -\pi(f\partial_\theta)$, where $\dagger$ is the adjoint, or "formal adjoint" when the underlying inner product space is not Cauchy-complete.)

Let $t \in (-\epsilon, \epsilon) \mapsto g_t \in \mathrm{Diff}^+(\mathbb{S}^1)$ be a smooth family of diffeomorphisms satisfying $g_0 = \mathbf{1}$. Then up to addition by a number of $\mathbf{i}\mathbb{R}$,

$$\frac{d}{dt}\pi(g_t)\Big|_{t=0} = \pi(\partial_t g_0) \tag{2.3}$$

where $\partial_t g_0 \in \mathrm{Vec}(\mathbb{S}^1)$, the derivative of $g$ at $t_0$, is the vector field determined by

$$(\partial_t g_0)(h) = \frac{d}{dt}(h \circ g_t)\Big|_{t=0} \tag{2.4}$$

for all smooth function $h$ on $\mathbb{S}^1$.

Let now $t \in \mathbb{R} \mapsto \exp(tf\partial_\theta) \in \mathrm{Diff}^+(\mathbb{S}^1)$ be the flow generated by $f\partial_\theta \in \mathrm{Vec}(\mathbb{S}^1)$. So its derivative at $t = 0$ is $f\partial_\theta$, and $\exp((t_1 + t_2)f\partial_\theta) = \exp(t_1 f\partial_\theta) \circ \exp(t_2 f\partial_\theta)$. Then (2.4) implies that up to $\mathbb{S}^1$-multiplication,

$$\pi(\exp(tf\partial_\theta)) = e^{t\pi(f\partial_\theta)}, \tag{2.5}$$

since the derivative of $\pi(\exp(tf\partial_\theta))e^{-t\pi(f\partial_\theta)}$ is $\pi(\exp(tf\partial_\theta))(\pi(f\partial_\theta) - \pi(f\partial_\theta))e^{-t\pi(f\partial_\theta)} = 0$.

## 2.5

The Witt algebra $\mathrm{Span}_{\mathbb{C}} = \{l_n : n \in \mathbb{Z}\}$ is a complex dense Lie subalgebra of the complexification $\mathrm{Vec}(\mathbb{S}^1) \otimes_{\mathbb{R}} \mathbb{C}$. Here,

$$l_n = z^{n+1}\partial_z = -\mathbf{i}e^{\mathbf{i}n\theta}\partial_\theta \tag{2.6}$$

where $z = e^{\mathbf{i}\theta}$ and $\partial_z = \frac{1}{\mathbf{i}e^{\mathbf{i}\theta}}\partial_\theta$. (We use the chain rule to "define" $\partial_z$.) One checks

$$[l_m, l_n] = (m - n)l_{m+n} \tag{2.7}$$

where the bracket is the negative of the usual one for vector fields.

Let us assume for simplicity that the CFT is unitary. In the decomposition $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, each $\mathbb{W}_i$ is a projective unitary representation $\pi_i$ of $\{l_n\}$ , and similarly $\widehat{\mathbb{W}}_i$ is one $\widehat{\pi}_i$ of $\{l_n\}$. We know that the choice of $\pi_i(l_n)$ is unique up to $\mathbf{i}\mathbb{R}$-scalar addition. Here is a well-known fact about projective representations of Witt algebra (cf. for instance [Was10, Sec. IV.1]): one can make a particular choice of $\pi_i(l_n)$ (for each $n$), denoted by $L_n$, such that the **Virasoro relation**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m,-n} \tag{2.8}$$

holds and $c \in \mathbb{C}$ is called the **central charge**. In the case that $\pi_i$ is projectively unitary, $L_n$ can be chosen such that $L_n^\dagger = L_{-n}$ also holds.

We have abused the notation by writing the actions of $l_n$ on all $\mathbb{V}$-modules $\mathbb{W}_i$ (as chiral halves of the CFT) as $L_n$. We are justified to do so because, as we will see later, the actions of $l_n$ come from those of $\mathbb{V}$. Technically: Virasoro algebra is inside the VOA. So the action of $\{l_n\}$ on $\mathbb{W}_i$ is the restriction of that of $\mathbb{V}$. In particular, all chiral halves $\mathbb{W}_i$ share the same central charge $c$.

Similarly, we write the actions of $l_n$ on all $\widehat{\mathbb{W}}_i$ as $\overline{L}_n$. (The bar over $L_n$ reflects the fact that $\overline{L}_n$ describes the conformal symmetries of the anti-chiral halves of the CFT. $\overline{L}_n$ is not related with $L_n$ by the CPT operator $\Theta$.) The central charge $\widehat{c}$ for $\{\overline{L}_n\}$ is independent of $\widehat{\mathbb{W}}_i$ and in general could be different from the one $c$ of $\{L_n\}$, although in most important cases they are equal. (E.g., when the CFT contains both closed and open strings.)

## 2.6

We shall generalize (2.5) to complex vector fields. First of all, we consider an element

$$f(z)\partial_z = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial_z$$

where the sum could be infinite. We treat $f(z) = \sum_n a_n z^{n+1}$ as a Laurent series. Let us now assume that *$f(z)$ is a holomorphic function on a neighborhood $U \subset \mathbb{C}$ of $\mathbb{S}^1$.*

$f\partial_z$ is a complex holomorphic vector field of $U$, which (after shrinking $U$) gives a **holomorphic flow** $\tau \in \Delta \mapsto \exp(\tau f\partial_z) \in \mathscr{O}(U)$ where $\Delta \subset \mathbb{C}$ is a neighborhood of $0$. (Recall from the notation section that $\mathscr{O}(U)$ is the space of holomorphic functions on $U$.) This means:

(1) $(\tau, z) \in \Delta \times U \mapsto \exp(\tau f\partial_z)(z)$ is holomorphic whose restriction to each slice $\tau \times U$ is injective (and hence, a biholomorphism onto its image).

(2) $\exp(0f\partial_z)(z) = z$.

(3) $\exp((\tau_1 + \tau_2)f\partial_z) = \exp(\tau_1 f\partial_z) \circ \exp(\tau_2 f\partial_z)$ on an open subset of $U$ containing $\mathbb{S}^1$.

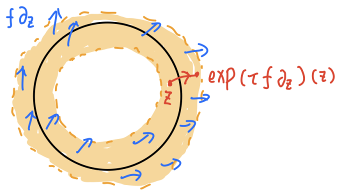
(4) For any holomorphic function $h$ defined on an open set inside $U$,

$$f\partial_z h = \frac{\partial}{\partial\tau} h \circ \exp(\tau f \partial_z)\Big|_{\tau=0}. \tag{2.9}$$

(Compare (2.4).) This condition is equivalent to

$$\frac{\partial}{\partial\tau} \exp(\tau f \partial_z)\Big|_{\tau=0} = f. \tag{2.10}$$

(To see the equivalence, set $h(z) = z$ for one direction, and use chain rule for the other one.)




**Remark 2.2.** A caveat: The notations $f\partial_z$ and $\exp(\tau f \partial_z)$ are not compatible with those in the real case. Indeed, if we assume that $\tau$ only takes real values $\tau = t$, then by taking the real and the imaginary parts of (2.10), we see that $\sigma_t$ is a real flow on the real surfaces $U$ generated by the real vector field $\mathrm{Re} f \cdot \partial_x + \mathrm{Im} f \cdot \partial_y$. Writing $\partial_x = \partial_z + \partial_{\overline{z}}, \partial_y = \mathbf{i}(\partial_z - \partial_{\overline{z}})$, we see that this vector field $f\partial_z$ should more precisely be written as $f\partial_z + \overline{f}\partial_{\overline{z}}$ where $\overline{f}(x) = \overline{f(x)}$.

This point is also justified by the fact that if $k$ is antiholomorphic, then

$$\overline{f}\partial_{\overline{z}} k = \frac{\partial}{\partial\overline{\tau}} k \circ \exp(\tau f \partial_z)\Big|_{\tau=0}. \tag{2.11}$$

(Proof: take $k = \overline{h}$ in (2.10).) Thus, a more precise notation for $\exp(\tau f \partial_z)$ should be $\exp(\tau f \partial_z + \overline{\tau}\overline{f}\partial_{\overline{z}})$. But we prefer to suppress the term $\overline{\tau}\overline{f}\partial_{\overline{z}}$ to keep the notations shorter.

## 2.7

One way to find the expression of $\sigma_\tau = \exp(\tau f \partial_z)$ is to solve the holomorphic nonlinear differential equation with initial condition:

$$\begin{aligned} \frac{\partial}{\partial\tau}\sigma_\tau(z) &= f(\sigma_\tau(z)), \\ \sigma_0(z) &= z. \end{aligned} \tag{2.12}$$

This is due to (2.10) and $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1} \circ \sigma_{\tau_2}$. (Indeed, the existence of holomorphic flows is due to that of the solutions of such equations.)

Alternatively, one may calculate the flow by brutal force using the formula

$$\begin{aligned} \exp(f\partial_z)(z) &= \sum_{k\in\mathbb{N}} \frac{1}{k!}(f(z)\partial_z)^k z \\ &= \sum_{k\in\mathbb{N}} \frac{1}{k!} \underbrace{f(z)\partial_z\Big(f(z)\partial_z\big(\cdots f(z)\partial_z}_{k \text{ times}}\, z\big)\cdots\Big)\Big). \end{aligned} \tag{2.13}$$

(One may treat this formula as a formal sum if one worries about the convergence issue.) To see why this formula is valid, check that such defined $\exp(\tau f\partial_z)(z) =: \sigma_\tau(z)$ satisfies that $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1}\circ\sigma_{\tau_2}$, that $\partial_\tau\sigma_\tau|_{\tau=0} = f$, and that $\sigma_0(z) = z$. This is easy.

## 2.8

**Example 2.3.** $\sigma_\tau(z) = e^\tau z$ is the holomorphic flow generated by the vector field $l_0 = z\partial_z$ since $\frac{\partial}{\partial\tau}e^\tau z|_{\tau=0} = z$. Namely,

$$\exp(\tau z\partial_z)(z) = e^\tau z.$$

Set $\lambda = e^\tau$. In view of the $A^g_{1,1}$ in 2.1, we consider the 2-pointed sphere $\mathfrak{X} = (\mathbb{P}^1; 0, \infty; \zeta, \lambda^{-1}\zeta^{-1})$ where $\zeta : z \mapsto z$ is the standard coordinate of $\mathbb{C}$. Then, when $|\lambda| \leqslant 1$, $\mathfrak{X}$ defines an annulus $A$, either genuine or thin, whose incoming circle has radius $1$ and outcoming $1/|\lambda|$. Thus, the conformal block $\pi_i(A)$ associated to this annulus, which is a linear operator on the Hilbert space completion $\mathcal{H}_i$, should be $e^{\tau L_0} = \lambda^{L_0}$ (by replacing $z\partial_z$ with $L_0$).

It is easy to check that $\overline{A}$ is isomorphic to the annulus defined by $(\mathbb{P}^1; 0, \infty; \zeta, \overline{\lambda^{-1}}\zeta^{-1})$. So the corresponding conformal block should be $\widehat{\pi}_i(\overline{A}) = \overline{\lambda}^{\overline{L}_0}$. Therefore, the interaction map $T_A : \mathcal{H} \to \mathcal{H}$ is determined by

$$T_A\big|_{\mathcal{H}_i\otimes\widehat{\mathcal{H}}_i} = \lambda^{L_0}\otimes\overline{\lambda}^{\overline{L}_0}. \tag{2.14}$$

In a unitary CFT, $L_0$ and $\overline{L}_0$ (or more precisely, their closures) are self-adjoint operators so that $\lambda^{L_0}$ and $\overline{\lambda}^{\overline{L}_0}$ can be defined and are unitary when $|\lambda| = 1$. Moreover, in a unitary CFT:

**Assumption 2.4** (Positive energy)**.** The spectra of $L_0$ and $\overline{L}_0$ are both positive (i.e. $\geqslant 0$). In these notes, we are mainly interested in the case that the spectra are discrete. We identify $L_0$ with $L_0\otimes\mathbf{1}$ and $\overline{L}_0$ with $\mathbf{1}\otimes\overline{L}_0$ so that $L_0, \overline{L}_0$ are commuting diagonalizable operators on $\mathcal{H}^{\mathrm{fin}}$ with $\geqslant 0$ eigenvalues.

Now we can explain what we meant by finite energy: A vector $\xi$ of $\mathcal{H}$ has **finite energy** if $\xi$ is a finite sum of eigenvectors of both $L_0$ and $\overline{L}_0$. (In general, a vector of $\mathcal{H}$ is an $l^2$-convergent sum, either finite or infinite, of eigenvectors.)

## 2.9

**Example 2.5.** Let $n \neq 0$. To understand the geometric meanings of $e^{\tau L_{-n}}$ and $e^{\overline{\tau}\overline{L}_{-n}}$, we find the expression of $\sigma_\tau = \exp(\tau z^{-n+1}\partial_z)$ by solving the differential equation $\partial_\tau\sigma_\tau = (\sigma_\tau)^{-n+1}$ with initial condition $\sigma_0(z) = z$ (cf. (2.12)). The solution is

$$\exp(\tau z^{-n+1}\partial_z)(z) = (z^n + n\tau)^{\frac{1}{n}}. \tag{2.15}$$

$\square$

Unfortunately, this flow does not give us any annulus in the usual sense. Take $n = 1$ for instance. Then the flow is just the translation by $\tau$. However, the circle after a small translation will intersect the original one. So there is no annulus whose outgoing circle is the translation of the incoming one. In fact, in most cases, $\exp(f\partial_z)$ is not the action of an annulus. We have to pursue another way of understanding this operator.

## 2.10

There are two ways to look at a group action $G \curvearrowright X$: (1) The action of $g \in G$ on $X$ is a transformation. So $gx \neq x$ in general. (2) $gx$ and $x$ are different expressions (under different coordinates) of the same element. The rule for change of coordinate is given by the action of $G$. We shall take the second viewpoint.

Let $\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ be an $N$-pointed compact Riemann surface with local coordinates satisfying Assumption 1.1. Assume the setting of 2.6. Write $\sigma_\tau = \exp(\tau f \partial_z)$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}$ be defined on $U \supset \mathbb{S}^1$. Let $\tau \in \Delta$ be close to $0$.

**Remark 2.6.** In case you want to know the precise meaning of "close": for the local coordinate $\eta_i$ we are to discuss in the following, we choose $\epsilon > 0$ such that $\sigma_\tau(U \cap \mathrm{Rng}(\eta_i))$ contains $\mathbb{S}^1$ for all $\tau \in \mathbb{D}_\epsilon$, where the open set $\mathrm{Rng}(\eta_i)$ is the range of $\eta_i$.

**Principle 2.7** (Change of boundary parametrizations)**.** Suppose that the local coordinate $\eta_i$ at $x_i$ is changed to the boundary parametrization $\sigma_\tau \circ \eta_i$ and the boundary string $\eta_i^{-1} \circ (\mathbb{S}^1)$ is gradually changed (with respect to the change of $\tau$) to $\eta_i^{-1}\big(\sigma_\tau^{-1}(\mathbb{S}^1)\big)$. Then, in the expressions of conformal blocks and correlation functions (without outputs), each $w_i \in \mathbb{W}_i$ is replaced by $e^{\tau \sum_n a_n L_n} w_i$, and each $\widehat{w}_i \in \widehat{\mathbb{W}}_i$ by $e^{\overline{\tau} \sum \overline{a_n} \overline{L}_n} \widehat{w}_i$.

To be more precise, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the correlation function where $\Sigma$ is obtained from $\mathfrak{X}$. Assume $i = 1$ for simplicity. Changing the local coordinate $\eta_1$ to $\sigma_\tau \circ \eta_1$ gives a new surface with parametrized boundary $\Sigma'$. Then up to scalar multiplication, $T_{\Sigma'}$ and $T_\Sigma$ are related by

$$T_\Sigma(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N) = T_{\Sigma'}\Big(\big(e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \overline{L}_n}\big)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N\Big) \tag{2.16}$$

for all $\xi_1, \dots, \xi_N$. Similarly, if $\Phi_\Sigma : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \to \mathbb{C}$ is a conformal block for $\Sigma$, then $\Phi_{\Sigma'}$ defined by

$$\Phi_\Sigma(w_1 \otimes w_2 \otimes \cdots \otimes w_N) = \Phi_{\Sigma'}\big(e^{\tau \sum_n a_n L_n} w_1 \otimes w_2 \otimes \cdots \otimes w_N\big) \tag{2.17}$$

is one for $\Sigma'$.

## 2.11

The geometric intuition in the above subsection is the following: $\xi_1$ in the $\eta_1$-parametrization is the same (up to scalar multiplication) vector as $(e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \cdot \overline{L}_n})\xi_1$ in the $\sigma_\tau \circ \eta_1$-parametrization. We call this same "abstract" vector $\widetilde{\xi}_1$, which is unique up to scalar multiplication. We write $\xi_1 = (\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*))\widetilde{\xi}_1$, understanding

$\mathcal{U}(\eta_1)\otimes\mathcal{U}(\eta_1^*)$ as the map sending an abstract vector to its concrete expression under the boundary parametrization $\eta_1$. Namely, $\mathcal{U}(\eta_1)\otimes\mathcal{U}(\eta_1^*)$ is a vector bundle trivialization. The transition function from the $\eta_1$-parametrization to the $\sigma_\tau\circ\eta_1$-parametrization is

$$\big(\mathcal{U}(\sigma_\tau\circ\eta_1)\otimes\mathcal{U}((\sigma_\tau\circ\eta_1)^*)\big)\big(\mathcal{U}(\eta_1)\otimes\mathcal{U}(\eta_1^*)\big)^{-1}=e^{\tau\sum_n a_nL_n}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\cdot\overline{L}_n}. \tag{2.18}$$

We have a parametrization independent $T$ (more precisely, independent of a small change of parametrizations) whose expressions under the concrete boundary parametrizations are (up to scalar multiplications)

$$\begin{aligned}T(\widetilde{\xi}_1\otimes\cdots)&=T_\Sigma\Big(\big(\mathcal{U}(\eta_1)\otimes\mathcal{U}(\eta_1^*)\big)^{-1}\widetilde{\xi}_1\otimes\cdots\Big)\\ &=T_{\Sigma'}\Big(\big(\mathcal{U}(\sigma_\tau\circ\eta_1)\otimes\mathcal{U}((\sigma_\tau\circ\eta_1)^*)\big)^{-1}\widetilde{\xi}_1\otimes\cdots\Big).\end{aligned}$$

## 2.12

Let us do an example to see how the change of parametrization formula works.

**Example 2.8.** Let $\mathfrak{X}=(\mathbb{P}^1;1/3,\infty;2(\zeta-1/3),\zeta^{-1})$ where $\zeta:z\mapsto z$ is the standard coordinate of $\mathbb{C}$. We choose $1/3$ to be the input point, and $\infty$ the outgoing one. The associated boundary parametrized surface $\Sigma$ is an annulus whose incoming circle $\{z:|2(z-1/3)|=1\}$ has center $1/3$ and radius $1/2$, and whose outgoing circle is $\mathbb{S}^1$. Let us find an expression for $T_\Sigma:\mathcal{H}\to\mathcal{H}$.

We know that the map for the standard thin annulus $A_{1,1}$ is $T_{A_{1,1}}=\mathbf{1}_\mathcal{H}$. Let $\mathfrak{X}_1=(\mathbb{P}^1;0,\infty;2\zeta,\zeta^{-1})$, which gives an annlus $\Sigma_1$ with incoming string $\frac{1}{2}\mathbb{S}^1$ and outgoing one $\mathbb{S}^1$. $A_{1,1}$ is changed to $\Sigma_1$ by changing the incoming boundary parametrization $\zeta$ to $2\zeta$. By Ex. 2.3, $2\zeta=\exp(\log 2\cdot z\partial_z)$. So, as $e^{\log 2L_0}=2^{L_0}$ and similarly $e^{\log 2\overline{L}_0}=2^{\overline{L}_0}$, by (2.16), $T_{\Sigma_1}$ could be $(1/2)^{L_0}\otimes(1/2)^{\overline{L}_0}$.

$\Sigma_1$ is changed to $\Sigma$ by adding $2\zeta$ by $-2/3$. According to Ex. 2.5, $\exp(-2/3\partial_z)(z)=z-2/3$. Therefore, up to a scalar multiplication, $T_{\Sigma_1}(\xi)=T_\Sigma((e^{-\frac{2}{3}L_{-1}}\otimes e^{-\frac{2}{3}\overline{L}_{-1}})\xi)$. Thus, the answer is

$$T_\Sigma=\big((1/2)^{L_0}\otimes(1/2)^{\overline{L}_0}\big)\cdot\big((e^{\frac{2}{3}L_{-1}}\otimes e^{\frac{2}{3}\overline{L}_{-1}})\big)=\big((1/2)^{L_0}e^{\frac{2}{3}L_{-1}}\big)\otimes\big((1/2)^{\overline{L}_0}e^{\frac{2}{3}\overline{L}_{-1}}\big).$$

$(1/2)^{L_0}e^{\frac{2}{3}L_{-1}}$ is a conformal block for $\Sigma$. □

## 2.13

What is the change of parametrization formula for $T_\Sigma$ (and hence $\Phi_\Sigma$) when some output strings are involved? Recall from Subsec. 1.15 that the correlation function $T_{A_{1,1}}:\mathcal{H}^{\otimes 2}\to\mathbb{C}$ is a symmetric bilinear form $\langle\xi,\nu\rangle=\langle\nu,\xi\rangle=\langle\Theta\nu|\xi\rangle$. With respect to this form, we actually have

$$(L_n\otimes\mathbf{1})^{\mathrm{t}}=L_{-n}\otimes\mathbf{1},\qquad(\mathbf{1}\otimes\overline{L}_n)^{\mathrm{t}}=\mathbf{1}\otimes\overline{L}_{-n}. \tag{2.19}$$

More precisely, for each $\xi,\nu\in\mathcal{H}^{\mathrm{fin}}$, we have

$$\langle(L_n\otimes\mathbf{1})\xi,\nu\rangle=\langle\xi,(L_{-n}\otimes\mathbf{1})\nu\rangle$$

and a similar relation for $\overline{L}_n$. Rewrite the above relation in terms of $\langle\cdot|\cdot\rangle$, we have $\langle\Theta(L_n\otimes\mathbf{1})\xi|\nu\rangle = \langle\Theta\xi|(L_{-n}\otimes\mathbf{1})\nu\rangle$, and noticing the unitarity property $L_n^\dagger = L_{-n}$, we get

$$\Theta(L_n\otimes\mathbf{1}) = (L_n\otimes\mathbf{1})\Theta, \qquad \Theta(\mathbf{1}\otimes\overline{L}_n) = (\mathbf{1}\otimes\overline{L}_n)\Theta. \tag{2.20}$$

These relations truly hold, not just up to scalar addition or multiply.

From this, we see that for the maps $T_\Sigma, T_{\Sigma'} : \mathcal{H}^{\otimes(N-1)} \to \mathcal{H}$ with $N-1$ inputs and $1$ output,

$$T_\Sigma = \left(e^{\tau\sum_n a_n L_{-n}} \otimes e^{\overline{\tau}\sum_n \overline{a_n}\overline{L}_{-n}}\right)\circ T_{\Sigma'}. \tag{2.21}$$

You can easily generalize this formula to the case of more than one outputs.

