

Operations on transcendental non-Archimedean metrics

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Dedicated to Bo Berndtsson.

ABSTRACT. We define the basic pluripotential-theoretic operations in terms of the transcendental theory of non-Archimedean metrics introduced in [DXZ23]. In particular, we establish that the analogue of Boucksom–Jonsson’s envelope conjecture holds in our theory.

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1. Introduction

Let X be a reduced irreducible unibranch compact Kähler space of dimension n and θ be a closed real $(1, 1)$ -form on X . Assume that $\text{PSH}(X, \theta) \neq \emptyset$. The purpose of this paper is to study the pluripotential theory of non-Archimedean metrics in the cohomology class of θ following the approach in [DXZ23].

1.1. Background. The non-Archimedean pluripotential theory has been developed by many authors for different purposes in the last three decades. The study of semi-positive metrics on line bundles on non-Archimedean spaces dates back to the breakthrough of Zhang [Zha95]. In [CL06], Chambert-Loir initiated the study of the non-Archimedean Monge–Ampère measures. The pluripotential theory on non-Archimedean curves was developed in detail in the thesis of Thuillier [Thu05]. All these developments together lead to the natural question: to what

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extent is there a pluripotential theory on Berkovich spaces similar to its complex counterpart?

This question has drawn the attention of many authors. In particular, so far we know that there are two general approaches in the literature.

The first approach is the local approach, initiated by Chambert-Loir–Ducros [CLD12] and Gubler [Gub16]. In this approach, one defines the notions of differential forms and currents on a Berkovich space based on local tropicalizations. Then one can define the Monge–Ampère operator using the usual differential formula. The problem is that this approach only works for regular plurisubharmonic functions and completely fails when a singularity is presented. This is an essential obstacle for applications to complex geometry.

The second approach was initiated by Boucksom–Favre–Jonsson in [BFJ15, BFJ16], following the original ideas of Zhang. In the trivially valued case, this theory was further developed by Boucksom–Jonsson [BJ21]. In this approach, one first considers metrics obtained from semipositive models and then defines general global plurisubharmonic metrics using approximations. This approach allows singularities but does not work in the local situation. It does not yield a sheaf of plurisubharmonic metrics. Both approaches have their own drawbacks. We do not have a completely satisfactory non-Archimedean pluripotential theory yet.

1.2. The transcendental theory. The non-Archimedean pluripotential theory is closely related to complex geometry. We refer to [BBJ21] and the references therein. The basic idea is that the degeneration in the space of Kähler potentials can be characterized using global plurisubharmonic metrics on the non-Archimedean space. This idea has been intensively studied in [BBJ21, Li20, DX21, Xia22b].

We observe that there is a strong obstacle in applying the non-Archimedean geometry to complex geometry: the envelope conjecture [BJ21, Conjecture 5.14] of Boucksom–Jonsson remains open. This conjecture together with its various consequences is indispensable for various applications.

In our previous work [DXZ23], we introduced a different framework for the non-Archimedean global pluripotential theory based on the series of works starting from [BBJ21, RWN14]. Roughly speaking, we represent a non-Archimedean metric in $\text{PSH}^{\text{NA}}(X, \theta)$ by a *test curve* $\mathfrak{T} = (\mathfrak{T}_\tau)_\tau$, a concave curve of \mathcal{I} -model potentials. Due to the pathology caused by zero-mass, the rigorous definition requires a projective limit construction as well, we refer to Definition 3.5 for the rigorous definition. In this short introduction, we will only talk about test curves to simplify our presentation.

In the algebraic setting, when Boucksom–Jonsson’s theory is applicable, there is an obvious comparison map sending \mathfrak{T} to a plurisubharmonic metric in the sense Boucksom–Jonsson. See Definition 3.11 for the details. The main result in [DXZ23] asserts that when X is smooth, this comparison map is in fact a bijection, hence giving a complex representation of the non-Archimedean pluripotential theory. When X is only unibranch, Boucksom–Jonsson’s theory is still applicable. In this case, we prove the following result:

THEOREM 1.1 (=Corollary 5.3). *Assume that X is a reduced irreducible unibranch projective variety over \mathbb{C} and L is a pseudoeffective line bundle on X . Then the following are equivalent:*

- (1) *Boucksom–Jonsson’s envelope conjecture holds on X ;*

- (2) *The comparison map extends to a bijection from the space of non-Archimedean potentials in the sense of [DXZ23] to the corresponding space in the sense of [BJ21].*

Note that Boucksom–Jonsson’s envelope conjecture is known to hold when X is smooth, as confirmed in [BJ22]. A different proof was given in [DXZ23].

Using the comparison map, we can effectively translate the pluripotential-theoretic operations in Boucksom–Jonsson’s theory to our transcendental theory. It turns out that in most cases, these translations can be extended to more general situations where the comparison fails or when Boucksom–Jonsson’s theory is not applicable. This is exactly what we do in this paper. The final result is summarized in Table 1. It can be regarded as a duality between non-Archimedean geometry and convex geometry.

NA potentials	Test curves
Addition by a constant	Translation
Maximum	Concave envelope of maximum
Decreasing limit	Decreasing limit
Increasing limit	Increasing limit
Regularized supremum	Increasing limit of concave envelopes of maxima
Rescaling	Rescaling and reparametrization
Addition	Infimal involution
Restriction to subvarieties	Trace operator

TABLE 1. Correspondence non-Archimedean/convex

Most results from this table are quite straightforward to experts. The only exception is perhaps the last line (=Theorem 4.21), where we establish a quantitative version of the fact that a non-Archimedean plurisubharmonic metric is determined by its restriction to divisorial points. We hope that this result, after suitable extension, could serve to give a proof of Boucksom–Jonsson’s envelope conjecture in general.

As a particular case from this table, we have

THEOREM 1.2 (=Theorem 4.13). *The transcendental analogue of the envelope conjecture holds in the theory of [DXZ23].*

This result slightly generalizes [DXZ23, Theorem 1.2] in that the latter only concerns the regularized supremum of increasing nets while the former works for a general bounded from above family.

Given the fact that Boucksom–Jonsson’s envelope conjecture is widely open in general, we expect that our theory is more useful on normal varieties. In particular, we hope that the non-Archimedean theory of K-stability could be effectively translated to our theory. The very first step in this direction is carried out in [Xia22b]. On the other hand, our theory works more generally for transcendental classes on compact Kähler manifolds. It should be useful to the transcendental applications as well, for example in the study of cscK metrics as in [SD18] or probably in the study of the transcendental Morse inequality.

Most results from the current paper have been announced in several conferences. The author hopes that the details presented in the current paper are of interest to

general readers as well. We assume that the readers are familiar with the complex pluripotential theory and Berkovich geometry.

1.3. Conventions. In the whole paper, the Monge–Ampère type products are always taken in the non-pluripolar sense. We adopt the convention that $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. A variety refers to a separated scheme of finite type over \mathbb{C} .

2. Pluripotential theory on unibranch spaces

Let X be a reduced irreducible unibranch compact Kähler space of dimension n and θ be a closed real $(1,1)$ -form on X . Here and in the sequel, a $(1,1)$ -form is always assumed to be smooth. Assume that $\text{PSH}(X, \theta) \neq \emptyset$. There is a well-developed pluripotential theory in this setting. For the Bedford–Taylor theory, we refer to [Dem85] and for the non-pluripolar theory, we refer to [Xia21a].

Note that we have assumed X to be irreducible. This is not a very severe restriction: since X is unibranch, its irreducible components are just its connected components. Every result in this paper extends in the obvious way to general reduced unibranch compact Kähler spaces.

2.1. Divisorial valuations.

Definition 2.1. A *prime divisor* over a reduced irreducible compact complex space Z is a connected smooth hypersurface $E \subseteq X'$, where $X' \rightarrow Z$ is a projective resolution.

Two prime divisors $E_1 \subseteq X'_1$ and $E_2 \subseteq X'_2$ over Z are *equivalent* if there is a common resolution $X'' \rightarrow X$ dominating both X'_1 and X'_2 such that the strict transforms of E_1 and E_2 coincide.

The set Z^{div} is the set of pairs (c, E) , where $c \in \mathbb{Q}_{>0}$ and E is an equivalence class of a prime divisor over Z . For simplicity, we will denote the pair (c, E) by cord_E , although one should not really think of this object as a valuation unless Z is projective and irreducible.

Definition 2.2. Given $\varphi \in \text{PSH}(X, \theta)$, we define its *associated non-Archimedean potential* as the map $\varphi^{\text{an}}: X^{\text{div}} \rightarrow \mathbb{R}$ given by

$$\varphi^{\text{an}}(\text{cord}_E) := -c\nu(\varphi, E).$$

Here associated with E , there is a projective resolution $\pi: Y \rightarrow X$ such that E is a connected smooth hypersurface in Y . The notation $\nu(\varphi, E)$ means the generic Lelong number of $\pi^*\varphi$ along E . By Siu’s semicontinuity theorem, this quantity is independent of the choice of π .

2.2. The pluripotential theory. For the notions of model, \mathcal{I} -model and \mathcal{I} -good potentials and the $P[\bullet]$ and $P[\bullet]_{\mathcal{I}}$ envelopes on a compact Kähler manifold, we refer to [DX21, DX22, Xia22a] for the details.

Definition 2.3. We say a potential $\varphi \in \text{PSH}(X, \theta)$ is *model* (resp. \mathcal{I} -model, \mathcal{I} -good) if for a projective resolution $\pi: Y \rightarrow X$, $\pi^*\varphi \in \text{PSH}(Y, \pi^*\theta)$ is model (resp. \mathcal{I} -model, \mathcal{I} -good).

All three notions are independent of the choice of π .

