Notes on Complex Analytic Geometry

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Chapter 1

Basic notions of complex spaces

1.1 Notations and conventions

The following notations and conventions are assumed throughout the monograph.

All commutative rings and algebras are assumed to have a unity 1. Their morphisms are assumed to map 1 to 1.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

$\mathbf{i} = \sqrt{-1}$.

$\{0\}, \mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \dots$ are called **(complex) number spaces**.

Unless otherwise stated, all vector spaces are over $\mathbb{C}$.

A **precompact subset** U of a topological space X is a subset such that the closure U^{cl} in X is compact.

$\mathbb{C}\{z_1, \dots, z_n\}$ denotes $\mathscr{O}_{\mathbb{C}^n,0}$, the algebra of convergent power series of $z_1, \dots, z_n$. It is clearly an integral domain. $\mathbb{C}[z_1, \dots, z_n]$ denotes the algebra of polynomials of $z_1, \dots, z_n$.

We assume the readers are familiar with the basic notions of sheaves and their maps (morphisms), sheafifications, image sheaves, kernels and cokernels of sheaves. For each presheaf $\mathscr{E}$ on a topological space X, we let $\mathscr{E}_x$ denote the stalk of $\mathscr{E}_x$ at x. If $\varphi : X \to Y$ is a continuous map of topological spaces, then the **direct image** $\varphi_* \mathscr{E}$ denotes the sheaf on Y whose space of sections over any open $V \subset Y$ is $\mathscr{E}(\varphi^{-1}(V))$, i.e.

$$(\varphi_* \mathscr{E})(V) = \mathscr{E}(\varphi^{-1}(V)).$$

If $\psi : Y \to Z$ is continuous, we clearly have

$$(\psi \circ \varphi)_* \mathscr{E} = \psi_*(\varphi_* \mathscr{E}).$$

If $f : \mathscr{E}_1 \to \mathscr{E}_2$ is an X-sheaf map, then we have a canonical $\varphi_* f : \varphi_* \mathscr{E}_1 \to \varphi_* \mathscr{E}_2$. φ_* is a **left exact functor** from the category of X-sheaves to that of Y-sheaves. (Cf. Rem. 1.9.6.)

If $\mathscr{F}$ is an $\mathscr{O}_Y$-module, the **inverse image** $\varphi^{-1}(\mathscr{F})$ is the sheafification of the presheaf on X associating to each open subsets of X:

$$U \mapsto \varinjlim_{V \supset \varphi(U)} \mathscr{F}(V)$$

where the direct limit is over all open subset $V \subset Y$ containing $\varphi(U)$. For each $x \in X$ there is a natural equivalence

$$(\varphi^{-1}\mathscr{F})_x \simeq \mathscr{F}_{\varphi(x)}. \tag{1.1.1}$$

$\mathscr{E}_U$, $\mathscr{E}|U$, $\mathscr{E}|_U$, $\mathscr{E} \upharpoonright_U$ all denote the restriction of an X-sheaf $\mathscr{E}$ to the open subset U. If Y is a subset of X and $\iota : Y \hookrightarrow X$ is the inclusion map, we define the **set theoretic restriction**

$$\mathscr{E} \upharpoonright_Y = \iota^{-1}(\mathscr{E}). \tag{1.1.2}$$

In particular, for each $y \in Y$, we have a canonical identification

$$(\mathscr{E} \upharpoonright_Y)_y = \mathscr{E}_y. \tag{1.1.3}$$

Warning: in the future, we will define the restriction $\mathscr{E}|_Y = \mathscr{E}|Y$ when Y is a complex subspace of a complex space X and $\mathscr{E}$ is an $\mathscr{O}_X$-module. $\mathscr{E}|Y$ will be different from $\mathscr{E} \upharpoonright_Y$. In particular, $(\mathscr{E}|_Y)_y$ is not $\mathscr{E}_y$.

We also write $\mathscr{E}(U)$ as $H^0(U, \mathscr{E})$.

Recall that the **support of an X-sheaf** $\mathscr{E}$, denoted by $\mathrm{Supp}(\mathscr{E})$, is the subset of all $x \in X$ such that $\mathscr{E}_x \neq 0$.

If U is an open subset of $\mathbb{C}^N$, then a **holomorphic function** f on U is, by definition, a continuous function $f : U \to \mathbb{C}$ which is separately holomorphic on each variable (i.e., if $z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_N$ are fixed, then $f(z_\bullet) = f(z_1, \dots, z_N)$ is holomorphic with respect to z_i).

Remark 1.1.1. The above definition agrees with our usual understanding of analytic functions, i.e., f has convergent power series expansions $f(z_\bullet) = \sum_{n_1,\dots,n_N \in \mathbb{N}} a_{n_1,\dots,n_N} (z_1 - w_1)^{n_1} \cdots (z_N - w_N)^{n_N}$ if $(w_\bullet) \in U$. To see this, choose a holomorphic f on U. Let us assume for simplicity $w_1 = \cdots = w_N = 0$, and U is the polydisc $\mathbb{D}_{R_\bullet} = \{(z_\bullet) \in \mathbb{C}^N : |z_1| < R_1, \dots, |z_N| < R_N\}$ where $R_1, \dots, R_N > 0$. Then for each $0 < r_i < R_i$ and $z_\bullet \in \mathbb{D}_{r_\bullet}$,

$$f(z_\bullet) = \oint_{|\zeta_1| = r_1} \cdots \oint_{|\zeta_N| = r_N} \frac{f(\zeta_\bullet)}{(\zeta_1 - z_1) \cdots (\zeta_N - z_N)} \cdot \frac{d\zeta_1 \cdots d\zeta_N}{(2\mathbf{i}\pi)^N}$$

by applying residue theorem successively to the variables $\zeta_1, \dots, \zeta_N$. Write each $(\zeta_i - z_i)^{-1}$ as $\sum_{n_i=0}^{\infty} z_i^{n_i} / \zeta_i^{n_i+1}$ which converges absolutely and uniformly on $|\zeta_i| = r_i$

and $z_\bullet$ on any compact subset of $\mathbb{D}_{r_\bullet}$, and substitute them into the above integral, we get the desired series expansion which is absolutely and uniformly convergent on $|z_1| \leqslant r_1, \ldots, |z_N| \leqslant r_N$ for all $0 < r_i < R_i$. This proves one direction. For the other direction, namely absolutely convergent power series give holomorphic functions, one simply applies Morera's theorem to each complex variable.

Lemma 1.1.2 (Identitätssatz). *If U is an open connected subset of $\mathbb{C}^n$, and if h is a non-zero (i.e. not constantly zero) holomorphic function on U, then h is non-zero when restricted to any open subset W of U.*

Proof. Consider the special case that U, W are open polydiscs. We know the lemma is true when $n = 1$ (by e.g. taking power series). For a general n, if $h|_W = 0$, we may enlarge successively the disc-shape domains of each variable $z_1, \ldots, z_n$ on which h is constantly zero to get $h = 0$.

In general, we let Ω be the (clearly open) subset of all $x \in U$ such that h is constantly zero on a neighborhood of x (i.e. the germ of h at x is zero). If $x \in U\backslash\Omega$, then every open polydisc in U containing x must be disjoint from Ω, according to the previous paragraph. So $U\backslash\Omega$ is open. Since U is connected, Ω must be either $\varnothing$ or U. Thus $\Omega = \varnothing$ since $h \neq 0$. □

1.2 $\mathbb{C}$-ringed spaces and sheaves of modules

1.2.1 $\mathbb{C}$-ringed spaces

Definition 1.2.1. A **$\mathbb{C}$-ringed space** is a topological space X together with a **sheaf of local $\mathbb{C}$-algebras** $\mathscr{O}_X$ on X (i.e., for each open $U \subset X$, $\mathscr{O}_X(U)$ is a $\mathbb{C}$-algebra with unity, and the additions and multiplications are compatible with the restriction to open subsets of U; each stalk $\mathscr{O}_{X,x}$ is a **local $\mathbb{C}$-algebra**).

By saying that $\mathscr{O}_{X,x}$ is a local $\mathbb{C}$-algebra, we mean that there is a unique maximal ideal $\mathfrak{m}_{X,x}$ of $\mathscr{O}_{X,x}$, and that we have an isomorphism of vector spaces

$$\mathbb{C} \xrightarrow{\simeq} \mathbb{C}_x := \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}, \qquad \lambda \mapsto \lambda 1.$$

We write $\mathfrak{m}_{X,x}$ as $\mathfrak{m}_x$ when no confusion arises. For each $f \in \mathscr{O}_{X,x}$, we let $f(x) \in \mathbb{C}$ denote the residue class of f in $\mathscr{O}_{X,x}/\mathfrak{m}_x$, called the **value** of f at x. In this way, any section of $\mathscr{O}_X$ can be viewed as a function.

$\mathscr{O}_X$ is called the **structure sheaf** of X. Each open subset $U \subset X$ is automatically a $\mathbb{C}$-ringed subspace of X with structure sheaf $\mathscr{O}_U := \mathscr{O}_X|_U$. □

For the sake of brevity, we write

$$\mathscr{O}(X) = \mathscr{O}_X(X) \tag{1.2.1}$$

The following important fact is obvious:

Proposition 1.2.2. *An element $f \in \mathscr{O}_{X,x}$ is a unit (i.e. invertible in the ring $\mathscr{O}_{X,x}$) iff $f(x) \neq 0$.*

Proof. $f(x) = 0$ iff $f \in \mathfrak{m}_{X,x}$ iff f is not a unit. □

Definition 1.2.3. A **morphism of $\mathbb{C}$-ringed spaces** $\varphi : X \to Y$ is a continuous map of topological spaces, together with a morphism of sheaves of $\mathbb{C}$-algebras $\varphi^\# : \mathscr{O}_Y \to \varphi_*\mathscr{O}_X$ (namely, $\varphi^\#$ is a sheaf map, and $\varphi^\# : \mathscr{O}_Y(V) \to \mathscr{O}_X(\varphi^{-1}(V))$ is a morphism of $\mathbb{C}$-algebras for each open $V \subset Y$), and for each $x \in X$ and $y = \varphi(x)$, the restriction $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a **morphism of local $\mathbb{C}$-algebras**, i.e. a morphism of $\mathbb{C}$-algebras such that

$$\varphi^\#(\mathfrak{m}_{Y,y}) \subset \mathfrak{m}_{X,x}. \tag{1.2.2}$$

The set of morphisms of $\mathbb{C}$-ringed spaces $X \to Y$ is denoted by $\mathrm{Mor}(X, Y)$. If $\varphi \in \mathrm{Mor}(X, Y)$ and $\psi \in \mathrm{Mor}(Y, Z)$, then their **composition** $\psi \circ \varphi \in \mathrm{Mor}(X, Z)$ is the usual composition of maps of sets, together with

$$(\psi \circ \varphi)^\# = \varphi^\# \circ \psi^\# : \mathscr{O}_{Z,\psi\circ\varphi(x)} \to \mathscr{O}_{X,x}$$

for all $x \in X$.

We leave it to the readers to define isomorphisms of $\mathbb{C}$-ringed spaces.

Proposition 1.2.4. *For each section $f \in \mathscr{O}_Y$ defined at $y = \varphi(x)$, we have*

$$(\varphi^\# f)(x) = f \circ \varphi(x). \tag{1.2.3}$$

Proof. This is true when $f = 1$ since $\varphi^\#$ preserves 1. It is also true when $f \in \mathfrak{m}_{Y,y}$. So it is true in general. □

Thus, $\varphi^\#$ may be viewed as the transpose of φ. When studying morphisms between complex spaces, we may write $\varphi^\# f$ as $f \circ \varphi$ (cf. Rem. 1.4.2).

Example 1.2.5. A complex manifold is a $\mathbb{C}$-ringed space if we define the structure sheaf $\mathscr{O}_X$ to be the sheaf of (germs of) holomorphic functions. If X and Y are complex manifolds, then a holomorphic map from X to Y is a morphism of $\mathbb{C}$-ringed spaces.

1.2.2 Modules over $\mathbb{C}$-ringed spaces

We begin this section with the following general observation:

Remark 1.2.6. If $\mathscr{M}, \mathscr{N}$ are two subsheaves of an X-sheaf such that $\mathscr{M}_x = \mathscr{N}_x$ for all $x \in X$, then $\mathscr{M} = \mathscr{N}$. (For any $s \in \mathscr{M}$, $s_x \in \mathscr{M}_x = \mathscr{N}_x$ for all x on which s is defined. So $s \in \mathscr{N}$. So $\mathscr{M} \subset \mathscr{N}$, and vice versa.) Thus, we can talk about" the *unique* subsheaf of a given sheaf whose stalks are..." where the unique part is automatic.

Definition 1.2.7. A **presheaf of $\mathscr{O}_X$-modules** $\mathscr{E}$ on a $\mathbb{C}$-ringed space X is a sheaf such that for each open $U \subset X$, $\mathscr{E}(U)$ is an $\mathscr{O}(U)$-module, and that the linear combination and the action of $\mathscr{O}(U)$ on $\mathscr{E}(U)$ are compatible with the restriction to open subsets of U. If $\mathscr{E}$ is a sheaf, we call $\mathscr{E}$ an $\mathscr{O}_X$**-module**. We call the vector space

$$\mathscr{E}|x = \mathscr{E}_x/\mathfrak{m}_{X,x}\mathscr{E}_x = \mathscr{E}_x \otimes (\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}) \tag{1.2.4}$$

the **fiber** of $\mathscr{E}$ at x. The right most expression of (1.2.4) will be explained in Rem. 1.9.3. The residue class of $s \in \mathscr{E}$ in $\mathscr{E}|x$ is denoted by $s(x)$ or $s|x$.

