Lectures on Vertex Operator Algebras and Conformal Blocks

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0 Notations

- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$.
- $\mathbf{i} = \sqrt{-1}$, $\mathbb{S}^1$ =unit circle, $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$, $\mathbb{D}_r^{\mathrm{cl}} = \{z \in \mathbb{C} : |z| \leqslant r\}$
- $\mathscr{O}(X)$ is the space of holomorphic functions on a complex manifold X.
- Configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}$.
- z and ζ could mean either points, or the standard coordinate of $\mathbb{C}$, or formal variables. We will give their meanings when the context is unclear.
- All vector spaces are over $\mathbb{C}$, unless otherwise stated. If W is a vector space equipped with a Hermitian form $\langle\cdot|\cdot\rangle$, we let $|\cdot\rangle$ be the linear variable and $\langle\cdot|$ be the antilinear (i.e. conjugate linear) one, following physicists' convention.
- If W, W' are vector spaces, then $\mathrm{Hom}(W, W')$ denote the space of linear operators from W to W'. We let $\mathrm{End}(W) = \mathrm{Hom}(W, W)$.
- We use symbols $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$ to denote bilinear forms (i.e., linear on both variables).
- Given a vector space W and a formal variable z,

$$W[z] = \{\text{polynomials of } z \text{ whose coefficients are elements of } W\}$$
$$W[[z]] = \Big\{\sum_{n\in\mathbb{N}} w_n z^n : w_n \in W\Big\}$$
$$W((z)) = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W, \text{ and } w_n = 0 \text{ when } n \text{ is sufficiently negative}\Big\}$$
$$W[[z^{\pm 1}]] = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W\Big\}.$$

  Each line is a subspace of the subsequent line. In case there are several formal variables, the spaces are defined in a similar way, expect $W((\cdots))$. For instance,

$$W[[z, \zeta^{\pm 1}]] := W[[z]][[\zeta^{\pm 1}]] = W[[\zeta^{\pm 1}]][[z]]$$

  consists of $\sum_{m\in\mathbb{N},n\in\mathbb{Z}} w_{m,n} z^m \zeta^n$ where each $w_{m,n} \in W$. However, note that $W((z))((\zeta))$ and $W((\zeta))((z))$ are not equal. (For instance, $\sum_{m\geqslant -n}\sum_{n\geqslant -1} z^m\zeta^n$ belongs to $\mathbb{C}((z))((\zeta))$ but not $\mathbb{C}((\zeta))((z))$.)

- We set

$$\mathrm{Res}_{z=0} \sum_{n\in\mathbb{Z}} w_n z^n dz = w_{-1}. \tag{0.2}$$

  This is in line with the complex analytic residue.

- A vector of $W_1 \otimes \cdots \otimes W_N$ writen as $w_\bullet$ means that it is of the form $w_1 \otimes \cdots \otimes w_N$ where each $w_i \in W_i$. Depending on the context, $w_\bullet$ will also mean a tuple $(w_1, \ldots, w_N)$.
- Unless otherwise stated, by a manifold, we mean one *without* boundaries. Also, "with boundaries" means "possibly with boundaries".

1 Segal's picture of 2d CFT; motivations of VOAs and conformal blocks

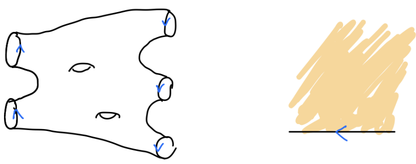
1.1

Vertex operator algebras (VOAs) are mathematical objects defined to understand and construct 2-dimensional conformal field theory (CFT for short). A CFT describes propagations and interactions of strings. The are two types of strings: closed strings $\simeq \mathbb{S}^1$ and open strings $\simeq [0,1]$. In this course, we will mainly focus on closed strings.

Let me explain how mathematicians understand CFT. Just like any quantum field theory (QFT), in CFT we must have a Hilbert space $\mathcal{H}$. The vectors in $\mathcal{H}$ are called "states", but unlike ordinary QFT, a vector $\xi \in \mathcal{H}$ is not a state of a particle, but a state of a closed string $\mathbb{S}^1$.

The most important and non-trivial part in CFT is to define/understand string interactions. According to Segal's picture [Seg88], an interaction is uniquely determined by a compact Riemann surface Σ with boundaries $\partial\Sigma$, where $\partial\Sigma$ is a disjoint union of some circles (strings). Each string is called either an incoming string or an outcomming one. Suppose $\partial\Sigma$ has N incoming strings and M outgoing ones, then this picture describes an interaction where N strings are going inside, and M strings are going outside.

Moreover, the boundary $\partial\Sigma$ must be **parametrized**. This means that to each connected component $\partial\Sigma_i$ a diffeomorphism $\eta_i : \partial\Sigma_i \xrightarrow{\simeq} \mathbb{S}^1$ is associated. The orientation on $\partial\Sigma_i$ defined by pulling back the one of $\mathbb{S}^1$ along η_i is assumed to be the opposite of the one defined in Stokes' theorem, shown as follows

 (1.1)

1.2

Unless otherwise stated, we assume that the boundary parametrization is also **analytic**. Roughly speaking, this means that Σ can be obtained by removing some open discs from a compact Riemann surface C (without boundary) such that the parametrizations of $\partial\Sigma$ are given by local holomorphic functions of C.

Here is a more rigorous explanation. By a **local coordinate** η of C at $x \in C$, we mean η is a holomorphic injective function on a neighborhood U of x such that $\eta(x) = 0$. So

η is a biholomorphism between U and a neighborhood $\eta(U)$ of 0. Now, suppose we have local coordinates $\eta_1, \dots, \eta_N$ at distinct points $x_1, \dots, x_N \in C$. The data

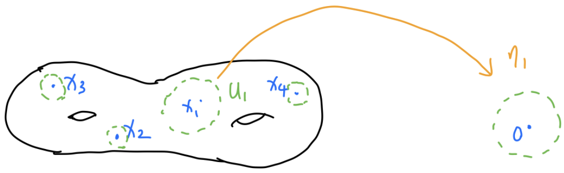
$$\mathfrak{X} := (C; x_\bullet; \eta_\bullet) = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N) \tag{1.2}$$

is called an **N-pointed compact Riemann surface with local coordinates**.

