# NOTE ON DUISTERMAAT–HECKMAN MEASURES OF NON-ARCHIMEDEAN METRICS

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<span style="color:red">This is an informal note. Please contact me at mingchen@imj-prg.fr for comments.</span>

## CONTENTS

## 1. INTRODUCTION

In this note, we define the Duistermaat–Heckman measure of a non-Archimedean metric using the theory of partial Okounkov bodies developped in [Xia21; DX24]. The main result <span style="color:red">Theorem 4.3</span> states that the Duistermaat–Heckman measure is canonical in two important cases.

Please let me know if you can prove <span style="color:red">Theorem 4.3</span> unconditionally.

## 2. PRELIMINARIES

In this section, we recall the theory of Hausdorff metrics on the set of convex bodies following [Sch14, Section 1.8]. Fix $n \in \mathbb{N}$. Recall that a convex body in $\mathbb{R}^n$ is a non-empty compact convex subset of $\mathbb{R}^n$, which may have empty interior. Let $\mathcal{K}_n$ denote the set of convex bodies in $\mathbb{R}^n$. We will fix the Lebesgue measure $\mathrm{d}\lambda$ on $\mathbb{R}^n$, normalized so that the unit cube has volume 1.

Recall the definition of the Hausdorff metric between $K_1, K_2 \in \mathcal{K}_n$:

$$d_n(K_1, K_2) := \max \left\{ \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

We extend $d_n$ to an extended metric on $\mathcal{K}_n \cup \{\emptyset\}$ by setting

$$d_n(K, \emptyset) = \infty$$

for all $K \in \mathcal{K}_n$.

**Theorem 2.1.** *The metric space $(\mathcal{K}_n, d_n)$ is complete.*

**Theorem 2.2** (Blaschke selection theorem)**.** *Every bounded sequence in $\mathcal{K}_n$ has a convergent subsequence.*

**Theorem 2.3.** *The Lebesgue volume* $\mathrm{vol} : \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ *is continuous.*

**Theorem 2.4.** *Let $K_i, K \in \mathcal{K}_n$ ($i \in \mathbb{N}$). Then $K_i \xrightarrow{d_n} K$ if and only if the following conditions hold*

    *(1) Each point $x \in K$ is the limit of a sequence $x_i \in K_i$.*

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(2) *The limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ lies in $K$, where $i_j$ is a subsequence of $1, 2, \ldots$.*

The proofs of all these results can be found in [Sch14, Section 1.8].

**Lemma 2.5.** *Let $K_0, K_1 \in \mathcal{K}_n$. Assume that $K_0 \subseteq K_1$ and*

$$\operatorname{vol} K_0 = \operatorname{vol} K_1 > 0.$$

*Then $K_0 = K_1$.*

*Proof.* In fact, if $K_1 \neq K_0$, then $K_1 \setminus K_0$ is a non-empty open subset of $K_1$. As $\operatorname{vol} K_1 > 0$, $(K_1 \setminus K_0) \cap \operatorname{Int} K_1 \neq \emptyset$. Thus, $\operatorname{vol} K_1 > \operatorname{vol} K_0$, which is a contradiction. $\square$

## 3. OKOUNKOV TEST CURVES

Let $\Delta \in \mathcal{K}^n$. Assume that $V = n! \operatorname{vol} \Delta > 0$.

**Definition 3.1.** An *Okounkov test curve* relative to $\Delta$ is an assignment $(\Delta_\tau)_{\tau < \tau^+}$ $(\tau^+ \in \mathbb{R})$ such that

(1) $\Delta_\tau$ is a decreasing assignment of convex bodies in $\mathbb{R}^n$ for $\tau < \tau^+$;
(2) $\Delta_\tau$ converges to $\Delta$ as $\tau \to -\infty$ with respect to the Hausdorff metric;
(3) $\Delta_\tau$ is concave in the $\tau$ variable.

The energy of the Okounkov test curve is defined as

$$\mathbf{E}(\Delta_\bullet) := \tau^+ V + V \int_{-\infty}^{\tau^+} \left( \frac{n!}{V} \operatorname{vol} \Delta_\tau - 1 \right) \, d\tau \in [-\infty, \infty).$$

**Proposition 3.2.** *Any Okounkov test curve $(\Delta_\tau)_{\tau \leq \tau^+}$ relative to $\Delta$ is continuous for $\tau < \tau^+$.*

This is proved in [Xia21] for finite energy curves, but the proof works in general as well.

**Definition 3.3.** A *test function* on $\Delta$ is a function $F : \Delta \to [-\infty, \infty)$ such that

(1) $F$ is concave;
(2) $F$ is finite on $\operatorname{Int} \Delta$;
(3) $F$ is usc.