Definition 1.2.8. A **morphism of (presheaves of) $\mathscr{O}_X$-modules** $\varphi : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are (presheaves of) $\mathscr{O}_X$-modules, is a sheaf map intertwining the actions of $\mathscr{O}_X$. More precisely, for each open $U \subset X$, $\varphi : s \in \mathscr{E}(U) \mapsto \varphi(s) \in \mathscr{F}(U)$ is a morphism of $\mathscr{O}(U)$-modules; if $V \subset U$ is open, then $\varphi(s|_U) = \varphi(s)|_U$.

φ is called **injective** resp. **surjective** if it is so as a sheaf map, namely $\varphi : \mathscr{E}_x \to \mathscr{F}_x$ is injective resp surjective for all $x \in X$. $\mathscr{E} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{G}$ is called **exact** if the corresponding sequence of stalk map $\mathscr{E}_x \xrightarrow{\varphi} \mathscr{F}_x \xrightarrow{\psi} \mathscr{G}_x$ is exact for all $x \in X$. φ is an **isomorphism** of $\mathscr{O}_X$-modules iff φ has an inverse iff φ is both injective and surjective. □

Remark 1.2.9. In the following diagrams, assume that all objects are $\mathscr{O}_X$-modules, that all horizontal arrows are morphisms of $\mathscr{O}_X$-modules, and that the two horizontal lines are exact.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{F} \arrow[r] \arrow[d, "\beta"'] & \mathscr{G} \arrow[d, "\gamma"'] \\ 0 \arrow[r] & \mathscr{E}' \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{G}' \end{tikzcd} \tag{1.2.5}$$

If there are morphisms β, γ such that the second square in (1.2.5) commutes, then β restricts to a (necessarily unique) morphism α such that the first square commutes.

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{F} \arrow[r] \arrow[d, "\beta"'] & \mathscr{G} \arrow[r] \arrow[d, "\gamma"'] & 0 \\ \mathscr{E}' \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{G}' \arrow[r] & 0 \end{tikzcd} \tag{1.2.6}$$

If there are morphisms α, β such that the first square in (1.2.6) commutes, then β descends to a (necessarily unique) morphism γ such that the second square commutes.

Of course, the same observations hold for morphisms of modules of any commutative ring/algebra, and for general sheaf maps. □

Remark 1.2.10 (Gluing construction of sheaves). Let $(V_\alpha)_{\alpha\in\mathfrak{A}}$ be an open cover of a topological space X. Suppose that for each $\alpha\in\mathfrak{A}$, we have a sheaf $\mathscr{E}^\alpha$, that for any $\alpha,\beta\in\mathfrak{A}$, we have a sheaf isomorphism $\phi_{\beta,\alpha}:\mathscr{E}^\alpha_{V_\alpha\cap V_\beta}\xrightarrow{\simeq}\mathscr{E}^\beta_{V_\alpha\cap V_\beta}$, that $\phi_{\alpha,\alpha}=\mathbf{1}$, and that $\phi_{\gamma,\alpha}=\phi_{\gamma,\beta}\phi_{\beta,\alpha}$ when restricted to $V_\alpha\cap V_\beta\cap V_\gamma$. Then we can define a sheaf $\mathscr{E}$ on X as follows. For any open $U\subset X$, $\mathscr{E}(U)$ is the set of all $(s_\alpha)_{\alpha\in\mathfrak{A}}\in\prod_{\alpha\in\mathfrak{A}}\mathscr{E}^\alpha(U\cap V_\alpha)$ (where each component s_α is in $\mathscr{E}^\alpha(U\cap V_\alpha)$) satisfying that $s_\beta|_{V_\alpha\cap V_\beta}=\phi_{\beta,\alpha}(s_\alpha|_{V_\alpha\cap V_\beta})$ for any $\alpha,\beta\in\mathfrak{A}$. If W is an open subset of U, then the restriction $\mathscr{E}(U)\to\mathscr{E}(W)$ is defined by that of $\mathscr{E}^\alpha(U\cap V_\alpha)\to\mathscr{E}^\alpha(W\cap V_\alpha)$. Then for each $\beta\in\mathfrak{A}$, we have a canonical isomorphism (trivialization) $\phi_\beta:\mathscr{E}_{V_\beta}\xrightarrow{\simeq}\mathscr{E}^\beta_{V_\beta}$ defined by $(s_\alpha)_{\alpha\in\mathfrak{A}}\mapsto s_\beta$. It is clear that for each $\alpha,\beta\in\mathfrak{A}$, we have $\phi_\beta=\phi_{\beta,\alpha}\phi_\alpha$ when restricted to $V_\alpha\cap V_\beta$.

In the case that X is a $\mathbb{C}$-ringed space, that each $\mathscr{E}^\alpha$ is an $\mathscr{O}_{V_\alpha}$-module, and that $\phi_{\beta,\alpha}$ is an equivalence of $\mathscr{O}_{V_\alpha\cap V_\beta}$-modules, then $\mathscr{E}$ is a sheaf of $\mathscr{O}_X$-modules. $\square$

Let X be a $\mathbb{C}$-ringed space.

Definition 1.2.11. A set of sections $\mathfrak{S}\subset\mathscr{O}_X(X)$ is said to **generate** the $\mathscr{O}_X$-module $\mathscr{E}$ if they generate each stalk $\mathscr{E}_x$, i.e., each element of $\mathscr{E}_x$ is an $\mathscr{O}_{X,x}$-linear combination of finitely many elements of $\mathfrak{S}$. Equivalently, this means that the $\mathscr{O}_X$-module morphism

$$\bigoplus_{s\in\mathfrak{S}}\mathscr{O}_X\to\mathscr{E},\qquad \oplus_s f_s\mapsto\sum_s f_s\cdot s \tag{1.2.7}$$

(where $f_s\in\mathscr{O}_X$) is surjective. If it is also injective, we say $\mathfrak{S}$ **generates freely** $\mathscr{E}$.

Definition 1.2.12. We say an $\mathscr{O}_X$-module $\mathscr{E}$ is of **finite type** if each $x\in X$ is contained in a neighborhood U such that the restriction $\mathscr{E}|_U$ is generated by finitely many elements of $\mathscr{E}(U)$, or equivalently, there is a surjective $\mathscr{O}_U$-module morphism $\mathscr{O}_U^n\to\mathscr{E}|_U$.

Exercise 1.2.13. Show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, then each stalk $\mathscr{E}_x$ is a finitely generated $\mathscr{O}_{X,x}$-module, and hence each fiber $\mathscr{E}|x$ is finite-dimensional.

Definition 1.2.14. If $\mathscr{E}_1,\mathscr{E}_2$ are $\mathscr{O}_X$-submodules of an $\mathscr{O}_X$-module $\mathscr{F}$. The sheafification of the presheaf

$$(\mathscr{E}_1+\mathscr{E}_2)^{\mathrm{pre}}(U)=\mathscr{E}_1(U)+\mathscr{E}_2(U) \tag{1.2.8}$$

is denoted by $\mathscr{E}_1+\mathscr{E}_2$. It is the unique subsheaf of $\mathscr{F}$ (cf. Rem. 1.2.6) whose stalks are $(\mathscr{E}_1+\mathscr{E}_2)_x=\mathscr{E}_1+\mathscr{E}_2$. It follows that if $\mathscr{E}_1$ is generated by $s_1,s_2,\dots\in\mathscr{E}_1(X)$ and $\mathscr{E}_2$ is generated by $t_1,t_2,\dots\in\mathscr{E}_2(X)$, then $\mathscr{E}_1+\mathscr{E}_2$ is generated by $s_1,s_2,\dots,t_1,t_2,\dots$.

We recall the well-known

Theorem 1.2.15 (Nakayama's lemma). *If A is a $\mathbb{C}$-local algebra with maximal ideal $\mathfrak{m}$, and if $\mathcal{M}$ is a finitely generated A-module. Then a finite set of elements $s_1, \dots, s_n \in \mathcal{M}$ generate the A-module $\mathcal{M}$ (i.e. elements of $\mathcal{M}$ are A-linear combinations of $s_1, \dots, s_n$) iff their residue classes in $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$ span the vector space $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$.*

Indeed, this is true when A is in general a local ring. In that case, $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$ is a vector space over the field $A/\mathfrak{m}$.

Proof. [AM, Prop. 2.8]. □

To apply Nakayama's lemma to sheaves of modules, we need the following observation:

Remark 1.2.16. Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Let $s_1, \dots, s_n$ be sections of $\mathscr{E}$ defined on a neighborhood of $x \in X$. Suppose (the germs of) $s_1, \dots, s_n$ generate the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$. Then there is a neighborhood U of x such that $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. In particular, "$\mathscr{E}_x$ generates $\mathscr{E}|_U$".

Proof. Since $\mathscr{E}$ is finite-type, we may find U such that $\mathscr{E}|_U$ is generated by $t_1, \dots, t_m \in \mathscr{E}(U)$. Since $s_1, \dots, s_n$ generate $\mathscr{O}_x$, the germs of $t_1, \dots, t_m$ are $\mathscr{O}_{X,x}$-linear combinations of $s_1, \dots, s_n$. Thus, on a possibly smaller U, $t_1, \dots, t_m$ are $\mathscr{O}_X(U)$-linear combinations of $s_1, \dots, s_n$. So $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. □

Corollary 1.2.17. *Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then* $\mathrm{Supp}(\mathscr{E})$ *is a closed subset of* X.

Proof. Assume the setting of Rem. 1.2.16. If $\mathscr{E}_x = 0$ then the stalks of $s_1, \dots, s_n$ are zero at x. So we may shrink U so that $s_1 = \cdots = s_n = 0$ in $\mathscr{E}(U)$. So $\mathscr{E}|_U = 0$. □

Exercise 1.2.18. Use Nakayama's lemma and Rem. 1.2.16 to show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, and if $s_1, \dots, s_n \in \mathscr{E}(U)$ (where U is a neighborhood of x) are such that $s_1(x), \dots, s_n(x)$ span the fiber $\mathscr{E}|x$, then they generate $\mathscr{E}|_V$ for a possibly smaller neighborhood V of x. (The opposite direction is obvious.) Nakayama's lemma is most often used in this form.

Corollary 1.2.19. *Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then the **rank function** $x \in X \mapsto \dim(\mathscr{E}|x)$ is upper-semicontinuous.*

Definition 1.2.20. We say that an $\mathscr{O}_X$-module $\mathscr{E}$ is **free** if it is isomorphic to $\mathscr{O}_X^n$ for some $n \in \mathbb{N}$. We say $\mathscr{E}$ is **locally free** if each $x \in X$ is contained in a neighborhood U such that $\mathscr{E}|_U$ is free (or equivalently, that $\mathscr{E}|_U$ is generated freely by finitely many elements of $\mathscr{E}(U)$).

Exercise 1.2.21. Show that for a complex manifold X, locally free $\mathscr{O}_X$-modules $\mathscr{E}$ are the same as holomorphic vector bundles on X. Describe local trivializations and transition functions in terms of local free generators of $\mathscr{E}$.

Definition 1.2.22. An **ideal sheaf** $\mathcal{I}$ on a $\mathbb{C}$-ringed space X is an $\mathscr{O}_X$-submodule of $\mathscr{O}_X$. In particular, each stalk $\mathcal{I}_x$ is an ideal of $\mathscr{O}_{X,x}$. The **zero set** $N(\mathcal{I})$ is defined to be

$$\begin{aligned} N(\mathcal{I}) &:= \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}_x\} = \{x \in X : \mathcal{I}_x \subset \mathfrak{m}_{X,x}\} \\ &= \{x \in X : \mathcal{I}_x \neq \mathscr{O}_{X,x}\} = \mathrm{Supp}(\mathscr{O}_U/\mathcal{I}). \end{aligned} \tag{1.2.9}$$

Note that this is a closed subset of X by Cor. 1.2.17.

Proof. Note that $(\mathscr{O}_U/\mathcal{I})_x = \mathscr{O}_{U,x}/\mathcal{I}_x$. So $x \in \mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ iff $\mathscr{O}_{U,x}/\mathcal{I}_x \neq 0$ iff $\mathcal{I}_x \subsetneq \mathscr{O}_{U,x}$ iff $\mathcal{I}_x \subset \mathfrak{m}_x$ (as $\mathfrak{m}_x$ is the unique maximal ideal) iff $f(x) = 0$ for all $f \in \mathfrak{m}_x$. $\square$

Remark 1.2.23. If $\mathcal{I}$ is generated by $f_1, \dots, f_n \in \mathscr{O}(X)$, written as

$$\mathcal{I} = f_1\mathscr{O}_X + \cdots + f_n\mathscr{O}_X,$$

then clearly

$$N(\mathcal{I}) = \{\text{The common zeros of } f_1, \dots, f_n\}. \tag{1.2.10}$$

We also write $N(\mathcal{I})$ as $N(f_1, \dots, f_n)$.

1.3 Complex spaces and subspaces

Definition 1.3.1. A **(complex) model space** is

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})}\big) \tag{1.3.1}$$

where U is an open subset of a number space $\mathbb{C}^n$, $\mathscr{O}_U$ is the sheaf of holomorphic functions on U, $\mathcal{I}$ is a *finite-type* ideal of $\mathscr{O}_U$. $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is called the **analytic spectrum** of the sheaf $\mathscr{O}_U/\mathcal{I}$. Its underlying topological space is $\mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ as a subset of U, and its structure sheaf is $(\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})}$, whose stalk at any $x \in N(\mathcal{I})$ is $\mathscr{O}_{U,x}/\mathcal{I}_x$ (cf. (1.1.3)). With abuse of notations, one also writes for simplicity

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), \mathscr{O}_U/\mathcal{I}\big). \tag{1.3.2}$$

The stalk at $x \in N(\mathcal{I})$ of the structure sheaf is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{U,x}/\mathcal{I}_x, \mathfrak{m}_{U,x}/\mathcal{I}_x\big)$$

Definition 1.3.2. A $\mathbb{C}$-ringed Hausdorff space X is called a **complex space** if each point $x \in X$ is contained in a neighborhood V such that the $\mathbb{C}$-ringed space V (whose structure sheaf is defined by $\mathscr{O}_V := \mathscr{O}_X|_V$) is isomorphic to a model space.

