# NOTE ON PARTIAL OKOUNKOV BODIES — THE POINT OF VIEW OF B-DIVISORS

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CONTENTS



## 1. INTRODUCTION

In this note, I will explain a more algebraic point view to the partial Okounkov bodies introduced in [Xia21].

The main theorem is the following:

thm:main **Theorem 1.1.** *Let $X$ be an irreducible complex projective manifold of dimension $n$ and $(L, \phi)$ be a Hermitian pseudo-effective line bundle on $X$ with positive volume. Fix a valuation $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ of rank $n$*[*].

*Then the partial Okounkov body $\Delta_\nu(L, \phi)$ admits the following expression:*

{eq:main}
$$\Delta_\nu(L,\phi) = \nu(\phi) + \lim_{\pi\colon Z\to X} \Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(\phi)\}\right), \tag{1.1}$$

*where $\pi$ runs over all smooth birational modifications of $X$.*

This theorem needs some explanations. Here $\mathrm{Sing}_Z(\phi)$ denotes the divisorial part of the Siu decomposition of $\mathrm{dd}^c\pi^*\phi$. The notation $\{\bullet\}$ means the associated numerical class. The limit is the Hausdorff limit. The valuation $\nu(\phi)$ is defined in [DX24] using the trace operator.

This theorem shows that the partial Okounkov bodies admit a natural interpretation in terms of the associated b-divisors.

One could easily generalize the argument below to the transcendental case, but I find no applications of such generalizations, so I will content myself to the algebraic setting.

One remark: in [DX24], we only explained how to define the trace operator when the subvariety is smooth. In general, the trace operator gives a $P$-equivalence class on the normalization of the subvariety. We will use these results freely.

## 2. SOME PRELIMINARIES

ationincseq **Lemma 2.1.** *Let $E_j$ be a countable family of distinct prime divisors on $X$. Consider $a_{ij} \in \mathbb{R}_{\geq 0}$ for all $i, j > 0$. We assume that the sequence $(a_{ij})$ for fixed $j$ is increasing in $i$. Moreover, assume that $a_j := \lim_{i\to\infty} a_{ij} < \infty$. Assume that the series $\sum_j a_j[E_j]$ converges, then*

$$\lim_{i\to\infty} \nu\left(\sum_j a_{ij}[E_j]\right) = \nu\left(\sum_j a_j[E_j]\right).$$

[*]Here implicitly, we assume that $\nu$ is surjective.

*Proof.* We may assume that the valuation $\nu$ is induced by a smooth flag $Y_\bullet$.

We argue by induction on the dimension $n$. When $n = 1$, there is nothing to argue. Assume that $n > 1$ and the case $n-1$ is known. We may assume that $Y_1$ is not among the $E_j$'s. Write $\mu$ for the valuation on $Y_1$ induced by the truncated flag. Then we need to prove the following:

$$\lim_{i\to\infty} \mu\left(\sum_j a_{ij}[E_j]|_{Y_1}\right) = \mu\left(\sum_j a_j[E_j]|_{Y_1}\right).$$

Note that $[E_j]|_{Y_1}$ is again the current of integration of an effective divisor on $Y_1$ (this can be seen using the Lelong–Poincaré formula for example), so the desired convergence follows by induction. □

**Lemma 2.2.** *Let $T$ be a closed positive $(1,1)$-current on $X$. Then we have*

$$\lim_{\pi\colon Z\to X} \nu(\mathrm{Sing}_Z(T)) = \nu(T). \tag{2.1}$$

*Proof.* We may assume that $\nu$ is induced by a smooth flag $Y_\bullet$ on $X$.

Given $\pi\colon Z \to X$, we let $W_1$ denote the strict transform of $Y_1$ in $Z$. The restriction $\pi_1\colon W_1 \to Y_1$ is necessarily birational.

We will argue by induction. The case $n = 1$ is trivial. Assume that $n > 1$ and the case $n-1$ is known.

We may clearly assume that $\nu(T, Y_1) = 0$. By definition, we have

$$\nu(T) = (0, \mu(\mathrm{Tr}_{Y_1}(T))),$$

where $\mu$ denotes the valuation induced by the flag on $Y_1$ induced by $Y_\bullet$.