*Proof.* Let $\xi_\bullet \in \mathcal{H}^{\otimes(N-1)}$ and $\nu\in\mathcal{H}$. By (1.8), the correlation function (with $N$-inputs and no outputs) for $\Sigma$ and $\Sigma'$ are $\langle\Theta\cdot|T_\Sigma\cdot\rangle$ and $\langle\Theta\cdot|T_{\Sigma'}\cdot\rangle$ respectively. So by (2.16),

$$\begin{aligned}
\langle\Theta\nu|T_\Sigma(\xi_\bullet)\rangle &= \langle\Theta(e^{\tau\sum_n a_n L_n}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\overline{L}_n})\nu|T_{\Sigma'}(\xi_\bullet)\rangle\\
&\overset{(2.20)}{=\!=}\langle(e^{\overline{\tau}\sum_n\overline{a_n}L_n}\otimes e^{\tau\sum_n a_n\overline{L}_n})\Theta\nu|T_{\Sigma'}(\xi_\bullet)\rangle\\
&\overset{\text{unitarity}}{=\!=\!=}\langle\Theta\nu|(e^{\tau\sum_n a_n L_{-n}}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\overline{L}_{-n}})T_{\Sigma'}(\xi_\bullet)\rangle.
\end{aligned}$$

□

**Exercise 2.9.** Show that the formula (2.14) in Example 2.3 follows from (2.21).

## 2.14

In case you want to know why $(L_{-n}\otimes\mathbf{1}) = (L_n\otimes\mathbf{1})^{\mathrm{t}}$, we give a geometric explanation below, in which we pretend to ignore the issue of the uniqueness up to scalar additions/multiplications.

*Proof.* Let $\mathfrak{X} = (\mathbb{P}^1; 0,\infty; z, z^{-1})$ where $z$ is the standard coordinate of $\mathbb{C}$, which gives the standard thin annulus $A_{1,1}$. Assume the two strings are incoming. We know the correlation function is $\langle\xi,\nu\rangle$, where we assume $\xi$ is associated to the string around $0$ and $\nu$ the one around $\infty$.

Change the local coordinate $z$ at $0$ to $\sigma_\tau$, and keep the other data of $\mathfrak{X}$. This changes $A_{1,1}$ to a new weird annulus $A$. By (2.16), the correlation function for $A$ is

$$T_A(\xi\otimes\nu) = \langle(e^{-\tau\sum_n a_n L_n}\otimes e^{-\overline{\tau}\sum_n\overline{a_n}\cdot\overline{L}_n})\xi,\nu\rangle.$$

Note that if we set $\zeta = \sigma_\tau(z)$, then $z^{-1} = 1/\sigma_\tau^{-1}(\zeta)$, which equals $1/\sigma_{-\tau}(\zeta)$ by the definition of flows. Namely, $A$ is equivalent to the weird annulus whose incoming boundary parametrization is $z$ and outcoming $1/\sigma_{-\tau}(z)$. To compute the correlation function for this choice of boundary parametrization, we note that the original $1/z$ at $\infty$ is changed to $1/\sigma_{-\tau}(z)$. Therefore, if we let $\gamma_\tau(z) = 1/\sigma_{-\tau}(1/z)$ which is a holomorphic flow generated by some $\sum_n b_n z^{n+1}$, then the expression for $T_A$ is

$$T_A(\xi\otimes\nu) = \langle\xi,(e^{-\tau\sum_n b_n L_n}\otimes e^{-\overline{\tau}\sum_n\overline{b_n}\cdot\overline{L}_n})\nu\rangle.$$

For the two expressions of $T_A$, we take the holomorphic derivative $-\partial_\tau$ at $\tau = 0$ to get

$$\sum a_n\langle (L_n \otimes \mathbf{1})\xi, \nu\rangle = \sum b_n \langle \xi, (L_n \otimes \mathbf{1})\nu\rangle.$$

To finish the proof, it suffices to prove $b_n = a_{-n}$.

Recall $\sum a_n z^{n+1} = \partial_\tau \sigma_\tau|_{\tau=0}$. Similarly, $\sum b_n z^{n+1} = \partial_\tau \gamma_\tau|_{\tau=0}$, which is

$$\begin{aligned} &\partial_\tau(1/\sigma_{-\tau}(1/z))\big|_{\tau=0} = -\frac{1}{\sigma_0(1/z)^2}\cdot \partial_\tau(\sigma_{-\tau}(1/z))\big|_{\tau=0} \\ =&z^2 \cdot \sum a_n (1/z)^{n+1} = \sum a_n z^{-n+1} = \sum a_{-n} z^{n+1}. \end{aligned}$$

□

## 2.15

As an easy application of our change of parametrization formula, we are able to describe the map $T_A : \mathcal{H} \to \mathcal{H}$ for an analytic annulus $A \in \mathbf{Ann}$ obtained from $(\mathbb{P}^1; 0, \infty; \eta_0, \eta_\infty)$ where $\eta_0$ and $\eta_\infty$ are local coordinates at $0, \infty$ respectively. Set $\varpi = 1/z$. One can write

$$\eta_0(z) = \exp\Big(\sum_{n\in\mathbb{N}} a_n z^{n+1}\partial_z\Big)(z), \qquad \eta_\infty(\varpi) = \exp\Big(\sum_{n\in\mathbb{N}} b_n \varpi^{n+1}\partial_\varpi\Big)(\varpi),$$

where the coefficients $a_n, b_n$ can be determined using (2.13). (We will say more about determining the coefficients in the future.) When $A$ is the standard thin annulus (i.e., when $\eta_0 : z \mapsto z, \eta_\infty : z \mapsto z^{-1}$), we know $T_A = \mathbf{1}$. Thus, in general, by (2.16) and (2.19), the map $T_A$ is (up to scalar multiplications)

$$T_A = \Big(e^{\sum_{n\in\mathbb{N}} -b_n L_{-n}} \otimes e^{\sum_{n\in\mathbb{N}} -\overline{b_n}\cdot \overline{L}_{-n}}\Big)\cdot\Big(e^{\sum_{n\in\mathbb{N}} -a_n L_n} \otimes e^{\sum_{n\in\mathbb{N}} -\overline{a_n}\cdot\overline{L}_n}\Big).$$

The reason that only $n \in \mathbb{N}$ are involved is because $\eta_0$ and $\eta_\infty$ can be defined near $0$ and send $0$ to $0$. Indeed, for $f(z) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}$, assume that $\exp(\tau f\partial_z)(z)$ is defined near $0$ and sends $0$ to $0$ for all small $\tau$. Then its derivative over $\tau$ at $z = 0$, which is $f(\exp(\tau f \partial_z)(0)) = f(0)$ by (2.14), should also be $0$. So $f$ must be of the form $\sum_{n\geqslant 0} a_n z^{n+1}$.

## 2.16

We call those in 2.10 and 2.11 **change of (boundary) parametrizations** in general, and those in 2.15 **change of (local) coordinates**. The former contains the latter.

When changing the boundary parametrizations, the standard coordinate $z$ could be changed to $\sigma_\tau$ not necessarily defined at $0$, or more generally, a local coordinate (say) $\eta_1$ of an $N$-pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ is changed to $\sigma_\tau \circ \eta_1$. This changes the boundary-parametrized Riemann surface $\Sigma$ to $\Sigma'$. Note that this process does not violate our definition of *analytic* boundary parametrizations in 1.2: The new surface $\Sigma'$ is obtained from a new $N$-pointed one $\mathfrak{X}' = (C'; x_\bullet; \sigma_\tau \circ \eta_1, \eta_1, \dots, \eta_N)$ where $C'$ is a new compact Riemann surface, which is defined by gluing $\Sigma$ with $N$ pieces of unit discs $\mathbb{D}_1$ using the maps $\sigma_\tau \circ \eta_1, \eta_2, \dots, \eta_N$. (If you use the maps $\eta_1, \dots, \eta_N$ instead, you simply get $C$.) Thus, *for the change of boundary parametrizations in general, the underlying compact Riemann surfaces $C$ could be changed.*

By change of coordinates, we mean $\mathfrak{X}$ is changed to $\mathfrak{X}' = (C; x_\bullet; \eta'_\bullet)$ with the same underlying compact Riemann surface $C$ and the same marked points $x_\bullet$ as the original ones but different local coordiates at these marked points. As mentioned in 2.15, in this process, only $L_0, L_1, L_2, \dots$ (and also $\overline{L}_0, \overline{L}_1, \overline{L}_2, \dots$) are involved, while in the change of boundary parametrizations, all $L_n$ are involved.

In the previous discussions, almost all formulas hold only up to scalar multiplications or additions. However, when only $L_{-1}, L_0, L_1, L_2, \dots$ are involved, the interaction maps $T_\Sigma$ can indeed be chosen such that all the formulas truly hold, not just up to scalar multiplications or additions. This is because the conformal anomaly is due to the central term $c \cdot (m^3 - m)\delta_{m,-n}/12$ in the Virasoro relation (2.8), which vanishes when $m, n \geqslant -1$. Note that $L_{-1}$ is responsible for translation. Thus:

**Principle 2.10.** $T_\Sigma$ can be chosen to have no ambiguity when changing the local coordinates, or when translating a marked point $x_i$ with respect to its local coordinate $\eta_i$.

To be more precise: We fix a compact Riemann surface $C$. Then for each choice of $N$ marked points $x_\bullet$ and local coordinates $\eta_\bullet$, we can choose the correlation function $T_\mathfrak{X} : \mathcal{H}^{\otimes N} \to \mathbb{C}$ associated to the boundary parametrized surface associated to $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ such that

- For another choice of $N$-pointed $\mathfrak{X}' = (C; x_\bullet; \eta'_\bullet)$ with the same marked points and different local coordinates $\eta'_\bullet$, $T_\mathfrak{X}$ and $T_{\mathfrak{X}'}$ are related by (2.16).
- If $\mathfrak{X}' = (C; x'_1, x_2, \dots, x_N; \eta'_1, \eta_2, \dots, \eta_N)$ where $\eta'_1 = \eta_1 - \eta_1(x'_1)$, and if $x'_1$ is inside an open disc $U_1$ centered at $x_1$ on which $\eta_1$ is holomorphically defined (more precisely, this means $\eta_1(U_1)$ is an open disc centered at $\eta_1(x_1) = 0$), then $T_\mathfrak{X}$ and $T_{\mathfrak{X}'}$ are related by (2.16), namely, (noticing (2.15) for $n = 1$)

$$T_\mathfrak{X}(\xi_1 \otimes \cdots \otimes \xi_N) = T_{\mathfrak{X}'}\Big(\big(e^{-\eta_1(x'_1)L_{-1}} \otimes e^{-\overline{\eta_1(x'_1)}\cdot \overline{L}_{-1}}\big)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N\Big). \tag{2.22}$$

A similar principle holds when $T_\mathfrak{X}$ has output strings. □

Recall the geometric picture described in 2.11. We see that when changing local coordinates, everything in 2.11 truly holds, not just up to scalar multiplications. In particular, the abstract vector $\widetilde{\xi}_1$ is uniquely determined when only the change of local coordinates are allowed.

## 2.17

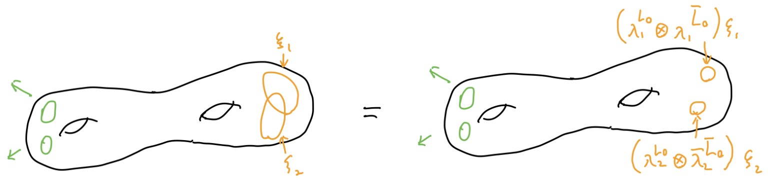
**Assumption 2.11.** We drop Assumption 1.1 for the incoming strings when we associate only finite energy vectors (i.e., vectors of $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, $\mathbb{V} \otimes \widehat{\mathbb{V}}$, etc.) to the incoming strings. Instead, we only assume that the (distinct) incoming points are outside the outgoing strings.

In this course, we will be mainly interested in finite energy vectors. Therefore, we do not assume that that each $\eta_i(U_i)$ contains $\mathbb{D}_1^{\mathrm{cl}}$, or that $U_i$ and $U_j$ are disjoint for different $i$ and $j$. In the latter case, the two boundary strings $\eta_i^{-1}(\mathbb{S}^1)$ and $\eta_j^{-1}(\mathbb{S}^1)$ possibly overlap. What does this picture actually mean?

Note that multiplying $\eta_i$ by $\lambda\eta_i$ amounts to shrinking the size of the string $\eta_i^{-1}(\mathbb{S}^1)$ by $|\lambda|$ and then rotating the string. If $\lambda > 0$ then there is only shrinking but not rotating. Thus, *for an local coordinated $N$-pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, we can find $\lambda_1, \ldots, \lambda_N \in \mathbb{C}^\times$ with large enough absolute values such that the new data $\mathfrak{X}' = (C; x_\bullet; \lambda_1\eta_1, \ldots, \lambda_N\eta_N)$ satisfies Assumption 1.1.* Then for finite energy vectors $\xi_1, \ldots, \xi_N \in \mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, $T_{\mathfrak{X}}(\xi_1 \otimes \cdots \otimes \xi_N)$ is understood as

$$T_{\mathfrak{X}}(\xi_1 \otimes \cdots \otimes \xi_N) := T_{\mathfrak{X}'}\Big(\big(\lambda_1^{L_0} \otimes \overline{\lambda_1}^{\overline{L}_0}\big)\xi_1 \otimes \cdots \otimes \big(\lambda_N^{L_0} \otimes \overline{\lambda_N}^{\overline{L}_0}\big)\xi_N\Big). \tag{2.23}$$

This definition is independent of the choice of sufficiently large $\lambda_1, \ldots, \lambda_N$. And each $\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}$ acts diagonally on $\mathcal{H}^{\mathrm{fin}}$ since $L_0 \otimes \overline{L}_0$ does. (Recall Assumption 2.4.)




In the spirit of the previous subsection, you should view the finite energy vectors $\xi_j$ and $\big(\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}\big)\xi_j$ not as different vectors, but as two coordinate representations of the same vector $\widetilde{\xi}_j$. When $|\lambda_j|$ becomes infinitely large, the string for $\xi_j$ shrinks to an infinitesimal one around $x_j$, i.e., it shrinks to $x_j$ as a **puncture**. It is very useful to view the abstract finite energy vector $\widetilde{\xi}_j$ not associated to any particular string, but associated to that puncture $x_j$. Thus, the marked points $x_\bullet$ of $\mathfrak{X}$ are also called punctures.

**Remark 2.12.** A side note: When we do local coordinate changes, finite energy vectors are changed to finite energy ones.

Therefore, in the above discussion, we don't have to stick to change of coordinates of the form $\eta_j \mapsto \lambda_j\eta_j$: any local coordinate change is valid. We will prove the above claim in later sections.

## 2.18

Let us choose $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ inside $\mathcal{H}^{\mathrm{fin}}$. According to Assumption 2.4, the eigenvalues of the diagonalizable operators $L_0$ (on $\mathbb{W}_i$) and $\overline{L}_0$ (on $\widehat{\mathbb{W}}_i$) are $\geqslant 0$. Now choose eigenvectors $w \in \mathbb{W}_i$ and $\widehat{w} \in \widehat{\mathbb{W}}_i$ with $L_0 w = \Delta w$, $\overline{L}_0\widehat{w} = \widehat{\Delta}\widehat{w}$ where $\Delta, \widehat{\Delta} \geqslant 0$.

Here is an important point about the two eigenvalues. They are not necessarily integers, which means that $\lambda^{L_0}w$ and $\overline{\lambda}^{\overline{L}_0}\widehat{w}$ might be *multivalued with respect to* $\lambda$, i.e., they may also depend on the choice of argument $\arg\lambda$. However, according to the No-Ambiguity Principle 2.10, the expression

$$\big(\lambda^{L_0} \otimes \lambda^{\overline{L}_0}\big)(w \otimes \widehat{w}) = \lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} \cdot w \otimes \widehat{w}$$

must be single-valued with respect to $\lambda$, namely, it does not rely on the choice of $\arg\lambda$. As $\lambda = |\lambda|e^{\mathbf{i}\arg\lambda}$ and hence $\lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} = |\lambda|^{\Delta+\widehat{\Delta}}e^{\mathbf{i}(\Delta-\widehat{\Delta})\arg\lambda}$, we conclude that

$$\Delta - \widehat{\Delta} \in \mathbb{Z}. \tag{2.24}$$

This gives a constraint on the possible $\mathbb{V}\otimes\widehat{\mathbb{V}}$-submodules of $\mathcal{H}^{\mathrm{fin}}$.

That $\lambda^{L_0}w$ could be multivalued is a crucial property in CFT, and it is not related to conformal anomaly. Indeed, it is related to the non-uniqueness of decomposing $T_\Sigma$ into conformal blocks. Thus, *the No-Ambiguity Principle 2.10 does not hold for conformal blocks*.

# 3 Definition of VOAs, I

## 3.1

We first give the rigorous definition of vertex operators algebras and a slightly weaker version, graded vertex algebras. Then we explain the meanings of the axioms.

**Definition 3.1.** A **graded vertex algebra** is a (complex) vector space $\mathbb{V}$ together with a diagonalizable operator $L_0$ acting on $\mathbb{V}$ whose eigenvalues are inside $\mathbb{N}$. We write the $L_0$-grading of $\mathbb{V}$ as $\mathbb{V} = \bigoplus_{n\in\mathbb{N}}\mathbb{V}(n)$. Any eigenvector $v$ of $L_0$ (including $0$) is called ($L_0$)-**homogeneous**, and if $v \in \mathbb{V}(n)$ (i.e. $L_0v = nv$), we write $\mathrm{wt}v = n$ and call $\mathrm{wt}v$ the **weight** of $v$. Moreover, we have a linear map

$$\begin{aligned} \mathbb{V} &\to \big(\mathrm{End}(\mathbb{V})\big)[[z^{\pm 1}]] \\ u &\mapsto Y(u,z) \equiv \sum_{n\in\mathbb{Z}} Y(u)_n z^{-n-1} \end{aligned} \tag{3.1}$$

where each $Y(u)_n \in \mathrm{End}(\mathbb{V})$ is called a **(Fourier) mode**. Here, $z$ is treated as a formal variable. Thus $Y(u,z)v \in \mathbb{V}[[z^{\pm 1}]]$ for each $v \in \mathbb{V}$. The reason for associating $z^{-n-1}$ to $Y(u)_n$ is because we could have (recalling (0.2))

$$\mathrm{Res}_{z=0}\, Y(u,z)z^n dz = Y(u)_n. \tag{3.2}$$

$Y(u,z)$ is called a **vertex operator**.

Moreover, the following axioms are satisfied:

- There is a distinguished vector $\mathbf{1} \in \mathbb{V}(0)$ called **vacuum vector** such that

  $$Y(\mathbf{1},z) = \mathbf{1}_{\mathbb{V}}.$$

  Namely $Y(\mathbf{1})_{-1} = \mathbf{1}_{\mathbb{V}}$ and $Y(\mathbf{1})_n = 0$ if $n \neq -1$.

- **Creation property**: For each $v \in \mathbb{V}$, $Y(v,z)\mathbf{1} = v + \bullet z + \bullet z^2 + \cdots$ where each $\bullet$ is in $\mathbb{V}$. Namely,

  $$Y(v)_{-1}\mathbf{1} = v, \tag{3.3}$$

  and $Y(v)_n\mathbf{1} = 0$ for all $n > -1$. This property is abbreviated to

  $$\lim_{z\to 0} Y(v,z)\mathbf{1} = v.$$

- **Grading property**: For each $v \in \mathbb{V}$,

$$[L_0, Y(v,z)] = Y(L_0 v, z) + z\frac{d}{dz}Y(v,z). \tag{3.4}$$

- **Translation property**: There is a distinguished linear operator $L_{-1}$ on $\mathbb{V}$ such that

$$L_{-1}\mathbf{1} = 0, \tag{3.5}$$

  and that for each $v \in \mathbb{V}$,

$$[L_{-1}, Y(v,z)] = \frac{d}{dz}Y(v,z). \tag{3.6}$$

- **Jacobi identity**: This is the most crucial yet complicated axiom. We postpone its definition to the next section. (See Def. 4.5.)

We say that $\mathbb{V}$ is a **vertex operator algebra** (VOA) if $L_0, L_{-1}$ can be extended to a sequence of linear operators $(L_n)_{n\in\mathbb{Z}}$ on $\mathbb{V}$ satisfying the Virasoro relation (2.8) for some central charge $c \in \mathbb{C}$, and if there is a distinguished vector $\mathbf{c} \in \mathbb{V}$, called the **conformal vector**, such that

$$Y(\mathbf{c})_n = L_{n-1}, \tag{3.7}$$

or equivalently,

$$Y(\mathbf{c}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{3.8}$$

□

You may wonder why the right hand side of (3.7) is not $L_n$ or $L_{n-a}$ for some constant $a \neq 1$. Indeed, if it were not $L_{n-1}$, then the Virasoro relation would be not compatible with the Jacobi identity. We will explain this in more details after defining the Jacobi identity. (See Exercise 5.4.)

We warn the readers that our definitions of graded vertex algebras and VOAs are slightly stronger than the usual ones in the VOA literature, which do not require $L_0$ to have non-negative eigenvalues. This positivity condition $L_0 \geqslant 0$ is very mild and satisfied by most interesting examples including all unitary ones. Since assuming this condition will simply proofs, we keep it in our definition.

Also, in most interesting cases, each $\mathbb{V}(n)$ is finite-dimensional. We do not include this in our definition of VOA here, but we will assume this fact in later sections.

Most VOA textbooks and articles use either $\omega$ or $\nu$ to denote the conformal vector $\mathbf{c}$. In our notes, $\omega$ and $\nu$ are reserved for other meanings and hence do not denote conformal vectors in order to avoid conflicts of notations.

The reason why we should assume that $\sum L_n z^{-n-2}$ can be written as $Y(\mathbf{c}, z)$ for some $\mathbf{c} \in \mathbb{V}$ will not be explained in this section. We will explain it in Subsec. 5.4.

There is a notion of **unitary VOA** which we do not define in this course (although our motivations are mainly from unitary CFTs). We refer the readers to [CKLW18, DL14] for details.

### 3.2

Before we give the motivations for these axioms, let us first derive some useful facts.

Expand the series (3.4) and take the coefficients before each $z^{-n-1}$. This gives us the following equivalent form of grading property:

$$[L_0, Y(v)_n] = Y(L_0 v)_n - (n+1)Y(v)_n. \tag{3.9}$$

To be more concrete, assuming that $v$ is homogeneous, then

$$[L_0, Y(v)_n] = (\mathrm{wt}v - n - 1)Y(v)_n. \tag{3.10}$$

Namely: $Y(v)_n$ *raises the weights by* $\mathrm{wt}v - n - 1$. It is useful to keep in mind that in the VOA theory, $Y(v)_n$ raises weights when $n$ is sufficiently negative, and lowers weights when $n$ is sufficiently positive. As a related fact, as

$$[L_0, L_n] = -nL_n \tag{3.11}$$

by the Virasoro relation (2.8), $L_{-n}$ raises (resp. $L_n$ lowers) the weights by $n$.

**Remark 3.2.** As an application of (3.11), we compute $L_n\mathbf{c}$ when $n \geqslant 0$. Since

$$\mathbf{c} = Y(\mathbf{c})_{-1}\mathbf{1} = L_{-2}\mathbf{1}, \tag{3.12}$$

and since $L_{-2}$ raises the weights by $2$, we see that

$$L_0\mathbf{c} = 2\mathbf{c}. \tag{3.13}$$

By $[L_1, L_{-2}] = 3L_{-1}$, $[L_2, L_{-2}] = 4L_0 + \frac{1}{2}c$, and that $L_n\mathbf{1} = 0$ whenever $n > 0$ (since its weight is $< 0$), we have

$$L_1\mathbf{c} = 0, \qquad L_2\mathbf{c} = \frac{c}{2}\mathbf{1}. \tag{3.14}$$

### 3.3

By (3.10), for each $u, v \in \mathbb{V}$, we know that $Y(u)_n v$ vanishes when $n$ is sufficiently large. Equivalently, we have

$$Y(u,z)v \in \mathbb{C}((z)). \tag{3.15}$$

This important fact is called the **lower truncation property**. It allows us to use meromorphic functions to study VOAs.

In the definition of graded vertex algebras, if the grading property is replaced by the lower truncation property, and if in particular the diagonalizable $L_0$ is not introduced, then $\mathbb{V}$ is called a **vertex algebra**. We will not address this most general notion in our notes.

## 3.4

We let

$$\mathbb{V}' = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)^*$$

where $\mathbb{V}(n)^*$ is the dual space of $\mathbb{V}$. $\mathbb{V}'$ is called the **graded dual space** of $\mathbb{V}$. We let $L_0$ act on $\mathbb{V}'$ such that $L_0 v' = nv'$ whenever $v' \in \mathbb{V}(n)^*$. Then $L_0^{\mathrm{t}} = L_0$. As before, a **homogeneous** vector of $\mathbb{V}'$ is either 0 or an eigenvector of $L_0$. From our definition, it is clear that the evaluation between $\mathbb{V}'(m) = \mathbb{V}(m)^*$ and $\mathbb{V}(n)$ vanishes if $m \neq n$.

