

# HAUSDORFF CONVERGENCE PROPERTY OF PARTIAL OKOUNKOV BODIES

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## CONTENTS

1. Introduction	1
2. Hausdorff convergence property	1
References	6

## 1. INTRODUCTION

This note is a refinement of [Xia21, Theorem A]. We prove the Hausdorff convergence property in full generality.

This note is motivated by a discussion with Sébastien Boucksom.

## 2. HAUSDORFF CONVERGENCE PROPERTY

Let  $X$  be a connected smooth projective variety of dimension  $n$ . Let  $(L, h)$  be a Hermitian pseudo-effective line bundle on  $X$  with  $\int_X (\mathrm{dd}^c h)^n > 0$ . Fix  $\nu : \mathbb{C}(X)^\times \rightarrow \mathbb{Z}^n$  a valuation of rank  $n$  and rational rank  $n$ . Take a smooth Hermitian metric  $h_0$  on  $L$  and set  $\theta = c_1(L, h_0)$ . We can then identify  $h$  with  $\varphi \in \mathrm{PSH}(X, \theta)$ .

For each  $k \in \mathbb{Z}_{>0}$ , we introduce

$$\Delta_\nu^k(\theta, \varphi) := \mathrm{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

Here  $\mathrm{Conv}$  denotes the closed convex hull.

For later use, we introduce a twisted version as well. If  $T$  is a holomorphic line bundle on  $X$ , we introduce

$$\Delta_\nu^{k,T}(\theta, \varphi) := \mathrm{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, T \otimes L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

We also write

$$\Delta_\nu^{k,T}(L) := \mathrm{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, T \otimes L^k) \right\} \subseteq \mathbb{R}^n$$

and

$$\Delta_\nu^k(L) := \mathrm{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, L^k) \right\} \subseteq \mathbb{R}^n$$

We write  $\mathcal{I}_\infty(\varphi) = \mathcal{I}_\infty(h)$  for the ideal sheaf on  $X$  locally consisting of holomorphic functions  $f$  such that  $|f|_h$  is locally bounded.

We first extend [Xia21, Theorem 3.13] to the twisted case.

**Proposition 2.1.** *For any holomorphic line bundle  $T$  on  $X$ ,*

$$\Delta_\nu^{k,T}(L) \rightarrow \Delta_\nu(L)$$

as  $k \rightarrow \infty$ .

Here and later on, we endow the space of convex bodies with the Hausdorff metric.

*Proof.* As  $L$  is big, we can take  $k_0 \in \mathbb{Z}_{>0}$  so that

- (1)  $T^{-1} \otimes L^{k_0}$  admits a non-zero global holomorphic section  $s_0$ ;
- (2)  $T \otimes L^{k_0}$  admits a non-zero global holomorphic section  $s_1$ .

Then for  $k \in \mathbb{Z}_{>k_0}$ , we have injective linear maps

$$H^0(X, L^{k-k_0}) \xrightarrow{\times s_1} H^0(X, T \otimes L^k) \xrightarrow{\times s_0} H^0(X, T \otimes L^{k+k_0}).$$

It follows that

$$(k - k_0)\Delta_\nu^{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_\nu^{k,T}(L) \subseteq (k + k_0)\Delta_\nu^{k+k_0}(L) - \nu(s_0).$$

By [Xia21, Theorem 3.13], we conclude.  $\square$

**Lemma 2.2.** *Let  $T$  be a holomorphic line bundle on  $X$ . Assume that  $\varphi$  has analytic singularities and  $\varphi$  has positive mass, then*

$$\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$$

as  $k \rightarrow \infty$ .

*Proof.* Up to replacing  $X$  by a birational model and twisting  $T$  accordingly, we may assume that  $\varphi$  has log singularities along a nc  $\mathbb{Q}$ -divisor  $D$ . Take  $\epsilon \in (0, 1) \cap \mathbb{Q}$ . In this case, by Ohsawa–Takegoshi theorem, for any  $k \in \mathbb{Z}_{>0}$  we have

$$H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

Take an integer  $N \in \mathbb{Z}_{>0}$  so that  $ND$  is a divisor and  $N\epsilon$  is an integer.