Definition 2.4. Given $\varphi \in \text{PSH}(X, \theta)$, we define $P_\theta[\varphi], P_\theta[\varphi]_{\mathcal{I}} \in \text{PSH}(X, \theta)$ as follows: let $\pi: Y \rightarrow X$ be a projective resolution, then we require the following holds:

$$\pi^* P_\theta[\varphi] = P_{\pi^*\theta}[\pi^*\varphi], \quad \pi^* P_\theta[\varphi]_{\mathcal{I}} = P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}}.$$

By Zariski's main theorem, both $P_\theta[\varphi]$ and $P_\theta[\varphi]_{\mathcal{I}}$ are uniquely determined.

Both operations are clearly independent of the choice of π .

Definition 2.5. Two quasi-psh functions φ and ψ on X are *P-equivalent* if for some Kähler form ω on X such that $\varphi, \psi \in \text{PSH}(X, \omega)$ and both ω_φ and ω_ψ have positive masses, we have $P_\omega[\varphi] = P_\omega[\psi]$. We write $\varphi \sim_P \psi$ in this case.

Here and in the sequel, when we say a potential has positive mass, we always refer to the non-pluripolar mass.

Definition 2.6. Two quasi-psh functions φ and ψ on X are *I-equivalent* if for some Kähler form ω on X such that both $\varphi, \psi \in \text{PSH}(X, \omega)$, we have $P_\omega[\varphi]_{\mathcal{I}} = P_\omega[\psi]_{\mathcal{I}}$. We write $\varphi \sim_{\mathcal{I}} \psi$ in this case.

As shown in [DX23], both conditions are independent of the choice of ω .

Proposition 2.7. Suppose that $\varphi^i \in \text{PSH}(X, \theta)$ ($i \in I$) is a decreasing net of \mathcal{I} -model potentials and $\inf_i \varphi^i$ has positive mass. Then for all Kähler form ω on X , we have

$$(2.1) \quad P_{\theta+\omega} \left[\inf_{i \in I} \varphi^i \right]_{\mathcal{I}} = \inf_{i \in I} P_{\theta+\omega}[\varphi^i]_{\mathcal{I}}.$$

The proof relies on the d_S -pseudometric introduced in [DDNL21].

PROOF. We may assume that X is smooth.

Observe that both sides of (2.1) are \mathcal{I} -model in $\text{PSH}(X, \theta + \omega)$, so it suffices to show that they have the same generic Lelong numbers. Fix a prime divisor E over X . As shown in [DDNL21, Proposition 4.8]¹, $P_{\theta+\omega}[\varphi^i]_{\mathcal{I}} \xrightarrow{d_S} \inf_{i \in I} P_{\theta+\omega}[\varphi^i]_{\mathcal{I}}$. Hence, by [Xia22a, Theorem 0.10],

$$\lim_{i \in I} \nu(\varphi^i, E) = \nu \left(\inf_{i \in I} P_{\theta+\omega}[\varphi^i]_{\mathcal{I}}, E \right).$$

By [DDNL21, Proposition 4.8] again, $\varphi^i \xrightarrow{d_S} \inf_{i \in I} \varphi^i$. It follows from [Xia22a, Theorem 0.10] (or its extension in [DX23]) that

$$\lim_{i \in I} \nu(\varphi^i, E) = \nu \left(\inf_{i \in I} \varphi^i, E \right).$$

We conclude (2.1). □

Lemma 2.8. Suppose that $\varphi^i \in \text{PSH}(X, \theta)$ ($i \in I$) is a family of \mathcal{I} -good potentials, uniformly bounded from above. Then $\sup_{i \in I} \varphi^i$ is also \mathcal{I} -good.

PROOF. We may assume that X is smooth. This property holds if I is finite by [DX23, Lemma 3.20]. By Choquet's lemma, we therefore assume that $I = \mathbb{Z}_{>0}$ and φ^i is an increasing sequence. In this case, the result follows from [Xia21b, Theorem 4.9]. □

¹The result is stated for a sequence, but the proof works for a net as well.

Lemma 2.9. *Let ω be a Kähler form on X . Let $\varphi^j \in \text{PSH}(X, \theta + \epsilon_j \omega)$ be a decreasing sequence of model potentials, where ϵ_j is a decreasing sequence converging to 0. Define $\varphi := \inf_j \varphi^j$.*

- (1) $\int_X \theta_\varphi^n = \lim_{j \rightarrow \infty} \int_X (\theta + \epsilon_j \omega + \text{dd}^c \varphi^j)^n$.
- (2) Assume that φ has positive mass. Then for any prime divisor E over X ,

$$\lim_{j \rightarrow \infty} \nu(\varphi^j, E) = \nu(\varphi, E).$$

In the proof below, $V_\theta := \sup\{\varphi \in \text{PSH}(X, \theta), \varphi \leq 0\}$.

PROOF. We may assume that X is smooth.

The argument of (1) is very similar to [DDNL21, Propostion 4.6]. We briefly recall the proof. First observe that as $j \rightarrow \infty$, $V_{\theta + \epsilon_j \omega}$ decreases to V_θ . So

$$\{V_\theta = 0\} = \bigcap_j \{V_{\theta + \epsilon_j \omega} = 0\}.$$

As $\{V_\theta = 0\}$ is contained in the ample locus of $\{\theta\}$, so is $\{V_{\theta + \epsilon_j \omega} = 0\}$ when j is large enough.

For any $C > 0$ and any $\beta > 0$,

$$\begin{aligned} & \overline{\lim}_{j \rightarrow \infty} \int_{\{\varphi^j = 0\}} (\theta + \epsilon_j \omega + \text{dd}^c \varphi^j)^n \\ &= \overline{\lim}_{j \rightarrow \infty} \int_{\{\varphi^j = 0\}} (\theta + \epsilon_j \omega + \text{dd}^c \max\{\varphi^j, V_{\theta + \epsilon_j \omega} - C\})^n \\ &\leq \overline{\lim}_{j \rightarrow \infty} \int_{\{V_{\theta + \epsilon_j \omega} = 0\}} \exp(\beta \varphi^j) (\theta + \epsilon_j \omega + \text{dd}^c \max\{\varphi^j, V_{\theta + \epsilon_j \omega} - C\})^n \\ &\leq \int_{\{V_{\theta + \epsilon_{j_0} \omega} = 0\}} \exp(\beta \varphi) (\theta + \epsilon_{j_0} \omega + \text{dd}^c \max\{\varphi, V_\theta - C\})^n \end{aligned}$$

The last step follows from [GZ17, Theorem 4.26]. For any fixed large enough j_0 so that $\{V_{\theta + \epsilon_{j_0} \omega} = 0\}$ is contained in locus where V_θ is locally bounded. We can find a constant $C_1 > 0$ independent of j and β so that

$$\begin{aligned} & \int_{\{V_{\theta + \epsilon_{j_0} \omega} = 0\}} \exp(\beta \varphi) (\theta + \epsilon_{j_0} \omega + \text{dd}^c \max\{\varphi, V_\theta - C\})^n \leq \\ & \int_{\{V_{\theta + \epsilon_{j_0} \omega} = 0\}} \exp(\beta \varphi) (\theta + \text{dd}^c \max\{\varphi, V_\theta - C\})^n + C_1 \epsilon_{j_0}. \end{aligned}$$

Now by Lebesgue's dominated convergence theorem, as $j_0 \rightarrow \infty$, the right-hand side tends to

$$\int_{\{V_\theta = 0\}} \exp(\beta \varphi) (\theta + \text{dd}^c \max\{\varphi, V_\theta - C\})^n.$$

Let $\beta \rightarrow \infty$, we therefore conclude that

$$\overline{\lim}_{j \rightarrow \infty} \int_{\{\varphi^j = 0\}} (\theta + \epsilon_j \omega + \text{dd}^c \varphi^j)^n \leq \int_{\{\varphi = 0\}} (\theta + \text{dd}^c \max\{\varphi, V_\theta - C\})^n = \int_{\{\varphi = 0\}} \theta_\varphi^n.$$

From this inequality, we can readily deduce (1).

Regarding (2), by [DDNL21, Lemma 4.3] and (1), for any $\varepsilon \in (0, 1)$ and j big enough there exists $\psi^j \in \text{PSH}(X, \theta + \varepsilon_j \omega)$ such that $(1 - \varepsilon)\varphi^j + \varepsilon\psi^j \leq \varphi$. This implies that for j big enough we have

$$(1 - \varepsilon)\nu(\varphi^j, E) + \varepsilon\nu(\psi^j, E) \geq \nu(\varphi, E) \geq \nu(\varphi^j, E).$$

However, $\nu(\chi, E)$ is uniformly bounded (by some Seshadri type constant) for any $\chi \in \text{PSH}(X, \theta + \omega)$ and E fixed. So letting $\varepsilon \searrow 0$ we conclude. \square

2.3. The trace operator. The results in this section are simple extensions of the results established in [DX23]. The proofs are almost identical, so we omit the details.

Let E be an irreducible reduced closed subspace of X of dimension m . In general, as E is not unibranch, the pluripotential theory on E is not well-behaved. So we will use the normalization \tilde{E} instead.

Unfortunately, due to the lack of Demailly regularization on singular spaces, the author cannot define the trace operator when E is contained in the singular locus of X . We will assume that X is smooth throughout this section for simplicity.

Let $\varphi \in \text{PSH}(X, \theta)$ such that $\nu(\varphi, E) = 0$. We will define $\text{Tr}_E(\varphi)$ as a qps function on \tilde{E} , well-defined up to P -equivalence.