**Proposition 3.3.** *For each $u, v \in \mathbb{V}, v' \in \mathbb{V}'$, $\langle v', Y(u,z)v\rangle := \sum_{n\in\mathbb{Z}}\langle v', Y(u)_n v\rangle z^{-n-1}$ is a **Laurent polynomial** of $z$, i.e.,*

$$\langle v', Y(u,z)v\rangle \in \mathbb{C}[z^{\pm 1}].$$

Thus, when evaluating between **finite energy vectors** (i.e., vectors of $\mathbb{V}$ and $\mathbb{V}'$), $Y(u,z)$ is not only a formal series, but a meromorphic function of $\mathbb{P}^1$ with poles at $0, \infty$.

*Proof.* We must show that $\sum_{n\in\mathbb{Z}}\langle v', Y(u)_n v\rangle z^{-n-1}$ is a finite sum. By linearity, it suffices to assume that $u, v, v'$ are homogeneous. Then $Y(u)_n v$ is homogeneous with weight $\mathrm{wt}u + \mathrm{wt}v - n - 1$. So $\langle v', Y(u)_n v\rangle$ is non-zero only if $\mathrm{wt}v' = \mathrm{wt}u + \mathrm{wt}v - n - 1$. Thus

$$\langle v', Y(u,z)v\rangle = \langle v', Y(u)_{\mathrm{wt}u+\mathrm{wt}v-\mathrm{wt}v'-1}\cdot v\rangle \cdot z^{\mathrm{wt}v'-\mathrm{wt}u-\mathrm{wt}v}.$$

□

**Remark 3.4.** The formula $\lim_{z\to 0} Y(u,z)\mathbf{1}$ can now be understood in an analytic sense: By the creation property, for each $v' \in \mathbb{V}$, $\langle v', Y(u,z)\mathbf{1}\rangle$ is a polynomial of $z$ since it has no negative powers of $z$. So

$$\lim_{z\to 0}\langle v', Y(u,z)\mathbf{1}\rangle = \langle v', u\rangle \tag{3.16}$$

where the left hand side is the limit of a polynomial function.

## 3.5

The grading and the translation properties were presented in the "derivative form". We shall present them in the integral form. To prepare for this task, we introduce

$$\mathbb{V}^{\mathrm{cl}} := \prod_{n\in\mathbb{N}} \mathbb{V}(n) = \big\{(v_0, v_1, v_2, \dots) : v_n \in \mathbb{V}(n)\big\}, \tag{3.17}$$

called the **algebraic completion** of $\mathbb{V}$. $\mathbb{V}^{\mathrm{cl}}$ is a naturally a subspace of the dual space $(\mathbb{V}')^*$ of $\mathbb{V}'$. (Indeed, we are mostly interested in the case that each $\mathbb{V}(n)$ is finite dimensional. In such case, one checks easily that $\mathbb{V}^{\mathrm{cl}} = (\mathbb{V}')^*$.) We let

$$P_n : \mathbb{V}^{\mathrm{cl}} \to \mathbb{V}(n), \qquad (v_0, v_1, v_2, \dots) \mapsto v_n \tag{3.18}$$

be the canonical projection onto the $n$-th component. Then for each $z \in \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, we have

$$Y(u,z)v \in \mathbb{V}^{\mathrm{cl}}$$

whose projection onto $\mathbb{V}(\mathrm{wt}u + \mathrm{wt}v - n - 1)$ is $Y(u)_n v \cdot z^{-n-1}$.

Note that $L_0$ and $\lambda^{L_0}$ act on $\mathbb{V}^{\mathrm{cl}}$ in an obvious way:

$$L_0(v_n)_{n\in\mathbb{N}} = (nv_n)_{n\in\mathbb{N}}, \qquad \lambda^{L_0}(v_n)_{n\in\mathbb{N}} = (\lambda^n v_n)_{n\in\mathbb{N}}.$$

## 3.6

**Proposition 3.5 (Scale covariance).** *For each $\lambda \in \mathbb{C}^\times$, we have*

$$\lambda^{L_0} Y(u,z)\lambda^{-L_0} v = Y(\lambda^{L_0}u, \lambda z)v \tag{3.19}$$

*on the level of $\mathbb{V}^{\mathrm{cl}}$. We drop the symbol $v$ and simply write the above relation as*

$$\lambda^{L_0} Y(u,z)\lambda^{-L_0} = Y(\lambda^{L_0}u, \lambda z).$$

The method in the following proof will appear repeatedly in our notes.

*Proof.* Recall $L_0^{\mathrm{t}} = L_0$. Fix $z \in \mathbb{C}^\times$. We prove that for each homogeneous $u, v, v'$,

$$\langle \lambda^{L_0} v', Y(u,z)\lambda^{-L_0} v\rangle = \langle v', Y(\lambda^{L_0}u, \lambda z)v\rangle. \tag{3.20}$$

The left hand side $f$ is a scalar times $\lambda^{\mathrm{wt}v' - \mathrm{wt}v}$, and the right hand side $g$ is a Laurent polynomial of $\lambda$. So both are holomorphic functions on $\mathbb{C}^\times$. Clearly these two expressions are equal when $\lambda = 1$. Let us prove that they are equal for all $\lambda \neq 0$ by showing that they satisfy the same differential equation.

From the form of $f$, it is clear that $\partial_\lambda f(\lambda) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} f(\lambda)$. To compute $\partial_\lambda g$, we first compute an easier derivative $\partial_\lambda \langle v', Y(u, \lambda z)v\rangle$. By the chain rule, we have

$$\frac{\partial}{\partial\lambda}\langle v', Y(u,\lambda z)v\rangle = z\frac{d}{d\zeta}\langle v', Y(u,\zeta)v\rangle\Big|_{\zeta=\lambda z},$$

which, due to the grading property, equals

$$\begin{aligned} &\lambda^{-1}\Big\langle v', \big([L_0, Y(u,\lambda z)] - Y(L_0 u, \lambda z)\big)v\Big\rangle \\ =&(\mathrm{wt}v' - \mathrm{wt}v - \mathrm{wt}u)\lambda^{-1}\big\langle v', Y(u,\lambda z)v\big\rangle. \end{aligned}$$

So

$$\partial_\lambda g(\lambda) = \partial_\lambda\big\langle v', Y(\lambda^{L_0}u, \lambda z)v\big\rangle = \partial_\lambda\big(\lambda^{\mathrm{wt}u}\big\langle v', Y(u,\lambda z)v\big\rangle\big) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} g(\lambda).$$

□

Informally, the integral form (3.19) (i.e., the scale covariance) also implies the derivative form (3.9) by taking partial derivative over $\lambda$. Thus, on a non-rigorous level, these two forms are equivalent. But the integral form has a clearer geometric meaning, which we shall give later.

In the above proof, we have done our first serious VOA calculation. You should be so familiar with these computations that you can "immediately see" the equivalence of the two forms.

The integral form of $[L_{-1}, Y(u,z)] = \partial_z Y(u,z)$ is

$$e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} = Y(u, z+\tau),$$

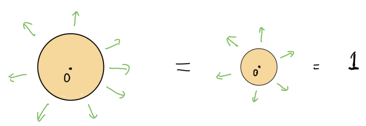
called the **translation covariance**. You may give an informal proof yourself by checking that both sides satisfy the same "linear differential equation". A rigorous treatment is more difficult than the scale covariance. So we leave it to the end of this section.

## 3.7

We now explain the motivations behind the definition of VOAs. Namely, we shall explain how the axioms are natural assumptions from the point of view of the previous two sections. The following explanations are heuristic and non-rigorous.

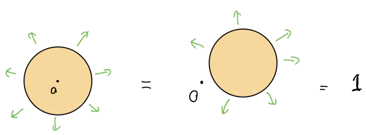
Recall the non-rigorous "definition" of $\mathbb{V}$ in (1.11). We know that $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are subspaces of $\mathcal{H}^{\mathrm{fin}}$, and the decomposition of $\mathcal{H}^{\mathrm{fin}}$ into $\mathbb{V} \otimes \widehat{\mathbb{V}}$-submodules contains a piece $\mathbb{V} \otimes \widehat{\mathbb{V}}$, which furthermore contains $\mathbb{V} \simeq \mathbb{V} \otimes \mathbf{1}$ and $\widehat{\mathbb{V}} \simeq \mathbf{1} \otimes \widehat{\mathbb{V}}$. The vacuum vector is $\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1}$.

We have said in Subsection 1.8 that the standard unit closed disc $\mathbb{D}_1^{\mathrm{cl}}$ with no input and whose boundary $\mathbb{S}^1$ is parametrized by $z \mapsto z^{-1}$ produces from nothing the vacuum vector $\mathbf{1} \otimes \mathbf{1}$. Namely, the vacuum vector comes from the data $(\mathbb{P}^1; \infty; \zeta^{-1})$ where $\zeta$ is the standard coordinate. This data is equivalent to $(\mathbb{P}^1; \infty; \lambda^{-1}\zeta^{-1})$ (where $\lambda \in \mathbb{C}^\times$) via the biholomorphism $z \in \mathbb{P}^1 \mapsto \lambda z \in \mathbb{P}^1$. By the change of local coordinate formula (Principle 2.10), the later geometric data produces uniquely the vector $(\lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0})\mathbf{1}$, which is equal to $\mathbf{1}$ by the equivalence of the two geometric data. Apply $\partial_\lambda$ and $\partial_{\overline{\lambda}}$ to $(\lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0})\mathbf{1} = \mathbf{1}$, we see that $L_0\mathbf{1} = \overline{L}_0\mathbf{1} = 0$. This explain $\mathbf{1} \in \mathbb{V}(0)$ in Def. 3.1.




Consequently, by (2.24), the eigenvalues of $L_0$ are integers, and hence $\geqslant 0$ integers by the positive energy Assumption 2.4. This explains $\mathrm{Spec}(L_0) \subset \mathbb{N}$.

Similarly, the standard disc $\mathbb{D}_1^{\mathrm{cl}}$ is equivalent to its translation by some $\tau \in \mathbb{C}$. So we must have $(e^{\tau L_{-1}} \otimes e^{\overline{\tau}\overline{L}_{-1}})\mathbf{1} = \mathbf{1}$ and hence, similarly, $L_{-1}\mathbf{1} = \overline{L}_{-1}\mathbf{1} = 0$. This explains part of the translation property.



### 3.8

Recall

$$[L_0, L_n] = -nL_n, \qquad [\overline{L}_0, \overline{L}_n] = -n\overline{L}_n. \tag{3.21}$$

As the $L_0$ and $\overline{L_0}$ spectral are $\geqslant 0$, and since $\mathbf{1}$ is a zero eigenvectors of them, we must have

$$L_n\mathbf{1} = \overline{L}_n\mathbf{1} = 0 \qquad (n \geqslant -1). \tag{3.22}$$

From (3.22), we see that for each $v \in \mathbb{V}$, if the change of boundary parametrization does not involve $L_{-2}, L_{-3}, \dots$ and $\overline{L}_{-2}, \overline{L}_{-3}, \dots$, then all $\overline{L}_n$ can be ignored:

$$\big(e^{\sum_{n\geqslant -1} a_n L_n} \otimes e^{\sum_{n\geqslant -1} \overline{a_n}\overline{L}_n}\big)v = e^{\sum_{n\geqslant -1} a_n L_n} v. \tag{3.23}$$

To see this, identify $v$ with $v \otimes \mathbf{1} \in \mathbb{V} \otimes \widehat{\mathbb{V}} \subset \mathcal{H}$ and note that $\mathbf{1}$ is fixed by $e^{\sum_{n\geqslant -1} \overline{a_n}\overline{L}_n}$.

Thus, we conclude: *The translation of the change of local coordinates formula for vectors of $\mathbb{V}$ does not involve $\overline{L}_n$.* In particular, note that the right hand side of (3.23) is almost a vector of $\mathbb{V}$. It is a genuine vector of $\mathbb{V}$ when it has finite energy. Thus, *the change of local coordinates and the translation almost preserve* $\mathbb{V}$. Indeed, the change of local coordinates truly preserve $\mathbb{V}$, as we will see in later sections.

A general change of *boundary parametrization* does not necessarily preserve $\mathbb{V}$ in any weak sense.

### 3.9

Let us describe the meaning of $Y(u, z)v$. For each $z \in \mathbb{C}^\times$, we define a local-coordinated 3-pointed sphere

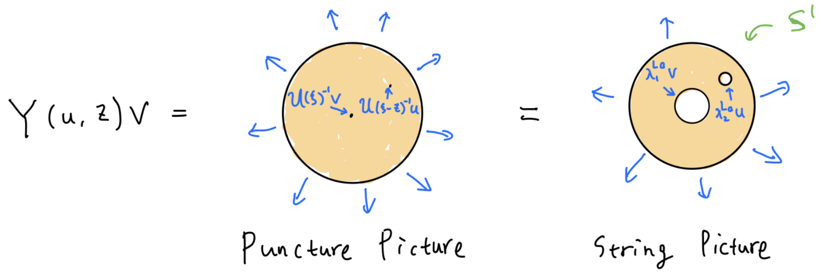
$$\mathfrak{P}_z = \{\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}\} \tag{3.24}$$

where $\zeta$ is the standard coordinate of $\mathbb{C}$.

Let us regard $0, z$ as incoming punctures and $\infty$ outgoing. Roughly speaking, $Y(u, z)v$ is just $T_{\mathfrak{P}_z}(v \otimes u)$ where $v$ is associate to $0$ and $u$ to $z$, understood in a suitable way by change of coordinates. Assume first of all that $0 < |z| < 1$. After scaling $\zeta$ and $\zeta - z$ to $\lambda_1\zeta, \lambda_2(\zeta - z)$ and hence shrinking the two incoming strings, Assumption 1.1 is satisfied. Let the new $N$-pointed sphere by denoted by $\mathfrak{P}_z^{\lambda_1,\lambda_2}$. Note that $v$ in the $\zeta$ coordinate becomes $(\lambda_1^{L_0} \otimes \overline{\lambda_1}^{\overline{L}_0})v = \lambda_1^{L_0}v$ in the $\lambda_1\zeta$ coordinate. Similarly, $u$ becomes $\lambda_2^{L_0}u$ in the new coordinate. Then $Y(u, z)v$ is (physically) defined as $T_{\mathfrak{P}_z^{\lambda_1,\lambda_2}}(\lambda_1^{L_0}v \otimes \lambda_2^{L_0}u)$.

As in Subsec. 2.17, we can use the *puncture picture* to view $u$ and $v$ as the states associated to the punctures $0, z$ with respect to the local coordinates $\zeta, \zeta - z$. Or moreover, formulated in a coordinate independent way as in Subsec. 2.11, we associate the abstract vector $\mathcal{U}(\zeta)^{-1}v$ (the one whose explicit expression under the coordinate $\zeta$ is $v$)

to the puncture $0$ and $\mathcal{U}(\zeta - z)^{-1}v$ to $z$. Then:




(3.25)

According to the notation in Subsec. 2.11, the abstract vectors should be written as $\big(\mathcal{U}(\zeta) \otimes \mathcal{U}(\zeta^*)\big)^{-1}v$ and $\big(\mathcal{U}(\zeta - z) \otimes \mathcal{U}((\zeta - z)^*)\big)^{-1}u$. Here we suppress the second tensor component because, by (3.23), the change of local coordinates for vectors of $\mathbb{V}$ does not involve $\overline{L}_n$.

### 3.10

In the string picture of (3.25), setting $u$ to be $\mathbf{1}$ means filling the hole around $z$ using the solid disc. The result we get is an annulus $\mathcal{A}_{\lambda_1,1}$ with inside parametrization $\lambda_1\zeta$ and outside one $\zeta^{-1}$.[1] According to the change of coordinate formula, the interaction map $\mathcal{H} \to \mathcal{H}$ for this annulus satisfies $T_{\mathcal{A}_{\lambda_1,1}}(\lambda_1^{L_0}v) = T_{A_{1,1}}v = v$. This explains $Y(\mathbf{1}, z)v = v$.

If we set $v = \mathbf{1}$ instead, then we fill the hold around $0$ with the solid disc. The result we get is an eccentric annulus $\mathcal{A}_{z,\lambda_2,1}$ with inside boundary parametrization $\lambda_2(\zeta - z)$ and outside one $\zeta^{-1}$. Let $T_{\mathcal{A}_{z,\lambda_2,1}} : \mathcal{H} \to \mathcal{H}$ be the interaction map. Then, by (3.25), $Y(u, z)\mathbf{1} = T_{\mathcal{A}_{z,\lambda_2,1}}(\lambda_2^{L_0}u)$. Let $z \to 0$. Then $\mathcal{A}_{z,\lambda_2,1}$ converges to $\mathcal{A}_{0,\lambda_2,1}$, which is just the concentric annulus $\mathcal{A}_{\lambda_2,1}$. We have $T_{\mathcal{A}_{\lambda_2,1}}(\lambda_2^{L_0}u) = u$. This explains $\lim_{z\to 0} Y(u, z)\mathbf{1} = u$.

### 3.11

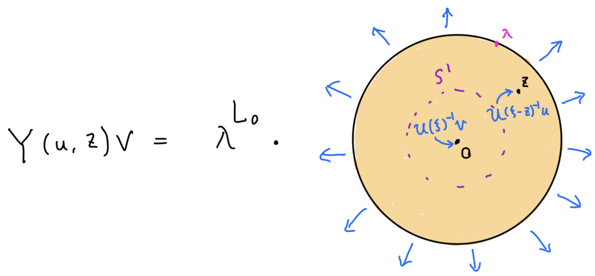
For a general $z \in \mathbb{C}^\times$, in the string picture, we must also shrink the outgoing string in order to get a true surface $\Sigma$. We thus choose $\lambda \in \mathbb{C}^\times$ with $|\lambda| > 1$. Let

$$\mathfrak{P}_z^{\lambda_1,\lambda_2,\lambda} = \{\mathbb{P}^1; 0, z, \infty; \lambda_1\zeta, \lambda_2(\zeta - z); \lambda\zeta^{-1}\}.$$

Then $Y(u, z)v$ is physically "defined" to be

$$Y(u, z)v = \lambda^{L_0} T_{\mathfrak{P}_z^{\lambda_1,\lambda_2,\lambda}}(\lambda_1^{L_0}v \otimes \lambda_2^{L_0}u). \tag{3.26}$$

In the puncture picture, it is




[1] We have previously defined an annulus $A_{r,R}$ with incoming string $|z| = r$ and outgoing $|z| = R$. According to that definition, $\mathcal{A}_{r^{-1},1} = A_{r,1}$.

The meaning of $Y(u)_n$ is clear:

$$\langle v', Y(u)_n v\rangle = \oint_0 \langle v', Y(u,z)v\rangle z^n \cdot \frac{dz}{2\mathbf{i}\pi} = \mathrm{Res}_{z=0}\, \langle v', Y(u,z)v\rangle z^n dz.$$

where the subscript under $\oint$ means that the integral is over any loop around $0$.

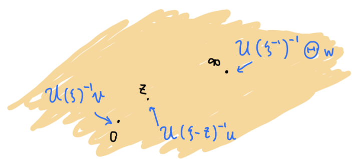
## 3.12

If we prefer not to scale $\zeta^{-1}$, we can make the output point $\infty$ input. To do this, note that from Subsec. 1.14 and 1.15, we know that each $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i)$ is equivalent to $\mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$, the space of finite energy dual vectors on $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$. In the case of $\mathbb{V}$, we get an equivalence

$$\Theta\mathbb{V} \xrightarrow{\simeq} \mathbb{V}', \qquad \Theta w \mapsto \langle \Theta w, \cdot\rangle\big|_{\mathbb{V}} = \langle w|\cdot\rangle\big|_{\mathbb{V}}$$

where $\langle\cdot,\cdot\rangle$ is the correlation function associated to $A_{1,1}$. (From $\langle w|\cdot\rangle\big|_{\mathbb{V}}$ you can see why this linear map is an isomorphism. Here, you may assume each $\mathbb{V}(n)$ is finite-dimensional, or even pretend that $\mathbb{V}$ is finite dimensional.) Then in the puncture picture, the vertex operator and the correlation function of $\mathfrak{P}_z$ (restricted to a linear functional on $\mathbb{V}\otimes\mathbb{V}\otimes\Theta\mathbb{V} \simeq \mathbb{V}\otimes\mathbb{V}\otimes\mathbb{V}'$) are related by

$$\langle \Theta w, Y(u,z)v\rangle = \langle w|Y(u,z)v\rangle \overset{(1.8)}{=\!=}$$




for all $u,v,w \in \mathbb{V}$ and hence $\Theta w \in \Theta\mathbb{V} \simeq \mathbb{V}'$.

## 3.13

We actually have

$$\Theta\mathbb{V} = \mathbb{V} \tag{3.27}$$

and similarly $\Theta\widehat{\mathbb{V}} = \widehat{\mathbb{V}}$. An explanation is as follows:

*Proof.* First of all, $\Theta$ maps finite energy vectors to finite energy ones since $\Theta$ commutes with the energy operators $L_0 \otimes \mathbf{1}, \mathbf{1}\otimes \overline{L}_0$. (See Subsec. 2.13.) By the physical definition of $\mathbb{V}$ in (1.11), for each $u \in \mathbb{V}$, the correlation function $T_z$ associated to $\mathfrak{P}_{z,r,R} = (\mathbb{P}^1; z, \infty; (\zeta - z)/r, R/\zeta)$ varies holomorphically if $u$ is associated to the puncture $z$. Namely, $T_z(u \otimes \nu)$ is holomorphic for all $\nu \in \mathcal{H}$. It is easy to see that the conjugate of $\mathfrak{P}_{z,r,R}$ is equivalent (via the standard conjugation of the complex plane) to $\mathfrak{P}_{\overline{z},r,R} = (\mathbb{P}^1; \overline{z}, \infty; (\zeta-\overline{z})/r, R/\zeta)$, whose correlation function is $T_{\overline{z}}$. Thus, by (1.9)

$$T_z(\Theta u \otimes \nu) = \overline{T_{\overline{z}}(u\otimes\Theta\nu)},$$

which is also holomorphic over $z$. This proves $\Theta u \in \mathbb{V}$ if $u \in \mathbb{V}$. $\square$

Consequently, $\mathbb{V} = \Theta\mathbb{V} \simeq \mathbb{V}'$. The equivalence is given by

$$\mathbb{V} \xrightarrow{\simeq} \mathbb{V}', \qquad u \mapsto \langle u, \cdot \rangle \tag{3.28}$$

Due to this equivalence, we call the VOA $\mathbb{V}$ to be **self-dual**.

So, in all unitary CFTs (and indeed, also in many non-unitary CFTs), the VOAs are self-dual. We remark that there is a mathematically rigorous definition of self-dualness, which plays an important role in the tensor categories of $\mathbb{V}$-modules. However, the definition of a general VOA does not require self-dualness, because many properties can be derived without assuming self-dualness.

## 3.14

Let $\zeta$ be the standard coordinate of $\mathbb{C}$ as usual. For each $\lambda \neq 0$, we have an equivalence

$$(\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}) \simeq (\mathbb{P}^1; 0, \lambda z, \infty; \lambda^{-1}\zeta, \lambda^{-1}\zeta - z, \lambda\zeta^{-1}) \tag{3.29}$$

realized by the biholomorphism $\gamma \mapsto \lambda\gamma$ of $\mathbb{P}^1$. (You should check that the pull-back of the local coordinates on the right hand side equal those on the left.) The correlation function for the left hand side, evaluating on $v \otimes u \otimes w \in \mathbb{V}^{\otimes 3}$, is $\langle w, Y(u,z)v \rangle$. The right hand side of (3.29) is obtained by scaling the local coordinates of $(\mathbb{P}^1; 0, \lambda z, \infty; \zeta, \zeta - \lambda z, \zeta^{-1})$ (whose correlation function on $\mathbb{V}^{\otimes 3}$ takes the form $\langle w, Y(u, \lambda z)v \rangle$) by $\lambda^{-1}, \lambda^{-1}, \lambda$ respectively. By the change of coordinate formula, the correlation function for the right hand side of (3.29), denoted temporarily by $\omega$, must satisfy

$$\langle w, Y(u, \lambda z)v \rangle = \omega(\lambda^{-L_0}v \otimes \lambda^{-L_0}u \otimes \lambda^{L_0}w),$$

namely, $\omega$ should be $\langle \lambda^{-L_0}w, Y(\lambda^{L_0}u, \lambda z)\lambda^{L_0}v \rangle$. This last equation must equal $\langle w, Y(u,z)v \rangle$ due to the equivalence (3.29). This explains the scale covariance.

