Convergence of Sewing Conformal Blocks

BIN GUI

Contents

Abstract

In recent work [DGT19b], Damiolini-Gibney-Tarasca showed that for a C_2-cofinite rational vertex operator algebra $\mathbb{V}$, sheaves of conformal blocks are locally free and satisfy the factorization property. In this article, we prove that if $\mathbb{V}$ is C_2-cofinite, the sewing of conformal blocks is convergent. This proves a conjecture proposed by Zhu [Zhu94] and Huang [Hua16].

Introduction

In conformal field theory (CFT), conformal blocks are linear functionals defined for N-pointed compact Riemann surfaces, together with a vertex operator algebra (VOA) $\mathbb{V}$ and $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$. One can sew a (possibly disconnected) N-pointed compact Riemann surface along pairs of points (with local coordinates) to obtain a new pointed compact Riemann surface with possibly higher genus or more marked points. Corresponding to this geometric sewing construction, one can also sew conformal blocks by taking contractions [Seg88, Vafa87, TK88, TUY89, BFM91, Zhu94, Hua97, Hua05a, Hua05b, NT05, Hua16, DGT19b]. In this article, we prove that for $\mathbb{V}$ satisfying natural conditions, sewing conformal blocks is convergent, solving a conjecture proposed in [Zhu94, Conj. 8.1] and [Hua16, Problem 2.2]. Our result provides a necessary step for the construction of higher genus rational (Euclidean) CFT in the sense of Segal [Seg88]. We hope it will also benefit other approaches to Euclidean CFT, such as the functional analytic one in [Ten17, Ten19a, Ten19b, Ten19c].

Conformal blocks were first studied by physicists [BPZ84, FS87, MS89]. In mathematics, conformal blocks were defined and explored by Tsuchiya-Kanie [TK88] for Weiss-Zumino-Witten (WZW) models and genus 0 Riemann surfaces, and were generalized by Tsuchiya-Ueno-Yamada to stable curves of all genera in [TUY89]. For minimal models, conformal blocks over any stable curve was studied by Beilinson-Feigein-Mazur [BFM91]. In particular, factorization was proved for these models in [TUY89, BFM91]. A definition of conformal blocks for quasi-primarily generated VOAs was given by Zhu in [Zhu94], and was generalized to any ($\mathbb{N}$-graded) VOA on families of compact Riemann surfaces by Frenkel and Ben-Zvi in [FB04]. In [DGT19a, DGT19b], Damiolini-Gibney-Tarasca defined conformal blocks for VOAs associated to (algebraic) families of stable curves, and showed that when $\mathbb{V}$ is C_2-cofinite and rational, sheaves of conformal blocks are vector bundles with projectively flat connections, and proved the factorization property and a (formal) sewing theorem. Their results generalize those of Tsuchiya-Ueno-Yamada [TUY89] for WZW models, those of Nagatomo-Tsuchiya [NT05] for genus 0 curves, and part of the results of Huang for genus 0 and 1 curves [Hua95, Hua05a, Hua05b]. In [DGT19c], the authors proved Chern characters of these bundles give cohomological field theories, and called CFT-type C_2-cofinite rational VOAs of CohFT-type. In the conformal net approach to CFT, conformal blocks were defined and investigated by Bartels-Douglas-Henriques [BDH17].

Sewing conjecture

Conformal blocks are the chiral halves of correlation functions of full CFT. Assume throughout this article that $\mathbb{V}$ is a positive-energy vertex operator algebra (VOA), i.e. the L_0-grading on $\mathbb{V}$ is $\mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}(n)$ where each $\mathbb{V}(n)$ is finite-dimensional. For an N-pointed compact Riemann surface $\mathfrak{X} = (C; x_1, \ldots, x_N)$ where $x_1, \ldots, x_N$ are distinct points of a compact Riemann surface C[1], one associates to these points $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$. Then a conformal block ψ is a linear functional on $\mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ "invariant" under the actions of certain global sections related to $\mathbb{V}$. If one has a family of N-pointed compact Riemann surfaces parametrized by a base manifold $\mathcal{B}$, then one allows ψ to vary holomorphically over $\mathcal{B}$. The vertex operator $Y(\cdot, z)$ can be regarded as a conformal block associated to $\mathbb{P}^1$ with distinct points $0, z, \infty$ and modules $\mathbb{V}, \mathbb{V}, \mathbb{V}'$ (where $\mathbb{V}'$ is the contragredient module of $\mathbb{V}$). More generally, if $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$ are $\mathbb{V}$-modules, an intertwining operator $\mathcal{Y}(\cdot, z)$ of type $\binom{\mathbb{W}_3}{\mathbb{W}_1 \mathbb{W}_2}$ (as defined in [FHL93]) corresponds to a conformal block associated to $(\mathbb{P}^1, 0, z, \infty)$ and $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3'$.

A fundamental problem in rational CFT is to prove the convergence of sewing conformal blocks. Suppose we have an $(N+2)$-pointed compact Riemann surface

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; x_1, \ldots, x_N, x', x'').$$

Then we can sew $\widetilde{\mathfrak{X}}$ along the pair of points x', x'' to obtain another Riemann surface with possibly higher genus. More precisely, we choose ξ, ϖ to be local coordinates of $\widetilde{C}$ at x', x''. Namely, they are univalent (i.e. holomorphic and injective) functions defined respectively in neighborhoods $U' \ni x', U'' \ni x''$ satisfying $\xi(x') = 0, \varpi(x'') = 0$. For each $r > 0$ we let $\mathcal{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathcal{D}_r^\times = \mathcal{D}_r - \{0\}$. We choose $r, \rho > 0$ so that the neighborhoods U', U'' can be chosen to satisfy that $\xi(U') = \mathcal{D}_r$ and $\varpi(U'') = \mathcal{D}_\rho$, that $U' \cap U'' = \varnothing$, and that none of $x_1, \ldots, x_N$ is in U' or U''. Then, for each $q \in \mathcal{D}_{r\rho}^\times$, we remove the closed subdiscs of U', U'' determined respectively by $|\xi| \leqslant \frac{|q|}{\rho}$ and $|\varpi| \leqslant \frac{|q|}{r}$, and glue the remaining part using the relation $\xi \varpi = q$. Then we obtain an N-pointed compact Riemann surface

$$\mathfrak{X}_q = (\mathcal{C}_q; x_1, \ldots, x_N)$$

which clearly depends on ξ and ϖ.

Now, if we associate finitely-generated $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N, \mathbb{M}, \mathbb{M}'$ (where $\mathbb{M}'$ is the contragredient (i.e. dual) module of $\mathbb{M}$) to $x_1, \ldots, x_N, x', x''$, and choose a conformal block ψ associated to $\widetilde{\mathfrak{X}}$ and these $\mathbb{V}$-modules, then its **sewing** $\mathcal{S}\psi$ is an $\mathbb{W}_\bullet^* = (\mathbb{W}_1 \otimes \cdots \mathbb{W}_N)^*$-valued formal series of q defined by sending each $w_\bullet = w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_\bullet$ to

$$\mathcal{S}\psi(w_\bullet) = \psi(w_\bullet \otimes q^{L_0} \blacktriangleright \otimes \blacktriangleleft) \quad \in \mathbb{C}\{q\}[\log q]$$

where $\blacktriangleright \otimes \blacktriangleleft$ is the element of the "algebraic completion" of $\mathbb{M} \otimes \mathbb{M}'$ corresponding to the identity element of $\mathrm{End}_\mathbb{C}(\mathbb{M})$, and L_0 is the zero mode of the Virasoro operators $\{L_n : n \in \mathbb{Z}\}$. (Cf. [Seg88, Vafa87, TUY89, Hua97, DGT19b].) The **sewing conjecture**, as proposed in [Zhu94, Conj. 8.1] and [Hua16, Problem 2.2], says that if $\mathbb{V}$ satisfies nice properties (such as the C_2-cofiniteness), then $\mathcal{S}\psi(w_\bullet)$ converges absolutely to a

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[1] We do not assume C to be connected

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(possibly) multivalued function on $\mathcal{D}_{r\rho}^{\times}$. Moreover, for each $q \in \mathcal{D}_{r\rho}^{\times}$, $\mathcal{S}\psi(\cdot, q)$ defines a conformal block associated to $\mathfrak{X}_q$ and $\mathbb{W}_1, \ldots, \mathbb{W}_N$. If we sew $\widetilde{C}$ along n pairs of points, and if we let $x_1, \ldots, x_N$ and $\widetilde{C}$ and ψ vary and be parametrized holomorphically by variables $\tau_\bullet = (\tau_1, \ldots, \tau_m)$, then sewing conformal blocks is also absolutely convergent with respect to $q_1, \ldots, q_n$ and (locally) uniform with respect to $\tau_\bullet$.

In this article, we give a complete proof of the sewing conjecture (see Section 13 for the main result), which complements the results of [DGT19b] on local freeness and factorization of sheaves of conformal blocks, and provides a necessary step of constructing rational conformal field theories on arbitrary (families of) compact Riemann surfaces. We remark that a sewing theorem (Thm. 8.5.1) was proved in [DGT19b]. In that theorem, one treats the (infinitesimal) formal disc $\mathrm{Spec}\mathbb{C}[[q]]$ instead of the analytic disc $\mathcal{D}_{r\rho}$, which is sufficient for application in the algebraic category. In particular, the convergence of sewing is not needed and not proved in [DGT19b]. In the analytic category, which is the one we are working in, the convergence is necessary. (Also, note that $\mathcal{D}_{r\rho}$ is not well-defined in the algebraic category.)

Historically, the sewing conjecture was proved in some special cases. Our result is general in the following aspects:

(a) We consider any $\mathbb{N}$-graded C_2-cofinite VOA $\mathbb{V}$ and any finitely-generated (admissible) $\mathbb{V}$-module.

(b) We consider compact N-pointed Riemann surfaces of all genera.

(c) We consider any (analytic) coordinates ξ, ϖ at x', x'', and prove the convergence on $\mathcal{D}_{r\rho}$ whenever r, ρ satisfy the previously described conditions. (Namely, we prove the convergence not only when q is small.)

(d) We consider sewing along several pairs of points, and allow $\widetilde{\mathfrak{X}}$ and ψ to be parametrized holomorphically by some $\tau_\bullet$.

To our knowledge, no previous results have covered all these aspects. Nevertheless, even those partial results have played extremely important roles in the development of a rigorous mathematical theory of conformal field theory. For instance: convergence in the genus 0 case is necessary for the statement of braiding and operator product expansions (fusion) of intertwining operators [TK88, Hua05a]; convergence of self-sewing a 3-pointed $\mathbb{P}^1$ (which leads to a 1-pointed elliptic curve) is necessary for the statement of modular invariance of VOA characters [Zhu96]; convergence of sewing a general N-pointed $\mathbb{P}^1$ is necessary for the proof of Verlinde conjecture and the rigidity and modularity of the tensor category of VOA modules [Hua05b, Hua08a, Hua08b].

History of the proof of convergence

In [TK88], Tsuchiya and Kanie proved for type A_1 WZW models the convergence of the products of intertwining operators, i.e. the convergence of sewing conformal blocks of such VOAs from (possibly disconnected) genus 0 to genus 0 Riemann surfaces. Their method applies directly to any WZW model. The local coordinates ξ, ϖ in their result are z, z^{-1} at $0, \infty$. Their method is to show that the formal series $\mathcal{S}\psi$ satisfies a differential equation (the Knizhnik–Zamolodchikov equation) with simple pole

at $q = 0$, and the coefficients of the differential equation is a (matrix-valued) analytic function of q and $\tau_\bullet$ (parametrizing $\widetilde{\mathfrak{X}}$ and ψ). Any result about convergence proved after [TK88], including ours, follows this pattern. The difficulty is, of course, to find such differential equations.

In [TUY89], Tsuchiya-Ueno-Yamada showed that for all WZW-models and all compact Riemann surfaces, there exist local coordinates ξ, ϖ at x', x'' (which are called z, w and described in [TUY89, Lemma 6.1.2] and [Ueno97, Lemma 5.3.1]) such that such differential equations exist for small q. This leads immediately to the convergence of $\mathcal{S}\psi$ under those conditions, which was later explicitly claimed in [Ueno97, Thm. 5.3.4]. Generalizing their result to any ξ, ϖ is not straightforward.

In [Zhu96], Zhu proved the convergence for any CFT-type C_2-cofinite VOA and ordinary $\mathbb{V}$-modules, for self-sewing an $(N + 2)$-pointed $\mathbb{P}^1$ along $0, \infty$ with respect to local coordinates z, z^{-1} to an N-pointed elliptic curve, assuming that the N $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$ not associated $0, \infty$ are the vacuum module $\mathbb{V}$. Its generalization to any $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$ is non-trivial and proved by Huang in [Hua05b]. When $\mathbb{W}_1, \ldots, \mathbb{W}_N$ are grading-restricted generalized $\mathbb{V}$-modules (equivalently, finitely-generated admissible $\mathbb{V}$-modules [Hua09]), the convergence was proved by Fiordalisi in [Fio16]. Meanwhile, for C_2-cofinite VOAs and ordinary $\mathbb{V}$-modules, for sewing several pointed $\mathbb{P}^1$ to a pointed $\mathbb{P}^1$, and assuming the local coordinates at sewing points are z or z^{-1}, Huang proved in [Hua05a] the convergence of sewing conformal blocks for any $\mathbb{V}$-modules. This result, with the help of [Hua98], can be generalized to any local coordinates ξ, ϖ at sewing points. When $\mathbb{W}_1, \ldots, \mathbb{W}_N$ are grading-restricted generalized $\mathbb{V}$-modules, the convergence is due to Huang-Lepowski-Zhang [HLZ11]. The solution of sewing conjecture is more or less complete in genus 0. When $\mathbb{V}$ is holomorphic, the convergence of sewing with respect to suitable local coordinates was proved by [Cod19].

Idea of the proof

Our proof of convergence is motivated by [DGT19b]. To explain the idea, we assume for simplicity that the $\mathbb{V}$-modules are semisimple. It was shown in [DGT19b] that by sewing a conformal block ψ, we get a formal conformal block $\mathcal{S}\psi$ as a formal series of q (Thm. 8.5.1) annihilated by $\nabla_{q\partial_q}$ (Rem. 8.3.3) where ∇ is a connection of the sheaf of conformal blocks on the infinitesimal disk $\mathrm{Spec}(\mathbb{C}[[q]])$ defined in [DGT19a]. The definition of ∇ is unique up to a projective term. In other words, if we choose ∇ to be an (analytic) connection defined on the analytic disc $\mathcal{D}_{r\rho}$, then $\mathcal{S}\psi$ is annihilated by $\nabla_{q\partial_q}$ plus a projective term f which is a priori only a formal power series of q. A key step of proving the convergence of $\mathcal{S}\psi$ is to show that f converges. Then, using a finiteness theorem 7.4 analogous to [DGT19b, Thm. 8.4.1], we obtain the desired differential equation.

It turns out that the connection ∇ is determined by a (relative) projective structure $\mathfrak{P}$ on $\widetilde{\mathfrak{X}}$. Moreover, when ξ, ϖ belong to $\mathfrak{P}$, the projective term f equals 0. Thus, for a chosen $\mathfrak{P}$ and the corresponding ∇, if we assume ξ, ϖ belong to $\mathfrak{P}$, then $\nabla_{q\partial_q}\mathcal{S}\psi = 0$, which will provide the differential equation.[2] The vanishing of f is due to that of

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[2]This observation is due to Liang Kong and Hao Zheng.

the Schwarzian derivatives between local coordinates belonging to $\mathfrak{P}$. In general, one cannot expect that ξ, ϖ belong to the same projective structure. To resolve this issue, we fix a projective structure $\mathfrak{P}$, and find an explicit formula of f in terms of $\mathfrak{P}, \xi, \varpi$, and the other local coordinates. That formula (see (11.18)) shows that f is analytic.

Outline

To carry out the above ideas, we first define and study some basic properties of sheaves of conformal blocks on complex curves and analytic families of curves. This is achieved in Sections 3 and 6. To prepare for this task, we first review Huang's change of coordinate formulas [Hua97] in Section 1. This formula is used to define sheaves of VOAs on curves and families of curves in Sections 2 and 5. In Section 4, we follow [TUY89, Sec. 6.1] and give a precise description of how to sew a compact Riemann surface and form a family of curves over the disc $\mathcal{D}_{r\rho}$. In fact, we describe the simultaneous sewing for a family of compact Riemann surfaces over a complex manifold $\widetilde{\mathcal{B}}$, which yield a family of complex curve over $\mathcal{B} := \widetilde{\mathcal{B}} \times \mathcal{D}_{r\rho}$.

In Section 7, we prove a finiteness theorem which will turn the relation $(\nabla_{q\partial_q} + f)\mathcal{S}\psi = 0$ into a differential equation with analytic coefficients (provided that f converges) and simple poles at $q = 0$. Section 8 recalls some basic facts about Schwarzian derivatives, and Section 9 prepares for the calculation of the projective term f. In Section 10, we give a proof that $\mathcal{S}\psi$ is a formal conformal block using an argument similar to but slightly different from the one in [DGT19b]. Then, in Section 11, we prove the convergence of sewing conformal blocks associated to a family $\widetilde{\mathfrak{X}}$ of compact Riemann surfaces along a pair of (families of) points. This result is generalized to sewing along several pairs of points in Section 13. In particular, our convergence theorem in the most general form is given in Theorem 13.1 of that section. In Section 12, we show that the sewing map $\psi \mapsto \mathcal{S}\psi$ (defined in a suitable and natural way) is injective, assuming $\mathbb{V}$ is C_2-cofinite and the modules for sewing are semisimple. If $\mathbb{V}$ is also CFT-type and rational, then this map is also bijective due to the factorization property proved in [DGT19b]. Using these results, we give an analytic version of the sewing theorem (Thm. 8.5.1) of [DGT19b].

We would like to point out that although many ideas are common in the analytical and algebraic settings, there are some subtle differences that will lead to different proof strategies. See Remark 7.5 for instance. Thus, we believe it is worthwhile to provide in our article a detailed account of the analytic theory of conformal blocks and a self-contained proof of convergence. In particular, our proof does not rely on the factorization or the local freeness of sheaves of conformal blocks.

Acknowledgment

1 Change of coordinates

Throughout this article, we let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. Also, $\mathbb{C}^\times = \mathbb{C} - \{0\}$. If W is a vector space and z is a (formal) variable, we define

$$W[[z]] = \left\{ \sum_{n \in \mathbb{N}} w_n z^n : \text{each } w_n \in W \right\},$$

$$W[[z^{\pm 1}]] = \left\{ \sum_{n \in \mathbb{Z}} w_n z^n : \text{each } w_n \in W \right\},$$

$$W((z)) = \left\{ f(z) : z^k f(z) \in W[[z]] \text{ for some } k \in \mathbb{Z} \right\},$$

$$W\{z\} = \left\{ \sum_{n \in \mathbb{C}} w_n z^n : \text{each } w_n \in W \right\}.$$

$W[\log z]\{z\}$ and $W\{z\}[\log z]$ are understood in the obvious way by treating z and $\log z$ as two unrelated formal variables. In particular, they are subspaces of $W\{\log z, z\}$. On $W\{\log z, z\}$, we define a linear operator ∂_z by (choosing $w \in W$)

$$\partial_z(w \cdot z^k (\log z)^l) = w \cdot (k z^{k-1} (\log z)^l + l z^{k-1} (\log z)^{l-1}). \tag{1.1}$$

Clearly $z \partial_z$ preserves the subspaces $W[\log z]\{z\}$ and $W\{z\}[\log z]$.

Let $\mathbb{V}$ be a vertex operator algebra (VOA for short) in the sense of [FHL93]. We let $\mathbf{1}$ and $\mathbf{c}$ be respectively the vacuum vector and the conformal vector of $\mathbb{V}$. For each $u \in \mathbb{V}$, we write the vertex operator as $Y(v, z) = \sum_{n \in \mathbb{Z}} Y(v)_n z^{-n-1}$ where each $Y(v)_n \in \mathrm{End}(\mathbb{V})$. Then $\{L_n = Y(\mathbf{c})_{n+1}\}$ are the Virasoro operators with a central charge $c \in \mathbb{C}$. We write $\mathbb{V} = \bigoplus_{n \in \mathbb{Z}} \mathbb{V}(n)$ where $\mathbb{V}(n)$ is the eigenspace of L_0 with eigenvalue n. We write $\mathrm{wt}(v) = n$ if $v \in \mathbb{V}(n)$. We assume L_0 has no negative eigenvalues (i.e., $\mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}(n)$), and each $\mathbb{V}(n)$ is finite dimensional.

In this article, a $\mathbb{V}$-module $\mathbb{W}$ (with vertex operators $Y_\mathbb{M}(v, z) = \sum_{n \in \mathbb{Z}} Y_\mathbb{M}(v)_n z^{-n-1}$ and Virasoro operators $L_n = Y_\mathbb{M}(\mathbf{c})_{n+1}$) means a **finitely-admissible $\mathbb{V}$-module**. This means that $\mathbb{W}$ is a weak $\mathbb{V}$-module in the sense of [DLM97], that $\mathbb{W}$ is equipped with a diagonalizable operator $\widetilde{L}_0$ satisfying $[\widetilde{L}_0, Y_\mathbb{W}(v)_n] = Y_\mathbb{W}(L_0 v)_n - (n+1) Y_\mathbb{W}(v)_n$, that the eigenvalues of $\widetilde{L}_0$ are in $\mathbb{N}$, and that each eigenspace $\mathbb{W}(n)$ is finite-dimensional. (Without the last finite-dimension condition, $\mathbb{W}$ is an admissible (i.e. $\mathbb{N}$-gradable) module in the usual sense.) Let

$$\mathbb{W} = \bigoplus_{n \in \mathbb{N}} \mathbb{W}(n)$$

be the grading given by $\widetilde{L}_0$, then

$$Y_\mathbb{W}(v)_m \mathbb{W}(n) \subset \mathbb{W}(n + \mathrm{wt}(v) - m - 1). \tag{1.2}$$

In (1.2), by taking $v = \mathbf{c}$, we see that each $\mathbb{W}(n)$ is L_0-invariant, or equivalently, $[\widetilde{L}_0, L_0] = 0$. We can decompose $L_0|_{\mathbb{W}(n)}$ and hence L_0 into mutually commuting the semi-simple part and nilpotent part $L_0 = L_{0,\mathrm{s}} + L_{0,\mathrm{n}}$. $\widetilde{L}_0$ clearly commutes with $L_{0,\mathrm{s}}$ and $L_{0,\mathrm{n}}$. We let

$$\mathbb{W} = \bigoplus_{n \in \mathbb{C}} \mathbb{W}_{[n]}$$

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be the $L_{0,s}$-grading of $\mathbb{W}$. Then we have (cf. [HZL14, Prop. 2.19])

$$Y_{\mathbb{W}}(v)_m \mathbb{W}_{[n]} \subset \mathbb{W}_{[n+\text{wt}(v)-m-1]}, \tag{1.3}$$

or equivalently, $[L_{0,s}, Y_{\mathbb{W}}(v)_n] = Y_{\mathbb{W}}(L_0 v)_n - (n+1)Y_{\mathbb{W}}(v)_n$. By Jacobi identity, the same relation holds when $L_{0,s}$ is replaced by L_0. So both $\widetilde{L}_0 - L_{0,s}$ and $L_{0,n} = L_0 - L_{0,s}$ commute with the action of $\mathbb{V}$ on $\mathbb{W}$. Thus, we see that if $\mathbb{W}$ is finitely generated, then there is $K \in \mathbb{N}$ such that $L_{0,n}{}^k w = 0$ for all $w \in \mathbb{W}$.

We say that $\mathbb{W}$ is $L_{0,s}$-**semisimple** if there exists a finite set $E \subset \mathbb{C}$ such that

$$\mathbb{W} = \bigoplus_{n \in E+\mathbb{N}} \mathbb{W}_{[n]}, \qquad \dim \mathbb{W}_{[n]} < +\infty.$$

In this case, we can choose E such that any two elements of E do not differ by an integer. Then for each $\alpha \in E$, $\mathbb{W}_\alpha = \bigoplus_{n \in \alpha+\mathbb{N}} \mathbb{W}_{[n]}$ is a submodule of $\mathbb{W}$. Such $\mathbb{W}_\alpha$ is called $L_{0,s}$-**simple**. Thus, every $L_{0,s}$-semisimple $\mathbb{V}$-module is a finite direct sum of $L_{0,s}$-quasi simple ones. It is clear that $\mathbb{W}$ is $L_{0,s}$-simple if and only if $\widetilde{L}_0 - L_{0,s}$ is a constant.

A vector $w \in \mathbb{W}$ is called $\widetilde{L}_0$- (resp. $L_{0,s}$-) homogeneous with weight n if $w \in \mathbb{W}(n)$ (resp. $w \in \mathbb{W}_{[n]}$). In this case, we write $\widetilde{\text{wt}}(w) = n$ (resp. $\text{wt}(w) = n$). Note that the $\widetilde{L}_0$-weights are natural numbers but the $L_{0,s}$-weights are not necessarily.

Convention 1.1. For the vacuum module $\mathbb{V}$, we choose $\widetilde{L}_0$ to be L_0. So $\mathbb{V}(n) = \mathbb{V}_{(n)}$. If $\mathbb{W}$ is a $\mathbb{V}$-module with simple L_0-grading, we choose $\widetilde{L}_0$ such that it equals a constant plus L_0.

We now recall the formula for changing coordinates discovered by Huang [Hua97]. To begin with, we let $\mathscr{O}_{\mathbb{C},0}$ be the stalk of the sheaf of holomorphic functions of $\mathbb{C}$ at 0. Namely, an element in $\mathscr{O}_{\mathbb{C},0}$ is precisely a formal power series $f(z) = \sum_{n \in \mathbb{N}} a_n z^n$ ($a_n \in \mathbb{C}$) converging absolutely in a neighborhood of 0. We consider the subset $\mathbb{G}$ of all $\rho \in \mathscr{O}_{\mathbb{C},0}$ satisfying $\rho(0) = 0$ and $\rho'(0) \neq 0$. Then $\mathbb{G}$ becomes a group if we define the multiplication of two elements ρ_1, ρ_2 to be their composition $\rho_1 \circ \rho_2$. The identity element of $\mathbb{G}$ is the standard coordinate z of $\mathbb{C}$.

For each $\rho \in \mathbb{G}$, we can find $c_0, c_1, c_2, \cdots \in \mathbb{C}$ such that

$$\boxed{\rho(z) = c_0 \cdot \exp\left(\sum_{n>0} c_n z^{n+1}\partial_z\right)z}$$

For instance, if we write

$$\rho(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.4}$$

then one has

$$c_0 = a_1,$$
$$c_1 c_0 = a_2,$$
$$c_2 c_0 + c_1^2 c_0 = a_3.$$

In particular, one has

$$c_0 = \rho'(0).$$

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We define $\mathcal{U}(\rho) \in \mathrm{End}(\mathbb{W})$ to be

$$\mathcal{U}(\rho) = \rho'(0)^{\tilde{L}_0} \cdot \exp\left(\sum_{n>0} c_n L_n \right) \tag{1.5}$$

Notice $a_n = \rho^{(n)}(0)/n!$, we have

$$c_1 = \frac{1}{2} \frac{\rho''(0)}{\rho'(0)},$$

$$c_2 = \frac{1}{6} \frac{\rho'''(0)}{\rho'(0)} - \frac{1}{4} \left(\frac{\rho''(0)}{\rho'(0)} \right)^2. \tag{1.6}$$

Remark 1.2. Considering the action of $\mathcal{U}(\varrho)$ on $\mathbb{W}$ might be inconvenient since $\mathbb{W}$ is not finite-dimensional. On the other hand, $\mathbb{W}(n)$ might not be preserved by $\mathcal{U}(\rho)$. Thus, it would be better to consider $\mathbb{W}^{\leqslant n} = \bigoplus_{k \leqslant n} \mathbb{W}(k)$ which is finite-dimensional and preserved by $\mathcal{U}(\varrho)$.

Since $L_m \mathbb{W}^{\leqslant n} \subset \mathbb{W}^{\leqslant n-1}$ when $m > 0$, from (1.5) it is easy to see that for any $w \in \mathbb{W}(n)$,

$$\mathcal{U}(\rho)w = \rho'(0)^n w \mod \mathbb{W}^{n-1}. \tag{1.7}$$

In other words, the action of $\mathcal{U}(\rho)$ on $\mathbb{W}^{\leqslant n}/\mathbb{W}^{\leqslant n-1}$ is $\rho'(0)^n \mathbf{1}$.

The following was (essentially) proved in [Hua97] section 4.2:

Theorem 1.3. *For each $\mathbb{V}$-module $\mathbb{W}$, $\mathcal{U}$ is a representation of $\mathbb{G}$ on $\mathbb{W}$. Namely, we have $\mathcal{U}(\rho_1 \circ \rho_2) = \mathcal{U}(\rho_1)\mathcal{U}(\rho_2)$ for each $\rho_1, \rho_2 \in \mathbb{G}$.*

Example 1.4. It is easy to see that $(c_1 z^2 \partial_z)^n z = n! c_1^n z^{n+1}$. Thus $\exp(c_1 z^2 \partial_z)z = \sum_{n=0}^{\infty} c_1^n z^{n+1} = z/(1 - c_1 z)$. For each $\xi \in \mathbb{C}^\times$, we set $\gamma_\xi \in \mathbb{G}$ to be

$$\gamma_\xi(z) = \frac{1}{\xi + z} - \frac{1}{\xi}. \tag{1.8}$$

If we set $\alpha(z) = -\xi^{-2}z$, then $\gamma_\xi(z) = \alpha/(1 - \xi\alpha) = \exp(c_1 \alpha^2 \partial_\alpha)(\alpha)$. Thus, by (1.5) and that $\mathcal{U}$ preserves composition, we obtain

$$\mathcal{U}(\gamma_\xi) = e^{\xi L_1}(-\xi^{-2})^{\tilde{L}_0}. \tag{1.9}$$

In particular,

$$\mathcal{U}(\gamma_1) = e^{L_1}(-1)^{\tilde{L}_0}. \tag{1.10}$$

It is easy to see $\gamma_\xi(\xi z) = \xi^{-1}\gamma_1(z)$. Thus

$$\mathcal{U}(\gamma_\xi)\xi^{\tilde{L}_0} = \xi^{-\tilde{L}_0}\mathcal{U}(\gamma_1). \tag{1.11}$$

Remark 1.5. Let X be a complex manifold and $\rho : X \to \mathbb{G}, x \mapsto \rho_x$ a function. We say that ρ is a **holomorphic family** of transformations if for any $x \in X$, there exists an open subset $V \subset X$ containing x and an open $U \subset \mathbb{C}$ containing 0 such that $(z, y) \in U \times V \mapsto \rho_y(z)$ is a holomorphic function on $U \times V$. Then it is clear that the coefficients $a_1, a_2, \ldots$ in (1.4) depend holomorphically on the parameter $x \in X$. Hence the same is true for $c_0, c_1, c_2, \ldots$. Thus, by the formula (1.5), for any $w \in \mathbb{W}^{\leqslant n}$, $x \in X \mapsto \mathcal{U}(\rho_x)w$ is a $\mathbb{W}^{\leqslant n}$-valued holomorphic function on X. Thus $\mathcal{U}(\rho)$ can be regarded as an isomorphism of $\mathscr{O}_X$-modules

$$\mathcal{U}(\rho) : \mathbb{W}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_X \xrightarrow{\simeq} \mathbb{W}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_X \tag{1.12}$$

sending each $\mathbb{W}^{\leqslant n}$-valued function w to the section $x \mapsto \mathcal{U}(\rho_x)w(x)$. Its inverse is $\mathcal{U}(\rho^{-1})$.

If $\mathbb{W}$ is a $\mathbb{V}$-module, then its **contragredient module** $\mathbb{W}'$ can be describe using $\mathcal{U}(\gamma_z)$. As a vector space,

$$\mathbb{W}' = \bigoplus_{n \in \mathbb{C}} \mathbb{W}(n)^*$$

where $\mathbb{W}(n)^*$ is the dual space of $\mathbb{W}(n)$. For each $v \in \mathbb{V}$, the vertex operator $Y_{\mathbb{W}'}(v, z)$ is defined such that if $w \in \mathbb{W}, w' \in \mathbb{W}'$, then, using $\langle, \rangle$ to denote the natural pairing of $\mathbb{W}$ and $\mathbb{W}'$, we have

$$\langle Y_{\mathbb{W}'}(v, z)w', w \rangle = \langle w', Y_{\mathbb{W}}(\mathcal{U}(\gamma_z)v, z^{-1})w \rangle, \tag{1.13}$$

recalling that $\mathcal{U}(\gamma_z) = e^{zL_1}(-z^{-2})^{L_0}$. That $(\mathbb{W}', Y_{\mathbb{W}'})$ satisfies the definition of a $\mathbb{V}$-module follows from [FHL93]. We have

$$\widetilde{L}_0^{\mathrm{t}} = \widetilde{L}_0, \tag{1.14}$$

i.e., $\langle \widetilde{L}_0 w, w' \rangle = \langle w, \widetilde{L}_0 w' \rangle$ for each $w \in \mathbb{W}, w' \in \mathbb{W}'$. Also, by choosing the v in (1.13) to be the conformal vector, we see $L_n^{\mathrm{t}} = L_{-n}$ and in particular $L_0^{\mathrm{t}} = L_0$. So $L_{0,\mathrm{s}}^{\mathrm{t}} = L_{0,\mathrm{s}}, L_{0,\mathrm{n}}{}^{\mathrm{t}} = L_{0,\mathrm{n}}$.

