LECTURES ON PLURIPOTENTIAL THEORY — LECTURE 1. BEDFORD–TAYLOR THEORY

Contents

1. Introduction

These are the lecture notes for a mini-course given in ShanghaiTech University in July 2023. The goal is to explain the results of [DX22; DX21]. In the first two lectures, we will explain the construction and the basic properties of the non-pluripolar theory of Monge–Ampère operators. These parts should be of interest to a general audience. In the third and the fourth lecture, we explain the results of [DX22; DX21] and various applications.

Due to the time limitation, I usually do not give full proofs.

2. Subharmonic functions

Let $\Omega \subseteq \mathbb{R}^2 = \mathbb{C}$ be a domain (i.e. a connected open subset). We recall a few well-known properties about subharmonic functions.

Definition 2.1. A smooth function $\phi : \Omega \to \mathbb{R}$ is *subharmonic* if

$$-\Delta\phi \leq 0. \tag{2.1}$$

For many purposes, it is important to allow singularities in ϕ. [1] Fortunately, it is not very difficult to translate this definition into a regularity-free definition.

lma:submean **Lemma 2.2.** *A smooth function $\phi : \Omega \to \mathbb{R}$ is subharmonic if and only if it has the following* sub-mean value property*: for any $a \in \Omega$, $r > 0$ such that $B(a, r) \Subset \Omega$, we have*

eq:submean}
$$\phi(a) \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{2.2}$$

Here $B(a, r)$ is the open ball of radius r centered as a.

Conversely, we can use the sub-mean value property as the definition of subharmonic functions.

Definition 2.3. A function $\phi : \Omega \to [-\infty, \infty)$ is *subharmonic* if the following two conditions are satisfied:

(1) ϕ is usc (i.e. upper semicontinuous);
(2) ϕ satisfies the sub-mean value property: for any $a \in \Omega$, $r > 0$ such that $B(a, r) \Subset \Omega$, (2.2) holds.

Usually we do not regard $\phi \equiv -\infty$ as a subharmonic function.

[1] Why? Because there are more singular plurisubharmonic functions! In higher dimension, on a compact Kähler manifold, there are pseudo-effective but not nef cohomology classes. In these classes, singularities necessarily occur.

Remark 2.4. For curious readers, there are two subtleties here.

Why do we require that ϕ be usc? If not, we can always change the values of ϕ at finitely many points, there is no hope to get any *rigidity* of ϕ.

Why do we forbid ϕ from taking the value $+\infty$? If not, (2.2) does not make sense as the integral is not necessarily defined. Moreover, it is important that ϕ is locally bounded from above in order to get some *compactness*.

Exercise 2.5. Suppose that (ϕ_i) is a decreasing sequence of subharmonic functions on Ω. Show that $\phi \coloneqq \inf_i \phi_i$ is either subharmonic or equal to $-\infty$.

Proposition 2.6. *Let ϕ be a subharmonic function on Ω. For any $x \in \Omega$, let δ be the maximum of the radius such that $B(x,\delta) \subseteq \Omega$. Then the function*

$$r \mapsto M(x,r) \coloneqq \frac{1}{2\pi}\int_0^{2\pi} \phi(x + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

is continuous increasing for $r \in (0,\delta)$. Moreover, $M(x,r)$ is a convex function in $\log r$.

When $r \to 0+$, the limit exists and is equal to $\phi(x)$.

See [GZ17, Proposition 1.13]. There are a few confusing typos in this reference moreover, the continuity is stated but not proved.

Proof. We leave the last assertion to the readers and prove only the previous assertions.

In the first two steps, we argue that $M(x,r)$ is increasing in r.

Take $0 < s < r < \delta$. We want to argue that

$$\int_0^{2\pi} \phi(x + s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leq \int_0^{2\pi} \phi(x + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{2.3}$$

Step 1. We argue the following auxiliary result: for any continuous function h on $\partial B(0,1)$ satisfying

$$\phi(x + r\mathrm{e}^{\mathrm{i}\theta}) \leq h(\mathrm{e}^{\mathrm{i}\theta})$$

for any $\theta \in \mathbb{R}$, we have

$$\int_0^{2\pi} \phi(x + s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leq \int_0^{2\pi} h(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{2.4}$$

Let H be the unique continuous harmonic function on $B(0,1)$ with boundary value h. Then $\phi - H$ is a subharmonic function on $B(a,r)$ with non-positive boundary value. We claim that is has non-positive value everywhere on $B(a,r)$. The proof is left to the readers. In particular,

$$u(x + r\xi) \leq H(\xi)$$

for all $\xi \in B(0,1)$. It follows that

$$\int_0^{2\pi} \phi(x + s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leq \int_0^{2\pi} H(r^{-1}s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_0^{2\pi} H(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \int_0^{2\pi} h(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Now (2.4) follows.

Step 2. As $\phi|_{\partial B(x,r)}$ is usc, we can find a decreasing sequence of continuous functions h_j on $\partial B(0,1)$ such that

$$\phi(x + r\mathrm{e}^{\mathrm{i}\theta}) = \lim_{j\to\infty} h_j(\mathrm{e}^{\mathrm{i}\theta}).$$

By Step 1, we have

$$\int_0^{2\pi} \phi(x + s\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leq \int_0^{2\pi} h_j(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Using monotone convergence theorem, we conclude (2.3).