## 3.15

Similarly, for each $\tau \in \mathbb{C}$, consider the equivalence

$$(\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}) \simeq \big(\mathbb{P}^1; \tau, z + \tau, \infty; \zeta - \tau, \zeta - z - \tau, \frac{1}{\zeta - \tau}\big) \tag{3.30}$$

induced by the biholomorphism $\gamma \mapsto \gamma + \tau$ of $\mathbb{P}^1$. The right hand side is a change of parametrization from $(\mathbb{P}^1; 0, z + \tau, \infty; \zeta, \zeta - z - \tau, \zeta^{-1})$ (whose correlation function is $\langle w, Y(u, z+\tau)v \rangle$), where $\zeta$ is changed to $\zeta - \tau$ (which is a translation), and $\eta := \zeta^{-1}$ is changed to $1/(\eta^{-1} - \tau)$. The translation corresponds to $e^{-\tau L_{-1}}$. The second change of coordinate is $\exp(\tau z^2 \partial_z)$ due to Ex. 2.5, which gives $e^{\tau L_1}$.

Let $\omega$ now be the correlation function (restricted to $\mathbb{V}^{\otimes 3}$) of the right hand side. Then we have

$$\langle w, Y(u, z + \tau)v \rangle = \omega(e^{-\tau L_{-1}}v \otimes u \otimes e^{\tau L_1}w).$$

So $\omega$ is $\langle e^{-\tau L_1}w, Y(u, z+\tau)e^{\tau L_1}v \rangle = \langle w, e^{-\tau L_{-1}}Y(u, z+\tau)e^{\tau L_1}v \rangle$, which must equal $\langle w, Y(u,z)v \rangle$ due to the equivalence (3.30). This explains the translation covariance.

**Exercise 3.6.** Find a geometric explanation of $Y(u, z + \tau) = Y(e^{\tau L_{-1}}u, z)$.

There is a another shorter geometric explanation of translation covariance: $e^{\tau L_{-1}}Y(u, z)v$ amounts to moving the outgoing large string in the string picture in (3.25) by $-\tau$. This is the same as fixing the outgoing string and translating the two incoming strings by $\tau$. Translating the one around $0$ changes $v$ to $e^{\tau L_{-1}}v$, and translating the one around $z$ just changes $z$ to $z + \tau$.

This second explanation is however less rigorous than the first one. But the first one is not rigorous anyway. So why should we care about the issue of rigor here? Well, our first geometric explanation for translation covariance, as well as the one in Subsec. 3.14 for rotation covariance, is much more rigorous in the sense that you can easily get the correct formulas using this method. You may try and give a short explanation for rotation covariance using our second method. Then you will realize that it is not easy to get the correct formula since the change of local coordinates is not so easy to visualize.

## 3.16

Now we return to rigorous mathematics. We are going to prove translation covariance rigorously. For that purpose, we need to generalize the differential equation method in the proof of scale covariance to the following vector-valued form:

**Lemma 3.7.** *Let $W$ be a (non-necessarily finite dimensional) vector space, and $f \in W[[z]]$. Suppose that $\frac{d}{dz}f(z) = Af(z)$ for some $A \in \mathrm{End}(W)$. Suppose also that $f(0) = 0$, namely, the constant term in the power series $f(z)$ is $0$. Then $f = 0$.*

*Proof.* Write $f(z) = \sum_{n\in\mathbb{N}} f_n z^n$ where each $f_n \in W$. The assumptions say that $f_0 = 0$ and

$$\sum_{n\in\mathbb{N}} n f_n z^{n-1} = \sum_{n\in\mathbb{N}} A f_n z^n.$$

So $nf_n = Af_{n-1}$ where $n > 0$. This proves that all $f_n$ are $0$. □

## 3.17

We have said that the integral form of $[L_{-1}, Y(u, z)] = \partial_z Y(u, z)$ is

$$\langle v', e^{\tau L_{-1}}Y(u, z)e^{-\tau L_{-1}}v\rangle = \langle v', Y(u, z + \tau)v\rangle. \tag{3.31}$$

This relation is more difficult to address than the scale covariance since both sides actually involve infinite sums of powers of $\tau$. Our goal is to understand: on which domain does this relation hold? Certainly we need $\tau \neq -z$. But this condition is far from enough.

Let us first understand the two sides as infinite series of $\tau$ and $z$. Assume without loss of generality that $u, v, v'$ are homogeneous. The right hand side is of the form $a(z + \tau)^m$ for some $a \in \mathbb{C}, m \in \mathbb{Z}$. Certainly this expression makes sense as a rational function, but we shall first regard it as a formal series of $\tau, z$ by expanding $(z + \tau)^m$ on the domain $|\tau| < |z|$, namely $(z+\tau)^m = \sum_{k\in\mathbb{N}} \binom{m}{k} z^{m-k}\tau^k$. Thus, the right hand side of (3.31), as an element of $\mathbb{C}[z^{\pm 1}][[\tau]]$, is understood as

$$\langle v', Y(u, z+\tau)v\rangle = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \binom{-n-1}{k} \langle v', Y(u)_n v\rangle \cdot z^{-n-1-k}\tau^k.$$

Here, the sum over $n \in \mathbb{Z}$ is finite, and when the vectors are homogeneous, there is only one possibly non-zero summand.

But why do we expand $(z+\tau)^m$ on $|\tau| < |z|$? Why not $|z| < |\tau|$? Well, this will give us $\sum_{k\in\mathbb{N}} \binom{m}{k} z^k \tau^{m-k}$ which contains negative powers of $\tau$. But the left hand side of (3.31) actually has only non-negative powers of $\tau$.

So let us turn to the left hand side of (3.31). It would be easier to first understand why

$$\langle v', e^{\lambda L_{-1}} Y(u,z) e^{-\mu L_{-1}} v\rangle \tag{3.32}$$

is an element of $\mathbb{C}[z^{\pm 1}][[\lambda, \mu]]$. We first want to move $e^{\lambda L_{-1}}$ to the left hand side of the bracket. In general, if $L_n$ is defined on $\mathbb{V}$, we define $L_{-n}$ on $\mathbb{V}'$ to be the transpose of $L_n$: $L_{-n} = L_n^{\mathrm{t}}$, or more precisely,

$$\langle L_{-n} v', v\rangle := \langle v', L_n v\rangle. \tag{3.33}$$

In case you doubt why this transpose exists, we can write the definition even more precisely: Assume $v' \in \mathbb{V}'(m)$. Then $L_{-n}v'$ is a linear functional on $\mathbb{V}(m+n)$ (so $L_{-n}$ raises the weights by $n$) whose value at any $v \in \mathbb{V}(m+n)$ is $\langle v', L_n v\rangle$. (Recall that $L_n$ lowers the weights by $n$ so $L_n v \in \mathbb{V}(m)$.) And $L_{-n}v'$ vanishes on $\mathbb{V}(a)$ if $a \neq m+n$.

Now, (3.32) equals

$$f(z, \lambda, \mu) := \langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} v\rangle = \sum_{n,l\in\mathbb{N}} \frac{\lambda^n (-\mu)^l}{n! l!} \langle L_1^n v', Y(u,z) L_{-1}^l v\rangle. \tag{3.34}$$

This is in $\mathbb{C}[z^{\pm 1}][[\lambda, \mu]]$. Indeed, it is in $\mathbb{C}[z^{\pm 1}][[\mu]][\lambda]$ since $L_1^n v'$ lowers the weight by $n$, and hence vanishes when $n > \mathrm{wt} v'$. But we will not need this fact here.

Now, the left hand side of (3.31) can be understood as $f(z, \tau, \tau)$, noting the following fact:

**Lemma 3.8.** *Let $W$ be a vector space. If $\varphi(z_1, \ldots, z_N) \in W[[z_1, \ldots, z_N]]$, then $\varphi(z, \ldots, z)$ naturally makes sense as an element of $W[[z]]$.*

*Proof.* Write $\varphi(z_\bullet) = \sum a_{n_1,\ldots,n_N} z_1^{n_1} \cdots z_N^{n_N}$. Then

$$\varphi(z, \ldots, z) = \sum_{n\in\mathbb{N}} \sum_{n_1+\cdots+n_N=n} a_{n_1,\ldots,n_N} z^n$$

where the inside sum is clearly finite. $\square$

## 3.18

**Proposition 3.9 (Translation covariance).** *For each $u, v \in \mathbb{V}$, $v' \in \mathbb{V}'$, the following equation holds on the level of $\mathbb{C}[z^{\pm 1}][[\tau]]$:*

$$\big\langle v', e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} v \big\rangle = \big\langle v', Y(u, z+\tau) v \big\rangle. \tag{3.35}$$

*Here, the right hand side, which is a priori a Laurent polynomial of $z+\tau$, is expanded as if $|\tau| < |z|$.*

*Proof.* Let $f_z(\tau)$ and $g_z(\tau)$ be the left and the right hand sides of (3.35), considered as formal power series of $\tau$ whose coefficients are elements of $\mathbb{C}[z^{\pm 1}]$. Then clearly $f_z(0) = g_z(0)$ as polynomials of $z^{\pm 1}$. So, it suffices to prove that $f_z$ and $g_z$ satisfy the same linear differential equation. The left hand side is $f_z(\tau, \tau)$ where

$$f_z(\lambda, \mu) = \big\langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} v \big\rangle \qquad \in \mathbb{C}[z^{\pm 1}][[\lambda, \mu]].$$

As a general result about multivariable formal power series, we have chain rule

$$\partial_\tau f_z(\tau, \tau) = (\partial_\lambda + \partial_\mu) f_z(\lambda, \mu)\big|_{\lambda = \mu = \tau}.$$

(It is reasonable to believe that this is true. But you can also give a rigorous proof by expanding the two series and check that their coefficients agree!) So, as

$$\begin{aligned} \partial_\lambda f_z(\lambda, \mu) &= \big\langle e^{\lambda L_1} L_1 v', Y(u,z) e^{-\mu L_{-1}} v \big\rangle, \\ \partial_\mu f_z(\lambda, \mu) &= -\big\langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} L_{-1} v \big\rangle, \end{aligned}$$

we have

$$\partial_\tau f_z(\tau) = \big\langle e^{\tau L_1} L_1 v', Y(u,z) e^{-\tau L_{-1}} v \big\rangle - \big\langle e^{\tau L_1} v', Y(u,z) e^{-\tau L_{-1}} L_{-1} v \big\rangle.$$

This expression is not a differential equation of the $\mathbb{C}[z^{\pm 1}]$-coefficients power series $f_z$. But we can make it an ODE by fixing $u$, varying $v, v'$, and view $f_z$ as a $\mathcal{V} := \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}', \mathbb{C}[z^{\pm 1}])$-valued power series of $\tau$. Then $\partial_\tau f_z = A f_z$ where $A \in \mathrm{End}\mathcal{V}$ is defined by sending each $\Phi : \mathbb{V} \otimes \mathbb{V}' \to \mathbb{C}[z^{\pm 1}]$ to

$$A\Phi = \Phi \circ (\mathbf{1} \otimes L_1 - L_{-1} \otimes \mathbf{1}).$$

Now, we compute (noting that the following sum is finite for each fixed $u, v$)

$$\begin{aligned} \partial_\tau g_z(\tau) &= \partial_\tau \big\langle v', Y(u, z+\tau) v \big\rangle = \partial_\tau \Big( \sum_n a_n (z+\tau)^n \Big) \\ &= \sum_n n a_n (z+\tau)^{n-1} = \partial_\zeta \Big( \sum_n a_n \zeta^n \Big)\Big|_{\zeta = z+\tau} = \partial_\zeta \big\langle v', Y(u,\zeta) v \big\rangle\big|_{\zeta = z+\tau}. \end{aligned}$$

By the translation property, the above equals

$$\partial_\tau g_z(\tau) = \big\langle v', [L_{-1}, Y(u,\zeta)] v \big\rangle\big|_{\zeta = z+\tau},$$

which also equals $A g_z(\tau)$ if we now vary $v, v'$ and regard $g$ as $\mathcal{V}$-valued. Therefore, $f_z(\tau) = g_z(\tau)$ due to Lemma 3.7. $\square$

### 3.19

Let us consider a useful variant of Prop. 3.9. Notice that (3.35) holds if $v'$ is replaced by $L_1^n$ and also both sides are multiplied by $\tau^n$. Thus, (3.35) holds on the level of $\mathbb{C}[z^{\pm 1}][[\tau]]$ if $v'$ is replaced by $e^{-\tau L_1}v'$. Namely:

$$\langle v', Y(u,z)e^{-\tau L_{-1}}v\rangle = \langle e^{-\tau L_1}v', Y(u,z+\tau)v\rangle. \tag{3.36}$$

**Remark 3.10.** The left hand sides of (3.35) and (3.36) converges absolutely when $|\tau| < |z|$ and $z \neq 0$ since the right hand side does. These right hand sides are linear combinations of $(z+\tau)^m$ for some $m \in \mathbb{Z}$, whose expansion $\sum_{j,k\in\mathbb{Z}} a_{j,k}z^j\tau^k := \sum_{n\in\mathbb{N}} \binom{m}{n} z^{m-n}\tau^n$ clearly satisfies

$$\sup_{(z,\tau)\in K} \sum_{j,k\in\mathbb{Z}} |a_{j,k}z^j\tau^k| < +\infty \tag{3.37}$$

on every compact subset $K$ of $\{(z,\tau) : |\tau| < |z|, z \neq 0\}$. Thus, the same convergence property holds for the left hand sides of (3.35) and (3.36). We call this property the **absolute and locally uniform converge**, which will be the focus of our study in this course.

Thus, we have actually proved our first convergence result in this course. The method used here is standard in the VOA theory: we show that a formal power series converges by identifying it with the power series expansion of a holomorphic function, which can be achieved with the help of linear differential equations.

### 3.20

Let us choose $v = \mathbf{1}$ in the formula (3.35). Then, as $L_{-1}\mathbf{1} = 0$, we obtain

$$\langle v', e^{\tau L_{-1}}Y(u,z)\mathbf{1}\rangle = \langle v', Y(u,z+\tau)\mathbf{1}\rangle \tag{3.38}$$

on the level of $\mathbb{C}[z,\tau]$, since, by Rem. 3.4, the right hand side is a polynomial of $z+\tau$. As $z \to 0$, the left hand side converges to $\langle e^{\tau L_1}v', u\rangle = \langle v', e^{\tau L_{-1}}u\rangle$ by (3.16). So we conclude:

**Corollary 3.11.** *For each $u \in \mathbb{V}, v \in \mathbb{V}'$, the equation*

$$\langle v', e^{\tau L_{-1}}u\rangle = \langle v', Y(u,\tau)\mathbf{1}\rangle$$

*holds as polynomials of $\tau$. Equivalently, the equation*

$$e^{\tau L_{-1}}u = Y(u,\tau)\mathbf{1}$$

*holds on the level of $\mathbb{V}[[\tau]]$, which is equivalent to that for each $n \in \mathbb{N}$,*

$$Y(u)_{-n-1}\mathbf{1} = \frac{1}{n!}L_{-1}^n u. \tag{3.39}$$

We leave it to the reader to find a geometric explanation of $e^{\tau L_{-1}}u = Y(u,\tau)\mathbf{1}$.

# 4 Definition of VOAs, II: Jacobi Identity.

## 4.1

**Principle 4.1.** When gluing Riemann spheres to get new spheres, the formula $T_{\Sigma_1} \circ T_{\Sigma_2} = T_{\Sigma_1 \# \Sigma_2}$ truely holds if the local coordinates at the points for sewing are Möbius transformations, i.e. of the form $z \mapsto \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

A rough reason for this No-Ambiguity Principle is that only $L_0, L_{\pm 1}$ are involved in the change of coordinate formulas between Möbius transformations, and the Lie bracket relations between them do not involve the central charge.

## 4.2

We shall give motivations for the Jacobi identity.

We first remark on the sewing of compact Riemann surfaces in Subsec. 1.4. Suppose we have data $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ and $\mathfrak{X}' = (C'; y_\bullet; \eta'_\bullet)$ and we sew them along $x_1$ and $x'_1$. For simplicity, we set $\xi = \eta_1, \varpi = \eta'_1$. From (1.4), we know that the gluing law is that any $x \in \xi^{-1}(\mathbb{S}^1)$ (recall that $\xi^{-1}(\mathbb{S}^1)$ is a boundary string of the corresponding surface $\Sigma$ for $\mathfrak{X}$) and any $y \in \varpi^{-1}(\mathbb{S}^1)$ are identified following the rule

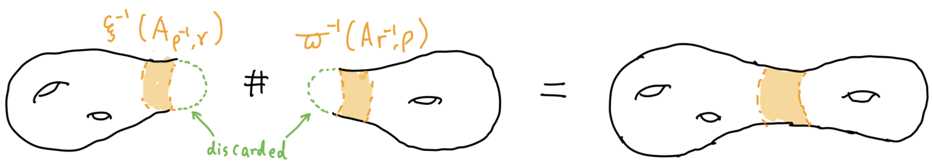
$$x = y \qquad \Longleftrightarrow \qquad \xi(x)\varpi(y) = 1. \tag{4.1}$$

This definition of gluing is topological, but not complex analytic. Analytically, we are actually gluing a neighborhood of $\xi^{-1}(\mathbb{S}^1)$ and one of $\varpi^{-1}(\mathbb{S}^1)$ using the rule (4.1) for all $x$ in the first neighborhood and $y$ in the second one. It is clear that a (locally defined) function on the first neighborhood is holomorphic if and only if it is so on the second one. This defines the complex analytic structure on $C\#C'$.

**Remark 4.2.** Let us be more precise on the shape of the neighborhoods. Let $\xi$ and $\varpi$ be defined (and injective) on $U, U'$ respectively. Choose $r > 1, \rho > 1$ such that $\xi(U) \supset \mathbb{D}_r, \varpi(U') \supset \mathbb{D}_\rho$. Then the following neighborhoods of $\xi^{-1}(\mathbb{S}^1)$ and $\varpi^{-1}(\mathbb{S}^1)$ are glued via the relation (4.1):

$$\begin{array}{c} \xi^{-1}(A_{\rho^{-1},r}) = \{x \in U : \rho^{-1} < |\xi(x)| < r\} \\ \updownarrow \text{identified via (4.1)} \\ \varpi^{-1}(A_{r^{-1},\rho}) = \{y \in U' : r^{-1} < |\varpi(y)| < \rho\} \end{array} \tag{4.2}$$

The parts $\{x \in U : |\xi(x)| \leqslant \rho^{-1}\}$ and $\{y \in U' : |\varpi(y)| \leqslant r^{-1}\}$ are discarded when gluing.



(4.3)

### 4.3

As pointed out before, when we associate finite energy vectors to the incoming strings/points, we may scale their local coordiates. However, for the local coordinates at the output points and the points to be sewn, an arbitrary scaling is not allowed. We thus assume that Assumption 1.1 holds after scaling (by some $\lambda$ with arbitrarily large $|\lambda|$) the local coordinates at the incoming points. This amounts to the following

**Assumption 4.3.** If $x_i$ is either an outgoing point or a point to be sewn with another point, then the local coordinate $\eta_i$ at $x_i$ defined on a neighborhood $U_i \ni x_i$ satisfies that $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$, that $\eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}}) \cap \eta_j^{-1}(\mathbb{D}_1^{\mathrm{cl}}) = \emptyset$ if $x_j$ is either outgoing or a point to be sewn, and that $x_j \in \eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}})$ if $x_j$ is incoming and not to be sewn.

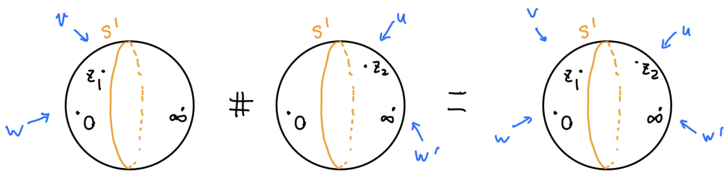
**Remark 4.4.** There is indeed one way we can slightly loosen the above assumption. Using the notation of (4.1). Then we may assume that Assumption 4.3 after scaling $\xi$ by some $\lambda \in \mathbb{C}^\times$ and $\varpi$ by $\lambda^{-1}$. Then the rule for gluing (4.1) is not changed. On the side of interaction maps $T_\Sigma$, the change $\xi \rightsquigarrow \lambda\xi$ adds a factor $\lambda^{-L_0} \otimes (\overline{\lambda})^{-\overline{L}_0}$ to one tensor component in $T_\Sigma$, and $\xi \rightsquigarrow \lambda^{-1}\xi$ adds a factor $\lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0}$. These two are canceled after taking contraction or composition.

### 4.4

We want to understand the product $\langle w', Y(u, z_2)Y(v, z_1)w\rangle$. Let $\zeta$ be the standard coordinate of $\mathbb{C}$. By the sewing property in Segal's picture, this expression should correspond to the sewing of

$$\mathfrak{P}_{z_1} = (\mathbb{P}_1^1; 0, z_1, \infty; \zeta, \zeta - z_1, \zeta^{-1}), \qquad \mathfrak{P}_{z_2} = (\mathbb{P}_2^1; 0, z_2, \infty; \zeta, \zeta - z_2, \zeta^{-1})$$

along the points $\infty$ of $\mathfrak{P}_{z_1}$ and $0$ of $\mathfrak{P}_{z_2}$. (Here, both $\mathbb{P}_1^1$ and $\mathbb{P}_2^1$ are $\mathbb{P}^1$. We assume the two $\infty$ are outgoing before sewing.) Assumption 4.3 is satisfied when $0 < |z_1| < 1 < |z_2| < +\infty$ if we consider all the points not for sewing as incoming. The sewing rule is that $\gamma_1 \in \mathbb{P}_1^1, 0 < |\gamma_1^{-1}| < +\infty$ is identified with $\gamma_2 \in \mathbb{P}_2^1, 0 < |\gamma_2| < +\infty$ if and only if $\gamma_1^{-1} \cdot \gamma_2 = 1$, namely $\gamma_1 = \gamma_2$. (Here, we set $r = \rho = \infty$ in order to apply Rem. 4.2. The discarded points are the $\infty$ of $\mathbb{P}_1^1$ and the $0$ of $\mathbb{P}_2^1$.) Thus, the sewing is just placing the first sphere onto the second one.




The result of sewing is

$$\mathfrak{P}_{z_1,z_2} = (\mathbb{P}^1; 0, z_1, z_2, \infty, \zeta - z_1, \zeta - z_2, \zeta^{-1}) \tag{4.4}$$

Assuming all the points of $\mathfrak{P}_{z_1,z_2}$ as incoming, for each $u, v, w, w' \in \mathbb{V}$,

$$T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, w') = \langle w', Y(u, z_2)Y(v, z_1)w\rangle \qquad (\text{if } 0 < |z_1| < |z_2| < +\infty). \tag{4.5}$$

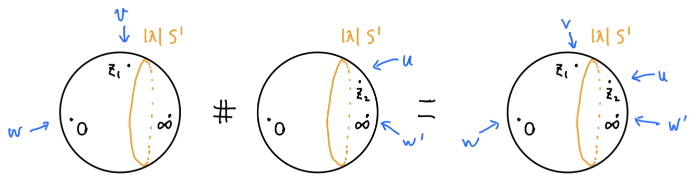
The reason why the conditions $|z_1| < 1$ and $1 < |z_2|$ can be dropped is explained below.

