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Singularities in global pluripotential theory

– Lectures at Zhejiang University –

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Preface

This book is an extended version of my lecture notes at Zhejiang university. The initial goal was to write a self-contained reference for the participants of the lectures. But I soon realized that many results have never been rigorously proved in any literature. When trying to fix these loose ends, the length of the notes becomes uncontrollable, eventually leading to the current book.

In this book, I would like to present my point of view towards the *global* pluripotential theories. There are three different but interrelated theories which deserve this name. They are

(1) the pluripotential theory on compact Kähler manifolds,
(2) the pluripotential theory on the Berkovich analytification of projective varieties, and
(3) the toric pluripotential theory on toric varieties.

We will begin by explaining the picture in the first case. Let us fix a connected compact Kähler manifold X. The central objects are the *quasi-plurisubharmonic functions* on X.

We are mostly interested in the *singularities* of such functions, that is, the places where a quasi-plurisubharmonic function φ tends to $-\infty$ and how it tends to $-\infty$.

Singularities occur naturally in mathematics. In geometric applications, X should be regarded as the compactified moduli space of certain geometric objects. A Zariski open subset $U \subseteq X$ would parametrize smooth objects. The natural metric on the associated polarizing line bundle is usually smooth only on U, not on X. In case we have suitable positivities, the classical Grauert–Remmert extension theorem allows us to extend the metric outside U, but at the cost of introducing singularities.

The classification of singularities is a huge project. Locally near the singularities we know that quasi-plurisubharmonic functions present very complicated behaviours. There are many local invariants associated with the singularities. The most notable ones are the Lelong numbers and the multiplier ideal sheaves. These invariants only reflect the rough behaviour of a quasi-plurisubharmonic function. As an example, a quasi-plurisubharmonic function with log-log singularities have the same local invariants as a bounded one.

The situation changes drastically in the global setting, namely on compact manifolds. In the global setting, there are three different ways to classify quasi-plurisubharmonic functions according to their singularities:

(1) The singularity type characterizing the singularities up to a bounded term.
(2) The P-singularity type associated with global masses.
(3) The $\mathcal{I}$-singularity type associated with all non-Archimedean data.

The classification becomes rougher and rougher as we go downward. In the first case, we say two quasi-plurisubharmonic functions have the same singularity type if their difference lies in L^∞. The corresponding equivalence class gives us essentially the finest information of the singularities we can expect. The other two relations are more delicate, we will study them in detail in Chapter 6.

A natural ideal to study the singularities would consist of the following steps:

(1) classify the $\mathcal{I}$-singularity types,
(2) classify the P-singularity types within a given $\mathcal{I}$-singularity class, and
(3) classify the singularity types within a given P-equivalence class.

The Step 3 is well-studied in the literature in the last decade under the name of pluripotential theory with prescribed singularities. There are numerous excellent results in this direction. In some sense, this step is already well-understood.

We will give a complete answer to Step 1 in Chapter 7, where we show that $\mathcal{I}$-singularity types can be described very explicitly.

It remains to consider Step 2. This is not an easy task. It is easy to construct examples where a given $\mathcal{I}$-equivalence class consists of a huge amount of P-equivalence classes.

On the other hand, by contrast, in the toric pluripotential theory and non-Archimedean pluripotential theory, Step 2 is essentially trivial: an $\mathcal{I}$-equivalence class consists of a single P-equivalence class.

This apparent anomaly and numerous examples show that in the pluripotential theory on compact Kähler manifolds, certain singularities are pathological. Within each $\mathcal{I}$-equivalence, we could pick up a canonical P-equivalence class, the quasi-plurisubharmonic functions in which are said to be *$\mathcal{I}$-good*. We will study the theory of $\mathcal{I}$-good singularities in Chapter 7. As we will see later on, almost all (if not all) singularities occurring naturally are $\mathcal{I}$-good.

My personal impression is that we are in a situation quite similar to the familiar one in real analysis. There are many non-measurable functions, but in real life, unless you construct a pathological function by force, you only encounter measurable functions. Similarly, although there exist many non-$\mathcal{I}$-good singularities, you would never encounter them in reality!

Having established this general principle, we could content ourselves in the framework of $\mathcal{I}$-good singularities. Then Step 2 is essentially solved, and we have a pretty good understanding of the classification of singularities.

Of course, this classification is a bit abstract. To put it into use, we will introduce two general techniques allowing us to make induction on $\dim X$. For a prime divisor Y in general position, we have the so-called analytic Bertini theorem relation quasi-plurisubharmonic functions on X and on Y. For a non-generic Y, we have the technique of trace operators. These techniques will be explained in Chapter 8.

In the toric situation, these constructions and methods are quite straightforward and are likely known to experts before I entered this field, see Chapter 5 for the toric pluripotential theory on ample line bundles.

The corresponding toric pluripotential theory on big line bundles has never been written down in the literature. A foundational result was proved in my paper on partial Okounkov bodies, which allows us to treat this problem rigorously. We will develop the theory of partial Okounkov bodies in Chapter 10 and the general toric pluripotential theory in Chapter 12.

Finally, we give applications to non-Archimedean pluripotential theory in Chapter 13 based on the theory of test curves developed in Chapter 9.

Minghen Xia
in Hangzhou, March 2024

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Contents

Conventions

In the whole book we adopt the following conventions:

- A complex space is always assumed to be *reduced* and *Hausdorff*.
- A *modification* of a complex space X is proper bimeromorphic morphism $\pi\colon Y \to X$ that is obtained from a finite composition of blow-ups with smooth centers.
- A *subnet* of a net refers to a cofinal subnet.
- A *domain* in $\mathbb{C}^n$ refers to a connected open subset.
- A *submanifold* of a complex manifold means a complex submanifold.

We will use the following notations throughout the book:

- If I is a non-empty set, then $\mathrm{Fin}(I)$ denote the net of finite non-empty subsets of I, ordered by inclusion.
- $\mathrm{dd^c}$ means $(2\pi)^{-1}\mathrm{i}\partial\overline{\partial}$.

Part I
Preliminaries

In this part, we recall a few preliminaries about the notion of plurisubharmonic functions.

Chapter 1
Plurisubharmonic functions

chap:psh

1.1 The definition of plurisubharmonic functions

sec:pshdef

In this section, we recall the notion of plurisubharmonic functions. We will also take care of the 0-dimensional case, which makes a number of induction arguments easier to carry out.

1.1.1 The 1-dimensional case

Let Ω be a domain (a connected non-empty open subset) in $\mathbb{C}$.

def:subhar1

Definition 1.1.1 A *subharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) φ satisfies the *sub-mean value inequality*: for any $a \in \Omega$ and $r > 0$ such that $B(a, r) \Subset \Omega$, we have

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta.$$

We will denote the set of subharmonic functions on Ω as $\mathrm{SH}(\Omega)$.

In fact, for each $a \in \Omega$, in 3, it suffices to require the sub-mean value inequality for all small enough r.

Intuitively, at a specific point $a \in \Omega$, the second condition gives a lower bound of the value of $\varphi(a)$ using the nearby values of φ, while the third condition gives an upper bound. This intuition leads to the following rigidity theorem:

thm:sh_rigid

Theorem 1.1.1 *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\Delta\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a subharmonic function ψ on Ω.*

Moreover, the subharmonic function ψ is unique.

Here in condition 1, $\Delta\varphi$ is the Laplacian in the sense of currents. This is a special case of Theorem 1.1.2 below.

This theorem gives a very useful way to construct subharmonic functions.

1.1.2 The higher dimensional case

We will fix $n \in \mathbb{N}$ and a domain Ω (non-empty connected open subset) in $\mathbb{C}^n$.

def:psh **Definition 1.1.2** When $n \geq 1$, a *plurisubharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) For any complex line $L \subseteq \mathbb{C}^n$ and any connected component U of $L \cap \Omega$, the restriction $\varphi|_U$ is subharmonic.

When $n = 0$, the only domain Ω is the singleton. A *plurisubharmonic function* on Ω is a real-valued function on Ω.

The set of plurisubharmonic functions on Ω is denoted by $\mathrm{PSH}(\Omega)$.

A plurisubharmonic function is also called a psh function for short.

Example 1.1.1 When $n = 0$, we have a canonical bijection $\mathrm{PSH}(\Omega) \cong \mathbb{R}$.

Example 1.1.2 When $n = 1$, we have $\mathrm{PSH}(\Omega) = \mathrm{SH}(\Omega)$.

Similar to Theorem 1.1.1, we have a rigidity theorem for plurisubharmonic functions as well.

thm:psh_rigid **Theorem 1.1.2** *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on Ω.*

Moreover, the plurisubharmonic function ψ is unique.

For the proof, we refer to [GZ17, Proposition 1.43].

Plurisubharmonic functions have nice functorialities.

prop:func_domain **Proposition 1.1.1** *Let $n' \in \mathbb{N}$ and $\Omega' \subseteq \mathbb{C}^{n'}$ be a domain. Given any holomorphic map $f\colon \Omega' \to \Omega$ and any $\varphi \in \mathrm{PSH}(\Omega')$ exactly one of the following cases occurs:*

(1) *$f^*\varphi \equiv -\infty$;*
(2) *$f^*\varphi \in \mathrm{PSH}(\Omega)$.*

We refer to [GZ17, Proposition 1.44] for the proof[1].

For each $n \in \mathbb{N}$, $a \in \mathbb{C}^n$ and $r > 0$, we write

$$B_n(a,r) = \{z \in \mathbb{C}^n : |z-a| < r\}\,.$$

prop:ballpshconvex

Proposition 1.1.2 *Let $\varphi \in \mathrm{PSH}(B_n(a,r_0))$ for some $r_0 > 0$. Then the function*

$$(-\infty, \log r_0) \to \mathbb{R}, \quad \log r \mapsto \sup_{B_n(a,r)} \varphi$$

is convex and increasing.

See [Bou17, Corollary 2.4].

1.1.3 The manifold case

Let X be a complex manifold.

def:pshmfd

Definition 1.1.3 A *plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ if for any $x \in X$, there is an open neighbourhood U of x in X, an integer $n \in \mathbb{N}$, a domain $\Omega \subseteq \mathbb{C}^n$ and a biholomorphic map $F\colon \Omega \to U$ such that $F^*(\varphi|_U) \in \mathrm{PSH}(X,\Omega)$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Example 1.1.3 When X is a domain in $\mathbb{C}^n$, the notions of plurisubharmonic functions in Definition 1.1.3 and in Definition 1.1.2 coincide.

Example 1.1.4 Write $\{X_i\}_{i\in I}$ for the set of connected components of X. Then we have a natural bijection

$$\mathrm{PSH}(X) \cong \prod_{i\in I} \mathrm{PSH}(X_i).$$

Here the product is in the category of sets. In particular, if $X = \varnothing$, then $\mathrm{PSH}(X) = \varnothing$.

This example allows us to reduce to the case of connected manifolds when studying general plurisubharmonic functions.

prop:pullbackpsh

Proposition 1.1.3 *Let Y be another complex manifold and $f\colon Y \to X$ be a holomorphic map. Then for any $\varphi \in \mathrm{PSH}(X)$, exactly one of the following cases occurs:*

(1) *$f^*\varphi$ is identically $-\infty$ on some connected component of Y;*
(2) $f^*\varphi \in \mathrm{PSH}(Y)$.

This proposition follows easily from Proposition 1.1.1. We leave the details to the readers.

Theorem 1.1.2 implies immediately the general form of the rigidity theorem.

[1] We remind the readers that the statement of [GZ17, Proposition 1.44] is flawed.

thm:psh_rigid_gen **Theorem 1.1.3** *Let $\varphi\colon X \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on X.*

Moreover, the plurisubharmonic function ψ is unique.

def:pluripolarsets **Definition 1.1.4** A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighbourhood U of x in X and a function $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{E\cap U} \equiv -\infty.$$

A subset $F \subseteq X$ is *co-pluripolar* if $X \setminus F$ is pluripolar.

prop:pluripolarunion **Proposition 1.1.4** *Let $\{E_i\}_{i\in\mathbb{Z}_{>0}}$ be a sequence of pluripolar sets in X. Then*

$$E \coloneqq \bigcup_{i=1}^{\infty} E_i$$

is pluripolar.

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain. In this case, by Josefson's theorem [GZ17, Corollary 4.41] that we can choose $\psi_i \in \mathrm{PSH}(\Omega)$ such that

$$\psi_i|_{E_i} \equiv -\infty, \quad \psi_i \leq 0$$

for all $i > 0$. After shrinking X, we may guarantee that $\psi_i \in L^1(\Omega)$ for all $i > 0$. After rescaling, we may also assume that $\|\psi_i\|_{L^1} \leq 1$ for all $i > 0$.

We then define

$$\psi = \sum_{i=1}^{\infty} 2^{-i}\psi_i.$$

Then $\psi \in \mathrm{PSH}(X, \theta)$ according to Proposition 1.2.1 and $\psi|_E = -\infty$. □

1.2 Properties of plurisubharmonic functions

In this section, we explore the basic properties of plurisubharmonic functions.

Let X be a complex manifold in this section.

prop:pshfunction_closedseq **Proposition 1.2.1**

(1) *Assume that $\{\varphi_i\}_{i\in I}$ is a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. Then $\sup^*_i \varphi_i \in \mathrm{PSH}(X)$;*
(2) *Assume that $\{\varphi_i\}_{i\in I}$ is a decreasing net in $\mathrm{PSH}(X)$ such that $\lim_{i\in I}\varphi_i$ is not identically $-\infty$ on each connected component of X, then $\lim_{i\in I}\varphi_i \in \mathrm{PSH}(X)$.*

Here sup* denotes the upper semicontinuous regularization of the supremum. When I is a finite family, observe that

$$\sup_{i\in I}{}^{*} \varphi_i = \sup_{i\in I} \varphi_i .$$

When $I = \{1, \dots, m\}$, we write

$$\varphi_1 \vee \cdots \vee \varphi_m \coloneqq \sup_{i\in I} \varphi_i .$$

We refer to [GZ17, Proposition 1.28, Proposition 1.40][2].

prop:Choquet

Proposition 1.2.2 (Choquet's lemma) *Assume that X admits a countable covering by open balls. Assume that $\{\varphi_i\}_{i\in I}$ is a non-empty family in* $\mathrm{PSH}(X)$ *that is locally uniformly bounded from above. There exists a countable subfamily $J \subseteq I$ such that*

$$\sup_{i\in I}{}^{*} \varphi_i = \sup_{j\in J}{}^{*} \varphi_j .$$

See [GZ17, Lemma 4.31] for the proof.

prop:supsupstardiff

Proposition 1.2.3 *Let $\{\varphi_i\}$ be a family in* $\mathrm{PSH}(X)$ *that is locally uniformly bounded from above. Then the set*

$$\left\{ x \in X : \sup_{i\in I} \varphi_i < \sup_{i\in I}{}^{*} \varphi_i \right\}$$

is pluripolar.

See [GZ17, Corollary 4.28].

prop:pshlocLp

Proposition 1.2.4 *Let $\varphi \in$* $\mathrm{PSH}(X)$*, then for any $p \geq 1$, $\varphi \in L^p_{\mathrm{loc}}(X)$.*

See [GZ17, Theorem 1.46, Theorem 1.48].

prop:pshfuncdetdense

Proposition 1.2.5 *Suppose that $\varphi, \psi \in$* $\mathrm{PSH}(X)$*. Assume that there is a dense subset $E \subseteq X$ such that $\varphi|_E \leq \psi|_E$, then $\varphi \leq \psi$.*

Proof The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$.

We may assume that $\varphi|_E = \psi|_E$ after replacing φ by $\varphi \vee \psi$. Then we need to show that

$$\varphi = \psi .$$

It follows from [GZ17, Theorem 4.20] that this holds outside a pluripolar set $Y \subseteq X$. In particular, $\varphi = \psi$ almost everywhere. It follows from the uniqueness statement in Theorem 1.1.3 that $\varphi = \psi$. □

[2] In [GZ17, Proposition 1.28], the second part is only stated for sequences, the net version is obvious using the sub-mean value inequality

thm:GRexten

Theorem 1.2.1 (Grauert–Remmert) *Let Z be an analytic subset in X and $\varphi \in \mathrm{PSH}(X \setminus Z)$. Then function φ admits an extension to $\mathrm{PSH}(X)$ in the following two cases:*

(1) *The set Z has codimension at least 2 everywhere;*
(2) *The set Z has codimension at least 1 everywhere and is locally bounded from above on an open neighbourhood of Z.*

In both cases, the extension is unique.

Proof The extension is unique thanks to Proposition 1.2.5.

(2). The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ and there is a non-zero holomorphic function f vanishing identically on Z. For each $\epsilon > 0$, we claim that the function φ_ϵ defined by

$$\varphi_\epsilon(x) := \begin{cases} \varphi(x) + \epsilon \log |f(x)|^2, & x \in X \setminus Z; \\ -\infty, & x \in Z \end{cases}$$

is plurisubharmonic on X. By Definition 1.1.2, it suffices to verify the case $n = 1$. In this case, we may assume that $Z = \{0\}$, It is clear that $\varphi_\epsilon \in \mathrm{PSH}(X \setminus Z)$. It suffices to verify the sub-mean value inequality at 0, which is immediate.

Next observe that the sequence φ_ϵ is increasing as $\epsilon \searrow 0$ and φ_ϵ is locally uniformly bounded from above. It follows from Proposition 1.2.1 that $\tilde{\varphi} := \sup^*_{\epsilon>0} \varphi_\epsilon \in \mathrm{PSH}(X)$. Moreover, $\tilde{\varphi}$ clearly extends φ.

(1). It suffices to verify that φ is locally bounded from above near each point of Z. The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$.

Assume that our assertion fails. Take $z \in Z$ so that there exists a sequence $(x_j)_j$ in $X \setminus Z$ such that

$$\lim_{j\to\infty} \varphi(x_j) = \infty.$$

Since Z has codimension at least 2, we could take a complex line L passing through z and intersects Z only on a discrete set. After shrinking X, we may assume that

$$L \cap Z = \{z\}.$$

Take an open ball $B_n(z, r) \Subset X$. After adding a constant to φ, we may guarantee that $\varphi < 0$ on $L \cap \partial B_n(z, r)$. Since φ is upper semi-continuous, we could find an open neighbourhood U of $L \cap \partial B_n(z, r)$ such that

$$\varphi|_U < 0.$$

For each $j \geq 1$, take a complex line L_j passing through x_j such that $L_j \to L$ as $j \to \infty$. Here the convergence is in the obvious sense. Then for large enough j, we know have

$$L_j \cap \partial B_n(z, r) \subseteq U.$$

It follows from the sub-mean value inequality that $\varphi(x_j) < 0$ for large enough j, which is a contradiction. □

lma:invariantpshfunfinite **Lemma 1.2.1** *Let $\varphi \in \mathrm{PSH}((\Delta^*)^n)$ be an $(S^1)^n$-invariant psh function. Then φ is finite everywhere.*

Proof It clearly suffices to handle the case $n = 1$. In this case, by [HK76, Theorem 2.12], the map

$$\log r \mapsto \int_0^1 \varphi(r\exp(2\pi \mathrm{i}\theta))\,\mathrm{d}\theta = \varphi(r)$$

is a convex function of $\log r$. So the set $\{r \in (0,1) : \varphi(r) = -\infty\}$ is convex. But φ is almost everywhere finite by Proposition 1.2.4. Since φ is S^1-invariant, $\varphi|_{(0,1)}$ is almost everywhere finite. It follows from the convexity that it is everywhere finite. □

cor:L1limipp **Corollary 1.2.1** *Let $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{PSH}(X)$ such that $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi \in \mathrm{PSH}(X)$. Then the set*

$$\left\{x \in X : \varphi(x) \neq \varlimsup_{j\to\infty} \varphi_j(x)\right\}$$

is pluripolar.

Proof We first observe that $(\varphi_j)_j$ is locally uniformly bounded from above. This follows from [GZ17, Exercise 1.20].

For each $j \geq 1$, let

$$\psi_j = \sup_{k\geq j}{}^* \varphi_k.$$

Then $\psi_j \in \mathrm{PSH}(X)$ by Proposition 1.2.1. Moreover, $(\psi_j)_j$ is a decreasing sequence and $\psi_j \geq \varphi_j$ for all j. So by Proposition 1.2.1 again, $\psi \coloneqq \inf_j \psi_j \in \mathrm{PSH}(X)$. On the other hand, by Proposition 1.2.3, there is a pluripolar set $Z \subseteq X$ such that for any $x \in X \setminus Z$, we have $\psi(x) = \varlimsup_j \varphi_j(x)$. Since $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi$, we can find a set $Y \subseteq X$ with zero Lebesgue measure such that $\varphi_j(x) \to \varphi(x)$ for all $x \in X \setminus Y$.

In particular, for any $x \in X \setminus (Y \cup Z)$, we have

$$\psi(x) = \varphi(x).$$

But thanks to Proposition 1.2.5, the equality holds everywhere. Therefore, for all $x \in X \setminus Z$,

$$\varphi(x) = \varlimsup_{j\to\infty} \varphi_j(x).$$

prop:Kis **Proposition 1.2.6 (Kiselman's principle)** *Let $\Omega \subseteq \mathbb{C}^m \times \mathbb{C}^n$ be a pseudoconvex domain. Assume that for each $z \in \mathbb{C}^m$, the set*

$$\Omega_z \coloneqq \{w \in \mathbb{C}^n : (z,w) \in \Omega\}$$

has the form $E + \mathrm{i}\mathbb{R}^n$, where $E \subseteq \mathbb{R}^n$ is a subset. Let $\varphi \in \mathrm{PSH}(\Omega)$, assume that φ is independent of the imaginary part of the variable in $\mathbb{C}^n$. Let Ω' be the projection of Ω to $\mathbb{C}^m$. Define $\psi\colon \Omega' \to [-\infty, \infty)$ as follows:

$$\psi(z) = \inf_{w\in\Omega_z} \varphi(z,w).$$

Then either $\psi \equiv -\infty$ *or* $\psi \in \mathrm{PSH}(\Omega')$.

DemBook
See [Dem12b, Theorem 7.5].

1.3 Plurifine topology

1.3.1 Plurifine topology on domains

Let $\Omega \subseteq \mathbb{C}^n$ $(n \in \mathbb{N})$ be a domain.

def:pftopologydomain **Definition 1.3.1** The *plurifine topology* on Ω is the weakest topology making all finite psh functions on Ω continuous.

We want to distinguish the Euclidean topology from the plurifine topology. In the whole book, topological notions without adjectives refer to those with respect to the Euclidean topology. We always include the word $\mathcal{F}$ in order to denote those with respect to the plurifine topology. For example, we will say $\mathcal{F}$-open subset, $\mathcal{F}$-neighbourhood, $\mathcal{F}$-closure, etc. The $\mathcal{F}$-closure of a set $E \subseteq \Omega$ will be denoted by $\bar{E}^{\mathcal{F}}$.

A priori, we should include Ω into the notations as well, but as we will see shortly in Corollary 1.3.1, this is usually unnecessary.

prop:pf_finer **Proposition 1.3.1** *The plurifine topology is finer than the Euclidean topology.*

Proof It suffices to show that the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ is $\mathcal{F}$-open. This follows from the observation that this set can be written as

$$\{\psi < 0\} \text{ with } \psi(z) := (\log|z|) \vee (-1).$$

Definition 1.3.2 A subset $E \subseteq \Omega$ is *thin* at $x \in \Omega$ if one of the following conditions holds:

(1) $x \notin \bar{E}$;
(2) $x \in \bar{E}$ and there is an open neighbourhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y\to x, y\in E} \varphi(y) < \varphi(x).$$

We say E is *thin* if it is thin at all $x \in \Omega$.

In the second case, the function φ can be very much improved.

prop:BTthin **Proposition 1.3.2 (Bedford–Taylor)** *Consider a set* $E \subseteq \Omega$ *and* $x \in \bar{E}$. *Assume that* E *is thin at* x, *then there is* $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ *satisfying the following properties:*

(1) *φ is locally bounded outside a neighbourhood of x;*
(2) $\underline{\varphi(x)} > -\infty$*;*
(3) $\overline{\lim}_{y\to x, y\in E}\,\varphi(y) = -\infty$.

Proof By definition, there is an open neighbourhood $U \subseteq \Omega$ of x and $\psi \in \mathrm{PSH}(U)$ such that

$$\overline{\lim_{y\to x, y\in E}}\,\psi(y) < \psi(x).$$

Without loss of generality, we may assume that $x = 0$, U is the unit ball in $\mathbb{C}^n$, $\psi < 0$ and $\psi|_{U\cap E} < -1$, while $\psi(0) = -\eta > -1$.

As ψ is upper semicontinuous, we may choose $\delta_j > 0$ for all large enough $j \in \mathbb{Z}_{>0}$ such that $\psi(y) < -\eta + 2^{-j-1}$ when $y \in \mathbb{C}^n$ satisfies $|y| < \delta_j$. Now we let

$$\varphi_j(z) := \begin{cases} \left(\dfrac{2^{-j-1}}{\log|\delta_j|}\log|z|\right) \vee \left(\psi(z) + 2^{-j}\right), & \text{if } |z| < \delta_j, \\ \dfrac{2^{-j-1}}{\log|\delta_j|}\log|z|, & \text{if } |z| \geq \delta_j. \end{cases}$$

Then $\varphi_j \in \mathrm{PSH}(\mathbb{C}^n)$ and $\varphi_j(0) = 2^{-j}$. It suffices to take $\varphi = \sum_j \varphi_j$.

thm:Cartan

Theorem 1.3.1 (Cartan) *Consider $x \in \Omega$ and a set $E \subseteq \Omega$. Assume that $x \in E$. Then the following are equivalent:*

(1) *E is an $\mathcal{F}$-neighbourhood of x;*
(2) *$\Omega \setminus E$ is thin at x.*

Proof (2) $\Longrightarrow$ (1). We may assume that $x \in \overline{\Omega \setminus E}$. Otherwise, our assertion follows from Proposition 1.3.1.

By Proposition 1.3.2, there is $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ and an open neighbourhood $U \subseteq \Omega$ of x such that

$$\varphi(x) > \sup_{y\in U\cap(\Omega\setminus E)} \varphi(y) =: \lambda.$$

Let $F = \{y \in \Omega : \varphi(y) > \lambda\}$. Then $x \in F$ and F is $\mathcal{F}$-open. Moreover, $U \cap F \subseteq E$. By Proposition 1.3.1, we conclude (1).

(1) $\Longrightarrow$ (2). We may always replace E by smaller $\mathcal{F}$-neighbourhoods of x. In particular, we may assume that E has the following form

$$\{y \in U : \varphi_1(y) > \lambda_1, \dots, \varphi_m(y) > \lambda_m\},$$

where $U \subseteq \Omega$ is an open neighbourhood of x, $\varphi_1, \dots, \varphi_m$ are finite psh functions on Ω and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since a finite union of thin sets is still thin, we may assume that $m = 1$. In this case, $\Omega \setminus E$ is clearly thin at x. □

thm:pf_basis

Theorem 1.3.2 *A basis of the plurifine topology on Ω is given by sets of the following from*

$$\{x \in U : \varphi(x) > 0\}, \tag{1.1}$$

{eq:basis_fine}

where $U \subseteq \Omega$ is an open subset and $\varphi \in \mathrm{PSH}(U)$.

Proof We first show that sets of the form (1.1) are $\mathcal{F}$-open. By Theorem 1.3.1, it suffices to show its complement in Ω is thin at x, which is obvious.

Now consider $x \in \Omega$ and an $\mathcal{F}$-open neighbourhood $V \subseteq \Omega$ of x. We want to find a set of the form (1.1) contained in V and containing x.

Write $E = \Omega \backslash V$. In case $a \in \operatorname{Int} V$, there is nothing to prove. So we may assume that $a \in \bar{E}$. By Theorem 1.3.1, E is thin at x. By definition, there is an open neighbourhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y \to x, y \in E \cap U} \varphi(y) < \varphi(x).$$

We may assume that $\varphi|_{E \cap U} \leq 0 < \varphi(x)$, Then the set $\{y \in U : \varphi(y) > 0\}$ suffices for our purpose. □

cor:pf_compatible

Corollary 1.3.1 *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n$ be two non-empty open subsets. Then the plurifine topology on Ω_1 is the same as the subspace topology induced from the plurifine topology on Ω_2.*

Corollary 1.3.2 *Let L be an affine subspace of $\mathbb{C}^n$, then the plurifine topology on L is the same as the subspace topology induced from the plurifine topology on $\mathbb{C}^n$.*

Proof We may assume that $L = \mathbb{C}^k \times \{0\}$ for some $k \leq n$. We write the coordinate z on $\mathbb{C}^n$ as (z', z'') with $z \in \mathbb{C}^k$ and $z'' \in \mathbb{C}^{n-k}$.

Consider an $\mathcal{F}$-open set $U \subseteq \mathbb{C}^n$ and $x = (x', 0) \in U \cap L$. We want to show that $U \cap L$ (identified with a subset of $\mathbb{C}^k$) is an $\mathcal{F}$-neighbourhood of x' in L. By Theorem 1.3.2, we may assume that there are open subsets $U' \subseteq \mathbb{C}^k$ containing x' and $U'' \subseteq \mathbb{C}^{n-k}$ containing 0 together with a psh function ψ on $U' \times U''$ such that

$$x \in \{(z', z'') \in U' \times U'' : \psi(z', z'') > 0\} \subseteq \Omega.$$

It follows that

$$x' \in \{z' \in U' : \psi(z', 0) > 0\} \subseteq U \cap L.$$

Conversely, if $U \subseteq \mathbb{C}^k$ is an $\mathcal{F}$-open subset, we claim that $U \times \mathbb{C}^{n-k}$ is $\mathcal{F}$-open in $\mathbb{C}^n$. In fact, suppose that $(x', x'') \in U \times \mathbb{C}^{n-k}$. By Theorem 1.3.1, we can find an open neighbourhood $V \subseteq \mathbb{C}^k$ of x' and a psh function φ on U such that

$$x' \in \{y \in U : \varphi(y) > 0\} \subseteq U.$$

We define $\psi(z', z'') \coloneqq \varphi(z')$. Then

$$(x', x'') \in \{y \in U \times \mathbb{C}^n : \psi(y) > 0\} \subseteq U \times \mathbb{C}^{n-k}.$$

cor:compactnhformbase

Corollary 1.3.3 *Let $\Omega \subseteq \mathbb{C}^n$ be an $\mathcal{F}$-open subset and $x \in \Omega$. Then x has a compact $\mathcal{F}$-neighbourhood contained in Ω.*

Proof By Theorem 1.3.2, we may assume that there is a locally compact open set $U \subseteq \mathbb{C}^n$ and a psh function φ on U such that $\Omega = \{y \in U : \varphi(y) > 0\}$.

Take a compact neighbourhood K of x in U. Now $\{y \in K : \varphi(y) \geq \varphi(x)/2\}$ is a compact $\mathcal{F}$-neighbourhood of x contained in Ω. □

cor:holomappfcont

Corollary 1.3.4 *Let $\Omega \in \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^{n'}$ be two domains and $F\colon \Omega' \to \Omega$ be a surjective holomorphic map. Then F is continuous with respect to the plurifine topology.*

Proof It suffices to show that the inverse image $F^{-1}(U)$ of each plurifine open subset $U \subseteq \Omega$ is plurifine open. By Theorem 1.3.2, after possibly shrinking Ω and Ω', we may assume that U has the form $\{x \in \Omega : \psi(x) > 0\}$, where $\psi \in \mathrm{PSH}(\Omega)$. Since $F^*\psi \in \mathrm{PSH}(\Omega')$ by Proposition 1.1.3, we find that

$$F^{-1}(U) = \{y \in \Omega' : F^*\psi(y) > 0\}$$

is a plurifine open subset. □

1.3.2 Plurifine topology on manifolds

Let X be a complex manifold.

def:pftopologygeneral

Definition 1.3.3 The *plurifine topology* on X is the topology with a basis consisting of sets of the form $F^{-1}(V)$, where $U \subseteq X$ is an open subset and $F\colon U \to \Omega$ is a biholomorphic morphism with $\Omega \subseteq \mathbb{C}^n$ for some $n \in \mathbb{N}$ and $V \subseteq \Omega$ is a plurifine open subset.

It follows from Corollary 1.3.4 that the plurifine topologies on domains defined in Definition 1.3.3 and in Definition 1.3.1 coincide.

prop:pshfunFcont

Proposition 1.3.3 *Let $\varphi \in \mathrm{QPSH}(X)$, then $\varphi|_{\{\varphi\neq-\infty\}}$ is $\mathcal{F}$-continuous.*

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain and $\varphi = \psi + g$, where $\psi \in \mathrm{PSH}(X)$ and $g \in C^\infty(X)$ and $|g| \leq C$ for some $C > 0$. Take an open interval $(a, b) \subseteq \mathbb{R}$, it suffices to show that

$$U \coloneqq \{x \in X : a < \varphi(x) < b\} = \{x \in X : a - g(x) < \psi(x) < b - g(x)\}$$

is $\mathcal{F}$-open. Take $x \in U$, we can find an open neighbourhood V of x in U such that

$$\sup_{y\in V}\,(a - g(y)) < \psi(x) < \inf_{y\in V}\,(b - g(y))\,.$$

Therefore,

$$\left\{z \in V : \sup_{y\in V}\,(a - g(y)) < \psi(z) < \inf_{y\in V}\,(b - g(y))\right\}$$

is an $\mathcal{F}$-open neighbourhood of z in U. We conclude that U is $\mathcal{F}$-open. □

ma:pshfunfinitelocuspfdense **Lemma 1.3.1** *Let $Z \subseteq X$ be a pluripolar subset. Then*

$$\overline{X \setminus Z}^{\mathcal{F}} = X.$$

Proof The problem is local, so we may assume that X be a domain in $\mathbb{C}^n$ and $Z = \{\varphi = -\infty\}$ for some $\varphi \in \mathrm{PSH}(X)$. We need to show that $\{\varphi > -\infty\}$ is $\mathcal{F}$-dense.

Let $x \in X$ such that $\varphi(x) = -\infty$ and $U \subseteq X$ be a plurifine open neighbourhood of x in X. We need to show that $U \cap \{\varphi > -\infty\} \neq \emptyset$.

Thanks to Theorem 1.3.2, after shrinking U, we may assume that there is $\psi \in \mathrm{PSH}(X)$ such that $U = \{\psi > 0\}$. Observe that U is not a pluripolar set: otherwise, $\psi \leq 0$ almost everywhere hence everywhere by Proposition 1.2.5. So $\varphi|_U \not\equiv -\infty$. We conclude. □

r:diffsupinfindeppluripolar **Corollary 1.3.5** *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Set*

$$W = \{x \in X : \min\{\varphi(x), \psi(x)\} = -\infty\}$$

Then for any pluripolar set $Z \subseteq X$, we have

$$\sup_{X\setminus W}(\varphi - \psi) = \sup_{X\setminus W\cup Z}(\varphi - \psi), \quad \inf_{X\setminus W}(\varphi - \psi) = \inf_{X\setminus W\cup Z}(\varphi - \psi).$$

Proof This is an immediate consequence of Lemma 1.3.1 and Proposition 1.3.3. □

1.4 Lelong numbers and multiplier ideal sheaves

There are two useful characterizations of the local singularities of plurisubharmonic functions. We will apply both of them in the sequel.

Let X be a complex manifold.

Definition 1.4.1 Let $\varphi \in \mathrm{PSH}(X)$ and $x \in X$. The *Lelong number* $\nu(\varphi, x)$ of φ at x is defined as follows: take an open neighbourhood U of x in X and a biholomorphic map $F\colon U \to \Omega$, where Ω is a domain in $\mathbb{C}^n$. Then we define

$$\nu(\varphi, x) := \sup\left\{\gamma \in \mathbb{R}_{\geq 0} : \varphi|_U(F^{-1}(y)) \leq \gamma \log|y - F(x)|^2 + O(1) \text{ as } y \to F(x)\right\}. \tag{1.2}$$ {eq:nuvarphix}

Observe that $\nu(\varphi, x)$ does not depend on the choice of F. Furthermore, it follows from Proposition 1.4.1 below that the supremum in (1.2) is a maximum.

Remark 1.4.1 Our definition of the Lelong number is not standard. It differs from the standard definition by a factor of 2.

prop:Lelongreform **Proposition 1.4.1** *Let $\varphi \in \mathrm{PSH}(B_n(0,1))$. Then*

$$\nu(\varphi, 0) = \lim_{r \to 0+} \frac{\sup_{B_n(0,r)} \varphi}{\log r^2} \in [0, \infty). \tag{1.3}$$ {eq:Lelongnewdef}

Proof It follows from Proposition 1.1.2 that the limit in (1.3) exists and is finite. We shall denote the limit by $\nu'(\varphi, 0)$ for the time being.

We first observe that by (1.3),

$$\varphi(x) \leq \nu'(\varphi, 0) \log |x|^2 + \sup_{B_n(0,1)} \varphi \tag{1.4}$$

{eq:varphixlocalupperbd}

when $x \in B_n(0, 1)$. In particular, $\nu(\varphi, x) \geq \nu'(\varphi, 0)$.

In order to argue the reverse inequality, we may assume that $\nu(\varphi, x) > 0$.

Next observe that by (1.2), for each small enough $\epsilon > 0$, we can find $r_0 \in (0, 1)$ and $C > 0$ so that for all $x \in B_n(0, r_0)$, we have

$$\varphi(x) \leq (\nu(\varphi, 0) - \epsilon) \log |x|^2 + C.$$

It follows that $\nu'(\varphi, 0) \geq \nu(\varphi, 0) - \epsilon$. Letting $\epsilon \to 0+$, we conclude. □

We recall Siu's semicontinuity theorem.

thm:Siusemi

Theorem 1.4.1 *Let $\varphi \in \mathrm{PSH}(X)$, then the map $X \ni x \mapsto \nu(\varphi, x)$ is upper semi-continuous with respect to the Zariski topology.*

For an elegant proof we refer to Dem12 [Dem12a, Theorem 2.10].

prop:Lelongmax

Proposition 1.4.2 *Let $\varphi, \psi \in \mathrm{PSH}(X)$, $\lambda \in \mathbb{R}_{>0}$ and $x \in X$, then*

$$\begin{aligned}
\nu(\varphi \vee \psi, x) &= \min\{\nu(\varphi, x), \nu(\psi, x)\}, \\
\nu(\varphi + \psi, x) &= \nu(\varphi, x) + \nu(\psi, x), \\
\nu(\lambda\varphi, x) &= \lambda\nu(\varphi, x).
\end{aligned}$$

Proof All properties are local, so we may assume that $X = B_n(0, 1)$ for some $n \in \mathbb{N}$. All properties follow directly from Proposition 1.4.1. □

cor:supsLelong

Corollary 1.4.1 *Let $(\varphi_i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}(X)$ uniformly bounded from above and $x \in X$, then*

$$\nu\left(\sup_{i\in I}{}^* \varphi_i, x\right) = \inf_{i\in I} \nu(\varphi_i, x).$$

Proof We observe that the $\leq$ inequality. It remains to argue the reverse inequality.

It follows from Proposition 1.2.2 that we may assume that I is countable. When I is finite, this is already proved in Proposition 1.4.2. Otherwise, we may further assume that $I = \mathbb{Z}_{>0}$. Thanks to Proposition 1.4.2, we may further assume that $(\varphi_i)_{i\in\mathbb{Z}_{>0}}$ is an increasing sequence. Furthermore, since the problem is local, we may assume that $X = B_n(0, 1)$ for some $n \in \mathbb{N}$. In this case, by (1.4), we have

$$\varphi_i(x) \leq \nu(\varphi_i, 0) \log |x|^2 + C$$

for all $x \in B_n(0, 1)$ and all $i \geq 1$ and C is a constant independent of i. In particular, thanks to Proposition 1.2.3, for almost all $x \in B_n(0, 1)$, we have

$$\varphi(x) \leq \lim_{i\to\infty} \nu(\varphi_i, 0) \log |x|^2 + C.$$

Thanks of Proposition 1.2.5, the same holds for all x and hence

$$\nu(\sup_{i\in\mathbb{Z}_{>0}}{}^* \varphi_i, x) \geq \lim_{i\to\infty} \nu(\varphi_i, x).$$

We conclude. □

Definition 1.4.2 Let $F \subseteq X$ be an analytic subset. Then we define the generic Lelong number of φ along F as

$$\nu(\varphi, F) := \min_{x\in F} \nu(\varphi, x).$$

Note that the minimum is obtained by Theorem 1.4.1.

Definition 1.4.3 Let $\varphi \in \mathrm{PSH}(X)$. Let E be a prime divisor over X (see Definition B.1.1). Take a proper bimeromorphic morphism $\pi : Y \to X$ from a complex manifold Y such that E is a prime divisor on Y, then we define the *generic Lelong number* of φ along E as

$$\nu(\varphi, E) := \nu(\pi^*\varphi, E).$$

It follows from Theorem 1.4.1 that $\nu(\varphi, E)$ does not depend on the choice of π.

Definition 1.4.4 Let $\varphi \in \mathrm{PSH}(X)$, the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ of φ is by definition the ideal sheaf given by

$$\Gamma(U, \mathcal{I}(\varphi)) = \left\{ f \in \mathcal{O}_X(U) : |f|^2 \exp(-\varphi) \in L^1_{\mathrm{loc}}(U) \right\}$$

for any open subset $U \subseteq X$.

Remark 1.4.2 This definition is different from a few standard references, where instead of $\exp(-\varphi)$, they use 2φ. The conventions adopted in the current book is the most convenient one as far as the author knows. It simplifies a number of formulae.

Proposition 1.4.3 (Nadel) *Let* $\varphi \in \mathrm{PSH}(X)$. *Then* $\mathcal{I}(\varphi)$ *is coherent.*

Dem12
See [Dem12a, Proposition 5.7].

thm:multipsubadd **Theorem 1.4.2** *Let* $\varphi, \psi \in \mathrm{PSH}(X)$, *then*

$$\mathcal{I}(\varphi + \psi) \subseteq \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi).$$

Dem12
See [Dem12a, Theorem 14.2].

The two invariants are related by the following simple result:

prop:Lelongnumfrommis **Proposition 1.4.4** *Let* $\varphi \in \mathrm{PSH}(X)$ *and* E *be a prime divisor over* X. *Then*

$$\nu(\varphi, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathcal{I}(k\varphi).$$

See [DX21, Proposition 2.14].

Also observe the following simple lemma:

lma:blowupLelong

Lemma 1.4.1 *Let $x \in X$ and $\varphi \in \mathrm{PSH}(X)$. Let $\pi \colon Y \to X$ be the blow-up of X at x with exceptional divisor E. Then*

$$\nu(\varphi, x) = \nu(\varphi, E),$$

See [Bou02a, Corollaire 1.1.8].

Conversely, the information of the generic Lelong numbers determines the multiplier ideal sheaves:

thm:valuativemulti

Theorem 1.4.3 *Let $\varphi \in \mathrm{PSH}(X)$. Let $x \in X$ and $f \in \mathcal{O}_{X,x}$. Then the following are equivalent:*

(1) *$f \in \mathcal{I}(\varphi)_x$;*

(2) *there exists $\epsilon > 0$ such that for any prime divisor E over X such that x is contained in the center of E on X, we have*

$$\mathrm{ord}_E(f) \geq (1+\epsilon)\nu(\varphi, E) - \frac{1}{2}A_X(E).$$

Here A_X denotes the log discrepancy. We refer to [Bou17, Corollary 10.18] for the proof and the precise definition of A_X.

thm:stongopen

Theorem 1.4.4 (Guan–Zhou) *Let $\varphi, \psi_j \in \mathrm{PSH}(X)$ ($j \in \mathbb{Z}_{>0}$) such that ψ_j is an increasing sequence converging to φ almost everywhere. Then for any $x \in X$, the germs satisfy*

$$\mathcal{I}(\psi_j)_x = \mathcal{I}(\varphi)_x$$

when j is large enough.

See [GZ15, Hie14] for the proof.

prop:pull-backmis

Proposition 1.4.5 *Let $\pi \colon Y \to X$ be a smooth morphism between complex manifolds. Assume that $\varphi \in \mathrm{PSH}(X)$, then*

$$\mathcal{I}(\pi^*\varphi) = \pi^*\mathcal{I}(\varphi).$$

Proof It follows from [Gro60, Théorème 3.10] that locally π can be written as the composition of an étale morphism and a projection. It suffices to handle the two cases separately.

Recall that in the complex analytic setting, an étale morphism is locally biholomorphic, so there is nothing to prove in this case.

Next, assume that $Y = X \times U$, where $U \subseteq \mathbb{C}^n$ is a domain and π is the natural projection. It follows from Fubini's theorem that

$$\mathcal{I}(\pi^*\varphi) \subseteq \pi^*\mathcal{I}(\varphi).$$

The reverse inequality is proved in Dem12 [Dem12a, Proposition 14.3][3]. □

def:restidealsheaf **Definition 1.4.5** Given a coherent ideal sheaf $\mathcal{I}$ on X, the *restriction* $\mathrm{Res}_Y\,\mathcal{I}$ is the inverse image ideal sheaf given by

$$\mathrm{Res}_Y\,\mathcal{I} := \mathcal{I}/(\mathcal{I}\cap\mathcal{I}_Y), \tag{1.5}$$ {eq:RestI}

where $\mathcal{I}_Y$ is the ideal sheaf defining Y.

In the literature, it is common to denote this sheaf by the misleading notation $\mathcal{I}|_Y$.
There is a natural morphism

$$i_Y^*\mathcal{I} = \mathcal{I}/(\mathcal{I}\cdot\mathcal{I}_Y) \to \mathrm{Res}_Y\,\mathcal{I}, \tag{1.6}$$ {eq:pullbacktoinverimage}

where $i_Y : Y \to X$ is the inclusion.

thm:OT **Theorem 1.4.5 (Ohsawa–Takegoshi)** *Let Y be a submanifold of X and $\varphi \in \mathrm{PSH}(X)$. Assume that $\varphi|_Y \not\equiv -\infty$, then*

$$\mathcal{I}(\varphi|_Y) \subseteq \mathrm{Res}_Y\,\mathcal{I}(\varphi).$$

See Dem12 [Dem12a, Theorem 14.1].

1.5 Quasi-plurisubharmonic functions

In practice, it is important to consider a variant of plurisubharmonic functions. We will fix a complex manifold X together with a closed real smooth $(1,1)$-form θ on X.

Definition 1.5.1 A *θ-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that for each $x \in X$ and each open neighbourhood U of x in X satisfying the condition that $\theta = \mathrm{dd^c} g$ for some smooth function g on U, we have $g+\varphi|_U \in \mathrm{PSH}(U)$. The set of θ-psh functions on X is denoted by $\mathrm{PSH}(X,\theta)$.

A *quasi-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that there exists a smooth closed real $(1,1)$-form θ' on X such that $\varphi \in \mathrm{PSH}(X,\theta')$. The set of quasi-plurisubharmonic functions on X is denoted by $\mathrm{QPSH}(X)$.

There is a natural non-strict partial order on $\mathrm{QPSH}(X)$ defined as follows:

def:parorder **Definition 1.5.2** Assume that X is compact. Given $\varphi,\psi \in \mathrm{QPSH}(X)$, we say that φ is *more singular* than ψ and write $\varphi \preceq \psi$ if there is $C \in \mathbb{R}$ such that $\varphi \leq \psi + C$. We also say ψ is less singular than φ and write $\psi \succeq \varphi$.

In case $\varphi \preceq \psi$ and $\psi \preceq \varphi$, we say φ and ψ has the same singularity types. We write $\varphi \sim \psi$ in this case.

[3] In Dem12 [Dem12a, Proposition 14.3], Demailly used the highly non-standard notation $f^*\mathcal{I}(\varphi)$ to denote the image of $f^*\mathcal{I}(\varphi) \to \mathcal{O}_X$.

Remark 1.5.1 The proceeding results concerning plurisubharmonic functions can be extended *mutatis mutandis* to quasi-plurisubharmonic functions. We will apply these extensions without further explanations.

prop:L1compa

Proposition 1.5.1 *Let θ be a closed real smooth $(1,1)$-form on X. Then for any $a, b \in \mathbb{R}$, $a \leq b$, the set*

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_X \varphi \in [a,b]\right\}$$

is compact with respect to the L^1-topology. Moreover, $\varphi \mapsto \sup_X \varphi$ is L^1-continuous for $\varphi \in \mathrm{PSH}(X,\theta)$.

This is an immediate consequence of [GZ17, Proposition 8.5, Exercise 1.20].

prop:Lelongnumberupperbound

Proposition 1.5.2 *Let θ be a closed real smooth $(1,1)$-form on X and E be a prime divisor over E. Then*

$$\sup\{\nu(\varphi, E) : \varphi \in \mathrm{PSH}(X,\theta)\} < \infty.$$

Proof It follows from the proof of Corollary 1.4.1 that $\nu(\bullet, E)$ is upper semi-continuous with respect to the L^1-topology on $\mathrm{PSH}(X,\theta)$. Thus, the desired upper bound follows from Proposition 1.5.1. □

prop:PSHpullbij

Proposition 1.5.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Let θ be a closed real smooth $(1,1)$-form on X. Then the pull-back gives a bijection*

$$\pi^*\colon \mathrm{PSH}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}(Y,\pi^*\theta).$$

This follows from a more general result Theorem B.1.1.

1.6 Analytic singularities

def:neatanasing

Definition 1.6.1 We say $\varphi \in \mathrm{QPSH}(X)$ has *analytic singularities* if for each $x \in X$, we can find an open neighbourhood U of x such that $\varphi|_U$ has the form:

$$c\log(|f_1|^2 + \cdots + |f_N|^2) + R, \tag{1.7}$$

{eq:anasinglocal}

where $f_1, \ldots, f_N$ are holomorphic functions on U, $c \in \mathbb{Q}_{>0}$ and R is a bounded function on U.

When R can be taken to be smooth, we say φ has *neat analytic singularities*.

Suppose that there is a coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ on X such that we can choose U so that the $f_1, \ldots, f_N$ can be chosen as the generators of $\Gamma(U, \mathcal{I})$ and c is independent of the choice of U, we say φ has analytic singularities of *type* $(c, \mathcal{I})$.

Each potential with analytic singularities has a type. We refer to [Bou02a] and [Bou02b] for the details.

prop:analysingclosed

Proposition 1.6.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$ be potentials with analytic singularities, then so are $\lambda\varphi$ ($\lambda \in \mathbb{Q}_{>0}$), $\varphi + \psi$ and $\varphi \vee \psi$.*

Proof The $\lambda\varphi$ assertion is trivial. The $\vee$ assertion is proved in [Dem15, Proposition 4.1.8]. The addition assertion is easy and is left to the readers. □

Definition 1.6.2 Let D be an effective $\mathbb{Q}$-divisor on X. We say $\varphi \in \mathrm{QPSH}(X)$ has *log singularities* (along D) on X if for each $x \in X$, there is an open neighbourhood U of x such that

(1) $D|_U$ has finitely many irreducible components and can be written as

$$D|_U = \sum_{i=1}^{N} a_i D_i$$

with D_i being prime divisors on D, $a_i \in \mathbb{Q}_{>0}$ and there is a holomorphic function s_i on U defining D_i, and

(2) we have

$$\varphi|_U = a_i \sum_i \log|s_i|^2 + R, \tag{1.8}$$

{eq:logsingreminder}

where R is a bounded function on U.

By Proposition 1.6.1, φ has analytic singularities.

lma:logsingrem

Lemma 1.6.1 *Suppose that θ is a closed smooth real $(1,1)$-form on X, a compact Kähler manifold and $\varphi \in \mathrm{PSH}(X, \theta)$. Suppose that φ has log singularities along an effective $\mathbb{Q}$-divisor D on X. Then the cohomology class $[\theta] - [D]$ is nef.*

Moreover, if in addition θ_φ is a Kähler current, then the cohomology class $[\theta] - [D]$ is ample.

Proof The first assertion follows immediately from the fact that R in (1.8) has bounded coefficients.

The second assertion follows immediately from the first. □

Proposition 1.6.2 *Let $\pi : Y \to X$ be a proper bimeromorphic morphism from a complex manifold Y. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities (resp. has log singularities along an effective $\mathbb{Q}$-divisor D). Then $\pi^*\varphi$ has analytic singularities (resp. has log singularities along π^*D).*

thm:resolvelogsing

Theorem 1.6.1 *Assume that X is compact. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then there is a modification $\pi : Y \to X$ such that $\pi^*\varphi$ has log singularities.*

For a proof, we refer to the arguments on [MM07, Page 104].

def:quasiequsing

Definition 1.6.3 Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. A sequence $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathrm{QPSH}(X)$ is *quasi-equisingular approximation* of φ if

(1) φ_j has analytic singularities for each j;
(2) φ_j is decreasing with limit φ;
(3) there is a decreasing sequence $\epsilon_j \geq 0$ with limit 0 and a Kähler form ω on X such that $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$;
(4) for each $\lambda' > \lambda > 0$, there is $j > 0$ such that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

We also say θ_{φ_j} is a quasi-equisingular approximation of θ_φ.

def:analy-sing

Definition 1.6.4 Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an analytic coherent ideal sheaf and $c \in \mathbb{Q}_{>0}$. A function $\varphi \in \mathrm{QPSH}(X)$ is said to have *gentle analytic singularities* (of type $(c,\mathcal{I})$) if

(1) φ has analytic singularities of type $(c,\mathcal{I})$,
(2) $\mathrm{e}^{\varphi/c}\colon X \to \mathbb{R}_{\geq 0}$ is a smooth function, and
(3) there is a proper bimeromorphic morphism $\pi\colon \tilde{X} \to X$ from a Kähler manifold $\tilde{X}$ and an effective $\mathbb{Z}$-divisor D on $\tilde{X}$ such that one can write $\pi^*\varphi$ locally as

$$\pi^*\varphi = c\log|g|^2 + h,$$

where g is a local equation of the divisor D and h is smooth.

thm:qequi

Theorem 1.6.2 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Then any $\varphi \in \mathrm{PSH}(X,\theta)$ admits a quasi-equisingular approximation $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$.*

Moreover, we can guarantee that φ_j has gentle analytic singularities of type $\left(2^{-j}, \mathcal{I}(2^j\varphi)\right)$.

We refer to [DPS01] for the proof.

Quasi-equisingular approximations are essentially unique in the following sense:

prop:compqequi

Proposition 1.6.3 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be two quasi-equisingular approximations of φ. Then for any $\epsilon > 0$ and any $j > 0$, we can find $k_0 > 0$ such that for any $k \geq k_0$, we have*

$$\psi_k \preceq (1-\epsilon)\varphi_j.$$

See [Dem15, Corollary 4.1.7].

def:Iinfty

Definition 1.6.5 Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then we define $\mathcal{I}_\infty(\varphi)$ as the ideal sheaf consisting of germs f of holomorphic functions such that $|f|^2\exp(-\varphi)$ is locally bounded.

Lemma 1.6.2 *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. The sheaf $\mathcal{I}_\infty(\varphi)$ is a coherent sheaf.*

Proof By Theorem 1.6.1, we may find a modification $\pi\colon Y \to X$ such that $\pi^*\varphi$ has log singularities. Observe that

$$\mathcal{I}_\infty(\varphi) = \pi_*\mathcal{I}(\pi^*\varphi),$$

so we may replace X and φ by Y and $\pi^*\varphi$ and assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. We decompose D into its irreducible components:

$$D = \sum_{i=1}^{N} a_i D_i.$$

In this case, observe that

$$\mathcal{I}_\infty(\varphi) = \mathcal{O}(-\sum_{i=1}^{N}(\lceil a_i \rceil D_i))$$

is clearly coherent. □

lma:IandIinf **Lemma 1.6.3** *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then for any $\epsilon > 0$, we can find $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$\mathcal{I}(k(1+\epsilon)\varphi) \subseteq \mathcal{I}_\infty(k\varphi).$$

See [Dem15, Proposition 4.1.6].

thm:CT-thm-refined' **Theorem 1.6.3** *Let X be a connected compact Kähler manifold and $Y \subseteq X$ be a connected positive dimensional submanifold. Take a Kähler form ω on X and $\varphi \in \mathrm{PSH}(Y, \omega|_Y)$ such that $\omega|_Y + \mathrm{dd^c}\varphi$ is a Kähler current and that e^{φ} is a Hölder continuous function on V. Then there exists $\tilde{\varphi} \in \mathrm{PSH}(X, \omega)$ satisfying*

(1) $\tilde{\varphi}|_Y = \varphi$.
(2) $\omega_{\tilde{\varphi}}$ *is a Kähler current.*

In addition, if φ has analytic singularities, then so does $\tilde{\varphi}$.

See [DRWN+23, Theorem 6.1].

1.7 The space of currents

Let X be a connected compact Kähler manifold of dimension n and $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$.

Definition 1.7.1 We say α is *pseudo-effective* if there is a closed positive $(1,1)$-current in α.

We say α is *big* if there is a closed positive $(1,1)$-current T in α dominating a Kähler form. Such currents are called *Kähler currents*.

def:spaceofcurrents **Definition 1.7.2** We introduce the following notations:

(1) $\mathcal{Z}_+(X)$ denotes the space of closed positive $(1,1)$-currents on X;
(2) Given a pseudo-effective $(1,1)$-class α on X, we write $\mathcal{Z}_+(X,\alpha)$ for the set of $T \in \mathcal{Z}_+(X)$ such that $[T] = \alpha$;

Given $T, T' \in \mathcal{Z}_+(X)$, we write

$$T \preceq T'$$

and say T is more singular than T' if when we write $T = \theta + \mathrm{dd^c}\varphi$, $T' = \theta' + \mathrm{dd^c}\varphi'$, we have $\varphi \preceq T'$. We write

$$T \sim T'$$

if $T \preceq T'$ and $T' \preceq T$. In this case, we say T and T' have the same singularity types.

rmk:qpshtocurrents *Remark 1.7.1* Observe that

$$\mathcal{Z}_+(X)/\!\sim\, \cong \mathrm{QPSH}(X)/\!\sim$$

canonically. We will adopt the following convention: whenever we have a notion for quasi-plurisubharmonic functions which depends only on the singularity type, we use the same notation and the same definition of a closed positive $(1,1)$-current.

1.8 Plurisubharmonic metrics on line bundles

A natural source of quasi-plurisubharmonic functions is the metrics on line bundles.

Let X be a connected compact Kähler manifold and L be a holomorphic line bundle on X. Usually, we do not distinguish L from the associated invertible sheaf $\mathcal{O}_X(L)$.

Definition 1.8.1 Let V be a 1-dimensional complex linear space. A *Hermitian form* h on V is a map $h\colon V \times V \to \mathbb{C}$ such that

(1) h is $\mathbb{C}$-linear in the second variable and conjugate linear in the first, and
(2)
$$|v|_h := h(v,v) \in \mathbb{R}_{>0}$$
for each $v \in V \setminus \{0\}$.

We usually identify h with the quadratic form $V \to \mathbb{R}$ sending v to $|v|_h$.

The *singular Hermitian form* on V is the map $V \to \{0,\infty\}$ sending 0 to 0 and other elements to ∞.

Definition 1.8.2 A *Hermitian metric* h on L is a family of Hermitian forms $(h_x)_{x\in X}$, such that

(1) for each $x \in X$, h_x is a Hermitian form on L_x, and

(2) for each local section s of $\mathcal{O}_X(L)$, the map $x \mapsto |s(x)|_{h_x}$ is smooth.

We shall write $c_1(L,h)$ for the first Chern form of h, normalized so that

$$[c_1(L,h)] = c_1(L).$$

The map $x \mapsto |s(x)|_{h_x}$ will be denoted by $|s|$.

prop:LelongPoincare **Proposition 1.8.1 (Lelong–Poincaré)** *Let $s \in H^0(X,L)$ be non-zero, h be a Hermitian metric on L. Then*

$$c_1(L,h) + \mathrm{dd^c} \log |s|_h^2 = [Z(s)],$$

where $Z(s)$ is the prime divisor defined by s and $[\bullet]$ denote the associated current of integration.

See [Dem12a, (3.11)].

Definition 1.8.3 A *plurisubharmonic metric* h on L is a family $(h_x)_x$ such that

(1) for each $x \in X$, h_x is either a Hermitian form on L_x or the singular Hermitian form, and
(2) there is a Hermitian metric h_0 on L and $\varphi \in \mathrm{PSH}(X, c_1(L,h_0))$ such that for each $x \in X$ and each $v \in L_x$, we have

$$|v|^2_{h_x} = \begin{cases} 0, \text{ if } v = 0; \\ |v|^2_{h_{0,x}} \mathrm{e}^{-\varphi(x)}, \text{ if } v \neq 0. \end{cases} \tag{1.9}$$ {eq:htwist}

The *(first) Chern current* of h is by definition

$$\mathrm{dd^c} h = c_1(L,h) \coloneqq c_1(L,h_0) + \mathrm{dd^c}\varphi.$$

We shall write the plurisubharmonic metric defined by (1.9) as $h\exp(-\varphi)$. As the readers can easily verify, our conventions guarantee that $c_1(L,h)$ does not depend on the choice of h_0.

Remark 1.8.1 In the literature, some people prefer the convention that in (1.9), neither side has the square.

thm: OT_ext **Theorem 1.8.1** *Assume that L is big and T is a holomorphic line bundle on X. Fix a Hermitian metric r on T. Take a Kähler form ω on X. Let $Y \subseteq X$ be a connected submanifold of dimension m. Suppose that $\varphi \in \mathrm{PSH}(X, \theta - \delta\omega)$ for some $\delta > 0$ and $\varphi|_Y \not\equiv -\infty$. Then there exists $k_0(\delta, r) > 0$ such that for all $k \geq k_0$ and $s \in H^0(Y, T \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y))$, there exists an extension $\tilde{s} \in H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ such that*

$$\int_X (h^k \otimes r)(\tilde{s}, \tilde{s}) \mathrm{e}^{-k\varphi}\, \omega^n \leq C \int_Y (h^k \otimes r)(s,s) \mathrm{e}^{-k\varphi|_Y}\, \omega|_Y^m,$$

where $C > 0$ is an absolute constant, independent of the data (φ, s, k).

This is a special case of [His12, Theorem 1.4].

Chapter 2
Non-pluripolar products

chap:npp

Let X be a complex manifold and $\varphi_1, \dots, \varphi_m \in \mathrm{PSH}(X)$ ($m \in \mathbb{Z}_{>0}$). When the functions $\varphi_1, \dots, \varphi_m$ are all smooth, there is an obvious definition of a current

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m \tag{2.1}$$

{eq:mixedMAtype}

by the usual differential calculus. It is of interest to extend this construction to the case where the φ_i's have worse regularities.

There are a number of different approaches to this problem. In this book, we will choose the so-called non-pluripolar theory due to Bedford–Taylor, Guedj–Zeriahi and Boucksom–Eyssidieux–Guedj–Zeriahi. The reason is that the non-pluripolar theory is the only known theory satisfying the following two features: it is defined for all psh singularities (at least in the global setting); it satisfies a monotonicity theorem.

2.1 Bedford–Taylor theory

Let X be a connected complex manifold of dimension n and $\varphi_1, \dots, \varphi_m \in \mathrm{PSH}(X)$ ($m \in \mathbb{Z}_{>0}$) be locally bounded plurisubharmonic functions on X. In this case, there is a canonical definition of the Monge–Ampère type product (2.1) as follows:

Definition 2.1.1 We define the closed positive (m, m)-current (2.1) on X as follows: we make an induction on $m \geq 1$. When $m = 1$, we define $\mathrm{dd^c}\varphi_1$ using the current calculus. Recall that φ_1 is locally integrable by Proposition 1.2.4, so we can regard it as a distribution on X. When $m > 1$ and the case $m - 1$ is defined, we let

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m \coloneqq \mathrm{dd^c}\left(\varphi_1\, \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m\right).$$

This definition is due to Bedford–Taylor and is usually called the Bedford–Taylor product.

Proposition 2.1.1 *The product* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m$ *is a closed positive*[1] (m,m)*-current on* X*. Moreover, the product is symmetric in the* φ_i*'s.*

See [GZ17, Proposition 3.3, Corollary 3.12].

The Bedford–Taylor theory has many satisfactory properties.

thm:contMA

Theorem 2.1.1 *Let* $(\varphi_i^j)_j$ *be decreasing sequences (resp. increasing sequences) of locally bounded psh functions on* X *converging (resp. converging a.e.) to locally bounded psh function* φ_i*, where* $i = 1, \dots, m$*. Then*

$$\varphi_0^j\,\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_m^j \rightharpoonup \varphi_0\,\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m$$

as $j \to \infty$*. In particular, if* φ_0^j *is the constant sequence* 1*, we have*

$$\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_m^j \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m.$$

We refer to [GZ17, Theorem 3.18, Theorem 3.23] for the proofs.

Theorem 2.1.2 *The Bedford–Taylor product* (2.1) *puts no mass on pluripolar sets (Definition 1.1.4) in* X*.*

Theorem 2.1.3 *The Bedford–Taylor product is local with respect to the plurifine topology.*

These results are also special cases of the more general results below.

2.2 The definition of non-pluripolar products

The proof of all results in this section can be found in [BEGZ10].

Let X be a complex manifold.

Definition 2.2.1 Let $\varphi_1, \dots, \varphi_p \in \mathrm{PSH}(X)$. We set

$$O_k := \bigcap_{j=1}^{p} \{\varphi_j > -k\}, \quad k \in \mathbb{Z}_{>0}.$$

We say that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is *well-defined* if for each open subset $U \subseteq X$ such that there is a Kähler form ω on U such that for each compact subset $K \subseteq U$, we have

[1] Recall that we say an (m,m)-current T on X is positive if either $m > n$ or for any smooth $(1,0)$-forms $\alpha_1, \dots, \alpha_{n-m}$ on X, the measure

$$T \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \mathrm{i}\alpha_{n-m} \wedge \overline{\alpha_{n-m}}$$

is positive.

$$\sup_{k\geq 0}\int_{K\cap O_k}\left(\bigwedge_{j=1}^{p}\mathrm{dd^c}\max\{\varphi_j,-k\}\right)\Bigg|_U\wedge\omega^{n-p}<\infty. \tag{2.2}$$

{eq:welldefinepluri}

In this case, we define $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_p$ by

$$\mathbb{1}_{O_k}\langle\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_p\rangle=\mathbb{1}_{O_k}\bigwedge_{j=1}^{p}\mathrm{dd^c}\max\left(\varphi_j\vee(-k)\right) \tag{2.3}$$

{eq:npp}

on $\bigcup_{k\geq 0}O_k$ and make a zero-extension to X.

prop:npp1 **Proposition 2.2.1** *Let* $\varphi_1,\dots,\varphi_p\in\mathrm{PSH}(X)$.

(1) *The product* $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_p$ *is local in plurifine topology. In the following sense: let* $O\subseteq X$ *be a plurifine open subset, let* $\psi_1,\dots,\psi_p\in\mathrm{PSH}(X)$*, assume that*

$$\varphi_j|_O=\psi_j|_O,\quad j=1,\dots,p.$$

Assume that

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}u_j\text{ and }\bigwedge_{j=1}^{p}\mathrm{dd^c}v_j$$

are both well-defined, then

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}\varphi_j\Bigg|_O=\bigwedge_{j=1}^{p}\mathrm{dd^c}\psi_j\Bigg|_O. \tag{2.4}$$

{eq:ppp1}

If O *is open in the usual topology, then the product*

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}\varphi_j|_O$$

on O *is well-defined and*

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}\varphi_j\Bigg|_O=\bigwedge_{j=1}^{p}\mathrm{dd^c}\psi_j|_O. \tag{2.5}$$

{eq:ppp2}

Let $\mathcal{U}$ *be an open covering of* X*. Then* $\mathrm{dd^c}u_1\wedge\cdots\wedge\mathrm{dd^c}u_p$ *is well-defined if and only if each of the following product is well-defined*

$$\bigwedge_{j=1}^{p}\mathrm{dd^c}u_j|_U,\quad U\in\mathcal{U}.$$

(2) *The current* $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_p$ *and the fact that it is well-defined depend only on the currents* $\mathrm{dd^c}\varphi_j$*, not on specific* φ_j.

(3) *When* $\varphi_1, \dots, \varphi_p \in L^\infty_{\mathrm{loc}}(X)$, $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is well-defined and is equal to the Bedford–Taylor product.*

(4) *Assume that* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is well-defined, then* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *puts not mass on pluripolar sets.*

(5) *Assume that* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is well-defined, then*

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j$$

is a closed positive (p, p)*-current on* X.

(6) *The product is multi-linear: let* $\psi_1 \in \mathrm{PSH}(X)$*, then*

$$\mathrm{dd^c}(\varphi_1 + \psi_1) \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j = \mathrm{dd^c}\varphi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j + \mathrm{dd^c}\psi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j \tag{2.6}$$

{eq:ppp6}

in the sense that left-hand side is well-defined if and only if both terms on right-hand side are well-defined, and the equality holds in that case.

Definition 2.2.2 Let $T_1, \dots, T_p$ be closed positive $(1, 1)$-currents on X. We say that $T_1 \wedge \cdots \wedge T_p$ is *well-defined* if there exists an open covering $\mathcal{U}$ of X, such that on each $U \in \mathcal{U}$, we can find $\varphi_j^U \in \mathrm{PSH}(U)$ $(j = 1, \dots, p)$ such that

$$\mathrm{dd^c}\varphi_j^U = T_j, \quad j = 1, \dots, p$$

and such that $\mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U$ is well-defined. In this case, we define $T_1 \wedge \cdots \wedge T_p$ as the closed positive (p, p)-current on X defined by

$$\left(T_1 \wedge \cdots \wedge T_p\right)|_U = \mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U, \quad U \in \mathcal{U}. \tag{2.7}$$

{eq:ppp5}

Proposition 2.2.1 can be formulated in terms of currents without any difficulty.