In Step 3, we argue the convexity of $M(x,r)$ in $\log r$. Note that this implies readily the continuity of $M(x,r)$ in r.

Step 3. We may assume that $\Omega = B(0,1)$ and $x = 0$ to simplify our notations. We observe that for each $\xi \in B(0,1)$, the function $\phi(\exp(z)\xi)$ is subharmonic in $z \in \{w \in \mathbb{C} : \operatorname{Re} w < 0\}$. It follows that

$$z \mapsto \int_0^{2\pi} \phi(\mathrm{e}^{z+\mathrm{i}\theta})\,\mathrm{d}\theta$$

is also subharmonic on the same domain. But this function is independent of $\operatorname{Im} z$, it follows that its restriction to the real axis is convex. We leave the details to the readers. □

Exercise 2.7. Fill in the omitted details in the proof.

It turns out that the class of functions defined in this way has many remarkable properties, as we will see later on in more general contexts.

log_one_var **Example 2.8.** *The key example: let $a \in \Omega$, then the function $\phi : \Omega \to [-\infty, \infty)$ defined by*

$$\phi(x) \coloneqq \log |x - a|^2$$

is subharmonic. Prove it as an exercise!

This example and its global variant are the central object in the third and the fourth lecture.
Although subharmonic functions usually have singularities, they can be locally approximated by smooth subharmonic functions. We need the standard Friedrichs mollifier technique.

Definition 2.9. A *(positive) mollifier* is a smooth function $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying:

(1) ρ is compactly supported;
(2) $\int_{\mathbb{R}^n} \rho(x)\mathrm{d}x = 1$;
(3) $\lim_{\epsilon \to 0+} \rho_\epsilon = \delta$, where

rhoepsilon}
$$\rho_\epsilon(x) \coloneqq \epsilon^{-n}\rho(x/\epsilon). \tag{2.5}$$

Here the convergence is in the sense of distributions and δ denotes the Dirac delta distribution at 0.

The standard example of a mollifier is given by the following function:

hostandard}
$$\rho(x) \coloneqq \begin{cases} \dfrac{\mathrm{e}^{-1/(1-|x|^2)}}{\int_{|x|<1} \mathrm{e}^{-1/(1-|x|^2)}\,\mathrm{d}x}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1. \end{cases} \tag{2.6}$$

Now fix the mollifier ρ on $\mathbb{R}^2$ as in (2.6) and define ρ_ϵ as in (2.5). This mollfier is special: it is supported on the unit disk and it is a radial function.
Consider a subharmonic function ϕ on Ω. We consider the convolution

$$\phi_\epsilon \coloneqq \phi * \rho_\epsilon : \Omega_\epsilon \to \mathbb{R}$$

for all small $\epsilon > 0$, where

$$\Omega_\epsilon \coloneqq \{x \in \Omega : B(x, \epsilon) \subseteq \Omega\}.$$

From distribution theory, we know that ϕ_ϵ is a smooth function and when ϵ decreases to 0, ϕ_ϵ converges to ϕ in the sense of distribution.

Exercise 2.10. Prove that ϕ_ϵ is subharmonic and as ϵ decreases to 0, ϕ_ϵ is decreasing and converges to ϕ pointwisely. Hint: for the latter, it is easy to begin with the case where ϕ is smooth. In this case, try to use Proposition 2.6. In general, first regularize ϕ.

In order to prepare for the higher dimensional generalization, let us reformulate the definition in a different way.

op:pshDelta **Proposition 2.11.** *Let $\phi : \Omega \to [-\infty, \infty)$ be a measurable function. Then ϕ coincides a.e. with a subharmonic function if and only if ϕ is locally integrable and*

{eq:Deltau}
$$-\Delta\phi \leq 0 \tag{2.7}$$

in the sense of distribution.

Sketch of the proof. For the direct implication, we may assume that ϕ is subharmonic. Then ϕ is locally integrable (prove it using Proposition 2.6!). Next we prove (2.7). When ϕ is smooth, $a \in \Omega$, by Taylor's expansion,

$$\Delta\phi(a) = \lim_{r\to 0+} \frac{2}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \phi(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta - \phi(a)\right) \geq 0.$$

In general, just choose a standard mollifier to regularize ϕ.

Conversely, suppose that ϕ is locally integrable and (2.7) holds. Again choose a standard mollifier ρ_ϵ. Show that $\phi * \rho_\epsilon$ is a smooth subharmonic function. It decreases as ϵ decreases to 0. Define ϕ as the pointwise limit of $\phi * \rho_\epsilon$. □

Exercise 2.12. Fill in the details.

3. Plurisubharmonic functions

Now let Ω be a domain in $\mathbb{C}^n$. We will extend the notion of suhharmonic functions to higher dimensions.

f:pshdomain **Definition 3.1.** A function $\phi : \Omega \to [-\infty, \infty)$ is *plurisubharmonic* (*psh* for short) if the following two conditions are satisfied:

(1) ϕ is usc;
(2) for any complex line $L \subseteq \mathbb{C}^n$, the restriction of ϕ to each connected component of $L \cap \Omega$ is subharmonic.

Similar to the subharmonic function case, we do not consider $\phi \equiv -\infty$ as a psh function. We write $\mathrm{PSH}(\Omega)$ for the set of psh functions on Ω.

We have a multi-directional version of Proposition 2.11.

