Lectures on Vertex Operator Algebras and Conformal Blocks

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Contents

0 Notations

- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$.
- $\mathrm{i} = \sqrt{-1}$, $\mathbb{S}^1$ =unit circle, $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$, $\mathbb{D}_r^{\mathrm{cl}} = \{z \in \mathbb{C} : |z| \leqslant r\}$
- $\mathscr{O}(X)$ is the space of holomorphic functions on a complex manifold X.
- Configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}$.
- z and ζ could mean either points, or the standard coordinate of $\mathbb{C}$, or formal variables. We will give their meanings when the context is unclear.
- All vector spaces are over $\mathbb{C}$, unless otherwise stated. If W is a vector space equipped with a Hermitian form $\langle\cdot|\cdot\rangle$, we let $|\cdot\rangle$ be the linear variable and $\langle\cdot|$ be the antilinear (i.e. conjugate linear) one, following physicists' convention.
- If W, W' are vector spaces, then $\mathrm{Hom}(W, W')$ denote the space of linear operators from W to W'. We let $\mathrm{End}(W) = \mathrm{Hom}(W, W)$.
- We use symbols $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$ to denote bilinear forms (i.e., linear on both variables).
- Given a vector space W and a formal variable z,

$$W[z] = \{\text{polynomials of } z \text{ whose coefficients are elements of } W\}$$
$$W[[z]] = \Big\{\sum_{n\in\mathbb{N}} w_n z^n : w_n \in W\Big\}$$
$$W((z)) = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W, \text{ and } w_n = 0 \text{ when } n \text{ is sufficiently negative}\Big\}$$
$$W[[z^{\pm 1}]] = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W\Big\}.$$

  Each line is a subspace of the subsequent line. In case there are several formal variables, the spaces are defined in a similar way, expect $W((\cdots))$. For instance,

$$W[[z, \zeta^{\pm 1}]] := W[[z]][[\zeta^{\pm 1}]] = W[[\zeta^{\pm 1}]][[z]]$$

  consists of $\sum_{m\in\mathbb{N}, n\in\mathbb{Z}} w_{m,n} z^m \zeta^n$ where each $w_{m,n} \in W$. However, note that $W((z))((\zeta))$ and $W((\zeta))((z))$ are not equal. (For instance, $\sum_{m\geqslant -n}\sum_{n\geqslant -1} z^m\zeta^n$ belongs to $\mathbb{C}((z))((\zeta))$ but not $\mathbb{C}((\zeta))((z))$.)

- We set

$$\mathrm{Res}_{z=0} \sum_{n\in\mathbb{Z}} w_n z^n dz = w_{-1}. \tag{0.2}$$

  This is in line with the complex analytic residue.

- A vector of $W_1 \otimes \cdots \otimes W_N$ writen as $w_\bullet$ means that it is of the form $w_1 \otimes \cdots \otimes w_N$ where each $w_i \in W_i$. Depending on the context, $w_\bullet$ will also mean a tuple $(w_1, \ldots, w_N)$.
- Unless otherwise stated, by a manifold, we mean one *without* boundaries. Also, "with boundaries" means "possibly with boundaries".

1 Segal's picture of 2d CFT; motivations of VOAs and conformal blocks

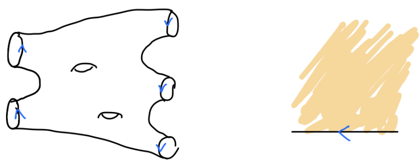
1.1

Vertex operator algebras (VOAs) are mathematical objects defined to understand and construct 2-dimensional conformal field theory (CFT for short). A CFT describes propagations and interactions of strings. The are two types of strings: closed strings $\simeq \mathbb{S}^1$ and open strings $\simeq [0,1]$. In this course, we will mainly focus on closed strings.

Let me explain how mathematicians understand CFT. Just like any quantum field theory (QFT), in CFT we must have a Hilbert space $\mathcal{H}$. The vectors in $\mathcal{H}$ are called "states", but unlike ordinary QFT, a vector $\xi \in \mathcal{H}$ is not a state of a particle, but a state of a closed string $\mathbb{S}^1$.

The most important and non-trivial part in CFT is to define/understand string interactions. According to Segal's picture [Seg88], an interaction is uniquely determined by a compact Riemann surface Σ with boundaries $\partial\Sigma$, where $\partial\Sigma$ is a disjoint union of some circles (strings). Each string is called either an incoming string or an outcomming one. Suppose $\partial\Sigma$ has N incoming strings and M outgoing ones, then this picture describes an interaction where N strings are going inside, and M strings are going outside.

Moreover, the boundary $\partial\Sigma$ must be **parametrized**. This means that to each connected component $\partial\Sigma_i$ a diffeomorphism $\eta_i : \partial\Sigma_i \xrightarrow{\simeq} \mathbb{S}^1$ is associated. The orientation on $\partial\Sigma_i$ defined by pulling back the one of $\mathbb{S}^1$ along η_i is assumed to be the opposite of the one defined in Stokes' theorem, shown as follows



(1.1)

1.2

Unless otherwise stated, we assume that the boundary parametrization is also **analytic**. Roughly speaking, this means that Σ can be obtained by removing some open discs from a compact Riemann surface C (without boundary) such that the parametrizations of $\partial\Sigma$ are given by local holomorphic functions of C.

Here is a more rigorous explanation. By a **local coordinate** η of C at $x \in C$, we mean η is a holomorphic injective function on a neighborhood U of x such that $\eta(x) = 0$. So

η is a biholomorphism between U and a neighborhood $\eta(U)$ of 0. Now, suppose we have local coordinates $\eta_1, \ldots, \eta_N$ at distinct points $x_1, \ldots, x_N \in C$. The data

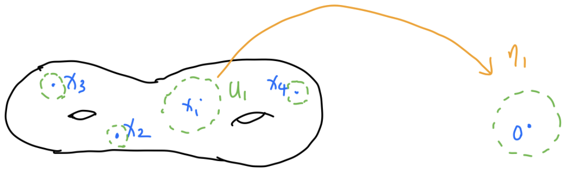
$$\mathfrak{X} := (C; x_\bullet; \eta_\bullet) = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N) \tag{1.2}$$

is called an **N-pointed compact Riemann surface with local coordinates**.

Let each η_i be defined on a neighborhood $U_i \ni x_i$. We assume moreover the following

Assumption 1.1. $U_i \cap U_j = \varnothing$ if $i \neq j$, and $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$ for each i. Here $\mathbb{D}_1^{\mathrm{cl}}$ is the closed unit disc.

By removing all $\eta_i^{-1}(\mathbb{D}_1)$, we get Σ with boundary strings $\eta_i^{-1}(\partial \mathbb{D}_1^{\mathrm{cl}}) = \eta_i^{-1}(\mathbb{S}^1)$ whose parametrizations are η_i.



1.3

Any Σ as above determines uniquely an interaction of strings. Suppose it has N incoming strings and M outgoing ones. Then mathematically, such an interaction is described by a bounded linear map $T = T_\Sigma : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes M}$. (The boundedness is automatic thanks to the uniform boundedness principle. But this is not an important point in this course.) Given $\xi_\bullet = \xi_1 \otimes \cdots \otimes \xi_N \in \mathcal{H}^{\otimes N}$ and $\eta_\bullet = \eta_1 \otimes \cdots \otimes \eta_M \in \mathcal{H}^{\otimes M}$, the value

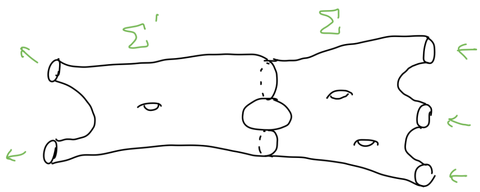
$$\langle \eta_\bullet | T \xi_\bullet \rangle \tag{1.3}$$

describes the probability amplitude that the N incoming closed strings with states $\xi_1, \ldots, \xi_N$ become $\eta_1, \ldots, \eta_M$ after interaction.

The word "conformal" in conformal field theory reflects the fact that T depends only on the complex structure of Σ and its parametrization, but not on the metric for instance. Thus, a CFT is more rigid than a topological quantum field theory (TQFT): in the latter case, T depends only on the topological structures of the manifolds.

1.4

Suppose we have another interaction $S : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes L}$ corresponding to the parametrized surface Σ', then the composition of them $S \circ T : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes L}$ corresponds to the **sewing** $\Sigma \# \Sigma'$ of Σ and Σ', where the j-th outgoing string $\partial_+ \Sigma_j$ of Σ is sewn with the j-th incoming one $\partial_- \Sigma'_j$ of Σ'.



It is important to specify how $\partial_+\Sigma_j$ (with parametrization η_j) is identified with $\partial_-\Sigma'_j$ (with parametrization η'_j). Pick $x \in \partial_+\Sigma_j$ and $y \in \partial_-\Sigma'_j$. Then

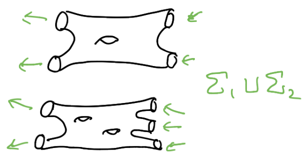
$$x = y \qquad \Longleftrightarrow \qquad \eta_j(x)\eta'_j(y) = 1. \tag{1.4}$$

It is clear from the picture that the orientations of $\partial_+\Sigma_j$ and $\partial_-\Sigma_j$ are opposite to each other. This is related to the fact that our rule for sewing is $\eta_j(x) = 1/\eta'_j(y)$ but not (say) $\eta_j(x) = \eta'_j(y)$.

Recall we assume that the parametrizations are analytic. We leave it to the readers to check that the sewing of Σ and Σ', a priori only a topological surface, has a natural complex analytic structure.

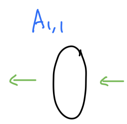
1.5

Suppose $T_1 : \mathcal{H}^{\otimes N_1} \to \mathcal{H}^{\otimes M_1}$ corresponds to Σ_1 and $T_2 : \mathcal{H}^{\otimes N_2} \to \mathcal{H}^{\otimes M_2}$ to Σ_2, then $T_1 \otimes T_2 : \mathcal{H}^{\otimes(N_1+N_2)} \to \mathcal{H}^{\otimes(M_1+M_2)}$ corresponds to the disjoint union $\Sigma_1 \sqcup \Sigma_2$.



1.6

Consider an annulus $A_{r,R} = \{z \in \mathbb{Z} : r < |z| < R\}$ obtained by removing two open discs from the compact Riemann sphere $\mathbb{P}^1$ via the local coordinate $\eta_1(z) = z/r$ at 0 and $\eta_2(z) = R/z$ at ∞. We call such $A_{r,R}$ (with the given boundary parametrization) a **standard annlus**. Let $r \nearrow 1, R \searrow 1$. The limit of this annulus is a "degenerate" Riemann surface with 1 incoming boundary circle and 1 outing one. Both circles are $\mathbb{S}^1$. The incoming one has parametrization $z \mapsto z$ and the outgoing one $z \mapsto z^{-1}$. We call this annulus the **standard thin annulus** and denote it by $A_{1,1}$. *The map $T : \mathcal{H} \to \mathcal{H}$ associated to $A_{1,1}$ is the identity map.* This reflects the fact that sewing any Σ with a disjoint union of $A_{1,1}$ gives Σ.



1.7

We give a fancy way to summarize what we have so far: Let $\mathscr{C}$ be the monoidal category of compact 1-dimensional smooth manifolds such that a morphism from an object S_1 to another S_2 is a compact Riemann surface with incomming parametrized boundary $\simeq S_1$ and outgoing one $\simeq S_2$, that the identity morphism for a union of N circles is a disjoint union of N pieces of $A_{1,1}$, that the unit object is the empty set, and that the tensor product of objects and morphisms are respectively the disjoint unions of strings and Riemann surfaces. Then a CFT is a monoidal functor from $\mathscr{C}$ to the monoidal category of Hilbert spaces. So, roughly speaking, a CFT is a representation of $\mathscr{C}$.

Since we choose Hilbert spaces as our underlying spaces, we should expect that the representation of $\mathscr{C}$ is unitary. Technically, the functor mentioned above should be a $*$-functor: this means that for each morphism Σ from N strings to M strings, we should define its adjoint morphism Σ^* from M strings to N ones whose corresponding map is the adjoint $T^* : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes N}$ of T. Σ^* is defined simply to be the **complex conjugate** $\overline{\Sigma}$ of Σ:

Definition 1.2. $\overline{\Sigma}$ consists of points $\overline{x}$ where $x \in \Sigma$; the local holomorphic functions on $\overline{\Sigma}$ are η^* where η is a locally defined holomorphic function on Σ and

$$\eta^*(\overline{x}) = \overline{\eta(x)} \tag{1.5}$$

whenever η is defined on $x \in \Sigma$; similarly, boundary parametrizations are given by η_j^*. Note that if Σ is obtained by removing open discs from an N pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, then $\overline{\Sigma}$ is obtained by removing discs from

$$\overline{\mathfrak{X}} := (\overline{C}; \overline{x}_1, \ldots, \overline{x}_N; \eta_1^*, \ldots, \eta_N^*) \tag{1.6}$$

η^* should not be confused with $\overline{\eta}$ defined on Σ by

$$\overline{\eta}(x) = \overline{\eta(x)}.$$

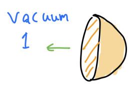
In the present context, we should assume that an incoming (resp. outgoing) string of Σ becomes an outgoing (resp. incoming) one of $\overline{\Sigma}$ via the conjugate map $\complement : x \in \Sigma \mapsto \overline{x} \in \overline{\Sigma}$. In the future, we will often consider all strings as incoming ones if necessary (cf. 1.9). In that case, we shall also assume all the boundary strings of $\overline{\Sigma}$ as incoming.

