HAUSDORFF CONVERGENCE PROPERTY OF PARTIAL OKOUNKOV BODIES

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Contents

1. Introduction

This note is an refinement of [Xia21, Theorem A]. We prove the Hausdorff convergence property in full generality.

This note is motivated by a discussion with Sébastien Boucksom.

2. Hausdorff convergence property

Let X be a connected smooth projective manifold of dimension n. Let (L, h) be a Hermitian pseudo-effective line bundle on X with L big. Fix $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ a valuation of rank n and rational rank n. Take a smooth Hermitian metric h_0 on L and set $\theta = c_1(L, h_0)$. We can then identify h with $\varphi \in \mathrm{PSH}(X, \theta)$.

For each $k \in \mathbb{Z}_{>0}$, we introduce

$$\Delta^k_\nu(\theta, \varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the closed convex hull.

For later use, we introduce a twisted version as well. If T is a holomorphic line bundle on X, we introduce

$$\Delta^{k,T}_\nu(\theta, \varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T \otimes L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

We also write

$$\Delta^{k,T}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T \otimes L^k) \right\} \subseteq \mathbb{R}^n$$

and

$$\Delta^{k}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k) \right\} \subseteq \mathbb{R}^n$$

We write $\mathcal{I}_\infty(\varphi) = \mathcal{I}_\infty(h)$ for the ideal sheaf on X locally consisting of holomorphic functions f such that $|f|_h$ is locally bounded.

We first extend [Xia21, Theorem 3.13] to the twisted case.

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Proposition 2.1. *For any holomorphic line bundle T on X,*

$$\Delta_\nu^{k,T}(L) \to \Delta_\nu(L)$$

as $k \to \infty$.

Here and later on, we endow the space of convex bodies with the Hausdorff metric.

Proof. As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0;
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$H^0(X, L^{k-k_0}) \xrightarrow{\times s_1} H^0(X, T \otimes L^k) \xrightarrow{\times s_0} H^0(X, T \otimes L^{k+k_0}).$$

It follows that

$$(k-k_0)\Delta_\nu^{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_\nu^{k,T}(L) \subseteq (k+k_0)\Delta_\nu^{k+k_0}(L) - \nu(s_0).$$

By [Xia21, Theorem 3.13], we conclude. □

Lemma 2.2. *Let T be a holomorphic line bundle on X. Assume that φ has analytic singularities and θ_φ is a Kähler current, then*

$$\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(\theta, \varphi).$$

Proof. Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along a nc $\mathbb{Q}$-divisor D. Take $\epsilon \in (0,1) \cap \mathbb{Q}$. In this case, by Ohsawa–Takegoshi theorem, for any $k \in \mathbb{Z}_{>0}$ we have

$$H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$, say the sequence defined by the indices $k_1, k_2, \ldots$. We want to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

There exists $t \in \{0, 1, \ldots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$.

Now we have

$$\Delta_\nu^{g_i, T\otimes L^t}(NL - ND) + N\nu(D) \subseteq N\Delta_\nu^{k,T}(\theta, \varphi) \subseteq \Delta_\nu^{g_i, T\otimes L^t}(NL - N(1-\epsilon)D) + N(1-\epsilon)\nu(D).$$

By Proposition 2.1,

$$\Delta_\nu(L-D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L - (1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Let $\epsilon \to 0+$, we find that

$$\Delta_\nu(L-D) + \nu(D) = \Delta'.$$

It follows from Blanschke selection theorem that

$$\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(L-D) + \nu(D) = \Delta_\nu(\theta, \varphi)$$

as $k \to \infty$. □

Lemma 2.3. *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \to 0+$, we have*

$$\Delta_\nu((1-\beta)\theta, \varphi) \to \Delta_\nu(\theta, \varphi).$$

Proof. By [Xia21, Proposition 5.15], we have

$$\Delta_\nu((1-\beta)\theta,\varphi)+\beta\Delta_\nu(L)\subseteq\Delta_\nu(\theta,\varphi).$$

In particular, if Δ' is a limit of a subsequence of $(\Delta_\nu((1-\beta)\theta,\varphi))_\beta$, then

$$\Delta'\subseteq\Delta_\nu(\theta,\varphi).$$

But

$$\operatorname{vol}\Delta'=\lim_{\beta\to 0+}\Delta_\nu((1-\beta)\theta,\varphi)=\lim_{\beta\to 0+}\int_X((1-\beta)\theta+\mathrm{dd^c}P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n.$$

We claim that

$$\lim_{\beta\to 0+}\int_X((1-\beta)\theta+\mathrm{dd^c}P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n=\int_X(\theta+\mathrm{dd^c}P^{\theta}[\varphi]_{\mathcal{I}})^n.$$

Note that this finishes the proof as $\operatorname{vol}\Delta_\nu(\theta,\varphi)$ is exactly equal to the right-hand side.