:pshcurrent **Proposition 3.2.** *Suppose that $\phi : \Omega \to [-\infty, \infty)$ is a psh function. Then ϕ is locally integrable and*

$$\mathrm{dd^c}\varphi \geq 0. \tag{3.1}$$

Conversely, if a function $\phi : \Omega \to [-\infty, \infty)$ satisfies these conditions, it coincides almost everywhere with a psh function.

The mollifier technique applies without any change. The proof is left as an exercise.

Here $\mathrm{dd^c}\phi$ is current $\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\phi$. Namely,

{eq:dc}
$$\mathrm{d^c} = \frac{1}{4\pi\mathrm{i}}(\partial - \bar{\partial}). \tag{3.2}$$

The advantage is that $\mathrm{d^c}$ is a real operator, unlike $\bar{\partial}$. There is no universally agreed convention about the constant in (3.2). One has to verify the convention in each paper separately. For example, our convention differs from that in [GZ17, Page 15].

Note that our convention guarantees that

{eq:ddcphi}
$$\mathrm{dd^c}\phi(x) = \delta_a \tag{3.3}$$

in Example 2.8. Here δ_a is the Dirac measure at a.

Exercise 3.3. Prove (3.3). You need Green's second identity.

In many references, when verifying an usc locally integrable function is psh, the authors only verify $\mathrm{dd^c}\varphi \geq 0$. This is not enough! For example, we can increase the value of a psh function arbitrarily at finitely many points to obtain such an example. On the contrary, a psh function is completely determined by its restriction to any dense subset of Ω.[2]

[2]This is non-trivial. Try to prove it only if you know what quasi-continuity means.

Lemma 2.2 also admits an extension:

Proposition 3.4. *A smooth function $\phi : \Omega \to \mathbb{R}$ is psh if and only if*
$$\mathrm{dd^c}\phi \geq 0.$$

Eventually we are interested in psh functions on complex manifolds. It is therefore important to know how they behave under biholomorphic maps. More generally, we have

Proposition 3.5. *Let $\Omega \subseteq \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^{n'}$ be two domains and $f : \Omega' \to \Omega$ is a holomorphic map. Consider $\phi \in \mathrm{PSH}(\Omega')$. Then either $f^*\phi \equiv -\infty$ or $f^*\phi \in \mathrm{PSH}(\Omega)$.*

In particular, if f is biholomorphic, then $f^*\phi \in \mathrm{PSH}(\Omega)$.

This corrects the imprecision of [GZ17, Proposition 1.44].

Exercise 3.6. Prove this proposition. The proof relies on the mollifier technique again and you need Proposition 3.11 as well.

Now it is clear how to define a psh function on a manifold.

Definition 3.7. Let X be a complex manifold and $\phi : X \to [-\infty, \infty)$ a function. We say ϕ is *plurisubharmonic* (*psh* for short) if for any $x \in X$, there is a connected neighbourhood U of x in X, a biholomorphism $i : U \to V$ to a domain V in some $\mathbb{C}^n$ (n may depend on x), a psh function ψ on V such that $\phi|_U = i^*\psi$.

The set of psh functions on X is denoted by $\mathrm{PSH}(\Omega)$.

Exercise 3.8. Prove that this definition coincides with the definition in Definition 3.1.

Exercise 3.9. Extend Proposition 2.6 to psh functions.

Example 3.10. *Consider a domain $\Omega \subseteq \mathbb{C}^n$, finitely many holomorphic function $f_1, \ldots, f_N : \Omega \to \mathbb{C}$ and $c \in \mathbb{R}_{>0}$. Then*
$$\phi \coloneqq c \log \sum_i |f_i|^2$$
is a psh function. Prove it as an exercise!

There are multiple elementary ways of producing psh functions from known ones. Fix a complex manifold X.

nctionbasic **Proposition 3.11.** *We have the following:*

1. *Suppose $\varphi, \psi \in \mathrm{PSH}(X)$, then $\varphi + \psi, \varphi \vee \psi \in \mathrm{PSH}(X)$. In particular, $\varphi + C \in \mathrm{PSH}(X)$ for any constant $C \in \mathbb{R}$;*
2. *Suppose $\varphi \in \mathrm{PSH}(X)$ and $c \in \mathbb{R}_{>0}$, then $c\varphi \in \mathrm{PSH}(X)$;*
3. *Suppose $(\varphi_j)_j$ is a decreasing sequence/net of psh functions on X, then the restriction of $\inf_j \varphi_j$ to each connected component of X is either constantly $-\infty$ or psh;*
4. *Suppose $\varphi \in \mathrm{PSH}(X)$ and $f : Y \to X$ is a holomorphic map from another complex manifold Y, then the restriction of $f^*\varphi$ to each connected component of Y is either constantly $-\infty$ or psh.*

Here $\vee$ denotes the maximum. By contrast, the pointwise minimum of two psh function is not psh in general. Try to find an example!

There is a less elementary way of producing psh functions:

Theorem 3.12. *Let $(\varphi_i)_{i \in I}$ be a non-empty family of psh functions on X. Assume that this family is locally uniformly bounded from above. Then*
$$\sup_{i \in I}{}^* \varphi_i : X \to [-\infty, \infty), \quad x \mapsto \varlimsup_{y \to x} \sup_{i \in I} \varphi_i(y)$$
is psh on X. Moreover, the set
$$\left\{ x \in X : \sup_{i \in I}{}^* \varphi_i(x) > \sup_{i \in I} \varphi_i(x) \right\}$$
is pluripolar.