Let each η_i be defined on a neighborhood $U_i \ni x_i$. We assume moreover the following

Assumption 1.1. $U_i \cap U_j = \varnothing$ if $i \neq j$, and $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$ for each i. Here $\mathbb{D}_1^{\mathrm{cl}}$ is the closed unit disc.

By removing all $\eta_i^{-1}(\mathbb{D}_1)$, we get Σ with boundary strings $\eta_i^{-1}(\partial\mathbb{D}_1^{\mathrm{cl}}) = \eta_i^{-1}(\mathbb{S}^1)$ whose parametrizations are η_i.



1.3

Any Σ as above determines uniquely an interaction of strings. Suppose it has N incoming strings and M outgoing ones. Then mathematically, such an interaction is described by a bounded linear map $T = T_\Sigma : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes M}$. (The boundedness is automatic thanks to the uniform boundedness principle. But this is not an important point in this course.) Given $\xi_\bullet = \xi_1 \otimes \cdots \otimes \xi_N \in \mathcal{H}^{\otimes N}$ and $\eta_\bullet = \eta_1 \otimes \cdots \otimes \eta_M \in \mathcal{H}^{\otimes M}$, the value

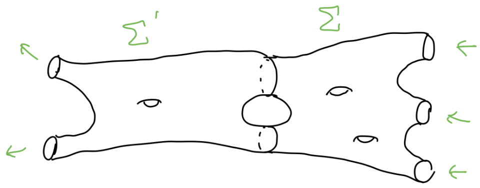
$$\langle \eta_\bullet | T\xi_\bullet \rangle \tag{1.3}$$

describes the probability amplitude that the N incoming closed strings with states $\xi_1, \dots, \xi_N$ become $\eta_1, \dots, \eta_M$ after interaction.

The word "conformal" in conformal field theory reflects the fact that T depends only on the complex structure of Σ and its parametrization, but not on the metric for instance. Thus, a CFT is more rigid than a topological quantum field theory (TQFT): in the latter case, T depends only on the topological structures of the manifolds.

1.4

Suppose we have another interaction $S : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes L}$ corresponding to the parametrized surface Σ', then the composition of them $S \circ T : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes L}$ corresponds to the **sewing** $\Sigma \# \Sigma'$ of Σ and Σ', where the j-th outgoing string $\partial_+\Sigma_j$ of Σ is sewn with the j-th incoming one $\partial_-\Sigma'_j$ of Σ'.



It is important to specify how $\partial_+\Sigma_j$ (with parametrization η_j) is identified with $\partial_-\Sigma'_j$ (with parametrization η'_j). Pick $x \in \partial_+\Sigma_j$ and $y \in \partial_-\Sigma'_j$. Then

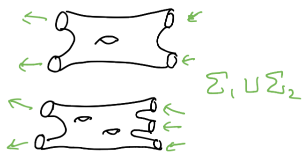
$$x = y \quad \Longleftrightarrow \quad \eta_j(x)\eta'_j(y) = 1. \tag{1.4}$$

It is clear from the picture that the orientations of $\partial_+\Sigma_j$ and $\partial_-\Sigma_j$ are opposite to each other. This is related to the fact that our rule for sewing is $\eta_j(x) = 1/\eta'_j(y)$ but not (say) $\eta_j(x) = \eta'_j(y)$.

Recall we assume that the parametrizations are analytic. We leave it to the readers to check that the sewing of Σ and Σ', a priori only a topological surface, has a natural complex analytic structure.

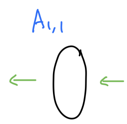
1.5

Suppose $T_1 : \mathcal{H}^{\otimes N_1} \to \mathcal{H}^{\otimes M_1}$ corresponds to Σ_1 and $T_2 : \mathcal{H}^{\otimes N_2} \to \mathcal{H}^{\otimes M_2}$ to Σ_2, then $T_1 \otimes T_2 : \mathcal{H}^{\otimes(N_1+N_2)} \to \mathcal{H}^{\otimes(M_1+M_2)}$ corresponds to the disjoint union $\Sigma_1 \sqcup \Sigma_2$.



1.6

Consider an annulus $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$ obtained by removing two open discs from the compact Riemann sphere $\mathbb{P}^1$ via the local coordinate $\eta_1(z) = z/r$ at 0 and $\eta_2(z) = R/z$ at ∞. We call such $A_{r,R}$ (with the given boundary parametrization) a **standard annlus**. Let $r \nearrow 1, R \searrow 1$. The limit of this annulus is a "degenerate" Riemann surface with 1 incoming boundary circle and 1 outing one. Both circles are $\mathbb{S}^1$. The incoming one has parametrization $z \mapsto z$ and the outgoing one $z \mapsto z^{-1}$. We call this annulus the **standard thin annulus** and denote it by $A_{1,1}$. *The map $T : \mathcal{H} \to \mathcal{H}$ associated to $A_{1,1}$ is the identity map.* This reflects the fact that sewing any Σ with a disjoint union of $A_{1,1}$ gives Σ.



1.7

We give a fancy way to summarize what we have so far: Let $\mathscr{C}$ be the monoidal category of compact 1-dimensional smooth manifolds such that a morphism from an object S_1 to another S_2 is a compact Riemann surface with incomming parametrized boundary $\simeq S_1$ and outgoing one $\simeq S_2$, that the identity morphism for a union of N circles is a disjoint union of N pieces of $A_{1,1}$, that the unit object is the empty set, and that the tensor product of objects and morphisms are respectively the disjoint unions of strings and Riemann surfaces. Then a CFT is a monoidal functor from $\mathscr{C}$ to the monoidal category of Hilbert spaces. So, roughly speaking, a CFT is a representation of $\mathscr{C}$.