The definition is given in [DX23], which we recall now. Suppose that φ has analytic singularities, then we let $\text{Tr}_E(\varphi) = \varphi|_{\tilde{E}}$. In general, fix a Kähler form ω on X and take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\text{PSH}(X, \theta + \omega)$. We define $\text{Tr}_E(\varphi)$ as the d_S -limit of $\varphi_j|_{\tilde{E}}$. The same arguments as in [DX23] show that the limit exists and is independent of the choices of $(\varphi_j)_j$ and ω up to P -equivalence. Moreover, $\text{Tr}_E(\varphi)$ depends only on the \mathcal{I} -equivalence class of φ .

3. The theories of non-Archimedean metrics

In this section, we briefly recall two different theories of non-Archimedean metrics.

3.1. Boucksom–Jonsson theory. In this section, we recall the basic concept in Boucksom–Jonsson’s theory as in [BJ21].

Let X be an irreducible reduced variety over \mathbb{C} of dimension n . Let X^{an} denote the Berkovich analytification X^{an} of X with respect to the trivial valuation on \mathbb{C} .

The set of real valuations on $\mathbb{C}(X)$ trivial on \mathbb{C} is denoted by X^{val} . The center of a valuation v is the scheme-theoretic point $c = c(v)$ of X such that $v \geq 0$ on $\mathcal{O}_{X,c}$ and $v > 0$ on the maximal ideal $\mathfrak{m}_{X,c}$ of $\mathcal{O}_{X,c}$. The center is unique if exists. It exists if X is proper.

In the remaining of this section, we assume that X is projective.

As a set, X^{an} is the set of semi-valuations on X , in other words, real-valued valuations v on irreducible reduced subvarieties Y in X that is trivial on \mathbb{C} . We call Y the *support* of the semi-valuation v . In other words,

$$X^{\text{an}} = \coprod_Y Y^{\text{val}}.$$

We will write $v_{\text{triv}} \in X^{\text{an}}$ for the trivial valuation on X : $v_{\text{triv}}(f) = 0$ for any $f \in \mathbb{C}(X)^\times$. See [Ber93] for more details.

We will be most interested in divisorial valuations. Recall that a divisorial valuation on X is a valuation of the form $c \text{ord}_E$, where $c \in \mathbb{Q}_{>0}$ and E is a prime

divisor over X . The set of divisorial valuations on X is denoted by X^{div} . This notation is compatible with that in Section 2.1.

Given any coherent ideal \mathfrak{a} on X and any $v \in X^{\text{an}}$, we define

$$(3.1) \quad v(\mathfrak{a}) := \min\{v(f) : f \in \mathfrak{a}_{c(v)}\} \in [0, \infty],$$

where $c(v)$ is the center of the valuation v on X .

Given any valuation v on X , the Gauss extension of v is a valuation $\sigma(v)$ on $X \times \mathbb{A}^1$:

$$\sigma(v) \left(\sum_i f_i t^i \right) := \min_i (v(f_i) + i).$$

Here t is the standard coordinate on $\mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$. The key property is that when v is a divisorial valuation, then so is $\sigma(v)$. See [BHJ17, Lemma 4.2].

Non-Archimedean plurisubharmonic functions. Let X be an irreducible complex projective variety of dimension n and L be a holomorphic pseudoeffective \mathbb{Q} -line bundle on X . Through the GAGA morphism $X^{\text{an}} \rightarrow X$ of ringed spaces, L can be pulled-back to an analytic line bundle L^{an} on X .

Following [BJ21, Definition 2.18], we define $\mathcal{H}_{\mathbb{Q}}^{\text{gf}}(L)$, the set of *(rational) generically finite Fubini-Study functions* $\phi : X^{\text{an}} \rightarrow [-\infty, \infty)$, that are of the following form:

$$(3.2) \quad \phi = \frac{1}{m} \max_j \{\log |s_j| + \lambda_j\}.$$

Here $m \in \mathbb{Z}_{>0}$ is an integer such that $L^{\otimes m}$ is a line bundle, the s_j 's are a finite collection of non-vanishing sections in $H^0(X, L^{\otimes m})$, and $\lambda_j \in \mathbb{Q}$. We followed the convention of Boucksom–Jonsson by writing $\log |s_j|(v) = -v(s_j)$.

Now we come to the main definition of this paragraph:

Definition 3.1 ([BJ21, Definition 4.1]). A psh metric on L^{an} is a function $\phi : X^{\text{an}} \rightarrow [-\infty, \infty)$ that is not identically $-\infty$, and is the pointwise limit of a decreasing net $(\phi_i)_{i \in I}$, where $\phi_i \in \mathcal{H}_{\mathbb{Q}}^{\text{gf}}(L_i^{\text{an}})$ for some \mathbb{Q} -line bundles L_i on X satisfying $c_1(L_i) \rightarrow c_1(L)$ in $\text{NS}^1(X)_{\mathbb{R}}$.

3.2. The transcendental theory. Let X be a reduced irreducible unibranch compact Kähler space of dimension n and θ be a closed real $(1,1)$ -form on X . Assume that $\text{PSH}(X, \theta)$ is non-empty.

We briefly recall the transcendental theory of non-Archimedean metrics introduced in [DXZ23].

Definition 3.2. Define $\text{PSH}^{\text{NA}'}(X, \theta)$ as the set of maps $\mathbb{T} : (-\infty, \tau_{\mathbb{T}}^+) \rightarrow \text{PSH}(X, \theta)$ for some $\tau_{\mathbb{T}}^+ \in \mathbb{R}$ satisfying the following conditions:

- (1) for each $\tau \in (-\infty, \tau_{\mathbb{T}}^+)$, \mathbb{T}_{τ} is \mathcal{I} -model;
- (2) the map $(-\infty, \tau_{\mathbb{T}}^+) \ni \tau \mapsto \mathbb{T}_{\tau}$ is decreasing and concave.

We write

$$\mathbb{T}_{-\infty} := \sup_{\tau \in (-\infty, \tau_{\mathbb{T}}^+)}^* \mathbb{T}_{\tau}.$$

We define $\text{PSH}^{\text{NA}}(X, \theta)_{>0}$ as the subset of $\text{PSH}^{\text{NA}'}(X, \theta)$ consisting of \mathbb{T} satisfying furthermore

$$(3) \quad \int_X \theta_{\mathbb{T}_{-\infty}}^n > 0.$$

The curves \mathfrak{T} are sometimes known as *test curves*.

Remark 3.3. Sometimes it is convenient to extend the domain of definition of $\mathfrak{T} \in \text{PSH}^{\text{NA}'}(X, \theta)$ to the whole \mathbb{R} as follows: for $\tau > \tau_{\mathfrak{T}}^+$, we simply set $\mathfrak{T}_{\tau} = -\infty$ while for $\tau = \tau_{\mathfrak{T}}^+$, we let

$$\mathfrak{T}_{\tau_{\mathfrak{T}}^+} = \inf_{\tau < \tau_{\mathfrak{T}}^+} \mathfrak{T}_{\tau}.$$

In particular, for each $x \in X$, the function $\mathbb{R} \ni \tau \mapsto \mathfrak{T}_{\tau}(x)$ is either constantly $-\infty$, or a proper usc concave function. We will frequently make use of this convention without further explanation.

Here we choose the Hebrew letter *dalet* \mathfrak{T} simply because it resembles the rotation of ℓ , the notation usually used to represent a geodesic ray, in view of their duality.

Lemma 3.4. *Assume that $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$, then $\int_X \theta_{\mathfrak{T}_{\tau}}^n > 0$ for any $\tau \in (-\infty, \tau_{\mathfrak{T}}^+)$.*

PROOF. Fix $\tau \in (-\infty, \tau_{\mathfrak{T}}^+)$, we want to show that

$$(3.3) \quad \int_X \theta_{\mathfrak{T}_{\tau}}^n > 0.$$

We may assume that X is smooth.

By assumption, $\mathfrak{T}_{-\infty}$ has positive mass. By [DDNL18, Theorem 2.3], we have

$$\int_X \theta_{\mathfrak{T}_{-\infty}}^n = \lim_{\tau \rightarrow -\infty} \int_X \theta_{\mathfrak{T}_{\tau}}^n.$$

In particular, for a sufficiently small $\tau_0 < \tau$, we have

$$\int_X \theta_{\mathfrak{T}_{\tau_0}}^n > 0.$$

Now take $\tau' \in (\tau, \tau_{\mathfrak{T}}^+)$ and $t \in (0, 1)$ so that

$$\tau = (1 - t)\tau' + t\tau_0.$$

From the concavity of \mathfrak{T} , we find that

$$\mathfrak{T}_{\tau} \geq (1 - t)\mathfrak{T}_{\tau'} + t\mathfrak{T}_{\tau_0}.$$

By the monotonicity theorem [WN19],

$$\int_X \theta_{\mathfrak{T}_{\tau}}^n \geq \int_X \theta_{(1-t)\mathfrak{T}_{\tau'} + t\mathfrak{T}_{\tau_0}}^n \geq t^n \int_X \theta_{\mathfrak{T}_{\tau_0}}^n > 0$$

and (3.3) follows. \square

Definition 3.5. Given any Kähler form ω on X , we define transition maps

$$\begin{aligned} p_{\theta, \theta + \omega} &: \text{PSH}^{\text{NA}'}(X, \theta) \rightarrow \text{PSH}^{\text{NA}'}(X, \theta + \omega), \\ p_{\theta, \theta + \omega} &: \text{PSH}^{\text{NA}}(X, \theta)_{>0} \rightarrow \text{PSH}^{\text{NA}}(X, \theta + \omega)_{>0} \end{aligned}$$

as follows: the image of $\mathfrak{T}: (-\infty, \tau_{\mathfrak{T}}^+) \rightarrow \text{PSH}(X, \theta)$ is given by

$$p_{\theta, \theta + \omega}(\mathfrak{T}): (-\infty, \tau_{\mathfrak{T}}^+) \rightarrow \text{PSH}(X, \theta + \omega), \quad \tau \mapsto P_{\theta + \omega}[\mathfrak{T}_{\tau}]_{\mathcal{I}}.$$

Note that $\{\text{PSH}^{\text{NA}'}(X, \theta + \omega), p_{\theta, \theta + \omega}\}_\omega$ and $\{\text{PSH}^{\text{NA}}(X, \theta + \omega)_{>0}, p_{\theta, \theta + \omega}\}_\omega$ form projective systems, where the set of Kähler forms ω 's is ordered by the reverse of the usual ordering. We define

$$(3.4) \quad \text{PSH}^{\text{NA}}(X, \theta) := \varprojlim_{\omega} \text{PSH}^{\text{NA}'}(X, \theta + \omega),$$

where the projective limit is taken in the category of sets.