Proposition 2.2.2 *Let* X *be a compact Kähler manifold and* $T_1, \dots, T_p$ *are closed positive* $(1, 1)$*-currents on* X*. Then* $T_1 \wedge \cdots \wedge T_p$ *is well-defined.*

2.3 Properties of non-pluripolar products

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta_1, \dots, \theta_n$ be closed real smooth $(1, 1)$-forms on X.

We write

$$\mathrm{PSH}(X, \theta)_{>0} = \left\{ \varphi \in \mathrm{PSH}(X, \theta) : \int_X \theta_\varphi^n > 0 \right\}. \tag{2.8}$$

{eq:PSHpos}

thm:semicon **Theorem 2.3.1** *Let* $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X, \theta_j)$ $(k \in \mathbb{Z}_{>0},\ j = 1, \dots, n)$*. Let* $\chi \geq 0$ *be a bounded function such that there are* $\eta_1, \eta_2 \in \mathrm{QPSH}(X)$ *such that* $\eta_1 + \chi = \eta_2$*.*

Assume that for any $j = 1, \dots, n$ and $i = 1, \dots, m$, as $k \to \infty$, either φ_j^k decreases to $\varphi_j \in \mathrm{PSH}(X, \theta)$ or increases to $\varphi_j \in \mathrm{PSH}(X, \theta)$ almost everywhere. Then for any open set $U \subseteq X$, we have

$$\varliminf_{k\to\infty} \int_U \chi\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_U \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.9}$$ {eq:semicon1}

DDNL18mono
See [DDNL18b, Theorem 2.3].

thm:mono **Theorem 2.3.2** *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X, \theta_j)$ for $j = 1, \dots, n$. Assume that $\varphi_j \geq \psi_j$ for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n}.$$

DDNL18mono
See [DDNL18b, Theorem 1.1].

As a corollary, we obtain that

cor:incseqnppcont **Corollary 2.3.1** *Fix a directed set I. For each $j = 1, \dots, n$, take an increasing net $(\varphi_j^i)_{i\in I}$ in $\mathrm{PSH}(X, \theta_j)$, uniformly bounded from above. Set*

$$\varphi_j := \sup_{i\in I}{}^* \varphi_j^i.$$

Then

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.10}$$ {eq:increseqnppcont}

Proof We may assume that I is infinite as there is nothing to prove otherwise. Thanks to Theorem 2.3.2, we already know the $\leq$ inequality in (2.10). We prove the reverse inequality. When $I \cong \mathbb{Z}_{>0}$ as directed sets, the reverse inequality follows from Theorem 2.3.1. In general, by Choquet's lemma Proposition 1.2.2, we can find a countable infinite subset $R \subseteq I$ such that

$$\sup_{r\in R}{}^* \varphi_j^r = \sup_{i\in I}{}^* \varphi_j^i$$

for all $j = 1, \dots, n$. We fix a bijection $R \cong \mathbb{Z}_{>0}$. We will then denote elements φ_k^r ($r \in R$) by $\varphi_k^1, \varphi_k^2, \dots$. We shall write

$$\psi_k^a = \varphi_k^1 \vee \cdots \vee \varphi_k^a$$

for each $a \in \mathbb{Z}_{>0}$.

It follows from the fact that I is a directed set and Theorem 2.3.2 that

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} \geq \lim_{a\to\infty} \int_X \theta_{1,\psi_1^a} \wedge \cdots \wedge \theta_{n,\psi_n^a}.$$

From the special case mentioned above, we know that the right-hand side is exactly the right-hand side of (2.10), so we conclude. □

lma:pathoenvelope **Lemma 2.3.1** *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, $\varphi \leq \psi$ and $\int_X \theta_\varphi^n > 0$. Then for any*

$$a \in \left(1, \left(\frac{\int_X \theta_\psi^n}{\int_X \theta_\psi^n - \int_X \theta_\varphi^n}\right)^{1/n}\right), \tag{2.11}$$ {eq:arangetemp}

there is $\eta \in \mathrm{PSH}(X,\theta)_{>0}$ such that

$$a^{-1}\eta + (1-a^{-1})\psi \leq \varphi.$$

The fraction in (2.11) is understood as ∞ if $\int_X \theta_\psi^n = \int_X \theta_\varphi^n$. We write

$$P(a\varphi + (1-a)\psi) = \sup{}^* \left\{\eta \in \mathrm{PSH}(X,\theta) : a^{-1}\eta + (1-a^{-1})\psi \leq \varphi\right\} \in \mathrm{PSH}(X,\theta). \tag{2.12}$$

Observe that

$$a^{-1}P(a\varphi + (1-a)\psi) + (1-a^{-1})\psi \leq \varphi. \tag{2.13}$$

In fact, this equation holds outside a pluripolar set by Proposition 1.2.3, hence it holds everywhere by Proposition 1.2.5.

Proof Without loss of generality, we may assume that $\varphi \leq \psi \leq 0$.

DDNLmetric
We refer to [DDNL21b, Lemma 4.3] for the proof of the existence of $\eta \in \mathrm{PSH}(X,\theta)$ satisfying the given inequality. Next we argue that $P(a\varphi + (1-a)\psi) \in \mathrm{PSH}(X,\theta)_{>0}$. Choose

$$a' \in \left(a, \left(\frac{\int_X \theta_\psi^n}{\int_X \theta_\psi^n - \int_X \theta_\varphi^n}\right)^{1/n}\right).$$

It follows that

$$P(a\varphi + (1-a)\psi) \geq \frac{a}{a'}P(a'\varphi + (1-a')\psi) + \frac{a'-a}{a'}\varphi.$$

Therefore, by Theorem 2.3.2, we have

$$\int_X \theta^n_{P(a\varphi+(1-a)\psi)} \geq \frac{(a'-a)^n}{a'^n}\int_X \theta_\varphi^n > 0.$$

lma:kahcurrentposmass **Lemma 2.3.2** *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ then there is $\psi \in \mathrm{PSH}(X,\theta)$ such that*

(1) *θ_ψ is a Kähler current;*
(2) *$\psi \leq \varphi$.*

Proof Using Lemma 2.3.1, we can find $\epsilon > 0$ and $\gamma \in \mathrm{PSH}(X,\theta)$ such that

$$\frac{\epsilon}{1+\epsilon}V_\theta + \frac{1}{1+\epsilon}\gamma \leq \varphi.$$

Take $\eta \in \mathrm{PSH}(X,\theta)$ such that θ_η is a Kähler current and $\eta \leq 0$. Then we may take

$$\psi = \frac{\epsilon}{1+\epsilon}\eta + \frac{1}{1+\epsilon}\gamma.$$

lma:existsecposmass

Lemma 2.3.3 *Let L be a holomorphic line bundle on X with $\theta \in c_1(L)$. Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then there exists $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$H^0(X, L^k \otimes \mathcal{I}(k\varphi)) \neq 0.$$

Proof By Lemma 2.3.2, we may further assume that θ_φ is a Kähler current. In this case, the result follows from [Dem12a, Theorem 13.21]. □

thm:logconc

Theorem 2.3.3 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. Then the map*

$$[0,1] \ni t \mapsto \log \int_X \theta^n_{t\varphi_1+(1-t)\varphi_0}$$

is concave.

See [DDNL21a] for the proof.

Remark 2.3.1 Here and in the sequel, when we write expressions like $t\varphi + (1-t)\psi$ for $\varphi, \psi \in \mathrm{QPSH}(X)$, we will follow the convention that when $t = 0$, the value is ψ and when $t = 1$, the value is φ.

Chapter 3
The envelope operators

chap:enve

3.1 The P-envelope

In this section, X will denote a connected compact Kähler manifold of dimension n.

3.1.1 The definition of the P-envelope

We recall that a non-strict partial order QPSH(X) is introduced in Definition 1.5.2. We will fix a smooth closed real $(1,1)$-form θ on X.

def:rooftop **Definition 3.1.1** Given $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we define their *rooftop operator* as follows:

$$\varphi \wedge \psi = \sup \{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq \varphi, \eta \leq \psi\}.$$

When we want to be more specific, we could also write $\varphi \wedge_\theta \psi$. Suppose that $\varphi \wedge \psi$ is not identically $-\infty$ on each connected component of X, we have $\varphi \wedge \psi \in \mathrm{PSH}(X,\theta)$ by Proposition 1.2.1.

def:Penv **Definition 3.1.2** Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define its *$P$-envelope* as follows

$$P_\theta[\varphi] \coloneqq \sup{}^* \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq \varphi\}. \tag{3.1}$$ {eq:Pthetavarphi}

Observe that by Proposition 1.2.1, we have $P_\theta[\varphi] \in \mathrm{PSH}(X,\theta)$. Moreover, the definition can be equivalently described as

$$P_\theta[\varphi] = \sup_{C \in \mathbb{Z}_{>0}}{}^* (\varphi + C) \wedge V_\theta. \tag{3.2}$$ {eq:Penvsups}

Here $\wedge$ is the rooftop operator defined in Definition 3.1.1. Observe that for any $C \in \mathbb{R}$, we have $(\varphi + C) \wedge V_\theta \in \mathrm{PSH}(X,\theta)$ and

$$(\varphi + C) \wedge V_\theta \sim \varphi.$$

prop:Penvindeptheta

Proposition 3.1.1 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X, \theta)$, we have $\varphi - g \in \mathrm{PSH}(X, \theta')$ and*

$$P_\theta[\varphi] \sim P_{\theta'}[\varphi'].$$

Proof By symmetry, it suffices to show that

$$P_\theta[\varphi] \leq P_{\theta'}[\varphi'].$$

We may assume that $g \geq 0$. Then for any $\psi \in \mathrm{PSH}(X, \theta)$ with $\psi \leq \varphi$ and $\psi \leq 0$, we set $\psi' := \psi - g$. Then $\psi' \leq \varphi'$ and $\psi' \leq 0$, so $\psi' \leq P_{\theta'}[\varphi']$. Since ψ is arbitrary, it follows that

$$P_\theta[\varphi] - g \leq P_{\theta'}[\varphi'].$$

prop:Ppresmass

Proposition 3.1.2 *Suppose that $\theta_1, \ldots, \theta_n$ be smooth closed real $(1,1)$-forms on X. Let $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ for each $i = 1, \ldots, n$. Then*

$$\int_X \theta_{1, P_{\theta_1}[\varphi_1]} \wedge \cdots \wedge \theta_{n, P_{\theta_n}[\varphi_n]} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.3}$$

{eq:Penvpremass}

Proof For each $C \in \mathbb{Z}_{>0}$ and each $i = 1, \ldots, n$, we have

$$(\varphi_i + C) \wedge V_{\theta_i} \sim \varphi_i.$$

It follows from Theorem 2.3.2 that

$$\int_X \theta_{1,(\varphi_1 + C)\wedge V_{\theta_1}} \wedge \cdots \wedge \theta_{n,(\varphi_n + C)\wedge V_{\theta_n}} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

So (3.3) follows from (3.2) and Corollary 2.3.1. □

thm:Pvarphidiffdef

Theorem 3.1.1 *Assume that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then*

$$P_\theta[\varphi] = \sup\left\{\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \varphi \leq \psi, \int_X \theta_\varphi^n = \int_X \theta_\psi^n\right\}. \tag{3.4}$$

{eq:Penvdef}

In particular, in this case,

$$P_\theta\left[P_\theta[\varphi]\right] = P_\theta[\varphi]. \tag{3.5}$$

{eq:Penvprojop}

We refer to [DDNL23, Theorem 3.14] for the proof. In general, we do not know if (3.5) holds when $\int_X \theta_\varphi^n > 0$. We expect it to be wrong. According to our general philosophy, the P-envelope operator is the correct object only when the non-pluripolar mass is positive. We will avoid using the degenerate case in the whole book.

Definition 3.1.3 If $\varphi = P_\theta[\varphi]$ and $\int_X \theta_\varphi^n > 0$, we say φ is a *model potential*.

We remind the readers that the notion of model potentials depends heavily on the choice of θ. When there is a risk of confusion, we also say φ is a model potential in $\mathrm{PSH}(X, \theta)$.

This definition is different from the common definition in the literature: we impose the extra condition $\int_X \theta_\varphi^n > 0$. The author believes that this is the only case where this notion is natural. We sometimes emphasize this point by saying $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$ is a model potential.

cor:Psendspotentialtomodel **Corollary 3.1.1** *Let $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then $P_\theta[\varphi]$ is a model potential in* $\mathrm{PSH}(X, \theta)$.

Proof This follows immediately from Theorem 3.1.1. □

3.1.2 Properties of the P-envelope

Let $\theta, \theta_1, \theta_2$ be smooth closed real $(1, 1)$-forms on X.

prop:Penvbimero **Proposition 3.1.3** *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for any $\varphi \in \mathrm{PSH}(X, \theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi] = \pi^* P_\theta[\varphi].$$

In particular, a potential $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$ is model if and only if $\pi^*\varphi \in \mathrm{PSH}(Y, \pi^*\theta)_{>0}$ is model.

Proof This follows immediately from Proposition 1.5.3. □

We have the following concavity property of the P-envelope.

prop:Pconc **Proposition 3.1.4**

(1) *Suppose that $\varphi \in \mathrm{PSH}(X, \theta)$ and $\lambda \in \mathbb{R}_{>0}$, then*

$$P_{\lambda\theta}[\lambda\varphi] = \lambda P_\theta[\varphi];$$

(2) *Suppose that $\varphi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X, \theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2] \geq P_{\theta_1}[\varphi_1] + P_{\theta_2}[\varphi_2].$$

Proof (1). This is obvious by definition.

(2). Suppose that $\psi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\psi_2 \in \mathrm{PSH}(X, \theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \preceq \varphi_i$$

for $i = 1, 2$. Then

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \preceq \varphi_1 + \varphi_2.$$

It follows from (3.1) that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2].$$

Since ψ_1 and ψ_2 are arbitrary, we conclude. □

prop:landpresmodel **Proposition 3.1.5** *Let $\varphi, \psi \in \operatorname{PSH}(X, \theta)$. Assume that*

$$\varphi = P_\theta[\varphi], \quad \psi = P_\theta[\psi], \quad \varphi \wedge \psi \not\equiv -\infty.$$

Then

$$P_\theta[\varphi \wedge \psi] = \varphi \wedge \psi. \tag{3.6}$$

{eq:Pthetaphilandpsi}

Proof Observe that we obviously have

$$P_\theta[\varphi \wedge \psi] \leq P_\theta[\varphi] = \varphi, \quad P_\theta[\varphi \wedge \psi] \leq P_\theta[\psi] = \psi.$$

So the $\leq$ direction in (3.6) holds. The reverse direction is trivial. □

thm:Pvarphisupport **Theorem 3.1.2** *Let $\varphi \in \operatorname{PSH}(X, \theta)$. Then*

$$\theta^n_{P_\theta[\varphi]} \leq \mathbb{1}_{\{P_\theta[\varphi]=0\}} \theta^n.$$

DDNL18mono
See [DDNL18b, Theorem 3.8] for the proof.

prop:landfinitecond1 **Proposition 3.1.6** *Assume that $\varphi, \psi, \eta \in \operatorname{PSH}(X, \theta)$ and*

$$\int_X \theta^n_\varphi + \int_X \theta^n_\psi > \int_X \theta^n_\eta, \quad \varphi \vee \psi \leq \eta.$$

Then $\varphi \wedge \psi \in \operatorname{PSH}(X, \theta)$.

DDNLmetric
We refer to [DDNL21b, Lemma 5.1] for the proof.

thm:diamond **Theorem 3.1.3** *Assume that $\varphi, \psi \in \operatorname{PSH}(X, \theta)$ and $\varphi \wedge \psi \in \operatorname{PSH}(X, \theta)$. Then*

$$\int_X \theta^n_\varphi + \int_X \theta^n_\psi \leq \int_X \theta^n_{\varphi\vee\psi} + \int_X \theta^n_{\varphi\wedge\psi}.$$

DDNLmetric
We refer to [DDNL21b, Theorem 5.4] for the proof.

prop:decseqmodel **Proposition 3.1.7** *Let $(\varphi_j)_{j\in I}$ be a decreasing net of potentials in $\operatorname{PSH}(X, \theta)$ satisfying $P_\theta[\varphi_j] = \varphi_j$ for each $j \in I$ and $\varphi := \inf_j \varphi_j \not\equiv -\infty$. Then $P_\theta[\varphi] = \varphi$.*

Proof It follows from Proposition 1.2.1 that $\varphi \in \operatorname{PSH}(X, \theta)$. Therefore, for each $j \in I$,

$$\varphi \leq P_\theta[\varphi] \leq P_\theta[\varphi_j] = \varphi_j.$$

Therefore, $\varphi = P_\theta[\varphi]$. □

prop:vol_limit_model **Proposition 3.1.8** *Let $(\epsilon_j)_{j\in I}$ be a decreasing net in $\mathbb{R}_{\geq 0}$ with limit 0. Take a Kähler form ω on X. Consider a decreasing net $\varphi_j \in \operatorname{PSH}(X, \theta + \epsilon_j\omega)$ ($j \in I$) satisfying*

$$P_{\theta+\epsilon_j\omega}[\varphi_j] = \varphi_j \tag{3.7}$$

{eq:Palmostmodeltemp}

with pointwise limit $\varphi \not\equiv -\infty$. Then

$$\lim_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}=\int_X\theta^n_{\varphi}. \tag{3.8}$$ {eq:massmodeldec}

Moreover, if $\int_X\theta^n_\varphi>0$, *then for any prime divisor* E *over* X, *we have*

$$\lim_{j\in I}\nu(\varphi_j,E)=\nu(\varphi,E). \tag{3.9}$$ {eq:Lelongcontdecseq}

Proof Observe that $\varphi\in\mathrm{PSH}(X,\theta)$. By Theorem 2.3.2, we have

$$\varliminf_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}\geq\varliminf_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi}=\int_X\theta^n_{\varphi}.$$

We now argue the reverse inequality.

Fix $j_0\in I$, we have

$$\begin{aligned}\varlimsup_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}&=\varlimsup_{j\in I}\int_{\{\varphi_j=0\}}(\theta+\epsilon_j\omega)^n_{\varphi_j}\\&\leq\varlimsup_{j\in I}\int_{\{\varphi_j=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi_j}\\&\leq\int_{\{\varphi=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi},\end{aligned}$$

where in the first line we used (3.7) and Theorem 3.1.2, and in the last line we have used the fact that $\varphi_j\searrow\varphi$ and [DDNL21b, Proposition 4.6] (see also [DDNL23, Lemma 2.11]). Taking limit with respect to j_0, we arrive at the desired conclusion:

$$\varlimsup_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}\leq\varliminf_{j_0\in I}\int_{\{\varphi=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi}=\int_{\{\varphi=0\}}\theta^n_{\varphi}\leq\int_X\theta^n_{\varphi}.$$

This finishes the proof of (3.8).

It remains to argue (3.9). By Lemma 2.3.1 and (3.8), for any $\epsilon\in(0,1)$ and j big enough there exists $\psi_j\in\mathrm{PSH}(X,\theta+\epsilon_j\omega)$ such that $(1-\epsilon)\varphi_j+\epsilon\psi_j\leq\varphi$. This implies that for j big enough we have

$$(1-\epsilon)\nu(\varphi_j,E)+\epsilon\nu(\psi_j,E)\geq\nu(\varphi,E)\geq\nu(\varphi_j,E).$$

On the other hand, the Lelong numbers $\nu(\psi_j,E)$ admit am upper bound for various j by Proposition 1.5.2. So taking limit with respect to j, we conclude (3.9). □

cor:Pprojdec **Corollary 3.1.2** *Let* $(\varphi_j)_{j\in I}$ *be a decreasing net of potentials in* $\mathrm{PSH}(X,\theta)$ *with pointwise limit* $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$. *Then*

$$P_\theta[\varphi]=\inf_{j\in I}P_\theta[\varphi_j].$$

Proof Let $\eta=\inf_{i\in I}P_\theta[\varphi_i]$. We clearly have $\eta\geq P_\theta[\varphi]$.

By Proposition 3.1.8, we have

$$\lim_{i\in I}\int_X \theta^n_{\varphi_i} = \int_X \theta^n_{\varphi} > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ $(i \in I)$ and $\psi_i \in \mathrm{PSH}(X,\theta)$ such that for all $i \in I$,

$$(1-\epsilon_i)\varphi_i + \epsilon_i\psi_i \leq \varphi.$$

By Proposition 3.1.4, we have

$$(1-\epsilon_i)\eta + \epsilon_i P_\theta[\psi_i] \leq (1-\epsilon_i)P_\theta[\varphi_i] + \epsilon_i P_\theta[\psi_i] \leq P_\theta[\varphi].$$

Taking limit with respect to $i \in I$, we conclude that $\eta \leq P_\theta[\varphi]$ outside a pluripolar set and hence everywhere by Proposition 1.2.5. □

Remark 3.1.1 The arguments like the last sentence in the proof of Corollary 3.1.2 is very common. We will usually omit the details.

prop:varphiperturbtheta

Corollary 3.1.3 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. Let ω be a Kähler form on X. Then*

$$\varphi = \inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi].$$

Proof Clearly, we have the $\leq$ direction and the right-hand side is non-positive. So by Theorem 3.1.1, it suffices to show that they have the same mass, which follows from Proposition 3.1.8. □

prop:incnetmodel

Proposition 3.1.9 *Let $(\varphi_i)_{i\in I}$ be an increasing net of potentials in $\mathrm{PSH}(X,\theta)_{>0}$ uniformly bounded from above. Let $\varphi := \sup^*_{i\in I}\varphi_i$. Then*

$$\sup^*_{i\in I} P_\theta[\varphi_i] = P_\theta[\varphi].$$

In particular, if φ_i is model for all $i \in I$, then so is φ.

Proof We write

$$\eta := \sup^*_{i\in I} P_\theta[\varphi_i].$$

Then it is clear that $\eta \leq P_\theta[\varphi]$.

By Corollary 2.3.1, we have

$$\lim_{i\in I}\int_X \theta^n_{\varphi_i} = \int_X \theta^n_{\varphi} > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ $(i \in I)$ and $\psi_i \in \mathrm{PSH}(X,\theta)$ such that for all $i \in I$,

$$(1-\epsilon_i)\varphi + \epsilon_i\psi_i \leq \varphi_i.$$

By Proposition 3.1.4, we have

$$(1-\epsilon_i)P_\theta[\varphi] + \epsilon_i P_\theta[\psi_i] \leq \eta \leq P_\theta[\varphi].$$

Taking limit with respect to i, we conclude that $P_\theta[\varphi] \leq \eta$. □

3.1.3 Relative full mass classes

subsec:fullmass

Let θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$. We shall write

$$V_\theta = \sup\left\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\right\}. \tag{3.10}$$

It follows from Proposition 1.2.1 that $V_\theta \in \mathrm{PSH}(X,\theta)$.

Definition 3.1.4 We define

$$\begin{aligned}
\mathrm{PSH}(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq \phi\right\},\\
\mathcal{E}^\infty(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathrm{PSH}(X,\theta) : \eta \sim \phi\right\},\\
\mathcal{E}(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathrm{PSH}(X,\theta;\phi) : \int_X \theta_\varphi^n = \int_X \theta_\phi^n\right\},\\
\mathcal{E}^1(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathcal{E}(X,\theta;\phi) : \int_X |\phi-\eta|\,\theta_\eta^n < \infty\right\}.
\end{aligned}$$

rmk:intwelldef

Remark 3.1.2 Note that this integral

$$\int_X |\phi-\eta|\,\theta_\eta^n$$

is defined: the locus where $\phi-\eta$ is undefined is a pluripolar set, while the product θ_η^n puts no mass on pluripolar sets (Proposition 2.2.1).

Similar remarks apply when we talk about similar integrals in the sequel.

When $\phi = V_\theta$, we usually write

$$\begin{aligned}
\mathcal{E}^\infty(X,\theta;V_\theta) =&\mathcal{E}^\infty(X,\theta),\\
\mathcal{E}(X,\theta;V_\theta) =&\mathcal{E}(X,\theta),\\
\mathcal{E}^1(X,\theta;V_\theta) =&\mathcal{E}^1(X,\theta).
\end{aligned}$$

Potentials in the three classes are said to have *minimal singularities*, *full mass* and *finite energy* respectively.

The P-envelope can be used to characterize the full mass class.

prop:fullmassP

Proposition 3.1.10 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}(X,\theta;\phi)$;
(2) $P_\theta[\varphi] = \phi$.

Proof (2) $\implies$ (1). This follows from Proposition 3.1.2.

(1) $\implies$ (2). Note that ϕ is a candidate of $P_\theta[\varphi]$ as in (3.4). So $P_\theta[\varphi] = \phi$. □

In order to handle the finite energy classes, it is convenient to introduce the following quantity:

def:MAenergy **Definition 3.1.5** We define the *Monge–Ampère energy* $E_\theta^\phi\colon \mathcal{E}^\infty(X,\theta;\phi)\to\mathbb{R}$ as follows

$$E_\theta^\phi(\varphi) \coloneqq \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\varphi-\phi)\,\theta_\varphi^j\wedge\theta_\phi^{n-j}. \tag{3.11}$$ {eq:Edefbdd}

More generally, we extend E_θ^ϕ to a functional $E_\theta^\phi\colon \mathrm{PSH}(X,\theta;\phi)\to[-\infty,\infty)$ as follows

$$E_\theta^\phi(\varphi) \coloneqq \inf\left\{E_\theta^\phi(\psi) : \psi\in\mathcal{E}^\infty(X,\theta;\phi), \varphi\leq\psi\right\}. \tag{3.12}$$ {eq:Eextendgeneral}

We write E_θ instead of E_θ^ϕ when $\phi = V_\theta$.

prop:cocycE1 **Proposition 3.1.11** *Let $\varphi\in\mathrm{PSH}(X,\theta;\phi)$. The following are equivalent:*

(1) *$\varphi\in\mathcal{E}^1(X,\theta;\phi)$;*
(2) *$E_\theta^\phi(\varphi)>-\infty$.*

When the conditions are satisfied, (3.11) holds.

Given $\varphi,\psi\in\mathcal{E}^1(X,\theta;\phi)$, we have the following cocycle equality

$$E_\theta^\phi(\psi)-E_\theta^\phi(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X(\psi-\varphi)\,\theta_\psi^j\wedge\theta_\varphi^{n-j}. \tag{3.13}$$ {eq:Ecocyc}

See [BEGZ10, Proposition 2.11] and [DDNL18a, Proposition 2.5] for the proofs.[1]

prop:relrooftopclosed **Proposition 3.1.12** *Assume that $\varphi,\psi\in\mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is $\varphi\wedge\psi$.*

Proof The case of $\mathcal{E}^\infty(X,\theta;\phi)$ is trivial.

We consider the case $\mathcal{E}(X,\theta;\phi)$. It follows from Proposition 3.1.6 that $\varphi\wedge\psi\in\mathrm{PSH}(X,\theta)$. By Theorem 3.1.3, we have

$$\int_X\theta_{\varphi\wedge\psi}^n\geq\int_X\theta_\phi^n.$$

By Theorem 2.3.2, equality holds. By Theorem 3.1.1, we conclude that

$$P_\theta[\varphi\wedge\psi]=\phi.$$

Finally, the case $\mathcal{E}^1(X,\theta;\phi)$ is proved in [Xia23a, Theorem 4.13] (the arXiv version). □

prop:relativeEupperclosed **Proposition 3.1.13** *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$ be potentials such that $\psi\preceq\phi$ and $\varphi\preceq\psi$. Assume that $\varphi\in\mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is ψ.*

Proof The case $\mathcal{E}^\infty(X,\theta;\phi)$ is trivial. The case $\mathcal{E}(X,\theta;\phi)$ follows from Theorem 2.3.2. The case $\mathcal{E}^1(X,\theta;\phi)$ follows from [Xia23a, Proposition 4.5] (arXiv version). □

[1] In these references, they took $\phi=V_\theta$, but the proof of the general case is almost identical.

prop:supsEE1

Proposition 3.1.14 *Let $(\varphi_i)_{i\in I}$ be a uniformly bounded from above non-empty family in $\mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is $\sup^*_i \varphi_i$.*

Proof It suffices to handle the case where $\varphi_i \in \mathcal{E}(X,\theta;\phi)$ for all $i \in I$. The remaining two cases follow from Proposition 3.1.13.

Step 1. We first assume that I is finite. In this case, we can easily further reduce to the case where $I = \{0,1\}$. Assume that $\varphi_0, \varphi_1 \in \mathcal{E}(X,\theta;\phi)$. Observe that $\varphi_0 \preceq \phi$ and $\varphi_1 \preceq \phi$, hence $\varphi_0 \vee \varphi_1 \preceq \phi$. On the other hand, by Theorem 2.3.2, $\varphi_0 \vee \varphi_1$ and ϕ have the same mass.

Step 2. We come back to the case where I is infinite.

By Proposition 1.2.2, we may assume that $I = \mathbb{Z}_{>0}$ as an ordered set. Moreover, by Step 1, we may assume that the sequence $(\varphi_i)_i$ is increasing. Furthermore, we may assume that $\varphi_i \leq 0$ for all i. Then we know that $\varphi_i \leq \phi$. Therefore, $\sup^*_i \varphi_i \leq \phi$. But they have the same mass as a consequence of Corollary 2.3.1. So we conclude using Theorem 3.1.1. □

prop:envrelfullmass

Proposition 3.1.15 *Let $\varphi, \psi \in \mathcal{E}(X,\theta;\phi)$. Then*

$$\sup_{C\geq 0}{}^*(\varphi + C) \wedge \psi = \psi.$$

Proof Since for each $C \geq 0$,

$$(\varphi \wedge \psi + C) \wedge \psi \leq (\varphi + C) \wedge \psi \leq \psi,$$

we may replace φ by $\varphi \wedge \psi$ (c.f. Proposition 3.1.12) and assume that $\varphi \leq \psi$. In this case, the result is proved in [DDNL18b, Theorem 3.8, Corollary 3.11]. □

3.2 The $\mathcal{I}$-envelope

From the algebraic point of view, a more natural envelope operator is given by the $\mathcal{I}$-envelope.

3.2.1 $\mathcal{I}$-equivalence

prop:Iequivchar

Proposition 3.2.1 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *for any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi),$$

(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) = \mathcal{I}(\lambda\psi),$$

(3) *for any modification $\pi \colon Y \to X$ and any $y \in Y$, we have*

$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y),$$

(4) *for any proper bimeromorphic morphism* $\pi\colon Y \to X$ *from a Kähler manifold and any* $y \in Y$*, we have*

$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y),$$

and

(5) *for any prime divisor* E *over* X*, we have*

$$\nu(\varphi, E) = \nu(\psi, E).$$

See Definition B.1.1 for the definition of prime divisors over X.

Proof 4 $\iff$ 5: this follows from Lemma 1.4.1.
3 $\iff$ 5: this follows from Corollary B.1.1.
1 $\implies$ 5: this follows from Proposition 1.4.4.
5 $\implies$ 2: this follows from Theorem 1.4.3.
2 $\implies$ 1: This is trivial. □

def:Iequiv **Definition 3.2.1** Given $\varphi, \psi \in \mathrm{QPSH}(X)$, we say they are $\mathcal{I}$*-equivalent* and write $\varphi \sim_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 3.2.1 are satisfied.

prop:Ienvbimero **Proposition 3.2.2** *Let* $\pi\colon Y \to X$ *be a proper bimeromorphic morphism from a connected Kähler manifold* Y *to* X*. Then for* $\varphi, \psi \in \mathrm{QPSH}(X)$*, we the following are equivalent:*

(1) $\varphi \sim_{\mathcal{I}} \psi$*;*
(2) $\pi^*\varphi \sim_{\mathcal{I}} \pi^*\psi$*.*

Proof 1 $\implies$ 2: This follows from 4 in Proposition 3.2.1.
2 $\implies$ 1: This follows from the simple fact that

$$\mathcal{I}(k\varphi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\varphi)\right), \quad \mathcal{I}(k\psi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\psi)\right).$$

prop:Iequivmax **Proposition 3.2.3** *Let* $\varphi, \varphi', \psi, \psi' \in \mathrm{QPSH}(X)$ *and* $\lambda > 0$*. Assume that* $\varphi \sim_{\mathcal{I}} \psi$ *and* $\varphi' \sim_{\mathcal{I}} \psi'$*, then*

$$\varphi \lor \varphi' \sim_{\mathcal{I}} \psi \lor \psi', \quad \varphi + \varphi' \sim_{\mathcal{I}} \psi + \psi', \quad \lambda\varphi \sim_{\mathcal{I}} \lambda\psi.$$

Proof This follows from Proposition 1.4.2. □

3.2.2 The definition the $\mathcal{I}$-envelope

We will fix a smooth closed real $(1,1)$-form θ on X.

def:Ienv **Definition 3.2.2** Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define its $\mathcal{I}$-envelope as follows:

$$P_\theta[\varphi]_{\mathcal{I}} \coloneqq \sup\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim_{\mathcal{I}} \varphi\}.$$

If $\varphi = P_\theta[\varphi]_{\mathcal{I}}$, we say φ is an *$\mathcal{I}$-model potential* (in $\mathrm{PSH}(X,\theta)$).

Note that by Proposition 1.2.1, $P_\theta[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$.

prop:Ienvindeptheta **Proposition 3.2.4** *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$P_\theta[\varphi]_{\mathcal{I}} \sim P_{\theta'}[\varphi']_{\mathcal{I}}.$$

The proof is similar to that of Proposition 3.1.1, so we omit it.

prop:Ienvelopebimero **Proposition 3.2.5** *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a connected Kähler manifold Y to X. Then for $\varphi \in \mathrm{PSH}(X,\theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}} = \pi^* P_\theta[\varphi]_{\mathcal{I}}.$$

Proof The proof is similar to that of Proposition 3.1.3 in view of Proposition 3.2.2. □

prop:Ienvprojection **Proposition 3.2.6** *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\varphi \sim_{\mathcal{I}} P_\theta[\varphi]_{\mathcal{I}}.$$

In particular,

$$P_\theta\left[P_\theta[\varphi]_{\mathcal{I}}\right]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

Proof In view of Proposition 3.2.1, it suffices to show that for $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}}). \tag{3.14}$$ {eq:IenvelopepreservLelong}

By Proposition 1.2.2, we can find $\psi_i \in \mathrm{PSH}(X,\theta)$ ($i \in \mathbb{Z}_{>0}$) such that $\psi_i \leq 0$, $\psi_i \sim_{\mathcal{I}} \varphi$ and

$$\operatorname*{sup*}_{i>0} \psi_i = P_\theta[\varphi]_{\mathcal{I}}.$$

By Proposition 3.2.3, we may replace ψ_i by $\psi_1 \vee \cdots \vee \psi_i$ and assume that the sequence ψ_i is increasing. In this case, it follows from the strong openness theorem Theorem 1.4.4 that for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi_j) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}})$$

for j large enough. □

def:volqpsh **Definition 3.2.3** Let $\varphi \in \mathrm{PSH}(X,\theta)$, we define the *volume* $\mathrm{vol}(\theta,\varphi)$ as

$$\mathrm{vol}(\theta,\varphi) = \int_X \left(\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}}\right)^n.$$

In view of the following proposition, we could write

$$\operatorname{vol}\theta_\varphi = \operatorname{vol}(\theta, \varphi).$$

Proposition 3.2.7 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$\operatorname{vol}(\theta, \varphi) = \operatorname{vol}(\theta', \varphi').$$

Proof This follows immediately from Proposition 3.2.4 and Theorem 2.3.2. □

The $\mathcal{I}$-envelope and the P-envelope are related in a simple manner.

prop:PandPI

Proposition 3.2.8 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, $\varphi \sim_{\mathcal{I}} P_\theta[\varphi]$.

Proof It suffices to show that $\varphi \sim_{\mathcal{I}} P_\theta[\varphi]$. Namely, for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}\left(kP_\theta[\varphi]\right). \tag{3.15}$$ {eq:IkvarphiIkP}

It follows from (3.2) and the strong openness theorem Theorem 1.4.4 that

$$\mathcal{I}\left(kP_\theta[\varphi]\right) = \mathcal{I}\left((k\varphi + C) \wedge V_{k\theta}\right)$$

when C is large enough. Since $(k\varphi + C) \wedge V_{k\theta} \sim k\varphi$, we have

$$\mathcal{I}\left((k\varphi + C) \wedge V_{k\theta}\right) = \mathcal{I}(k\varphi)$$

and (3.15) follows. □

cor:compnppmassandvol

Corollary 3.2.1 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\int_X \theta_\varphi^n \leq \operatorname{vol}\theta_\varphi.$$

Proof This follows from Proposition 3.2.8, Theorem 2.3.2 and Proposition 3.1.2. □

We note the following special case.

prop:analysingcompPandPI

Proposition 3.2.9 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that φ has analytic singularities, then*

$$\varphi \sim P_\theta[\varphi] \sim_P P_\theta[\varphi]_{\mathcal{I}}.$$

Proof In view of Proposition 3.2.8, it suffices to show that

$$P_\theta[\varphi]_{\mathcal{I}} \preceq \varphi. \tag{3.16}$$ {eq:Pprecvarphitemp1}

By Proposition 3.2.5 and Theorem 1.6.1, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. By rescaling using Proposition 3.2.10, we may assume that D is a divisor. Take quasi-equisingular approximations η_j and φ_j of $P_\theta[\varphi]_{\mathcal{I}}$ and of φ respectively. Recall that by Theorem 1.6.2, we can guarantee that η_j and φ_j both have the singularity type $(2^{-j}, \mathcal{I}(2^j\varphi))$ and hence $\eta_j \sim \varphi_j$. On the other hand, it is clear that $\varphi_j \sim \varphi$. So (3.16) follows. □

3.2.3 Properties of the $\mathcal{I}$-envelope

Let $\theta, \theta_1, \theta_2$ be smooth closed real $(1,1)$-forms on X.

We have the following concavity property of the P-envelope.

prop:PIconc **Proposition 3.2.10**

(1) *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ and $\lambda \in \mathbb{R}_{>0}$, then*

$$P_{\lambda\theta}[\lambda\varphi]_{\mathcal{I}} = \lambda P_\theta[\varphi]_{\mathcal{I}};$$

(2) *Suppose that $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1+\varphi_2]_{\mathcal{I}} \geq P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(3) *Suppose that $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1+\varphi_2]_{\mathcal{I}} \sim_{\mathcal{I}} P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(4) *Suppose that $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi \vee \varphi]_{\mathcal{I}} \sim_{\mathcal{I}} P_\theta[\varphi_1]_{\mathcal{I}} + P_\theta[\varphi_2]_{\mathcal{I}}.$$

Proof 1. This is obvious by definition.

2. Suppose that $\psi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\psi_2 \in \mathrm{PSH}(X,\theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \sim_{\mathcal{I}} \varphi_i$$

for $i = 1, 2$. Then

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \sim_{\mathcal{I}} \varphi_1 + \varphi_2.$$

It follows that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1+\varphi_2]_{\mathcal{I}}.$$

Since ψ_1 and ψ_2 are arbitrary, we conclude.

3. This follows easily from Proposition 1.4.2 and 3.2.1.

4. The proof is similar to that of 3. We omit the details. □

prop:decnetmodelPI **Proposition 3.2.11** *Consider a decreasing net $(\varphi_i)_{i\in I}$ of model potentials in $\mathrm{PSH}(X,\theta)_{>0}$. Suppose that $\varphi := \inf_{i\in I}\varphi_i \not\equiv -\infty$ and $\int_X \theta_\varphi^n > 0$. Then*

$$\inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

Proof Let $\eta = \inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}}$. We clearly have $\eta \geq P_\theta[\varphi]_{\mathcal{I}}$.

By Proposition 3.1.8, we have

$$\lim_{i\in I} \int_X \theta^n_{\varphi_i} = \int_X \theta^n_\varphi > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) and $\psi_i \in \mathrm{PSH}(X,\theta)$ such that for all $i \in I$,

$$(1-\epsilon_i)\varphi_i + \epsilon_i\psi_i \leq \varphi.$$

By Proposition 3.2.10, we have

$$(1-\epsilon_i)\eta + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq (1-\epsilon_i)P_\theta[\varphi_i]_{\mathcal{I}} + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq P_\theta[\varphi]_{\mathcal{I}}.$$

Taking limit with respect to i, we conclude that $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$. □

prop:incnetmodelPI

Proposition 3.2.12 *Let $(\varphi_i)_{i\in I}$ be an increasing net in $\mathrm{PSH}(X,\theta)_{>0}$ uniformly bounded from above. Let $\varphi := \sup^*_{i\in I} \varphi_i$. Then*

$$\sup_{i\in I}{}^* P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

Proof Let $\eta = \sup^*_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}}$. We clearly have $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$.

By Corollary 2.3.1, we have

$$\lim_{i\in I} \int_X \theta^n_{\varphi_i} = \int_X \theta^n_\varphi > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) and $\psi_i \in \mathrm{PSH}(X,\theta)$ such that for all $i \in I$,

$$(1-\epsilon_i)\varphi + \epsilon_i\psi_i \leq \varphi_i.$$

By Proposition 3.2.10, we have

$$(1-\epsilon_i)P_\theta[\varphi]_{\mathcal{I}} + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq P_\theta[\varphi_i]_{\mathcal{I}} \leq \eta.$$

Taking limit with respect to i, we conclude that $\eta \geq P_\theta[\varphi]_{\mathcal{I}}$. □

Chapter 4
Geodesic rays in the space of potentials

chap:rays

4.1 Subgeodesics

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

def:subgeod

Definition 4.1.1 Let us fix $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a curve $(\varphi_t)_{t\in(0,1)}$ in $\mathrm{PSH}(X,\theta)$ such that

(1) if we define

$$\Phi\colon X\times\{z\in\mathbb{C}:\mathrm{e}^{-1}<|z|<1\}\to[-\infty,\infty),\quad (x,z)\mapsto\varphi_{-\log|z|}(x),$$

then Φ is $p_1^*\theta$-psh, where $p_1 : X\times\{z\in\mathbb{C}:\mathrm{e}^{-1}<|z|<1\}\to X$ is the natural projection;

(2) When $t\to 0+$ (resp. to $1-$), φ_t converges to φ_0 (resp. φ_1) with respect to the L^1-topology.

By abuse of notation, we also say $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

We say Φ is the *complexification* of the subgeodesic $(\varphi_t)_t$.

prop:convexsubgeod

Proposition 4.1.1 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$ and $(\varphi_t)_{t\in(0,1)}$ be a subgeodesic from φ_0 to φ_1. Then for each $x\in X$, $[0,1]\ni t\mapsto\varphi_t(x)$ is a convex function.*

Proof The convexity on the interval $(0,1)$ follows simply from Definition 4.1.1 1. In order to verify the convexity at the boundary, let us fix $s\in(0,1)$. We need to show that

$$\varphi_s(x)\le s\varphi_1(x)+(1-s)\varphi_0(x) \tag{4.1}$$

{eq:varphisconvextemp1}

for all $x\in X$. Thanks to Proposition 1.2.5, it suffices to prove this for almost all x.

Take a set $Z\subseteq X$ with zero Lebesgue measure such that for all $x\in X\setminus Z$, we have

(1) $\varphi_t(x)\neq-\infty$ for all $t\in[0,1]\cap\mathbb{Q}$;
(2) $\varphi_t(x)\to\varphi_0(x)$ as $t\to 0+$ and $\varphi_t(x)\to\varphi_1(x)$ as $t\to 1-$.

For all such x, the convexity of φ guarantees that $\varphi_t(x) \neq -\infty$ for all $t \in [0,1]$ and $t \mapsto \varphi_t(x)$ is convex for $t \in [0,1]$. In particular, (4.1) holds. □

prop:maxsubgeod

Proposition 4.1.2 *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two non-empty uniformly bounded from above families in* $\mathrm{PSH}(X,\theta)$. *Let $(\varphi_t^i)_{t\in(0,1)}$ be subgeodesics from φ_0^i to φ_1^i for each $i \in I$. Then*

$$\left(\sup_{i\in I}{}^* \varphi_t^i\right)_{t\in(0,1)}$$

is a subgeodesic from $\sup^*_i \varphi_0^i$ to $\sup^*_i \varphi_0^i$.

Proof We may assume that $\varphi_0^i, \varphi_1^i \leq 0$ for all $i \in I$. Then it follows that $\varphi_t^i \leq 0$ for all $t \in (0,1)$ and all $i \in I$ from Proposition 4.1.1.

We define

$$\varphi_t \coloneqq \sup_{i\in I}{}^* \varphi_t^i \in \mathcal{E}(X,\theta;\phi)$$

for all $t \in [0,1]$. Observe that $[0,1] \ni t \mapsto \varphi_t$ by the same argument leading to (4.1).

Let $(\psi_t)_{t\in(0,1)}$ be the subgeodesic whose complexification Φ_ψ corresponds to $\sup^*_i \Phi_{\varphi^i}$, the complexification of $(\varphi_t^i)_{t\in(0,1)}$. Then clearly, $\varphi_t \leq \psi_t$ for each $t \in (0,1)$. On the other hand, by Proposition 1.2.3,

$$\psi_t = \sup_{i\in I} \varphi_t^i = \varphi_t \quad \text{almost everywhere}$$

for almost all $t \in (0,1)$. Therefore, using Proposition 1.2.5, $\psi_t = \varphi_t$ for almost all $t \in (0,1)$. Since both functions are convex in t, we conclude that $\psi_t = \varphi_t$ for all $t \in (0,1)$.

It remains to argue that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$ and $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. By symmetry, it suffices to argue the former. In fact, we know that for any $t \in (0,1)$ and any $j \in I$,

$$\varphi_t^j \leq \varphi_t \leq t\varphi_1 + (1-t)\varphi_0,$$

where the latter inequality follows from Proposition 4.1.1. Letting $t \to 0+$ and then taking limit with respect to j, we conclude. □

4.2 Geodesics in the space of potentials

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

Definition 4.2.1 Let $\varphi_0, \varphi_1 \in \mathcal{E}(X,\theta)$. The *geodesic* $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 is a collection of potentials $\varphi_t \in \mathrm{PSH}(X,\theta)$ such that

$$\begin{aligned}\varphi_t = \sup\{\eta_t : (\eta_s)_s \text{ is a subgeodesic from } \psi_0 \text{ to } \psi_1, \\ \psi_0, \psi_1 \in \mathrm{PSH}(X,\theta), \psi_0 \leq \varphi_0, \psi_1 \leq \varphi_1\}.\end{aligned} \tag{4.2}$$

{eq:Perron}

The construction is known as the *Perron–Bremermann envelope*.

def:geod **Definition 4.2.2** Let $(\varphi_t)_{t\in[a,b]}$ ($a,b\in\mathbb{R}$, $a\leq b$) be a curve in $\mathcal{E}^1(X,\theta)$. We say $(\varphi_t)_{t\in[a,b]}$ is a *geodesic* if the curve $(\psi_t)_{t\in(0,1)}$ is a geodesic from φ_a to φ_b, where

$$\psi_t=\varphi_{t(b-a)+a},\quad t\in[0,1].$$

We also say $(\varphi_t)_{t\in[a,b]}$ is a geodesic in $\mathcal{E}(X,\theta)$ or is the geodesic in $\mathcal{E}(X,\theta)$ from φ_a to φ_b.

prop:perronenvissubgeod **Proposition 4.2.1** *Given $\varphi_0,\varphi_1\in\mathcal{E}(X,\theta)$, the geodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 is a subgeodesic from φ_0 to φ_1 and $\varphi_t\in\mathcal{E}(X,\theta)$ for each $t\in(0,1)$.*

Moreover, for any $0\leq a\leq b\leq 1$, the restriction $(\varphi_t)_{t\in[a,b]}$ is a geodesic.

If furthermore $\varphi_0,\varphi_1\in\mathcal{E}^1(X,\theta)$ (resp. $\mathcal{E}^\infty(X,\theta)$), then $\varphi_t\in\mathcal{E}^1(X,\theta)$ (resp. $\mathcal{E}^\infty(X,\theta)$) for all $t\in(0,1)$.

We will prove a more general result in Proposition 4.3.1.

prop:energylinear **Proposition 4.2.2** *Let $(\varphi_t)_{t\in[a,b]}$ be a geodesic in $\mathcal{E}^1(X,\theta)$, then $t\mapsto E_\theta(\varphi_t)$ is a linear function of $t\in[a,b]$.*

Proof This follows from DDNL18fullmass [DDNL18c, Theorem 3.12] and DDNL18big [DDNL18a, Proposition 3.13]. □

Definition 4.2.3 Let $\ell=(\ell_t)_{t\geq 0}$ be a curve in $\mathcal{E}(X,\theta)$. We say ℓ is a *geodesic ray* in $\mathcal{E}(X,\theta)$ emanating from ℓ_0 if for each $0\leq a\leq b$, the restriction $(\ell_t)_{t\in[a,b]}$ is a geodesic.

The set of geodesic rays in $\mathcal{E}(X,\theta)$ emanating from V_θ is denoted by $\mathcal{R}(X,\theta)$.

We say $\ell\in\mathcal{R}(X,\theta)$ has *finite energy* if $\ell_t\in\mathcal{E}^1(X,\theta)$ for all $t>0$. The set of finite energy rays in $\mathcal{R}(X,\theta)$ is denoted by $\mathcal{R}^1(X,\theta)$. The set of rays $\ell\in\mathcal{R}^1(X,\theta)$ such that $\ell_t\in\mathcal{E}^\infty(X,\theta)$ for all $t>0$ is denoted by $\mathcal{R}^\infty(X,\theta)$.

Given $\ell,\ell'\in\mathcal{R}(X,\theta)$, we write $\ell\leq\ell'$ if for each $t\geq 0$, $\ell_t\geq\ell'_t$.

prop:supsgeod **Proposition 4.2.3** *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two uniformly bounded from above increasing nets in $\mathcal{E}^\infty(X,\theta)$. Let $(\varphi_t^i)_{t\in(0,1)}$ be the geodesic from φ_0^i to φ_1^i for each $i\in I$. Then*

$$\left(\sup_{i\in I}{}^*\varphi_t^i\right)_{t\in(0,1)}$$

is the geodesic from $\sup^*_i\varphi_0^i$ to $\sup^*_i\varphi_0^i$.

Proof By Proposition 1.2.2 and Proposition 4.1.2, we may assume that I is countable. In this case, the assertion follows from DDNL18fullmass [DDNL18c, Proposition 3.3] and Theorem 2.1.1. □

Definition 4.2.4 We define the *radial Monge–Ampère energy* $\mathbf{E}\colon\mathcal{R}^1(X,\theta)\to\mathbb{R}$ as follows: $\mathbf{E}(\ell)$ is the slope of $\mathbb{R}_{\geq 0}\ni t\mapsto E_\theta(\ell_t)$.

The energy $E_\theta(\ell_t)$ is linear in t by Proposition 4.2.2.

Recall that the d_1-metric on $\mathcal{E}^1(X,\theta)$ is introduced in Definition 4.3.5.

Proposition 4.2.4 *Let $\ell, \ell' \in \mathcal{R}^1(X, \theta)$. Then the map*

$$t \mapsto d_1(\ell_t, \ell'_t)$$

is convex.

DDNLmetric
See [DDNL21b, Proposition 2.10] for the proof. In particular, we can introduce

def:d1rays
Definition 4.2.5 Let $\ell, \ell' \in \mathcal{R}^1(X, \theta)$. We define

$$d_1(\ell, \ell') := \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t, \ell'_t).$$

thm:d1raycomplete
Theorem 4.2.1 *The function d_1 defined in Definition 4.2.5 is a metric and $(\mathcal{R}^1(X, \theta), d_1)$ is a complete metric space.*

DDNLmetric
See [DDNL21b, Theorem 2.14] for the proof.

prop:d1geod_diff_E
Proposition 4.2.5 *Let $\ell, \ell' \in \mathcal{R}^1(X, \theta)$ and $\ell \leq \ell'$. Then*

$$d_1(\ell, \ell') = \mathbf{E}(\ell') - \mathbf{E}(\ell). \tag{4.3}$$ {eq:d1rayscompa}

Proof This is a direct consequence of (4.14). □

ex:rayasspsh
Example 4.2.1 Let $\varphi \in \mathrm{PSH}(X, \theta)$. For each $C > 0$, let $(\ell_t^{\varphi,C})_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta - C) \vee \varphi$. For each $t \geq 0$, the potential $\ell_t^{\varphi,C}$ is increasing in $C \in [t, \infty)$. We let

$$\ell_t^\varphi := \sup_{C\geq t}{}^* \ell_t^{\varphi,C}. \tag{4.4}$$ {eq:ellvarphiraydef}

Then $\ell^\varphi \in \mathcal{R}^\infty(X, \theta)$ and

$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \left(\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_{V_\theta}^n \right). \tag{4.5}$$ {eq:Elphi}

Proof We first show that for each fixed $t \geq 0$, $\ell_t^{\varphi,C}$ is increasing in $C \geq t$.

To see this, choose $t \leq C_1 < C_2$. We need to show that

$$\ell_t^{\varphi,C_1} \leq \ell_t^{\varphi,C_2}.$$

Since both sides are geodesics for $t \in [0, C_1]$, it suffices to show that

$$(V_\theta - C_1) \vee \varphi \leq \ell_{C_1}^{\varphi,C_2}. \tag{4.6}$$ {eq:VthetaminusC1temp1}

Then $((V_\theta - t) \vee \varphi)_{t\in[0,C_2]}$ is a subgeodesic from V_θ to $(V_\theta - C_2) \vee \varphi$ by Proposition 4.1.2. At $t = 0$ and $t = C_1$, it is dominated by the geodesic ℓ_t^{φ,C_2}, hence by (4.2.1), we conclude that the same holds at $t = C_1$, which is exactly (4.6).

From Proposition 4.1.1, we know that for any $C \geq t > 0$, we have

$$\ell_t^{\varphi,C} \leq t\left((V_\theta - C) \vee \varphi\right) + (1 - t)V_\theta \leq 0.$$

So in (4.4), $\ell_t^\varphi \in \mathrm{PSH}(X,\theta)$ for any $t>0$. Also observe that by Proposition 4.3.1, we have $\ell_t^\varphi \in \mathcal{E}^\infty(X,\theta)$ for all $t>0$. It follows from Proposition 4.2.3 that $\ell^\varphi \in \mathcal{R}^1(X,\theta)$.

It remains to compute the energy of ℓ^φ.

We first fix $C \geq t > 0$ and compute

$$E_\theta(\ell_t^{\varphi,C}) = \frac{t}{C} E_\theta\left((V_\theta - C)\vee\varphi\right).$$

Letting $C\to\infty$ and applying Theorem 4.3.1, we find that

$$E_\theta(\ell_t^{\varphi}) = \lim_{C\to\infty}\frac{t}{C} E_\theta\left((V_\theta - C)\vee\varphi\right).$$

It follows that

$$\mathbf{E}(\ell^{\varphi}) = \lim_{C\to\infty}\frac{1}{C} E_\theta\left((V_\theta - C)\vee\varphi\right).$$

Using the definition of E_θ, it suffices to show that for each $j=0,\dots,n$, we have

$$\lim_{C\to\infty}\int_X \frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta} = \int_X \theta^j_\varphi\wedge\theta^{n-j}_{V_\theta} - \int_X\theta^n_{V_\theta}. \tag{4.7}$$

{eq:limCintXtemp1}

For this purpose, for each $C>0$, we decompose X as $\{\varphi > V_\theta - C\}$ and $\{\varphi\leq V_\theta - C\}$. We have

$$\begin{aligned}&\int_{\{\varphi>V_\theta-C\}}\frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\ =&\int_{\{\varphi>V_\theta-C\}}\frac{\varphi - V_\theta}{C}\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\{\varphi\leq V_\theta-C\}}\frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\ =&-\int_{\{\varphi\leq V_\theta-C\}}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\ =&-\int_X\theta^n_{V_\theta}+\int_{\{\varphi>V_\theta-C\}}\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}.\end{aligned}$$

Observe that for $C>0$, the functions $\mathbb{1}_{\{\varphi>V_\theta-C\}}C^{-1}(\varphi-V_\theta)$ is defined almost everywhere and is bounded. When $C\to\infty$, these functions converge to 0 almost everywhere. Therefore, (4.7) follows. □

prop:raysupsublinear1

Proposition 4.2.6 *Let $\ell\in\mathcal{R}(X,\theta)$, then there is $C>0$ such that*

$$\sup_X \ell_t \leq Ct.$$

A more general result will be proved in Proposition 4.3.4.

Next we recall that $\vee$ operator at the level of geodesic rays.

def:lorray1 **Definition 4.2.6** Let $\ell,\ell'\in\mathcal{R}(X,\theta)$. We define $\ell\vee\ell'$ as the minimal ray in $\mathcal{R}(X,\theta)$ lying above both ℓ and ℓ'.

prop:lorrays **Proposition 4.2.7** *Given $\ell,\ell'\in\mathcal{R}(X,\theta)$. Then $\ell\vee\ell'\in\mathcal{R}(X,\theta)$ exists. Moreover, if $\ell,\ell'\in\mathcal{R}^1(X,\theta)$, then so is $\ell\vee\ell'$ and*

$$\mathbf{E}(\ell\vee\ell')=\lim_{t\to\infty}\frac{1}{t}E_\theta(\ell_t\vee\ell'_t). \tag{4.8}$$ {eq:Elor}

Furthermore, if both $\ell,\ell'\in\mathcal{R}^\infty(X,\theta)$, then so is $\ell\vee\ell'$.

Proof For each $t>0$, let $(\ell''^t_s)_{s\in[0,t]}$ be the geodesic from V_θ to $\ell_t\vee\ell'_t$. Then clearly, for each fixed $s\geq 0$, ℓ''^t_s is increasing in $t\in[s,\infty)$. Moreover, Proposition 4.2.6 guarantees that $(\sup_X\ell''^t_s)_t$ is bounded from above for a fixed s. Let $(\ell\vee\ell')_s=\sup^*_{t\geq s}\ell''^t_s$. Then Proposition 4.2.3 guarantees that $\ell\vee\ell'$ is a geodesic ray. It is clear that this ray is minimal among all rays dominating ℓ and ℓ'.

Assume that $\ell,\ell'\in\mathcal{R}^1(X,\theta)$, it follows from Proposition 3.1.13 that $\ell\vee\ell'\in\mathcal{R}^1(X,\theta)$. Next we compute its energy:

$$\mathbf{E}(\ell\vee\ell')=E_\theta(\ell\vee\ell')_1=\lim_{t\to\infty}E_\theta(\ell''^t_1)=\frac{1}{t}E_\theta(\ell_t\vee\ell'_t),$$

where we applied Proposition 4.2.2 and Theorem 4.3.1.

The last assertion is trivial. □

lma:d1rayineq **Lemma 4.2.1** *For any $\ell,\ell'\in\mathcal{R}^1(X,\theta)$, we have*

$$d_1(\ell,\ell')\leq d_1(\ell,\ell\vee\ell')+d_1(\ell',\ell\vee\ell')\leq C_n d_1(\ell,\ell'), \tag{4.9}$$ {eq:d1maxineq}

where $C_n=3(n+1)2^{n+2}$.

Proof The first inequality is trivial. As for the second, we estimate

$$\begin{aligned} d_1(\ell,\ell\vee\ell') =&\mathbf{E}(\ell\vee\ell')-\mathbf{E}(\ell)\\ =&\lim_{t\to\infty}\frac{1}{t}\mathbf{E}(\ell_t\vee\ell'_t)-\mathbf{E}(\ell)\\ =&\lim_{t\to\infty}\frac{1}{t}d_1(\ell_t\vee\ell'_t,\ell_t), \end{aligned}$$

where one the first line, we applied Proposition 4.2.5, on the second line, we used (4.8), the first and the third lines follow from Proposition 4.2.5. In all, we find

$$d_1(\ell,\ell\vee\ell')+d_1(\ell',\ell\vee\ell')\leq\lim_{t\to\infty}\frac{1}{t}\left(d_1(\ell_t\vee\ell'_t,\ell_t)+d_1(\ell_t\vee\ell'_t,\ell'_t)\right).$$

By [DDNL18big DDNL18a, Theorem 3.7],

$$d_1(\ell_t\vee\ell'_t,\ell_t)+d_1(\ell_t\vee\ell'_t,\ell'_t)\leq 3(n+1)2^{n+2}d_1(\ell_t,\ell'_t).$$

Now (4.9) follows. □

4.3 The relative setting

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

The proceeding discussions can also be carried out in this setting. The proofs can be modified *mutadis mutandis*. We leave the details to the readers.

Definition 4.3.1 Let $\varphi_0, \varphi_1 \in \mathcal{E}(X,\theta;\phi)$. The *geodesic* $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 is a collection of potentials $\varphi_t \in \mathrm{PSH}(X,\theta)$ such that

$$\begin{aligned} \varphi_t = \sup\{\eta_t : (\eta_s)_s \text{ is a subgeodesic from } \psi_0 \text{ to } \psi_1, \\ \psi_0,\psi_1 \in \mathrm{PSH}(X,\theta),\ \psi_0 \le \varphi_0, \psi_1 \le \varphi_1\}\,. \end{aligned} \tag{4.10}$$ {eq:Perron2}

def:geod2 **Definition 4.3.2** Let $(\varphi_t)_{t\in[a,b]}$ $(a,b \in \mathbb{R}\ a \le b)$ be a curve in $\mathcal{E}(X,\theta;\phi)$. We say $(\varphi_t)_{t\in[a,b]}$ is a *geodesic* if the curve $(\psi_t)_{t\in(0,1)}$ is a geodesic from φ_a to φ_b, where

$$\psi_t = \varphi_{t(b-a)+a}, \quad t \in [0,1].$$

We also say $(\varphi_t)_{t\in[a,b]}$ is a geodesic in $\mathcal{E}(X,\theta;\phi)$ or is the geodesic in $\mathcal{E}(X,\theta;\phi)$ from φ_a to φ_b.

prop:perronenvissubgeod2 **Proposition 4.3.1** *Given $\varphi_0,\varphi_1 \in \mathcal{E}(X,\theta;\phi)$, the geodesic $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic from φ_0 to φ_1 and $\varphi_t \in \mathcal{E}(X,\theta;\phi)$ for each $t \in (0,1)$.*

Moreover, for any $0 \le a \le b \le 1$, the restriction $(\varphi_t)_{t\in[a,b]}$ is a geodesic.

If furthermore $\varphi_0,\varphi_1 \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$), then $\varphi_t \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$) for all $t \in (0,1)$.

Proof Without loss of generality, we may assume that $\varphi_0,\varphi_1 \le \phi$. It follows from Proposition 4.1.1 that $\varphi_t \le \phi$ for all $t \in (0,1)$. In fact,

$$\varphi_t \le t\varphi_1 + (1-t)\varphi_0 \tag{4.11}$$ {eq:geodesicconvextemp1}

for all $t \in (0,1)$.

We first observe that when $\varphi_0,\varphi_1 \in \mathcal{E}(X,\theta;\phi)$, so is $\varphi_0\wedge\varphi_1$, see Proposition 3.1.12. In particular, the constant subgeodesic $t \mapsto \varphi_0 \wedge \varphi_1$ is a candidate in (4.10). So $\varphi_t \ge \varphi_0\wedge\varphi_1$ for all $t \in (0,1)$. It follows from Proposition 3.1.13 that $\varphi_t \in \mathcal{E}(X,\theta;\phi)$ for all $t \in (0,1)$. By Proposition 4.1.2, $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic.

Next, we show that as $t \to 0+$, $\varphi_t \xrightarrow{L^1} \varphi_0$. The corresponding result at $t=1$ is similar.

We first argue the special case where $\varphi_0 \le \varphi_1$. Take a constant $C>0$ such that

$$\varphi_0 - C \le \varphi_1.$$

Then $(\varphi_0 - Ct)_{t\in(0,1)}$ is clearly a candidate in (4.10). Therefore, for all $t \in (0,1)$,

$$\varphi_0 - Ct \le \varphi_t \le t\varphi_1 + (1-t)\varphi_0. \tag{4.12}$$ {eq:varphi0andvarphit}

It is clear that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$.

Let us come back to the general case. By (4.11), we know that for all $t \in (0,1)$,

$$\sup_X \varphi_t \leq (\sup_X \varphi_0) \vee (\sup_X \varphi_1)$$

On the other hand, $\sup_X \varphi_t \geq \sup_X \varphi_0 \wedge \varphi_1$. It follows from Proposition 1.5.1 that $\{\varphi_t : t \in (0,1)\}$ is a relatively compact subset of $\mathrm{PSH}(X,\theta)$ with respect to the L^1-topology.

Let ψ be an L^1-cluster point of φ_t as $t \to 0$, it suffices to show that $\psi = \varphi_0$.

For each $M \in \mathbb{N}$, we write

$$\varphi_0^M = \varphi_0 \wedge (\varphi_1 + M).$$

Let $(\varphi_t^M)_{t\in(0,1)}$ be the geodesic from φ_0^M to φ_1. Then it is clear that

$$\varphi_t^M \leq \varphi_t$$

for all $t \in (0,1)$. Therefore,

$$\psi \geq \varphi_0 \wedge (\varphi_1 + M).$$

On the other hand, by (4.11), $\psi \leq \varphi_0$. So it suffices to show that

$$\varphi_0 \wedge (\varphi_1 + M) \xrightarrow{L^1} \varphi_0$$

as $M \to \infty$. This is shown in Proposition 3.1.15.

Next, take $0 \leq a \leq b \leq 1$. We want to show that the restriction $(\varphi_t)_{t\in[a,b]}$ is the geodesic from φ_a to φ_b. We may assume that $a < b$. The argument is the standard *balayage* argument.

Let $(\psi_t)_{t\in(a,b)}$ be the (rescaled) geodesic from φ_a to φ_b. It is easy to see that the curve $(\eta_t)_{t\in(0,1)}$ defined by $\eta_t = \psi_t$ for $t \in (a,b)$ and $\eta_t = \varphi_t$ otherwise is a candidate in (4.10). So we conclude that $\eta_t = \varphi_t = \psi_t$ for $t \in (a,b)$.

Finally, assume furthermore that $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta;\phi)$. Thanks to Proposition 3.1.13, it suffices to show that $\varphi_0 \wedge \varphi_1 \in \mathcal{E}^1(X,\theta;\phi)$. This is proved in Proposition 3.1.12.

If furthermore $\varphi_0, \varphi_1 \in \mathcal{E}^\infty(X,\theta;\phi)$, then an argument as (4.12) shows that $\varphi_t \in \mathcal{E}^\infty(X,\theta;\phi)$ for all $t \in (0,1)$. □

prop:geodsupsublinear

Proposition 4.3.2 *Let $\varphi_1, \varphi_0 \in \mathcal{E}(X,\theta;\phi)$ with $\varphi_1 \leq \varphi_0$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$t \sup_{\{\varphi_0 \neq -\infty\}} (\varphi_1 - \varphi_0) = \sup_{\{\varphi_0 \neq -\infty\}} (\varphi_t - \varphi_0)$$

for all $t \in (0,1]$.

Proof After replacing φ_t by $\varphi_t - C't$ for some large enough $C' > 0$, we may assume that $\varphi_1 \leq \varphi_0$. It follows that $\varphi_1 \leq \varphi_t$ for all $t \in [0, 1]$.

Let

$$C = \sup_{\varphi_1 \neq -\infty} (\varphi_1 - \varphi_0) .$$

Then by Proposition 1.2.5, we have

$$\varphi_1 \leq \varphi_0 + C.$$

So $\varphi_1 - C(1-t)$ is a candidate in (4.10) and hence

$$\varphi_1 - C(1-t) \leq \varphi_t \tag{4.13}$$

{eq:varphi1leqvarphittemp}

for all $t \in (0, 1)$.

By Proposition 4.3.1, we have $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. Therefore, we can find a pluripolar set $Z \subseteq X$ such that $\varphi_t(x) \to \varphi_1(x) > -\infty$ as $t \to 1-$ for all $x \in X \setminus Z$. Here we applied Corollary 1.2.1 and the convexity of $t \mapsto \varphi_t(x)$. Observe that $\varphi_0 = \sup^*{}_{t\in(0,1)} \varphi_t$, therefore, after enlarging Z, we may also guarantee that $\varphi_t(x) \to \varphi_0(x) > -\infty$ as $t \to 0+$ for all $x \in X \setminus Z$ by Proposition 1.2.3.

For any such $x \in X \setminus Z$, $\varphi_t(x) \neq -\infty$ for any $t \in [0, 1]$. Therefore, $t \mapsto \varphi_t(x)$ is a real-valued continuous convex function on $[0, 1]$. Hence,

$$\varphi_1(x) - \varphi_0(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\,\mathrm{d}t \leq \lim_{t\to 1-} \frac{\varphi_1(x) - \varphi_t(x)}{1-t} \leq \lim_{t\to 1-} \frac{C(1-t)}{1-t} = C,$$

the inequality follows from (4.13).

Fix an arbitrary pluripolar set $Z' \supseteq Z$. Taking supremum, we find that

$$\begin{aligned}\sup_{x\in X\setminus Z'} \varphi_1(x) - \varphi_0(x) &= \sup_{x\in X, \varphi_0(x)\neq-\infty} \varphi_1(x) - \varphi_0(x) \\ &= \sup_{x\in X\setminus Z'} \lim_{t\to 1-} \frac{\varphi_1(x) - \varphi_t(x)}{1-t} = C.\end{aligned}$$

The first equality follows from Corollary 1.3.5.