We should point out that although unitarity is a very important condition, there are important non-unitary CFTs, for instance, the logarithmic CFTs. (In such cases, $\mathcal{H}$ is a vector space without inner products.) Also, many VOA results and techniques do not rely on the unitarity. Nevertheless, assuming unitarity will often reasonably simply discussions or give motivations.

Example 1.3. Let $\mathfrak{X} = (\mathbb{P}^1; 0; \lambda\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$ and $\lambda \in \mathbb{C}^\times$. We can identify the conjugate of $\mathbb{P}^1$ with $\mathbb{P}^1$ by letting $x \in \mathbb{P}^1 \mapsto \overline{x}$ be the standard conjugate of $\mathbb{C}$: $z \mapsto \overline{z}$. Then $(\lambda\zeta)^*(\overline{z}) = \overline{\lambda\zeta(z)} = \overline{\lambda} \cdot \overline{z} = \overline{\lambda}\zeta(\overline{z})$. So the conjugate of $\mathfrak{X}$ is isomorphic to $\overline{\mathfrak{X}} = (\mathbb{P}^1; 0; \overline{\lambda}\zeta)$.

1.8

An interaction process could have no incoming or outgoing strings. *The Hilbert space for the empty string $\varnothing$ is $\mathbb{C}$.* The most elementary and important example with no incoming boundary is the closed unit disc $\mathbb{D}_1^{\mathrm{cl}}$ with 1 outgoing boundary parametrized by $z \mapsto z^{-1}$. The corresponding map $\mathbb{C} \to \mathcal{H}$ can be identified with its value at 1. This element in $\mathcal{H}$ is denoted by **1** and called the **vacuum vector**.



(1.7)

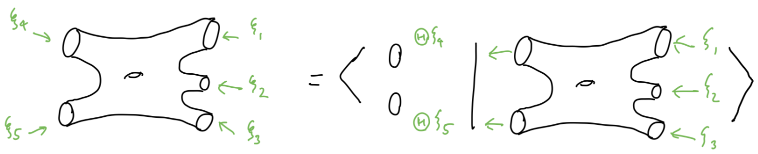
Assume as before that out theory is unitary. Then conjugate of the above disk is the same disk and boundary parametrization, but the original outgoing string is now the incoming one. The corresponding map $\mathcal{H} \to \mathbb{C}$ is, according to 1.7, the linear functional $\langle \Omega | \cdot \rangle$.

1.9

In general, one may wonder what the interaction $T : \mathcal{H}^{\otimes N} \to \mathbb{C}$ means physically for a surface Σ with N incoming strings but no outgoing ones. Choose $0 < M < N$, and make M of the N strings of $\partial\Sigma$ be outgoing strings. Then the corresponding interaction is a map $\widetilde{T} : \mathcal{H}^{\otimes(N-M)} \to \mathcal{H}^{\otimes M}$. In unitary CFT, T can be related to $\widetilde{T}$ by a anti-unitary (i.e. conjugate-unitary) map Θ on $\mathcal{H}$, called the **CPT operator**, such that for $\xi_1, \ldots, \xi_N \in \mathcal{H}$ (where the last M vectors are associated to the outgoing strings), we have

$$T(\xi_1 \otimes \cdots \otimes \xi_N) = \langle \Theta\xi_{N-M+1} \otimes \cdots \otimes \Theta\xi_N | \widetilde{T}(\xi_1 \otimes \cdots \otimes \xi_{N-M}) \rangle, \tag{1.8}$$

interpreted pictorially as



The operator Θ is an involution, i.e., $\Theta^2 = \mathbf{1}_{\mathcal{H}}$.

Such a linear functional T corresponding to an interaction with no outgoing strings is called a **correlation function** (or an **N-point function**). These functions are the central objects in CFT (and indeed, in any quantum field theory). Relation (1.8) teaches us that: (1) correlation functions can be interpreted as probability amplitudes in string interactions with the help of Θ, and (2) to study arbitrary interactions, it suffices to study those with no outgoing strings.

Let me close this subsection by mentioning an important fact: suppose the complex structure of Σ and the (assumed analytic) boundary parametrizations are parametrized holomorphically by some complex variables $\tau_\bullet = (\tau_1, \ldots, \tau_k)$, then the value of $T(\xi_\bullet)$ is now a *real analytic function* of $\tau_\bullet$, i.e., it is locally a power series of $\tau_1, \ldots, \tau_k$ and their conjugates. Actually, the word "function" in "correlation function" means a function of $\tau_\bullet$, but not of $\xi_\bullet$.

1.10

You must be curious what CPT means. Indeed, Θ is responsible for the simultaneous symmetry of charge conjugation (C), parity transformation (P), and time reversal (T). P+T together means an *anti-biholomorphism* $\Sigma \to \Sigma'$. Now we have arrived at a point that we missed previously: since anti-holomorphic maps are also conformal maps, should we expect that the interaction maps (or the correlation functions) for anti-biholomorphic surfaces are equal? The answer is no. (Namely, P+T are not preserved.) Indeed, if we let Σ has N incomes and no outcomes, let $\overline{\Sigma}$ be its complex conjugate (cf. 1.7) but still with N incomes, and let $T_\Sigma, T_{\overline{\Sigma}}$ be the correlation functions associated to them, then from 1.7 and relation (1.8), one can in fact check that

$$T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N) = \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \tag{1.9}$$

This relation explains CPT symmetry: the symmetries of charge (taking complex conjugate of the values of correlation functions) and parity+time (the conjugate biholomophism $\complement : \Sigma \to \overline{\Sigma}$) are preserved, and the operator realizing this simultaneous symmetry is Θ.

Note that mathematically, charge conjugation C is related to taking complex conjugate of numbers (but not of Σ). Physically, it means making a string into its "anti-string", or (in general QFT) making a particle (e.g. an election with negative charge) to its anti-particle (e.g. an antielectron with positive charge).

1.11

The CFT we have described so far is actually very special: it has no conformal anomaly. There are indeed no nontrivial CFTs which are both unitary and without anomaly. In this course, we will be mainly interested in CFTs with conformal anomaly. Technically, the conformal anomaly is determined by a complex number c (positive for unitary CFT), called **central charge**. To describe such CFT, we modify the previous descriptions as follows: The map (or the correlation function) T_Σ for Σ is only up to a positive scalar multiplication depending on Σ. $T_{\Sigma_1} \circ T_{\Sigma_2} = \lambda T_{\Sigma_1 \# \Sigma_2}$ where $\lambda > 0$. (The constants are not necessarily positive in non-unitary CFT.) If Σ is parametrized holomorphically by some complex variables $\tau_\bullet$, then by shrinking the domain of $\tau_\bullet$, we can choose T_Σ depending real analytically on $\tau_\bullet$.

There are many important cases where a real analytic (or even a holomorphic) T_Σ can be chosen globally for $\tau_\bullet$. This will be studied later in details.

Unless otherwise stated, a CFT always means one with (possible) conformal anomaly. Using the fancy language of 1.7, one can say that a unitary CFT is a *projective* monoidal $*$-functor from the category $\mathscr{C}$ in 1.7 to the category of Hilbert spaces. Namely, it is a projective unitary representation of $\mathscr{C}$.

1.12

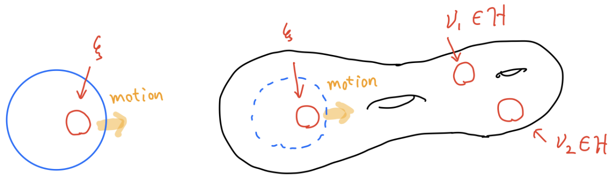
To study the representations of a topological group G, one must first understand very well the topological and the algebraic structures of G. Similarly, the study of CFTs relies heavily on the geometric and analytic structures of compact Riemann surfaces. However, from what we have discussed, there is a huge obstacle for studying CFTs: the correlation functions are real analytic, but not complex analytic (i.e. holomorphic) functions of the parameters $\tau_\bullet$. Thus, in order to study CFTs using the powerful tools of complex analysis (residue theorem, for instance), we make the following Ansatz: A correlation function T is a sum : $T_\Sigma = \sum_j \Phi^j_\Sigma \cdot \Psi^j_{\overline{\Sigma}}$, where each Φ^j and Ψ^j relies holomorphically on Σ and $\overline{\Sigma}$ respectively (so $\Psi^j_{\overline{\Sigma}}$ relies anti-holomorphically on Σ).

This Ansatz is very vague. Let me explain it in more details. Consider the annulus $A_{r,R}$ with boundary parametrization as in 1.6. We move the inside circle to another one centered at z (where $z \in A_{r,R}$ is reasonably small), still with radius r. The new eccentric annulus $A_{z,r,R}$ has larger outgoing string parametrized by $\zeta \mapsto R/\zeta$ and the smaller incoming one parametrized by $\zeta \mapsto (\zeta - z)/r$. Let $T_z : \mathcal{H} \to \mathcal{H}$ be the corresponding map. As we have said, for general vectors $\xi, \eta \in \mathcal{H}$, the expression $\langle \eta | T_z \xi \rangle = \langle \Theta\eta, T_z \xi \rangle$ can be chosen to be real analytic with respect to z. We now let

$$\begin{aligned} \mathbb{V} = \{\xi \in \mathcal{H} : & \text{ For all } r, R, \text{ the map } T \text{ can be chosen such that} \\ & z \mapsto \langle \nu | T_z \xi \rangle \text{ is holomorphic for all } \nu \in \mathcal{H}, \text{ and} \\ & \xi \text{ has "finite energy"}\} \end{aligned} \tag{1.10}$$

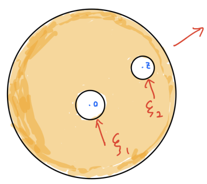
"Finite energy" is a minor condition to be explained later. (See 2.8.)

We can sew $A_{z,r,R}$ with any Σ, and the motion of the smaller string inside the annulus becomes, after sewing, the motion of a boundary string of Σ:



Therefore, if a vector $\xi \in \mathbb{V}$ is assigned to an incoming string of Σ with (analytic) boundary parametrization η_i, then, when translating this parametrized string with respect to η_i, the correlation function $T_\Sigma(\xi \otimes \cdots)$ should be holomorphic with respect to the motion, whatever states we assign to the other strings. We can therefore study $\mathbb{V}$ with the help of complex analysis. $\mathbb{V}$ is called a **vertex operator algebra** (VOA).

We have only described $\mathbb{V}$ as a vector space. But in which sense is $\mathbb{V}$ an algebra? An obvious candidate is as follows: consider $\mathbb{P}^1$ with three marked points $0, z, \infty$ and usual coordinates, e.g. $\eta_0(\zeta) = \zeta/r_1, \eta_z(\zeta) = (\zeta - z)/r_2, \eta_\infty(\zeta) = R/\zeta$ at $0, z, \infty$ where $r_1, r_2 > 0$ are small and $R > 0$ is large. We assume the strings around 0 and z are ingoing and that around ∞ outgoing. If we assign $\xi_1, \xi_2 \in \mathbb{V}$ to the incoming strings, then the outcome can be viewed as a product of ξ_1 and ξ_2.



Although this product does not have finite energy, it does satisfy the statement before the last line in (1.10). Thus, this product is almost a vector in $\mathbb{V}$. By modifying this product suitably, we can ensure that the products of vectors in $\mathbb{V}$ are always in $\mathbb{V}$. Details will be give in later sections.

Similarly to (1.10), we define $\widehat{\mathbb{V}} \subset \mathcal{H}$ to be the set of finite energy vectors ξ such that $\langle \nu | T_z \xi \rangle$ is anti-holomorphic over z. The vacuum vector $\mathbf{1}$ belongs to $\mathbb{V} \cap \widehat{\mathbb{V}}$: The result of gluing the unit disc into the inside of $A_{z,r,R}$ is just the disc with radius R and parametrization $\zeta \mapsto R/\zeta$, which is independent of z. So $T_z \mathbf{1}$ and hence $\langle \nu | T_z \mathbf{1} \rangle$ are constant over z, and hence both holomorphic and anti-holomorphic over z.

1.13

Now we can give a more detailed presentation of our Ansatz. We let $\mathcal{H}^{\mathrm{fin}}$ be the (indeed dense) subspace of vectors in $\mathcal{H}$ with "finite energy", which is acted on by $\mathbb{V}\otimes\widehat{\mathbb{V}}$. Ansatz:

1. $\mathcal{H}^{\mathrm{fin}}$ as a $\mathbb{V}\otimes\widehat{\mathbb{V}}$-module has decomposition
$$\mathcal{H}^{\mathrm{fin}}=\bigoplus_{i\in\mathfrak{I}}\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i\qquad\supset\mathbb{V}\otimes\widehat{\mathbb{V}}\tag{1.11}$$
where each $\mathbb{W}_i,\widehat{\mathbb{W}}_i$ are respectively irreducible $\mathbb{V}$-modules and $\widehat{\mathbb{V}}$-modules. $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are (according to their definition cf. (1.10)) subspaces of $\mathcal{H}^{\mathrm{fin}}$ by identifying them with $\mathbb{V}\otimes\mathbf{1}$ and $\mathbf{1}\otimes\widehat{\mathbb{V}}$ respectively. The vacuum vector $\mathbf{1}$ of $\mathcal{H}$ is identified with $\mathbf{1}\otimes\mathbf{1}$ (which belongs to $\mathbb{V}\otimes\widehat{\mathbb{V}}$).