This is not at exercise level. The latter part is known as *Choquet's lemma*. You may find a proof in [GZ17, Corollary 4.28].

We made use of the following definition:

Definition 3.13. A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighbourhood U of x in X, a psh function ϕ on U such that

$$E \cap U \subseteq \{x \in U : \phi(x) = -\infty\}.$$

This notion is sometimes known as *locally pluripolar* in the literature.

The next non-trivial result is Grauert–Remmert's extension theorem.

Theorem 3.14. *Let $Z \subseteq X$ be an analytic set and $\phi \in \mathrm{PSH}(X \setminus Z)$. We assume that one of the following conditions is satisfied:*

(1) Z has codimension ≥ 1 and for any $x \in Z$, there is an open neighbourhood U of x in X such that $\phi|_{U\setminus Z}$ is bounded from above;

(2) Z has codimension ≥ 2.

Then there is a unique $\tilde{\phi} \in \mathrm{PSH}(X)$ such that $\phi = \tilde{\phi}|_{X\setminus Z}$.

I learned this result from their original paper [GR56]. Maybe someone could include a reference in English.

Let us point out that in the global setting, there are very few psh functions.

pshfunction **Proposition 3.15.** *Assume that X is compact. Then $\mathrm{PSH}(X)$ consists of locally constant functions $\varphi : X \to \mathbb{R}$.*

The proof is again left as an exercise.

In view of Proposition 3.15, it is of interest to extend the notion of psh functions. This leads to the following definition:

Definition 3.16. Given a closed real smooth $(1,1)$-form θ on X, we say a function $\varphi : X \to [-\infty, \infty)$ is *θ-plurisubharmonic* (*θ-psh* for short) if

(1) for each $x \in X$, there is an open neighbourhood U of x in X a psh function ϕ on U, a smooth function $p : U \to \mathbb{R}$ such that $\varphi|_U = \phi + p$;

(2) $\theta_\varphi := \theta + \mathrm{dd^c}\varphi \geq 0$.

We denote the set of θ-psh functions by $\mathrm{PSH}(X, \theta)$.

A function $\varphi : X \to [-\infty, \infty)$ is *quasi-plurisubharmonic* (*qpsh* for short) if condition (1) is satisfied.

There are plenty of qpsh functions in general.

eq:singP1 **Example 3.17.** *Take $X = \mathbb{P}^1$, $\theta = \omega$ is the Fubini–Study form: denote the homogeneous coordinate by $[X_0 : X_1]$, then ω is*

$$\omega := \mathrm{dd^c} \log\left(|X_0|^2 + |X_1|^2\right).$$

Show that there is $\varphi \in \mathrm{PSH}(X, \omega)$ with $\omega + \mathrm{dd^c}\varphi = \delta_0$. Write φ down explicitly in local coordinates.

Qpsh functions are stable under various operations as before.

Proposition 3.18. *Let θ be a smooth real closed $(1,1)$-form on X.*

(1) Suppose $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ and $t \in (0,1)$, $C \in \mathbb{R}$, then $t\varphi + (1-t)\psi, \varphi + C, \varphi \vee \psi \in \mathrm{PSH}(X, \theta)$;

(2) Suppose $(\varphi_j)_j$ is a decreasing sequence/net in $\mathrm{PSH}(X, \theta)$, then the restriction of $\inf_j \varphi_j$ to each connected component of X is either constantly $-\infty$ or θ-psh[3];

(3) Suppose $\varphi \in \mathrm{PSH}(X, \theta)$ and $f : Y \to X$ is a holomorphic map from another complex manifold Y, then the restriction of $f^*\varphi$ to each connected component of Y is either constantly $-\infty$ or $f^*\theta$-psh;

[3]Here and in the sequel we are abusing the language by denoting the restriction of θ as θ.

(4) Let $(\varphi_i)_{i\in I}$ be a non-empty family of θ-psh functions on X. Assume that this family is locally uniformly bounded from above. Then

$$\sup_{i\in I}{}^* \varphi_i : X \to [-\infty,\infty), \quad x \mapsto \overline{\lim_{y\to x}} \sup_{i\in I} \varphi_i(y)$$

is θ-psh on X. Moreover, the set

$$\left\{ x\in X : \sup_{i\in I}{}^* \varphi_i(x) > \sup_{i\in I} \varphi_i(x) \right\}$$

is pluripolar;

(5) Let $Z \subseteq X$ be an analytic set and $\phi \in \mathrm{PSH}(X \setminus Z, \theta|_{X\setminus Z})$. We assume that one of the following conditions is satisfied:

(a) Z has codimension ≥ 1 and for any $x \in Z$, there is an open neighbourhood U of x in X such that $\phi|_{U\setminus Z}$ is bounded from above;

(b) Z has codimension ≥ 2.

Then there is a unique $\tilde{\phi} \in \mathrm{PSH}(X,\theta)$ such that $\phi = \tilde{\phi}|_{X\setminus Z}$.