We will denote the components of an element $\Upsilon \in \text{PSH}^{\text{NA}}(X, \theta)$ by $\Upsilon^{\theta + \omega} \in \text{PSH}^{\text{NA}'}(X, \theta + \omega)$. Observe that $\tau_{\Upsilon^{\theta + \omega}}^+$ is independent of the choice of ω , we will denote this common value by τ_{Υ}^+ .

Remark 3.6. The extension in Remark 3.3 is compatible with transition maps in the following sense: Suppose that $\Upsilon \in \text{PSH}^{\text{NA}}(X, \theta)$, then for any Kähler forms ω and ω' on X ,

$$P_{\theta + \omega + \omega'} \left[\Upsilon_{\tau_{\Upsilon}^+}^{\theta + \omega} \right] = \Upsilon_{\tau_{\Upsilon}^+}^{\theta + \omega + \omega'}.$$

Lemma 3.7. *The natural maps*

$$\varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)_{>0} \rightarrow \text{PSH}^{\text{NA}}(X, \theta) \rightarrow \varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)$$

are both bijective.

In the sequel, we will be constantly making use of these identifications without further mentioning.

PROOF. It is clear that both maps are injective, so it suffices to verify that the map

$$\varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)_{>0} \rightarrow \varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)$$

is surjective. Consider $\Upsilon \in \varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)$ with components $\Upsilon^{\theta + \omega} \in \text{PSH}^{\text{NA}}(X, \theta + \omega)$. For each ω , $\Upsilon^{\theta + \omega}$ has components $\Upsilon^{\theta + \omega, \theta + \omega + \omega'} \in \text{PSH}^{\text{NA}'}(X, \theta + \omega + \omega')$. It is easy to verify that Υ is the image of the following element in $\varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega)_{>0}$ with components $\Upsilon^{\theta + \omega/2, \theta + \omega}$. \square

Also note that the transition maps give natural injections

$$(3.5) \quad \text{PSH}^{\text{NA}}(X, \theta) \hookrightarrow \text{PSH}^{\text{NA}}(X, \theta + \omega).$$

In particular, we can define

$$\text{QPSH}^{\text{NA}}(X) := \varprojlim_{\omega} \text{PSH}^{\text{NA}}(X, \theta + \omega),$$

where ω runs over all Kähler forms on X . This definition is clearly independent of the choice of θ .

Lemma 3.8. *The natural map*

$$(3.6) \quad \varprojlim_{\omega'} \text{PSH}^{\text{NA}}(X, \theta + \omega')_{>0} \rightarrow \text{PSH}^{\text{NA}}(X, \theta)$$

is a bijection, where ω' runs over the set of semipositive closed real $(1, 1)$ -forms on X with positive total mass.

PROOF. The injectivity is trivial. In order to prove the surjectivity, let $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$. We want to show that \mathfrak{T} can be extended to an element in the domain of (3.6). For this purpose, take a semipositive closed real $(1, 1)$ -form ω' on X with positive total mass. Fix a Kähler form ω on X . We define

$$\mathfrak{T}_{\tau}^{\theta+\omega'} := \inf_{k \in \mathbb{Z}_{>0}} \mathfrak{T}_{\tau}^{\theta+\omega'+k^{-1}\omega}$$

for any $\tau \in (-\infty, \tau_{\mathfrak{T}}^+)$. By Lemma 2.9,

$$\int_X (\theta + \omega' + \text{dd}^c \mathfrak{T}_{\tau}^{\theta+\omega'})^n > 0$$

and

$$\nu(\mathfrak{T}_{\tau}^{\theta+\omega'}, E) = \nu(\mathfrak{T}_{\tau}^{\theta+\omega'+\omega}, E)$$

for any prime divisor E over X and any $\tau \in (-\infty, \tau_{\mathfrak{T}}^+)$. It is clear that $\mathfrak{T}^{\omega'} \in \text{PSH}^{\text{NA}}(X, \theta + \omega')_{>0}$. The $(\mathfrak{T}^{\theta+\omega'})_{\omega'}$ defined in this way is clearly an element in the domain of (3.6) with image \mathfrak{T} . \square

Definition 3.9. The *non-pluripolar mass* of $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ is defined as the limit

$$\lim_{\omega} \int_X (\theta + \omega + \text{dd}^c \mathfrak{T}_{-\infty}^{\theta+\omega})^n,$$

where ω runs over all Kähler forms on X .

Since the net is decreasing, the limit exists.

Lemma 3.10. *The image of the canonical injection*

$$\text{PSH}^{\text{NA}}(X, \theta)_{>0} \hookrightarrow \text{PSH}^{\text{NA}}(X, \theta)$$

is given by the set of $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ with positive non-pluripolar mass.

PROOF. It is clear that the image of an element in $\text{PSH}^{\text{NA}}(X, \theta)_{>0}$ has positive non-pluripolar mass. Conversely, take $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ with positive non-pluripolar mass. We want to construct $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$ representing \mathfrak{T} .

Fix a Kähler form ω on X . Using the same arguments as Lemma 3.4, we find that

$$\lim_{k \rightarrow \infty} \int_X (\theta + k^{-1}\omega + \text{dd}^c \mathfrak{T}_{\tau}^{\theta+\omega})^n > 0$$

for any $\tau < \tau_{\mathfrak{T}}^+$. We define

$$\mathfrak{T}'_{\tau} := \lim_{k \rightarrow \infty} \mathfrak{T}_{\tau}^{\theta+k^{-1}\omega}$$

for any $\tau < \tau_{\mathfrak{T}}^+$. It follows from Lemma 2.9 that \mathfrak{T}' represents \mathfrak{T} . \square

3.3. Comparison between the two theories. Let X be a reduced irreducible unibranch projective variety over \mathbb{C} and L be a pseudoeffective line bundle on X . Recall that in this case, the Zariski unibranchness is equivalent to the analytic unibranchness, as proved in [Xia21a], so there is no ambiguity in the adjective *unibranch*.

Choose a smooth Hermitian metric h_0 on L and let $\theta = c_1(L, h_0)$. In this case, both the Boucksom–Jonsson theory and the transcendental theory make sense, so we can consider the problem of comparison.

3.3.1. The general case.

Definition 3.11. Given $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$, we define its *associated non-Archimedean potential* $\mathfrak{T}^{\text{an}}: X^{\text{div}} \rightarrow \mathbb{R}$ as follows:

$$(3.7) \quad \mathfrak{T}^{\text{an}}(c \text{ord}_E) := \sup_{\tau < \tau_+^+} (\mathfrak{T}_\tau^{\theta+\omega, \text{an}}(c \text{ord}_E) + \tau)$$

for any Kähler form ω on X . Clearly, this map is independent of the choice of ω .

The map $\mathfrak{T} \mapsto \mathfrak{T}^{\text{an}}$ is compatible with the inclusions (3.5). In particular, given $\mathfrak{T} \in \text{QPSH}^{\text{NA}}(X)$, we can define \mathfrak{T}^{an} by choosing any representative of \mathfrak{T} .

It is clear that the map $\mathfrak{T} \mapsto \mathfrak{T}^{\text{an}}$ is injective.

3.3.2. On a smooth variety. In the sequel we assume that X is smooth. In this case, we have the techniques of multiplier ideal sheaves.

A potential $\varphi \in \text{PSH}(X, \theta)$ defines a potential $\varphi^{\text{an}} \in \text{PSH}(L^{\text{an}})$ as follows:

$$\varphi^{\text{an}}(v) := - \lim_{k \rightarrow \infty} \frac{1}{k} v(\mathcal{I}(k\varphi))$$

for any $v \in X^{\text{an}}$. As explained in [BBJ21, DXZ23], the limit exists and lies in $\text{PSH}(L^{\text{an}})$ and this function extends the non-Archimedean potential defined in Definition 2.2 on X^{div} .

We can then define a map

$$(3.8) \quad \text{an}: \text{PSH}^{\text{NA}}(X, \theta) \rightarrow \text{PSH}(L^{\text{an}})$$

as follows: take $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$, its image is given by

$$\mathfrak{T}^{\text{an}}: X^{\text{an}} \rightarrow [-\infty, \infty), \quad v \mapsto \sup_{\tau < \tau_+^+} (\mathfrak{T}_\tau^{\theta+\omega, \text{an}}(v) + \tau)$$

for any Kähler form ω on X . Clearly, this map is independent of the choice of ω . The map \mathfrak{T}^{an} extends the corresponding definition in Definition 3.11.

THEOREM 3.12 ([DXZ23]). *Assume that X is smooth, then the map (3.8) is a bijection.*

Motivated by this bijection, we could translate the pluripotential-theoretic operations of $\text{PSH}(L^{\text{an}})$ to operations on the space $\text{PSH}^{\text{NA}}(X, \theta)$. Furthermore, in most cases, the resulting operations have natural extensions to the transcendental case as well. This is what we will carry out in this paper.

4. Pluripotential-theoretic operations

Let X be a reduced irreducible unibranch compact Kähler space of dimension n and θ be a closed real $(1, 1)$ -form on X . Assume that $\text{PSH}(X, \theta) \neq \emptyset$. When necessary, we use θ' and θ'' to denote similar forms with the same properties as θ .