Let  $\Delta'$  be the limit of a subsequence of  $(\Delta_\nu^{k,T}(\theta, \varphi))_k$ , say the sequence defined by the indices  $k_1, k_2, \dots$ . We want to show that  $\Delta' = \Delta_\nu(\theta, \varphi)$ .

There exists  $t \in \{0, 1, \dots, N-1\}$  such that  $k_i \equiv t$  modulo  $N$  for infinitely many  $i$ , up to replacing  $k_i$  by a subsequence, we may assume that  $k_i \equiv t$  modulo  $N$  for all  $i$ . Write  $k_i = Ng_i + t$ .

Now we have

$$\Delta_\nu^{g_i, T \otimes L^t}(NL - ND) + N\nu(D) \subseteq N\Delta_\nu^{k_i, T}(\theta, \varphi) \subseteq \Delta_\nu^{g_i, T \otimes L^t}(NL - N(1-\epsilon)D) + N(1-\epsilon)\nu(D).$$

By Proposition 2.1,

$$\Delta_\nu(L - D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L - (1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Let  $\epsilon \rightarrow 0+$ , we find that

$$\Delta_\nu(L - D) + \nu(D) = \Delta'.$$

It follows from Blanschke selection theorem that

$$\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(L - D) + \nu(D) = \Delta_\nu(\theta, \varphi)$$

as  $k \rightarrow \infty$ .  $\square$

**Lemma 2.3.** *Assume that  $\theta_\varphi$  is a Kähler current, then as  $\mathbb{Q} \ni \beta \rightarrow 0+$ , we have*

$$\Delta_\nu((1 - \beta)\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi).$$

*Proof.* By [Xia21, Proposition 5.15], we have

$$\Delta_\nu((1 - \beta)\theta, \varphi) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, if  $\Delta'$  is a limit of a subsequence of  $(\Delta_\nu((1 - \beta)\theta, \varphi))_\beta$ , then

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

But

$$\text{vol } \Delta' = \lim_{\beta \rightarrow 0+} \Delta_\nu((1 - \beta)\theta, \varphi) = \lim_{\beta \rightarrow 0+} \int_X ((1 - \beta)\theta + \text{dd}^c P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n.$$

We claim that

$$\lim_{\beta \rightarrow 0+} \int_X ((1 - \beta)\theta + \text{dd}^c P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n.$$

Note that this finishes the proof as  $\text{vol } \Delta_\nu(\theta, \varphi)$  is exactly equal to the right-hand side.

Next we prove our claim. We make use of the b-divisors introduced in [Xia22b; Xia22a]. By [Xia22a, Theorem 0.6], the claim is equivalent to

$$\lim_{\beta \rightarrow 0+} \text{vol } \mathbb{D}((1 - \beta)\theta, \varphi) = \text{vol } \mathbb{D}(\theta, \varphi).$$

This is a special case of [Xia22a, Theorem 9.6] □

**Theorem 2.4.** *Let  $T$  be a holomorphic line bundle on  $X$ . As  $k \rightarrow \infty$ ,  $\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$ .*

*Proof.* Fix a Kähler form  $\omega \geq \theta$  on  $X$ .

**Step 1.** We first handle the case where  $\text{dd}^c h$  is a Kähler current, say  $\text{dd}^c h \geq \beta_0 \omega$  for some  $\beta_0 \in (0, 1)$ .

Take a decreasing quasi-equisingular approximation  $\varphi_j$  of  $\varphi$ . Up to replacing  $\beta_0$  by  $\beta_0/2$ , we may assume that

$$\theta_{\varphi_j} \geq \beta_0 \omega$$

for all  $j \geq 1$ .