Sections of $\mathscr{O}_X(X)$ are called **holomorphic functions on** X. $\mathscr{O}_{X,x}$ is called an **analytic local $\mathbb{C}$-algebra**. Equivalently, an analytic local $\mathbb{C}$-algebra is $\mathbb{C}\{z_1,\dots,z_n\}/I$ for some finitely generated ideal I.[1]

If X,Y are complex spaces, a morphism $\varphi : X \to Y$ of $\mathbb{C}$-ringed spaces is called a **holomorphic map**. If φ has an inverse morphism $Y \to X$, we say that φ is a **biholomorphism**. Clearly, a holomorphic map φ is a biholomorphism iff it is a homeomorphism of topological spaces and induces isomorphisms $\varphi^\# : \mathscr{O}_{Y,\varphi(x)} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ for each $x \in X$. □

Definition 1.3.3. A **morphism of (analytic) local $\mathbb{C}$-algebras** $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a homomorphism of unital algebras sending $\mathfrak{m}_{Y,y}$ into $\mathfrak{m}_{X,x}$.

Definition 1.3.4. Let X be a complex space. An **open complex subspace** is $(U, \mathscr{O}_X|_U)$ where U is an open subset of X. A **closed complex subspace** is

$$\mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_X/\mathcal{I}) \upharpoonright_{N(\mathcal{I})}\big) \tag{1.3.3}$$

where $\mathcal{I}$ is a finite type ideal of $\mathscr{O}_X$. The stalk of the structure sheaf at $x \in N(\mathcal{I})$ is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{X,x}/\mathcal{I}_x, \mathfrak{m}_x/\mathcal{I}_x\big).$$

Remark 1.3.5. Let $X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. The construction of $\mathscr{O}_{X_0} = (\mathscr{O}_X/\mathcal{I}) \upharpoonright_{N(\mathcal{I})}$ involves two sheafifications: one for quotient, and one for set-theoretic restriction. It would be convenient to combine these two into one: $\mathscr{O}_{X_0}$ is the sheafification of the presheaf $\mathscr{O}_{X_0}^{\mathrm{pre}}$ sending each open $U_0 \subset X_0$ (more precisely, $U_0 \subset N(\mathcal{I})$) to

$$\mathscr{O}_{X_0}^{\mathrm{pre}}(U_0) = \varinjlim_{U \supset U_0} \mathscr{O}_X(U)/\mathcal{I}(U) \tag{1.3.4}$$

where the direct limit is over all open $U \subset X$ containing U_0. Indeed, one can also take the direct limit over all open U satisfying $U \cap N(\mathcal{I}) = U_0$.

Remark 1.3.6. We have an obvious inclusion map which is holomorphic:

$$\iota : X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$$

such that $\iota^\# : \mathscr{O}_X \to \iota_*\mathscr{O}_{X_0} = \iota_*\iota^{-1}(\mathscr{O}_X/\mathcal{I})$ restricts to the quotient maps $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathcal{I}_x = (\iota_*\iota^{-1}(\mathscr{O}_X/\mathcal{I}))_x$ for all $x \in X$.

Proof. We explain the existence of such sheaf map $\iota^\#$. Choose any open $U \subset X$. Then by passing to direct limits (1.3.4), the quotient map $\mathscr{O}_X(U) \to \mathscr{O}_X(U)/\mathcal{I}(U)$ becomes a map $\mathscr{O}_X(U) \to \mathscr{O}_{X_0}^{\mathrm{pre}}(U \cap N(\mathcal{I}))$ whose composition with $\mathscr{O}_{X_0}^{\mathrm{pre}} \to \mathscr{O}_{X_0}$ gives $\mathscr{O}_X(U) \to \mathscr{O}_{X_0}(U \cap N(\mathcal{I})) = (\iota_*\mathscr{O}_{X_0})(U)$. □

[1] As we shall see, $\mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian. So the condition that I is finitely generated is redundant.

Complex spaces arise from

Remark 1.3.7 (Gluing construction of complex spaces). Suppose X is a Hausdorff space with an open cover $\mathfrak{V} = (V_\alpha)$. Suppose that for each V_α there is a homoemorphism $\varphi_\alpha : V_\alpha \to U_\alpha$ where U_α is a complex space. Suppose also that for each α, β, the homeomorphism $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$ (where the source and the target are regarded as open subspaces of U_α and U_β respectively) can be extended to an isomorphism $\varphi_{\beta,\alpha}$ of $\mathbb{C}$-ringed spaces satisfying the **cocycle condition**: for all α, β, γ, we have $\varphi_{\alpha,\alpha} = \mathbf{1}$ and $\varphi_{\gamma,\alpha} = \varphi_{\gamma,\beta}\varphi_{\beta,\alpha}$ (from $\varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$ to $\varphi_\gamma(V_\alpha \cap V_\beta \cap V_\gamma)$). Then X is naturally a complex space such that $\varphi_\alpha : V_\alpha \to U_\alpha$ is extended to an isomorphism of $\mathbb{C}$-ringed spaces such that $\varphi_\beta = \varphi_{\beta,\alpha}\varphi_\alpha$ (from $V_\alpha \cap V_\beta$ to $\varphi_\beta(V_\alpha \cap V_\beta)$). Indeed, $\mathscr{O}_X$ is constructed by gluing all the V_α-sheaves $\varphi_\alpha^{-1}\mathscr{O}_{U_\alpha}$ (cf. Rem. 1.2.10).

Let us see some examples of complex spaces. We begin with an easier class of examples:

Definition 1.3.8. Let X be a complex space, and let $\mathscr{C}_X$ be the sheaf of complex valued continuous functions on X. Then there is a natural **morphism of sheaves of local $\mathbb{C}$-algebras** (i.e. a morphism of X-sheaves which preserve the linear structures and algebra multiplications when restricted to each open subset, and whose stalk maps send the maximal ideals into maximal ones)

$$\mathrm{red} : \mathscr{O}_X \to \mathscr{C}_X \tag{1.3.5}$$

sending each $f \in \mathscr{O}_X$ to f as a function (cf. Def. 1.2.1). red is called the **reduction map** of X. If $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective, we say that X is **reduced at** $x \in X$, or equivalently that x is a **reduced point** of X. If X is reduced everywhere, X is called a **reduced complex space**.

Thus, a holomorphic function on a reduced complex space can be viewed as a genuine continuous function without losing information. (Formally speaking: $\mathscr{O}_X$ is a subsheaf of $\mathscr{C}_X$.) For non-reduced spaces, holomorphic functions cannot be viewed as genuine functions.

Remark 1.3.9. In commutative algebra, there is a notion of reducedness: $\mathscr{O}_{X,x}$ is called reduced if it has no non-zero nilpotent element. We will see later that a complex space X is reduced at x iff $\mathscr{O}_{X,x}$ is a reduced ring. This is the famous Nullstellensatz.

Example 1.3.10. Let $U \subset \mathbb{C}^m \times \mathbb{C}^n$ be open, and let $\mathcal{I} = z_1\mathscr{O}_U + \cdots + z_m\mathscr{O}_U$. Then $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally equivalent to the complex submanifold $U \cap (0 \times \mathbb{C}^n) \simeq U \cap \mathbb{C}^n$ (whose structure sheaf is the sheaf of holomorphic functions $f(\zeta_1, \ldots, \zeta_n)$).

Proof. Clearly $N(\mathcal{I}) = U \cap \mathbb{C}^n$ (cf. Rem. 1.2.23). Consider the identity map $\varphi : U \cap \mathbb{C}^n \to X$ as a homeomorphism of topological spaces. In particular, we have an isomorphism $\mathrm{red}\varphi^\# : \mathscr{C}_X \to \mathscr{C}_{U\cap\mathbb{C}^n}$. We shall construct $\varphi^\# : \mathscr{O}_X = \mathscr{O}_U/\mathcal{I} \restriction_{N(\mathcal{I})} \to \mathscr{O}_{U\cap\mathbb{C}^n}$ such that φ is an isomorphism of $\mathbb{C}$-ringed spaces.

By (1.1.3), for each $x \in U \cap \mathbb{C}^n$,

$$\mathscr{O}_{X,x} = ((\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})})_x \simeq \mathscr{O}_{\mathbb{C}^{m+n},x}/\mathcal{I}_x \simeq \mathscr{O}_{\mathbb{C}^n,x}$$

where the last isomorphism can be seen by taking power series expansions of $f(z_\bullet, \zeta_\bullet) = f(z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_n)$ at n and throwing away every terms containing powers of $\zeta_\bullet$. Define a sheaf map

$$\varphi^\# : \mathscr{O}_X \xrightarrow{\mathrm{red}} \mathscr{C}_X \xrightarrow[\simeq]{\mathrm{red}\varphi^\#} \mathscr{C}_{U\cap\mathbb{C}^n}.$$

Its stalk map is $\mathscr{O}_{\mathbb{C}^n,x} \to \mathscr{C}_{U\cap\mathbb{C}^n,x}$ sending each f to the function f itself. From this we see that the stalk map is injective and has image $\mathscr{O}_{U\cap\mathbb{C}^n,x}$. This shows that $\varphi^\#$ is an injective sheaf map with image $\mathscr{O}_{U\cap\mathbb{C}^n}$. So $\varphi^\#$ restricts to an isomorphism of sheaves of local $\mathbb{C}$-algebras $\mathscr{O}_X \to \mathscr{O}_{U\cap\mathbb{C}^n}$. □

Remark 1.3.11. The proof of Exp. 1.3.10 suggests a way of understanding a *reduced* model space $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$: 1. Find the underlying topological space $N(\mathcal{I})$. 2. Understand each stalk $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}/\mathcal{I}_x$ and show that $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective. 3. Find a familiar sheaf of local $\mathbb{C}$-subalgebras $\mathscr{A} \subset \mathscr{C}_X$ such that $\mathscr{A}_x = \mathrm{red}(\mathscr{O}_{X,x})$. Then $X \simeq (N(\mathcal{I}), \mathscr{A})$.

Exercise 1.3.12. Let U be a neighborhood of $0 \in \mathbb{C}^2$. Let z, w be the standard coordinates of $\mathbb{C}^2$. Let $\mathcal{I} = zw \cdot \mathscr{O}_U$, the ideal sheaf generated by the function zw. Show that $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is equivalent to the $\mathbb{C}$-ringed space whose underlying topological space is $N(\mathcal{I}) = \{(z,w) \in U : z = 0 \text{ or } w = 0\}$, and whose structure sheaf is the sheaf of continuous functions on open subsets of $N(\mathcal{I})$ that are holomorphic when restricted respectively to the z-axis and to the w-axis.

Example 1.3.13. Let $k \in \mathbb{Z}_+$. Let U be a neighborhood of $0 \in \mathbb{C}$. We call $\mathrm{Specan}(\mathscr{O}_U/z^k\mathscr{O}_U) = (0, \mathbb{C}\{z\}/z^k\mathbb{C}\{z\}) = (0, \mathbb{C}[z]/z^k\mathbb{C}[z])$ the ***k*-fold point**. It is not reduced when $k > 1$. A single reduced point is precisely a 1-fold point, which is the same as the connected 0-dimensional complex manifold $\mathbb{C}^0$.

We close this section by discussing a useful relationship between local-freeness and rank functions. A locally-free sheaf clearly has locally constant rank. The converse holds under some conditions which are often easy to verify:

Proposition 1.3.14. *Let X be a* ***reduced*** *complex space, and let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Then $\mathscr{E}$ is locally free if and only if the rank function $\mathbf{R} : x \in X \mapsto \dim(\mathscr{E}|x)$ is locally constant. Moreover, if $\mathbf{R}$ has constant value n, and if $s_1, \ldots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$, then $s_1, \ldots, s_n$ generate $\mathscr{E}$ freely.*

Proof. Suppose $\mathbf{R}$ has constant value n and $s_1, \ldots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$. Then for each open $U \subset X$ and $f_1, \ldots, f_n \in \mathscr{O}(U)$ satisfying $f_1 s_1 + \cdots + f_n s_n = 0$, we have for each $x \in U$ that $f_1(x)s_1(x) + \cdots + f_n(x)s_n(x) = 0$ where $s_i(x)$ is the restriction of s_i to the fiber $\mathscr{E}|x$. Clearly $s_1(x), \ldots, s_n(x)$ span $\mathscr{E}|x$. Since $\dim(\mathscr{E}|x) = n$, $s_1(x), \ldots, s_n(x)$ form a basis of $\mathscr{E}|x$. So $f_1(x) = \cdots = f_n(x) = 0$. As holomorphic functions on a reduced space are determined by their values, we have $f_1 = \cdots = f_n = 0$. This proves that $s_1, \ldots, s_n$ are $\mathscr{O}_X$-free.

Assume in general that $\mathscr{E}$ is finite-type and $\mathbf{R}$ is locally constant. By shrinking X to a neighborhood of $x \in X$ we may assume $\mathbf{R}$ has constant value n. Choose $s_1, \ldots, s_n \in \mathscr{E}_x$ whose values at x form a basis of $\mathscr{E}|x$. By Nakayam's lemma (Exe. 1.2.18), we may shrink X so that $s_1, \ldots, s_n \in \mathscr{E}(X)$ generate $\mathscr{E}$. So by the first paragraph, $\mathscr{E}$ is locally-free. □

1.4 Holomorphic maps

In order to construct complex spaces by gluing model spaces (Rem. 1.3.7), and to understand holomorphic maps between complex spaces, we need to understand morphisms (i.e. holomorphic maps) between model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) \to \mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ (where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open). This is a main goal of this section.