4.1. The partial order.

Definition 4.1. We define a partial order \leq on $\text{PSH}^{\text{NA}}(X, \theta)$ as follows: given $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$, we say $\mathfrak{T} \leq \mathfrak{T}'$ if for some (hence for all) Kähler form ω on X , we have $\mathfrak{T}_\tau^{\theta+\omega} \leq \mathfrak{T}'_\tau^{\theta+\omega}$ for all $\tau < \tau_+^+$.

It is trivial to verify that this is indeed a partial order. Note that this partial order is compatible with the inclusions (3.5), so it defines a partial order on $\text{QPSH}^{\text{NA}}(X)$.

Lemma 4.2. *Suppose that $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$, then the following are equivalent:*

- (1) $\mathfrak{T} \leq \mathfrak{T}'$;
- (2) $\mathfrak{T}^{\text{an}}(\text{cord}_E) \leq \mathfrak{T}'^{\text{an}}(\text{cord}_E)$ for all $\text{cord}_E \in X^{\text{div}}$.

The proof is almost identical to [DXZ23, Theorem 3.12], we omit the details.

4.2. Addition by a constant.

Definition 4.3. Given $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and $C \in \mathbb{R}$, we define the addition $\mathfrak{T} + C \in \text{PSH}^{\text{NA}}(X, \theta)$ as the element with components $(\mathfrak{T} + C)^{\theta+\omega} : (-\infty, \tau_{\mathfrak{T}}^+ + C) \rightarrow \text{PSH}(X, \theta + \omega)$ given by

$$(\mathfrak{T} + C)_{\tau}^{\theta+\omega} := \mathfrak{T}_{\tau-C}^{\theta+\omega}.$$

The associated non-Archimedean potential behaves as expected:

Proposition 4.4. *Given $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and $C \in \mathbb{R}$, we have*

$$(4.1) \quad (\mathfrak{T} + C)^{\text{an}}(\text{cord}_E) = \mathfrak{T}^{\text{an}}(\text{cord}_E) + C$$

for all $\text{cord}_E \in X^{\text{div}}$ and

$$\tau_{\mathfrak{T}+C}^+ = \tau_{\mathfrak{T}}^+ + C.$$

This follows trivially from (3.7). We will omit similar proofs in the sequel.

Of course, this operation has the obvious compatibility in itself: given another constant $C' \in \mathbb{R}$, we have

$$(\mathfrak{T} + C) + C' = \mathfrak{T} + (C + C').$$

It is compatible with the partial order in the following sense: given another $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$, the following are equivalent:

- (1) $\mathfrak{T} \leq \mathfrak{T}'$;
- (2) $\mathfrak{T} + C \leq \mathfrak{T}' + C$.

4.3. Maximum.

Definition 4.5. Suppose that $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$, we define their *maximum* $\mathfrak{T} \vee \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$ as the map $(-\infty, \tau_{\mathfrak{T}}^+ \vee \tau_{\mathfrak{T}'}^+) \rightarrow \text{PSH}(X, \theta)$:

$$\tau \mapsto P_{\theta}[\text{EC}(\tau' \mapsto \mathfrak{T}_{\tau'} \vee \mathfrak{T}'_{\tau'})_{\tau}]_{\mathcal{I}},$$

where EC denotes the concave hull.

We need to verify that $\mathfrak{T} \vee \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$. The only non-trivial part is to show that the concave envelope $\text{EC}(\tau' \mapsto \mathfrak{T}_{\tau'} \vee \mathfrak{T}'_{\tau'})$ is a curve of θ -psh functions. For this purpose, we may assume that X is smooth and connected. According to the Ross–Witt Nyström correspondence [DXZ23, Theorem 2.6], \mathfrak{T} and \mathfrak{T}' correspond to geodesic rays ℓ and ℓ' in $\text{PSH}(X, \theta)$: for any $t \geq 0$, we have

$$\ell_t = \sup_{\tau \in \mathbb{R}} (\mathfrak{T}_{\tau} + t\tau), \quad \ell'_t = \sup_{\tau \in \mathbb{R}} (\mathfrak{T}'_{\tau} + t\tau).$$

It follows that

$$\ell_t \vee \ell'_t = \sup_{\tau \in \mathbb{R}} (\mathfrak{T}_{\tau} \vee \mathfrak{T}'_{\tau} + t\tau).$$

Hence,

$$\text{EC}(\tau \mapsto \mathfrak{T}_{\tau} \vee \mathfrak{T}'_{\tau})_{\tau'} = \inf_{t \geq 0} (\ell_t \vee \ell'_t - t\tau')$$

and $\text{EC}(\tau \mapsto \mathfrak{T}_{\tau} \vee \mathfrak{T}'_{\tau})_{\tau'} \in \text{PSH}(X, \theta) \cup \{-\infty\}$ by Kiselman's principle.

By a similar argument, we have

Corollary 4.6. *Suppose that $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(\theta)_{>0}$. For any $c \text{ord}_E \in X^{\text{div}}$, we have*

$$-(\mathfrak{T} \vee \mathfrak{T}')^{\text{an}}(c \text{ord}_E) = \text{EC}(\tau' \mapsto (-\mathfrak{T}_{\tau'}^{\text{an}}(c \text{ord}_E)) \vee (-\mathfrak{T}'^{\text{an}}(c \text{ord}_E)))_{\tau}.$$

In particular,

$$(4.2) \quad (\mathfrak{T} \vee \mathfrak{T}')^{\text{an}}(c \text{ord}_E) = \mathfrak{T}^{\text{an}}(c \text{ord}_E) \vee \mathfrak{T}'^{\text{an}}(c \text{ord}_E).$$

It follows from Corollary 4.6 and Lemma 4.2 that the maximum operation is compatible with the projective system (3.4). Hence, we can make the following definition in general:

Definition 4.7. Suppose that $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$, we define their *maximum* $\mathfrak{T} \vee \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$ as the element with components

$$(\mathfrak{T} \vee \mathfrak{T}')^{\theta+\omega} = \mathfrak{T}^{\theta+\omega} \vee \mathfrak{T}'^{\theta+\omega}.$$

The associated non-Archimedean potential behaves as expected:

Corollary 4.8. *Suppose that $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(\theta)$. For any $c \text{ord}_E \in X^{\text{div}}$, we have*

$$(4.3) \quad (\mathfrak{T} \vee \mathfrak{T}')^{\text{an}}(c \text{ord}_E) = \mathfrak{T}^{\text{an}}(c \text{ord}_E) \vee \mathfrak{T}'^{\text{an}}(c \text{ord}_E)$$

and

$$\tau_{\mathfrak{T} \vee \mathfrak{T}'}^+ = \tau_{\mathfrak{T}}^+ \vee \tau_{\mathfrak{T}'}^+.$$

The maximum operation is commutative and associative. In particular, finite (non-empty) maximum makes sense in the obvious way.

The maximum operation is compatible with the partial order in the following sense: given $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(\theta)$, the following are equivalent

- (1) $\mathfrak{T} \vee \mathfrak{T}' = \mathfrak{T}'$;
- (2) $\mathfrak{T} \leq \mathfrak{T}'$.

The maximum operation is compatible with the addition by a constant: if $C \in \mathbb{R}$, then

$$(\mathfrak{T} \vee \mathfrak{T}') + C = (\mathfrak{T} + C) \vee (\mathfrak{T}' + C).$$

4.4. Decreasing limit along a net.

Definition 4.9. Let $(\mathfrak{T}_i)_{i \in I}$ be a decreasing net in $\text{PSH}^{\text{NA}}(X, \theta)$. Assume that $\inf_{i \in I} \tau_{\mathfrak{T}_i}^+ > -\infty$. We define $\inf_{i \in I} \mathfrak{T}_i \in \text{PSH}^{\text{NA}}(X, \theta)$ as the element with components

$$\left(\inf_{i \in I} \mathfrak{T}_i \right)_{\tau}^{\theta+\omega} = \inf_{i \in I} \mathfrak{T}_{i, \tau}^{\theta+\omega}.$$

This gives an element in $\text{PSH}^{\text{NA}}(X, \theta)$ by Proposition 2.7 together with [DX22, Lemma 2.21(i)]².

When $\inf_{i \in I} \tau_{\mathfrak{T}_i}^+ = -\infty$, we could still formally define $\inf_{i \in I} \mathfrak{T}_i$ as a symbol $-\infty$. As one could easily verify, all of our pluripotential-theoretic operations admit natural extensions to $-\infty$.

Using [DXZ23, Lemma 3.13], we have

²The result is stated for a sequence, but the proof extends to a net.