Next we prove our claim. We make use of the b-divisors introduced in [Xia20; Xia22]. By [Xia22, Theorem 0.6], the claim is equivalent to

$$\lim_{\beta\to 0+}\operatorname{vol}\mathbb{D}((1-\beta)\theta,\varphi)=\operatorname{vol}\mathbb{D}(\theta,\varphi).$$

This is a special case of [Xia22, Theorem 9.6] □

Theorem 2.4. *Let T be a holomorphic line bundle on X. As $k\to\infty$, $\Delta_\nu^{k,T}(\theta,\varphi)\to\Delta_\nu(\theta,\varphi)$.*

Proof. Fix a Kähler form $\omega\geq\theta$ on X.

Step 1. We first handle the case where $\mathrm{dd^c}h$ is a Kähler current, say $\mathrm{dd^c}h\geq\beta_0\omega$ for some $\beta_0\in(0,1)$.

Take a decreasing quasi-equisingular approximation φ_j of φ. Up to replacing β_0 by $\beta_0/2$, we may assume that

$$\theta_{\varphi_j}\geq\beta_0\omega$$

for all $j\geq 1$.

Take $\beta\in(0,\beta_0)\cap\mathbb{Q}$. Write $\beta=p/q$ with $p,q\in\mathbb{Z}_{>0}$. Fix $t\in\{0,1,\dots,q-1\}$.

By [DX21, Lemma 4.2], we can find $k_0\in\mathbb{Z}_{>0}$ such that for all $k\geq k_0$, there is $v_{\beta,k}\in\mathrm{PSH}(X,\theta)$ satisfying

(1)
$$P[\varphi]_{\mathcal{I}}\geq(1-\beta)\varphi_k+\beta v_{\beta,k};$$

(2) $v_{\beta,k}$ has positive mass.

Observe that for any $j\geq 1$,

$$\theta_{\varphi_j}\geq\beta\omega\geq\beta\theta.$$

Namely, $\varphi_j\in\mathrm{PSH}(X,(1-\beta)\theta)$.

Fix $k\geq k_0$. Let $\pi:Y\to X$ be a log resolution of the singularities of φ_k. By the proof of [DX21, Proposition 4.3], there is $j_0=j_0(\beta,k)\in\mathbb{Z}_{>0}$ such that for any $j\geq j_0$, we can find a non-zero section $s_j\in H^0(Y,\pi^*L^{pj}\otimes\mathcal{I}(jp\pi^*v_{\beta,k}))$ such that we get an injective linear map

$$H^0(Y,\pi^*T\otimes\pi^*L^t\otimes K_{Y/X}\otimes\pi^*L^{(q-p)j}\otimes\mathcal{I}(jq\pi^*\varphi_k))\xrightarrow{\times s_j}H^0(X,T\otimes L^t\otimes L^{jq}\otimes\mathcal{I}(jq\varphi)).$$

It follows that
$$\Delta_\nu^{j,\pi^*T\otimes\pi^*L^t\otimes K_{Y/X}}((1-\beta)q\pi^*\theta,q\pi^*\varphi_k)+j^{-1}\nu(s_j)\subseteq q\Delta_\nu^{qj,T\otimes L^t}(\theta,\varphi).$$
We observe that $j^{-1}\nu(s_j)$ is bounded as the right-hand side is bounded when j varies.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{qj,T\otimes L^t}(\theta,\varphi))_j$, then by Lemma 2.2, there is a vector $v'_k\in\mathbb{R}^n_{\geq 0}$ such that
$$\Delta_\nu((1-\beta)\pi^*\theta,\pi^*\varphi_k)+v'_k\subseteq\Delta'.$$
By the birational invariance of the partial Okounkov bodies,
$$\Delta_\nu((1-\beta)\theta,\varphi_k)+v'_k\subseteq\Delta'.$$
Let $k\to\infty$, by [Xia21, Theorem A],
$$\Delta_\nu((1-\beta)\theta,\varphi)+v'\subseteq\Delta'$$
for some vector $v'\in\mathbb{R}^n_{\geq 0}$ depending on β.

Let $\beta\to 0+$, by Lemma 2.3, we have
$$\Delta_\nu(\theta,\varphi)+v''\subseteq\Delta'$$
for some vector $v''\in\mathbb{R}^n_{\geq 0}$.