2   Sheaves of VOAs on complex curves

In this article, for any complex manifold or complex space X, we let $\mathscr{O}_X$ denote the sheaf (of germs) of holomorphic functions of X. So for each open subset $U \subset X$, $\mathscr{O}_X(U)$ (written also as $\mathscr{O}(U)$ for short) is the algebra of holomorphic functions on U. For every (sheaf of) $\mathscr{O}_X$-modules $\mathscr{E}$, recall the usual notation that $\mathscr{E}_x$ is the stalk of $\mathscr{E}$ at $x \in X$. The dual $\mathscr{O}_X$-module of $\mathscr{E}$ is denoted by $\mathscr{E}^*$, or $\mathscr{E}^{-1}$ when $\mathscr{E}$ is a line bundle. For two $\mathscr{O}_X$-modules $\mathscr{E}, \mathscr{F}$, we write their tensor product $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ as $\mathscr{E} \otimes \mathscr{F}$ for short. If Y is a complex submanifold or complex subspace of X, we let $\mathscr{E}|Y$ (also written as $\mathscr{E}|_Y$) denote the restriction of $\mathscr{E}$ to X, namely, the pullback of $\mathscr{E}$ along the inclusion map $Y \hookrightarrow X$. The restriction of a section s of $\mathscr{E}$ is denoted by $s|X$ or $s|_X$. In the case Y is a single point $\{x\}$, the restriction $\mathscr{E}|x$ can be naturally identified with $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ where $\mathfrak{m}_x$ the ideal of all $f \in \mathscr{O}_{X,x}$ vanishing at x. If s is a section defined near x, we let $s(x)$

be the restriction $s|x$. If we consider $s(x)$ as an element of $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$, then $s(x)$ equals $s_x + \mathfrak{m}_x\mathscr{E}_x$, where $s_x \in \mathscr{E}_x$ is the germ of s at x.

By a complex curve C, we mean either a compact Riemann surface[3] or a (simple) nodal curve. For simplicity, we assume throughout this article that a nodal curve has only one (simple) node. We let ω_C denote the dualizing sheaf of C, which is the sheaf of holomorphic 1-forms when C is smooth (i.e., a compact Riemann surface). Its dual sheaf is denoted by $\Theta_C = \omega_C^{-1}$, which is the (holomorphic) tangent bundle when C is smooth. In the case that C is nodal, $\mathscr{O}_C, \omega_C, \Theta_C$ are described as follows.

Assume C has only one simple node. Then C can be obtained by gluing two distinct points y', y'' of a compact Riemann surface $\widetilde{C}$ (the normalization of C).[4] The gluing map is denoted by $\nu : \widetilde{C} \to C$. We identify $\widetilde{C} - \{y', y''\}$ with $C - \{x'\}$ (where $x' = \nu(y') = \nu(y'')$) via ν. Then $\mathscr{O}_C(U), \omega_C(U), \Theta_C(U)$ agree with $\mathscr{O}_{\widetilde{C}}(U), \omega_{\widetilde{C}}(U), \Theta_{\widetilde{C}}(U)$ when $x' \neq U$. If $x' \in U$ and U is small enough such that $\nu^{-1}(U)$ is a disjoint union of neighborhoods $V' \ni y', V'' \ni y''$, and that there exist univalent functions $\xi \in \mathscr{O}(V'), \varpi \in \mathscr{O}(V'')$ satisfying $\xi(y') = \varpi(y'') = 0$, then $\mathscr{O}_C(U)$ consists of all $f \in \mathscr{O}_{\widetilde{C}}(V' \cup V'')$ satisfying $f(y') = f(y'')$; $\Theta_C(U)$ is the (free) $\mathscr{O}_C(U)$-submodule of $\Theta_{\widetilde{C}}(V' \cup V'')$ generated by the tangent fields whose restrictions to $V' \cup V''$ are

$$\xi\partial_\xi, \qquad \text{resp.} \qquad -\varpi\partial_\varpi; \tag{2.1}$$

$\omega_C(U)$ is the (free) $\mathscr{O}_C(U)$-submodule of $\omega_{\widetilde{C}}(\nu^{-1}(U - \{x'\}))$ generated by

$$\xi^{-1}d\xi, \qquad \text{resp.} \qquad -\varpi^{-1}d\varpi.$$

We refer the reader to [ACG11, Chap.X] for basic facts about nodal curves.

Definition of $\mathscr{V}_C$

Let $\mathbb{V}$ be a VOA. The **sheaf of VOA** $\mathscr{V}_C$ on a complex curve C is defined when C is smooth by [FB04] and generalized to (simple) nodal curves by [DGT19a]. Let us recall the definition. $\mathscr{V}_C$ is defined by the filtration

$$\mathscr{V}_C = \varinjlim_{n\in\mathbb{N}} \mathscr{V}_C^{\leqslant n},$$

where each $\mathscr{V}_C^{\leqslant n}$ is a locally free $\mathscr{O}_C$-module (i.e., a vector bundle) of rank $\dim \mathbb{V}^{\leqslant n}$. We need some preparation before we describe $\mathscr{V}_C^{\leqslant n}$.

We first assume C is a smooth curve or the smooth open subset of a nodal curve. Let U, V be open subsets of C, equipped with univalent functions $\eta \in \mathscr{O}(U), \mu \in \mathscr{O}(V)$. Define a holomorphic family $\varrho(\eta|\mu) : U \cap V \to \mathbb{G}$ as follows. For any $p \in U \cap V, \eta - \eta(p)$ and $\mu - \mu(p)$ are local coordinates at p. We set $\varrho(\eta|\mu)_p \in \mathbb{G}$ satisfying

$$\eta - \eta(p) = \varrho(\eta|\mu)_p(\mu - \mu(p)). \tag{2.2}$$

Let $z \in \mathscr{O}_{\mathbb{C},0}$ be the standard coordinate. Then, by composing both sides of (2.2) with $\mu^{-1}(z + \mu(p))$, we find the equivalent formula

$$\varrho(\eta|\mu)_p(z) = \eta \circ \mu^{-1}(z + \mu(p)) - \eta(p), \tag{2.3}$$

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[3]Unless otherwise stated, compact Riemann surfaces are *not* assumed to be connected.
[4]To simply the following discussions, when C is smooth, we let $\widetilde{C}$ be C and ν be the identity map.

which justifies that the family of transformation $\varrho(\eta|\mu)$ is holomorphic. It is also clear that if η_1, η_2, η_3 are three local coordinates, then on their common domain the following cocycle condition holds:

$$\varrho(\eta_3|\eta_1) = \varrho(\eta_3|\eta_2)\varrho(\eta_2|\eta_1). \tag{2.4}$$

By Remark 1.5, for each $n \in \mathbb{N}$ we have an isomorphism of $\mathscr{O}_{U \cap V}$-modules

$$\mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V} \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}.$$

The vector bundle $\mathscr{V}_C^{\leqslant n}$ is defined such that its transition functions are given by $\mathcal{U}(\varrho(\eta|\mu))$. Thus, for any open subset $U \subset C$ and a univalent $\eta \in \mathscr{O}(U)$, we have a trivilization, i.e., an isomorphism of $\mathscr{O}_U$-modules

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_C^{\leqslant n}|_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U. \tag{2.5}$$

If $V \subset C$ is also open and $\mu \in \mathscr{O}(V)$ is univalent, then on $U \cap V$ we have

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} = \mathcal{U}(\varrho(\eta|\mu)). \tag{2.6}$$

From (1.7), we can compute that for any section v of $\mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}$,

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} \cdot v = (\partial_\mu \eta)^n \cdot v \quad \mod \quad \mathbb{V}^{\leqslant n-1} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}. \tag{2.7}$$

By comparing the transition functions, we see that $\mathscr{V}_C^{\leqslant n}/\mathscr{V}_C^{\leqslant n-1}$ is naturally equivalent to $\mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes n}$ (cf. [FB04, Sec. 6.5.9]).

We now assume C has a node x'. We shall define $\mathscr{V}_C^{\leqslant n}$ to be an $\mathscr{O}_C$-submodule of $\mathscr{V}_{C-\{x'\}}^{\leqslant n}$ as follows. Let $\mathscr{V}_C^{\leqslant n}$ be equal to $\mathscr{V}_{C-\{x'\}}^{\leqslant n}$ outside x'. To describe $\mathscr{V}_C^{\leqslant n}$ near x', we choose a neighborhood U of x', and choose $y', y'', V', V'', \xi \in \mathscr{O}(V'), \varpi \in \mathscr{O}(V'')$ as previously. Then $U - \{x'\}$ can be identified with $(V' - \{y'\}) \cup (V'' - \{y''\})$ via ν. We let $\mathscr{V}_C^{\leqslant n}(U)$ be the $\mathscr{O}_C(U)$-submodule of $\mathscr{V}_{C-\{x'\}}^{\leqslant n}(U - \{x'\})$ generated by

$$\boxed{\mathcal{U}_\varrho(\xi)^{-1}\big(\xi^{L_0}v\big) + \mathcal{U}_\varrho(\varpi)^{-1}\big(\varpi^{L_0}\mathcal{U}(\gamma_1)v\big)} \qquad (\forall v \in \mathbb{V}^{\leqslant n}). \tag{2.8}$$

To be more precise, (2.8) defines a section on $(V' - \{y'\}) \sqcup (V'' - \{y''\})$ which equals $\mathcal{U}_\varrho(\xi)^{-1}\big(\xi^{L_0}v\big)$ on $(V' - \{y'\})$ and $\mathcal{U}_\varrho(\varpi)^{-1}\big(\varpi^{L_0}\mathcal{U}(\gamma_1)v\big)$ on V''. Also, ξ^{L_0} is an element of $\mathscr{O}_{\widetilde{C}}(V' - \{y'\})$ acting on the constant section $v \in \mathbb{V}^{\leqslant n} \subset \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{\widetilde{C}}(V' - \{y'\})$, and $\varpi^{L_0}\mathcal{U}(\gamma_1)v$ is understood in a similar way. By $\mathcal{U}(\gamma_1) = e^{L_1}(-1)^{L_0}$ and induction on n, it is easy to see that $\mathscr{V}_C^{\leqslant n}(U)$ is $\mathscr{O}_C(U)$-generated freely by

$$\mathcal{U}_\varrho(\xi)^{-1}\big(\xi^{\mathrm{wt}(v)}v\big) + \mathcal{U}_\varrho(\varpi)^{-1}\big((-\varpi)^{\mathrm{wt}(v)}v\big) \tag{2.9}$$

for all $v \in E$ where E is any basis of $\mathbb{V}^{\leqslant n}$ whose elements are homogenous. By (2.9) and (2.7), it is easy to check that the definition of $\mathscr{V}_C^{\leqslant n}(U)$ is independent of the choice of the coordinates ξ, ϖ at y', y''.

If we compare (2.9) with (2.1), and notice (2.7) and the sentence after that, we see the following fact (cf. [FB04, Sec.6.5.9] and [DGT19b]):

Proposition 2.1. *Let C be a complex curve and $n \in \mathbb{N}$. Then we have the following isomorphism of $\mathscr{O}_C$-modules:*

$$\mathscr{V}_C^{\leqslant n}/\mathscr{V}_C^{\leqslant n-1} \simeq \mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes n}.$$

Under this isomorphism, if $U \subset C$ is open and smooth, and $\eta \in \mathscr{O}(U)$ is univalent, then for any $v \in \mathbb{V}(n)$, $v \otimes \partial_\eta^n$ is identified with the equivalence class of $\mathcal{U}_\varrho(\eta)^{-1}v$.

A vanishing theorem

In the remaining part of this section, we use Proposition 2.1 to prove a vanishing theorem for $\mathcal{V}_C^{\leqslant n} \otimes \omega_C$. By an (analytic) **local coordinate** η **of** C **at** a smooth point x, we mean a (holomorphic) univalent function η defined on a smooth neighborhood of x satisfying $\eta(x) = 0$. By an N-**pointed complex curve with local coordinates**

$$\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$$

we mean a complex curve C together with N distinct smooth points $x_1, x_2, \ldots, x_N \in C$ such that each x_i is associated with a local coordinate η_i at x_i. If local coordinates are not specified, we simply say an N-pointed complex curve $\mathfrak{X} = (C; x_1, \ldots, x_N)$. Unless otherwise stated, *we assume that each irreducible component of C* (equivalently, each connected component of $\widetilde{C}$) *contains at least one of $x_1, \ldots, x_N$*.

We set

$$S_{\mathfrak{X}} = x_1 + x_2 + \cdots + x_N,$$

which can be considered as a divisor both of C and of $\widetilde{C}$. Note that by Kodaira vanishing theorem, if C is smooth and connected with genus g, and $\mathcal{L}$ is a line bundle on C, then $H^1(C, \mathcal{L} \otimes \omega_C) = 0$ and (by Serre duality) equivalently $H^0(C, \mathcal{L}^{-1}) = 0$ whenever $\deg \mathcal{L} > 0$. Since $\deg \omega_C = 2g - 2$, we conclude that $H^1(C, \mathcal{L}) = 0$ whenever $\deg \mathcal{L} > 2g - 2$.

Recall that $\widetilde{C}$ is the normalization of the complex curve C. We let $\widetilde{g}$ be the largest genus of the connected components of $\widetilde{C}$. Let $M \in \{0, 1\}$ be the number of nodes. Notice the following elementary fact:

Lemma 2.2. *Choose any integer $n \geqslant -1$. Then $H^1(C, \Theta_C^{\otimes n}(kS_{\mathfrak{X}})) = 0$ whenever $k > (n + 1)(2\widetilde{g} - 2) + 2M$.*

Recall that for any locally free $\mathscr{E}$ (i.e., $\mathscr{E}$ is a vector bundle), $\mathscr{E}(kS_{\mathfrak{X}}) \simeq \mathscr{E} \otimes \mathcal{O}(kS_{\mathfrak{X}})$ is the sheaf of meromorphic sections of $\mathscr{E}$ whose only possible poles are at $x_1, \ldots, x_N$ and of orders at most k. Also, this lemma can be easily generalized to curves with more than one node.

Proof. Assume first of all that C is smooth. Then it suffices to assume C is connected. Then $\widetilde{g}$ is the genus g of C. Since $\deg \Theta_C = 2 - 2g$, the degree of $\Theta_C^{\otimes n}(kS_{\mathfrak{X}})$ is no less than $n(2 - 2g) + k$, which is greater than $2g - 2$ if $k > (n + 1)(2g - 2)$. In that case, we have $H^1(C, \Theta_C^{\otimes n}(kS_{\mathfrak{X}})) = 0$ by the discussion before the lemma.

Next, assume C has one node $x' = \nu(y') = \nu(y'')$. Choose any $k > (n + 1)(2\widetilde{g} - 2) + 2$. To prove the vanishing of H^1, it suffices, by Serre duality, to prove $H^0(C, \omega_C^{\otimes(n+1)}(-kS_{\mathfrak{X}})) = 0$. Note that $H^0(C, \omega_C^{\otimes(n+1)}(-kS_{\mathfrak{X}}))$ can be viewed as a subspace of $H^0(\widetilde{C}, \mathscr{L})$ where $\mathscr{L} = \omega_{\widetilde{C}}^{\otimes(n+1)}(-kS_{\mathfrak{X}} + y' + y'')$. We shall prove that for each connected component $\widetilde{C}_0$ of $\widetilde{C}$, $H^0(\widetilde{C}_0, \mathscr{L}|_{\widetilde{C}_0}) = 0$. This is true since

$$\deg(\mathscr{L}|_{\widetilde{C}_0}) \leqslant (n + 1)\deg(\omega_{\widetilde{C}_0}) - k + 2 \leqslant (n + 1)(2\widetilde{g} - 2) - k + 2 < 0.$$

$\square$

Theorem 2.3. *Choose any $n \in \mathbb{N}$, and assume $k > n \cdot \max\{0, 2\widetilde{g} - 2\} + 2M$. Then*

$$H^1(C, \mathscr{V}_C^{\leqslant n} \otimes \omega_C(kS_{\mathfrak{X}})) = 0.$$

Proof. Since $\mathbb{V}^{\leqslant -1}$ is trivial, we have $\mathscr{V}_C^{\leqslant 0} \simeq \mathscr{V}_C^{\leqslant 0}/\mathscr{V}_C^{\leqslant -1} \simeq \mathbb{V}(0) \otimes_{\mathbb{C}} \mathscr{O}_C$ by Proposition 2.1. Thus, the claim follows from Lemma 2.2. Now, suppose the claim is true for $n-1$. We shall prove that it is also true for n. Assume $k > n \cdot \max\{0, 2\widetilde{g} - 2\} + 2M$. By Proposition 2.1, we have an exact sequence

$$H^1(C, \mathscr{V}_C^{\leqslant n-1} \otimes \omega_C(kS_{\mathfrak{X}})) \to H^1(C, \mathscr{V}_C^{\leqslant n} \otimes \omega_C(kS_{\mathfrak{X}}))$$
$$\to H^1\big(C, \mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes(n-1)}(kS_{\mathfrak{X}})\big).$$

The first term vanishes by induction, and the last term vanishes by Lemma 2.2. Thus the middle term vanishes. $\square$

3   Conformal blocks associated to complex curves

We recall the definition of conformal blocks associated to complex curves in [FB04] and [DGT19a, DGT19b], but rephrase it in the analytic language suitable for our purpose in this article.

Action of $H^0(C, \mathscr{V}_C \otimes \omega_C(\bullet S_{\mathfrak{X}}))$ on $\mathbb{W}_{\bullet}$

Let $\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$ be an N-pointed complex curve with local coordinates. If $\mathbb{W}$ is a $\mathbb{V}$-module, then, by considering $\mathbb{V}$ as the subspace of constant sections of $\mathbb{V}((z))$, we can extend the linear map $\mathbb{V} \otimes_{\mathbb{C}} \mathbb{W} \to \mathbb{W}((z)), v \otimes w \mapsto Y_{\mathbb{W}}(v, z)w$ uniquely to an $\mathbb{C}((z))$-module homomorphism $\mathbb{V}((z)) \otimes_{\mathbb{C}} \mathbb{W} \to \mathbb{W}((z))$.

Let $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_N$ be $\mathbb{V}$-modules. Set $\mathbb{W}_{\bullet} = \mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \cdots \otimes \mathbb{W}_N$.

Convention 3.1. By $w \in \mathbb{W}_{\bullet}$, we mean a vector of $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$. By $w_{\bullet} \in \mathbb{W}_{\bullet}$, we mean a vector of the form $w_1 \otimes w_2 \otimes \cdots \otimes w_N$, where $w_1 \in \mathbb{W}_1, \ldots, w_N \in \mathbb{W}_N$.

For each $\mathscr{O}_C$-module $\mathscr{E}$, we set

$$\mathscr{E}(\bullet S_{\mathfrak{X}}) = \varinjlim_{k \in \mathbb{N}} \mathscr{E}(kS_{\mathfrak{X}}),$$

whose sections are meromorphic sections of $\mathscr{E}$ whose only possible poles are at $x_1, \ldots, x_N$. For each $1 \leqslant i \leqslant N$, if v is a section of $\mathscr{V}_C \otimes \omega_C(\bullet S_{\mathfrak{X}})$ defined near x_i, we define a linear action of v on $\mathbb{W}_i$ as follows. Choose a neighborhood U_i of x_i on which η_i is defined. By tensoring with the identity map of ω_{U_i}, the map (2.5) induces naturally an $\mathscr{O}_{U_i}$-module isomorphism (also denoted by $\mathcal{U}_{\varrho}(\eta_i)$):

$$\mathcal{U}_{\varrho}(\eta_i) : \mathscr{V}_C|_{U_i} \otimes \omega_{U_i}(\bullet S_{\mathfrak{X}}) \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \omega_{U_i}(\bullet S_{\mathfrak{X}}).$$

Identify U_i and $\eta_i(U_i)$ via η_i. Then η_i as a variable equals the standard variable z of $\mathbb{C}$. The action of v on any $w_i \in \mathbb{W}_i$ is then

$$\boxed{v \cdot w_i = \mathrm{Res}_{z=0} Y_{\mathbb{W}_i}(\mathcal{U}_{\varrho}(\eta_i)v, z)w_i} \tag{3.1}$$

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(Here the η_i in $\mathcal{U}_\varrho(\eta_i)$ is understood as a coordinate but not a variable. So it is different from z.)

Define a linear action of $H^0(C, \mathcal{V}_C \otimes \omega_C(\bullet S_{\mathfrak{x}}))$ on $\mathbb{W}_\bullet$ as follows. If $v \in H^0(C, \mathcal{V}_C \otimes \omega_C(\bullet S_{\mathfrak{x}}))$ and $w_\bullet = w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_\bullet$, then

$$v \cdot w_\bullet = \sum_{i=1}^{N} w_1 \otimes w_2 \otimes \cdots \otimes (v|_{U_i}) \cdot w_i \otimes \cdots \otimes w_N. \tag{3.2}$$

Coordinate-independent definition

Let $\mathfrak{X} = (C; x_1, \ldots, x_N)$ be an N-pointed complex curve without specifying local coordinates, and let $\mathbb{W}_1, \ldots, \mathbb{W}_N$ be $\mathbb{V}$-modules. Define a vector space $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ isomorphic to $\mathbb{W}_\bullet$ as follows. $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is a (possibly infinite rank) vector bundle on the 0-dimensional manifold $\{C\}$ (consider as the base manifold of the family $C \to \{C\}$). For any choice of local coordinates $\eta_\bullet = (\eta_1, \ldots, \eta_N)$ of $x_1, \ldots, x_N$ respectively, we have a trivialization

$$\mathcal{U}(\eta_\bullet) : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \xrightarrow{\simeq} \mathbb{W}_\bullet \tag{3.3}$$

such that if $\mu_\bullet$ is another set of local coordinates, then the transition function is

$$\begin{aligned}
\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} &= \mathcal{U}(\eta_\bullet \circ \mu_\bullet^{-1}) \\
&:= \mathcal{U}(\eta_1 \circ \mu_1^{-1}) \otimes \mathcal{U}(\eta_2 \circ \mu_2^{-1}) \otimes \cdots \otimes \mathcal{U}(\eta_N \circ \mu_N^{-1}).
\end{aligned} \tag{3.4}$$

If $v \in H^0(C, \mathcal{V}_C \otimes \omega_C(\bullet S_{\mathfrak{x}}))$ and $w \in \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$, we set

$$v \cdot w = \mathcal{U}(\eta_\bullet)^{-1} \cdot v \cdot \mathcal{U}(\eta_\bullet) \cdot w, \tag{3.5}$$

where the action of v on $\mathcal{U}(\eta_\bullet)w$ (which depends on $\eta_\bullet$) is defined by (3.1) and (3.2).

Theorem 3.2 (Cf. [FB04] Thm. 6.5.4). *The action of $H^0(C, \mathcal{V}_C \otimes \omega_C(\bullet S_{\mathfrak{x}}))$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ defined by (3.5) is independent of the choice of $\eta_\bullet$.*

Proof. We prove this theorem for the case $N = 1$. The general cases can be proved in a similar way. Choose local coordinates η, μ at $x = x_1$ defined on a neighborhood U. We identify U with $\mu(U)$ via μ. So μ as a coordinate is identified with the standard coordinate $\mathbf{1}_{\mathbb{C}}$ of $\mathbb{C}$, and $\eta \in \mathbb{G}$. As a variable, μ is identified with the standard one z of $\mathbb{C}$. Also, identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W})$ (where $\mathbb{W} = \mathbb{W}_1 = \mathbb{W}_\bullet$) with $\mathbb{W}$ via $\mathcal{U}(\mu)$. So $\mathcal{U}(\mu) = \mathcal{U}(\mathbf{1}_{\mathbb{C}}) = \mathbf{1}$. Choose any $w \in \mathbb{W}$, choose any section v of $\mathcal{V}_C \otimes \omega_C(\bullet S_{\mathfrak{x}})$ defined on U. So

$$\mathcal{U}_\varrho(\mu)v = u(z)dz$$

for some $u = u(z) \in \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{C}}(\bullet 0)(U)$. Then, by (2.6),

$$\mathcal{U}_\varrho(\eta)v = \mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}}))u(z)dz = \mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}})_z)u(z)dz.$$

Set variable $\zeta = \eta(z)$. Then

$$\mathcal{U}(\eta)^{-1} \cdot v \cdot \mathcal{U}(\eta) \cdot w = \operatorname{Res}_{\zeta=0} \mathcal{U}(\eta)^{-1} Y_{\mathbb{W}}\big(\mathcal{U}_\varrho(\eta)v, \zeta\big)\mathcal{U}(\eta)w$$

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where the ζ in $Y_{\mathbb{W}}$ is due to the fact that η (as a variable) equals ζ when U is identified with $\eta(U)$ via η. (Such identification is needed in the definition of the coordinate-dependent action (3.1).) This expression equals

$$\mathrm{Res}_{z=0}\, \mathcal{U}(\eta)^{-1} Y_{\mathbb{W}}\big(\mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}})_z)u(z), \eta(z)\big)\mathcal{U}(\eta)w \cdot dz,$$

which by Theorem 3.3 (with $\alpha = \eta$) equals

$$\mathrm{Res}_{z=0}\, Y_{\mathbb{W}}\big(u(z), z\big)w \cdot dz = \mathcal{U}(\mu)^{-1} \cdot v \cdot \mathcal{U}(\mu) \cdot w.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In the above proof, we have used the following theorem of Huang [Hua97]; see also [FB04, Lemma 6.5.6].

Theorem 3.3. *Let $\mathbb{W}$ be a $\mathbb{V}$-module. Let $U \subset \mathbb{C}$ be a neighborhood of 0. Let $\alpha \in \mathcal{O}(U)$ be a local coordinate at 0 (so $\alpha(0) = 0$). Let $\mathbf{1}_{\mathbb{C}} \in \mathbb{G}$ be the standard coordinate of $\mathbb{C}$ (i.e. the identity element of $\mathbb{G}$). Then for any $v \in \mathbb{V}$ and $w \in \mathbb{W}$, we have the following equation of elements in $\mathbb{W}((z))$:*

$$\mathcal{U}(\alpha)Y_{\mathbb{W}}(v, z)\mathcal{U}(\alpha)^{-1} \cdot w = Y_{\mathbb{W}}\big(\mathcal{U}(\varrho(\alpha|\mathbf{1}_{\mathbb{C}}))v, \alpha(z)\big) \cdot w. \qquad (3.6)$$

Note that $U(\varrho(\alpha|\mathbf{1}_{\mathbb{C}}))v$ is in $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}(U)$ and hence can be regarded as an element of $\mathbb{V}((z))$. Of course, (3.6) also holds in an obvious way for any $v \in \mathbb{V}((z))$.

For instance, take $\alpha(z) = \lambda z$ where $\lambda \in \mathbb{C}^{\times}$. Then $\varrho(\alpha|\mathbf{1}_{\mathbb{C}})$ is constantly λ. It follows that

$$\lambda^{\tilde{L}_0}Y_{\mathbb{W}}(v, z)\lambda^{-\tilde{L}_0} = Y_{\mathbb{W}}(\lambda^{L_0}v, \lambda z). \qquad (3.7)$$

We now define **space of covacua**

$$\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_{\bullet}) = \frac{\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_{\bullet})}{H^0(C, \mathscr{V}_C \otimes \omega_C(\bullet S_{\mathfrak{X}})) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_{\bullet})} \qquad (3.8)$$

(we have omitted $\mathrm{Span}_{\mathbb{C}}$ in the denominator), whose dual vector space is denoted by $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_{\bullet})$ and called **space of conformal blocks**. Elements of $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_{\bullet})$ are called **conformal blocks** associated to $\mathbb{W}_{\bullet}$ and $\mathfrak{X}$. They are the linear functionals of $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_{\bullet})$ vanishing on the denominator of (3.8).

4   Sewing families of compact Riemann surfaces

By a (holomorphic) **family of compact Riemann surfaces** $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ we mean $\mathcal{B}, \mathcal{C}$ are complex manifolds, $\mathcal{B}$ has finitely many connected components,[5] π is a proper surjective holomorphic submersion, and each fiber $\mathcal{C}_b := \pi^{-1}(b)$ (where $b \in \mathcal{B}$) is a compact Riemann surface.

---

[5]This is assumed only for simplicity.

Note that by Ehresmann's theorem, when $\mathcal{B}$ is connected, as a family of differential manifolds, $\mathfrak{X}$ is trivial, i.e., $\mathfrak{X}$ is equivalent to the projection $\mathcal{C}_b \times \mathcal{B} \to \mathcal{B}$ onto the $\mathcal{B}$-component. In particular, all fibers are diffeomorphic.

We say that

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$$

is a **family of N-pointed compact Riemann surfaces with local coordinates**, if

(a) $\pi : \mathcal{C} \to \mathcal{B}$ is a family of compact Riemann surfaces.

(b) Each ς_i is a section of the family, namely, each $\varsigma_i : \mathcal{B} \to \mathcal{C}$ is a holomorphic map satisfying $\pi \circ \varsigma_i = \mathbf{1}_{\mathcal{B}}$.

(c) Each η_i is a **local coordinate at** $\varsigma_i(\mathcal{B})$, which means that there is an open subset $U_i \subset \mathcal{C}$ containing $\varsigma_i(\mathcal{B})$ such that $\eta_i \in \mathscr{O}(U_i)$ is univalent on each fiber $U_{i,b} = U_i \cap \mathcal{C}_b$ of U_i. In that case, (η_i, π) is a biholomorphic map from U_i to an open subset of $\mathbb{C} \times \mathcal{B}$. Moreover, we assume the restriction of η_i to $\varsigma_i(\mathcal{B})$ is 0.

(d) $\varsigma_i(\mathcal{B}) \cap \varsigma_j(\mathcal{B}) = \varnothing$ whenever $i \neq j$.

(e) For each $b \in \mathcal{B}$, every connected component of the fiber $\mathcal{C}_b$ contains at lease one of the marked points $\varsigma_1(b), \ldots, \varsigma_N(b)$.

In the case that the local coordinates $\eta_1, \ldots, \eta_N$ are not assigned (and hence condition (c) is not assumed), we say that $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N)$ is a family of N-pointed compact Riemann surfaces. We set

$$S_{\mathfrak{X}} = \sum_{i=1}^{N} \varsigma_i(\mathcal{B})$$

to be a divisor of $\mathcal{C}$. Then for each $b \in \mathcal{B}$,

$$S_{\mathfrak{X}}(b) := \sum_{i=1}^{N} \varsigma_i(b)$$

is a divisor of $\mathcal{C}_b$. The definitions of $S_{\mathfrak{X}}$ and $S_{\mathfrak{X}}(b)$ will also apply to the case that $\mathfrak{X}$ is formed by sewing a smooth family.

We say that $\mathfrak{X}$ is a **family of N-pointed complex curves (resp. with local coordinates)**, if $\mathfrak{X}$ is either a family of N-pointed compact Riemann surfaces (resp. with local coordinates) (in that case we say $\mathfrak{X}$ is a smooth family), or if $\mathfrak{X}$ is formed by sewing a smooth family $\widetilde{\mathfrak{X}}$, whose meaning is explained below.

Sewing open discs

We first describe how to sew a pair of open discs $\mathcal{D}_r, \mathcal{D}_\rho$.