Lemma 4.10. *Let $(\mathfrak{T}_i)_{i \in I}$ be a decreasing net in $\text{PSH}^{\text{NA}}(X, \theta)$ with $\inf_{i \in I} \mathfrak{T}_i \in \text{PSH}^{\text{NA}}(X, \theta)$. Then for any $c \text{ord}_E \in X^{\text{div}}$,*

$$(4.4) \quad \left(\inf_{i \in I} \mathfrak{T}_i \right)^{\text{an}} (c \text{ord}_E) = \inf_{i \in I} \mathfrak{T}_i^{\text{an}} (c \text{ord}_E).$$

Moreover,

$$\tau_{\inf_{i \in I} \mathfrak{T}_i}^+ = \inf_{i \in I} \tau_{\mathfrak{T}_i}^+.$$

This operation is compatible with itself: if I and J are two non-empty directed set and \mathfrak{T}_{ij} is a decreasing net in $\text{PSH}^{\text{NA}}(X, \theta)$ indexed by $I \times J$ with $\inf_{i,j} \mathfrak{T}_{ij} \in \text{PSH}^{\text{NA}}(X, \theta)$, then

$$\inf_{i,j} \mathfrak{T}_{ij} = \inf_{i \in I} \inf_{j \in J} \mathfrak{T}_{ij} = \inf_{j \in J} \inf_{i \in I} \mathfrak{T}_{ij}.$$

This operation is compatible with the partial order in the following senses: if $\mathfrak{T}_i \in \text{PSH}^{\text{NA}}(X, \theta)$ ($i \in I$) is a decreasing net with $\inf_{i \in I} \mathfrak{T}_i \in \text{PSH}^{\text{NA}}(X, \theta)$. Then $\inf_{j \in I} \mathfrak{T}_j \leq \mathfrak{T}_i$ for all $i \in I$ and it is the biggest element in $\text{PSH}^{\text{NA}}(X, \theta)$ with this property. On the other hand, if $\mathfrak{T}'_i \in \text{PSH}^{\text{NA}}(X, \theta)$ ($i \in I$) is another decreasing net with $\inf_{i \in I} \mathfrak{T}'_i \in \text{PSH}^{\text{NA}}(X, \theta)$ such that $\mathfrak{T}_i \leq \mathfrak{T}'_i$ for all $i \in I$, then

$$\inf_{i \in I} \mathfrak{T}_i \leq \inf_{i \in I} \mathfrak{T}'_i.$$

This operation is compatible with the addition by a constant: if $C \in \mathbb{R}$, then

$$\inf_{i \in I} (\mathfrak{T}_i + C) = \left(\inf_{i \in I} \mathfrak{T}_i \right) + C.$$

4.5. Regularized supremum.

Definition 4.11. Let $(\mathfrak{T}^i)_{i \in I}$ be an increasing net in $\text{PSH}^{\text{NA}}(X, \theta)_{>0}$. We assume that $\sup_{i \in I} \tau_{\mathfrak{T}^i}^+ < \infty$. We define a map $\sup_{i \in I} \tau_{\mathfrak{T}^i}^+ : (-\infty, \sup_{i \in I} \tau_{\mathfrak{T}^i}^+) \rightarrow \text{PSH}(X, \theta)$ as follows:

$$\left(\sup_{i \in I}^* \mathfrak{T}^i \right)_{\tau} = P_{\theta} \left[\sup_{i \in I}^* \mathfrak{T}_{\tau}^i \right].$$

By Lemma 2.8, it is easy to see that $\sup_{i \in I}^* \mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$.

More generally, suppose that $\mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta)$ ($i \in I$) is an increasing net with $\sup_{i \in I} \tau_{\mathfrak{T}^i}^+ < \infty$. We define $\sup_{i \in I}^* \mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta)$ as the element with components

$$(4.5) \quad \left(\sup_{i \in I}^* \mathfrak{T}^i \right)^{\theta + \omega} = \sup_{i \in I}^* \mathfrak{T}^{i, \theta + \omega}$$

for any Kähler form ω on X . The compatibility follows from [DX21, Lemma 3.14].

Definition 4.12. Let $(\mathfrak{T}^i)_{i \in I}$ be a (non-empty) family in $\text{PSH}^{\text{NA}}(X, \theta)$ such that $\sup_{i \in I} \tau_{\mathfrak{T}^i}^+ < \infty$. Let \mathcal{J} denote the directed set of finite non-empty subsets of I ordered by inclusion. Then we define

$$\sup_{i \in I}^* \mathfrak{T}^i := \sup_{J \in \mathcal{J}}^* \left(\max_{j \in J} \mathfrak{T}^j \right).$$

Recall that the maximum is defined right after Corollary 4.8.

THEOREM 4.13. *Let $(\mathbb{T}^i)_{i \in I}$ be a (non-empty) family in $\text{PSH}^{\text{NA}}(X, \theta)$ with $\sup_{i \in I} \tau_{\mathbb{T}^i}^+ < \infty$. Then for any $c \text{ord}_E \in X^{\text{div}}$, we have*

$$\left(\sup_{i \in I}^* \mathbb{T}^i \right)^{\text{an}} (c \text{ord}_E) = \sup_{i \in I} \mathbb{T}^{i, \text{an}} (c \text{ord}_E).$$

Moreover,

$$\tau_{\sup_{i \in I}^* \mathbb{T}^i}^+ = \sup_{i \in I} \tau_{\mathbb{T}^i}^+.$$

This result is the transcendental analogue of Boucksom–Jonsson’s envelope conjecture.

PROOF. By Corollary 4.8, we may assume that $(\mathbb{T}^i)_{i \in I}$ is an increasing net. Furthermore, it is easy to reduce to the case where $\mathbb{T}^i \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$. In this case, the result follows from [DXZ23, Lemma 3.13]. \square

In particular, this operation extends the finite maximum.

This operation is compatible with itself: if $(\mathbb{T}^{i,j})_{i \in I, j \in J}$ is a (non-empty) family in $\text{PSH}^{\text{NA}}(X, \theta)$ with $\sup_{i \in I, j \in J} \tau_{\mathbb{T}^{i,j}}^+ < \infty$. Then

$$\sup_{i \in I}^* \sup_{j \in J}^* \mathbb{T}^{i,j} = \sup_{j \in J}^* \sup_{i \in I}^* \mathbb{T}^{i,j}.$$

It is compatible with the partial order: Suppose that $(\mathbb{T}^i)_{i \in I}, (\mathbb{T}'^i)_{i \in I}$ are two families in $\text{PSH}^{\text{NA}}(X, \theta)$ with $\sup_{i \in I}^* \tau_{\mathbb{T}^i}^+ < \infty$ and $\sup_{i \in I}^* \tau_{\mathbb{T}'^i}^+ < \infty$ and $\mathbb{T}^i \leq \mathbb{T}'^i$ for each $i \in I$, then

$$\sup_{i \in I}^* \mathbb{T}^i \leq \sup_{i \in I}^* \mathbb{T}'^i.$$

It is compatible with addition by a constant: if $C \in \mathbb{R}$, then

$$\sup_{i \in I}^* (\mathbb{T}^i + C) = \left(\sup_{i \in I}^* \mathbb{T}^i \right) + C.$$

There is no non-trivial compatibility with the decreasing limits, as expected.

4.6. Rescaling and addition.

Definition 4.14. Let $\mathbb{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and $\lambda \in \mathbb{R}_{>0}$, we define the rescaling $\lambda \mathbb{T} \in \text{PSH}^{\text{NA}}(X, \lambda \theta)$ as the element with components $(\lambda \mathbb{T})^{\lambda \theta + \omega} : (-\infty, \lambda \tau_{\mathbb{T}}^+) \rightarrow \text{PSH}(X, \lambda \theta + \omega)$:

$$(\lambda \mathbb{T})_{\tau}^{\lambda \theta + \omega} := \lambda \mathbb{T}_{\lambda^{-1} \tau}^{\theta + \lambda^{-1} \omega}$$

for any Kähler form ω on X .

Definition 4.15. Let $\mathbb{T} \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$ and $\mathbb{T}' \in \text{PSH}^{\text{NA}}(X, \theta')_{>0}$. We define $\mathbb{T} + \mathbb{T}' : (-\infty, \tau_{\mathbb{T}}^+ + \tau_{\mathbb{T}'}^+) \rightarrow \text{PSH}(X, \theta + \theta')$ as

$$(\mathbb{T} + \mathbb{T}')_{\tau} := P_{\theta} \left[\sup_{t \in \mathbb{R}} (\mathbb{T}_t + \mathbb{T}'_{\tau-t}) \right].$$

Observe that $\tau \mapsto \sup_{t \in \mathbb{R}} (\mathbb{T}_t + \mathbb{T}'_{\tau-t})$ is concave. In fact, this curve is just the concave involution of \mathbb{T} and \mathbb{T}' . From the general properties of infimal involution, we know that this map is concave and usc. For each fixed $x \in X$, $\tau \mapsto \sup_{t \in \mathbb{R}} (\mathbb{T}_t(x) + \mathbb{T}'_{\tau-t}(x))$ is either constantly $-\infty$ or proper. In particular, by [Roc70, Theorem 16.4] and Legendre–Fenchel duality, $\tau \mapsto \sup_{t \in \mathbb{R}} (\mathbb{T}_t + \mathbb{T}'_{\tau-t})$ is the same as the inverse Legendre transform of $\ell + \ell'$, where ℓ and ℓ' are the geodesic rays defined by \mathbb{T} and \mathbb{T}' . In particular, $\sup_{t \in \mathbb{R}} (\mathbb{T}_t + \mathbb{T}'_{\tau-t})$ is either in

PSH(X, θ) or constantly $-\infty$ by Kiselman's principle. By Lemma 2.8 and [Xia22a, Proposition 2.8], we find that $\mathfrak{T} + \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$.

This argument also shows that for any $c \text{ord}_E \in X^{\text{div}}$, we have

$$(4.6) \quad (\mathfrak{T} + \mathfrak{T}')^{\text{an}}(c \text{ord}_E) = \mathfrak{T}^{\text{an}}(c \text{ord}_E) + \mathfrak{T}'^{\text{an}}(c \text{ord}_E).$$

Definition 4.16. Let $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta')$. We define $\mathfrak{T} + \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta + \theta')$ as the element with components

$$(\mathfrak{T} + \mathfrak{T}')^{\theta+\omega} = \mathfrak{T}^{\theta+\omega} + \mathfrak{T}'^{\theta+\omega}.$$

Note that $\mathfrak{T} + \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta + \theta')$ by (4.6).