## 4.5

We explain why (4.5) holds provided $0 < |z_1| < |z_2| < +\infty$.

Pick $\lambda \in \mathbb{C}$ such that $|z_1| < |\lambda| < |z_2|$. Following the guide of Rem. 4.4, we replace the local coordinate $\zeta^{-1}$ of $\mathfrak{P}_{z_1}$ by $\lambda\zeta^{-1}$ and the one $\zeta$ of $\mathfrak{P}_{z_2}$ by $\zeta/\lambda$. Then Assumption 4.3 is again satisfied. In particular, the outgoing string of $\mathbb{P}^1_1$ around $\infty$ and the incoming one of $\mathbb{P}^1_2$ around $0$ are both $|\lambda|\mathbb{S}^1$.

The interaction map $T_{\mathfrak{P}_{z_1}} : \mathcal{H}^{\otimes 2} \to \mathcal{H}$ acting on $w \otimes v$ is $\lambda^{-L_0}Y(v, z_1)w$. $T_{\mathfrak{P}_{z_2}}$ sends $u \otimes \_ \in \mathbb{V} \otimes \mathbb{V}$ to $Y(u, z_2)\lambda^{L_0}\_$. The composition of these two expressions, evaluated with $w' \in \mathbb{V}$, is again the right hand side of (4.5). And the result of sewing is again $\mathfrak{P}_{z_1,z_2}$. So (4.5) holds in general.




## 4.6

According to the physical definition of $\mathbb{V}$ in Subsec. 1.12 as well as the No-Ambiguity Principle 2.10, we know that when the vectors of $\mathbb{V}$ are inserted, the correlation functions change holomorphically with respect to the translation of the marked points and their local coordinates. Thus $T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, v')$ is a holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times) = \{(z_1, z_2) \in \mathbb{C}^\times : z_1 \neq z_2\}$. Since, similar to (4.5), we also have

$$T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, w') = \langle w', Y(v, z_1)Y(u, z_2)w\rangle \qquad (\text{if } 0 < |z_2| < |z_1| < +\infty), \tag{4.6}$$

we conclude that $\langle w', Y(u, z_2)Y(v, z_1)w\rangle$ defined on $0 < |z_1| < |z_2|$ and $\langle w', Y(v, z_1)Y(u, z_2)w\rangle$ defined on $0 < |z_2| < |z_1|$ can be continued to the same holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times)$. That this fact is true for all $w, w' \in \mathbb{V}$ (or more generally, all $w \in \mathbb{V}, w' \in \mathbb{V}'$ if $\mathbb{V} \simeq \mathbb{V}'$ is not assumed) is simply written as

$$Y(u, z_2)Y(v, z_1) \sim Y(v, z_1)Y(u, z_2). \tag{4.7}$$

This property is called **commutativity**.

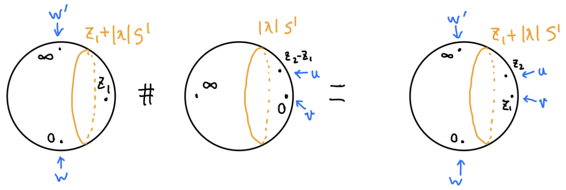
## 4.7

We now consider the sewing of

$$\mathfrak{P}_{z_1} = (\mathbb{P}^1_1; 0, z_1, \infty; \zeta, \zeta - z_1, \zeta^{-1}), \qquad \mathfrak{P}_{z_2-z_1} = (\mathbb{P}^1_{21}; 0, z_2 - z_1, \infty; \zeta, \zeta - z_2 + z_1, \zeta^{-1})$$

(where $\mathbb{P}^1_{21} = \mathbb{P}^1$) along the points $z_1 \in \mathbb{P}^1_1$ and $\infty \in \mathbb{P}^1_{21}$. We assume $0 < |z_2 - z_1| < |z_1| < +\infty$. Choose $\lambda \in \mathbb{C}$ satisfying $|z_2 - z_1| < |\lambda| < |z_1|$. Replace the local coordinate $\zeta - z_1$ of $\mathfrak{P}_{z_1}$ by $\lambda^{-1}(\zeta - z_1)$ and the one $\zeta^{-1}$ of $\mathfrak{P}_{z_2-z_1}$ by $\lambda\zeta^{-1}$. Then Assumption 4.3 is satisfied. The rule for sewing is identifying $\gamma_1 \in \mathbb{P}^1_1, 0 < |\lambda^{-1}(\gamma_1 - z_1)| < +\infty$ with $\gamma_{21} \in \mathbb{P}^1_{21}, 0 < |\lambda\gamma_{21}^{-1}| < +\infty$ if and only if $(\gamma_1 - z_1) = \gamma_{21}$. Thus, gluing $\mathfrak{P}_{z_1-z_1}$ to $\mathfrak{P}_{z_1}$

amounts to translating $\mathfrak{P}_{z_2-z_1}$ to $\mathfrak{P}_{z_2}$. After sewing, the points $0$ and $z_2-z_1$ of $\mathfrak{P}_{z_2-z_1}$ become $z_1$ and $z_2$. (The points $z_1$ of $\mathfrak{P}_{z_1}$ and $\infty$ of $\mathfrak{P}_{z_2-z_1}$ are discarded.)



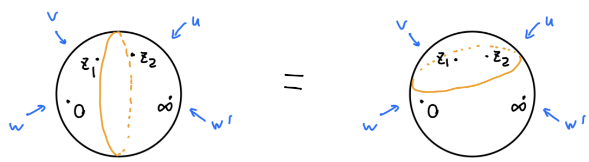
This sewing picture gives

$$T_{\mathfrak{P}_{z_1,z_2}}(w,v,u,w')=\langle w',Y(Y(u,z_2-z_1)v,z_1)w\rangle\quad(\text{if } 0<|z_2-z_1|<|z_1|<+\infty).\tag{4.8}$$

We therefore have the **associativity** property

$$\begin{gathered}\langle w',Y(u,z_2)Y(v,z_1)w\rangle=\langle w',Y(Y(u,z_2-z_1)v,z_1)w\rangle\\ \text{if } 0<|z_2-z_1|<|z_1|<|z_2|.\end{gathered}\tag{4.9}$$

Geometrically, it means the equivalence of sewing spheres in the following way:



## 4.8

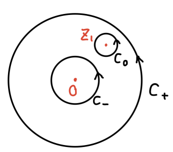
The fact that for all $u,v,w\in\mathbb{V},w'\in\mathbb{V}'$, (4.5), (4.6), and (4.8) can be defined as holomorphic functions of $z_1,z_2$ on the given domain (the precise meaning will be given later), and that these three expressions can be extended to the same holomorphic function (namely $T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$) on $\mathrm{Conf}^2(\mathbb{C}^\times)$ is called the **Jacobi identity** in the complex analytic form. Roughly speaking,

$$\text{Jacobi identity}=\text{Commutativity}+\text{Associativity}.\tag{4.10}$$

For the moment, we will derive an algebraic version, and use it as the formal definition of Jacobi identity in Def. 3.1.

Write $f(z_1,z_2)=T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$. Fix $z_1\in\mathbb{C}^\times$, and consider $f$ as a holomorphic function of $z_2$ on $\mathbb{C}^\times\backslash\{z_1\}$. (Moreover, from (4.5), (4.6), (4.8), and by the lower truncation property (3.15), it is easy to see that $f$ has finite poles at $z_1=0,z_2,\infty$. So $f$ is a meromorphic function.) By the residue theorem, for each meromorphic 1-form $\mu$ on $\mathbb{P}^1$ with possible poles only at $0,z_1,\infty$, we must have $(\mathrm{Res}_{z_2=0}+\mathrm{Res}_{z_2=z_1}+\mathrm{Res}_{z_2=\infty})f\mu=0$. It is easy to see that such $\mu$ are linear combinations of those of the form $z_2^m(z_2-z_1)^n dz_2$.

Equivalently, choose $C_+$ to be a circle around $0$ whose radius is $>|z_1|$, $C_-$ is one around $0$ whose radius is $<|z_1|$, and $C_0$ a small circle around $z_1$ between $C_+$ and $C_-$.



Let $f_+, f_-, f_0$ be respectively the right hand sides of (4.5), (4.6), (4.8). Then, when $z_2$ is on $C_+, C_-, C_0$ respectively, $f$ equals $f_+, f_-, f_0$. Then the fact that $f_+, f_-, f_0$ defined on their domains extend to the same meromophic function on $\mathbb{P}^1$ with poles $0, z_1, \infty$ implies for any $m, n \in \mathbb{Z}$ and $\mu = z_2^m (z_2 - z_1)^n dz_2$ that

$$\oint_{C_+} \frac{f_+\mu}{2\mathbf{i}\pi} - \oint_{C_-} \frac{f_-\mu}{2\mathbf{i}\pi} = \oint_{C_0} \frac{f_0\mu}{2\mathbf{i}\pi}. \tag{4.11}$$

Indeed, the latter one also implies the previous one. This is guaranteed by the so called *strong residue theorem*, which will be discussed in later sections. The strong residue theorem will imply that the analytic form and the algebraic form of Jacobi identity are equivalent.

Recall the general formula $\oint_C Y(u,z) z^k \frac{dz}{2\mathbf{i}\pi} = Y(u)_k$ if $C$ is a circle around the origin. When $z_2 \in C_+$, $\mu$ has absolutely convergent expansion $\mu = \sum_{l \in \mathbb{N}} \binom{n}{l} (-z_1)^l z_2^{m+n-l} dz_2$. So

$$\begin{aligned}
&\oint_{C_+} \frac{f_+\mu}{2\mathbf{i}\pi} = \sum_{l\in\mathbb{N}} \oint_{C_+} \binom{n}{l} (-z_1)^l z_2^{m+n-l} \langle w', Y(u,z_2) Y(v,z_1) w\rangle \frac{dz_2}{2\mathbf{i}\pi} \\
=& \sum_{l\in\mathbb{N}} \binom{n}{l} (-z_1)^l \langle w', Y(u)_{m+n-l} Y(v,z_1) w\rangle =: a(z_1)
\end{aligned}$$

When $z_2 \in C_-$,

$$\begin{aligned}
&\oint_{C_-} \frac{f_-\mu}{2\mathbf{i}\pi} = \sum_{l\in\mathbb{N}} \oint_{C_+} \binom{n}{l} (-z_1)^{n-l} z_2^{m+l} \langle w', Y(v,z_1) Y(u,z_2) w\rangle \frac{dz_2}{2\mathbf{i}\pi} \\
=& \sum_{l\in\mathbb{N}} \binom{n}{l} (-z_1)^{n-l} \langle w', Y(v,z_1) Y(u)_{m+l} w\rangle =: b(z_1)
\end{aligned}$$

When $z_2 \in C_0$, since $0 < |z_2 - z_1| < |z_1|$, we have the absolutely convergent expansion $\mu = (z_1 + (z_2 - z_1))^m (z_2 - z_1)^n dz_2 = \sum_{l\in\mathbb{N}} \binom{m}{l} z_1^{m-l} (z_2 - z_1)^{n+l} dz_2$. So

$$\begin{aligned}
&\oint_{C_0} \frac{f_0\mu}{2\mathbf{i}\pi} = \sum_{l\in\mathbb{N}} \oint_{C_0} \binom{m}{l} z_1^{m-l} (z_2 - z_1)^{n+l} \langle w', Y(Y(u, z_2 - z_1) v, z_1) w\rangle \frac{dz_2}{2\mathbf{i}\pi} \\
=& \sum_{l\in\mathbb{N}} \binom{m}{l} z_1^{m-l} \langle w', Y(Y(u)_{n+l} v, z_1) w\rangle := c(z_1)
\end{aligned}$$

Now we have $c(z_1) = a(z_1) - b(z_1)$. We vary $z_1$. For each $k \in \mathbb{Z}$, multiply both sides by $z_1^k \frac{dz_1}{2\mathbf{i}\pi}$ and apply the residue at $z_1 = 0$. We then get (by suppressing $w'$ and $w$)

**Definition 4.5 (Jacobi identity** (algebraic version)**).** For each $u, v, w \in \mathbb{V}$, and each $m, n, k \in \mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{l\in\mathbb{N}} \binom{m}{l} Y\big(Y(u)_{n+l} v\big)_{m+k-l} \\
=& \sum_{l\in\mathbb{N}} (-1)^l \binom{n}{l} Y(u)_{m+n-l} Y(v)_{k+l} - \sum_{l\in\mathbb{N}} (-1)^{n+l} \binom{n}{l} Y(v)_{n+k-l} Y(u)_{m+l}.
\end{aligned} \tag{4.12}$$

This completes Definition 3.1.

In the above three terms, when acting on every $w \in \mathbb{V}$, each sum over $l \in \mathbb{N}$ is finite thanks to the lower truncation property.

# 5 Consequences of Jacobi identity; reconstruction theorem

## 5.1

The algebraic form of Jacobi identity is very complicated. Very few people can write down exactly the right formula without checking the references or reproving this formula using the long argument in Subsec. 4.8. But we shall try our best to explain how to use this formula and what this formula implies.

First of all, if (4.12) holds whenever $m = 0$ or $n = 0$, then it holds in general. We will not give a rigorous proof for this statement. But, since (4.12) is derived from (4.11) for all $\mu = z_2^m(z_2 - z_1)^n dz_2$, the readers can be convinced of this statement by the following elementary fact:

**Exercise 5.1.** Show that $z_2^m(z_2 - z_1)^n$ is a $\mathbb{C}[z_1^{\pm 1}]$-linear combination of $z_2^k$ and $(z_2 - z_1)^l$ where $k, l \in \mathbb{Z}$ and $l < 0$. (Hint: Assume without loss of generality that $m, n < 0$. Prove the statement by induction on $|m|$ and $|n|$.)

Thus, we may understand (4.12) by restricting to the special cases $m = 0, n < 0$ or $n = 0$.

## 5.2

We now return to rigorous mathematics. Consider the case that $n = 0$, i.e., $\mu = z_2^m dz_2$. Then (4.12) reads

$$\left[Y(u)_m, Y(v)_k\right] = \sum_{l \in \mathbb{N}} \binom{m}{l} Y\left(Y(u)_l v\right)_{m+k-l}. \tag{5.1}$$

This is a Lie bracket relation. Interestingly, this general formula does not come from Lie groups, but from the residue theorem. However, in many concrete examples, such Lie bracket relations do have Lie-theoretic origins.

Let me take this chance to say a few words about the similarity and the difference between the VOA theory and the Lie theory. In the VOA theory, the residue theorem is the standard way of passing from the complex analytic world to the algebraic world. The opposite direction is through the strong residue theorem. This is strikingly different from the Lie theory, in which one passes from the differential geometric formulation (i.e. Lie groups) to the algebraic one (i.e. Lie algebras) by taking derivatives, and vice versa by taking exponentiation/integral. Thus, although Lie brackets do appear in VOAs, it is not always fruitful to think of VOAs as generalizations of Lie algebras. These two mathematical objects have very different geometric intuitions. Also, if we view VOAs in the complex analytic way, then by (4.10), VOAs are more like commutative algebras. Thus, VOAs can be viewed as a quantum version of both the Lie algebras and the commutative algebras.

## 5.3

Take $u$ to be the conformal vector $\mathbf{c}$ in (5.1) and recall that $Y(\mathbf{c})_{m+1} = L_m$. We obtain

$$
\begin{aligned}
[L_m, Y(v)_k] &= \sum_{l \in \mathbb{N}} \binom{m+1}{l} Y(L_{l-1}v)_{m+k+1-l} \\
&= Y(L_{-1}v)_{m+k+1} + \sum_{l \in \mathbb{N}} \binom{m+1}{l+1} Y(L_l v)_{m+k-l}.
\end{aligned} \tag{5.2}
$$

Multiply $z^{-k-1}$ to both sides and take the sum over all $k \in \mathbb{Z}$, we obtain

$$
[L_m, Y(v,z)] = z^{m+1} Y(L_{-1}v, z) + \sum_{l \in \mathbb{N}} \binom{m+1}{l+1} z^{m-l} Y(L_l v, z) \tag{5.3}
$$

either on the level of $\mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$, or as Laurent polynomials of $z$ when evaluating between any $w \in \mathbb{V}$ and $w' \in \mathbb{V}'$. Then the cases $m = -1$ and $m = 0$ imply

$$
[L_{-1}, Y(v,z)] = Y(L_{-1}v, z) \tag{5.4a}
$$

$$
[L_0, Y(v,z)] = zY(L_{-1}v, z) + Y(L_0 v, z). \tag{5.4b}
$$

Note that these two equations follow solely from the Jacobi identity. By the translation property, we have

$$
Y(L_{-1}v, z) = \frac{d}{dz} Y(v,z). \tag{5.5}
$$

Equivalently, by applying $\mathrm{Res}_{z=0}(\cdot) z^n dz$, we get a crucial relation

$$
Y(L_{-1}v)_n = -nY(v)_{n-1}. \tag{5.6}
$$

(The quickest way to get the formula on the right hand side is integration by parts.)

**Exercise 5.2.** Show that (3.39) follows from (5.6) and the creation property.

**Exercise 5.3.** Assume that $\mathbb{V}$ satisfies the lower truncation property (3.15) and all the axioms of VOAs in Def. 3.1 except the grading and the translation property. Use (5.4) to prove that the following conditions are equivalent.

1. The grading property.
2. $Y(L_{-1}v, z) = \partial_z Y(v,z)$ for all $v \in \mathbb{V}$.
3. The translation property.
4. The translation property without assuming $L_{-1}\mathbf{1} = 0$.

Thus, we may use the lower truncation property and any of these four conditions to replace the grading and the translation properties in the definition of VOAs.

**Exercise 5.4.** In (5.2), set $v = \mathbf{c}$, and show that this formula is compatible with the Virasoro relation.

### 5.4

We see that (5.3) for $m = 0, -1$ (together with (5.5)) means the grading and the translation properties, which integrate to the rotation and the translation covariance. For general $m$, (5.3) also has a geometric explanation. To simplify discussions, we give such an explanation by assuming that $v$ is primary.

**Definition 5.5.** A vector $v \in \mathbb{V}$ is called a **primay vector** if it is homogeneous and $L_n v = 0$ for all $n > 0$.

Some important VOAs (affine VOAs for instance) are generated by primary vectors. And many important formulas in CFT were first proved by physics who assumed that their theories are generated by primary vectors in the following sense:

**Definition 5.6.** We say that a VOA $\mathbb{V}$ is **generated** by a subset $E \subset \mathbb{V}$ if $\mathbb{V}$ is spanned by vectors of the form $Y(v_1)_{n_1} \cdots Y(v_k)_{n_k} \mathbf{1}$ where $k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{Z}$, and $v_1, \dots, v_k \in E$.

Indeed, formula (5.3) for any primary vector $v$ is one such example, which (combined with (5.5)) reads

$$[L_m, Y(v,z)] = z^{m+1}\partial_z Y(v,z) + (m+1)\mathrm{wt}v \cdot z^m Y(v,z). \tag{5.7}$$

This is called by physicists (or more precisely, is equivalent to what physicists call) the **conformal Ward identity**.

Choose a holomorphic vector field $f(z)\partial_z = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial_z$ on a neighborhood of $\mathbb{S}^1$. Let $\sigma_\tau = \exp(\tau f \partial_z)$ be the holomorphic flow. Then (5.7) (with $L_m, z^m$ replaced by $\sum_m a_m L_m, \sum_m a_m z_m$) integrates to

$$e^{\tau \sum_{n\in\mathbb{Z}} a_n L_n} Y(v,z) e^{-\tau \sum_{n\in\mathbb{Z}} a_n L_n} = \big(\partial_z \sigma_\tau(z)\big)^{\mathrm{wt}v} Y\big(v, \sigma_\tau(z)\big), \tag{5.8}$$

called **conformal covariance**. For now, we do not treat this formula in a rigorous way. But the readers can convince themselves by checking that both sides satisfy the same linear differential equation over $\tau$.

The right hand side of (5.8) looks familiar to us. Set $\tau = 1$, $\sigma = \sigma_1$, and $\Delta = \mathrm{wt}v$. Then formula (5.8) resembles the change of variable formula $\big(\partial(\varphi\circ\sigma)\big)^\Delta = \big(\partial\varphi\circ\sigma\big)^\Delta \cdot (\partial_z\sigma)^\Delta$ for a function $\varphi = \varphi(z)$ and $\partial$ is the standard holomorphic derivative. Indeed, the primary field $Y(v,z)$ can be viewed as the quantization of $(\partial\varphi)^\Delta$, or more generally, of $\partial\varphi_1 \cdots \partial\varphi_\Delta$. It is also interesting to write (5.8) in the form

$$e^{\sum a_n L_n}\big(Y(v,z)dz^\Delta\big)e^{-\sum a_n L_n} = Y(v,\sigma)d\sigma^\Delta. \tag{5.9}$$

Conformal covariance (5.8) can be interpreted in a similar geometric way as we did for rotation and translation covariance in Subsec. 3.14 and 3.15. (We will give this explanation in the future assuming $f = \sum_{n\geqslant 0} a_n z^{n+1}\partial_z$.) So, from the CFT point of view, this formula follows naturally from our change of parametrization formula in Sec. 2 and the physical definition of the vertex operator $Y(v,z)$ in Sec. 3 (if we ignore the issue of uniqueness up to scalar multiplications). In particular, the geometric intuition we are using for formula (5.7) is Lie theoretic, because the relationship between Virasoro algebras and change of parametrization formula is the one between the representations of Lie algebras and Lie groups. But we have also derived (5.7) from the

Jacobi identity, whose geometric intuition relies on the residue theorem. How should we view this coincidence of the two geometric pictures?

My answer is that we should regard the Lie theoretic explanation as the fundamental one for conformal covariance/Ward identity. In fact, to use the Jacobi identity to obtain (5.7), we have assumed that $\sum L_n z^{-n-2}$ is the vertex operator of a vector of $\mathbb{V}$, namely the conformal vector $\mathbf{c}$. But the reason that this assumption should be included in the definition of VOA was not explained in Sec. 3. Here we give a short explanation: we will see later (cf. the reconstruction Thm. 5.12 and Rem. 5.13) that if the Fourier modes $A_m \in \mathrm{End}(\mathbb{V})$ of a field $A(z)$ satisfy the correct Jacobi identity (such as (5.1) or (5.7)) with the modes $Y(v)_k$ for $v$ inside a generating subset $E \subset \mathbb{V}$, then $A(z)$ must be $Y(u, z)$ for some $u \in \mathbb{V}$. Thus, (in my opinion) the better point of view is that we use the conformal Ward identity (whose geometric intuition relies on the change of parametrization formula and the physical meaning of $Y(u, z)$) and the Jacobi identity to explain the fact that $\sum L_n z^{-n-2}$ is represented by a vector $\mathbf{c}$ in $\mathbb{V}$, but not that we explain the Ward identity using the VOA Jacobi identity.

## 5.5

We say that $\mathbb{V}$ is of **CFT-type** if $\dim \mathbb{V}(n) < +\infty$ for each $n$, and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$. The CFT-type condition is a very natural and mild one satisfied by all the examples in our notes. It says that the only quantum states with zero energy are the vacuum.

In this subsection, we assume $\mathbb{V}$ is CFT-type, and study (5.1) for vectors in $\mathbb{V}(1)$. For each $u \in \mathbb{V}(1)$, we write $Y(u)_m$ as $u_m$ for short. By (3.10), $u_l$ lowers the weights by $l$. Then (5.1) says $[u_m, v_n] = (u_0 v)_{m+n} + m(u_1 v)_{m+n-1}$, where $u_l v$ vanishes when $l > 1$ since its weight is $1 - l$. Since $u_1 v \in \mathbb{V}(0) \in \mathbb{C}$, we may write

$$u_1 v = (u, v)\mathbf{1} \tag{5.10}$$

where $(\cdot, \cdot)$ is a bilinear form on $\mathbb{V}(1)$. Thus $(u_1 v)_{m+n-1} = (u, v)\delta_{m,-n}$ since $Y(\mathbf{1}, z) = \mathbf{1}$. Set

$$[u, v] := u_0 v. \tag{5.11}$$

Then

$$[u_m, v_n] = [u, v]_{m+n} + m(u, v)\delta_{m,-n}. \tag{5.12}$$

**Proposition 5.7.** $[\cdot, \cdot]$ *defines a Lie algebra structure on* $\mathbb{V}(1)$*, and* $(\cdot, \cdot)$ *is an invariant symmetric bilinear form, namely,* $(u, v) = (v, u)$ *and* $([w, u], v) = -(u, [w, v])$.