4. The Monge–Ampère operator — Local theory

One reason why we are interested in regular qpsh functions is their relation to certain geometric PDE, known as the *complex Monge–Ampère equation*:

$$(\omega + \mathrm{dd^c}\phi)^n = \mu.$$

Here ω is a Kähler form on a compact manifold of pure dimension n, ϕ is a smooth ω-psh function and μ is a smooth volume form such that $\int_X \mu = \int_X \omega^n$. Yau's solution to the Calabi conjecture implies that this equation is always solvable. However, for most applications, it is of interest to investigate rougher data, where μ is no longer smooth. In this case, we can no longer expect ϕ to be smooth or even continuous. It is therefore important to have a definition of $(\omega + \mathrm{dd^c}\phi)^n$ for rough ϕ.

There are many different definitions of $(\omega + \mathrm{dd^c}\phi)^n$ in general, but when ϕ is locally bounded, there is only one reasonable definition. This leads to the Bedford–Taylor theory.

We first recall the notion of positive forms.

Definition 4.1. Let V be a complex vector space of dimension n. An (n,n)-form α on V is *positive* if it is a non-negative multiple of a volume form on V: if $\mathrm{d}z_1,\ldots,\mathrm{d}z_n$ denote a basis of V, then

$$\alpha = \lambda \mathrm{i}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{i}\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n, \quad \lambda \geq 0.$$

A (p,p)-form $(0 \leq p \leq n)$ is *strongly positive* if it is a finite linear combination with non-negative coefficients of forms of the following form:

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha_p},$$

where $\alpha_1,\ldots,\alpha_p$ are $(1,0)$-forms.

A (p,p)-form $(0 \leq p \leq n)$ β is *weakly positive* if for all $(1,0)$-forms $\alpha_1,\ldots,\alpha_{n-p}$, the following form is positive:

$$\beta \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha_1} \wedge \cdots \wedge \mathrm{i}\alpha_{n-p} \wedge \overline{\alpha_{n-p}}.$$

A (p,p)-form $(0 \leq p \leq n)$ is *positive* if it is a finite linear combination with non-negative coefficients of forms of the following form:

$$\mathrm{i}^{p^2} \alpha \wedge \bar{\alpha},$$

where α is a $(p,0)$-form.

A form on a complex manifold is *positive* (resp. *strongly positive*, *weakly positive*) if it is so at each point.

Definition 4.2. A (p,p)-current $(0 \leq p \leq n)$ T on a complex manifold is *strongly positive* if for all weakly positive $(n-p,n-p)$-form α with compact support, $\langle T, \alpha\rangle \geq 0$.

A (p,p)-current $(0 \leq p \leq n)$ T is *weakly positive* if for all strongly positive $(n-p,n-p)$-form α, $\langle T, \alpha\rangle \geq 0$.

A (p,p)-current $(0 \leq p \leq n)$ T is *positive* if for all positive $(n-p,n-p)$-form α, $\langle T, \alpha\rangle \geq 0$.

In this case of $(1,1)$-currents, these notions are equivalent.

Example 4.3. *If ϕ is a psh function on a complex manifold, then $\mathrm{dd^c}\phi$ is a closed positive $(1,1)$-current.*

We will fix a domain $\Omega \subseteq \mathbb{C}^n$.

Definition 4.4. Let T be a closed positive (p,p)-current on Ω and $\phi \in \mathrm{PSH}(\Omega)$. Assume that ϕ is locally bounded. Then we define a $(p+1,p+1)$-current $\mathrm{dd^c}\phi \wedge T$ on Ω by

$$\mathrm{dd^c}\phi \wedge T \coloneqq \mathrm{dd^c}(\phi T). \tag{4.1}$$

In other words, if we take a smooth $(n-p-1,n-p-1)$-form α with compact support on Ω, we have

$$\langle \mathrm{dd^c}\phi \wedge T, \alpha\rangle \coloneqq \langle \phi T, \mathrm{dd^c}\alpha\rangle.$$

Note that by definition, $\mathrm{dd^c}\phi \wedge T$ is a closed current and is continuous along decreasing limits of ϕ. In particular, if ϕ_j denotes a local regularization of ϕ, we can write $\mathrm{dd^c}\phi \wedge T$ as the weak limit of $\mathrm{dd^c}\phi_j \wedge T$.

Exercise 4.5. Show the positivity of $\mathrm{dd^c}\phi_j \wedge T$ and deduce the positivity of $\mathrm{dd^c}\phi \wedge T$.

More generally, by iteration, we get the following definition:

def:MABT **Definition 4.6.** Let T be a closed positive (p,p)-current on Ω and $\phi_1,\dots,\phi_k \in \mathrm{PSH}(\Omega)$. Then we define a closed positive $(p+k,p+k)$-current $\mathrm{dd^c}\phi_1 \wedge \cdots \mathrm{dd^c}\phi_k \wedge T$ on Ω by

$$\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_k \wedge T \coloneqq \mathrm{dd^c}\left(\phi_1 \mathrm{dd^c}\phi_2 \wedge \cdots \mathrm{dd^c}\phi_k \wedge T\right).$$

When T is the current of integration along Ω, we just write

$$\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_k$$

instead. These operators are know as the *Monge–Ampère operators*.