For any $r > 0$, let $\mathcal{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathcal{D}_r^\times = \mathcal{D}_r - \{0\}$. If $r, \rho > 0$, we define

$$\pi_{r,\rho} : \mathcal{D}_r \times \mathcal{D}_\rho \to \mathcal{D}_{r\rho}, \qquad (\xi, \varpi) \mapsto \xi \varpi. \tag{4.1}$$

$d\pi_{r,\rho}$ is surjective at (ξ, ϖ) whenever $\xi \neq 0$ or $\varpi \neq 0$. Denote also by ξ and ϖ the standard coordinates of $\mathcal{D}_r$ and $\mathcal{D}_\rho$, which can be extended constantly to $\xi : \mathcal{D}_r \times \mathcal{D}_\rho \to \mathcal{D}_r, \varpi : \mathcal{D}_r \times \mathcal{D}_\rho \to \mathcal{D}_\rho$. Set $q = \pi_{r,\rho}$, i.e.,

$$q : \mathcal{D}_r \times \mathcal{D}_\rho \to \mathbb{C}, \qquad q = \xi\varpi.$$

Then $(\xi, \varpi), (\xi, q), (\varpi, q)$ are coordinates of $\mathcal{D}_r \times \mathcal{D}_\rho, \mathcal{D}_r^\times \times \mathcal{D}_\rho, \mathcal{D}_r \times \mathcal{D}_\rho^\times$ respectively. The standard tangent vectors of the coordinates $(\xi, \varpi), (\xi, q)$ are related by

$$\begin{cases} \partial_\xi = \partial_\xi - \xi^{-1}\varpi \cdot \partial_\varpi \\ \partial_q = \xi^{-1}\partial_\varpi \end{cases} \qquad \begin{cases} \partial_\xi = \partial_\xi + \xi^{-1}q \cdot \partial_q \\ \partial_\varpi = \xi\partial_q \end{cases} \tag{4.2}$$

The formulae between $(\xi, \varpi), (\varpi, q)$ are similar.

It is easy to see that $(\xi, q)(\mathcal{D}_r^\times \times \mathcal{D}_\rho)$ (resp. $(\varpi, q)(\mathcal{D}_r \times \mathcal{D}_\rho^\times)$) is precisely the subset of all $(\xi_0, q_0) \in \mathcal{D}_r \times \mathcal{D}_{r\rho}$ (resp. $(\varpi_0, q_0) \in \mathcal{D}_\rho \times \mathcal{D}_{r\rho}$) satisfying

$$\frac{|q_0|}{\rho} < |\xi_0| < r \qquad \text{resp.} \qquad \frac{|q_0|}{r} < |\varpi_0| < \rho. \tag{4.3}$$

We choose closed subsets $E'_{r,\rho} \subset \mathcal{D}_r \times \mathcal{D}_{r\rho}$ and $E''_{r,\rho} \subset \mathcal{D}_\rho \times \mathcal{D}_{r\rho}$ such that

$$(\xi, q) : \mathcal{D}_r^\times \times \mathcal{D}_\rho \xrightarrow{\simeq} \mathcal{D}_r \times \mathcal{D}_{r\rho} - E'_{r,\rho},$$
$$(\varpi, q) : \mathcal{D}_r \times \mathcal{D}_\rho^\times \xrightarrow{\simeq} \mathcal{D}_\rho \times \mathcal{D}_{r\rho} - E''_{r,\rho} \tag{4.4}$$

are bijective.

Sewing a family of compact Riemann surfaces

We discuss how to simultaneously sew a family of compact Riemann surfaces. This construction is also known as smoothing in the world of algebraic geometry. See [TUY89, Sec. 6.1], [Ueno97, Sec. 5.3], or [ACG11, Sec. XI.3]. Its algebraic version is given in [Loo10, Sec. 6] and [DGT19b, Sec. 8.1].

Consider a family of $(N + 2)$-pointed compact Riemann surfaces with local coordinates

$$\widetilde{\mathfrak{X}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}; \varsigma_1, \ldots, \varsigma_N, \varsigma', \varsigma''; \eta_1, \ldots \eta_N, \xi, \varpi), \tag{4.5}$$

assuming that for every $b \in \widetilde{\mathcal{B}}$, each connected component of $\widetilde{\mathcal{C}}_b = \widetilde{\pi}^{-1}(b)$ contains one of $\varsigma_1(b), \ldots, \varsigma_N(b)$. Choose $r, \rho > 0$ and a neighborhood U' (resp. U'') of $\varsigma'(\widetilde{\mathcal{B}})$ (resp. $\varsigma''(\widetilde{\mathcal{B}})$) on which ξ (resp. ϖ) is defined, such that

$$(\xi, \widetilde{\pi}) : U' \xrightarrow{\simeq} \mathcal{D}_r \times \widetilde{\mathcal{B}} \qquad \text{resp.} \qquad (\varpi, \widetilde{\pi}) : U'' \xrightarrow{\simeq} \mathcal{D}_\rho \times \widetilde{\mathcal{B}} \tag{4.6}$$

is a biholomorphic map. We also assume that U' and U'' are disjoint and do not intersect $\varsigma_1(\widetilde{\mathcal{B}}), \ldots, \varsigma_N(\widetilde{\mathcal{B}})$. Identify

$$U' = \mathcal{D}_r \times \widetilde{\mathcal{B}} \qquad \text{resp.} \qquad U'' = \mathcal{D}_\rho \times \widetilde{\mathcal{B}}$$

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via the above maps. Then ξ, ϖ (when restricted to the first components) become the standard coordinates of $\mathcal{D}_r, \mathcal{D}_\rho$ respectively, and $\tilde{\pi}$ is the projection onto the $\tilde{\mathcal{B}}$-component. Set $q = \xi\varpi = \pi_{r,\rho} : \mathcal{D}_r \times \mathcal{D}_\rho \to \mathcal{D}_{r\rho}$ as previously.

Set

$$\mathcal{B} = \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}}. \tag{4.7}$$

We now **sew the smooth family** $\tilde{\mathfrak{X}}$ to obtain a family

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$$

of N-pointed complex curves with local coordinates. We first explain how to obtain $\mathcal{C}$ and $\pi : \mathcal{C} \to \mathcal{B}$. We shall freely switch the orders of Cartesian products. Note that

$$
\begin{aligned}
F' &:= E'_{r,\rho} \times \tilde{\mathcal{B}} &&\subset \mathcal{D}_r \times \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}} \quad (= U' \times \mathcal{D}_{r\rho}), \\
F'' &:= E''_{r,\rho} \times \tilde{\mathcal{B}} &&\subset \mathcal{D}_\rho \times \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}} \quad (= U'' \times \mathcal{D}_{r\rho})
\end{aligned}
$$

are subsets of $\tilde{\mathcal{C}} \times \mathcal{D}_{r\rho}$. They are the subsets we should discard in the sewing process. Then $\mathcal{C}$ is obtained by gluing $\tilde{\mathcal{C}} \times \mathcal{D}_{r\rho}$ (with F', F'' all removed) with

$$W := \mathcal{D}_r \times \mathcal{D}_\rho \times \tilde{\mathcal{B}}. \tag{4.8}$$

To be more precise, we define

$$\mathcal{C} = W \bigsqcup (\tilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F'') \Big/ \sim \tag{4.9}$$

where the equivalence $\sim$ is described as follows. Consider the following subsets of W:

$$W' = \mathcal{D}_r^\times \times \mathcal{D}_\rho \times \tilde{\mathcal{B}}, \tag{4.10}$$

$$W'' = \mathcal{D}_r \times \mathcal{D}_\rho^\times \times \tilde{\mathcal{B}}. \tag{4.11}$$

Then the relation $\sim$ identifies W' and W'' respectively via $(\xi, q, \mathbf{1}_{\tilde{\mathcal{B}}})$ and $(\varpi, q, \mathbf{1}_{\tilde{\mathcal{B}}})$ to

$$\mathcal{D}_r \times \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}} - F' \quad (\subset U' \times \mathcal{D}_{r\rho}), \tag{4.12}$$

$$\mathcal{D}_\rho \times \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}} - F'' \quad (\subset U'' \times \mathcal{D}_{r\rho}) \tag{4.13}$$

(recall (4.4)), which are subsets of $\tilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F''$. (In particular, certain open subsets of (4.12) and (4.13) are glued together and identified with $W' \cap W''$.)

We now define π. It is easy to see that the projection

$$\tilde{\pi} \times \mathbf{1} : \tilde{\mathcal{C}} \times \mathcal{D}_{r\rho} \to \tilde{\mathcal{B}} \times \mathcal{D}_{r\rho} = \mathcal{B}, \tag{4.14}$$

agrees with

$$\pi_{r,\rho} \times \mathbf{1} : W = \mathcal{D}_r \times \mathcal{D}_\rho \times \tilde{\mathcal{B}} \to \mathcal{D}_{r\rho} \times \tilde{\mathcal{B}} = \mathcal{B} \tag{4.15}$$

when restricted to W', W''. Thus, we have a well-defined surjective holomorphic map $\pi : \mathcal{C} \to \mathcal{B}$ whose restrictions to $\tilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F''$ and to W are $\tilde{\pi} \otimes \mathbf{1}$ and $\pi_{r,\rho} \otimes \mathbf{1}$ respectively.

Finally, we extend each $\varsigma_i : \tilde{\mathcal{B}} \to \tilde{\mathcal{C}}$ constantly to $\tilde{\mathcal{B}} \times \mathcal{D}_{r\rho} \to \tilde{\mathcal{C}} \times \mathcal{D}_{r\rho}$, whose image is disjoint from F', F''. Thus ς_i can be regarded as a section $\varsigma_i : \mathcal{B} \to \mathcal{C}$. Likewise, we extend η_i constantly over $\tilde{\mathcal{B}}$ to a local coordinate at $\varsigma_i(\mathcal{B})$. We say that the N-points $\varsigma_1, \ldots, \varsigma_N$ and the local coordinates $\eta_1, \ldots, \eta_N$ are **constant with respect to sewing**.

A short exact sequence

The goal of this subsection is to recall a short exact sequence (4.20) (cf. [Ueno97, Eq. (4.2.3)] or [AU07, Eq. (4.6)]) which plays an important role in defining a logarithmic connection on sheaves of conformal blocks. In Section 11, we will use this exact sequence to find the differential equations that sewn conformal blocks satisfy.

Consider the discriminant locus Δ and the critical locus Σ:

$$\Delta = \{0\} \times \widetilde{\mathcal{B}} \qquad (\subset \mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}} = \mathcal{B}),$$
$$\Sigma = \{0\} \times \{0\} \times \widetilde{\mathcal{B}} \qquad (\subset \mathcal{D}_r \times \mathcal{D}_\rho \times \widetilde{\mathcal{B}} = W).$$

Then Δ is the set of all $b \in \mathcal{B}$ such that the fiber $\mathcal{C}_b = \pi^{-1}(b)$ is a nodal curve with one node. Outside Δ, the fibers are compact Riemann surfaces. Σ is the set of all points of $\mathcal{C}$ at which $d\pi$ is not surjective, and is also the set of all nodes of the fibers. Note that

$$\Sigma = W - (W' \cup W'').$$

Also, Σ is the set of points of $\mathcal{C}$ not coming from $\widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F''$. The union of nodal fibers is therefore

$$\mathcal{C}_\Delta = \pi^{-1}(\Delta).$$

Let $\Theta_{\mathcal{B}}(-\log \Delta)$ and $\Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta)$ be the sheaves of sections of $\Theta_{\mathcal{B}}$ and $\Theta_{\mathcal{C}}$ tangent to Δ and $\mathcal{C}_\Delta$ respectively. Then the differential $d\pi : \Theta_{\mathcal{C}} \to \pi^* \Theta_{\mathcal{B}}$ of the map π restricts to an $\mathscr{O}_{\mathcal{C}}$-module homomorphism

$$d\pi : \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta) \to \pi^* \Theta_{\mathcal{B}}(-\log \Delta) \tag{4.16}$$

(the later is short for $\pi^*(\Theta_{\mathcal{B}}(-\log \Delta))$), which is indeed surjective. To understand the meaning and to see the claimed facts, let us describe the two sheaves and the morphism $d\pi$ using coordinates.

We assume $\widetilde{\mathcal{B}}$ is small enough to admit a coordinate $\tau_\bullet = (\tau_1, \ldots, \tau_m) : \widetilde{\mathcal{B}} \to \mathbb{C}^m$. Then $(q, \tau_\bullet)$ is a coordinate of $\mathcal{B}$ if we let q be the standard coordinate of $\mathcal{D}_{r\rho}$. Then $\Theta_{\mathcal{B}}(-\log \Delta)$ is an $\mathscr{O}_{\mathcal{B}}$-module generated freely by

$$q\partial_q, \partial_{\tau_1}, \ldots, \partial_{\tau_m}.$$

Their pullback under π will also be denoted by the same symbols $q\partial_q, \partial_{\tau_1}, \ldots, \partial_{\tau_m}$, for simplicity. Choose any $x \in \mathcal{C}$.

Case I. $x \notin \Sigma$. Then x can be regarded as a point $(\widetilde{x}, q_0)$ of $\widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho}$ disjoint from F', F''. Choose a neighborhood $\widetilde{U} \subset \widetilde{\mathcal{C}}$ of $\widetilde{x}$ together with $\eta \in \mathscr{O}(\widetilde{U})$ univalent on each fiber of $\widetilde{U}$. Choose a neighborhood V of $q_0 \in \mathcal{D}_{r\rho}$ such that $U := \widetilde{U} \times V$ is disjoint from F', F''. Write $\tau_\bullet \circ \widetilde{\pi}$ also as $\tau_\bullet$ for short. Then $(\eta, \tau_\bullet, q)$ is a coordinate of $U \ni x$. Note that $U \cap \mathcal{C}_\Delta$ is described by $q = 0$. The $\mathscr{O}_V$-module $\Theta_{\mathcal{C}}(-\log \Delta)|_V$ is generated freely by

$$\partial_\eta, q\partial_q, \partial_{\tau_1}, \ldots, \partial_{\tau_m}, \tag{4.17}$$

and the morphism $d\pi$ in (4.16) sends ∂_η to 0 and keeps the other elements of (4.17).

Case II. $x \in \Sigma$. Then $W = \mathcal{D}_r \times \mathcal{D}_\rho \times \tilde{\mathcal{B}}$ is a neighborhood of x, and has coordinate $(\xi, \varpi, \tau_\bullet)$. Note that $W \cap \mathcal{C}_\Delta$ is described by $\xi \varpi = 0$. The $\mathcal{O}_W$-module $\Theta_{\mathcal{C}}(-\log \Delta)|_W$ is generated freely by

$$\xi \partial_\xi, \varpi \partial_\varpi, \partial_{\tau_1}, \ldots, \partial_{\tau_m}, \tag{4.18}$$

and the morphism $d\pi$ satisfies

$$d\pi(\xi \partial_\xi) = d\pi(\varpi \partial_\varpi) = q \partial_q \tag{4.19}$$

(note that $q \partial_q$ is short for $\pi^*(q \partial_q)$) and keeps the other elements of (4.18).

It is clear that in both cases, $d\pi$ (in (4.16)) is surjective. Thus, by letting $\Theta_{\mathcal{C}/\mathcal{B}}$ be the kernel of $d\pi$, we obtain an exact sequence of $\mathcal{O}_{\mathcal{C}}$-modules

$$\boxed{0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta) \xrightarrow{d\pi} \pi^* \Theta_{\mathcal{B}}(-\log \Delta) \to 0} \tag{4.20}$$

In case I resp. case II, $\Theta_{\mathcal{C}/\mathcal{B}}|_V$ resp. $\Theta_{\mathcal{C}/\mathcal{B}}|_W$ is generated (freely) by

$$\partial_\eta, \qquad \text{resp.} \qquad \xi \partial_\xi - \varpi \partial_\varpi. \tag{4.21}$$

So $\Theta_{\mathcal{C}/\mathcal{B}}$ is locally free of rank 1, whose dual module is the relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{B}}$. Using (4.2), we see that when restricted to W' (resp. W'') and under the coordinate $(\xi, q, \tau_\bullet)$ (resp. $(\varpi, q, \tau_\bullet)$), the section $\xi \partial_\xi - \varpi \partial_\varpi$ in (4.21) equals

$$\xi \partial_\xi \qquad \text{resp.} \qquad -\varpi \partial_\varpi. \tag{4.22}$$

Compare this with (2.1), we see that for each $b \in \mathcal{B}$, there are natural equivalences

$$\Theta_{\mathcal{C}/\mathcal{B}}|_{\mathcal{C}_b} \simeq \Theta_{\mathcal{C}_b}, \qquad \omega_{\mathcal{C}/\mathcal{B}}|_{\mathcal{C}_b} \simeq \omega_{\mathcal{C}_b}.$$

Finally, we remark that when $\mathfrak{X}$ is a smooth family, $d\pi : \Theta_{\mathcal{C}} \to \pi^* \Theta_{\mathcal{B}}$ is surjective. Thus, (4.20) becomes

$$0 \to \Theta_{\mathcal{C}/\mathcal{B}} \to \Theta_{\mathcal{C}} \xrightarrow{d\pi} \pi^* \Theta_{\mathcal{B}} \to 0$$

where $\Theta_{\mathcal{C}/\mathcal{B}}$ is the kernel of $d\pi$.

A theorem of Grauert

Recall that if C is a complex curve and $\mathscr{E}$ is a coherent $\mathcal{O}_C$-module, then $H^p(C, \mathscr{E})$ is always finite-dimensional by (for instance) the direct image theorem of Grauert. Also $H^p(C, \mathscr{E})$ vanishes when $p > 1$, which follows for instance from the fact that C can be covered by two Stein open subsets. Thus, the character of $\mathscr{E}$ is $\chi(C, \mathscr{E}) = \dim H^0(C, \mathscr{E}) - \dim H^1(C, \mathscr{E})$.

Given a family $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ of complex curves, recall that we have assumed π is proper when $\mathfrak{X}$ is a smooth family. If $\mathfrak{X}$ is formed by sewing $\tilde{\mathfrak{X}}$, then properness still holds and is not hard to check. Finally, in both smooth and singular cases, the map π is clearly open. So π is flat by [Fis76, Sec. 3.20] (see also [GPR94, Thm. II.2.13] or [BS76, Thm. V.2.13]). If $\mathscr{E}$ is locally free (of finite rank), then $\mathscr{E}$ is flat over $\mathcal{B}$. Therefore, by a theorem of Grauert [Gra60] (cf. [GPR94, Thm. III.4.7] or [BS76, Thm. III.4.12] or [EP96, Thm. 9.4.8]), we have

Theorem 4.1. *Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ be a family of complex curves. Let $\mathscr{E}$ be a locally free $\mathscr{O}_{\mathcal{C}}$-module.*

(a) The function

$$\mathcal{B} \to \mathbb{Z}, \qquad b \mapsto \chi(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b) = \dim H^0(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b) - \dim H^1(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b)$$

is locally constant.

(b) For any $p \in \mathbb{N}$, if the function $b \mapsto \dim H^p(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b)$ is locally constant, then the $\mathscr{O}_{\mathcal{B}}$-module $R^p\pi_*\mathscr{E}$ is locally free of rank $\dim H^p(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b)$, and for any $b \in \mathcal{B}$, the linear map $(R^p\pi_*\mathscr{E})_b \to H^p(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b)$ defined by restricting the sections $s \mapsto s|\mathcal{C}_b$ induces an isomorphism of vector spaces

$$\frac{(R^p\pi_*\mathscr{E})_b}{\mathfrak{m}_b \cdot (R^p\pi_*\mathscr{E})_b} \simeq H^p(\mathcal{C}_b, \mathscr{E}|\mathcal{C}_b).$$

Recall $R^0\pi_* = \pi_*$, and the (higher) direct image sheaf $R^p\pi_*\mathscr{E}$ is an $\mathscr{O}_{\mathcal{B}}$-module associated to the presheaf $V \mapsto H^p(V, \mathscr{E}|_V)$ (for all open $V \subset \mathcal{B}$). As a consequence of this theorem, we see $R^p\pi_*\mathscr{E} = 0$ when $\mathscr{E}$ is locally free and $p > 1$.

5   Sheaves of VOAs on families of complex curves

Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ be a family of complex curves. Recall that Σ is the critical locus, which is empty when the family is smooth. Let U, V be open subsets of $\mathcal{C} - \Sigma$, and let $\eta \in \mathscr{O}(U), \mu \in \mathscr{O}(V)$ be univalent on each fiber of U and V respectively. Then (η, π) and (μ, π) are biholomorphic maps from U resp. V to open subsets of $\mathbb{C} \times \mathcal{B}$. For each $p \in U \cap V$, we define $\varrho(\eta|\mu)_p \in \mathscr{O}_{\mathbb{C},0}$ by

$$\varrho(\eta|\mu)_p(z) = \eta \circ (\mu, \pi)^{-1}\big(z + \mu(p), \pi(p)\big) - \eta(p). \tag{5.1}$$

Then $\varrho(\eta|\mu)_p$ is a holomorphic function of z on $\mu\big((U \cap V)_{\pi(p)}\big)$ where $(U \cap V)_{\pi(p)}$ is the fiber $U \cap V \cap \pi^{-1}(\pi(p))$. It is easy to check that for each $n \in \mathbb{N}$,

$$\partial_z^n \varrho(\eta|\mu)_p(0) = \partial_\mu^n \eta(p), \tag{5.2}$$

where the partial derivative ∂_μ is defined to be vertical to $d\pi$. From this, we see that $\varrho(\eta|\mu)_p(0) = 0$ and $\partial_z\varrho(\eta|\mu)_p(0) \neq 0$. So $\varrho(\eta|\mu)_p$ is an element of $\mathbb{G}$. We thus obtain a family of transformations $\varrho(\eta|\mu) : U \cap V \to \mathbb{G}, p \mapsto \varrho(\eta|\mu)_p$, which is clearly holomorphic. According to Remark 1.5, we have an $\mathscr{O}_{U \cap V}$-module isomorphism

$$\mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V} \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}.$$

As in Section 2, $\varrho(\eta|\mu)$ is also described by

$$\eta - \eta(p)\big|_{(U \cap V)_{\pi(p)}} = \varrho(\eta|\mu)_p\big(\mu - \mu(p)\big|_{(U \cap V)_{\pi(p)}}\big). \tag{5.3}$$

To see this, one composes both sides of (5.3) with $(\mu, \pi)^{-1}\big(z + \mu(p), \pi(p)\big)$. Thus, we can get $\varrho(\eta|\mu)_p$ by restricting η, μ to the fiber $(U \cap V)_{\pi(p)}$ and then using definition (2.2). Therefore, the cocycle relation (2.4) still holds for holomorphic functions η_1, η_2, η_3 univalent on each fiber.

Definition of $\mathscr{V}_{\mathfrak{X}}$

Sheaves of VOAs were introduced in [FB04] for algebraic families of smooth curves, and were generalized in [DGT19a] to algebraic families of stable curves. Similar to their construction, we now define the sheaf of VOA $\mathscr{V}_{\mathfrak{X}}$ associated to a VOA $\mathbb{V}$ and the analytic family $\mathfrak{X}$.

We set

$$\mathscr{V}_{\mathfrak{X}} = \varinjlim_{n \in \mathbb{N}} \mathscr{V}_{\mathfrak{X}}^{\leqslant n}$$

where each $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}$ is a locally free $\mathscr{O}_{\mathcal{C}}$-module of rank $\dim \mathbb{V}^{\leqslant n}$ defined as follows. Outside Σ, $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}$ is a vector bundle with transition function $\mathcal{U}(\varrho(\eta|\mu))$. Thus, for each open $U \subset \mathcal{C} - \Sigma$ and $\eta \in \mathscr{O}(U)$ univalent on each fiber, we have an isomorphism of $\mathscr{O}_U$-modules (a trivilization)

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_{\mathfrak{X}}^{\leqslant n}|_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U. \tag{5.4}$$

If V is another open subset of $\mathcal{C} - \Sigma$ and $\mu \in \mathscr{O}(V)$ is also univalent on each fiber, then on $U \cap V$ we have

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} = \mathcal{U}(\varrho(\eta|\mu)). \tag{5.5}$$

Since (2.7) holds when restricting to each fiber, we again have that for any section v of $\mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}$,

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} \cdot v = (\partial_\mu \eta)^n \cdot v \quad \mod \quad \mathbb{V}^{\leqslant n-1} \otimes_{\mathbb{C}} \mathscr{O}_{U \cap V}. \tag{5.6}$$

We now assume $\mathfrak{X}$ is formed by sewing a smooth family $\widetilde{\mathfrak{X}}$ as in section 4. Let $W, W', W'', \xi, \varpi, q$ be as in that section. (See the discussion near (4.8).) Then $(\xi, q, \mathbf{1}_{\widetilde{\mathcal{B}}})$ and $(\varpi, q, \mathbf{1}_{\widetilde{\mathcal{B}}})$ are respectively biholomorphic maps from W' and W'' to complex manifolds, and the projection π equals $(q, \mathbf{1}_{\widetilde{\mathcal{B}}})$ when restricted to W' or W''. Thus, ξ, ϖ are univalent on fibers of W', W'' respectively.

We shall define $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}|_W$ to be an $\mathscr{O}_W$-submodule of $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}|_{W-\Sigma}$ generated (freely) by some sections on W whose restrictions to W' and W'' are described under the trivilizations $\mathcal{U}_\varrho(\xi)$ and $\mathcal{U}_\varrho(\varpi)$ respectively. For that purpose, we need to first calculate the transition function

$$\mathcal{U}(\varrho(\varpi|\xi)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{W' \cap W''} \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{W' \cap W''}.$$

Lemma 5.1. *Choose any $p \in W' \cap W''$. Then we have*

$$\varrho(\varpi|\xi)_p(z) = q(p)\gamma_{\xi(p)}(z)$$

and hence

$$\mathcal{U}(\varrho(\varpi|\xi)_p) = q(p)^{L_0}\mathcal{U}(\gamma_{\xi(p)}).$$

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Proof. Choose any $x \in (W' \cap W'')_{\pi(p)}$. Then $\pi(x) = \pi(p)$ and hence $q(x) = q(p)$. Since $\varpi = \xi^{-1}q$, we have

$$\varpi(x) - \varpi(p) = q(p)(\xi(x)^{-1} - \xi(p)^{-1}).$$

By (5.3), we have

$$\varpi(x) - \varpi(p) = \varrho(\varpi|\xi)_p(\xi(x) - \xi(p)).$$

If we compare these two equations and set $z = \xi(x) - \xi(p)$, we obtain

$$\varrho(\varpi|\xi)_p(z) = \varrho(\varpi|\xi)_p(\xi(x) - \xi(p)) = q(p)(\xi(x)^{-1} - \xi(p)^{-1})$$
$$= q(p)\big((\xi(p) + z)^{-1} - \xi(p)^{-1}\big) = q(p)\gamma_{\xi(p)}(z).$$

$\square$

We define $\mathscr{V}_{\mathfrak{x}}^{\leqslant n}|_W$ to be the $\mathscr{O}_W$-submodule of $\mathscr{V}_{\mathfrak{x}-\Sigma}^{\leqslant n}|_{W-\Sigma}$ generated by any section on $W - \Sigma$ whose restrictions to W' and W'' are

$$\boxed{\mathcal{U}_\varrho(\xi)^{-1}\big(\xi^{L_0}v\big) \qquad \text{resp.} \qquad \mathcal{U}_\varrho(\varpi)^{-1}\big(\varpi^{L_0}\mathcal{U}(\gamma_1)v\big)} \qquad (5.7)$$

where $v \in \mathbb{V}^{\leqslant n}$. Since $\gamma_1 = \gamma_1^{-1}$ and hence $\mathcal{U}(\gamma_1) = \mathcal{U}(\gamma_1)^{-1}$, this definition is symmetric with respect to ξ and ϖ. To check that (5.7) is well-defined, we need:

Lemma 5.2. *The two sections defined in* (5.7) *agree on* $W' \cap W''$.

Proof. Using (1.11) and Lemma 5.1, we check that

$$\mathcal{U}_\varrho(\varpi)\mathcal{U}_\varrho(\xi)^{-1}\xi^{L_0}v = \mathcal{U}(\varrho(\varpi|\xi))\xi^{L_0}v = q^{L_0}\mathcal{U}(\gamma_\xi)\xi^{L_0}v$$
$$= q^{L_0}\xi^{-L_0}\mathcal{U}(\gamma_1)v = \varpi^{L_0}\mathcal{U}(\gamma_1)v.$$

$\square$

It is easy to see that, if we take $v \in E$ where E is a basis of $\mathbb{V}^{\leqslant n}$ consisting of homogeneous vectors, then $\mathscr{V}_{\mathfrak{x}}^{\leqslant n}|_W$ is generated freely by sections defined by (5.7) for all $v \in E$. We have completed the definition of the locally free $\mathscr{O}_{\mathcal{C}}$-module $\mathscr{V}_{\mathfrak{x}}^{\leqslant n}$.

Remark 5.3. Since the vacuum vector $\mathbf{1}$ is annihilated by L_n ($n \geqslant 0$), we see that $\mathbf{1}$ is fixed by any transition function $\mathcal{U}(\varrho(\eta|\mu))$. Thus, we can define unambiguously an element $\mathbf{1} \in \mathscr{V}_{\mathfrak{x}}(\mathcal{C} - \Sigma)$ (the **vacuum section**) such that for any open $U \subset \mathcal{C} - \Sigma$ and any $\eta \in \mathscr{O}(U)$ univalent on each fiber, $\mathcal{U}_\varrho(\eta)\mathbf{1}$ is the vaccum vector $\mathbf{1}$ (considered as a constant function). Also, by (5.7), it is clear that

$$\mathbf{1} \in \mathscr{V}_{\mathfrak{x}}(\mathcal{C}).$$

Restriction to fibers

By comparing the transition functions and looking at the generating sections near the nodes, it is easy to see:

Proposition 5.4. *For any $n \in \mathbb{N}$ and $b \in \mathcal{B}$, we have a natural isomorphism of $\mathscr{O}_{\mathcal{C}_b}$-modules*

$$\mathscr{V}_{\mathfrak{X}}^{\leq n}|\mathcal{C}_b \simeq \mathscr{V}_{\mathcal{C}_b}^{\leq n}.$$

Let now $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \dots, \varsigma_N)$ be a family of N-pointed complex curves.

Theorem 5.5. *Let $n \in \mathbb{N}$. Then there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$, the $\mathscr{O}_{\mathcal{B}}$-module $\pi_*\big(\mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big)$ is locally free, and for any $b \in \mathcal{B}$ there is a natural isomorphism of vector spaces*

$$\frac{\pi_*\big(\mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big)_b}{\mathfrak{m}_b \cdot \pi_*\big(\mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big)_b} \simeq H^0\big(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b}^{\leq n} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}}(b))\big) \tag{5.8}$$

defined by restriction of sections. In particular, $\dim H^0\big(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b}^{\leq n} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}}(b))\big)$ is locally constant over b.

Recall that the left hand side of (5.8) is the fiber $\pi_*\big(\mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big)\big|b$, which, by Cartan's Theorem A, is formed by the restrictions of global sections if $\mathcal{B}$ is Stein.

Proof. By Theorem 2.3, we can find $k_0 \in \mathbb{N}$ such that for any $k > k_0$, $H^1\big(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b}^{\leq n} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}}(b))\big)$ vanishes for any $b \in \mathcal{B}$. Since the restriction of $\omega_{\mathcal{C}/\mathcal{B}}$ to $\mathcal{C}_b$ is $\omega_{\mathcal{C}_b}$, by Proposition 5.4, the restriction of $\mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})$ to $\mathcal{C}_b$ is naturally equivalent to $\mathscr{V}_{\mathcal{C}_b}^{\leq n} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}}(b))$. Thus, our theorem follows easily from Grauert's Theorem 4.1. $\square$

For any $\mathscr{O}_{\mathcal{C}}$-module $\mathscr{E}$, we set

$$\mathscr{E}(\bullet S_{\mathfrak{X}}) = \varinjlim_{k \in \mathbb{N}} \mathscr{E}(kS_{\mathfrak{X}}),$$

whose sections are meromorphic sections of $\mathscr{E}$ whose only possible poles are in $\varsigma_1(\mathcal{B}), \dots, \varsigma_N(\mathcal{B})$.

Corollary 5.6. *If $\mathcal{B}$ is a Stein manifold, then for any $n \in \mathbb{N}$ and any $b \in \mathcal{B}$, the restrictions to $\mathcal{C}_b$ of the elements of $H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{X}}^{\leq n} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)$ form the vector space $H^0\big(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b}^{\leq n} \otimes \omega_{\mathcal{C}_b}(\bullet S_{\mathfrak{X}}(b))\big)$.*

Proof. Choose k_0 as in Theorem 5.5. Then by that theorem and Cartan's Theorem A, we see the claim holds if $\bullet S_{\mathfrak{X}}$ is replaced by $kS_{\mathfrak{X}}$ for all $k > k_0$. The original claim thus follows. $\square$

The subsheaf $\mathcal{V}ir_c$

If we compare (5.6) with the transition functions of $\Theta_{\mathcal{C}/\mathcal{B}}$, and compare (5.7) with (4.22), we immediately see that

Proposition 5.7. *For any $n \in \mathbb{N}$, we have the following isomorphism of $\mathscr{O}_C$-modules:*

$$\mathscr{V}_{\mathfrak{X}}^{\leqslant n}/\mathscr{V}_{\mathfrak{X}}^{\leqslant n-1} \simeq \mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_{C/\mathcal{B}}^{\otimes n}. \tag{5.9}$$

Under this isomorphism, if $U \subset C - \Sigma$ is open and smooth, and $\eta \in \mathscr{O}(U)$ is univalent on each fiber of U, then for any $v \in \mathbb{V}(n)$, $v \otimes \partial_{\eta}^n$ is identified with the equivalence class of $\mathcal{U}_{\varrho}(\eta)^{-1}v$.