Since we choose Hilbert spaces as our underlying spaces, we should expect that the representation of $\mathscr{C}$ is unitary. Technically, the functor mentioned above should be a $*$-functor: this means that for each morphism Σ from N strings to M strings, we should define its adjoint morphism Σ^* from M strings to N ones whose corresponding map is the adjoint $T^* : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes N}$ of T. Σ^* is defined simply to be the **complex conjugate** $\overline{\Sigma}$ of Σ:

Definition 1.2. $\overline{\Sigma}$ consists of points $\overline{x}$ where $x \in \Sigma$; the local holomorphic functions on $\overline{\Sigma}$ are η^* where η is a locally defined holomorphic function on Σ and

$$\eta^*(\overline{x}) = \overline{\eta(x)} \tag{1.5}$$

whenever η is defined on $x \in \Sigma$; similarly, boundary parametrizations are given by η_j^*. Note that if Σ is obtained by removing open discs from an N pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, then $\overline{\Sigma}$ is obtained by removing discs from

$$\overline{\mathfrak{X}} := (\overline{C}; \overline{x}_1, \dots, \overline{x}_N; \eta_1^*, \dots, \eta_N^*) \tag{1.6}$$

η^* should not be confused with $\overline{\eta}$ defined on Σ by

$$\overline{\eta}(x) = \overline{\eta(x)}.$$

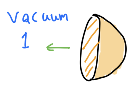
In the present context, we should assume that an incoming (resp. outgoing) string of Σ becomes an outgoing (resp. incoming) one of $\overline{\Sigma}$ via the conjugate map $\complement : x \in \Sigma \mapsto \overline{x} \in \overline{\Sigma}$. In the future, we will often consider all strings as incoming ones if necessary (cf. 1.9). In that case, we shall also assume all the boundary strings of $\overline{\Sigma}$ as incoming.

We should point out that although unitarity is a very important condition, there are important non-unitary CFTs, for instance, the logarithmic CFTs. (In such cases, $\mathcal{H}$ is a vector space without inner products.) Also, many VOA results and techniques do not rely on the unitarity. Nevertheless, assuming unitarity will often reasonably simply discussions or give motivations.

Example 1.3. Let $\mathfrak{X} = (\mathbb{P}^1; 0; \lambda\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$ and $\lambda \in \mathbb{C}^\times$. We can identify the conjugate of $\mathbb{P}^1$ with $\mathbb{P}^1$ by letting $x \in \mathbb{P}^1 \mapsto \overline{x}$ be the standard conjugate of $\mathbb{C}$: $z \mapsto \overline{z}$. Then $(\lambda\zeta)^*(\overline{z}) = \overline{\lambda\zeta(z)} = \overline{\lambda} \cdot \overline{z} = \overline{\lambda}\zeta(\overline{z})$. So the conjugate of $\mathfrak{X}$ is isomorphic to $\overline{\mathfrak{X}} = (\mathbb{P}^1; 0; \overline{\lambda}\zeta)$.

1.8

An interaction process could have no incoming or outgoing strings. *The Hilbert space for the empty string $\varnothing$ is $\mathbb{C}$.* The most elementary and important example with no incoming boundary is the closed unit disc $\mathbb{D}_1^{\mathrm{cl}}$ with 1 outgoing boundary parametrized by $z \mapsto z^{-1}$. The corresponding map $\mathbb{C} \to \mathcal{H}$ can be identified with its value at 1. This element in $\mathcal{H}$ is denoted by $\mathbf{1}$ and called the **vacuum vector**.



(1.7)

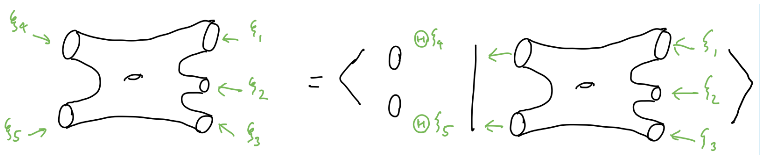
Assume as before that out theory is unitary. Then conjugate of the above disk is the same disk and boundary parametrization, but the original outgoing string is now the incoming one. The corresponding map $\mathcal{H} \to \mathbb{C}$ is, according to 1.7, the linear functional $\langle \mathbf{1} | \cdot \rangle$.

1.9

In general, one may wonder what the interaction $T : \mathcal{H}^{\otimes N} \to \mathbb{C}$ means physically for a surface Σ with N incoming strings but no outgoing ones. Choose $0 < M < N$, and make M of the N strings of $\partial\Sigma$ be outgoing strings. Then the corresponding interaction is a map $\widetilde{T} : \mathcal{H}^{\otimes(N-M)} \to \mathcal{H}^{\otimes M}$. In unitary CFT, T can be related to $\widetilde{T}$ by a anti-unitary (i.e. conjugate-unitary) map Θ on $\mathcal{H}$, called the **CPT operator**, such that for $\xi_1, \dots, \xi_N \in \mathcal{H}$ (where the last M vectors are associated to the outgoing strings), we have

$$T(\xi_1 \otimes \cdots \otimes \xi_N) = \langle \Theta\xi_{N-M+1} \otimes \cdots \otimes \Theta\xi_N | \widetilde{T}(\xi_1 \otimes \cdots \otimes \xi_{N-M}) \rangle, \tag{1.8}$$

interpreted pictorially as



The operator Θ is an involution, i.e., $\Theta^2 = \mathbf{1}_{\mathcal{H}}$.

Such a linear functional T corresponding to an interaction with no outgoing strings is called a **correlation function** (or an **N-point function**). These functions are the central objects in CFT (and indeed, in any quantum field theory). Relation (1.8) teaches us that: (1) correlation functions can be interpreted as probability amplitudes in string interactions with the help of Θ, and (2) to study arbitrary interactions, it suffices to study those with no outgoing strings.

Let me close this subsection by mentioning an important fact: suppose the complex structure of Σ and the (assumed analtytic) boundary parametrizations are parametrized holomorphically by some complex variables $\tau_\bullet = (\tau_1, \dots, \tau_k)$, then the value of $T(\xi_\bullet)$ is now a *real analytic function* of $\tau_\bullet$, i.e., it is locally a power series of $\tau_1, \dots, \tau_k$ and their conjugates. Actually, the word "function" in "correlation function" means a function of $\tau_\bullet$, but not of $\xi_\bullet$.