It remains to show that this operator behaves in a good way.

thm:CLN **Theorem 4.7** (Chern–Levine–Nirenberg)**.** *Let T be a closed positive $(n-k,n-k)$-current on Ω and $\phi_1,\dots,\phi_k \in \mathrm{PSH}(\Omega)$ be locally bounded functions. Then for any open subsets $\Omega_1 \Subset \Omega_2 \Subset \Omega$, there is a constant $C = C(\Omega_1,\Omega_2) > 0$ such that for any compact subset $K \subseteq \Omega_1$, we have*

$$\int_K \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_k \wedge T \leq C\|\phi_1\|_{L^\infty(E)} \cdots \|\phi_k\|_{L^\infty(E)}\|T\|_E,$$

where $E = (\Omega_2 \setminus \Omega_1) \cap \operatorname{Supp} T$.

Here $\|T\|_E$ need further explanation. We expand T as

$$\mathrm{i}^{(n-k)^2} \sum_{|I|=|J|=n-k} T_{I,J}\, \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J.$$

The positivity of T implies that $T_{I,J}$ are of order 0 and the total variation of the off diagonal elements are dominated by the diagonal ones. So we can define

$$\|T\|_E = \sum_{|I|=n-k} |T_{I,I}|_E,$$

where $|T_{I,I}|_E$ is the total variation of the positive measure $T_{I,I}\mu$ on E, where μ is the standard Lebesgue measure. [4]

[4]The definition in [GZ17, Proposition 2.18] is not quite precise. In particular, $T_{I,J}$ themselves are not measures.

Proof. We follow the elegant proof in [GZ17, Theorem 3.9]. By induction, we may assume that $k = 1$. We write ϕ instead of ϕ_1. We may assume that $\phi \leq 0$ on Ω_2 up to replacing ϕ by $\phi - \sup_{\Omega_2} \phi$.

Let $\chi : \Omega_2 \to \mathbb{R}$ be a smooth function with compact support such that $\chi = 1$ on Ω_1. Then

$$\int_{\Omega_1} T \wedge \mathrm{dd^c}\phi \leq \int_{\Omega_2} \chi T \wedge \mathrm{dd^c}\phi.$$

As $\mathrm{dd^c}\chi = 0$ on Ω_1, we have

$$\int_{\Omega_2} \chi T \wedge \mathrm{dd^c}\phi = \int_{\Omega_2 \backslash \Omega_1} \phi\, \mathrm{dd^c}\chi \wedge T.$$

Fix $A > 0$ so that $\mathrm{dd^c}\chi \leq A\beta$, where $\beta = \mathrm{i} \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$. Then

$$\int_{\Omega_2 \backslash \Omega_1} \phi\, \mathrm{dd^c}\chi \wedge T \leq A \|\phi\|_{\Omega_2 \backslash \Omega_1} \int_E T \wedge \beta.$$

Putting everything together, we conclude the proof. □

Exercise 4.8. Fill in the omitted details in the proof.

The Chern–Levine–Nirenberg inequality is the cornerstone of the whole pluripotential theory. It allows us to reduce problems related to bounded psh functions to the simpler problem of smooth psh functions. We will see its power in the following example.

Corollary 4.9. *Let T be a closed positive (p,p)-current on Ω. Let ϕ_j be a decreasing sequence of bounded psh functions in Ω with limit $\phi \in \mathrm{PSH}(\Omega)$. We assume that ϕ is locally bounded. Then for any continuous psh function h on Ω, we have*

$$h\, \mathrm{dd^c}\varphi_j \wedge T \rightharpoonup h\, \mathrm{dd^c}\varphi \wedge T, \quad \mathrm{dd^c}h \wedge \mathrm{dd^c}\varphi_j \wedge T \rightharpoonup \mathrm{dd^c}h \wedge \mathrm{dd^c}\varphi \wedge T$$

as measures *in the following sense: multiplying everything by a test form of suitable bidegree, then the corresponding measures converge weakly.*

Proof. We know that $\mathrm{dd^c}\phi_j \wedge T \rightharpoonup \mathrm{dd^c}\phi \wedge T$ as currents. But Theorem 4.7 also guarantees that the masses are locally uniformly bounded. It follows that the convergence holds as measures. Now using the continuity of h, we conclude the first convergence.

The second is left as an exercise. □

Corollary 4.10. *Under the notations of Definition 4.6, if σ is a permutation of the set $\{1, \dots, k\}$, then*

$$\mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_k \wedge T = \mathrm{dd^c}\phi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\phi_{\sigma(k)} \wedge T. \tag{4.2}$$

Proof. We reduce immediately to the case $k = 2$. Everything is obvious when ϕ_1 and ϕ_2 are both smooth.

Next we treat the less trivial case where only ϕ_1 is smooth. Locally write ϕ_2 as a decreasing limit of smooth psh functions ψ_j, then

$$\mathrm{dd^c}\phi_1 \wedge \mathrm{dd^c}\psi_j \wedge T = \mathrm{dd^c}\psi_j \wedge \mathrm{dd^c}\phi_1 \wedge T.$$

Letting $j \to \infty$ and applying Corollary 4.9, we conclude (4.2).

When neither of ϕ_1 and ϕ_2 are smooth, the proof is slightly more involved. We refer to [GZ17, Corollary 3.12]. □

The Monge–Ampère operators are not continuous in general, but we still have the following:

Theorem 4.11. *Let $(\phi_i^j)_j$ be decreasing sequences (resp. increasing sequences) of locally bounded psh functions on Ω converging (resp. converging a.e.) to locally bounded psh function ϕ_i, where $i = 1, \dots, q$. Then*

$$\phi_0^j\, \mathrm{dd^c}\phi_1^j \wedge \cdots \wedge \mathrm{dd^c}\phi_q^j \rightharpoonup \phi_0\, \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_q$$

as $j \to \infty$. In particular, if ϕ_0^j is the constant sequence 1, we have

$$\mathrm{dd^c}\phi_1^j \wedge \cdots \wedge \mathrm{dd^c}\phi_q^j \rightharpoonup \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_q.$$

These results are highly non-trivial consequences of Theorem 4.7. We refer to [GZ17, Theorem 3.18, Theorem 3.23] for the proofs.