On the other hand, take $j\geq 1$, as $\varphi\leq\varphi_j$,
$$\Delta_\nu^{k,T}(\theta,\varphi)\subseteq\Delta_\nu^{k,T}(\theta,\varphi_j).$$
By Lemma 2.2,
$$\Delta'\subseteq\Delta_\nu(\theta,\varphi_j).$$
So
$$\Delta_\nu(\theta,\varphi)+v''\subseteq\Delta_\nu(\theta,\varphi_j).$$
Let $j\to\infty$, we find that $v''=0$. Namely,
$$\Delta_\nu(\theta,\varphi)\subseteq\Delta' \tag{2.1}$$

Next we compute
$$\operatorname{vol}\Delta'\leq\operatorname{vol}\Delta_\nu(\theta,\varphi_j)=\int_X\theta^n_{\varphi_j}.$$
Let $j\to\infty$, we find
$$\operatorname{vol}\Delta'\leq\int_X\theta^n_{P[\varphi]_{\mathcal{I}}}=\operatorname{vol}\Delta_\nu(\theta,\varphi).$$
It follows that equality holds in (2.1). Namely,
$$\Delta_\nu^{qj,T\otimes L^t}(\theta,\varphi)\to\Delta_\nu(\theta,\varphi)$$
as $j\to\infty$. It follows that
$$(qj+t)^{-1}\operatorname{Conv}\{\nu(s):s\in H^0(X,T\otimes L^{qj+t}\otimes\mathcal{I}(qj\varphi))\}\to\Delta_\nu(\theta,\varphi)$$
as $j\to\infty$. Observe that
$$\begin{aligned}&(qj+t)^{-1}\operatorname{Conv}\{\nu(s):s\in H^0(X,T\otimes L^{qj+t}\otimes\mathcal{I}(qj\varphi))\}\supseteq\Delta_\nu^{qj+t,T}(\theta,\varphi)\\ \supseteq&(qj+t)^{-1}\operatorname{Conv}\{\nu(s):s\in H^0(X,T\otimes L^{-q}\otimes L^{q(j+1)+t}\otimes\mathcal{I}(q(j+1)\varphi))\}.\end{aligned}$$
It follows that $\Delta_\nu^{qj+t,T}(\theta,\varphi)\to\Delta_\nu(\theta,\varphi)$ as $j\to\infty$. As t is arbitrary, we conclude.

Step 2. Next we handle the general case.

Take $\psi \in \mathrm{PSH}(X,\theta)$ such that

(1) θ_ψ is a Kähler current;
(2) $\psi \leq \varphi$.

The existence of ψ is proved in [DX21, Proposition 3.6].

Then for any $\epsilon \in \mathbb{Q} \cap (0,1)$,

$$\Delta_\nu^{k,T}(\theta,\varphi) \supseteq \Delta_\nu^{k,T}(\theta,(1-\epsilon)\varphi+\epsilon\psi)$$

for all k. It follows from Step 1 that for any limit Δ' of any subsequence of $\{\Delta_\nu^{k,T}(\theta,\varphi)\}_k$, we have

$$\Delta' \supseteq \Delta_\nu(\theta,(1-\epsilon)\varphi+\epsilon\psi).$$

For later use, we denote the indices defining the subsequence as $k_1, k_2, \ldots$.

Letting $\epsilon \to 0$ and applying [Xia21, Theorem A], we have

$$\Delta' \supseteq \Delta_\nu(\theta,\varphi).$$

We claim that

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(\theta,\varphi).$$

From this claim, the theorem follows.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in H^0(X,H)$. Take $N \in \mathbb{Z}_{>0}$ then at least one element among $\{0,\ldots,N-1\}$ occurs infinitely many times as the residues of k_i modulo N for various i. Up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ for all i, where $t \in \{0,\ldots,N-1\}$. Up to changing T to $T \otimes L^{-t}$, we may assume that $t=0$.

We have an injective linear map

$$H^0(X, T \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)) \xrightarrow{\times s^k} H^0(X, T \otimes H^k \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)).$$

In particular, for each $i \geq 1$,

$$k_i\Delta_\nu^{k_i,T}(N\theta,N\varphi) + k_i\nu(s) \subseteq k_i\Delta_\nu^{k_i,T}(N\theta+\omega,N\varphi).$$

Let $i \to \infty$, by Step 1,

$$N\Delta' + \nu(s) \subseteq \Delta_\nu(\omega+N\theta,N\varphi).$$

So

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(N^{-1}\omega+\theta,\varphi) = \int_X (N^{-1}\omega+\theta+\mathrm{dd^c}P^{N^{-1}\omega+\theta}[\varphi]_{\mathcal{I}})^n.$$

By [Xia21, Corollary 4.4], the right-hand side is equal to

$$\int_X (N^{-1}\omega+\theta+\mathrm{dd^c}P^{\theta}[\varphi]_{\mathcal{I}})^n.$$

Let $N \to \infty$, we find

$$\operatorname{vol}\Delta' \leq \int_X (\theta+\mathrm{dd^c}P^{\theta}[\varphi]_{\mathcal{I}})^n = \operatorname{vol}\Delta_\nu(\theta,\varphi).$$

Our claim holds. $\square$

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