The first step is to understand the case that target is just V. As one may expect, holomorphic maps in this case are described by an n-tuple of holomorphic functions. Recall that $\mathrm{Mor}(X, Y)$ is the set of holomorphic maps from the complex space X to Y. Let $z_1, \ldots, z_n$ be the standard coordinates of $\mathbb{C}^n$.

Theorem 1.4.1. *Let X be a complex space. Then the following map is bijective:*

$$\mathrm{Mor}(X, \mathbb{C}^n) \to \mathscr{O}(X)^n, \qquad \varphi \mapsto (\varphi^\# z_1, \ldots, \varphi^\# z_n). \tag{1.4.1}$$

Remark 1.4.2. Due to this theorem, if $\psi : X \to Y$ is a holomorphic map and $f \in \mathscr{O}(Y)$, then we may write

$$f \circ \psi = \psi^\# f \tag{1.4.2}$$

by viewing f as a holomorphic map $Y \to \mathbb{C}$.

The proof of Thm. 1.4.1 relies on the Noetherian property of $\mathscr{O}_{X,x}$, whose proof is deferred to the next section.

Proof that (1.4.1) is surjective assuming (1.4.1) is injective. Assume (1.4.1) is injective for all complex spaces. Fix X and $F = (f_1, \ldots, f_n) \in \mathscr{O}(X)^n$. We claim that each $x \in X$ is contained in a neighborhood U_x such that $F|_{U_x} \in \mathscr{O}(U_x)^n$ corresponds to some $\varphi_x \in \mathrm{Mor}(U_x, \mathbb{C}^n)$. By the injectivity, for every $x, y \in X$, φ_x and φ_y agree on $U_x \cap U_y$. Gluing all φ_x together gives the desired φ corresponding to F.

To prove the claim, we may assume U_x is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathcal{I})$ where $V \subset \mathbb{C}^m$ is open and $\mathcal{I}$ is finite-type. Since the stalk $(\mathscr{O}_V/\mathcal{I})|_x$ equals $\mathscr{O}_{V,x}/\mathcal{I}_x$, we can further shrink U_x so that $F|_{U_x}$ can be lifted to $\widetilde{F}|_V \in \mathscr{O}(V)^n$. $\widetilde{F}$ can be viewed as a holomorphic map $V \to \mathbb{C}^n$. Its composition with the inclusion $\iota : \mathrm{Specan}(\mathscr{O}_V/\mathcal{I}) \hookrightarrow V$ gives the desired holomorphic map φ. □

Proof that (1.4.1) *is injective.* Let $\varphi_1, \varphi_2 \in \mathrm{Mor}(X, \mathbb{C}^n)$ correspond to the same element $(f_1, \dots, f_n)$ of $\mathscr{O}(X)^n$. By (1.2.3), $z_i \circ \varphi_\bullet(x) = (\varphi_\bullet^\# z_i)(x) = f_i(x)$. So φ_1 equals φ_2 as set maps, i.e. $\varphi_\bullet(x) = (f_1(x), \dots, f_n(x))$. Checking that they are equal as morphisms of $\mathbb{C}$-ringed spaces is equivalent to showing for any x that $\varphi_1^\# = \varphi_2^\#$ as maps from $\mathscr{O}_{\mathbb{C}^n, \varphi_\bullet(x)} = \mathscr{O}\{z_1 - f_1(x), \dots, z_n - f_n(x)\}$ to $\mathscr{O}_{X,x}$. We know that they both send each $z_i - f_i(x)$ to $f_i - f_i(x)$. So they are equal by the uniqueness part of the following proposition. □

The following proposition can be viewed as the infinitesimal version of Thm. 1.4.1. (This will become clear after the readers read Thm. 1.6.2.)

Proposition 1.4.3. *Let $\mathscr{O}_{X,x}$ be an analytic local $\mathbb{C}$-algebra. Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathscr{O}_{X,x}$. Then there is a unique morphism of local $\mathbb{C}$-algebras satisfying*

$$\Phi : \mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\} \to \mathscr{O}_{X,x}, \qquad z_i \mapsto f_i - f_i(x). \tag{1.4.3}$$

Note that as a morphism of *local* rings, Φ is assumed to send $\mathfrak{m}_{\mathbb{C}^n,0} = \sum_{j=1}^n z_j \mathbb{C}\{z_1, \dots, z_n\}$ into $\mathfrak{m}_{X,x}$.

Proof. Existence: By the second paragraph of the proof that (1.4.1) is surjective (which does not rely on the injectivity of (1.4.1)), by shrinking X, we may choose a holomorphic map $\phi : X \to \mathbb{C}^n$ corresponding to $(f_1 - f_1(x), \dots, f_n - f_n(x))$. Then the stalk map $\phi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ gives Φ.

Injectivity: Assume Φ_1, Φ_2 both satisfy the requirement. Then they clearly agree when restricted to the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. Now we choose $g \in \mathbb{C}\{z_\bullet\}$. For each $k \in \mathbb{N}$, we may write g as a polynomial of $z_\bullet$ plus $g_k \in \mathfrak{m}_{\mathbb{C}^n,0}^k$. So $\Phi_1(g) - \Phi_2(g)$ equals $\Phi_1(g_k) - \Phi_2(g_k)$, which belongs to $\mathfrak{m}_{X,x}^k$ since Φ_i sends $\mathfrak{m}_{\mathbb{C},0}$ into $\mathfrak{m}_{X,x}$. So $\Phi_1(g) - \Phi_2(g)$ belongs to $\bigcap_{k \in \mathbb{N}} \mathfrak{m}_{X,x}^k$, which is 0 due to the following theorem *and the fact that $\mathscr{O}_{X,x}$ is Noetherian.* □

Theorem 1.4.4 (Krull's intersection theorem). *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{M}$ be a finitely-generated A-module. Then $\bigcap_{k \in \mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M} = 0$.*

Proof. The submodule $\mathcal{N} = \bigcap_{k \in \mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M}$ is also finitely generated as A is Noetherian. Then $\mathcal{N} = 0$ will follow from $\mathfrak{m}\mathcal{N} = \mathcal{N}$ (equivalently, 0 spans the "fiber" $\mathcal{N}/\mathfrak{m}\mathcal{N}$) and Nakayama's lemma. That $\mathfrak{m}\mathcal{N} = \mathcal{N}$ is due to Artin-Rees lemma (applied to the $\mathfrak{m}$-stable filtration $(\mathfrak{m}^k\mathcal{M})_{k \in \mathbb{N}}$ to show that $(\mathcal{N} \cap \mathfrak{m}^k\mathcal{M})_{k \in \mathbb{N}} = (\mathcal{N})_{k \in \mathbb{N}}$ is $\mathfrak{m}$-stable). □

Recall that if I is an ideal of a ring A, an I**-filtration** $(\mathcal{M}_n)_{n\in\mathbb{N}}$ (of $\mathcal{M}_0$) is a descending chain of A-modules $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$ satisfying $I\mathcal{M}_n \subset \mathcal{M}_{n+1}$ for all $n \in \mathbb{N}$. It is called I**-stable** if for some $N \in \mathbb{N}$ we have $I\mathcal{M}_n = \mathcal{M}_{n+1}$ for all $n \geqslant N$.

Theorem 1.4.5 (Artin-Rees lemma). *Let I be an ideal of a Noetherian ring A. Then for any I-stable filtration $(\mathcal{M}_n)_{n\in\mathbb{N}}$ inside a finitely-generated A-module $\mathcal{M}$, and for any submodule $\mathcal{N} \subset \mathcal{M}$, $(\mathcal{N} \cap \mathcal{M}_n)_{n\in\mathbb{N}}$ is I-stable.*

Proof. This follows from two ingredients: 1. The graded ring $A_\bullet = (A, I, I^2, \cdots)$ is a quotient of the Noetherian ring $A[z_1, \ldots, z_m]$ if I is generated by m elements. So $A_\bullet$ is Noetherian. 2. An I-filtration $(\mathcal{M}_0)_{n\in\mathbb{N}}$ of finitely-generated A-modules is I-stable iff the graded $A_\bullet$-module $\mathcal{M}_\bullet = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \cdots)$ is finitely-generated. See [AM, Sec. 10.3] for details. □

The uniqueness part of Thm. 1.4.1 can be formulated in the following form.

Remark 1.4.6 (Substitution rule). Let X be a complex space, let $\mathcal{I}$ be a finite type ideal of $\mathscr{O}_X$ containing $f_1 - g_1, \ldots, f_n - g_n$ where $f_\bullet, g_\bullet \in \mathscr{O}(X)$. Let $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$. Let $h \in \mathscr{O}_{\mathbb{C}^n}$. Then $F^\# h$ and $G^\# h$ restrict to the same (locally defined) holomorphic function of $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$, i.e. they are equal as elements of $\mathscr{O}_X/\mathcal{I}$.

Proof. f_i and g_i are equal as holomorphic functions of Y. So by Thm. 1.4.1, F and G are the same holomorphic map $X \to \mathbb{C}^n$. So $F^\# h$ equals $G^\# h$ as elements of $\mathscr{O}_Y$. □

Example 1.4.7. Let $U \subset \mathbb{C}^2$ be open, let $f \in \mathscr{O}(U)$, and let $\mathcal{I}$ be the ideal sheaf of $\mathscr{O}_U$ generated by $z_2 - f(z_1, z_2)$. Then for each $h \in \mathscr{O}_{\mathbb{C}^2}$, $h(z_1, z_2)$ and $h(z_1, f(z_1, z_2))$ are equal as elements of $\mathscr{O}_U/\mathcal{I}$.

We have seen how a holomorphic map from a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ to $V \subset \mathbb{C}^n$ looks like. The next question is when this map "has image in $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$"? This is answered by the following theorem whose proof does not rely on the Noetherian property.

Theorem 1.4.8. *Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Let $X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ and $Y_0 = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$ be closed complex subspaces of X and Y respectively. Then the following are equivalent:*

(a) There is a (necessarily unique) holomorphic map $\psi : X_0 \to Y_0$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "\psi"] \arrow[d, hook] & Y_0 \arrow[d, hook] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \tag{1.4.4}$$

(b) For each $x \in X$ and $y = \varphi(x)$, the stalk map $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ satisfies

$$\varphi^\#(\mathcal{J}_y) \subset \mathcal{I}_x$$

Proof. Assume (a). If $x \in X_0$, then each $f \in \mathcal{J}_y \subset \mathscr{O}_{Y,y}$ is sent by the transpose $\iota^\#_{Y_0,Y}$ to 0. Also f is sent by $\varphi^\#$ to $\varphi^\#(f) \in \mathscr{O}_{X,x}$, and then sent by $\iota^\#_{X_0,X}$ to $\varphi^\#(f) + \mathcal{I}_x$ in $\mathscr{O}_{X_0,x} = \mathscr{O}_{X,x}/\mathcal{I}_x$, which must be 0 since (1.4.4) commutes. So $\varphi^\#(f) \in \mathcal{I}_x$.

If $x \in X \backslash X_0$, then $x \neq N(\mathcal{I})$. So $\mathcal{I}_x = \mathscr{O}_{X,x_0}$. Then clearly $\varphi^\#(\mathcal{J}_y) \subset \mathcal{I}_x$. (b) is proved.

Now assume (b). If $y \notin N(\mathcal{J})$, then $\mathcal{J}_y = \mathscr{O}_{Y,y}$. So $1 \in \mathcal{J}_y$, and so $1 = \varphi^\#(1)$ belongs to $\mathcal{I}_x$. Therefore $x \notin N(\mathcal{I})$. This proves $\varphi(N(\mathcal{I})) \subset N(\mathcal{J})$. So ψ exists as a continuous map of topological spaces, and such a map is clearly unique.

Choose $x \in X_0$ i.e. $x \in N(\mathcal{I})$. By (b), we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X_0,x} = \mathscr{O}_{X,x}/\mathcal{I}_x & \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \arrow[l, "\psi^\#"'] \\
\mathscr{O}_{X,x} \arrow[u, two heads] & \mathscr{O}_{Y,y} \arrow[l, "\varphi^\#"] \arrow[u, two heads]
\end{tikzcd}$$

for a unique stalk map $\psi^\# : \mathscr{O}_{Y_0,y} \to \mathscr{O}_{X_0,x}$, which is clearly a morphism of local $\mathbb{C}$-algebras. It remains to show that these stalk maps can be assembled into a sheaf map.

Recall the presheaves in Rem. 1.3.5. For each open $V \subset Y$, (b) implies $\varphi^\#(\mathcal{J}(V)) \subset \mathcal{I}(\varphi^{-1}(V))$. So the map $\varphi^\# : \mathscr{O}_Y(V) \to (\varphi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\varphi^{-1}(V))$ descends to

$$\mathscr{O}_Y(V)/\mathcal{J}(V) \to \mathscr{O}_X(\varphi^{-1}(V))/\mathcal{I}(\varphi^{-1}(V)).$$

By taking direct limit over all V containing a fixed open $V_0 \subset Y_0$, we obtain

$$\mathscr{O}^{\mathrm{pre}}_{Y_0}(V_0) \to \mathscr{O}^{\mathrm{pre}}_{X_0}(\psi^{-1}(V_0))$$

Its composition with

$$\mathscr{O}^{\mathrm{pre}}_{X_0}(\psi^{-1}(V_0)) \to \mathscr{O}_{X_0}(\psi^{-1}(V_0)) = (\psi_*\mathscr{O}_{X_0})(V_0)$$

gives a presheaf map $\mathscr{O}^{\mathrm{pre}}_{Y_0} \to \psi_*\mathscr{O}_{X_0}$ whose sheafification is the desired $\psi^\# : \mathscr{O}_{Y_0} \to \psi_*\mathscr{O}_{X_0}$. $\square$

1.5 Weierstrass division theorem and Noetherian property of $\mathscr{O}_{X,x}$

1.5.1 Main results

Now that we have seen the importance of the Noetherian property, we prove this in this section. Since $\mathscr{O}_{X,x}$ is a quotient of $\mathscr{O}_{\mathbb{C}^n,0}$, it suffices to prove that $\mathscr{O}_{\mathbb{C}^n,0}$ is Noetherian. The proof relies on Weierstrass division theorem, which we state below.