1.10

You must be curious what CPT means. Indeed, Θ is responsible for the simultaneous symmetry of charge conjugation (C), parity transformation (P), and time reversal (T). P+T together means an *anti-biholomorphism* $\Sigma \to \Sigma'$. Now we have arrived at a point that we missed previously: since anti-holomorphic maps are also conformal maps, should we expect that the interaction maps (or the correlation functions) for anti-biholomorphic surfaces are equal? The answer is no. (Namely, P+T are not preserved.) Indeed, if we let Σ have N incomes and no outcomes, let $\overline{\Sigma}$ be its complex conjugate (cf. 1.7) but still with N incomes, and let $T_\Sigma, T_{\overline{\Sigma}}$ be the correlation functions associated to them. Then from 1.7 and relation (1.8), we have

$$T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N) = \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \tag{1.9}$$

Proof. By the description in Subsec. 1.7, the interaction map $\widetilde{T}_{\overline{\Sigma}}$ associated $\overline{\Sigma}$ with no input and N outputs is $T_\Sigma^* : \mathbb{C} \to \mathcal{H}^{\otimes N}$, the adjoint of T_Σ. By $\Theta^2 = 1$, we have

$$\begin{aligned} &T_\Sigma(\xi_1 \otimes \cdots \xi_N) = \langle 1 | T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N)\rangle = \langle T_\Sigma^* 1 | \xi_1 \otimes \cdots \otimes \xi_N\rangle \\ =&\overline{\langle \xi_1 \otimes \cdots \otimes \xi_N | \widetilde{T}_{\overline{\Sigma}} 1\rangle} \xlongequal{(1.8)} \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \end{aligned}$$

Note that mathematically, the point of formula (1.9) is to translate (using (1.8)) the relation $\widetilde{T}_{\overline{\Sigma}} = T_\Sigma^*$ (regarding all the strings of $\overline{\Sigma}$ as outgoing) to the case that all the strings of $\overline{\Sigma}$ are incoming. $\square$

Formula (1.9) explains CPT symmetry: the symmetries of charge (taking complex conjugate of the values of correlation functions) and parity+time (the conjugate biholomophism $\complement : \Sigma \to \overline{\Sigma}$) are preserved, and the operator realizing this simultaneous symmetry is Θ.

Note that mathematically, charge conjugation C is related to taking complex conjugate of numbers (but not of Σ). Physically, it means making a string into its "anti-string", or (in general QFT) making a particle (e.g. an election with negative charge) to its anti-particle (e.g. an antielectron with positive charge).

1.11

The CFT we have described so far is actually very special: it has no conformal anomaly. There are indeed no nontrivial CFTs which are both unitary and without anomaly. In this course, we will be mainly interested in CFTs with conformal anomaly. Technically, the conformal anomaly is determined by a complex number c (positive for unitary CFT), called **central charge**. To describe such CFT, we modify the previous descriptions as follows: The map (or the correlation function) T_Σ for Σ is only up to a positive scalar multiplication depending on Σ. $T_{\Sigma_1} \circ T_{\Sigma_2} = \lambda T_{\Sigma_1 \# \Sigma_2}$ where $\lambda > 0$. (The constants are not necessarily positive in non-unitary CFT.) If Σ is parametrized holomorphically by some complex variables $\tau_\bullet$, then by shrinking the domain of $\tau_\bullet$, we can choose T_Σ depending real analytically on $\tau_\bullet$.

There are many important cases where a real analytic (or even a holomorphic) T_Σ can be chosen globally for $\tau_\bullet$. This will be studied later in details.

Unless otherwise stated, a CFT always means one with (possible) conformal anomaly. Using the fancy language of 1.7, one can say that a unitary CFT is a *projective* monoidal $*$-functor from the category $\mathscr{C}$ in 1.7 to the category of Hilbert spaces. Namely, it is a projective unitary representation of $\mathscr{C}$.

1.12

To study the representations of a topological group G, one must first understand very well the topological and the algebraic structures of G. Similarly, the study of CFTs relies heavily on the geometric and analytic structures of compact Riemann surfaces. However, from what we have discussed, there is a huge obstacle for studying CFTs: the correlation functions are real analytic, but not complex analytic (i.e. holomorphic) functions of the parameters $\tau_\bullet$. Thus, in order to study CFTs using the powerful tools of complex analysis (residue theorem, for instance), we make the following Ansatz: A correlation function T is a sum : $T_\Sigma = \sum_j \Phi^j_\Sigma \cdot \Psi^j_{\overline{\Sigma}}$, where each Φ^j and Ψ^j relies holomorphically on Σ and $\overline{\Sigma}$ respectively (so $\Psi^j_{\overline{\Sigma}}$ relies anti-holomorphically on Σ).

This Ansatz is very vague. Let me explain it in more details. Consider the annulus $A_{r,R}$ with boundary parametrization as in 1.6. We move the inside circle to another one centered at z (where $z \in A_{r,R}$ is reasonably small), still with radius r. The new eccentric annulus $A_{z,r,R}$ has larger outgoing string parametrized by R/ζ and the smaller incoming one parametrized by $(\zeta - z)/r$, where ζ is the standard coordinate of $\mathbb{P}^1$. Namely, it is determined by the data

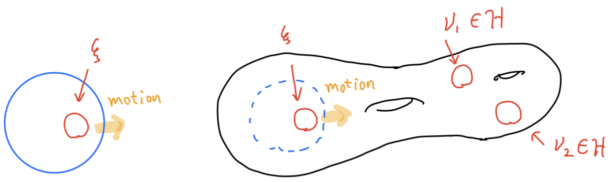
$$(\mathbb{P}^1; z, \infty; (\zeta - z)/r, R/\zeta). \tag{1.10}$$

Let $T_z : \mathcal{H} \to \mathcal{H}$ be the corresponding map. As we have said, for general vectors $\xi, \eta \in \mathcal{H}$, the expression $\langle \eta | T_z \xi \rangle = \langle \Theta\eta, T_z\xi \rangle$ can be chosen to be real analytic with respect to z. We now let

$$\begin{aligned} \mathbb{V} = \{\xi \in \mathcal{H} : &\text{ For all } r, R, \text{ the map } T \text{ can be chosen such that} \\ &z \mapsto \langle \nu | T_z \xi \rangle \text{ is holomorphic for all } \nu \in \mathcal{H}, \text{ and} \\ &\xi \text{ has ``finite energy''}\} \end{aligned} \tag{1.11}$$

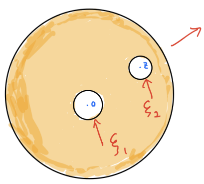
"Finite energy" is a minor condition to be explained later. (See **??**.)