The energy of the test function is defined by

$$(3.1) \qquad \mathbf{E}(F) := n! \int_\Delta F \, d\lambda \in [-\infty, \infty).$$

Let $\tau^+ = \sup_\Delta F$, then

$$(3.2) \qquad \mathbf{E}(F) = \tau^+ V + V \int_{-\infty}^{\tau^+} \left( \frac{n!}{V} \operatorname{vol}\{F \geq \tau\} - 1 \right) \, d\tau.$$

Let $\Delta_\bullet$ be an Okounkov test curve relative to $\Delta$. We define the *Legendre transform* of $\Delta_\bullet$ as

$$G[\Delta_\bullet] : \Delta \to [-\infty, \infty), \quad a \mapsto \sup \left\{ \tau < \tau^+ : a \in \Delta_\tau \right\}.$$

Conversely, a test function $F$ on $\Delta$, set $\tau^+ = \sup_\Delta F$. We define the *inverse Legendre transform* of $F$ as

$$\Delta[F] : (-\infty, \tau^+] \to \mathcal{K}_n, \quad \Delta[F]_\tau = \{F \geq \tau\}.$$

**Theorem 3.4.** *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between the set of Okounkov test curves relative to $\Delta$ and test functions on $\Delta$. Given any Okounkov test curve $\Delta_\bullet$, we have*

$$\mathbf{E}(\Delta_\bullet) = \mathbf{E}(G[\Delta_\bullet]).$$

The proof is essentially contained in [Xia21].

**Definition 3.5.** Let $\Delta_\bullet$ be an Okounkov test curve relative to $\Delta$. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) := G[\Delta_\bullet]_*(\mathrm{d}\lambda).$$

It is a Radon measure on $\mathbb{R}$.

In other words, $\mathrm{DH}(\Delta_\bullet)$ is the probability distribution of the random variable $G[\Delta_\bullet]$ on the measure space $(\Delta, \mathrm{d}\lambda)$.

**Lemma 3.6.** *Suppose that $\Delta_\bullet^k$ is a decreasing sequence of finite energy Okounkov test curves relative to $\Delta$ with the same $\tau^+$. Assume that the pointwise Hausdorff limit $\Delta_\bullet$ is still a Okounkov test curve relative to $\Delta$ and has finite energy. Then $\mathrm{DH}(\Delta_\bullet^k) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ as $k \to \infty$.*

*Proof.* Observe that

$$G[\Delta_\bullet^k] \to G[\Delta_\bullet]$$

pointwisely as $k \to \infty$. It follows from the dominated convergence theorem that $\mathrm{DH}(\Delta_\bullet^k) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$ as $k \to \infty$. $\qquad\square$

Observe that

(3.3)
$$\int_{\mathbb{R}} \mathrm{DH}(\Delta_\bullet) = \mathrm{vol}\,\Delta.$$

More generally, we compute the characteristic function of $G[\Delta_\bullet]$ as follows: for any $t \in \mathbb{C}$,

(3.4)
$$\int_\Delta e^{itG[\Delta_\bullet]}\,\mathrm{d}\lambda = e^{it\tau^+}\,\mathrm{vol}\,\Delta - it \int_{-\infty}^{\tau^+} (\mathrm{vol}\,\Delta - \mathrm{vol}\,\Delta_\tau)\,e^{it\tau}\,\mathrm{d}\tau.$$

In particular, the moments are given by

$$\int_{\mathbb{R}} x^m \mathrm{DH}(\Delta_\bullet)(x) = \int_\Delta G[\Delta_\bullet]^m\,\mathrm{d}\lambda = (\tau^+)^m\,\mathrm{vol}\,\Delta - \int_{-\infty}^{\tau^+} m\tau^{m-1}(\mathrm{vol}\,\Delta - \mathrm{vol}\,\Delta_\tau)\,\mathrm{d}\tau.$$

## 4. The Duistermaat–Heckman measure of a non-Archimedean metric

Let $X$ be an connected compact Kähler manifold of dimension $n$ and $\theta$ be a closed real smooth $(1,1)$-form on $X$ such that $\mathrm{PSH}(X,\theta) \neq \varnothing$. We will define the Duistermaat–Heckman measure of elements in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as studied in [DXZ23; Xia23]. We will follow the notations in [Xia23].