Definition 1.5.1. We say that $f(z) \in \mathbb{C}\{z\}$ has **order** $k \in \mathbb{N} \cup \{\infty\}$ if $f(z) = z^k(a_k + a_{k+1}z + a_{k+2}z^2 + \cdots)$ and $a_k \neq 0$; f has order ∞ iff $f = 0$. More generally, for $m \in \mathbb{N}$, we say that $f(w_\bullet, z) = f(w_1, \dots, w_m, z) \in \mathbb{C}\{w_\bullet, z\}$ has **order** k **(in** z**)** if $f(0, z) \in \mathbb{C}\{z\}$ has order k. Equivalently, $f(w_\bullet, z) = \sum_{i=0}^{\infty} a_k(w_\bullet)z^k$ where

$$a_0(0) = \cdots = a_{k-1}(0) = 0, \qquad a_k(0) \neq 0. \tag{1.5.1}$$

That f has order ∞ in z means $a_i(0) = 0$ for all i.

Recall that the **degree** of a polynomial $p(w_\bullet, z) \in \mathbb{C}\{w_\bullet\}[z]$ is the smallest power of z whose coefficient is a non-zero element of $\mathbb{C}\{w_\bullet\}$. The degree of zero polynomial is set to be $-\infty$. □

Remark 1.5.2. Let $f(w_\bullet, z)$ have order $k < \infty$ in z, defined on a neighborhood of 0. Then inside this neighborhood we can find a smaller one $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ such that $f(0, z)$ has one zero in V^{cl} (namely $z = 0$) with multiplicity k. By Rouché's theorem, we may shrink U such that for each fixed $w_\bullet \in U$, the holomorphic function $f(w_\bullet, z)$ of z has k zeros in V counting multiplicities; see Fig. 1.5.1.

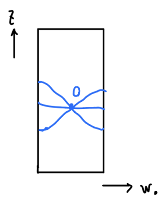

Figure 1.5.1

In the following, we suppress the variable $w_\bullet$ when necessary.

Theorem 1.5.3 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\}$ has order $k < \infty$ in z. Then for each $f \in \mathbb{C}\{w_\bullet, z\}$, there exist unique $q \in \mathbb{C}\{w_\bullet, z\}$ and $r \in \mathbb{C}\{w_\bullet\}[z]$ with degree $< k$ such that $f = gq + r$.*

We shall prove the Noetherian property using the following (almost) equivalent form of WDT.

Theorem 1.5.4 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\} = \mathscr{O}_{\mathbb{C}^{m+1}}$ has order $k < \infty$ in z. Then $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a rank-k free $\mathscr{O}_{\mathbb{C}^m}$-module. $1, z, \dots, z^{k-1}$ are a set of free generators.*

Theorem 1.5.5. *Every analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. It suffices to discuss $\mathscr{O}_{\mathbb{C}^n,0}$. We prove this by induction on n. The case $n = 0$ is trivial. Suppose the case $m = n - 1$ is known. We prove the case $m + 1$. Choose any ideal non-zero $I \subset \mathscr{O}_{\mathbb{C}^{m+1},0}$. Choose $0 \neq g \in I$. Then on a complex line passing through 0, 0 must be an isolated zero of h. (Otherwise, on each line, g vanishes on a neighborhood of 0. So g vanishes on each line (and hence each domain containing 0) by complex analysis.) By choosing new coordinates, we may assume the last coordinate axis is that line. Namely, writing $g = g(w_1, \dots, w_m, z)$, g has finite order in z.

By WDT, $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a finitely-generated $\mathscr{O}_{\mathbb{C}^m,0}$-module. Its submodule $I/I \cap g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is generated by finitely many elements $f_1, \dots, f_N \in I$, thanks to the assumption that $\mathscr{O}_{\mathbb{C}^m,0}$ is Noetherian. So elements of I are $\mathscr{O}_{\mathbb{C}^{m+1},0}$-linear combinations of $f_1, \dots, f_N, g$. □

1.5.2 Proof of WDT

We prove the first version of WDT following [GR].

Proof of the uniqueness. Let $f = gq_1 + r_1 = gq_2 + r_2$. Then $g(q_1 - q_2) = r_2 - r_1$. Choose a small enough neighborhood $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ as in Rem. 1.5.2 such that for all fixed $w_\bullet \in U$, $g(z)$ has k zeros in V (counting multiplicities). So $g(q_1 - q_2)$ has $\geqslant k$ zeros in z. Since $r_2 - r_1$ has degree $< k$ in z, for the fixed $w_\bullet$, the number of zeros of $r_2 - r_1$ is either $< k$ (which is impossible), or is ∞. Since the latter is the only possible case, we conclude $(r_1 - r_2)(z) = 0$ for all $w_\bullet$. And $(q_1 - q_2)(z) = 0$ since it is so outside the (finitely many) zeros of g. (One can also deduce $q_1 = q_2$ from the fact that $\mathscr{O}_{\mathbb{C}^{m+1},0}$ is an integral domain.) □

Discussion. We now discuss the proof of the existence part. Let $\widehat{f}, \widehat{g}$ be the first k terms in their power series expansions of z. So

$$g(w_\bullet, z) = \underbrace{a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}}_{\widehat{g}} + z^k(a_k + a_{k+1} z + a_{k+2} z^2 + \cdots)$$

where all $a_i = a_i(w_\bullet) \in \mathbb{C}\{w_\bullet\}$ and $a_0(0) = \cdots = a_{k-1}(0) = 0$, $a_k(0) \neq 0$. So $(g - \widehat{g})z^{-k}$ and similarly $(f - \widehat{f})z^{-k}$ are naturally elements of $\mathbb{C}\{w_\bullet, z\}$. Moreover, $(g - \widehat{g})z^{-k}$ is a unit.

A naïve attempt to find the decomposition $f = gq + r$ is to write

$$f = g \cdot \frac{f - \widehat{f}}{g} + \widehat{f}$$

since clearly $\widehat{f} \in \mathbb{C}\{w_\bullet\}[z]$ has degree $< k$ in z. This certainly works for single-variable functions. However, when $m > 0$, the expression $(f - \widehat{f})/g$ might not be continuous at the origin. (Take for instance the quotient to be $z^2/(wz + z^2)$.) We can only divide $f - \widehat{f}$ by $g - \widehat{g}$, which gives an element of $\mathbb{C}\{w_\bullet, z\}$. So we write

$$f = (g - \widehat{g}) \cdot \frac{f - \widehat{f}}{g - \widehat{g}} + \widehat{f} = g \cdot \frac{f - \widehat{f}}{g - \widehat{g}} + \widehat{f} + \underbrace{\left(-\widehat{g} \cdot \frac{f - \widehat{f}}{g - \widehat{g}} \right)}_{f_1}$$

We then decompose f_1, find f_2, and then repeat this procedure again and again to produce an infinite series, which we hope would converge to the expected decomposition. Namely, we let $f_0 = f$. So the above defines f_1 in terms of f_0. We define in a similar way f_{n+1} in terms of f_n:

$$f_n = g \cdot \frac{f_n - \widehat{f}_n}{g - \widehat{g}} + \widehat{f}_n + f_{n+1}. \tag{1.5.2}$$

Substituting $f_0, f_1, \ldots, f_n$ into f, we get

$$\begin{aligned} f &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0 \right) + f_1 \\ &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0 \right) + \left(g \cdot \frac{f_1 - \widehat{f}_1}{g - \widehat{g}} + \widehat{f}_1 \right) + f_2 = \cdots \\ &= g \cdot \sum_{i=0}^{n} \frac{f_i - \widehat{f}_i}{g - \widehat{g}} + \sum_{i=0}^{n} \widehat{f}_i + f_{n+1}. \end{aligned} \tag{1.5.3}$$

In the following formal proof, we give careful analysis when $n \to \infty$. $\square$

Finishing the proof of WDT. For each $(r_\bullet, \rho) = (r_1, \ldots, r_m, \rho) \in \mathbb{R}^m_{>0} \times \mathbb{R}_{>0}$, define a norm $\|\cdot\|_{r_\bullet,\rho}$ on $\mathbb{C}\{w_\bullet, z\}$ as follows: if $h = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} b_{i_\bullet,j} w_1^{i_1} \cdots w_m^{i_m} z^j$ then

$$\|h\|_{r_\bullet,\rho} = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} |b_{i_\bullet,j}| r_1^{i_1} \cdots r_m^{i_m} \rho^j,$$

which might take value ∞. We have

$$\|h_1 h_2\|_{r_\bullet,\rho} \leqslant \|h_1\|_{r_\bullet,\rho} \cdot \|h_2\|_{r_\bullet,\rho} \qquad \|h - \widehat{h}\|_{r_\bullet,\rho} \leqslant \|h\|_{r_\bullet,\rho}. \tag{1.5.4}$$

We write (1.5.2) as

$$
\begin{aligned}
-f_{n+1} &= \frac{\widehat{g}}{(g-\widehat{g})} \cdot (f_n - \widehat{f}_n) \\
&= \frac{\widehat{g}}{z^{-k}(g-\widehat{g})} \cdot z^{-k}(f_n - \widehat{f}_n) =: \beta \cdot \alpha_n. \qquad (1.5.5)
\end{aligned}
$$

By the first paragraph in the previous *Discussion*, we have $\beta, \alpha_n \in \mathbb{C}\{w_\bullet, z\}$. Choose $r_\bullet, \rho$ such that f, g are defined (and holomorphic) and $g - \widehat{g}$ has no zeros in the polydisc D with multiradii $r_\bullet, \rho$ except at the origin. Then (1.5.5) shows that all f_n are defined in this domain.

Slightly shrink ρ so that $C := \|f\|_{r_\bullet,\rho} < \infty$. *Now we use the condition that g has order k in z in full power*: it tells us that $\beta(0, z) = 0$. So we may shrink $r_\bullet$ such that $\|\beta\|_{r_\bullet,\rho} < \frac{1}{2}\rho^k$. Clearly $\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} = \rho^k \|\alpha_n\|_{r_\bullet,\rho}$. So by (1.5.4),

$$
\|f_{n+1}\|_{r_\bullet,\rho} < \frac{1}{2}\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} \leqslant \frac{1}{2}\|f_n\|_{r_\bullet,\rho}.
$$

Thus $\|f_n\|_{r_\bullet,\rho} < 2^{-n}C$. So $\|z^{-k}(f_n - \widehat{f}_n)\|_{r_\bullet,\rho} < 2^{-n}\rho^{-k}C$ and $\|\widehat{f}_n\|_{r_\bullet,\rho} < 2^{-n}C$.

The uniform norm on the polydisc with multi-radii $(r_\bullet, \rho)$ is clearly $\leqslant \|\cdot\|_{r_\bullet,\rho}$. So $f_n \to 0$ uniformly on the polydisc D. The infinite series $\sum_{i=0}^{\infty} \frac{z^{-k}(f_i - \widehat{f}_i)}{z^{-k}(g-\widehat{g})}$ converges uniformly to a continuous function q on any compact subset of D. q is holomorphic, since it is so on each variable by Morera's theorem. Similarly, $\sum_{i=0}^{\infty} \widehat{f}_i$ converges uniformly to a holomorphic r. Residue theorem and the fact that contour integrals commute with (uniformly convergent) infinite sum show that r does not have $\geqslant k$ powers of z (since each $\widehat{f}_n$ does not). Thus, we obtain the decomposition $f = gq + r$ by letting $n \to \infty$ in (1.5.3). □

1.6 Germs of complex spaces

Definition 1.6.1. The **category of germs of complex spaces** denotes the one whose objects are (X, x) where X is a complex space and x is a marked point. If $U \subset X$ is a neighborhood of x then (X, x) is identified with (U, x). A **morphism of germs** from (X, x) to (Y, y) is a holomorphic map $\varphi : U \to Y$ where $U \subset X$ is a neighborhood of x such that $\varphi(x) = y$. Two morphisms $\varphi_1, \varphi_2 : (X, x) \to (Y, y)$ are regarded equal if there is a neighborhood U of x such that $\varphi_1|_U$ equals $\varphi_2|_U$ as holomorphic maps $U \to Y$. Composition of morphisms are the usual one for holomorphic functions (i.e. for $\mathbb{C}$-ringed spaces).

An **isomorphism of germs of complex spaces** $\varphi : (X, x) \to (Y, y)$ is a morphism of germs with inverses, namely, there is a morphism $\psi : (Y, y) \to (X, x)$

such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are $\mathbf{1}$ on neighborhoods of x and y respectively. Equivalently, there are neighborhoods $U \ni x$ and $V \ni y$ such that $\varphi : U \to V$ is a biholomorphism, and that $\varphi(x) = y$. $\square$

The category of analytic local $\mathbb{C}$-algebras is understood in the obvious way: the morphisms are defined by Def. 1.3.3.