We can sew $A_{z,r,R}$ with any Σ, and the motion of the smaller string inside the annulus becomes, after sewing, the motion of a boundary string of Σ:



Therefore, if a vector $\xi \in \mathbb{V}$ is assigned to an incoming string of Σ with (analytic) boundary parametrization η_i, then, when translating this parametrized string with respect to η_i, the correlation function $T_\Sigma(\xi \otimes \cdots)$ should be holomorphic with respect to the motion, whatever states we assign to the other strings. We can therefore study $\mathbb{V}$ with the help of complex analysis. $\mathbb{V}$ is called a **vertex operator algebra** (VOA).

We have only described $\mathbb{V}$ as a vector space. But in which sense is $\mathbb{V}$ an algebra? An obvious candidate is as follows: consider $\mathbb{P}^1$ with three marked points $0, z, \infty$ and usual coordinates, e.g. $\eta_0 = \zeta/r_1, \eta_z = (\zeta - z)/r_2, \eta_\infty = R/\zeta$ at $0, z, \infty$ where $r_1, r_2 > 0$ are small and $R > 0$ is large, and ζ is again the standard coordinate of $\mathbb{C}$. We assume the strings around 0 and z are ingoing and that around ∞ outgoing. If we assign $\xi_1, \xi_2 \in \mathbb{V}$ to the incoming strings, then the outcome can be viewed as a product of ξ_1 and ξ_2.



Although this product does not have finite energy, it does satisfy the statement before the last line in (1.11). Thus, this product is almost a vector in $\mathbb{V}$. By modifying this product suitably, we can ensure that the products of vectors in $\mathbb{V}$ are always in $\mathbb{V}$. Details will be give in later sections.

Similarly to (1.11), we define $\widehat{\mathbb{V}} \subset \mathcal{H}$ to be the set of finite energy vectors ξ such that $\langle \nu | T_z \xi \rangle$ is anti-holomorphic over z. The vacuum vector $\mathbf{1}$ belongs to $\mathbb{V} \cap \widehat{\mathbb{V}}$: The result of gluing the unit disc into the inside of $A_{z,r,R}$ is just the disc with radius R and parametrization R/ζ, which is independent of z. So $T_z \mathbf{1}$ and hence $\langle \nu | T_z \mathbf{1} \rangle$ are constant over z, and hence both holomorphic and anti-holomorphic over z.

1.13

Now we can give a more detailed presentation of our Ansatz. We let $\mathcal{H}^{\mathrm{fin}}$ be the (indeed dense) subspace of vectors in $\mathcal{H}$ with "finite energy", which is acted on by $\mathbb{V} \otimes \widehat{\mathbb{V}}$. Ansatz:

1. $\mathcal{H}^{\mathrm{fin}}$ as a $\mathbb{V} \otimes \widehat{\mathbb{V}}$-module has decomposition
$$\mathcal{H}^{\mathrm{fin}} = \bigoplus_{i \in \mathfrak{I}} \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i \qquad \supset \mathbb{V} \otimes \widehat{\mathbb{V}} \tag{1.12}$$
where each $\mathbb{W}_i, \widehat{\mathbb{W}}_i$ are respectively irreducible $\mathbb{V}$-modules and $\widehat{\mathbb{V}}$-modules. $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are (according to their definition cf. (1.11)) subspaces of $\mathcal{H}^{\mathrm{fin}}$ by identifying them with $\mathbb{V} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \widehat{\mathbb{V}}$ respectively. The vacuum vector $\mathbf{1}$ of $\mathcal{H}$ is identified with $\mathbf{1} \otimes \mathbf{1}$ (which belongs to $\mathbb{V} \otimes \widehat{\mathbb{V}}$).

2. For some Σ without outgoing boundaries, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the corresponding map. Then, corresponding to the above direct sum decomposition, we have
$$T_\Sigma \Big|_{(\mathcal{H}^{\mathrm{fin}})^{\otimes N}} = \sum_{i_1, \dots, i_N \in \mathfrak{I}} \Phi_{\Sigma, i_\bullet} \otimes \Psi_{\overline{\Sigma}, i_\bullet} \tag{1.13}$$
where
$$\Phi_{\Sigma, i_\bullet} : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \to \mathbb{C},$$
$$\Psi_{\overline{\Sigma}, i_\bullet} : \widehat{\mathbb{W}}_{i_1} \otimes \cdots \otimes \widehat{\mathbb{W}}_{i_N} \to \mathbb{C}$$
are linear. Moreover, when the complex structure and boundary parametrization are parametrized analytically by complex variables $\tau_\bullet$, then locally (with respect to the domain of $\tau_\bullet$), $T_\Sigma, \Psi_{\Sigma, i_\bullet}, \Psi_{\overline{\Sigma}, i_\bullet}$ can be chosen such that $\Psi_{\Sigma, i_\bullet}$ is holomorphic over $\tau_\bullet$ (for all input vectors), and $\Psi_{\overline{\Sigma}, i_\bullet}$ holomorphic over $\overline{\tau}_\bullet$. $\Phi_{\Sigma, i_\bullet}$ and $\Psi_{\overline{\Sigma}, i_\bullet}$ are called **conformal blocks** associated to Σ (resp. $\overline{\Sigma}$) and $\mathbb{V}$ (resp. $\widehat{\mathbb{V}}$).

In part one, $\bigoplus$ could be finite (our main focus in this course), infinite but discrete, or continuous.

The second part can be summarized by saying that the CFT is separated into the **chiral halves** (those Φ or $\mathbb{W}_i$) and the **anti-chiral halves** (those Ψ or $\widehat{\mathbb{W}}_i$). Here, "chiral"="holomorphic".

When physicists say a CFT is **rational**, they usually mean that the above direct sum is finite, and each $\mathbb{W}_{i_k}, \widehat{\mathbb{W}}_{i_k}$ are semi-simple (hence, by further decomposition, can be irreducible). So far, the mathematical theory of conformal blocks is complete almost only for rational CFTs. These will be the main examples of this course. For non-rational logarithmic CFTs, even the above Ansatz needs to be modified. (So far, it is not even clear how to do it.)

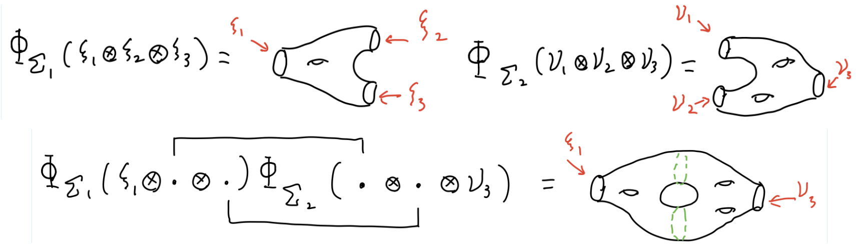
Physicists more or less consider the above description as the definition of conformal blocks. We mathematicians should do the opposite: define conformal blocks in a different way, and use them to *construct* CFTs following the above Ansatz.