Observe that modifications of the form $\pi_1\colon W_1 \to Y_1$ is cofinal in the directed set of modifications of $Y_1$. This is obvious since the modifications given by compositions of blow-ups with smooth centers on $Y_1$ are cofinal.

Therefore, by induction, it suffices to argue that for any $\pi\colon Z \to X$, we have

$$\nu(\mathrm{Sing}_Z(T)) = \left(0, \mu(\mathrm{Sing}_{\widetilde{W}_1}(\mathrm{Tr}_{Y_1}(T)))\right), \tag{2.2}$$

where $\widetilde{W}_1$ is the normalization of $W_1$.

In order to prove (2.2), we may assume that $\pi$ is the identity map. This follows from the birational behaviour of the trace operator established in [DX24]. So we reduce to show the following:

$$\nu(\mathrm{Sing}_X(T)) = \left(0, \mu(\mathrm{Sing}_{Y_1}(\mathrm{Tr}_{Y_1}(T)))\right).$$

Adding a Kähler form to $T$, we may assume that $T$ is a Kähler current. Take a quasi-equisingular approximation $T_j$ of $T$. By the decreasing continuity of the trace operator proved in [DX24], the $d_S$-continuity of Lelong numbers proved in [Xia22] and Lemma 2.1, both sides are continuous along quasi-equisingular approximations, we reduce to the case where $T$ has analytic singularities. In this case, argue as before, we may assume that $T = [D]$ for a snc $\mathbb{Q}$-divisor $D$. By additivity, we finally reduce to the case where $D$ is a prime divisor on $X$ different from $Y_1$. The problem is reduced to

$$\nu([D]) = (0, \mu([D]|_{Y_1})),$$

which is clear by definition. □

## 3. THE PROOF

Now let us begin the argument of Theorem 1.1. We argue by induction on $n$. The case $n = 1$ is of course trivial. Let us assume that $n > 1$ and the result is known in dimension $n-1$.

We first make a few simplifications. Observe that (1.1) is birationally invariant, so we may assume that $\nu$ is equivalent the the valuation induced by a smooth flag. Furthermore, we reduce to the case that $\nu$ is the valuation induced by a smooth flag $Y_\bullet$.

It would be more convenient to use the language of currents. We shall write $T = \mathrm{dd}^{\mathrm{c}}\phi$. Then one needs to prove two things: first of all, the limit in (1.1) exists; secondly,

$$\Delta_\nu(T) = \nu(T) + \lim_{\pi\colon Z\to X} \Delta_\nu(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}). \tag{3.1}$$

We may replace $T$ by $T - \nu(T, Y_1)[Y_1]$ and $L$ by the numerical class $L - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$. But now $L$ is replaced by a big numerical class $\alpha$ on $X$ in the real Néron–Severi group of $X$. By perturbation, we may assume $\alpha$ lies in the rational Néron–Severi group. After a rescaling, we reduce back to the case where $\alpha$ is represented by a line bundle $L$.

Eventually we want to show (3.1) assuming that $\nu(T, Y_1) = 0$.

Let us prove (3.1). As shown in [DX24], we have

$$\Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\right) = \overline{\{\nu(S) : S \in c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\}}.$$

Therefore,

$$\Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\right) + \nu(\mathrm{Sing}_Z(T)) \subseteq \overline{\{\nu(S) : S \in c_1(L), \pi^*S \geq \mathrm{Sing}_Z(T)\}}.$$

We observe that the right-hand side is decreasing with respect to $\pi$, which together with Lemma 2.2 implies that the net of convex bodies $\Delta_\nu(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\})$ for various $Z$ is uniformly bounded. Suppose that $\Delta$ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in c_1(L), S \preceq_{\mathcal{I}} T\}}.$$

As shown in [DX24], the right-hand side is exactly $\Delta_\nu(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_\nu(T).$$

But observe that both sides have the same volume, as computed in [DX24] and [Xia22]. So equality holds.

It follows from the Blaschke selection theorem that the limit in (3.1) exists and (3.1) holds.

## References

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