*Proof.* $w \in \mathbb{V}(1) \mapsto w_{-1}$ is injective since $w_{-1}\mathbf{1} = w$ by the creation property. By (5.12), $[u, v]_{-1} = [u_0, v_{-1}] = -[v_{-1}, u_0] = -[v, u]_{-1}$. This proves $[u, v] = -[v, u]$. By calculating $[u_1, v_{-1}]$ and $[v_{-1}, u_1]$ using (5.12), we obtain $(u, v) = (v, u)$. (5.12) implies

$$[w_k, [u_m, v_n]] = [w, [u, v]]_{k+m+n} + k(w, [u, v])\delta_{k+m+n,0}.$$

Apply the Jacobi identity for the Lie bracket of linear operators, we obtain the Jacobi identity for $[\cdot, \cdot]$ on $\mathbb{V}(1)$ if we set $k = -1, m = n = 0$, and we obtain the invariance of $(\cdot, \cdot)$ if we set $k = 0, m = 1, n = -1$. $\square$

The vector space $\mathrm{Span}_{\mathbb{C}}\{v_n, \mathbf{1}_{\mathbb{V}} : n \in \mathbb{Z}\}$ is a Lie algebra whose bracket is the standard one for linear operators. Since it satisfies (5.12), we call it an **affine Lie algebra** associated to the finite-dimensional complex Lie algebra $\mathbb{V}(1)$. When $\mathbb{V}$ is generated by $\mathbb{V}(1)$, we say $\mathbb{V}$ is an **affine VOA**.

We are mostly interested in the case that $(\cdot,\cdot)$ is non-degenerate. This is always true when the CFT (or the VOA) is unitary, since $(\cdot,\cdot)$ is indeed the negative of the correlation function $\langle\cdot,\cdot\rangle = \langle\Theta\cdot|\cdot\rangle$ of $A_{1,1}$ restricted to $\mathbb{V}^{\otimes 2}$. Moreover, a unitary affine VOA $\mathbb{V}$ is indeed uniquely determined by its Lie subalgebra $\mathbb{V}(1)$, where $\mathbb{V}(1)$ is a direct sum of an abelian Lie algebra and a semisimple one. (We refer the readers to [Gui19, Sec. 1 and 2] for a detailed account of the relationship between unitary VOAs and their "unitary" Lie subalgebras $\mathbb{V}(1)$.) Affine Lie algebras and affine VOAs in the strict sense are those such that $\mathbb{V}(1)$ are simple Lie algebras. If on the other hand $\mathbb{V}(1)$ is abelian, then $\mathbb{V}$ is called a **free boson VOA** or a **Heisenberg VOA**.

If $\mathbb{V}$ is generated by $\mathbf{c}$, we call $\mathbb{V}$ a **Virasoro VOA**.

## 5.6

We now turn to the case $m = 0, n < 0$ in the VOA Jacobi identity (4.12). First consider $n = -1$. Then (4.12) reads

$$Y\big(Y(u)_{-1}v\big)_k = \sum_{l\in\mathbb{N}} Y(u)_{-1-l}Y(v)_{k+l} + \sum_{l\in\mathbb{N}} Y(v)_{k-1-l}Y(u)_l. \tag{5.13}$$

This formula can be written in a compact way. For a general series $f(z) = \sum_{l\in\mathbb{Z}} a_l z^{-l-1} \in W[[z^{\pm 1}]]$ where $W$ is a vector space, we let

$$f(z)_+ = \sum_{l\in\mathbb{N}} a_l z^{-1-l}, \qquad f(z)_- = \sum_{l\in\mathbb{N}} a_{-l-1} z^l \tag{5.14}$$

(so we have $f(z) = f(z)_+ + f(z)_-$). Define the **normal-ordered product**

$$:\!Y(u,z)Y(v,z)\!: = Y(u,z)_- Y(v,z) + Y(v,z)Y(u,z)_+ \tag{5.15}$$

which is non-commutative in general. Then (5.13) can be abbreviated to

$$Y\big(Y(u)_{-1}v, z\big) = \,:\!Y(u,z)Y(v,z)\!: \tag{5.16}$$

By (5.6) we have

$$Y(u)_{-j-1} = \frac{1}{j!} Y(L_{-1}^j u)_{-1} \tag{5.17}$$

when $j \geqslant 0$. Combine this with $Y(L_{-1}^j u, z) = \partial_z^j Y(u,z)$, we obtain

$$Y\big(Y(u)_{-j-1}v, z\big) = \frac{1}{j!}:\!\big(\partial_z^j Y(u,z)\big)Y(v,z)\!: \tag{5.18}$$

where the normal-ordered product is defined in a similar way using the positive and the negative parts of $\partial_z^j Y(u,z)$. We leave it to the readers to check that this formula agrees with the Jacobi identity (4.12) when $m = 0, n < 0$.

Thus, once we know how $Y(u,z)$ looks like for all $u$ in a small generating subset $E$ of $\mathbb{V}$, we can write down the formula of $Y(w,z)$ for any $w\in\mathbb{V}$ using the formula

$$Y\big(Y(u_1)_{-j_1-1}\cdots Y(u_k)_{-j_k-1}v,z\big)=\frac{1}{j_1!\cdots j_k!}{:}\partial_z^{j_1}Y(u_1,z)\cdots\partial_z^{j_k}Y(u_k,z)\cdot Y(v,z){:} \tag{5.19}$$

where the normal-ordered product for several operators is defined inductively by

$${:}A_1A_2\cdots A_n{:}={:}A_1({:}A_2\cdots A_n{:}){:} \tag{5.20}$$

## 5.7

One can also write down the explicit formula of $Y(Y(u)_nv,z)$ for $n\geqslant 0$ using (4.12) where $m=0,n\geqslant 0$. But as I said, (4.12) is determined by the special cases $m=0,n<0$ and $n=0$. So we hope that $Y(Y(u)_nv,z),n\geqslant 0$ can be calculated using (5.1). This is true.

Write (5.1) in the equivalent form

$$\big[Y(u)_m,Y(v,z)\big]=\sum_{l\in\mathbb{N}}\binom{m}{l}z^{m-l}Y\big(Y(u)_lv,z\big). \tag{5.21}$$

Thus, for $m\geqslant 0$, $Y(Y(u)_mv,z)$ can be computed inductively by

$$\begin{gathered}Y\big(Y(u)_0v,z\big)=\big[Y(u)_0,Y(v,z)\big]\\ Y\big(Y(u)_mv,z\big)=\big[Y(u)_m,Y(v,z)\big]-\sum_{l=0}^{m-1}\binom{m}{l}z^{m-l}Y\big(Y(u)_lv,z\big).\end{gathered} \tag{5.22}$$

We now see the close relation between the Lie brackets of vertex operators and the data $Y\big(Y(u)_mv,z\big),m\geqslant 0$. The latter plays a very different role from $Y\big(Y(u)_mv,z\big),m<0$. To understand this relation better, we write the associativity relation (4.9) as

$$Y(u,z_2)Y(v,z_1)=\sum_{m\in\mathbb{Z}}(z_2-z_1)^{-m-1}Y(Y(u)_mv,z_1) \tag{5.23}$$

when $0<|z_2-z_1|<|z_1|$. Here, we understand $Y(u,z_2)Y(v,z_1)$ as $Y(v,z_1)Y(u,z_2)$ when $0<|z_1|<|z_2|$ or more generally, as a linear functional on $\mathbb{V}^{\otimes 2}$ sending $w\otimes w'$ to $T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$ (the correlation function associated to (4.4)) for all $(z_1,z_2)\in\mathrm{Conf}^2(\mathbb{C}^\times)$. Then the part $m\geqslant 0$ in (5.23) accounts for the poles of $T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$ at $z_2=z_1$.

The summand in (5.23) vanishes for sufficiently positive $m$. In physics, a series expansion of the form

$$A(z_2)B(z_1)=\sum_{m\geqslant -N}(z_2-z_1)^mC^m(z_1)$$

is called the **operator product expansion (OPE)** of the fields $A(z_2),B(z_1)$. Thus, in the VOA context, *OPEs are just the associativity property* (4.9). OPE is useful to physicists because it allows them to reduce the calculation of $4$-point correlations functions to that of $3$-point ones, or in general, $N$-point to $(N-1)$-point.

We split the right hand side of (5.23) into two parts: $m \geqslant 0$, which is called the **regular terms** since it has no poles at $z_2 = z_1$, and $m < 0$ called the **singular terms**. Thus

$$Y(u, z_2)Y(v, z_1) = \frac{Y(Y(u)_{N-1}v, z_1)}{(z_2 - z_1)^N} + \cdots + \frac{Y(Y(u)_0 v, z_1)}{(z_2 - z_1)} + \text{regular terms},$$

or, written in physics language,

$$Y(u, z_2)Y(v, z_1) \sim \frac{Y(Y(u)_{N-1}v, z_1)}{(z_2 - z_1)^N} + \cdots + \frac{Y(Y(u)_0 v, z_1)}{(z_2 - z_1)}. \tag{5.24}$$

Thus, to summarize, (5.1) *establishes a close relationship between the Lie brackets of vertex operators, the finite poles of the correlation function $T_{\mathfrak{P}_{z_1,z_2}}$ at $z_1 = z_2$, and the finitely may singular terms in the OPE of vertex operators*. As a special case, from (5.21) and (5.22) one sees that two vertex operators $Y(u, z_2), Y(v, z_1)$ commute (namely, their Fourier modes $Y(u)_m, Y(v)_k$ commute) iff there are no singular terms in the OPE of $Y(u, z_2)Y(v, z_1)$, iff $T_{\mathfrak{P}_{z_1,z_2}}(\cdot \otimes v \otimes u \otimes \cdot)$ is holomorphic on a neighborhood of $z_2 = z_1$.

## 5.8

In the previous subsection, we derived the relationship from the definition of VOAs (in particular, from the VOA Jacobi identity). So one may ask this natural question: does this relationship rely on the full Jacobi identity? For instance, does it rely on (5.18)?

The answer is no. In a very vague sense, any of the following three implies the others without assuming the full Jacobi identity.

1. Suitable Lie bracket relations hold for a pair of field operators $A(z_2), B(z_1)$.

2. The finite poles of (the analytic continuation of) $\langle w', A(z_2)B(z_1)w\rangle$ at $z_2 = z_1$.

3. The finitely many singular terms in the OPE of $A(z_2)B(z_1)$ and, in particular, *the existence of such OPE*.

Clearly, the third one a priori implies the second one, since the second does not assume the existence of OPE. Thus, as we have said that OPEs are roughly the same as associativity, we see that the associativity (and indeed, the full Jacobi identity) can be derived from the first or the second statement above. This is called the **reconstruction theorem** because it allows us to build examples of VOAs by checking only a small part of the Jacobi identity, namely the Lie bracket relations. This theorem is the most important one for constructing examples of VOAs.

A rigorous and detailed discussion of the equivalence of the above three statements will be given in the later section on local fields. The first and the second statements correspond to three seeming different but indeed equivalent definitions of the **locality** of $A(z_2), B(z_1)$. (There are two ways to describe the second one, a formal variable way and a complex analytic way.) Here, we first state the rigorous definition of the first one.

## 5.9

We let $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ be an $\mathbb{N}$-graded vector space, graded by a diagonalizable operator $L_0$. We do not assume that $\mathbb{V}$ and $L_0$ are from any graded vertex algebra.

**Definition 5.8.** An **($L_0$-)homogeneous field (operator)** on $\mathbb{V}$ is an element

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$$

(where each $A_n$ is in $\mathrm{End}(\mathbb{V})$) satisfying

$$[L_0, A(z)] = \Delta_A \cdot A(z) + z\partial_z A(z) \tag{5.25}$$

or equivalently,

$$[L_0, A_n] = (\Delta_A - n - 1)A_n \qquad (\forall n \in \mathbb{Z}). \tag{5.26}$$

$\Delta_A$ is called the **weight** of $A(z)$.

Clearly, a homogeneous field $A(z)$ satisfies the **lower truncation property** $A(z)w \in \mathbb{C}((z))$ (for all $w \in \mathbb{V}$).

**Definition 5.9** (**Local fields** (Lie algebraic version))**.** Given homogeneous fields $A(z)$ and $B(z)$, we say $A(z)$ is **local** to $B(z)$ if there exist $C^j(z) = \sum_{n\in\mathbb{Z}} C^j_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$ (where $j = 0, 1, \ldots, N-1$ for some $N \in \mathbb{N}$) satisfying

$$[A_m, B_k] = \sum_{l=0}^{N-1} \binom{m}{l} C^l_{m+k-l} \tag{5.27}$$

for all $m, k \in \mathbb{Z}$. We consider the right hand side of (5.27) as 0 if $N = 0$.

**Remark 5.10.** $A(z)$ is local to $B(z)$ if and only if there exist $D^0(z), \ldots, D^{N-1}(z) \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$ satisfying for all $m, k \in \mathbb{Z}$ that

$$[A_m, B_k] = \sum_{l=0}^{N-1} m^l D^l_{m+k}. \tag{5.28}$$

This is because $\widetilde{C}^l_j := C^l_{j-l}$ and $D^l_j$ are related by $\widetilde{C}^l_j + \sum_{p=l+1}^{N-1} a_{p,l} \cdot \widetilde{C}^p_j = D^l_j$ where each $a_{p,l} \in \mathbb{R}$ is determined by $\binom{m}{p} = m^p + \sum_{l=1}^{p-1} a_{p,l} \cdot m^l$.

**Exercise 5.11.** Use (5.28) to show that if $A(z)$ is local to $B(z)$ then $B(z)$ is local to $A(z)$.

## 5.10

Roughly speaking, reconstruction theorem says that if we have a small set $\mathcal{E}$ of operators $A(z) \in \mathrm{End}(\mathbb{V})$ that generates $\mathbb{V}$ and satisfies all the axioms in the definition of graded vertex algebras/VOAs, except that the Jacobi identity is replaced by the weaker condition that the operators in $\mathcal{E}$ are mutually local, then the Jacobi identity is automatically satisfies, and hence $\mathbb{V}$ is a graded vertex algebra/VOA. This theorem will be proved in a later section.

**Theorem 5.12 (Reconstruction theorem).** *Let $\mathcal{E}$ be a set of $L_0$-homogeneous fields on $\mathbb{V}$. Assume that the following conditions are satisfied. Then $\mathbb{V}$ has a unique graded vertex algebra structure such that each $A(z) \in \mathcal{E}$ is a vertex operator (namely, is of the form $Y(u, z)$ for some $u \in \mathbb{V}$), and that the vacuum vector $\mathbf{1}$ and the operator $L_{-1}$ are those described in the following.*

- *Creation property: There is a distinguished vector $\mathbf{1} \in \mathbb{V}(0)$ such that $A(z)\mathbf{1}$ has no negative powers of $z$ for all $A(z) \in \mathcal{E}$.*
- *Translation property: There is a distinguished $L_{-1} \in \mathrm{End}(\mathbb{V})$ such that $L_{-1}\mathbf{1} = 0$, and that for each $A(z) \in \mathcal{E}$ we have $[L_{-1}, A(z)] = \partial_z A(z)$.*
- *Generating property: Vectors of the form $A^1_{n_1} \cdots A^k_{n_k}\mathbf{1}$ (where $k \in \mathbb{N}$, $A^1(z), \dots, A^k(z) \in \mathcal{E}$, and $n_1, \dots, n_k \in \mathbb{Z}$) span $\mathbb{V}$.*
- *Locality: Any two fields of $\mathcal{E}$ are local.*

*Moreover, if $L_0, L_{-1}$ can be extended to a sequence of operators $(L_n)_{n\in\mathbb{Z}}$ on $\mathbb{V}$ such that $\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ belongs to $\mathcal{E}$, and that the Virasoro relation (2.8) is satisfied for some $c \in \mathbb{C}$, then $\mathbb{V}$ is a VOA whose conformal vector $\mathbf{c}$ satisfies $Y(\mathbf{c}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.*

Note that the uniqueness of the graded vertex algebra/VOA structure follows directly from (5.19). The non-trivial part of this theorem is of course the existence of such structure.

**Remark 5.13.** The end of the reconstruction Thm. 5.12 means that in order to show that a graded vertex algebra $\mathbb{V}$ is a VOA, it suffices to show that $L_0, L_{-1}$ can be extended to $(L_n)_{n\in\mathbb{Z}}$ satisfying the Virasoro relation, that $T(z) = \sum L_n z^{-n-2}$ satisfies the creation property (namely, $L_n\mathbf{1} = 0$ for all $n \geqslant -1$), and that $T(z)$ is local with any field in $\mathcal{E}$ (by showing for instance the conformal Ward identity $[L_m, A(z)] = z^{m+1}\partial_z A(z) + \Delta_A \cdot z^m A(z)$ for all $A(z) \in \mathcal{E}$ if one expects that all $A(z)$ are "primary"). The translation property is automatically satisfied due to the Virasoro relation $[L_{-1}, L_n] = -(n+1)L_{n-1}$.

# 6 Constructing examples of VOAs

## 6.1

In the previous section, we have mentioned some important examples of VOAs: affine VOAs and Virasoro VOAs. But we didn't explain why they exist. This is the task of this section. The standard references for this section are [LL, Chapter 6] and [Was10] (with emphasis on the unitarity aspect).

The style of this section is different from the previous ones: it has a strong flavor of Lie theory. The methods in this section will not be used in the future (except when we discuss examples of VOA modules). So the readers can safely skip this section if they do not want to bother with the existence issue. (But they should at least read Subsec. 6.17 on tensor product VOAs.)

Our first class of examples are Virasoro VOAs, namely, those generated by the conformal vector $\mathbf{c}$. To begin with, the **Virasoro algebra** is a Lie algebra $\mathrm{Vir} = \mathrm{Span}_{\mathbb{C}}\{L_n, K : n \in \mathbb{Z}\}$ satisfying the bracket relation

$$\begin{gathered}[L_m, L_n] = (m-n)L_{m+n} + \frac{K}{12}(m+1)m(m-1)\delta_{m,-n},\\ [K, L_n] = 0.\end{gathered}$$

We know that any VOA must satisfy $L_n\mathbf{1} = 0$ for all $n \geqslant -1$. Motivated by this fact, we have:

**Proposition 6.1.** *Let $\mathbb{V}$ be a representation of $\mathrm{Vir}$ such that $L_0$ is diagonalizable and has $\mathbb{N}$-spectrum. Assume that $\mathbb{V}$ has a distinguished vector $\mathbf{1}$ killed by $L_n$ for all $n \geqslant -1$, that vectors of the form $L_{n_1}\cdots L_{n_k}\mathbf{1}$ (where $k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{Z}$) span $\mathbb{V}$, and that $K$ acts as a constant $c \in \mathbb{C}$. Then $\mathbb{V}$ has a unique natural structure of a Virasoro VOA. Its central charge is $c$.*

*Proof.* This follows immediately from the reconstruction Thm. 5.12. Note that by (5.28), $\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is local to itself due to the Virasoro relation. $\square$

### 6.2

Thus, it remains to construct Vir-modules satisfying the conditions in Prop. 6.1. Let us first find a "largest" such module. We expect that this module should have basis $L_{-n_1}\cdots L_{-n_k}\mathbf{1}$ where $n_1 \geqslant \cdots \geqslant n_k \geqslant 2$, because:

**Exercise 6.2.** Let $\mathbb{V}$ be as in Prop. 6.1. Prove by induction on $k$ that $L_{n_1}\cdots L_{n_k}\mathbf{1}$ (for any $n_1, \dots, n_k$) can be written as a linear combination of $L_{-m_1}\cdots L_{-m_l}\mathbf{1}$ where $l \in \mathbb{N}, m_1, \dots, m_l \geqslant 2$. (Hint: if $n_j \leqslant -2$, move $L_{n_j}$ to the rightmost by using the Virasoro relation.)

Now let us construct this largest module $V_{\mathrm{Vir}}(c, 0)$ for each $c \in \mathbb{C}$. Its basis consists of $(-n_1, \dots, -n_k)$ where $k \in \mathbb{N}$ and $n_1 \geqslant \cdots \geqslant n_k \geqslant 2$. The one with $k = 0$ is denoted by $\mathbf{1}$. If $n \geqslant n_1$, we simply define the action of $L_{-n}$ on each $(-n_1, \dots, -n_k)$ to be $(-n, -n_1, \dots, -n_k)$. But we also want to define the action of $L_n$ on $(-n_1, \dots, -n_k) = L_{-n_1}\cdots L_{-n_k}\mathbf{1}$ for all $n \in \mathbb{Z}$. In practice, we can write down the formula explicitly using the Virasoro relation. For instance: $L_0L_{-n_1}\cdots L_{-n_k}\mathbf{1} = (n_1 + \cdots + n_k)L_{-n_1}\cdots L_{-n_k}\mathbf{1}$, and

$$\begin{aligned}&L_3L_{-4}L_{-3}\mathbf{1} = [L_3, L_{-4}]L_{-3}\mathbf{1} + L_{-4}[L_3, L_{-3}]\mathbf{1}\\ =&7L_{-1}L_{-3}\mathbf{1} + 6L_{-4}L_0\mathbf{1} + 2cL_{-4}\mathbf{1} = (14 + 2c)L_{-4}\mathbf{1}.\end{aligned} \tag{6.2}$$

There is a natural question about this approach: how do we verify that such defined action of $\mathrm{Vir}$ on $V_{\mathrm{Vir}}(c, 0)$ preserves the Lie bracket relations of $\mathrm{Vir}$?

### 6.3

The standard way to deal with is issue is to use the **Poincaré–Birkhoff–Witt (PBW)** theorem, which says the following: Let $\mathfrak{g}$ be a Lie algebra (over any field). Let $U(\mathfrak{g})$ be its universal enveloping algebra, i.e., the largest unital associative algebra containing and generated by the vector space $\mathfrak{g}$ such that $xy - yx = [x, y]$ for all $x, y \in \mathfrak{g}$. If $E$ is a basis of $U(\mathfrak{g})$ with a total order $\leqslant$, then vectors of the form

$$x_1 x_2 \cdots x_k \qquad (k \in \mathbb{N}, x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k \in E) \tag{6.3}$$

(when $k = 0$, we understand this expression as $1$) form a basis of $U(\mathfrak{g})$.

The remarkable point about the PBW theorem is that if we define a vector space $V$ to have a basis of vectors as in (6.3), and if we define the action of $x \in \mathfrak{g}$ using the Lie bracket relations of $\mathfrak{g}$ (similar to the argument in (6.2)), then this gives a well defined action of $\mathfrak{g}$ on $V$ preserving the bracket relations of $\mathfrak{g}$, i.e., this gives a well defined representation of $\mathfrak{g}$.

To apply the PBW theorem to our construction of VOAs, we need the following result:

**Lemma 6.3.** *Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie subalgebras of $\mathfrak{g}$. Use the PBW theorem to show that there is an isomorphism of vector spaces $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \to U(\mathfrak{g})$ sending each $x_1 \cdots x_k \otimes y_1 \cdots y_l$ to $x_1 \cdots x_k y_1 \cdots y_k$ where $x_\bullet \in \mathfrak{g}_1, y_\bullet \in \mathfrak{g}_2$.*

The proof is an easy application of the PBW theorem, which we leave to the readers.