Theorem 1.6.2. *The contravariant functor $\mathfrak{F}$ from the category of germs of complex spaces to the category of analytic local $\mathbb{C}$-algebras, sending (X, x) to $\mathscr{O}_{X,x}$ and sending $\varphi : (X, y) \to (Y, y)$ to $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$, is an* ***antiequivalence of categories****. Namely:*

(1) For each (X, x) and (Y, y), the following map is bijective

$$\mathfrak{F} : \mathrm{Mor}\big((X, x), (Y, y)\big) \to \mathrm{Mor}\big(\mathscr{O}_{Y,y}, \mathscr{O}_{X,x}\big), \qquad \varphi \mapsto \varphi^\#. \tag{1.6.1}$$

(2) Each analytic local $\mathbb{C}$-algebra is isomorphic to $\mathfrak{F}((X, x))$ for some germ of complex space (X, x).

Part (2) is obvious. Let us prove part (1).

Proof. Assume without loss of generality that Y is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathscr{J})$ where $V \subset \mathbb{C}^n$ is open and $y = 0$.

Suppose $\varphi_1^\#, \varphi_2^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ are equal. Then for each $j = 1, \dots, n$, $\varphi_1^\# z_j$ equals $\varphi_2^\# z_j$ as elements of $\mathscr{O}_{X,x}$. So they are equal on X if we shrink X to a smaller neighborhood of x. By Thm. 1.4.1, φ_1 and φ_2 are equal as holomorphic maps $X \to V$, and hence are equal as $X \to Y$. So the map $\mathfrak{F}$ in (1.6.1) is injective.

Next, we choose a morphism $\Phi : \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$. Let $f_1 = \Phi(z_1), \dots, f_n = \Phi(z_n)$, which are elements of $\mathscr{O}(X)$ if we shrink X to a smaller neighborhood of x. View $F = (f_1, \dots, f_n) \in \mathscr{O}(X)^n$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$. Replace X by $\varphi^{-1}(V)$ such that $\varphi : X \to V$. Note that $\varphi(x) = 0$. So $h \in \mathscr{O}_{\mathbb{C}^n,0} \mapsto h \circ \varphi = \varphi^\# h \in \mathscr{O}_{X,x}$ is a morphism of local $\mathbb{C}$-algebras. It agrees with $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\Phi} \mathscr{O}_{X,x}$ on $z_1, \dots, z_n$ by the very definition of F. So they agree on any element of $\mathscr{O}_{\mathbb{C}^n,0}$ due to Prop. 1.4.3. We conclude $\varphi^\#(h) = \Phi([h])$ for all $h \in \mathscr{O}_{\mathbb{C}^n,0}$ (where $[h]$ denotes the residue class of h in $\mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0$). In particular, we have $\varphi^\# \mathscr{J}_0 = 0$ in $\mathscr{O}_{X,x}$.

Shrink V and $X \subset \varphi^{-1}(V)$, and choose $g_1, \dots, g_k \in \mathscr{O}_{\mathbb{C}^n}(V)$ generating the ideal $\mathscr{J}_0$ and sent by $\varphi^\#$ to $0 \in \mathscr{O}(X)$. Since $\mathscr{J}$ is finite-type, by Rem. 1.2.16, we can shrink V such that $g_1, \dots, g_k$ generate $\mathscr{J}$. Thus $\varphi^\# \mathscr{J} = 0$ in $\varphi_* \mathscr{O}_X$. By Thm. 1.4.8, φ restricts to a holomorphic map $\widetilde{\varphi} : X \to Y$. $\widetilde{\varphi}^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ equals Φ since $\varphi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ factors as $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\widetilde{\varphi}^\#} \mathscr{O}_{X,x}$. This proves that $\mathfrak{F}$ is surjective. $\square$

Corollary 1.6.3. *Let X, Y be complex spaces, $x \in Y, y \in Y$, and $\Phi : \mathscr{O}_{Y,y} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ be an isomorphism of local algebras. Then there are neighborhoods $U \ni x, V \ni y$ and a biholomorphism $\varphi : U \xrightarrow{\simeq} V$ whose transpose $\varphi^\# : \mathscr{O}_{V,y} \to \mathscr{O}_{U,x}$ equals Φ.*

Definition 1.6.4. An analytic local $\mathbb{C}$-algebra is called **regular** if it is isomorphic to $\mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}$ for some n.

Corollary 1.6.5. *Let X be a complex space and $x \in X$. If $\mathscr{O}_{X,x}$ is regular, then there is a neighborhood U of x biholomorphic to an open subset of $\mathbb{C}^n$ for some n.*

Definition 1.6.6. We say that X is **smooth at** x (equivalently, x is a **smooth point** of X) if $\mathscr{O}_{X,x}$ is regular. We say that X is **smooth** (equivalently, X is a complex manifold) if it is smooth everywhere.

1.7 Immersions and closed embeddings; generating $\mathscr{O}_{X,x}$ analytically

Definition 1.7.1. A holomorphic map $\varphi : X \to Y$ is called an **immersion at** $x \in X$ if $\varphi^\# : \mathscr{O}_{Y,\varphi(y)} \to \mathscr{O}_{X,x}$ is surjective. φ is called an **immersion** if it is an immersion at every $x \in X$. φ is called a **closed (resp. open) embedding** if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\simeq"'] & & Y \\ & Y_0 \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{1.7.1}$$

where Y_0 is a closed (resp. open) complex subspace of Y and $X \xrightarrow{\simeq} Y_0$ is a biholomorphic map.

A closed embedding is clearly an immersion. Moreover, an immersion is locally a closed embedding:

Proposition 1.7.2. *Let $\varphi : X \to Y$ be an immersion at x. Then there are neighborhoods V of $y = \varphi(x)$ and $U \subset \varphi^{-1}(V)$ of x such that $\varphi : U \to V$ is a closed embedding. In particular, φ is an immersion on U.*

Proof. By assumption, $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective. Let J be its kernel, and choose generating elements $g_1, \dots, g_k \in J$. By shrinking Y to a neighborhood of y (and shrink X accordingly), we assume $g_1, \dots, g_k \in \mathscr{O}_Y(Y)$. Let $\mathcal{J} = g_1\mathscr{O}_Y + \cdots + g_k\mathscr{O}_Y$. Then $\mathcal{J}_x = J$. Define a closed subspace $Z = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$ of Y. Then φ factors as

$$\varphi^\# : \mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{Y,y}/J = \mathscr{O}_{Z,y} \xrightarrow[\simeq]{\Psi} \mathscr{O}_{X,x}.$$

By Cor. 1.6.3, we may shrink X so that there is an open embedding $\widetilde{\varphi} : X \to Z$, $\widetilde{\varphi}(x) = y$, such that $\widetilde{\varphi}^\# : \mathscr{O}_{Z,y} \to \mathscr{O}_{X,x}$ equals Ψ. Let $\iota : Z \to Y$ be the inclusion. Then $(\iota\widetilde{\varphi})^\# = \widetilde{\varphi}^\#\iota^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ equals $\varphi^\#$. By Thm. 1.6.2, we may find open $U \ni x$ such that $\varphi = \iota\widetilde{\varphi}$ on U. Since $\widetilde{\varphi}(U)$ is an open subset of Z, we may find open $V \subset Y$ such that $\widetilde{\varphi}(U) = V \cap Z = V \cap N(\mathcal{J})$. So φ restricts to the biholomorphism $\widetilde{\varphi} : U \to \widetilde{\varphi}(U)$ where $\widetilde{\varphi}(U)$ is a closed subspace of V. □

We now discuss when an immersion is a closed embedding and give some examples.

Proposition 1.7.3. *Let X be complex spaces and $\varphi : X \to Y$ a holomorphic immersion. Assume that φ is an injective and closed map*[2] *of topological spaces. Suppose we have a finite type ideal $\mathcal{J}$ of $\mathscr{O}_Y$ such that $N(\mathcal{J})$ equals the image of φ, and that*

$$\mathcal{J}_y = \mathrm{Ker}(\mathscr{O}_{Y,y} \xrightarrow{\varphi^\#} \mathscr{O}_{X,x}) \tag{1.7.2}$$

for all $x \in X$ and $y = \varphi(x)$. Then φ is a closed embedding. More precisely, φ restricts to a biholomorphism

$$\widetilde{\varphi} : X \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J}). \tag{1.7.3}$$

We will see in Cor. 2.7.7 that the assumption on the existence of $\mathcal{J}$ is redundant.

Proof. Let $Y_0 := \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$. By Thm. 1.4.8, the restriction (1.7.3) as a holomorphic map exists, i.e., we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & Y_0 \arrow[d, hook] \\ & Y \end{tikzcd}$$

The underlying topological space of $Y_0 := \mathrm{Specan}(\mathscr{O}_X/\mathcal{J})$ is $N(\mathcal{J})$. So $\widetilde{\varphi}$ is a continuous closed bijection from X to $N(\mathcal{J})$, which is therefore a homeomorphism. For each $x \in X, y = \varphi(x)$, the stalk map $\widetilde{\varphi}^\# : \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \to \mathscr{O}_{X,x}$ is surjective since φ is an immersion, and is injective by (1.7.2). So $\widetilde{\varphi}$ is a biholomorphism. □

Example 1.7.4. The holomorphic map $\iota : 0 \times \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n$ is an immersion and a closed injective map, and the kernels of $\iota^\#$ at the level of stalks are the stalks of the ideal $\mathcal{I} = z_1\mathscr{O}_{\mathbb{C}^{m+n}} + \cdots + z_m\mathscr{O}_{\mathbb{C}^{m+n}}$. Thus, by Prop. 1.7.3, ι restricts to a biholomorphism $0 \times \mathbb{C}^n \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^{m+n}}/\mathcal{I})$. This reproves Exp. 1.3.10.

[2] φ is called closed if it maps closed subsets to closed subsets.

Example 1.7.5. Let X be a complex space, and let $\mathcal{I}, \mathcal{J}$ be finite-type ideals of $\mathscr{O}_X$. Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. So $\mathscr{O}_Y = (\mathscr{O}_X/\mathcal{I})|_{N(\mathcal{I})}$. Then

$$\widetilde{\mathcal{J}} = \big((\mathcal{I}+\mathcal{J})/\mathcal{I}\big) \restriction_{N(\mathcal{I})}$$

is a finite-type ideal of $\mathscr{O}_Y$, and is the unique ideal whose stalk at each $x \in N(\mathcal{I})$ equals $(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x$. Then there is a biholomorphism

$$\mathrm{Specan}(\mathscr{O}_X/(\mathcal{I}+\mathcal{J})) \xrightarrow[\simeq]{\varphi} \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}). \tag{1.7.4}$$

which equals $N(\mathcal{I}+\mathcal{J}) \xrightarrow{=} N(\mathcal{I}) \cap N(\mathcal{J})$ as maps of topological spaces, and whose stalk maps are

$$\mathscr{O}_{Y,x}/\widetilde{\mathcal{J}}_x = \frac{\mathscr{O}_{X,x}/\mathcal{I}_x}{(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x} \quad \xrightarrow{\simeq} \quad \mathscr{O}_{X,x}/(\mathcal{I}_x + \mathcal{J}_x).$$

Proof. The key point is to show that the above stalk isomorphisms can be assembled into a sheaf isomorphism. Consider the diagram

$$\begin{tikzcd}
& \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}) \arrow[d, hook] \\
\mathrm{Specan}(\mathscr{O}_X/(\mathcal{I}+\mathcal{J})) \arrow[ur, "\varphi"] \arrow[r, "\alpha"] \arrow[dr, hook] & Y \arrow[d, hook] \\
& X
\end{tikzcd} \tag{1.7.5}$$

By Thm. 1.4.8, there is a holomorphic map α such that the lower triangle commutes. The stalk maps are $\alpha^\# : \mathscr{O}_{X,x}/\mathcal{I}_x \to \mathscr{O}_{X,x}/(\mathcal{I}_x + \mathcal{J}_x)$, with kernel $(\mathcal{I}_x + \mathcal{J}_x/\mathcal{I}_x)$. These kernels can be assembled into the ideal sheaf $\widetilde{\mathcal{J}}$ on $N(\mathcal{I})$. Thus, Prop. 1.7.3 guarantees that there is a biholomorphism making the upper triangle in (1.7.5) commutes. □

Exp. 1.7.5 shows that a closed complex subspace of a closed subspace is again a closed subspace of the original space. Thus, we have more generally:

Corollary 1.7.6. *If $\alpha : X \to Y$ and $\beta : Y \to Z$ are closed embeddings, then so is the composition $\beta \circ \alpha : X \to Z$.*

Let us consider the special case $\varphi : X \to \mathbb{C}^n$, where φ is represented by $(f_1, \dots, f_n) \in \mathscr{O}_X^n$ (cf. Thm. 1.4.1). Then φ is an immersion at x iff the morphism of analytic local $\mathbb{C}$-algebras defined in Prop. 1.4.3, namely $\mathbb{C}\{z_\bullet\} \to \mathscr{O}_{X,x}$ sending z_j to $f_j - f_j(x)$, is surjective. This actually mean that $f_1, \dots, f_n$ *generate (analytically)* *the analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$.* (They certainly do not generate the ring $\mathscr{O}_{X,x}$ algebraically. But one can imagine that the subalgebra generated algebraically by $f_\bullet$ is "dense" in $\mathscr{O}_{X,x}$, where the density means approximation by power series of $f_1,\dots,f_n$.) The situation is similar to the case of a surjective morphism of $\mathbb{C}$-algebras $\mathbb{C}[z_\bullet] \to A$, whose algebro-geometric meaning is that the affine scheme $\mathrm{Spec}(A)$ is embedded into the affine plane $\mathbb{C}^n$.