Lemma 4.17. Suppose that $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$, $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta')$, $\lambda \in \mathbb{R}_{>0}$. Then for any $c \text{ord}_E \in X^{\text{div}}$, we have

$$(\lambda \mathfrak{T})^{\text{an}}(c \text{ord}_E) = \lambda \mathfrak{T}^{\text{an}}(c \text{ord}_E), \quad (\mathfrak{T} + \mathfrak{T}')^{\text{an}}(c \text{ord}_E) = \mathfrak{T}^{\text{an}}(c \text{ord}_E) + \mathfrak{T}'^{\text{an}}(c \text{ord}_E).$$

Moreover,

$$\tau_{\lambda \mathfrak{T}}^+ = \lambda \tau_{\mathfrak{T}}^+, \quad \tau_{\mathfrak{T} + \mathfrak{T}'}^+ = \tau_{\mathfrak{T}}^+ + \tau_{\mathfrak{T}'}^+.$$

Both operations are compatible with themselves: if $\lambda, \lambda' \in \mathbb{R}_{>0}$ and $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$, $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta')$ and $\mathfrak{T}'' \in \text{PSH}^{\text{NA}}(X, \theta'')$, then

$$\lambda(\lambda' \mathfrak{T}) = (\lambda \lambda') \mathfrak{T}$$

and

$$(\mathfrak{T} + \mathfrak{T}') + \mathfrak{T}'' = \mathfrak{T} + (\mathfrak{T}' + \mathfrak{T}'').$$

The addition is clear commutative. Moreover, these two operations are compatible with each other:

$$\lambda(\mathfrak{T} + \mathfrak{T}') = \lambda \mathfrak{T} + \lambda \mathfrak{T}', \quad (\lambda + \lambda') \mathfrak{T} = \lambda \mathfrak{T} + \lambda' \mathfrak{T}.$$

Both operations are compatible with the partial order: if $\mathfrak{T}, \mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta)$, $\lambda \in \mathbb{R}_{>0}$, then $\mathfrak{T} \leq \mathfrak{T}'$ if and only if $\lambda \mathfrak{T} \leq \lambda \mathfrak{T}'$. Suppose that $\mathfrak{T}^1, \mathfrak{T}^2 \in \text{PSH}^{\text{NA}}(X, \theta)$, $\mathfrak{T}'^1, \mathfrak{T}'^2 \in \text{PSH}^{\text{NA}}(X, \theta')$ and $\mathfrak{T}^1 \leq \mathfrak{T}^2$, $\mathfrak{T}'^1 \leq \mathfrak{T}'^2$, then $\mathfrak{T}^1 + \mathfrak{T}'^1 \leq \mathfrak{T}^2 + \mathfrak{T}'^2$.

Both operations are compatible with addition by a constant: if $C \in \mathbb{R}$, $\lambda \in \mathbb{R}_{>0}$, $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and $\mathfrak{T}' \in \text{PSH}^{\text{NA}}(X, \theta')$, then

$$\lambda(\mathfrak{T} + C) = \lambda \mathfrak{T} + \lambda C$$

and

$$(\mathfrak{T} + \mathfrak{T}') + C = \mathfrak{T} + (\mathfrak{T}' + C).$$

Both operations satisfy obvious inequalities with respect to the limit operations. We leave the details to the readers.

4.7. The trace operator. We assume in addition that X is smooth. Recall that in this case, we have a natural bijection (3.8). Fix an irreducible reduced closed analytic subspace E of X .

Definition 4.18. We say $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ has *non-trivial restriction* to E if for any Kähler form ω on X , we have $\nu(\mathfrak{T}_{-\infty}^{\theta+\omega}, E) = 0$.

This definition is clearly independent of the choice of ω . As Lelong numbers are d_S -continuous, this assumption can be equivalently formulated as follows: there is a small enough $C \in \mathbb{R}$ so that $\nu(\mathfrak{T}_C^{\theta+\omega}, E) = 0$ for any Kähler form ω on X .

This definition is motivated by the following observation.

Lemma 4.19. *Assume in addition that X is projective. Let $\mathbb{T} \in \text{PSH}^{\text{NA}}(X, \theta)$. Then the following are equivalent:*

- (1) $\mathbb{T}^{\text{an}}(v_{E, \text{triv}}) \neq -\infty$;
- (2) $\mathbb{T}^{\text{an}}|_{E^{\text{an}}} \neq -\infty$;
- (3) \mathbb{T} has non-trivial restriction to E .

Here $v_{E, \text{triv}}$ denotes the trivial valuation of $\mathbb{C}(E)$.

PROOF. The equivalence between (1) and (2) is a simple consequence of the maximum principle [BJ21, Lemma 1.4(i)].

To see the equivalence between (1) and (3), it suffices to observe that for any $\varphi \in \text{PSH}(X, \theta)$,

$$\varphi^{\text{an}}(v_{E, \text{triv}}) = \begin{cases} -\infty, & \text{if } \nu(\varphi, E) > 0; \\ 0, & \text{if } \nu(\varphi, E) = 0. \end{cases}$$

□

Definition 4.20. Assume that \mathbb{T} has non-trivial restriction to E , we define the *trace operator* of \mathbb{T} along E as the element $\text{Tr}_E(\mathbb{T}) \in \text{PSH}^{\text{NA}}(\tilde{E}, \theta|_{\tilde{E}})$ defined as follows: for any Kähler form ω on X , the component $\text{Tr}_E(\mathbb{T})^{(\theta+\omega)|_{\tilde{E}}} \in \text{PSH}^{\text{NA}'}(\tilde{E}, \theta|_{\tilde{E}})$ is given by

$$\tau \mapsto P_{(\theta+\omega)|_{\tilde{E}}}[\text{Tr}_E(\mathbb{T}_\tau^{\theta+\omega/2})],$$

where $\tau \in (-\infty, \tau_{\text{Tr}_E(\mathbb{T})}^+)$ and $\tau_{\text{Tr}_E(\mathbb{T})}^+ = \sup\{\tau \in \mathbb{R} : \nu(\mathbb{T}_\tau^{\theta+\omega}, E) = 0\}$ for any Kähler form ω on X .

We remind the readers that \tilde{E} is the normalization of E . The fact that $\text{Tr}_E(\mathbb{T}) \in \text{PSH}^{\text{NA}}(\tilde{E}, \theta|_{\tilde{E}})$ follows from the fact that Kähler forms of the form $\omega|_{\tilde{E}}$ are cofinal in the directed set of Kähler forms on \tilde{E} .

The non-Archimedean potential associated with $\text{Tr}_E(\mathbb{T})$ is not very easy to describe. We will only do this when X is smooth projective and $\{\theta\}$ is the first Chern class of a pseudoeffective line bundle L and there is a smooth Hermitian metric h_0 with $c_1(L, h_0) = \theta$.

THEOREM 4.21. *Assume that the above assumptions hold. Consider an element $\mathbb{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ with non-trivial restriction to E . Then*

$$(4.7) \quad \text{Tr}_E(\mathbb{T})^{\text{an}}|_{E^{\text{div}}} = \mathbb{T}^{\text{an}}|_{E^{\text{div}}}.$$

Observe that there is a canonical identification $E^{\text{div}} = \tilde{E}^{\text{div}}$.

In order to prove the theorem, we need to recall the following notion introduced in [DXZ23].

Definition 4.22. A *piecewise linear curve* ψ_\bullet in $\text{PSH}(X, \theta)$ associated with $\psi_{\tau_j} \in \text{PSH}(X, \theta)$, for a finite number of parameters $\tau_0 > \tau_1 > \dots > \tau_N$ is the affine interpolation of this data:

- (1) $\psi_\tau = \psi_{\tau_N}$ for $\tau \leq \tau_N$;
- (2) For $t \in (0, 1)$ and $i = 0, \dots, N-1$, we have $\psi_{(1-t)\tau_i + t\tau_{i+1}} = (1-t)\psi_{\tau_i} + t\psi_{\tau_{i+1}}$;
- (3) $\psi_\tau = -\infty$ for $\tau > \tau_0$.

Similarly, an *essentially piecewise linear curve* in $\text{PSH}(X, \theta)$ is a curve η_\bullet in $\text{PSH}(X, \theta) \cup \{-\infty\}$ parameterized by \mathbb{R} such that there is a piecewise linear curve ψ_\bullet in $\text{PSH}(X, \theta)$ as above such that $\eta_\tau \sim_{\mathcal{I}} \psi_\tau$ for all $\tau \in \mathbb{R}$.

The analytification ψ^{an} of a piecewise linear or essentially piecewise linear curve ψ_\bullet as above is defined as

$$(4.8) \quad \psi^{\text{an}}(v) := \sup_{\tau \leq \tau_0} (\psi_\tau^{\text{an}}(v) + \tau) = \max_{\tau_i} (\psi_{\tau_i}^{\text{an}}(v) + \tau_i) \quad \text{for } v \in X^{\text{an}}.$$

PROOF OF THEOREM 4.21. We may assume that $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$. Let $\phi = \mathfrak{T}^{\text{an}} \in \text{PSH}(L^{\text{an}})$. By Lemma 4.19, $\phi(v_{E, \text{triv}}) \neq -\infty$.

Take an ample line bundle S on X , a Kähler form ω in $c_1(S)$. Write ϕ as the decreasing limit of a sequence ϕ^i of elements in $\mathcal{H}_{\mathbb{Q}}^{\text{gf}}(L + i^{-1}S)$. This is possible by [BJ21, Corollary 12.18].

Take $\mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta + i^{-1}\omega)$ such that $\mathfrak{T}^{i, \text{an}} = \phi^i$. Note that by Lemma 3.10, $\mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta + i^{-1}\omega)_{>0}$.