We now assume for simplicity that $\mathfrak{X}$ is a smooth family, and define an important $\mathscr{O}_C$-submodule $\mathcal{V}ir_c$ of $\mathscr{V}_{\mathfrak{X}}^{\leqslant 2}$ related to the conformal vector $\mathbf{c} \in \mathbb{V}(2)$. See [FB04, Sec. 8.2]. If U is an open subset of C equipped with $\eta \in \mathscr{O}(U)$ univalent on each fiber, then $\mathcal{V}ir_c|_U$ is the $\mathscr{O}_U$-submodule of $\mathscr{V}_{\mathfrak{X}}|_U$ generated (freely) by $\mathcal{U}_{\varrho}(\eta)^{-1}\mathbf{c}$ and the vacuum section $\mathbf{1}$, which is locally free of rank 2. This definition is independent of the choice of η. Indeed, if $\mu : U \to \mathbb{C}$ is also univalent on each fiber, then $\mathcal{U}_{\varrho}(\mu)\mathcal{U}_{\varrho}(\eta)^{-1}\mathbf{c} = \mathcal{U}(\varrho(\mu|\eta))\mathbf{c}$, which can be calculated using the actions of L_n ($n \geqslant 0$) on $\mathbf{c}$, is an $\mathscr{O}_U$-linear combination of $\mathbf{c}$ and $\mathbf{1}$. Thus, by gluing all such U, we get $\mathcal{V}ir_c$.

By Proposition 5.7, we have a short exact sequence

$$0 \to \mathscr{V}_{\mathfrak{X}}^{\leqslant 1} \to \mathscr{V}_{\mathfrak{X}}^{\leqslant 2} \xrightarrow{\lambda} \mathbb{V}(2) \otimes_{\mathbb{C}} \Theta_{C/\mathcal{B}}^{\otimes 2} \to 0$$

where λ is described locally by sending $\mathcal{U}_{\varrho}^{-1}(\eta)v$ (where $v \in \mathbb{V}(2)$) to $v \cdot \partial_{\eta}^2$ and sending the submodule $\mathscr{V}_{\mathfrak{X}}^{\leqslant 1}$ to 0. Using this description of λ, it is easy to see that the restriction of λ to the subsheaf $\mathcal{V}ir_c$ has image $\mathbf{c} \otimes_{\mathbb{C}} \Theta_{C/\mathcal{B}}^{\otimes 2} \simeq \Theta_{C/\mathcal{B}}^{\otimes 2}$, and that its kernel is the $\mathscr{O}_C$-submodule generated by $\mathbf{1}$. Thus, we obtain an exact sequence

$$0 \to \mathscr{O}_C \to \mathcal{V}ir_c \xrightarrow{\lambda} \Theta_{C/\mathcal{B}}^{\otimes 2} \to 0. \tag{5.10}$$

where $\mathscr{O}_C \simeq \mathscr{O}_C \cdot \mathbf{1} \subset \mathcal{V}ir_c$. If we choose $U \subset C$ and $\eta \in \mathscr{O}(U)$ holomorphic on each fiber, then

$$\lambda : \quad \mathcal{U}_{\varrho}(\eta)^{-1}\mathbf{c} \mapsto \partial_{\eta}^2, \qquad \mathbf{1} \mapsto 0.$$

By tensoring with $\omega_{C/\mathcal{B}}$, we get an exact sequence

$$0 \to \omega_{C/\mathcal{B}} \to \mathcal{V}ir_c \otimes \omega_{C/\mathcal{B}} \xrightarrow{\lambda} \Theta_{C/\mathcal{B}} \to 0 \tag{5.11}$$

whose local expression is

$$\lambda : \quad \mathcal{U}_{\varrho}(\eta)^{-1}\mathbf{c} \, d\eta \mapsto \partial_{\eta}, \qquad \mathbf{1} \, d\eta \mapsto 0. \tag{5.12}$$

6 Conformal blocks associated to families of complex curves

Let $\mathfrak{X} = (\pi : C \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N)$ be a family of N-pointed complex curves. For each $b \in \mathcal{B}$,

$$\mathfrak{X}_b = (C_b; \varsigma_1(b), \ldots, \varsigma_N(b))$$

is an N-pointed complex curve. If $\mathfrak{X}$ is equipped with local coordinates $\eta_1, \ldots, \eta_N$ at $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$, then, by restricting to $\mathcal{C}_b$, $\mathfrak{X}_b$ also has local coordinates.

Let $\mathbb{W}_1, \ldots, \mathbb{W}_N$ be V-modules. Then for each $b \in \mathcal{B}$, $\mathscr{T}^*_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ is the space of conformal blocks associated to $\mathbb{W}_\bullet$ and $\mathfrak{X}_b$. In this section, we shall define conformal blocks ϕ associated to the family $\mathfrak{X}$. ϕ is a function on $\mathcal{B}$ whose value $\phi(b)$ at each $b \in \mathcal{B}$ is an element of $\mathscr{T}^*_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$. Moreover, we require ϕ is a holomorphic function in a certain sense. Since this property is local, we assume that $\mathcal{B}$ is small enough such that $\mathfrak{X}$ admits local coordinates. This assumption will be dropped in the later half of this section.

Definition of conformal blocks

We need to explain how the vector spaces $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ (for all $b \in \mathcal{B}$) form an (infinite rank) holomorphic vector bundle $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$.

Define $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ to be an infinite rank locally free $\mathscr{O}_\mathcal{B}$-module as follows. For any local coordinates $\eta_1, \ldots, \eta_N$ of $\mathfrak{X}$ at $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$ respectively, we have a trivialization

$$\mathcal{U}(\eta_\bullet) \equiv \mathcal{U}(\eta_1) \otimes \cdots \otimes \mathcal{U}(\eta_N) : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \xrightarrow{\simeq} \mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_\mathcal{B} \tag{6.1}$$

such that if $\mu_\bullet$ is another set of local coordinates, then the transition function

$$\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} : \mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_\mathcal{B} \xrightarrow{\simeq} \mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_\mathcal{B}$$

is defined such that for any constant section $w_\bullet = w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_\bullet$, $\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} w_\bullet$, as a $\mathbb{W}_\bullet$-valued holomorphic function, satisfies

$$\left(\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} w_\bullet \right)(b) \equiv \left(\mathcal{U}(\eta_\bullet | \mu_\bullet)^{-1} w_\bullet \right)(b)$$
$$= \mathcal{U}\left((\eta_1 | \mu_1)_b \right) w_1 \otimes \mathcal{U}\left((\eta_2 | \mu_2)_b \right) w_2 \otimes \cdots \otimes \mathcal{U}\left((\eta_N | \mu_N)_b \right) w_N \tag{6.2}$$

for any $b \in \mathcal{B}$. Here, for each $1 \leqslant i \leqslant N$, $(\eta_i | \mu_i)_b$ is the element in $\mathbb{G}$ satisfying

$$(\eta_i | \mu_i)_b(z) = \eta_i \circ (\mu_i, \pi)^{-1}(z, b). \tag{6.3}$$

If we compare the transition functions (3.4) and (6.2), we see that there is a natural and coordinate-independent isomorphism of vector spaces

$$\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|b \simeq \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$$

where $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ is defined near (3.3). We shall identify these two spaces in the following.

Definition 6.1. A **conformal block** ϕ associated to $\mathbb{W}_\bullet$, $\mathfrak{X}$ (and defined over $\mathcal{B}$) is an $\mathscr{O}_\mathcal{B}$-module homomorphism $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_\mathcal{B}$ whose restriction to each fiber $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ is an element of $\mathscr{T}^*_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$, i.e., vanishes on $H^0\left(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(\bullet S_{\mathfrak{X}}(b))\right) \cdot \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$. The vector space of all such ϕ is denoted by $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$.

Remark 6.2. The holomorphicity of a conformal block ϕ as a function on $\mathcal{B}$ would be easier to understand by shrinking $\mathcal{B}$ and choosing local coordinates $\eta_1, \ldots, \eta_N$, and identifying $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ with $\mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_\mathcal{B}$ via $\mathcal{U}(\eta_\bullet)$. That ϕ is a morphism $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_\mathcal{B}$ means precisely that ϕ is an $(\mathbb{W}_\bullet)^*$-valued function on $\mathcal{B}$ whose evaluation with any vector of $\mathbb{W}_\bullet$ is a holomorphic function on $\mathcal{B}$; also, ϕ is determined by and can be reconstructed from the corresponding $\mathscr{O}(\mathcal{B})$-module homomorphism $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}) \to \mathscr{O}(\mathcal{B})$. Moreover, ϕ is in $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)(\mathcal{B})$ if and only if for each $b \in \mathcal{B}$, $\phi(b)$ as a linear functional on $\mathbb{W}_\bullet \simeq \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ is in $\mathscr{T}_{\mathfrak{X}_b}^*(\mathbb{W}_\bullet)$.

Action of $H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$

The above definition of conformal blocks is fiberwise, i.e., the restriction of ϕ to each fiber of the family is a conformal block defined in Section 3. We now give a global description which relates conformal blocks to the sheaf of VOA $\mathscr{V}_{\mathfrak{X}}$.

Recall there are natural equivalences $\mathscr{V}_{\mathfrak{X}}|_{\mathcal{C}_b} \simeq \mathscr{V}_{\mathcal{C}_b}$ and $\omega_{\mathcal{C}/\mathcal{B}}|_{\mathcal{C}_b} \simeq \omega_{\mathcal{C}_b}$. Thus, we can define a linear action of $H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$ whose restriction to each fiber $H^0(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(\bullet S_{\mathfrak{X}}(b)))$ acts on $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$. To see that the image of this action is holomorphic on $\mathcal{B}$, let us write down the action more explicitly.

Let z be the standard variable of $\mathbb{C}$. If $\mathbb{W}$ is a $\mathbb{V}$-module, we have a linear map

$$Y_\mathbb{W} : \big(\mathbb{V} \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})((z))\big) \otimes_{\mathscr{O}(\mathcal{B})} \big(\mathbb{W} \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})\big) \to \mathbb{W} \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})((z)),$$
$$v(b,z) \otimes w(b) \mapsto Y_\mathbb{W}(v(b,z), z)w(b) \tag{6.4}$$

which is clearly an $\mathscr{O}(\mathcal{B})((z))$-module homomorphism. Note that a section of $\mathbb{V} \otimes_\mathbb{C} \mathscr{O}_{\mathbb{C} \times \mathcal{B}}$ on a neighborhood of $\{0\} \times \mathcal{B}$ can be regarded as an element of $\mathbb{V} \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})((z))$ by taking series expansion.

Now, choose local coordinates $\eta_\bullet$ of $\mathfrak{X}$. For each $1 \leqslant i \leqslant N$, choose a neighborhood U_i of $\varsigma_i(\mathcal{B})$ on which η_i is defined. Identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ with $\mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_\mathcal{B}$ via $\mathcal{U}(\eta_\bullet)$. By tensoring with the identity map of $\omega_{\mathcal{C}/\mathcal{B}}$, the map (5.4) induces naturally an $\mathscr{O}_{U_i}$-module isomorphism (also denoted by $\mathcal{U}_\varrho(\eta_i)$):

$$\mathcal{U}_\varrho(\eta_i) : \mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big|_{U_i} \xrightarrow{\simeq} \mathbb{V} \otimes_\mathbb{C} \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big|_{U_i}.$$

Identify U_i with $(\eta_i, \pi)(U_i)$ via (η_i, π). Then η_i as a variable equals the standard variable z. If v is a section of $\mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})$ defined near $\varsigma_i(\mathcal{B})$ and $w_i \in \mathbb{W}_i \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})$, then again we have

$$v \cdot w_i = \mathrm{Res}_{z=0} Y_{\mathbb{W}_i}(\mathcal{U}_\varrho(\eta_i)v, z)w_i, \tag{6.5}$$

which is clearly also an element of $\mathbb{W}_i \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})$. If $v \in H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)$, then as in (3.2), v acts on $\mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}(\mathcal{B})$ by summing up the actions on the tensor components.

Another description of conformal blocks

We now give a global description of conformal blocks, followed by an application. We shall *not* assume $\mathfrak{X}$ admits local coordinates. Then $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is still an infinite-rank locally free $\mathscr{O}_\mathcal{B}$-module whose local trivializations are given by (6.1). Again, we define conformal blocks using Definition 6.1.

Theorem 6.3. *Let $V \subset \mathcal{B}$ be an open subset of $\mathcal{B}$, and let $\phi : \mathscr{W}_{\mathfrak{X}_V}(\mathbb{W}_\bullet) \to \mathscr{O}_V$ be an $\mathscr{O}_V$-module homomorphism. Assume that $\mathcal{B}$ is Stein. Then ϕ is in $\mathscr{T}^*_{\mathfrak{X}_V}(\mathbb{W}_\bullet)(V)$ if and only if the evaluation of ϕ with the restriction of any element of*

$$\mathscr{J}(\mathcal{B}) = H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{X}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}) \tag{6.6}$$

to V is the zero function.

Note that we have omitted $\mathrm{Span}_{\mathbb{C}}$ in (6.6).

Proof. The "only if" part is obvious by the definition of conformal blocks and does not require $\mathcal{B}$ to be Stein. The "if" part follows from Corollary 5.6. $\qquad\square$

For any open subset $V \subset \mathcal{B}$, let $\mathcal{C}_V = \pi^{-1}(V)$. Then $\mathfrak{X}$ restricts to

$$\mathfrak{X}_V := (\pi : \mathcal{C}_V \to V; \varsigma_1|_V, \ldots, \varsigma_N|_V).$$

Any set of local coordinates $\eta_\bullet$ of $\mathfrak{X}$ restricts to one of $\mathfrak{X}_V$. We also write $\mathscr{T}^*_{\mathfrak{X}_V}(\mathbb{W}_\bullet)(V)$ as $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)(V)$. The reason for this notation will be explained shortly.

Proposition 6.4. *Let $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ be an $\mathscr{O}_{\mathcal{B}}$-module homomorphism. Assume that $\mathcal{B}$ is connected and contains a non-empty open subset V such that $\phi|_V : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|_V \to \mathscr{O}_V$ is an element of $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)(V)$. Then ϕ is an element of $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$.*

Proof. We first assume $\mathcal{B}$ is also Stein. The evaluation of ϕ with any element of $\mathscr{J}(\mathcal{B})$ is a holomorphic function on $\mathcal{B}$ vanishing on V. So it must be constantly 0. So ϕ is a conformal block by Theorem 6.3.

Now, we do not assume $\mathcal{B}$ is Stein. Let A be the set of all $b \in \mathcal{B}$ such that b has a neighborhood U satisfying that the restriction $\phi|_U$ is a conformal block. Then A is open and non-empty. For any $b \in \mathcal{B} - A$, let U be a connected Stein neighborhood of b. Then, by the first paragraph, $\phi|_U$ is a conformal block if U has a non-zero open subset V such that $\phi|_V$ is a conformal block. Therefore U must be disjoint from A. This shows that $\mathcal{B} - A$ is open. So $\mathcal{B} = A$. $\qquad\square$

Remark 6.5. It is clear that the collection of all $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)(V)$ (where $V \subset \mathcal{B}$) form a sheaf of $\mathscr{O}_{\mathcal{B}}$-modules, which we denote by $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)$ and call the **sheaf of conformal blocks** associated to $\mathbb{W}_\bullet$ and $\mathfrak{X}$. Let $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ be the sheaf of $\mathscr{O}_{\mathcal{B}}$-modules associated to the presheaf

$$V \mapsto \frac{\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(V)}{H^0\big(\mathcal{C}_V, \mathscr{V}_{\mathfrak{X}_V} \otimes \omega_{\mathcal{C}_V/V}(\bullet S_{\mathfrak{X}_V})\big) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(V)}$$

and call it the **sheaf of covacua**. Then, by Theorem 6.3, $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is the dual $\mathscr{O}_{\mathcal{B}}$-module of $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$. Moreover, using Theorem 5.5 or Corollary 5.6, it is easy to see that for each $b \in \mathcal{B}$, the clearly surjective linear map

$$\frac{\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)_b}{\mathfrak{m}_b \cdot \mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)_b} \to \mathscr{T}_{\mathfrak{X}_b}(\mathbb{W}_\bullet) \tag{6.7}$$

defined by the restriction $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)_b \to \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|b$ is injective. Thus, the above two vector spaces have the same dimension, and are clearly equal to that of $\mathscr{T}^*_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$.

If $\mathbb{V}$ is C_2-cofinite, rational, and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$, then by the main results of [DGT19b], the function $b \mapsto \dim \mathscr{T}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ has finite values and is locally constant.[6] By Theorem 7.4 (applied to any subfamily $\mathfrak{X}_V$ where V is Stein), the $\mathscr{O}_{\mathcal{B}}$-module $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is finitely-generated. Thus, $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is locally free by an easy consequence of Nakayama's Lemma. (Cf. for instance [BS76, Lemma III.1.6].) Thus, $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)$ is also locally free, and by (6.7), its fibers are naturally equivalent to the spaces of conformal blocks. As a consequence, any conformal block associated to a fiber $\mathfrak{X}_b$ can be extended to a conformal block associated to a family $\mathfrak{X}_V$ whenever V is Stein.

If $\mathbb{V}$ is only C_2-cofinite and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$, but $\mathfrak{X}$ is a smooth family, then our claims about $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ and $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)$ and their fibers in the last paragraph are still true since, by [DGT19b], the above result about dimensions still holds.

7 A finiteness theorem

Recall that $\mathbb{V}$ is called C_2**-cofinite** if the subspace of $\mathbb{V}$ spanned by $C_2(\mathbb{V}) := \{Y(u)_{-2}v : u, v \in \mathbb{V}\}$ has finite codimension. The following important result is due to Miyamoto [Miy04, Lemma 2.4]; see also [GN03] or [Buhl02, Thm. 1].

Theorem 7.1. *Assume $\mathbb{V}$ is C_2-cofinite. Then there exist a finite set $\mathbb{E}$ of homogeneous vectors of $\mathbb{V}$ such that any weak $\mathbb{V}$-module $\mathbb{W}$ generated by a vector w_0 is spanned by*

$$Y_{\mathbb{W}}(v_k)_{-n_k} Y_{\mathbb{W}}(v_{k-1})_{-n_{k-1}} \cdots Y_{\mathbb{W}}(v_1)_{-n_1} w_0 \tag{7.1}$$

where $k \in \mathbb{N}$, $v_1, \ldots, v_k \in \mathbb{E}$, and $n_1 < n_2 < \cdots < n_k$.

Note that by the lower-truncation property (i.e., for each v, w, $Y_{\mathbb{W}}(v)_n w = 0$ for sufficiently large n) and that $\mathbb{E}$ is finite, we can assume in the above theorem that $n_1 > -L$ for some constant $L \in \mathbb{N}$ depending only on $\mathbb{V}, \mathbb{W}, \mathbb{E}$.

Remark 7.2. It follows that if $\mathbb{V}$ is C_2-cofinite, a $\mathbb{V}$-module $\mathbb{W}$ is finitely-generated if and only if it is $L_{0,s}$-semisimple. Indeed, if $\mathbb{W}$ is $L_{0,s}$-simple, then [Hua09, Cor. 3.16] shows that it has finite length, and hence is finitely-generated. In particular, since $L_{0,n} \in \mathrm{End}_{\mathbb{V}}(\mathbb{W})$, we see that $L_{0,n}{}^k$ vanishes on $\mathbb{M}$ for sufficiently large k since it vanishes on those generating vectors. Conversely, if $\mathbb{W}$ is finitely generated, then it is finitely generated by $L_{0,s}$-eigenvectors. Then the above theorem proves the $L_{0,s}$-semisimplicity.

We will fix this $\mathbb{E}$ in this section. Theorem 7.1 will be used in the following form.

Corollary 7.3. *Assume that $\mathbb{V}$ is C_2-cofinite. Let $\mathbb{W}$ be a finitely-generated $\mathbb{V}$-module. Then for any $n \in \mathbb{N}$, there exists $\nu(n) \in \mathbb{N}$ such that any $\widetilde{L}_0$-homogeneous vector $w \in \mathbb{W}$ whose weight $\widetilde{\mathrm{wt}}(w) > \nu(n)$ is a sum of vectors in $Y_{\mathbb{W}}(v)_{-l}\mathbb{W}(\widetilde{\mathrm{wt}}(w) - \mathrm{wt}(v) - l + 1)$ where $v \in \mathbb{E}$ and $l > n$.*

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[6]Since each fiber $\mathcal{C}_b$ is a projective variety, and since each $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}$ is locally free and (by comparing the two constructions) equivalent to the analytification of the corresponding one in [DGT19a, DGT19b], the algebraic results of $\mathscr{T}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ also hold in the analytic setting.

Proof. Assume without loss of generality that $\mathbb{W}$ is generated by a single $\tilde{L}_0$-homogeneous vector w_0. Let T be the set of all vectors of the form (7.1) where $n_k \leqslant n$. Then, by the above theorem, T is a finite subset of $\mathbb{W}$. Set $\nu(n) = \max\{\widetilde{\mathrm{wt}}(w_1) : w_1 \in T\}$. If $w \in \mathbb{W}$ is $\tilde{L}_0$-homogeneous with weight $\widetilde{\mathrm{wt}}(w) > \nu(n)$, then we can also write w as a sum of $\tilde{L}_0$-homogeneous vectors (with weight $\widetilde{\mathrm{wt}}(w)$) of the form (7.1), but now the n_k must be greater than ν since such vector is not in T. This proves that w is a sum of $\tilde{L}_0$-homogeneous vectors of the form $Y_{\mathbb{W}}(v)_{-l}w_2$ where $v \in \mathbb{E}$, $l > n$, and $w_2 \in \mathbb{W}$ is $\tilde{L}_0$-homogeneous. The same is true if $\mathbb{W}$ is finitely-generated. By (1.2), we have $\widetilde{\mathrm{wt}}(w_2) = \widetilde{\mathrm{wt}}(w) - \mathrm{wt}(v) - l + 1$. $\qquad\square$

Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$ be a family of N-pointed complex curves with local coordinates, and let $\mathbb{W}_1, \ldots, \mathbb{W}_N$ be finitely-generated $\mathbb{V}$-modules. Notice Theorem 6.3 and recall the definition of $\mathscr{J}(\mathcal{B})$ in (6.6).

Theorem 7.4. *Let $\mathbb{V}$ be C_2-cofinite, and assume that $\mathcal{B}$ is a Stein manifold. Then $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$ is a finitely-generated $\mathscr{O}(\mathcal{B})$-module.*

Proof. Since local coordinates are chosen, we identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ with $\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_{\mathcal{B}}$. Let $E = \max\{\mathrm{wt}(v) : v \in \mathbb{E}\}$. By Theorem 2.3, there exists $k_0 \in \mathbb{N}$ such that

$$H^1(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b}^{\leqslant E} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}})) = 0 \tag{7.2}$$

for any $b \in \mathcal{B}$ and $k \geqslant k_0$. We fix an arbitrary $k \in \mathbb{N}$ satisfying $k \geqslant E + k_0$.

Introduce a weight $\widetilde{\mathrm{wt}}$ on $\mathbb{W}_\bullet$ such that $\widetilde{\mathrm{wt}}(w_\bullet) = \widetilde{\mathrm{wt}}(w_1) + \widetilde{\mathrm{wt}}(w_2) + \cdots + \widetilde{\mathrm{wt}}(w_N)$ when $w_1, \ldots, w_N$ are $\tilde{L}_0$-homogeneous. For each $n \in \mathbb{N}$, $\mathbb{W}_\bullet^{\leqslant n}$ (resp. $\mathbb{W}_\bullet(n)$) denotes the (finite-dimensional) subspace spanned by all $\tilde{L}_0$-homogeneous homogeneous vectors $w \in \mathbb{W}_\bullet$ satisfying $\widetilde{\mathrm{wt}}(w) \leqslant n$ (resp. $\widetilde{\mathrm{wt}}(w) = n$). We shall prove by induction that for any $n > N\nu(k)$, any vector of $\mathbb{W}_\bullet(n)$ (considered as constant sections of $\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$) is a (finite) sum of elements of $\mathbb{W}_\bullet^{\leqslant n-1} \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$ mod $\mathscr{J}(\mathcal{B})$. Then the claim of our theorem follows.

Choose any $w_\bullet = w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_\bullet(n)$ such that $w_1, \ldots, w_N$ are $\tilde{L}_0$-homogeneous. Then one of $w_1, \ldots, w_N$ must have $\tilde{L}_0$-weight greater than $\nu(k)$. Assume, without loss of generality, that $\widetilde{\mathrm{wt}}(w_1) > \nu(k)$. Then, by Corollary 7.3, w_1 is a sum of non-zero $\tilde{L}_0$-homogeneous vectors of the form $Y_{\mathbb{W}_1}(u)_{-l}w_1^\circ$ where $u \in \mathbb{E}$, $l > k$, $w_1^\circ \in \mathbb{W}_1$, and $\widetilde{\mathrm{wt}}(w_1^\circ) = \widetilde{\mathrm{wt}}(w_1) - \mathrm{wt}(u) - l + 1$. Thus $\widetilde{\mathrm{wt}}(w_1) - \widetilde{\mathrm{wt}}(w_1^\circ) \geqslant l - 1 \geqslant k \geqslant E + k_0$.

It suffices to show that each $Y_{\mathbb{W}_1}(u)_{-l}w_1^\circ \otimes w_2 \otimes \cdots \otimes w_N$ is a sum of elements of $\mathbb{W}_\bullet^{\leqslant n-1} \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$ mod $\mathscr{J}(\mathcal{B})$. Thus, we assume for simplicity that $w_1 = Y_{\mathbb{W}_1}(u)_{-l}w_1^\circ$. Then

$$w_\bullet = Y_{\mathbb{W}_1}(u)_{-l}w_1^\circ \otimes w_2 \otimes \cdots \otimes w_N.$$

Set also

$$w_\bullet^\circ = w_1^\circ \otimes w_2 \otimes \cdots \otimes w_N.$$

Then $n - \widetilde{\mathrm{wt}}(w_\bullet^\circ) = \widetilde{\mathrm{wt}}(w_\bullet) - \widetilde{\mathrm{wt}}(w_\bullet^\circ) \geqslant E + k_0$. Thus

$$\widetilde{\mathrm{wt}}(w_\bullet^\circ) \leqslant n - E - k_0. \tag{7.3}$$

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Consider the short exact sequence of $\mathscr{O}_{\mathcal{C}}$-modules

$$0 \to \mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}}) \to \mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(l S_{\mathfrak{X}}) \to \mathscr{G} \to 0$$

where $\mathscr{G}$ is the quotient of the previous two sheaves. By (7.2), Proposition 5.4, and Grauert's Theorem 4.1, we see that $R^1\pi_*(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}})) = 0$, and $\pi_*(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}}))$ is locally free. Thus, we obtain an exact sequence of $\mathscr{O}_{\mathcal{B}}$-modules

$$0 \to \pi_*\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}})\big) \to \pi_*\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(l S_{\mathfrak{X}})\big) \to \pi_*\mathscr{G} \to 0.$$

Since $\mathcal{B}$ is Stein, by Cartan's Theorem B, $H^1(\mathcal{B}, \pi_*(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}}))) = 0$. Thus, there is an exact sequence

$$0 \to H^0\big(\mathcal{B}, \pi_*\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(k_0 S_{\mathfrak{X}})\big)\big) \to H^0\big(\mathcal{B}, \pi_*\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(l S_{\mathfrak{X}})\big)\big)$$
$$\to H^0\big(\mathcal{B}, \pi_*\mathscr{G}\big) \to 0. \tag{7.4}$$

Note that $H^0\big(\mathcal{B}, \pi_*\mathscr{G}\big)$ is exactly $\mathscr{G}(\mathcal{C})$. Choose mutually disjoint neighborhoods $W_1, \ldots, W_N$ of $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$ respectively. For each $1 \leqslant i \leqslant N$, identify $\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}|_{W_i}$ with $\mathbb{V}^{\leqslant E} \otimes_{\mathbb{C}} \omega_{\mathcal{C}/\mathcal{B}}|_{W_i}$ via $\mathcal{U}_{\varrho}(\eta_i)$, and identify η_i with the standard coordinate z by identifying W_i with $(\eta_i, \pi)(W_i)$. Define an element $\upsilon \in \mathscr{G}(\mathcal{C})$ as follows. $\upsilon|_{W_1}$ is the equivalence class represented by $u z^{-l} dz$, and $\upsilon|_{\mathcal{C} - \varsigma_1(\mathcal{B})} = 0$. Since the second map in the above exact sequence is surjective, υ lifts to an element $\hat{\upsilon}$ of $H^0\big(\mathcal{B}, \pi_*\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(l S_{\mathfrak{X}})\big)\big)$, i.e., of $\big(\mathscr{V}_{\mathcal{C}}^{\leqslant E} \otimes \omega_{\mathcal{C}/\mathcal{B}}(l S_{\mathfrak{X}})\big)(\mathcal{C})$. Moreover, by the definition of $\mathscr{G}$ as a quotient, for each $1 \leqslant i \leqslant N$ we have an element v_i of $\mathbb{V}^{\leqslant E} \otimes_{\mathbb{C}} \mathscr{O}_{\mathcal{C}}(k_0 S_{\mathfrak{X}})(W_i)$ (and hence of $\mathbb{V}^{\leqslant E} \otimes_{\mathbb{C}} \mathscr{O}_{W_i}(k_0 \varsigma_i(\mathcal{B}))(W_i)$) such that

$$\hat{\upsilon}|_{W_1} = u z^{-l} dz + v_1 dz,$$
$$\hat{\upsilon}|_{W_i} = v_i dz \qquad (2 \leqslant i \leqslant N).$$

Notice that $\mathrm{Res}_{z=0} Y(\cdot, z) z^n dz = Y(\cdot)_n$. It follows that the element $\hat{\upsilon} \cdot w_{\bullet}^{\circ}$, which is in $\mathscr{J}(\mathcal{B})$, equals $w_{\bullet} + w_{\triangle}$ where

$$w_{\triangle} = (v_1 dz) \cdot w_1^{\circ} \otimes w_2 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_N + \sum_{i=2}^{N} w_1^{\circ} \otimes w_2 \otimes \cdots \otimes (v_i dz) \cdot w_i \otimes \cdots \otimes w_N.$$

Thus $w_{\bullet}$ equals $-w_{\triangle}$ mod $\mathscr{J}(\mathcal{B})$. For each $1 \leqslant i \leqslant N$, v_i has pole at $z = 0$ with order at most k_0. Thus, by (1.2), the action of $v_i dz$ on $\mathbb{W}_i$ increases the $\widetilde{L}_0$-weight by at most $E + k_0 - 1$. It follows from (7.3) that $w_{\triangle} \in \mathbb{W}_{\bullet}^{\leqslant n-1} \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$. The proof is complete. $\qquad \square$

Remark 7.5. Theorem 7.4 is the complex-analytic analogue of [DGT19b] Thm. 8.4.1, which says that for an algebraic family of complex curves, the sheaf of covacua is coherent (assuming that $\mathbb{V}$ is C_2-cofinite and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$). The key ideas in our proof are similar to theirs.

It is clear that Theorem 7.4 implies $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_{\bullet})$ is a finitely-generated $\mathscr{O}_{\mathcal{B}}$-module. However, Theorem 7.4 does not seem to imply that $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_{\bullet})$ is analytically coherent. This is different from the algebraic setting in which the sheaves of covacua are algebraically coherent since they are quasi-coherent. Nevertheless, one can show that $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_{\bullet})$ is locally free (if $\mathbb{V}$ is also rational) by combining the algebraic results with Theorem 7.4, as explained in Remark 6.5.

We remark that certain forms of Theorem 7.4 are well-known in the low genus cases: see [Zhu96, Lemma 4.4.1], [Hua05a, Cor. 1.2], [Hua05b, Cor. 3.4].

8 Schwarzian derivatives

Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ be a family of compact Riemann surfaces. Choose an open subset $U \subset \mathcal{C}$ and holomorphic functions $\eta, \mu : U \to \mathbb{C}$ univalent on each fiber. If $f \in \mathcal{O}(U)$ and $\partial_\eta f$ is nowhere zero, we define the (partial) **Schwarzian derivative** of f over η to be

$$\mathbf{S}_\eta f = \frac{\partial_\eta^3 f}{\partial_\eta f} - \frac{3}{2}\left(\frac{\partial_\eta^2 f}{\partial_\eta f}\right)^2 \tag{8.1}$$

where the partial derivative ∂_η is defined with respect to (η, π), i.e., it is annihilated by $d\pi$ and restricts to $d/d\eta$ on each fiber. Similarly, one can define $\mathbf{S}_\mu f$.