We must find a criterion on whether $f_1,\dots,f_n$ generate $\mathscr{O}_{X,x}$ (analytically). At first sight, this problem seems not easy even if X is smooth. (For instance, take $f_1,\dots,f_n$ to be some arbitrary holomorphic functions and deduce whether they generate $\mathscr{O}_{X,x}$.) There is indeed a simple criterion, which is proved using the (holomorphic version of) inverse function theorem. To begin with, we define:

Definition 1.7.7. If X is a complex space and $x \in X$, the vector space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is called the **cotangent space** of X at x, and its dual space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the **tangent space**. Since $\mathscr{O}_{X,x}$ is Noetherian, $\mathfrak{m}_{X,x}$ is finitely-generated, and hence $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is finite-dimensional.

It is inspiring to write the residue class of $f - f(x)$ (where $f \in \mathscr{O}(X)$) in the cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ as $d_x f$.

Theorem 1.7.8. *Let X be a complex space and $x \in X$. Let $f_1,\dots,f_n \in \mathscr{O}(X)$. Consider $(f_1,\dots,f_n)$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$ (cf. Thm. 1.4.1). The following are equivalent.*

(1) φ is an immersion at x.

(2) The morphism of analytic local $\mathbb{C}$-algebras $\Phi : \mathscr{O}_{\mathbb{C}^n,\varphi(x)} \to \mathscr{O}_{X,x}$ sending each z_i to f_i (cf. Prop. 1.4.3) is surjective.

(3) (The residue classes of) $f_1 - f_1(x),\dots,f_n - f_n(x)$ span $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$.

(4) (The germs of) $f_1 - f_1(x),\dots,f_n - f_n(x)$ generate the ideal $\mathfrak{m}_{X,x}$.

*If any of these conditions holds, we say that $f_1,\dots,f_n$ **generate (the algebra) $\mathscr{O}_{X,x}$ analytically**.*

Proof. Assume for simplicity that $\varphi(x) = 0$. Clearly (1)⇔(2) and (3)⇔(4). (Note that (3)⇒(4) follows from Nakayama's lemma.) It remains to prove (2)⇔(3).

Assume (2). Choose any $g \in \mathfrak{m}_{X,x}$. Then there is $h(z_\bullet) \in \mathscr{O}_{\mathbb{C}^n,0}$ sent by Φ to g. We may write $h(z_\bullet) = \sum_i a_i z_i +$ an element of $\mathfrak{m}_{\mathbb{C}^n,0}^2$ where $a_i \in \mathbb{C}$. Since $\Phi(z_i) = f_i$ and $\Phi(\mathfrak{m}_{\mathbb{C}}^2) \subset \mathfrak{m}_{X,x}^2$, we have $g \in \sum_i a_i f_i + \mathfrak{m}_{X,x}^2$. This proves (3).

Asume (3). By discarding some elements, we may assume that $f_1,\dots,f_n$ form a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. Assume X is a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ where $U \subset \mathbb{C}^N$ is open and $x = 0$. So $\mathscr{O}_{X,x} = \mathscr{O}_{\mathbb{C}^N,0}/\mathcal{I}_0$, $\mathfrak{m}_{X,x} = \mathfrak{m}_{\mathbb{C}^N,0}/\mathcal{I}_0$, and hence

$$\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = \mathfrak{m}_{\mathbb{C}^N,0}/(\mathfrak{m}_{\mathbb{C}^N,0}^2 + \mathcal{I}_0). \tag{1.7.6}$$

Lift $f_\bullet$ to elements of $\mathscr{O}_{\mathbb{C}^N,0}$, still denoted by $f_\bullet$. Then we can extend $f_1,\dots,f_n$ to a list $f_1,\dots,f_N$ whose residue classes form a basis of $\mathfrak{m}_{\mathbb{C}^N,0}/\mathfrak{m}^2_{\mathbb{C}^N,0}$ such that $f_{n+1},\dots,f_N\in\mathcal{I}_0$. By the inverse function theorem, we may assume $x=0$ and $f_1,\dots,f_N$ are the standard coordinates $z_1,\dots,z_N$ of $\mathbb{C}^N$. By shrinking U, we may assume $z_{n+1},\dots,z_N\in\mathcal{I}(U)$.

Assume for simplicity that $\mathcal{I}$ is generated by $z_{n+1},\dots,z_N$ together with $g_1,\dots,g_k\in\mathcal{I}(U)$. Let $\mathcal{I}_1=z_{n+1}\mathscr{O}_U+\cdots+z_N\mathscr{O}_U$. Then by Exp. 1.7.5, $X=\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally a closed subspace of $X_1=\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}_1)$ (defined by $g_1,\dots,g_k$). By Exp. 1.7.4, X_1 is naturally equivalent to $U\cap(\mathbb{C}^n\times 0)$. So the map $(z_1,\dots,z_n):X_1\to\mathbb{C}^n$ is an open embedding. φ is its restriction to X, which is therefore an immersion at 0. This proves (1) and hence (2). □

We give an application of analytically generating elements.

Proposition 1.7.9.

Let $\Phi,\Psi:\mathscr{O}_{Y,y}\to\mathscr{O}_{X,x}$ be morphisms of analytic local $\mathbb{C}$-algebras. Assume $f_1,\dots,f_n\in\mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically.

(1) If $\Phi(f_i)=\Psi(f_i)$ for all $i=1,\dots,n$, then $\Phi=\Psi$.

(2) Let I be the ideal of $\mathscr{O}_{X,x}$ generated by $\Phi(f_i)-\Psi(f_i)$ for all i. Then I contains $\Phi(h)-\Psi(h)$ for every $h\in\mathscr{O}_{Y,y}$.

Proof. (1): By Prop. 1.4.3, we have a (unique) morphism $\Upsilon:\mathscr{O}_{\mathbb{C}^n,0}\to\mathscr{O}_{Y,y}$ sending z_i to $f_i-f_i(x)$. So $\Phi\circ\Upsilon$ and $\Psi\circ\Upsilon$ agree at $z_1,\dots,z_n$. So $\Phi\circ\Upsilon=\Psi\circ\Upsilon$ by Prop. 1.4.3. By assumption, Υ is surjective. So $\Phi=\Psi$.

(2): Apply (1) to the restriction $\Phi,\Psi:\mathscr{O}_{Y,y}\to\mathscr{O}_{X,x}/I$. □

Prop. 1.7.9-(2) is the stalk version of a geometric construction called equalizer.

1.8 Equalizers of $X\rightrightarrows Y$

Definition 1.8.1. Let $\varphi,\psi:X\to Y$ be holomorphic maps of complex spaces. A **kernel** or an **equalizer of the double arrow** $X\overset{\varphi}{\underset{\psi}{\rightrightarrows}}Y$ is a complex space E and a holomorphic map $\iota:E\to X$ such that $\varphi\circ\iota=\psi\circ\iota$, and that for every complex space S and holomorphic map $\mu:S\to X$ satisfying $\varphi\circ\mu=\psi\circ\mu$ there is a unique holomorphic $\widetilde{\mu}:S\to E$ such that $\mu=\iota\circ\widetilde{\mu}$.

$$\begin{tikzcd} S \arrow[d, dashed, "\widetilde{\mu}"'] \arrow[dr, "\mu"] & & \\ E \arrow[r, "\iota"'] & X \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & Y \end{tikzcd} \tag{1.8.1}$$

It is easy to see that equalizers are unique up to isomorphisms.

The main result of this section is:

Theorem 1.8.2. *Every double arrow* $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ *of holomorphic maps has an equalizer which is the inclusion map of a closed subspace* $\iota : E = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$*. This is called the* ***canonical equalizer****. The finite-type ideal* $\mathcal{I}$ *is uniquely determined by the fact that for all* $x \in X$*:*

(a) If $\varphi(x) \neq \psi(x)$*, then* $\mathcal{I}_x = \mathscr{O}_{X,x}$*.*

(b) If $\varphi(x) = \psi(x)$*, then by considering* $\varphi^\#, \psi^\#$ *as stalk maps* $\mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$*,* $\mathcal{I}_x$ *is the ideal of* $\mathscr{O}_{X,x}$ *generated by all* $\varphi^\#(f) - \psi^\#(f)$ *(where* $f \in \mathscr{O}_{Y,\varphi(x)}$*).*

Moreover, $N(\mathcal{I})$*, the underlying set of* E*, is* $\Delta = \{x \in X : \varphi(x) = \psi(x)\}$*.*

From Prop. 1.7.9, it is clear that $\mathcal{I}_x$ is generated by $\varphi^\#(f_i) - \psi^\#(f_i)$ if $f_1, \dots, f_n \in \mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically, e.g. $z_1, \dots, z_n$ if Y is a model space in $\mathbb{C}^n$.

Remark 1.8.3. From Thm. 1.8.2, it is clear that if $E_0 \to X$ is an equalizer of $X \rightrightarrows Y$, then it is a closed embedding, and equals the composition of a unique biholomorphism $E_0 \xrightarrow{\simeq} E$ and the inclusion map $E \hookrightarrow X$ where E is the canonical equalizer.

Construction of E*.* We define a finite-type ideal $\mathcal{I}$ satisfying (a) and (b). We shall first define it locally and then glue the pieces. Then $\mathcal{I}$ gives E.

Let $\Omega = X \backslash \Delta$ which is open. We set $\mathcal{I}_\Omega = \mathscr{O}_X|_\Omega$. For each $x \in \Delta$, we choose a neighborhood $V_y \subset Y$ of $y = \varphi(x)$ biholomorphic to a model space. So we can choose finitely many $f_1, \dots f_n \in \mathscr{O}_Y(V_y)$ embedding V_y onto a closed subspace of an open subset of $\mathbb{C}^n$. $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$ is a neighborhood of x, and we set $\mathcal{I}_{U_x}$ to be the ideal of $\mathscr{O}_{U_x}$ generated by $\varphi^\#(f_1) - \psi^\#(f_1), \dots, \varphi^\#(f_n) - \psi^\#(f_n)$ (defined on U_x).

We claim that these locally defined finitely-generated ideals are compatible. If $p \in U_x \cap \Delta$ then, as $\varphi(p) = \psi(p)$, by Prop. 1.7.9 or by substitution rule (Rem. 1.4.6), the stalk $(\mathcal{I}_{U_x})_p$ is the ideal generated by all $\varphi^\#(f) - \psi^\#(f) \in \mathscr{O}_{X,p}$ where $f \in \mathscr{O}_{Y,\varphi(p)}$. If $p \in U_x \cap \Omega$, then as $\varphi(p) \neq \psi(p)$ and $(f_1, \dots, f_n)$ is an embedding, there is some f_i among $f_1, \dots, f_n$ such that $\varphi^\#(f_i) - \psi^\#(f_i)$ has non-zero value at p, and hence its germ at p is not in $\mathfrak{m}_{X,p}$. This proves $(\mathcal{I}_{U_x})_p = \mathscr{O}_{X,p}$. Combining these two cases together, we see that $\mathcal{I}_\Omega$ and $\mathcal{I}_{U_x}$ (for all $x \in \Delta$) are compatible. This defines $\mathcal{I}$.

If $\varphi(x) \neq \psi(x)$, then $\mathcal{I}_x = \mathscr{O}_{X,x}$ shows $x \notin N(\mathcal{I})$. If $\varphi(x) = \psi(x)$, then $\varphi^\#(f) - \psi^\#(f)$ vanishes at x by (1.2.3). So $\mathcal{I}_x$ vanishes at x. So $x \in N(\mathcal{I})$. This proves $\Delta = N(\mathcal{I})$. □

Proof that E is an equalizer. It is easy to check $\varphi\circ\iota = \psi\circ\iota$. Choose any holomorphic $\mu : S \to X$ such that $\varphi \circ \mu = \psi \circ \mu$. For any $s \in S$, let $x = \mu(s)$. Then $\varphi(x) = \psi(x)$. Choose any $f \in \mathscr{O}_{Y,\varphi(x)}$. Then $\varphi \circ \mu = \psi \circ \mu$ shows that $\mu^\#$ sends $\varphi^\#(f) - \psi^\#(f)$ to $0 \in \mathscr{O}_{S,s}$. Thus $\mu^\# : \mathscr{O}_{X,x} \to \mathscr{O}_{S,s}$ vanishes on $\mathcal{I}_x$. Thus, by Thm. 1.4.8, there is a unique holomorphic $\widetilde{\mu} : S \to E$ such that the triangle in (1.8.1) commutes. □

The proof of Thm. 1.8.2 is finished. From the proof, we know:

Remark 1.8.4. Assume the setting of Thm. 1.8.2. Assume $\varphi(x) = \psi(x) =: y$. Let V_y be a neighborhood of y biholomorphic to a model space. More precisely, we choose $(f_1, \dots, f_n) \in \mathscr{O}_Y(V_y)^n$ which, considered as a holomorphic map $V_y \to \mathbb{C}^n$, is a closed embedding of V_y into an open subset of $\mathbb{C}^n$. Let $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$. Then the ideal sheaf $\mathcal{I}|_{U_x}$ is generated by $\varphi^\#(f_1) - \psi^\#(f_1), \dots, \varphi^\#(f_n) - \psi^\#(f_n) \in \mathscr{O}(U_x)$.

1.9 $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$, and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$

We fix a $\mathbb{C}$-ringed space X.

1.9.1 Tensor product

Definition 1.9.1. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{O}_X$-modules. Consider the presheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules defined by $\mathscr{G}(U) = \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$. The tensor product of restriction maps $\mathscr{E}(U) \to \mathscr{E}(V)$ and $\mathscr{F}(U) \to \mathscr{F}(V)$ gives the restriction map $\mathscr{G}(U) \to \mathscr{G}(V)$. The sheafification of $\mathscr{G}$ is denoted by $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ or simply $\mathscr{E} \otimes \mathscr{F}$ and called the **tensor product** of $\mathscr{E}$ and $\mathscr{F}$.