Take  $\beta \in (0, \beta_0) \cap \mathbb{Q}$ . Write  $\beta = p/q$  with  $p, q \in \mathbb{Z}_{>0}$ . Fix  $t \in \{0, 1, \dots, q-1\}$ .

By [DX21, Lemma 4.2], we can find  $k_0 \in \mathbb{Z}_{>0}$  such that for all  $k \geq k_0$ , there is  $v_{\beta,k} \in \text{PSH}(X, \theta)$  satisfying

(1)

$$P[\varphi]_{\mathcal{I}} \geq (1 - \beta)\varphi_k + \beta v_{\beta,k};$$

(2)  $v_{\beta,k}$  has positive mass.

Observe that for any  $j \geq 1$ ,

$$\theta_{\varphi_j} \geq \beta \omega \geq \beta \theta.$$

Namely,  $\varphi_j \in \text{PSH}(X, (1 - \beta)\theta)$ .

Fix  $k \geq k_0$ . Let  $\pi : Y \rightarrow X$  be a log resolution of the singularities of  $\varphi_k$ . By the proof of [DX21, Proposition 4.3], there is  $j_0 = j_0(\beta, k) \in \mathbb{Z}_{>0}$  such that

for any  $j \geq j_0$ , we can find a non-zero section  $s_j \in H^0(Y, \pi^* L^{pj} \otimes \mathcal{I}(jp\pi^* v_{\beta,k}))$  such that we get an injective linear map

$$H^0(Y, \pi^* T \otimes \pi^* L^t \otimes K_{Y/X} \otimes \pi^* L^{(q-p)j} \otimes \mathcal{I}(jq\pi^* \varphi_k)) \xrightarrow{\times s_j} H^0(X, T \otimes L^t \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)).$$

It follows that

$$\Delta_\nu^{j, \pi^* T \otimes \pi^* L^t \otimes K_{Y/X}}((1-\beta)q\pi^* \theta, q\pi^* \varphi_k) + j^{-1}\nu(s_j) \subseteq q\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi).$$

We observe that  $j^{-1}\nu(s_j)$  is bounded as the right-hand side is bounded when  $j$  varies.

Let  $\Delta'$  be the limit of a subsequence of  $(\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi))_j$ , then by [Lemma 2.2](#), there is a vector  $v'_k \in \mathbb{R}_{\geq 0}^n$  such that

$$\Delta_\nu((1-\beta)\pi^* \theta, \pi^* \varphi_k) + v'_k \subseteq \Delta'.$$

By the birational invariance of the partial Okounkov bodies,

$$\Delta_\nu((1-\beta)\theta, \varphi_k) + v'_k \subseteq \Delta'.$$

Let  $k \rightarrow \infty$ , by [\[Xia21, Theorem A\]](#),

$$\Delta_\nu((1-\beta)\theta, \varphi) + v' \subseteq \Delta'$$

for some vector  $v' \in \mathbb{R}_{\geq 0}^n$  depending on  $\beta$ .

Let  $\beta \rightarrow 0+$ , by [Lemma 2.3](#), we have

$$\Delta_\nu(\theta, \varphi) + v'' \subseteq \Delta'$$

for some vector  $v'' \in \mathbb{R}_{\geq 0}^n$ .

On the other hand, take  $j \geq 1$ , as  $\varphi \leq \varphi_j$ ,

$$\Delta_\nu^{k, T}(\theta, \varphi) \subseteq \Delta_\nu^{k, T}(\theta, \varphi_j).$$

By [Lemma 2.2](#),

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi_j).$$

So

$$\Delta_\nu(\theta, \varphi) + v'' \subseteq \Delta_\nu(\theta, \varphi_j).$$

Let  $j \rightarrow \infty$ , we find that  $v'' = 0$ . Namely,

$$(2.1) \quad \Delta_\nu(\theta, \varphi) \subseteq \Delta'$$

Next we compute

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(\theta, \varphi_j) = \int_X \theta_{\varphi_j}^n.$$