We refer the reader to [Ahl, Gun] for the basic facts about Schwarzian derivatives. The change of variable formula is easy to calculate:

$$\mathbf{S}_\mu f = (\partial_\mu \eta)^2 \mathbf{S}_\eta f + \mathbf{S}_\mu \eta, \tag{8.2}$$

Take $f = \mu$ and notice $\mathbf{S}_\mu \mu = 0$, we have

$$\mathbf{S}_\mu \eta = -(\partial_\mu \eta)^2 \mathbf{S}_\eta \mu. \tag{8.3}$$

Assuming f is also univalent on each fiber, we obtain the cocycle relation.

$$\mathbf{S}_\mu \eta \cdot d\mu^2 = -\mathbf{S}_\eta \mu \cdot d\eta^2, \qquad \mathbf{S}_\mu f \cdot d\mu^2 + \mathbf{S}_f \eta \cdot df^2 + \mathbf{S}_\eta \mu \cdot d\eta^2 = 0. \tag{8.4}$$

The transition functions of $\mathcal{V}ir_c$ (which is a subsheaf of $\mathscr{V}_{\mathfrak{X}}$) defined in Section 5 can be expressed by Schwarzian derivatives. Note that $L_0 \mathbf{c} = 2\mathbf{c}$, $L_1 \mathbf{c} = 0$, $L_2 \mathbf{c} = \frac{c}{2}\mathbf{1}$,[7] and $L_n \mathbf{c} = 0$ for all $n > 2$. Thus, if $\rho = \rho(z) \in \mathbb{G}$, then using the formula (1.5), we have $\mathcal{U}(\rho)\mathbf{c} = \rho'(0)^{L_0} e^{c_2 L_2}\mathbf{c} = \rho'(0)^{L_0}(\mathbf{c} + \frac{c}{2}c_2\mathbf{1}) = \rho'(0)^2 \mathbf{c} + \frac{c}{2}c_2\mathbf{1}$ where c is the central charge of $\mathbb{V}$, and c_2, which is given by (1.6), is $\frac{1}{6}\mathbf{S}_z\rho(0)$. Replace ρ by $\varrho(\eta|\mu) : U \to \mathbb{G}$. Then $\rho^{(n)}(0)$ should be replaced by $\partial_\mu^n \eta$. Thus the transition function $\mathcal{U}(\varrho(\eta|\mu))$ is described by

$$\mathcal{U}(\varrho(\eta|\mu))\mathbf{1} = \mathbf{1}, \qquad \mathcal{U}(\varrho(\eta|\mu))\mathbf{c} = (\partial_\mu \eta)^2 \mathbf{c} + \frac{c}{12}\mathbf{S}_\mu \eta \cdot \mathbf{1}. \tag{8.5}$$

We recall some well-known properties of Schwarzian derivatives. See [Hub81].

Proposition 8.1. *The following are true.*

(1) *If the restriction of η to each fiber $U_b = U \cap \pi^{-1}(b)$ (where $b \in \mathcal{B}$) is a Möbius transformation of μ, i.e., of the form $\frac{a\mu+b}{c\mu+d}$ where $ad - bc \neq 0$, then $\mathbf{S}_\mu \eta = 0$.*

(2) *Let $Q \in \mathcal{O}(U)$. Then, for each $x \in U$, one can find a neighborhood $V \subset U$ of x and a function $f \in \mathcal{O}(V)$ univalent on each fiber $V_b = V \cap \pi^{-1}(b)$, such that $\mathbf{S}_\eta f = Q$.*

(3) *If $f, g \in \mathcal{O}(U)$ are univalent on each fiber, then $\mathbf{S}_\eta f = \mathbf{S}_\eta g$ if and only if $\mathbf{S}_f g = 0$.*

We remark that the converse of (1) is also true: If f is univalent on each fiber, and if $S_\eta f = 0$, then the restriction of f to each fiber is a Möbius transformation of η.

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[7]$L_2 \mathbf{c} = L_2 L_{-2}\mathbf{1} = [L_2, L_{-2}]\mathbf{1} = 4L_0\mathbf{1} + \frac{c}{2}\mathbf{1} = \frac{c}{2}\mathbf{1}.$

Proof. (1) can be verified directly. To prove (2), we identify U with an open subset of $\mathbb{C} \times \mathcal{B}$ via (η, π). So η is identified with the standard coordinate z. We choose a neighborhood $V \subset U$ of x of the form $\mathcal{D} \times T$ where $T \subset \mathcal{B}$ is open, and $\mathcal{D}$ is an open disc centered at point $p = \eta(x) \in \mathbb{C}$. By basic theory of ODE, the differential equation

$$\partial_z^2 h + Qh/2 = 0.$$

have solutions $h_1, h_2 \in \mathscr{O}(V)$ satisfying the initial conditions $h_1(\cdot, p) = 1, \partial_z h_1(\cdot, p) = 0$ and $h_2(\cdot, p) = 0, \partial_z h_2(\cdot, p) = 1$. It is easy to check that $f := h_2/h_1$ satisfies $\mathbf{S}_z f = Q$, and is defined and satisfies $\partial_z f \neq 0$ near $\{p\} \times T$.

(3) follows from (8.2), which says $\mathbf{S}_\eta g = (\partial_\eta f)^2 \mathbf{S}_f g + \mathbf{S}_\eta f$. $\qquad\square$

Definition 8.2. An open cover $(U_\alpha, \eta_\alpha)_{\alpha \in \mathfrak{A}}$ of $\mathcal{C}$, where each open set U_α is equipped with a function $\eta_\alpha \in \mathscr{O}(U_\alpha)$ holomorphic on each fiber, is called a (relative) **projective chart** of $\mathfrak{X}$, if for any $\alpha, \beta \in \mathfrak{A}$, we have $\mathbf{S}_{\eta_\beta} \eta_\alpha = 0$ on $U_\alpha \cap U_\beta$. Two projective charts are called equivalent if their union is a projective chart. An equivalence class of projective charts is called a (relative) **projective structure**. Equivalently, a projective structure is a maximal projective chart.

If $\mathcal{B}$ is Stein, then $\mathfrak{X}$ has a projective structure. See Section B.

Remark 8.3. Let $\mathfrak{P}$ be a projective chart on $\mathfrak{X}$. Choose an open subset $U \subset \mathcal{C}$ and a fiberwisely univalent $\eta \in \mathscr{O}(U)$. One can define an element

$$\mathbf{S}_\eta \mathfrak{P} \in \mathscr{O}(U)$$

as follows. Choose any $(U_1, \mu) \in \mathfrak{P}$. Then $\mathbf{S}_\eta \mathfrak{P} = \mathbf{S}_\eta \mu$ on $U \cap U_1$. To check that $\mathbf{S}_\eta \mathfrak{P}$ is well defined, suppose there is another $(U_2, \zeta) \in \mathfrak{P}$. Then $\mathbf{S}_\mu \zeta = 0$ on $U_1 \cap U_2$. Thus $\mathbf{S}_\eta \mu = \mathbf{S}_\eta \zeta$ on $U \cap U_1 \cap U_2$ by Proposition 8.1-(3).

9 Actions of $H^0(\mathcal{C}, \Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}}))$

In this section, we fix $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$ to be a family of N-pointed compact Riemann surfaces with local coordinates. We assume for simplicity that $\mathcal{B}$ is a Stein manifold with coordinates $\tau_\bullet = (\tau_1, \ldots, \tau_N)$. Let $\mathbb{W}_1, \ldots, \mathbb{W}_N$ be $\mathbb{V}$-modules.

By Lemma 2.2, there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$ and $b \in \mathcal{B}$, we have $H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}})) = 0$. Thus $R^1\pi_*\omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}}) = 0$ (and also $\pi_*\omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})$ is locally free) due to Grauert's Theorem 4.1. Therefore, (5.11) implies an exact sequence

$$0 \to \pi_*\omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}}) \to \pi_*\big(\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big) \xrightarrow{\lambda} \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}}) \to 0.$$

By Cartan's Theorem B, $H^1(\mathcal{B}, \pi_*\omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})) = 0$. So we have an exact sequence

$$0 \to H^0\big(\mathcal{B}, \pi_*\omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big) \to H^0\big(\mathcal{B}, \pi_*\big(\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big)\big) \xrightarrow{\lambda} H^0\big(\mathcal{B}, \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(kS_{\mathfrak{X}})\big) \to 0.$$

Take the direct limit over all $k > k_0$, we get an exact sequence

$$0 \to H^0\big(\mathcal{B}, \pi_*\omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big) \to H^0\big(\mathcal{B}, \pi_*\big(\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)\big) \xrightarrow{\lambda} H^0\big(\mathcal{B}, \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big) \to 0. \tag{9.1}$$

According to Section 6, $H^0\big(\mathcal{B}, \pi_*\big(\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)\big)$ acts on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$ which clearly descends to the trivial action on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$. We shall use the above exact sequence to define an action of $H^0\big(\mathcal{B}, \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big) = H^0(\mathcal{C}, \Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}}))$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$, which turns out to be an $\mathscr{O}(\mathcal{B})$-scalar multiplication. This action depends on the local coordinates $\eta_\bullet$.

Choose mutually disjoint neighborhoods $U_1, \ldots, U_N$ of $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$ on which $\eta_1, \ldots, \eta_N$ are defined respectively. Write each $\tau_j \circ \pi$ as τ_j for short, so that $(\eta_i, \tau_\bullet)$ is a set of coordinates of U_i. Set $U = U_1 \cup \cdots \cup U_N$. Choose any $\theta \in H^0(\mathcal{C}, \Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}}))$, which, in each U_i, is expressed as

$$\theta|_{U_i} = a_i(\eta_i, \tau_\bullet)\partial_{\eta_i}. \tag{9.2}$$

Define

$$\nu(\theta) \in \big(\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big)(U)$$

such that

$$\mathcal{U}_\varrho(\eta_i)\nu(\theta)|_{U_i} = a_i(\eta_i, \tau_\bullet)\mathbf{c}\, d\eta_i. \tag{9.3}$$

The action of θ on $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is defined to be the action of $\nu(\theta)$ as in Section 6, namely, is determined by

$$\nu(\theta) \cdot w_\bullet = \sum_{i=1}^N w_1 \otimes \cdots \otimes \nu(\theta) \cdot w_i \otimes \cdots \otimes w_N \tag{9.4}$$

for any $w_\bullet = w_1 \otimes \cdots \otimes w_N \in \mathbb{W}_\bullet$, where $\nu(\theta) \cdot w_i$ is described by (6.5).

Lemma 9.1. *Assume that $(U_1, \eta_1), \ldots, (U_N, \eta_N)$ belong to a projective structure $\mathfrak{P}$. Then the action of $\nu(\theta)$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$ is zero.*

Proof. By (8.5), the transition function for $\mathbf{c} \otimes \omega_{\mathcal{C}/\mathcal{B}}$ between two projective coordinates is the same as that for $\Theta_{\mathcal{C}/\mathcal{B}}$, namely, when $\mathbf{S}_\mu \eta = 0$, ∂_μ changes to $\partial_\mu \eta \cdot \partial_\eta$, and $\mathbf{c}d\mu$ changes to $\partial_\mu \eta \cdot \mathbf{c}d\eta$, sharing the same transition function $\partial_\mu \eta$. Thus, as θ is over $\mathcal{C}$, $\nu(\theta)$ can be extended to a global section of $\mathcal{V}ir_c \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})$ on $\mathcal{C}$. Thus, $\nu(\theta)$ acts trivially since $\nu(\theta) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}) \subset \mathscr{J}(\mathcal{B})$. $\qquad\square$

Proposition 9.2. *Let $\mathfrak{P}$ be a projective structure of $\mathfrak{X}$. Choose $\theta \in H^0(\mathcal{C}, \Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}}))$ whose local expression is given by (9.2). Then the action of $\nu(\theta)$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$ (defined by the local coordinates $\eta_\bullet$) is the $\mathscr{O}(\mathcal{B})$-scalar multiplication by*

$$\#(\theta) := \frac{c}{12} \sum_{i=1}^N \mathrm{Res}_{\eta_i=0} \mathbf{S}_{\eta_i}\mathfrak{P} \cdot a_i(\eta_i, \tau_\bullet)\, d\eta_i. \tag{9.5}$$

Note that each $\mathbf{S}_{\eta_i}\mathfrak{P}$ (defined in Remark 8.3) is an element of $\mathscr{O}(U_i)$.

Proof. It suffices to prove that the claim is locally true. Thus, we may shrinking $\mathcal{B}$ and $U_1, \ldots, U_N$ so that for each $1 \leqslant i \leqslant N$, there exists a coordinate $\mu_i \in \mathscr{O}(U_i)$ at $\varsigma_i(\mathcal{B})$ such that $(U_i, \mu_i) \in \mathfrak{P}$. Then

$$\theta|_{U_i} = a_i(\eta_i, \tau_\bullet) \cdot (\partial_{\mu_i}\eta_i)^{-1}\partial_{\mu_i}.$$

Our strategy is to compare the action $\widetilde{\nu}(\theta)$ of θ defined by the coordinates $\mu_\bullet$ (which is trivial by Lemma 9.1) with the one $\nu(\theta)$ defined by $\eta_\bullet$. We have $\mathcal{U}_\varrho(\mu_i)\widetilde{\nu}(\theta)|_{U_i} = a_i(\eta_i, \tau_\bullet) \cdot (\partial_{\mu_i}\eta_i)^{-1}\mathbf{c}\, d\mu_i$ on each U_i. Then

$$\mathcal{U}_\varrho(\mu_i)\widetilde{\nu}(\theta)|_{U_i} = a_i(\eta_i, \tau_\bullet) \cdot (\partial_{\mu_i}\eta_i)^{-2}\mathbf{c}\, d\eta_i.$$

By Lemma 9.1, the action of $\widetilde{\nu}(\theta)$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$ is zero. Notice that the action of $\widetilde{\nu}(\theta)$ is independent of the choice of local coordinates. (See Theorem 3.2.) By (8.5), we have

$$\mathcal{U}_\varrho(\eta_i)\widetilde{\nu}(\theta)|_{U_i} = \mathcal{U}(\varrho(\eta_i|\mu_i))\mathcal{U}_\varrho(\mu_i)\widetilde{\nu}(\theta)|_{U_i}$$
$$= a_i(\eta_i, \tau_\bullet)\mathbf{c}\, d\eta_i + \frac{c}{12}a_i(\eta_i, \tau_\bullet) \cdot (\partial_{\mu_i}\eta_i)^{-2}\mathbf{S}_{\mu_i}\eta_i \cdot \mathbf{1}\, d\eta_i$$

By (9.3) and (8.3), we have

$$\mathcal{U}_\varrho(\eta_i)\widetilde{\nu}(\theta)|_{U_i} = \mathcal{U}_\varrho(\eta_i)\nu(\theta)|_{U_i} - \frac{c}{12}a_i(\eta_i, \tau_\bullet) \cdot \mathbf{S}_{\eta_i}\mu_i \cdot \mathbf{1}\, d\eta_i$$
$$= \mathcal{U}_\varrho(\eta_i)\nu(\theta)|_{U_i} - \frac{c}{12}a_i(\eta_i, \tau_\bullet) \cdot \mathbf{S}_{\eta_i}\mathfrak{P} \cdot \mathbf{1}\, d\eta_i.$$

Since the action of $\widetilde{\nu}(\theta)$ is zero, the action of $\nu(\theta)$ equals the sum over i of the actions of $\frac{c}{12}a_i(\eta_i, \tau_\bullet) \cdot \mathbf{S}_{\eta_i}\mathfrak{P} \cdot \mathbf{1}\, d\eta_i$, which is exactly the scalar multiplication by (9.5). $\qquad\square$

10　Sewing conformal blocks

In this and the following sections, we let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$ be a family of N-pointed complex curves with local coordinates obtained by sewing the following smooth family

$$\widetilde{\mathfrak{X}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}; \varsigma_1, \ldots, \varsigma_N, \varsigma', \varsigma''; \eta_1, \ldots, \eta_N, \xi, \varpi).$$

(See Section 4.) Recall that the N-points $\varsigma_1, \ldots, \varsigma_N$ and the local coordinates $\eta_1, \ldots, \eta_N$ of $\mathfrak{X}$ are constant with respect to sewing, and that each connected component of each fiber $\widetilde{\mathcal{C}}_b$ of $\widetilde{\mathfrak{X}}$ contains at least one $\varsigma_1(b), \ldots, \varsigma_N(b)$. Choose $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N, \mathbb{M}$, which together with the contragredient module $\mathbb{M}'$ are associated to $\varsigma_1, \ldots, \varsigma_N, \varsigma', \varsigma''$ respectively.

Sewing conformal blocks

Note $\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}'$ is $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M} \otimes \mathbb{M}'$. Note also $(\mathbb{M}' \otimes \mathbb{M})^*$ can be regarded as the algebraic completion of $\mathbb{M} \otimes \mathbb{M}'$. Define

$$\blacktriangleright \otimes \blacktriangleleft \in (\mathbb{M}' \otimes \mathbb{M})^*$$

such that for any $m' \in \mathbb{M}', m \in \mathbb{M}$,

$$\langle \blacktriangleright \otimes \blacktriangleleft, m' \otimes m \rangle = \langle m', m \rangle. \tag{10.1}$$

Let $A \in \mathrm{End}(\mathbb{M})$ whose transpose $A^{\mathrm{t}} \in \mathrm{End}(\mathbb{M}')$ exists, i.e.,

$$\langle Am, m' \rangle = \langle m, A^{\mathrm{t}} m' \rangle \tag{10.2}$$

for any $m' \in \mathbb{M}', m \in \mathbb{M}$. Then we have an element

$$A \blacktriangleright \otimes \blacktriangleleft \equiv \blacktriangleright \otimes A^{\mathrm{t}} \blacktriangleleft \quad \in (\mathbb{M}' \otimes \mathbb{M})^* \tag{10.3}$$

whose value at each $m' \otimes m$ is (10.2).

More explicitly, for each $n \in \mathbb{N}$ we choose a basis $\{m(n, a)\}_a$ of the finite-dimensional vector space $\mathbb{M}(n)$. Its dual basis $\{\breve{m}(n, a)\}_a$ is a basis of $\mathbb{M}'(n) = \mathbb{M}(n)^*$ satisfying $\langle m(n, a), \breve{m}(n, b) \rangle = \delta_{a,b}$. Then we have

$$\blacktriangleright \otimes \blacktriangleleft = \sum_{n \in \mathbb{N}} \sum_a m(n, a) \otimes \breve{m}(n, a),$$

and

$$A \blacktriangleright \otimes \blacktriangleleft = \sum_{n \in \mathbb{N}} \sum_a A \cdot m(n, a) \otimes \breve{m}(n, a)$$
$$= \blacktriangleright \otimes A^{\mathrm{t}} \blacktriangleleft = \sum_{n \in \mathbb{N}} \sum_a m(n, a) \otimes A^{\mathrm{t}} \cdot \breve{m}(n, a).$$

For each $n \in \mathbb{N}$, let $P(n)$ be the projection of $\mathbb{M}$ onto $\mathbb{M}(n)$. Its transpose, which is the projection of $\mathbb{M}'$ onto $\mathbb{M}'(n)$, is also denoted by $P(n)$. Then we clearly have

$$P(n) \blacktriangleright \otimes \blacktriangleleft = \blacktriangleright \otimes P(n) \blacktriangleleft = \sum_a m(n, a) \otimes \breve{m}(n, a) \qquad \in \mathbb{M} \otimes \mathbb{M}'.$$

Recall $\widetilde{L}_0^{\mathrm{t}} = \widetilde{L}_0$ by (1.14). Define

$$q^{\widetilde{L}_0} = \sum_{k \in \mathbb{N}} P(n) q^n \qquad \in \mathrm{End}(\mathbb{M})[[q]].$$

Then we have

$$q^{\widetilde{L}_0} \blacktriangleright \otimes \blacktriangleleft = \blacktriangleright \otimes q^{\widetilde{L}_0} \blacktriangleleft \qquad \in (\mathbb{M} \otimes \mathbb{M}')[[q]]. \tag{10.4}$$

We set

$$q^{L_0} \blacktriangleright \otimes \blacktriangleleft \qquad \in (\mathbb{M}' \otimes \mathbb{M})^* [\log q]\{q\} \tag{10.5}$$

sending each $w' \otimes w$ to $\langle q^{L_0} w, w' \rangle = \langle w, q^{L_0} w' \rangle$. When $\mathbb{M}$ is $L_{0,\mathrm{s}}$-semisimple, $L_{0,\mathrm{s}} - \widetilde{L}_0$ is a scalar $\lambda \in \mathbb{C}$. Notice

$$L_{0,\mathrm{n}} = L_0 - \widetilde{L}_0 - \lambda. \tag{10.6}$$

For each vector w in $\mathbb{M}(n)$, we set

$$q^{L_{0,\mathrm{n}}} w = e^{L_{0,\mathrm{n}} \log q} w = \sum_{n \in \mathbb{N}} \frac{1}{n!} L_{0,\mathrm{n}}{}^n w \cdot (\log q)^n \in \mathbb{M}(n)[\log q].$$

Then

$$q^{L_0} \blacktriangleright \otimes \blacktriangleleft \; = q^\lambda \cdot q^{L_{0,\mathrm{n}}} \cdot q^{\tilde{L}_0} \blacktriangleright \otimes \blacktriangleleft \; = \sum_{n \in \mathbb{N}} q^{\lambda+n} \sum_a q^{L_{0,\mathrm{n}}} m(n,a) \otimes \check{m}(n,a) \tag{10.7}$$

is in $q^\lambda(\mathbb{M} \otimes \mathbb{M}')[\log q][[q]]$. Thus, when $\mathbb{M}$ is in general only $L_{0,\mathrm{s}}$-semisimple, we have

$$q^{L_0} \blacktriangleright \otimes \blacktriangleleft \; \in (\mathbb{M} \otimes \mathbb{M}')[\log q]\{q\}. \tag{10.8}$$

From (1.1), we have

$$q\partial_q(q^{L_0} \blacktriangleright \otimes \blacktriangleleft) = L_0 q^{L_0} \blacktriangleright \otimes \blacktriangleleft . \tag{10.9}$$

Note that $\mathscr{O}(\mathcal{B})$ can be viewed as a subring of $\mathscr{O}(\tilde{\mathcal{B}})[[q]]$ by taking power series expansions. So $\mathscr{O}(\tilde{\mathcal{B}})[[q]]$ and $\mathscr{O}(\tilde{\mathcal{B}})[\log q]\{q\}$ are $\mathscr{O}(\mathcal{B})$-modules. For any $\psi \in \mathscr{T}^*_{\tilde{\mathfrak{X}}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')(\tilde{\mathcal{B}})$, we define its (normalized) **sewing** $\tilde{\mathcal{S}}\psi$ which is an $\mathscr{O}(\mathcal{B})$-module homomorphism

$$\tilde{\mathcal{S}}\psi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}) = \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B}) \to \mathscr{O}(\tilde{\mathcal{B}})[[q]],$$

and, in the case that $\mathbb{M}$ is $L_{0,\mathrm{s}}$-semisimple, the (standard) **sewing**

$$\mathcal{S}\psi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}) = \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B}) \to \mathscr{O}(\tilde{\mathcal{B}})[\log q]\{q\},$$

as follows. Regard ψ as an $\mathscr{O}(\tilde{\mathcal{B}})$-module homomorphism $\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}' \otimes_{\mathbb{C}} \mathscr{O}(\tilde{\mathcal{B}}) \to \mathscr{O}(\tilde{\mathcal{B}})$. $\tilde{\mathcal{S}}\psi$ is defined such that for any $w \in \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\tilde{\mathcal{B}})$,

$$\tilde{\mathcal{S}}\psi(w) = \psi\big(w \otimes q^{\tilde{L}_0} \blacktriangleright \otimes \blacktriangleleft \big). \tag{10.10}$$

When $\mathbb{M}$ (and hence $\mathbb{M}'$) is $L_{0,\mathrm{s}}$-semisimple, $\mathcal{S}\psi$ is defined in the same way except that $\tilde{L}_0$ is replaced by L_0, i.e.,

$$\mathcal{S}\psi(w) = \psi\big(w \otimes q^{L_0} \blacktriangleright \otimes \blacktriangleleft \big). \tag{10.11}$$

Formal conformal blocks

Our goal is to show that $\tilde{\mathcal{S}}\psi$ (and hence $\mathcal{S}\psi$) is a **formal conformal block** associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$, which means $\tilde{\mathcal{S}}\psi$ vanishes on $\mathscr{J}(\mathcal{B})$ (defined by (6.6)).[8] To prove this, we first need:

Lemma 10.1. *Let R be a unital commutative $\mathbb{C}$-algebra. For any $u \in \mathbb{V}$ and $f \in R[[\xi, \varpi]]$, the following two elements of $(\mathbb{M} \otimes \mathbb{M}' \otimes R)[[q]]$ (where the tensor products are over $\mathbb{C}$) are equal:*

$$\operatorname{Res}_{\xi=0} Y_{\mathbb{M}}\big(\xi^{L_0} u, \xi\big) q^{\tilde{L}_0} \blacktriangleright \otimes \blacktriangleleft \cdot f(\xi, q/\xi)\frac{d\xi}{\xi}$$

$$= \operatorname{Res}_{\varpi=0} q^{\tilde{L}_0} \blacktriangleright \otimes Y_{\mathbb{M}'}\big(\varpi^{L_0} \mathcal{U}(\gamma_1)u, \varpi\big) \blacktriangleleft \cdot f(q/\varpi, \varpi)\frac{d\varpi}{\varpi}. \tag{10.12}$$

When $\mathbb{M}$ is $L_{0,\mathrm{s}}$-semisimple, the same relation holds on the level of $(\mathbb{M} \otimes \mathbb{M}' \otimes R)[\log q]\{q\}$ if $\tilde{L}_0$ is replaced by L_0.

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[8]Due to Theorem 6.3, it would be proper to use this definition only when $\tilde{\mathcal{B}}$ and hence $\mathcal{B}$ are Stein.

Remark 10.2. We explain the meaning of the left hand side; the other side can be understood in a similar way. In the case that $\mathbb{M}$ is $L_{0,\mathrm{s}}$-semisimple, as $q^{L_0} {\blacktriangleright}{\otimes}{\blacktriangleleft}$ is an element of $(\mathbb{M} \otimes \mathbb{M}')[\log q]\{q\}$, $Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{L_0} {\blacktriangleright}{\otimes}{\blacktriangleleft}$ is an element of $(\mathbb{M} \otimes \mathbb{M}')((\xi))[\log q]\{q\}$.

Identify $(\mathbb{M} \otimes \mathbb{M}')((\xi))[\log q]\{q\} \simeq (\mathbb{M} \otimes \mathbb{M}' \otimes 1)((\xi))[\log q]\{q\}$, which is a subspace of the $R((\xi))[\log q]\{q\}$-module $(\mathbb{M} \otimes \mathbb{M}' \otimes R)((\xi))[\log q]\{q\}$. On the other hand, write $f(\xi, \varpi) = \sum_{m,n \in \mathbb{N}} f_{m,n} \xi^m \varpi^n$ where each $f_{m,n}$ is in R. Then

$$f(\xi, q/\xi) = \sum_{n \geqslant 0} \sum_{k \geqslant -n} f_{n+k,n} \xi^k q^n,$$

which shows $f(\xi, q/\xi) \in R((\xi))[[q]]$. Thus, the term in the residue on the left hand side is an element in

$$(\mathbb{M} \otimes \mathbb{M}' \otimes R)((\xi))[\log q]\{q\},$$

whose residue is in $(\mathbb{M} \otimes \mathbb{M}' \otimes R)((\xi))[\log q]\{q\}$. In the case of L_0, the explanation is the same if we replace $[\log q]\{q\}$ by $[[q]]$.

We note that once (10.12) has been proved for $\widetilde{L}_0$, when $\mathbb{M}$ is $L_{0,\mathrm{s}}$-simple, we multiply both sides of (10.12) by $q^{L_{0,\mathrm{n}}}$ and notice that $q^{L_{0,\mathrm{n}}}$ commutes with the action of $Y_{\mathbb{M}}$. So (10.12) holds on the level of $(\mathbb{M} \otimes \mathbb{M}' \otimes R)[\log q][[q]]$ if $\widetilde{L}_0$ is replaced by $\widetilde{L}_0 + L_{0,\mathrm{n}}$. So (10.12) holds, for every $L_{0,\mathrm{s}}$-simple and hence $L_{0,\mathrm{s}}$-semisimple $\mathbb{M}$, if $\widetilde{L}_0$ is replaced by L_0.

Proof of Lemma 10.1. Consider $Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{\widetilde{L}_0}$ as an element of $\mathrm{End}(\mathbb{M})[[\xi^{\pm 1}, q]]$. Since $\widetilde{L}_0^{\mathrm{t}} = \widetilde{L}_0$, we have the following relations of elements of $\mathrm{End}(\mathbb{M}')[[\xi^{\pm 1}, q^{\pm 1}]]$:

$$\big(Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{\widetilde{L}_0}\big)^{\mathrm{t}} = q^{\widetilde{L}_0}\big(Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big)\big)^{\mathrm{t}} \overset{(1.13)}{=\!=\!=\!=} q^{\widetilde{L}_0} Y_{\mathbb{M}'}\big(\mathcal{U}(\gamma_\xi)\xi^{L_0}u, \xi^{-1}\big)$$

$$\overset{(1.11)}{=\!=\!=\!=} q^{\widetilde{L}_0} Y_{\mathbb{M}'}\big(\xi^{-L_0}\mathcal{U}(\gamma_1)u, \xi^{-1}\big) \overset{(3.7)}{=\!=\!=\!=} Y_{\mathbb{M}'}\big((q/\xi)^{L_0}\mathcal{U}(\gamma_1)u, q/\xi\big) q^{\widetilde{L}_0}.$$

Thus, by (10.3), we have the following equations of elements in $(\mathbb{M}' \otimes \mathbb{M})^*[[\xi^{\pm 1}, q^{\pm 1}]]$:

$$Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{\widetilde{L}_0} {\blacktriangleright}{\otimes}{\blacktriangleleft} = {\blacktriangleright}{\otimes} \big(Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{\widetilde{L}_0}\big)^{\mathrm{t}}{\blacktriangleleft}$$

$$= {\blacktriangleright}{\otimes} Y_{\mathbb{M}'}\big((q/\xi)^{L_0}\mathcal{U}(\gamma_1)u, q/\xi\big) q^{\widetilde{L}_0}{\blacktriangleleft} = q^{\widetilde{L}_0} {\blacktriangleright}{\otimes} Y_{\mathbb{M}'}\big((q/\xi)^{L_0}\mathcal{U}(\gamma_1)u, q/\xi\big) {\blacktriangleleft}. \qquad (10.13)$$

Since for each n, $P(n) {\blacktriangleright}{\otimes}{\blacktriangleleft}$ is in $\mathbb{M} \otimes \mathbb{M}'$, (10.13) is actually an element in $(\mathbb{M} \otimes \mathbb{M}')[[\xi^{\pm 1}, q^{\pm 1}]]$.

Let

$$A(\xi, q) = Y_{\mathbb{M}}\big(\xi^{L_0}u, \xi\big) q^{\widetilde{L}_0} {\blacktriangleright}{\otimes}{\blacktriangleleft},$$

$$B(\varpi, q) = q^{\widetilde{L}_0} {\blacktriangleright}{\otimes} Y_{\mathbb{M}'}\big(\varpi^{L_0}\mathcal{U}(\gamma_1)u, \varpi\big){\blacktriangleleft},$$

considered as elements of $(\mathbb{M} \otimes \mathbb{M}')[[\xi^{\pm 1}, q^{\pm 1}]]$ and $(\mathbb{M} \otimes \mathbb{M}')[[\varpi^{\pm 1}, q^{\pm 1}]]$ respectively. Then (10.13) says $A(\xi, q) = B(q/\xi, q)$. Let $C(\xi, \varpi) \in (\mathbb{M} \otimes \mathbb{M}')[[\xi^{\pm 1}, \varpi^{\pm 1}]]$ be $A(\xi, \xi\varpi)$, which also equals $B(\varpi, \xi\varpi)$. Since $A(\xi, q)$ contains only non-negative powers of q, so does $A(\xi, \xi\varpi)$ for ϖ. Similarly, since $B(\varpi, q)$ contains only non-negative powers of q, so does $B(\varpi, \xi\varpi)$ for ξ. Therefore $C(\xi, \varpi)$ is an element in $(\mathbb{M} \otimes \mathbb{M}')[[\xi, \varpi]]$, where the

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latter can be identified with the subspace $(\mathbb{M} \otimes \mathbb{M}' \otimes 1)[[\xi, \varpi]]$ of the $R[[\xi, \varpi]]$-module $(\mathbb{M} \otimes \mathbb{M}' \otimes R)[[\xi, \varpi]]$. Thus $D(\xi, \varpi) := f(\xi, \varpi)C(\xi, \varpi)$ is well-defined as an element in $(\mathbb{M} \otimes \mathbb{M}' \otimes R)[[\xi, \varpi]]$. It is easy to check that

$$\mathrm{Res}_{\xi=0}\left(D(\xi, q/\xi)\frac{d\xi}{\xi}\right) = \mathrm{Res}_{\varpi=0}\left(D(q/\varpi, \varpi)\frac{d\varpi}{\varpi}\right).$$

(Indeed, they both equal $\sum_{n \in \mathbb{N}} D_{n,n} q^n$ if we write $D(\xi, \varpi) = \sum_{m,n \in \mathbb{N}} D_{m,n} \xi^m \varpi^n$.) This proves (10.12). $\qquad\square$

Recall $\mathcal{B} = \widetilde{\mathcal{B}} \times \mathcal{D}_{r\rho} = \mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}}$, and the order of Cartesian products will be switched when necessary.