Fix $s \in (0, 1)$. The same argument shows that after enlarging Z', we may guarantee that

$$\begin{aligned}\sup_{x\in X, \varphi_0(x)\neq-\infty} \varphi_1(x) - \varphi_0(x) &= \sup_{x\in X\setminus Z'} \lim_{t\to 1-} \frac{\varphi_1(x) - \varphi_t(x)}{1-t} \\ &= \sup_{x\in X, \varphi_0(x)\neq-\infty} \frac{\varphi_1(x) - \varphi_s(x)}{1-s}.\end{aligned}$$

On the other hand,

$$\sup_{\varphi_1\neq-\infty} (\varphi_1 - \varphi_0) \leq s \sup_{\varphi_1\neq-\infty} \frac{\varphi_s - \varphi_0}{s} + (1-s) \sup_{\varphi_1\neq-\infty} \frac{\varphi_1 - \varphi_s}{1-s}.$$

Using the convexity, we clearly have

$$\sup_{\varphi_1\neq-\infty}\frac{\varphi_s-\varphi_0}{s}=\sup_{\varphi_1\neq-\infty}(\varphi_1-\varphi_0).$$

Since the locus where φ_0,φ_1 or φ_s is identical to $-\infty$ is pluripolar, using Corollary 1.3.5, we find

$$\sup_{\varphi_0\neq-\infty}\frac{\varphi_s-\varphi_0}{s}=\sup_{\varphi_0\neq-\infty}\frac{\varphi_s-\varphi_0}{s}.$$

With an almost identical proof, we find

prop:geodinfsublinear

Proposition 4.3.3 *Let $\varphi_1,\varphi_0\in\mathcal{E}^\infty(X,\theta;\phi)$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$t\inf_{\{\phi\neq-\infty\}}(\varphi_1-\varphi_0)=\inf_{\{\phi\neq-\infty\}}(\varphi_t-\varphi_0)$$

for all $t\in(0,1]$.

Definition 4.3.3 Let $\ell=(\ell_t)_{t\geq0}$ be a curve in $\mathcal{E}(X,\theta;\phi)$. We say ℓ is a *geodesic ray* in $\mathcal{E}(X,\theta;\phi)$ emanating from ℓ_0 if for each $0\leq a\leq b$, the restriction $(\ell_t)_{t\in[a,b]}$ is a geodesic.

The set of geodesic rays in $\mathcal{E}(X,\theta;\phi)$ emanating from ϕ is denoted by $\mathcal{R}(X,\theta;\phi)$.

We say a geodesic ray $\ell\in\mathcal{R}(X,\theta;\phi)$ has finite energy if $\ell_t\in\mathcal{E}^1(X,\theta;\phi)$ for all $t>0$. The set of geodesic rays with finite energy is denoted by $\mathcal{R}^1(X,\theta;\phi)$.

Given $\ell,\ell'\in\mathcal{R}(X,\theta;\phi)$, we write $\ell\leq\ell'$ if for each $t\geq0$, $\ell_t\geq\ell'_t$.

prop:raysuplinear

Proposition 4.3.4 *Let $\ell\in\mathcal{R}(X,\theta;\phi)$. Then there is a constant $C>0$ such that*

$$\sup_X\ell_t\leq Ct,\quad t\geq0.$$

Proof We first observe that for any $t>0$, the set $Z=\{x\in X:\ell_t(x)=-\infty\}$ is the same. It follows from Proposition 4.3.2 that

$$\varphi_s\leq\phi+s\sup_{X\setminus Z}(\varphi_1-\phi).$$

Since $\varphi_1\in\mathcal{E}(X,\theta;\phi)$, we have $\varphi_1\leq\phi+C$ for some constant C and our conclusion follows. □

prop:energylinear2

Proposition 4.3.5 *Let $(\varphi_t)_{t\in[a,b]}$ be a geodesic in $\mathcal{E}^1(X,\theta;\phi)$, then $t\mapsto E^\phi_\theta(\varphi_t)$ is a convex function of $t\in[a,b]$.*

If $\phi=V_\theta$, the map is in fact linear.

We expect that $t\mapsto E^\phi_\theta(\varphi_t)$ is linear in general. The author does not know how to prove this.

Proof The first assertion is clear.

The second follows from the proofs of [DDNL18c, Theorem 3.12] and [DDNL18a, Proposition 3.13]. □

DDNL18fullmass DDNL18big

def:radialMAenergy2 **Definition 4.3.4** We define the *radial Monge–Ampère energy* $\mathbf{E}^{\phi}\colon \mathcal{R}^1(X,\theta;\phi) \to \mathbb{R}$ as follows:

$$\mathbf{E}^{\phi}(\ell) \coloneqq \lim_{t\to\infty} \frac{E^{\phi}_{\theta}(\ell_t)}{t}.$$

Thanks to Proposition 4.3.2, $\mathbf{E}^{\phi}(\ell) \in \mathbb{R}$.

def:d1onE12 **Definition 4.3.5** Let $\varphi, \psi \in \mathcal{E}^1(X,\theta;\phi)$, we define

$$d_1(\varphi,\psi) = E^{\phi}_{\theta}(\varphi) + E^{\phi}_{\theta}(\psi) - 2E^{\phi}_{\theta}(\varphi \wedge \psi).$$

In particular, if $\varphi \leq \psi$, we have

$$d_1(\varphi,\psi) = E^{\phi}_{\theta}(\psi) - E^{\phi}_{\theta}(\varphi). \tag{4.14}$$ {eq:d1asEdiff}

thm:d1complete **Theorem 4.3.1** *The function d_1 defined in Definition 4.3.5 is a complete metric on $\mathcal{E}^1(X,\theta;\phi)$.*

The function $E^{\phi}_{\theta}\colon \mathcal{E}^1(X,\theta;\phi) \to \mathbb{R}$ is continuous with respect to d_1.

Moreover, given a decreasing (resp. increasing) sequence $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathcal{E}^1(X,\theta;\phi)$ converging (resp. converging almost everywhere) to $\varphi \in \mathcal{E}^1(X,\theta;\phi)$, then $\varphi_j \xrightarrow{d_1} \varphi$.

See DDNL18big [DDNL18a, Theorem 1.1, Proposition 2.9, Proposition 2.7]. The readers should have no difficulty in generalizing all arguments to the current setting.

thm:d1lor **Theorem 4.3.2** *Let $\varphi, \psi, \eta \in \mathcal{E}^1(X,\theta;\phi)$. Then*

$$d_1(\varphi \vee \eta, \psi \vee \eta) \leq d_1(\varphi,\psi).$$

See Xia23Mabuchi [Xia23a, Proposition 4.12] (Proposition 6.8 in the arXiv version).

Chapter 5
Toric pluripotential theory on ample line bundles

chap:toric_ample

In this chapter, we develop the toric pluripotential theory relative to an ample line bundle. The general case of big line bundles will be handled after developing the powerful machinery of partial Okounkov bodies.

Let T be a complex torus of dimension n and $T_c \subset T(\mathbb{C})$ denotes the corresponding compact torus. Write M for its character lattice, which is a free Abelian group of rank n. Similarly, let N be cocharacter lattice of T. Let $P \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be a full-dimensional *smooth*[1] lattice polytope.

Let Σ be the normal fan of P and $\Sigma(1)$ denotes the set of rays in Σ. For each $\rho \in \Sigma(1)$, let $u_\rho \in N$ denote the ray generator of ρ, namely the first non-zero element in $N \cap \rho$. We write

$$P = \left\{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1) \right\}.$$

Let $\mathrm{Supp}_P \colon N_{\mathbb{R}} \to \mathbb{R}$ denote the support function of P. Recall that the support function (Example A.1.2) of P is defined as

$$\mathrm{Supp}_P(n) = \max \left\{ (m, n) : m \in P \right\}.$$

Our convention differs from [CLS11, Proposition 4.2.14] by a minus sign. Let $X = X_\Sigma$ be the corresponding smooth projective toric variety. There is a canonical embedding $T \subseteq X$ as a dense Zariski open subset. Let D be the Cartier divisor on X defined by P:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where D_ρ is the toric prime divisor defined by ρ under the orbit–cone correspondence. Let L be the toric line bundle induced by P, namely $L = \mathcal{O}_X(D_\rho)$. Since P has full dimension, L^k is very ample for each $k \geq n-1$ by [CLS11, Corollary 2.2.19], we actually know that L is ample.

[1] Recall that *smooth* means that for every vertex $v \in P$, if we take the first lattice point w_E apart from v as one transverses each edge E of P containing v from v, then $\{w_E - v\}_E$ forms a basis of M. See [CLS11, Definition 2.4.2]. We also say P is a *Delzant polytope* in this case.

We will choose the base e for the log map

$$\mathbb{C}^* \to \mathbb{R}, \quad z \mapsto \log |z|^2.$$

This choice will be fixed throughout the whole section. Since we have a canonical identification $T(\mathbb{C}) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$, we obtain an identification $T(\mathbb{C})/T_c \cong N_{\mathbb{R}}$. This gives a tropicalization map

$$\mathrm{Trop}\colon T(\mathbb{C}) \to N_{\mathbb{R}}.$$

5.1 Toric plurisubharmonic functions

lma:convextopsh

Lemma 5.1.1 *Let $F\colon N_{\mathbb{R}} \to [-\infty, \infty]$ be a function. Then the following are equivalent:*

(1) *F is convex and takes values in $\mathbb{R}$;*
(2) *$\mathrm{Trop}^* F$ is plurisubharmonic on $T(\mathbb{C})$.*

Proof We may choose an identification $N \cong \mathbb{Z}^n$ so that we have an identification $T(\mathbb{C}) \cong \mathbb{C}^{*n}$. Then Trop is identified with the map

$$\mathrm{Trop}\colon \mathbb{C}^{*n} \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto \left(\log |z_1|^2, \dots, \log |z_n|^2\right).$$

(1) $\implies$ (2). Let $F_k \in C^\infty(\mathbb{R}^n) \cap \mathrm{Conv}(\mathbb{R}^n)$ be a decreasing sequence with limit F (see Proposition A.3.3). It follows from a straightforward computation that

$$\mathrm{dd^c}\, \mathrm{Trop}^* F_k(z_1, \dots, z_n) = \frac{\mathrm{i}}{2\pi} \sum_{i,j=1}^n \partial_{ij} F_k \left(\log |z_1|^2, \dots, \log |z_n|^2\right) z_i^{-1} \overline{z_j}^{-1} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j}. \tag{5.1}$$

{eq:ddctrop}

So $\mathrm{Trop}^* F_k$ is plurisubharmonic. It follows from Proposition 1.2.1 that $\mathrm{Trop}^* F$ is plurisubharmonic.

(2) $\implies$ (1). It follows from Lemma 1.2.1 that F is finite. Moreover, take a radial mollifier, we may find a decreasing sequence φ_k of smooth psh functions on $\mathbb{C}^{*n}$ with limit $\mathrm{Trop}^* F$. Write $\varphi_k = \mathrm{Trop}^* F_k$ for some function $F_k\colon \mathbb{R}^n \to \mathbb{R}$, it follows from (5.1) that F_k is convex for all k. Therefore, F is convex by Lemma A.1.2. □

Let $G_0\colon M_{\mathbb{R}} \to (-\infty, \infty]$ be defined as

$$G_0(m) := \begin{cases} \frac{1}{2} \sum_{\rho \in \Sigma(1)} \left(\langle m, u_\rho \rangle + a_\rho\right) \log \left(\langle m, u_\rho \rangle + a_\rho\right), & \text{if } m \in P, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.2}$$

{eq:G0def}

This is a closed proper convex function and $G_0 \sim \chi_P$. Let

$$F_0 = G_0^* \in \mathcal{E}^\infty(N_{\mathbb{R}}, P). \tag{5.3}$$

{eq:F0def}

By Guillemin's theorem [Gui94, CDG03], $\mathrm{dd^c}\,\mathrm{Trop}^* F_0$ can be extended to a unique Kähler form ω in $c_1(L)$.

Let $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ denote the set of T_c-invariant ω-psh functions.

thm:toricpsh **Theorem 5.1.1** *There is a canonical bijection between the following three sets:*

(1) *the set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$,
(2) *the set* $\mathcal{P}(N_{\mathbb{R}}, P)$ *in Definition A.3.1, namely, the set of convex functions* $F\colon N_{\mathbb{R}} \to \mathbb{R}$ *satisfying* $F \leq \mathrm{Supp}_P$, *and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G|_{M_{\mathbb{R}}\setminus P} \equiv \infty.$$

Proof The bijection between (2) and (3) is the classical Legendre duality. Given F as in (2), we construct $G = F^*$. The bijection is proved in Proposition A.2.4.

The map from (1) to (2) is given as follows: given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, since φ is T_c-invariant, we can find $f\colon N_{\mathbb{R}} \to [-\infty,\infty)$ such that

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^* f.$$

We then define $F = f + F_0$. By Lemma 5.1.1, $F(n)$ is finite for any $n \in N_{\mathbb{R}}$ and F is convex. Moreover, $F \leq \mathrm{Supp}_P$ since this holds for F_0.

Conversely, given a map $F \in \mathcal{P}(N_{\mathbb{R}}, P)$, then

$$\mathrm{Trop}^*(F - F_0) \in \mathrm{PSH}(T(\mathbb{C}), \omega|_{T(\mathbb{C})}).$$

It follows from Theorem 1.2.1 that this function can be extended uniquely to an ω-psh function on X. The uniqueness of the extension guarantees its T_c-invariance.

The two maps are clearly inverse to each other. □

Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, we will write F_φ and G_φ for the convex functions given by Theorem 5.1.1.

Proposition 5.1.1 *Given* $\varphi,\psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. *The following are equivalent:*

(1) $\varphi \leq \psi$;
(2) $F_\varphi \leq F_\psi$;
(3) $G_\varphi \geq G_\psi$.

In particular, $\varphi \in \mathcal{E}^\infty(X,\theta)$ *if and only if* $F_\varphi \in \mathcal{E}^\infty(N_{\mathbb{R}}, P)$.

prop:toricpluscst **Proposition 5.1.2** *Given* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and* $C \in \mathbb{R}$. *We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

Both results follow immediately from the constructions of F and G. We leave the details to the readers.

prop:toricrooftop **Proposition 5.1.3** *Given* $\varphi,\psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, *then* $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and*

$$F_{\varphi\wedge\psi} = F_\varphi \wedge F_\psi, \quad G_{\varphi\wedge\psi} = G_\varphi \vee G_\psi.$$

Proof It is clear that $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. The claim for G is obvious and the claim for F follows from Proposition A.2.2. □

prop:toricseq

Proposition 5.1.4 *Let $\{\varphi_i\}_{i\in I}$ be a family in $\mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ uniformly bounded from above. Then $\sup^*_{i\in I} \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ and*

$$F_{\sup^*_{i\in I} \varphi_i} = \sup_{i\in I} F_{\varphi_i}, \quad G_{\sup^*_{i\in I} \varphi_i} = \mathrm{cl} \bigwedge_{i\in I} G_{\varphi_i}.$$

Moreover, if I is finite, then

$$G_{\max_{i\in I} \varphi_i} = \bigwedge_{i\in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i\in I}$ is a decreasing net in $\mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ such that $\inf_{i\in I} \varphi_i \not\equiv -\infty$, then $\inf_{i\in I} \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ and

$$F_{\inf_{i\in I} \varphi_i} = \inf_{i\in I} F_{\varphi_i}, \quad G_{\inf_{i\in I} \varphi_i} = \sup_{i\in I} G_{\varphi_i}.$$

Proof In both cases, the statement for F is clear. The corresponding statement for G is obtained via Proposition A.2.2. □

prop:toricMAandrealMA

Proposition 5.1.5 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$, then*

$$\mathrm{Trop}_* \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi). \tag{5.4}$$

{eq:tropMAmea}

In particular,

$$\int_X \omega_\varphi^n = \int_{N_\mathbb{R}} \mathrm{MA}_{\mathbb{R}}(F_\varphi) = n! \,\mathrm{vol}\, \overline{\{G_\varphi < \infty\}}$$

and

$$\int_X \omega^n = n! \,\mathrm{vol}\, P.$$

Proof We first prove (5.4). By Proposition A.3.3, we can find a decreasing sequence of smooth convex functions F_j on $N_\mathbb{R}$ with limit F_φ. We write $F_j = F_{\varphi_j}$ for some $\varphi_j \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. By Theorem 2.1.1 and Theorem A.4.1, we may reduce to the case where F_φ is smooth. Then it suffices to carry out the straightforward computation using (5.1). □

5.2 Envelopes

sec:envelopestoric

Let us begin by consider the P-envelope.

Definition 5.2.1 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. We define its *Newton body* as

$$\Delta(\omega, \varphi) := \overline{\{G_\varphi < \infty\}} \subseteq P.$$

By Proposition A.2.1, we have

$$\Delta(\omega,\varphi)=\overline{\nabla F_{\varphi}(N_{\mathbb{R}})}.$$

prop:GPenvelope **Proposition 5.2.1** *Let $\varphi\in\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Then $P_{\omega}[\varphi]\in\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and*

$$G_{P_{\omega}[\varphi]}(x)=\begin{cases}G_0(x), & \text{if } x\in\Delta(\omega,\varphi);\\ \infty, & \text{otherwise.}\end{cases} \tag{5.5}$$

{eq:toricPenv}

Proof By (3.2), we have

$$P_{\omega}[\varphi]=\sup_{C\in\mathbb{R}}{}^{*}\left((\varphi+C)\wedge 0\right).$$

It follows from Proposition 5.1.2, Proposition 5.1.3 and Proposition 5.1.4 that $P_{\omega}[\varphi]\in\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Moreover, by the same propositions, we have

$$G_{P_{\omega}[\varphi]}=\inf_{C\in\mathbb{R}}\left(G_0\vee(G_{\varphi}-C)\right),$$

which is clearly equal to the right-hand side of (5.5).

Next we prove a result of Yi Yao claiming that in the toric setting, all potentials are $\mathcal{I}$-good.

thm:Yao **Theorem 5.2.1** *Let $\varphi\in\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, then*

$$h^0(X,L\otimes\mathcal{I}(\varphi))=\#\left(\Delta(\omega,\varphi)\cap M\right).$$

Proof It is well-known that $H^0(X,L)$ can be identified with the vector space generated by χ^m for all $m\in P\cap M$, see [CLS11, Proposition 4.3.3]. We will show that

$$H^0(X,L\otimes\mathcal{I}(\varphi))=\bigoplus_{m\in\Delta(\omega,\varphi)\cap M}\mathbb{C}\chi^m. \tag{5.6}$$

{eq:toricL2sec}

It is convenient to use explicit coordinates. We will identify N with $\mathbb{Z}^n$ after choosing a basis. In this way, we get an identification $M=\mathbb{Z}^n$ and $T(\mathbb{C})=\mathbb{C}^{*n}$. In this case, we have

$$\chi^m(z)=z^m$$

with the multi-index notation.

Observe that $H^0(X,L\otimes\mathcal{I}(\varphi))$ is a $\mathbb{C}^{*n}$-invariant subspace of $H^0(X,L)$, it follows that $H^0(X,L\otimes\mathcal{I}(\varphi))$ is the direct sum of suitable χ^m's.

We first show that $\chi^m\in H^0(X,L\otimes\mathcal{I}(\varphi))$ for each $m\in\Delta(\omega,\varphi)\cap M$. We need to show that

$$\int_{\mathbb{C}^{*n}}|\chi^m|^2\exp(-P_{\omega}[\varphi])\,\omega^n<\infty.$$

Using Proposition 5.2.1 and Proposition 5.1.5, we find that the latter holds if and only if

$$\int_{\mathbb{R}^n} \exp\left(\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n)\right) \mathrm{MA}_{\mathbb{R}}(F_0)(n) < \infty,$$

which is obvious since

$$\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n) \leq 0.$$

Next we show that for any $m \in M \cap (P \setminus \Delta(\omega,\varphi))$, χ^m does not lie in $H^0(X, L^k \otimes \mathcal{I}(k\varphi))$. Again, this means

$$\int_{\mathbb{R}^n} \exp\left(\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n)\right) \mathrm{MA}_{\mathbb{R}}(F_0)(n) = \infty.$$

Since m does not lie in $\Delta(\omega,\varphi)$, we can find $n_0 \in \mathbb{R}^n$ such that

$$\langle m, n_0\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n_0) > 0.$$

We may take a small enough closed ball B containing n_0 such that

$$\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n) > 0$$

for all $n \in B$. Let C be the closed convex cone generated by B. Then there exists $\epsilon > 0$ such that

$$\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n) \geq \epsilon|n|$$

for all $n \in C$. Take a polyhedral cone D of full dimension contained in C and containing n_0 in the interior. Then D is defined by finitely many linear inequalities.

It therefore suffices to show that

$$\int_{D} \exp\left(\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n)\right) \mathrm{MA}_{\mathbb{R}}(F_0)(n) = \infty.$$

By change of variable, this holds if and only if

$$\int_{P\cap\{\nabla G_0 \subseteq D\}} \exp\left(\langle m, \nabla G_0(m')\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(\nabla G_0(m'))\right) \mathrm{d}m' = \infty,$$

which would follow if

$$\int_{P\cap\{\nabla G_0 \subseteq D\}} \exp\left(\epsilon|\nabla G_0(m')|\right) \mathrm{d}m' = \infty.$$

We shall write

$$n_0 = \sum_{\rho\in\Sigma} a_\rho u_\rho, \quad a_\rho < 0,$$

where $\Sigma \subseteq \Sigma(1)$ is a linearly independent subset. Let $\Sigma' \subseteq \Sigma(1)$ be a basis containing Σ. Let Q be the domain

$$Q = \left\{x \in P : \langle m', u_\rho\rangle + a_\rho \leq \epsilon' \text{ for } \rho \in \Sigma, \langle m', u_\rho\rangle + a_\rho \geq \delta \text{ for } \rho \in \Sigma(1) \setminus \Sigma\right\}$$

for suitable small $\epsilon', \delta > 0$. We will show that

$$\int_{Q\cap\{\nabla G_0 \subseteq D\}} \exp\left(\epsilon |\nabla G_0(m')|\right) \,\mathrm{d}m' = \infty. \tag{5.7}$$

{eq:intQfinitetemp}

It follows from (5.2) that

$$\nabla G_0(m') = \frac{1}{2} \sum_{\rho\in\Sigma(1)} \left(\log\left(\langle m', u_\rho\rangle + a_\rho\right) + 1\right) u_\rho.$$

So we could need to show

$$\int_{Q\cap\{\nabla G_0 \subseteq D\}} \exp\left(2^{-1}\epsilon \left|\sum_{\rho\in\Sigma} \left(\log\left(\langle m', u_\rho\rangle + a_\rho\right) + 1\right) u_\rho\right|\right) \mathrm{d}m' = \infty.$$

After possible replacing ϵ by a smaller constant, this would follow from the following estimate, for any $\rho \in \Sigma$, we have

$$\int_{Q\cap\{\nabla G_0 \subseteq D\}} \exp\left(-\epsilon \log\left(\langle m', u_\rho\rangle + a_\rho\right)\right) \,\mathrm{d}m' = \infty.$$

Next we change the coordinates from to $\log\langle m', u_\rho\rangle + a_\rho$ for all $\rho \in \Sigma'$, the above equation is obvious. □

cor:DXmaintoric

Corollary 5.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$, then*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) = n! \operatorname{vol} \Delta(\omega, \varphi).$$

In view of Corollary 5.2.1 and Theorem 7.3.1 proved later, we know that

$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}$$

always holds when $\int_X \theta_\varphi^n > 0$ in the toric setting. So we do not need to bother to study the $\mathcal{I}$-envelope separately in the toric setting.

5.3 Full mass potentials

We interpret the full mass potentials studied in Section 3.1.3 in the toric setting.

We have the following straightforward observation in the full mass case.

Proposition 5.3.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^\infty(X, \omega)$;
(2) $F_\varphi \sim F_0$;
(3) $G_\varphi \sim G_0$.

Proposition 5.3.2 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}(X,\omega)$;
(2) $F_\varphi \in \mathcal{E}(N_\mathbb{R}, P)$;
(3) $\overline{\mathrm{Dom}\, G_\varphi} = P$.

Proof (1) $\iff$ (3). By Proposition 5.1.5

$$\int_X \omega_\varphi^n = \int_{T(\mathbb{C})} \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = n!\,\mathrm{vol}\,\overline{\mathrm{Dom}\, G_\varphi}, \quad \int_X \omega^n = n!\,\mathrm{vol}\, P.$$

Therefore, (1) and (3) are equivalent.

(2) $\iff$ (3). This follows from Proposition A.2.1. □

Proposition 5.3.3 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, then*

$$E_\omega(\varphi) = n! \int_P \left(G_0 - G_\varphi\right)\,\mathrm{d}\,\mathrm{vol}\,.$$

Proof It suffices to consider the case where φ is bounded. In this case, one could apply [BB13, Proposition 2.9]. □

Corollary 5.3.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X,\omega)$;
(2) $F_\varphi \in \mathcal{E}^1(N_\mathbb{R}, P)$;
(3) $G_\varphi \in L^1(P)$.

Definition 5.3.1 We define

$$\begin{aligned}
\mathcal{E}^\infty_{\mathrm{tor}}(X,\omega) =& \mathcal{E}^\infty(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega),\\
\mathcal{E}^1_{\mathrm{tor}}(X,\omega) =& \mathcal{E}^1(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega),\\
\mathcal{E}_{\mathrm{tor}}(X,\omega) =& \mathcal{E}(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega).
\end{aligned}$$

cor:toricd1 **Corollary 5.3.2** *Let $\varphi,\psi \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$, then*

$$d_1(\varphi,\psi) = -n! \int_P \left(G_\varphi + G_\psi - 2G_{\varphi\vee\psi}\right)\,\mathrm{d}\,\mathrm{vol}\,.$$

5.4 Geodesics

prop:toricgeodseg **Proposition 5.4.1** *Let $\varphi_0,\varphi_1 \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$. The geodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 satisfies the following: for each $t \in (0,1)$, $\varphi_t \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$ and*

$$G_{\varphi_t} = (1-t)G_{\varphi_0} + tG_{\varphi_1}.$$

This will be proved more generally in Corollary 12.2.2.

Definition 5.4.1 We define

$$\mathcal{R}^1_{\mathrm{tor}}(X,\omega) := \left\{\ell \in \mathcal{R}^1(X,\omega) : \ell_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega) \text{ for all } t \geq 0\right\}.$$

Corollary 5.4.1 *Let $\ell \in \mathcal{R}^1_{\mathrm{tor}}(X,\omega)$. Then there is an integrable convex function $G' \in \mathrm{Conv}(N_{\mathbb{R}})$ with $\overline{\mathrm{Dom}\, G'} = P$ such that*

$$G_{\ell_t} = G_0 + tG'$$

for all $t \geq 0$.

We could also make Example 4.2.1 concrete.

Proposition 5.4.2 *Suppose that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Then the ray ℓ^φ defined in Example 4.2.1 satisfies:*

$$G_{\ell_t} = G_0 + tf_\ell, \quad f_\ell(x) = \min_{\substack{\lambda\in[0,1] \\ x_1\in P, x_0\in\Delta(\omega,\varphi) \\ \lambda x_1+(1-\lambda)x_0 = x}} \lambda$$

for any $t \geq 0$ and $x \in M_{\mathbb{R}}$.

Proof Recall that for each $C > 0$, we defined $(\ell_t^{\varphi,C})_t$ as the geodesic from 0 to $-C \vee \varphi$. By Proposition 5.1.2, Proposition 5.1.4, we have $G_{-C\vee\varphi} = (G_0 + C) \wedge G_\varphi$. So by Proposition 5.4.1, we have

$$G_{\ell_t^{\varphi,C}} = \frac{t}{C}\left((G_0 + C) \wedge G_\varphi\right) + \frac{C-t}{C} G_0$$

for each $t \in [0, C]$.

Recall that for all $t \geq 0$,

$$\ell_t = \sup_{C\geq t}{}^* \ell_t^{\varphi,C}.$$

It follows from Proposition 5.1.4 that

$$G_{\ell_t} = \mathrm{cl} \inf_{C\geq t} \frac{t}{C}\left((G_0 + C) \wedge G_\varphi\right) + \frac{C-t}{C} G_0.$$

Since the infimum is clearly linear, the closure operation is not needed and G_{ℓ_t} is linear in t. So it suffices to compute the slope f:

$$f_\ell := \inf_{C>0} \frac{1}{C}\left((G_0 + C) \wedge G_\varphi\right) - \frac{1}{C} G_0.$$

We compute this limit using Proposition A.1.2: for $x \in M_{\mathbb{R}}$, we compute the slope as follows

$$
\begin{aligned}
f_\ell(x) &= \inf_{C>0} \inf_{\substack{\lambda\in(0,1)\\ x_1,x_0\in M_{\mathbb{R}}\\ \lambda x_1+(1-\lambda)x_0=x}} \lambda\left(\frac{G_0(x_1)}{C}+1\right)+\frac{1-\lambda}{C}G_\varphi(x_0)-\frac{G_0(x)}{C} \\
&= \inf_{\substack{\lambda\in(0,1)\\ x_1,x_0\in M_{\mathbb{R}}\\ \lambda x_1+(1-\lambda)x_0=x}} \inf_{C>0} \lambda\left(\frac{G_0(x_1)}{C}+1\right)+\frac{1-\lambda}{C}G_\varphi(x_0)-\frac{G_0(x)}{C} \\
&= \min_{\substack{\lambda\in[0,1]\\ x_1\in P,\,x_0\in\Delta(\omega,\varphi)\\ \lambda x_1+(1-\lambda)x_0=x}} \lambda.
\end{aligned}
$$

Part II
The theory of $\mathcal{I}$-good singularities

In this part, we will develop the theory of $\mathcal{I}$-good singularities.

Chapter 6
Comparison of singularities

chap:comp

6.1 The P- and $\mathcal{I}$-partial orders

sec:PIpartialorder

Let X be a connected compact Kähler manifold of dimension n.

Recall that we have defined a partial order on $\mathrm{QPSH}(X)$ in Definition 1.5.2 to compare the singularity types of quasi-plurisubharmonic functions. The problem with this partial order is that it is too fine. In general, for our interest, it is helpful to consider rougher relations.

6.1.1 The definitions of the partial orders

Recall that the P-envelope is defined in Definition 3.1.2.

def:Pmoresing

Definition 6.1.1 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is *P-more singular than* ψ and write $\varphi \preceq_P \psi$ if for some closed smooth real $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, we have

$$P_\theta[\varphi] \leq P_\theta[\psi].$$

Suppose that $\varphi \preceq_P \psi$ and $\psi \preceq_P \varphi$, we shall write $\varphi \sim_P \psi$ and say φ and ψ have the same *P-singularity type*.

We need to show that the definition is independent of the choice of θ.

lma:Pproj_insens_omega

Lemma 6.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. For any Kähler form ω on X, the following are equivalent:*

(1) $P_\theta[\varphi] \leq P_\theta[\psi]$;
(2) $P_{\theta+\omega}[\varphi] \leq P_{\theta+\omega}[\psi]$.

Proof (1) implies (2): Observe that

$$P_\theta[\varphi] \leq P_{\theta+\omega}[\varphi], \quad \varphi \leq P_\theta[\varphi].$$

It follows that

$$P_{\theta+\omega}[\varphi] = P_{\theta+\omega}[P_\theta[\varphi]]. \tag{6.1}$$

{eq:doubleP}

A similar formula holds for ψ. So we see that (2) holds.

(2) implies (1): By (6.1), we may assume that φ and ψ are both model potentials in $\mathrm{PSH}(X,\theta)$.

Observe that $\varphi \vee \psi \leq P_{\theta+\omega}[\psi]$. It follows that $P_{\theta+\omega}[\varphi\vee\psi] \leq P_{\theta+\omega}[\psi]$. The reverse inequality is trivial, so

$$P_{\theta+\omega}[\varphi\vee\psi] = P_{\theta+\omega}[\psi].$$

From the direction we have proved, for any $C \geq 1$,

$$P_{\theta+C\omega}[\varphi\vee\psi] = P_{\theta+C\omega}[\psi].$$

So by Proposition 3.1.2,

$$\int_X \left(\theta + C\omega + \mathrm{dd^c}(\varphi\vee\psi)\right)^n = \int_X \left(\theta + C\omega + \mathrm{dd^c}\psi\right)^n .$$

Since both sides are polynomials in C, the equality extends to $C = 0$, namely,

$$\int_X \theta^n_{\varphi\vee\psi} = \int_X \theta^n_\psi .$$

As φ and ψ are both model, it follows that $\varphi\vee\psi = \psi$. So (1) follows. □

prop:Pequivchar2 **Proposition 6.1.1** *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \leq \psi$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$*;*
(2) *For each $j = 0,\dots,n$, we have*

$$\int_X \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_\psi \wedge \theta^{n-j}_{V_\theta}. \tag{6.2}$$

{eq:mixedmassequal}

Assume furthermore that $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$, then these conditions are equivalent to the following:

(3) *we have*

$$\int_X \theta^n_\varphi = \int_X \theta^n_\psi .$$

Proof We first prove the equivalence between 1 and 3 when $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$.

(1) $\Longrightarrow$ (3). Assume that $\varphi \sim_P \psi$. By Definition 6.1.1, we have

$$P_\theta[\varphi] = P_\theta[\psi].$$

So (3) follows from Proposition 3.1.2.

(3) $\Longrightarrow$ (1). It follows from Theorem 3.1.1 that $P_\theta[\varphi] = P_\theta[\psi]$, so (1) follows.

Let us come back to the general case.

(1) $\implies$ (2). Fix $j \in \{0, \dots, n\}$, we argue (6.2).

Take a Kähler form ω on X. By Definition 6.1.1, for each $\epsilon > 0$, we have

$$P_{\theta+\epsilon\omega}[\varphi] = P_{\theta+\epsilon\omega}[\psi].$$

It follows from Proposition 3.1.2 that

$$\begin{aligned}\int_X (\theta + \epsilon\omega + \mathrm{dd^c}\psi)^j \wedge \theta_{V_\theta}^{n-j} &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\psi])^j \wedge \theta_{V_\theta}^{n-j} \\ &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi])^j \wedge \theta_{V_\theta}^{n-j} \\ &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c}\varphi)^j \wedge \theta_{V_\theta}^{n-j}.\end{aligned}$$

Since the two extremes are both polynomials in ϵ, we conclude that the same holds when $\epsilon = 0$, that is, (6.2) holds.

(2) $\implies$ (1). Assume (6.2) holds for all j. For each $t \in (0, 1)$, we have

$$\int_X \theta^n_{t\varphi+(1-t)V_\theta} = \int_X \theta^n_{t\psi+(1-t)V_\theta}$$

by the binomial expansion. By the implication (3) $\implies$ (1), we have

$$t\varphi + (1-t)V_\theta \sim_P t\psi + (1-t)V_\theta$$

for each $t \in (0, 1)$.

Fix a Kähler form ω on X. From the implication (1) $\implies$ (3), we have

$$\int_X (\theta + \omega)^n_{t\varphi+(1-t)V_\theta} = \int_X (\theta + \omega)^n_{t\psi+(1-t)V_\theta}.$$

Since both sides are polynomials in t, the same holds when $t = 1$. From the implication (3) $\implies$ (1) again, we have $\varphi \sim_P \psi$. □

prop:Iequivchar2

Proposition 6.1.2 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *for any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) \subseteq \mathcal{I}(k\psi),$$

(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) \subseteq \mathcal{I}(\lambda\psi),$$

(3) *for any modification $\pi\colon Y \to X$ and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y),$$

(4) *for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold and any $y \in Y$, we have*

$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y),$$

and

(5) *for any prime divisor E over X, we have*

$$\nu(\varphi, E) \geq \nu(\psi, E).$$

Proof The proof is almost identical to that of Proposition 3.2.1, we omit the details. □

Definition 6.1.2 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is *$\mathcal{I}$-more singular than* ψ and write $\varphi \preceq_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 3.2.1 are satisfied.

Note that $\varphi \preceq_{\mathcal{I}} \psi$ and $\psi \preceq_{\mathcal{I}} \varphi$ both hold if and only if $\varphi \sim_{\mathcal{I}} \psi$ in the sense of Definition 3.2.1.

prop:Icomparandenvelope **Proposition 6.1.3** *Suppose that $\varphi, \psi \in \mathrm{QPSH}(X)$ and θ is a closed real smooth $(1,1)$-form on X such that $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Then the following are equivalent:*

(1) *$\varphi \preceq_{\mathcal{I}} \psi$;*
(2) *$P_\theta[\varphi]_{\mathcal{I}} \leq P_\theta[\psi]_{\mathcal{I}}$.*

Proof (1) $\implies$ (2). This follows immediately from Definition 3.2.2.
(2) $\implies$ (1). This follows from Proposition 3.2.6. □

lma:reform_preceqP **Lemma 6.1.2** *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Then the following are equivalent:*

(1) *$\varphi \preceq_P \psi$ (resp. $\varphi \preceq_{\mathcal{I}} \psi$);*
(2) *$\varphi \lor \psi \sim_P \psi$ (resp. $\varphi \lor \psi \sim_{\mathcal{I}} \psi$).*

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. We only prove the P case, the $\mathcal{I}$ case is similar.
(2) $\implies$ (1). By (2), $P_\theta[\varphi \lor \psi] = P_\theta[\psi]$. But $\varphi \leq P_\theta[\varphi \lor \psi]$, so (1) follows.
(1) $\implies$ (2). We may assume that φ, ψ are both model in $\mathrm{PSH}(X, \theta)_{>0}$ as

$$P_\theta[\varphi \lor \psi] = P_\theta[P_\theta[\varphi] \lor P_\theta[\psi]].$$

Then $\varphi \leq \psi$ and (2) follows. □

cor:PimpliesI **Corollary 6.1.1** *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Assume that $\varphi \preceq_P \psi$, then $\varphi \preceq_{\mathcal{I}} \psi$.*

Proof This follows from Lemma 6.1.2 and Proposition 3.2.8. □

cor:Pvarphidef3 **Corollary 6.1.2** *Assume that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then*

$$\begin{aligned} P_\theta[\varphi] &= \sup\{\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \psi \sim_P \varphi\} \\ &= \sup\{\psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \psi \preceq_P \varphi\}. \end{aligned}$$

Proof Note that $\psi \sim_P \varphi$ implies that $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ by Proposition 6.1.4. So the first equality is a direct consequence of Proposition 6.1.1 and Theorem 3.1.1.

Next we prove the second equality. We only need to show that for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \leq 0$ and $\psi \preceq_P \varphi$, we have $\psi \leq P_\theta[\varphi]$.

By Lemma 6.1.2, we know that $P_\theta[\varphi] \vee \psi \sim_P \varphi$ and $P_\theta[\varphi] \vee \psi \leq 0$. It follows from the first equality that $\psi \leq P_\theta[\varphi]$. □

Similarly, we have

cor:Ienvelopedef2 **Corollary 6.1.3** *Assume that $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq_{\mathcal{I}} \varphi\right\}.$$

6.1.2 Properties of the partial orders

Now we state a more natural version of the monotonicity theorem Theorem 2.3.2.

prop:mono2 **Proposition 6.1.4** *Let $\theta_1,\dots,\theta_n$ be closed real smooth $(1,1)$-forms on X. Let $\varphi_i,\psi_i \in \mathrm{PSH}(X,\theta_i)$ for $i=1,\dots,n$. Assume that $\varphi_i \preceq_P \psi_i$ for each i. Then*

$$\int_X \theta_{\varphi_1} \wedge \dots \wedge \theta_{\varphi_n} \leq \int_X \theta_{\psi_1} \wedge \dots \wedge \theta_{\psi_n}.$$

Proof Fix a Kähler form ω on X. For each $i=1,\dots,n$, since $\varphi_i \preceq_P \psi_i$, we have

$$P_{\theta+\epsilon\omega}[\varphi_i] \leq P_{\theta+\epsilon\omega}[\psi_i]$$

for all $\epsilon > 0$. Therefore, by Proposition 3.1.2 and Theorem 2.3.2, we have

$$\int_X (\theta+\epsilon\omega)_{\varphi_1} \wedge \dots \wedge (\theta+\epsilon\omega)_{\varphi_n} \leq \int_X (\theta+\epsilon\omega)_{\psi_1} \wedge \dots \wedge (\theta+\epsilon\omega)_{\psi_n}.$$

Since both sides are polynomials in ϵ, we find that the same holds at $\epsilon = 0$, which is the desired inequality. □

prop:Ppartialsum **Proposition 6.1.5** *Let $\varphi,\psi,\varphi',\psi' \in \mathrm{QPSH}(X)$. Assume that*

$$\varphi \preceq_P \psi, \quad \varphi' \preceq_P \psi'.$$

Then

$$\varphi + \varphi' \preceq_P \psi + \psi'.$$

The same holds with $\preceq_{\mathcal{I}}$ in place of $\preceq_P$.

Proof Take a Kähler form ω on X such that $\varphi,\psi,\varphi',\psi' \in \mathrm{PSH}(X,\omega)_{>0}$. The statement for $\preceq_{\mathcal{I}}$ is a simple consequence of Proposition 1.4.2. We only need to handle the case of $\preceq_P$.

Step 1. We first show that

$$P_\omega[\varphi] + P_\omega[\varphi'] \sim_P \varphi + \varphi'.$$

In fact, we clearly have

$$P_\omega[\varphi] + P_\omega[\varphi'] \geq \varphi + \varphi'.$$

So it suffices to show that they have the same volume. We compute

$$\begin{aligned}
&\int_X (2\omega + \mathrm{dd^c} P_\omega[\varphi] + \mathrm{dd^c} P_\omega[\varphi'])^n \\
=& \sum_{j=0}^n \binom{n}{j} \int_X (\omega + \mathrm{dd^c} P_\omega[\varphi])^j \wedge (\omega + \mathrm{dd^c} P_\omega[\varphi'])^{n-j} \\
=& \sum_{j=0}^n \binom{n}{j} \int_X \omega_\varphi^j \wedge \omega_{\varphi'}^{n-j} \\
=& \int_X (2\omega + \varphi + \varphi')^n ,
\end{aligned}$$

where we applied Proposition 3.1.2 on the third line.

Step 2. By Step 1, we may assume that $\varphi, \psi, \varphi', \psi'$ are all model potentials. So $\varphi \leq \psi$ and $\varphi' \leq \psi'$. Our assertion follows. □

prop:Ppartialsup

Proposition 6.1.6 *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be uniformly bounded from above non-empty families in $\mathrm{QPSH}(X)$. Assume that there exists a closed smooth real $(1,1)$-form θ such that $\varphi_i, \psi_i \in \mathrm{PSH}(X,\theta)$ and $\varphi_i \preceq_P \psi_i$ for all $i \in I$. Then*

$$\sup_{i\in I}{}^* \varphi_i \preceq_P \sup_{i\in I}{}^* \psi_i.$$

The same holds with $\preceq_I$ in place of $\preceq_P$.

Proof By increasing θ, we may assume that $\varphi_i, \psi_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$. The statement for $\preceq_I$ is a simple consequence of Corollary 1.4.1, we only have to consider the statement for $\preceq_P$.

Step 1. We first handle the case where I is a directed set and $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are increasing nets.

In this case, our assertion follows simply from Proposition 3.1.9.

Step 2. We handle the case where I is finite. We may assume that $I = \{0, 1\}$. It suffices to show that

$$P_\theta[\varphi_0] \vee P_\theta[\varphi_1] \sim_P \varphi_0 \vee \varphi_1.$$

For this purpose, it suffices to prove the following:

$$P_\theta[\varphi_0] \vee \varphi_1 \sim_P \varphi_0 \vee \varphi_1.$$

The $\succeq_P$ direction is obvious. So it suffices to argue that they have the same mass. We may assume that $\varphi_0 \leq 0$. Thanks to Lemma 2.3.1, for each $\epsilon \in (0,1)$, we can find $\eta_\epsilon \in \mathrm{PSH}(X,\theta)_{>0}$ such that

$$(1-\epsilon)P_\theta[\varphi_0] + \epsilon\eta \le \varphi_0.$$

Observe that $\eta \le \varphi_0 \le P_\theta[\varphi_0]$. In particular,

$$(1-\epsilon)\left(P_\theta[\varphi_0] \vee \varphi_1\right) + \epsilon\eta \le \varphi_0 \vee \varphi_1.$$

It follows from Theorem 2.3.2 that

$$(1-\epsilon)^n \int_X \theta^n_{P_\theta[\varphi_0]\vee\varphi_1} \le \int_X \theta^n_{\varphi_0\vee\varphi_1}.$$

Letting $\epsilon \to 0+$ and using Theorem 2.3.2 again, we conclude that

$$\theta^n_{P_\theta[\varphi_0]\vee\varphi_1} = \int_X \theta^n_{\varphi_0\vee\varphi_1}.$$

Our assertion is proved.

Step 3. The general case can be reduced to the two cases handled in Step 1 and Step 2. More precisely, by Proposition 1.2.2, we could find a countable subset $J \subseteq I$ such that

$$\sup_{j\in J}{}^* \varphi_j = \sup_{i\in I}{}^* \varphi_i, \quad \sup_{i\in I}{}^* \psi_j = \sup_{i\in I}{}^* \psi_i.$$

We may replace I by J and assume that I is countable. We may assume that I is infinite, as otherwise, we could apply Step 2 directly. So let us assume that $J = \mathbb{Z}_{>0}$. In this case, by Step 2 again, we may assume that both $(\varphi_i)_i$ and $(\psi_i)_i$ are increasing, which is the situation of Step 1.

6.2 The d_S-pseudometric

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. The goal of this section is to study a pseudometric on the space $\mathrm{PSH}(X,\theta)$.

6.2.1 The definition of the d_S-pseudometric

Recall that for any $\varphi \in \mathrm{PSH}(X,\theta)$, the geodesic ray $\ell^\varphi \in \mathcal{R}^1(X,\theta)$ is defined in Example 4.2.1.

def:dS **Definition 6.2.1** For $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we define

$$d_S(\varphi,\psi) := d_1(\ell^\varphi, \ell^\psi).$$

When we want to be more specific, we write $d_{S,\theta}$ instead of d_S.

Proposition 6.2.1 *The function d_S defined in Definition 6.2.1 is a pseudometric on* $\mathrm{PSH}(X,\theta)$.

Proof This follows immediately from Theorem 4.2.1. □

When styding a pseudometric, the first thing is to understand when the distance between two elements vanishes.

We first prove a preparation:

lma:dSalmostriang **Lemma 6.2.1** *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)$. Then*

$$d_S(\varphi,\psi) \leq d_S(\varphi,\varphi\vee\psi) + d_S(\psi,\varphi\vee\psi) \leq C_n d_S(\varphi,\psi),$$

where $C_n = 3(n+1)2^{n+2}$.

Proof Observe that

$$\ell^{\varphi} \vee \ell^{\psi} = \ell^{\varphi\vee\psi}. \tag{6.3}$$ {eq:elllorsingtype}

In fact, it is clear that

$$\ell^{\varphi} \leq \ell^{\varphi\vee\psi}, \quad \ell^{\psi} \leq \ell^{\varphi\vee\psi},$$

so the $\leq$ direction in (6.3) holds.

Conversely, if $\ell' \in \mathcal{R}^1(X,\theta)$ and $\ell' \geq \ell^{\varphi}\vee\ell^{\psi}$, then for each $t \geq 0$,

$$\ell'_t \geq ((V_\theta - t)\vee\varphi)\vee((V_\theta - t)\vee\psi) = (V_\theta - t)\vee(\varphi\vee\psi).$$

It follows that $\ell' \geq \ell^{\varphi\vee\psi}$.

So our assertion follows from Lemma 4.2.1. □

prop:ds0char **Proposition 6.2.2** *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) $d_S(\varphi,\psi) = 0$.

In particular, $d_S(\varphi, P_\theta[\varphi]) = 0$ for all $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proof By Lemma 6.1.2, we have $\varphi \sim_P \psi$ if and only if $\varphi \sim_P \varphi\vee\psi$ and $\psi \sim_P \varphi\vee\psi$. By Lemma 6.2.1, $d_S(\varphi,\psi) = 0$ if and only if $d_S(\varphi,\varphi\vee\psi) = 0$ and $d_S(\psi,\varphi\vee\psi) = 0$. So it suffices to prove the assertion when $\varphi \leq \psi$. Assuming this, by Proposition 4.2.5 we have that 2 holds if and only if

$$\mathbf{E}(\ell^{\varphi}) = \mathbf{E}(\ell^{\psi}),$$

But using (4.5), this holds if and only if

$$\sum_{j=0}^{n} \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \sum_{j=0}^{n} \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}.$$

But by Theorem 2.3.2, this holds if and only if for all $j = 0,\dots,n$,

$$\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j},$$

which is equivalent to 1 by Proposition 6.1.1. □

lma:varphileqpsi_metric **Lemma 6.2.2** *Suppose that $\varphi,\psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi$, then*

$$d_S(\varphi,\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}\right).$$

Proof This follows trivially from (4.5). □

cor:dsthreeterm **Corollary 6.2.1** *Suppose that $\varphi,\psi,\eta \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi \preceq_P \eta$. Then*

$$d_S(\varphi,\eta) \geq d_S(\varphi,\psi), \quad d_S(\varphi,\eta) \geq d_S(\psi,\eta).$$

Proof This is an immediate consequence of Lemma 6.2.2 and Proposition 6.1.4. □

cor:dsmetricdoubleineq **Corollary 6.2.2** *For any $\varphi,\psi \in \mathrm{PSH}(X,\theta)$, we have*

$$\begin{aligned} d_S(\varphi,\psi) \leq & \sum_{j=0}^{n}\left(2\int_X \theta_{\varphi\vee\psi}^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}\right) \\ \leq & C_n d_S(\varphi,\psi), \end{aligned} \tag{6.4}$$ {eq:ds_biineq}

where $C_n = 3(n+1)2^{n+2}$.

In particular, if $(\varphi_i)_{i\in I}$ is a net in $\mathrm{PSH}(X,\theta)$ with d_S-limit φ, then for each $j = 0,\dots,n$,

$$\lim_{i\in I}\int_X \theta_{\varphi_i}^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_{\varphi_i\vee\varphi}^j \wedge \theta_{V_\theta}^{n-j}$$

Proof The estimates (6.4) follows from the combination of Lemma 6.2.2 and Lemma 6.2.1.

The last assertion follows from (6.4) and Theorem 2.3.2. □

cor:incseqdSconv **Corollary 6.2.3** *Suppose that $\varphi_i \in \mathrm{PSH}(X,\theta)$ ($i \in I$) be an increasing net, uniformly bounded from above. Then*

$$\varphi_i \xrightarrow{d_S} \sup_{j\in I}{}^* \varphi_j.$$

Proof Write $\varphi = \sup^*_{j\in I}\varphi_j$. Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X,\theta)$. By Lemma 6.2.2, it suffices to show that for each $k = 0,\dots,n$, we have

$$\lim_{j\in I}\int_X \theta_{\varphi_j}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}.$$

The latter follows from Corollary 2.3.1. □

By constrast, for decreasing nets, the situation is different:

cor:decnetdS **Corollary 6.2.4** *Suppose that $\varphi_i \in \mathrm{PSH}(X,\theta)$ is a decreasing net such that $\varphi := \inf_{i\in I}\varphi_i \not\equiv -\infty$. Then the following are equivalent:*

(1) *we have*

$$\varphi_i \xrightarrow{d_S} \varphi;$$

(2) *for each $k = 0,\ldots,n$, we have*

$$\lim_{j\in I}\int_X \theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta} = \int_X \theta^k_{\varphi}\wedge\theta^{n-k}_{V_\theta}. \tag{6.5}$$ {eq:mixedmasslim}

If we assume furthermore that $\int_X \theta^n_\varphi > 0$, then the above conditions are equivalent to

(3) *we have*

$$\lim_{j\in I}\int_X \theta^n_{\varphi_j} = \int_X \theta^n_\varphi.$$

In the latter case, we also have

$$P_\theta[\varphi] = \inf_{j\in I} P_\theta[\varphi_j]. \tag{6.6}$$ {eq:Pcontdecseq}

Proof Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X,\theta)$.

(1) $\iff$ (2). This follows immediately from Lemma 6.2.2.

(2) $\implies$ (3). This is trivial.

(3) $\implies$ (2). Let $(b_j)_{j\in I}$ be a net converging to ∞ such that

$$b_j \in \left(1, \left(\frac{\int_X \theta^n_{\varphi_j}}{\int_X \theta^n_{\varphi_j} - \int_X \theta^n_\varphi}\right)^{1/n}\right).$$

By Lemma 2.3.1, for each $j \in I$, we can find $\eta_j \in \mathrm{PSH}(X,\theta)$ such that

$$b_j^{-1}\eta_j + (1-b_j^{-1})\varphi_j \le \varphi.$$

It follows from Theorem 2.3.2 that for any $k = 0,\ldots,n$,

$$\int_X \theta^k_\varphi \wedge \theta^{n-k}_{V_\theta} \ge (1-b_j^{-1})^k \int_X \theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta}.$$

Taking the limit, we conclude the $\le$ direction in (6.5). The $\ge$ direction follows from Theorem 2.3.2.

Finally, we argue (6.6).

Let $\psi_j = P_\theta[\varphi_j]$. It follows from Corollary 3.1.1 that ψ_j is a model potential. Let

$$\psi = \inf_{j\in I}\psi_j.$$

It follows from Proposition 3.1.2 and Proposition 3.1.8 that

$$\int_X \theta_\psi^n = \lim_{j\in I}\int_X \theta_{\psi_j}^n = \lim_{j\in I}\int_X \theta_{\varphi_j}^n = \int_X \theta_\varphi^n.$$

By Proposition 3.1.7, ψ is a model potential. So by Proposition 6.1.1, we have $\varphi \sim_P \psi$ and hence $\psi = P_\theta[\varphi]$ by Corollary 6.1.2. □

Having understood the increasing and decreasing cases, we shall handle more general convergent sequences. In fact, since d_S is a pseudometric, the topology is completely determined by convergent sequences, so we do not need to consider nets in general.

prop:incanddec

Proposition 6.2.3 *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \geq 1$), $\varphi_j \xrightarrow{d_S} \varphi$. Assume that there is $\delta > 0$ such that*

$$\int_X \theta_{\varphi_j}^n \geq \delta, \quad \int_X \theta_\varphi^n \geq \delta$$

for all j and the φ_j's and φ are all model potentials. Then up to replacing $(\varphi_j)_j$ by a subsequence, there is a decreasing sequence $\psi_j \in \mathrm{PSH}(X,\theta)$ and an increasing sequence $\eta_j \in \mathrm{PSH}(X,\theta)$ such that

(1) $\psi_j \xrightarrow{d_S} \varphi$, $\eta_j \xrightarrow{d_S} \varphi$;
(2) $\psi_j \geq \varphi_j \geq \eta_j$ *for all j.*

In fact, for any $j \geq 1$, we will take

$$\eta_j = \inf_{k\in\mathbb{N}} \varphi_j \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_{j+k}, \quad \psi_j = \sup_{k\geq j}{}^* \varphi_k.$$

Proof We are free to replace $(\varphi_j)_j$ by a subsequence. So we may assume that

$$d_S(\varphi_j,\varphi_{j+1}) \leq C_n^{-2j}, \quad d_S(\varphi,\varphi_j) \leq \frac{2^{-j-2}}{(n+1)C_n}, \tag{6.7}$$

{eq:conditiononvarphijtemp1}

where C_n is the constant in Corollary 6.2.2.

Step 1. We handle the ψ_j's. For each $j \geq 1$ and $k \geq 1$, by Corollary 6.2.2 we have

$$\begin{aligned} d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \leq & C_n d_S(\varphi_j, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \\ \leq & C_n d_S(\varphi_j,\varphi_{j+1}) + C_n d_S(\varphi_{j+1}, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}). \end{aligned}$$

By iteration, we find

$$\begin{aligned} d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) &\leq \sum_{a=j}^{j+k-1} C_n^{a+1-j} d_S(\varphi_a,\varphi_{a+1}) \\ &\leq \sum_{a=j}^{j+k-1} C_n^{a+1-j} C_n^{-2a} = \frac{C_n^{1-2j}}{1-C_n^{-1}}. \end{aligned}$$

Using Corollary 6.2.3, we have

$$\varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k} \xrightarrow{d_S} \psi_j$$

as $k \to \infty$ and hence when $j \geq j_0$ for some j_0, we have

$$d_S(\varphi_j, \psi_j) \leq \frac{C_n^{1-2j}}{1 - C_n^{-1}} \leq \frac{1}{(n+1)C_n 2^{2+j}}. \tag{6.8}$$

{eq:dsvarphijpsijesttemp1}

We conclude that $\psi_j \xrightarrow{d_S} \varphi$.

Moreover, we observe that

$$\varphi = \inf_j P_\theta[\psi_j] \tag{6.9}$$

{eq:varphiexpressiontemp1}

by Corollary 6.2.4.

Step 2. We consider the η_j's.

For each $j \geq 1$ and $k \geq 0$, we let

$$\eta_j^k \coloneqq \varphi_j \wedge \cdots \wedge \varphi_{j+k}.$$

Using the assumption (6.7) and Corollary 6.2.2, we have

$$\left| \int_X \theta_{\varphi_j}^n - \int_X \theta_\varphi^n \right| \leq 2^{-j}.$$

Similarly, using (6.8), we have

$$\left| \int_X \theta_{\psi_j}^n - \int_X \theta_\varphi^n \right| \leq 2^{-j}.$$

Step 2-1. Take j_1 so that for $j \geq j_1$, $2^{3-j} < \delta$. We claim that for a fixed $j \geq j_0 \vee j_1$, for any $k \in \mathbb{N}$, we have $\eta_j^k \in \mathrm{PSH}(X, \theta)$ and

$$\int_X \theta_{\eta_j^k} \geq \int_X \theta_{\varphi_j}^n - \sum_{a=0}^{k} 2^{-j-a+2}.$$

We argue by induction on $k \geq 0$. The case $k = 0$ follows from Theorem 2.3.2. When $k > 0$, assume that the case $k - 1$ is known. Then

$$\begin{aligned} \int_X \theta_{\eta_j^{k-1}}^n + \int_X \theta_{\varphi_{j+k}}^n &> \int_X \theta_{\varphi_j}^n - \sum_{a=0}^{k-1} 2^{2-j-a} + \int_X \theta_{\psi_{j+k-1}}^n - 2^{2-j-k} \\ &\geq \int_X \theta_{\varphi_j}^n - 2^{3-j} + \int_X \theta_{\psi_{j+k-1}}^n > \int_X \theta_{\psi_{j+k-1}}^n. \end{aligned}$$

It follows from Proposition 3.1.6 that $\eta_j^k \in \mathrm{PSH}(X, \theta)$. By Theorem 3.1.3, we deduce that

$$\int_X \theta^n_{\varphi_{j+k}} + \int_X \theta^n_{\eta_j^{k-1}} \leq \int_X \theta^n_{\psi_{j+k-1}} + \int_X \theta^n_{\eta_j^k}.$$

Our claim therefore follows.

Step 2-2. It follows from Proposition 3.1.5 that

$$P_\theta[\eta_j^k] = \eta_k^j.$$

By Proposition 3.1.8, we have

$$\lim_{k\to\infty} \int_X \theta^n_{\varphi_j^k} = \int_X \theta^n_{\eta_j}.$$

By Step 1, for large enough j, we have

$$\int_X \theta^n_{\eta_j} \geq \int_X \theta^n_{\varphi_j} - 2^{3-j} > 0.$$

Let $\eta = \sup^*{}_j \eta^j$. Observe that we also have

$$\int_X \theta^n_{\eta_j} \leq \int_X \theta^n_{\psi_j}$$

by Theorem 2.3.2. It follows that

$$\int_X \theta^n_\eta = \lim_{j\to\infty} \int_X \theta^n_{\varphi_j} = \lim_{j\to\infty} \int_X \theta^n_{\psi_j} = \int_X \theta^n_\varphi.$$

Since $\eta_j \leq \varphi_j \leq \psi_j \leq 0$, we also have that $\eta_j \leq P_\theta[\psi_j]$. Therefore, by Corollary 6.2.4, we also have $\eta \leq \varphi$. It follows from Proposition 6.1.1 that $\eta \sim_P \varphi$. By Corollary 6.2.3 and Proposition 6.2.2, we have $\eta^j \xrightarrow{d_S} \varphi$. □

cor:completenessdS

Corollary 6.2.5 *Let $(\varphi_j)_{j\in I}$ be a net in* PSH(X,θ)*. Assume that there is $\delta > 0$ such that $\int_X \theta^n_{\varphi_j} \geq \delta$ for all $j \in I$. Then $(\varphi_j)_{j\in I}$ has a d_S-convergent subnet.*

If moreover $(\varphi_j)_{j\in I}$ is decreasing, then $(\varphi_j)_{j\in I}$ itslef is convergent.

Proof Since the space of $\varphi \in$ PSH(X,θ) with $\int_X \theta^n_\varphi \geq \delta$ is a pseudometric space, its completeness can be characterized using sequences instead of nets. So we may assume that $(\varphi_j)_{j\in I}$ is a sequence.

Replacing φ_j by a subsequence, we may assume that (6.7) holds. By the proof of Proposition 6.2.3 Step 1, we may assume that φ_j is a decreasing sequence. In this case, by Proposition 6.2.2 and Corollary 6.1.2, we may assume that each φ_j is a model potential. Then φ_j converges by Corollary 6.2.4 and Proposition 3.1.8.

On the other hand, if $(\varphi_j)_{j\in I}$ is decreasing, then it is convergent by Corollary 6.2.4 and Proposition 3.1.8. □

lma:dSsmallmult

Lemma 6.2.3 *There is a constant $C > 0$ such that for any $\varphi \in$ PSH(X,θ) satisfying that θ_φ is a Kähler current, we have*

$$d_{S,\theta}((1-\epsilon)\varphi,\varphi)\leq C\epsilon$$

for $\epsilon>0$ *such that* $(1-\epsilon)\varphi\in \mathrm{PSH}(X,\theta)$.

Proof By Lemma 6.2.2, we can compute

$$\begin{aligned}
d_{S,\theta}((1-\epsilon)\varphi,\varphi)=&\frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta^j_{(1-\epsilon)\varphi}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}\right)\\
=&\frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X (1-\epsilon)^j\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}\right)\\
&+\sum_{j=0}^{n}\sum_{k=0}^{j-1}\binom{j}{k}(1-\epsilon)^k\epsilon^{j-k}\int_X\theta^{j-k}\wedge\theta^k_\varphi\wedge\theta^{n-j}_{V_\theta}.
\end{aligned}$$

Both terms are of the order of $O(\epsilon)$. □

6.2.2 Convergence theorems

lma:dsconvpertV **Lemma 6.2.4** *Let* $(\varphi_i)_{i\in I}$ *be a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi\in\mathrm{PSH}(X,\theta)$. *Assume that* $\varphi_i\xrightarrow{d_S}\varphi$. *Then for any* $t\in(0,1]$,

$$(1-t)\varphi_i+tV_\theta\xrightarrow{d_S}(1-t)\varphi+tV_\theta.$$

Proof Fix $t\in(0,1]$, we write

$$\varphi_{i,t}=(1-t)\varphi_i+tV_\theta,\quad \varphi_t=(1-t)\varphi+tV_\theta$$

for any $i\in I$. By Corollary 6.2.2, it suffices to show that for each $j=0,\dots,n$,

$$2\int_X\theta^j_{\varphi_{i,t}\vee\varphi_t}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_{\varphi_{i,t}}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_{\varphi_t}\wedge\theta^{n-j}_{V_\theta}\to 0. \tag{6.10}$$ {eq:massconvafterpert}

Observe that

$$\varphi_{i,t}\vee\varphi_t=(1-t)(\varphi\vee\varphi_i)+tV_\theta.$$

So after binary expansion, (6.10) follows from Corollary 6.2.2. □

Similarly,

lma:linearpertbyVtheta **Lemma 6.2.5** *Let* $\varphi\in\mathrm{PSH}(X,\theta)$. *For each* $t\in(0,1)$, *let* $\varphi_t=(1-t)\varphi+tV_\theta$. *Then*

$$\varphi_t\xrightarrow{d_S}\varphi$$

as $t\to 0+$.

Proof By Lemma 6.2.2, we need to show that for each $j = 1, \ldots, n$, we have

$$\lim_{t\to 0+} \int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}.$$

For this purpose, we compute

$$\begin{aligned}&\int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}\\ =&\sum_{i=0}^{j-1} \binom{j}{i} (1-t)^i t^{j-i}\, \theta^i_{\varphi} \wedge \theta^{n-i}_{V_\theta}.\end{aligned}$$

As $t \to 0+$, the right-hand side clearly tends to 0. □

The following convergent theorem lies at the heart of the whole theory.

thm:convdS

Theorem 6.2.1 *Let $\theta_1, \ldots, \theta_n$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi^k_j)_{k\in I}$ are nets in $\mathrm{PSH}(X, \theta_j)$ for $j = 1, \ldots, n$ and $\varphi_1, \ldots, \varphi_n \in \mathrm{PSH}(X, \theta)$. We assume that $\varphi^k_j \xrightarrow{d_S} \varphi_j$ for each $j = 1, \ldots, n$. Then*

$$\lim_{k\in I} \int_X \theta_{1,\varphi^k_1} \wedge \cdots \wedge \theta_{n,\varphi^k_n} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{6.11}$$

{eq:convmixedmassds}

Proof Since d_S is a pseudometric, in order to establish the continuity of mixed masses, it suffices to consider sequences instead of nets. So we may assume that $I = \mathbb{Z}_{>0}$ as ordered sets.

Step 1. We reduce to the case where φ^k_j, φ_j all have positive masses and there is a constant $\delta > 0$, such that for all j and k,

$$\int_X \theta^n_{j,\varphi^k_j} > \delta.$$

Take $t \in (0,1)$. By Lemma 6.2.4, we have

$$(1-t)\varphi^k_j + tV_{\theta_j} \xrightarrow{d_S} (1-t)\varphi_j + tV_{\theta_j}$$

for each j. Assume that we have proved the special case of the theorem, we have

$$\begin{aligned}&\lim_{k\in I} \int_X \theta_{1,(1-t)\varphi^k_1+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi^k_n+tV_{\theta_n}}\\ =&\int_X \theta_{1,(1-t)\varphi_1+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n+tV_{\theta_n}}.\end{aligned}$$

Since both sides are polynomials in t, it follows that the same holds at $t = 0$. From this, (6.11) follows.

Step 2. Next we may assume that φ^k_j, φ_j are model potentials by Proposition 6.2.2 and Corollary 3.1.1.

It suffices to prove that any subsequence of $\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}$. Thus, by Proposition 6.2.3 and Theorem 2.3.2, we may assume that for each fixed i, φ_i^k is either increasing or decreasing. We may assume that for $i\leq i_0$, the sequence is decreasing and for $i>i_0$, the sequence is increasing.

Recall that in (6.11) the $\geq$ inequality always holds by Theorem 2.3.2, it suffices to prove

$$\varlimsup_{k\in I}\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\leq\int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}. \tag{6.12}$$

{eq:limsup}

By Theorem 2.3.2 in order to prove (6.12), we may assume that for $j>i_0$, the sequences φ_j^k are constant. Thus, we are reduced to the case where for all i, φ_i^k are decreasing.

In this case, for each i we may take an increasing sequence $b_i^k>1$, tending to ∞, such that

$$(b_i^k)^n\int_X \theta_{i,\varphi_i}^n\geq\left((b_i^k)^n-1\right)\int_X\theta_{i,\varphi_i^k}^n.$$

Let ψ_i^k be the maximal θ_i-psh function such that

$$(b_i^k)^{-1}\psi_i^k+\left(1-(b_i^k)^{-1}\right)\varphi_i^k\leq\varphi_i\,,$$

whose existence is guaranteed by Lemma 2.3.1.

Then by Theorem 2.3.2 again,

$$\prod_{i=1}^n\left(1-(b_i^k)^{-1}\right)\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\leq\int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}.$$

Letting $k\to\infty$, we conclude (6.12). □

cor:dsconvcrit

Corollary 6.2.6 *Suppose that $(\varphi_i)_{i\in I}$ is a net in $\mathrm{PSH}(X,\theta)$ and $\varphi\in\mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi_i\xrightarrow{d_S}\varphi$;

(2) $\varphi_i\vee\varphi\xrightarrow{d_S}\varphi$ *and*

$$\lim_{i\in I}\int_X\theta_{\varphi_i}^j\wedge\theta_{V_\theta}^{n-j}=\int_X\theta_\varphi^j\wedge\theta_{V_\theta}^{n-j} \tag{6.13}$$

{eq:massconv_varphii}

for each $j=0,\dots,n$.

The corollary allows us to reduce a number of convergence problems related to d_S to the case $\varphi_i\geq\varphi$, which is much easier to handle by Lemma 6.2.2. This is the most handy way of establishing d_S-convergence in practice.

Proof (1) $\implies$ (2). $\varphi_i\vee\varphi\xrightarrow{d_S}\varphi$ follows from Corollary 6.2.2. While (6.13) follows from Theorem 6.2.1.

(2) $\implies$ (1). By (6.4), we need to show that for each $j = 0, \dots, n$, we have

$$2\int_X \theta^j_{\varphi_i \vee \varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} \to 0.$$

This follows from Theorem 6.2.1 and (6.13). □

cor:dSconv_changetheta

Corollary 6.2.7 *Let $(\varphi_i)_{i\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)$. Let ω be a Kähler form on X. Then the following are equivalent:*

(1) $\varphi_i \xrightarrow{d_{S,\theta}} \varphi$;

(2) $\varphi_i \xrightarrow{d_{S,\theta+\omega}} \varphi$.