2. For some Σ without outgoing boundaries, let $T_\Sigma:\mathcal{H}^{\otimes N}\to\mathbb{C}$ be the corresponding map. Then, corresponding to the above direct sum decomposition, we have
$$T_\Sigma\Big|_{(\mathcal{H}^{\mathrm{fin}})^{\otimes N}}=\sum_{i_1,\dots,i_N\in\mathfrak{I}}\Phi_{\Sigma,i_\bullet}\otimes\Psi_{\overline{\Sigma},i_\bullet}\tag{1.12}$$
where
$$\begin{aligned}\Phi_{\Sigma,i_\bullet}&:\mathbb{W}_{i_1}\otimes\cdots\otimes\mathbb{W}_{i_N}\to\mathbb{C},\\ \Psi_{\overline{\Sigma},i_\bullet}&:\widehat{\mathbb{W}}_{i_1}\otimes\cdots\otimes\widehat{\mathbb{W}}_{i_N}\to\mathbb{C}\end{aligned}$$
are linear. Moreover, when the complex structure and boundary parametrization are parametrized analytically by complex variables $\tau_\bullet$, then locally (with respect to the domain of $\tau_\bullet$), T_Σ, $\Psi_{\Sigma,i_\bullet}$, $\Psi_{\overline{\Sigma},i_\bullet}$ can be chosen such that $\Psi_{\Sigma,i_\bullet}$ is holomorphic over $\tau_\bullet$ (for all input vectors), and $\Psi_{\overline{\Sigma},i_\bullet}$ holomorphic over $\overline{\tau}_\bullet$. $\Phi_{\Sigma,i_\bullet}$ and $\Psi_{\overline{\Sigma},i_\bullet}$ are called **conformal blocks** associated to Σ (resp. $\overline{\Sigma}$) and $\mathbb{V}$ (resp. $\widehat{\mathbb{V}}$).

In part one, $\bigoplus$ could be finite (our main focus in this course), infinite but discrete, or continuous.

The second part can be summarized by saying that the CFT is separated into the **chiral halves** (those Φ or $\mathbb{W}_i$) and the **anti-chiral halves** (those Ψ or $\widehat{\mathbb{W}}_i$). Here, "chiral"="holomorphic".

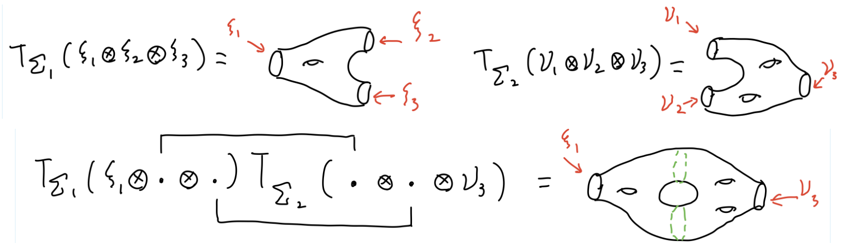
When physicists say a CFT is **rational**, they usually mean that the above direct sum is finite, and each $\mathbb{W}_{i,k}$ is semi-simple (hence, by further decomposition, can be irreducible). So far, the mathematical theory of conformal blocks is complete almost only for rational CFTs. These will be the main examples of this course. For non-rational logarithmic CFTs, even the above Ansatz needs to be modified. (So far, it is not even clear how to do it.)

Physicists more or less consider the above description as the definition of conformal blocks. We mathematicians should do the opposite: define conformal blocks in a different way, and use them to *construct* CFTs following the above Ansatz.

1.14

You may notice that to make this Ansatz compatible with 1.4 and 1.5, it is necessarily to assume that

1. The tensor product of conformal blocks $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ associated to Σ_1, Σ_2 respectively should be a conformal block associated to $\Sigma_1 \sqcup \Sigma_2$.
2. The composition of $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ (or more precisely, their contractions) should be conformal blocks associated to the sewings of Σ_1 and Σ_2, where the pair of $\mathbb{V}$-modules to be contracted must be dual to each other.



(A side note on linear algebra: If $V^\vee$ is the dual space (or a suitable dense subspace of the dual space) of a vetor space V, we choose a basis $\{v_\alpha\}_{\alpha\in\mathfrak{A}}$ labeled by elements of $\mathfrak{A}$, and choose a dual basis $\{v_\alpha^\vee\}_{\alpha\in\mathfrak{A}}$ of $V^\vee$ (i.e. the one determined by $\langle v_\alpha, v_\beta\rangle = \delta_{\alpha,\beta}$), then taking contraction means substituting $\sum_{\alpha\in\mathfrak{A}} v_\alpha \otimes v_\alpha^\vee$ inside the linear functional on a tensor product of vector spaces such that $V, V^\vee$ are tensor components.)

After we define conformal blocks rigorously, we will see that the first point is obvious, while the second one is a non-trivial theorem.

We briefly explain the meaning of "dual": For instance, in the above picture, the unitary $\mathbb{V}$-module containing ξ_2 is dual to the one containing η_1. As vector spaces, they are ("graded") dual spaces of each other. (It is a dense subspace of the full dual space. We will talk about this in future sections.) In unitary CFTs, all $\mathbb{V}$ and $\widehat{\mathbb{V}}$ modules are unitary, and the CPT operator Θ maps each $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ to some $\mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$ in $\mathcal{H}^{\mathrm{fin}}$ where $\mathbb{W}_i'$ is a $\mathbb{V}$-module dual to $\mathbb{W}_i$, and $\widehat{\mathbb{W}}_i'$ a $\widehat{\mathbb{V}}$-module dual to $\widehat{\mathbb{W}}_i$. ($\mathbb{V}$ is the dual of itself, and $\widehat{\mathbb{V}}$ similarly. So Θ restricts to involutions on $\mathbb{V}$ and $\widehat{\mathbb{V}}$ respectively. Such $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are called **unitary VOAs**.) The formal name for dual module is **contragredient module**, to be defined rigorously in later sections.

From this description, we know that for each $w_i \otimes \widehat{w}_i \in \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, $\Theta(w_i \otimes \widehat{w}_i)$ is a linear functional on $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$. This linear functional is easy to define: its value on each $m_i \otimes \widehat{m}_i \in \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, denoted by $\langle \Theta(w_i \otimes \widehat{w}_i), m_i \otimes \widehat{m}_i\rangle$, is

$$\langle \Theta(w_i \otimes \widehat{w}_i), m_i \otimes \widehat{m}_i\rangle = \langle w_i \otimes \widehat{w}_i | m_i \otimes \widehat{m}_i\rangle. \tag{1.13}$$

(Using (1.8), it is not hard to check that (1.13) is the right formula relating the composition of the interaction maps and the contraction of correlation functions.)

1.15

Note that a conformal block with $M+N$ inputs $\Phi_\Sigma : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \otimes \mathbb{W}_{j_1} \otimes \cdots \otimes \mathbb{W}_{j_N} \to \mathbb{C}$ can be regarded as one with N inputs and M outputs $\Phi_\Sigma : \mathbb{W}_{j_1} \otimes \cdots \otimes$

$\mathbb{W}_{j_N} \to \mathcal{H}'_{i_1} \otimes \cdots \otimes \mathcal{H}'_{i_M}$ where $\mathcal{H}'_{i_k}$ is the Hilbert space completion of $\mathbb{W}'_{i_k}$ and $\mathbb{W}'_{i_k}$ is the contragredient $\mathbb{V}$-module of $\mathbb{W}_{i_k}$.

2 Virasoro algebras; change of boundary parametrizations; strings vs. punctures

2.1

The goal of this section is to understand conformal blocks associated to 2-pointed Riemann sphere, equivalently, genus-0 surfaces with two boundary strings. We simply call them **annuli**, although their complex structures and boundary parametrizations are not necessarily the standard ones as in 1.6.

Let us first consider some degenerate examples whose boundary parametrizations are not necessarily analytic. Let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the topological group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. For each $g \in \mathrm{Diff}^+(\mathbb{S}^1)$, we let $A^g_{1,1}$ be the thin annulus whose incoming and outgoing strings are both $\mathbb{S}^1$ with parametrizations

$$\text{Incoming} : z \mapsto z, \qquad \text{Outgoing} : z \mapsto 1/g(z).$$

Lemma 2.1. *If $h \in \mathrm{Diff}^+(\mathbb{S}^1)$, then $A^{gh}_{1,1}$ is obtained by gluing the incoming circle of $A^g_{1,1}$ with the outgoing one of $A^h_{1,1}$.*

Proof. By (1.4), a point $z \in A^h_{1,1}$ is glued with $\zeta \in A^h_{1,1}$ iff $\zeta \cdot 1/h(z) = 1$, i.e., $\zeta = h(z)$. Now, a point z of $A^h_{1,1}$ becomes the point $h(z)$ of $A^g_{1,1}$ after gluing, which is sent by the outgoing parametrization of $A^g_{1,1}$ to $1/g(h(z))$. □

This proof is not rigorous since we are considering degenerate annuli. A rigorous one would be approximating $A^g_{1,1}$ and $A^h_{1,1}$ by genuine annuli, identifying the sewn annuli, and then taking the limit. This proof is not easy, unless when g and h are real-analytic (e.g., rotations). Nevertheless, we only need this lemma to motivate our following discussions.

2.2

Thus, we may consider $\mathrm{Diff}^+(\mathbb{S}^1)$ as the group of thin annuli whose product is the sewing. The merit of this viewpoint is that it convinces us to *consider the semi-group* **Ann** *of annuli as the complexification of* $\mathrm{Diff}^+(\mathbb{S}^1)$. The multiplication A_1A_2 of $A_1, A_2 \in$ **Ann** is the sewing of A_1, A_2 defined by gluing the inside of A_1 with the outside of A_2 using their parametrizations.

As an example, consider $\mathbb{P}^1$ with marked points $0, \infty$ and local coordinates $\eta_0(z) = z, \eta_\infty(z) = e^{-\mathrm{i}\tau}/z$, which gives a thin annulus corresponding to the rotation $z \mapsto e^{\mathrm{i}\tau}z$ when τ is real. Now consider τ as a complex variable $\tau = s + \mathrm{i}t$. Then the outgoing circle is the one with radius e^t. This gives a genuine annulus whenever $t > 0$.

The Ansatz in 1.13 should be expanded to include the following point: for each annulus $A \in$ **Ann**, the comformal block decomposition of the interaction $T_A : \mathcal{H} \to \mathcal{H}$

(with one income and one outcome) with respect to $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ is of the form

$$T_A = \sum_i \pi_i(A) \otimes \widehat{\pi}_i(\overline{A}) \tag{2.1}$$

where $\pi_i(A)$ is a bounded linear operator on the Hilbert space completion $\mathcal{H}_i$ of $\mathbb{W}_i$, and $\widehat{\pi}_i(\overline{A})$ is one on the completion $\widehat{\mathcal{H}}_i$ of $\widehat{\mathbb{W}}_i$. ($\overline{A}$ is the complex conjugate of A; see Def. 1.2. We assume the conjugate of the incomming string of $\overline{A}$ is the incoming of A, and similarly for the outcoming strings.) The choice of $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are unique up to scalar multiplications, and if A vary holomorphically over some complex variable $\tau_\bullet$, then locally $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ can be chosen to vary holomorphically with respect to $\tau_\bullet$ and $\overline{\tau}_\bullet$ respectively. Finally, if $A_1, A_2 \in \mathbf{Ann}$, then $\pi_i(A_1A_2)$ equals $\pi_i(A_1)\pi_i(A_2)$ up to scalar multiplication, and a similar thing can be said about $\widehat{\pi}_i$.

Namely, each π_i is a projective representation of $\mathbf{Ann}$ on $\mathcal{H}_i$, and so is $\widehat{\pi}_i$ on $\mathcal{H}_{\bar{i}}$. They should extend analytically to projective unitary representations of $\mathrm{Diff}^+(\mathbb{S}^1)$.

We emphasize that $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are conformal blocks associated to A and $\overline{A}$ respectively. Roughly speaking, π_i describes the conformal symmetries of chiral halves and $\widehat{\pi}_i$ the anti-chiral halves. A and $\overline{A}$ have to act jointly on the full space $\mathcal{H}$.

2.3

Thus, the study of CFT interactions for annuli reduces to that of the projective representations of $\mathbf{Ann}$. Our goal is to describe such representations in terms of Lie algebras.

Let $\mathrm{Vec}(\mathbb{S}^1)$ be the Lie algebra of smooth real vector fields of $\mathbb{S}^1$, whose elements are of the form $f\partial_\theta$ where ∂_θ is the pushforward of the standard unit vector of the real line under the map $\theta \mapsto e^{\mathbf{i}\theta}$, and $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$. The action of $f\partial_\theta$ on $h \in C^\infty(\mathbb{S}^1, \mathbb{R})$ is the negative of the usual one, $-f(e^{\mathbf{i}\theta}) \cdot \frac{\partial}{\partial\theta} h(e^{\mathbf{i}\theta})$. This is because the action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on h should be $h \circ g^{-1}$ in order to respect the order of group multiplication. Therefore, the Lie bracket in $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the usual one:

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta. \tag{2.2}$$

2.4

A projective unitary representation π of $\mathrm{Vec}(\mathbb{S}^1)$ and the corresponding one π of $\mathrm{Diff}^+(\mathbb{S}^1)$ (if exists) are related as follows. (Here unitary means that for each vector field $f\partial_\theta$, we have $\pi(f\partial_\theta)^\dagger = -\pi(f\partial_\theta)$, where $\dagger$ is the adjoint, or "formal adjoint" when the underlying inner product space is not Cauchy-complete.)