Remark 4.12. It is natural to wonder we can allow part of the sequences $(\phi_i^j)_j$ be decreasing and part of them be increasing. This turns out to be true. For this purpose, we need the more sophisiticated notion of convergence in capacity. We refer to [GZ17, Theorem 4.26].

Finally, we mention two key properties of the Bedford–Taylor theory.

Proposition 4.13. *Let $\phi_1, \dots, \phi_p \in \mathrm{PSH}(\Omega)$ be bounded. Then for any smooth $(n-p, n-p)$-form α and any pluripolar set $E \subseteq \Omega$, we have*

$$\int_E (\mathrm{dd^c}\phi_1 \wedge \dots \wedge \mathrm{dd^c}\phi_p) \wedge \alpha = 0.$$

In other words, the Bedford–Taylor product does not put mass on pluripolar sets.

Theorem 4.14. *Let $\phi_1, \dots, \phi_p, \varphi_1, \dots, \varphi_p \in \mathrm{PSH}(\Omega)$ be locally bounded. Consider two psh functions $\psi_1, \psi_2 \in \mathrm{PSH}(\Omega)$ (not necessarily bounded). Define $E = \{\psi_1 > \psi_2\}$. Assume that $\phi_i|_E = \varphi_i|_E$ for all $i = 1, \dots, p$. Then*

$$\mathbb{1}_E \, \mathrm{dd^c}\phi_1 \wedge \dots \wedge \mathrm{dd^c}\phi_p = \mathbb{1}_E \, \mathrm{dd^c}\varphi_1 \wedge \dots \wedge \mathrm{dd^c}\varphi_p.$$

This result is usually expressed by saying that the Bedford–Taylor product is local in the pluri-fine topology.

5. The Monge–Ampère operator — Global theory

Eventually we are interested in the global picture, namely the pluripotential theory on compact Kähler manifolds. We will see how the Bedford–Taylor theory can be extended to this setting.

We first begin with a relatively general setting. Let X be a complex manifold of pure dimension n. Consider a closed positive (p, p)-current T on X, $\theta_1, \dots, \theta_k$ be smooth real closed $(1,1)$-forms on X. Consider locally bounded functions $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ for $i = 1, \dots, k$. Then we want to define a $(p+k, p+k)$-current $\theta_{\varphi_1} \wedge \dots \wedge \theta_{\varphi_k} \wedge T$ on X.

For this purpose, take $x \in X$, an open neighbourhood U of x in X biholomorphic to a simply connected domain $\Omega \subseteq \mathbb{C}^n$ thorough a map $\eta : U \to \Omega$. Then we can write $\theta_i = \mathrm{dd^c} g_i$ for some smooth functions g_i on U. Then we define

$$(\theta_{\varphi_1} \wedge \dots \wedge \theta_{\varphi_k} \wedge T)|_U \coloneqq \eta_*^{-1}\left(\mathrm{dd^c}(g_1 \circ \eta^{-1} + \varphi_1 \circ \eta^{-1}) \wedge \dots \wedge \mathrm{dd^c}(g_1 \circ \eta^{-1} + \varphi_k \circ \eta^{-1}) \wedge \eta_* T\right).$$

Exercise 5.1. Show that this definition is independent of the choices we made and gives a closed positive $(p+k, p+k)$ current on X.

The continuity theorem Theorem 4.11 holds with suitable modification, which we left to the readers.

So far, we have not imposed any Kähler condition. Now let us see what happens if X is compact Kähler. From now on, we assume that X is a compact Kähler manifold of pure dimension n.

Theorem 5.2 ($\partial\bar{\partial}$-lemma)**.** *Let α be a d-exact smooth real (p, q)-form (resp. real d-exact (p, q)-current) on X $(p, q \geq 1)$. Then there is a smooth real $(p-1, q-1)$-form (resp. real $(p-1, q-1)$-current) β on X such that*

$$\alpha = \mathrm{dd^c}\beta.$$

In particular, when $p = q = 1$, this tells us that every form or current in a given cohomology class $\{\alpha\}$ can always be written as $\alpha + \mathrm{dd^c} g$ for some $(0,0)$-current. If moreover we impose the positivity assumption, this yields the following corollary:

Corollary 5.3. *Let θ be a smooth real closed $(1,1)$-form on X. The map $\varphi \mapsto \theta_\varphi$ sending $\varphi \in \mathrm{PSH}(X, \theta)$ to the current θ_φ induces an identification $\mathrm{PSH}(X, \theta)/\mathbb{R}$ with the set of closed positive $(1,1)$-currents in the cohomology class $\{\theta\}$.*

We close this lecture by an easy observation:

Proposition 5.4. *Let θ be a smooth real closed $(1,1)$-form on X and $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Assume that φ, ψ are both bounded, then*

{eq:massBT} (5.1)
$$\int_X \theta_\varphi^n=\int_X \theta_\psi^n. \tag{5.1}$$

Proof. Take a Kähler form ω on X. It suffices to show that for sufficiently small $\epsilon>0$,

$$\int_X (\epsilon\omega+\theta)_\varphi^n=\int_X (\epsilon\omega+\theta)_\psi^n.$$

In particular, we may assume in the beginning that θ_φ and θ_ψ are Kähler currents.