Let  $j \rightarrow \infty$ , we find

$$\text{vol } \Delta' \leq \int_X \theta_{P[\varphi]_X}^n = \text{vol } \Delta_\nu(\theta, \varphi).$$

It follows that equality holds in (2.1). Namely,

$$\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$$

as  $j \rightarrow \infty$ . It follows that

$$(qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{qj+t} \otimes \mathcal{I}(qj\varphi))\} \rightarrow \Delta_\nu(\theta, \varphi)$$

as  $j \rightarrow \infty$ . Observe that

$$\begin{aligned} (qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{qj+t} \otimes \mathcal{I}(qj\varphi))\} &\supseteq \Delta_\nu^{qj+t, T}(\theta, \varphi) \\ &\supseteq (qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{-q} \otimes L^{q(j+1)+t} \otimes \mathcal{I}(q(j+1)\varphi))\}. \end{aligned}$$

It follows that  $\Delta_\nu^{gj+t,T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$  as  $j \rightarrow \infty$ . As  $t$  is arbitrary, we conclude.

**Step 2.** Next we handle the general case.

Take  $\psi \in \text{PSH}(X, \theta)$  such that

- (1)  $\theta_\psi$  is a Kähler current;
- (2)  $\psi \leq \varphi$ .

The existence of  $\psi$  is proved in [DX21, Proposition 3.6].

Then for any  $\epsilon \in \mathbb{Q} \cap (0, 1)$ ,

$$\Delta_\nu^{k,T}(\theta, \varphi) \supseteq \Delta_\nu^{k,T}(\theta, (1 - \epsilon)\varphi + \epsilon\psi)$$

for all  $k$ . It follows from Step 1 that for any limit  $\Delta'$  of any subsequence of  $\{\Delta_\nu^{k,T}(\theta, \varphi)\}_k$ , we have

$$\Delta' \supseteq \Delta_\nu(\theta, (1 - \epsilon)\varphi + \epsilon\psi).$$

For later use, we denote the indices defining the subsequence as  $k_1, k_2, \dots$

Letting  $\epsilon \rightarrow 0$  and applying [Xia21, Theorem A], we have

$$\Delta' \supseteq \Delta_\nu(\theta, \varphi).$$

We claim that

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(\theta, \varphi).$$

From this claim, the theorem follows.

Take a very ample line bundle  $H$  on  $X$  and fix a Kähler form  $\omega \in c_1(H)$ , take a non-zero section  $s \in H^0(X, H)$ . Take  $N \in \mathbb{Z}_{>0}$  then at least one element among  $\{0, \dots, N-1\}$  occurs infinitely many times as the residues of  $k_i$  modulo  $N$  for various  $i$ . Up to replacing  $k_i$  by a subsequence, we may assume that  $k_i \equiv t$  for all  $i$ , where  $t \in \{0, \dots, N-1\}$ . Up to changing  $T$  to  $T \otimes L^{-t}$ , we may assume that  $t = 0$ .

We have an injective linear map

$$H^0(X, T \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)) \xrightarrow{\times s^k} H^0(X, T \otimes H^k \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)).$$

In particular, for each  $i \geq 1$ ,

$$k_i \Delta_\nu^{k_i, T}(N\theta, N\varphi) + k_i \nu(s) \subseteq k_i \Delta_\nu^{k_i, T}(N\theta + \omega, N\varphi).$$

Let  $i \rightarrow \infty$ , by Step 1,

$$N\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + N\theta, N\varphi).$$

So

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(N^{-1}\omega + \theta, \varphi) = \int_X (N^{-1}\omega + \theta + \text{dd}^c P^{N^{-1}\omega + \theta}[\varphi]_{\mathcal{I}})^n.$$

By [Xia21, Corollary 4.4], the right-hand side is equal to

$$\int_X (N^{-1}\omega + \theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n.$$

Let  $N \rightarrow \infty$ , we find

$$\text{vol } \Delta' \leq \int_X (\theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n = \text{vol } \Delta_\nu(\theta, \varphi).$$

Our claim holds.  $\square$

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