Let $t \in (-\epsilon, \epsilon) \mapsto g_t \in \mathrm{Diff}^+(\mathbb{S}^1)$ be a smooth family of diffeomorphisms satisfying $g_0 = \mathbf{1}$. Then up to addition by a number of $\mathbf{i}\mathbb{R}$,

$$\frac{d}{dt}\pi(g_t)\Big|_{t=0} = \pi(\partial_t g_0) \tag{2.3}$$

where $\partial_t g_0 \in \mathrm{Vec}(\mathbb{S}^1)$, the derivative of g at t_0, is the vector field determined by

$$(\partial_t g_0)(h) = \frac{d}{dt}(h \circ g_t)\Big|_{t=0} \tag{2.4}$$

for all smooth function h on $\mathbb{S}^1$.

Let now $t \in \mathbb{R} \mapsto \exp(tf\partial_\theta) \in \mathrm{Diff}^+(\mathbb{S}^1)$ be the flow generated by $f\partial_\theta \in \mathrm{Vec}(\mathbb{S}^1)$. So its derivative at $t = 0$ is $f\partial_\theta$, and $\exp((t_1 + t_2)f\partial_\theta) = \exp(t_1 f\partial_\theta) \circ \exp(t_2 f\partial_\theta)$. Then (2.4) implies that up to $\mathbb{S}^1$-multiplication,

$$\pi(\exp(tf\partial_\theta)) = e^{t\pi(f\partial_\theta)}, \tag{2.5}$$

since the derivative of $\pi(\exp(tf\partial_\theta))e^{-t\pi(f\partial_\theta)}$ is $\pi(\exp(tf\partial_\theta))(\pi(f\partial_\theta)-\pi(f\partial_\theta))e^{-t\pi(f\partial_\theta)} = 0$.

2.5

The Witt algebra $\{l_n : n \in \mathbb{Z}\}$ is a complex dense Lie subalgebra of the complexification $\mathrm{Vec}(\mathbb{S}^1) \otimes_{\mathbb{R}} \mathbb{C}$. Here,

$$l_n = z^{n+1}\partial_z \tag{2.6}$$

where $z = e^{\mathrm{i}\theta}$ and $\partial_z = \frac{1}{\mathrm{i}e^{\mathrm{i}\theta}}\partial_\theta$. (We use the chain rule to "define" ∂_z.) One checks

$$[l_m, l_n] = (m-n)l_{m+n} \tag{2.7}$$

where the bracket is the negative of the usual one for vector fields.

Let us assume for simplicity that the CFT is unitary. In the decomposition $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, each $\mathbb{W}_i$ is a projective unitary representation π_i of $\{l_n\}$, and similarly $\widehat{\mathbb{W}}_i$ is one $\widehat{\pi}_i$ of $\{l_n\}$. We know that the choice of $\pi_i(l_n)$ is unique up to $\mathrm{i}\mathbb{R}$-scalar addition. Here is a well-known fact about projective representations of Witt algebra: one can make a particular choice of $\pi_i(l_n)$ (for each n), denoted by L_n, such that the **Virasoro relation**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m,-n} \tag{2.8}$$

holds and $c \in \mathbb{C}$ is called the **central charge**. In the case that π_i is projective unitary, L_n can be chosen such that $L_n^\dagger = L_{-n}$ also holds.

We have abused the notation by writing the actions of l_n on all $\mathbb{V}$-modules $\mathbb{W}_i$ (as chiral halves of the CFT) as L_n. We are justified to do so because, as we will see later, the actions of l_n come from those of $\mathbb{V}$. Technically: Virasoro algebra is inside the VOA. So the action of $\{l_n\}$ on $\mathbb{W}_i$ is the restriction of that of $\mathbb{V}$. In particular, all chiral halves $\mathbb{W}_i$ share the same central charge c.

Similarly, we write the actions of l_n on all $\widehat{\mathbb{W}}_i$ as $\overline{L}_n$. (The bar over L_n reflects the fact that $\overline{L}_n$ describes the conformal symmetries of the anti-chiral halves of the CFT. $\overline{L}_n$ is not related with L_n by the CPT operator Θ.) The central charge $\widehat{c}$ for $\{\overline{L}_n\}$ is independent of $\widehat{\mathbb{W}}_i$ and in general could be different from the one c of $\{L_n\}$, although in most important cases they are equal. (E.g., when the CFT contains both closed and open strings.)

2.6

We shall generalize (2.5) to complex vector fields. First of all, we consider an element

$$f(z)\partial_z = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial_z$$

where the sum could be infinite. We treat $f(z)=\sum_n a_n z^{n+1}$ as a Laurent series. Let us now assume that $f(z)$ *is a holomorphic function on a neighborhood* $U\subset\mathbb{C}$ *of* $\mathbb{S}^1$.

$f\partial_z$ is a complex holomorphic vector field of U, which (after shrinking U) gives a **holomorphic flow** $\tau\in\Delta\mapsto\exp(\tau f\partial_z)\in\mathscr{O}(U)$ where $\Delta\subset\mathbb{C}$ is a neighborhood of 0. (Recall from the notation section that $\mathscr{O}(U)$ is the space of holomorphic functions on U.) This means:

(1) $(\tau,z)\in\Delta\times U\mapsto\exp(\tau f\partial_z)(z)$ is holomorphic whose restriction to each slice $\tau\times U$ is injective (and hence, a biholomorphism onto its image).

(2) $\exp(0f\partial_z)(z)=z$.

(3) $\exp((\tau_1+\tau_2)f\partial_z)=\exp(\tau_1 f\partial_z)\circ\exp(\tau_2 f\partial_z)$ on an open subset of U containing $\mathbb{S}^1$.

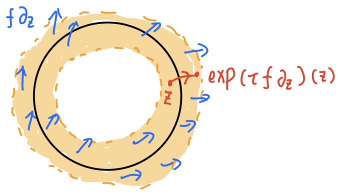
(4) For any holomorphic function h defined on an open set inside U,

$$f\partial_z h=\frac{\partial}{\partial\tau}h\circ\exp(\tau f\partial_z)\Big|_{\tau=0}. \tag{2.9}$$

(Compare (2.4).) This condition is equivalent to

$$\frac{\partial}{\partial\tau}\exp(\tau f\partial_z)\Big|_{\tau=0}=f. \tag{2.10}$$

(To see the equivalence, set $h(z)=z$ for one direction, and use chain rule for the other one.)



Remark 2.2. A caveat: The notations $f\partial_z$ and $\exp(\tau f\partial_z)$ are not compatible with those in the real case. Indeed, if we assume that τ only takes real values $\tau=t$, then by taking the real and the imaginary parts of (2.10), we see that σ_t is a real flow on the real surfaces U generated by the real vector field $\mathrm{Re}f\cdot\partial_x+\mathrm{Im}f\cdot\partial_y$. Writing $\partial_x=\partial_z+\partial_{\overline{z}},\partial_y=\mathbf{i}(\partial_z-\partial_{\overline{z}})$, we see that this vector field $f\partial_z$ should more precisely be written as $f\partial_z+\overline{f}\partial_{\overline{z}}$ where $\overline{f}(x)=\overline{f(x)}$.

This point is also justified by the fact that if k is antiholomorphic, then

$$\overline{f}\partial_{\overline{z}}k=\frac{\partial}{\partial\overline{\tau}}k\circ\exp(\tau f\partial_z)\Big|_{\tau=0}. \tag{2.11}$$

(Proof: take $k=\overline{h}$ in (2.10).) Thus, a more precise notation for $\exp(\tau f\partial_z)$ should be $\exp(\tau f\partial_z+\overline{\tau}\overline{f}\partial_{\overline{z}})$. But we prefer to suppress the term $\overline{\tau}\overline{f}\partial_{\overline{z}}$ to keep the notations shorter.

2.7

One way to find the expression of $\sigma_\tau = \exp(\tau f \partial_z)$ is to solve the holomorphic nonlinear differential equation with initial condition:

$$\begin{gathered}\frac{\partial}{\partial \tau}\sigma_\tau(z) = f(\sigma_\tau(z)),\\ \sigma_0(z) = z.\end{gathered} \tag{2.12}$$

This is due to (2.10) and $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1} \circ \sigma_{\tau_2}$. (Indeed, the existence of holomorphic flows is due to that of the solutions of such equations.)

Alternatively, one may calculate the flow by brutal force using the formula

$$\begin{aligned}\exp(f\partial_z)(z) &= \sum_{k\in\mathbb{N}} \frac{1}{k!}(f(z)\partial_z)^k z\\ &= \sum_{k\in\mathbb{N}} \frac{1}{k!} \underbrace{f(z)\partial_z\Big(f(z)\partial_z\big(\cdots f(z)\partial_z}_{k \text{ times}}\, z\big)\cdots\Big).\end{aligned} \tag{2.13}$$

(One may treat this formula as a formal sum if one worries about the convergence issue.) To see why this formula is valid, check that such defined $\exp(\tau f \partial_z)(z) =: \sigma_\tau(z)$ satisfies that $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1} \circ \sigma_{\tau_2}$, that $\partial_\tau \sigma_\tau|_{\tau=0} = f$, and that $\sigma_0(z) = z$. This is easy.

2.8

Example 2.3. $\sigma_\tau(z) = e^\tau z$ is the holomorphic flow generated by the vector field $l_0 = z\partial_z$ since $\frac{\partial}{\partial \tau} e^\tau z|_{\tau=0} = z$. Namely,

$$\exp(\tau z \partial_z)(z) = e^\tau z.$$

Set $\lambda = e^\tau$. In view of the $A^g_{1,1}$ in 2.1, we consider the 2-pointed sphere $\mathfrak{X} = (\mathbb{P}^1; 0, \infty; \zeta, \lambda^{-1}\zeta^{-1})$ where $\zeta : z \mapsto z$ is the standard coordinate of $\mathbb{C}$. Then, when $|\lambda| \leqslant 1$, $\mathfrak{X}$ defines an annulus A, either genuine or thin, whose incoming circle has radius 1 and outcoming $1/|\lambda|$. Thus, the conformal block $\pi_i(A)$ associated to this annulus, which is a linear operator on the Hilbert space completion $\mathcal{H}_i$, should be $e^{\tau L_0} = \lambda^{L_0}$ (by replacing $z\partial_z$ with L_0).

It is easy to check that $\overline{A}$ is isomorphic to the annulus defined by $(\mathbb{P}^1; 0, \infty; \zeta, \overline{\lambda^{-1}}\zeta^{-1})$. So the corresponding conformal block should be $\widehat{\pi}_i(\overline{A}) = \overline{\lambda}^{\overline{L}_0}$. Therefore, the interaction map $T_A : \mathcal{H} \to \mathcal{H}$ is determined by

$$T_A\big|_{\mathcal{H}_i \otimes \widehat{\mathcal{H}}_i} = \lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0}. \tag{2.14}$$

In a unitary CFT, L_0 and $\overline{L}_0$ (or more precisely, their closures) are self-adjoint operators so that λ^{L_0} and $\overline{\lambda}^{\overline{L}_0}$ can be defined and are unitary when $|\lambda| = 1$. Moreover, in a unitary CFT:

Assumption 2.4 (Positive energy)**.** The spectra of L_0 and $\overline{L}_0$ are both positive (i.e. $\geqslant 0$). In these notes, we are mainly interested in the case that the spectra are discrete. We identify L_0 with $L_0 \otimes \mathbf{1}$ and $\overline{L}_0$ with $\mathbf{1} \otimes \overline{L}_0$ so that $L_0, \overline{L}_0$ are commuting diagonalizable operators on $\mathcal{H}^{\mathrm{fin}}$ with $\geqslant 0$ eigenvalues.

Thus, the right hand side of (2.14) can be written as $\lambda^{L_0} \cdot \overline{\lambda}^{\overline{L}_0}$.

Now we can explain what we meant by finite energy: A vector ξ of $\mathcal{H}$ has **finite energy** if ξ is a finite sum of eigenvectors of both L_0 and $\overline{L}_0$. (In general, a vector of $\mathcal{H}$ is an l^2-convergent sum, either finite or infinite, of eigenvectors.)

2.9

Example 2.5. Let $n \neq 0$. To understand the geometric meanings of $e^{\tau L_{-n}}$ and $e^{\overline{\tau}\overline{L}_{-n}}$, we find the expression of $\sigma_\tau = \exp(\tau z^{-n+1}\partial_z)$ by solving the differential equation $\partial_\tau \sigma_\tau = (\sigma_\tau)^{-n+1}$ with initial condition $\sigma_0(z) = z$ (cf. (2.12)). The solution is

$$\exp(\tau z^{-n+1}\partial_z)(z) = (z^n + n\tau)^{\frac{1}{n}}. \tag{2.15}$$

□

Unfortunately, this flow does not give us any annulus in the usual sense. Take $n = 1$ for instance. Then the flow is just the translation by τ. However, the circle after a small translation will intersect the original one. So there is no annulus whose outgoing circle is the translation of the incoming one. In fact, in most cases, $\exp(f\partial_z)$ is not the action of an annulus. We have to pursue another way of understanding this operator.

2.10

There are two ways to look at a group action $G \curvearrowright X$: (1) The action of $g \in G$ on X is a transformation. So $gx \neq x$ in general. (2) gx and x are different expressions (under different coordinates) of the same element. The rule for change of coordinate is given by the action of G. We shall take the second viewpoint.

Let $\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$ be an N-pointed compact Riemann surface with local coordinates satisfying Assumption 1.1. Assume the setting of 2.6. Write $\sigma_\tau = \exp(\tau f \partial_z)$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}$ be defined on $U \supset \mathbb{S}^1$. Let $\tau \in \Delta$ be close to 0.