Theorem 10.3. *Let* $\psi \in \mathscr{T}^*_{\widetilde{\mathfrak{x}}}(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')(\widetilde{\mathcal{B}})$. *Then* $\widetilde{\mathcal{S}}\psi$ *vanishes on* $\mathscr{J}(\mathcal{B})$. *If* $\mathbb{M}$ *is* $L_{0,s}$-*semisimple, then* $\mathcal{S}\psi$ *also vanishes on* $\mathscr{J}(\mathcal{B})$.

Proof. Step 1. Note that we have divisors $S_{\mathfrak{x}} = \sum_{i=1}^N \varsigma_i(\mathcal{B})$ and $S_{\widetilde{\mathfrak{x}}} = \sum_{i=1}^N \varsigma_i(\widetilde{\mathcal{B}}) + \varsigma'(\widetilde{\mathcal{B}}) + \varsigma''(\widetilde{\mathcal{B}})$ of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ respectively. Choose any v in $H^0\big(\mathcal{C}, \mathscr{V}_{\mathfrak{x}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{x}})\big)$. In this first step, we would like to construct a formal power series expansion

$$v = \sum_{n \in \mathbb{N}} v_n q^n \tag{10.14}$$

where each v_n is in $H^0\big(\widetilde{\mathcal{C}}, \mathscr{V}_{\widetilde{\mathfrak{x}}} \otimes \omega_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{x}}})\big)$.

First, choose any precompact open subset U of $\widetilde{\mathcal{C}}$ disjoint from the double points $\varsigma'(\widetilde{\mathcal{B}})$ and $\varsigma''(\widetilde{\mathcal{B}})$. Then one can find small enough positive numbers $\epsilon < r, \lambda < \rho$ such that $U \times \mathcal{D}_{\epsilon\lambda}$ is an open subset of $\widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F''$ in (4.9), and hence an open subset of $\mathcal{C}$. Moreover, by (4.14), the projection $\pi : \mathcal{C} \to \mathcal{B}$ equals $\widetilde{\pi} \times \mathbf{1} : \widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho} \to \widetilde{\mathcal{B}} \times \mathcal{D}_{r\rho}$ when restricted to $U \times \mathcal{D}_{\epsilon\lambda}$. It follows that the section $v|_{U \times \mathcal{D}_{\epsilon\lambda}}$ of $\mathscr{V}_{\mathfrak{x}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{x}})$ can be regarded as a section of $\mathscr{V}_{\widetilde{\mathfrak{x}} \times \mathcal{D}_{r\rho}} \otimes \omega_{\widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho}/\widetilde{\mathcal{B}} \times \mathcal{D}_{r\rho}}(\bullet S_{\mathfrak{x}})$, which, by taking power series expansions at $q = 0$, is in turn an element of $\big(\mathscr{V}_{\widetilde{\mathfrak{x}}} \otimes \omega_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{x}}})\big)(U)[[q]]$. The coefficient before q^n defines $v_n|_U$. This defines the section v_n of $\mathscr{V}_{\widetilde{\mathfrak{x}}} \otimes \omega_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{x}}})$ on $\widetilde{\mathcal{C}} - \varsigma'(\widetilde{\mathcal{B}}) - \varsigma''(\widetilde{\mathcal{B}})$ satisfying (10.14).

We now show that v_n has poles of orders at most n at $\varsigma'(\widetilde{\mathcal{B}})$ and $\varsigma''(\widetilde{\mathcal{B}})$. Let W, W', W'' be as described near (4.8). By (5.7) and (4.22), $v|_{W-\Sigma}$ is a sum of those whose restrictions to W', W'' under the trivializations $\mathcal{U}_\varrho(\xi), \mathcal{U}_\varrho(\varpi)$ are

$$f(\xi, q/\xi, \cdot)\xi^{L_0} u \cdot \frac{d\xi}{\xi} \qquad \text{resp.} \qquad -f(q/\varpi, \varpi, \cdot)\varpi^{L_0} \mathcal{U}(\gamma_1) u \cdot \frac{d\varpi}{\varpi} \tag{10.15}$$

where $u \in \mathbb{V}$ and $f = f(\xi, \varpi, \cdot) \in \mathscr{O}(W)$, and the coordinates of $\widetilde{\mathcal{B}}$ are suppressed as the dot. (Recall $q = \xi\varpi$.) In the above two terms, if we take power series expansions of q, then it is obvious that the coefficients before q^n have poles of orders at most n at $\xi = 0$ and $\varpi = 0$ respectively. This proves the claim.

Step 2. Let us assume $\mathbb{M}$ is $L_{0,s}$-semisimple (or just $L_{0,s}$-simple, for the sake of simplicity) and prove that $\mathcal{S}\psi$ vanishes on $\mathscr{J}(\mathcal{B})$. A similar method proves that $\widetilde{\mathcal{S}}$ vanished on $\mathscr{J}(\mathcal{B})$ in the general case.

By (4.8), we can regard $f(\xi, \varpi, \cdot)$ as an element of $\mathscr{O}(\widetilde{\mathcal{B}})[[\xi, \varpi]]$. Thus, by Lemma 10.1 (applied to $R = \mathscr{O}(\widetilde{\mathcal{B}})$) and the fact that $v|_{W-\Sigma}$ is a (finite) sum of those of the form (10.15), we have the following equation of elements in $(\mathbb{M} \otimes \mathbb{M}' \otimes \mathscr{O}(\widetilde{\mathcal{B}}))[\log q]\{q\}$:

$$\sum_{n \in \mathbb{N}} \left(v_n \cdot q^{L_0} \blacktriangleright \otimes \blacktriangleleft + q^{L_0} \blacktriangleright \otimes v_n \cdot \blacktriangleleft \right) q^n = 0 \tag{10.16}$$

where the actions of v_n on $\mathbb{M}$ and $\mathbb{M}'$ are as in (6.5) using the local coordinates ξ, ϖ of $\widetilde{\mathfrak{X}}$. On the other hand, since ψ is conformal block, for each n and each $w \in \mathbb{W}_\bullet$ (considered as a constant section of $\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$), the element $A_n \in \mathscr{O}(\widetilde{\mathcal{B}})[\log q]\{q\}$ defined by

$$A_n := \psi\big(v_n \cdot w \otimes (q^{L_0} \blacktriangleright \otimes \blacktriangleleft)\big) + \psi\big(w \otimes (v_n \cdot q^{L_0} \blacktriangleright \otimes \blacktriangleleft)\big) + \psi\big(w \otimes (q^{L_0} \blacktriangleright \otimes v_n \cdot \blacktriangleleft)\big)$$

equals 0. Here, similarly, the action of v_n on w is defined by summing up the componentwise actions described by (6.5) using the local coordinates $\eta_\bullet$. By (10.16), we have

$$0 = \sum_{n \in \mathbb{N}} A_n q^n = \sum_{n \in \mathbb{N}} \psi\big(v_n \cdot w \otimes (q^{L_0} \blacktriangleright \otimes \blacktriangleleft)\big) q^n,$$

which is exactly $\mathcal{S}\psi(v \cdot w)$. This finishes the proof that $\mathcal{S}\psi$ vanishes on $\mathscr{J}(\mathcal{B})$. $\qquad\square$

Remark 10.4. The algebraic version of Theorem 10.3 for $\widetilde{\mathcal{S}}\psi$ (i.e. assuming $\widetilde{\mathfrak{X}}$ is an algebraic family and replacing $\mathcal{D}_{r\rho}$ with $\mathrm{Spec}(\mathbb{C}[[q]])$) was proved in [DGT19b, Thm. 8.5.1] and its proof can be easily adapted to the analytic setting. We have provided a complete proof of Theorem 10.3 for the reader's convenience. We remark that [DGT19b] proved a version of Lemma 10.1 (for $\widetilde{L}_0$). Their proof uses [NT05, Lemma 8.7.1] and is different from ours.

In low genus cases, similar versions of Theorem 10.3 were proved in [Zhu96, Prop. 4.3.6], [Hua05a, Thm. 1.4], [Hua05b, Prop. 3.6].

11 Convergence of sewing

We continue our discussions and assume the setting in Section 10. Recall that $\mathcal{B}$ equals $\mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}}$. Then $\mathcal{B}$ is Stein if $\widetilde{\mathcal{B}}$ is so. Set $\mathcal{B}^\times = \mathcal{D}_{r\rho}^\times \times \widetilde{\mathcal{B}}$. We identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ with $\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_{\mathcal{B}}$ via $\mathcal{U}(\eta_\bullet)$.

Absolute and locally uniform convergence

Definition 11.1. We say that $\widetilde{\mathcal{S}}\psi$ **converges absolutely and locally uniformly (a.l.u.)** if it sends each element of $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$ to an element of $\mathscr{O}(\mathcal{B})$.

In the case that $\mathbb{M}$ and hence $\mathbb{M}'$ are $L_{0,s}$-semisimple, since $\mathcal{S}\psi$ is possibly multivalued over q, we need to define its a.l.u. convergence in another way. For each $w \in \mathbb{W}_\bullet$, considered as a constant section of $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$, we write

$$\mathcal{S}\psi(w) = \sum_{n \in \mathbb{C}, l \in \mathbb{N}} \mathcal{S}\psi(w)_{n,l} \cdot (\log q)^l q^n.$$

where $\mathcal{S}\psi(w)_{n,l}$ is a holomorphic function on $\widetilde{\mathcal{B}}$.

Definition 11.2. Assume $\mathcal{S}\psi(w) \in \mathcal{O}(\tilde{\mathcal{B}})\{q\}[\log q]$ (namely, $\mathcal{S}\psi(w)_{n,l} = 0$ when $l \geqslant L$ for some L and all n). We say that $\mathcal{S}\psi$ **converges a.l.u.** if for any $w \in \mathbb{W}_\bullet$, any compact subsets $K \subset \tilde{\mathcal{B}}$ and $Q \subset \mathcal{D}_{r\rho}^\times$, there exists $C > 0$ such that

$$\sum_{n \in \mathbb{C}} \left| \mathcal{S}\psi(w)_{n,l}(b) \right| \cdot |q^n| \leqslant C \tag{11.1}$$

for any $b \in K$, $q \in Q$, and $l \in \mathbb{N}$.

It is clear that $\tilde{\mathcal{S}}\psi$ converges a.l.u. if $\mathcal{S}\psi$ does, since $\tilde{\mathcal{S}}\psi$ is precisely the part of $\mathcal{S}\psi$ without $\log q$.

Theorem 11.3. *If $\tilde{\mathcal{S}}\psi$ converges a.l.u., then it is an element of $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)(\mathcal{B})$. Similarly, if $\mathbb{M}$ is $L_{0,\mathrm{s}}$-semisimple and $\mathcal{S}\psi$ converges a.l.u., then it is an element of $\mathscr{T}_{\mathfrak{X}_{\mathcal{B}^\times}}^*(\mathbb{W}_\bullet)(\mathcal{B}^\times)$.*

Proof. We may shrink $\tilde{\mathcal{B}}$ so that $\tilde{\mathcal{B}}$ and hence $\mathcal{B}$ are Stein. Assume the a.l.u. convergence. Note that, since $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is generated freely by some global sections, the $\mathcal{O}(\mathcal{B}^\times)$-module homomorphism $\tilde{\mathcal{S}}\psi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B}^\times) \to \mathcal{O}(\mathcal{B}^\times)$ (defined by taking the limit of the infinite series (10.11)) can be regarded as an $\mathscr{O}_{\mathcal{B}^\times}$-module homomorphism $\tilde{\mathcal{S}}\psi : \mathscr{W}_{\mathfrak{X}_{\mathcal{B}^\times}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}^\times}$. Now, the fact that $\tilde{\mathcal{S}}\psi$ is a conformal block follows from Theorems 10.3 and 6.3. The proof for $\mathcal{S}\psi$ is similar. $\square$

Convergence and differential equations

In the remaining part of this section, *we assume $\mathbb{V}$ is C_2-cofinite, the $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$ are finitely-generated (equivalently, $L_{0,\mathrm{s}}$-semisimple cf. Rem. 7.2), and $\mathbb{M}$ and hence $\mathbb{M}'$ are $L_{0,\mathrm{s}}$-simple.* Since $L_{0,\mathrm{n}}{}^k = 0$ on $\mathbb{M}$ when k is sufficiently large (cf. Rem. 7.2), the powers of $\log q$ in the sewing $\mathcal{S}\psi$ are uniformly bounded from above.

As in the proof of Theorem 7.4, for each $k \in \mathbb{N}$, $\mathbb{W}_\bullet^{\leqslant k}$ (resp. $\mathbb{W}_\bullet(k)$) denotes the (finite dimensional) subspace spanned by all $\tilde{L}_0$-homogeneous homogeneous vectors $w \in \mathbb{W}_\bullet$ satisfying $\widetilde{\mathrm{wt}}(w) \leqslant k$ (resp. $\widetilde{\mathrm{wt}}(w) = k$). This gives a filtration (resp. grading) of $\mathbb{W}_\bullet$. We define

$$\mathcal{S}\psi^{\leqslant k} \in (\mathbb{W}_\bullet^{\leqslant k})^* \otimes_{\mathbb{C}} \mathcal{O}(\tilde{\mathcal{B}})\{q\}[\log q]$$

whose evaluation with each $w \in \mathbb{W}_\bullet^{\leqslant k}$ is $\mathcal{S}\psi(w)$.

Theorem 11.4. *Assume $\tilde{\mathcal{B}}$ is a Stein manifold. There exists $k_0 \in \mathbb{Z}_+$ such that for any $k \geqslant k_0$, there exists*

$$\Omega \in \mathrm{End}_{\mathbb{C}}\big((\mathbb{W}_\bullet^{\leqslant k})^*\big) \otimes_{\mathbb{C}} \mathcal{O}(\mathcal{B})$$

not depending on $\mathbb{M}$, such that

$$q\partial_q(\mathcal{S}\psi^{\leqslant k}) = \Omega \cdot \mathcal{S}\psi^{\leqslant k}. \tag{11.2}$$

Using this theorem, it is easy to prove:

Theorem 11.5. *$\tilde{\mathcal{S}}\psi$ and $\mathcal{S}\psi$ converge a.l.u..*

Proof. It suffices to assume $\mathcal{B}$ is a Stein open subset of $\mathbb{C}^m$. Then by Theorem A.1, $\mathcal{S}\psi^{\leqslant k}$ converges a.l.u.. $\square$

It follows that $\tilde{\mathcal{S}}\psi$ is a conformal block associated to $\mathbb{W}_\bullet$ and $\mathfrak{X}$. Outside the discriminant locus $\Delta = \{0\} \times \tilde{\mathcal{B}}$, $\mathcal{S}\psi$ is also a conformal block.

Proof of Theorem 11.4

In this subsection, we assume $\widetilde{\mathcal{B}}$ and hence $\mathcal{B}$ are Stein manifolds. By Theorem B.2, we are allowed to fix a projective structure $\mathfrak{P}$ of $\widetilde{\mathfrak{X}}$. Recall $S_{\mathfrak{X}} = \sum_{i=1}^{N} \varsigma_i(\mathcal{B})$; set $S_{\widetilde{\mathfrak{X}}} = \sum_{i=1}^{N} \varsigma_i(\widetilde{\mathcal{B}}) + \varsigma'(\widetilde{\mathcal{B}}) + \varsigma''(\widetilde{\mathcal{B}})$. As argued for (9.1), we may use (4.20) to obtain an exact sequence

$$0 \to H^0\big(\mathcal{B}, \pi_* \Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})\big) \to H^0\big(\mathcal{B}, \pi_* \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta + \bullet S_{\mathfrak{X}})\big)$$
$$\xrightarrow{d\pi} H^0\big(\mathcal{B}, \pi_* \big(\pi^* \Theta_{\mathcal{B}}(-\log \Delta)(\bullet S_{\mathfrak{X}})\big)\big) \to 0. \tag{11.3}$$

$\mathfrak{y} = q\partial_q$ is a section of $\Theta_{\mathcal{B}}(-\log \Delta)$ and hence of $\pi_*\big(\pi^*\Theta_{\mathcal{B}}(-\log \Delta)(\bullet S_{\mathfrak{X}})\big)$ over $\mathcal{B}$. Thus, we have

$$\widetilde{\mathfrak{y}} \in H^0\big(\mathcal{C}, \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta + \bullet S_{\mathfrak{X}})\big)$$

satisfying $d\pi(\widetilde{\mathfrak{y}}) = q\partial_q$. We let

$$\Gamma = \varsigma'(\widetilde{\mathcal{B}}) \cup \varsigma''(\widetilde{\mathcal{B}}).$$

Our first step is to take the series expansion $\sum \widetilde{\mathfrak{y}}_n^{\perp} q^n$ (as in the proof of Theorem 10.3) of the "vertical part" of $\widetilde{\mathfrak{y}}$. Choose any precompact open subset $U \subset \widetilde{\mathcal{C}} - \Gamma$ together with a fiberwisely univalent $\eta \in \mathscr{O}(U)$. Then as in that proof, we may find a small subdisc $\mathcal{D} = \mathcal{D}_{\epsilon\lambda}$ of $\mathcal{D}_{r\rho}$ centered at 0 such that $\mathcal{D} \times U \simeq U \times \mathcal{D}$ is an open subset of $\widetilde{\mathcal{C}} \times \mathcal{D}_{r\rho} - F' - F''$ and hence of $\mathcal{C}$. Extend η constantly (over $\mathcal{D}$) to a fiberwise univalent function on $\mathcal{D} \times U$. Then we may write

$$\widetilde{\mathfrak{y}}|_{\mathcal{D} \times U} = h\partial_\eta + q\partial_q \tag{11.4}$$

for some $h \in \mathscr{O}(\bullet S_{\mathfrak{X}})(\mathcal{D} \times U)$. Write $h = \sum_{n \in \mathbb{N}} h_n q^n$ where $h_n \in \mathscr{O}(\bullet S_{\widetilde{\mathfrak{X}}})(U)$. For each $n \in \mathbb{N}$, set an element $\widetilde{\mathfrak{y}}_n^{\perp} \in \Theta_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{X}}})(U)$ by

$$\widetilde{\mathfrak{y}}_n^{\perp}|_U = h_n \partial_\eta. \tag{11.5}$$

Lemma 11.6. *The locally defined $\widetilde{\mathfrak{y}}_n^{\perp}$ is independent of the choice of η, and hence can be extended to an element of $H^0(\widetilde{\mathcal{C}} - \Gamma, \Theta_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{X}}}))$*

Proof. Suppose we have another $\mu \in \mathscr{O}(U)$ univalent on each fiber, which is extended constantly to $\mathcal{D} \times U$. So $\partial_q \mu = 0$ and hence $\widetilde{\mathfrak{y}}|_{\mathcal{D} \times U} = h \cdot \partial_\eta \mu \cdot \partial_\mu + q\partial_q$. Note that $\partial_\eta \mu$ is constant over q. Thus, if we define $\widetilde{\mathfrak{y}}_n^{\perp}|_U$ using μ, then $\widetilde{\mathfrak{y}}_n^{\perp}|_U = h_n \cdot \partial_\eta \mu \cdot \partial_\mu$, which agrees with (11.5). $\square$

We shall show that $\widetilde{\mathfrak{y}}_n^{\perp}$ has poles of finite orders at Γ. For that purpose, we need to describe explicitly $\widetilde{\mathfrak{y}}$ near the critical locus Σ. Recall the open subsets W, W', W'' of $\mathcal{C}$ described near (4.8) and U', U'' of $\widetilde{\mathcal{C}}$ described near (4.6). Note also $q = \xi\varpi$. In the following, we let $\tau_\bullet$ be any biholomorphic map from $\widetilde{\mathcal{B}}$ to an open subset of a complex manifold. If $\widetilde{\mathcal{B}}$ is small enough, then $\tau_\bullet$ can be a set of coordinates of $\widetilde{\mathcal{B}}$. The only purpose of introducing $\tau_\bullet$ is to indicate the dependence of certain functions on the points of $\widetilde{\mathcal{B}}$. Thus, $(\xi, q, \tau_\bullet)$ and $(\varpi, q, \tau_\bullet)$ are respectively biholomorphic maps of

$$W' = \mathcal{D}_r^{\times} \times \mathcal{D}_\rho \times \widetilde{\mathcal{B}}, \qquad W'' = \mathcal{D}_r \times \mathcal{D}_\rho^{\times} \times \widetilde{\mathcal{B}}$$

to complex manifolds. By (4.18), we can find $a, b \in \mathscr{O}((\xi, \varpi, \tau_\bullet)(W))$ such that

$$\widetilde{\mathfrak{y}}|_W = a(\xi, \varpi, \tau_\bullet)\xi\partial_\xi + b(\xi, \varpi, \tau_\bullet)\varpi\partial_\varpi.$$

Since $d\pi(\xi\partial_\xi) = d\pi(\varpi\partial_\varpi) = q\partial_q$ by (4.19), we must have

$$a + b = 1. \tag{11.6}$$

This relation, together with (4.2), shows that under the coordinates $(\xi, q, \tau_\bullet)$ and $(\varpi, q, \tau_\bullet)$ respectively,

$$\widetilde{\mathfrak{y}}|_{W'} = a(\xi, q/\xi, \tau_\bullet)\xi\partial_\xi + q\partial_q, \qquad \widetilde{\mathfrak{y}}|_{W''} = b(q/\varpi, \varpi, \tau_\bullet)\varpi\partial_\varpi + q\partial_q. \tag{11.7}$$

Lemma 11.7. *For each $n \in \mathbb{N}$, $\widetilde{\mathfrak{y}}_n^\perp$ has poles of orders at most $n - 1$ at $\varsigma'(\widetilde{\mathcal{B}})$ and $\varsigma''(\widetilde{\mathcal{B}})$. Consequently, $\widetilde{\mathfrak{y}}_n^\perp$ is an element of $H^0(\widetilde{\mathcal{C}}, \Theta_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{X}}}))$.*

Proof. Let us write

$$a(\xi, \varpi, \tau_\bullet) = \sum_{m,n\in\mathbb{N}} a_{m,n}(\tau_\bullet)\xi^m\varpi^n, \qquad b(\xi, \varpi, \tau_\bullet) = \sum_{m,n\in\mathbb{N}} b_{m,n}(\tau_\bullet)\xi^m\varpi^n$$

where $a_{m,n}, b_{m,n} \in \mathscr{O}(\tau_\bullet(\widetilde{\mathcal{B}}))$. Then

$$a(\xi, q/\xi, \tau_\bullet) = \sum_{n\geqslant 0, l\geqslant -n} a_{l+n,n}(\tau_\bullet)\xi^l q^n, \qquad b(q/\varpi, \varpi, \tau_\bullet) = \sum_{m\geqslant 0, l\geqslant -m} b_{m,l+m}(\tau_\bullet)\varpi^l q^m. \tag{11.8}$$

Combine these two relations with (11.5) and (11.7), and take the coefficients before q^n. We obtain

$$\widetilde{\mathfrak{y}}_n^\perp\Big|_{U'-\varsigma'(\widetilde{\mathcal{B}})} = \sum_{l\geqslant -n} a_{l+n,n}(\tau_\bullet)\xi^{l+1}\partial_\xi, \qquad \widetilde{\mathfrak{y}}_n^\perp\Big|_{U''-\varsigma''(\widetilde{\mathcal{B}})} = \sum_{l\geqslant -n} b_{n,l+n}(\tau_\bullet)\varpi^{l+1}\partial_\varpi, \tag{11.9}$$

which finishes the proof. $\qquad\square$

The description (11.9) of $\widetilde{\mathfrak{y}}_n^\perp$ near Γ can be found in [Loo10, Lemma 33]. Next, we shall apply the results of Section 9 to the smooth family $\widetilde{\mathfrak{X}}$. In particular, $\mathcal{V}ir_c$ is defined for $\widetilde{\mathfrak{X}}$ and is an $\mathscr{O}_{\widetilde{\mathcal{C}}}$-module. We let $\nu(\widetilde{\mathfrak{y}}_n^\perp)$ be a section of $\mathcal{V}ir_c \otimes \omega_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\widetilde{\mathfrak{X}}})$ defined on $U' \cup U''$ (near $\varsigma'(\widetilde{\mathcal{B}}), \varsigma''(\widetilde{\mathcal{B}})$) and near $\varsigma_1(\widetilde{\mathcal{B}}), \ldots, \varsigma_N(\widetilde{\mathcal{B}})$ as in Section (9), *which relies on the local coordinates* $\eta_1, \ldots, \eta_N, \xi, \varpi$ *of* $\widetilde{\mathfrak{X}}$. Recall the correspondence $\partial_\xi \mapsto \mathbf{c}d\xi, \partial_\varpi \mapsto \mathbf{c}d\varpi$. We calculate the actions of $\nu(\widetilde{\mathfrak{y}}_n^\perp)$ on $\mathbb{M}$ and on $\mathbb{M}'$ to be respectively

$$\mathrm{Res}_{\xi=0} \sum_{l\geqslant -n} a_{l+n,n} Y_{\mathbb{M}}(\mathbf{c}, \xi)\xi^{l+1}d\xi, \qquad \mathrm{Res}_{\varpi=0} \sum_{l\geqslant -n} b_{n,l+n} Y_{\mathbb{M}'}(\mathbf{c}, \varpi)\varpi^{l+1}d\varpi. \tag{11.10}$$

In the following proofs, we will suppress the symbol $\tau_\bullet$ when necessary.

The next lemma is crucial to finding the differential equation (11.2), and was observed in [DGT19b, Rem. 8.5.3] when there is not $\log q$.

Lemma 11.8. *The following equation of elements of $(\mathbb{M} \otimes \mathbb{M}')\{q\}[\log q]$ is true.*

$$L_0 q^{L_0} \blacktriangleright \otimes \blacktriangleleft = \sum_{n \in \mathbb{N}} \nu(\tilde{\mathfrak{y}}_n^\perp) q^{n+L_0} \blacktriangleright \otimes \blacktriangleleft + \sum_{n \in \mathbb{N}} q^{n+L_0} \blacktriangleright \otimes \nu(\tilde{\mathfrak{y}}_n^\perp) \blacktriangleleft \tag{11.11}$$

Proof. It is obvious that $\mathcal{U}(\gamma_1)\mathbf{c} = \mathbf{c}$, $\xi^{L_0}\mathbf{c} = \xi^2 \mathbf{c}$, $\varpi^{L_0}\mathbf{c} = \varpi^2 \mathbf{c}$. Notice Remark 10.2. We have

$$Y_{\mathbb{M}}(\xi^{L_0}\mathbf{c}, \xi) q^{L_0} \blacktriangleright \otimes \blacktriangleleft \cdot a(\xi, q/\xi) \frac{d\xi}{\xi}$$
$$= \sum_{n \geqslant 0} \sum_{l \geqslant -n} Y_{\mathbb{M}}(\mathbf{c}, \xi) q^{n+L_0} \blacktriangleright \otimes \blacktriangleleft \cdot a_{l+n,n} \xi^{l+1} d\xi$$

as elements of $(\mathbb{M} \otimes \mathbb{M}' \otimes \mathscr{O}(\tilde{\mathcal{B}}))((\xi))\{q\}[\log q] d\xi$. Take $\mathrm{Res}_{\xi=0}$ and notice (11.10). Then, the above expression becomes the first summand on the right hand side of (11.11). A similar thing could be said about the second summand. Thus, the right hand side of (11.11) equals

$$\mathrm{Res}_{\xi=0} Y_{\mathbb{M}}(\xi^{L_0}\mathbf{c}, \xi) q^{L_0} \blacktriangleright \otimes \blacktriangleleft \cdot a(\xi, q/\xi) \frac{d\xi}{\xi}$$
$$+ \mathrm{Res}_{\varpi=0} q^{L_0} \blacktriangleright \otimes Y_{\mathbb{M}'}(\varpi^{L_0}\mathcal{U}(\gamma_1)\mathbf{c}, \varpi) \blacktriangleleft \cdot b(q/\varpi, \varpi) \frac{d\varpi}{\varpi}.$$

By Lemma 10.1 and that $a + b = 1$, it equals

$$\mathrm{Res}_{\xi=0} Y_{\mathbb{M}}(\xi^{L_0}\mathbf{c}, \xi) q^{L_0} \blacktriangleright \otimes \blacktriangleleft \cdot \frac{d\xi}{\xi} = \mathrm{Res}_{\xi=0} Y_{\mathbb{M}}(\mathbf{c}, \xi) q^{L_0} \blacktriangleright \otimes \blacktriangleleft \cdot \xi d\xi$$
$$= Y_{\mathbb{M}}(\mathbf{c})_1 q^{L_0} \blacktriangleright \otimes \blacktriangleleft = L_0 q^{L_0} \blacktriangleright \otimes \blacktriangleleft.$$

$\square$

Lemma 11.9. *For any $w_\bullet \in \mathbb{W}_\bullet$, we have the following relation of elements of $\mathscr{O}(\tilde{\mathcal{B}})\{q\}[\log q]$.*

$$q\partial_q \mathcal{S}\psi(w_\bullet) = \sum_{n \in \mathbb{N}} \psi(w_\bullet \otimes \nu(\tilde{\mathfrak{y}}_n^\perp) q^{n+L_0} \blacktriangleright \otimes \blacktriangleleft) + \sum_{n \in \mathbb{N}} \psi(w_\bullet \otimes q^{n+L_0} \blacktriangleright \otimes \nu(\tilde{\mathfrak{y}}_n^\perp) \blacktriangleleft).$$

Proof. By (10.9), we have

$$q\partial_q \mathcal{S}\psi(w_\bullet) = q\partial_q \psi(w_\bullet \otimes q^{L_0} \blacktriangleright \otimes \blacktriangleleft) = \psi(w_\bullet \otimes L_0 q^{L_0} \blacktriangleright \otimes \blacktriangleleft).$$

By the Lemma 11.8, the desired equation is proved. $\square$

As usual, we let $\nu(\tilde{\mathfrak{y}}_n^\perp) w_\bullet$ denote $\sum_i w_1 \otimes \cdots \otimes \nu(\tilde{\mathfrak{y}}_n^\perp) w_i \otimes \cdots \otimes w_N$. For any $w_\bullet \in \mathbb{W}_\bullet$, define $\nabla_{q\partial_q} w_\bullet \in \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\tilde{\mathcal{B}})[[q]]$ to be

$$\nabla_{q\partial_q} w_\bullet = - \sum_{n \in \mathbb{N}} q^n \nu(\tilde{\mathfrak{y}}_n^\perp) w_\bullet. \tag{11.12}$$

Proposition 11.10. *There exists $\#(\tilde{\mathfrak{y}}_n^\perp) \in \mathscr{O}(\tilde{\mathcal{B}})$ for each $n \in \mathbb{N}$, such that for any $w_\bullet \in \mathbb{W}_\bullet$, we have the following equation of elements of $\mathscr{O}(\tilde{\mathcal{B}})\{q\}[\log q]$:*

$$q\partial_q \mathcal{S}\psi(w_\bullet) = \mathcal{S}\psi(\nabla_{q\partial_q} w_\bullet) + \sum_{n \in \mathbb{N}} \#(\tilde{\mathfrak{y}}_n^\perp) q^n \cdot \mathcal{S}\psi(w_\bullet).$$

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Proof. $\#(\widetilde{\mathfrak{y}}_n^\perp)$ is defined by Proposition 9.2. Moreover, by that proposition, we have

$$w_\bullet \otimes \nu(\widetilde{\mathfrak{y}}_n^\perp) q^{L_0} \blacktriangleright \otimes \blacktriangleleft + w_\bullet \otimes q^{L_0} \blacktriangleright \otimes \nu(\widetilde{\mathfrak{y}}_n^\perp) \blacktriangleleft + \nu(\widetilde{\mathfrak{y}}_n^\perp) w_\bullet \otimes q^{L_0} \blacktriangleright \otimes \blacktriangleleft$$
$$= \#(\widetilde{\mathfrak{y}}_n^\perp) \cdot w_\bullet \otimes q^{L_0} \blacktriangleright \otimes \blacktriangleleft.$$

By Lemma 11.9 and relation (11.12), it is easy to prove the desired equation. $\qquad\square$

To prove Theorem 11.4, it remains to check that (11.12) and the projective term $\sum_n \#(\widetilde{\mathfrak{y}}_n^\perp) q^n$ converge a.l.u.. To treat the first one, we choose mutually disjoint neighborhoods $U_1, \ldots, U_N$ of $\varsigma_1(\widetilde{\mathcal{B}}), \ldots, \varsigma_N(\widetilde{\mathcal{B}})$ on which $\eta_1, \ldots, \eta_N$ are defined respectively, and assume they are disjoint from U', U''. Then $\mathcal{D}_{r\rho} \times U_1, \ldots, \mathcal{D}_{r\rho} \times U_N$ are neighborhoods of $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$ disjoint from F', F''. Write $\tau_\bullet \circ \pi$ and $\tau_\bullet \circ \widetilde{\pi}$ as $\tau_\bullet$ for simplicity. Then $(\eta_i, \tau_\bullet)$ is a set of coordinates of U_i. Recall (11.4). We may write

$$\widetilde{\mathfrak{y}}|_{\mathcal{D}_{r\rho} \times U_i} = h_i(q, \eta_i, \tau_\bullet) \partial_{\eta_i} + q \partial_q \tag{11.13}$$

where $h_i(q, \eta_i, \tau_\bullet) \in \mathscr{O}_{\mathcal{C}}(\bullet S_{\mathfrak{x}})(\mathcal{D}_{r\rho} \times U_i)$. Let $\nu(\widetilde{\mathfrak{y}}^\perp)$ be a section of $\mathscr{V}_{\mathfrak{x}} \otimes \omega_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{x}})$ on $\mathcal{D}_{r\rho} \times (U_1 \cup \cdots \cup U_N)$ satisfying

$$\mathcal{U}(\eta_i) \nu(\widetilde{\mathfrak{y}}^\perp)|_{\mathcal{D}_{r\rho} \times U_i} = h_i(q, \eta_i, \tau_\bullet) \mathbf{c} d\eta_i. \tag{11.14}$$

Write $h_i = \sum_n h_{i,n} q^n$. Then by (11.5),

$$\widetilde{\mathfrak{y}}_n^\perp|_{U_i} = h_{i,n}(\eta_i, \tau_\bullet) \partial_{\eta_i}. \tag{11.15}$$

So we have $\nabla_{q\partial_q} w_\bullet \in \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}}) = \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$ due to the obvious fact:

Lemma 11.11. *We have*

$$\nabla_{q\partial_q} w_\bullet = -\nu(\widetilde{\mathfrak{y}}^\perp) w_\bullet. \tag{11.16}$$

We now prove the convergence of the projective term.