1.14

You may notice that to make this Ansatz compatible with 1.4 and 1.5, it is necessarily to assume that

1. The tensor product of conformal blocks $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ associated to Σ_1, Σ_2 respectively should be a conformal block associated to $\Sigma_1 \sqcup \Sigma_2$.

2. The composition of $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ (or more precisely, their contractions) should be conformal blocks associated to the sewings of Σ_1 and Σ_2, where the pair of $\mathbb{V}$-modules to be contracted must be dual to each other.



(A side note on linear algebra: If $V^\vee$ is the dual space (or a suitable dense subspace of the dual space) of a vetor space V, we choose a basis $\{v_\alpha\}_{\alpha\in\mathfrak{A}}$ labeled by elements of $\mathfrak{A}$, and choose a dual basis $\{v_\alpha^\vee\}_{\alpha\in\mathfrak{A}}$ of $V^\vee$ (i.e. the one determined by $\langle v_\alpha, v_\beta\rangle = \delta_{\alpha,\beta}$), then taking contraction means substituting $\sum_{\alpha\in\mathfrak{A}} v_\alpha \otimes v_\alpha^\vee$ inside the linear functional on a tensor product of vector spaces such that $V, V^\vee$ are tensor components.)

After we define conformal blocks rigorously, we will see that the first point is obvious, while the second one is a non-trivial theorem.

We briefly explain the meaning of "dual", and why the dual modules appear in $\mathcal{H}$. For instance, in the above picture, the unitary $\mathbb{V}$-module containing ξ_2 is dual to the one containing η_1. As vector spaces, they are "graded" dual spaces of each other. (It is a dense subspace of the full dual space, the subspace of "finite energy" linear functionals. We will talk about this in future sections.) In unitary CFTs, all $\mathbb{V}$ and $\widehat{\mathbb{V}}$ modules are unitary, and $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i)$ is equivalent to $\mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$ where $\mathbb{W}_i'$ is a $\mathbb{V}$-module dual to $\mathbb{W}_i$, and $\widehat{\mathbb{W}}_i'$ a $\widehat{\mathbb{V}}$-module dual to $\widehat{\mathbb{W}}_i$. The formal name for dual module is **contragredient module**, to be defined rigorously in later sections.

1.15

Let us describe the equivalence $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i) \simeq \mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$ in more details.

For each $w_i \otimes \widehat{w}_i \in \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, the vector $\Theta(w_i \otimes \widehat{w}_i)$ is regarded as a linear functional on $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ in the following way. Let the (clearly symmetric) bilinear form $\langle \cdot, \cdot \rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ be the correlation function $T_{A_{1,1}}$ for the standard thin annulus $A_{1,1}$ (with two inputs and no outputs). Note that by (1.8), for each $\xi, \nu \in \mathcal{H}$, we have

$$\langle \Theta\xi, \nu \rangle = \langle \xi | \nu \rangle. \tag{1.14}$$

Then $\Theta(w_i \otimes \widehat{w}_i)$ is equivalent to the linear functional

$$\langle \Theta(w_i \otimes \widehat{w}_i), \cdot \rangle = \langle w_i \otimes \widehat{w}_i | \cdot \rangle \tag{1.15}$$

restricted onto $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$.

A conformal block with $M+N$ inputs $\Phi_\Sigma : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \otimes \mathbb{W}_{j_1} \otimes \cdots \otimes \mathbb{W}_{j_N} \to \mathbb{C}$ can be regarded as one with N inputs and M outputs $\Phi_\Sigma : \mathbb{W}_{j_1} \otimes \cdots \otimes \mathbb{W}_{j_N} \to \mathcal{H}_{i_1}' \otimes \cdots \otimes \mathcal{H}_{i_M}'$ where $\mathcal{H}_{i_k}'$ is the Hilbert space completion of $\mathbb{W}_{i_k}'$ and $\mathbb{W}_{i_k}'$ is the contragredient $\mathbb{V}$-module of $\mathbb{W}_{i_k}$. Using (1.14), it is not hard to show that taking compositions of conformal blocks with outputs is equivalent to taking contractions for conformal blocks without outputs.

2 Virasoro algebras; change of boundary parametrizations; strings vs. punctures

2.1

The goal of this section is to understand conformal blocks associated to 2-pointed Riemann spheres, equivalently, genus-0 surfaces with two boundary strings. We simply call them **annuli**, although their complex structures and boundary parametrizations are not necessarily the standard ones as in 1.6.

Let us first consider some degenerate examples whose boundary parametrizations are not necessarily analyic. Let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the topological group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. For each $g \in \mathrm{Diff}^+(\mathbb{S}^1)$, we let $A_{1,1}^g$ be the thin annulus whose incoming and outgoing strings are both $\mathbb{S}^1$ with parametrizations

$$\text{Incoming} : z \mapsto z, \qquad \text{Outgoing} : z \mapsto 1/g(z).$$

Lemma 2.1. *If $h \in \mathrm{Diff}^+(\mathbb{S}^1)$, then $A_{1,1}^{gh}$ is obtained by gluing the incoming circle of $A_{1,1}^g$ with the outgoing one of $A_{1,1}^h$.*

Proof. By (1.4), a point $z \in A_{1,1}^h$ is glued with $\zeta \in A_{1,1}^g$ iff $\zeta \cdot 1/h(z) = 1$, i.e., $\zeta = h(z)$. Now, a point z of $A_{1,1}^h$ becomes the point $h(z)$ of $A_{1,1}^g$ after gluing, which is sent by the outgoing parametrization of $A_{1,1}^g$ to $1/g(h(z))$. □

This proof is not rigorous since we are considering degenerate annuli. A rigorous one would be approximating $A_{1,1}^g$ and $A_{1,1}^h$ by genuine annuli, identifying the sewn annuli, and then taking the limit. This proof is not easy, unless when g and h are real-analytic (e.g., rotations). Nevertheless, we only need this lemma to motivate our following discussions.