Remark 2.6. In case you want to know the precise meaning of "close": for the local coordinate η_i we are to discuss in the following, we choose $\epsilon > 0$ such that $\sigma_\tau(U \cap \mathrm{Rng}(\eta_i))$ contains $\mathbb{S}^1$ for all $\tau \in \mathbb{D}_\epsilon$, where the open set $\mathrm{Rng}(\eta_i)$ is the range of η_i.

Principle 2.7 (Change of boundary parametrizations)**.** Suppose that the local coordinate η_i at x_i is changed to the boundary parametrization $\sigma_\tau \circ \eta_i$ and the boundary string $\eta_i^{-1} \circ (\mathbb{S}^1)$ is gradually changed (with respect to the change of τ) to $\eta_i^{-1}\big(\sigma_\tau^{-1}(\mathbb{S}^1)\big)$. Then, in the expressions of conformal blocks and correlation functions (without outputs), each $w_i \in \mathbb{W}_i$ is replaced by $e^{\tau \sum_n a_n L_n} w_i$, and each $\widehat{w}_i \in \widehat{\mathbb{W}}_i$ by $e^{\overline{\tau}\sum \overline{a_n}\overline{L}_n}\widehat{w}_i$.

To be more precise, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the correlation function where Σ is obtained from $\mathfrak{X}$. Assume $i = 1$ for simplicity. Changing the local coordinate η_1 to $\sigma_\tau \circ \eta_1$ gives a new surface with parametrized boundary Σ'. Then up to scalar multiplication, $T_{\Sigma'}$ and T_Σ are related by

$$T_\Sigma(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N) = T_{\Sigma'}\Big(\big(e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau}\sum_n \overline{a_n}\overline{L}_n}\big)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N\Big) \tag{2.16}$$

for all $\xi_1,\dots,\xi_N$. Similarly, if $\Phi_\Sigma:\mathbb{W}_{i_1}\otimes\cdots\otimes\mathbb{W}_{i_N}\to\mathbb{C}$ is a conformal block for Σ, then $\Phi_{\Sigma'}$ defined by

$$\Phi_\Sigma(w_1\otimes w_2\otimes\cdots\otimes w_N)=\Phi_{\Sigma'}\big(e^{\tau\sum_n a_nL_n}w_1\otimes w_2\otimes\cdots\otimes w_N\big) \tag{2.17}$$

is one for Σ'.

2.11

The geometric intuition in the above subsection is the following: ξ_1 in the η_i-parametrization is the same (up to scalar multiplication) vector as $(e^{\tau\sum_n a_nL_n}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\cdot\overline{L}_n})\xi_1$ in the $\sigma_\tau\circ\eta_i$-parametrization. We call this same "abstract" vector $\widetilde{\xi}_1$, which is unique up to scalar multiplication. We write $\xi_1=(\mathcal{U}(\eta_i)\otimes\mathcal{U}(\eta_i^*))\widetilde{\xi}_1$, understanding $\mathcal{U}(\eta_i)\otimes\mathcal{U}(\eta_i^*)$ as the map sending an abstract vector to its concrete expression under the boundary parametrization η_i. Namely, $\mathcal{U}(\eta_i)\otimes\mathcal{U}(\eta_i^*)$ is a vector bundle trivialization. The transition function from the η_i-parametrization to the $\sigma_\tau\circ\eta_i$-parametrization is

$$\big(\mathcal{U}(\sigma_\tau\circ\eta_i)\otimes\mathcal{U}((\sigma_\tau\circ\eta_i)^*)\big)\big(\mathcal{U}(\eta_i)\otimes\mathcal{U}(\eta_i^*)\big)^{-1}=e^{\tau\sum_n a_nL_n}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\cdot\overline{L}_n}. \tag{2.18}$$

We have a parametrization independent T whose expressions under the concrete boundary parametrizations are (up to scalar multiplications)

$$\begin{aligned}T(\widetilde{\xi}_1\otimes\cdots)&=T_\Sigma\Big(\big(\mathcal{U}(\eta_i)\otimes\mathcal{U}(\eta_i^*)\big)^{-1}\widetilde{\xi}_1\otimes\cdots\Big)\\&=T_{\Sigma'}\Big(\big(\mathcal{U}(\sigma_\tau\circ\eta_i)\otimes\mathcal{U}((\sigma_\tau\circ\eta_i)^*)\big)^{-1}\widetilde{\xi}_1\otimes\cdots\Big).\end{aligned}$$

2.12

Let us do an example to see how the change of parametrization formula works.

Example 2.8. Let $\mathfrak{X}=(\mathbb{P}^1;1/3,\infty;2(\zeta-1/3),\zeta^{-1})$ where $\zeta:z\mapsto z$ is the standard coordinate of $\mathbb{C}$. We choose $1/3$ to be the input point, and ∞ the outgoing one. The associated boundary parametrized surface Σ is an annulus whose incoming circle $\{z:|2(z-1/3)|=1\}$ has center $1/3$ and radius $1/2$, and whose outgoing circle is $\mathbb{S}^1$. Let us find an expression for $T_\Sigma:\mathcal{H}\to\mathcal{H}$.

We know that the map for the standard thin annulus $A_{1,1}$ is $T_{A_{1,1}}=\mathbf{1}_\mathcal{H}$. Let $\mathfrak{X}_1=(\mathbb{P}^1;0,\infty;2\zeta,\zeta^{-1})$, which gives an annlus Σ_1 with incoming string $\frac{1}{2}\mathbb{S}^1$ and outgoing one $\mathbb{S}^1$. $A_{1,1}$ is changed to Σ_1 by changing the incoming boundary parametrization ζ to 2ζ. By Ex. 2.3, $2\zeta=\exp(\log 2\cdot z\partial_z)$. So, as $e^{\log 2L_0}=2^{L_0}$ and similarly $e^{\log 2\overline{L}_0}=2^{\overline{L}_0}$, by (2.16), T_{Σ_1} could be $(1/2)^{L_0}\otimes(1/2)^{\overline{L}_0}$.

Σ_1 is changed to Σ by adding 2ζ by $-2/3$. According to Ex. 2.5, $\exp(-2/3\partial_z)(z)=z-2/3$. Therefore, up to a scalar multiplication, $T_{\Sigma_1}(\xi)=T_\Sigma((e^{-\frac{2}{3}L_{-1}}\otimes e^{-\frac{2}{3}\overline{L}_{-1}})\xi)$. Thus, the answer is

$$T_\Sigma=\big((1/2)^{L_0}\otimes(1/2)^{\overline{L}_0}\big)\cdot\big((e^{\frac{2}{3}L_{-1}}\otimes e^{\frac{2}{3}\overline{L}_{-1}})\big)=\big((1/2)^{L_0}e^{\frac{2}{3}L_{-1}}\big)\otimes\big((1/2)^{\overline{L}_0}e^{\frac{2}{3}\overline{L}_{-1}}\big).$$

$(1/2)^{L_0}e^{\frac{2}{3}L_{-1}}$ is a conformal block for Σ. □

2.13

What is the change of parametrization formula for T_Σ and Φ_Σ when some output strings are involved? Let's turn the conformal block Φ_Σ in 2.10 to $\Phi_\Sigma : \mathbb{W}_{i_2}\otimes\cdots\otimes\mathbb{W}_{i_N} \to \mathcal{H}'_{i_1}$ with $N-1$ inputs and one output. Here $\mathcal{H}'_{i_1}$ is the Hilbert space completion of $\mathbb{W}'_i$ where $\mathbb{W}'_{i_1}$ is contragredient to $\mathbb{W}_{i_1}$. L_{-n} is indeed the transpose of L_n (up to scalar addition):

$$L_{-n} = L_n^{\mathrm{t}}. \tag{2.19}$$

Namely, for the standard pairing $\mathbb{W}_{i_1}\otimes\mathbb{W}'_{i_1}\to\mathbb{C}$, we have

$$\langle L_n w_{i_1}, w'_{i_1}\rangle = \langle w_{i_1}, L_{-n}w'_{i_1}\rangle. \tag{2.20}$$

(The same can be said about $\overline{L}_n$ and $\widehat{\mathbb{V}}$-modules.) So

$$\Phi_\Sigma = e^{\tau\sum_n a_n L_{-n}}\circ\Phi_{\Sigma'}. \tag{2.21}$$

Therefore, for the map $T_\Sigma : \mathcal{H}^{\otimes(N-1)}\to\mathcal{H}$ with $N-1$ inputs and 1 output,

$$T_\Sigma = \left(e^{\tau\sum_n a_n L_{-n}}\otimes e^{\overline{\tau}\sum_n \overline{a_n}\overline{L}_{-n}}\right)\circ T_{\Sigma'}. \tag{2.22}$$

Exercise 2.9. Show that the formula (2.14) in Example 2.3 follows from (2.22).

2.14

In case you want to know why $L_{-n} = L_n^{\mathrm{t}}$ up to scalar addition, we give a geometric explanation below. A more serious (but less intuitive) proof will be given in later sections. This subsection can be skipped on first reading.

Proof. Set $\mathbb{W} = \mathbb{W}_{i_1}$ and $\mathbb{W}' = \mathbb{W}'_{i_1}$. Let $\mathfrak{X} = (\mathbb{P}^1; 0,\infty; z, z^{-1})$ where z is the standard coordinate of $\mathbb{C}$, which gives the standard thin annulus $A_{1,1}$. In the 1-input-1-output picture, $T_{A_{1,1}} : \mathcal{H}\to\mathcal{H}$ is the identity map. Hence the conformal block $\Phi_A = \pi(A) : \mathbb{W}\to\mathbb{W}$ is the identity. Thus, in the 2-inputs-0-outputs picture, $\Phi_{A_{1,1}}$ is the contraction, whose restriction to $\mathbb{W}\otimes\mathbb{W}'$ is the standard pairing $\langle w, w'\rangle$ defined by the value of w' (as a linear functional on $\mathbb{W}$) at w.

Change the local coordinate z at 0 to σ_τ, and keep the other data of $\mathfrak{X}$. This changes $A_{1,1}$ to a new weird annulus A. By (2.17), the conformal block for A is

$$\Phi_A(w\otimes w') = \langle e^{-\tau\Sigma_n a_n L_n}w, w'\rangle.$$

Note that if we set $\zeta = \sigma_\tau(z)$, then $z^{-1} = 1/\sigma_\tau^{-1}(\zeta)$, which equals $1/\sigma_{-\tau}(\zeta)$ by the definition of flows. Namely, A is equivalent to the weird annulus whose incoming boundary parametrization is z and outcoming $1/\sigma_{-\tau}(z)$. To compute the conformal block for this choice of boundary parametrization, we note that the original $1/z$ at ∞ is changed to $1/\sigma_{-\tau}(z)$. Therefore, if we let $\gamma_\tau(z) = 1/\sigma_{-\tau}(1/z)$ which is a holomorphic flow generated by some $\sum_n b_n z^{n+1}$, then the expression for Φ_A is

$$\Phi_A(w\otimes w') = \langle w, e^{-\tau\Sigma_n b_n L_n}w'\rangle.$$

For the two expressions of Φ_A, we take the negative derivative of τ at $\tau = 0$ to get

$$\sum a_n \langle L_n w, w' \rangle = \sum b_n \langle w, L_n w' \rangle.$$

To finish the proof, it suffices to prove $b_n = a_{-n}$.

Recall $\sum a_n z^{n+1} = \partial_\tau \sigma_\tau|_{\tau=0}$. Similarly, $\sum b_n z^{n+1} = \partial_\tau \gamma_\tau|_{\tau=0}$, which is

$$\begin{aligned} &\partial_\tau (1/\sigma_{-\tau}(1/z))\big|_{\tau=0} = -\frac{1}{\sigma_0(1/z)^2} \cdot \partial_\tau (\sigma_{-\tau}(1/z))\big|_{\tau=0} \\ =& z^2 \cdot \sum a_n (1/z)^{n+1} = \sum a_n z^{-n+1} = \sum a_{-n} z^{n+1}. \end{aligned}$$

□

2.15

As an easy application of our change of parametrization formula, we are able to describe the correlator $T_A : \mathcal{H} \to \mathcal{H}$ for an analytic annulus $A \in \mathbf{Ann}$ obtained from $(\mathbb{P}^1; 0, \infty; \eta_0, \eta_\infty)$ where η_0 and η_∞ are local coordinates at $0, \infty$ respectively. Set $\varpi = 1/z$. One can write

$$\eta_0(z) = \exp\Big(\sum_{n \in \mathbb{N}} a_n z^{n+1} \partial_z\Big)(z), \qquad \eta_\infty(\varpi) = \exp\Big(\sum_{n \in \mathbb{N}} b_n \varpi^{n+1} \partial_\varpi\Big)(\varpi),$$

where the coefficients a_n, b_n can be determined using (2.13). (We will say more about determining the coefficients in the future.) When A is the standard thin annulus (i.e., when $\eta_0 : z \mapsto z, \eta_\infty : z \mapsto z^{-1}$), we know $T_A = \mathbf{1}$. Thus, in general, by (2.16) and (2.19), the map T_A is (up to scalar multiplications)

$$T_A = \Big(e^{\sum_{n \in \mathbb{N}} -b_n L_{-n}} \otimes e^{\sum_{n \in \mathbb{N}} -\overline{b_n} \cdot \overline{L}_{-n}}\Big) \cdot \Big(e^{\sum_{n \in \mathbb{N}} -a_n L_n} \otimes e^{\sum_{n \in \mathbb{N}} -\overline{a_n} \cdot \overline{L}_n}\Big).$$

The reason that only $n \in \mathbb{N}$ are involved is because η_0 and η_∞ can be defined near 0 and send 0 to 0. Indeed, for $f(z) = \sum_{n \in \mathbb{Z}} a_n z^{n+1}$, assume that $\exp(\tau f \partial_z)(z)$ is defined near 0 and sends 0 to 0 for all small τ. Then its derivative over τ at $z = 0$, which is $f(\exp(\tau f \partial_z)(0)) = f(0)$ by (2.14), should also be 0. So f must be of the form $\sum_{n \geqslant 0} a_n z^{n+1}$.