In this case, we have the celebrated regularization theorem of Demailly: we can find decreasing sequences of smooth functions $\varphi_j,\psi_j\in\mathrm{PSH}(X,\theta)$ converging pointwise to φ and ψ respectively. Using the continuity result Theorem 4.11, it suffices to prove (5.1) with φ_j, ψ_j in place of φ and ψ. So we may assume that φ and ψ are smooth.

In this case, it suffices to observe that $\theta_\varphi^n-\theta_\psi^n$ is exact. □

It therefore makes sense to call this common value the *volume* of the class $\{\theta\}$. When θ is semi-positive, we can take $\varphi=0$, then the volume of the class is nothing but $\int_X\theta^n$. We have seen a very important phenomenon: the Bedford–Taylor products do not lose mass.

6. Lelong numbers of psh functions

We include a brief introduction of the Lelong numbers.

We begin with the situation on a domain $\Omega\subseteq\mathbb{C}^N$. The Lelong number is a rough measure of the singularities of a psh function.

Definition 6.1. Given $x\in\Omega$, $\varphi\in\mathrm{PSH}(X)$, define the *Lelong number* of φ at x by

$$\nu(\varphi,x):=\lim_{r\to 0+}\frac{\sup_{B(x,r)}\varphi}{\log r}.$$

Exercise 6.2. Show that the limit exists and

$$\nu(\varphi,x)=\sup\{c\geq 0:\varphi(y)\leq c\log|y-x|+\mathcal{O}(1)\text{ as }y\to x\}.$$

Exercise 6.3. Compute $\nu(\varphi,0)$ in Example 3.17.

Exercise 6.4. As a challenging exercise, prove that

$$\nu(\varphi,x):=\lim_{r\to 0+}\frac{\int_{\partial B(x,r)}\varphi}{|\partial B(x,r)|\log r}.$$

Make sure that you know what Harnack inequality is before doing this exercise.

As a corollary, show that if ψ is another psh function on Ω, we have

$$\nu(\max\{\varphi,\psi\},x)=\min\{\nu(\varphi,x),\nu(\psi,x)\},\quad \nu(\varphi+\psi,x)=\nu(\varphi,x)+\nu(\psi,x).$$

Exercise 6.5. Show that the Lelong number is a biholomorphic invariant notion. Explain why and how it can be extended to quasi-psh function on manifolds.

Now given this exercise, we can take talk about Siu's semi-continuity theorem:

Theorem 6.6. *Let X be a complex manifold and φ be a qpsh function on X. Then for any $c>0$,*

$$\{x\in X:\nu(\varphi,x)\geq c\}$$

is an analytic subset of X.

This result allows us to define the generic Lelong number: given a prime divisor E on X, we define

$$\nu(\varphi,E):=\min_{x\in E}\nu(\varphi,x).$$

Note that $\nu(\varphi,E)=\nu(\varphi,x)$ for a general $x\in E$. More generally, for a prime divisor E over X, we can take a bimeromorphically modification $\pi:Y\to X$ from a connected compact Kähler manifold Y to X such that E is a prime divisor on E. In this case, we define

$$\nu(\varphi,E):=\nu(\pi^*\varphi,E).$$

Exercise 6.7. Show that this definition is independent of the choice of π.

Another useful and closely related measure of singularities is given by the multiplier ideal sheaf.

Definition 6.8. Let X be a complex manifold and φ be a qpsh function on X. Then multiplier ideal sheaf $\mathcal{I}(\varphi)$ of φ is the analytic ideal sheaf on X locally generated by holomorphic functions f such that [5]

$$\int |f|^2 e^{-\varphi} < \infty.$$

It is a celebrated theorem of Nadel that $\mathcal{I}(\varphi)$ is coherent. It can be used to give a vast extension of the Kodaira–Kawamata–Vieweg vanishing theorem. The vanishing theorem is known as Nadel–Cao vanishing theorem. We will return to this result later on.

The multiplier ideal sheaf is smaller when the qpsh function is more singular. It is also a measure of singularity.

Of course, the two measures of singularities should be related. The precise relation is given by the following theorem:

Theorem 6.9. *A local holomorphic function f belongs to $\mathcal{I}(\varphi)$ if and only if there is $\epsilon > 0$ such that*

$$\operatorname{ord}_E(f) \geq (1+\epsilon)\nu(\varphi, E) - A_X(E), \tag{6.1}$$

where $A_X(E) \in \mathbb{R}_{\geq 0}$ is the log-discrepancy of E.

So the Lelong numbers determine the multiplier ideal sheaf. Conversely, $\mathcal{I}(k\varphi)$ for all $k \in \mathbb{Z}_{>0}$ uniquely determines the Lelong numbers as well, try to give a proof!

Scholie: The information of all multiplier ideal sheaves $\mathcal{I}(k\varphi)$ is equivalent to the information of all generic Lelong numbers $\nu(\varphi, E)$.

[5] Some authors use $e^{-2\varphi}$ instead.

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