In particular, there is no risk when we simply write $\varphi_i \xrightarrow{d_S} \varphi$.

Proof (1) $\implies$ (2). It suffices to show that for each $j = 0, \dots, n$, we have

$$\begin{aligned}2\int_X (\theta+\omega)^j_{\varphi_i \vee \varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X (\theta+\omega)^j_{\varphi_i} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\ - \int_X (\theta+\omega)^j_{\varphi} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} \to 0.\end{aligned}$$

Note that this quantity is a linear combination of terms of the following form:

$$\begin{aligned}2\int_X \theta^r_{\varphi_i \vee \varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}} - \int_X \theta^r_{\varphi_i} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\ - \int_X \theta^r_{\varphi} \wedge \omega^{j-r} \wedge (\theta+\omega)^{n-j}_{V_{\theta+\omega}},\end{aligned}$$

where $r = 0, \dots, j$. By Theorem 6.2.1, it suffices to show that $\varphi \vee \varphi_i \xrightarrow{d_S} \varphi$. But this follows from Corollary 6.2.6.

(2) $\implies$ (1). From the direction we already proved, for each $C \geq 1$, we have that

$$\varphi_i \xrightarrow{d_{S,\theta+C\omega}} \varphi.$$

By Theorem 6.2.1, it follows that

$$\lim_{i\in I} \int_X (\theta + C\omega)^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X (\theta + C\omega)^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}$$

for all $j = 0, \dots, n$. It follows that

$$\lim_{i\in I} \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}. \tag{6.14}$$

{eq:varphijmass_limit}

By Corollary 6.2.6, it remains to show that $\varphi_i \vee \varphi \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 6.2.6 again, we know that $\varphi_i \vee \varphi \xrightarrow{d_{S,\theta+\omega}} \varphi$. So it suffices to apply (6.14) to $\varphi_i \vee \varphi$ instead of φ_i, and we conclude by Lemma 6.2.2. □

We sometimes need a slightly more general form.

cor:dsequivalenceindep **Corollary 6.2.8** *Let $(\varphi_j)_{j\in I}$, $(\psi_j)_{j\in I}$ be nets in* $\mathrm{PSH}(X,\theta)$. *Consider a Kähler form ω on X. Then the following are equivalent:*

(1) $d_{S,\theta}(\varphi_i,\psi_i) \to 0$;
(2) $d_{S,\theta+\omega}(\varphi_i,\psi_i) \to 0$.

In particular, we can write $d_S(\varphi_i,\psi_i) \to 0$ without ambiguity.

Proof The proof is similar to that of Corollary 6.2.7, which is therefore left to the readers. □

We have the following sandwich criterion:

lma:dsconvupplower **Corollary 6.2.9** *Let $(\varphi_i)_{i\in I}, (\psi_i)_{i\in I}, (\eta_i)_{i\in I}$ be three nets in* $\mathrm{PSH}(X,\theta)$ *and $\varphi \in$* $\mathrm{PSH}(X,\theta)$. *Assume that*

(1) $\psi_i \preceq_P \varphi_i \preceq_P \eta_i$ *for each $i \in I$;*
(2) $\eta_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \varphi$.

Then $\varphi_i \xrightarrow{d_S} \varphi$.

Proof By Corollary 6.2.7, we may replace θ by $\theta+\omega$, where ω is a Kähler form on X. In particular, we may assume that $\varphi_i,\psi_i,\eta_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$. By Proposition 6.2.2, we may assume that φ_i,ψ_i,η_i are model potentials for all $i \in I$ and hence $\varphi_i \leq \psi_i \leq \eta_i$ for all $i \in I$.

It follows from Theorem 2.3.2 that for each $k = 0,\dots,n$, we have

$$\int_X \theta_{\psi_i}^k \wedge \theta_{V_\theta}^{n-k} \leq \int_X \theta_{\varphi_i}^k \wedge \theta_{V_\theta}^{n-k} \leq \int_X \theta_{\eta_i}^k \wedge \theta_{V_\theta}^{n-k}$$

for all $i \in I$. By Theorem 6.2.1, the limits of the both ends are $\int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}$ as $j \to \infty$. It follows that

$$\lim_{i\in I} \int_X \theta_{\varphi_i}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}. \tag{6.15}$$ {eq:thetak_conv}

By Corollary 6.2.6, it remains to prove that $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$. By Corollary 6.2.6, up to replacing ψ_i (resp. φ_i,η_i) by $\psi_i \vee \varphi$ (resp. $\varphi_i \vee \varphi$, $\eta_i \vee \varphi$), we may assume from the beginning that $\psi_i,\varphi_i,\eta_i \geq \varphi$. Now $\varphi_i \xrightarrow{d_S} \varphi$ by (6.15) and Lemma 6.2.2. □

prop:dsconvpresorder **Proposition 6.2.4** *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be nets in* $\mathrm{PSH}(X,\theta)$ *such that $\varphi_i \xrightarrow{d_S} \varphi \in$* $\mathrm{PSH}(X,\theta)$ *and $\psi_i \xrightarrow{d_S} \psi \in$* $\mathrm{PSH}(X,\theta)$. *Assume that $\varphi_i \preceq_P \psi_i$ for all $i \in I$. Then $\varphi \preceq_P \psi$.*

Proof It follows from Proposition 6.2.5 that

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

By Lemma 6.1.2, we have $\varphi_i \vee \psi_i \sim_P \psi_i$ for all $i \in I$. In particular, by Proposition 6.2.2,

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \psi.$$

By Proposition 6.2.2 again, $\varphi \vee \psi \sim_P \psi$ and hence $\varphi \leq_P \psi$ by Lemma 6.1.2. □

lma:dslor **Lemma 6.2.6** *Let* $\varphi, \psi, \eta \in \mathrm{PSH}(X, \theta)$, *then*

$$d_S(\varphi \vee \eta, \psi \vee \eta) \leq C_n d_S(\varphi, \psi), \tag{6.16}$$ {eq:dSmax}

where $C_n = 3(n+1)2^{n+2}$.

Proof According to Corollary 6.2.2, we may assume that $\varphi \leq \psi$.

We will show that for each $C \geq t \geq 0$,

$$d_1(\ell_t^{\varphi \vee \eta, C}, \ell_t^{\psi \vee \eta, C}) \leq d_1(\ell_t^{\varphi, C}, \ell_t^{\psi, C}). \tag{6.17}$$ {eq:d1maxcomp}

When $C \to \infty$, by Corollary 2.3.1 and Theorem 4.3.1, it follows that

$$d_1(\ell_t^{\varphi \vee \eta}, \ell_t^{\psi \vee \eta}) \leq d_1(\ell_t^{\varphi}, \ell_t^{\psi}),$$

which implies (6.16).

It remains to argue (6.17). As $\varphi \leq \psi$, we know that

$$d_1(\ell_t^{\varphi}, \ell_t^{\psi}) = \frac{t}{C} d_1(\ell_C^{\varphi}, \ell_C^{\psi}), \quad d_1(\ell_t^{\varphi \vee \eta}, \ell_t^{\psi \vee \eta}) = \frac{t}{C} d_1(\ell_C^{\varphi \vee \eta}, \ell_C^{\psi \vee \eta}).$$

It suffices to handle the case $t = C$, namely,

$$d_1(\varphi \vee \eta \vee (V_\theta - C), \psi \vee \eta \vee (V_\theta - C)) \leq d_1(\varphi \vee (V_\theta - C), \psi \vee (V_\theta - C)).$$

This is a consequence of Theorem 4.3.2. □

prop:lor_dS_conv **Proposition 6.2.5** *Let* $(\varphi_i)_{i \in I}$ *(resp.* $(\psi_i)_{i \in I}$*) be a net in* $\mathrm{PSH}(X, \theta)$ *such that* $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X, \theta)$ *(resp.* $\varphi_i \xrightarrow{d_S} \psi \in \mathrm{PSH}(X, \theta)$*). Then*

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

Proof We compute

$$\begin{aligned} d_S(\varphi_i \vee \psi_i, \varphi \vee \psi) \leq& d_S(\varphi_i \vee \psi_i, \varphi_i \vee \psi) + d_S(\varphi_i \vee \psi, \varphi \vee \psi) \\ \leq& C_n \left(d_S(\psi_i, \psi) + d_S(\varphi_i, \varphi) \right), \end{aligned}$$

where the second inequality follows from Lemma 6.2.6. The right-hand side converges to 0 by our hypothesis. □

thm:dSadditivity

Theorem 6.2.2 *Let θ_1, θ_2 be smooth real closed $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi_i)_{i\in I}$ (resp. $(\psi_i)_{i\in I}$) be a net in $\mathrm{PSH}(X,\theta_1)$ (resp. $\mathrm{PSH}(X,\theta_2)$) and $\varphi\in\mathrm{PSH}(X,\theta_1)$ (resp. $\psi\in\mathrm{PSH}(X,\theta_2)$). Consider the following three conditions:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;
(2) $\psi_i \xrightarrow{d_S} \psi$;
(3) $\varphi_i+\psi_i \xrightarrow{d_S} \varphi+\psi$.

Then any two of these conditions imply the third.

Proof By Corollary 6.2.7, we may assume that θ_1,θ_2 are both Kähler forms. We denote them by ω_1,ω_2 instead. Let $\omega=\omega_1+\omega_2$.

(1)+(2) $\implies$ (3). It suffices to show that for each $r=0,\dots,n$,

$$2\int_X \omega^r_{(\varphi_j+\psi_j)\vee(\varphi+\psi)}\wedge\omega^{n-r}-\int_X\omega^r_{\varphi_j+\psi_j}\wedge\omega^{n-r}-\int_X\omega^r_{\varphi+\psi}\wedge\omega^{n-r}\to 0.$$

Observe that for each $j\in I$,

$$(\varphi_j+\psi_j)\vee(\varphi+\psi)\leq\varphi_j\vee\varphi+\psi_j\vee\psi.$$

Thus, it suffices to show that

$$2\int_X \omega^r_{\varphi_j\vee\varphi+\psi_j\vee\psi}\wedge\omega-\int_X\omega^r_{\varphi_j+\psi_j}\wedge\omega^{n-r}-\int_X\omega^r_{\varphi+\psi}\wedge\omega^{n-r}\to 0.$$

The left-hand side is a linear combination of

$$2\int_X \omega^a_{1,\varphi_j\vee\varphi}\wedge\omega^{r-a}_{2,\psi_j\vee\psi}\wedge\omega^{n-r}-\int_X\omega^a_{1,\varphi_j}\wedge\omega^{r-a}_{2,\psi_j}\wedge\omega^{n-r}-\int_X\omega^a_{1,\varphi}\wedge\omega^{r-a}_{2,\psi}\wedge\omega^{n-r}$$

with $a=0,\dots,r$. Observe that $\varphi_j\vee\varphi\xrightarrow{d_S}\varphi$ and $\psi_j\vee\psi\xrightarrow{d_S}\psi$ by Corollary 6.2.2, each term tends to 0 by Theorem 6.2.1.

(2)+(3) $\implies$ (1). This is similar.

(1)+(3) $\implies$ (2). For each $C\geq 1$, from the direction we already proved,

$$C\varphi_i+\psi_i\xrightarrow{d_S}C\varphi+\psi.$$

By Theorem 6.2.1, for each $j=0,\dots,n$,

$$\begin{aligned}&\lim_{i\in I}\int_X(C\omega_1+\omega_2+\mathrm{dd^c}(C\varphi_i+\psi_i))^j\wedge\omega_2^{n-j}\\ =&\int_X(C\omega_1+\omega_2+\mathrm{dd^c}(C\varphi+\psi))^j\wedge\omega_2^{n-j}.\end{aligned}$$

It follows that

$$\lim_{i\in I}\int_X \omega^j_{2,\psi_i}\wedge\omega_2^{n-j}=\int_X \omega^j_{2,\psi}\wedge\omega_2^{n-j}. \tag{6.18}$$ {eq:psii_quant_conv}

Therefore, 2 follows if $\psi_i \geq \psi$ for each i by Lemma 6.2.2.

Next we prove the general case. By the direction that we already proved, we know that $\varphi_i+\psi \xrightarrow{d_S} \varphi+\psi$. By Proposition 6.2.5, we have that

$$\varphi_i+\psi_i\vee\psi \xrightarrow{d_S} \varphi+\psi.$$

It follows from the special case above that $\psi_i\vee\psi \xrightarrow{d_S} \psi$. It follows from (6.18) and Corollary 6.2.6 that (2) holds. □

thm:contPI **Theorem 6.2.3** *The map*

$$P_\theta[\bullet]_{\mathcal{I}}:\mathrm{PSH}(X,\theta)_{>0}\to\mathrm{PSH}(X,\theta)_{>0}$$

is continuous with respect to d_S.

Proof Let $(\varphi_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{PSH}(X,\theta)_{>0}$ such that $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)_{>0}$. We want to show that

$$P[\varphi_i]_{\mathcal{I}} \xrightarrow{d_S} P[\varphi]_{\mathcal{I}}. \tag{6.19}$$

We may assume that the φ_i's and φ are all model potentials by Proposition 6.2.2.

By Proposition 6.2.3 and Corollary 6.2.9, we may assume that $(\varphi_i)_i$ is either increasing or decreasing. The two cases are handled by Proposition 3.2.12 and Proposition 3.2.11 respectively. □

6.2.3 Continuity of invariants

thm:Lelongcont **Theorem 6.2.4** *Let $(\varphi_j)_{j\in I}$ be a net in* $\mathrm{PSH}(X,\theta)$ *and $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. Then for any prime divisor E over X, we have*

$$\lim_{j\in I}\nu(\varphi_j,E)=\nu(\varphi,E). \tag{6.20}$$ {eq:convnu}

Proof First observe that since d_S is a pseudometric, it suffices to prove (6.20) when $I=\mathbb{Z}_{>0}$ as partially ordered sets.

By Corollary 6.2.7, we may assume that the masses of φ_j and of φ are bounded from below by a positive constant.

By Theorem 6.2.3, we may assume that φ_i and φ are both $\mathcal{I}$-model. When proving (6.20), we are free to pass to subsequences.

By Proposition 6.2.3, we may assume that the sequence (φ_i) is either increasing or decreasing. In the increasing case, there is nothing to prove. In the decreasing case, (6.20) follows from Proposition 3.1.8. □

thm:contvolu **Theorem 6.2.5** *Let $(\varphi_j)_{j\in I}$ be a net in* $\mathrm{PSH}(X,\theta)$ *such that* $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. *Assume that $\int_X \theta_\varphi^n > 0$, we have*

$$\operatorname{vol}\theta_{\varphi_j} \to \operatorname{vol}\theta_\varphi. \tag{6.21}$$ {eq:Ivolcont}

Recall the volume is defined in Definition 3.2.3.

Proof It follows from Theorem 6.2.1 that

$$\int_X \theta_{\varphi_j}^n \to \int_X \theta_\varphi^n.$$

We may therefore assume that $\int_X \theta_{\varphi_j}^n$ for all $j \in I$. Then by Theorem 6.2.3, we have

$$P_\theta[\varphi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}.$$

Therefore, (6.21) follows from Theorem 6.2.1. □

thm:equising_cond_general **Theorem 6.2.6** *Let $\varphi_j, \varphi \in$* $\mathrm{PSH}(X,\theta)$ *($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \xrightarrow{d_S} \varphi$. Then for each $\lambda' > \lambda > 0$, there is $j_0 > 0$ so that for $j \geq j_0$,*

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi). \tag{6.22}$$ {eq:quasi_equi_cond}

Proof Fix $\lambda' > \lambda > 0$, we want to find $j_0 > 0$ so that for $j \geq j_0$, (6.22) holds.

Step 1. We first assume that φ has analytic singularities.

Let $\pi : Y \to X$ be a log resolution of φ and let $E_1, \dots, E_N$ be all prime divisors of the singular part of φ on Y. Recall that a local holomorphic function f lies in the right-hand side of (6.22) if and only if

$$\operatorname{ord}_{E_i}(f) > \lambda \operatorname{ord}_{E_i}(\varphi) - A_X(E_i) \tag{6.23}$$ {eq:ordEif}

whenever they make sense. Here A_X denotes the log discrepancy. Similarly, f lies in the left-hand side of (6.22) implies that there is $\epsilon > 0$ so that

$$\operatorname{ord}_{E_i}(f) \geq (1+\epsilon)\lambda' \operatorname{ord}_{E_i}(\varphi_j) - A_X(E_i).$$

As Lelong numbers are continuous with respect to d_S by Theorem 6.2.4, we can find $j_0 > 0$ so that when $j \geq j_0$, $\lambda' \operatorname{ord}_{E_i}(\varphi_j) \geq \lambda \operatorname{ord}_{E_i}(\varphi)$ for all i. In particular, (6.23) follows.

Step 2. We handle the general case.

By Corollary 6.2.7, we are free to increase θ and assume that θ_φ is a Kähler current.

Take a quasi-equisingular approximation ψ_k of φ. The existence is guaranteed by Theorem 1.6.2. Take $\lambda'' \in (\lambda, \lambda')$, then by definition, we can find $k > 0$ so that

$$\mathcal{I}(\lambda''\psi_k) \subseteq \mathcal{I}(\lambda\varphi).$$

Observe that $\varphi_j \vee \psi_k \xrightarrow{d_S} \psi_k$ as $j \to \infty$ by Proposition 6.2.5. By Step 1, we can find $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}(\lambda'(\varphi_j \vee \psi_k)) \subseteq \mathcal{I}(\lambda''\psi_k).$$

It follows that for $j \geq j_0$,

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

Chapter 7
$\mathcal{I}$-good singularities

chap:Igood

7.1 The notion of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold of dimension n.

thm:charIgoodasclosure

Theorem 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) *there exists a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$,*

(2) *we have*

$$\int_X \theta_\varphi^n = \mathrm{vol}\,\theta_\varphi, \tag{7.1}$$

{eq:nppmassequalvolume}

and

(3) *we have*

$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}.$$

In (1), we could in addition require that each θ_{φ_j} is a Kähler current.

Moreover, if θ_φ is a Kähler current, the sequence in (1) can be taken as any quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

Proof (1) $\implies$ (2). By Theorem 6.2.1, we may assume that $\int_X \theta_{\varphi_j}^n > 0$ for all j. It follows from Proposition 3.2.9 that

$$\int_X \theta_{\varphi_j}^n = \mathrm{vol}\,\theta_{\varphi_j}$$

for any $j \geq 1$. Using Theorem 6.2.5 and Theorem 6.2.1, we conclude (7.1).

(2) $\iff$ (3). This follows from Theorem 3.1.1.

(3) $\implies$ (1). Note that the condition in (1) characterizes the closure of analytic singularities in $\mathrm{PSH}(X,\theta)$.

Step 1. We first reduce to the case where θ_φ is a Kähler current.

By Lemma 2.3.2, we can find $\psi \in \mathrm{PSH}(X,\theta)$ so that θ_ψ is a Kähler current and $\psi \leq \varphi$. We let

$$\psi_j = (1-j^{-1})\varphi + j^{-1}\psi$$

for each $j \in \mathbb{Z}_{>0}$. Then $(\psi_j)_j$ is an increasing sequence converging almost everywhere to φ. Then

$$P_\theta[\psi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} = P_\theta[\varphi]$$

by Proposition 3.2.12, Corollary 6.2.3. So it suffices to show that $P_\theta[\psi_j]_{\mathcal{I}}$ lies in the closure of analytic singularities.

Step 2. We assume that θ_φ is a Kähler current. We show that $P_\theta[\varphi]_{\mathcal{I}}$ lies in the closure of analytic singularities.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We will show that $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$. Let

$$\psi = \inf_{j\in\mathbb{Z}_{>0}} P_\theta[\varphi_j].$$

We know that $\varphi_j \xrightarrow{d_S} \psi$ by Proposition 6.2.2, Proposition 3.1.8 and Corollary 6.2.4.

Moreover, observe that ψ is $\mathcal{I}$-model by Proposition 3.2.11 and Example 7.1.1. So it suffices to show that $\varphi \sim_{\mathcal{I}} \psi$.

It is clear that $\psi \geq \varphi$. Conversely, it remains to argue that $\psi \leq_{\mathcal{I}} \varphi$. For this purpose, take $\lambda > 0$, we need to show that

$$\mathcal{I}(\lambda\psi) \subseteq \mathcal{I}(\lambda\varphi).$$

By the strong openness Theorem 1.4.4, we may take $\lambda' > \lambda$ such that $\mathcal{I}(\lambda\psi) = \mathcal{I}(\lambda'\psi)$, then it follows from the definition of the quasi-equisingular approximation that

$$\mathcal{I}(\lambda'\psi) \subseteq \mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi)$$

for large enough j. Our assertion follows. □

def:Igoodpot **Definition 7.1.1** We say a potential $\varphi \in \mathrm{QPSH}(X)$ is *$\mathcal{I}$-good* if for some smooth closed real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, we have

$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}. \tag{7.2}$$ {eq:envelopeeq}

An immediate question is to verify that this definition is in dependent of the choice of θ.

lma:Igoodinsenspert **Lemma 7.1.1** *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ for some smooth closed real $(1,1)$-form θ on X. Take a Kähler form ω on X. Then the following are equivalent:*

(1) $P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}$;
(2) $P_{\theta+\omega}[\varphi] = P_\theta[\varphi+\omega]_{\mathcal{I}}$.

Proof (1) $\Longrightarrow$ (2). By Theorem 7.1.1, we can find $\varphi_j \in \mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 6.2.7, we have $\varphi_j \xrightarrow{d_{S,\theta+\omega}} \varphi$. Therefore, by Theorem 7.1.1 again, 2 holds.

(2) $\implies$ (1). Suppose that (1) fails, so that

$$\int_X (\theta + \mathrm{dd^c}\varphi)^n < \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

It follows that

$$\begin{aligned}
\int_X (\theta + \omega + \mathrm{dd^c}\varphi)^n &= \sum_{i=0}^n \binom{n}{i} \int_X \theta_\varphi^i \wedge \omega^{n-i} \\
&< \sum_{i=0}^n \binom{n}{i} \int_X \theta_{P_\theta[\varphi]_{\mathcal{I}}}^i \wedge \omega^{n-i} \\
&= \int_X (\theta + \omega + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n \\
&\leq \int_X (\theta + \omega + \mathrm{dd^c} P_{\theta+\omega}[\varphi]_{\mathcal{I}})^n.
\end{aligned}$$

So (2) fails as well. □

cor:Igoodclosed

Corollary 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. Let $(\varphi_j)_{j\in I}$ be a net of $\mathcal{I}$-good potentials in $\mathrm{PSH}(X,\theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi$. Then φ is $\mathcal{I}$-good.*

Proof By Corollary 6.2.7, we may assume that $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)_{>0}$ for all $j \in I$. It follows from Theorem 7.1.1 that

$$\int_X \theta_{\varphi_j}^n = \operatorname{vol}\theta_{\varphi_j}$$

for all $j \in I$. Taking limit with respect to j with the help of Theorem 6.2.5 and Theorem 6.2.1, we conclude that

$$\int_X \theta_\varphi^n = \operatorname{vol}\theta_\varphi.$$

Therefore, by Theorem 7.1.1 again, we find that φ is $\mathcal{I}$-good. □

ex:analyIgood

Example 7.1.1 Assume that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then φ is $\mathcal{I}$-good. This is proved in Proposition 3.2.9.

ex:ImodelIgood

Example 7.1.2 Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ is an $\mathcal{I}$-model potential for some closed real smooth $(1,1)$-form θ on X. Then φ is $\mathcal{I}$-good.

cor:quasi-equichar

Corollary 7.1.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ and $(\epsilon_j)_j$ be a decreasing sequence in $\mathbb{R}_{\geq 0}$ with limit 0. Fix a Kähler form ω on X. Consider a decreasing sequence $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$ of potentials with analytic singularities for each $j \geq 1$. Assume that $\varphi = \inf_j \varphi_j$. Then the following are equivalent:*

(1) $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$, *and*

(2) $(\varphi_j)_j$ *is a quasi-equisingular approximation of* φ.

Proof By Corollary 6.2.7 and Example 7.1.2, we may replace θ by $\theta + C\omega$ for some large constant $C > 0$ and assume that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta - \omega)$ for all $j \geq 1$.

(2) $\implies$ (1). This is already proved in the proof of Theorem 7.1.1.

(1) $\implies$ (2). This follows from Theorem 6.2.6. □

7.2 Properties of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold.

prop:Igoodlinear

Proposition 7.2.1 *Let* $\varphi, \psi \in \mathrm{QPSH}(X)$ *be* $\mathcal{I}$*-good and* $\lambda > 0$*. Then the following potentials are all* $\mathcal{I}$*-good.*

(1) $\varphi + \psi$;
(2) $\varphi \vee \psi$;
(3) $\lambda\varphi$.

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. It follows from Theorem 7.1.1 that there are sequences φ_j, ψ_j in $\mathrm{PSH}(X, \theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$ and $\psi_j \xrightarrow{d_S} \psi$.

By Theorem 6.2.2, Proposition 6.2.5, we have

$$\varphi_j + \psi_j \xrightarrow{d_S} \varphi + \psi, \quad \varphi_j \vee \psi_j \xrightarrow{d_S} \varphi \vee \psi.$$

On the other hand, it is clear that

$$\lambda\varphi_j \xrightarrow{d_S} \lambda\varphi.$$

Therefore, our assertions follow from Theorem 7.1.1. □

prop:Igoodsup

Proposition 7.2.2 *Let* $\{\varphi_j\}_{j\in I}$ *be a non-empty family of* $\mathcal{I}$*-good potentials. Assume that the family is uniformly bounded from above and there exists a closed real smooth* $(1,1)$*-form* θ *on* X *such that* $\varphi_j \in \mathrm{PSH}(X, \theta)$ *for all* $j \in I$*. Then* $\sup^*_{j\in I} \varphi_j$ *is* $\mathcal{I}$*-good.*

Proof Without loss of generality, we may assume that $\varphi_j \in \mathrm{PSH}(X, \theta)_{>0}$ for all $j \in I$.

When I is finite, this result follows from Proposition 7.2.1. When I is infinite, we may assume that $I = \mathbb{Z}_{>0}$ by Proposition 1.2.2. By Proposition 7.2.1, we may assume that the sequence $(\varphi_j)_j$ is increasing. In this case, as shown in Corollary 6.2.3,

$$\varphi_j \xrightarrow{d_S} \sup_{i\in\mathbb{Z}_{>0}}{}^* \varphi_i.$$

Therefore, $\sup^*_{i\in\mathbb{Z}_{>0}} \varphi_i$ is $\mathcal{I}$-good by Theorem 7.1.1. □

thm:contvolu2

Theorem 7.2.1 *Let $(\varphi_j)_{j\in I}$ be a net in* $\mathrm{PSH}(X,\theta)$ *such that* $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. *Assume that φ is $\mathcal{I}$-good, then we have*

$$\operatorname{vol}\theta_{\varphi_j} \to \operatorname{vol}\theta_{\varphi}. \tag{7.3}$$

{eq:Ivolcont2}

Proof Fix a Kähler form ω on X. Then for any $\epsilon > 0$, we have

$$\begin{aligned}\operatorname{vol}(\theta+\epsilon\omega)_{\varphi} &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}\right)^n \\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}\varphi\right)^n .\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_X \left(\theta+\epsilon\omega+\mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}\right)^n &\geq \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c} P_{\theta}[\varphi]_{\mathcal{I}}\right)^n \\ &\geq \int_X \left(\theta+\mathrm{dd^c} P_{\theta}[\varphi]_{\mathcal{I}}\right)^n \\ &\geq \int_X \theta_{\varphi}^n .\end{aligned}$$

Therefore,

$$\operatorname{vol}(\theta+\epsilon\omega)_{\varphi} - \operatorname{vol}\theta_{\varphi} \leq \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}\varphi\right)^n - \int_X \theta_{\varphi}^n .$$

The difference can be controled by a polynomial in ϵ without constant term independent of the choice of φ. We have a similar estimate for φ_j as well. So our assertion follows from Theorem 6.2.5. ◻

prop:vollinearlimit

Proposition 7.2.3 *Let $\varphi,\psi \in$* $\mathrm{PSH}(X,\theta)_{>0}$. *Then*

(1) *We have*

$$\lim_{\epsilon\to 0+} \operatorname{vol}(\theta,(1-\epsilon)\varphi+\epsilon\psi) = \operatorname{vol}(\theta,\varphi);$$

(2) *Let ω be a Kähler form on X, then*

$$\operatorname{vol}\theta_{\varphi} = \lim_{\epsilon\to 0+} \operatorname{vol}(\theta+\epsilon\omega)_{\varphi};$$

(3) *Consider a prime divisor E on X. Then*

$$\operatorname{vol}\theta_{\varphi} = \operatorname{vol}(\theta_{\varphi} - \nu(\varphi,E)[E]).$$

Proof (1) We need to show that

$$\lim_{\epsilon\to 0+} \int_X \left(\theta+\mathrm{dd^c} P_{\theta}[(1-\epsilon)\varphi+\epsilon\psi]_{\mathcal{I}}\right)^n = \int_X \left(\theta+\mathrm{dd^c} P_{\theta}[\varphi]_{\mathcal{I}}\right)^n .$$

By Proposition 3.2.10, for any $\epsilon \in (0,1)$,

$$(1-\epsilon)\varphi+\epsilon\psi \sim_{\mathcal{I}} (1-\epsilon)P_\theta[\varphi]_{\mathcal{I}}+\epsilon P_\theta[\psi]_{\mathcal{I}}.$$

In particular, we may replace φ and ψ by $P_\theta[\varphi]_{\mathcal{I}}$ and $P_\theta[\psi]_{\mathcal{I}}$ respectively. By Proposition 7.2.1, it remains to show that

$$\lim_{\epsilon\to 0+}\int_X \left(\theta+\mathrm{dd^c}\left((1-\epsilon)\varphi+\epsilon\psi\right)\right)^n=\int_X (\theta+\mathrm{dd^c}\varphi)^n\,,$$

which is obvious.

(2) For each $\epsilon>0$,

$$\begin{aligned}\operatorname{vol}(\theta+\epsilon\omega)_\varphi &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}\right)^n\\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}\left[P_\theta[\varphi]_{\mathcal{I}}\right]\right)^n\\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta}[\varphi]_{\mathcal{I}}\right)^n\,,\end{aligned}$$

where the third equality follows from Example 7.1.2. Letting $\epsilon\to 0+$, we conclude.

(3) By (2), we may assume that θ_φ is a Kähler current. Take a quasi-equisingular approximation $(S_j)_j$ of $\theta_\varphi-\nu(\varphi,E)[E]$. By Theorem 6.2.2,

$$S_j+\nu(\varphi,E)[E]\xrightarrow{d_S}\theta_\varphi.$$

For each $j\geq 1$, the currents $S_j+\nu(\varphi,E)[E]$ and S_j are $\mathcal{I}$-good as follows from Proposition 7.2.1, we have

$$\operatorname{vol}(S_j+\nu(\varphi,E)[E])=\int_X (S_j+\nu(\varphi,E)[E])^n=\int_X S_j^n=\operatorname{vol}S_j.$$

Letting $j\to\infty$, we conclude by Theorem 6.2.6. ◻

7.3 The volume of Hermitian big line bundles

sec:volHermitianbig

Let X be a connected compact Kähler manifold of dimension n.

Definition 7.3.1 A *Hermitian pseudoeffective line bundle* (L,h) on X consists of a pseudoeffective line bundle L on X together with a plurisubharmonic metric h on L.

A *Hermitian big line bundle* (L,h) on X is a big line bundle L on X together with a plurisubharmonic metric h on L such that $\operatorname{vol}(\mathrm{dd^c}h)>0$.

When X admits a big line bundle, it is necessarily projective. See MM07 [MM07, Theorem 2.2.26].

thm:DXmain1

Theorem 7.3.1 *Let (L,h) be a Hermitian big line bundle and T be a holomorphic line bundle on X. We have*

$$\lim_{k\to\infty}\frac{n!}{k^n}h^0(X, T\otimes L^k\otimes \mathcal{I}(h^k)) = \operatorname{vol}(\mathrm{dd^c}h). \tag{7.4}$$ {eq:DXmain1}

In particular, the limit exists.

Remark 7.3.1 This theorem also holds for a general Hermitian pseudoeffective line bundle. The proof is more involved. We would have to apply the singular holomorphic Morse inequality of Bonavero [Bon98]. See [DX21, Theorem 1.1].

For the proof, let us fix a smooth Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$. We identify h with $h_0\exp(-\varphi)$ for some $\varphi\in \mathrm{PSH}(X,\theta)$.

We first handle the case where φ has analytic singularities.

prop:DXmainanalytic **Proposition 7.3.1** *Under the assumptions of Theorem 7.3.1, assume furthermore that φ has analytic singularities, then (7.4) holds.*

Proof **Step 1**. Reduce to the case of log singularities.

Let $\pi\colon Y\longrightarrow X$ be a modification such that $\pi^*\varphi$ has log singularities. In this case, for each $k\in\mathbb{Z}_{>0}$, we have

$$h^0(X, T\otimes L^k\otimes\mathcal{I}(kh)) = h^0(Y, K_{Y/X}\otimes \pi^*T\otimes\pi^*L^k\otimes\mathcal{I}(k\pi^*h)).$$

By Proposition 3.2.5, we have

$$\operatorname{vol}(\mathrm{dd^c}h) = \operatorname{vol}(\mathrm{dd^c}\pi^*h).$$

Therefore, it suffices to argue (7.4) with $K_{Y/X}\otimes\pi^*T$, π^*L and π^*h in place of T, L and h.

Step 2. Assume that D has log singularities along an effective $\mathbb{Q}$-divisor D, we decompose D into irreducible components, say

$$D=\sum_{i=1}^N a_iD_i.$$

In this case, we can easily compute

$$\mathcal{I}(k\varphi) = \mathcal{O}_X\left(-\sum_{i=1}^N \lfloor ka_i\rfloor D_i\right)$$

for each $k\in\mathbb{Z}_{>0}$. Observe that $L-D$ is nef (see Lemma 1.6.1), so we could apply the asymptotic Riemann–Roch theorem to conclude that

$$\lim_{k\to\infty}\frac{n!}{k^n}h^0\left(X, T\otimes L^k\otimes\mathcal{O}_X\left(-\sum_{i=1}^N\lfloor ka_i\rfloor D_i\right)\right) = (L-D)^n.$$

Observe that by Proposition 1.8.1,

$$\theta_\varphi = [D]+T,$$

where T is a closed positive $(1,1)$-current with bounded potential. Therefore,

$$(L-D)^n = \int_X T^n = \int_X \theta_\varphi^n.$$

By Example 7.1.1, we know that the right-hand side is exactly $\operatorname{vol}\theta_\varphi$. □

Proof (Proof of Theorem 7.3.1) **Step 1**. We first handle the case where θ_φ is a Kähler current. Fix a Kähler form $\omega \geq \theta$ on X such that $\theta_\varphi \geq 2\delta\omega$ for some $\delta \in (0,1)$.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j} \geq \delta\omega$ for all j. From Proposition 7.3.1, we know that for each $j \geq 1$,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi_j)) = \operatorname{vol}\theta_{\varphi_j}.$$

It follows from Theorem 7.1.1 and Theorem 6.2.5 that the right-hand side converges to $\operatorname{vol}\theta_\varphi$ as $j\to\infty$. Therefore,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \operatorname{vol}\theta_\varphi.$$

Conversely, fix an integer $N > \delta^{-1}$. From Theorem 7.1.1 and Theorem 6.2.1, we know that

$$\lim_{j\to\infty} \int_X \theta_{\varphi_j}^n = \int_X \theta^n_{P_\theta[\varphi]_\mathcal{I}} > 0. \tag{7.5}$$

{eq:quasiequmassconvtemp1}

Therefore, by Lemma 2.3.1, we can find $j_0 > 0$ such that for $j \geq j_0$, there is $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ with

$$(1-N^{-1})\varphi_j + N^{-1}\psi \leq P_\theta[\varphi]_\mathcal{I}. \tag{7.6}$$

{eq:linearlowerbdPItemp1}

For each $k > 0$, we write $k = k'N - r$, where $k' \in \mathbb{N}$ and $r \in \{0, 1, \dots, N-1\}$. Then we compute for $j > j_0$ and large enough k that

$$\begin{aligned}
&h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\\
\geq& h^0(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}(k'N\varphi))\\
\geq& h^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}\left(k'(\psi+(N-1)\varphi_j)\right)\right)\\
\geq& h^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes L^{k'(N-1)}\otimes \mathcal{I}\left(k'N\varphi_j\right)\right),
\end{aligned}$$

where the third line follows from (7.6), the fourth line can be argued as follows: for large enough k, there is a non-zero section $s \in H^0(X, L^{k'}\otimes\mathcal{I}(k'\psi))$ by Lemma 2.3.3; It follows from Lemma 1.6.3 that for large enough k,

$$\mathcal{I}\left(k'N\varphi_j\right) \subseteq \mathcal{I}_\infty\left(k'(N-1)\varphi_j\right).$$

It follows that multiplication by s gives an injective map

$$\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'(N-1)}\otimes \mathcal{I}\left(k'N\varphi_j\right)\right)\hookrightarrow$$
$$\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}\left(k'\psi+k'(N-1)\varphi_j\right)\right).$$

Next observe that

$$(N-1)\theta+N\mathrm{dd^c}\varphi_j\geq 0.$$

So Proposition 7.3.1 is applicable. We let $k\to\infty$ to conclude that

$$\begin{aligned}\varliminf_{k\to\infty} h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\geq&\frac{1}{n!\cdot N^{-n}}\int_X\left((N-1)\theta+N\mathrm{dd^c}\varphi_j\right)^n\\ =&\frac{1}{n!}\int_X\left((1-N^{-1})\theta+\mathrm{dd^c}\varphi_j\right)^n.\end{aligned}$$

Letting $j\to\infty$ and then $N\to\infty$ and using (7.5), we find that

$$\varliminf_{k\to\infty} h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\geq\int_X\theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

Step 2. We handle the general case. We may assume that φ is $\mathcal{I}$-model.

Take an ample line bundle A on X and a Kähler form ω in $c_1(A)$. Then for any fixed $N\in\mathbb{Z}_{>0}$, we apply Step 1 to $L^N\otimes A$ in place of L and $T\otimes L^i$ with $i=0,\dots,N-1$ in place of T, we have

$$\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\leq\int_X\left(N^{-1}\omega+\theta+\mathrm{dd^c}P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}}\right)^n.$$

On the other hand, since φ is $\mathcal{I}$-good by Example 7.1.2, we have

$$P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}}=P_{\theta+N^{-1}\omega}[\varphi].$$

It follows from Proposition 3.1.2 that

$$\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\leq\int_X\left(\theta+N^{-1}\omega+\mathrm{dd^c}\varphi\right)^n.$$

Letting $N\to\infty$, we conclude

$$\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\leq\int_X\theta^n_\varphi.$$

It remains to argue the reverse inequality.

Choose $\psi\in\mathrm{PSH}(X,\theta)$ such that θ_ψ is a Kähler current and $\psi\leq\varphi$. The existence of ψ is guaranteed by Lemma 2.3.2. Then for any $t\in(0,1)$, we set

$$\varphi_t=(1-t)\varphi+t\psi.$$

It follows again from Step 1 that

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \geq \varliminf_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_t)) = \operatorname{vol}\theta_{\varphi_t}.$$

On the other hand, by Corollary 6.2.3, we have $\varphi_t \xrightarrow{d_S} \varphi$ as $t \to 0+$. It follows from Theorem 6.2.5 that

$$\lim_{t\to 0+} \operatorname{vol}\theta_{\varphi_t} = \operatorname{vol}\theta_{\varphi}.$$

So we find

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \geq \operatorname{vol}\theta_{\varphi}.$$

ex:toricIgood *Example 7.3.1* If X is a toric smooth projective variety and θ is invariant under the action of the compact torus. Suppose that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ is also invariant under the action of the compact torus, then φ is $\mathcal{I}$-good.

Proof Thanks to Lemma 7.1.1, we may assume that $\theta \in c_1(L)$ for some toric invariant ample line bundle L. In this case, the result follows from Theorem 7.1.1, Theorem 7.3.1 and Theorem 5.2.1. □

cor:volbigL **Corollary 7.3.1** *We have*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k) = \int_X \theta^n_{V_\theta}. \tag{7.7}$$ {eq:volbig}

This common quantity is the *volume* of L, usually denoted by $\operatorname{vol} L$.

Chapter 8
The trace operator

chap:trace

8.1 The definition of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset. The trace operator gives a way to restrict a quasi-plurisubharmonic function on X to $\tilde{Y}$, the normalization of Y. It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space. We refer to Appendix B for the pluripotential theory on unibranch Kähler spaces.

For later applications, we need this generality even if initially we are only interested in the smooth case.

We first observe that given $\varphi \in \mathrm{QPSH}(X)$ with analytic singularities such that $\nu(\varphi, Y) = 0$, then $\varphi|_Y \not\equiv -\infty$. This observation will be crucial in the sequel.

op:traceindquasiequisingapp

Proposition 8.1.1 *Let $\varphi \in \mathrm{QPSH}(X)$. Consider a smooth closed real $(1,1)$-form on X and $\varphi \in \mathrm{PSH}(X, \theta)$ such that $\nu(\varphi, Y) = 0$. Let $(\varphi_i)_i$, $(\psi_i)_i$ be quasi-equisingular approximations of φ. Then*

$$\lim_{i\to\infty} d_S\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) = 0. \tag{8.1}$$

{eq:dsequivtemp1}

The meaning of (8.1) is explained in Corollary 6.2.8.

Proof Take a Kähler form ω on X. By Corollary 6.2.8, we may assume that $\varphi, \varphi_i, \psi_i \in \mathrm{PSH}(X, \theta - \omega)$ for all $i \geq 1$. Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good. It follows from Corollary 7.1.2 and Proposition 6.2.5 that we can assume $\varphi_i \leq \psi_i$ for all $i \geq 1$.

Take a decreasing sequence $(\epsilon_j)_j$ in $\mathbb{R}_{>0}$ with limit 0 such that $(1-\epsilon_j)\varphi_j \in \mathrm{PSH}(X, \theta)$. We first observe that

$$\lim_{i\to\infty} d_S(\varphi_i|_{\tilde{Y}}, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}) = 0.$$

This is a consequence of Lemma 6.2.3.

Next by Proposition 1.6.3, we could find a subsequence $(\psi_{j_i})_{i\in\mathbb{Z}_{>0}}$ of $(\psi_j)_j$ such that for each $i \geq 1$,

$$\varphi_{j_i} \leq \psi_{j_i} \leq (1-\epsilon_i)\varphi_i.$$

Therefore, (8.1) follows from Corollary 6.2.1. □

def:traceop

Definition 8.1.1 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. We say a potential $\psi \in \mathrm{QPSH}(\tilde{Y})$ is a *trace operator* of φ along Y if there is a smooth closed real $(1,1)$-form θ on X such that $\varphi \in \mathrm{PSH}(X, \theta)$ and a quasi-equisingular approximation $(\varphi_j)_j$ of φ such that

$$\varphi_j|_{\tilde{Y}} \xrightarrow{d_S} \psi. \tag{8.2}$$

{eq:deftrace}

By Corollary 6.2.5, the trace operator is always defined. Observe that by Proposition 8.1.1, the condition (8.2) is independent of the choice of $(\varphi_j)_j$. It is also independent of the choice of θ by Corollary 6.2.7.

prop:traceunique

Proposition 8.1.2 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Suppose that ψ and ψ' are trace operators of φ along Y. Then ψ and ψ' are $\mathcal{I}$-good and $\psi \sim_P \psi'$.*

Proof That ψ and ψ' are $\mathcal{I}$-good follows from Theorem 7.1.1. The fact that $\psi \sim_P \psi'$ follows from Proposition 8.1.1 and Proposition 6.2.2. □

Definition 8.1.2 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. We write $\mathrm{Tr}_Y(\varphi)$ for any trace operator of φ along Y.

Given a closed smooth real $(1,1)$-form θ on X. When $\mathrm{Tr}_Y(\varphi)$ can be chosen to lie in $\mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})_{>0}$, we write

$$\mathrm{Tr}_Y^{\theta}(\varphi) := P_{\theta|_{\tilde{Y}}}\left[\mathrm{Tr}_Y(\varphi)\right] = P_{\theta|_{\tilde{Y}}}\left[\mathrm{Tr}_Y(\varphi)\right]_{\mathcal{I}}.$$

The trace operator $\mathrm{Tr}_Y(\varphi)$ is therefore well-defined only up to P-equivalence by Proposition 8.1.2.

rmk:tracecurrent

Remark 8.1.1 As in Remark 1.7.1, the trace operator could also be applied to closed positive $(1,1)$-currents on X. If $T \in \mathcal{Z}_+(X, \alpha)$ (see Definition 1.7.2) and $\beta \in \mathrm{H}^{1,1}(\tilde{Y}, \mathbb{R})$, then we write

$$\mathrm{Tr}_Y^{\beta}(T)$$

for any closed positive $(1,1)$-current in β representing $\mathrm{Tr}_Y(T)$ when $\nu(T, Y) = 0$.

prop:Trdominarest

Proposition 8.1.3 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that $\varphi|_Y \not\equiv -\infty$. Then*

$$\varphi|_{\tilde{Y}} \preceq_P \mathrm{Tr}_Y(\varphi).$$

Proof Take a Kähler form ω such that ω_φ is a Kähler current. Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X, \omega)$. We may assume that $\varphi_j \leq 0$ for all $j \geq 1$.

Then

$$\varphi_j|_{\tilde{Y}} \leq P_{\theta|_{\tilde{Y}}}\left[\varphi_j|_{\tilde{Y}}\right] \tag{8.3}$$

{eq:varphijrestrleqPtemp}

for all $j \geq 1$.

Thanks to Corollary 6.2.4,

$$\mathrm{Tr}_Y(\varphi) \sim_P \inf_{j\geq 1} P_{\theta|_{\tilde{Y}}}[\varphi_j|_{\tilde{Y}}]. \tag{8.4}$$ {eq:TrYnewexpression}

Letting $j \to \infty$ in (8.3), we conclude our assertion. □

ex:resanalyt *Example 8.1.1* Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that φ has analytic singularities, then

$$\mathrm{Tr}_Y(\varphi) \sim_P \varphi|_{\tilde{Y}}.$$

Example 8.1.2 Let $\varphi \in \mathrm{QPSH}(X)$. Take a closed real smooth $(1,1)$-form θ on X such that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then

$$\mathrm{Tr}_X(\varphi) \sim_P P_\theta[\varphi]_{\mathcal{I}}, \quad \mathrm{Tr}_X^\theta(\varphi) = P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, the trace operator can be regarded as a generalization of the $\mathcal{I}$-envelope.

ex:tracedefinedposmass *Example 8.1.3* Assume that $\varphi \in \mathrm{PSH}(X,\theta)$ for some closed smooth real $(1,1)$-form θ on X and

$$\lim_{\epsilon\searrow 0}\int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right)^m > 0 \tag{8.5}$$ {eq:traceposmasscond}

for any arbitrary choice of a Kähler form ω on X. Then it follows from Proposition 3.1.8 that $\mathrm{Tr}_Y^\theta(\varphi)$ is defined, and its mass is exact the above limit.

In particular, if θ_φ is a Kähler current, $\mathrm{Tr}_Y^\theta(\varphi)$ is always defined.

8.2 Properties of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset.

prop:tracelinear **Proposition 8.2.1** *Let $\varphi, \psi \in \mathrm{QPSH}(X)$, $\lambda > 0$. Assume that $\nu(\varphi, Y) = \nu(\psi, Y) = 0$. Then we have the following:*

(1) *suppose that $\varphi \leq_{\mathcal{I}} \psi$, then $\mathrm{Tr}_Y(\varphi) \leq_P \mathrm{Tr}_Y(\psi)$;*
(2) *We have*
$$\mathrm{Tr}_Y(\varphi+\psi) \sim_P \mathrm{Tr}_Y(\varphi) + \mathrm{Tr}_Y(\psi);$$
(3) *We have*
$$\mathrm{Tr}_Y(\lambda\varphi) \sim_P \lambda\,\mathrm{Tr}_Y(\varphi);$$
(4) *We have*
$$\mathrm{Tr}_Y(\varphi \vee \psi) \sim_P \mathrm{Tr}_Y(\varphi) \vee \mathrm{Tr}_Y(\psi).$$

Proof Take a closed smooth real $(1,1)$-form θ on X such that $\theta_\varphi, \theta_\psi$ are both Kähler currents. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be quasi-equisingular approximations of φ and ψ in $\mathrm{PSH}(X,\theta)$ respectively.

(1). By Corollary 7.1.2 and Proposition 6.2.5, we may assume that $\varphi_j \leq \psi_j$ for all j. Then our assertion follows from Proposition 6.2.4.

(2). It follows from Theorem 6.2.2 that $\varphi_j + \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}}$. However, by Proposition 3.2.10 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi+\psi]_{\mathcal{I}}.$$

Therefore, by Proposition 6.2.2, Corollary 7.1.2 and Proposition 1.6.1, $\varphi_j + \psi_j$ is a quasi-equisingular approximation of $\varphi+\psi$. We conclude using Theorem 6.2.2.

(3). Let $(\lambda_j)_j$ be an increasing sequence of positive rational numbers with limit λ. Then $(\lambda_j\varphi_j)_j$ is a quasi-equisingular approximation of φ. Our assertion follows Lemma 6.2.3.

(4). By Proposition 6.2.5, we have

$$\varphi_j \vee \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}}.$$

By Proposition 3.2.10 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi \vee \psi]_{\mathcal{I}}.$$

Therefore, our assertion follows exactly as in the proof of (2). □

prop:tracedeclimit

Proposition 8.2.2 *Let $(\varphi_j)_{j\in I}$ be a decreasing net in* $\mathrm{QPSH}(X)$. *Assume that there exists a closed real smooth $(1,1)$-form θ such that $\varphi_j \in \mathrm{PSH}(X,\theta)$ for each $j \in I$. Assume that $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{QPSH}(X)$ and $\nu(\varphi, Y) = 0$. Then*

$$\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \mathrm{Tr}_Y(\varphi).$$

Proof By Corollary 6.2.7, we may assume that there is a Kähler form ω on X such that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta-\omega)$ for all $j \in I$. Note that for each $j \geq 1$,

$$\mathrm{Tr}_Y(\varphi_{j+1}) \preceq_P \mathrm{Tr}_Y(\varphi_j).$$

It follows from Proposition 8.2.1 and Corollary 6.2.5 that there exists $\psi \in \mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})$ such that $\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \psi$.

For each j, we take a quasi-equisingular approximation $(\varphi_j^k)_k$ in $\mathrm{PSH}(X,\theta)$ of φ_j. Using Theorem 1.6.2, we may guarantee that

$$\varphi_{j+1}^k \preceq \varphi_j^k$$

for each $j, k \geq 1$. In particular, $(\varphi_j^j)_j$ is a quasi-equisingular approximation of φ. By Proposition 6.2.4, we have $\psi \preceq_P \mathrm{Tr}_Y(\varphi)$.

Conversely, by Proposition 8.2.1, $\mathrm{Tr}_Y(\varphi_j) \succeq_P \mathrm{Tr}_Y(\varphi)$. It follows again from Proposition 6.2.4 that $\mathrm{Tr}_Y(\varphi) \preceq_P \psi$. □

Example 8.2.1 The trace operator is not continuous along increasing sequences. Let us consider the case $X = \mathbb{P}^2$ with coordinates (z_1, z_2). Let ω_{FS} denote the Fubini–Study metric. The subvariety $Y \cong \mathbb{P}^1$ is defined by $z_2 = 0$. Consider an increasing sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X, \omega_{\mathrm{FS}})$, whose potentials near $(0,0)$ are given by

$$\log|z_1|^2 \vee \left(k^{-1}\log|z_2|^2\right) + O(1).$$

The pointwise restriction of these potentials to Y are given locally by

$$\log|z_1|^2 + O(1).$$

On the other hand, locally

$$\log|z_1|^2 \vee \left(k^{-1}\log|z_2|^2\right) \to 0$$

almost everywhere as $k \to \infty$. So the trace operator is not continuous along the sequence $(\varphi_j)_j$.

lma:rescommpullback **Lemma 8.2.1** *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a connected Kähler manifold. Assume that W (resp. Y) be analytic subsets in Z (resp. X) of codimension 1 such that the restriction $\Pi\colon W \to Y$ of π is defined and is bimeromorphic, so that we have the following commutative diagram*

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d, "\tilde{\Pi}"] & W \arrow[r, hook] \arrow[d, "\Pi"] & Z \arrow[d, "\pi"] \\
\tilde{Y} \arrow[r] & Y \arrow[r, hook] & X.
\end{tikzcd}$$

Then for any $\varphi \in \mathrm{QPSH}(X)$ with $\nu(\varphi, Y) = 0$, we have

$$\tilde{\Pi}^* \operatorname{Tr}_Y(\varphi) \sim_P \operatorname{Tr}_W(\pi^*\varphi). \tag{8.6}$$ {eq:rescommpullback}

Proof We first observe that by Zariski's main theorem, $\nu(\pi^*\varphi, W) = 0$. So the right-hand side of (8.6) makes sense.

Step 1. Assume that T has analytic singularities. It suffices to apply Example 8.1.1 to reformulate (8.6) as

$$\tilde{\Pi}^*(\varphi|_{\tilde{Y}}) \sim_P (\pi^*\varphi)|_{\tilde{W}}.$$

In fact, the strict equality holds, which is nothing but the functoriality of pullbacks.

Step 2. Next we handle the general case. Up to replacing θ by $\theta + \omega$ for some Kähler form ω on X, we may assume that T is a Kähler current. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\mathrm{PSH}(X, \theta)$. By Corollary 7.1.2, $(\pi^*\varphi_j)_j$ is a quasi-equisingular approximation of $\pi^*\varphi$. From Step 1, we know that for each j,

$$\tilde{\Pi}^* \operatorname{Tr}_Y(\varphi_j) \sim_P \operatorname{Tr}_W(\pi^*\varphi_j).$$

Letting $j \to \infty$, we conclude (8.6) using Proposition 8.2.2. □

prop:OT2 **Proposition 8.2.3** *Let $\varphi \in \mathrm{QPSH}(X)$ with $\nu(\varphi, Y) = 0$. Assume that Y is smooth. Then for any $\lambda > 0$, we have*

$$\mathcal{I}(\lambda \operatorname{Tr}_Y(\varphi)) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi). \tag{8.7}$$

{eq:OT}

Proof Take a Kähler form ω on X such that ω_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\operatorname{PSH}(X,\omega)$.

By definition, for each $j \geq 1$, we get that

$$\operatorname{Tr}_Y(\varphi) \preceq_P \varphi_j|_Y.$$

For any $\lambda' > \lambda > 0$, we can find $j > 0$ so that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

By Theorem 1.4.5, we have

$$\mathcal{I}(\lambda' \operatorname{Tr}_Y(\varphi)) \subseteq \mathcal{I}(\lambda'\varphi_j|_Y) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda'\varphi_j) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi).$$

Thanks to Theorem 1.4.4, we conclude (8.7). □

Lastly, we turn our attention to global sections. For this we will need the following global Ohsawa–Takegoshi extension theorem for the trace operator:

thm: OT_ext_global

Theorem 8.2.1 *Let L be a big line bundle on X and θ is a closed real smooth $(1,1)$-form on X representing $c_1(L)$. Suppose that $\varphi \in \operatorname{PSH}(X,\theta)$ and θ_φ is a Kähler current. Assume that $\nu(\varphi, Y) = 0$. Let T be a holomorphic line bundle on X. Then there exists k_0 such that for all $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k \operatorname{Tr}_Y^\theta(\varphi)))$, there exists an extension $\tilde{s} \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$.*

It is of interest to know if one could control the L^2-norm of $\tilde{s}$ in the above result.

Proof Fix a Kähler form ω on X. We may assume that $Y \neq X$ and that $\theta_\varphi \geq 3\delta\omega$ for some $\delta > 0$. Let $(\varphi_j)_j$ be the decreasing quasi-equisingular approximation of φ in $\operatorname{PSH}(X,\theta)$. We can assume that $\theta_{\varphi_j} \geq 2\delta\omega$ for all $j \geq 1$. Also, there exists $\epsilon_0 > 0$ such that $\theta_{(1+\epsilon)\varphi_j} \geq \delta\omega$ for any $\epsilon \in (0, \epsilon_0)$. Take $k_0 = k_0(\delta)$ as in Theorem 1.8.1.

We fix $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k \operatorname{Tr}_Y^\theta(\varphi)))$. By Theorem 1.4.4, there exists $\epsilon \in (0, \epsilon_0)$ such that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon) \operatorname{Tr}_Y^\theta(\varphi)))$.

Since $\operatorname{Tr}_Y^\theta(\varphi) \preceq \varphi_j|_Y$, we obtain that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j|_Y))$. Due to Theorem 1.8.1 there exists $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j))$ such that $\tilde{s}_j|_Y = s$, for all j.

But by definition of quasi-equisingular approximation, we obtain that for high enough j the inclusion $\mathcal{I}(k(1+\epsilon)\varphi_j) \subseteq \mathcal{I}(k\varphi)$ holds. As a result, $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ for high enough j, finishing the argument. □

thm:exttracegeneral

Conjecture 8.2.1 Assume that Y is smooth and has positive dimension. Fix a Kähler form ω on X. For each $\varphi \in \operatorname{PSH}(Y, \omega|_Y)$ such that $\omega|_Y + \mathrm{dd^c}\varphi$ is a Kähler current, we can find $\tilde{\varphi} \in \operatorname{PSH}(X,\omega)$ such that $\omega + \mathrm{dd^c}\tilde{\varphi}$ is a Kähler current and

$$\operatorname{Tr}_Y(\tilde{\varphi}) \sim_{\mathcal{I}} \varphi.$$

8.3 Restricted volumes

Let X be a connected projective manifold of dimension n and $Y \subseteq$ be a connected submanifold of dimension m. Consider a big line bundle L on X, a Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$. Let A be a very ample line bundle on X. Take a Hermitian metric h_A on A such that $\omega = \mathrm{dd^c} h_A$ is a Kähler form.

Using the trace operator, one could prove the following generalization of Theorem 7.3.1.

thm: rest_volume **Theorem 8.3.1** *Let h be a singular plurisubharmonic metric on L with $\nu(\mathrm{dd^c}h, Y) = 0$. Assume that*

$$\lim_{\epsilon \searrow 0} \left(\mathrm{Tr}_Y^{c_1(L|_Y)+\epsilon\omega}(c_1(L,h)) \right)^m > 0. \tag{8.8}$$

{eq:traceposmasscond2}

Then for any holomorphic line bundle T on X we have that

$$\int_Y \left(\mathrm{Tr}_Y^{c_1(L|_Y)}(c_1(L,h)) \right)^m = \lim_{k\to\infty} \frac{m!}{k^m} h^0 \left(Y, T|_Y \otimes L|_Y^k \otimes \mathrm{Res}_Y(\mathcal{I}(h^k)) \right). \tag{8.9}$$

{eq:DXmainrelative}

Recall that Res_Y is defined in Definition 1.4.5. Observe that by Example 8.1.3, (8.8) implies that $\mathrm{Tr}_Y^{c_1(L|_Y)}(c_1(L,h))$ is defined. So (8.9) is defined.

We will identify h with $\varphi \in \mathrm{PSH}(X, \theta)$ as in (1.9).

We only need to consider the case $Y \neq X$, since otherwise, the result is proved in Theorem 7.3.1. We will always assume $Y \neq X$ in the sequel.

Lemma 8.3.1 *There is $\psi_Y \in \mathrm{QPSH}(X)$ with neat analytic singularities such that $\{\psi_Y = -\infty\} = Y$ and in an open neighbourhood of Y, we have*

$$\psi_Y(x) = 2(n-m) \log \mathrm{dist}(x, Y) \tag{8.10}$$

{eq: Psi_Y_def}

for some Riemannian distance function $\mathrm{dist}(\cdot, Y)$.

See Definition 1.6.1 for the definition of neat analytic singularities.

See [Fin22, Lemma 2.3] for the proof.

lma:IpsiY **Lemma 8.3.2** *The multiplier ideal sheaf of ψ_Y can be calculated as*

$$\mathcal{I}(\psi_Y) = \mathcal{I}_Y. \tag{8.11}$$

{eq:mis_psi}

Moreover, given $y \in Y$ and $\epsilon > 0$, for any germ $f \in \mathcal{I}_{Y,y}$ we have

$$\int_U |f|^\epsilon \mathrm{e}^{-\psi_Y} \omega^n < \infty, \tag{8.12}$$

{eq:integrabilitypsiY}

where U is an open neighbourhood of y in X.

In other words, ψ_Y has *log canonical singularities*.

Proof Since ψ_Y is locally bounded away from Y, it suffices to prove (8.11) along Y. Fix $y \in Y$, and we will verify (8.11) germ-wise at y.

Take an open neighbourhood $U \subset X$ of y and a biholomorphic map $F\colon U \to V \times W$, where V is an open neighbourhood of y in Y and W is a connected open subset in $\mathbb{C}^{n-m}$ containing 0, such that $F(Y \cap U) = V \times \{0\}$. For any $x \in U$, write x_V, x_W for the two components of $F(x)$ in V and W respectively. We denote the coordinates in $\mathbb{C}^{n-m}$ as $w_1, \dots, w_{n-m}$.

Due to (8.10), after possibly shrinking U, we may assume that

$$\exp(-\psi_Y(x)) = |x_W|^{2m-2n} + O(1)$$

for any $x \in U \setminus Y$.

Given $f \in \mathcal{I}_{Y,y}$, after shrinking U, we may assume that there exists $g_1, \dots, g_{n-m} \in \mathrm{H}^0(V \times W, \mathcal{O}_{V\times W})$ such that

$$f = \sum_{i=1}^{n-m} w_i g_i .$$

In order to verify $f \in \mathcal{I}(\psi_Y)_y$, it suffices to show $w_i g_i \in \mathcal{I}\left(\left(\sum_{i=1}^{n-m} |w_i|^2\right)^{m-n}\right)_{F(y)}$, which follows from Fubini's theorem. The proof of (8.12) is similar.

Conversely, take $f \in \mathcal{I}(\psi_Y)$, the similar application of Fubini's theorem shows that after possible shrinking U, we have $f|_Y = 0$. By Rückert's Nullstellensatz [GR84, Page 67], it follows that $f \in \mathcal{I}_Y$. □

lem: analytic_formula

Lemma 8.3.3 *Assume that φ has analytic singularity type and θ_u is a Kähler current. Suppose that $\varphi|_Y \not\equiv -\infty$. Then*

$$\int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m = \lim_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\}. \quad (8.13)$$

{eq:asymanasing}

Recall that $\mathcal{I}_\infty$ is defined in Definition 1.6.5.

Proof Suppose that $\epsilon \in (0,1)$ is small enough so that $(1-\epsilon)u \in \mathrm{PSH}(X,\theta)$.

Using Theorem 7.3.1 we can start to write the following sequence of inequalities:

$$
\begin{aligned}
&\frac{1}{m!}\int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m \\
=&\lim_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}(k\varphi|_Y))\\
\leq&\varliminf_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\right\}\quad\text{by Theorem 1.8.1}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\right\}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X,T\otimes L^k\otimes\mathcal{I}_\infty((1-\epsilon)k\varphi))\right\}\quad\text{by Lemma 1.6.3}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim_{\mathbb{C}}\left\{s\in \mathrm{H}^0(Y,T|_Y\otimes L|_Y^k) : \log h^k(s,s)\leq (1-\epsilon)k\varphi|_Y\right\}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}h^0\left(Y,T|_Y\otimes L|_Y^k\otimes\mathcal{I}((1-\epsilon)k\varphi|_Y)\right)\\
=&\frac{1}{m!}\int_Y(\theta|_Y+(1-\epsilon)\mathrm{dd^c}\varphi|_Y)^m\quad\text{by Theorem 7.3.1.}
\end{aligned}
$$

Letting $\epsilon\to 0$, (8.13) follows from multi-linearity of the non-pluripolar product. □

prop: rest_volume

Proposition 8.3.1 *In the setting of Theorem 8.3.1, assume that* $\mathrm{dd^c}h$ *is a Kähler current. Then* (8.9) *holds.*

Proof Let $(\varphi_j)_j$ a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. After possibly replacing $(\varphi_j)_j$ by a subsequence, there exists $\epsilon_0\in(0,1)\cap\mathbb{Q}$ such that $\theta_{(1-\epsilon)^2\varphi_j}$ and $\theta_{(1-\epsilon)\varphi_j}$ are also Kähler currents for any $\epsilon\in(0,\epsilon_0)$.

We claim that for any $j\geq 1$ and $k\in\mathbb{N}$, we have

$$
\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\cap\mathcal{I}(\psi_Y)\subseteq\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y). \tag{8.14}
$$

{eq:JcapI}

Take $x\in X$, and it suffices to argue (8.14) along the germ of x. Since ψ_Y is locally bounded outside Y, we may assume that $x\in Y$. Recall that by Lemma 8.3.2, $\mathcal{I}(\psi_Y)=\mathcal{I}_Y$.

Let $f\in\mathcal{I}_\infty((1-\epsilon)k\varphi_j)_x\cap\mathcal{I}(\psi_Y)_x$. Then there is an open neighbourhood U of x in X such that $|f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j}\leq C$ holds on $U\setminus\{\varphi_j=-\infty\}$ for some $C>0$, hence

$$
\begin{aligned}
\int_U |f|^2\mathrm{e}^{-k(1-\epsilon)^2\varphi_j-\psi_Y}\,\omega^n=&\int_U|f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j}|f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n\\
\leq& C\int_U|f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n<\infty,
\end{aligned}
$$

where the last inequality follows from Lemma 8.3.2. We have proved the claim (8.14).

Next we consider the following composition morphism of coherent sheaves on Y:

$$
\mathrm{Res}_Y\,\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\hookrightarrow\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\cap\mathcal{I}_Y}\to\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}. \tag{8.15}
$$

{eq: sheaf_injection}

Here we have identified the coherent $\mathcal{O}_X$-modules supported on Y with coherent $\mathcal{O}_Y$-modules. Note that the target of (8.15) is also supported on Y as ψ_Y is locally bounded outside Y. We denote the coherent $\mathcal{O}_Y$-module whose pushforward to X gives $\frac{\mathcal{I}((1-\epsilon)^2 k\varphi_j)}{\mathcal{I}((1-\epsilon)^2 k\varphi_j+\psi_Y)}$ by $\mathcal{I}_{k,j}$.

In (8.15), the first map is the inclusion and the second one is the obvious projection induced by (8.14). Although in general the second map fails to be injective, we observe that the composition is still injective as $\mathcal{I}((1-\epsilon)^2 k\varphi_j+\psi_Y) \subseteq \mathcal{I}(\psi_Y)=\mathcal{I}_Y$. Therefore, for any $k\in\mathbb{N}$, we have an injective morphism of coherent $\mathcal{O}_Y$-modules:

$$L|_Y^k \otimes T|_Y \otimes \mathrm{Res}_Y\, \mathcal{I}_\infty((1-\epsilon)k\varphi_j) \hookrightarrow L|_Y^k \otimes T|_Y \otimes \mathcal{I}_{k,j}. \tag{8.16}$$

{eq:injLkTideal}

Using Theorem 7.3.1 we can start the following inequalities:

$$\begin{aligned}
&\frac{1}{m!}\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m \\
=&\lim_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}(k\,\mathrm{Tr}_Y^\theta(\varphi))) \quad \text{by Theorem 7.3.1}\\
\leq&\varliminf_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathrm{Res}_Y(\mathcal{I}(k\varphi))) \quad \text{by Theorem 1.4.5}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathrm{Res}_Y(\mathcal{I}(k\varphi)))\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}(k\varphi_j)|_Y)\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}_\infty((1-\epsilon)k\varphi_j)|_Y) \quad \text{by Lemma 1.6.3}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}_{k,j}) \quad \text{by (8.16)}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} \dim_{\mathbb{C}}\left\{ s|_Y : s\in \mathrm{H}^0\left(X, T\otimes L^k\otimes \frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}\right)\right\}\\
=&\varlimsup_{k\to\infty}\frac{1}{k^m} \dim_{\mathbb{C}}\left\{ s|_Y : s\in \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}((1-\epsilon)^2k\varphi_j))\right\} \quad \text{(see below)}\\
=&\frac{1}{m!}\int_Y \left(\theta|_Y + (1-\epsilon)^2\mathrm{dd^c}\varphi_j|_Y\right)^m \quad \text{by Lemma 8.3.3,}
\end{aligned}$$

where in the penultimate line we used [CDM17, Theorem 1.1(6)] for $q=0$. Letting $\epsilon\to\infty$ and then $j\to\infty$ the result follows. □

Proof (Proof of Theorem 8.3.1) Using Proposition 8.2.3 and Theorem 7.3.1 we obtain that

$$\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\,\mathrm{Tr}_Y^\theta(\varphi)))$$
$$\leq \varliminf_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathrm{Res}_Y(\mathcal{I}(k\varphi))).$$

{eq:DX_cor}

Now we address the other direction in (8.9). Let $\phi \in \mathrm{H}^0(X, A)$ be a section that does not vanish identically on Y. Such ϕ exists since A is very ample.