2.16

We call those in 2.10 and 2.11 **change of (boundary) parametrizations** in general, and those in 2.15 **change of (local) coordinates**. The former contains the latter.

When changing the boundary parametrizations, the standard coordinate z could be changed to σ_τ not necessarily defined at 0, or more generally, a local coordinate (say) η_1 of an N-pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ is changed to $\sigma_\tau \circ \eta_1$. This changes the boundary-parametrized Riemann surface Σ to Σ'. Note that this process does not violate out definition of *analytic* boundary parametrizations in 1.2: The new surface Σ' is obtained from a new N-pointed one $\mathfrak{X}' = (C'; x_\bullet; \sigma_\tau \circ \eta_1, \eta_1, \ldots, \eta_N)$ where C' is a new compact Riemann surface, which is defined by gluing Σ with N pieces of unit discs $\mathbb{D}_1$ using the maps $\sigma_\tau \circ \eta_1, \eta_2, \dots, \eta_N$. (If you use the maps $\eta_1, \dots, \eta_N$ instead, you simply get C.) Thus, *for the change of boundary parametrizations in general, the underlying compact Riemann surfaces C could be changed.*

By change of coordinates, we mean $\mathfrak{X}$ is changed to $\mathfrak{X}' = (C; x_\bullet; \eta'_\bullet)$ with the same underlying compact Riemann surface C and the same marked points $x_\bullet$ as the original ones but different local coordiates at these marked points. As mentioned in 2.15, in this process, only $L_0, L_1, L_2, \dots$ (and also $\overline{L}_0, \overline{L}_1, \overline{L}_2$) are involved, while in the change of boundary parametrizations, all L_n are involved.

In the previous discussions, almost all formulas hold only up to scalar multiplications or additions. However, when only $L_{-1}, L_0, L_1, L_2, \dots$ are involved, the interaction maps T_Σ can indeed be chosen such that all the formulas truely hold, not just up to scalar multiplications or additions. This is because the conformal anomaly is due to the central term $c \cdot (m^3 - m)\delta_{m,-n}/12$ in the Virasoro relation (2.8), which vanishes when $m, n \geqslant -1$. Note that L_{-1} is responsible for translation. Thus:

Principle 2.10. T_Σ can be chosen to have no ambiguity when changing the local coordinates, or when translating a marked point x_i with respect to its local coordinate η_i.

To be more precise: We fix a compact Riemann surface C. Then for each choice of N marked points $x_\bullet$ and local coordinates $\eta_\bullet$, we can choose the correlation function $T_{\mathfrak{X}} : \mathcal{H}^{\otimes N} \to \mathbb{C}$ associated to the boundary parametrized surface associated to $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ such that

- For another choice of N-pointed $\mathfrak{X}' = (C; x_\bullet; \eta'_\bullet)$ with the same marked points and different local coordinates $\eta'_\bullet$, $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}'}$ are related by (2.16).
- If $\mathfrak{X}' = (C; x'_1, x_2, \dots, x_N; \eta'_1, \eta_2, \dots, \eta_N)$ where $\eta'_1 = \eta_1 - \eta_1(x'_1)$, and if x'_1 is inside an open disc U_1 centered at x_1 on which η_1 is holomorphically defined (more precisely, this means $\eta_1(U_1)$ is an open disc centered at $\eta_1(x_1) = 0$), then $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}'}$ are related by (2.16), namely, (noticing (2.15) for $n = 1$)

$$T_{\mathfrak{X}}(\xi_1 \otimes \cdots \otimes \xi_N) = T_{\mathfrak{X}'}\Big(\big(e^{-\eta_1(x'_1)L_{-1}} \otimes e^{-\overline{\eta_1(x'_1)}\cdot \overline{L}_{-1}}\big)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N\Big). \tag{2.23}$$

A similar principle holds when $T_{\mathfrak{X}}$ has output strings. □

Recall the geometric picture described in 2.11. We see that when changing local coordinates, everything in 2.11 truly holds, not just up to scalar multiplications. In particular, the abstract vector $\widetilde{\xi}_1$ is uniquely determined when only the change of local coordinates are allowed.

2.17

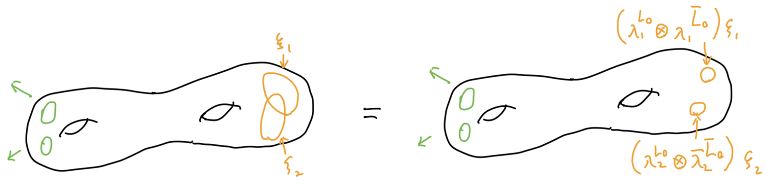
Assumption 2.11. We drop Assumption 1.1 for the incoming strings when we associate only finite energy vectors (i.e., vectors of $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, $\mathbb{V} \otimes \widehat{\mathbb{V}}$, etc.) to the incoming strings.

In this course, we will be mainly interested in finite energy vectors. Therefore, we do not assume that that each $\eta_i(U_i)$ contains $\mathbb{D}_1^{\mathrm{cl}}$, or that U_i and U_j are disjoint for different i and j. In the latter case, the two boundary strings $\eta_i^{-1}(\mathbb{S}^1)$ and $\eta_j^{-1}(\mathbb{S}^1)$ possibly overlap. What does this picture actually mean?

Note that multiplying η_i by $\lambda\eta_i$ amounts to shrinking the size of the string $\eta_i^{-1}(\mathbb{S}^1)$ by $|\lambda|$ and then rotating the string. If $\lambda > 0$ then there is only shrinking but not rotating. Thus, *for an local coordinated N-pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, we can find $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ with large enough absolute values such that the new data $\mathfrak{X}' = (C; x_\bullet; \lambda_1\eta_1, \dots, \lambda_N\eta_N)$ satisfies Assumption 1.1.* Then for finite energy vectors $\xi_1, \dots, \xi_N \in \mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, $T_{\mathfrak{X}}(\xi \otimes \cdots \otimes \xi_N)$ is understood as

$$T_{\mathfrak{X}}(\xi_1 \otimes \cdots \otimes \xi_N) := T_{\mathfrak{X}'}\Big(\big(\lambda_1^{L_0} \otimes \overline{\lambda_1}^{\overline{L}_0}\big)\xi_1 \otimes \cdots \otimes \big(\lambda_N^{L_0} \otimes \overline{\lambda_N}^{\overline{L}_0}\big)\xi_N\Big). \tag{2.24}$$

This definition is independent of the choice of sufficiently large $\lambda_1, \dots, \lambda_N$. And each $\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}$ acts diagonally on $\mathcal{H}^{\mathrm{fin}}$ since $L_0 \otimes \overline{L}_0$ do. (Recall Assumption 2.4.)



In the spirit of the previous subsection, you should view the finite energy vectors ξ_j and $\big(\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}\big)\xi_j$ not as different vectors, but as two coordinate representations of the same vector $\widetilde{\xi}_j$. When $|\lambda_1|$ becomes infinitely large, the string for ξ_j shrinks to an infinitesimal one around x_i, i.e., it shrinks to x_1 as a **puncture**. It is very useful to view the abstract finite energy vector $\widetilde{\xi}_j$ not associated to any particular strings, but associated to that puncture x_i. Thus, the marked points $x_\bullet$ of $\mathfrak{X}$ are also called punctures.

Remark 2.12. A side note: When we do local coordinate changes, finite energy vectors are changed to finite energy ones.

Therefore, in the above discussion, we don't have to stick to change of coordinates of the form $\eta_j \mapsto \lambda_j\eta_j$: any local coordinate change is valid. We will prove the above claim in later sections.

2.18

Let us choose $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ inside $\mathcal{H}^{\mathrm{fin}}$. According to Assumption 2.4, the eigenvalues of the diagonalizable operators L_0 (on $\mathbb{W}_i$) and $\overline{L}_0$ (on $\widehat{\mathbb{W}}_i$) are $\geqslant 0$. Now choose eigenvectors $w \in \mathbb{W}_i$ and $\widehat{w} \in \widehat{\mathbb{W}}_i$ with $L_0 w = \Delta w$, $\overline{L}_0\widehat{w} = \widehat{\Delta}\widehat{w}$ where $\Delta, \widehat{\Delta} \geqslant 0$.

Here is an important point about the two eigenvalues. They are not necessarily integers, which means that $\lambda^{L_0}w$ and $\overline{\lambda}^{\overline{L}_0}\widehat{w}$ might be *multivalued with respect to* λ, i.e., they may also depend on the choice of argument $\arg\lambda$. However, according to the No-Ambiguity Principle 2.10, the expression

$$\big(\lambda^{L_0} \otimes \lambda^{\overline{L}_0}\big)(w \otimes \widehat{w}) = \lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} \cdot w \otimes \widehat{w}$$

must be single-valued with respect to λ, namely, it does not rely on the choice of $\arg\lambda$. As $\lambda = |\lambda| e^{\mathbf{i}\arg\lambda}$ and hence $\lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} = |\lambda|^{\Delta+\widehat{\Delta}} e^{\mathbf{i}(\Delta-\widehat{\Delta})\arg\lambda}$, we conclude that

$$\Delta - \widehat{\Delta} \in \mathbb{Z}. \tag{2.25}$$

This gives a constraint on the possible $\mathbb{V}\otimes\widehat{\mathbb{V}}$-submodules of $\mathcal{H}^{\mathrm{fin}}$.

That $\lambda^{L_0}w$ could be multivalued is a crucial property in CFT, and it is not related to conformal anomaly. Indeed, it is related to the non-uniqueness of decomposing T_Σ into conformal blocks. Thus, *the No-Ambiguity Principle 2.10 does not hold for conformal blocks*.

3 Definition of VOAs, I

3.1

We first give the rigorous definition of vertex operators algebras and a slightly weaker version, graded vertex algebras. Then we explain the meanings of the axioms.

Definition 3.1. A **graded vertex algebra** is a (complex) vector space $\mathbb{V}$ together with a diagonalizable operator L_0 acting on $\mathbb{V}$ whose eigenvalues are inside $\mathbb{N}$. We write the L_0-grading of $\mathbb{V}$ as $\mathbb{V} = \bigoplus_{n\in\mathbb{N}}\mathbb{V}(n)$. Any eigenvector v of L_0 (including 0) is called (L_0)-**homogeneous**, and if $v \in \mathbb{V}(n)$ (i.e. $L_0 v = nv$), we write $\mathrm{wt}v = n$ and call $\mathrm{wt}v$ the **weight** of v. Moreover, we have a linear map

$$\begin{aligned} \mathbb{V} &\to \big(\mathrm{End}(\mathbb{V})\big)[[z^{\pm 1}]] \\ u &\mapsto Y(u,z) \equiv \sum_{n\in\mathbb{Z}} Y(u)_n z^{-n-1} \end{aligned} \tag{3.1}$$

where each $Y(u)_n \in \mathrm{End}(\mathbb{V})$ is called a **mode**. Here, z is treated as a formal variable. Thus $Y(u,z)v \in \mathbb{V}[[z^{\pm}]]$ for each $v \in \mathbb{V}$. The reason for associating z^{-n-1} to $Y(u)_n$ is because we could have (recalling (0.2))

$$\mathrm{Res}_{z=0}\, Y(u,z) z^n dz = Y(u)_n. \tag{3.2}$$

$Y(u,z)$ is called a **vertex operator**.

Moreover, the following axioms are satisfied:

- There is a distinguished vector $\mathbf{1} \in \mathbb{V}(0)$ called **vacuum vector** such that

  $$Y(\mathbf{1},z) = \mathbf{1}_{\mathbb{V}}.$$

  Namely $Y(\mathbf{1})_{-1} = \mathbf{1}_{\mathbb{V}}$ and $Y(\mathbf{1})_n = 0$ if $n \neq -1$.

- **Creation property**: For each $v \in \mathbb{V}$, $Y(v,z)\mathbf{1} = v + \bullet z + \bullet z^2 + \cdots$ where each $\bullet$ is in $\mathbb{V}$. Namely,

  $$Y(v)_{-1}\mathbf{1} = v, \tag{3.3}$$

  and $Y(v)_n\mathbf{1} = 0$ for all $n \neq -1$. This property is abbreviated to

  $$\lim_{z\to 0} Y(v,z)\mathbf{1} = v.$$

- **Grading property**: For each $v \in \mathbb{V}$,

$$[L_0, Y(v,z)] = Y(L_0 v, z) + z\frac{d}{dz}Y(v,z). \tag{3.4}$$

- **L_{-1}-derivative property**: There is a distinguished linear operator L_{-1} on $\mathbb{V}$ such that

$$L_{-1}\mathbf{1} = 0, \tag{3.5}$$

  and that for each $v \in \mathbb{V}$,

$$[L_{-1}, Y(v,z)] = \frac{d}{dz}Y(v,z). \tag{3.6}$$

- **Jacobi identity**: This is the most crucial yet complicated axiom. We postpone its definition to Subsection

We say that $\mathbb{V}$ is a **vertex operator algebra** (VOA) if L_0, L_{-1} can be extended to a countable set of linear operators $\{L_n : n \in \mathbb{Z}\}$ on $\mathbb{V}$ satisfying the Virasoro relation (2.8) for some central charge $c \in \mathbb{C}$, and if there is a distinguished vector $\mathbf{c} \in \mathbb{V}$, called the **conformal vector**, such that

$$Y(\mathbf{c})_n = L_{n-1}, \tag{3.7}$$

or equivalently,

$$Y(\mathbf{c}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{3.8}$$

□

You may wonder why the right hand side of (3.7) is not L_n or L_{n-a} for some constant $a \neq 1$. Indeed, if it were not L_{n-1}, then the Virasoro relation would be not compatible with the Jacobi identity. We will explain this in more details after we formally introduce the Jacobi identity.