## 6.4

Consider the following Lie subalgebras of Vir:

$$V_- = \mathrm{Span}\{L_n : n \leqslant -2\}, \qquad V_+ = \mathrm{Span}\{K, L_n ; n \geqslant -1\}.$$

$\mathbb{C}_c = \mathbb{C}$ is a representation of $V_+$ if we let $L_n$ act as $0$ and $K$ as $c$. So $\mathbb{C}_c$ is also a $U(V_+)$-module. Now $U(\mathrm{Vir})$ is clearly a right $U(V_+)$-module. So

$$\mathrm{Ind}_{U(V_+)}^{U(V)} \mathbb{C}_c := U(\mathrm{Vir}) \otimes_{U(V_+)} \mathbb{C}_c$$

is a (left) $U(\mathrm{Vir})$-module, called the **induced representation** of $\mathbb{C}_c$. This is a Vir-module, and by Lemma 6.3, this vector space is isomorphic to $U(V_-) \otimes_{\mathbb{C}} U(V_+) \otimes_{(U(V_+))} \mathbb{C}_c \simeq U(V_-)$, which by the PBW theorem has a basis of vectors the form $L_{-n_1} \cdots L_{-n_k} \mathbf{1}$ where $\mathbf{1}$ is the unit $1$ and $n_1 \geqslant \cdots \geqslant n_k \geqslant 2$. So we can view $V_{\mathrm{Vir}}(c, 0)$ as $\mathrm{Ind}_{U(V_+)}^{U(V)} \mathbb{C}_c$. In particular, this proves that $V_{\mathrm{Vir}}(c, 0)$ carries a ntural structure of representation of Vir. Hence, by Prop. 6.1, $V_{\mathrm{Vir}}(c, 0)$ is a Virasoro VOA with central charge $c$.

**Exercise 6.4.** Find an explicit expression of $Y(L_{-4}\mathbf{c}, z)$ on $V_{\mathrm{Vir}}(c, 0)$ in terms of the Virasoro operators $L_n$.

## 6.5

$V_{\mathrm{Vir}}(c, 0)$ is not always an irreducible Vir-module. But the irreducible cases are the most interesting one. For instance, every CFT-type unitary VOA is irreducible. (See [CKLW18].)

The method of getting irreducible examples is quite standard in Lie theory: We shall take the largest quotient of $V_{\mathrm{Vir}}(c,h)$. To be more precise, note that for any proper $\mathrm{Vir}$-invariant subspace $W$ of $V_{\mathrm{Vir}}(c,h)$, note that $L_0$ is diagonalizable on $W$,[2] i.e., $W$ has a $L_0$-grading, whose lowest weight must not be $0$ since otherwise it contains $\mathbf{1}$ and hence must be $V_{\mathrm{Vir}}(c,0)$. Let $I$ be the span of all such $W$, then $I$ is the largest proper $\mathrm{Vir}$-subspace since $I$ has no non-zero weight-$0$ vectors. Then

$$L_{\mathrm{Vir}}(c,0) := V_{\mathrm{Vir}}(c,0)/I$$

is an irreducible $\mathrm{Vir}$-module, which is also a Virasoro VOA of CFT type by Prop. 6.1.

## 6.6

One may wonder when $L_{\mathrm{Vir}}(c,0)$ equals $V_{\mathrm{Vir}}(c,0)$, i.e., when $I$ is trivial. Indeed, $I$ is non-trivial if and only if

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \tag{6.4}$$

where $p,q \in \{2,3,4,\dots\}$ are relatively prime. (Cf. [LL, Rem. 6.1.13] and the reference therein.) In this case, $L_{\mathrm{Vir}}(c,0)$ is called a **minimal model**. It has finitely many irreducible modules. Minimal models are an important class of "rational" VOAs. More precisely: **rational and $C_2$-cofinite** VOAs. We will give precise meanings of these terms in later sections. The theory of conformal blocks for such VOAs is well-established.

It is a deep result that $L_{\mathrm{Vir}}(c,0)$ is a unitary $\mathrm{Vir}$-module if and only if $c \geqslant 1$ or $c$ satisfies (6.4) with $|p-q| = 1$, namely,

$$c = 1 - \frac{1}{m(m+1)} \tag{6.5}$$

for some integer $m \geqslant 2$. We refer the readers to [FMS, Chapter 8] and [Was10, Chapter IV] for details.

## 6.7

We now turn to affine VOAs. We fix a finite dimensional complex Lie algebra $\mathfrak{g}$ together with a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$. (Indeed, we will not use the non-degeneracy until we define the Virasoro operators.) Recall that invariance means

$$([X,Y],Z) = -(Y,[X,Z]). \tag{6.6}$$

[2] In general, if $D$ is a diagonalizable linear operator on a vector space $M$ and $W$ is an $D$-invariant subspace of $M$, then $D|_W$ is diagonalizable. To see this, choose any $w \in M$ which is a finite sum $m_1 + \cdots + m_k$ where each summand is an eigenvector of $D$ in $M$, and they have distinct eigenvalues $\lambda_1,\dots,\lambda_k$. Use polynomial interpolation to find a polynomial $p$ such that $p(\lambda_j) = \delta_{1,j}\lambda_1$. So $w_1 = p(D)w \in M$.

An **affine Lie algebra** is $\widehat{\mathfrak{g}}$ with basis $X_n, K$ (where $X \in \mathfrak{g}, n \in \mathbb{Z}$) satisfying the Lie bracket relation

$$[X_m, Y_n] = [X,Y]_{m+n} + m(X,Y)\delta_{m,-n}K,$$
$$[K, X_m] = 0.$$

It is more convenient to add a basis element $D$ (which will be the $L_0$ of our VOA) to $\widehat{\mathfrak{g}}$ to get a slightly larger Lie algebra $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}D$ such that

$$[D, X_m] = -mX_m, \qquad [D, K] = 0.$$

$\widetilde{\mathfrak{g}}$ is also called an affine Lie algebra.

## 6.8

$\widetilde{\mathfrak{g}}$ decomposes into Lie subalgebras $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_- \oplus \widetilde{\mathfrak{g}}_+$ where

$$\widetilde{\mathfrak{g}}_- = \mathrm{Span}\{X_n : X \in \mathfrak{g}, n < 0\}, \qquad \widetilde{\mathfrak{g}}_+ = \mathrm{Span}\{X_n, K, D : X \in \mathfrak{g}, n \geqslant 0\}.$$

Then $U(\widetilde{\mathfrak{g}}) \simeq U(\widetilde{\mathfrak{g}}_-) \otimes U(\widetilde{\mathfrak{g}}_+)$ by Lemma 6.3. For each $l \in \mathbb{C}$ called the **level**, we let $\mathbb{C}_l = \mathbb{C}$ be an $\widetilde{\mathfrak{g}}_+$-module such that $K$ acts as $l$ and $X_n, D$ act trivially. We are interested in two types of associated VOAs:

$$V_{\mathfrak{g}}(l,0) := \mathrm{Ind}_{U(\widetilde{\mathfrak{g}}_+)}^{U(\widetilde{\mathfrak{g}})} \mathbb{C}_l = U(\widetilde{\mathfrak{g}})_{\otimes U(\widetilde{\mathfrak{g}}_+)} \mathbb{C}_l \tag{6.7}$$

which as a vector space is naturally equivalent to $U(\widetilde{\mathfrak{g}}_-)$. Let $\mathbf{1}$ be the $1 \otimes 1$ in $U(\widetilde{\mathfrak{g}})_{\otimes U(\widetilde{\mathfrak{g}}_+)} \mathbb{C}_l$. Then $V_{\mathfrak{g}}(l,0)$ has a basis of vectors

$$X^{i_1}_{-n_1} \cdots X^{i_k}_{-n_k} \mathbf{1}$$

(which has $D$-weight $n_1 + \cdots + n_k$) written in the lexicon order where $\{X^1, X^2, \dots\}$ is a basis of $\mathfrak{g}$ and $n_1, \dots, n_k > 0$. Thus, $D$ is diagonaizable on $V_{\mathfrak{g}}(l,0)$ with non-negative spectrum, and each eigenspace is finite dimensional. Similar to the argument in Subsec. 6.5, we can take a simple quotient

$$L_{\mathfrak{g}}(l,0) = V_{\mathfrak{g}}(l,0)/I \tag{6.8}$$

where $I$ is the largest proper $\widetilde{\mathfrak{g}}$-submodule.

$V_{\mathfrak{g}}(0,0)$ and $L_{\mathfrak{g}}(0,0)$ are never equal, because:

**Exercise 6.5.** Show that $L_{\mathfrak{g}}(0,0)$ is spanned by $\mathbf{1}$. Equivalently, show that if $l = 0$, then $I$ contains all $D$-eigenvectors with eigenvalues $> 0$.

In the following, we discuss how to make $L_{\mathfrak{g}}(l,0)$ a VOA since $L_{\mathfrak{g}}(l,0)$ is our main interest. The same method applies to $V_{\mathfrak{g}}(l,0)$.

For each $X \in \mathfrak{g}$, $X_n$ acts on $L_{\mathfrak{g}}(l,0)$ in an obvious way. We define $X(z) \in \mathrm{End}(L_{\mathfrak{g}}(l,0))[[z^{\pm 1}]]$ to be

$$X(z) = \sum_{n \in \mathbb{Z}} X_n z^{-n-1}.$$

It is a homogeneous field (with respect to $D$) with weight $1$ since $[D, X_n] = -nX_n$. One checks easily that these fields satisfy the creation property and locality, and that they generate $L_{\mathfrak{g}}(l,0)$. So it remains to construct $L_{-1}$ and verify the translation property. We shall actually construct all $L_n$ in a uniform way.

## 6.9

Choose a basis $E$ of $\mathfrak{g}$, which gives a dual basis $\{\check{e} : e \in E\}$, namely, for each $e, f \in E$, $(e, \check{f}) = \delta_{e,f}$ with respect to the given non-degenerate symmetric bilinear form $(\cdot, \cdot)$. By linear algebra,

$$\sum_{e \in E} e \otimes \check{e} \in \mathfrak{g} \otimes \mathfrak{g} \tag{6.9}$$

is independent of the choice of basis $E$. As an immediate consequence, we have

$$\sum_{e \in E} \check{e} \otimes e = \sum_{e \in E} e \otimes \check{e}. \tag{6.10}$$

With the help of $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $X \otimes Y \mapsto [X, Y]$, this shows $\sum[\check{e}, e] = \sum[e, \check{e}] = -\sum[\check{e}, e]$, i.e.,

$$\sum_{e \in E} [\check{e}, e] = 0. \tag{6.11}$$

**Lemma 6.6.** *For each $X \in \mathfrak{g}$, we have*

$$\sum_{e \in E} \check{e} \otimes [e, X] = -\sum_{e \in E} [\check{e}, X] \otimes e. \tag{6.12}$$

*Proof.* Evaluate both sides by $Y \otimes Z$ using $(\cdot, \cdot)$, and use the invariance condition (6.6) to show that both sides equal $(Y, [X, Z])$. $\square$

Thus, on each $\mathfrak{g}$-module $V$, we have $\sum \check{e}[e, X] + \sum[\check{e}, X]e = 0$, namely,

$$\sum_{e} [\check{e}e, \mathfrak{g}] = 0. \tag{6.13}$$

So when $V$ is finite dimensional and is either irreducible or trivial, $\Omega = \sum \check{e}e \in \mathrm{End}(V)$ is a constant by Schur's lemma, called **Casimir element**. The operator $\Omega$ in general gives the nagative Laplactian of the Lie group action.

**Assumption 6.7.** We assume that for the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}$, $X \mapsto [X, \cdot]$, the Casimir element is a constant $2h^\vee \in \mathbb{C}$, i.e.,

$$\sum_{e \in E} [\check{e}, [e, \cdot]] = 2h^\vee \mathbf{1}_{\mathfrak{g}}. \tag{6.14}$$

This is always true when $\mathfrak{g}$ is abelian (in which case $h^\vee = 0$) or simple. We assume

$$l + h^\vee \neq 0.$$

### 6.10

We define the Virasoro operator "as if" the conformal vector is

$$\mathbf{c} = \gamma^{-1} \sum_e \breve{e}_{-1} e_{-1} \mathbf{1} \qquad \big(\text{where } \gamma = 2(l + h^\vee)\big). \tag{6.15}$$

Thus, using (5.13) and $L_m = Y(\mathbf{c})_{m+1}$, and noting that $\breve{e}_i e_j = e_i \breve{e}_j$ by (6.10), we write down the definition

$$L_m = \gamma^{-1} \sum_e \Big( \sum_{k \leqslant -1} \breve{e}_k e_{m-k} + \sum_{k \geqslant 0} \breve{e}_{m-k} e_k \Big) \tag{6.16}$$

acting on $L_{\mathfrak{g}}(l, 0)$. This is called **Sugawara construction**. One checks that this sum is finite when acting on any vector.

To use the reconstruction theorem, we need the following crucial fact:

**Proposition 6.8.** *For each $m, n \in \mathbb{Z}$ and $X \in \mathfrak{g}$,*

$$[L_m, X_n] = -n X_{m+n}. \tag{6.17}$$

(Note that if we assume the existence of the VOA structure, then (6.17) can be derived from the conformal Ward identity (5.7) and the fact that $X_{-1}\mathbf{1}$ is indeed primary.)

**Convention 6.9.** In the remaining part of this section, we suppress $\sum_e$ if possible.

From this proposition, we know that $T(z) = \sum_m L_m z^{-m-2}$ and $X(z)$ are local, and $X(z)$ satisfies the translation property. To use the reconstruction theorem, we need to check the following facts:

**Lemma 6.10.** *The following are true.*

*(a) $T(z)$ satisfies the creation property, namely, $L_n \mathbf{1} = 0$ if $n \geqslant -1$.*

*(b) $L_0$ agrees with $D$.*

*(c) $\{L_n\}$ satisfy the Virasoro relation.*

*Proof.* (a) Assume $m \geqslant -1$. $\sum_{k \geqslant 0} \breve{e}_{m-k} e_k \mathbf{1}$ is $0$ since all $X_0 \mathbf{1}$ are zero by our construction. $\sum_{k \leqslant -1} \breve{e}_k e_{m-k} \mathbf{1}$ is $0$ because $m - k \geqslant m + 1 \geqslant 0$.

(b) Since $L_0 \mathbf{1} = 0$ and $[L_0, X_n] = -n X_n = [D, X_n]$, $L_0$ and $D$ act the same on any $X^1_{n_1} \cdots X^k_{n_k} \mathbf{1}$. So $L_0 = D$.

(c) By the reconstruction theorem, $L_{\mathfrak{g}}(l, 0)$ is a graded vertex algebra. Clearly $L_m = Y(\mathbf{c})_{m+1}$ by our definition of $L_m$ and $\mathbf{c}$. We can use (5.1) or (5.2) to show

$$[L_m, L_n] = Y(L_{-1}\mathbf{c})_{m+n+2} + \sum_{l \geqslant 0} \binom{m+1}{l+1} Y(L_l \mathbf{c})_{m+n+1-l}. \tag{6.18}$$

By the expression $\mathbf{c}$, clearly $L_0 \mathbf{c} = D\mathbf{c} = 2\mathbf{c}$. Also, from the Sugawara construction, we clearly have $[D, L_m] = -m L_m$, i.e., $[L_0, L_m] = -m L_m$. So $L_l \mathbf{c} = 0$ if $l > 2$. To find $[L_m, L_n]$, we need to find $L_1 \mathbf{c}$ and $L_2 \mathbf{c}$.

Using (6.17), we calculate that $\gamma L_1\mathbf{c}$ equals

$$L_1\check{e}_{-1}e_{-1}\mathbf{1} = [L_1, \check{e}_{-1}]e_{-1}\mathbf{1} + \check{e}_{-1}[L_1, e_{-1}]\mathbf{1} = \check{e}_0e_{-1}\mathbf{1} + \check{e}_{-1}e_0\mathbf{1} = \check{e}_0e_{-1}\mathbf{1}.$$

And $\check{e}_0e_{-1}\mathbf{1} = [\check{e}_0, e_{-1}]\mathbf{1} = [\check{e}, e]_{-1}\mathbf{1}$ equals $0$ by (6.11). Recall $K$ acts as $l$ on $L_{\mathfrak{g}}(l,0)$. Then $\gamma L_2\mathbf{c}$ equals

$$\begin{aligned} L_2\check{e}_{-1}e_{-1}\mathbf{1} &= [L_2, \check{e}_{-1}]e_{-1}\mathbf{1} + \check{e}_{-1}[L_2, e_{-1}]\mathbf{1} = \check{e}_1e_{-1}\mathbf{1} + \check{e}_{-1}e_1\mathbf{1} \\ =&\check{e}_1e_{-1}\mathbf{1} = [\check{e}_1, e_{-1}]\mathbf{1} = [\check{e}, e]_0\mathbf{1} + l(\check{e}, e)\mathbf{1}, \end{aligned}$$

which equals $l\cdot\dim\mathfrak{g}\cdot\mathbf{1}$. Therefore, using (5.6), we find that (6.18) becomes the Virasoro relation where $\frac{c}{2} = \gamma^{-1}l\cdot\dim\mathfrak{g}$. □

Thus, by the reconstruction Thm. 5.12, we conclude:

**Theorem 6.11.** *For $l \neq -h^\vee$, $V_{\mathfrak{g}}(l,0)$ and $L_{\mathfrak{g}}(l,0)$ are VOAs satisfying $Y(X_{-1}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$ (for all $X \in \mathfrak{g}$) if we define the conformal vector $\mathbf{c}$ as in (6.15). The central charge is $\frac{l\dim\mathfrak{g}}{l+h^\vee}$.*

## 6.11

It remains to prove Prop. 6.8. Recall Convention 6.9 that we are suppressing $\sum_e$. The following discussions focus on $L_{\mathfrak{g}}(l,0)$, though the same argument works for $V_{\mathfrak{g}}(l,0)$.

**Lemma 6.12.** *For all $i,j,n \in \mathbb{Z}$, on $L_{\mathfrak{g}}(l,0)$ we have $[\check{e}_ie_j, X_n] = A_{i,j,n} + B_{i,j,n}$ where*

$$A_{i,j,n} = \check{e}_i[e,X]_{j+n} - \check{e}_{i+n}[e,X]_j \tag{6.19a}$$
$$B_{i,j,n} = -nl(\delta_{j,-n}X_i + \delta_{i,-n}X_j). \tag{6.19b}$$

*In particular, $B_{i,j,n} = B_{j,i,n}$.*

*Proof.* We compute

$$[\check{e}_ie_j, X_n] = \check{e}_i[e_j, X_n] + [\check{e}_i, X_n]e_j = A_{i,j,n} + B_{i,j,n}$$

where

$$\begin{aligned} A_{i,j,n} &= \check{e}_i[e,X]_{j+n} + [\check{e},X]_{i+n}e_j \\ B_{i,j,n} &= -nl\delta_{j,-n}\cdot\check{e}_i(e,X) - nl\delta_{i,-n}(\check{e},X)e_j. \end{aligned}$$

$B_{i,j,n}$ clearly equals (6.19b) by the basic property of (dual) basis. Note that in general, for all $i,j \in \mathbb{Z}$, by Lemma 6.6 and the map $\mathfrak{g}\otimes\mathfrak{g} \to \mathrm{End}(L_{\mathfrak{g}}(l,0))$ sending $Y\otimes Z$ to $Y_iZ_j$, we have

$$[\check{e},X]_ie_j = -\check{e}_i[e,X]_j. \tag{6.20}$$

This proves that $A_{i,j,n}$ equals (6.19a). □

*Proof of Prop. 6.8.* We compute

$$[\gamma L_m, X_n] = \sum_{k\leqslant -1}[\check{e}_k e_{m-k}, X_n] + \sum_{k\geqslant 0}[\check{e}_{m-k}e_k, X_n]$$
$$= \sum_{k\leqslant -1}(A_{k,m-k,n} + B_{k,m-k,n}) + \sum_{k\geqslant 0}(A_{m-k,k,n} + B_{m-k,k,n}).$$

By Lemma 6.12, the sum of the two $B$ is

$$\sum_{k\in\mathbb{Z}} B_{k,m-k,n} = -nl\sum_{k\in\mathbb{Z}}(\delta_{m-k,-n}X_k + \delta_{k,-n}X_{m-k}) = -2nlX_{m+n}.$$

Also,

$$\sum_{k\geqslant 0} A_{m-k,k,n} = \sum_{k\geqslant 0}\check{e}_{m-k}[e,X]_{k+n} - \sum_{k\geqslant 0}\check{e}_{m+n-k}[e,X]_k$$

where the two sums are both finite when acting on any vector. But the first summand is just (setting $j = k+n$) $\sum_{j\geqslant n}\check{e}_{m+n-j}[e,X]_j$. So

$$\sum_{k\geqslant 0} A_{m-k,k,n} = -\big(\check{e}_{m+n}[e,X]_0 + \check{e}_{m+n-1}[e,X]_1 + \cdots + \check{e}_{m+1}[e,X]_{n-1}\big). \tag{6.21}$$

Simiarly, setting $i = m-k$,

$$\sum_{k\leqslant -1} A_{k,m-k,n} = \sum_{i\geqslant m+1}\check{e}_{m-i}[e,X]_{i+n} - \sum_{i\geqslant m+1}\check{e}_{m+n-i}[e,X]_i$$
$$= -\big(\check{e}_{n-1}[e,X]_{m+1} + \cdots + \check{e}_0[e,X]_{m+n}\big). \tag{6.22}$$

By Lemma 6.13, the sum of (6.21) and (6.22) is $-2nh^\vee X_{m+n}$. This finishes the proof. □

**Lemma 6.13.** *For each $i, j \in \mathbb{Z}$ and $X \in \mathfrak{g}$,*

$$\check{e}_i[e,X]_j + \check{e}_j[e,X]_i = 2h^\vee X_{i+j}. \tag{6.23}$$

This is the only place we use the definition of $h^\vee$ (cf. Assumption 6.7).

*Proof.* By (6.20),

$$\check{e}_i[e,X]_j + \check{e}_j[e,X]_i = \check{e}_i[e,X]_j - [\check{e},X]_j e_i,$$

which, according to (6.10) and the map $\mathfrak{g}\otimes\mathfrak{g}\to\mathrm{End}(L_{\mathfrak{g}}(l,0)), Y\otimes Z\mapsto [Y,X]_j Z_i$, is

$$\check{e}_i[e,X]_j - [e,X]_j\check{e}_i = [\check{e}_i,[e,X]_j] = [\check{e},[e,X]]_{i+j} + il\delta_{i,-j}(\check{e},[e,X]).$$

Now, by the invariance of $(\cdot,\cdot)$, $(\check{e},[e,X]) = ([\check{e},e],X)$, which equals $0$ by (6.11). By the definition of $h^\vee$, $[\check{e},[e,X]] = 2h^\vee X$. We are done with the proof. □

## 6.12

We now discuss the unitarity problem for affine VOAs. We first look at Heisenberg VOAs, namely, we assume $\mathfrak{g}$ is abelian. We assume that $\mathfrak{g}$ is equipped with an inner product $(\cdot|\cdot)$ (antilinear on the first variable) and an anti-unitary involution $X \in \mathfrak{g} \mapsto X^* \in \mathfrak{g}$. Recall that "anti-unitary" means that $*$ is conjugate linear, bijective, and satisfies

$$(X^*|Y^*) = (Y|X).$$

Involution means $X^{**} = X$. By considering $\mathfrak{g}$ as an (abelian) **unitary Lie algebra**, we regard $*$ and $(\cdot|\cdot)$ as part of the data of $\mathfrak{g}$.

**Exercise 6.14.** Show that $\mathfrak{g}$ is unitarily isomorphic to $\mathbb{C}^n$ with the standard inner product, where the involution is $(z_1, \dots, z_n) \mapsto (\overline{z_1}, \dots, \overline{z_n})$, the unique one fixing $\mathbb{R}^n$. (Hint: First find an real isomorphism from $\{X \in \mathfrak{g} : X^* = X\}$ to $\mathbb{R}^n$ preserving the inner products.)

It is easy to check that the bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ defined by

$$(X, Y) = (X^*|Y) \tag{6.24}$$

is symmetric. (It is obviously invariant.) We define $V_{\mathfrak{g}}(l, 0)$ using this bilinear form.