4.1. **Non-Archimedean metrics.** Consider an element $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, recall that by definition, $\Gamma$ is an inverse system $(\Gamma^{\theta+\omega})_\omega$ indexed by the directed set of Kähler forms on $X$ ordered by reverse of the usual comparison. For each $\omega$,

$$\Gamma^{\theta+\omega}\colon (-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X, \theta+\omega)$$

is a decreasing concave curve of $\mathcal{I}$-model potentials. The number $\Gamma_{\max} \in \mathbb{R}$ is independent of the choice of $\omega$. The transition map from the index $\omega$ to $\omega + \omega'$ sends $\Gamma^{\theta+\omega}$ to the following map

$$(-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X, \theta+\omega+\omega'), \quad \tau \mapsto P_{\theta+\omega+\omega'}\left[\Gamma_\tau^{\theta+\omega}\right]_{\mathcal{I}}.$$

The volume of $\Gamma$ is defined as the limit

$$\lim_\omega \left(\theta + \omega + \mathrm{dd}^c \Gamma_{-\infty}^{\theta+\omega}\right)^n.$$

Here $\Gamma_{-\infty}^{\theta+\omega} = \sup^*_{\tau < \Gamma_{\max}} \Gamma_\tau^{\theta+\omega}$.

The subset $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ of $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ consisting of elements with positive volume can be identified with the set of concave curves of $\mathcal{I}$-model potentials $(\Gamma_\tau)_{\tau < \Gamma_{\max}}$ in $\mathrm{PSH}(X,\theta)$ for some $\Gamma_{\max} \in \mathbb{R}$ such that the volume $\int_X (\theta + \mathrm{dd}^c \Gamma_{-\infty})^n > 0$.

4.2. **The Duistermaat–Heckman measure.** We fix a smooth flag $Y_\bullet$ on $X$.

Now suppose that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. We define the Okounkov test curve $(\Delta_{Y_\bullet}(\Gamma)_\tau)_{\tau < \Gamma_{\max}}$ associated with $\Gamma$ as follows: given $\tau < \Gamma_{\max}$, we set

$$\Delta_{Y_\bullet}(\Gamma)_\tau := \Delta_{Y_\bullet}(\theta + \mathrm{dd}^c \Gamma_\tau).$$

The right-hand side is the partial Okounkov body studied in [DX24].

**Proposition 4.1.** *Given* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, *the curve* $(\Delta_{Y_\bullet}(\Gamma)_\tau)_{\tau < \Gamma_{\max}}$ *is an Okounkov test curve relative to* $\Delta_{Y_\bullet}(\theta + \mathrm{dd}^c \Gamma_{-\infty})$.

*Proof.* This is a simple consequence of the properties proved in [DX24]. $\qquad\square$

**Definition 4.2.** The *Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined as the Duistermaat–Heckman measure of the Okounkov test curve $\Delta_{Y_\bullet}(\Gamma)$.

The energy of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined as in [DXZ23]:

$$\mathbf{E}(\Gamma) := \tau^+ V + \int_{-\infty}^{\tau^+} \left( \int_X \theta_{\Gamma_\tau}^n - V \right) \mathrm{d}\tau \in [-\infty, \infty),$$

where $V$ denotes the volume of the cohomology class $\{\theta\}$. From the volume formula of partial Okounkov bodies established in [DX24], we find that

$$\mathbf{E}(\Gamma) = \mathbf{E}\left( \Delta_{Y_\bullet}(\Gamma) \right).$$

**Theorem 4.3.** *The Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ *of* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *is independent of the choice of the flag* $Y_\bullet$ *in the following two cases:*

*(1)* $\Gamma$ *has finite energy;*
*(2)* $X$ *is projective and the cohomology class of* $\theta$ *is the first Chern class of a big line bundle.*

Can one prove the same result in general?

*Proof.* Case 2 is proved using the same Boucksom–Chen type argument as in [Xia21].

In Case 1, assume further more that $\Gamma$ is bounded ($\Gamma_\tau = V_\theta$ for small enough $\tau$), we observe that the characteristic function of the random variable $G[\Delta_{Y_\bullet}(\Gamma)]$ as computed in (3.4) is independent of the choice of the flag and is entire. It is a classical result that in this case, the corresponding probability distribution is determined by the moments.

In general, $\Gamma$ is the decreasing limit of the sequence $\Gamma \vee \Gamma^k$ as $k \to \infty$, where $\Gamma^k \colon (-\infty, -k) \to \mathrm{PSH}(X,\theta)$ takes the constant value $V_\theta$. It follows from the general continuity result proved in [DX24] that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of $\Delta_{Y_\bullet}(\Gamma \vee \Gamma^k)_\tau$ for any $\tau < \Gamma_{\max}$. So $\mathrm{DH}(\Gamma \vee \Gamma^k) \rightharpoonup \mathrm{DH}(\Gamma)$ by Lemma 3.6. It follows that $\mathrm{DH}(\Gamma)$ is independent of the choice of the flag. $\qquad\square$

## References

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