We warn the readers that our definitions of graded vertex algebras and VOAs are slightly stronger than the usual ones in the VOA literature, which do not require L_0 to have non-negative eigenvalues. This positivity condition $L_0 \geqslant 0$ is very mild and satisfied by most interesting examples including all unitary ones. Since assuming this condition will simply proofs, we keep it in our definition.

Also, most VOA textbooks and articles use either ω or ν to denote the conformal vector $\mathbf{c}$. In our notes, ω and ν are reserved for other meanings and hence do not denote conformal vectors in order to avoid conflicts of notations.

3.2

Before we give the motivations for these axioms, let us first derive some useful facts.

Expand the series (0.2) and take the coefficients before each z^{-n-1}. This gives us the following equivalent form of grading property:

$$[L_0, Y(v)_n] = Y(L_0v)_n - (n+1)Y(v)_n. \tag{3.9}$$

To be more concrete, assuming that v is homogeneous, then

$$[L_0, Y(v)_n] = (\mathrm{wt}v - n - 1)Y(v)_n. \tag{3.10}$$

Namely: $Y(v)_n$ *raises the weights by* $\mathrm{wt}v - n - 1$. It is useful to keep in mind that in the VOA theory, $Y(v)_n$ raises weights when n is sufficiently negative, and lowers weights when n is sufficiently positive. As a related fact, as

$$[L_0, L_n] = -nL_n \tag{3.11}$$

by the Virasoro relation (2.8), L_n raises (resp. L_{-n} lowers) the weights by n.

Notice that $[L_0, Y(\mathbf{c})_n] = [L_0, L_{n-1}] = (-n+1)L_{n-1} = (-n+1)Y(\mathbf{c})_n$. Compare this result with (3.9), we get an important relation which you should remember:

$$L_0\mathbf{c} = 2\mathbf{c}. \tag{3.12}$$

3.3

By (3.10), for each $u, v \in \mathbb{V}$, we know that $Y(u)_n v$ vanishes when n is sufficiently large. Equivalently, we have

$$Y(u,z)v \in \mathbb{C}((z)). \tag{3.13}$$

This important fact is called the **lower truncation property**. It allows us to use meromorphic functions to study VOAs.

In the definition of graded vertex algebras, if the grading property is replaced by the lower truncation property, and if in particular the diagonalizable L_0 is not introduced, then $\mathbb{V}$ is called a **vertex algebra**. We will not address this most general notion in our notes.

3.4

We let

$$\mathbb{V}' = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)^*$$

where $\mathbb{V}(n)^*$ is the dual space of $\mathbb{V}$. $\mathbb{V}'$ is called the **graded dual space** of $\mathbb{V}$. We let L_0 act on $\mathbb{V}'$ such that $L_0 v' = nv'$ whenever $v' \in \mathbb{V}(n)$. Then $L_0^{\mathrm{t}} = L_0$. As before, a **homogeneous** vector of $\mathbb{V}'$ is either 0 or an eigenvector of L_0. From our definition, it is clear that the evaluation between $\mathbb{V}(m)$ and $\mathbb{V}(n)$ vanishes if $m \neq n$.

Proposition 3.2. *For each $u, v \in \mathbb{V}, v' \in \mathbb{V}'$, $\langle v', Y(u,z)v\rangle := \sum_{n\in\mathbb{Z}}\langle v', Y(u)_n v\rangle z^{-n-1}$ is a **Laurent polynomial** of z, i.e.,*

$$\langle v', Y(u,z)v\rangle \in \mathbb{C}[z^{\pm 1}].$$

Thus, when evaluating between **finite energy vectors** (i.e., vectors of $\mathbb{V}$ and $\mathbb{V}'$), $Y(u,z)$ is not only a formal series, but a meromorphic function of $\mathbb{P}^1$ with poles at $0,\infty$.

Proof. We must show that $\sum_{n\in\mathbb{Z}}\langle v',Y(u)_n v\rangle z^{-n-1}$ is a finite sum. By linearity, it suffices to assume that u,v,v' are homogeneous. Then $Y(u)_n v$ is homogeneous with weight $\mathrm{wt}u+\mathrm{wt}v-n-1$. So $\langle v',Y(u)_n v\rangle$ is non-zero only if $\mathrm{wt}v'=\mathrm{wt}u+\mathrm{wt}v-n-1$. Thus

$$\langle v',Y(u,z)v\rangle=\langle v',Y(u)_{\mathrm{wt}u+\mathrm{wt}v-\mathrm{wt}v'-1}\cdot v\rangle\cdot z^{\mathrm{wt}v'-\mathrm{wt}u-\mathrm{wt}v}.$$

□

3.5

The grading and the L_{-1}-derivative properties were presented in the "derative form". We shall present them in the integral form. To prepare for this task, we introduce

$$\mathbb{V}^{\mathrm{cl}}:=\prod_{n\in\mathbb{N}}\mathbb{V}(n)=\big\{(v_0,v_1,v_2,\dots):v_n\in\mathbb{V}(n)\big\},\tag{3.14}$$

called the **algebraic completion** of $\mathbb{V}$. $\mathbb{V}^{\mathrm{cl}}$ is a naturally a subspace of the dual space $(\mathbb{V}')^*$ of $\mathbb{V}'$. (Indeed, we are most interested in the case that each $\mathbb{V}(n)$ is finite dimensional. In such case, one checks easily that $\mathbb{V}^{\mathrm{cl}}=(\mathbb{V}')^*$.) We let

$$P(n):\mathbb{V}^{\mathrm{cl}}\to\mathbb{V}(n),\qquad(v_0,v_1,v_2,\dots)\mapsto v_n\tag{3.15}$$

be the canonical projection onto the n-th component. Then for each $z\in\mathbb{C}^\times=\mathbb{C}\backslash\{0\}$, we have

$$Y(u,z)v\in\mathbb{V}^{\mathrm{cl}}$$

whose projection onto $\mathbb{V}(\mathrm{wt}u+\mathrm{wt}v-n-1)$ is $Y(u)_n v\cdot z^{-n-1}$.

Note that L_0 and λ^{L_0} act on $\mathbb{V}^{\mathrm{cl}}$ in an obvious way:

$$L_0(v_n)_{n\in\mathbb{N}}=(nv_n)_{n\in\mathbb{N}},\qquad\lambda^{L_0}(v_n)_{n\in\mathbb{N}}=(\lambda^n v_n)_{n\in\mathbb{N}}.$$

3.6

Proposition 3.3 (Scale covariance). *For each $\lambda\in\mathbb{C}^\times$, we have*

$$\lambda^{L_0}Y(u,z)\lambda^{-L_0}v=Y(\lambda^{L_0}u,\lambda z)v\tag{3.16}$$

on the level of $\mathbb{V}^{\mathrm{cl}}$. We drop the symbol v and simply write the above relation as

$$\lambda^{L_0}Y(u,z)\lambda^{-L_0}=Y(\lambda^{L_0}u,\lambda z).$$

The method in the following proof will appear repeatedly in our notes.

Proof. Recall $L_0^{\mathrm{t}} = L_0$. Fix $z \in \mathbb{C}^\times$. We prove that for each homogeneous u, v, v',

$$\langle \lambda^{L_0} v', Y(u,z)\lambda^{-L_0} v\rangle = \langle v', Y(\lambda^{L_0} u, \lambda z)v\rangle. \tag{3.17}$$

The left hand side f is a scalar times $\lambda^{\mathrm{wt}v' - \mathrm{wt}v}$, and the right hand side g is a Laurent polynomial of λ. So both are holomorphic functions on $\mathbb{C}^\times$. Clearly these two expressions are equal when $\lambda = 1$. Let us prove that they are equal for all $\lambda \neq 0$ by showing that they satisfy the same differential equation.

From the form of f, it is clear that $\partial_\lambda f(\lambda) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} f(\lambda)$. To compute $\partial_\lambda g$, we first compute an easier derivative $\partial_\lambda \langle v', Y(u, \lambda z)v\rangle$. By the chain rule, we have

$$\frac{\partial}{\partial \lambda}\langle v', Y(u,\lambda z)v\rangle = z\frac{d}{d\zeta}\langle v', Y(u,\zeta)v\rangle\Big|_{\zeta=\lambda z},$$

which, due to the grading property, equals

$$\begin{aligned}&\lambda^{-1}\Big\langle v', \big([L_0, Y(u,\lambda z)] - Y(L_0 u, \lambda z)\big)v\Big\rangle\\ =&(\mathrm{wt}v' - \mathrm{wt}v - \mathrm{wt}u)\lambda^{-1}\big\langle v', Y(u,\lambda z)v\big\rangle.\end{aligned}$$

So

$$\partial_\lambda g(\lambda) = \partial_\lambda \big\langle v', Y(\lambda^{L_0}u, \lambda z)v\big\rangle = \partial_\lambda\big(\lambda^{\mathrm{wt}u}\big\langle v', Y(u,\lambda z)v\big\rangle\big) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} g(\lambda).$$

□

Informally, the integral form (3.16) (i.e., the scale covariance) also implies the derivative form (3.9) by taking partial derivative over λ. Thus, on a non-rigorous level, these two forms are equivalent. But the integral form has a clearer geometric meaning, which we shall give later.

In the above proof, we have done our first serious VOA calculation. You should be so familiar these computations that you can "immediately see" the equivalence of the two forms.

The integral form of $[L_{-1}, Y(u,z)] = \partial_z Y(u,z)$ is

$$e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} = Y(u, z+\tau),$$

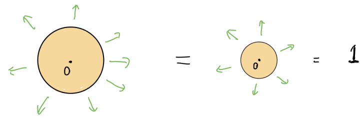
called the **translation covariance**. You may give an informal proof yourself by checking that both sides satisfy the same "linear differential equation". A rigorous treatment is more difficult than the scale covariance. So we leave it to later subsections.

3.7

We now explain the motivations behind the definition of VOAs. Namely, we shall explain how the axioms are natural assumptions from the point of view of the previous two sections. The following explanations are heuristic and non-rigorous.

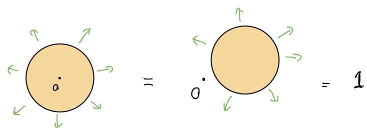
Recall the non-rigorous "definition" of $\mathbb{V}$ in (1.10). We know that $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are subspaces of $\mathcal{H}^{\mathrm{fin}}$, and the decomposition of $\mathcal{H}^{\mathrm{fin}}$ into $\mathbb{V} \otimes \widehat{\mathbb{V}}$-submodules contains a piece $\mathbb{V}\otimes\widehat{\mathbb{V}}$, which furthermore contains $\mathbb{V} \simeq \mathbb{V}\otimes\mathbf{1}$ and $\widehat{\mathbb{V}} \simeq \mathbf{1}\otimes\widehat{\mathbb{V}}$. The vacuum vector is $\mathbf{1} \simeq \mathbf{1}\otimes\mathbf{1}$.

We have said in Subsection 1.8 that the standard unit closed disc $\mathbb{D}_1^{\mathrm{cl}}$ with no input and whose boundary $\mathbb{S}^1$ is parametrized by $z \mapsto z^{-1}$ produces from nothing the vacuum vector $\mathbf{1}\otimes\mathbf{1}$. Namely, the vacuum vector comes from the data $(\mathbb{P}^1; \infty; \zeta^{-1})$ where ζ is the standard coordinate. This data is equivalent to $(\mathbb{P}^1; \infty; \lambda^{-1}\zeta^{-1})$ (where $\lambda \in \mathbb{C}^\times$) via the biholomorphism $z \in \mathbb{P}^1 \mapsto \lambda z \in \mathbb{P}^1$. By the change of local coordinate formula (Principle 2.10), the later geometric data produces uniquely the vector $(\lambda^{L_0}\otimes\overline{\lambda}^{\overline{L}_0})\mathbf{1}$, which is equal to $\mathbf{1}$ by the equivalence of the two geometric data. Apply ∂_λ and $\partial_{\overline{\lambda}}$ to $(\lambda^{L_0}\otimes\overline{\lambda}^{\overline{L}_0})\mathbf{1} = \mathbf{1}$, we see that $L_0\mathbf{1} = \overline{L}_0\mathbf{1} = 0$. This explain $\mathbf{1} \in \mathbb{V}(0)$ in Def. 3.1.



Consequently, by (2.25), the eigenvalues of L_0 are integers, and hence $\geqslant 0$ integers by the positive energy Assumption 2.4.

Similarly, the standard disc $\mathbb{D}_1^{\mathrm{cl}}$ is equivalent to its translation by some $\tau \in \mathbb{C}$. So we must have $(e^{\tau L_{-1}}\otimes e^{\overline{\tau}\overline{L}_{-1}})\mathbf{1} = \mathbf{1}$ and hence, similarly, $L_{-1}\mathbf{1} = \overline{L}_{-1}\mathbf{1} = 0$. This explains part of the L_{-1}-derivative property.