We fix $k_0 \in \mathbb{N}$. For any $k \geq 0$, we have that $k = qk_0 + r$ with $q, r \in \mathbb{N}$ and $r \in \{0, \dots, k_0 - 1\}$. Also, we have an injective linear map

$$\mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)) \xrightarrow{\cdot\phi^{\otimes q}} \mathrm{H}^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}(k\varphi|_Y)\right).$$

Therefore,

$$\begin{aligned}
&\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)\right)\\
\leq &\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}(k\varphi|_Y)\right)\\
= &\frac{1}{k_0^m} \varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}(k\varphi|_Y)\right)\\
\leq &\frac{1}{k_0^m} \varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}(k_0 q\varphi|_Y)\right)\\
= &\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta + k_0^{-1}\omega}(\varphi)\right)^m\\
= &\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m,
\end{aligned}$$

where in the fourth line we have used that $k_0 q \leq k$ and in the last line we have used Proposition 8.3.1 for the big line bundle $L^{k_0} \otimes A$, the Kähler current $k_0\theta_u - \mathrm{dd^c}\log g = k_0\theta_u + \omega$, and twisting bundle $T \otimes L^r$. Letting $k_0 \to \infty$, we conclude that

$$\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)\right) \leq \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m.$$

thm: rest_volume_2

Theorem 8.3.2 *Let $\varphi \in \mathrm{PSH}(X, \theta)$ such that $\nu(\varphi, Y) = 0$. Assume that θ_φ is a Kähler current. Then*

$$\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\}.$$

Proof This is a consequence of Theorem 7.3.1, Theorem 8.2.1 and Theorem 8.3.1:

$$
\begin{aligned}
\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m &= \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\,\mathrm{Tr}_Y^\theta(\varphi))) \\
&\leq \varliminf_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\} \\
&\leq \varlimsup_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\} \\
&\leq \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi)|_Y) \\
&= \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m .
\end{aligned}
$$

Remark 8.3.1 One could also show that when (8.8) fails, the right-hand side of (8.9) is 0. See [DX24].

8.4 Analytic Bertini theorem

The analytic Bertini theorem handles the restriction along a generic subvariety.

thm:Bert

Theorem 8.4.1 *Let X be a connected projective manifold of dimension $n \geq 1$ and $\varphi \in \mathrm{QPSH}(X)$. Let $p\colon X \to \mathbb{P}^N$ be a morphism ($N \geq 1$). Define*

$$
\mathcal{G} := \{ H \in |\mathcal{O}_{\mathbb{P}^N}(1)| : H' := H \cap X \text{ is smooth and } \mathcal{I}(\varphi|_{H'}) = \mathrm{Res}_{H'}(\mathcal{I}(\varphi)) \} .
$$

Then $\mathcal{G} \subseteq |\mathcal{O}_{\mathbb{P}^N}(1)|$ is co-pluripolar.

Recall that co-pluripolar sets are defined in Definition 1.1.4.

Remark 8.4.1 Here and in the sequel, we slightly abuse the notation by writing $H \cap X$ for $p^{-1}H$, the scheme-theoretic inverse image of H. In other words, $H \cap X := H \times_{\mathbb{P}^N} X$.

By definition, any $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ such that $p^{-1}H = \emptyset$ lies in $\mathcal{G}$.

Proof Take an ample line bundle L with a smooth Hermitian metric h such that $c_1(L, h) + \mathrm{dd^c}\varphi \geq 0$, where $c_1(L, h)$ is the first Chern form of (L, h), namely the curvature form of h. We introduce $\Lambda := |\mathcal{O}_{\mathbb{P}^N}(1)|$ to simplify our notations.

Step 1. We prove that the following set is co-pluripolar:

$$
\begin{aligned}
\mathcal{G}_L := \Big\{ H \in \Lambda : H \cap X \text{ is smooth and } &\mathrm{H}^0\left(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})\right) = \\
&\mathrm{H}^0\left(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))\right) \Big\} .
\end{aligned}
$$

Here $\omega_{H\cap X}$ denotes the dualizing sheaf of $H \cap X$.

Let $U \subseteq \Lambda \times X$ be the closed subvariety whose $\mathbb{C}$-points correspond to pairs $(H, x) \in \Lambda \times X$ with $p(x) \in H$. Let $\pi_1 : U \to \Lambda$ be the natural projection. We may assume that π_1 is surjective, as otherwise there is nothing to prove.

Observe that U is a local complete intersection scheme by *Krulls Hauptidealsatz* and *a fortiori* a Cohen–Macaulay scheme. It follows from miracle flatness [Mat89, Theorem 23.1] that the natural projection $\pi_2 \colon U \to X$ is flat. As the fibers of π_2 over closed points of X are isomorphic to $\mathbb{P}^{N-1}$, it follows that π_2 is smooth. Thus, U is smooth as well. Moreover, observe that

$$\mathcal{I}(\pi_2^*\varphi) = \pi_2^*\mathcal{I}(\varphi) \tag{8.17}$$

by Proposition 1.4.5.

In the following, we will construct pluripolar sets $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \Lambda$ such that the behaviour of π_1 is improved successively on the complement of Σ_i.

Step 1.1. The usual Bertini theorem shows that there is a proper Zariski closed set $\Sigma_1 \subseteq \Lambda$ such that π_1 has smooth fibres outside Σ_1. Enlarging Σ_1, we could guarantee that π_1 is flat

Moreover, we could guarantee that $\mathcal{I}(\pi_2^*\varphi)$ is flat over $\Lambda \setminus \Sigma_1$. Then after further enlarging Σ_1, we could arrive at

$$\mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi)) = i_H^*\mathcal{I}(\pi_2^*\varphi)$$

for all $H \in \Lambda \setminus \Sigma_1$. Here $\pi_{1,H}$ denotes the fibre of π_1 at H and $i_H \colon \pi_{1,H} \to U$ is the inclusion morphism. This is a consequence of [Sta20, Tag 05DB].

Step 1.2. By Grauert's coherence theorem,

$$\mathcal{F}^i \coloneqq R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L \otimes \mathcal{I}(\pi_2^*\varphi)\right)$$

is coherent for all i. Here $\omega_{U/\Lambda}$ denotes the relative dualizing sheaf of the morphism $U \to \Lambda$. Thus, there is a proper Zariski closed set $\Sigma_2 \subseteq \Lambda$ such that

(1) $\Sigma_2 \supseteq \Sigma_1$.
(2) The $\mathcal{F}^i$'s are locally free outside Σ_2.
(3) $\omega_{U/\Lambda} \otimes \pi_2^*L \otimes \mathcal{I}(\pi_2^*\varphi)$ is π_1-flat on $\pi_1^{-1}(\Lambda \setminus \Sigma_2)$ [DG65, Théorème 6.9.1].

We write $\mathcal{F} = \mathcal{F}^0$. By cohomology and base change [Har13, Theorem III.12.11], for any $H \in \Lambda \setminus \Sigma_2$, the fibre $\mathcal{F}|_H$ of $\mathcal{F}$ is given by

$$\mathcal{F}|_H = \mathrm{H}^0\left(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \pi_2^*L|_{\pi_{1,H}} \otimes \mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi))\right).$$

Step 1.3. In order to proceed, we need to make use of the Hodge metric h_H on $\mathcal{F}$ defined in [HPS18]. We briefly recall its definition in our setting. By [HPS18, Section 22], we can find a proper Zariski closed set $\Sigma_3 \subseteq \Lambda$ such that

(1) $\Sigma_3 \supseteq \Sigma_2$,
(2) π_1 is smooth outside Σ_3,
(3) both $\mathcal{F}$ and $\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)/\mathcal{F}$ are locally free outside Σ_3, and
(4) for each i,

$$R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)$$

is locally free outside Σ_3.

Then for any $H \in \Lambda \setminus \Sigma_3$,

$$\mathrm{H}^0(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes \mathcal{I}(\varphi|_{H\cap X}))\subseteq \mathcal{F}|_H\subseteq \mathrm{H}^0(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}).$$

See [HPS18, Lemma 22.1].

Now we can give the definition of the Hodge metric on $\Lambda\setminus\Sigma_3$. Given any $H\in\Lambda\setminus\Sigma_3$, any $\alpha\in\mathcal{F}|_H$, the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha,\alpha):=\int_{X\cap H}|\alpha|_h^2 \mathrm{e}^{-\varphi}\in[0,\infty].$$

Observe that $h_{\mathcal{H}}(\alpha,\alpha)<\infty$ if and only if $\alpha\in\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))$. Moreover, $h_{\mathcal{H}}(\alpha,\alpha)>0$ if $\alpha\neq 0$. It is shown in [HPS18] (c.f. [PT18, Theorem 3.3.5]) that $h_{\mathcal{H}}$ is indeed a singular Hermitian metric, and it extends to a positive metric on $\mathcal{F}$.

Step 1.4. The determinant $\det h_{\mathcal{H}}$ is singular at all $H\in\Lambda\setminus\Sigma_3$ such that

$$\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))\neq\mathcal{F}|_H.$$

As the map π_2 is smooth, we have $\pi_2^*\mathcal{I}(\varphi)=\mathcal{I}(\pi_2^*\varphi)$ by Proposition 1.4.5. Under the identification $\pi_{1,H}\cong H\cap X$, we have

$$\mathrm{Res}_{\pi_{1,H}}\left(\pi_2^*\mathcal{I}(\varphi)\right)\cong\mathrm{Res}_{H\cap X}\left(\mathcal{I}(\varphi)\right).$$

Thus, we have the following inclusions:

$$\begin{aligned}&\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))\\ \subseteq&\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))),\end{aligned}$$

the right-hand side being $\mathcal{F}|_H$.

Recall that the first inclusion follows from Theorem 1.4.5. Hence, $\det h_{\mathcal{H}}$ is singular at all $H\in|\mathcal{O}_{\mathbb{P}^N}(1)|\setminus\Sigma_3$ such that

$$\begin{aligned}&\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))\\ \neq&\mathrm{H}^0(H\cap X,\omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))).\end{aligned}$$

Let Σ_4 be the union of Σ_3 and the set of all such H. Since the Hodge metric $h_{\mathcal{H}}$ is positive ([PT18, Theorem 3.3.5] and [HPS18, Theorem 21.1]), its determinant $\det h_{\mathcal{H}}$ is also positive ([Rau15, Proposition 1.3] and [HPS18, Proposition 25.1]), it follows that Σ_4 is pluripolar. As a consequence, $\mathcal{G}_L$ is co-pluripolar.

Step 2.

Fix an ample invertible sheaf S on X. The same result holds with $L\otimes S^{\otimes a}$ in place of L. Thus, the set

$$A:=\bigcap_{a=0}^{\infty}\mathcal{G}_{L\otimes S^{\otimes a}}$$

is co-pluripolar. For each $H\in W$ such that $X\cap H$ is smooth and $\mathcal{I}(\varphi|_{X\cap H})\neq\mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))$, let $\mathcal{K}$ be the following cokernel:

$$0 \to \mathcal{I}(\varphi|_{X\cap H}) \to \operatorname{Res}_{H\cap X}(\mathcal{I}(\varphi)) \to \mathcal{K} \to 0.$$

By Serre vanishing theorem, taking a large enough, we may guarantee that

$$H^1(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{I}(\varphi|_{X\cap H})) = 0$$

and

$$\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{K}) \neq 0.$$

Then

$$\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{I}(\varphi|_{X\cap H})) \neq$$
$$\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \operatorname{Res}_{H\cap X}(\mathcal{I}(\varphi))).$$

Thus, $H \notin A$. We conclude that $\mathcal{G}$ is co-pluripolar. □

cor:ABTfortrace

Corollary 8.4.1 *Let X be a connected projective manifold of dimension $n \geq 1$ and Λ be a base-point free linear system on X. Fix $\varphi \in \mathrm{QPSH}(X)$.*

Then there is a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that any $H \in \Lambda'$ is smooth, $\nu(\varphi, H) = 0$ and we have

$$\operatorname{Tr}_H(\varphi) \sim_{\mathcal{I}} \varphi|_H.$$

Proof First observe that the set $\{x \in X : \nu(\varphi, x) > 0\}$ is a countable union of proper analytic subsets by Theorem 1.4.1. It follows that a very general element in Λ is not contained in this set.

Fix an ample line bundle L so that there is a smooth psh metric h_L such that $c_1(L, h_L) + \mathrm{dd^c}\varphi$ is a Kähler current. Thanks to Theorem 8.4.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that each $H \in \Lambda'$ satisfies the following:

(1) H is smooth;
(2) $\nu(\varphi, H) = 0$;
(3) $\mathcal{I}(k\varphi|_H) = \operatorname{Res}_H(\mathcal{I}(\varphi))$ for all $k > 0$.

It follows from Theorem 8.3.1 and Theorem 7.3.1 that

$$\int_H \left(c_1(L, h_L)|_H + \mathrm{dd^c}\operatorname{Tr}_Y^{c_1(L,h_L)}(\varphi)\right)^{n-1} = \int_H (c_1(L, h_L)|_H + \mathrm{dd^c}\varphi|_H)^{n-1}.$$

Since $\varphi|_H \preceq \operatorname{Tr}_Y(\varphi)$ by Proposition 8.1.3, our assertion follows. □

Chapter 9
Test curves

chap:testcurve

9.1 The notion of test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

def:testcur **Definition 9.1.1** A *test curve* Γ in $\mathrm{PSH}(X,\theta)$ consists of a real number $\Gamma_{\max}$ together with a map $(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta)$ denoted by $\tau \mapsto \Gamma_\tau$ satisfying the following conditions:

(1) The map $\tau \mapsto \Gamma_\tau$ is concave and decreasing;
(2) Each Γ_τ is a model potential;
(3) The potential

$$\Gamma_{-\infty} \coloneqq \sup_{\tau<\Gamma_{\max}}{}^{*}\, \Gamma_\tau \tag{9.1}$$ {eq:Gammaminf}

satisfies

$$\int_X (\theta + \mathrm{dd^c}\Gamma_{-\infty})^n > 0.$$

Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. The set of test curves Γ with $\Gamma_{-\infty} = \phi$ is denoted by $\mathrm{TC}(X,\theta;\phi)$.

The set of all $\mathrm{TC}(X,\theta;\phi)$'s for various model potentials $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ is denoted by $\mathrm{TC}(X,\theta)_{>0}$.

By 2, $\sup_X \Gamma_\tau = 0$ for each $\tau < \Gamma_{\max}$. So $\Gamma_{-\infty} \in \mathrm{PSH}(X,\theta)$ defined in (9.1) by Proposition 1.2.1. Moreover, $\Gamma_{-\infty}$ is a model potential by Proposition 3.1.9.

Remark 9.1.1 Sometimes it is convenient to extend Γ_τ to $\tau \geq \Gamma_{\max}$ as well. This can be done as follows: for $\tau > \Gamma_{\max}$, we set $\Gamma_\tau \equiv -\infty$. For $\tau = \Gamma_{\max}$, we set

$$\Gamma_\tau \coloneqq \inf_{\tau'<\Gamma_{\max}} \Gamma_{\tau'} \in \mathrm{PSH}(X,\theta).$$

We will always make this extension in the sequel.

Recall that according to our general principle, we only talk about model potentials when a potential has positive mass. Fortunately, this principle is not violated in the above definition, as shown below:

lma:testcurvposmass

Lemma 9.1.1 *Assume that* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. *Then for each* $\tau < \Gamma_{\max}$, *we have*

$$\int_X (\theta + \mathrm{dd^c}\Gamma_\tau)^n > 0. \tag{9.2}$$

{eq:dalethtauposmass}

Proof Fix $\tau \in (-\infty, \Gamma_{\max})$.

By assumption, $\Gamma_{-\infty}$ has positive mass. By Corollary 2.3.1, we have

$$\int_X \theta^n_{\Gamma_{-\infty}} = \lim_{\tau\to-\infty} \int_X \theta^n_{\Gamma_\tau}.$$

In particular, for a sufficiently small $\tau_0 < \tau$, we have

$$\int_X \theta^n_{\Gamma_{\tau_0}} > 0.$$

Now take $\tau' \in (\tau, \Gamma_{\max})$ and $t \in (0,1)$ so that

$$\tau = (1-t)\tau' + t\tau_0.$$

From the concavity of Γ, we find that

$$\Gamma_\tau \geq (1-t)\Gamma_{\tau'} + t\Gamma_{\tau_0}.$$

By Theorem 2.3.2,

$$\int_X \theta^n_{\Gamma_\tau} \geq \int_X \theta^n_{(1-t)\Gamma_{\tau'}+t\Gamma_{\tau_0}} \geq t^n \int_X \theta^n_{\Gamma_{\tau_0}} > 0$$

and (9.2) follows. □

prop:testcurvmasslogconc

Proposition 9.1.1 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. *Then the map*

$$[-\infty, \Gamma_{\max}) \to \mathbb{R}, \quad \tau \mapsto \log \int_X \theta^n_{\Gamma_\tau}$$

is concave and continuous.

Proof The concavity of this function follows from Theorem 2.3.3 and Theorem 2.3.2. The continuity at $-\infty$ is a consequence of Corollary 2.3.1. □

Definition 9.1.2 Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential.

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to be *bounded* if for τ small enough, $\Gamma_\tau = \phi$. The subset of bounded test curves is denoted by $\mathrm{TC}^\infty(X,\theta;\phi)$. In this case, we write

$$\Gamma_{\min} := \{\tau \in \mathbb{R} : \Gamma_\tau = \phi\}.$$

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to have *finite energy* if

$$\mathbf{E}^{\phi}(\Gamma) \coloneqq \Gamma_{\max} \int_X \theta_\phi^n + \int_{-\infty}^{\Gamma_{\max}} \left(\int_X \theta_{\Gamma_\tau}^n - \int_X \theta_\phi^n \right) \mathrm{d}\tau > -\infty. \tag{9.3}$$ {eq:tcfiniteenergy}

The subset of test curves with finite energy is denoted by $\mathrm{TC}^1(X,\theta;\phi)$.

We first observe that the notion of test curves does not really depend on the choice of θ within its cohomology class.

prop:testcurveindeptheta **Proposition 9.1.2** *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. Let $\phi' \in \mathrm{PSH}(X,\theta')_{>0}$ be the unique model potential satisfying $\phi \sim \phi'$.*

Then there is a canonical bijection

$$\mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(X,\theta';\phi').$$

This bijection induces the following bijections:

$$\mathrm{TC}^1(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^1(X,\theta';\phi'), \quad \mathrm{TC}^\infty(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^\infty(X,\theta';\phi').$$

These bijections satisfy the obvious cocycle conditions.

Proof Choose $g \in C^\infty(X)$ such that $\theta' = \theta + \mathrm{dd^c} g$. Given any $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we observe that $\Gamma' \colon (-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X,\theta')$ defined as

$$\tau \mapsto P_{\theta'}[\Gamma_\tau - g]$$

lies in $\mathrm{TC}(X,\theta';\phi')$. Moreover, the choice of g is irrelevant since for any other choice of g, say g', we have

$$\Gamma_\tau - g \sim \Gamma_\tau - g'.$$

All assertions follow directly from the definition. □

prop:ETCbimero **Proposition 9.1.3** *Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold. Then the pointwise pull-back induces a bijection*

$$\pi^* \colon \mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This follows immediately from Proposition 3.1.3. □

prop:Gammaclosed **Proposition 9.1.4** *Let Γ be a test curve in $\mathrm{PSH}(X,\theta)$. For each $x \in X$, the map $\mathbb{R} \ni \tau \mapsto \Gamma_\tau(x)$ is a closed concave function. Moreover, the map is proper as long as $\Gamma_{\Gamma_{\max}}(x) \neq -\infty$.*

The notion of closedness is recalled in Definition A.1.6.

Proof We argue the closedness. Fix $x \in X$. Assume that $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. We only need to argue the upper-semicontinuity of $\tau \mapsto \Gamma_\tau(x)$. The upper semi-continuity is clear at $\tau \geq \Gamma_{\max}$, so we are reduced to prove the following:

$$\Gamma_\tau = \inf_{\tau'<\tau} \Gamma_{\tau'} \tag{9.4}$$ {eq:Gammatautemp1}

for any $\tau < \Gamma_{\max}$. Take $\tau'' \in (\tau, \Gamma_{\max})$. Outside the polar locus of $\Gamma_{\tau''}$, we know that (9.4) holds by continuity. So (9.4) holds everywhere by Proposition 1.2.5.

The final assertion is trivial. □

def:Ptestcurve **Definition 9.1.3** Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a smooth closed real positive $(1,1)$-form. Then we define $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$ as follows:

(1) Define
$$P_{\theta+\omega}[\Gamma]_{\max} = \Gamma_{\max};$$

(2) For each $\tau < \Gamma_{\max}$, define
$$P_{\theta+\omega}[\Gamma]_\tau = P_{\theta+\omega}[\Gamma_\tau].$$

It follows form Proposition 3.1.4 that $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$.

9.2 Ross–Witt Nyström correspondence

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proposition 9.1.4 allows us to talk about the Legendre transforms in the expected way.

The general definition of the Legendre transform Definition A.2.1 can be translated as follows:

def:Legtrans **Definition 9.2.1** Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$. We define its *Legendre transform* as $\Gamma^*\colon [0,\infty) \longrightarrow \mathrm{PSH}(X,\theta)$ given by

$$\Gamma^*_t = \sup_{\tau\in\mathbb{R}} \left(t\tau + \Gamma_\tau\right). \tag{9.5}$$ {eq:testcurveLegtran}

rmk:negativeray *Remark 9.2.1* Here we do not talk about the case $t < 0$ because its behaviour there pretty trivial: take $x \in X$, if $\Gamma_\tau(x) = -\infty$ for all τ, then $\Gamma^*_t = -\infty$; otherwise, $\Gamma^*_t = \infty$.

As we will see later on, the information about $t \geq 0$ suffices to characterize Γ.

We have made a non-trivial claim that $\Gamma^*_t \in \mathrm{PSH}(X,\theta)$ for all $t \geq 0$. Let us prove this.

lma:testcurvelegusc **Lemma 9.2.1** *Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$. Then $\Gamma^*_t \in \mathrm{PSH}(X,\theta)$ for all $t \geq 0$. In fact, Γ is upper semicontinuous as a function of $X \times (0,\infty)$.*

Proof We first observe that for each $x \in X$, we have

$$\Gamma^*_t(x) \leq t\Gamma_{\max} < \infty.$$

Let $R = \{a + \mathrm{i}b \in \mathbb{C} : a > 0\}$. We consider

$$F\colon X \times R \to [-\infty, \infty), \quad (x, a + \mathrm{i}b) \mapsto \Gamma_a^*(x).$$

Let $\pi\colon X \times R \to X$ be the natural projection. Observe that the upper semicontinuous envelope G of F is $\pi^*\theta$-psh by Proposition 1.2.1. It suffices to show that $F = G$. We let

$$E := \{(x, z) \in X \times R : F(x, z) < G(x, z)\}.$$

We want to argue that $E = \emptyset$. Clearly, E can be written as $B \times \mathrm{i}\mathbb{R}$ for some set $B \subseteq X \times (0, \infty)$. Since E is a pluripolar set by Proposition 1.2.3, it has zero Lebesgue measure. Hence, B has zero Lebesgue measure. For each $x \in X$, write

$$B_x = \{t \in (0, \infty) : (t, x) \in B\}.$$

By Fubini theorem, B_x has zero 1-dimensional Lebesgue measure for all $x \in X \setminus Z$, where $Z \subseteq X$ is a subset of measure 0. We may assume that $Z \supseteq \{\Gamma_{-\infty} = 0\}$ so that for $x \in X \setminus Z$, $\Gamma_t(x) \neq -\infty$ for all $t > 0$.

For any $x \in X \setminus Z$, both $t \mapsto F(x, t)$ and $G(x, t)$ are convex functions with values in $\mathbb{R}$ on $(0, \infty)$. They agree almost everywhere, hence everywhere by their continuity. It follows that for $x \in X \setminus Z$, we have $B_x = 0$.

By Theorem A.2.1, for any $x \in X$, we have

$$\Gamma_\tau(x) = \inf_{t>0}(F(t, x) - t\tau), \quad \tau < \Gamma_{\max}.$$

On the other hand, let

$$\chi_\tau(x) = \inf_{t>0}(G(t, x) - t\tau), \quad \tau < \Gamma_{\max}, x \in X.$$

By Kiselman's principle Proposition 1.2.6, $\chi_\tau \in \mathrm{PSH}(X, \theta)$. But on $X \setminus Z$, we already know that $\Gamma_\tau = \chi_\tau$ for all $\tau < \Gamma_{\max}$. By Proposition 1.2.5, they are equal everywhere. By Theorem A.2.1 again, we find that $F = G$. ◻

lma:suplegenlinear

Lemma 9.2.2 *Let $\Gamma \in \mathrm{TC}(X, \theta; \phi)$, then*

$$\sup_X \Gamma_t^* = t\Gamma_{\max}$$

for all $t \geq 0$.

In particular, $t \mapsto \Gamma_t^* - t\Gamma_{\max}$ is a decreasing function in $t \geq 0$.

Proof Choose $x \in X$ such that $\Gamma_{\Gamma_{\max}}(x) = 0$. Then

$$\Gamma_t^*(x) = t\Gamma_{\max}$$

by definition. On the other hand, since $\Gamma_\tau \leq 0$ for all $\tau < \Gamma_{\max}$, we have

$$\sup_X \Gamma_t^* \leq t\Gamma_{\max}.$$

lma:LegsendsTCtoR **Lemma 9.2.3** *Given $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we have $\Gamma^* \in \mathcal{R}(X,\theta;\phi)$.*

Proof It follows from Lemma 9.2.1, (9.5) and Proposition 1.2.1 that Γ^* is a subgeodesic (in the sense that for each $0 \le a \le b$, the restriction $(\Gamma^*_t)_{t\in(a,b)}$ is a subgeodesic from Γ^*_a to Γ^*_b).

First observe that as $t \to 0+$, we have

$$\Gamma^*_t \xrightarrow{L^1} \phi. \tag{9.6}$$ {eq:GammatophiL1temp}

To see this, first observe that by (9.5), for any fixed $t > 0$ and any $x \in X$ with $\phi(x) \neq -\infty$, we have

$$\Gamma^*_t(x) \le t\Gamma_{\max} + \phi(x).$$

By Proposition 1.2.5, the same holds everywhere. Therefore, any L^1-cluster point ψ of Γ^*_t as $t \to 0$ satisfies $\psi \le \phi$. On the other hand, for any fixed $\tau < \Gamma_{\max}$, by (9.5), we have

$$\Gamma^*_t \ge \Gamma_\tau + t\tau$$

for any $t > 0$. So $\psi \ge \Gamma_\tau$ almost everywhere and hence everywhere by Proposition 1.2.5. It follows that $\psi \ge \phi$. Therefore, $\psi = \phi$. On the other hand, from the above estimates and Proposition 1.5.1 that $(\Gamma^*_t)_{t\in(0,1)}$ is a relative compact subset in $\mathrm{PSH}(X,\theta)$ with respect to the L^1-topology. We therefore conclude (9.6).

Assume that Γ^* is not a geodesic ray. Then we can find $0 \le a < b$ such that $(\Gamma^*_t)_{t\in(a,b)}$ differs from the geodesic $(\eta_t)_{t\in(a,b)}$ from Γ^*_a to Γ^*_b. We consider the subgeodesic $(\ell_t)_{t>0}$ given by $\ell_t = \eta_t$ for $t \in (a,b)$ and $\ell_t = \Gamma^*_t$ otherwise. Consider the Legendre transform

$$\Gamma'_\tau = \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}.$$

Then $\Gamma'_\tau \ge \Gamma_\tau$ and $\Gamma'_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$ by Proposition 1.2.6 for all $\tau \in \mathbb{R}$.

We claim that

$$\Gamma'_\tau \le \Gamma_\tau + (b-a)(\Gamma_{\max} - \tau), \quad \tau \in \mathbb{R}.$$

Observe that $\Gamma'_\tau \equiv -\infty$ when $\tau > \Gamma_{\max}$ by Lemma 9.2.2. So it suffices to consider $\tau \le \Gamma_{\max}$. In this case, we compute

$$\inf_{t\in[a,b]}(\ell_t - t\tau) \le \Gamma^*_b - b\tau \le (b-a)(\Gamma_{\max} - \tau) \inf_{t\in[a,b]}(\Gamma^*_t - t\tau),$$

where we applied Lemma 9.2.2. In particular, for any $\tau < \Gamma_{\max}$, we have

$$\Gamma'_\tau \le \Gamma_\tau.$$

On the other hand, by definition of Γ'_τ, we clearly have $\Gamma'_\tau \le 0$ for all $\tau < \Gamma_{\max}$. It follows from the fact that Γ_τ is a model potential that $\Gamma_\tau = \Gamma'_\tau$ for all $\tau < \Gamma_{\max}$. Therefore, by Theorem A.2.1, we have $\Gamma^*_t = \ell'_t$ for all $t > 0$, which is a contradiction. □

thm:Legenbij **Theorem 9.2.1** *The Legendre transform in Definition 9.2.1 is a bijection*

$$\mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathcal{R}(X,\theta;\phi).$$

Moreover, this bijection restricts to the following bijections:

$$\mathrm{TC}^1(X,\theta;\phi) \xrightarrow{\sim} \mathcal{R}^1(X,\theta;\phi), \quad \mathrm{TC}^\infty(X,\theta;\phi) \xrightarrow{\sim} \mathcal{R}^\infty(X,\theta;\phi).$$

For any $\Gamma \in \mathrm{TC}^1(X,\theta;\phi)$*, we have*

$$\mathbf{E}^\phi(\Gamma) = \mathbf{E}^\phi(\Gamma^*). \tag{9.7}$$ {eq:RWNenergy}

Proof It follows from Lemma 9.2.3 that the forward map is well-defined.

The inverse map is of course also given by the Legendre transform: given $\ell \in \mathcal{R}(X,\theta;\phi)$, its Legendre transform is given by

$$\ell^*_\tau \coloneqq \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}. \tag{9.8}$$ {eq:invLeg}

By Proposition 4.3.4, there is a constant $C > 0$ such that $\ell_t \leq Ct$.

Note that it follows from Proposition 1.2.6 that $\ell^*_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$ for all $\tau \in \mathbb{R}$.

We need to argue for any $\tau \in \mathbb{R}$ such that $\ell^*_\tau \not\equiv -\infty$, we have $P_\theta[\ell^*_\tau] = \ell^*_\tau$. Fix such τ and some $C > 0$. It suffices to show that

$$(\ell^*_\tau + C) \wedge \phi \leq \ell^*_\tau. \tag{9.9}$$ {eq:ellstarleqetemp1}

For this purpose, let us consider the following geodesics: for any $M > 0$ and $t \in [0,1]$, let

$$\ell^{1,M}_t = \ell_{tM} - tM\tau, \quad \ell^{2,M}_t = (\ell^*_\tau + C) \wedge \phi - Ct.$$

It is clear that at $t = 0, 1$, we have $\ell^{2,M}_t \leq \ell^{1,M}_t$. Hence, the same holds for all $t \in [0,1]$. In particular, for any fixed $s \in [0,1]$, we have

$$(\ell^*_\tau + C) \wedge \phi - Cs \leq \ell_{sM} - sM.$$

Take infimum with respect to $M \geq 1$ and then the supremum with respect to s, we conclude (9.9).

The two operations are inverse to each other thanks to Theorem A.2.1.

Next we consider the bounded situation. Suppose that $\Gamma \in \mathrm{TC}^\infty(X,\theta;\phi)$. Take $\tau_0 \in \mathbb{R}$ so that $\Gamma_\tau = \phi$ for all $\tau \leq \tau_0$. It follows from that

$$\Gamma^*_t \geq \phi + t\tau_0$$

for all $t > 0$. Therefore, $\Gamma^*_t \sim \phi$ for all $t > 0$ and hence $\Gamma^* \in \mathcal{R}^\infty(X,\theta;\phi)$.

Conversely, suppose that $\ell \in \mathcal{R}^\infty(X,\theta;\phi)$. Thanks to Proposition 4.3.3, there is a constant $C > 0$ such that

$$\ell_t \geq \phi - Ct.$$

Therefore, according to (9.8), we have

$$\ell^*_\tau \geq \inf_{t>0} \phi - (C+\tau)t = \phi$$

if $\tau \leq -C$. Therefore, $\ell^*_\tau = \phi$ for all $\tau \leq -C$.

Finally, it remains to handle (9.7). Take $\Gamma \in \mathrm{TC}^\infty(X,\theta;\phi)$. We may assume that $\Gamma_{\max} = 0$ after a translation.

For $N \in \mathbb{Z}_{>0}$, $M \in \mathbb{Z}$, we introduce the following:

$$\Gamma^{*,N,M}_t := \max_{\substack{k\in\mathbb{Z}\\ k\leq M}} \left(\Gamma_{k/2^N} + tk/2^N\right) \in \mathcal{E}^\infty(X,\theta;\phi), \quad t>0.$$

Moreover, we now argue that

$$\frac{t}{2^N}\int_X \theta^n_{\Gamma_{(M+1)/2^N}} \leq E^\phi_\theta(\Gamma^{*,N,M+1}_t) - E^\phi_\theta(\Gamma^{*,N,M}_t) \leq \frac{t}{2^N}\int_X \theta^n_{\Gamma_{M/2^N}}. \tag{9.10}$$

{eq: diff_eq_I}

Indeed, for elementary reasons:

$$\begin{aligned}\int_X \left(\Gamma^{*,N,M+1}_t - \Gamma^{*,N,M}_t\right)\theta^n_{\Gamma^{*,N,M+1}_t} \leq & E^\phi_\theta(\Gamma^{*,N,M+1}_t) - E^\phi_\theta(\Gamma^{*,N,M}_t) \\ \leq & \int_X \left(\Gamma^{*,N,M+1}_t - \Gamma^{*,N,M}_t\right)\theta^n_{\Gamma^{*,N,M}_t}.\end{aligned} \tag{9.11}$$

{eq: first_I_ineq}

Clearly $\Gamma^{*,N,M+1}_t \geq \Gamma^{*,N,M}_t$, and using τ-concavity, we notice that

$$U_t := \left\{\Gamma^{*,N,M+1}_t - \Gamma^{*,N,M}_t > 0\right\} = \left\{\Gamma_{(M+1)/2^N} + 2^{-N}t - \Gamma_{M/2^N} > 0\right\}.$$

Moreover, on U_t we have

$$\Gamma^{*,N,M+1}_t = \Gamma_{(M+1)/2^N} + t(M+1)/2^N, \quad \Gamma^{*,N,M}_t = \Gamma_{M/2^N} + tM/2^N.$$

We also note that U_t is an open set in the plurifine topology, implying that

$$\begin{aligned}\theta^n_{\Gamma_{(M+1)/2^N}}|_{U_t} =& \theta^n_{\Gamma^{*,N,M+1}_t}|_{U_t},\\ \theta^n_{\Gamma_{M/2^N}}|_{U_t} =& \theta^n_{\Gamma^{*,N,M}_t}|_{U_t}.\end{aligned}$$

Recall that $\theta^n_{\Gamma_{M/2^N}}$ and $\theta^n_{\Gamma_{(M+1)/2^N}}$ are supported on the sets $\{\Gamma_{M/2^N} = 0\}$ and $\{\Gamma_{(M+1)/2^N} = 0\}$ respectively, see Theorem 3.1.2. Since $\{\Gamma_{(M+1)/2^N} = 0\} \subseteq U_t$ and $\{\Gamma_{(M+1)/2^N} = 0\} \subseteq \{\Gamma_{M/2^N} = 0\}$, applying the above to (9.11), we arrive at (9.10).

Fixing N, let $M = \lfloor 2^N \Gamma_{\min} \rfloor$. Then repeated application of (9.10) yields

$$\sum_{M+1\leq j\leq 0} \frac{t}{2^N}\int_X \theta^n_{\Gamma_{j/2^N}} \leq E^\phi_\theta(\Gamma^{*,N,0}_t) - E^\phi_\theta(E^{*,N,M}_t) \leq \sum_{M\leq j\leq -1} \frac{t}{2^N}\int_X \theta^n_{\Gamma_{j/2^N}}.$$

Since $M \leq 2^N\Gamma_{\min}$, we have that

$$\Gamma_t^{*,N,M} = \Gamma_{M/2^N} + tM/2^N = \phi + tM/2^N,$$

we can continue to write

$$\sum_{j=M+1}^{0} \frac{t}{2^N} \left(\int_X \theta^n_{\Gamma_{j/2^N}} - \int_X \theta^n_\phi \right) \leq E^\theta_\phi(\Gamma_t^{*,N,0}) \leq \sum_{j=M}^{-1} \frac{t}{2^N} \left(\int_X \theta^n_{\Gamma_{j/2^N}} - \int_X \theta^n_\phi \right).$$

We now notice that we have Riemann sums on both the left and right of the above inequality. Using Proposition 9.1.1, it is possible to let $N \to \infty$ and obtain

$$E^\theta_\phi(\Gamma^*_t) = t\mathbf{E}^\phi(\Gamma)$$

So (9.7) follows as desired. Note that we have furthermore shown that $t \mapsto E^\theta_\phi(\Gamma^*_t)$ is linear.

Finally, let us come back to the general case. Let $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. Again, we may assume that $\Gamma_{\max} = 0$. For each $\epsilon > 0$, we introduce $\Gamma^\epsilon \in \mathrm{TC}^\infty(X, \theta; \phi)$ as follows:

(1) we let $\Gamma^\epsilon_{\max} = 0$;
(2) for each $\tau < 0$, we set

$$\Gamma^\epsilon_\tau = P_\theta \left[((1 + \epsilon\tau) \vee 0)\, \Gamma_\tau + (1 - (1 + \epsilon\tau) \vee 0))\, \phi \right].$$

It follows from Corollary 3.1.2 that for each $\tau < 0$, the sequence Γ^ϵ_τ is a decreasing sequence with limit Γ_τ as $\epsilon \searrow 0$. Therefore, by Proposition 3.1.8, we have

$$\lim_{\epsilon \to 0+} \int_X \left(\theta + \mathrm{dd^c}\Gamma^\epsilon_\tau\right)^n = \int_X (\theta + \mathrm{dd^c}\Gamma_\tau)^n$$

for all $\tau < 0$. Hence, by the monotone convergence theorem, we find

$$\mathbf{E}^\phi(\Gamma) = \lim_{\epsilon \to 0+} \mathbf{E}^\phi(\Gamma^\epsilon) = \lim_{\epsilon \to 0+} \mathbf{E}^\phi(\Gamma^{\epsilon,*}). \tag{9.12}$$

{eq:EphiGammatemp1}

Furthermore, according to Proposition A.2.2, we have

$$\Gamma^*_t = \inf_{\epsilon > 0} \Gamma^{\epsilon,*}_t$$

for all $t > 0$.

Now suppose that $\Gamma \in \mathrm{TC}^1(X, \theta; \phi)$. Then it follows from Theorem 4.3.1 that for each $t > 0$,

$$E^\phi_\theta(\Gamma^*_t) = \lim_{\epsilon \to 0+} E^\phi_\theta(\Gamma^{\epsilon,*}_t) = t\mathbf{E}^\phi(\Gamma).$$

Hence, $\Gamma^* \in \mathcal{E}^1(X, \theta; \phi)$.

Conversely, suppose that $\Gamma^* \in \mathcal{E}^1(X, \theta; \phi)$. Then (9.12) implies that $\Gamma \in \mathrm{TC}^1(X, \theta; \phi)$. □

As an immediate consequence of the proof, we have

Corollary 9.2.1 *Let $\ell \in \mathcal{R}^1(X, \theta; \phi)$, then $[0, \infty) \ni t \mapsto E^\phi_\theta(\ell_t)$ is linear.*

cor:reltestcursuplinear **Corollary 9.2.2** *Let $\ell \in \mathcal{R}(X,\theta;\phi)$. Then $\sup_X \ell_t = \ell^*_{\max} t$.*

Proof This follows from Lemma 9.2.2 and Theorem 9.2.1. □

9.3 $\mathcal{I}$-model test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Definition 9.3.1 A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is *$\mathcal{I}$-model* if for any $\tau < \Gamma_{\max}$, the potential Γ_τ is $\mathcal{I}$-model.

The subset of $\mathcal{I}$-model test curves in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathrm{PSH}^{\mathrm{NA}}(X,\theta;\phi)$.

The set of $\mathcal{I}$-model test curves in $\mathrm{PSH}(X,\theta)$ for any model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ is denoted by $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

prop:GammaminfImodel **Proposition 9.3.1** *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Then $\Gamma_{-\infty}$ is an $\mathcal{I}$-model potential.*

Proof This follows from Proposition 3.2.12. □

p:Imodeltestcurveindeptheta **Proposition 9.3.2** *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Then there is a canonical bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}.$$

This bijection satisfies the obvious cocycle condition.

Proof This is an immediate consequence of Proposition 9.1.2 and Example 7.1.2. □

prop:ETCIbimero **Proposition 9.3.3** *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold. Then the pointwise pull-back induces a bijection*

$$\pi^*\colon \mathrm{PSH}^{\mathrm{NA}}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This is an immediate consequence of Proposition 9.1.3 and Proposition 3.2.5. □

def:TCIenvelope **Definition 9.3.2** Given $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we define its *$\mathcal{I}$-envelope* $P_\theta[\Gamma]_{\mathcal{I}}$ as the map $(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta)$ given by

$$\tau \mapsto P_\theta\left[\Gamma_\tau\right]_{\mathcal{I}}.$$

prop:transitionPI **Proposition 9.3.4** *Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, then*

$$P_\theta[\Gamma]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta;P_\theta[\phi]_{\mathcal{I}}).$$

More generally, for any closed real smooth positive $(1,1)$-form ω on X, we have

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega;P_{\theta+\omega}[\phi]_{\mathcal{I}}).$$

Proof The only non-trivial point is to show that

$$\sup_{\tau<\Gamma_{\max}}{}^{*} P_\theta[\Gamma_\tau]_{\mathcal{I}} = P_\theta[\phi]_{\mathcal{I}}, \quad \sup_{\tau<\Gamma_{\max}}{}^{*} P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}} = P_{\theta+\omega}[\phi]_{\mathcal{I}}.$$

This follows from Proposition 3.2.12. □

9.4 Operations on test curves

sec:operationtc

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta', \theta''$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes.

def:potestcurve

Definition 9.4.1 Given $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$, we say $\Gamma \leq \Gamma'$ if for all $\Gamma_{\max} \leq \Gamma'_{\max}$ and for all $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \leq \Gamma'_\tau. \tag{9.13}$$

{eq:GammatauGammap}

Observe that (9.13) actually holds for all $\tau \in \mathbb{R}$. It is easy to verify that for all $\leq$ defines a partial order on $\mathrm{TC}(X,\theta)_{>0}$.

lma:testcurord1

Lemma 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a closed real smooth positive $(1,1)$-form on X. Then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma] = P_{\theta+\omega}[\Gamma']$.

Proof It suffices to observe that we could rewrite (9.13) as

$$\Gamma_\tau \leq_P \Gamma'_\tau,$$

since both potentials are model. □

def:sumtestcur

Definition 9.4.2 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then we define $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$ as follows:

(1) we set
$$(\Gamma+\Gamma')_{\max} \coloneqq \Gamma_{\max} + \Gamma'_{\max};$$
(2) for any $\tau < (\Gamma+\Gamma')_{\max}$, we define

$$(\Gamma+\Gamma')_\tau \coloneqq P_\theta\left[\sup_{t\in\mathbb{R}}\left(\Gamma_t + \Gamma'_{\tau-t}\right)\right]. \tag{9.14}$$

{eq:GammaGammapsum}

lma:testcurvplus

Lemma 9.4.2 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then for any $\tau < (\Gamma + \Gamma')_{\max}$, we have*

$$\sup_{t\in\mathbb{R}}\left(\Gamma_t + \Gamma'_{\tau-t}\right) \in \mathrm{PSH}(X,\theta).$$

This potential is $\mathcal{I}$-good if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

In particular, (9.14) in Definition 9.4.2 makes sense.

Proof Let
$$\eta_\tau = \sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-t}\right) = \sup_{t<\Gamma_{\max},\tau-t<\Gamma'_{\max}} \left(\Gamma_t + \Gamma'_{\tau-t}\right)$$
for all $\tau \in \mathbb{R}$. Set
$$Z = \left\{x \in X : \Gamma_{-\infty}(x) = -\infty \text{ or } \Gamma'_{-\infty}(x) = -\infty\right\}.$$
It follows from Proposition A.2.3 that for any $x \in X \setminus Z$, we have
$$\eta^*_t(x) = \Gamma^*_t(x) + \Gamma'^*_t(x)$$
for all $t > 0$. The same trivially holds when $x \in Z$, so the equation holds everywhere. In particular, by Theorem A.2.1 and Proposition 1.2.6, we have
$$\eta_\tau = (\Gamma^* + \Gamma'^*)^*_\tau \in \mathrm{PSH}(X, \theta + \theta') \cup \{-\infty\}.$$

Next, assume that Γ and Γ' are $\mathcal{I}$-model. We need to argue that so is $\Gamma + \Gamma'$. Fix $\tau < \Gamma_{\max} + \Gamma'_{\max}$. Then for each $t \in \mathbb{R}$ such that $t < \Gamma_{\max}$ and $\tau - t < \Gamma'_{\max}$, we know that $\Gamma_t \in \mathrm{PSH}(X,\theta)_{>0}$ and $\Gamma'_{\tau-t} \in \mathrm{PSH}(X,\theta')_{>0}$ by Lemma 9.1.1. It follows from Example 7.1.2 that Γ_t and $\Gamma'_{\tau-t}$ are both $\mathcal{I}$-good, hence so is $\Gamma_t + \Gamma'_{\tau-t} \in \mathrm{PSH}(X,\theta+\theta')_{>0}$ by Proposition 7.2.1. Therefore, η_τ is $\mathcal{I}$-good by Proposition 7.2.2. Therefore, $\Gamma + \Gamma'$ is $\mathcal{I}$-model. □

prop:testcurvesumproperty **Proposition 9.4.1** *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$. Moreover,*
$$(\Gamma + \Gamma')_{-\infty} = P_{\theta+\theta'}[\Gamma_{-\infty} + \Gamma'_{-\infty}]. \tag{9.15}$$
{eq:sumGammaGammap}

When $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$, we have $\Gamma + \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$.

The operation $+$ is commutative and associative.

Proof It follows immediately from Lemma 9.4.2 that $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$ if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

We argue (9.15). By definition, for any small enough τ, we have
$$(\Gamma + \Gamma')_{-\infty} \geq (\Gamma + \Gamma')_{2\tau} \geq_P \Gamma_\tau + \Gamma'_\tau.$$
Letting $\tau \to -\infty$ and applying Proposition 6.2.4 and Theorem 6.2.2, we find that
$$(\Gamma + \Gamma')_{-\infty} \geq_P \Gamma_{-\infty} + \Gamma'_{-\infty}.$$
On the other hand, for each small enough τ, we have
$$(\Gamma + \Gamma')_\tau \sim_P \sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-t}\right) \leq_P \Gamma_{-\infty} + \Gamma'_{-\infty}$$
by Proposition 6.1.5 and Proposition 6.2.4. We apply Proposition 6.2.4 again, we conclude that

$$(\Gamma+\Gamma')_{-\infty} \leq_P \Gamma_{-\infty}+\Gamma'_{-\infty}.$$

So (9.15) follows.

Finally, let us show that $+$ is commutative and associative. Commutativity is obvious. Let $\Gamma'' \in \mathrm{TC}(X,\theta'')_{>0}$. Then we want to show that

$$(\Gamma+\Gamma')+\Gamma'' = \Gamma+(\Gamma'+\Gamma'').$$

First observe that

$$((\Gamma+\Gamma')+\Gamma'')_{\max} = (\Gamma+(\Gamma'+\Gamma''))_{\max}\,.$$

Fix τ less than this common value. We observe that

$$\begin{aligned}
&((\Gamma+\Gamma')+\Gamma'')_\tau\\
=&P_\theta\left[\sup_{t_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{t_1}+\Gamma''_{\tau-t_1}\right)\right]\\
\sim_P&\sup_{t_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{t_1}+\Gamma''_{\tau-t_1}\right)\\
\sim_P&\sup_{t_1,t_2\in\mathbb{R}}\left(\Gamma_{t_2}+\Gamma'_{t_1-t_2}+\Gamma''_{\tau-t_1}\right),
\end{aligned}$$

where in the last line, we applied Proposition 6.2.4 and Proposition 6.1.5. Similarly, for $(\Gamma+(\Gamma'+\Gamma''))_\tau$, we get the same expression. The associativity follows. □

lma:testcursumcomp **Lemma 9.4.3** *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then for any closed smooth positive $(1,1)$-forms ω and ω' on X, we have*

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma'] = P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma].$$

Proof Observe that

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma']_{\max} = \left(P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma]\right)_{\max} = \Gamma_{\max}+\Gamma'_{\max}.$$

Take $\tau\in\mathbb{R}$ less than this common value, we need to verify that

$$(\Gamma+\Gamma')_\tau \sim_P \left(P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma]\right)_\tau\,.$$

By definition, this means that

$$\sup_{t\in\mathbb{R}}\left(\Gamma_t+\Gamma'_{\tau-t}\right) \sim_P \sup_{t\in\mathbb{R}}\left(P_{\theta+\omega}[\Gamma_t]+P_{\theta'+\omega'}[\Gamma'_{\tau-t}]\right).$$

This is a consequence of Proposition 6.1.5 and Proposition 6.1.6. □

def:testcurvplusC **Definition 9.4.3** Let $\Gamma\in\mathrm{TC}(X,\theta)_{>0}$ and $C\in\mathbb{R}$, we define $\Gamma+C\in\mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) we set
$$(\Gamma+C)_{\max} := \Gamma_{\max}+C,$$

and

(2) for any $\tau < (\Gamma + C)_{\max}$, we set

$$\Gamma_\tau \coloneqq \Gamma_{\tau - C}.$$

It is obvious that if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$, then so is $\Gamma + C$.

prop:testcurveplusC

Proposition 9.4.2 *Let $\Gamma \in \mathrm{TC}(X, \theta)_{>0}$, $\Gamma \in \mathrm{TC}(X, \theta')_{>0}$ and $C, C' \in \mathbb{R}$, then*

(1) $(\Gamma + \Gamma') + C = \Gamma + (\Gamma' + C) = (\Gamma + C) + \Gamma'$;
(2) $\Gamma + (C + C') = (\Gamma + C) + C'$.

Proof (1) We first observe that

$$((\Gamma + \Gamma') + C)_{\max} = (\Gamma + (\Gamma' + C))_{\max} = ((\Gamma + C) + \Gamma')_{\max} = \Gamma_{\max} + \Gamma'_{\max} + C.$$

Take any $\tau \in \mathbb{R}$ less than this common value. We compute

$$\begin{aligned}
((\Gamma + \Gamma') + C)_\tau =&(\Gamma + \Gamma')_{\tau - C} = P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-C-t}\right)\right],\\
(\Gamma + (\Gamma' + C))_\tau =&P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + (\Gamma' + C)_{\tau-t}\right)\right] = P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-C-t}\right)\right],\\
((\Gamma + C) + \Gamma')_\tau =&P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left((\Gamma + C)_{C+t} + \Gamma'_{\tau-C-t}\right)\right]\\
=&P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-C-t}\right)\right].
\end{aligned}$$

(2) Observe that

$$(\Gamma + (C + C'))_{\max} = ((\Gamma + C) + C')_{\max} = \Gamma_{\max} + C + C'.$$

For any $\tau \in \mathbb{R}$ less than this value, we have

$$(\Gamma + (C + C'))_\tau = \Gamma_{\tau - C - C'} = ((\Gamma + C) + C')_\tau .$$

def:testcurlor

Definition 9.4.4 Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$. We define $\Gamma \vee \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$ as follows:

(1) We set

$$(\Gamma \vee \Gamma')_{\max} \coloneqq \Gamma_{\max} \vee \Gamma'_{\max};$$

(2) for any $\tau < (\Gamma \vee \Gamma')_{\max}$, we define

$$(\Gamma \vee \Gamma')_\tau \coloneqq P_\theta\left[\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)\right]. \tag{9.16}$$

{eq:testcurlordef}

Recall that the upper convex hull CE is defined in Definition A.1.4. Trivially, we have $\Gamma \vee \Gamma' \geq \Gamma$ and $\Gamma \vee \Gamma' \geq \Gamma'$.

lma:testcurlor

Lemma 9.4.4 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$. Then for any $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$, we have*

$$\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau \in \mathrm{PSH}(X,\theta).$$

This potential is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

In particular, (9.16) *in Definition 9.4.4 makes sense.*

Proof To simply the notations, we write

$$\psi_\tau = \mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau$$

for all $\tau \in \mathbb{R}$. Thanks to Proposition A.2.2, we have

$$\psi^*_t(x) = \Gamma^*_t(x) \vee \Gamma'^*_t(x) \tag{9.17}$$

{eq:psistartemp1}

for all $t > 0$ as long as $\Gamma_\tau(x) \neq -\infty$ and $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. Otherwise, assume that $x \in X$ is such that $\Gamma_\tau = -\infty$ for all $\tau \in \mathbb{R}$, then by definition, $\psi_\tau(x) = \Gamma'_\tau(x)$ for all $\tau \in \mathbb{R}$. Therefore, $\Gamma^*_t(x) = -\infty$ for all $t > 0$ and hence (9.17) continues to hold. Therefore, we have shown that

$$\psi^*_t = \Gamma^*_t \vee \Gamma'^*_t \in \mathrm{PSH}(X,\theta).$$

It follows from Proposition 4.1.2 that $(\psi^*_t)_{t\in[a,b]}$ is a subgeodesic for any $0 < a < b$.

Next we observe that $\psi_\bullet$ is closed by definition. So it follows from Proposition A.2.2 and Proposition 1.2.6 that

$$\psi_\tau = (\psi^*_\bullet)^*_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}.$$

Due to Proposition 9.1.4 and Proposition A.1.2, there is a pluripolar set $Z \subseteq X$ such that for $x \in X \setminus Z$, we have

$$\psi_\tau(x) = \sup\left\{\lambda\Gamma_\rho(x) + (1-\lambda)\Gamma'_{\rho'}(x) : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\}$$

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$. It follows from Proposition 1.2.5 that

$$\psi_\tau = \sup{}^*\left\{\lambda\Gamma_\rho + (1-\lambda)\Gamma'_{\rho'} : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\} \tag{9.18}$$

{eq:psitausupslineartemp}

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$.

It follows from (9.18) that ψ_τ is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, thanks to Proposition 7.2.1 and Proposition 7.2.2. □

cor:testcurvlorprop

Corollary 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$. Then $\Gamma \vee \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ and*

$$(\Gamma \vee \Gamma')_{-\infty} = P_\theta\left[\Gamma_{-\infty} \vee \Gamma'_{-\infty}\right]. \tag{9.19}$$

{eq:GammalorGammapminfty}

If $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

For each $\Gamma'' \in \mathrm{TC}(X,\theta)_{>0}$ and each $\Gamma'' \geq \Gamma$ and $\Gamma'' \geq \Gamma'$, we have $\Gamma'' \geq \Gamma \vee \Gamma'$.

Moreover, the operation $\vee$ *is associative and commutative.*

Proof It follows immediately from Lemma 9.4.4 that $\Gamma\vee\Gamma'\in \mathrm{TC}(X,\theta)_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ if $\Gamma,\Gamma'\in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

The argument of (9.19) is very similar to that of (9.15), which we leave to the readers.

Take Γ'' as in the statement of the proposition. First observe that

$$\Gamma''_{\max}\geq \Gamma_{\max}\vee\Gamma'_{\max}=(\Gamma\vee\Gamma')_{\max}.$$

Take $\tau<(\Gamma\vee\Gamma')_{\max}$, we argue that

$$\Gamma''_\tau\geq(\Gamma\vee\Gamma')_\tau.$$

By the concavity of Γ'', this is equivalent to

$$\Gamma''_\tau\geq\Gamma_\tau\vee\Gamma'_\tau.$$

Therefore,

$$\Gamma''\geq\Gamma\vee\Gamma'.$$

The commutativity and associativity of $\vee$ are trivial. □

lma:testcurlorPthetapomega **Lemma 9.4.5** *Let* $\Gamma,\Gamma'\in\mathrm{TC}(X,\theta)_{>0}$ *and* ω *be a closed smooth positive* $(1,1)$*-form on* X*. Then*

$$P_{\theta+\omega}[\Gamma\vee\Gamma']=P_{\theta+\omega}[\Gamma]\vee P_{\theta+\omega}[\Gamma'].$$

Proof We first observe that

$$(P_{\theta+\omega}[\Gamma\vee\Gamma'])_{\max}=(P_{\theta+\omega}[\Gamma]\vee P_{\theta+\omega}[\Gamma'])_{\max}=\Gamma_{\max}\vee\Gamma'_{\max}.$$

Let $\tau\in\mathbb{R}$ be less than this common value. We need to show that

$$(\Gamma\vee\Gamma')_\tau\sim_P(P_{\theta+\omega}[\Gamma]\vee P_{\theta+\omega}[\Gamma'])_\tau\,.$$

We need the formula (9.18) proved in the proof of Lemma 9.4.4:

$$(\Gamma\vee\Gamma')_\tau=\sup{}^*\left\{\lambda\Gamma_\rho+(1-\lambda)\Gamma'_{\rho'}:\lambda\in(0,1),\rho,\rho'\in\mathbb{R},\lambda\rho+(1-\lambda)\rho'=\tau\right\}.$$

A similar result holds with $P_{\theta+\omega}[\Gamma]$ and $P_{\theta+\omega}[\Gamma']$ in place of Γ and Γ'. So our assertion is a direct consequence of Proposition 6.1.5 and Proposition 6.1.6. □

def:testcursup **Definition 9.4.5** Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$. Assume that

$$\sup_{i\in I}\Gamma^i_{\max}<\infty. \tag{9.20}$$ {eq:Gammaisupfinite1}

Then we define $\sup^*_{i\in I}\Gamma^i\in\mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) we set

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max};$$

(2) For any $\tau < \sup_{i\in I}\Gamma^i_{\max}$, we let

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{\tau} := \sup_{i\in I}{}^*\Gamma^i_{\tau}.$$

prop:supsincnetteestcur

Proposition 9.4.3 *Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Then $\sup^*_{i\in I}\Gamma^i$ as defined in Definition 9.4.5 lies in $\sup^*_{i\in I}\Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$. Moreover, if $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ for all $i \in I$, then $\sup^*_{i\in I}\Gamma^i$ lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ as well.*

Moreover, we have

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{-\infty} = \sup_{i\in I}{}^*\Gamma^i_{-\infty}. \tag{9.21}$$

{eq:Gammiminf}

Proof The first assertion follows easily from Proposition 3.1.9, while the second follows from Proposition 3.2.12.

It remains to argue (9.21). Without loss of generality, we may assume that I contains a minimal element i_0.

By Proposition 1.2.3, there is a pluripolar set $Z \subseteq X$ such that for any $x \in X \setminus Z$,

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{-\infty}(x) = \sup_{\tau<\Gamma^{i_0}_{\max}}\left(\sup_{i\in I}{}^*\Gamma^i_{\tau}\right)(x) = \sup_{\tau<\Gamma^{i_0}_{\max},\, i\in I}\Gamma^i_{\tau}(x) = \sup_{i\in I}\Gamma^i_{-\infty}(x).$$

So they are equal everywhere by Proposition 1.2.5. $\square$

lma:suptestcurvcompatible

Lemma 9.4.6 *Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right] = \sup_{i\in I}{}^* P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof Observe that

$$\left(P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right]\right)_{\max} = \left(\sup_{i\in I}{}^* P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max}.$$

Fix $\tau \in \mathbb{R}$ less than this common value.

It suffices to show that

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{\tau} = \left(\sup_{i\in I}{}^* P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\tau}.$$

This is an immediate consequence of Proposition 6.1.6. $\square$

def:testcurvsupsgeneral

Definition 9.4.6 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Then we define

$$\sup_{i\in I}{}^{*}\Gamma^i := \sup_{J\in \mathrm{Fin}(I)}{}^{*}\left(\bigvee_{j\in J}\Gamma^j\right). \tag{9.22}$$

{eq:generalsupstestcurv}

Observe that by Definition 9.4.4, we have

$$\sup_{J\in\mathrm{Fin}(I)}\left(\bigvee_{j\in J}\Gamma^j\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max} < \infty.$$

So (9.22) makes sense. In particular,

$$\left(\sup_{i\in I}\Gamma^i\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max}. \tag{9.23}$$

{eq:testcursupmax}

It is clear that Definition 9.4.6 extends both Definition 9.4.5 and Definition 9.4.4.

Proposition 9.4.4 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Then $\sup^*_{i\in I}\Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$. Moreover, if $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then so is $\sup^*_{i\in I}\Gamma^i$.*

Finally, we have

$$\left(\sup_{i\in I}{}^{*}\Gamma^i\right)_{-\infty} = P_\theta[\sup_{i\in I}{}^{*}\Gamma^i_{-\infty}]. \tag{9.24}$$

{eq:supsminfty}

Proof The first assertion and the second follow from Proposition 9.4.3 and Corollary 9.4.1.

It remains to argue (9.24). For this purpose, it suffices to show that

$$\left(\sup_{i\in I}{}^{*}\Gamma^i\right)_{-\infty} \sim_P \sup_{i\in I}{}^{*}\Gamma^i_{-\infty}.$$

For any $J\in\mathrm{Fin}(I)$, it follows from Corollary 9.4.1 and Proposition 6.1.6 that

$$\left(\bigvee_{j\in J}\Gamma^j\right)_{-\infty} \sim_P \bigvee_{j\in J}\Gamma^j_{-\infty}.$$

From this, applying Proposition 6.1.6 and Proposition 9.4.3, we conclude our assertion. □

lma:testcursupcompatible

Lemma 9.4.7 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^{*}\Gamma^i\right] = \sup_{i\in I}{}^{*}P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof This is a direct consequence of Lemma 9.4.6 and Lemma 9.4.5. □

prop:testcurvChoquet

Proposition 9.4.5 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Then there is a countable subset $I'\subseteq I$ such that*

$$\sup_{i\in I}{}^*\,\Gamma^i=\sup_{i\in I'}{}^*\,\Gamma^i.$$

Proof We may assume that I is infinite.

It follows from Proposition 1.2.2 that we can find a countable subset $I'\subseteq I$ such that for each

$$\tau\in\left(-\infty,\sup_{i\in I}{}^*\,\Gamma^i_{\max}\right)\cap\mathbb{Q},$$

we have

$$\sup_{i\in I}{}^*\,\Gamma^i_\tau=\sup_{i\in I'}{}^*\,\Gamma^i_\tau.$$

Let $\Gamma'=\sup^*_{i\in I'}\Gamma^i$. Then clearly, $\Gamma'\leq\Gamma$. We claim that they are actually equal. For this purpose, it suffices to show that for any $\tau<\sup^*_{i\in I}\Gamma^i_{\max}$, we have

$$\int_X\left(\theta+\mathrm{dd^c}\Gamma'_\tau\right)^n=\int_X\left(\theta+\mathrm{dd^c}\Gamma_\tau\right)^n.$$

Since we know that this holds on a dense subset of τ, this holds everywhere by Theorem 2.3.3. □

prop:supGammiotherprop

Proposition 9.4.6 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Let $C\in\mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i+C)=\sup_{i\in I}{}^*\,\Gamma^i+C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.20). Suppose that $\Gamma^i\leq\Gamma'^i$ for all $i\in I$, then

$$\sup_{i\in I}{}^*\,\Gamma^i\leq\sup_{i\in I}{}^*\,\Gamma'^i.$$

Proof This is immediate by definition. □

def:res

Definition 9.4.7 Let $\Gamma\in\mathrm{TC}(X,\theta)_{>0}$ and $\lambda>0$, we define $\lambda\Gamma\in\mathrm{TC}(X,\lambda\theta)_{>0}$ as follows:

(1) we set

$$(\lambda\Gamma)_{\max}=\lambda\Gamma_{\max};$$

(2) For any $\tau<\lambda\Gamma_{\max}$, we set

$$(\lambda\Gamma)_\tau=\lambda\Gamma_{\lambda^{-1}\tau}.$$

prop:testcurrescaling

Proposition 9.4.7 *Let $\Gamma\in\mathrm{TC}(X,\theta)_{>0}$ and $\lambda>0$, then $\lambda\Gamma$ as defined in Definition 9.4.7 lies in $\mathrm{TC}(X,\lambda\theta)_{>0}$. Moreover, if $\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then $\lambda\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)_{>0}$.*

We have

$$(\lambda\Gamma)_{-\infty}=\lambda\Gamma_{-\infty}.\tag{9.25}$$

prop:resclacompat

Proposition 9.4.8 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, $C \in \mathbb{R}$ *and* $\lambda, \lambda' > 0$, *we have*

$$\begin{aligned}\lambda(\Gamma+\Gamma') =&\lambda\Gamma+\lambda\Gamma',\\ (\lambda\lambda')\Gamma =&\lambda(\lambda'\Gamma),\\ \lambda(\Gamma+C) =&\lambda\Gamma+\lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ *is a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.20), *then*

$$\lambda\left(\sup_{i\in I}{}^* \Gamma^i\right) = \sup_{i\in I}{}^*(\lambda\Gamma^i).$$

lma:testcurvrescompatible

Lemma 9.4.8 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ *and* $\lambda > 0$. *Then for any closed smooth positive* $(1,1)$*-form* ω *on* X, *we have*

$$P_{\lambda(\theta+\omega)}[\lambda\Gamma] = \lambda P_{\theta+\omega}[\Gamma].$$

Proof This is clear by definition. □

Chapter 10
The theory of Okounkov bodies

chap:Okou

10.1 Flags and valuations

10.1.1 The algebraic setting

subsec:flagvalalgebraic

Let X be an irreducible normal projective variety of dimension n.

def:admfl

Definition 10.1.1 An *admissible flag* $(Y_\bullet)$ on X is a flag of subvarieties

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

such that Y_i is irreducible of codimension i and is smooth at the point Y_n.

Given any admissible flag $(Y_\bullet)$, we can define a rank n valuation $\nu_{Y_\bullet} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$. Here we consider $\mathbb{Z}^n$ as a totally ordered Abelian group with the lexicographic order. We sometimes write $\mathbb{Z}^n_{\mathrm{lex}}$ to emphasize this point.

The automorphism group $\mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ of $\mathbb{Z}^n_{\mathrm{lex}}$ is then identified with the subgroup of $\mathrm{GL}(n,\mathbb{Z})$ consisting of matrices of the form $\mathrm{I} + U$, where I is the identity matrix and U is a strictly upper triangular matrix with elements in $\mathbb{Z}$.

We recall the definition: let $s \in \mathbb{C}(X)^\times$. Let $\nu(s)_1 = \mathrm{ord}_{Y_1} s$. After localization around Y_n, we can take a local defining equation t^1 of Y_1, set $s_1 = (s(t^1)^{-\nu_1(s)})|_{Y_1}$. Then $s_1 \in \mathbb{C}(Y_1)^\times$. We can repeat this construction with Y_2 in place of Y_1 to get $\nu(s)_2$ and s_2. Repeating this construction n times, we get

$$\nu_{Y_\bullet}(s) = \nu(s) = (\nu(s)_1, \nu(s)_2, \ldots, \nu(s)_n) \in \mathbb{Z}^n.$$

It is easy to verify that ν is indeed a rank n valuation.

The same construction can be applied to define $\nu_{Y_\bullet}(s)$ when $s \in H^0(X, L)$ or $\nu_{Y_\bullet}(D)$ when D is an effective divisor on X.

rmk:Abhyankar

Remark 10.1.1 Conversely, by a theorem of Abhyankar, any valuation of $\mathbb{C}(X)$ with Noetherian valuation ring of rank n is equivalent to a valuation taking value in $\mathbb{Z}^n$, see [FK18, Chapter 0, Theorem 6.5.2]. As shown in [CFK+17, Theorem 2.9], any such valuation is equivalent[1] to (but not necessarily equal to) a valuation induced by an admissible flag on a modification of X.

10.1.2 The transcendental setting

Let X be a connected compact Kähler manifold of dimension n.

Definition 10.1.2 A *smooth flag* $Y_\bullet$ on X consists of a flag of connected submanifolds of X:

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n,$$

where Y_i has dimension $n-i$.

In this section, we will fix a smooth flag $Y_\bullet$ on X.

def:valcurr **Definition 10.1.3** Let T be a closed positive $(1,1)$-current on X. We define the *valuation* of T along $Y_\bullet$ as

$$\nu_{Y_\bullet}(T) = \left(\nu_{Y_\bullet}(T)_1, \ldots, \nu_{Y_\bullet}(T)_n\right) \in \mathbb{R}^n_{\geq 0}$$

by induction on n. When $n=0$, we define $\nu_{Y_\bullet}(T)$ as the unique point in $\mathbb{R}^0$. When $n>1$, we define

$$\nu_{Y_\bullet}(T)_1(T) = \nu(T, Y_1);$$

Then for $i = 2, \ldots, n$, we define

$$\nu_{Y_\bullet}(T)_i = \nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T - \nu(T,Y_1)[Y_1])\right)_{i-1}.$$

Proposition 10.1.1 *Let T be a closed positive $(1,1)$-current on X. Then $\nu_{Y_\bullet}(T) \in \mathbb{R}^n_{\geq 0}$ defined in Definition 10.1.3 is independent of the choices of the trace operators in the definition. Moreover, $\nu_{Y_\bullet}(T)$ depends only on the $\mathcal{I}$-equivalence class of T.*

Proof We will prove both statements at the same time by induction on $n \geq 0$. The case $n=0$ is trivial.