2.2

Thus, we may consider $\mathrm{Diff}^+(\mathbb{S}^1)$ as the group of thin annuli whose product is the sewing. The merit of this viewpoint is that it convinces us to *consider the semi-group* $\mathbf{Ann}$ *of annuli as the complexification of* $\mathrm{Diff}^+(\mathbb{S}^1)$. The multiplication A_1A_2 of $A_1, A_2 \in \mathbf{Ann}$ is the sewing of A_1, A_2 defined by gluing the inside of A_1 with the outside of A_2 using their parametrizations.

As an example, consider $\mathbb{P}^1$ with marked points $0, \infty$ and local coordinates $\eta_0(z) = z, \eta_\infty(z) = e^{-\mathbf{i}\tau}/z$, which gives a thin annulus corresponding to the rotation $z \mapsto e^{\mathbf{i}\tau}z$ when τ is real. Now consider τ as a complex variable $\tau = s + \mathbf{i}t$. Then the outgoing circle is the one with radius e^t. This gives a genuine annulus whenever $t > 0$.

The Ansatz in 1.13 should be expanded to include the following point: for each annulus $A \in \mathbf{Ann}$, the comformal block decomposition of the interaction $T_A : \mathcal{H} \to \mathcal{H}$ (with one income and one outcome) with respect to $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ is of the form

$$T_A = \sum_i \pi_i(A) \otimes \widehat{\pi}_i(\overline{A}) \tag{2.1}$$

where $\pi_i(A)$ is a bounded linear operator on the Hilbert space completion $\mathcal{H}_i$ of $\mathbb{W}_i$, and $\widehat{\pi}_i(\overline{A})$ is one on the completion $\widehat{\mathcal{H}}_i$ of $\widehat{\mathbb{W}}_i$. ($\overline{A}$ is the complex conjugate of A; see Def. 1.2. We assume the conjugate of the incomming string of $\overline{A}$ is the incoming of A, and similarly for the outcoming strings.) The choice of $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are unique up to scalar multiplications, and if A vary holomorphically over some complex variable $\tau_\bullet$, then locally $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ can be chosen to vary holomorphically with respect to $\tau_\bullet$ and $\overline{\tau}_\bullet$ respectively. Finally, if $A_1, A_2 \in \mathbf{Ann}$, then $\pi_i(A_1A_2)$ equals $\pi_i(A_1)\pi_i(A_2)$ up to scalar multiplication, and a similar thing can be said about $\widehat{\pi}_i$.

Namely, each π_i is a projective representation of $\mathbf{Ann}$ on $\mathcal{H}_i$, and so is $\widehat{\pi}_i$ on $\mathcal{H}_{\bar{i}}$. They should be the analytic extensions of projective unitary representations of $\mathrm{Diff}^+(\mathbb{S}^1)$.

We emphasize that $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are conformal blocks associated to A and $\overline{A}$ respectively. Roughly speaking, π_i describes the conformal symmetries of chiral halves and $\widehat{\pi}_i$ the anti-chiral halves. A and $\overline{A}$ have to act jointly on the full space $\mathcal{H}$.

2.3

Thus, the study of CFT interactions for annuli reduces to that of the projective representations of $\mathbf{Ann}$. Our goal is to describe such representations in terms of Lie algebras.

Let $\mathrm{Vec}(\mathbb{S}^1)$ be the Lie algebra of smooth real vector fields of $\mathbb{S}^1$, whose elements are of the form $f\partial_\theta$ where ∂_θ is the pushforward of the standard unit vector of the real line under the map $\theta \mapsto e^{\mathbf{i}\theta}$, and $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$. The action of $f\partial_\theta$ on $h \in C^\infty(\mathbb{S}^1, \mathbb{R})$ is the negative of the usual one, $-f(e^{\mathbf{i}\theta}) \cdot \frac{\partial}{\partial\theta} h(e^{\mathbf{i}\theta})$. This is because the action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on h should be $h \circ g^{-1}$ in order to respect the order of group multiplication. Therefore, the Lie bracket in $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the usual one:

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta. \tag{2.2}$$

2.4

A projective unitary representation π of $\mathrm{Vec}(\mathbb{S}^1)$ and the corresponding one π of $\mathrm{Diff}^+(\mathbb{S}^1)$ (if exists) are related as follows. (Here unitary means that for each vector field $f\partial_\theta$, we have $\pi(f\partial_\theta)^\dagger = -\pi(f\partial_\theta)$, where $\dagger$ is the adjoint, or "formal adjoint" when the underlying inner product space is not Cauchy-complete.)

Let $t \in (-\epsilon, \epsilon) \mapsto g_t \in \mathrm{Diff}^+(\mathbb{S}^1)$ be a smooth family of diffeomorphisms satisfying $g_0 = \mathbf{1}$. Then up to addition by a number of $\mathbf{i}\mathbb{R}$,

$$\frac{d}{dt}\pi(g_t)\Big|_{t=0} = \pi(\partial_t g_0) \tag{2.3}$$

where $\partial_t g_0 \in \mathrm{Vec}(\mathbb{S}^1)$, the derivative of g at t_0, is the vector field determined by

$$(\partial_t g_0)(h) = \frac{d}{dt}(h \circ g_t)\Big|_{t=0} \tag{2.4}$$

for all smooth function h on $\mathbb{S}^1$.

Let now $t \in \mathbb{R} \mapsto \exp(tf\partial_\theta) \in \mathrm{Diff}^+(\mathbb{S}^1)$ be the flow generated by $f\partial_\theta \in \mathrm{Vec}(\mathbb{S}^1)$. So its derivative at $t = 0$ is $f\partial_\theta$, and $\exp((t_1 + t_2)f\partial_\theta) = \exp(t_1 f\partial_\theta) \circ \exp(t_2 f\partial_\theta)$. Then (2.4) implies that up to $\mathbb{S}^1$-multiplication,

$$\pi(\exp(tf\partial_\theta)) = e^{t\pi(f\partial_\theta)}, \tag{2.5}$$

since the derivative of $\pi(\exp(tf\partial_\theta))e^{-t\pi(f\partial_\theta)}$ is $\pi(\exp(tf\partial_\theta))(\pi(f\partial_\theta) - \pi(f\partial_\theta))e^{-t\pi(f\partial_\theta)} = 0$.