**Proposition 6.15.** *$l > 0$ if and only if there exists an inner product $\langle\cdot|\cdot\rangle$ on $V_{\mathfrak{g}}(l, 0)$ satisfying $\langle \mathbf{1}|\mathbf{1}\rangle = 1$ such that the representation of $\widetilde{\mathfrak{g}}$ on $V_{\mathfrak{g}}(l, 0)$ is unitary, namely, for each $X \in \mathfrak{g}, u, v \in V_{\mathfrak{g}}(l, 0), n \in \mathbb{Z}$,*

$$\langle u|X_n v\rangle = \langle (X^*)_{-n} u|v\rangle, \qquad \langle u|Kv\rangle = \langle Ku|v\rangle, \qquad \langle u|Dv\rangle = \langle Dv|u\rangle,$$

*or simply $(X_n)^\dagger = X^*_{-n}$, $K^\dagger = K$, $D^\dagger = D$ for short. Such $\langle\cdot|\cdot\rangle$ is unique if it exists.*

The if part is easy to explain: We compute that $\langle X_{-1}\mathbf{1}|X_{-1}\mathbf{1}\rangle = \langle \mathbf{1}|X_1^* X_{-1}\mathbf{1}\rangle = \langle \mathbf{1}|[X^*, X]_0 \mathbf{1}\rangle + l(X^*, X) = l(X|X)$. So if $\langle\cdot|\cdot\rangle$ is an inner product, then for each $X \neq 0$, $l(X|X)$ is $> 0$. So $l > 0$. We now explain the only if part. To simplify discussions, by scaling $(\cdot|\cdot)$ and hence $(\cdot, \cdot)$ by $l$ and $K$ by $l^{-1}$, it suffices to assume $l = 1$. (Indeed, people usually just assume $l = 1$ when discussing Heisenberg VOAs.)

## 6.13

Assume $l = 1$. The uniqueness of $\langle\cdot|\cdot\rangle$ is easy to prove:

$$\langle X^1_{n_1} \cdots X^k_{n_k} \mathbf{1}|Y^1_{m_1} \cdots Y^l_{m_l}\mathbf{1}\rangle = \langle \mathbf{1}|(X^k)^*_{-n_k} \cdots (X^1)^*_{-n_1} Y^1_{m_1} \cdots Y^l_{m_l}\mathbf{1}\rangle =: \langle \mathbf{1}|w\rangle.$$

If $n_1 + \cdots + n_k = m_1 + \cdots + m_l$, then $w$ has $D$-weight $0$. But the weight-0 homogeneous vectors are $\mathbb{C}\mathbf{1}$. So $w = \lambda\mathbf{1}$, and $\lambda$ uniquely determined by the Lie bracket relations. If $n_1 + \cdots + n_k \neq m_1 + \cdots + m_l$, then the weight of $w$ is not $0$. So $w = 0$ since $\langle D\mathbf{1}|w\rangle = \langle \mathbf{1}|Dw\rangle$.

The existence part follows from the general construction of symmetric Fock spaces. Let $W$ be a (complex) inner product space together with an antiunitary involution $*$.

Note that for each $N \in \mathbb{N}$, $W^{\otimes N}$ is naturally an inner product space. We assume $W$ has an orthonormal basis $\{e_i : i \in \mathfrak{I}\}$ (which spans $W$ algebraically). Let $\mathfrak{S}_N$ be the set of permutations on $\{1, \ldots, N\}$. For each $v_1, \ldots, v_N \in W$, we define

$$v_1 \cdots v_N := \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathfrak{S}_N} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)},$$

and let $S^N(W) \subset W^{\otimes N}$ be spanned by all such vectors. We understand $S^0(W)$ to be the standard one dimensional inner product space $\mathbb{C}$. In particular, it has a unit vector 1. $S^N(W)$ has an orthonormal basis consisting of vectors

$$\frac{(e_{i_1})^{m_1} \cdots (e_{i_k})^{m_k}}{\sqrt{m_1! \cdots m_k!}} \qquad (\text{where } i_1, \ldots, i_k \in \mathfrak{I} \text{ are distinct and } \sum_{j=1}^{k} m_j = N). \tag{6.25}$$

Define an inner product space

$$S^\bullet(W) = \bigoplus_{N \in \mathbb{N}} S^N(W), \tag{6.26}$$

called the **symmetric Fock space** associated to $W$. For each $v \in W$, define linear maps $a^+(v), a^-(v)$ on $S^\bullet(W)$ determined by

$$a^+(v)1 = v, \qquad a^+(v)v_1 \cdots v_N = vv_1 \cdots v_N. \tag{6.27a}$$

$$a^-(v)1 = 0, \qquad a^-(v)v_1 \cdots v_N = \sum_{j=1}^{N} \langle v^* | v_j \rangle \cdot v_1 \cdots v_{j-1} v_{j+1} \cdots v_N. \tag{6.27b}$$

The maps $a^\pm(v)$ are well-defined, thanks to the basis (6.25).

**Exercise 6.16.** Prove the following relations.

1. $a^+(v)^\dagger = a^-(v^*)$, namely, $\langle \xi | a^+(v)\nu \rangle = \langle a^-(v^*)\xi | \nu \rangle$ for all $\xi, \nu \in S^\bullet(W)$. (Hint: write $\xi, \nu, v$ in terms of the previously mentioned orthonormal basis vectors.)
2. $[a^-(u), a^+(v)] = \langle u^* | v \rangle \mathbf{1}_{S^\bullet(W)}$. This is called the **canonical commutation relation (CCR)**.

Now let $W = t^{-1} \cdot \mathfrak{g}[t^{-1}]$ with inner product

$$\langle Xt^{-m} | Yt^{-n} \rangle = m(X|Y)\delta_{m,n}$$

for all $m, n \in \mathbb{Z}_+$. The involution is defined to be $(Xt^{-m})^* = X^* t^{-m}$. According to the description of the basis of $S^\bullet(W)$, $V_\mathfrak{g}(1, 0)$ is linearly equivalent to $S^\bullet(W)$ by identifying $\mathbf{1}$ with 1 and

$$X^1_{-n_1} \cdots X^k_{-n_k} \mathbf{1} \qquad \text{with} \qquad X^1 t^{-n_1} \cdots X^k t^{-n_k}. \tag{6.28}$$

We use the inner product on $S^\bullet(W)$ to define the one on $V_\mathfrak{g}(1, 0)$. Using CCR, it is not hard to check that the action of $X_n$ on $V_\mathfrak{g}(1, 0) \simeq S^\bullet(W)$ is

$$X_n = \begin{cases} a^+(Xt^{-|n|}) & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ a^-(Xt^{-n}) & \text{if } n > 0. \end{cases} \tag{6.29}$$

Thus, the representation of $\widetilde{\mathfrak{g}}$ on $V_\mathfrak{g}(1, 0)$ is unitary.

## 6.14

When $l > 0$, $L_{\mathfrak{g}}(l, 0)$ and $V_{\mathfrak{g}}(l, 0)$ share the same unitarity property, because:

**Proposition 6.17.** *If $l \in \mathbb{C}^\times$, then $V_{\mathfrak{g}}(l, 0)$ is an irreducible $\widetilde{\mathfrak{g}}$-module, i.e., $V_{\mathfrak{g}}(l, 0) = L_{\mathfrak{g}}(l, 0)$.*

*Proof.* We assume $l > 0$ and prove the irreducibility using the unitarity. Choose any non-zero $\widetilde{\mathfrak{g}}$-submodule $W$ of $V_{\mathfrak{g}}(l, 0)$. We shall show $W = V_{\mathfrak{g}}(l, 0)$.

Since $W$ is a $D$-invariant subspace, $D$ is diagonalizable on $W$. So $W$ has $D$-grading $W = \bigoplus_{n \geqslant a} W(n)$ where $a$ is the smallest eigenvalue of $D$ on $W$. We claim that $a = 0$. Then, as the $D$-weight $0$ subspace of $V_{\mathfrak{g}}(l, 0)$ is clearly spanned by $\mathbf{1}$, we must have $\mathbf{1} \in W$. From this one sees that $W = V_{\mathfrak{g}}(l, 0)$.

Suppose $a > 0$. We choose a non-zero $w \in W(a)$, which must be a sum of vectors of the form $X^1_{-n_1} \cdots X^k_{-n_k} \mathbf{1}$ where the sum of the positive integers $n_1, \ldots, n_k$ is $a$. Then by the unitarity, $\langle w|w \rangle$ (which is non-zero) is a sum of $\langle \mathbf{1}|(X^k)^*_{n_k} \cdots (X^1)^*_{n_1} w \rangle$. So for some $X^1_{n_1}$, the vector $v = (X^1)^*_{n_1} w$ must be nonzero. But $v$ has $D$-weight $a - n_1 < a$, and clearly $v \in W$. This is a contradiction.

Now, for a general $l = |l| e^{\mathbf{i}\theta} \in \mathbb{C}^\times$, we may replace $(\cdot, \cdot)$ by $e^{\mathbf{i}\theta}(\cdot, \cdot)$ and $K$ by $e^{-\mathbf{i}\theta} K$. Then $(\cdot|\cdot)$ and the new $(\cdot, \cdot)$ are related by $(X|Y) = (e^{\mathbf{i}\theta} X^*, Y)$, and $X \mapsto e^{\mathbf{i}\theta} X^*$ is clearly an antiunitary involution. So $V_{\mathfrak{g}}(l, 0)$ becomes $V_{\mathfrak{g}}(|l|, 0)$ under the new involution and bilinear form, and the latter has been proved irreducible. □

## 6.15

In general, we say a finite-dimensional (complex) Lie algebra $\mathfrak{g}$ is **unitary** if it is equipped with an inner product $(\cdot|\cdot)$ and an antiunitary involution $*$ satisfying the following conditions:

1. $[X, Y]^* = [Y^*, X^*]$.
2. The inner product is **invariant**, namely, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is unitary:
$$([X, Y]|Z) = (Y|[X^*, Z]).$$

Then $(X, Y) := (X^*|Y)$ defines a symmetric invariant bilinear form on $\mathfrak{g}$.

**Exercise 6.18.** Let $\mathfrak{k}$ be an $\mathfrak{g}$-invariant and $*$-invariant (i.e. $[\mathfrak{g}, \mathfrak{k}] \subset \mathfrak{k}$, $\mathfrak{k}^* = \mathfrak{k}$) subspace of $\mathfrak{g}$. Let $\mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

1. Show that $\mathfrak{k}^\perp$ is also $\mathfrak{g}$-invariant and $*$-invariant.
2. Show that $[\mathfrak{k}, \mathfrak{k}^\perp] = 0$ and hence $[\mathfrak{g}, \mathfrak{k}] = [\mathfrak{k}, \mathfrak{k}]$. Consequently, if $\mathfrak{k}$ is an irreducible $\mathfrak{g}$-submodule, then $\mathfrak{k}$ is an irreducible $\mathfrak{k}$-module, which is (by definition) a simple Lie algebra if moreover $[\mathfrak{k}, \mathfrak{k}] \neq 0$.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, which is clearly $\mathfrak{g}$- and $*$-invariant. Let $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{z}^\perp$ so that $\mathfrak{g} = \mathfrak{z} \oplus^\perp \mathfrak{g}_{\mathrm{ss}}$. Then the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}_{\mathrm{ss}}$ (equivalently, $\mathfrak{g}_{\mathrm{ss}} \curvearrowright \mathfrak{g}_{\mathrm{ss}}$) has orthogonal irreducible $*$-invariant decomposition $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}_1 \oplus^\perp \cdots \oplus^\perp \mathfrak{g}_N$. Each $\mathfrak{g}_j$ is a simple unitary Lie algebra, which is classified by the type A-G Dynkin diagrams.

Conversely, suppose $\mathfrak{g}$ is a complex simple Lie algebra, which is the complexification of $\mathfrak{g}_{\mathbb{R}}$ which is the real Lie algebra of a finite dimensional compact real Lie group $G$. It is well known in Lie theory that the real vector space $\mathfrak{g}_{\mathbb{R}}$ has a unique up to $\mathbb{R}_{>0}$-scalar multiplication $G$-invariant (equivalently, $\mathfrak{g}_{\mathbb{R}}$-invariant) inner product, which extends to a complex invariant inner product $(\cdot|\cdot)$ on $\mathfrak{g}$ thanks to the real direct sum $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \mathbf{i}\mathfrak{g}_{\mathbb{R}}$. The antiunitary involution on $\mathfrak{g}$ is defined to be the unique one fixing $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Thus $\mathfrak{g}$ is unitary.

Therefore, in general, if $\mathfrak{z}$ is abelian and $\mathfrak{g}_1, \dots, \mathfrak{g}_N$ are simple, then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is naturally a unitary Lie algebra. So the study of unitary affine VOAs for unitary Lie algebras reduces to that of the abelian case (which we have finished) and the simple case.

## 6.16

When $\mathfrak{g}$ is simple, the unitarity properties of $V_{\mathfrak{g}}(l,0)$ and $L_{\mathfrak{g}}(l,0)$ are very different from the abelian case. Indeed, in the abelian case, scaling the inner product does not change the unitary equivalence class of abelian unitary Lie algebras. (This is because scaling the vectors by a non-zero constant is an isomorphism of abelian Lie algebras.) But this is no longer true for non-abelian Lie algebras. Also, it turns out that for a simple $\mathfrak{g}$, $V_{\mathfrak{g}}(l,0)$ is never a unitary $\widetilde{\mathfrak{g}}$-module, and $L_{\mathfrak{g}}(l,0)$ is unitary for a discrete set of levels $l$ if one fixes the invariant inner product, or for a discrete set of invariant inner product if one fixes the level $l$.

Assume $\mathfrak{g}$ is a simple Lie algebra with compact form decomposition $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Let $*$ be the unique involution fixing $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. As we have said, the invariant bilinear forms on $\mathfrak{g}_{\mathbb{R}}$ (and hence on $\mathfrak{g}$) are unique up to scalar multiplication. So it would be better to fix one. The one that people usually choose is:

**Convention 6.19.** We choose the invariant inner product on $\mathfrak{g}$ (under which $*$ is antiunitary) to be the unique one such that the longest roots of $\mathfrak{g}$ have length $\sqrt{2}$.

It follows from the invariance of $(\cdot|\cdot)$ that $h^\vee$ (defined in Assumption 6.7) is a positive number. (To see this, one may choose $E$ to be an orthonormal basis of $\mathfrak{g}$, and check that its dual basis $\{\breve{e} : e \in E\}$ satisfies $\breve{e} = e^*$.) The $h^\vee$ corresponding to the inner product in Convention 6.19 is called the **dual Coxeter number**. We have said that $L_{\mathfrak{g}}(l,0)$ and $V_{\mathfrak{g}}(l,0)$ are VOAs if $l \neq -h^\vee$. So this is true when $l \geqslant 0$.

**Theorem 6.20.** *$L_{\mathfrak{g}}(l,0)$ is unitary if and only if $l \in \mathbb{N}$. For such $l$, $L_{\mathfrak{g}}(l,0)$ is called a **Weiss-Zumino-Witten (WZW)** model.*

This is a highly non-trivial result whose proof relies on deep Lie theory. We refer the readers to [Was10, Chapter III, Sec. 2 and 10] for a proof. Moreover, just like minimal models, WZW models are $C_2$-cofinite and rational. So their representation categories are extremely nice. Due to these properties, WZW models are central objects in the study of CFT and VOAs. (However, Heisenberg VOAs are neither $C_2$-cofinite nor rational.)

## 6.17

We have shown the existence of affine VOAs when the unitary Lie algebra $\mathfrak{g}$ is abelian or simple. The general case can be addressed by tensor product VOAs.

Let $\mathbb{V}_1, \mathbb{V}_2$ be VOAs. We use the diagonalizable operator $L_0 \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_0$ to define the grading on $\mathbb{V}_1 \otimes \mathbb{V}_2$. The vacuum vector is $\mathbf{1} \otimes \mathbf{1}$. $\mathbb{V}_1 \otimes \mathbb{V}_2$ is clearly generated by $Y(v_1)_m \otimes \mathbf{1}_{\mathbb{V}_2}$ and $\mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2)_n$ where $v_j \in \mathbb{V}_j$, and $Y(v_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}$ is clearly local to $Y(u_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}$ (where $u_1 \in \mathbb{V}_1$) and $\mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2, z)$. One checks that $L_{-1} \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_{-1}$ satisfies the translation property. So $\mathbb{V} \otimes \mathbb{V}$ is naturally a graded vertex algebra by the reconstruction theorem. Its vertex operator satisfies

$$Y(v_1 \otimes \mathbf{1}, z) = Y(v_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}, \qquad Y(\mathbf{1} \otimes v_2, z) = \mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2, z). \tag{6.30}$$

**Exercise 6.21.** Use (5.13) or (5.16) to show

$$Y(v_1 \otimes v_2, z) = Y(v_1, z) \otimes Y(v_2, z). \tag{6.31}$$

Equivalently,

$$Y(v_1 \otimes v_2)_n = \sum_{n \in \mathbb{Z}} \sum_{n_1 + n_2 = n-1} Y(v_1)_{n_1} Y(v_2)_{n_2}. \tag{6.32}$$

When $\mathbb{V}_1, \mathbb{V}_2$ are VOAs with conformal vectors $\mathbf{c}_1, \mathbf{c}_2$ and central charges $c_1, c_2$, it is easy to check that $\mathbb{V}_1 \otimes \mathbb{V}_2$ is a VOA with conformal vector $\mathbf{c}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{c}_2$. In particular, its Virasoro operators are $Y(\mathbf{c}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{c})_{n+1} = L_n \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_n$. We call $\mathbb{V}_1 \otimes \mathbb{V}_2$ the **tensor product VOA** of $\mathbb{V}_1$ and $\mathbb{V}_2$.

**Exercise 6.22.** Show that $\mathbb{V}_1 \otimes \mathbb{V}_2$ has central charge $c_1 + c_2$.

We remark that if $\mathbb{V}_1$ and $\mathbb{V}_2$ are unitary, then their tensor product is also unitary (cf. [DL14, CKLW18]).

**Exercise 6.23.** Let $\mathfrak{g}_1, \dots, \mathfrak{g}_N$ be either abelian or simple. Let $\mathbb{V} = L_{\mathfrak{g}_1}(l_1, 0) \otimes \cdots \otimes L_{\mathfrak{g}_N}(l_N, 0)$. Show that the weight-1 subspace $\mathbb{V}(1)$, as a Lie algebra (cf. Subsec. 5.5), is naturally isomorphic to $\mathfrak{g} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$. Show that $\mathbb{V}(1)$ generates $\mathbb{V}$.

**Exercise 6.24.** Show that $L_{\mathbb{C}^n}(1, 0) \simeq \underbrace{L_{\mathbb{C}}(1, 0) \otimes \cdots \otimes L_{\mathbb{C}}(1, 0)}_{n \text{ times}}$.

# 7 Local fields

## 7.1

Having explored some important examples, we now return to the general theory. The goal of this section is to understand the close relationship between the three statements in Subsec. 5.8. The precise formulation of statement 1 is the Lie bracket version of local fields, as defined in Def. 5.9 or Rem. 5.10. For statement 2 we give two rigorous descriptions: the complex analytic version and the formal variable version of local fields. We first give the complex analytic version, which is more intuitive.

We first need to define:

**Definition 7.1.** Let $\Omega$ be a locally compact Hausdorff space. A series of functions $\sum_n f_n$ is said to **converge absolutely and locally uniformly (a.l.u.) on** $\Omega$ if each $x_0 \in \Omega$ is contained in a neighborhood $U$ such that

$$\sup_{x\in U}\sum_n |f_n(x)| < +\infty.$$

Equivalently, for each compact subset $K \subset \Omega$, we have $\sup_{x\in K}\sum_n |f_n(x)| < +\infty$

Clearly, if each $\sum f_n$ converges a.l.u. and each $f_n$ is continuous (resp. holomorphic), then so is the limit $\sum f_n$.

## 7.2

Now let $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ be graded by a diagonalizable $L_0$. Recall the projection $P_n : \mathbb{V}^{\mathrm{cl}} = \prod_{m\in\mathbb{N}} \mathbb{V}(m) \to \mathbb{V}(n)$ (cf. (3.18)). Let $A(z) = \sum A_n z^{-n-1}$, $B(z) = \sum B_n z^{-n-1}$ be homogeneous fields with weights $\Delta_A, \Delta_B$ (cf. Def. 5.8). For each $n \in \mathbb{N}$ and $v, v' \in \mathbb{V}$, we have

$$\langle v', A(z_1)P_nB(z_2)v\rangle \quad \in \mathscr{O}(\mathbb{C}^\times \times \mathbb{C}^\times) \tag{7.1}$$

since, when $v, v'$ are homogeneous, this expression equals

$$\langle v', A_{n_1}B_{n_2}v\rangle z_1^{-n_1-1}z_2^{-n_2-1}$$

where $n_2, n_1$ are determined by $\Delta_B + \mathrm{wt}v - n_2 - 1 = n$ and $\Delta_A + n - n_1 - 1 = \mathrm{wt}v'$.

**Definition 7.2 (Local fields** (complex analytic version)**).** We say $A(z)$ and $B(z)$ are **local** to each other if for each $v \in \mathbb{V}, v' \in \mathbb{V}'$ the following hold.

1. The series

$$\langle v', A(z_1)B(z_2)v\rangle := \sum_{n\in\mathbb{N}}\langle v', A(z_1)P_nB(z_2)v\rangle \tag{7.2a}$$

$$\langle v', B(z_2)A(z_1)v\rangle := \sum_{n\in\mathbb{N}}\langle v', B(z_2)P_nA(z_1)v\rangle \tag{7.2b}$$

   converge a.l.u. respectively on the open sets $\Omega_1 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < |z_1|\}$ and $\Omega_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < |z_2|\}$. So (7.2a) and (7.2b) are automatically holomorphic functions on $\Omega_1$ and $\Omega_2$.

2. (7.2a) and (7.2b) can be analytically continued to the same holomorphic function $f_{v,v'}$ on $\mathrm{Conf}^2(\mathbb{C}^\times)$. Moreover, there exists $N \in \mathbb{N}$ depending only on $A, B$ but not on $v, v'$ such that the function

$$(z_1 - z_2)^N f_{v,v'}(z_1, z_2) \tag{7.3}$$

   is holomorphic on $\mathbb{C}^\times \times \mathbb{C}^\times$.

Roughly speaking, this definition says that (7.2a) and (7.2b) converge a.l.u on $\Omega_1, \Omega_2$ and extend to the same holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times)$ which has poles of order at most $N$ at $z_1 - z_2$, where $N$ is independent of $v, v'$.

### 7.3

The readers will immediately notice that there is another natural convergence condition on $A(z_1)B(z_2)$: that $\langle v', A(z_1)B(z_2)v\rangle$ as a formal Laurent series of $z_1, z_2$ converges a.l.u. on $\Omega_1$. Or more precisely, the joint series

$$\sum_{m,n\in\mathbb{Z}} \langle v', A_m B_n v\rangle z_1^{-m-1} z_2^{-n-1} \tag{7.4}$$

converges a.l.u. on $\Omega$. Is this equivalent to the convergence statement in Def. 7.2? The answer is yes. But people will easily overlook the need to justify this equivalence. And we need both versions of convergence since they are useful in different situations. For instance, to prove that formal variable implies complex analytic, it is easier to prove the a.l.u. convergence of the formal Laurent series; to prove the other direction, it is better to use the a.l.u. convergence of the RHS of (7.2a) and (7.2b).

There is (unfortunately) one more way to understand the convergence (7.2a): we regard the RHS as a series of formal Laurent series of $z_1, z_2$, which converges formally to the LHS also as a formal Laurent series in the following sense:

**Definition 7.3.** We say that a sequence (indexed by $k$)

$$f_k(z_1, \dots, z_M) = \sum_{n_1,\dots,n_M\in\mathbb{Z}} f_{k,n_\bullet} z_1^{n_1} \cdots z_M^{n_M}$$

of elements of $W[[z_1^{\pm 1}, \dots, z_M^{\pm 1}]]$ **converges formally** to

$$f(z_1, \dots, z_M) = \sum_{n_1,\dots,n_M\in\mathbb{Z}} f_{n_\bullet} z_1^{n_1} \cdots z_M^{n_M}$$

if for each $n_k$, the coefficient $f_{k,n_\bullet}$ equals $f_{n_\bullet}$ except for finitely many $k$.

Note that in applications, $k$ can be in any countable set: $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^2$, etc.

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