Let us consider the case $n>0$ and assume that the result is known in dimension $n-1$. We first observe that $\nu_{Y_\bullet}(T)$ is independent of the choice of the trace operator: different choices of $\mathrm{Tr}_{Y_1}(T - \nu(T,Y_1)[Y_1])$ are $\mathcal{I}$-equivalent by Proposition 8.1.2. Therefore, by induction, its valuation is well-defined.

Next, let T' be another closed positive $(1,1)$-current such that $T \sim_{\mathcal{I}} T'$. Using Proposition 3.2.1, we know that $\nu(T,Y_1) = \nu(T',Y_1)$. Therefore,

$$T - \nu(T,Y_1)[Y_1] \sim_{\mathcal{I}} T' - \nu(T',Y_1)[Y_1].$$

It follows by induction that

[1] Two valuations ν, ν' with value in $\mathbb{Z}^n$ are equivalent if one can find a matrix G of the form $\mathrm{I}+N$, where N is strictly upper triangular with integral entries, such that $\nu' = \nu G$.

$$\nu_{Y_1\supseteq\cdots\supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T-\nu(T,Y_1)[Y_1])\right)=\nu_{Y_1\supseteq\cdots\supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T'-\nu(T',Y_1)[Y_1])\right).$$

ex:valuationdivcompatible *Example 10.1.1* When X is projective, we have

$$\nu_{Y_\bullet}([D])=\nu_{Y_\bullet}(D),$$

where the right-hand side is defined in Section 10.1.1.

prop:nuvaluationlinear **Proposition 10.1.2** *Let T, S be closed positive $(1,1)$-currents on X, $\lambda\in\mathbb{R}_{\geq 0}$. Then*

(1) *if $T\preceq_{\mathcal{I}} S$, we have*

$$\nu_{Y_\bullet}(T)\geq_{\mathrm{lex}}\nu_{Y_\bullet}(S); \tag{10.1}$$ {eq:nuTS}

(2) *We have the following additivity property:*

$$\nu_{Y_\bullet}(T+S)=\nu_{Y_\bullet}(T)+\nu_{Y_\bullet}(S),\quad \nu_{Y_\bullet}(\lambda T)=\lambda\nu_{Y_\bullet}(T). \tag{10.2}$$ {eq:nuvaluationlinear}

Proof (1) We make an induction on $n\geq 0$. The case $n=0,1$ is trivial. Assume that $n\geq 2$ and the case $n-1$ is known. Observe that $\nu(T,Y_1)\geq\nu(S,Y_1)$, if the inequality is strict, we are done. So let us assume that $\nu(T,Y_1)=\nu(S,Y_1)$. By Proposition 8.2.1, we find that

$$\mathrm{Tr}_{Y_1}(T-\nu(T,Y_1)[Y_1])\preceq_{\mathcal{I}}\mathrm{Tr}_{Y_1}(S-\nu(T,Y_1)[Y_1]).$$

By the inductive hypothesis, we conclude (10.1).

(2) We make an induction on $n\geq 0$. The cases $n=0,1$ are trivial. Assume that $n\geq 2$ and the case $n-1$ is known. By Proposition 1.4.2, we have

$$\nu(T+S,Y_1)=\nu(T,Y_1)+\nu(S,Y_1),\quad \nu(\lambda T,Y_1)=\lambda\nu(T,Y_1).$$

By Proposition 8.2.1, we have

$$\begin{aligned}\mathrm{Tr}_{Y_1}(T+S-\nu(T+S,Y_1)[Y_1])\sim_P\mathrm{Tr}_{Y_1}(T-\nu(T,Y_1)[Y_1])+\mathrm{Tr}_{Y_1}(S-\nu(S,Y_1)[Y_1]),\\ \mathrm{Tr}_{Y_1}(\lambda T-\nu(\lambda T,Y_1)[Y_1])\sim_P\lambda\,\mathrm{Tr}_{Y_1}(T-\nu(T,Y_1)[Y_1]).\end{aligned}$$

By the inductive hypothesis, we conclude (10.2).

Definition 10.1.4 Let $\pi\colon Z\rightarrow X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. We say that a smooth flag $W_\bullet$ on Z is a *lifting* of $Y_\bullet$ to Z if the restriction of π to $W_i\rightarrow Y_i$ is defined and bimeromorphic for each $i=0,\ldots,n$.

In this case, we define $\mathrm{cor}(Y_\bullet,\pi)\in\mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ inductively as follows:

$$\mathrm{cor}(Y_\bullet,\pi):=\begin{bmatrix}1 & -\nu_{W_1\supseteq\cdots\supseteq W_n}((\pi^*[Y_1]-[W_1])|_{W_1})\\ 0 & \mathrm{cor}(Y_1\supseteq\cdots\supseteq Y_n,\pi|_{W_1}\colon W_1\rightarrow Y_1)\end{bmatrix}. \tag{10.3}$$ {eq:correcur}

We observe that a lifting $W_\bullet$ of $Y_\bullet$ on Z is unique if it exists. For each $i=0,\ldots,n-1$, the component W_{i+1} is necessarily the strict transform of Y_{i+1} with respect to the bimeromorphic morphism $W_i \to Y_i$. We shall also say that $(W_\bullet, \operatorname{cor}(Y_\bullet, \pi))$ is *the lifting* of $Y_\bullet$ to Z.

prop:cormult

Proposition 10.1.3 *Let $\pi\colon Z \to X$, $p\colon Z' \to Z$ be proper bimeromorphic morphisms with Z and Z' being Kähler manifolds. Assume that $Y_\bullet$ admits a lifting $W_\bullet$ (resp. $W'_\bullet$) to Z (resp. Z'). Then*

$$\operatorname{cor}(Y_\bullet, \pi \circ p) = \operatorname{cor}(Y_\bullet, \pi) \operatorname{cor}(W_\bullet, p). \tag{10.4}$$

{eq:cormul}

Proof We let $\pi' = \pi \circ p$:

$$\begin{tikzcd} Z' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd} .$$

We make induction on $n \geq 1$. The case $n = 1$ is trivial. Assume that $n \geq 2$ and the case $n-1$ has been solved. Then by (10.3), the desired formula (10.4) can be reformulated as

$$\begin{bmatrix} 1 & -\nu_{W'_1 \supseteq \cdots \supseteq W'_n}((\pi'^*[Y_1] - [W'_1])|_{W'_1}) \\ 0 & \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi'|_{W'_1} : W'_1 \to Y_1) \end{bmatrix} =$$
$$\begin{bmatrix} 1 & -\nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) \\ 0 & \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi|_{W_1} : W_1 \to Y_1) \end{bmatrix} \cdot$$
$$\begin{bmatrix} 1 & -\nu_{W'_1 \supseteq \cdots \supseteq W'_n}((p^*[W_1] - [W'_1])|_{W'_1}) \\ 0 & \operatorname{cor}(W_1 \supseteq \cdots \supseteq W_n, p|_{W'_1} : W'_1 \to W_1) \end{bmatrix}$$

By the inductive hypothesis, this is equivalent to

$$\begin{aligned} \nu_{W'_1 \supseteq \cdots \supseteq W'_n}((\pi'^*[Y_1] - [W'_1])|_{W'_1}) &= \nu_{W'_1 \supseteq \cdots \supseteq W'_n}((p^*[W_1] - [W'_1])|_{W'_1}) + \\ \nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) &\operatorname{cor}(W_1 \supseteq \cdots \supseteq W_n, p|_{W'_1} : W'_1 \to W_1), \end{aligned}$$

which can be further rewritten as

$$\begin{aligned} \nu_{W'_1 \supseteq \cdots \supseteq W'_n}((\pi'^*[Y_1] - [W'_1])|_{W'_1}) = \nu_{W'_1 \supseteq \cdots \supseteq W'_n}&((p^*[W_1] - [W'_1])|_{W'_1}) + \\ &\nu_{W'_1 \supseteq \cdots \supseteq W'_n}(p|_{W'_1}^*(\pi^*[Y_1] - [W_1])|_{W_1}). \end{aligned}$$

This follows from Proposition 10.1.2. ☐

prop:cormatrix

Proposition 10.1.4 *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. Let $W_\bullet$ be a lifting of $Y_\bullet$, then for any closed positive $(1,1)$-current T on X, we have*

$$\nu_{W_\bullet}(\pi^* T) = \nu_{Y_\bullet}(T) \operatorname{cor}(Y_\bullet, \pi). \tag{10.5}$$

Proof We make induction on $n \geq 0$. The case $n = 0$ is trivial. In general, assume that $n \geq 1$ and the result is proved in dimension $n-1$.

For simplicity, we write $\nu = \nu_{Y_\bullet}$ and $\nu' = \nu_{W_\bullet}$. Let μ (resp. μ') be the valuation of currents defined by the truncated flag $Y_1 \supseteq \cdots \supseteq Y_n$ (resp. $W_1 \supseteq \cdots \supseteq W_n$). Then we need to show that

$$\begin{aligned}&\left[\nu'(\pi^*T)_1\ \mu'(\mathrm{Tr}_{W_1}(\pi^*T - \nu'(\pi^*T)_1[W_1]))\right]\\ =&\left[\nu(T)_1\ \mu(\mathrm{Tr}_{Y_1}(T-\nu(T)_1[Y_1]))\right]\mathrm{cor}(Y_\bullet,\pi).\end{aligned} \tag{10.6}$$

{eq:mubiration}

By Zariski's main theorem,

$$\nu'(\pi^*T)_1 = \nu(T)_1 =: c.$$

By the inductive hypothesis, we have

$$\mu'(\Pi^*\,\mathrm{Tr}_{Y_1}(T-c[Y_1])) = \mu(\mathrm{Tr}_{Y_1}(T-c[Y_1]))\,\mathrm{cor}(Y_1\supseteq\cdots\supseteq Y_n,\Pi), \tag{10.7}$$

{eq: ind_hypos}

where $\Pi\colon W_1 \longrightarrow Y_1$ is the restriction of π. By Lemma 8.2.1 and Proposition 8.2.1,

$$\begin{aligned}\Pi^*\,\mathrm{Tr}_{Y_1}(T-c[Y_1]) &\sim_P \mathrm{Tr}_{W_1}(\pi^*(T-c[Y_1]))\\ &\sim_P \mathrm{Tr}_{W_1}(\pi^*T - c[W_1]) + c\,\mathrm{Tr}_{W_1}(\pi^*[Y_1]-[W_1]).\end{aligned}$$

So

$$\mu'(\Pi^*\,\mathrm{Tr}_{Y_1}(T-c[Y_1])) = \mu'(\mathrm{Tr}_{W_1}(\pi^*T-c[W_1])) + c\mu'(\mathrm{Tr}_{W_1}(\pi^*[Y_1]-[W_1])).$$

Combining the above with (10.7), we see that (10.6) follows. □

thm:liftableflag

Theorem 10.1.1 *Let $\pi\colon Z \longrightarrow X$ be a proper bimeromorphic morphism from a reduced complex space Z. Then there is a modification $W \longrightarrow X$ dominating $Z \longrightarrow X$ such that $Y_\bullet$ admits a lifting to W.*

Proof By Hironaka's Chow lemma, we may assume that π is a modification.

We begin by setting $W_0 = Z$. We will construct W_i inductively for each i. Assume that for $0 \le i < n$ a smooth partial flag $W_0 \supset \cdots \supset W_i$ has been constructed on a modification $\pi_i : Z_i \longrightarrow Z$ so that $\pi \circ \pi_i$ restricts to bimeromorphic morphisms $W_j \longrightarrow Y_j$ for each $j = 0, \dots, i$.

By Zariski's main theorem, $W_i \longrightarrow Y_i$ is an isomorphism outside a codimension 2 subset of Y_i. We let W_{i+1} be the strict transform of Y_{i+1} in W_i. The problem is that W_{i+1} is not necessarily smooth.

We will further modify Z_i and lift $W_1, \dots, W_{i+1}$ in order to make the flag smooth. Take the embedded resolution of (W_j, W_{i+1}), say $W'_j \longrightarrow W_j$ for each $j = 0, \dots, i$.

We have canonical embeddings $W'_i \hookrightarrow W'_{i-1} \hookrightarrow \cdots \hookrightarrow W'_0$ making the following diagram commutative:

$$\begin{tikzcd} W'_i \arrow[r, hook] \arrow[d] & W'_{i-1} \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & W'_0 \arrow[d] \\ W_i \arrow[r, hook] & W_{i-1} \arrow[r, hook] & \cdots \arrow[r, hook] & W_0 \end{tikzcd}$$

Let W'_{i+1} be the strict transform of W_{i+1} in W'_i. It suffices to define π_{i+1} as the morphism $W'_0 \to Z_i \to Z$ and replace $W_0 \supset \cdots \supset W_{i+1}$ by $W'_0 \supset \cdots \supset W'_{i+1}$. □

10.2 Algebraic partial Okounkov bodies

sec:PoB

Let X be a connected smooth complex projective variety of dimension n and (L, h) be a Hermitian big line bundle on X.

Let h_0 be a smooth Hermitian metric on L. Let $\theta = c_1(L, h_0)$. Then we can identify h with a function $\varphi \in \mathrm{PSH}(X, \theta)$. We will use interchangeably the notations (θ, φ) and (L, h).

Fix a rank n valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$, which without loss of generality can be assumed to be surjective.

We will adopt the notations of Appendix C.2.

10.2.1 The spaces of sections

Definition 10.2.1 We will write

$$\begin{aligned}\Gamma(\theta,\varphi) &:= \left\{(\nu(s),k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))^\times\right\},\\ \Delta_k(\theta,\varphi) &:= \mathrm{Conv}\left\{k^{-1}\nu(f) : f \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))^\times\right\} \subseteq \mathbb{R}^n, \quad k \geq 0.\end{aligned}$$

When $\theta = V_\theta$, we simply write $\Gamma(L)$ and $\Delta_k(L)$ instead.

Here Conv denotes the convex hull. For large enough k, $\Delta_k(\theta, \varphi)$ is non-empty thanks to Theorem 7.3.1.

Definition 10.2.2 Assume that φ has analytic singularities. We define

$$\Gamma^\infty(\theta,\varphi) := \left\{(\nu(s),k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi))^\times\right\}. \tag{10.8}$$

{eq:Weps1}

For later use, we introduce a twisted version as well.

Definition 10.2.3 If T is a holomorphic line bundle on X, we introduce

$$\begin{aligned}\Delta_{k,T}(\theta,\varphi) &:= \mathrm{Conv}\left\{k^{-1}\nu(f) : f \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))^\times\right\} \subseteq \mathbb{R}^n,\\ \Delta_{k,T}(L) &:= \mathrm{Conv}\left\{k^{-1}\nu(f) : f \in \mathrm{H}^0(X, T \otimes L^k)^\times\right\} \subseteq \mathbb{R}^n.\end{aligned}$$

10.2.2 Algebraic Okounkov bodies

prop:Okounbiglbdl

Proposition 10.2.1 *There is a convex body $\Delta \in \mathcal{K}_n$ such that $\Gamma(L) \in \mathcal{S}'(\Delta)$.*

Proof **Step 1**. We first show that there is $\Delta \in \mathcal{K}_n$ such that $\Delta_k(L) \subseteq \Delta$. For this purpose, using Remark 10.1.1, we may assume that ν is induced by an admissible flag $Y_\bullet$ on X.

Fix $s \in H^0(X, L^k)^\times$ for some $k \in \mathbb{Z}_{>0}$. Assume that $s \neq 0$. We need to show that for each $i = 1, \ldots, n$, $\nu(s)_i \leq Ck$ for some constant $C > 0$, independent of the choices of k and s.

Fix an ample divisor H on X. Take a large enough integer $b_1 > 0$ such that

$$(L - b_1 Y_1) \cdot H^{n-1} < 0.$$

Then $\nu(s)_1 \leq b_1 k$. Next take a large enough integer b_2 such that

$$\big((L - aY_1)|_{Y_1} - b_2 Y_2\big) \cdot H^{n-2} < 0.$$

It follows that $\nu(s)_2 \leq b_2 k$. Continue in this manner, we conclude that $\nu(s)_i/k$ is bounded for each i.

Step 2. Observe that $\Gamma(L)$ is clearly a semigroup. It remains to show that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$ as an Abelian group.

For this purpose, take two very ample divisors A and B so that $L = \mathcal{O}_X(A - B)$. After choosing A and B ample enough, we may guarantee that there exist sections $s_0 \in \mathrm{H}^0(X, A)$, $t_i \in \mathrm{H}^0(X, B)$ for $i = 0, \ldots, n$ such that

$$\nu(s_0) = \nu(t_0) = 0$$

and $\nu(t_i)$ is the i-th unit vector $e_i \in \mathbb{R}^n$ for $i = 1, \ldots, n$.

Since L is big, we can find $m_0 > 0$ such that for any $m \geq m_0$ we can find an effective divisor F_m on X linearly equivalent to $mL - B$. Let $f_m = \nu([F_m])$. Then we find that

$$(f_m, m), (f_m + e_1, m), \ldots, (f_m + e_n, m) \in \Gamma(L).$$

Since $(m+1)L$ is linearly equivalent to $A + F_m$, so

$$(f_m, m+1) \in \Gamma(L).$$

It follows that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$. □

Thanks to Proposition 10.2.1, we can introduce the next definition.

Definition 10.2.4 We define the *Okounkov body* of L with respect to the valuation ν as

$$\Delta_\nu(L) := \Delta(\Gamma(L)).$$

prop:Okounonlydepnum

Proposition 10.2.2 *The Okounkov body $\Delta_\nu(L)$ depends only on the numerical class of L.*

See [LM09, Proposition 4.1] for the elegant proof.

cor:Okounvol

Corollary 10.2.1 *We have*

$$\operatorname{vol} \Delta_\nu(L) = \frac{1}{n!} \operatorname{vol} L. \tag{10.9}$$

Proof This follows immediately from Proposition 10.2.1 and Theorem C.2.1. □

prop:GammaepsSp **Proposition 10.2.3** *Assume that φ has analytic singularities and θ_φ is a Kähler current. Then we have*

$$\Gamma^\infty(\theta, \varphi) \in \mathcal{S}'(X, \theta)$$

and

$$\operatorname{vol} \Gamma^\infty(\theta, \varphi) = \frac{1}{n!} \int_X \theta_\varphi^n.$$

Proof Replacing X by a modification, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. See Theorem 1.6.1.

In this case,

$$\Gamma^\infty(\theta, \varphi) = \left\{(\nu(s), k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-\lfloor kD \rfloor)).\right\}$$

Since $L - D$ is ample by Lemma 1.6.1, our assertion follows from the same argument as Proposition 10.2.1. □

We first extend Theorem C.2.1 to the twisted case.

prop-Deltaconvtwisted **Proposition 10.2.4** *For any holomorphic line bundle T on X, as $k \to \infty$*

$$\Delta_{k,T}(L) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(L).$$

Proof As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0, and
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$\mathrm{H}^0(X, L^{k-k_0}) \xrightarrow{\times s_1} \mathrm{H}^0(X, T \otimes L^k) \xrightarrow{\times s_0} \mathrm{H}^0(X, L^{k+k_0}).$$

It follows that

$$(k - k_0)\Delta_{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_{k,T}(L) \subseteq (k + k_0)\Delta_{k+k_0}(L) - \nu(s_0).$$

Using Theorem C.2.1, we conclude. □

prop:subaddOkoun **Proposition 10.2.5** *Let L' be another big line bundle on X. Then*

$$\Delta_\nu(L) + \Delta_\nu(L') \subseteq \Delta_\nu(L \otimes L').$$

Proof Observe that for each $k \in \mathbb{N}$, we have

$$\Delta_k(L) + \Delta_k(L') \subseteq \Delta_k(L \otimes L').$$

So our assertion follows immediately from Theorem C.2.1. □

prop:Okourescaling

Proposition 10.2.6 *For any $a \in \mathbb{Z}_{>0}$, we have*

$$\Delta_\nu(L^a) = a\Delta_\nu(L).$$

Proof This is an immediate consequence of Theorem C.2.1. □

10.2.3 Construction of partial Okounkov bodies

thm:Gammaasg

Theorem 10.2.1 *We have*

$$\Gamma(\theta, \varphi) \in \overline{\mathcal{S}'(\Delta_\nu(L))}_{>0}.$$

This theorem allows us to give the following definition:

Definition 10.2.5 The *partial Okounkov body* of (L, h) is defined as

$$\Delta_\nu(L, h) = \Delta_\nu(\theta, \varphi) := \Delta\left(\Gamma(\theta, \varphi)\right). \tag{10.10}$$

{eq:Deltalbdef}

When ν is induced by an admissible flag $(Y_\bullet)$ on X (see Definition 10.1.1), we also say that $\Delta_\nu(\theta, \varphi)$ the *partial Okounkov body* of (L, h) or of (θ, φ) with respect to $(Y_\bullet)$. In this case, we also write $\Delta_{Y_\bullet}$ instead of Δ_ν.

cor:POBvolume

Corollary 10.2.2 *We have*

$$\operatorname{vol}\Delta_\nu(\theta, \varphi) = \frac{1}{n!}\operatorname{vol}\theta_\varphi. \tag{10.11}$$

{eq:Okov}

Proof This follows immediately from Theorem 10.2.1, Theorem 7.3.1 and Theorem C.2.2. □

We will prove Theorem 10.2.1 and Corollary 10.2.2 at the same time.

Proof **Step 1**. We first assume that φ has analytic singularities and θ_φ is a Kähler current.

We claim that

$$d_{\mathrm{sg}}(\Gamma^\infty(\theta, \varphi), \Gamma(\theta, \varphi)) = 0. \tag{10.12}$$

{eq:Gamma0Gammaanalytic}

Observe that for each $\epsilon \in \mathbb{Q}_{>0}$, we have

$$H^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq H^0(X, L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

for all large enough k. This is a consequence of Lemma 1.6.3. Therefore, it suffices to show that

$$\lim_{\mathbb{Q}\ni\epsilon\to 0+}\operatorname{vol}\Gamma^\infty(\theta, (1-\epsilon)\varphi) = \operatorname{vol}\Gamma^\infty(\theta, \varphi).$$

This follows from the explicit formula in Proposition 10.2.3.

Step 2. We next handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. Then $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$ by Corollary 7.1.2.

In this case, it suffices to prove that

$$\Gamma(\theta,\varphi_j) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta,\varphi). \tag{10.13}$$

{eq:WtoWclaim}

In fact, by Theorem 7.3.1, we have

$$\begin{aligned}
&d_{\mathrm{sg}}(\Gamma(\theta,\varphi_j),\Gamma(\theta,\varphi))\\
=&\varlimsup_{k\to\infty} k^{-n}\left(h^0(X,L^k\otimes\mathcal{I}(k\varphi_j))-h^0(X,L^k\otimes\mathcal{I}(k\varphi))\right)\\
=&\lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi_j))-\lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi))\\
=&\frac{1}{n!}\operatorname{vol}\theta_{\varphi_j}-\frac{1}{n!}\operatorname{vol}\theta_{\varphi}.
\end{aligned}$$

Letting $j\to\infty$, we conclude (10.13) by Theorem 6.2.5.

Step 3. Now we only assume that $\operatorname{vol}\theta_\varphi>0$. We may replace φ with $P_\theta[\varphi]_{\mathcal{I}}$ and then assume that $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$.

Take a potential $\psi\in\mathrm{PSH}(X,\theta)$ such that $\psi\leq\varphi$ and θ_ψ is a Kähler current. The existence of ψ is proved in Lemma 2.3.2. For each $\epsilon\in(0,1)$, let $\varphi_\epsilon=(1-\epsilon)\varphi+\epsilon\psi$. It suffices to show that

$$\Gamma(\theta,\varphi_\epsilon)\xrightarrow{d_{\mathrm{sg}}}\Gamma(\theta,\varphi)$$

as $\epsilon\to 0+$. We compute using Theorem 7.3.1:

$$\begin{aligned}
&d_{\mathrm{sg}}\left(\Gamma(\theta,\varphi_\epsilon),\Gamma(\theta,\varphi)\right)\\
=&\varlimsup_{k\to\infty} k^{-n}\left(h^0(X,L^k\otimes\mathcal{I}(k\varphi))-h^0(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon))\right)\\
=&\lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi))-\lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon))\\
=&\frac{1}{n!}\operatorname{vol}\theta_{\varphi}-\frac{1}{n!}\operatorname{vol}\theta_{\varphi_\epsilon}\\
\to&0
\end{aligned}$$

by Theorem 6.2.5, as $\epsilon\to 0+$. □

rmk:DeltaanaW0

Remark 10.2.1 It follows from the proof that if φ has analytic singularities and θ_φ is a Kähler current, then (10.12) holds.

If we take a modification $\pi\colon Y\to X$ such that $\pi^*\varphi$ has log singularities along an effective $\mathbb{Q}$-divisor D on Y, then

$$\Delta_\nu(\theta,\varphi)=\Delta_\nu(\pi^*L-D)+\nu(D).$$

10.2.4 Basic properties of partial Okounkov bodies

cor:Okocurrent

Proposition 10.2.7 *The partial Okounkov body $\Delta_\nu(L,h)$ depends only on $\mathrm{dd^c} h$, not on the explicit choices of L, h_0, h.*

Thanks to this result, given a closed positive $(1,1)$-current $T \in c_1(L)$ on X with $\int_X T^n > 0$, we can write

$$\Delta_\nu(T) \coloneqq \Delta_\nu(\theta,\varphi)$$

if $T = \theta + \mathrm{dd^c}\varphi$ for some $\varphi \in \mathrm{PSH}(X,\theta)$.

Proof There are two different claims to prove, as detailed in the two steps below.

Step 1. Let h_0' be another Hermitian metric on L. Set $\theta' = c_1(L, h_0')$. Write $\mathrm{dd^c} f = \theta - \theta'$. Let $\varphi' = \varphi + f \in \mathrm{PSH}(X,\theta')$. Then

$$\Delta_\nu(\theta,\varphi) = \Delta_\nu(\theta',\varphi'). \tag{10.14}$$

{eq:DeltaDelta1}

This is obvious since $\Gamma(\theta,\varphi) = \Gamma(\theta',\varphi')$.

Step 2. Let L' be another big line bundle on X. By Step 1, we may assume that the reference Hermitian metric h_0' on L' is such that $c_1(L', h_0') = \theta$.

Let h' be a plurisubharmonic metric on L' with $c_1(L,h) = c_1(L',h')$. Then

$$\Delta_\nu(L,h) = \Delta_\nu(L',h').$$

From our construction, we may assume that $c_1(L,h)$ has analytic singularities. After taking a birational resolution, it suffices to deal with the case where $c_1(L,h)$ has analytic singularities along an effective $\mathbb{Q}$-divisors D. By rescaling, we may also assume that D is a divisor. By Remark 10.2.1, we further reduce to the case where $c_1(L,h)$ is not singular.

In this case, the assertion is proved in Proposition 10.2.2. □

prop:IcompimplyDeltacomp

Proposition 10.2.8 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. Assume that $\varphi \preceq_{\mathcal{I}} \psi$, then*

$$\Delta_\nu(\theta,\varphi) \subseteq \Delta_\nu(\theta,\psi). \tag{10.15}$$

{eq:Deltacomp}

Proof This follows from Corollary C.2.2. □

thm:Okoucont

Theorem 10.2.2 *The Okounkov body map*

$$\Delta_\nu(\theta,\bullet) : (\mathrm{PSH}(X,\theta)_{>0}, d_S) \longrightarrow (\mathcal{K}_n, d_{\mathrm{Haus}})$$

is continuous.

Proof Let $\varphi_j \to \varphi$ be a d_S-convergent sequence in $\mathrm{PSH}(X,\theta)_{>0}$. We want to show that

$$\Delta_\nu(\theta,\varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi). \tag{10.16}$$

{eq:Deltavjv}

By Proposition 10.2.8, we may assume that all φ_j's and φ are model potentials.

By Theorem C.1.1 and Proposition 6.2.3, we may assume that $(\varphi_j)_j$ is either decreasing or increasing. By Theorem 6.2.3, we may further assume that the φ_j's are $\mathcal{I}$-model. In both cases, we claim that

$$\Gamma(\theta, \varphi_j) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta, \varphi)$$

as $j \to \infty$. In fact, using Theorem 7.3.1, we can compute

$$\begin{aligned} d_{\mathrm{sg}}\left(\Gamma(\theta, \varphi_j), \Gamma(\theta, \varphi)\right) = & \varlimsup_{k\to\infty} k^{-n} \left| h^0(X, L^k \otimes \mathcal{I}(k\varphi_j)) - h^0(X, L^k \otimes \mathcal{I}(k\varphi)) \right| \\ = & \frac{1}{n!} \left| \operatorname{vol} \theta_{\varphi_j} - \operatorname{vol} \theta_{\varphi} \right|, \end{aligned}$$

which converges to 0 by Theorem 6.2.5. □

prop:birinv0 **Proposition 10.2.9** *Let $\pi\colon Y \to X$ be a modification. Then*

$$\Delta_\nu(\pi^* L, \pi^* h) = \Delta_\nu(L, h).$$

Proof Thanks to Proposition 3.2.5, we may assume that φ is $\mathcal{I}$-model. By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ with analytic singularities in $\mathrm{PSH}(X, \theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi$. It is clear that $\pi^*\varphi_j \xrightarrow{d_S} \pi^*\varphi$. By Theorem 10.2.2, we may then reduce to the case where φ has analytic singularities. In this case, it suffices to apply Remark 10.2.1. □

prop:suba **Proposition 10.2.10** *Let (L', h') be another Hermitian big line bundle on X. Then*

$$\Delta_\nu(L, h) + \Delta_\nu(L', h') \subseteq \Delta_\nu(L \otimes L', h \otimes h').$$

Proof Take a smooth metric h'_0 on L' and let $\theta' = c_1(L', h'_0)$. We identify h' with $\varphi' \in \mathrm{PSH}(X, \theta')$. Then we need to show

$$\Delta_\nu(\theta, \varphi) + \Delta_\nu(\theta', \varphi') \subseteq \Delta_\nu(\theta + \theta', \varphi + \varphi'). \tag{10.17}$$ {eq:suba}

By Theorem 7.1.1, we can find sequences $(\varphi_j)_j$ and $(\varphi'_j)_j$ in $\mathrm{PSH}(X, \theta)_{>0}$ and $\mathrm{PSH}(X, \theta')_{>0}$ respectively such that

(1) φ_j and φ'_j both have analytic singularities for all $j \geq 1$, and

(2) $\varphi_j \xrightarrow{d_S} \varphi$, $\varphi'_j \xrightarrow{d_S} \varphi'$.

Then $\varphi_j + \varphi'_j \in \mathrm{PSH}(X, \theta + \theta')_{>0}$ and $\varphi_j + \varphi'_j \xrightarrow{d_S} \varphi + \varphi'$ by Theorem 6.2.2. Thus, by Theorem 10.2.2, we may assume that φ and ψ both have analytic singularities. Taking a birational resolution, we may further assume that they have log singularities. By Remark 10.2.1, we reduce to the case without singularities, in which case the result is just Proposition 10.2.5. □

thm:concOko **Theorem 10.2.3** *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. Then for any $t \in (0, 1)$,*

$$\Delta_\nu(\theta, t\varphi + (1-t)\psi) \supseteq t\Delta_\nu(\theta, \varphi) + (1-t)\Delta_\nu(\theta, \psi). \tag{10.18}$$ {eq:Deltaconcave}

Proof We may assume that t is rational as a consequence of Theorem 10.2.2. Similarly, as in the proof of Proposition 10.2.10, we could reduce to the case where both φ and ψ have analytic singularities. In this case, let $N > 0$ be an integer such that Nt is an integer. Then for any $s \in H^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi))$ and $r \in H^0(X, L^k \otimes \mathcal{I}_\infty(k\psi))$, we have

$$s^{tN} \otimes r^{N-tN} \in H^0(X, L^{kN} \otimes \mathcal{I}_\infty(Nt\varphi + (N - Nt)\psi)).$$

By Theorem C.2.1 and Remark 10.2.1, (10.18) follows. □

prop:res **Proposition 10.2.11** *For any $a \in \mathbb{Z}_{>0}$,*

$$\Delta_\nu(a\theta, a\varphi) = a\Delta_\nu(\theta, \varphi).$$

Proof As in the proof of Proposition 10.2.10, we may assume that φ has log singularities. Using Remark 10.2.1, we reduce to the case without the singularity φ, which is proved in Proposition 10.2.6. □

In particular, if T is a closed positive $(1, 1)$-current on X with $\int_X T^n > 0$ and such that

$$[T] \in \mathrm{NS}^1(X)_{\mathbb{Q}},$$

we can define

$$\Delta_\nu(T) \coloneqq a^{-1}\Delta_\nu(aT) \tag{10.19}$$ {eq:DeltanuTalgebraic1}

for a sufficiently divisible positive integer a.

We also need the following perturbation. Let A be an ample line bundle on X. Fix a Hermitian metric h_A on A such that $\omega \coloneqq c_1(A, h_A)$ is a Kähler form on X.

prop:Deltapert **Proposition 10.2.12** *As $\delta \searrow 0$, the convex bodies $\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi)$ are decreasing and*

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta_\varphi).$$

Proof Let $0 \leq \delta < \delta'$ be two rational numbers. Take $C \in \mathbb{N}_{>0}$ divisible enough, so that $C\delta$ and $C\delta'$ are both integers. Then by Proposition 10.2.10,

$$\Delta_\nu(C\theta + C\delta\omega + C\mathrm{dd^c}\varphi) \subseteq \Delta_\nu(C\theta + C\delta'\omega + C\mathrm{dd^c}\varphi).$$

It follows that

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \subseteq \Delta_\nu(\theta + \delta'\omega + \mathrm{dd^c}\varphi).$$

On the other hand,

$$\operatorname{vol}\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) = \frac{1}{n!}\operatorname{vol}(\theta + \delta\omega)_\varphi = \frac{1}{n!}\int_X (\theta + \delta\omega)^n_{P_\theta[\varphi]_{\mathcal{I}}},$$

where we applied Example 7.1.2. As $\delta \to 0+$, the right-hand side converges to

$$\operatorname{vol}\Delta_\nu(\theta, \varphi) = \frac{1}{n!}\operatorname{vol}\theta_\varphi.$$

Our assertion therefore follows. □

10.2.5 The Hausdorff convergence property of partial Okounkov bodies

Let T be a holomorphic line bundle on X.

thm:HCP **Theorem 10.2.4** *As $k \to \infty$, we have $\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi)$.*

Although we are only interested in the untwisted case, the proof given below requires twisted case.

lma:twistedHcp **Lemma 10.2.1** *Assume that φ has analytic singularities and θ_φ is a Kähler current, then as $k \to \infty$,*

$$\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi).$$

Proof Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D, see Proposition 10.2.9 and Theorem 1.6.1.

Take a small enough $\epsilon \in \mathbb{Q}_{>0}$. In this case, for large enough $k \in \mathbb{Z}_{>0}$ we have

$$\mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}_\infty(k\varphi)) \subseteq \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi)) \subseteq \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi)).$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$, say the sequence defined by the indices $k_1, k_2, \dots$. We want to show that $\Delta' = \Delta(\theta,\varphi)$.

There exists $t \in \{0, 1, \dots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$. Then for large enough i, we have

$$\begin{aligned}\mathrm{H}^0(X, T\otimes L^{-N+t}\otimes L^{N(g_i+1)}\otimes \mathcal{I}_\infty(N(g_i+1)\varphi)) &\subseteq \mathrm{H}^0(X, T\otimes L^{k_i}\otimes \mathcal{I}(k_i\varphi))\\ &\subseteq \mathrm{H}^0(X, T\otimes L^t\otimes L^{Ng_i}\otimes \mathcal{I}_\infty(g_iN(1-\epsilon)\varphi)).\end{aligned}$$

So

$$\begin{aligned}(g_i+1)\Delta_{g_i+1, T\otimes L^{-N+t}}(NL-ND) + N(g_i+1)\nu(D) &\subseteq (Ng_i+t)\Delta_{k,T}(\theta,\varphi)\\ &\subseteq g_i\Delta_{g_i, T\otimes L^t}(NL - N(1-\epsilon)D) + Ng_i(1-\epsilon)\nu(D).\end{aligned}$$

Letting $i \to \infty$, by Proposition 10.2.4,

$$\Delta_\nu(L-D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L-(1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Letting $\epsilon \to 0+$, we find that

$$\Delta_\nu(L-D) + \nu(D) = \Delta'.$$

It follows from Theorem C.1.1 that

$$\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(L-D) + \nu(D) = \Delta_\nu(\theta,\varphi)$$

as $k \to \infty$. □

lma-Hausconvbetato0 **Lemma 10.2.2** *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \to 0+$, we have*

$$\Delta_\nu((1-\beta)\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi).$$

Here and in the sequel, $\Delta_\nu((1-\beta)\theta,\varphi) = \Delta_\nu((1-\beta)\theta + \mathrm{dd^c}\varphi)$.

Proof By Proposition 10.2.10, we have

$$\Delta_\nu\left((1-\beta)\theta,\varphi\right) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta,\varphi).$$

In particular, if Δ' is the Hausdorff limit of a subsequence of $(\Delta((1-\beta)\theta,\varphi))_\beta$, then $\Delta' \subseteq \Delta_\nu(\theta,\varphi)$. But

$$\begin{aligned}\operatorname{vol}\Delta' = \lim_{\beta\to 0+}\Delta_\nu((1-\beta)\theta,\varphi) &= \lim_{\beta\to 0+}\int_X ((1-\beta)\theta + \mathrm{dd^c} P_{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n\\ &= \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n,\end{aligned}$$

where the last step follows easily from Theorem 11.2.1. It follows that $\Delta' = \Delta_\nu(\theta,\varphi)$. We conclude by Theorem C.1.1. □

Proof (Proof of Theorem 10.2.4) Fix a Kähler form $\omega \geq \theta$ on X.

Step 1. We first handle the case where θ_φ is a Kähler current, say $\theta_\varphi \geq 2\delta\omega$ for some $\delta \in (0,1)$. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\mathrm{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j} \geq \delta\omega$ for all $j \geq 1$.

Let Δ' be a limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$. Let us say the indices of the subsequence are $k_1 < k_2 < \cdots$. By Theorem C.1.1, it suffices to show that $\Delta' = \Delta_\nu(\theta,\varphi)$.

Observe that for each $j \geq 1$, we have $\Delta' \subseteq \Delta_\nu(\theta,\varphi_j)$ by Lemma 10.2.1. Letting $j \to \infty$, we find $\Delta' \subseteq \Delta_\nu(\theta,\varphi)$. Therefore, it suffices to prove that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu(\theta,\varphi). \tag{10.20}$$

Fix an integer $N > \delta^{-1}$. Observe that for any $j \geq 1$, we have $\varphi_j \in \mathrm{PSH}(X,(1-N^{-1})\theta)$. Similarly, $\varphi \in \mathrm{PSH}(X,(1-N^{-1})\theta)$. By Lemma 10.2.2, it suffices to argue that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu((1-N^{-1})\theta,\varphi). \tag{10.21}$$ {eq:volDeltatoprove}

For this purpose, we are free to replace k_i's by a subsequence, so we may assume that $k_i \equiv a$ modulo q for all $i \geq 1$, where $a \in \{0,1,\ldots,q-1\}$. We write $k_i = g_iq + a$. Observe that for each $i \geq 1$,

$$\mathrm{H}^0(X, T\otimes L^{k_i}\otimes\mathcal{I}(k_i\varphi)) \supseteq \mathrm{H}^0\left(X, T\otimes L^{-q+a}\otimes L^{g_iq+q}\otimes\mathcal{I}((g_iq+q)\varphi)\right).$$

Up to replacing T by $T\otimes L^{-q+a}$, we may therefore assume that $a = 0$.

By Lemma 2.3.1, we can find $k' \in \mathbb{Z}_{>0}$ such that for all $k \geq k'$, there is $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ satisfying

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1-N^{-1})\varphi_k + N^{-1}\psi_k.$$

Fix $k \geq k'$. It suffices to show that

$$\Delta_\nu((1-N^{-1})\theta, \varphi_k) + \nu' \subseteq \Delta' \tag{10.22}$$

{eq:DeltatransinDeltaprime}

for some $\nu' \in \mathbb{R}^n$. In fact, if this is true, we have

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta((1-N^{-1})\theta, \varphi_k).$$

Letting $k \to \infty$ and applying Theorem 10.2.2, we conclude (10.21).

It remains to prove (10.22). By the proof of Theorem 7.3.1, there is $j_0 > 0$ such that for any $j \geq j_0$, we can find a non-zero section $s_j \in \mathrm{H}^0(X, L^j \otimes \mathcal{I}(j\psi_k))$ such that we get an injective linear map

$$\mathrm{H}^0(X, T \otimes L^{(N-1)j} \otimes \mathcal{I}(jN\varphi_k)) \xrightarrow{\times s_j} \mathrm{H}^0(X, T \otimes L^{jN} \otimes \mathcal{I}(jN\varphi)).$$

In particular, when $j = k_i$ for some i large enough, we then find

$$\Delta_{k_i,T}((N-1)\theta, N\varphi_k) + (k_i)^{-1}\nu(s_{k_i}) \subseteq N\Delta_{k_i,T}(\theta, \varphi).$$

We observe that $(k_i)^{-1}\nu(s_{k_i})$ is bounded as both convex bodies appearing in this equation are bounded when i varies. Then by Lemma 10.2.1, there is a vector $\nu' \in \mathbb{R}^n$ such that (10.22) holds.

Step 2. Next we handle the general case.

Let Δ' be the Hausdorff limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$, say the subsequence with indices $k_1 < k_2 < \cdots$. By Theorem C.1.1, it suffices to prove that $\Delta' = \Delta_\nu(\theta, \varphi)$.

Take $\psi \in \mathrm{PSH}(X, \theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. The existence of ψ follows from Lemma 2.3.2.

Then for any $\epsilon \in \mathbb{Q} \cap (0,1)$,

$$\Delta_{k,T}(\theta, \varphi) \supseteq \Delta_{k,T}(\theta, (1-\epsilon)\varphi + \epsilon\psi)$$

for all $k \geq 1$. It follows from Step 1 that

$$\Delta' \supseteq \Delta_\nu(\theta, (1-\epsilon)\varphi + \epsilon\psi).$$

Letting $\epsilon \to 0$ and applying Theorem 10.2.2, we have $\Delta' \supseteq \Delta_\nu(\theta, \varphi)$. It remains to establish that

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(\theta, \varphi). \tag{10.23}$$

{eq:Deltapvolumeupp}

For this purpose, we are free to replace $k_1 < k_2 < \cdots$ by a subsequence. Fix $q > 0$, we may then assume that $k_i \equiv a$ modulo q for all $i \geq 1$ for some $a \in \{0, 1, \ldots, q-1\}$. We write $k_i = g_i q + a$. Observe that

$$\mathrm{H}^0(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi)) \subseteq \mathrm{H}^0(X, T \otimes L^a \otimes L^{g_i q} \otimes \mathcal{I}(g_i q\varphi)).$$

Up to replacing T by $T \otimes L^a$, we may assume that $a = 0$.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in \mathrm{H}^0(X, H)$.

We have an injective linear map

$$\mathrm{H}^0(X, T \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)) \xrightarrow{\times s^j} \mathrm{H}^0(X, T \otimes H^j \otimes L^{jq} \otimes \mathcal{I}(jq\varphi))$$

for each $j \geq 1$. In particular, for each $i \geq 1$,

$$k_i \Delta_{k_i, T}(q\theta, q\varphi) + k_i \nu(s) \subseteq k_i \Delta_{k_i, T}(\omega + q\theta, q\varphi).$$

Letting $i \to \infty$, by Step 1, we have

$$q\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + q\theta, q\varphi).$$

So

$$\operatorname{vol} \Delta' \leq \operatorname{vol} \Delta_\nu(q^{-1}\omega + \theta, \varphi) = \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P_{q^{-1}\omega+\theta}[\varphi]_{\mathcal{I}})^n.$$

By Example 7.1.2,

$$\operatorname{vol} \Delta' \leq \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

Letting $q \to \infty$, we conclude (10.23). □

10.2.6 Recover Lelong numbers from partial Okounkov bodies

thm:nuOk **Theorem 10.2.5** *Let E be a prime divisor on X. Let $Y_\bullet$ be an admissible flag with $E = Y_1$. Then*

$$\nu(\varphi, E) = \min_{x \in \Delta_{Y_\bullet}(\theta, \varphi)} x_1. \tag{10.24}$$ {eq:numinOk}

Here x_1 denotes the first component of x.

Proof Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good.

Step 1. We first reduce to the case where φ has analytic singularities.

By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X, \theta)_{>0}$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$. It follows from Theorem 10.2.2 that

$$\Delta_{Y_\bullet}(\theta, \varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta, \varphi).$$

Therefore,

$$\lim_{j\to\infty} \min_{x \in \Delta_{Y_\bullet}(\theta, \varphi_j)} x_1 = \min_{x \in \Delta_{Y_\bullet}(\theta, \varphi)} x_1.$$

In view of Theorem 6.2.4, it suffices to prove (10.24) with φ_j in place of φ.

Step 2. Assume that φ has analytic singularities. In view of Proposition 10.2.9 and Theorem 1.6.1, after replacing X by a birational model, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor F.

Perturbing L by an ample $\mathbb{Q}$-line bundle by Proposition 10.2.12, we may assume that θ_φ is a Kähler current. Therefore, $L - F$ is ample by Lemma 1.6.1. Finally, by rescaling, we may assume that F is a divisor and L is a line bundle.

By Theorem 10.2.4, we know that

$$\min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1 = \lim_{k\to\infty} \min_{x\in\Delta_k(\theta,\varphi)} x_1.$$

By definition,

$$\min_{x\in\Delta_k(\theta,\varphi)} x_1 = k^{-1} \operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)).$$

It remains to show that

$$\lim_{k\to\infty} k^{-1} \operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \lim_{k\to\infty} k^{-1} \operatorname{ord}_E \mathcal{I}(k\varphi). \tag{10.25}$$ {eq:temp1}

The $\geq$ direction is trivial, we prove the converse. Observe that

$$\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-kF)), \quad \mathcal{I}(k\varphi) = \mathcal{O}(-kF).$$

As $L - F$ is ample, for large enough k, we have

$$\operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-kF)) = \operatorname{ord}_E(kF).$$

Thus, (10.25) is clear. □

cor:Deltacontimplyvarphi **Corollary 10.2.3** *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. If*

$$\Delta_{W_\bullet}(\pi^*\theta, \pi^*\varphi) \subseteq \Delta_{W_\bullet}(\pi^*\theta, \pi^*\psi)$$

for all birational models $\pi : Y \to X$ and all admissible flags $W_\bullet$ on Y, then $\varphi \preceq_{\mathcal{I}} \psi$.

Proof This follows immediately from Theorem 10.2.5. □

cor:numin **Corollary 10.2.4** *Let E be a prime divisor over X. Then*

$$\nu(V_\theta, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathrm{H}^0(X, L^k). \tag{10.26}$$

Proof This follows from Theorem 10.2.5 and the fact that $\Delta_{Y_\bullet}(\theta, V_\theta) = \Delta_{Y_\bullet}(L)$ for any admissible flag $Y_\bullet$ on X. □

10.3 Transcendental partial Okounkov bodies

Let X be a connected compact Kähler manifold of dimension n. Fix a smooth flag $Y_\bullet$ on X.

10.3.1 The traditional approach to the Okounkov body problem

Definition 10.3.1 Let α be a big cohomology class on X. We define

$$\Delta_{Y_\bullet}(\alpha) \coloneqq \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X,\alpha), S \text{ has gentle analytic singularities}\right\}}. \tag{10.27}$$

{eq:twodefspob}

See Definition 1.6.4 for the definition of gentle analytic singularities.

DRWNXZ

The results of [DRWN+23] can be summarized as follows:

thm:Okounkovtranmain

Theorem 10.3.1 *For any big cohomology class α on X, the set $\Delta_{Y_\bullet}(\alpha) \subseteq \mathbb{R}^n$ is a convex body satisfying the following properties:*

(1) *we have*

$$\operatorname{vol}\Delta_{Y_\bullet}(\alpha) = \frac{1}{n!}\operatorname{vol}\alpha;$$

(2) *Given another big cohomology class α' on X, we have*

$$\Delta_{Y_\bullet}(\alpha) + \Delta_{Y_\bullet}(\alpha') \subseteq \Delta_{Y_\bullet}(\alpha+\alpha');$$

(3) *Let $\pi\colon Y \longrightarrow X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that $(W_\bullet, g)$ is the lifting of $Y_\bullet$ to Y, then*

$$\Delta_{W_\bullet}(\pi^*\alpha) = \Delta_{Y_\bullet}(\alpha)g;$$

(4) *The map $\alpha \mapsto \Delta_{Y_\bullet}(\alpha)$ is continuous in the big cone with respect to the Hausdorff metric;*

(5) *For any small enough $t>0$, we have*

$$\left\{y \in \mathbb{R}^{n-1} : (t,y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}((\beta - t[Y_1])|_{Y_1}).$$

10.3.2 Definitions of partial Okounkov bodies

Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class α.

Let $T = \theta_\varphi \in \mathcal{Z}_+(X,\alpha)$. We shall define a convex body $\Delta_{Y_\bullet}(T) \subseteq \mathbb{R}^n$, which is also written as $\Delta_{Y_\bullet}(\theta,\varphi)$. This convex body is called the *partial Okounkov body* of T with respect to the flag $Y_\bullet$.

10.3.2.1 The case of analytic singularities

def:POBanalsing **Definition 10.3.2** When T is a Kähler current with analytic singularities, we take a modification $\pi\colon Y \longrightarrow X$ so that

(1)
$$\pi^* T = [D] + R, \tag{10.28}$$
{eq:resolveanalytic}

where D is an effective $\mathbb{Q}$-divisor on Y and R is a closed positive $(1,1)$-current with bounded potential, and

(2) the lifting $(Z_\bullet, g)$ of $Y_\bullet$ to Y exists.

Define
$$\Delta_{Y_\bullet}(T) \coloneqq \Delta_{Z_\bullet}([R])g^{-1} + \nu_{Z_\bullet}([D])g^{-1}.$$

The existence of π is guaranteed by Theorem 1.6.1 and Theorem 10.1.1.

Lemma 10.3.1 *The convex body $\Delta_{Y_\bullet}(T)$ defined in Definition 10.3.2 is independent of the choice of π.*

Proof Take another map $\pi'\colon Y' \longrightarrow X$ with the same properties. We want to show that π and π' defines the same $\Delta_{Y_\bullet}(T)$. We may assume that π' dominates π through $p\colon Y' \longrightarrow Y$, so that we have a commutative diagram

$$\begin{tikzcd}
Y' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\
& X. &
\end{tikzcd}$$

We take D and R as in (10.28). Then
$$\pi'^* T = [p^* D] + p^* R.$$

Write $(Z_\bullet, g)$ and $(Z'_\bullet, g')$ for the liftings of $Y_\bullet$ to Y and Y' respective. We need to prove that
$$\Delta_{Z_\bullet}([R])g^{-1} + \nu_{Z_\bullet}([D])g^{-1} = \Delta_{Z'_\bullet}([p^*R])g'^{-1} + \nu_{Z'_\bullet}([p^*D])g'^{-1}.$$

This follows Theorem 10.3.1, Proposition 10.1.4 and Proposition 10.1.3. □

Note that from the above proof, we could describe the bimeromorphic behaviour of $\Delta_{Y_\bullet}(T)$ as follows:

lma:liftOkounana **Lemma 10.3.2** *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current with analytic singularities. Let $\pi\colon Y \longrightarrow X$ be a proper bimeromorphic morphism and $(W_\bullet, g)$ be the lifting of $Y_\bullet$ to Y. Then*
$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T)g.$$

lma:Okounkovanalycomp **Lemma 10.3.3** *Assume that $T, S \in \mathcal{Z}_+(X, \alpha)$ are two Kähler currents with analytic singularities and $T \preceq S$, then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Moreover,

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!}\int_X T^n. \tag{10.29}$$

{eq:volpobanaly}

Proof We first show that

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S).$$

Using Lemma 10.3.2, we may assume that T and S have log singularities along effective $\mathbb{Q}$-divisors E and F respectively. By assumption, $E \geq F$. Replacing T and S by $T - [F]$ and $S - [F]$ respectively, we may assume that $F = 0$.

In this case, we need to show that

$$\Delta_{Y_\bullet}(\alpha) \supseteq \Delta_{Y_\bullet}(\alpha - [E]) + \nu_{Y_\bullet}([E]),$$

which is obvious.

Next we prove that

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

By Lemma 10.3.2 and Theorem 10.3.1 again, we may assume that T has log singularities. We take D and β as in (10.28). We need to show that

$$\Delta_{Y_\bullet}(\alpha - [D]) + \nu_{Y_\bullet}([D]) \subseteq \Delta_{Y_\bullet}(\alpha),$$

which is again obvious.

Finally, (10.29) follows immediately from Theorem 10.3.1. □

10.3.2.2 The case of Kähler currents

def:POBKahcurr

Definition 10.3.3 Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. Then we define

$$\Delta_{Y_\bullet}(T) \coloneqq \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T_j).$$

Lemma 10.3.4 *The convex body $\Delta_{Y_\bullet}(T)$ in Definition 10.3.3 is independent of the choices of the T_j's.*

In particular, if T also has analytic singularities, then the $\Delta_{Y_\bullet}(T)$'s defined in Definition 10.3.3 and in Definition 10.3.2 coincide.

Proof Let $(S_j)_j$ be another quasi-equisingular approximation of T in $\mathcal{Z}_+(X, \alpha)$. By Proposition 1.6.3, for any small rational $\epsilon > 0$, $j > 0$, we can find $k > 0$ so that

$$S_k \preceq (1 - \epsilon) T_j.$$

It is more convenient to use the language of θ-psh functions at this point. Let ψ_k (resp. φ_k) denote the potentials in $\mathrm{PSH}(X,\theta)$ corresponding to S_k (resp. T_k) for each $k \geq 1$. Note that ψ_k and φ_k are unique up to additive constants.

By Lemma 10.3.3,

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \Delta_{Y_\bullet}(\theta, (1-\epsilon)\varphi_j).$$

On the other hand, observe that

$$\bigcap_{\epsilon \in \mathbb{Q}_{>0} \text{ small enough}} \Delta_{Y_\bullet}(\theta, (1-\epsilon)\varphi_j) = \Delta_{Y_\bullet}(\theta, \varphi_j).$$

In fact, the $\supseteq$ direction follows from Lemma 10.3.3, so it suffices to show that the two sides have the same volume, which follows from (10.29).

It follows that

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\theta, \varphi_j).$$

The other inclusion follows by symmetry. □

The same argument shows that

cor:Kahlercurrentcase

Corollary 10.3.1 *Suppose that $T, S \in \mathcal{Z}_+(X,\alpha)$ are two Kähler currents satisfying $T \preceq_{\mathcal{I}} S$. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proposition 10.3.1 *Let $T \in \mathcal{Z}_+(X,\alpha)$ be a Kähler current. Then*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T. \tag{10.30}$$ {eq:volOkocur}

Proof Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X,\alpha)$. Note that $\Delta_{Y_\bullet}(T_j)$ is decreasing in j, as follows from Lemma 10.3.3. Our assertion follows from (10.29) and Theorem 6.2.5. □

lma:Okomonotone

Lemma 10.3.5 *Let $T \in \mathcal{Z}_+(X,\alpha)$ be a Kähler current and ω be a Kähler form on X. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T+\omega). \tag{10.31}$$ {eq:DeltaTincreaseomegatemp1}

Moreover,

$$\Delta_{Y_\bullet}(T) = \bigcap_{\epsilon>0} \Delta_{Y_\bullet}(T+\epsilon\omega). \tag{10.32}$$ {eq:DeltaTincreaseomegatemp2}

Proof We first prove (10.31). Taking quasi-equisingular approximations, we reduce immediately to the case where T has analytic singularities. By Lemma 10.3.2, we may assume that T has log singularities. Take D and R as in (10.28). By definition again, it suffices to show that

$$\Delta_{Y_\bullet}([\beta]) \subseteq \Delta_{Y_\bullet}([\beta+\omega]),$$

which is clear by definition.

Next we prove (10.32). Thanks to (10.31), it remains to prove that both sides have the same volume:

$$\lim_{\epsilon\to 0+} \operatorname{vol}(T+\epsilon\omega) = \operatorname{vol} T.$$

This is proved in Proposition 7.2.3. □

10.3.2.3 The general case

def:generalPOB **Definition 10.3.4** Let $T \in \mathcal{Z}_+(X,\alpha)$. Take a Kähler form ω on X, we define

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega). \tag{10.33}$$ {eq:DeltaTgeneral}

This definition is clearly independent of the choice of ω by Lemma 10.3.5. Moreover, it extends Definition 10.3.3 and Definition 10.3.2 as a result of Lemma 10.3.5.

The main properties of $\Delta_{Y_\bullet}(T)$ are summarized as follows:

thm:pobmain **Theorem 10.3.2** *The convex bodies $\Delta_{Y_\bullet}(T)$'s satisfies the following properties:*

(1) *Suppose that $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, We have*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T; \tag{10.34}$$ {eq:volpobgeneral}

(2) *For $T, S \in \mathcal{Z}_+(X,\alpha)$ satisfying $T \preceq_{\mathcal{I}} S$, we have*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha);$$

(3) *For any current $T \in \mathcal{Z}_+(X,\alpha)$ with minimal singularities, we have*

$$\Delta_{Y_\bullet}(T) = \Delta_{Y_\bullet}(\alpha);$$

(4) *The map $\mathcal{Z}_+(X,\alpha)_{>0} \to \mathcal{K}_n$ given by $T \mapsto \Delta_{Y_\bullet}(T)$ is continuous, where we endow the d_S-pseudometric on $\mathcal{Z}_+(X,\alpha)_{>0}$ and the Hausdorff topology on $\mathcal{K}_n$;*

(5) *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that the lifting $(W_\bullet, g)$ of $Y_\bullet$ to Y exists, then for any $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, we have*

$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T) g;$$

(6) *For $T, S \in \mathcal{Z}_+(X,\alpha)$, we have*

$$\Delta_{Y_\bullet}(T) + \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(T+S). \tag{10.35}$$ {eq:pobadditiv}

Proof (1) By (10.33) and (10.30), for any Kähler form ω on X,

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \lim_{j\to\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega) = \frac{1}{n!} \lim_{j\to\infty} \operatorname{vol}(T + j^{-1}\omega).$$

The right-hand side is computed in Proposition 7.2.3. Hence, (10.34) follows.

(2) Fix a Kähler form ω on X. By Corollary 10.3.1, for each $j \geq 1$,

$$\Delta_{Y_\bullet}(T + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(S + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

It remains to show that

$$\Delta_{Y_\bullet}(\alpha) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

The $\subseteq$ direction is clear. Comparing the volumes using Theorem 10.3.1, we conclude that equality holds.

(3) This follows from (1) and (2).

(4) Let $(T_j)_j$ be a sequence in $\mathcal{Z}_+(X,\alpha)_{>0}$ converging to $T \in \mathcal{Z}_+(X,\alpha)_{>0}$ with respect to d_S. We want to show that $\Delta_{Y_\bullet}(T_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(T)$. By Proposition 6.2.3 and (2), we may assume that the singularity type of T_j is either increasing or decreasing. In both cases, the continuity follows from (1).

(5) We may assume that T is $\mathcal{I}$-good. It follows from (4) and Theorem 7.1.1 that we could reduce to the case where T has analytic singularities. Our assertion follows from Lemma 10.3.2.

(6) By (10.33), in order to prove (10.35), we may assume that T and S are both Kähler currents. Take quasi-equisingular approximations $(T_j)_j$ and $(S_j)_j$ of T and S respectively. By Theorem 6.2.2, $T_j + S_j \xrightarrow{d_S} T + S$. By (4), we may therefore assume that T and S have analytic singularities. Replacing X by a suitable modification, we may assume that T and S both have log singularities, say

$$T = [D] + R, \quad S = [D'] + R',$$

where D and D' are $\mathbb{Q}$-divisors on X and β and β' are closed positive $(1,1)$-currents with bounded potentials. We need to show that

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) + \nu_{Y_\bullet}([D]) + \nu_{Y_\bullet}([D']) \subseteq \Delta_{Y_\bullet}([R+R']) + \nu_{Y_\bullet}([D+D']).$$

By Proposition 10.1.2, this is equivalent to

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) \subseteq \Delta_{Y_\bullet}([R+R']),$$

which is already proved in Theorem 10.3.1. □

Corollary 10.3.2 *Assume that L is a big line bundle on X and h is a plurisubharmonic metric on L with positive volume. Then*

$$\Delta_{Y_\bullet}(\mathrm{dd^c} h) = \Delta_{Y_\bullet}(L,h). \tag{10.36}$$

{eq:tranOkounandalgOkoun}

Similarly, the definition (10.19) is compatible with the definition in Definition 10.3.4.

Proof We may assume that $\mathrm{dd^c}h$ has positive mass and is $\mathcal{I}$-good. By the d_S-continuity of both sides of (10.36) as proved in Theorem 10.3.2 and Theorem 10.2.2, together with Theorem 7.1.1, we may assume that $\mathrm{dd^c}h$ has analytic singularities.

In this case, using the birational invariance of both sides of (10.36) as proved in Proposition 10.2.9 and Theorem 10.3.2, we may assume that $\mathrm{dd^c}h$ has log singularities. Finally, after all these reductions, the equality (10.36) holds by construction. □

10.3.3 The valuative characterization

In this section, we will characterize the partial Okounkov bodies using valuations of currents.

lma:Kahlerclassokounrest **Lemma 10.3.6** *Let β be a nef class on X. Then*

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1}). \tag{10.37}$$ {eq:Deltaresttox10}

Proof **Step 1**. We first reduce to the case where β is a Kähler class.

Take a Kähler class α on X. It follows from the volume formula in Theorem 10.3.1 that

$$\Delta_{Y_\bullet}(\beta) = \bigcap_{\epsilon>0} \Delta_{Y_\bullet}(\beta + \epsilon\alpha), \quad \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1}) = \bigcap_{\epsilon>0} \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1} + \epsilon\alpha|_{Y_1}).$$

So it suffices to prove (10.37) with $\beta + \epsilon\alpha$ in place of β.

Step 2. Assume that α is a Kähler class. The $\supseteq$ direction in (10.37) follows from the extension theorem Theorem 1.6.3. To prove the other direction, recall that by Theorem 10.3.1, for $t > 0$ small enough, we have

$$\left\{y \in \mathbb{R}^{n-1} : (t, y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left((\beta - t[Y_1])|_{Y_1}\right).$$

As $t \to 0+$, the right-hand side converges to $\Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1})$ with respect to the Hausdorff metric as a consequence of Theorem 10.3.1, while the left-hand side converges to

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(\beta)\right\}$$

by Lemma C.1.2. We conclude our assertion. □

lma:slicepob **Lemma 10.3.7** *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Assume that $\nu(T, Y_1) = 0$, then*

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(T)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right). \tag{10.38}$$ {eq:Deltaslice}

Note that $\Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right)$ is independent of the choice of the representative $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$.

Proof **Step 1**. We first handle the case where T has analytic singularities. Let $\pi\colon Z \to X$ be a modification such that

(1) $Y_\bullet$ admits a lifting $(W_\bullet, g)$, and
(2) $\pi^*T = [D] + R$, where D is an effective $\mathbb{Q}$-divisor on Z and R is closed positive $(1,1)$-current with bounded potential.

This is possible by Theorem 1.6.1 and Theorem 10.1.1.

By Lemma 8.2.1,

$$\Pi^* \operatorname{Tr}_{Y_1}(T) \sim_P \operatorname{Tr}_{W_1}(\pi^*T),$$

where $\Pi\colon W_1 \longrightarrow Y_1$ is the restriction of π. It follows from Theorem 10.3.2 that

$$\begin{aligned}\Delta_{W_1\supseteq\cdots\supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^*T)) =&\Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\operatorname{Tr}_{Y_1}(T))\operatorname{cor}(Y_1\supseteq\cdots\supseteq Y_n,\Pi),\\ \Delta_{W_\bullet}(\pi^*T) =&\Delta_{Y_\bullet}(T)g.\end{aligned}$$

Taking (10.3) into account, we find that it suffices to show that

$$\left\{y\in\mathbb{R}^{n-1} : (0,y)\in\Delta_{W_\bullet}(\pi^*T)\right\} = \Delta_{W_1\supseteq\cdots\supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^*T)).$$

We may assume that π is the identity map. Then we have

$$T = [D] + R,\quad T|_{Y_1} = [D]|_{Y_1} + R|_{Y_1}.$$

Note that $[D]|_{Y_1}$ is the current of integration along an effective $\mathbb{Q}$-divisor on Y_1.

In particular,

$$\begin{aligned}\Delta_{Y_\bullet}(T) =&\Delta_{Y_\bullet}([R]) + \nu_{Y_\bullet}([D]),\\ \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(T|_{Y_1}) =&\Delta_{Y_1\supseteq\cdots\supseteq Y_n}([R]|_{Y_1}) + \nu_{Y_1\supseteq\cdots\supseteq Y_n}([D]|_{Y_1}).\end{aligned}$$

So it suffices to show that

$$\left\{y\in\mathbb{R}^{n-1} : (0,y)\in\Delta_{Y_\bullet}([R])\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}([R]|_{Y_1}),$$

which is exactly Lemma 10.3.6.

Step 2. Next we consider the case where T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X,\alpha)$. From Step 1, we know that for large $j\geq 1$,

$$\left\{y\in\mathbb{R}^{n-1} : (0,y)\in\Delta_{Y_\bullet}(T_j)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\operatorname{Tr}_{Y_1}(T_j)).$$

Letting $j\longrightarrow\infty$ and applying Theorem 10.3.2 and Proposition 8.2.2, we conclude (10.38). □

thm:KahcurrminOkoun **Theorem 10.3.3** *Assume that $T\in\mathcal{Z}_+(X,\alpha)_{>0}$ is a Kähler current. We have*

$$\min_{\text{lex}}\Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.39}$$ {eq:minOkounkov}

Here the minimum is with respect to the lexicographic order.

Proof We make induction on $n\geq 0$. The case $n=0$ is of course trivial. Let us assume that $n>0$ and the case $n-1$ has been proved.

We first observe that by Theorem 10.3.2,

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \ldots, 0) \subseteq \Delta_{Y_\bullet}(T).$$

Comparing the volumes of both sides using Theorem 10.3.2 and Proposition 7.2.3, we find that equality holds:

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \ldots, 0) = \Delta_{Y_\bullet}(T).$$

Replacing T by $T - \nu(T, Y_1)[Y_1]$, we may therefore assume that $\nu(T, Y_1) = 0$. It suffices to apply Lemma 10.3.7 and the inductive hypothesis. □

cor:valuationcurrentinPOB **Corollary 10.3.3** *For any* $T \in \mathcal{Z}_+(X, \alpha)$,

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proof When T is a Kähler current, this follows from Theorem 10.3.3.

In general, by definition, $\nu_{Y_\bullet}(T) = \nu_{Y_\bullet}(T + \omega)$ for any Kähler form ω on X. It follows that

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T + \omega)$$

for any Kähler form ω. It follows that $\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T)$. □

thm:Deltapartialint **Theorem 10.3.4** *For any* $T \in \mathcal{Z}_+(X, \alpha)_{>0}$,

$$\Delta_{Y_\bullet}(T) = \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \le_{\mathcal{I}} T\right\}}. \tag{10.40}$$ {eq:DeltaTequalallval}

In particular,

$$\Delta_{Y_\bullet}(\alpha) = \overline{\left\{\nu_{Y_\bullet}(T) : T \in \mathcal{Z}_+(X, \alpha)\right\}}.$$

We expect that the closure operation is not necessary.

Proof The $\supseteq$ direction in (10.40) follows from Corollary 10.3.3 and Theorem 10.3.2(2).

Let us write

$$D_{Y_\bullet}(T) = \left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \le_{\mathcal{I}} T\right\}$$

for the time being.

Step 1. Assume that T has analytic singularities. We have

$$\begin{aligned}\Delta_{Y_\bullet}(T) \supseteq & \overline{D_{Y_\bullet}(T)} \\ \supseteq & \overline{\left\{\nu_{Y_\bullet}(S) : \mathcal{Z}_+(X, \alpha) \ni S \text{ has gentle analytic singularities}, S \le T\right\}}.\end{aligned}$$

It follows easily from Theorem 10.3.1 that the volume of the right-hand side is equal to the volume of $\Delta_{Y_\bullet}(T)$, so (10.40) holds.

Step 2. Assume that T is a Kähler current. Take a quasi-equisingular approximation $T_j \in \mathcal{Z}_+(X, \alpha)$ of T. Next we use the language of psh functions. Let $\varphi_j, \varphi \in \mathrm{PSH}(X, \theta)$ be the potentials corresponding to T_j, T for each $j \ge 1$.