2.5

The Witt algebra $\mathrm{Span}_{\mathbb{C}} = \{l_n : n \in \mathbb{Z}\}$ is a complex dense Lie subalgebra of the complexification $\mathrm{Vec}(\mathbb{S}^1) \otimes_{\mathbb{R}} \mathbb{C}$. Here,

$$l_n = z^{n+1}\partial_z = -\mathbf{i}e^{\mathbf{i}n\theta}\partial_\theta \tag{2.6}$$

where $z = e^{\mathbf{i}\theta}$ and $\partial_z = \frac{1}{\mathbf{i}e^{\mathbf{i}\theta}}\partial_\theta$. (We use the chain rule to "define" ∂_z.) One checks

$$[l_m, l_n] = (m-n)l_{m+n} \tag{2.7}$$

where the bracket is the negative of the usual one for vector fields.

Let us assume for simplicity that the CFT is unitary. In the decomposition $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, each $\mathbb{W}_i$ is a projective unitary representation π_i of $\{l_n\}$, and similarly $\widehat{\mathbb{W}}_i$ is one $\widehat{\pi}_i$ of $\{l_n\}$. We know that the choice of $\pi_i(l_n)$ is unique up to $\mathbf{i}\mathbb{R}$-scalar addition. Here is a well-known fact about projective representations of Witt algebra (cf. for instance [Was10, Sec. IV.1]): one can make a particular choice of $\pi_i(l_n)$ (for each n), denoted by L_n, such that the **Virasoro relation**

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m+1)m(m-1)\delta_{m,-n} \tag{2.8}$$

holds and $c \in \mathbb{C}$ is called the **central charge**. In the case that π_i is projectively unitary, L_n can be chosen such that $L_n^\dagger = L_{-n}$ also holds.

We have abused the notation by writing the actions of l_n on all $\mathbb{V}$-modules $\mathbb{W}_i$ (as chiral halves of the CFT) as L_n. We are justified to do so because, as we will see later, the actions of l_n come from those of $\mathbb{V}$. Technically: Virasoro algebra is inside the VOA. So the action of $\{l_n\}$ on $\mathbb{W}_i$ is the restriction of that of $\mathbb{V}$. In particular, all chiral halves $\mathbb{W}_i$ share the same central charge c.

Similarly, we write the actions of l_n on all $\widehat{\mathbb{W}}_i$ as $\overline{L}_n$. (The bar over L_n reflects the fact that $\overline{L}_n$ describes the conformal symmetries of the anti-chiral halves of the CFT. $\overline{L}_n$ is not related with L_n by the CPT operator Θ.) The central charge $\widehat{c}$ for $\{\overline{L}_n\}$ is independent of $\widehat{\mathbb{W}}_i$ and in general could be different from the one c of $\{L_n\}$, although in most important cases they are equal. (E.g., when the CFT contains both closed and open strings.)

2.6

We shall generalize (2.5) to complex vector fields. First of all, we consider an element

$$f(z)\partial_z = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial_z$$

where the sum could be infinite. We treat $f(z) = \sum_n a_n z^{n+1}$ as a Laurent series. Let us now assume that *$f(z)$ is a holomorphic function on a neighborhood $U \subset \mathbb{C}$ of $\mathbb{S}^1$.*

$f\partial_z$ is a complex holomorphic vector field of U, which (after shrinking U) gives a **holomorphic flow** $\tau \in \Delta \mapsto \exp(\tau f\partial_z) \in \mathscr{O}(U)$ where $\Delta \subset \mathbb{C}$ is a neighborhood of 0. (Recall from the notation section that $\mathscr{O}(U)$ is the space of holomorphic functions on U.) This means:

(1) $(\tau, z) \in \Delta \times U \mapsto \exp(\tau f\partial_z)(z)$ is holomorphic whose restriction to each slice $\tau \times U$ is injective (and hence, a biholomorphism onto its image).

(2) $\exp(0f\partial_z)(z) = z$.

(3) $\exp((\tau_1 + \tau_2)f\partial_z) = \exp(\tau_1 f\partial_z) \circ \exp(\tau_2 f\partial_z)$ on an open subset of U containing $\mathbb{S}^1$.

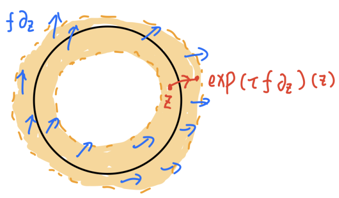
(4) For any holomorphic function h defined on an open set inside U,

$$f\partial_z h = \frac{\partial}{\partial\tau} h \circ \exp(\tau f\partial_z)\Big|_{\tau=0}. \tag{2.9}$$

(Compare (2.4).) This condition is equivalent to

$$\frac{\partial}{\partial\tau}\exp(\tau f\partial_z)\Big|_{\tau=0} = f. \tag{2.10}$$

(To see the equivalence, set $h(z) = z$ for one direction, and use chain rule for the other one.)



Remark 2.2. A caveat: The notations $f\partial_z$ and $\exp(\tau f\partial_z)$ are not compatible with those in the real case. Indeed, if we assume that τ only takes real values $\tau = t$, then by taking the real and the imaginary parts of (2.10), we see that σ_t is a real flow on the real surfaces U generated by the real vector field $\mathrm{Re} f \cdot \partial_x + \mathrm{Im} f \cdot \partial_y$. Writing $\partial_x = \partial_z + \partial_{\overline{z}}, \partial_y = \mathbf{i}(\partial_z - \partial_{\overline{z}})$, we see that this vector field $f\partial_z$ should more precisely be written as $f\partial_z + \overline{f}\partial_{\overline{z}}$ where $\overline{f}(x) = \overline{f(x)}$.

This point is also justified by the fact that if k is antiholomorphic, then

$$\overline{f}\partial_{\overline{z}} k = \frac{\partial}{\partial_{\overline{\tau}}} k \circ \exp(\tau f \partial_z)\Big|_{\tau=0}. \tag{2.11}$$

(Proof: take $k = \overline{h}$ in (2.10).) Thus, a more precise notation for $\exp(\tau f\partial_z)$ should be $\exp(\tau f\partial_z + \overline{\tau}\overline{f}\partial_{\overline{z}})$. But we prefer to suppress the term $\overline{\tau}\overline{f}\partial_{\overline{z}}$ to keep the notations shorter.

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