Proposition 11.12. $\sum_{n \in \mathbb{N}} \#(\widetilde{\mathfrak{y}}_n^\perp) q^n$ *is an element of* $\mathscr{O}(\mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}}) = \mathscr{O}(\mathcal{B})$.

Proof. Combine (11.9) and (11.15), and apply Proposition 9.2 to the family $\widetilde{\mathfrak{x}}$. We obtain

$$\#(\widetilde{\mathfrak{y}}_n^\perp) = \frac{c}{12}\left(A_n + B_n + \sum_{i=1}^N C_{i,n} \right)$$

where

$$A_n = \sum_{l \geqslant -n} \operatorname{Res}_{\xi=0} \mathbf{S}_\xi \mathfrak{P} \cdot a_{l+n,n}(\tau_\bullet) \xi^{l+1} d\xi,$$
$$B_n = \sum_{l \geqslant -n} \operatorname{Res}_{\varpi=0} \mathbf{S}_\varpi \mathfrak{P} \cdot b_{n,l+n}(\tau_\bullet) \varpi^{l+1} d\varpi,$$
$$C_{i,n} = \operatorname{Res}_{\eta_i=0} \mathbf{S}_{\eta_i} \mathfrak{P} \cdot h_{i,n}(\eta_i, \tau_\bullet) d\eta_i.$$

Notice that $\mathbf{S}_{\eta_i} \mathfrak{P} = \mathbf{S}_{\eta_i} \mathfrak{P}(\eta_i, \tau_\bullet)$, $\mathbf{S}_\varpi \mathfrak{P} = \mathbf{S}_\varpi \mathfrak{P}(\varpi, \tau_\bullet)$, $\mathbf{S}_\xi \mathfrak{P} = \mathbf{S}_\xi \mathfrak{P}(\xi, \tau_\bullet)$ are holomorphic functions on U_i, U', U'' which are identified with their images under $(\eta_i, \tau_\bullet), (\xi, \tau_\bullet), (\varpi, \tau_\bullet)$ respectively.

We have

$$\sum_{n \geqslant 0} A_n q^n = \sum_{n \geqslant 0} \sum_{l \geqslant -n} \text{Res}_{\xi=0} \, \mathbf{S}_\xi \mathfrak{P} \cdot a_{l+n,n}(\tau_\bullet) \xi^{l+1} q^n d\xi. \tag{11.17}$$

We claim that (11.17) is an element of $\mathscr{O}(\mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}})$. Note that $a(\xi, q/\xi, \tau_\bullet)$ is defined when $|q|/\rho < |\xi| < r$. Choose any $\epsilon \in (0, r\rho)$. Choose a circle γ' surrounding $\mathcal{D}_{\epsilon/\rho}$ and inside $\mathcal{D}_r$. Then, when ξ is on γ, $a(\xi, q/\xi, \tau_\bullet)$ can be defined whenever $|q| < \epsilon$. Thus,

$$A := \oint_{\gamma'} \mathbf{S}_\xi \mathfrak{P}(\xi, \tau_\bullet) \cdot a(\xi, q/\xi, \tau_\bullet) \xi d\xi$$

is a holomorphic function defined whenever $|q| < \epsilon$. Recall the first equation of (11.8), and note that the series converges absolutely and uniformly when $\xi \in \gamma'$ and $|q| \leqslant \epsilon$, by the double Laurent series expansion of $a(\xi, q/\xi, \tau_\bullet)$. Therefore, the above contour integral equals

$$\sum_{n \geqslant 0} \sum_{l \geqslant -n} \oint_{\gamma'} \mathbf{S}_\xi \mathfrak{P}(\xi, \tau_\bullet) \cdot a_{l+n,n}(\tau_\bullet) \xi^{l+1} q^n d\xi,$$

which clearly equals (11.17) as an element of $\mathscr{O}(\widetilde{\mathcal{B}})[[q]]$. Thus (11.17) is an element of $\mathscr{O}(\mathcal{D}_\epsilon \times \widetilde{\mathcal{B}})$ whenever $\epsilon < r\rho$, and hence when $\epsilon = r\rho$.

A similar argument shows $\sum B_n q^n$ converges a.l.u. to

$$B := \oint_{\gamma''} \mathbf{S}_\varpi \mathfrak{P}(\varpi, \tau_\bullet) \cdot b(q/\varpi, \varpi, \tau_\bullet) \varpi d\varpi$$

where γ'' is any circle in $\mathcal{D}_\rho$ surrounding 0. Finally, we compute

$$C_i := \sum_{n \geqslant 0} C_{i,n} q^n = \sum_{n \geqslant 0} \text{Res}_{\eta_i=0} \, \mathbf{S}_{\eta_i} \mathfrak{P}(\eta_i, \tau_\bullet) \cdot h_{i,n}(\eta_i, \tau_\bullet) q^n d\eta_i$$
$$= \text{Res}_{\eta_i=0} \, \mathbf{S}_{\eta_i} \mathfrak{P}(\eta_i, \tau_\bullet) \cdot h_i(q, \eta_i, \tau_\bullet) d\eta_i$$

which is clearly inside $\mathscr{O}(\mathcal{D}_{r\rho} \times \widetilde{\mathcal{B}})$. The proof is now complete. We summarize that the projective term equals

$$\sum_{n \in \mathbb{N}} \#(\widetilde{\mathfrak{y}}_n^\perp) q^n = \frac{c}{12} \Big(A + B + \sum_{i=1}^N C_i \Big). \tag{11.18}$$

$\square$

We can now finish the

Proof of Theorem 11.4. By Theorem 7.4, we may find $k_0 \in \mathbb{N}$ such that the vectors in $\mathbb{W}_\bullet^{\leqslant k_0}$, considered as constant sections of $\mathbb{W}_\bullet^{\leqslant k_0} \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$, generate the $\mathscr{O}(\mathcal{B})$-module $\mathscr{W}_{\mathfrak{x}}(\mathbb{W}_\bullet)(\mathcal{B})/\mathscr{J}(\mathcal{B})$. Fix any $k \geqslant k_0$. We choose a basis $s_1, s_2, \dots$ of $\mathbb{W}_\bullet^{\leqslant k}$.

By propositions 11.10 and 11.12, for each s_i of $s_1, s_2, \ldots$, we have the following equation of elements of $\mathscr{O}(\tilde{\mathcal{B}})\{q\}[\log q]$:

$$q\partial_q \mathcal{S}\psi(s_i) = \mathcal{S}\psi(\nabla_{q\partial_q} s_i) + g\mathcal{S}\psi(s_i)$$

where $g \in \mathscr{O}(\mathcal{D}_{r\rho} \times \tilde{\mathcal{B}}) = \mathscr{O}(\mathcal{B})$ equals (11.18). Since we have $\nabla_{q\partial_q} s_i \in \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B}) = \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{B})$ due to Lemma 11.11, we can find $f_{i,j} \in \mathscr{O}(\mathcal{B})$ such that $\nabla_{q\partial_q} s_i$ equals $\sum_j f_{i,j} s_j$ mod elements of $\mathscr{J}(\mathcal{B})$. Since, by Theorem 10.3, $\mathcal{S}\psi$ vanish on $\mathscr{J}(\mathcal{B})$, we must have

$$q\partial_q \mathcal{S}\psi(s_i) = \sum_j f_{i,j}\mathcal{S}\psi(s_j) + g\mathcal{S}\psi(s_i).$$

The proof is completed by setting the matrix-valued holomorphic function Ω to be $(f_{i,j} + g\delta_{i,j})_{i,j}$. $\qquad\qquad\square$

Remark 11.13. Following [TUY89] or [BK01, Chapter 7] or [DGT19a], one can define locally a (logarithmic) connection ∇ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ as follows. Assume first of all that $\mathfrak{X}$ admits local coordinates $\eta_1, \ldots, \eta_N$, and $\mathcal{B}$ is a small enough Stein manifold such that $\Theta_{\mathcal{B}}(-\log \Delta)$ is $\mathscr{O}_{\mathcal{B}}$-generated freely by finitely many global sections $\mathfrak{y}_1, \mathfrak{y}_2, \ldots$. Then each $\mathfrak{y}_j$ lifts to an element $\tilde{\mathfrak{y}}_j \in H^0(\mathcal{C}, \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta + \bullet S_{\mathfrak{X}}))$, i.e. $d\pi(\tilde{\mathfrak{y}}_j) = \mathfrak{y}_j$. As in (11.14), one can define a section $\tilde{\mathfrak{y}}_j^\perp$ of $\Theta_{\mathcal{C}/\mathcal{B}}(\bullet S_{\mathfrak{X}})$ (near $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$) to be the vertical part of $\tilde{\mathfrak{y}}_j$, and define $\nu(\tilde{\mathfrak{y}}_j^\perp)$ as in Section 9 using the local coordinates $\eta_\bullet$. Then ∇ is determined by

$$\nabla_{\mathfrak{y}_j} w_\bullet = -\nu(\tilde{\mathfrak{y}}_j^\perp) w_\bullet.$$

for any constant section $w_\bullet \in \mathbb{W}_\bullet$. As argued in [DGT19a], ∇ preserves $\mathscr{J}(\mathcal{B})$. So ∇ descends to a connection of $\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet)$. Its dual connection on $\mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)$ is also denoted by ∇. Thus, Proposition 11.10 says that for the $\mathfrak{X}$ obtained by sewing $\tilde{\mathfrak{X}}$, and for any $\psi \in \mathscr{T}_{\tilde{\mathfrak{X}}}^*(\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')(\tilde{\mathcal{B}})$, its sewing $\mathcal{S}\psi \in \mathscr{T}_{\mathfrak{X}}^*(\mathbb{W}_\bullet)(\mathcal{B})$ satisfies

$$\nabla_{q\partial_q} \mathcal{S}\psi = g\mathcal{S}\psi \qquad\qquad (11.19)$$

where $g \in \mathscr{O}(\mathcal{B})$ is given by (11.18). This observation is the analytic analog of [DGT19b, Rem. 8.5.3].

12 Injectivity of sewing for semisimple modules

We continue the study of sewing, but assume that $\mathfrak{X}$ is formed by sewing a single $(N + 2)$-pointed compact Riemann surface with local coordinates

$$\tilde{\mathfrak{X}} = (\tilde{C}; x_1, \ldots, x_N, x', x''; \eta_1, \ldots, \eta_N, \xi, \varpi).$$

Namely, we assume $\tilde{\mathcal{B}}$ is a single point. As usual, each connected component of $\tilde{C}$ contains one of $x_1, \ldots, x_N$. So

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{D}_{r\rho}; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$$

where $x_1, \ldots, x_N, \eta_1, \ldots, \eta_N$ are extended from those of $\widetilde{C}$ and are constant over $\mathcal{D}_{r\rho}$.

Assume $\mathbb{V}$ is C_2-cofinite. Let $\mathcal{E}$ be a complete list of mutually inequivalent irreducible (ordinary) $\mathbb{V}$-modules. "Complete" means that any simple ordinary $\mathbb{V}$-module is equivalent to an object of $\mathcal{E}$. $\mathbb{W}_1, \ldots, \mathbb{W}_N$ are finitely-generated $\mathbb{V}$-modules. Then by Theorems 11.3 and 11.5, for each $q \in \mathcal{D}_{r\rho}^{\times}$, we can define a linear map

$$\mathfrak{S}_q : \bigoplus_{\mathbb{M} \in \mathcal{E}} \mathscr{T}_{\widetilde{\mathfrak{X}}}^*(\mathbb{W}_{\bullet} \otimes \mathbb{M} \otimes \mathbb{M}') \to \mathscr{T}_{\mathfrak{X}_q}^*(\mathbb{W}_{\bullet}), \tag{12.1}$$

$$\bigoplus_{\mathbb{M}} \psi_{\mathbb{M}} \mapsto \sum_{\mathbb{M}} \mathcal{S}\psi_{\mathbb{M}}(q)$$

where $\mathfrak{X}_q = (\mathcal{C}_q; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$. Similarly, one can define $\widetilde{\mathfrak{S}}_q$ by replacing $\mathcal{S}$ with $\widetilde{\mathcal{S}}$. Notice that $\sum_{\mathbb{M}} \widetilde{\mathcal{S}}\psi_{\mathbb{M}}(q) = \sum_{\mathbb{M}} q^{\lambda_{\mathbb{M}}} \mathcal{S}\psi_{\mathbb{M}}(q)$ for some constants $\lambda_{\mathbb{M}}$ depending only on $\mathbb{M}$. Thus $\mathfrak{S}_q$ is injective (resp. bijective) if and only if $\widetilde{\mathfrak{S}}_q$ is. Also, $\mathfrak{S}_q$ depends on the argument of q.

Theorem 12.1. *Assume $\mathbb{V}$ is C_2-cofinite, and choose any $q \in \mathcal{D}_{r\rho}^{\times}$. Then $\mathfrak{S}_q$ and $\widetilde{\mathfrak{S}}_q$ are injective linear maps. If $\mathbb{V}$ is also rational and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$, then $\mathfrak{S}_q$ and $\widetilde{\mathfrak{S}}_q$ are bijective.*

Proof. Let us fix $q_0 \in \mathcal{D}_{r\rho}^{\times}$ and let q denote a complex variable. Let us prove that $\mathfrak{S}_{q_0}$ is injective. Suppose that the finite sum $\sum_{\mathbb{M}} \mathcal{S}\psi_{\mathbb{M}}(q_0)$ equals 0. We shall prove by contradiction that $\psi_{\mathbb{M}} = 0$ for any $\mathbb{M} \in \mathcal{E}$.

Suppose this is not true. Let $\mathcal{F}$ be the (finite) subset of all $\mathbb{M} \in \mathcal{E}$ satisfying $\psi_{\mathbb{M}} \neq 0$. Then $\mathcal{F}$ is not an empty set. We first show that $\phi := \sum_{\mathbb{M}} \mathcal{S}\psi_{\mathbb{M}}$ (which is a multivalued holomorphic function on $\mathcal{D}_{r\rho}^{\times}$) satisfies $\phi(q) = 0$ for each $q \in \mathcal{D}_{r\rho}^{\times}$. Choose any large enough $k \in \mathbb{N}$. Then, by Theorem 11.4, $\phi^{\leqslant k}$ satisfies a linear differential equation on $\mathcal{D}_{r\rho}^{\times}$ of the form $\partial_q \phi^{\leqslant k} = q^{-1}\Omega \cdot \phi^{\leqslant k}$. Moreover, it satisfies the initial condition $\phi^{\leqslant k}(q_0) = 0$. Thus, $\phi^{\leqslant k}$ is constantly 0. So is ϕ.

Consider the $\mathbb{V} \otimes \mathbb{V}$-module $\mathbb{X} := \bigoplus_{\mathbb{M} \in \mathcal{F}} \mathbb{M} \otimes \mathbb{M}'$. Define a linear map $\kappa : \mathbb{X} \to \mathbb{W}_{\bullet}^*$ as follows. If $m \otimes m' \in \mathbb{M} \otimes \mathbb{M}'$, then the evaluation of $\kappa(m \otimes m')$ with any $w_{\bullet} \in \mathbb{W}_{\bullet}$ is

$$\langle \kappa(m \otimes m'), w_{\bullet} \rangle = \sum_{\mathbb{M} \in \mathcal{F}} \psi_{\mathbb{M}}(w_{\bullet} \otimes m \otimes m').$$

We claim that $\mathrm{Ker}(\kappa)$ is a non-zero subspace of $\mathbb{X}$ invariant under the action of $\mathbb{V} \otimes \mathbb{V}$. If this can be proved, then $\ker(\kappa)$ is a semi-simple $\mathbb{V} \otimes \mathbb{V}$-submodule of $\mathbb{X}$, which must contain $\mathbb{M} \otimes \mathbb{M}'$ for some $\mathbb{M} \in \mathcal{F}$. Therefore, $\psi_{\mathbb{M}}(w_{\bullet} \otimes m \otimes m') = 0$ for any $w_{\bullet} \in \mathbb{W}_{\bullet}$ and $m \otimes m' \in \mathbb{M} \otimes \mathbb{M}'$. Namely, $\psi_{\mathbb{M}} = 0$. So $\mathbb{M} \notin \mathcal{F}$, which gives a contradiction.

For any $n \in \mathbb{C}$, let $P_{(n)}$ be the projection of $\mathbb{M}$ onto its L_0-weight n subspace. Then

$$\phi(w_{\bullet})(q) = \sum_{\mathbb{M} \in \mathcal{F}} \sum_{n \in \mathbb{C}} \psi_{\mathbb{M}}(w_{\bullet} \otimes P_{(n)} \blacktriangleright \otimes \blacktriangleleft)q^n.$$

Since this multivalued function is always 0, by [Hua17, Prop. 2.1], any coefficient before q^n is 0. Thus $P_{(n)} \blacktriangleright \otimes \blacktriangleleft$ (which is an element of $\mathbb{M}_{(n)} \otimes \mathbb{M}'_{(n)}$) is in $\ker(\kappa)$ for any n. Thus $\ker(\kappa)$ must be non-empty.

Suppose now that $\sum_j m_j \otimes m'_j \in \mathrm{Ker}(\kappa)$ where each $m_j \otimes m'_j$ belongs to some $\mathbb{M} \otimes \mathbb{M}'$. We set $\psi_{\widetilde{\mathbb{M}}}(w_{\bullet} \otimes m_j \otimes m'_j) = 0$ if $\mathbb{M}, \widetilde{\mathbb{M}} \in \mathcal{F}$ and $\mathbb{M} \neq \widetilde{\mathbb{M}}$. Choose any $n \in \mathbb{N}$ and $l \in \mathbb{Z}$. We

shall show that $\sum_j Y(u)_l m_j \otimes m'_j \in \mathrm{Ker}(\kappa)$ for any $u \in \mathbb{V}^{\leqslant n}$. (Here Y denotes $Y_{\mathbb{M}}$ for a suitable $\mathbb{M}$.) Thus $\mathrm{Ker}(\kappa)$ is $\mathbb{V} \otimes \mathbf{1}$-invariant. A similar argument will show that $\mathrm{Ker}(\kappa)$ is $\mathbf{1} \otimes \mathbb{V}$-invariant, and hence $\mathbb{V} \otimes \mathbb{V}$-invariant.

Set divisors $D_1 = x_1 + \cdots + x_N$ and $D_2 = x' + x''$. Choose a natural number $k_2 \geqslant l$ such that $Y(u)_k m_j = Y(u)_k m'_j = 0$ for any j, any $k \geqslant k_2$, and any $u \in \mathbb{V}^{\leqslant n}$. This is possible by the lower truncation property. By Serre's vanishing Theorem, we can find $k_1 \in \mathbb{N}$ such that $H^1\big(\widetilde{C}, \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - k_2 D_2)\big) = 0$. Thus, the short exact sequence

$$0 \to \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - k_2 D_2) \to \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - l D_2) \to \mathscr{G} \to 0$$

(where $\mathscr{G}$ is the quotient of the previous two sheaves) induces another one

$$0 \to H^0\big(\widetilde{C}, \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - k_2 D_2)\big) \to H^0\big(\widetilde{C}, \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - l D_2)\big)$$
$$\to H^0(\widetilde{C}, \mathscr{G}) \to 0.$$

Choose any $u \in \mathbb{V}^{\leqslant n}$. Choose $v \in H^0(\widetilde{C}, \mathscr{G})$ to be $\mathcal{U}_\varrho(\xi)^{-1} u \xi^l d\xi$ near x' and 0 in $\widetilde{C} - \{x'\}$. Then v has a lift ν in $H^0\big(\widetilde{C}, \mathscr{V}_{\widetilde{C}}^{\leqslant n} \otimes \omega_{\widetilde{C}}(k_1 D_1 - l D_2)\big)$, which must be of the form

$$\mathcal{U}_\varrho(\xi)\nu|_{U'} = u \xi^l d\xi + \xi^{k_2}(\text{elements of } \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathcal{O}(U'))d\xi,$$
$$\mathcal{U}_\varrho(\varpi)\nu|_{U''} = \varpi^{k_2}(\text{elements of } \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathcal{O}(U''))d\varpi,$$

where $U' \ni x', U'' \ni x''$ are open subsets of $\widetilde{C}$ (see (4.6)). It is clear that $\nu \cdot m_j = Y(u)_l m_j$ and $\nu \cdot m'_j = 0$. Thus, as each $\psi_{\mathbb{M}}$ vanishes on $\nu \cdot (\mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}')$, we have

$$\sum_{\mathbb{M} \in \mathcal{F}} \sum_j \psi_{\mathbb{M}}(w_\bullet \otimes Y(u)_l m_j \otimes m'_j) = - \sum_{\mathbb{M} \in \mathcal{F}} \sum_j \psi_{\mathbb{M}}((\nu \cdot w_\bullet) \otimes m_j \otimes m'_j)$$
$$= - \sum_j \langle \kappa(m_j \otimes m'_j), \nu \cdot w_\bullet \rangle = 0.$$

So $\sum_j Y(u)_l m_j \otimes m'_j \in \mathrm{Ker}(\kappa)$.

We have proved the injectivity when $\mathbb{V}$ is C_2-cofinite and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$. If $\mathbb{V}$ is also rational, the surjectivity follows by comparing the dimensions of both sides and using the factorization property proved in [DGT19b]. $\square$

As a consequence of this theorem, we obtain again the well-known fact that any C_2-cofinite $\mathbb{V}$ with $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$ has finitely many equivalence classes of simple modules. Indeed, we let $\widetilde{\mathfrak{X}} = (\mathbb{P}^1; 1, 0, \infty; z - 1, z, z^{-1})$, and let $\mathbb{W}_1 = \mathbb{V}$. Then $\mathscr{T}^*_{\widetilde{\mathfrak{X}}}(\mathbb{V} \otimes \mathbb{M} \otimes \mathbb{M}')$ is nontrivial since $Y_{\mathbb{M}}$ defines a non-zero element of it. Thus, as the range of $\mathfrak{S}_q$ is finite-dimensional, $\mathcal{E}$ is finite.

13   Convergence of multiple sewing

In this section, we prove the convergence of sewing conformal blocks along several pairs of points, which generalizes Theorem 11.5. Let $N, M \in \mathbb{Z}_+$. Let

$$\widetilde{\mathfrak{X}} = (\widetilde{\pi} : \widetilde{C} \to \widetilde{B}; \varsigma_1, \ldots, \varsigma_N; \varsigma'_1, \ldots, \varsigma'_M; \varsigma''_1, \ldots, \varsigma''_M; \eta_1, \ldots, \eta_N; \xi_1, \ldots, \xi_M; \varpi_1, \ldots, \varpi_M)$$

be a family of $(N + 2M)$-pointed compact Riemann surfaces with local coordinates. So each η_i, ξ_j, ϖ_j are local coordinates at $\varsigma_i(\widetilde{\mathcal{B}}), \varsigma_j'(\widetilde{\mathcal{B}}), \varsigma_j''(\widetilde{\mathcal{B}})$ respectively. We assume that for every $b \in \widetilde{\mathcal{B}}$, each connected component of the fiber $\widetilde{\mathcal{C}}_b = \widetilde{\pi}^{-1}(b)$ contains one of $\varsigma_1(b), \ldots, \varsigma_N(b)$.

For each $1 \leqslant j \leqslant M$ we choose $r_j, \rho_j > 0$ and a neighborhood U_j' (resp. U_j'') of $\varsigma_j'(\widetilde{\mathcal{B}})$ (resp. $\varsigma_j''(\widetilde{\mathcal{B}})$) such that

$$(\xi_j, \widetilde{\pi}) : U_j' \xrightarrow{\simeq} \mathcal{D}_{r_j} \times \widetilde{\mathcal{B}} \qquad \text{resp.} \qquad (\varpi_j, \widetilde{\pi}) : U_j'' \xrightarrow{\simeq} \mathcal{D}_{\rho_j} \times \widetilde{\mathcal{B}} \qquad (13.1)$$

is a biholomorphic map. We also assume that these r_i and ρ_j are small enough such that the neighborhoods $U_1', \ldots, U_M', U_1'', \ldots, U_M''$ are mutually disjoint and are also disjoint from $\varsigma_1(\widetilde{\mathcal{B}}), \ldots, \varsigma_N(\widetilde{\mathcal{B}})$.

Assume $\mathbb{V}$ is C_2-cofinite. Associate to $\varsigma_1, \ldots, \varsigma_N$ finitely-generated (equivalently, $L_{0,s}$-semisimple) $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$. Associate to $\varsigma_1', \ldots, \varsigma_M'$ finitely-generated $\mathbb{V}$-modules $\mathbb{M}_1, \ldots, \mathbb{M}_M$ whose contragredient modules $\mathbb{M}_1', \ldots, \mathbb{M}_M'$ are associated to $\varsigma_1'', \ldots, \varsigma_M''$. Each $\mathbb{M}_j'$ is $L_{0,s}$-semisimple and hence, equivalently, finitely generated. We understand $\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}_\bullet'$ as

$$\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M}_1 \otimes \mathbb{M}_1' \otimes \cdots \otimes \mathbb{M}_M \otimes \mathbb{M}_M',$$

where the order has been changed so that each $\mathbb{M}_j'$ is next to $\mathbb{M}_j$.

For any $\psi \in \mathscr{T}_{\widetilde{\mathfrak{X}}}^*(\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}_\bullet')(\widetilde{\mathcal{B}})$ and $w \in \mathbb{W}_\bullet$, we define

$$\mathcal{S}\psi(w) = \psi\left(w \otimes (q_1^{L_0} \blacktriangleright \otimes_1 \blacktriangleleft) \otimes \cdots \otimes (q_M^{L_0} \blacktriangleright \otimes_M \blacktriangleleft)\right) \qquad \in \mathscr{O}(\widetilde{\mathcal{B}})\{q_\bullet\}[\log q_\bullet]. \qquad (13.2)$$

(Recall that the powers the $\log q_j$ are bounded above since each $\mathbb{M}_j$ and $\mathbb{M}_j'$ are finitely-generated; see the paragraph above Thm. 11.4.) Here, each $q_j^{L_0} \blacktriangleright \otimes_j \blacktriangleleft \in (\mathbb{M}_j \otimes \mathbb{M}_j')\{q_j\}[\log q_j]$ is defined as in (10.5). $\widetilde{\mathcal{S}}\psi(w)$ is defined similarly, except that L_0 is replaced by $\widetilde{L}_0$.

Let $\mathcal{D}_{r_\bullet \rho_\bullet} = \mathcal{D}_{r_1 \rho_1} \times \cdots \times \mathcal{D}_{r_M \rho_M}$ and $\mathcal{D}_{r_\bullet \rho_\bullet}^\times = \mathcal{D}_{r_1 \rho_1}^\times \times \cdots \times \mathcal{D}_{r_M \rho_M}^\times$. Let q_j be the standard coordinate of $\mathcal{D}_{r_j \rho_j}$. We say that $\widetilde{\mathcal{S}}\psi$ **converges a.l.u.** if $\widetilde{\mathcal{S}}\psi(w) \in \mathscr{O}(\widetilde{\mathcal{B}} \times \mathcal{D}_{r_\bullet \rho_\bullet})$ for each $w \in \mathbb{W}_\bullet$. Write

$$\mathcal{S}\psi(w) = \sum_{n_\bullet \in \mathbb{C}^M, l_\bullet \in \mathbb{N}^M} \mathcal{S}\psi(w)_{n_\bullet, l_\bullet} q_\bullet^{n_\bullet} (\log q_\bullet)^{l_\bullet}$$

where

$$q_\bullet^{n_\bullet} = q_1^{n_1} \cdots q_M^{n_M}, \qquad (\log q_\bullet)^{l_\bullet} = (\log q_1)^{l_1} \cdots (\log q_M)^{l_M}.$$

We say $\mathcal{S}\psi$ **converges a.l.u.** if for each compact subsets $K \subset \widetilde{\mathcal{B}}$ and $Q \subset \mathcal{D}_{r_\bullet \rho_\bullet}^\times = \mathcal{D}_{r_1 \rho_1}^\times \times \cdots \times \mathcal{D}_{r_M \rho_M}^\times$, there exists $C > 0$ such that

$$\sum_{n_\bullet \in \mathbb{C}^M} \left| \mathcal{S}\psi(w)_{n_\bullet, l_\bullet}(b) \right| \cdot |q_\bullet^{n_\bullet}| \leqslant C$$

for any $b \in K, q_\bullet = (q_1, \ldots, q_M) \in Q$, and $l_\bullet \in \mathbb{N}^M$. Clearly, if $\mathcal{S}\psi$ converges a.l.u. then so does $\widetilde{\mathcal{S}}\psi$.

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Theorem 13.1. *Assume that $\mathbb{V}$ is C_2-cofinite and all the $\mathbb{V}$-modules $\mathbb{W}_i, \mathbb{M}_j$ are finitely-generated. Then $\mathcal{S}\psi$ and $\widetilde{\mathcal{S}}\psi$ converge a.l.u..*

Proof. The main idea is the same as single sewing. We sketch the proof below.

Since the convergence property is local with respect to $\widetilde{\mathcal{B}}$, we may identify $\widetilde{\mathcal{B}}$ as an open subset of $\mathbb{C}^m$ and let $\tau_\bullet = (\tau_1, \dots, \tau_m)$ denote the set of m standard coordinates.

Similar to Sec. 4, we may define a new family of curves $\mathfrak{X}$ with base manifold $\mathcal{B} = \mathcal{D}_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}} = \mathcal{D}_{r_1 \rho_1} \times \cdots \times \mathcal{D}_{r_M \rho_M} \times \widetilde{\mathcal{B}}$ to be the simultaneous sewing of $\widetilde{\mathfrak{X}}$ along ς_j' and ς_j'' using the relation $\xi_j \varpi_j = q_j$ for all $1 \leqslant j \leqslant M$. Let q_j be the standard coordinate of $\mathcal{D}_{r_j \rho_j}$. We let $\Sigma \subset \mathcal{C}$ be the set of critical points of $\pi : \mathcal{C} \to \mathcal{B}$, i.e., the set of all points at which π is not submersive. Then $\Delta := \pi(\Sigma)$ equals $\Delta_1 \cup \cdots \cup \Delta_M$ where Δ_j is the hypersurface of $\mathcal{B}$ defined by $q_j = 0$. We let $\mathcal{C}_\Delta = \pi^{-1}(\Delta)$. $\Theta_{\mathcal{B}}(-\log \Delta)$ is defined to be the free $\mathscr{O}_{\mathcal{B}}$-module generated by

$$q_1 \partial_{q_1}, \dots, q_M \partial_{q_M}, \partial_{\tau_1}, \dots, \partial_{\tau_m}.$$

The partial coordinates $\tau_\bullet \circ \widetilde{\pi}$ of $\widetilde{\mathcal{C}}$ and $\tau_\bullet \circ \pi$ of $\mathcal{C}$ are also denoted by $\tau_\bullet$ for simplicity.