Fix an integer $N > 0$. For large enough $j \geq 1$, we can find $\psi \in \mathrm{PSH}(X, \theta)_{>0}$ such that

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1 - N^{-1})\varphi_j + N^{-1}\psi_j.$$

The existence of ψ_j follows from Lemma 2.3.1. It follows that

$$\begin{aligned} D_{Y_\bullet}(T) \supseteq & D_{Y_\bullet}\left(\theta + \mathrm{dd^c}\left((1 - N^{-1})\varphi_j + N^{-1}\psi_j\right)\right) \\ \supseteq & (1 - N^{-1}) D_{Y_\bullet}(T_j) + N^{-1} D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j). \end{aligned}$$

By Theorem C.1.1, up to replacing T_j by a subsequence, we may guarantee that $\overline{D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j)}$ admits a Hausdorff limit contained in $\Delta_{Y_\bullet}(\alpha)$ as $j \to \infty$. Let $j \to \infty$ and $N \to \infty$ then it follows that

$$\overline{D_{Y_\bullet}(T)} \supseteq \bigcap_{j=1}^{\infty} D_{Y_\bullet}(T_j).$$

By Lemma C.1.3,

$$\overline{D_{Y_\bullet}(T)} \supseteq \overline{\bigcap_{j=1}^{\infty} D_{Y_\bullet}(T_j)} = \bigcap_{j=1}^{\infty} \overline{D_{Y_\bullet}(T_j)}.$$

Therefore, by Step 1, we conclude that

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^{\infty} \overline{\Delta_{Y_\bullet}(T_j)} = \bigcap_{j=1}^{\infty} \overline{D_{Y_\bullet}(T_j)} \subseteq \overline{D_{Y_\bullet}(T)}.$$

The reverse direction is already known.

Step 3. Finally, consider the general case. Take a Kähler current $T' \in \mathcal{Z}_+(X, \alpha)$ more singular than T. For each $\epsilon \in (0, 1)$. The existence of T' is proved in Lemma 2.3.2. We know that

$$\Delta_{Y_\bullet}((1 - \epsilon)T + \epsilon T') = \overline{D_{Y_\bullet}((1 - \epsilon)T + \epsilon T')} \subseteq \overline{D_{Y_\bullet}(T)}.$$

Letting $\epsilon \to 0+$ and using Proposition 7.2.3, we find that

$$\Delta_{Y_\bullet}(T) \subseteq \overline{D_{Y_\bullet}(T)}.$$

As the other inclusion is already known, we conclude. □

cor:KahcurrminOkoun **Corollary 10.3.4** *Assume that $T \in \mathcal{Z}_+(X, \alpha)_{>0}$. We have*

$$\min_{\mathrm{lex}} \Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.41}$$ {eq:minOkounkov3}

Proof By Theorem 10.3.4, it is clear that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \leq_{\text{lex}} \nu_{Y_\bullet}(T).$$

On the other hand, we clearly have

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T+\omega)$$

for any Kähler form ω on X. It follows that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \geq_{\text{lex}} \min_{\text{lex}} \Delta_{Y_\bullet}(T+\omega).$$

By Theorem 10.3.3, the right-hand side is just $\nu_{Y_\bullet}(T+\omega) = \nu_{Y_\bullet}(T)$. We conclude the proof. □

10.4 Okounkov test curves

Let $\Delta \subseteq \mathbb{R}^n$ be a convex body with positive volume.

def:Otc **Definition 10.4.1** An *Okounkov test curve* relative to Δ consists of

(1) a number $\Delta_{\max} \in \mathbb{R}$ and
(2) an assignment $(-\infty, \Delta_{\max}) \ni \tau \mapsto \Delta_\tau \in \mathcal{K}_n$ satisfying

  a. the assignment $\tau \mapsto \Delta_\tau$ is a decreasing and concave;
  b. the convex bodies Δ_τ converge to Δ as $\tau \to -\infty$ with respect to the Hausdorff metric.

The set of Okounkov test curves relative to Δ is denoted by $\mathrm{TC}(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ is *bounded* if $\Delta_\tau = \Delta$ when Δ is small enough. The subset of bounded Okounkov test curves is denoted by $\mathrm{TC}^\infty(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ is said to have *finite energy* if

$$\mathbf{E}(\Delta_\bullet) \coloneqq n!\Delta_{\max} \operatorname{vol}\Delta + n! \int_{-\infty}^{\Delta_{\max}} (\operatorname{vol}\Delta_\tau - \operatorname{vol}\Delta)\,\mathrm{d}\tau > -\infty.$$

The subset of Okounkov test curves with finite energy is denoted by $\mathrm{TC}^1(\Delta)$.

Here concavity refers to the concavity with respect to the Minkowski sum.

prop:Otccont **Proposition 10.4.1** *Any Okounkov test curve* $(\Delta_\tau)_{\tau<\Delta_{\max}}$ *relative to* Δ *is continuous in* τ. *Moreover,* $\operatorname{vol}\Delta_\tau > 0$ *for all* $\tau < \Delta_{\max}$.

Proof We first claim that $\operatorname{vol}\Delta_{\tau'} > 0$ for all $\tau' < \Delta_{\max}$. By Condition 2.b in Definition 10.4.1 and Theorem C.1.2, we know that $\operatorname{vol}\Delta_{\tau''} > 0$ when τ'' is small enough. Fix one such τ''. Any $\tau' < \tau^+$ can be written as a convex combination of τ^+ and τ'', thus $\Delta_{\tau'}$ has positive volume by the concavity.

Next we claim that $\operatorname{vol}\Delta_\tau$ is continuous for $\tau < \Delta_{\max}$. In fact, by the Minkowski inequality, we know that $\log\operatorname{vol}\Delta_\tau$ is concave for $\tau < \Delta_{\max}$. The continuity follows.

Next we show that

$$\Delta_\tau = \bigcap_{\tau'<\tau} \Delta_{\tau'}.$$

The $\supseteq$ direction is obvious. By the continuity of the volume, both sides have the same volume and the volume is positive, hence, they are actually equal.

Similarly, we have

$$\Delta_\tau = \overline{\bigcup_{\tau'>\tau} \Delta_{\tau'}}.$$

The continuity of Δ_τ at $\tau < \Delta_{\max}$ is proved. □

def:tf **Definition 10.4.2** A *test function* on Δ is a function $F\colon \Delta \to [-\infty, \infty)$ such that

(1) F is concave,
(2) F is finite on $\operatorname{Int}\Delta$, and
(3) F is upper semicontinuous.

A test function F is *bounded* if F is bounded from below.

A test function F has *finite energy* if

$$\mathbf{E}(F) \coloneqq n! \int_\Delta F \,\mathrm{d}\lambda > -\infty. \tag{10.42}$$ {eq:EF}

def:LegOkoun **Definition 10.4.3** Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We define its *Legendre transform* as

$$G[\Delta_\bullet]\colon \Delta \to [-\infty, \infty), \quad a \mapsto \sup\left\{\tau < \Delta_{\max} : a \in \Delta_\tau\right\}.$$

Given a test function $F\colon \Delta \to [-\infty, \infty)$, we define its inverse Legendre transform $\Delta[F]_\bullet$ as the Okounkov test curve relative to Δ defined as follows:

(1) $\Delta[F]_{\max} = \sup_\Delta F$, and
(2) For each $\tau < \sup_\Delta F$, we set

$$\Delta[F]_\tau = \{x \in \Delta : F \geq \tau\}.$$

lma:convbodyLegendre **Lemma 10.4.1** *Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. Then $G[\Delta_\bullet]$ defined in Definition 10.4.3 is a test function.*

Similar, if $F\colon \Delta \to [-\infty, \infty)$ is a test function, then $\Delta[F]_\bullet$ is an Okounkov test curve.

Proof First suppose that $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We want to verify that $G[\Delta_\bullet]$ satisfies the conditions in Definition 10.4.2.

We first verify the concavity. Take $a, b \in \Delta$. We want to prove that for any $t \in (0, 1)$,

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b). \tag{10.43}$$ {eq:GDeltaconc}

There is nothing to prove if $G[\Delta_\bullet](a)$ or $G[\Delta_\bullet](b)$ is $-\infty$. So we assume that both are finite. Take $\epsilon > 0$, then $a \in \Delta_{G[\Delta_\bullet](a)-\epsilon}$ and $b \in \Delta_{G[\Delta_\bullet](b)-\epsilon}$. Thus,

$$ta + (1-t)b \in t\Delta_{G[\Delta_\bullet](a)-\epsilon} + (1-t)\Delta_{G[\Delta_\bullet](b)-\epsilon} \subseteq \Delta_{tG[\Delta_\bullet](a)+(1-t)G[\Delta_\bullet](b)-\epsilon}.$$

We deduce that

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b) - \epsilon.$$

Since $\epsilon > 0$, (10.43) follows.

It is clear that F is finite on the interior of Δ. So it remains to argue that F is upper semicontinuous.

Let $a_i \in \Delta$ with $a_i \to a \in \Delta$. Define $\tau_i = G[\Delta_\bullet](a_i)$. Let $\tau = \overline{\lim}_i \tau_i$. We need to show that

$$G[\Delta_\bullet](a) \geq \tau. \tag{10.44}$$

{eq:ainDelta1}

There is nothing to prove if $\tau = -\infty$. We assume that it is not this case. Up to subtracting a subsequence we may assume that $\tau_i \to \tau$. In particular, we can assume that $\tau_i \neq -\infty$ for all i. Fix $\epsilon > 0$, then $a_i \in \Delta_{\tau_i - \epsilon}$. Observe that $\Delta_{\tau_i-\epsilon} \xrightarrow{d_{\mathrm{Haus}}} \Delta_{\tau-\epsilon}$. By Theorem C.1.3 it follows that $a \in \Delta_{\tau-\epsilon}$. Thus,(10.44) follows since $\epsilon > 0$ is arbitrary.

Conversely, suppose that $F\colon \Delta \to [-\infty, \infty)$ is a test function. We argue that $\Delta[F]_\bullet$ is an Okounkov test curve. We verify the conditions in Definition 10.4.1.

Firstly, for each $\tau < \sup_\Delta F$, $\Delta[F](\tau)$ is a convex body as F is concave and usc. Moreover, $\Delta[F]_\tau$ is clearly decreasing in τ.

Secondly, for each $a \in \Delta$, we can write $a = \lim_i a_i$ with $a_i \in \operatorname{Int}\Delta$. By assumption, F is finite at a_i. Thus,

$$a \in \overline{\{F > -\infty\}} = \overline{\bigcup_\tau \Delta[F]_\tau}.$$

By Theorem C.1.3, $\Delta[F]_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta$ as $\tau \to -\infty$.

Thirdly, $\Delta[F]$ is concave. To see, take $\tau, \tau' < \tau^+$, we need to prove that for any $t \in (0,1)$,

$$\Delta[F]_{t\tau+(1-t)\tau'} \supseteq t\Delta[F]_\tau + (1-t)\Delta[F]_{\tau'}. \tag{10.45}$$

{eq:Deconc}

Let $a \in \Delta[F]_\tau$ and $b \in \Delta[F]_{\tau'}$. We have $F(a) \geq \tau$ and $F(b) \geq \tau'$. As F is concave, we have $F(ta + (1-t)b) \geq t\tau + (1-t)\tau'$. Thus,

$$ta + (1-t)b \in \Delta[F]_{t\tau+(1-t)\tau'}$$

and (10.45) follows. □

thm:Okotestcurve

Theorem 10.4.1 *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between* $\mathrm{TC}(\Delta)$ *and test functions on* Δ.

Under this bijection, $\mathrm{TC}^1(\Delta)$ *corresponds to test functions on* Δ *with finite energy and* $\mathrm{TC}^\infty(\Delta)$ *corresponds to bounded test functions.*

Proof Thanks to Lemma 10.4.1, in order to prove the first assertion, it only remains to see that the Legendre transform and the inverse Legendre transform are inverse to each other, which is immediate by definition.

It is obvious that $\mathrm{TC}^\infty(\Delta)$ corresponds to bounded test curves. Moreover, a direct computation shows that if $\Delta_\bullet \in \mathrm{TC}(\Delta)$, then

$$\mathbf{E}(\Delta_\bullet) = \mathbf{E}(G[\Delta_\bullet]),$$

concluding the $\mathrm{TC}^1(\Delta)$ case. □

The main source of Okounkov test curves is the following:

thm:Okountescurvex **Theorem 10.4.2** *Let θ be a closed smooth real $(1,1)$-form on X representing a big cohomology class α. Let $Y_\bullet$ be a smooth flag on X and $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. Then the map*

$$(-\infty, \Gamma_{\max}) \ni \tau \mapsto \Delta_{Y_\bullet}(\theta,\Gamma)_\tau := \Delta_{Y_\bullet}(\theta,\Gamma_\tau)$$

defines an Okounkov test curve.

Moreover, if $\Gamma \in \mathrm{TC}^1(X,\theta)$ (resp. $\mathrm{TC}^\infty(X,\theta)$), then $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\alpha))$ (resp. $\mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$).

Proof Consider $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. We need to verify that $\Delta_{Y_\bullet}(\theta,\Gamma)$ is an Okounkov test curve.

First observe that $\tau \mapsto \Gamma_\tau$ is concave and decreasing for $\tau < \Gamma_{\max}$. This is a direct consequence of Theorem 10.3.4.

Next we show that as $\tau \to -\infty$, we have

$$\Delta_{Y_\bullet}(\theta,\Gamma_\tau) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta,\Gamma_{-\infty})$$

as $\tau \to -\infty$.

It suffices to compute

$$\begin{aligned}\lim_{\tau\to-\infty} \operatorname{vol} \Delta_{Y_\bullet}(\theta,\Gamma_\tau) = \frac{1}{n!}\lim_{\tau\to-\infty} \operatorname{vol}(\theta + \mathrm{dd^c}\Gamma_\tau) &= \frac{1}{n!}\operatorname{vol}(\theta+\mathrm{dd^c}\Gamma_{-\infty})\\ &= \operatorname{vol}\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}),\end{aligned}$$

where we applied Theorem 10.3.2 and Theorem 6.2.5.

When $\Gamma \in \mathrm{TC}^\infty(X,\theta)$, it is clear that $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$.

When $\Gamma \in \mathrm{TC}^1(X,\theta)$, by Theorem 10.3.2, we have

$$\mathbf{E}(\Gamma) = \mathbf{E}(\Delta_{Y_\bullet}(\theta,\Gamma)).$$

So $\Gamma \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$. □

Definition 10.4.4 Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) := G[\Delta_\bullet]_*(\mathrm{d}\operatorname{vol}).$$

It is a Radon measure on $\mathbb{R}$.

In other words, $\mathrm{DH}(\Delta_\bullet)$ is the probability distribution of the random variable $G[\Delta_\bullet]$ on the measure space $(\Delta, \mathrm{d}\lambda)$.

For each $m \in \mathbb{N}$, the moments are given by

$$\int_{\mathbb{R}} x^m \, \mathrm{DH}(\Delta_\bullet)(x) = \int_\Delta G[\Delta_\bullet]^m \, \mathrm{d}\lambda = \Delta_{\max}^m \operatorname{vol}\Delta - \int_{-\infty}^{\Delta_{\max}} m\tau^{m-1} (\operatorname{vol}\Delta - \operatorname{vol}\Delta_\tau) \, \mathrm{d}\tau. \tag{10.46}$$

{eq:momentcalc}

lma:DHmconv

Lemma 10.4.2 *Suppose that* $(\Delta_\bullet^k)_k$ *is a decreasing sequence in* $\mathrm{TC}(\Delta)$. *Assume that the pointwise Hausdorff limit* $(\Delta_\tau)_{\tau < \inf_k \Delta_{\max}^k}$ *is still an Okounkov test curve relative to* Δ. *Then* $\mathrm{DH}(\Delta_\bullet^k) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ *as* $k \to \infty$.

Proof Observe that

$$G[\Delta_\bullet^k] \to G[\Delta_\bullet]$$

as $k \to \infty$. It follows from the dominated convergence theorem that $\mathrm{DH}(\Delta_\bullet^k) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ as $k \to \infty$. □

Chapter 11
The theory of b-divisors

chap:bdiv

11.1 The intersection theory of b-divisors

In this section, we briefly recall the intersection theory of Dang–Favre DF20 [DF22].

Let X be a connected smooth projective variety of dimension n.

Definition 11.1.1 A *birational model* of X is a projective birational morphism $\pi\colon Y \to X$ from a *smooth* variety Y. A morphism between two birational models $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$ is a morphism $Y \to Y'$ over X.

We write $\mathrm{Bir}(X)$ for the isomorphism classes of birational models of X. It is a directed set under the partial ordering of domination.

We will usually be sloppy by omitting π and say Y is a birational model of X.

We write $\mathrm{NS}^1(X)$ for the Néron–Severi group of X and $\mathrm{NS}^1(X)_K$ for $\mathrm{NS}^1(X)\otimes_{\mathbb{Z}} K$ for any subfield K of $\mathbb{R}$. Given $\alpha, \beta \in \mathrm{NS}^1(X)_K$, we write $\alpha \leq \beta$ if $\beta - \alpha$ is pseudo-effective.

Definition 11.1.2 A *Weil b-divisor* $\mathbb{D}$ on X is an assignment that associates with each $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ a class $\mathbb{D}_Y = \mathbb{D}_\pi \in \mathrm{NS}^1(Y)_{\mathbb{R}}$ such that when $\pi' : Y' \to X$ dominates π through $p : Y' \to Y$, we have

$$p_*\mathbb{D}_{Y'} = \mathbb{D}_Y .$$

The set of Weil b-divisors on X is denoted by $\mathrm{bWeil}(X)$.

A Weil b-divisor $\mathbb{D}$ on X is *Cartier* if there is $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ such that for any $(\pi' : Y' \to X) \in \mathrm{Bir}(X)$ which dominates π through $p : Y' \to Y$, we have

$$\mathbb{D}_{Y'} = p^*\mathbb{D}_Y .$$

In this case we say $\mathbb{D}$ is *determined* on Y or $\mathbb{D}$ has an *incarnation* $\mathbb{D}_Y$ on Y and write $\mathbb{D} = \mathbb{D}(\mathbb{D}_Y)$. We also say $\mathbb{D}$ is a Cartier b-divisor. The linear space of Cartier b-divisors is denoted by $\mathrm{bCart}(X)$.

Our definition simply means

$$\begin{aligned} \mathrm{bWeil}(X) &= \varprojlim_{(\pi\colon Y\to X)\in \mathrm{Bir}(X)} \mathrm{NS}^1(Y)_{\mathbb{R}},\\ \mathrm{bCart}(X) &= \varinjlim_{(\pi\colon Y\to X)\in \mathrm{Bir}(X)} \mathrm{NS}^1(Y)_{\mathbb{R}}, \end{aligned} \tag{11.1}$$

{eq:bdivprojlim}

in the category of vector spaces.

We endow $\mathrm{bWeil}(X)$ with the projective limit topology, then the first equation in (11.1) becomes a projective limit in the category of locally convex linear spaces. Clearly, $\mathrm{bCart}(X)$ is dense in $\mathrm{bWeil}(X)$.

def:nef **Definition 11.1.3** A Cartier b-divisor $\mathbb{D}$ on X is *nef* (resp. *big*) if some incarnation is (equivalently all incarnations are) nef (resp. *big*).

A Weil b-divisor $\mathbb{D}$ on X is *nef* if it lies in the closure of the set of nef Cartier b-divisors.

Write $\mathrm{bWeil}_{\mathrm{nef}}(X)$ for the set of nef Weil b-divisors on X.

A Weil b-divisor $\mathbb{D}$ on X is *pseudo-effective* if for all $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$, $\mathbb{D}_Y \geq 0$.

We introduce a partial ordering on $\mathrm{bWeil}(X)$:

$$\mathbb{D} \leq \mathbb{D}' \text{ if and only if } \mathbb{D}_Y \leq \mathbb{D}'_Y \text{ for all } (\pi\colon Y \to X) \in \mathrm{Bir}(X).$$

We summarise Dang–Favre's results:

thm:DF1 **Theorem 11.1.1 ([DF22, Theorem 2.1])** *Let $\mathbb{D} \in \mathrm{bWeil}(X)$ be a nef Weil b-divisor. Then there is a* decreasing *net $(\mathbb{D}_i)_{i\in I}$ of nef Cartier b-divisors such that*

$$\mathbb{D} = \lim_{i\in I} \mathbb{D}_i.$$

def:nefint **Definition 11.1.4** Let $\mathbb{D}_i \in \mathrm{bWeil}(X)$ $(i = 1, \ldots, n)$ be nef Cartier b-divisors on X. We define $(\mathbb{D}_1, \ldots, \mathbb{D}_n) \in \mathbb{R}$ as follows: take $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ such that all $\mathbb{D}_i's$ are determined on Y. Then define

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) := (\mathbb{D}_{1,Y}, \ldots, \mathbb{D}_{n,Y}). \tag{11.2}$$

The intersection number $(\mathbb{D}_1, \ldots, \mathbb{D}_n)$ does not depend on the choice of Y.

thm:DF2 **Theorem 11.1.2 ([DF22, Proposition 3.1,Theorem 3.2])** *There is a unique pairing*

$$(\mathrm{bWeil}_{\mathrm{nef}}(X))^n \to \mathbb{R}_{\geq 0}$$

extending the pairing in Definition 11.1.4 such that

(1) *The pairing is monotonically increasing in each variable.*
(2) *The pairing is continuous along decreasing nets in each variable.*

Moreover, this pairing has the following properties:

(1) *It is symmetric, multilinear.*
(2) *It is usc in each variable.*

Definition 11.1.5 We define the *volume* of $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$ by

$$\operatorname{vol} \mathbb{D} = (\mathbb{D}, \dots, \mathbb{D}). \tag{11.3}$$

{eq:volbdivdef}

We say $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$ is *big* if $\operatorname{vol} \mathbb{D} > 0$.

Note that the definition of bigness is compatible with the definition in Definition 11.1.3 in the case of Cartier b-divisors.

lma:volbdivaslim

Lemma 11.1.1 *Let* $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$*, then*

$$\operatorname{vol} \mathbb{D} = \inf_{(Y \to X) \in \mathrm{Bir}(X)} \operatorname{vol} \mathbb{D}_Y = \lim_{(Y \to X) \in \mathrm{Bir}(X)} \operatorname{vol} \mathbb{D}_Y .$$

Proof By Theorem 11.1.1, we can find a decreasing net $\mathbb{D}^\alpha$ of nef Cartier b-divisors on X converging to $\mathbb{D}$. Clearly,

$$\operatorname{vol} \mathbb{D}^\alpha = \inf_{Y \to X} \operatorname{vol} \mathbb{D}^\alpha_Y .$$

It follows from Theorem 11.1.2 and the continuity of the volume functional ELMNP05 [ELM+05, Corollary 2.6] that

$$\operatorname{vol} \mathbb{D} = \inf_\alpha \inf_{Y \to X} \operatorname{vol} \mathbb{D}^\alpha_Y = \inf_{Y \to X} \operatorname{vol} \mathbb{D}_Y .$$

On the other hand, as in general push-forward will increase the volume, we see that $\operatorname{vol} \mathbb{D}_Y$ is decreasing in Y, so we conclude. □

11.2 The singularity b-divisors

sec:bdiv1

Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension n. Let $\alpha \in \mathrm{NS}^1(X)_\mathbb{R}$ be a big class and T be a closed positive $(1,1)$-current in α.

Fix a closed real smooth $(1,1)$-form θ in $c_1(L)$ and we can write $T = \theta_\varphi$ for some $\varphi \in \mathrm{PSH}(X, \theta)$.

Definition 11.2.1 Define the *singularity divisor* $\mathrm{Sing}_X T$ of T as the formal sum

$$\mathrm{Sing}_X T \coloneqq \sum_E \nu(T, E) E, \tag{11.4}$$

{eq:singhatL}

where E runs over all prime divisors contained in X.

The singularity divisor is *not* a Weil divisor in general.

Note that this is a countable sum by Siu's semicontinuity theorem. Although $\mathrm{Sing}_X T$ is not a divisor in general, it does define a closed positive $(1,1)$-current due to

Siu's decomposition. Moreover, the numerical class $[\mathrm{Sing}_X T]$ in $\mathrm{NS}^1(X)_{\mathbb{R}}$ is also well-defined by treating the sum in (11.4) as a sum of numerical classes [BFJ09, Proposition 1.3].

def:singbdiv **Definition 11.2.2** The *singularity b-divisor* $\mathrm{Sing}\, T$ of T is the b-divisor over X defined by

$$(\mathrm{Sing}\, T)_Y \coloneqq [\mathrm{Sing}_Y \pi^* T],$$

where $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$.

Define

$$\mathbb{D}(T) \coloneqq \mathbb{D}(\alpha) - \mathrm{Sing}\, T.$$

Here $\mathbb{D}(\alpha)$ is the Cartier b-divisor determined by α on X.

We are ready to derive the first version of the Chern–Weil formula.

thm:nefbvolume **Theorem 11.2.1** *The b-divisor* $\mathbb{D}(T)$ *is a nef b-divisor and if in addition* $\mathrm{vol}\, T > 0$*,*

$$\mathrm{vol}\, \mathbb{D}(T) = \mathrm{vol}\, T. \tag{11.5}$$
{eq:volbandline}

Proof **Step 1**. We first handle the case where T has analytic singularities. After replacing X by a modification, we may assume that T has log singularities along an effective $\mathbb{Q}$-divisor D on X. Namely, we can write

$$T = [D] + R,$$

where R is a closed positive $(1,1)$-current with bounded potential. In this case, $\mathbb{D}(T) = \mathbb{D}(\alpha - D)$, which is nef. In order to prove (11.5), it suffices to show that

$$\int_X T^n = ((\alpha - D)^n), \tag{11.6}$$
{eq:temp14}

which is obvious.

Step 2. Assume that T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X,\theta)$. By Theorem 6.2.5, we have

$$\lim_{j\to\infty} \mathrm{vol}\, T_j = \mathrm{vol}\, T.$$

In view of Step 1 and Theorem 11.1.2, it remains to show that $\mathbb{D}(T_j) \to \mathbb{D}(T)$ as $j \to \infty$. In more concrete terms, this means that for any $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$,

$$[\mathrm{Sing}_Y(\pi^* T_j)] \to [\mathrm{Sing}_Y(\pi^* T)]$$

in $\mathrm{NS}^1(Y)_{\mathbb{R}}$. This obviously follows from Theorem 6.2.4 if $\mathrm{Sing}(\pi^* T)$ has only finitely many components. In general, fix an ample class ω in $\mathrm{NS}^1(Y)$. We want to show that for any $\epsilon > 0$, we can find $j_0 > 0$ so that when $j \geq j_0$,

$$[\mathrm{Sing}_Y(\pi^* T_j)] \geq [\mathrm{Sing}_Y(\pi^* T)] - \epsilon\omega. \tag{11.7}$$
{eq:temp55}

Write

$$[\mathrm{Sing}_Y(\pi^*T)] = \sum_{i=1}^{\infty} a_i E_i, \quad [\mathrm{Sing}(\pi^*T_j)] = \sum_{i=1}^{\infty} a_i^j E_i.$$

Then $a_i^j \leq a_i$. We can find $N > 0$ large enough, so that

$$[\mathrm{Sing}_Y(\pi^*T)] \leq \sum_{i=1}^{N} a_i E_i + \frac{\epsilon}{2}\omega.$$

By Theorem 6.2.4, we can take j_0 large enough so that for $j > j_0$,

$$(a_i - a_i^j)E_i \leq \frac{\epsilon}{2N}\omega, \quad i = 1, \dots, N.$$

Then (11.7) follows.

Step 3. Assume that $\mathrm{vol}\, T > 0$.

By Lemma 2.3.2, we can take a Kähler current $S \in \alpha$ such that $S \preceq T$. Consider $\epsilon S + (1-\epsilon)T$ for $\epsilon \in (0,1)$. When $\epsilon \to 0+$, we have $\epsilon S + (1-\epsilon)T \xrightarrow{d_S} T$. Using Theorem 6.2.5, we reduce immediately to the situation of Step 2.

Step 4. We handle the general case.

Take a Kähler form ω on X From Step 3, we know that for any $\epsilon > 0$, $\mathbb{D}(T) + \epsilon\mathbb{D}(\omega)$ is a nef b-divisor. It follows immediately that $\mathbb{D}(T)$ is nef. □

cor:Imodcharbdiv **Corollary 11.2.1** *Assume that* $\mathrm{vol}\, T > 0$, *then* T *is* $\mathcal{I}$*-good if and only if*

$$\mathrm{vol}\, \mathbb{D}(T) = \int_X T^n.$$

Proof This follows from Theorem 11.2.1 and Theorem 7.3.1. □

thm:pshbdivcont **Theorem 11.2.2** *The map* $\mathbb{D}\colon \mathrm{PSH}(X,\theta) \to \mathrm{bWeil}(X)$ *is continuous. Here on* $\mathrm{PSH}(X,\theta)$ *we take the* d_S*-pseudometric.*

Proof Let $\varphi_i \in \mathrm{PSH}(X,\theta)$ be a sequence converging to $\varphi \in \mathrm{PSH}(X,\theta)$ with respect to d_S. We want to show that

$$\mathbb{D}(\theta + \mathrm{dd^c}\varphi_i) \to \mathbb{D}(T).$$

As $\varphi_i \xrightarrow{d_S} \varphi$ implies that $\pi^*\varphi_i \xrightarrow{d_S} \pi^*\varphi$ for any $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$, it suffices to prove

$$[\mathrm{Sing}_X \varphi_i] \to [\mathrm{Sing}_X \varphi] \quad \text{in } \mathrm{NS}^1(X)_{\mathbb{R}}. \tag{11.8}$$ {eq:temp7}

Write

$$\mathrm{Sing}_X \varphi_i = \sum_E a_i^E E, \quad \mathrm{Sing}_X \varphi = \sum_E a^E E,$$

where E runs over all prime divisors on X. By Theorem 6.2.4, $a_i^E \to a^E$ as $i \to \infty$. When the number of E's is finite, (11.8) follows trivially. Otherwise, we write the prime divisors on X having positive coefficients in either $\mathrm{Sing}_X \varphi_i$ or $\mathrm{Sing}_X \varphi$ as $E_1, E_2, \dots$.

We fix a basis $e_1, \ldots, e_N$ of the finite-dimensional vector space $\mathrm{NS}^1(X)_{\mathbb{R}}$, so that the pseudo-effective cone is contained in the cone $\sum_d \mathbb{R}_{\geq 0} e_d$. Write

$$E_i = \sum_{d=1}^{N} f_i^d e_d, \quad i = 1, 2, \ldots.$$

Then we need to show that for any $d = 1, \ldots, N$,

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_i^{E_j} f_j^d = \sum_{j=1}^{\infty} a^{E_j} f_j^d.$$

This follows from the dominated convergence theorem, since

$$\sum_{j=1}^{\infty} a_i^{E_j} [E_j] \leq c_1(L), \quad \sum_{j=1}^{\infty} a^{E_j} [E_j] \leq c_1(L).$$

A mixed version of Theorem 11.2.1 is also true:

thm:nefbvolume2

Theorem 11.2.3 *Let $T_1, \ldots, T_n \in \mathcal{Z}_+(X)$ such that $\operatorname{vol} T_i > 0$ for each $i = 1, \ldots, n$. Then*

$$\frac{1}{n!} (\mathbb{D}(T_1), \ldots, \mathbb{D}(T_n)) \geq \frac{1}{n!} \int_X c_1(T_1) \wedge \cdots \wedge c_1(T_n). \tag{11.9}$$

{eq:bdivmixint}

If the T_i's are $\mathcal{I}$-good, then equality holds.

Proof This follows from Theorem 11.2.1 and Proposition 7.2.1. □

11.3 Okounkov bodies of b-divisors

sec:Okounkovbdiv

Let X be a connected projective manifold of dimension n and (L, h) be a Hermitian big line bundle on X.

Fix a smooth flag $Y_\bullet$ on X. Let $\nu = \nu_{Y_\bullet} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ be the valuation associated with $Y_\bullet$.

thm:pobbd

Theorem 11.3.1 *The partial Okounkov body $\Delta_{Y_\bullet}(L, h)$ admits the following expression:*

$$\Delta_{Y_\bullet}(L, h) = \nu_{Y_\bullet}(\mathrm{dd^c} h) + \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet} \left(c_1(\pi^* L) - [\mathrm{Sing}_Z(\pi^* h)]\right), \tag{11.10}$$

{eq:DeltaasHlim}

where π runs over the directed set of projective birational morphisms to X with Z normal.

Here the limit is a Hausdorff limit.

This theorem suggests that we define

$$\Delta_{Y_\bullet}(\mathbb{D}(\mathrm{dd^c} h)) \coloneqq \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet} \left(c_1(\pi^* L) - [\mathrm{Sing}_Z(\pi^* h)]\right).$$

Then one could rewrite (11.10) as

$$\Delta_{Y_\bullet}(L,h) = \Delta_{Y_\bullet}(\mathbb{D}(\mathrm{dd}^{\mathrm{c}}h)) + \nu_{Y_\bullet}(\mathrm{dd}^{\mathrm{c}}h).$$

lma:valuationT **Lemma 11.3.1** *Let T be a closed positive $(1,1)$-current on X. Then we have*

$$\lim_{\pi\colon Z\to X} \nu(\mathrm{Sing}_Z(\pi^*T)) = \nu(T), \tag{11.11}$$

{eq:nuTaslimit}

where π runs over the directed set of projective birational morphisms to X with Z normal.

Proof Given $\pi\colon Z \to X$, we let W_1 denote the strict transform of Y_1 in Z. The restriction $\pi_1\colon W_1 \to Y_1$ is necessarily birational. Let $\widetilde{W_1}$ be the normalization of W_1. Let $\widetilde{\pi_1}$ denote the normalization of π_1 so that we have a commutative diagram

$$\begin{tikzcd}
\widetilde{W_1} \arrow[r] \arrow[d, "\widetilde{\pi_1}"] & W_1 \arrow[r, hook] \arrow[d, "\pi_1"] & Z \arrow[d, "\pi"] \\
Y_1 \arrow[r, equal] & Y_1 \arrow[r, hook] & X.
\end{tikzcd}$$

We will argue by induction. The case $n = 0$ is trivial. Assume that $n > 0$ and the case $n-1$ is known.

We may clearly assume that $\nu(T, Y_1) = 0$. By definition, we have

$$\nu(T) = \left(0, \mu(\mathrm{Tr}_{Y_1}(T))\right),$$

where μ denotes the valuation induced by the flag $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$.

Observe that birational morphisms of the form $\pi_1\colon \widetilde{W_1} \to Y_1$ are cofinal in the directed set of projective birational morphisms of Y_1. This is obvious since the modifications given by compositions of blow-ups with smooth centers on Y_1 are cofinal. It suffices to blow-up X with the same centers.

Therefore, by the inductive hypothesis applied to $\mathrm{Tr}_{Y_1} T$, it suffices to argue that

$$\nu(\mathrm{Sing}_Z(\pi^*T)) = \left(0, \mu\left(\mathrm{Sing}_{\widetilde{W_1}} \widetilde{\pi_1}^*(\mathrm{Tr}_{Y_1}(T))\right)\right). \tag{11.12}$$

{eq:indstep}

From Lemma 8.2.1, we know that

$$\widetilde{\pi_1}^* \mathrm{Tr}_{Y_1}(T) \sim_P \mathrm{Tr}_{W_1}(\pi^*T).$$

So we only need to prove

$$\nu(\mathrm{Sing}_Z(\pi^*T)) = \left(0, \mu(\mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{W_1}(\pi^*T))\right),$$

This is reduced to the following statement:

$$\mathrm{Tr}_{W_1} \mathrm{Sing}_Z(\pi^*T) \sim_P \mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{W_1}(\pi^*T)). \tag{11.13}$$

{eq:nusingzpistarTtemp1}

In order to prove this, we may add a Kähler form to T and assume that T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T. Then $(\pi^* T_j)_j$ is a quasi-equisingular approximation of $\pi^* T$. Thanks to Proposition 8.2.2, we have

$$\mathrm{Tr}_{W_1}(\pi^* T_j) \xrightarrow{d_S} \mathrm{Tr}_{W_1}(\pi^* T)$$

Therefore, as in the proof of Theorem 11.2.2, we find that Sing_Z and $\mathrm{Sing}_{\overline{W_1}}$ are both continuous along this sequence as well. So we finally reduce to the case where T has analytic singularities.

In this case, arguing as before, we may assume replace π by a modification dominating it so that $\pi^* T \sim [D]$ for an effective $\mathbb{Q}$-divisor D on Z, in which case (11.13) is clear. □

Proof (The proof of Theorem 11.3.1) It would be more convenient to use the language of currents. We shall write $T = \mathrm{dd^c} h$.

Instead of arguing (11.10), we shall argue a slightly more general version: for any $\alpha \in \mathrm{NS}^1(X)_{\mathbb{R}}$, we have

$$\Delta_{Y_\bullet}(T) = \nu(T) + \lim_{\pi\colon Z \to X} \Delta_{Y_\bullet}(\alpha - [\mathrm{Sing}_Z(\pi^* T)]). \tag{11.14}$$

{eq:mainvar}

We argue by induction on n. The case $n = 0$ is of course trivial. Let us assume that $n > 0$ and the result is known in dimension $n - 1$.

We may replace T by $T - \nu(T, Y_1)[Y_1]$ and α by $\alpha - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$.

For any projective birational morphism $\pi\colon Z \to X$ with Z normal, it follows from Theorem 10.3.4 (which also holds for a normal variety, as can be seen after passing to a resolution) that we have

$$\Delta_{Y_\bullet}\left(\pi^*\alpha - [\mathrm{Sing}_Z(\pi^* T)]\right) = \overline{\left\{\nu(S) : S \in \pi^*\alpha - [\mathrm{Sing}_Z(\pi^* T)]\right\}}.$$

Therefore,

$$\Delta_{Y_\bullet}\left(\pi^*\alpha - [\mathrm{Sing}_Z(\pi^* T)]\right) + \nu(\mathrm{Sing}_Z(\pi^* T)) \subseteq \overline{\left\{\nu(S) : S \in \alpha, \pi^* S \geq \mathrm{Sing}_Z(\pi^* T)\right\}}.$$

We observe that the right-hand side is decreasing with respect to π, which together with Lemma 11.3.1 implies that the net of convex bodies $\Delta_{Y_\bullet}(c_1(\pi^* L) - [\mathrm{Sing}_Z(\pi^* T)])$ for various Z is uniformly bounded. Suppose that Δ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in c_1(L), S \preceq_{\mathcal{I}} T\}}.$$

As shown in Theorem 10.3.4, the right-hand side is exactly $\Delta_{Y_\bullet}(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_{Y_\bullet}(T).$$

But observe that both sides have the same volume, as computed in Theorem 10.3.2 and Theorem 11.2.1. So equality holds.

It follows from the Blaschke selection theorem Theorem C.1.1 that the limit in (11.14) exists and (11.14) holds. □

Part III
Applications

In this part, we explain a few applications of the theory developed in this book.

Chapter 12
Toric pluripotential theory on big line bundles

chap:toricbig

Let T be a complex torus of dimension n with character lattice M and cocharacter lattice N. Consider a rational polyhedral fan Σ in $N_{\mathbb{R}}$ corresponding to an n-dimensional smooth toric variety X.

Let D be a T-invariant big divisor on X. Then $P_D \subseteq M_{\mathbb{R}}$ be the lattice polytope generated by $u \in M$ such that

$$D + \operatorname{div} \chi^u \geq 0.$$

Let $L = \mathcal{O}_X(D)$.

We shall fix a smooth T_c-invariant metric h_0 on L. Let $\theta = c_1(L, h_0)$. Fix a smooth function $F_\theta \colon N_{\mathbb{R}} \to \mathbb{R}$ such that

$$\theta = \mathrm{dd^c} \operatorname{Trop}^* F_\theta.$$

Note that F_θ is well-defined up to a linear term.

12.1 Toric partial Okounkov bodies

12.1.1 Newton bodies

Let $\mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ be the set of T_c-invariant functions in $\mathrm{PSH}(X, \theta)$.

Definition 12.1.1 A function $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ can be written as

$$\varphi|_{T(\mathbb{C})} = \operatorname{Trop}^* f$$

for some unique $f \colon N_{\mathbb{R}} \to [-\infty, \infty)$. Then we define

$$F_\varphi \colon N_{\mathbb{R}} \to \mathbb{R}$$

as follows:

$$F_\varphi = F_\theta + f. \tag{12.1}$$

Observe that F_φ is a convex function and takes finite values by Lemma 5.1.1. It is well-defined up to a linear term.

Definition 12.1.2 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$, we define its *Newton body* as

$$\Delta(\theta, \varphi) := \overline{\nabla F_\varphi(N_\mathbb{R})} \subseteq M_\mathbb{R}.$$

Observe that $\Delta(\theta, \varphi)$ depends only on the current θ_φ, not on the choices of θ, F_θ and D.

prop:toricMAandrealMA2

Proposition 12.1.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$, then*

$$\mathrm{Trop}_* \left(\theta|_{T(\mathbb{C})} + \mathrm{dd}^\mathrm{c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_\mathbb{R}(F_\varphi). \tag{12.2}$$

{eq:tropMAmea2}

In particular,

$$\int_X \theta_\varphi^n = \int_{N_\mathbb{R}} \mathrm{MA}_\mathbb{R}(F_\varphi) = n!\,\mathrm{vol}\,\Delta(\theta, \varphi) \tag{12.3}$$

{eq:toricmass2}

and

$$\int_X \theta_{V_\theta}^n = n!\,\mathrm{vol}\,P. \tag{12.4}$$

{eq:toricminsingmass}

Proof Take F_0 as in (5.3) and ω denotes the corresponding Kähler form.

Then for any large enough $C > 0$, $\theta + C\omega$ is a Kähler form. So we conclude from Proposition 5.1.5 that

$$\mathrm{Trop}_* \left((\theta + C\omega)|_{T(\mathbb{C})} + \mathrm{dd}^\mathrm{c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_\mathbb{R}(F_\varphi + CF_0).$$

Since both sides are polynomials in C, we conclude that the same holds for $C = 0$. Therefore, (12.2) follows.

(12.3) is a direct consequence, while (12.4) follows from Theorem 12.2.2. □

12.1.2 Partial Okounkov bodies

subsec:pobtorgeneral

There are some canonical choices of smooth flags in the toric setting.

Recall that for each $\rho \in \Sigma(1)$, u_ρ denotes the ray generator of ρ. Since X is smooth and projective, we could choose $\rho_1, \ldots, \rho_n \in \Sigma(1)$ such that $u_{\rho_1}, \ldots, u_{\rho_n}$ form a basis of N. Define

$$Y_i = D_{\rho_1} \cap \cdots \cap D_{\rho_i}, \quad i = 1, \ldots, n.$$

Then $Y_\bullet$ is a smooth flag on X. Let

$$\Phi\colon M \to \mathbb{Z}^n, \quad m \mapsto \left(\langle m, u_{\rho_1}\rangle, \ldots, \langle m, u_{\rho_n}\rangle\right). \tag{12.5}$$

{eq:isoMZncanonical}

Then Φ is an isomorphism of Abelian groups. It induces an $\mathbb{R}$-linear isomorphism

$$\Phi_{\mathbb{R}}\colon M_{\mathbb{R}} \to \mathbb{R}^n.$$

prop:toricusualOko **Proposition 12.1.2** *We have*

$$\nu_{Y_\bullet}\left(H^0(X, L^k)^\times\right) = \Phi\left((kP_D) \cap M\right) \tag{12.6}$$

{eq:DeltakLtoric}

for any $k \in \mathbb{Z}_{>0}$. In particular,

$$\Delta_{Y_\bullet}(L) = \Phi_{\mathbb{R}}(P_D). \tag{12.7}$$

Proof It suffices to prove (12.6) for $k = 1$. Let $s \in H^0(X, L)$ be a non-zero section, say χ^u for some $u \in P_D \cap M$. The zero-locus of s is given by

$$D + \sum_{i=1}^n \langle u, u_{\rho_i}\rangle D_{\rho_i}.$$

Therefore,

$$\nu_{Y_\bullet}(s) = \left(\langle u, u_{\rho_1}\rangle, \dots, \langle u, u_{\rho_n}\rangle\right) = \Phi(u).$$

So (12.6) follows. □

thm:toricpob **Theorem 12.1.1** *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$, then*

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi)\right) = \Delta_{Y_\bullet}(\theta, \varphi). \tag{12.8}$$

{eq:toricOkounkovcomp}

The proof follows from a simple but tedious computation based on Example 7.3.1, we refer to [Xia21, Theorem 8.3].

Proof **Step 1**. We first reduce to the case where θ_φ is a Kähler current.

By Lemma 2.3.2, we can find $\psi \in \mathrm{PSH}(X, \theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. Taking the average along T_c, we may assume that ψ is T_c-invariant.

For each $t \in (0, 1)$, we let

$$\varphi_t = (1 - t)\psi + t\varphi.$$

Suppose that Kähler current case is known. Then we get

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi_t)\right) = \Delta_{Y_\bullet}(\theta, \varphi_t)$$

for any $t \in (0, 1)$. It follows from Theorem A.4.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi)\right) \supseteq \Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi_t)\right) \supseteq \Delta_{Y_\bullet}(\theta, \varphi_t)$$

for any $t \in (0, 1)$. Thanks to Theorem 10.2.2, we have

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \supseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Compare the volumes of both sides using Proposition 12.1.1 and (10.11), we find that

$$n!\operatorname{vol}\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) = \int_X \theta_\varphi^n = \operatorname{vol}\theta_\varphi = n!\operatorname{vol}\Delta_{Y_\bullet}(\theta,\varphi).$$

In particular, we conclude (12.8).

Step 2. We handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

We may assume that φ_j is T_c-invariant for each $j \geq 1$ from the construction of [Dem12a, Theorem 13.21].

Now assume that the result is known for each φ_j. Then

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_j)\right) = \Delta_{Y_\bullet}(\theta,\varphi_j).$$

In particular, by Proposition 12.1.1 again,

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi_j)$$

for each $j \geq 1$. It follows from Theorem 10.2.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Compare the volumes of both sides using Proposition 12.1.1, (10.11) and Theorem 5.2.1, we conclude (12.8).

Step 3. It remains to handle the case where φ has analytic singularities and θ_φ is a Kähler current. In fact, we may assume that φ has the form

$$\varphi = \log \sum_{i=1}^{a} |s_i|^2_{h_0} + O(1),$$

where $s_1,\dots,s_a \in H^0(X,L)$. This follows from the proof of Step 2 and the construction of [Dem12a, Theorem 13.21].

Let $u_1,\dots,u_a \in P_D \cap M$ be the lattice points corresponding to $s_1,\dots,s_a$. Observe that $\Delta(\theta,\varphi)$ is the convex envelope of $u_1,\dots,u_a$ by Lemma A.5.2.

Then for any $m \in M$ and $k \in \mathbb{Z}_{>0}$, $m \in kP_D$ if and only if

$$|\chi^m|^2_{h_0^k} \mathrm{e}^{-k\varphi}$$

is bounded from above. It follows that

$$\Phi\left(k\Delta(\theta,\varphi)\cap M\right) \subseteq k\Delta_k(\theta,\varphi).$$

The notation Δ_k is defined Section 10.2. Letting $k \to \infty$ and applying Theorem 10.2.4, we find that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta(\theta,\varphi).$$

Compare the volumes of both sides using Proposition 12.1.1 and (10.11), we conclude that the equality holds and (12.8) follows. □

As another consequence we have

cor:toricLelong

Corollary 12.1.1 *Let E be a T-invariant prime divisor on X corresponding to a ray with ray generator $n \in N$. Then for any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$, we have*

$$\nu(\varphi, E) = \inf \{\langle m, n\rangle : m \in \Delta(\theta, \varphi)\} .$$

Proof This follows immediately from Theorem 12.1.1 and Theorem 10.2.5. In fact, since X is projective and smooth, there is always a T-invariant smooth flag $Y_\bullet$ with $Y_1 = E$. □

cor:toricLelong2

Corollary 12.1.2 *For any T-invariant subvariety $Y \subseteq X$ and any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$ corresponding to a cone σ in Σ. Then the following are equivalent:*

(1) $\nu(\varphi, Y) = 0$;
(2) *There is a point $m \in \Delta(\theta, \varphi)$ such that $m \cdot u_\rho = 0$ for any 1-dimensional face ρ of σ.*

Proof This follows immediately from Corollary 12.1.1 after blowing-up Y. □

12.2 The pluripotential theory

thm:toricpshbig

Theorem 12.2.1 *There is a canonical bijection between the following sets:*

(1) *the set of $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$;*
(2) *the set of $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$ satisfying $F \leq F_{V_\theta}$, and*
(3) *the set of closed proper convex functions $G \in \mathrm{Conv}(M_{\mathbb{R}})$ satisfying*

$$G \geq F^*_{V_\theta}.$$

As before, we write F_φ, G_φ for the functions determined by this construction.

Proof The proof is similar to that of Theorem 5.1.1, but due to its importance, we give the proof. Again, the correspondence between (2) and (3) is proved in Proposition A.2.4.

Given φ, we can construct F_φ in (2) as explained earlier. Conversely, given $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$ such that $F \leq F_{V_\theta}$. Then

$$\mathrm{Trop}^*(F - F_\theta) \in \mathrm{PSH}(T(\mathbb{C}), \theta|_{T(\mathbb{C})}).$$

Since $F \leq F_{V_\theta}$, we see that $\mathrm{Trop}^*(F - F_\theta)$ is bounded from above. It follows that Grauert–Remmert's extension theorem Theorem 1.2.1 is applicable, and this function extends to a unique θ-psh function φ. The uniqueness of the extension guarantees that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$.

The two maps are clearly inverse to each other. □

We fix a model potential $\phi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$ with Newton body $\Delta(\theta,\phi)$.

A similar argument guarantees the folloiwng:

Corollary 12.2.1 *There is a canonical bijection between the following sets:*

(1) *the set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta;\phi)$,
(2) *the set of* $F \in \mathcal{P}(N_{\mathbb{R}}, \Delta(\theta,\phi))$ *satisfying* $F \leq F_{V_\theta}$, *and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G \geq F^*_{V_\theta}, \quad G|_{M_{\mathbb{R}}\setminus\Delta(\theta,\phi)} = \infty.$$

With an almost identical argument, we arrive at

prop:toricsubgeod **Proposition 12.2.1** *Let* $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$. *There is a canonical bijection between the following sets:*

(1) *the set of* T_c*-invariant subgeodesics from* φ_0 *to* φ_1,
(2) *the set of convex functions* $F\colon N_{\mathbb{R}} \times (0,1) \to \mathbb{R}$ *such that for each* $r \in (0,1)$, *the function*

$$F_r\colon N_{\mathbb{R}} \to \mathbb{R}, \quad n \mapsto F(n,r)$$

satisfies $F_r \to F_{\varphi_1}$ *(resp.* $F_r \to F_{\varphi_0}$*) everywhere as* $r \to 1-$ *(resp.* $r \to 0+$*), and*
(3) *the set of convex functions* Ψ *on* $M_{\mathbb{R}} \times \mathbb{R}$ *such that*

$$\Psi(m,s) \geq G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right).$$

Note that Ψ in (3) is nothing but the Legendre transform of F.

As an immediate corollary,

cor:toricgeodgeneral **Corollary 12.2.2** *Let* $\varphi_0, \varphi_1 \in \mathcal{E}_{\mathrm{tor}}(X,\theta)$. *Then the geodesic* $(\varphi_t)_{t\in(0,1)}$ *from* φ_0 *to* φ_1 *corresponds to the lower convex envelope Definition A.1.4 of the function*

$$N_{\mathbb{R}} \times [0,1] \to \mathbb{R}, \quad (n,t) \mapsto tF_{\varphi_1}(n) + (1-t)F_{\varphi_0}(n).$$

Moreover, we have

$$G_{\varphi_t} = (1-t)G_{\varphi_1} + tG_{\varphi_0}. \tag{12.9}$$ {eq:Glinear}

Proof The first assertion follows immediately from Proposition 12.2.1. It remains to argue (12.9).

Let $F\colon N_{\mathbb{R}} \times [0,1]$ be the map $(n,t) \mapsto F_{\varphi_t}(n)$.

It follows from the correspondence in Proposition 12.2.1 that the Legendre transform of F is given by $G_{\varphi_0} \vee (G_{\varphi_1} + s)$. From this we conclude that

$$G_{\varphi_t}(m) = -\sup_{s\in\mathbb{R}} \left(st - G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right)\right) = (1-t)G_{\varphi_1}(m) + tG_{\varphi_0}(m).$$

thm:FVtheta **Theorem 12.2.2** *We have*

$$F_{V_\theta} \in \mathcal{E}(N_{\mathbb{R}}, P_D).$$

Proof We will use the notations of Section 12.1.2. Take $\varphi = V_\theta$ in Theorem 12.1.1, we find

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, V_\theta)\right) = \Delta_{Y_\bullet}(\theta, V_\theta) = \Phi_{\mathbb{R}}(P_D),$$

where we applied Proposition 12.1.2 in the second equality. Therefore,

$$\Delta(\theta, V_\theta) = P_D.$$

The proofs of the following results are similar to the ample case studied in Chapter 5. We omit the details.

prop:toricpluscstbig **Proposition 12.2.2** *Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and $C \in \mathbb{R}$. We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

prop:toricrooftopbig **Proposition 12.2.3** *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$, then $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and*

$$F_{\varphi\wedge\psi} = F_\varphi \wedge F_\psi, \quad G_{\varphi\wedge\psi} = G_\varphi \vee G_\psi.$$

prop:toricseqbig **Proposition 12.2.4** *Let $\{\varphi_i\}_{i\in I}$ be a family in $\mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ uniformly bounded from above. Then $\sup^*_{i\in I} \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and*

$$F_{\sup^*_{i\in I}\varphi_i} = \sup_{i\in I} F_{\varphi_i}, \quad G_{\sup^*_{i\in I}\varphi_i} = \mathrm{cl} \bigwedge_{i\in I} G_{\varphi_i}.$$

Moreover, if I is finite, then

$$G_{\max_{i\in I}\varphi_i} = \bigwedge_{i\in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i\in I}$ is a decreasing net in $\mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ such that $\inf_{i\in I}\varphi_i \not\equiv -\infty$, then $\inf_{i\in I}\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and

$$F_{\inf_{i\in I}\varphi_i} = \inf_{i\in I} F_{\varphi_i}, \quad G_{\inf_{i\in I}\varphi_i} = \sup_{i\in I} G_{\varphi_i}.$$

prop:GPenvelopebig **Proposition 12.2.5** *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$. Then $P_\theta[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and*

$$G_{P_\theta[\varphi]}(x) = \begin{cases} G_{V_\theta}(x), & \text{if } x \in \overline{\{G_\varphi(x) < \infty\}}; \\ \infty, & \text{otherwise}. \end{cases} \tag{12.10}$$ {eq:toricPenvbig}

As a consequence, we have

Corollary 12.2.3 *Let $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) $\Delta(\theta, \varphi) = \Delta(\theta, \psi)$.

Next we consider the trace operator. For this purpose, we will need to fix a T-invariant subvariety $Y \subseteq X$. Since X is smooth, so is Y. Let σ be the cone in Σ corresponding to Y and Q be the face of P corresponding to Y.

Recall that the cocharacter lattice $N(\sigma)$ of Y is given by $N/N\cap\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the linear span of σ. See [CLS11, (3.2.6)]. In particular, the character lattice $M(\sigma)$ of Y can be naturally identified with the linear span of Q. Let $i_\sigma\colon M(\sigma)\to M$ be the corresponding inclusion.

Take $m_\sigma\in M$ so that Supp_{P_D} coincides with m_σ on σ.

prop:traceoptoric

Proposition 12.2.6 *Let $\varphi\in\mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$. Consider a T-invariant subvariety Y corresponding to a face Q of P. Suppose that $\nu(\varphi,Y)=0$ and $\mathrm{vol}(\theta|_Y,\mathrm{Tr}^\theta_Y(\varphi))>0$. Then*

$$\Delta(\theta|_Y,\mathrm{Tr}^\theta_Y(\varphi))=(i_\sigma+m_\sigma)^*_{\mathbb{R}}\left(\Delta(\theta,\varphi)\cap Q\right). \tag{12.11}$$

{eq:tracetoricNewton}

In particular, $\mathrm{Tr}_Y(\varphi)\sim_{\mathcal{P}}\varphi|_Y$ *if* $\varphi|_Y\not\equiv-\infty$.

Observe that the condition $\nu(\varphi,Y)=0$ means exactly that $\Delta(\theta,\varphi)\cap Q\neq\varnothing$ by Corollary 12.1.2.

Proof Perturbing θ slightly, we may assume that θ_φ is a Kähler current. Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$. It follows from the continuity of the partial Okounkov bodies Theorem 10.2.2 and the continuity of the trace operator Proposition 8.2.2 that it suffices to handle the case where φ has analytic singularities. We need to show that

$$\Delta(\theta|_Y,\varphi|_Y)=(i_\sigma+m_\sigma)^*_{\mathbb{R}}\left(\Delta(\theta,\varphi)\cap Q\right).$$

It is enough to observe that

$$G_{\varphi|_Y}=(i_\sigma+m_\sigma)^*_{\mathbb{R}}G_\varphi|_Q.$$

The argument is contained in [BGPS14, Proof of Proposition 4.8.9].

Finally, observe that if $\varphi|_Y\not\equiv-\infty$, the right-hand side of (12.11) is nothing but $\Delta(\theta|_Y,\varphi|_Y)$ using [BGPS14, Proof of Proposition 4.8.9]. So we conclude that $\varphi|_Y\sim_P\mathrm{Tr}_Y(\varphi)$. □

Chapter 13
Non-Archimedean pluripotential theory

chap:NAapp

13.1 The definition of non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n. Let $\mathrm{Käh}(X)$ be the set of Kähler forms on X with the partial order given as follows: we say $\omega \leq \omega'$ if $\omega \geq \omega'$. Note that the ordered set $\mathrm{Käh}(X)$ is a directed set.

Let θ be a closed smooth real $(1,1)$-form.

Definition 13.1.1 We define

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) = \varprojlim_{\omega \in \mathrm{Käh}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}$$

in the category of sets, where the transition maps are given as follows: suppose that $\omega, \omega' \in \mathrm{Käh}$ and $\omega \geq \omega'$, then the transition map is defined in Proposition 9.3.4:

$$P_{\theta+\omega'}[\bullet]_{\mathcal{I}} \colon \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega')_{>0} \longrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}. \tag{13.1}$$

{eq:PItransPSHNApositive}

In general, we denote the components of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$ by $P_{\theta+\omega'}[\Gamma]_{\mathcal{I}}$.

Remark 13.1.1 Thanks to Proposition 9.3.2, for any other θ' representing $[\theta]$, we have a canonical bijection

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(X,\theta').$$

Moreover, these bijections satisfy the cocycle condition. If we view the set of closed real smooth $(1,1)$-forms representing $[\theta]$ as a category with a unique morphism between any two objects, then we can define

$$\mathrm{PSH}^{\mathrm{NA}}(X,[\theta]) = \varprojlim_{\theta} \mathrm{PSH}^{\mathrm{NA}}(X,\theta).$$

This definition is independent of the choice of the explicit representative of the cohomology class $[\theta]$.

However, given the fact that our notations are already quite heavy, we decide to stick to the set $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. The readers should verify that all constructions below are independent of the choice of θ within its cohomology class.

prop:testcminftyPrela **Proposition 13.1.1** *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Then given $\omega, \omega' \in \mathrm{K\ddot{a}h}(X)$ with $\omega \leq \omega'$, we have*

$$P_{\theta+\omega}\left[P_{\theta+\omega'}[\Gamma]_{\mathcal{I},-\infty}\right] = P_{\theta+\omega}[\Gamma]_{\mathcal{I},-\infty}.$$

Proof Since $P_{\theta+\omega'}[\Gamma]_{\mathcal{I},-\infty}$ is $\mathcal{I}$-good by Example 7.1.2, it follows that

$$P_{\theta+\omega}\left[P_{\theta+\omega'}[\Gamma]_{\mathcal{I},-\infty}\right] = P_{\theta+\omega}\left[P_{\theta+\omega'}[\Gamma]_{\mathcal{I},-\infty}\right]_{\mathcal{I}}.$$

Our assertion follows from Proposition 3.2.12. □

prop:NAposNAemb **Proposition 13.1.2** *There is a natural injective map*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \hookrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta), \quad \Gamma \mapsto \left(P_{\theta+\omega}[\Gamma]_{\mathcal{I}}\right)_{\omega\in\mathrm{K\ddot{a}h}(X)}.$$

In the sequel, we will not distinguish an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ with its image in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

Proof It is obvious that this map is well-defined. It suffices to argue its injectivity. Suppose that $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} = P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$$

for some Kähler form ω on X. Then for any $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \sim_{\mathcal{I}} \Gamma'_\tau$$

by Proposition 6.1.3. It follows again from Proposition 6.1.3 that

$$\Gamma_\tau = \Gamma'_\tau.$$

Definition 13.1.2 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. We define $\Gamma_{\max}$ as $P_{\theta+\omega}[\Gamma]_{\mathcal{I},\max}$ for any Kähler form ω on X.

Note that under the identification of Proposition 13.1.2, for any $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, this definition is compatible with the notion of $\Gamma_{\max}$ in Definition 9.1.1.

Definition 13.1.3 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define its volume as follows:

$$\operatorname{vol}\Gamma := \lim_{\omega\in\mathrm{K\ddot{a}h}(X)} \int_X \left(\theta+\omega+\mathrm{dd^c} P_{\theta+\omega'}[\Gamma]_{\mathcal{I},-\infty}\right)^n \in [0,\infty).$$

Observe that the net is decreasing, so the limit exists.

Proposition 13.1.3 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. *Then*

$$\operatorname{vol}\Gamma = \int_X \left(\theta + \mathrm{dd^c}\Gamma_{-\infty}\right)^n .$$

Proof This follows from Proposition 3.1.8, Corollary 3.1.3 and Proposition 13.1.1. □

def:PSHNAtrangeneral

Definition 13.1.4 Let ω be a closed real smooth positive $(1,1)$-form on X. We define the map

$$P_{\theta+\omega}[\bullet]_{\mathcal{I}} : \mathrm{PSH}^{\mathrm{NA}}(X,\theta) \to \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$$

as follows: given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define $P_{\theta+\omega}[\Gamma]_{\mathcal{I}}$ as the element such that for any $\omega' \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega+\omega'}\left[P_{\theta+\omega}[\Gamma]_{\mathcal{I}}\right]_{\mathcal{I}} = P_{\theta+\omega+\omega'}[\Gamma]_{\mathcal{I}} .$$

It is straightforward to check that under the identification of Proposition 13.1.2, the map $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ extends the map (13.1).

Proposition 13.1.4 *The maps* $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ *in Definition 13.1.4 together induce a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega\in\mathrm{K\ddot{a}h}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega). \tag{13.2}$$ {eq:PSHNAprojlimigeneral2}

Proof It is a tautology that the maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.4 are compatible with the transition maps. So the map (13.2) is well-defined. It is injective by the same argument as Proposition 13.1.2. We argue the surjectivity.

By unfolding the definitions, an object in the target of (13.2) is an assignment: with each $\omega \in \mathrm{K\ddot{a}h}(X)$, we associate a family $(\Gamma^{\omega,\omega'})_{\omega'\in\mathrm{K\ddot{a}h}(X)}$ satisfying:

(1) $\Gamma^{\omega,\omega'} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega+\omega')_{>0}$ for each $\omega,\omega' \in \mathrm{K\ddot{a}h}(X)$;
(2) for each $\omega,\omega',\omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega'' \geq \omega'$, we have

$$P_{\theta+\omega+\omega''}\left[\Gamma^{\omega,\omega'}\right]_{\mathcal{I}} = \Gamma^{\omega,\omega''};$$

(3) for each $\omega,\omega',\omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega \leq \omega'$, we have

$$P_{\theta+\omega'+\omega''}\left[\Gamma^{\omega,\omega''}\right]_{\mathcal{I}} = \Gamma^{\omega',\omega''}.$$

The preimage of such an object is given by the family $(\Gamma^{\omega})_{\omega\in\mathrm{K\ddot{a}h}(X)}$ given by

$$\Gamma^{\omega} = \Gamma^{\omega/2,\omega/2}.$$

The fact that the image of Γ is as expected is a tautology, which we leave to the readers. □

With an almost identical argument involving Proposition 3.1.8, we get prop:PSHNAreform1 **Proposition 13.1.5** *The maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.4 and the injective maps Proposition 13.1.2 together induce bijections*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0} \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega), \tag{13.3}$$ {eq:PSHNAprojlimigeneral}

where ω runs over either the partially ordered set of all smooth closed real positive $(1,1)$-forms with positive volume on X or $\mathrm{Käh}(X)$.

cor:PSHNAbimero **Corollary 13.1.1** *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then π^* induces a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta).$$

Proof This follows immediately from Proposition 13.1.5. □

It is immediate to verify that π^* in Corollary 13.1.1 extends the map Proposition 9.3.3.

13.2 Operations on non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n and θ,θ',θ'' be closed real smooth $(1,1)$-forms on X representing big cohomology classes.

Definition 13.2.1 Let $\Gamma,\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. We say $\Gamma \leq \Gamma'$ if $\Gamma_{\max} \leq \Gamma'_{\max}$ and for some $\omega \in \mathrm{Käh}(X)$, we have

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \geq P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

This notion is independent of the choice of ω thanks to (9.13).

Moreover, we have the following:

Proposition 13.2.1 *Let $\Gamma,\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and ω be a closed smooth positive $(1,1)$-form on X, then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \leq P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$.

Proof This follows immediately from (9.13). □

Observe that this definition coincides with the corresponding definition in Definition 9.4.1 when $\Gamma,\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

def:sumNAmetrics **Definition 13.2.2** Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. Then we define $\Gamma+\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$P_{\theta+\omega}[\Gamma+\Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

This definition yields an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')$ by Lemma 9.4.3.

Proposition 13.2.2 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. Suppose that ω, ω' are two smooth closed positive $(1,1)$-forms on X. Then*

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + P_{\theta'+\omega'}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.3. □

Proposition 13.2.3 *The operation $+$ is commutative and associative: for any $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ and $\Gamma'' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta'')$, we have*

$$\Gamma+\Gamma' = \Gamma'+\Gamma, \quad (\Gamma+\Gamma')+\Gamma'' = \Gamma+(\Gamma'+\Gamma'').$$

Proof This is a direct consequence of Proposition 9.4.1. □

Definition 13.2.3 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $C \in \mathbb{R}$. We define $\Gamma+C \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega}[\Gamma+C] = P_{\theta+\omega}[\Gamma]+C.$$

It is obvious from Definition 9.4.3 that $\Gamma + C \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. It is also obvious that this definition extends Definition 9.4.3.

Proposition 13.2.4 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $C \in \mathbb{R}$. Suppose that ω is a smooth closed positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}}+C = P_{\theta+\omega}[\Gamma+C]_{\mathcal{I}}.$$

Proof This is clear by definition. □

prop:NAmetricplusC **Proposition 13.2.5** *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ and $C, C' \in \mathbb{R}$, then*

(1) $(\Gamma+\Gamma')+C = \Gamma+(\Gamma'+C) = (\Gamma+C)+\Gamma'$;
(2) $\Gamma+(C+C') = (\Gamma+C)+C'$.

Proof This is a direct consequence of Proposition 9.4.2. □

def:PSHNAlor **Definition 13.2.4** Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega}[\Gamma \vee \Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \vee P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

It follows from Lemma 9.4.5 that $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and this definition extends the corresponding definition in Definition 9.4.4.

Proposition 13.2.6 *Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and ω be a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma \vee \Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \vee P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.5. □

Proposition 13.2.7 *The operation $\vee$ is commutative and associative.*

In particular, given a finite non-empty family $(\Gamma^i)_{i\in I}$ in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we then define $\bigvee_{i\in I}\Gamma^i$ in the obvious way.

Proof This is a direct consequence of Corollary 9.4.1. □

Definition 13.2.5 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that

$$\sup_{i\in I}\Gamma^i_{\max}<\infty. \tag{13.4}$$

{eq:supPSHNAmaxfinite}

Then we define $\sup^*_{i\in I}\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for any $\omega\in\mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega}\left[\sup_{i\in I}\Gamma^i\right]=\sup_{i\in I}{}^*\,P_{\theta+\omega}\left[\Gamma^i\right].$$

It follows immediately from Lemma 9.4.7 that $\sup^*_{i\in I}\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and this definition extends Definition 9.4.6. Moreover, this definition clearly extends Definition 13.2.4 as well.