3.8

Let us describe the meaning of $Y(u,z)v$. For each $z \in \mathbb{C}^\times$, we define a local-coordinated 3-pointed sphere

$$\mathfrak{P}_z = \{\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}\}$$

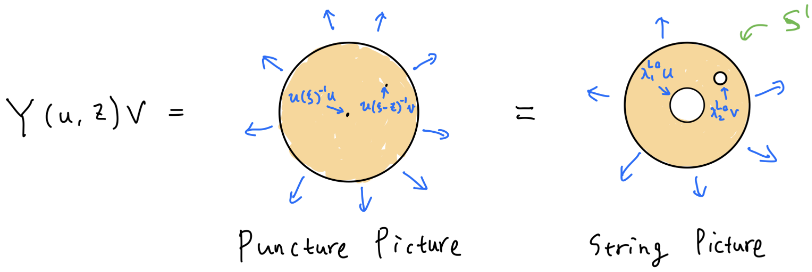
where ζ is the standard coordinate of $\mathbb{C}$.

Let us regard $0, z$ as incoming punctures and ∞ outgoing. Roughly speaking, $Y(u,z)v$ is just $T_{\mathfrak{P}_z}(u\otimes v)$, understood in a suitable way by change of coordinates. Assume first of all that $0 < |z| < 1$. After scaling ζ and $\zeta - z$ to $\lambda_1\zeta, \lambda_2(\zeta - z)$ and hence shrinking the two incoming strings, Assumption 1.1 is satisfied. Let the new N-pointed sphere by denoted by $\mathfrak{P}_z^{\lambda_1,\lambda_2}$. Note that u in the ζ coordinate becomes $(\lambda_1^{L_0}\otimes\overline{\lambda_1}^{\overline{L}_0})u$ in the $\lambda_1\zeta$ coordinate. The latter vector equals $\lambda_1^{L_0}u$ since $\overline{L}_0$ kills $\mathbf{1}$; to be more detailed:

$$(\lambda_1^{L_0}\otimes\overline{\lambda_1}^{\overline{L}_0})u = (\lambda_1^{L_0}\otimes\overline{\lambda_1}^{\overline{L}_0})(u\otimes\mathbf{1}) = \lambda_1^{L_0}u\otimes\mathbf{1} = \lambda_1^{L_0}u.$$

Similarly, v becomes $\lambda_2^{L_0}v$ in the new coordinate. Then $Y(u,z)v$ is (physically) defined as $T_{\mathfrak{P}_z^{\lambda_1,\lambda_2}}(\lambda_1^{L_0}u\otimes\lambda_2^{L_0}v)$.

As in Subsec. 2.17, we can use the *puncture picture* to view u and v as the states associated to the punctures $0, z$ with respect to the local coordinates $\zeta, \zeta - z$. Or moreover, formulated in a coordinate independent way as in Subsec. 2.11, we associate the abstract vector $\mathcal{U}(\zeta)^{-1}v$ (the one whose explicit expression under the coordinate ζ is v) to the puncture 0 and $\mathcal{U}(\zeta - z)^{-1}v$ to z.



3.9

We are going to prove translation covariance rigorously. For that purpose, we need to generalize the differential equation method in the proof of scale covariance to the following vector-valued form:

Lemma 3.4. *Let W be a (non-necessarily finite dimensional) vector space, and $f \in W[[z]]$. Suppose that $\frac{d}{dz}f(z) = Af(z)$ for some $A \in \mathrm{End}(W)$. Suppose also that $f(0) = 0$, namely, the constant term in the power series $f(z)$ is 0. Then $f = 0$.*

Proof. Write $f(z) = \sum_{n\in\mathbb{N}} f_n z^n$ where each $f_n \in W$. The assumptions say that $f_0 = 0$ and

$$\sum_{n\in\mathbb{N}} n f_n z^{n-1} = \sum_{n\in\mathbb{N}} A f_n z^n.$$

So $nf_n = Af_{n-1}$ where $n > 0$. This proves that all f_n are 0. □

3.10

We have said that the integral form of $[L_{-1}, Y(u,z)] = \partial_z Y(u,z)$ is

$$\langle v', e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} v\rangle = \langle v', Y(u, z+\tau)v\rangle. \tag{3.18}$$

This relation is more difficult to address than the scale covariance since both sides actually involve infinite sums of powers of τ. Our goal is to understand: on which domain does this relation hold? Certainly we need $\tau \neq -z$. But this condition is far from enough.

Let us first understand the two sides as infinite series of τ. Assume without loss of generality that u, v, v' are homogeneous. The right hand side is of the form $a(z+\tau)^m$ for some $a \in \mathbb{C}, m \in \mathbb{Z}$. Certainly this expression makes sense as a rational function, but we shall first regard it as a power series of τ by expanding $(z+\tau)^m$ on the domain $|\tau| < |z|$, namely $(z+\tau)^m = \sum_{k\in\mathbb{N}} \binom{m}{k} z^{m-k}\tau^k$. Thus, the right hand side of (3.18), as an element of $\mathbb{C}[[\tau]]$, is understood as

$$\langle v', Y(u, z+\tau)v\rangle = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{N}} \binom{-n-1}{k} \langle v', Y(u)_n v\rangle \cdot z^{-n-1-k}\tau^k.$$

Here, the sum over $n \in \mathbb{Z}$ is finite, and when the vectors are homogeneous, there is only one possibly non-zero summand.

But why do we expand $(z+\tau)^m$ on $|\tau| < |z|$? Why not $|z| < |\tau|$? Well, this will give us $\sum_{k\in\mathbb{N}} \binom{m}{k} z^k \tau^{m-k}$, an element of $\mathbb{C}[[\tau^{\pm 1}]]$ but not $\mathbb{C}[[\tau]]$. (Namely, we will have infinitely many negative powers of τ). But why at all do we want it to be in $\mathbb{C}[[\tau]]$? Because the left hand side of (3.18) can only be regarded as in $\mathbb{C}[[\tau]]$.

So let us turn to the left hand side of (3.18). It would be easier to first understand why

$$\langle v', e^{\lambda L_{-1}} Y(u,z) e^{-\mu L_{-1}} v \rangle \tag{3.19}$$

is an element of $\mathbb{C}[[\lambda,\mu]]$. We first want to move $e^{\lambda L_{-1}}$ to the left hand side of the bracket. In general, if L_n is defined on $\mathbb{V}$, we define L_{-n} on $\mathbb{V}'$ to be the transpose of L_n: $L_{-n} = L_n^{\mathrm{t}}$, or more precisely,

$$\langle L_{-n}v', v\rangle := \langle v', L_n v\rangle. \tag{3.20}$$

In case you doubt why this transpose exists, we can write the definition even more precisely: Assume $v' \in \mathbb{V}'(m)$. Then $L_{-n}v'$ is a linear functional on $\mathbb{V}(m+n)$ (so L_{-n} raises the weights by n) whose value at any $v \in \mathbb{V}(m+n)$ is $\langle v', L_n v\rangle$. (Recall that L_n lowers the weights by n so $L_n v \in \mathbb{V}(m)$.)

Now, for each $z \in \mathbb{C}^\times$, (3.19) equals

$$f(\lambda,\mu) := \langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} v\rangle = \sum_{n,l\in\mathbb{N}} \frac{\lambda^n(-\mu)^l}{n!l!} \langle L_1^n v', Y(u,z) L_{-1}^l v\rangle. \tag{3.21}$$

This is in $\mathbb{C}[[\lambda,\mu]]$. Indeed, it is in $\mathbb{C}[[\mu]][\lambda]$ since $L_1^n v'$ lowers the weight by n, and hence vanishes when $n > \mathrm{wt} v'$. But we will not need this fact here.

Now, the left hand side of (3.18) can be understood as $f(\tau,\tau)$, noting the following fact:

Lemma 3.5. *If $f(z_1,\dots,z_N) \in \mathbb{C}[[z_1,\dots,z_N]]$, then $f(z,\dots,z)$ naturally makes sense as an element of $\mathbb{C}[[z]]$.*

Proof. Write $f(z_\bullet) = \sum a_{n_1,\dots,n_N} z_1^{n_1}\cdots z_N^{n_N}$. Then

$$f(z,\dots,z) = \sum_{n\in\mathbb{N}} \sum_{n_1+\cdots+n_N=n} a_{n_1,\dots,n_N} z^n$$

where the inside sum is clearly finite. □

3.11

Proposition 3.6 (Translation covariance). *For each $u,v \in \mathbb{V}$, $v' \in \mathbb{V}'$ and each $z \in \mathbb{C}^\times$, the following power series of τ are equal and converge absolutely on the domain $\mathbb{D}_{|z|} = \{\tau \in \mathbb{C} : |\tau| < |z|\}$:*

$$\langle v', e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} v\rangle = \langle v', Y(u, z+\tau) v\rangle. \tag{3.22}$$

Here, the right hand side, which is a priori a Laurent polynomial of $z+\tau$, is expanded as a (clearly absolutely convergent) power series of τ on $\mathbb{D}_{|z|}$.

Proof. Let $f(\tau)$ and $g(\tau)$ be the left and the right hand sides of (3.22). If we can show that they are equal as formal power series of τ, then the left hand side converges absolutely on $\mathbb{D}_{|z|}$ since the right hand side does.

Since $f(0) = g(0)$, it suffices to prove that f and g satisfy the same linear differential equation. The left hand side is $f(\tau, \tau)$ where

$$f(\lambda, \mu) = \big\langle e^{\lambda L_1}v', Y(u, z)e^{-\mu L_{-1}}v\big\rangle \qquad \in \mathbb{C}[[\lambda, \mu]].$$

As a general result about multivariable formal power series, we have chain rule

$$\frac{d}{d\tau}f(\tau, \tau) = (\partial_\lambda + \partial_\mu)f(\lambda, \mu)\big|_{\lambda=\mu=\tau}.$$

(It is reasonable to believe that this is true. But you can also give a rigorous proof by expanding the two series and check that their coefficients agree!) So, as

$$\begin{aligned}\partial_\lambda f(\lambda, \mu) &= \big\langle e^{\lambda L_1}L_1v', Y(u, z)e^{-\mu L_{-1}}v\big\rangle,\\ \partial_\mu f(\lambda, \mu) &= -\big\langle e^{\lambda L_1}v', Y(u, z)e^{-\mu L_{-1}}L_{-1}v\big\rangle,\end{aligned}$$

we have

$$\partial_\tau f(\tau) = \big\langle e^{\tau L_1}L_1v', Y(u, z)e^{-\tau L_{-1}}v\big\rangle - \big\langle e^{\tau L_1}v', Y(u, z)e^{-\tau L_{-1}}L_{-1}v\big\rangle.$$

This expression is not a differential equation of the scalar-valued power series f. But we can make it an ODE by fixing u, varying v, v', and view f as a $(\mathbb{V}\otimes\mathbb{V}')^*$-valued power series of τ. Then $\partial_\tau = Af$ where $A \in \mathrm{End}\big((\mathbb{V}\otimes\mathbb{V}')^*\big)$ is defined to be the transpose of $\mathbf{1}\otimes L_1 - L_{-1}\otimes\mathbf{1}$. Namely, for each $\xi \in (\mathbb{V}\otimes\mathbb{V}')^*$,

$$\langle A\xi, v\otimes v'\rangle = \langle \xi, v\otimes L_1v' - L_{-1}v\otimes v'\rangle.$$

Now, we compute (noting that the following sum is finite for each fixed v, v')

$$\begin{aligned}&\partial_\tau g(\tau) = \partial_\tau\big\langle v', Y(u, z+\tau)v\big\rangle = \partial_\tau\Big(\sum_n a_n(z+\tau)^n\Big)\\ &= \sum_n na_n(z+\tau)^{n-1} = \partial_\zeta\Big(\sum_n a_n\zeta^n\Big)\Big|_{\zeta=z+\tau} = \partial_\zeta\big\langle v', Y(u,\zeta)v\big\rangle\big|_{\zeta=z+\tau}.\end{aligned}$$

By the L_{-1}-derivative property, the above equals

$$\partial_\tau g(\tau) = \big\langle v', [L_{-1}, Y(u,\zeta)]v\big\rangle\big|_{\zeta=z+\tau},$$

which also equals $Ag(\tau)$ if we now vary v, v' and regard g as $(\mathbb{V}\otimes\mathbb{V}')^*$-valued. Therefore, $f(\tau) = g(\tau)$ due to Lemma 3.4. $\square$

3.12

Remark 3.7. Let us consider a useful variant of Prop. 3.6. Notice that (3.22) holds if v' is replaced by L_{-1}^n and also both sides are multiplied by τ^n. Thus, (3.22) holds on the level of $\mathbb{C}[[\tau]]$ if v' is replaced by $e^{-\tau L_{-1}}v'$. Namely: $\big\langle v', Y(u, z)e^{-\tau L_{-1}}v\big\rangle = \big\langle e^{-\tau L_1}v', Y(u, z+\tau)v\big\rangle$. We prefer to replace τ by $-\tau$:

$$\big\langle v', Y(u, z)e^{\tau L_{-1}}v\big\rangle = \big\langle e^{\tau L_1}v', Y(u, z-\tau)v\big\rangle. \tag{3.23}$$

Again, the left hand side converges absolutely on $|\tau| < |z|$ since the right hand side (which is a polynomial of τ and $(z-\tau)^{\pm 1}$) does. $\square$

We have just proved our first *convergence property* in this course.

Index

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