It follows from Lemma 4.10 (applied to the images of \mathfrak{T}^i in $\text{PSH}^{\text{NA}}(X, \theta + \omega)$) that for any $\tau < \tau_{\mathfrak{T}}^+$, we have

$$\lim_{i \rightarrow \infty} \mathfrak{T}_\tau^i = \mathfrak{T}_\tau.$$

In particular, by Lemma 2.9, the Lelong numbers of \mathfrak{T}_τ^i converge to those of \mathfrak{T}_τ . It follows that $\mathfrak{T}_\tau^i \xrightarrow{d_{S, \theta + \omega}} \mathfrak{T}_\tau$ for all $\tau < \tau_{\mathfrak{T}}^+$.

By Lemma 4.19 again, each \mathfrak{T}^i has non-trivial restriction to E . By [DX23, Proposition 4.14], we have

$$\text{Tr}_E(\mathfrak{T}_\tau^i)^{(\theta + \omega)|_{\bar{E}}} \xrightarrow{d_S} \text{Tr}_E(\mathfrak{T}_\tau)^{(\theta + \omega)|_{\bar{E}}}$$

for any $\tau < \tau_{\text{Tr}_E(\mathfrak{T})}^+$. Thanks to [Xia22a, Theorem 0.10] (or its generalization in [DX23]), the Lelong numbers of $\text{Tr}_E(\mathfrak{T}_\tau^i)^{(\theta + \omega)|_{\bar{E}}}$ converge to those of $\text{Tr}_E(\mathfrak{T}_\tau)^{(\theta + \omega)|_{\bar{E}}}$. Therefore, by [DXZ23, Lemma 3.13],

$$\text{Tr}_E(\mathfrak{T})^{\text{an}}(c \text{ord}_F) = \inf_i \text{Tr}_E(\mathfrak{T})^{i, \text{an}}(c \text{ord}_F)$$

for any $c \text{ord}_F \in E^{\text{div}}$. In particular, it suffices to prove (4.7) with \mathfrak{T}^i in place of \mathfrak{T} .

In other words, we have reduced to the case where $\phi \in \mathcal{H}_{\mathbb{Q}}^{\text{gf}}(L)$ and L is big.

Under this extra assumption, as proved in the proof of [DXZ23, Theorem 3.12], there is a piecewise linear curve ψ connecting potentials of the form

$$(4.9) \quad m^{-1} \log \max_i (|s_i|_{h_0^m}^2 + c),$$

where $m \in \mathbb{Z}_{>0}$, the s_i 's are finitely many holomorphic sections of $L^{\otimes m}$ and $c \in \mathbb{R}$ such that \mathfrak{T} is \mathcal{I} -equivalent to the concave envelope of ψ (in the sense that at each $\tau < \tau_{\mathfrak{T}}^+$, the two corresponding potentials are \mathcal{I} -equivalent). Thanks to the linearity of the trace operator proved in [DX23], the trace operator is \mathcal{I} -equivalent to the naive restriction for any ψ_τ . Note that $\tau \mapsto \mathfrak{T}_\tau|_{\bar{E}}$ is \mathcal{I} -equivalent to the concave envelope of $\tau \mapsto \psi_\tau|_{\bar{E}}$ (This follows from the general fact that the trace operator preserves \mathcal{I} -equivalence relation. In the current case, this is trivial as \mathfrak{T} and the concave envelope of ψ have the same singularity type for all $\tau < \tau_{\mathfrak{T}}^+$). In particular, by [DXZ23, Lemma 3.2], for any $c \text{ord}_F \in E^{\text{div}}$,

$$\begin{aligned} \text{Tr}_E(\mathfrak{T})^{\text{an}}(c \text{ord}_F) &= \sup_{\tau < \tau_{\mathfrak{T}}^+} ((\psi_\tau|_{\bar{E}})^{\text{an}}(c \text{ord}_F) + \tau) \\ &= \sup_{\tau < \tau_{\mathfrak{T}}^+} (\psi_\tau^{\text{an}}(c \text{ord}_F) + \tau) = \mathfrak{T}^{\text{an}}(c \text{ord}_F). \end{aligned}$$

Here we adopt the convention that $(-\infty)^{\text{an}}(c \text{ord}_F) = -\infty$ to simplify the notations. In order to justify the second equality, it suffices to show that for $\tau < \tau_{\text{Tr } E}^+(\mathfrak{T})$, if ψ_τ has the form (4.9), then

$$(\psi_\tau|_{\bar{E}})^{\text{an}}(\text{ord}_F) = \psi_\tau^{\text{an}}(\text{ord}_F).$$

We could easily reduce to the case $m = 1$ and $\psi_\tau = \log |s|_{h_0}^2$ for a single holomorphic section s of L . In this case, the above equality is trivial. \square

In particular, the trace operator is compatible with the partial order and the addition by a constant.

As an interesting corollary,

Corollary 4.23. *Assume the same assumptions as in Theorem 4.21. Suppose that $\phi \in \text{PSH}(L^{\text{an}})$ is a non-Archimedean metric represented by $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)$ and E is an irreducible reduced subvariety of X . Then*

$$\phi(v_{E, \text{triv}}) = \sup \{ \tau \in \mathbb{R} : \nu(\mathfrak{T}_\tau^{\theta+\omega}, E) = 0 \}$$

for any Kähler form ω on X .

4.8. Functoriality. Let $\pi: Y \rightarrow X$ be a proper bimeromorphic morphism from a reduced irreducible unibranch compact Kähler space of dimension n . By Zariski's main theorem, the obvious pull-back map induces a bijection

$$\pi^*: \text{PSH}(X, \theta) \xrightarrow{\sim} \text{PSH}(Y, \pi^*\theta).$$

In particular, it induces a canonical bijection

$$\pi^*: \text{PSH}^{\text{NA}}(X, \theta)_{>0} \xrightarrow{\sim} \text{PSH}^{\text{NA}}(Y, \pi^*\theta)_{>0}$$

as follows: the image of $\mathfrak{T} \in \text{PSH}^{\text{NA}}(X, \theta)_{>0}$ is given by

$$\pi^*\mathfrak{T}: (-\infty, \tau_{\mathfrak{T}}^+) \rightarrow \text{PSH}(Y, \pi^*\theta), \quad (\pi^*\mathfrak{T})_\tau := \pi^*\mathfrak{T}_\tau.$$

This induces a canonical bijection

$$\pi^*: \text{PSH}^{\text{NA}}(X, \theta) \xrightarrow{\sim} \text{PSH}^{\text{NA}}(Y, \pi^*\theta).$$

The injectivity is trivial and the surjectivity follows from Lemma 3.8.

Note that there is a canonical identification $Y^{\text{div}} \xrightarrow{\sim} X^{\text{div}}$, so that we can identify $(\pi^*\mathfrak{T})^{\text{an}}$ and \mathfrak{T}^{an} .

One trivially verifies that all the previous operations are functorial with respect to π^* .

5. Comparison with Boucksom–Jonsson's theory

Let X be a reduced irreducible unibranch projective variety and L be a pseudoeffective line bundle on X with $\xi = c_1(L)$. Fix a smooth Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$.

Recall the following basic result:

THEOREM 5.1 ([BJ21, Theorem 4.22]). *Let $\phi, \psi \in \text{PSH}(L^{\text{an}})$. Assume that $\phi \leq \psi$ on X^{div} , then the same holds on X^{an} .*

In particular, as we have computed the effect of our operations on X^{div} , their behaviours on X^{an} is in fact completely determined.

Corollary 5.2. *The order relation, addition by a constant, maximum, decreasing limit, rescaling and addition correspond to the corresponding operations in Boucksom–Jonsson’s theory. Assume that the envelope conjecture holds for X , then the same holds for the supremum operation.*

Briefly speaking, we have established the correspondence between the non-Archimedean pluripotential theory and the convex geometry of complex potentials as in Table 1.

Finally, we have the following comparison result between our theory of non-Archimedean potentials and Boucksom–Jonsson’s.

Corollary 5.3. *The following are equivalent:*

- (1) *The envelope conjecture of Boucksom–Jonsson holds for X ;*
- (2) *The comparison map $\mathfrak{T} \mapsto \mathfrak{T}^{\text{an}}$ defined in Definition 3.11 admits an extension as a bijection $\text{PSH}^{\text{NA}}(X, \theta) \xrightarrow{\sim} \text{PSH}(L^{\text{an}})$.*

PROOF. By Theorem 5.1, if the map in (2) exists, it is necessarily unique. Let $\pi: Y \rightarrow X$ be a projective resolution. Consider the following diagram:

$$\begin{array}{ccc} \text{PSH}^{\text{NA}}(X, \theta) & \xrightarrow{\text{an}} & \text{PSH}^{\text{NA}}(L^{\text{an}}) \\ \downarrow \pi^* & & \downarrow \pi^* \\ \text{PSH}^{\text{NA}}(Y, \pi^* \theta) & \xrightarrow{\text{an}} & \text{PSH}^{\text{NA}}(\pi^* L^{\text{an}}). \end{array}$$

Clearly, (2) means that we can complete the dotted map as a bijection so that the diagram is commutative. We already know that the left-vertical map is a bijection, see Section 4.8. The lower-horizontal map is also a bijection by Theorem 3.12.

Now assume (1), then the right-vertical map is also a bijection by [BJ21, Lemma 5.13]. Therefore, (2) follows.

Conversely, assume that (2) holds. Take an increasing net $\phi_i \in \text{PSH}^{\text{NA}}(L^{\text{an}})$ with $\phi_i \leq 0$. Recall that the envelope conjecture means that ϕ_i converges to some $\phi \in \text{PSH}^{\text{NA}}(L^{\text{an}})$ with respect to the pointwise convergence on X^{div} . We write $\phi_i = \mathfrak{T}^{i, \text{an}}$ for some $\mathfrak{T}^i \in \text{PSH}^{\text{NA}}(X, \theta)$. It suffices to take $\phi = (\sup_i^* \mathfrak{T}^i)^{\text{an}}$ and apply Theorem 4.13. \square

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