Proposition 13.2.8 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\,\Gamma^i\right]=\sup_{i\in I}{}^*\,P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof This is a direct consequence of Lemma 9.4.7. □

prop:NAChoquet

Proposition 13.2.9 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Then there exists a countable subfamily $I'\subseteq I$ such that*

$$\sup_{i\in I}{}^*\,\Gamma^i=\sup_{i\in I'}{}^*\,\Gamma^i.$$

Proof For any fixed $\omega\in\mathrm{K\ddot{a}h}(X)$, thanks to Proposition 9.4.5, we could find a countable subfamily $I'\subseteq I$ such that

$$\sup_{i\in I}{}^*\,P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}}=\sup_{i\in I'}{}^*\,P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}}.$$

It suffices to show that for any other $\omega'\in\mathrm{K\ddot{a}h}(X)$, we have

$$\sup_{i\in I}{}^*\,P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}}=\sup_{i\in I'}{}^*\,P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}}.$$

This is an immediate consequence of Proposition 6.1.6. □

prop:supGammiotherprop2

Proposition 13.2.10 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Let $C\in\mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i + C) = \sup_{i\in I}{}^* \Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ *is another family in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.4). *Suppose that* $\Gamma^i \leq \Gamma'^i$ *for all* $i \in I$, *then*

$$\sup_{i\in I}{}^* \Gamma^i \leq \sup_{i\in I}{}^* \Gamma'^i.$$

Proof This is an immediate consequence of Proposition 9.4.6. □

Definition 13.2.6 Let $(\Gamma_i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that

$$\inf_{i\in I} \Gamma_{i,\max} > -\infty, \tag{13.5}$$

{eq:decnetcontition}

then we define $\inf_{i\in I} \Gamma_i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for each $\omega \in \mathrm{Käh}(X)$, the component

$$P_{\theta+\omega}\left[\inf_{i\in I} \Gamma_i\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}$$

is defined as follows:

(1) we set

$$\left(P_{\theta+\omega}\left[\inf_{i\in I} \Gamma_i\right]_{\mathcal{I}}\right)_{\max} = \inf_{i\in I} \Gamma_{i,\max};$$

(2) For any $\tau < \inf_{i\in I} \Gamma_{i,\max}$, we define

$$\left(P_{\theta+\omega}\left[\inf_{i\in I} \Gamma_i\right]_{\mathcal{I}}\right)_{\tau} = \inf_{i\in I} P_{\theta+\omega}[\Gamma_{i,\tau}]_{\mathcal{I}}. \tag{13.6}$$

{eq:decnettestcurdef}

We observe that

$$P_{\theta+\omega}\left[\inf_{i\in I} \Gamma_i\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}.$$

This follows from Proposition 3.2.11. Now it is clear that $\inf_{i\in I} \Gamma_i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

prop:infGammiotherprop2

Proposition 13.2.11 *Let* $(\Gamma^i)_{i\in I}$ *be a decreasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.5). *Let* $C \in \mathbb{R}$. *Then*

$$\inf_{i\in I}(\Gamma^i + C) = \inf_{i\in I} \Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ *is another decreasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.5). *Suppose that* $\Gamma^i \leq \Gamma'^i$ *for all* $i \in I$, *then*

$$\inf_{i\in I} \Gamma^i \leq \inf_{i\in I} \Gamma'^i.$$

Proof This is clear by definition. □

Definition 13.2.7 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\lambda \in \mathbb{R}_{>0}$, then we define $\lambda\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\lambda\theta+\omega}[\lambda\Gamma]_{\mathcal{I}} = \lambda P_{\theta+\lambda^{-1}\omega}[\Gamma]_{\mathcal{I}}.$$

It follows immediately from Lemma 9.4.8 that $\lambda\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ and this definition extends Definition 9.4.7.

Proposition 13.2.12 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\lambda \in \mathbb{R}_{>0}$. Then for any closed smooth positive $(1,1)$-form ω on X, we have*

$$P_{\lambda\theta+\omega}[\lambda\Gamma]_{\mathcal{I}} = \lambda P_{\theta+\lambda^{-1}\omega}[\Gamma]_{\mathcal{I}}.$$

Proof This follows immediately from Lemma 9.4.8. □

prop:resclacompat2

Proposition 13.2.13 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$, $C \in \mathbb{R}$ and $\lambda,\lambda' > 0$, we have*

$$\begin{aligned}\lambda(\Gamma+\Gamma') &= \lambda\Gamma + \lambda\Gamma',\\ (\lambda\lambda')\Gamma &= \lambda(\lambda'\Gamma),\\ \lambda(\Gamma+C) &= \lambda\Gamma+\lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ is a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4), then

$$\lambda\left(\sup_{i\in I}{}^{*}\,\Gamma^i\right) = \sup_{i\in I}{}^{*}(\lambda\Gamma^i).$$

If $(\Gamma^i)_{i\in I}$ is a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.5), then

$$\lambda\left(\inf_{i\in I}\Gamma^i\right) = \inf_{i\in I}(\lambda\Gamma^i).$$

Proof Everything except the last assertion follows from Proposition 9.4.8. The last assertion is obvious by definition. □

Definition 13.2.8 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Let $Y \subseteq X$ be an irreducible analytic subset. We say that the trace operator of Γ along Y is *well-defined* if

$$\nu\left(P_{\theta+\omega''}[\Gamma_\tau]_{\mathcal{I}}, Y\right) = 0$$

for small enough τ and any $\omega'' \in \mathrm{K\ddot{a}h}(X)$. We define

$$(\mathrm{Tr}_Y(\Gamma))_{\max} \coloneqq \sup\left\{\tau < \Gamma_{\max} : \nu\left(P_{\theta+\omega''}[\Gamma_\tau]_{\mathcal{I}}, Y\right) = 0\right\}.$$

In this case, we define $\mathrm{Tr}_Y(\Gamma) \in \mathrm{PSH}^{\mathrm{NA}}(\tilde{Y},\theta|_{\tilde{Y}})$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(\tilde{Y})$, the component

$$P_{\theta|_{\tilde{Y}}+\omega}\left[\mathrm{Tr}_Y(\Gamma)\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(Y,\theta|_{\tilde{Y}}+\omega)_{>0}$$

is defined as follows:

(1) we let

$$\left(P_{\theta|_{\tilde{Y}}+\omega}\left[\operatorname{Tr}_Y(\Gamma)\right]_{\mathcal{I}}\right)_{\max} = \left(\operatorname{Tr}_Y(\Gamma)\right)_{\max}\,; \tag{13.7}$$

{eq:tracemax}

(2) For each $\tau \in \mathbb{R}$ less than the common value (13.7), we define

$$P_{\theta|_{\tilde{Y}}+\omega}\left[\operatorname{Tr}_Y(\Gamma)\right]_{\mathcal{I},\tau} \coloneqq P_{\theta|_{\tilde{Y}}+\omega}\left[\operatorname{Tr}_Y^{\theta+\tilde{\omega}}\left(P_{\theta+\tilde{\omega}}[\Gamma]_{\mathcal{I},\tau}\right)\right],$$

where $\tilde{\omega}$ is an arbitrary Kähler form on X such that $\omega \geq \tilde{\omega}|_{\tilde{Y}}$.

It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space. We observe that the choice of the trace operator $\operatorname{Tr}_Y^{\theta+\tilde{\omega}}\left(P_{\theta+\tilde{\omega}}[\Gamma]_{\mathcal{I},\tau}\right)$ is irrelevant since two different choice are $\mathcal{I}$-equivalent. Moreover,

$$\left(P_{\theta|_{\tilde{Y}}+\omega}\left[\operatorname{Tr}_Y(\Gamma)\right]_{\mathcal{I}}\right)_{\tau}$$

is $\mathcal{I}$-model by Proposition 8.1.2.

Furthermore,

$$P_{\theta|_{\tilde{Y}}+\omega}\left[\operatorname{Tr}_Y(\Gamma)\right]_{\mathcal{I}} \in \operatorname{PSH}^{\mathrm{NA}}(Y, \theta|_{\tilde{Y}}+\omega)_{>0}$$

is a consequence of Proposition 8.2.1. It is therefore clear that $\operatorname{Tr}_Y(\Gamma) \in \operatorname{PSH}^{\mathrm{NA}}(X,\theta)$.

Proposition 13.2.14 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then all definitions in this section are invariant under pulling-back to Y.*

The meaning is clear in most cases. In the case of the trace operator, this means the following: suppose that $Z \subseteq X$ is an analytic subset and $\Gamma \in \operatorname{PSH}^{\mathrm{NA}}(X,\theta)$ has non-trivial restriction to Z. Suppose that Z is not contained in the non-isomorphism locus of π so that the strict transform W of Z is defined. If we write $\Pi\colon W \to Z$ for the restriction of π and $\tilde{\Pi}\colon \tilde{W} \to \tilde{Z}$ the strict transform of Π, then we have

$$\tilde{\Pi}^* \operatorname{Tr}_Z(\Gamma) = \operatorname{Tr}_W(\pi^*\Gamma).$$

Proof We only prove the assertion for the trace operator, as the other proofs are similar.

We shall use the notations above. Observe that for any closed positive smooth $(1,1)$-form on X with positive mass, we have

$$\left(\tilde{\Pi}^*\operatorname{Tr}_Z(\Gamma)\right)_{\max} = \left(\operatorname{Tr}_Z(\Gamma)\right)_{\max} = \sup\left\{\tau < \Gamma_{\max} : \nu(P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}}, Z) = 0\right\}$$

and

$$\begin{aligned}
\left(\operatorname{Tr}_W(\pi^*\Gamma)\right)_{\max} &= \sup\left\{\tau < \Gamma_{\max} : \nu(P_{\pi^*\theta+\pi^*\omega}[\pi^*\Gamma_\tau]_{\mathcal{I}}, W) = 0\right\}\\
&= \sup\left\{\tau < \Gamma_{\max} : \nu(\pi^* P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}}, W) = 0\right\}\\
&= \sup\left\{\tau < \Gamma_{\max} : \nu(P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}}, Z) = 0\right\}.
\end{aligned}$$

Here we applied implicitly Proposition 13.1.5. Therefore,

$$\left(\tilde{\Pi}^* \operatorname{Tr}_Z(\Gamma)\right)_{\max} = \left(\operatorname{Tr}_W(\pi^*\Gamma)\right)_{\max}.$$

Let $\tau \in \mathbb{R}$ be less than this common value. Take a closed smooth Kähler form ω (resp. ω') on $\tilde{Z}$ (resp. $\tilde{W}$) with positive mass. We may assume that $\omega' \geq \tilde{\Pi}^*\omega$. Take a Kähler form $\tilde{\omega}$ on Y (resp. $\tilde{\omega}'$ on X) such that

$$\omega' \geq \tilde{\omega}'|_{\tilde{W}}, \quad \omega \geq \tilde{\omega}|_{\tilde{Z}}.$$

Without loss of generality, we may assume that

$$\tilde{\omega}' \geq \pi^*\tilde{\omega}.$$

It suffices to show that

$$\operatorname{Tr}_W^{\pi^*\theta+\tilde{\omega}'}\left(P_{\pi^*\theta+\tilde{\omega}'}[\pi^*\Gamma]_{\mathcal{I},\tau}\right) \sim_P \tilde{\Pi}^* \operatorname{Tr}_Z^{\theta+\tilde{\omega}}\left[P_{\theta+\tilde{\omega}}[\Gamma]_{\mathcal{I},\tau}\right].$$

Using Proposition 8.2.1, this is equivalent to

$$\operatorname{Tr}_W\left(P_{\pi^*\theta+\pi^*\omega}[\pi^*\Gamma]_{\mathcal{I},\tau}\right) \sim_P \tilde{\Pi}^* \operatorname{Tr}_Z\left[P_{\theta+\tilde{\omega}}[\Gamma]_{\mathcal{I},\tau}\right].$$

This is a consequence of Lemma 8.2.1. □

13.3 Duistermaat–Heckman measures

sec:DHmeasure

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class.

We fix a smooth flag $Y_\bullet$ on X.

Now suppose that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Recall that $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}(\Delta_{Y_\bullet}(\theta,V_\theta))$ is defined in Theorem 10.4.2.

Definition 13.3.1 The *Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ of an element $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined as the Duistermaat–Heckman measure of the Okounkov test curve $\Delta_{Y_\bullet}(\Gamma)$.

thm:DHindep

Theorem 13.3.1 *The Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ *of* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *is independent of the choice of the flag* $Y_\bullet$.

Proof Assume furthermore that Γ is bounded, we observe that the moments of the random variable $G[\Delta_{Y_\bullet}(\Gamma)]$ as computed in (10.46) are independent of the choice of the flag. Since the Duistermaat–Heckman measure has bounded support in this case, we conclude that $\mathrm{DH}(\Gamma)$ is uniquely determined.

In general, Γ is the decreasing limit of the sequence $\Gamma \vee \Gamma^k$ as $k \to \infty$, where $\Gamma^k : (-\infty,-k) \to \mathrm{PSH}(X,\theta)$ takes the constant value $\Gamma_{-\infty}$. It follows from the general continuity result Theorem 10.3.2 that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of

$\Delta_{Y_\bullet}(\Gamma \vee \Gamma^k)_\tau$ for any $\tau < \Gamma_{\max}$. So $\mathrm{DH}(\Gamma \vee \Gamma^k) \rightharpoonup \mathrm{DH}(\Gamma)$ by Lemma 10.4.2. It follows that $\mathrm{DH}(\Gamma)$ is independent of the choice of the flag. $\square$

More generally, when X does not admit a smooth flag, we could make a modification $\pi\colon Y \to X$ so that Y admits a flag. We define

$$\mathrm{DH}(\Gamma) = \mathrm{DH}(\pi^*\Gamma).$$

It follows from Theorem 10.3.2 that this measure is independent of the choice of π.

Appendix A
Convex functions and convex bodies

chap:convex

We study convex functions in this section. Our basic reference is Roc70 [Roc70].

A.1 The notion of convex functions

Let N be a real vector space of finite dimension.

Definition A.1.1 Let $F\colon N \to [-\infty, \infty]$ be a function. The *epigraph* of F is defined as the following set

$$\operatorname{epi} F \coloneqq \{(n, r) \in N \times \mathbb{R} : r \geq F(n)\}\,.$$

Definition A.1.2 A *convex function* on N is a function $F\colon N \to [-\infty, \infty]$ such that the epigraph $\operatorname{epi} F$ is a convex subset of $N \times \mathbb{R}$.

The *effective domain* of F is the set

$$\operatorname{Dom} F \coloneqq \{n \in N : F(n) < \infty\}\,.$$

A convex function F on N such that $\operatorname{Dom} F \neq \varnothing$ and $F(n) \neq -\infty$ for all $n \in N$ is said to be *proper*.

The set of convex functions on N is denoted by $\operatorname{Conv}(N)$. The subset set of proper convex functions is denoted by $\operatorname{Conv}^{\mathrm{prop}}(N)$.

The following characterization of convex functions is well-known.

lma:charconvex

Lemma A.1.1 *Let $F\colon N \to [-\infty, \infty]$. Then F is convex if and only if the following condition holds: suppose that $n, r \in N$ and $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$, then for any $t \in (0, 1)$, we have*

$$F(tn + (1 - t)r) < ta + (1 - t)b.$$

See Roc70 [Roc70, Theorem 4.2] for the proof.

Example A.1.1 Let $A \subseteq N$ be a convex subset. Then the *characteristic function* $\chi_A \colon N \to \{0, \infty\}$ of A is defined by

$$\chi_A(n) \coloneqq \begin{cases} 0, & n \in A; \\ \infty, & n \notin A. \end{cases}$$

The function χ_A lies in $\mathrm{Conv}(N)$.

ex:suppfun *Example A.1.2* Let M be the dual vector space of N and $P \subseteq M$ be a convex subset. The *support function* $\mathrm{Supp}_P \in \mathrm{Conv}(N)$ of P is defined as follows:

$$\mathrm{Supp}_P(n) \coloneqq \sup\{\langle m, n\rangle : m \in P\}.$$

It is well-known that convexity is preserved by a number of natural operations. We recall a few to fix the notation.

Definition A.1.3 Let $F_1, \dots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$ $(m \in \mathbb{Z}_{>0})$. We define their *infimal convolution* $F_1 \Box \cdots \Box F_m \in \mathrm{Conv}(N)$ as follows:

$$F_1 \Box \cdots \Box F_m(n) \coloneqq \inf\left\{\sum_{i=1}^m F_i(n_i) : n_i \in N, \sum_{i=1}^m n_i = n\right\}.$$

The fact $F_1 \Box \cdots \Box F_m \in \mathrm{Conv}(N)$ is proved in [Roc70, Theorem 5.4]. One should note that $F_1 \Box \cdots \Box F_m$ is not always proper.

prop:supconv **Proposition A.1.1** *Let $\{F_i\}_{i \in I}$ be a non-empty family in* $\mathrm{Conv}(N)$*. Then* $\sup_{i \in I} F_i \in \mathrm{Conv}(N)$.

This follows from [Roc70, Theorem 5.5]. In particular, this allows us to introduce

def:LCE **Definition A.1.4** Let $f \colon N \to [-\infty, \infty]$. The *lower convex envelope* of f is defined as

$$\mathrm{CE}\, f \coloneqq \sup\{F \in \mathrm{Conv}(N) : F \leq f\}.$$

It follows from Proposition A.1.1 that $\mathrm{CE}\, f \in \mathrm{Conv}(N)$.

def:convwedge **Definition A.1.5** Given a non-empty family $\{F_i\}_{i \in I}$ in $\mathrm{Conv}(N)$, we define

$$\bigwedge_{i \in I} F_i \coloneqq \mathrm{CE}\left(\inf_{i \in I} F_i\right).$$

When the family I is finite, say $I = \{1, \dots, m\}$, we also write

$$F_1 \wedge \cdots \wedge F_m = \bigwedge_{i \in I} F_i.$$

prop:concavhull **Proposition A.1.2** *Let $F_1, \dots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$, then*

$$F_1 \wedge \cdots \wedge F_m(x) = \inf \left\{ \sum_{i=1}^m \lambda_i F_i(x_i) : x_i \in \mathrm{Dom}(F_i), \right.$$
$$\left. \lambda_i \in [0,1], \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \right\}.$$

See [Roc70, Theorem 5.6] for the more general result.

lma:convdecnet **Lemma A.1.2** *Let* $\{F_i\}_{i\in I}$ *be a decreasing net in* $\mathrm{Conv}(N)$*. Then* $\inf_{i\in I} F_i \in \mathrm{Conv}(N)$.

Proof Write $F = \inf_{i\in I} F_i$. We shall apply the characterization in Lemma A.1.1. Take $n, r \in N$, $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$ and $t \in (0,1)$. We need to show that

$$F(tn + (1-t)r) < ta + (1-t)b. \tag{A.1}$$ {eq:convtemp1}

By definition, there exists $j \in I$ such that for any $i \geq I$ with $i \geq j$, we have

$$a > F_i(n), \quad b > F_i(r).$$

It follows from Lemma A.1.1 that

$$F_i(tn + (1-t)r) < ta + (1-t)b$$

for any $i \geq j$. Since F_i is decreasing in i, we conclude (A.1). □

def:convexclosure **Definition A.1.6** Let $F \in \mathrm{Conv}(N)$. The *closure* $\mathrm{cl}\, F \in \mathrm{Conv}(N)$ of F is defined as follows: if $F(n) = -\infty$ for some $n \in N$, then $\mathrm{cl}\, F := -\infty$. Otherwise, we define $\mathrm{cl}\, F$ as the lower semicontinuity regularization fo F.

A convex function $F \in \mathrm{Conv}(N)$ is *closed* if $F = \mathrm{cl}\, F$. In other words, $F \in \mathrm{Conv}(N)$ if one of the following conditions hold:

(1) $F \equiv -\infty$;
(2) $F \equiv \infty$;
(3) F is proper and lower semi-continuous.

Proposition A.1.3 *Let* $F \in \mathrm{Conv}(N)$ *be a closed convex function. Then* F *is the supremum of all affine functions lying below* F.

See [Roc70, Theorem 12.1].

Theorem A.1.1 *Let* $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$*. Then* $\mathrm{cl}\, F$ *is a closed proper convex function. Moreover,* $\mathrm{cl}\, F$ *agrees with* F *except possibly on the relative boundary of* $\mathrm{Dom}\, F$.

See [Roc70, Theorem 7.4].

def:partialorderconv **Definition A.1.7** Given $F, F' \in \mathrm{Conv}(N)$, we write $F \preceq F'$ if there is $C \in \mathbb{R}$ such that

$$F \leq F' + C.$$

We say $F \sim F'$ if $F \preceq F'$ and $F' \preceq F$ both hold.

A.2 Legendre transform

Let N be a real vector space of finite dimension and M be the dual vector space. The pairing $M \times N \longrightarrow \mathbb{R}$ will be denoted by $\langle \bullet, \bullet \rangle$.

def:Legendregeneral **Definition A.2.1** Let $F \in \mathrm{Conv}(N)$ be a convex function. We define the *Legendre transform* of F as the function $F^* \in \mathrm{Conv}(M)$:

$$F^*(m) := \sup_{n \in N} (\langle m, n \rangle - F(n)) = \sup_{n \in \mathrm{RelInt}\,\mathrm{Dom}\,F} (\langle m, n \rangle - F(n)) \,.$$

The latter equality follows from [Roc70, Corollary 12.2.2].

Recall the well-known Legendre–Fenchel duality [Roc70, Theorem 12.2].

thm:Legendredual **Theorem A.2.1** *Let $F \in \mathrm{Conv}(N)$. Then F^* is a closed convex function. The function F^* is proper if and only if F is.*

Moreover, we have $(\mathrm{cl}\,F)^* = F^*$ and

$$F^{**} = \mathrm{cl}\,F.$$

ex:suppfundual *Example A.2.1* Let $P \subseteq M$ be a closed convex subset. Then

$$\mathrm{Supp}_P^* = \chi_P, \quad \chi_P^* = \mathrm{Supp}_P \,.$$

See [Roc70, Theorem 13.2].

Definition A.2.2 Let $F \in \mathrm{Conv}(N)$ and $n \in N$. An element $m \in M$ is a *subgradient* of F at n if

$$F(n') \geq F(n) + \langle n' - n, m \rangle, \quad \forall n' \in N. \tag{A.2}$$ {eq:subgrad}

The set of subgradients of F at n is denoted by $\nabla F(n)$.

More generally, for any subset $E \subseteq N$, we write

$$\nabla F(E) = \bigcup_{n \in E} \nabla F(n).$$

def:convexPorder **Definition A.2.3** Given $F, F' \in \mathrm{Conv}(N)$, we write $F \preceq_P F'$ if

$$\overline{\nabla F(N)} \subseteq \overline{\nabla F'(N)}.$$

We write $F \sim_P F'$ if $F \preceq_P F'$ and $F' \preceq_P F$.

Theorem A.2.2 *Suppose that $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. Then the following hold:*

(1) *for any $n \notin \mathrm{Dom}\,F$, $\nabla F(n) = \emptyset$;*

(2) *for any $n \in \mathrm{RelInt}\,\mathrm{Dom}\,F$, $\nabla F(n) \neq \emptyset$; Moreover, for any $n' \in N$, we have*

$$\partial_{n'} F(n) = \sup \{ \langle n', m \rangle : m \in \nabla F(n) \} \,;$$

(3) *for $n \in N$, the set $\nabla F(n)$ is bounded if and only if $n \in \mathrm{Int}\,\mathrm{Dom}\,F$.*

For the proof, we refer to [Roc70, Theorem 23.4].

prop:gradDomFstar

Proposition A.2.1 *Let* $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. *Then*

$$\nabla F(N) \subseteq \mathrm{Dom}\, F^*.$$

If moreover F *is closed, we have*

$$\mathrm{RelInt}\,\mathrm{Dom}\, F^* \subseteq \nabla F(N). \tag{A.3}$$

{eq:relintdomFstar}

In particular, if F *is a proper closed convex function on* N, *then*

$$\overline{\nabla F(N)} = \overline{\mathrm{Dom}\, F^*}.$$

Proof Suppose that $m \in \nabla F(n)$ for some $n \in N$, it follows that (A.2) holds. In particular,

$$\langle m, n' \rangle - F(n') \leq \langle m, n \rangle - F(n).$$

It follows that

$$F^*(m) \leq \langle m, n \rangle - F(n) < \infty.$$

(A.3) is proved in [Roc70, Corollary 23.5.1]. For the last assertion, it suffices to observe that $\overline{\mathrm{RelInt}\,\mathrm{Dom}\, F^*} = \overline{\mathrm{Dom}\, F^*}$. □

prop:Legendretranssup

Proposition A.2.2 *Let* $\{F_i\}_{i\in I}$ *be a non-empty family in* $\mathrm{Conv}^{\mathrm{prop}}(N)$. *Then*

$$\left(\bigwedge_{i\in I} F_i\right)^* = \sup_{i\in I} F_i^*, \qquad \left(\sup_{i\in I} \mathrm{cl}\, F_i\right)^* = \mathrm{cl} \bigwedge_{i\in I} F_i^*.$$

If I *is finite and* $\overline{\mathrm{Dom}\, F_i}$ *is independent of the choice of* $i \in I$, *then*

$$\left(\sup_{i\in I} F_i\right)^* = \bigwedge_{i\in I} F_i^*.$$

Recall that $\wedge$ is defined in Definition A.1.5. See [Roc70, Theorem 16.5] for the proof.

prop:sumLegendre

Proposition A.2.3 *Let* $F_1, \ldots, F_r \in \mathrm{Conv}^{\mathrm{prop}}(N)$ $(r \in \mathbb{Z}_{>0})$. *Assume that*

$$\bigcap_{i=1}^{r} \mathrm{RelInt}\,\mathrm{Dom}(F_i) \neq \emptyset,$$

then

$$\left(\sum_{i=1}^{r} F_i\right)^* (m) = \inf\left\{\sum_{i=1}^{r} F_i^*(m_i) : m_1, \ldots, m_r \in M, \sum_{i=1}^{r} m_i = m\right\}.$$

prop:Fsuppchar

Proposition A.2.4 *Let $P \subseteq M$ be a convex body*[1] *and $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. The following are equivalent:*

(1) $F \preceq \mathrm{Supp}_P$;
(2) $\mathrm{Dom}\, F = N$ *and* $F^*|_{M\setminus P} \equiv \infty$;
(3) $\mathrm{Dom}\, F = N$ *and* $\nabla F(N) \subseteq P$.

Moreover, under these conditions,

$$F(n) - \mathrm{Supp}_P(n) \leq F(0), \quad \forall n \in N. \tag{A.4}$$

{eq:Fsupequal}

Proof (1) $\implies$ (2). It is clear that $\mathrm{Dom}\, F = N$ since $\mathrm{Dom}\,\mathrm{Supp}_P = N$. From $F \preceq \mathrm{Supp}_P$ and Example A.2.1, we know that

$$\chi_P = \mathrm{Supp}_P^* \preceq F^*.$$

So ii follows.

(2) $\implies$ (3). This follows from Proposition A.2.1.

(3) $\implies$ (1). Taken $n \in N$, we know that F is locally Lipschitz Roc70 [Roc70, Theorem 10.4], so we can compute

$$\begin{aligned} F(n) - F(0) = \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(tn)\,\mathrm{d}t &= \int_0^1 \langle \nabla F(tn), n\rangle\,\mathrm{d}t \\ &\leq \int_0^1 \mathrm{Supp}_P(n)\,\mathrm{d}t = \mathrm{Supp}_P(n). \end{aligned}$$

In particular, (A.4) also follows. □

A.3 Classes of convex functions

Let N be a real vector space of finite dimension and M be the dual vector space.

We shall fix a convex body $P \subseteq M$.

The following classes are introduced in BB13 [BB13].

def:convexPfucntions

Definition A.3.1 We define the set $\mathcal{P}(N, P)$ as the set of proper convex functions $F \in \mathrm{Conv}(N)$ such that $F \preceq \mathrm{Supp}_P$.

We define the set $\mathcal{E}^\infty(N, P)$ as the set of closed convex functions $F \in \mathrm{Conv}(N)$ such that $F \sim \mathrm{Supp}_P$.

We define the set $\mathcal{E}(N, P)$ as follows: suppose that $\mathrm{Int}\, P = \emptyset$, then $\mathcal{E}(N, P) := \mathcal{P}(N, P)$; otherwise, let

$$\mathcal{E}(N, P) = \left\{ F \in \mathcal{P}(N, P) : P = \overline{\nabla F(N)} \right\}.$$

[1] Here a convex body refers to a non-empty closed convex subset, not necessarily having non-empty interior.

Observe that for any $F \in \mathcal{P}(N, P)$, we have $\operatorname{Dom} F = N$ and F is necessarily closed.

Proposition A.3.1 *We have*

$$\mathcal{E}^\infty(N, P) \subseteq \mathcal{E}(N, P) \subseteq \mathcal{P}(N, P).$$

Proof When $\operatorname{Int} P = \varnothing$, the assertion is clear. We assume that $\operatorname{Int} P \neq \varnothing$. The second inclusion follows from definition. We only hand the first inequality. Take $F \in \mathcal{E}^\infty(N, P)$. By definition, $F \sim \operatorname{Supp}_P$ and hence $F^* \sim \chi_P$. It follows that $P = \operatorname{Dom} F^*$.

By Proposition A.2.4, we already know that

$$\nabla F(N) \subseteq P = \operatorname{Dom} F^*.$$

On the other hand, by Proposition A.2.1, we have

$$\operatorname{Int} P \subseteq \nabla F(N).$$

So it follows that

$$P = \overline{\nabla F(N)}.$$

Proposition A.3.2 *For any $F \in \mathcal{E}^\infty(N, P)$, we have $F^*|_{M \setminus P} \equiv \infty$ and F^* is bounded on P.*

Proof From $F \sim \operatorname{Supp}_P$, we take the Legendre transform to get $F^* \sim \operatorname{Supp}_P^* = \chi_P$, where we applied Example A.2.1. □

Definition A.3.2 We endow the topology of pointwise convergence on $\mathcal{P}(N, P)$. Note that this topology coincides with the compact-open topology.

prop:regularizationconvex **Proposition A.3.3** *Let $F \in \mathcal{P}(N, P)$. Then there is a decreasing sequence $F_j \in \mathcal{E}^\infty(N, P) \cap C^\infty(N)$ converging to F.*

BB13
See [BB13, Lemma 2.2].

We observe that the point $0 \in N$ plays a special role since it does in the definition of the support function.

Proposition A.3.4 *For any $F \in \operatorname{Conv}(N, P)$, we have*

$$\max_N (F - \operatorname{Supp}_P) = F(0).$$

Proof It follows from (A.4) that

$$\sup_N (F - \operatorname{Supp}_P) \leq F(0).$$

The equality is clearly obtained at $0 \in N$. □

A.4 Monge–Ampère measures

Let N be a free Abelian group of finite rank (i.e. a lattice) and M be its dual lattice. There is a canonical Lebesgue type measure on $M_{\mathbb{R}}$, denoted by $\mathrm{d}\,\mathrm{vol}$, normalized so that the smallest cubes in M have volume 1. Similarly, the canonical measure on $N_{\mathbb{R}}$ is normalized in the same way and is denoted by $\mathrm{d}\,\mathrm{vol}$ as well.

We will write

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition A.4.1 Let $F \in \mathrm{Conv}(N_{\mathbb{R}})$, we define $\mathrm{MA}_{\mathbb{R}} F$ as the Borel measure on $N_{\mathbb{R}}$ given as follows: for each Borel measurable set $E \subseteq N_{\mathbb{R}}$, define

$$\mathrm{MA}_{\mathbb{R}} F(E) \coloneqq n! \int_{\nabla F(E)} \mathrm{d}\,\mathrm{vol}\,.$$

Proposition A.4.1 *Let $P \in M_{\mathbb{R}}$ be a convex body and $F \in \mathcal{P}(N_{\mathbb{R}}, P)$. Then $F \in \mathcal{E}(N_{\mathbb{R}}, P)$ if and only if*

$$\int_{M_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}} F = n!\,\mathrm{vol}\,P. \tag{A.5}$$ {eq:cvxfullmass}

Proof By definition of $\mathrm{MA}_{\mathbb{R}}$, (A.5) is equivalent to

$$\mathrm{vol}\,\overline{\nabla F(N_{\mathbb{R}})} = \mathrm{vol}\,P.$$

We first handle the case where $\mathrm{Int}\,P \neq \varnothing$. By Proposition A.2.4, the latter is equivalent to

$$\overline{\nabla F(N_{\mathbb{R}})} = P.$$

Now assume that $\mathrm{Int}\,P = \varnothing$, then $\mathrm{vol}\,\overline{\nabla F(N)} = \mathrm{vol}\,P = 0$ by Proposition A.2.4. The assertion is clear. □

thm:realMAcont **Theorem A.4.1** *Let $F, F_j \in \mathcal{P}(N_{\mathbb{R}}, P)$ ($j \in \mathbb{Z}_{>0}$). Assume that $F_j \to F$, then $\mathrm{MA}_{\mathbb{R}}(F_j)$ converges to $\mathrm{MA}_{\mathbb{R}}(F)$ weakly.*

See [Fig17, Proposition 2.6].

There is a well-known comparison principle.

thm:convcomp **Theorem A.4.2** *Let $F, F' \in \mathcal{P}(N_{\mathbb{R}}, P)$. Assume that $F \leq F'$, then*

$$\overline{\nabla F(N_{\mathbb{R}})} \subseteq \overline{\nabla F'(N_{\mathbb{R}})}.$$

$$\int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F) \leq \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F').$$

See [BB13, Lemma 2.5].

A.5 Separation lemmata

lma:polybdd **Lemma A.5.1** *Let $\alpha, \beta_1, \ldots, \beta_m \in \mathbb{Z}^n$. Let Δ be the polytope generated by $\beta_1, \ldots, \beta_m$. Then the following are equivalent:*

(1)
$$|z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \tag{A.6}$$
{eq:zalpha}

is a bounded function on $\mathbb{C}^{*n}$.

(2) $\alpha \in \Delta$.

Proof (2) $\implies$ (1). Write $\alpha = \sum_i t_i \beta_i$, where $t_i \in [0, 1]$, $\sum_i t_i = 1$. Then

$$\begin{aligned} |z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} &= \prod_i |z^{\beta_i}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \\ &\leq \prod_i \sum_j |z^{\beta_j}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \leq 1. \end{aligned}$$

(1) $\implies$ (2). Assume that $\alpha \notin \Delta$. Let H be a hyperplane that separates α and Δ. Say H is defined by $a_1 x_1 + \cdots + a_n x_n = C$. Set

$$z(t) := (t^{a_1}, \ldots, t^{a_n}).$$

Then clearly (A.6) evaluated at $z(t)$ is not bounded. □

lma:polybdd2 **Lemma A.5.2** *Let $\beta_1, \ldots, \beta_m \in \mathbb{N}^n$ and $\beta \in \mathbb{R}^n$. Then the following are equivalent*

(1) $\log \sum_{i=1}^m \mathrm{e}^{x \cdot \beta_i} - (x, \beta)$ *is bounded from below.*

(2) β *is in the convex hull of the β_i's.*

Proof The proof follows the same pattern as Lemma A.5.1. □

Appendix B
Pluripotential theory on unibranch spaces

chap:unib

In this appendix, we extend the theory in the book to compact unibranch Kähler spaces.

B.1 Complex spaces

A complex space is assumed to be reduced, Hausdorff and paracompact in the whole book.

def:primdiv

Definition B.1.1 A *prime divisor* over an irreducible complex space Z is a connected smooth hypersurface $E \subseteq X'$, where $X' \to Z$ is a proper bimeromorphic morphism with X' smooth. Such a morphism $X' \to Z$ is also called a *resolution* of Z.

Two prime divisors $E_1 \subseteq X'_1$ and $E_2 \subseteq X'_2$ over Z are *equivalent* if there is a common resolution $X'' \to X$ dominating both X'_1 and X'_2 such that the strict transforms of E_1 and E_2 coincide.

The set Z^{div} is the set of pairs (c, E), where $c \in \mathbb{Q}_{>0}$ and E is an equivalence class of a prime divisor over Z. For simplicity, we will denote the pair (c, E) by $c \operatorname{ord}_E$, although one should not really think of this object as a valuation unless Z is projective and irreducible.

Note that a prime divisor on Z does not always define a prime divisor over Z if Z is singular.

Definition B.1.2 A complex space X is *unibranch* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is unibranch.

Xia23Mabuchi

It is shown in the arXiv version of [Xia23a, Remark 2.7] that when X is a projective variety, this notion coincides with the corresponding algebraic notion of unibranchness.

thm:Zariskimain

Theorem B.1.1 (Zariski's main theorem) *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism between complex spaces. Assume that X is unibranch, then π has connected fibers.*

We refer to [Dem85, Proof of Théorème 1.7].

def:modif **Definition B.1.3** A *modification* of a compact complex space X is a finite composition of blow-ups with smooth centers.

thm:HironakaChow **Theorem B.1.2 (Hironaka's Chow lemma)** *Suppose that X is a compact complex space. Then every proper bimeromorphic morphism to X can be dominated by a modification.*

This follows from the proof of [Hir75, Corollary 2].

thm:res **Theorem B.1.3** *Let X be a compact complex space. Then there is a modification $\pi\colon Y \to X$ such that Y is smooth.*

See [BM97, Wło09].

cor:primerealization **Corollary B.1.1** *Let X be a compact complex space and E be a prime divisor over X. Then there is a modification $\pi\colon Y \to X$ such that Y is smooth and E can be realized as a prime divisor on Y.*

B.2 Plurisubharmonic functions

Let X be a complex space.

Given a function $f\colon X \to [-\infty, \infty)$, we define

$$f^*\colon X \to [-\infty, \infty], \quad f^*(x) = \varlimsup_{X^{\mathrm{Reg}} \ni y \to x} f(y)$$

Definition B.2.1 A function $\varphi\colon X \to [-\infty, \infty)$ is *plurisubharmonic* if

(1) φ is not identically $-\infty$ on any irreducible component of X;
(2) For any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a plurisubharmonic function $\tilde{\varphi} \in \mathrm{PSH}(\Omega)$ such that $\varphi|_{\Omega\cap V} = \tilde{\varphi}|_{\Omega\cap V}$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Similarly, if θ is a smooth closed[1] real $(1,1)$-form on X, then a function $\varphi\colon X \to [-\infty, \infty)$ is *θ-plurisubharmonic* if for any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a smooth function g on Ω such that $\theta = (\mathrm{dd^c} g)|_{V\cap\Omega}$ and $g + \varphi|_V \in \mathrm{PSH}(V)$.

thm:FN **Theorem B.2.1 (Fornaess–Narasimhan)** *Let $\varphi\colon X \to [-\infty, \infty)$ be a function. Assume that φ is not identically $-\infty$ on any irreducible component of X, then the following are equivalent:*

(1) *φ is psh;*

[1] Here *closed* means that locally θ is defined by a closed form under a local embedding.

(2) *φ is usc and for any morphism $f\colon \Delta \to X$ from the open unit disk Δ in $\mathbb{C}$ to X such that $f^*\varphi$ is not identically $-\infty$, the pull-back $f^*\varphi$ is psh.*

If further more X is unibranch, then these conditions are equivalent to

(3) *$\varphi \in \mathrm{PSH}(X^{\mathrm{Reg}})$, locally bounded from above near X^{Sing} and $\varphi = \varphi^*$.*

See [FsN80] and [Dem85, Section 1.8].

Corollary B.2.1 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism between compact Kähler spaces. Let θ be a smooth closed real $(1,1)$-form on X. Assume that X is unibranch, then the pull-back induces a bijection*

$$\pi^*\colon \mathrm{PSH}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}(Y,\pi^*\theta).$$

See [Dem85, Théorème 1.7] for the details.

B.3 Extension of the results in the smooth setting

Let X be an irreducible unibranch compact Kähler space of dimension n. Let θ be a closed real smooth $(1,1)$-form on X. We say *the cohomology class* $[\theta]$ is big if for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a compact Kähler manifold Y, $[\pi^*\theta]$ is big.

The non-pluripolar products can be defined exactly as in Chapter 2 and the results in that chapter holds *mutadis mutandis*.

The results in Chapter 3 can be also be easily extended. The definition of the P-envelope remains unchanged. As for the $\mathcal{I}$-envelope, we define

Definition B.3.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define $P_\theta[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$ as the unique element with the following property: if $\pi\colon Y \to X$ is a proper bimeromorphic morphism from a compact Kähler manifold Y, then

$$\pi^* P_\theta[\varphi]_{\mathcal{I}} = P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}}.$$

It follows from Corollary B.2.1 and Proposition 3.2.5 that $P_\theta[\varphi]_{\mathcal{I}}$ is independent of the choice of π and is well-defined. The other results can be easily extended.

Chapter 4 and Chapter 6 can be extended without big changes. The only exception is Theorem 6.2.6, where we do not have the notion of multiplier ideal sheaves. So we do not know how to extend this theorem.

Chapter 7 can be extended execpt for Section 7.3 for the same reason as above.

The trace operator defined in Chapter 8 can be extended as long as Y is not contained in X^{Sing} using the embedded resolution. In general, due to the lack of Demailly regularization, we do not know how to define the trace operator.

Chapter 11 is unchanged, since we always take projective limits with respect to all models in that section.

Chapter 9 can be extended easily.

Chapter 10 is easy to extend since the partial Okounkov bodies are bimeromorphically invariant in the sense of Theorem 10.3.2.

Chapter 13 can be extended except for the parts involving the trace operator.

I do not know how to extend the results in Chapter 5 and Chapter 12 to the singular setting.

Appendix C
Almost semigroups

chap:almostsg

C.1 Convex bodies

Fix $n \in \mathbb{N}$.

def:convbodies

Definition C.1.1 A *convex body* in $\mathbb{R}^n$ is a non-empty compact convex set.

We allow a convex body to have empty interior.

We write $\mathcal{K}_n$ for the set of convex bodies in $\mathbb{R}^n$.

def:Hausdorffmetric

Definition C.1.2 The *Hausdorff metric* between $K_1, K_2 \in \mathcal{K}_n$ is given by

$$d_{\mathrm{Haus}}(K_1, K_2) := \max\left\{ \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

It is well-known that the metric space $(\mathcal{K}_n, d_{\mathrm{Haus}})$ is complete. We will need the following fundamental theorem:

thm:Blaschke

Theorem C.1.1 (Blaschke selection theorem) *The metric space $(\mathcal{K}_n, d_{\mathrm{Haus}})$ is locally compact.*

We refer to Sch14 [Sch93, Theorem 1.8.7] for details.

thm:contvol

Theorem C.1.2 *The Lebesgue volume* $\mathrm{vol}\colon \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ *is continuous.*

See Sch14 [Sch93, Theorem 1.8.20].

thm:Hausconvcond

Theorem C.1.3 *Let $K_i, K \in \mathcal{K}_n$ ($i \in \mathbb{N}$). Then $K_i \xrightarrow{d_{\mathrm{Haus}}} K$ if and only if the following conditions hold*

(1) *Each point $x \in K$ is the limit of a sequence $x_i \in K_i$.*
(2) *The limit of any convergent sequence $(x_{i_j})_{j \in \mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ lies in K, where i_j is a strictly increasing sequence in $\mathbb{Z}_{>0}$.*

See Sch14 [Sch93, Theorem 1.8.8].

lma:latcvb **Lemma C.1.1** *Let $K \in \mathcal{K}_n$ be a convex body with positive volume and $K' \in \mathcal{K}_n$. Assume that for some large enough $k \in \mathbb{Z}_{>0}$, K' contains $K \cap (k^{-1}\mathbb{Z})^n$, then $K' \supseteq K^{n^{1/2}k^{-1}}$.*

Proof Let $x \in K^{n^{1/2}k^{-1}}$, by assumption, the closed ball B with center x and radius $n^{1/2}k^{-1}$ is contained in K. Observe that x can be written as a convex combination of points in $B \cap (k^{-1}\mathbb{Z})^n$, which are contained in K' by assumption. It follows that $x \in K'$. □

Given a sequence of convex bodies K_i ($i \in \mathbb{N}$), we set

$$\varliminf_{i\to\infty} K_i = \overline{\bigcup_{i=0}^{\infty} \bigcap_{j\geq i} K_j}.$$

Suppose K is the limit of a subsequence of K_i, we have

$$\varliminf_{i\to\infty} K_i \subseteq K. \tag{C.1}$$ {eq:liminflimsup}

This is a simple consequence of Theorem C.1.3.

lma:Hausdorffconvslice **Lemma C.1.2** *Let $K \subseteq \mathbb{R}^n$ be a convex body. Let*

$$t_{\min} := \min\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}, \quad t_{\max} := \max\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}.$$

Then for $t \in [t_{\min}, t_{\max}]$, the map

$$t \mapsto \{x_1 = t\} \cap K$$

is continuous with respect to the Hausdorff metric.

Here x_1 denotes the first coordinate in $\mathbb{R}^n$.

Proof We may assume that $t_{\min} < t_{\max}$ as otherwise there is nothing to prove.

For each $t \in [t_{\min}, t_{\max}]$, we write $K_t = \{x_1 = t\} \cap K$. Let $t_j \to t$ be a convergent sequence in $[t_{\min}, t_{\max}]$, we want to show that K_{t_j} converges to K_t with respect to the Hausdorff metric. Recall that this amounts to the following two assertions:

(1) For each convergent sequence $x_j \in K_{t_j}$ with limit x, we have $x \in K_t$;
(2) Given any $x \in K_t$, up to replacing t_j by a subsequence, we can find $x_j \in K_{t_j}$ converging to x. □

The first assertion is obvious. Let us prove the second. Take $x = (t, x') \in K_t$. Up to replacing t_j by a subsequence and taking the symmetry into account, we may assume that $t_j > t$ for all t. In particular, $t < t_{\max}$.

We can find a point $y = (y^1, y') \in K$ such that $y^1 > t$ (for example, there is always such a point with $y^1 = t_{\max}$). Replacing t_j by a subsequence, we may assume that $t_j \in (t, y^1)$ for all j. Then it suffices to take

$$x_j = \frac{y^1 - t_j}{y^1 - t} x + \frac{t_j - t}{y^1 - t} y.$$

lma:intconvexset

Lemma C.1.3 *Let $D_j \subseteq \mathbb{R}^n$ ($j \geq 1$) be a decreasing sequence of convex sets. Assume that* $\operatorname{vol} \bigcap_j D_j > 0$*, then*

$$\overline{\bigcap_{j=1}^{\infty} D_j} = \bigcap_{j=1}^{\infty} \overline{D_j}.$$

Proof The $\subseteq$ direction is clear. By convexity, it suffices to show that both sides have the same positive volume. As the boundary of convex sets has zero Lebesgue measure, it follows that the volumes of both sides are equal to $\lim_{j\to\infty} \operatorname{vol} D_j$. □

C.2 The Okounkov bodies of almost semigroups

sec:clo

Fix an integer $n \geq 0$. Fix a closed convex cone $C \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ such that $C \cap \{x_{n+1} = 0\} = \{0\}$. Here x_{n+1} is the last coordinate of $\mathbb{R}^{n+1}$.

C.2.1 Generalities on semigroups

Write $\hat{\mathcal{S}}(C)$ for the set of subsets of $C \cap \mathbb{Z}^{n+1}$ and $\mathcal{S}(C)$ for the set of sub-semigroups $S \subseteq C \cap \mathbb{Z}^{n+1}$. For each $k \in \mathbb{N}$ and $S \in \hat{\mathcal{S}}(C)$, we write

$$S_k \coloneqq \{x \in \mathbb{Z}^n : (x, k) \in S\}.$$

Note that S_k is a finite set by our assumption on C.

We introduce a pseudometric on $\hat{\mathcal{S}}(C)$ as follows:

$$d_{\mathrm{sg}}(S, S') \coloneqq \varlimsup_{k\to\infty} k^{-n} (|S_k| + |S'_k| - 2|(S \cap S')_k|).$$

Here $|\bullet|$ denotes the cardinality of a finite set.

lma:dps

Lemma C.2.1 *The above defined d_{sg} is a pseudometric on $\hat{\mathcal{S}}(C)$.*

Proof Only the triangle inequality needs to be argued. Take $S, S', S'' \in \hat{\mathcal{S}}(C)$. We claim that for any $k \in \mathbb{N}$,

$$|S_k| + |S'_k| - 2|S_k \cap S'_k| + |S''_k| + |S'_k| - 2|S''_k \cap S'_k| \geq |S_k| + |S''_k| - 2|S_k \cap S''_k|.$$

From this the triangle inequality follows. To argue the claim, we rearrange it to the following form:

$$|S'_k| - |S_k \cap S'_k| \geq |S'_k \cap S''_k| - |S_k \cap S''_k|,$$

which is obvious. □

Given $S, S' \in \hat{\mathcal{S}}(C)$, we say S is equivalent to S' and write $S \sim S'$ if $d_{\mathrm{sg}}(S, S') = 0$. This is an equivalence relation by Lemma C.2.1.

lma:dBil **Lemma C.2.2** *Given $S, S', S'' \in \hat{\mathcal{S}}(C)$, we have*

$$d_{\mathrm{sg}}(S \cap S'', S' \cap S'') \leq d_{\mathrm{sg}}(S, S').$$

In particular, if $S^i, S'^i \in \hat{\mathcal{S}}(C)$ ($i \in \mathbb{N}$) and $S^i \to S$, $S'^i \to S'$, then

$$S^i \cap S'^i \to S \cap S'.$$

Proof Observe that for any $k \in \mathbb{N}$,

$$|S_k \cap S''_k| - |S_k \cap S'_k \cap S''_k| \leq |S_k| - |S_k \cap S'_k|.$$

The same holds if we interchange S with S'. It follows that

$$|S_k \cap S''_k| + |S'_k \cap S''_k| - 2|S_k \cap S'_k \cap S''_k| \leq |S_k| + |S'_k| - 2|S_k \cap S'_k|.$$

The first assertion follows.

Next we compute

$$\begin{aligned} d_{\mathrm{sg}}(S^i \cap S'^i, S \cap S') \leq & d_{\mathrm{sg}}(S^i \cap S'^i, S^i \cap S') + d_{\mathrm{sg}}(S^i \cap S', S \cap S') \\ \leq & d_{\mathrm{sg}}(S'^i, S') + d_{\mathrm{sg}}(S^i, S) \end{aligned}$$

and the second assertion follows. □

The volume of $S \in \mathcal{S}(C)$ is defined as

$$\operatorname{vol} S := \lim_{k \to \infty} (ka)^{-n} |S_{ka}| = \varlimsup_{k \to \infty} k^{-n} |S_k|,$$

where a is a sufficiently divisible positive integer. The existence of the limit and its independence from a both follow from the more precise result [KK12, Theorem 2].

lma:vollip **Lemma C.2.3** *Let $S, S' \in \mathcal{S}(C)$, then*

$$|\operatorname{vol} S - \operatorname{vol} S'| \leq d_{\mathrm{sg}}(S, S').$$

Proof By definition, we have

$$d_{\mathrm{sg}}(S, S') \geq \operatorname{vol} S + \operatorname{vol} S' - 2\operatorname{vol}(S \cap S').$$

It follows that $\operatorname{vol} S - \operatorname{vol} S' \leq d_{\mathrm{sg}}(S, S')$ and $\operatorname{vol} S' - \operatorname{vol} S \leq d_{\mathrm{sg}}(S, S')$. □

We define $\overline{\mathcal{S}}(C)$ as the closure of $\mathcal{S}(C)$ in $\hat{\mathcal{S}}(C)$ with respect to the topology defined by the pseudometric d. By Lemma C.2.3, $\operatorname{vol}\colon \mathcal{S}(C) \to \mathbb{R}$ admits a unique 1-Lipschitz extension to

$$\operatorname{vol}\colon \overline{\mathcal{S}}(C) \to \mathbb{R}. \tag{C.2}$$ {eq:volex}

lma:volcompa **Lemma C.2.4** *Suppose that $S, S' \in \overline{\mathcal{S}}(C)$ and $S \subseteq S'$. Then*

$$\operatorname{vol} S \leq \operatorname{vol} S'.$$

Proof Take sequences S^j, S'^j in $\mathcal{S}(C)$ such that $S^j \to S$, $S'^j \to S'$. By Lemma C.2.2, after replacing S^j by $S^j \cap S'^j$, we may assume that $S^j \subseteq S'^j$ for each j. Then our assertion follows easily. □

C.2.2 Okounkov bodies of semigroups

Given $S \in \hat{\mathcal{S}}(C)$, we will write $C(S) \subseteq C$ for the closed convex cone generated by $S \cup \{0\}$. Moreover, for each $k \in \mathbb{Z}_{>0}$, we define

$$\Delta_k(S) := \operatorname{Conv}\left\{k^{-1}x \in \mathbb{R}^n : x \in S_k\right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the convex hull.

Definition C.2.1 Let $\mathcal{S}'(C)$ be the subset of $\mathcal{S}(C)$ consisting of semigroups S such that S generates $\mathbb{Z}^{n+1}$ (as an Abelian group).

Note that for any $S \in \mathcal{S}'(C)$, the cone $C(S)$ has full dimension (i.e. the topological interior is non-empty). Given a full-dimensional subcone $C' \subseteq C$, it is clear that $C' \cap \mathbb{Z}^{n+1} \in \mathcal{S}'(C)$.

This class behaves well under intersections:

lma:intersecS' **Lemma C.2.5** *Let $S, S' \in \mathcal{S}'(C)$. Assume that $\operatorname{vol}(S \cap S') > 0$, then $S \cap S' \in \mathcal{S}'(C)$.*

The lemma obviously fails if $\operatorname{vol}(S \cap S') = 0$.

Proof We first observe that the cone $C(S) \cap C(S')$ has full dimension since otherwise $\operatorname{vol}(S \cap S') = 0$. Take a full-dimensional subcone C' in $C(S) \cap C(S')$ such that C' intersects the boundary of $C(S) \cap C(S')$ only at 0. It follows from [KK12, Theorem 1] that there is an integer $N > 0$ such that for any $x \in \mathbb{Z}^{n+1} \cap C'$ with Euclidean norm no less than N lies in $S \cap S'$. Therefore, $S \cap S' \in \mathcal{S}'(C)$. □

We recall the following definition from [KK12].

def:Okokk **Definition C.2.2** Given $S \in \mathcal{S}'(C)$, its *Okounkov body* is defined as follows

$$\Delta(S) := \{x \in \mathbb{R}^n : (x, 1) \in C(S)\}.$$

thm:HausOkoun **Theorem C.2.1** *For each $S \in \mathcal{S}'(C)$, we have*

$$\operatorname{vol} S = \lim_{k \to \infty} k^{-n} |S_k| = \operatorname{vol} \Delta(S) > 0. \tag{C.3}$$ {eq:volWvolDelta}

Moreover, as $k \to \infty$,

$$\Delta_k(S) \xrightarrow{d_{\text{Haus}}} \Delta(S). \tag{C.4}$$ {eq:HausconvDeltaGLS}

This is essentially proved in [WN14, Lemma 4.8], which itself follows from a theorem of Khovanskii [Kho92]. We remind the readers that (C.3) fails for a general $W \in \mathcal{S}(C)$, see [KK12, Theorem 2].

Proof The equalities (C.3) follow from the general theorem [KK12, Theorem 2].

It remains to prove (C.4). By the argument of [WN14, Lemma 4.8], for any compact set $K \subseteq \operatorname{Int}\Delta(S)$, there is $k_0 > 0$ such that for any $k \geq k_0$, $\alpha \in K \cap (k^{-1}\mathbb{Z})^n$ implies that $\alpha \in \Delta_k(S)$.

In particular, taking $K = \Delta(S)^\delta$ for any $\delta > 0$ and applying Lemma C.1.1, we find

$$d_{\mathrm{Haus}}(\Delta(S), \Delta_k(S)) \leq n^{1/2}k^{-1} + \delta$$

when k is large enough. This implies (C.4). □

cor:dist

Corollary C.2.1 *Let $S, S' \in \mathcal{S}'(C)$. Assume that $\operatorname{vol}(S \cap S') > 0$, then we have*

$$d_{\mathrm{sg}}(S, S') = \operatorname{vol}(S) + \operatorname{vol}(S') - 2\operatorname{vol}(S \cap S').$$

Proof This is a direct consequence of Lemma C.2.5 and (C.3). □

lma:regularizat

Lemma C.2.6 *Given $S \in \mathcal{S}'(C)$, we have $S \sim \operatorname{Reg}(S)$.*

Recall that the regularization $\operatorname{Reg}(S)$ of S is defined as $C(S) \cap \mathbb{Z}^{n+1}$.

Proof Since S and $\operatorname{Reg}(S)$ have the same Okounkov body, we have $\operatorname{vol} S = \operatorname{vol}\operatorname{Reg}(S)$ by Theorem C.2.1. By Corollary C.2.1 again,

$$d_{\mathrm{sg}}(\operatorname{Reg}(S), S) = \operatorname{vol}\operatorname{Reg}(S) - \operatorname{vol} S = 0.$$

lma:Deltaindclass

Lemma C.2.7 *Let $S, S' \in \mathcal{S}'(C)$. Assume that $d_{\mathrm{sg}}(S, S') = 0$, then $\Delta(S) = \Delta(S')$.*

Proof Observe that $\operatorname{vol}(S \cap S') > 0$, as otherwise

$$d_{\mathrm{sg}}(S, S') \geq \operatorname{vol} S + \operatorname{vol} S' > 0,$$

which is a contradiction.

It follows from Lemma C.2.5 that $S \cap S' \in \mathcal{S}'(C)$. It suffices to show that $\Delta(S) = \Delta(S \cap S')$. In fact, suppose that this holds, since $\operatorname{vol}\Delta(S') = \operatorname{vol} S' = \operatorname{vol} S = \operatorname{vol}\Delta(S)$, the inclusion $\Delta(S') \supseteq \Delta(S \cap S') = \Delta(S)$ is an equality.

By Lemma C.2.2, we can therefore replace S' by $S \cap S'$ and assume that $S \supseteq S'$. Then clearly $\Delta(S) \supseteq \Delta(S')$. By (C.3),

$$\operatorname{vol}\Delta(S) = \operatorname{vol}\Delta(S') > 0.$$

Thus, $\Delta(S) = \Delta(S')$. □

lma:Sprimeint

Lemma C.2.8 *Suppose that $S^i \in \mathcal{S}'(C)$ is a decreasing sequence such that*

$$\lim_{i\to\infty} \operatorname{vol} S^i > 0.$$

Then there is $S \in \mathcal{S}'(C)$ such that $S^i \to S$.

In general, one cannot simply take $S = \bigcap_i S^i$. For example, consider the sequence $S^i = S^1 \cap \{x_{n+1} \geq i\}$.

Proof By Lemma C.2.6, we may replace S^i by its regularization and assume that $S^i = C(S^i) \cap \mathbb{Z}^{n+1}$. We define

$$S = \left(\bigcap_{i=1}^{\infty} C(S^i) \right) \cap \mathbb{Z}^{n+1}.$$

Since $\bigcap_{i=1}^{\infty} C(S^i)$ is a full-dimensional cone by assumption, we have $S \in \mathcal{S}'(C)$. By Corollary C.2.1 and Theorem C.2.1, we can compute the distance

$$d_{\mathrm{sg}}(S, S^i) = \operatorname{vol} S^i - \operatorname{vol} S = \operatorname{vol} \Delta(S^i) - \operatorname{vol} \Delta(S),$$

which tends to 0 by construction. □

C.2.3 Okounkov bodies of almost semigroups

subsec:Okobalmosg

Definition C.2.3 We define $\overline{\mathcal{S}'(C)}_{>0}$ as elements in the closure of $\mathcal{S}'(C)$ in $\hat{\mathcal{S}}(C)$ with positive volume. An element in $\overline{\mathcal{S}'(C)}_{>0}$ is called an *almost semigroup* in C.

Recall that the volume here is defined in (C.2).

Our goal is to prove the following theorem:

thm:Okocont

Theorem C.2.2 *The Okounkov body map $\Delta\colon \mathcal{S}'(C) \to \mathcal{K}_n$ as defined in Definition C.2.2 admits a unique continuous extension*

$$\Delta\colon \overline{\mathcal{S}'(C)}_{>0} \to \mathcal{K}_n. \tag{C.5}$$ {eq:Deltagensg}

Moreover, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, we have

$$\operatorname{vol} S = \operatorname{vol} \Delta(S). \tag{C.6}$$ {eq:volWfinal}

Proof The uniqueness of the extension is clear as long as it exists. Moreover, (C.6) follows easily from Theorem C.2.1 and Theorem C.1.2 by continuity. It remains to argue the existence of the continuous extension. We first construct an extension and prove its continuity.

Step 1. We construct the desired map (C.5). Let $S \in \overline{\mathcal{S}'(C)}_{>0}$. We wish to construct a convex body $\Delta(S) \in \mathcal{K}_n$.

Let $S^i \in \mathcal{S}'(C)$ be a sequence that converges to S such that

$$d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}.$$

For each $i, j \geq 0$, we introduce

$$S^{i,j} = S^i \cap S^{i+1} \cdots \cap S^{i+j}.$$

Then by Lemma C.2.2,

$$d_{\mathrm{sg}}(S^{i,j}, S^{i,j+1}) \leq 2^{-i-j}.$$

Take $i_0 > 0$ large enough so that for $i \geq i_0$, $\operatorname{vol} S^i > 2^{-1} \operatorname{vol} S$ and $2^{2-i} < \operatorname{vol} S$ and hence

$$\operatorname{vol} S^i - \operatorname{vol} S^{i,j} \leq d_{\mathrm{sg}}(S^{i,0}, S^{i,1}) + d_{\mathrm{sg}}(S^{i,1}, S^{i,2}) + \cdots + d_{\mathrm{sg}}(S^{i,j-1}, S^{i,j}) \leq 2^{1-i}.$$

It follows that $\operatorname{vol} S^{i,j} > 2^{-1} \operatorname{vol} S - 2^{1-i} > 0$ whenever $i \geq i_0$. In particular, by Lemma C.2.5, $S^{i,j} \in \mathcal{S}'(C)$ for $i \geq i_0$.

By Lemma C.2.8, for $i \geq i_0$, there exists $T^i \in \mathcal{S}'(C)$ such that $S^{i,j} \to T^i$ as $j \to \infty$. Moreover,

$$d_{\mathrm{sg}}(T^i, S) = \lim_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S) \leq \lim_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S^i) + d_{\mathrm{sg}}(S^i, S) \leq 2^{1-i} + d_{\mathrm{sg}}(S^i, S).$$

Therefore, $T^i \to S$. We then define

$$\Delta(S) \coloneqq \overline{\bigcup_{i=i_0}^{\infty} \Delta(T^i)}.$$

In other words, we have defined

$$\Delta(S) \coloneqq \varliminf_{i\to\infty} \Delta(S^i).$$

This is an honest limit: if Δ is the limit of a subsequence of $\Delta(S^i)$, then $\Delta(S) \subseteq \Delta$ by (C.1). Comparing the volumes, we find that equality holds. So by Theorem C.1.1,

$$\Delta(S) = \lim_{i\to\infty} \Delta(S^i). \tag{C.7}$$ {eq:deltawtemp}

Next we claim that $\Delta(S)$ as defined above does not depend on the choice of the sequence S^i. In fact, suppose that $S'^i \in \mathcal{S}'(C)$ is another sequence satisfying the same conditions as S^i. The same holds for $R^i \coloneqq S^{i+1} \cap S'^{i+1}$. It follows that

$$\lim_{i\to\infty} \Delta(R^i) \subseteq \lim_{i\to\infty} \Delta(S^i).$$

Comparing the volumes, we find that equality holds. The same is true with S'^i in place of S^i. So we conclude that $\Delta(S)$ as in (C.7) does not depend on the choices we made.

Step 2. It remains to prove the continuity of Δ defined in Step 1. Suppose that $S^i \in \overline{\mathcal{S}'(C)}_{>0}$ is a sequence with limit $S \in \overline{\mathcal{S}'(C)}_{>0}$. We want to show that

$$\Delta(S^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S). \tag{C.8}$$ {eq:temp5}

We first reduce to the case where $S^i \in \mathcal{S}'(C)$. By (C.7), for each i, we can choose $T^i \in \mathcal{S}'(C)$ such that $d_{\mathrm{sg}}(S^i, T^i) < 2^{-i}$ and $d_{\mathrm{Haus}}(\Delta(S^i), \Delta(T^i)) < 2^{-i}$. If we have shown $\Delta(T^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S)$, then (C.8) follows immediately.

Next we reduce to the case where $d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}$. In fact, thanks to Theorem C.1.1, in order to prove (C.8), it suffices to show that each subsequence of $\Delta(S^i)$ admits a subsequence that converges to $\Delta(S)$. Hence, we easily reduce to the required case.

After these reductions, (C.8) is nothing but (C.7). □

Remark C.2.1 As the readers can easily verify from the proof, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, there is $S' \in \mathcal{S}'(C)$ such that $S \sim S'$.

cor:Okocomp

Corollary C.2.2 *Suppose that $S, S' \in \overline{\mathcal{S}'(C)}_{>0}$ with $S \subseteq S'$, then*

$$\Delta(S) \subseteq \Delta(S'). \tag{C.9}$$

{eq:Deltacontain}

Proof Let $S^j, S'^j \in \mathcal{S}'(C)$ be elements such that $S^j \to S$, $S'^j \to S'$. Then it follows from Lemma C.2.2 that $S^j \cap S'^j \to S$. Since vol is continuous, for large j, $S^j \cap S'^j$ has positive volume and hence lies in $\mathcal{S}'(C)$ by Lemma C.2.5. We may therefore replace S^j by $S^j \cap S'^j$ and assume that $S^j \subseteq S'^j$. Hence, (C.9) follows from the continuity of Δ proved in Theorem C.2.2. □

Remark C.2.2 As the readers can easily verify, the construction of Δ is independent of the choice of C in the following sense: Suppose that C' is another cone satisfying the same assumptions as C and $C' \supseteq C$, then the Okounkov body map $\Delta\colon \overline{\mathcal{S}'(C')}_{>0} \to \mathcal{K}_n$ is an extension of the corresponding map (C.5). We will constantly use this fact without further explanations.

Comments

Here we recall the origin of various results.

Chapter 1.

The extension theorem Theorem 1.2.1 was proved in [GR56]. In fact, they proved a more general version for complex spaces. See their Satz 3 and Satz 4. Here we reproduce their arguments almost word by word for the convenience of the readers.

The plurifine topology was introduced by Bedford–Taylor [BT87] based on Cartan's works on the fine topology. This area lacks a rigorous foundation until the appearance of [EMW06], giving the first proof of Theorem 1.3.2.

The strong openness was first established by Guan–Zhou [GZ15]. The first proof which I can understand was due to Hiep [Hie14].

The idea of Theorem 1.4.3 first appeared in the ground-breaking work of Boucksom–Favre–Jonsson [BFJ08].

Proposition 1.2.6 was due to Kiselman [Kis78].

The semicontinuity theorem was due to Siu [Siu74]

Chapter 2 The Monge–Ampère operators for bound plurisubharmonic functions were introduced by Bedford–Taylor [BT76, BT82]. The non-pluripolar product is due to Bedford–Taylor [BT87], Guedj–Zeriahi [GZ07] and Boucksom–Eyssidieux–Guedj–Zeriahi [BEGZ10].

Chapter 3

The notion of the P-envelope is due to Ross–Witt Nyström [RWN14] based on the ideas of Rashkovskii–Sigurdsson [RS05].

The $\mathcal{I}$-envelope was introduced by Darvas–Xia [DX22], inspired by the work of Dano Kim [Kim15] and Boucksom–Favre–Jonsson [BFJ08].

Chapter 4

The notion of weak geodesics was studied in detail by Darvas [Dar17] in the Kähler case.

The case of general big classes was partly handled in [DDNL18c], [DDNL18a]. However, the key fact that the geodesics between two full mass potentials have the correct limit at the end points does not seem to have been proved in any references. We give a proof in Proposition 4.3.1. We also extend the relevant results to the relative setting.

Previously, Proposition 4.3.2 and Proposition 4.3.4 were only known in the Kähler case. The proofs in the big case are kind of involved. The original treatment of Darvas in [Dar17, Lemma 3.1] in the Kähler setting is slightly flawed. In the Kähler setting, [Dar17, Lemma 3.1] can be fixed by requiring better regularity of u_0 and u_1. In the big setting, the hidden difficulty becomes essential. This explains our long proof of Proposition 4.3.2.

Chapter 5

The toric framework was first written down by Coman–Guedj–Sahin–Zeriahi in [CGSZ19].

The beautiful theorem Theorem 5.2.1 was first proved by Yi Yao, who did not publish the result. Later on, a new proof was found by Botero–Burgos Gil–Holmes–de Jong [BBGHdJ21]. We chose to present the approach of Yao, which integrates naturally with our framework.

Chapter 6

The notion of P- and $\mathcal{I}$-partial orders are new, as well as most results in Section 6.1.

The d_S-pseudometric was introduced in [DDNL21b]. The basic properties are proved in [DDNL21b] and [Xia21].

Theorem 6.2.4 is proved in [Xia22b]. Theorem 6.2.6 and Theorem 6.2.5 appear to be new. These results appeared previously in the form of lecture notes.

Chapter 7

The notion of $\mathcal{I}$-good singularities was due to [DX21]. The name *$\mathcal{I}$-good* was chosen in [Xia22b].

Theorem 7.1.1 and Eq. (7.4) are due to [DX21, DX22].

Chapter 8

The trace operator was introduced in [DX24]. Here we present a different point of view. Theorem 8.3.1 was proved in [DX24].

The analytic Bertini theorem Theorem 8.4.1 was proved in [Xia22a], based on the works of Matsumura–Fujino [FM21] and [Fuj23]. A weaker result was established by Meng–Zhou [MZ23].

Chapter 11

The application of b-divisors in pluripotential theory begins with [BFJ09]. The intersection theory of nef b-divisors was introduced by Dang–Favre [DF22]. The technique of singularity b-divisors was due to [Xia23c] and [Xia22b].

Chapter 9

The technique of test curves originates from [RWN14]. It was generalized by Darvas–Di Nezza–Lu [DDNL18a], [DX21], [DZ22] and [DXZ23]. The proofs in these literatures omit some non-trivial details when the underlying cohomology class is not ample. We give the full details.

Test curves in Definition 9.1.1 is called *maximal test curves* in the literature, a terminology which I do not like. I prefer to call the usual notion of test curves in the literature *sub-test curves*.

Results in Section 9.4 are easy generalizations of the results proved in [Xia23b].

Chapter 10

The algebraic theory of partial Okounkov bodies was developed in [Xia21]. The transcendental Okounkov body was first defined by Deng [Den17] as suggested by

Demailly. The volume identity was proved in [DRWN+23]. The transcendental theory of partial Okounkov bodies is new. Results in Section 11.3 are also new.

Chapter 12

The whole chapter appears to be new. The study of toric pluripotential theory on big line bundles was made possible by the development of partial Okounkov bodies. The key result is Theorem 12.2.2.

The toric situation of the trace operator Proposition 12.2.6 resulted from a discussion with Yi Yao.

Chapter 13

Most results from this chapter are from [Xia23b]. Results from Section 13.3 are new, although the main idea was already contained in [Xia21].

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