To understand π locally at $x \in \mathcal{C}$, we consider the following two cases:

1. $x \notin \Sigma$. The fiber $\mathcal{C}_{\pi(x)}$ is either smooth or nodal, and x is a smooth point of $\mathcal{C}_{\pi(x)}$. Then we may find $r_0 > 0$ and $r_\bullet' < r_\bullet, \rho_\bullet' < \rho_\bullet$ such that π on a neighborhood of x is equivalent to part of the projection

$$\mathcal{D}_{r_0} \times \mathcal{D}_{r_\bullet' \rho_\bullet'} \times \widetilde{\mathcal{B}} \to \mathcal{D}_{r_\bullet' \rho_\bullet'} \times \widetilde{\mathcal{B}}.$$

The $\mathscr{O}_{\mathcal{C}}$-module $\Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta)$ near x is generated freely by

$$\partial_z, q_1 \partial_{q_1}, \dots, q_M \partial_{q_M}, \partial_{\tau_1}, \dots, \partial_{\tau_m}$$

where z is the standard coordinate of $\mathcal{D}_{r_0}$. The morphism

$$d\pi : \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta) \to \pi^* \Theta_{\mathcal{B}}(-\log \Delta) \tag{13.3}$$

is defined by killing ∂_z, and by keeping all the other generating elements.

2. $x \in \Sigma$. The fiber $\mathcal{C}_{\pi(x)}$ is nodal, and x is a node of $\mathcal{C}_{\pi(x)}$. Denote by $\mathcal{D}_{r_\bullet \rho_\bullet \backslash j}$ the product of $\mathcal{D}_{r_1 \rho_1}, \dots, \mathcal{D}_{r_M \rho_M}$ except $\mathcal{D}_{r_j \rho_j}$. Then π at x is locally equivalent to the map

$$\mathcal{D}_{r_j} \times \mathcal{D}_{\rho_j} \times \mathcal{D}_{r_\bullet \rho_\bullet \backslash j} \times \widetilde{\mathcal{B}} \to \mathcal{D}_{r_j \rho_j} \times \mathcal{D}_{r_\bullet \rho_\bullet \backslash j} \times \widetilde{\mathcal{B}} = \mathcal{B},$$
$$(\xi_j, \varpi_j, q_{\bullet \backslash j}, \tau_\bullet) \mapsto (\xi_j \varpi_j, q_{\bullet \backslash j}, \tau_\bullet).$$

Near x, the $\mathscr{O}_{\mathcal{C}}$-module $\Theta_C(-\log \Delta)$ is generated freely by

$$\xi_j \partial_{\xi_j}, \varpi_j \partial_{\varpi_j}, q_1 \partial_{q_1}, \dots, q_{j-1} \partial_{q_{j-1}}, q_{j+1} \partial_{q_{j+1}}, \dots, q_M \partial_{q_M}, \tau_1, \dots, \tau_m.$$

(13.3) is defined near x by setting

$$d\pi(\xi_j \partial_{\xi_j}) = d\pi(\varpi_j \partial_{\varpi_j}) = q_j \partial_{q_j} \tag{13.4}$$

and keeping the other generators.

We now obtain again an exact sequence (4.20) where $\Theta_{\mathcal{C}/\mathcal{B}}$ is defined to be the kernal of $d\pi$, and we again have a natural isomorphism $\Theta_{\mathcal{C}/\mathcal{B}}|_{\mathcal{C}_b} \simeq \Theta_{\mathcal{C}_b}$ for each fiber $\mathcal{C}_b$ of $\mathfrak{X}$. So we obtain the long exact sequence (11.3). Similar to the proof of Thm. 11.4, one may lift $q_k \partial_{q_k} \in \Theta_{\mathcal{B}}(-\log \Delta)$ to a section

$$\widetilde{\mathfrak{y}} \in H^0(\mathcal{C}, \Theta_{\mathcal{C}}(-\log \mathcal{C}_\Delta + \bullet S_{\mathfrak{X}}))$$

which has multi series expansion

$$\widetilde{\mathfrak{y}} = \sum_{n_\bullet \in \mathbb{N}^M} \widetilde{\mathfrak{y}}_{n_\bullet}^\perp \cdot q_\bullet^{n_\bullet} + q_k \partial_{q_k}$$

on $\mathcal{C} \backslash \Sigma$. Here, ∂_{q_k} should be understood as the tangent field $\partial_{q_k \circ \pi}$ of $\mathcal{C}$ which vanishes when pullbacked to each fiber of $\mathcal{C}$. And $\widetilde{\mathfrak{y}}_{n_\bullet}^\perp \in H^0(\widetilde{\mathcal{C}} - \Gamma, \Theta_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}})$ where $\Gamma = \bigcup_{1 \leqslant j \leqslant M}(\varsigma_j'(\widetilde{\mathcal{B}}) \cup \varsigma_j''(\widetilde{\mathcal{B}}))$.

Choose $\widetilde{V}_1, \ldots, \widetilde{V}_N$ to be neighborhoods of $\varsigma_1(\widetilde{\mathcal{B}}), \ldots, \varsigma_N(\widetilde{\mathcal{B}})$. Assume that they are disjoint from $U_1', \ldots, U_M', U_1'', \ldots, U_M''$. Then $\mathcal{D}_{r_\bullet \rho_\bullet} \times \widetilde{V}_1, \ldots, \mathcal{D}_{r_\bullet \rho_\bullet} \times \widetilde{V}_N$ are neighborhoods of $\varsigma_1(\mathcal{B}), \ldots, \varsigma_N(\mathcal{B})$ in $\mathcal{C}$. By the descriptions of $d\pi$ in (13.3) and (13.4), for each $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant M$, we can write

$$\widetilde{\mathfrak{y}}\big|_{\mathcal{D}_{r_\bullet \rho_\bullet} \times \widetilde{V}_i} = h_i(q_\bullet, \eta_i, \tau_\bullet)\partial_{\eta_i} + q_k \partial_{q_k},$$

$$\widetilde{\mathfrak{y}}\big|_{\mathcal{D}_{r_j} \times \mathcal{D}_{\rho_j} \times \mathcal{D}_{r_\bullet \rho_{\bullet \backslash j}} \times \widetilde{\mathcal{B}}} = a_j(\xi_j, \varpi_j, q_{\bullet \backslash j}, \tau_\bullet)\xi_j \partial_{\xi_j} + b_j(\xi_j, \varpi_j, q_{\bullet \backslash j}, \tau_\bullet)\varpi_j \partial_{\varpi_j} + (1 - \delta_{j,k})q_k \partial_{q_k}.$$

where h_i, a_j, b_j are holomorphic functions on suitable domains, and

$$a_j + b_j = \delta_{j,k}.$$

This shows in particular that $\widetilde{\mathfrak{y}}_{n_\bullet}^\perp \in H^0(\widetilde{\mathcal{C}}, \Theta_{\widetilde{\mathcal{C}}/\widetilde{\mathcal{B}}}(\bullet S_{\mathfrak{X}}))$. Fix a projective structure $\mathfrak{P}$ on $\widetilde{\mathfrak{X}}$. Then we can define $\nu(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp)$ and $\#(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp)$ as in Sec. 9.

For $q_j^{L_0} \blacktriangleright \otimes_j \blacktriangleleft \in \mathbb{M}_j \otimes \mathbb{M}_j'\{q_j\}[\log q_j]$, (11.11) can be generalized to

$$\delta_{j,k} \cdot L_0 q_k^{L_0} \blacktriangleright \otimes_k \blacktriangleleft = \sum_{n_\bullet \in \mathbb{N}^M} \nu(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp)q_\bullet^{n_\bullet} q_j^{L_0} \blacktriangleright \otimes_j \blacktriangleleft + \sum_{n_\bullet \in \mathbb{N}^M} q_\bullet^{n_\bullet} q_j^{L_0} \blacktriangleright \otimes_j \nu(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp) \blacktriangleleft \qquad (13.5)$$

by showing that the right hand side equals

$$\mathrm{Res}_{\xi_j=0} Y_{\mathbb{M}_j}(\xi_j^{L_0}\mathbf{c}, \xi_j)q_j^{L_0} \blacktriangleright \otimes_j \blacktriangleleft \cdot a_j(\xi_j, q_j/\xi_j, q_{\bullet \backslash j}, \tau_\bullet)\frac{d\xi_j}{\xi_j}$$

$$+\mathrm{Res}_{\varpi_j=0} q_j^{L_0} \blacktriangleright \otimes_j Y_{\mathbb{M}_j'}(\varpi_j^{L_0}\mathcal{U}(\gamma_1)\mathbf{c}, \varpi_j) \blacktriangleleft \cdot b_j(q_j/\varpi_j, \varpi_j, q_{\bullet \backslash j}, \tau_\bullet)\frac{d\varpi_j}{\varpi_j}$$

and using Lemma 10.1. Then, similar to Prop. 11.10, we can prove

$$q_k \partial_{q_k} \mathcal{S}\psi(w_\bullet) = \mathcal{S}\psi(\nabla_{q_k \partial_{q_k}} w_\bullet) + \sum_{n_\bullet \in \mathbb{N}^M} \#(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp)q_\bullet^{n_\bullet} \cdot \mathcal{S}\psi(w_\bullet) \qquad (13.6)$$

where $\nabla_{q_k \partial_{q_k}} w_\bullet = -\sum_{n_\bullet \in \mathbb{N}^M} q_\bullet^{n_\bullet} \nu(\widetilde{\mathfrak{y}}_{n_\bullet}^\perp)w_\bullet$ is indeed an element of $\mathbb{W}_\bullet \otimes \mathscr{O}(\mathcal{B})$.

Consider holomorphic functions $\mathbf{S}_{\eta_i}\mathfrak{P} = \mathbf{S}_{\eta_i}\mathfrak{P}(\eta_i, \tau_\bullet)$, $\mathbf{S}_{\xi_j}\mathfrak{P} = \mathbf{S}_{\xi_j}\mathfrak{P}(\xi_j, \tau_\bullet)$, $\mathbf{S}_{\varpi_j}\mathfrak{P} = \mathbf{S}_{\varpi_j}\mathfrak{P}(\varpi_j, \tau_\bullet)$ on $\tilde{V}_i, U'_j, U''_j$ which are identified with their images under $(\eta_i, \tau_\bullet), (\xi_j, \tau_\bullet), (\varpi_j, \tau_\bullet)$ respectively. Under these identifications, choose circles γ'_j in the ξ_j-plane $\mathcal{D}_{r_j}$ of U'_j and γ''_j in the ϖ_j-plane $\mathcal{D}_{\rho_j}$ of U''_j around the origins (see (13.1)). Then the projective term $\sum_{n_\bullet \in \mathbb{N}^M} \#(\tilde{\mathfrak{H}}^\perp_{n_\bullet}) q_\bullet^{n_\bullet}$ equals

$$f_k = \frac{c}{12} \cdot \left(\sum_{1 \leqslant j \leqslant M} A_j + \sum_{1 \leqslant j \leqslant M} B_j + \sum_{1 \leqslant i \leqslant N} C_i \right), \tag{13.7}$$

where

$$\begin{cases} A_j = \oint_{\gamma'_j} \mathbf{S}_{\xi_j}\mathfrak{P}(\xi_j, \tau_\bullet) \cdot a_j(\xi_j, q_j/\xi_j, q_{\bullet \backslash j}, \tau_\bullet) \cdot \xi_j d\xi_j, \\[2mm] B_j = \oint_{\gamma''_j} \mathbf{S}_{\varpi_j}\mathfrak{P}(\varpi_j, \tau_\bullet) \cdot b_j(q_j/\varpi_j, \varpi_j, q_{\bullet \backslash j}, \tau_\bullet) \cdot \varpi_j d\varpi_j, \\[2mm] C_i = \mathrm{Res}_{\eta_i=0} \, \mathbf{S}_{\eta_i}\mathfrak{P}(\eta_i, \tau_\bullet) \cdot h_i(q_\bullet, \eta_i, \tau_\bullet) d\eta_i \end{cases} \tag{13.8}$$

whenever $|q_j|/\rho_j$ and $|q_j|/r_j$ are respectively less than the radii of γ'_j and γ''_j. This proves that the projective term is an element of $\mathscr{O}(\mathcal{B})$.

Thus, as in the last step of the proof of Thm. 11.4, we conclude that for all large enough l, there are $\Omega_1, \cdots, \Omega_M \in \mathrm{End}_{\mathbb{C}}((\mathbb{W}^{\leqslant l}_\bullet)^*) \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$ independent on $\mathbb{M}_1, \ldots, \mathbb{M}_M$ such that

$$q_k \partial_{q_k}(\mathcal{S}\psi^{\leqslant l}) = \Omega_k \cdot \mathcal{S}\psi^{\leqslant l} \tag{13.9}$$

for every $1 \leqslant k \leqslant M$. This finishes the proof of that $\mathcal{S}\psi$ converges a.l.u., thanks to Thm. A.1. $\qquad\square$

In the above proof, we give the formulas (13.7) (13.8) for the projective term since we think it might be useful in the future.

A Differential equations with simple poles and parameters

Let V be a connected open subset of $\mathbb{C}^m$ with coordinates $\tau_\bullet = (\tau_1, \ldots, \tau_m)$. Let $\mathcal{D}_{r_\bullet} = \mathcal{D}_{r_1} \times \cdots \times \mathcal{D}_{r_M}$. We let q_j be the standard variable of $\mathcal{D}_{r_j}$. For each $1 \leqslant j \leqslant M$, choose

$$\begin{aligned} A^j &\in \mathrm{End}(\mathbb{C}^N) \otimes \mathscr{O}(\mathcal{D}_{r_\bullet} \times V), \\ \omega^j &\in \mathbb{C}^N \otimes \mathscr{O}(\mathcal{D}_{r_\bullet} \times V). \end{aligned}$$

Recall $\mathcal{D}_r$ is the open disc at 0 with radius r. Consider the following system of differential equations

$$q_j \partial_{q_j} \psi = A^j \psi + \omega^j \tag{A.1}$$

for all j, where

$$\psi = \sum_{l_1,\ldots,l_M=0}^{L} \sum_{n_\bullet \in \mathbb{N}^M} \psi_{n,l}(\tau_\bullet) q_\bullet^{n\bullet} (\log q_\bullet)^{l\bullet}$$

$$= \sum_{l_1,\ldots,l_M=0}^{L} \sum_{n_\bullet \in \mathbb{N}^M} \psi_{n,l}(\tau_\bullet) q_1^{n_1} \cdots q_M^{n_M} (\log q_1)^{l_1} \cdots (\log q_M)^{l_M} \tag{A.2}$$

is in $\mathbb{C}^N \otimes_\mathbb{C} \mathcal{O}(V)[[q_\bullet]][\log q_\bullet]$ and is a formal solution of this system of equations. The following result is well-known although the reference is not easy to find. Thus, we provide a proof for completeness purpose. (See also [McR21] Appendix A.)

Theorem A.1. *Suppose that the formal series ψ satisfies* (A.1) *for all j. Then ψ converges a.l.u., namely, for each $l_1, \ldots, l_M$ and $|q_j| < r_j$, and each compact subset $K \subset V$,*

$$\sup_{\tau_\bullet \in K} \sum_{n_\bullet \in \mathbb{N}^M} \|\psi_{n_\bullet, l_\bullet}(\tau_\bullet)\| \cdot |q_\bullet^{n\bullet}| < +\infty. \tag{A.3}$$

To prove this theorem, we first need:

Lemma A.2. *Suppose $M = 1$ and we suppress the subscript 1. Suppose that the formal series ψ does not contain $\log q$ and is a formal solution of* (A.1), *namely,*

$$q\partial_q \psi = A\psi + \omega.$$

Then ψ is an element of $\mathbb{C}^N \otimes_\mathbb{C} \mathcal{O}(\mathcal{D}_r \times V)$. Equivalently, ψ converges a.l.u..

Proof. It suffices to prove that ψ is an $\mathbb{C}^N$-valued holomorphic function on a neighborhood of $\{0\} \times V \subset \mathcal{D}_r \times V$. Then, by basic theory of linear differential equations, there is a (possibly multivalued) $\mathbb{C}^N$-valued holomorphic function on $\mathcal{D}_r^\times \times V$ whose "initial value" is given by ψ. (See [Kna] the remark after Thm. B.1.) This function must be single-valued since ψ is so. So ψ is holomorphic on $\mathcal{D}_r \times V$.

Consider the series expansion of A and ω:

$$A(q, \tau_\bullet) = \sum_{n \in \mathbb{N}} A_n(\tau_\bullet) q^n, \qquad \omega(q, \tau_\bullet) = \sum_{n \in \mathbb{N}} \omega_n(\tau_\bullet) q^n$$

where each A_n is in $\mathrm{End}(\mathbb{C}^N) \otimes_\mathbb{C} \mathcal{O}(V)$, and each ω_n is in $\mathbb{C}^N \otimes_\mathbb{C} \mathcal{O}(V)$. Then for each $n \in \mathbb{N}$,

$$n\psi_n = \omega_n + \sum_{j=0}^{n} A_{n-j}\psi_j.$$

Choose any open subset U of V with compact closure, and choose $B > 0$ such that $\|A_0(\tau_\bullet)\| \leqslant B$ whenever $\tau_\bullet \in U$. (Here $\|\cdot\|$ is the operator norm.) Then for any $n > B$, $n\mathbf{1} - A_0(\tau_\bullet)$ is invertible (with inverse $n^{-1} \sum_{j=0}^{\infty} (A_0(\tau_\bullet)/n)^j$). Thus, whenever $n > B$,

$$\psi_n = (n - A_0)^{-1} \Big(\omega_n + \sum_{j=0}^{n-1} A_{n-j}\psi_j \Big). \tag{A.4}$$

Choose any $r_1 < r$ and set

$$\alpha = \sup_{(q,\tau_\bullet) \in \overline{\mathcal{D}_{r_1}} \times U} \max\{\|A(q,\tau_\bullet)\|, \|\omega(q,\tau_\bullet)\|\}. \tag{A.5}$$

Using $A_n(\tau_\bullet) = \oint_{\partial \mathcal{D}_{r_1}} A(q,\tau_\bullet) q^{-n-1} \frac{dq}{2\mathbf{i}\pi}$ and a similar relation for ω_n, we have

$$\|A_n(\tau_\bullet)\|, \|\omega_n(\tau_\bullet)\| \leqslant \alpha r_1^{-n} \tag{A.6}$$

for all n and all $\tau_\bullet$ in U.

Choose $\beta > 0$ such that $\|(n - A_0(\tau_\bullet))^{-1}\| \leqslant \beta n^{-1}$ for any $n > B$ and $\tau_\bullet \in U$. (Such β can be found using the explicit formula of inverse matrix given above.) Set $\gamma \geqslant \max\{1, \alpha\beta\}$. Then, from (A.4) and (A.6), we see that for any $n > B$ and $\tau_\bullet \in U$,

$$r_1^n \|\psi_n(\tau_\bullet)\| \leqslant \gamma n^{-1} \Big(1 + \sum_{j=0}^{n-1} r_1^j \|\psi_j(\tau_\bullet)\| \Big). \tag{A.7}$$

By induction, one can show that there exists $c > 0$ such that

$$r_1^n \|\psi_n(\tau_\bullet)\| \leqslant c\gamma^n$$

for any $n \in \mathbb{N}$ and $\tau_\bullet \in U$. Indeed, if this is true for $0, 1, 2, \dots, n-1$ where $n > B$, and if we assume $c(\gamma^{n-1} - 1) \geqslant 1$, then by (A.7),

$$r_1^n \|\psi_n(\tau_\bullet)\| \leqslant \gamma n^{-1} \Big(1 + \sum_{j=0}^{n-1} c\gamma^j \Big) = \gamma n^{-1} \Big(1 + c + \sum_{j=1}^{n-1} c\gamma^j \Big)$$

$$\leqslant \gamma n^{-1} \Big(c\gamma^{n-1} + \sum_{j=1}^{n-1} c\gamma^{n-1} \Big) = c\gamma^n.$$

Thus $\|\psi_n(\tau_\bullet)\| \leqslant c\gamma^n r_1^{-n}$ for all n and $\tau_\bullet \in U$. Therefore, if we choose any $r_0 \in (0, \gamma^{-1} r_1)$, then the series $\sum_n \|\hat{\psi}_n(\tau_\bullet)\| \cdot |q|^n$ is uniformly bounded by some positive number for all $|q| \leqslant r_0$ and $\tau_\bullet \in U$. Since each $\psi_n(\tau_\bullet)$ is holomorphic over $\tau_\bullet$, the series (A.2) must converge uniformly to a holomorphic function on $\mathcal{D}_{r_0} \times U$. This proves ψ is holomorphic on a neighborhood of $\{0\} \times U$, and hence of $\{0\} \times V$ by choosing arbitrary U. $\qquad\square$

Lemma A.3. *Thm. A.1 holds if ψ has no log terms.*

Proof. We prove this by induction on M. The case $M = 1$ has been proved. Suppose the case is proved for $M - 1$. In the case M, write $n_{\bullet \setminus 1} = (n_2, \dots, n_M)$. Set

$$\psi(q_1, \dots, q_M) = \sum_{n_{\bullet \setminus 1} \in \mathbb{N}^{M-1}} \psi_{n_{\bullet \setminus 1}} \psi(q_1) q_2^{n_2} \cdots q_M^{n_M} \tag{A.8}$$

and define $A^1_{n_{\bullet \setminus 1}}, \omega^1_{n_{\bullet \setminus 1}}$ in a similar way. Then

$$q_1 \partial_{q_1} \psi_{n_{\bullet \setminus 1}}(q_1) = \sum_{k_{\bullet \setminus 1} \leqslant n_{\bullet \setminus 1}} A_{n_2 - k_2, \dots, n_M - k_M} \psi_{k_2, \dots, k_M}(q_1) + \omega^1_{n_{\bullet \setminus 1}}(q_1).$$

Therefore, for each $n_{\bullet\backslash 1}$, $\bigoplus_{k_{\bullet\backslash 1} \leqslant n_{\bullet\backslash 1}} \psi_{k_{\bullet\backslash 1}} \in \mathbb{C}^{N(n_2+1)\cdots(n_M+1)}$ satisfies a differential equation of ∂_{q_1} similar to that in Lemma A.2. So $\psi_{n_{\bullet\backslash 1}} \in \mathbb{C}^N \otimes \mathscr{O}(\mathcal{D}_{r_1} \times V)$.

Now consider (A.8) as a formal series of $q_2, \ldots, q_M$ whose coefficients are holomorphic over $\mathcal{D}_{r_1} \times V$ and which satisfies a system of $M-1$ differential equations as in (A.1). Thus, by the case $M-1$, (A.8) is the series expansion of an element of $\mathbb{C}^N \otimes \mathscr{O}(\mathcal{D}_{r_\bullet} \times V)$. This finishes the proof. $\qquad\square$

Proof of Thm. A.1. Write

$$\psi = \sum_{n_\bullet, l_\bullet} \psi_{n_\bullet, l_\bullet} q_\bullet^{n\bullet} (\log q_\bullet)^{l\bullet} = \sum_{l_\bullet} \mu_{l_\bullet} (\log q_\bullet)^{l\bullet}$$

where $\mu_{l_\bullet} = \sum_{n_\bullet} \psi_{n_\bullet, l_\bullet} q_\bullet^{n\bullet}$. Recall that we assume (A.1) holds. By looking at the $(\log q_\bullet)^{l\bullet}$ part of this equation, we see

$$q_j \partial_{q_j} \mu_{l_\bullet} + (l_j + 1) \mu_{l_1, \ldots, l_j+1, \ldots, l_M} = A \mu_{l_\bullet}$$

when one of $l_1, \ldots, l_M$ is non-zero; if $l_1 = \cdots = l_M = 0$, the right hand side is $A\mu_{l_\bullet} + \omega^j$. By induction on $l_1 + \cdots + l_M$ (from large to small), and by Lemma A.3, we see that each $\mu_{l_\bullet}$ belongs to $\mathbb{C}^N \otimes \mathscr{O}(\mathcal{D}_{r_\bullet} \times V)$. So ψ satisfies (A.3). $\qquad\square$

B   Existence of projective structures

Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ be a family of compact Riemann surfaces. In [Hub81, Lemma 5] Hubbard showed that $\mathfrak{X}$ has a projective structure when $\mathcal{B}$ is Stein and the connected components of the fibers have genera > 1. In this section, we cover the low genus cases.

Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathfrak{A}}$ be a Stein cover [9] of $\mathcal{C}$ such that each member U_α admits $\eta_\alpha \in \mathscr{O}(U_\alpha)$ univalent on each fiber of U_α. Define a Čech 1-cochain $\sigma = (\sigma_{\alpha,\beta})_{\alpha,\beta \in \mathfrak{A}} \in C^1(\mathfrak{U}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ such that

$$\sigma_{\alpha,\beta} = \mathbf{S}_{\eta_\alpha} \eta_\beta \cdot d\eta_\alpha^2 \qquad \in \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}(U_\alpha \cap U_\beta).$$

By (8.4), σ is a cocycle and hence can be viewed as an element of $H^1(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$. The proof of the following Lemma is in [Hub81, Lemma 5]. We recall the proof for the reader's convenience.

Lemma B.1. *$\mathfrak{X}$ admits a projective structure if and only if σ is the zero element of $H^1(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$.*

Note that $H^1(\mathcal{C}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ equals $H^1(\mathfrak{U}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ by Leray's theorem.

Proof. "If": We have a 0-cochain $s = (s_\alpha)_{\alpha \in \mathfrak{A}} \in C^0(\mathfrak{U}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ such that $\delta s = \sigma$. By Proposition 8.1-(2) and by passing to a finer Stein cover (still denoted by $\mathfrak{U}$ for simplicity), we can find $f_\alpha \in \mathscr{O}(U_\alpha)$ univalent on each fiber, such that

$$\mathbf{S}_{\eta_\alpha} f_\alpha \cdot d\eta_\alpha^2 = s_\alpha. \tag{B.1}$$

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[9] An open cover $\mathfrak{U}$ of a complex manifold X is called Stein if each open set $U \in \mathfrak{U}$ is a Stein manifold. Then any finite intersection of open sets of $\mathfrak{U}$ is also Stein (see [GR84, Sec. 1.4.4]).

Thus, on $U_\alpha \cap U_\beta$ we have

$$\mathbf{S}_{\eta_\alpha}\eta_\beta \cdot d\eta_\beta^2 = \sigma_{\alpha,\beta} = s_\alpha - s_\beta = \mathbf{S}_{\eta_\alpha}f_\alpha \cdot d\eta_\alpha^2 - \mathbf{S}_{\eta_\beta}f_\beta \cdot d\eta_\beta^2.$$

By (8.4), we have

$$\mathbf{S}_{\eta_\alpha}\eta_\beta \cdot d\eta_\beta^2 = \mathbf{S}_{\eta_\alpha}f_\beta \cdot d\eta_\alpha^2 - \mathbf{S}_{\eta_\beta}f_\beta \cdot d\eta_\beta^2.$$

These two imply $\mathbf{S}_{\eta_\alpha}f_\alpha = \mathbf{S}_{\eta_\alpha}f_\beta$. Thus, by Proposition 8.1-(3), $\mathbf{S}_{f_\alpha}f_\beta = 0$. So $(U_\alpha, f_\alpha)_{\alpha \in \mathfrak{A}}$ is a projective chart.

"Only if": Assume $(U_\alpha, f_\alpha)_{\alpha \in \mathfrak{A}}$ is a projective chart. Define $s = (s_\alpha)_{\alpha \in \mathfrak{A}}$ using (B.1). Notice $\mathbf{S}_{\eta_\alpha}f_\alpha = \mathbf{S}_{\eta_\alpha}f_\beta$. One can reverse the argument in the first paragraph to show $\delta s = \sigma$. □

Theorem B.2. *Assume $\mathcal{B}$ is a Stein manifold. Then $\mathfrak{X}$ admits a projective structure.*

Proof. Assume without loss of generality that the fibers are connected and have genus g. (Recall they are diffeomorphic by Ehresmann's theorem.)

Assume $g = 1$. Each fiber $\mathcal{C}_b$ is a complex torus and hence obviously has a projective structure (e.g. the one from the standard local coordinates of $\mathbb{C}$). Thus, by Lemma B.1, for each $b \in \mathcal{B}$ the restriction $\sigma|\mathcal{C}_b$ is the zero element of $H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}^{\otimes 2})$. Since $b \mapsto \dim H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}^{\otimes 2})$ is constantly 1 (because $\omega_{\mathcal{C}_b} \simeq \mathcal{O}_{\mathcal{C}_b}$), by Grauert's Theorem 4.1, $R^1\pi_*\omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}$ is locally free and each fiber is equivalent to $H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}^{\otimes 2})$. So σ is the zero section of $R^1\pi_*\omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}$ over $\mathcal{B}$. Thus, we may find a Stein cover $\mathfrak{V} = (V^i)_{i \in \mathcal{I}}$ of $\mathcal{B}$ such that the restriction of σ to $\mathcal{C}_{V^i} = \pi^{-1}(V^i)$ is the zero element of $H^1(\mathcal{C}_{V^i}, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$.

Let $W_\alpha^i = U_\alpha \cap \mathcal{C}_{V^i}$ which is Stein (cf. [GR84, Sec. 1.4.4]). Then for each $i \in \mathcal{I}$, $\mathfrak{W}^i = (W_\alpha^i)_{\alpha \in \mathfrak{A}}$ is a Stein cover of $\mathcal{C}_{V^i}$. So $\sigma|\mathcal{C}_{V^i}$ is the zero element of $H^1(\mathfrak{W}^i, \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$. Choose $t^i = (t_\alpha^i)_{\alpha \in \mathfrak{A}}$ (where each $t_\alpha^i \in \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}(W_\alpha^i)$) such that $\delta t^i = \sigma|\mathcal{C}_{V^i}$. On $U_\alpha \cap U_\beta \cap \mathcal{C}_{V^i} \cap \mathcal{C}_{V^j}$ we have $t_\alpha^i - t_\beta^i = \sigma_{\alpha,\beta} = t_\alpha^j - t_\beta^j$ and hence $t_\alpha^i - t_\alpha^j = t_\beta^i - t_\beta^j$. Therefore, we have a well-defined element $u = (u^{i,j})_{i,j \in \mathcal{I}}$ of $H^1(\mathfrak{V}, \pi_*\omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ such that $u^{i,j}$, which is an element of $(\pi_*\omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})(V^i \cap V^j) = \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}(\mathcal{C}_{V^i} \cap \mathcal{C}_{V^j})$, equals $t_\alpha^i - t_\alpha^j$ when restricted to $\mathcal{C}_{V^i} \cap \mathcal{C}_{V^j} \cap U_\alpha$ for any α. Since $\mathcal{B}$ is Stein, by Cartan's Theorem B, $H^1(\mathfrak{V}, \pi_*\omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2})$ is trivial. So there exists $v = (v^i)_{i \in \mathcal{I}}$ (where each $v^i \in \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}(\mathcal{C}_{V^i})$) such that $v^i - v^j = u^{i,j}$ on $\mathcal{C}_{V^i} \cap \mathcal{C}_{V^j}$. So $v^i - v^j = t_\alpha^i - t_\alpha^j$ on $\mathcal{C}_{V^i} \cap \mathcal{C}_{V^j} \cap U_\alpha$. So there is a well-defined $s_\alpha \in \omega_{\mathcal{C}/\mathcal{B}}^{\otimes 2}(U_\alpha)$ which equals $t_\alpha^i - v^i$ on $U_\alpha \cap \mathcal{C}_{V^i}$ for each i. Let $s = (s_\alpha)_{\alpha \in \mathfrak{A}}$. One checks easily $\delta s = \sigma$. Thus, by Lemma B.1, $\mathfrak{X}$ has a projective structure.

Assume $g > 1$. Then $H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}^{\otimes 2})$ is trivial (since $\deg \omega_{\mathcal{C}_b} = 2g - 2 > 0$). A similar argument shows that σ is a coboundary. Assume $g = 0$. Then $b \mapsto \dim H^1(\mathcal{C}_b, \omega_{\mathcal{C}_b}^{\otimes 2})$ is still constant since any fiber $\mathcal{C}_b$ is biholomorphic to $\mathbb{P}^1$. Moreover, $\mathcal{C}_b$ clearly has a projective structure. The above argument for $g = 1$ applies to this case. □

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, USA.

E-mail: bin.gui@rutgers.edu   binguimath@gmail.com