Remark 1.9.2. Let A be a commutative ring, and fix an A-module $\mathcal{N}$. Recall the following basic facts:

1. **Tensor products commute with direct limits**. More precisely, let $(\mathcal{M}_\alpha)$ be a direct system of A-modules. Then the canonical map $\mathcal{M}_\beta \otimes_A \mathcal{N} \to (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}$ (for each fixed β) defines, by passing to the direct limit, an isomorphism
$$\varinjlim_\alpha (\mathcal{M}_\alpha \otimes_A \mathcal{N}) \xrightarrow{\simeq} (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}. \tag{1.9.1}$$
(Proof: Construct the inverse map explicitly.)

2. **The tensor product functor $-\otimes\mathcal{N}$ is right exact**. Namely, if
$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0$$

is an exact sequence of A-modules, then so is

$$\mathcal{M}_1 \otimes \mathcal{N} \xrightarrow{f\otimes\mathbf{1}} \mathcal{M}_2 \otimes \mathcal{N} \xrightarrow{g\otimes\mathbf{1}} \mathcal{M}_3 \otimes \mathcal{N} \to 0.$$

Identify $\mathcal{M}_3$ with $\mathrm{Coker} f = \mathcal{M}_2/f(\mathcal{M}_1)$. Then the right exactness of tensor product is equivalent to that **tensor products commute with cokernels**: we have an equivalence of A-modules

$$\mathrm{Coker}\big(\mathcal{M}_1 \otimes_A \mathcal{N} \xrightarrow{f\otimes\mathbf{1}} \mathcal{M}_2 \otimes_A \mathcal{N}\big) \quad \xrightarrow{\simeq} \quad \mathrm{Coker}\big(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2\big) \otimes_A \mathcal{N} \tag{1.9.2}$$

descended from the canonical morphism

$$\mathcal{M}_2 \otimes_A \mathcal{N} \longrightarrow \frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \otimes_A \mathcal{N}. \tag{1.9.3}$$

□

Proof. We have a well-defined map sending $\frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \times \mathcal{N}$ to $\frac{\mathcal{M}_2 \otimes_A \mathcal{N}}{(f\otimes\mathbf{1})(\mathcal{M}_1 \otimes_A \mathcal{N})}$ (i.e. the LHS of (1.9.2)) sending $[\xi] \times \eta$ to $[\xi \otimes_A \eta]$, where $[\cdots]$ stands for the residue classes, and $\xi \in \mathcal{M}_2, \eta \in \mathcal{N}$. This map is clearly A-biinvariant. So it gives an A-module morphism from the RHS to the LHS of (1.9.2), which is clearly the inverse of the map in (1.9.2) from LHS to RHS. So (1.9.2) is an isomorphism. □

Remark 1.9.3. We can now use (1.9.2) to explain the last equality of (1.2.4):

$$\begin{aligned}&\mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x}/\mathfrak{m}_x) = \mathscr{E}_x \otimes \mathrm{Coker}(\mathfrak{m}_x \hookrightarrow \mathscr{O}_{X,x})\\ \simeq&\mathrm{Coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x \otimes \mathscr{O}_{X,x}) \simeq \mathrm{Coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x) = \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x\end{aligned}$$

since the image of the multiplication map $\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x$ is $\mathfrak{m}_x\mathscr{E}_x$.

Proposition 1.9.4. *The canonical morphism of $\mathscr{O}(U)$-modules*

$$\mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \to \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$$

(where $U \ni x$ is open and the map is the tensor product of $\mathscr{E}(U) \to \mathscr{E}_x$ and $\mathscr{F}(U) \to \mathscr{F}_x$) induces an isomorphism

$$(\mathscr{E} \otimes \mathscr{F})_x = \varinjlim_{U \ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \quad \xrightarrow{\simeq} \quad \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x. \tag{1.9.4}$$

Proof. Define a canonical map from $\mathscr{E}_x \times \mathscr{F}_x$ to $\varinjlim_{U\ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$ and show that it is $\mathscr{O}_{X,x}$-biinvariant. This descends to the inverse map of (1.9.4). □

Corollary 1.9.5. *For each $\mathscr{O}_X$-module $\mathscr{F}$, the functor $-\otimes\mathscr{F}$ on the abelian category of $\mathscr{O}_X$-modules is right exact: if*

$$\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$$

is exact, then so is

$$\mathscr{E}_1 \otimes \mathscr{F} \to \mathscr{E}_2 \otimes \mathscr{F} \to \mathscr{E}_3 \otimes \mathscr{F} \to 0.$$

Proof. Exactness of sheaves can be checked at the level of stalks. Then this follows from the isomorphism (1.9.4) and the right exactness of $-\otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$. □

1.9.2 Hom

We leave it to the readers to check the following easy facts:

Remark 1.9.6. Let A be a commutative ring, and fix an A-module $\mathcal{N}$:

1. $\mathrm{Hom}_A(\mathcal{N}, -)$ **is a left exact functor**. Namely, for any exact sequence of A-modules
$$0 \to \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3, \tag{1.9.5}$$
we have an exact sequence
$$0 \to \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_1) \xrightarrow{f_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)$$
where f_* sends T to $f \circ T$ and g_* is defined similarly. Equivalently, $\mathrm{Hom}_A(\mathcal{N}, -)$ **commutes with kernels**: there is a equivalence
$$\mathrm{Hom}_A\big(\mathcal{N}, \mathrm{Ker}(\mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3)\big) \simeq \mathrm{Ker}\big(\mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)\big) \tag{1.9.6}$$
induced by the obvious inclusion
$$\mathrm{Hom}_A\big(\mathcal{N}, \mathrm{Ker}(\mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3)\big) \hookrightarrow \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2).$$

2. $\mathrm{Hom}_A(-, \mathcal{N})$ **is a left exact contravariant functor**. for any exact sequence of A-modules
$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0 \tag{1.9.7}$$
we have an exact sequence
$$0 \to \mathrm{Hom}_A(\mathcal{M}_3, \mathcal{N}) \xrightarrow{g^*} \mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1, \mathcal{N})$$
where f^* sends T to $T \circ f$ and g^* is defined similarly. Equivalently, $\mathrm{Hom}_A(-, \mathcal{N})$ **turns cokernels into kernels**: there is a canonical equivalence
$$\mathrm{Hom}_A\big(\mathrm{Coker}(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2), \mathcal{N}\big) \simeq \mathrm{Ker}\big(\mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1, \mathcal{N})\big) \tag{1.9.8}$$
induced by the obvious inclusion
$$\mathrm{Hom}_A\big(\mathrm{Coker}(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2), \mathcal{N}\big) \hookrightarrow \mathrm{Hom}_A(\mathcal{M}_2, \mathcal{N}).$$

Definition 1.9.7. Let $\mathscr{E},\mathscr{F}$ be $\mathscr{O}_X$-modules. The **hom space** $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ is defined to be the space of all $\mathscr{O}_X$-module morphims from $\mathscr{E}$ to $\mathscr{F}$.

The presheaf of $\mathscr{O}_X$-modules sending each open $U\subset X$ to the $\mathscr{O}(U)$-module $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}_U,\mathscr{F}_U)$, and whose restriction map is the obvious restriction of sheaf morphisms, is automatically a sheaf of $\mathscr{O}_X$-modules. It is called the **hom sheaf** and denoted by $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$.

The dual and the double dual of $\mathscr{E}$ is defined by

$$\mathscr{E}^\vee=\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{O}_X),\qquad \mathscr{E}^{\vee\vee}=(\mathscr{E}^\vee)^\vee. \tag{1.9.9}$$

□

Exercise 1.9.8. Describe canonical equivalences

$$\mathscr{E}\simeq\mathscr{E}\otimes_{\mathscr{O}_X}\mathscr{O}_X\simeq\mathscr{O}_X\otimes_{\mathscr{O}_X}\mathscr{E}\simeq\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{E}). \tag{1.9.10}$$

In general, the stalks of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ cannot be identified with $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x)$. But good things happen when $\mathscr{E}$ is coherent, as we will see in Cor. 2.2.4.

1.10 $(\mathscr{O}_X\mathrm{-mod})\otimes_{\mathscr{O}_S}(\mathscr{O}_S\mathrm{-mod})$; pullback sheaves

Definition 1.10.1. Let $\varphi:X\to S$ be a holomorphic map of complex spaces. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module and $\mathscr{M}$ an $\mathscr{O}_S$-module. Then $\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M}=\mathscr{M}\otimes_{\mathscr{O}_S}\mathscr{E}$ denotes the sheafification of the presheaf of $\mathscr{O}_X$-modules sending each open $U\subset X$ to

$$(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M})^{\mathrm{pre}}(U)=\varinjlim_{V\supset\varphi(U)}\mathscr{E}(U)\otimes_{\mathscr{O}_S(V)}\mathscr{M}(V) \tag{1.10.1}$$

where the direct limit is over all open subset $V\subset S$ containing $\varphi(U)$, and $g\in\mathscr{O}_S(V)$ acts on $\varsigma\in\mathscr{E}(U)$ as

$$g\cdot\varsigma:=\varphi^\#(g)\cdot\varsigma. \tag{1.10.2}$$

For each $x\in X$, we have a canonical equivalence

$$(\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M})_x\simeq\mathscr{E}_x\otimes_{\mathscr{O}_{S,\varphi(x)}}\mathscr{M}_{\varphi(x)}. \tag{1.10.3}$$

Thus $\mathscr{M}\mapsto\mathscr{E}\otimes_{\mathscr{O}_S}\mathscr{M}$ is a right exact functor.

Definition 1.10.2. The **pullback sheaf** of $\mathscr{M}$ along φ is the $\mathscr{O}_X$-module defined by

$$\varphi^*\mathscr{M}:=\mathscr{O}_X\otimes_{\mathscr{O}_S}\mathscr{M} \tag{1.10.4}$$

whose stalk at x is $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,\varphi(x)}} \mathscr{M}_x$. It can be viewed as the induced representation of $\mathscr{M}$. Thus we may write

$$\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M} = \mathscr{E} \otimes_{\mathscr{O}_X} \varphi^* \mathscr{M}. \tag{1.10.5}$$

If $V \subset S$ is open and $\sigma \in \mathscr{M}(V)$, then the **pullback section** $\varphi^*(\sigma) \in \varphi^*\mathscr{M}(\varphi^{-1}(V))$ is the image of

$$1 \otimes \sigma \in \mathscr{O}(\varphi^{-1}(V)) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to (\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{M})(\varphi^{-1}(V)) = (\varphi_* \varphi^* \mathscr{M})(V). \tag{1.10.6}$$

This gives a canonical morphism of $\mathscr{O}_S$-modules

$$\mathscr{M} \to \varphi_* \varphi^* \mathscr{M}. \tag{1.10.7}$$

If $g : \mathscr{M}_1 \to \mathscr{M}_2$ is a morphism of $\mathscr{O}_S$-modules, we define an $\mathscr{O}_X$-module morphism

$$\varphi^* g := \mathbf{1} \otimes g : \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_1 \to \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_2, \tag{1.10.8}$$

called the **pullback of** g.

The notation $\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}$ is a generalization of $\mathscr{E} \otimes_{\mathbb{C}} W$ for a ($\mathbb{C}$-)vector space W by viewing $\mathbb{C}$ as the structure sheaf of the single reduced point $\{0\}$, and by viewing the holomorphic map as the obvious one $X \to \{0\}$.

Proposition 1.10.3. *(φ^*, φ_*) is a pair of* ***adjoint functors*** *between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_S$-modules (with φ^* the left adjoint and φ_* the right one). Namely, there is a functorial isomphism*

$$\mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}, \mathscr{E}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}, \varphi_* \mathscr{E}). \tag{1.10.9}$$

The word ***functorial*** *(also called* ***natural****) means that for any morphisms $g : \mathscr{M}_2 \to \mathscr{M}_1$ of $\mathscr{O}_S$-modules and $f : \mathscr{E}_1 \to \mathscr{E}_2$ of $\mathscr{O}_X$-modules, $\varphi^* g$ and $\varphi_* f$ induce a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}_1, \mathscr{E}_1) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_1, \varphi_* \mathscr{E}_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{O}_X}(\varphi^* \mathscr{M}_2, \mathscr{E}_2) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_2, \varphi_* \mathscr{E}_2) \end{array} \tag{1.10.10}$$

Proof. Given a morphism $F : \varphi^* \mathscr{M} \to \mathscr{E}$, the composition of $\mathscr{M} \to \varphi_* \varphi^* \mathscr{M}$ with $\varphi_* F : \varphi_* \varphi^* \mathscr{M} \to \varphi_* \mathscr{E}$ gives a morphism $G : \mathscr{M} \to \varphi_* \mathscr{E}$. Informally,

$$G(\sigma) = F(1 \otimes \sigma). \tag{1.10.11}$$

We leave it to the readers to check that $F \mapsto G$ is functorial.

Conversely, given $G : \mathscr{M} \to \varphi_* \mathscr{E}$. The $\mathscr{O}(U)$-module morphisms

$$\mathscr{O}(U) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to \mathscr{E}(U), \qquad f \otimes \sigma \mapsto f \cdot G(\sigma)|_U$$

for all open $U \subset X$ and $V \supset \varphi(U)$ pass to $F : \varphi^* \mathscr{M} \to \mathscr{E}$. This gives the inverse of the above construction. $\square$

Definition 1.10.4. Let $\iota : Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$ be a closed subspace of X. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the **(sheaf theoretic) restriction of $\mathscr{E}$ to Y**, denoted by $\mathscr{E}|_Y$ or $\mathscr{E}|Y$ is

$$\mathscr{E}|_Y = \iota^* \mathscr{E} = (\mathscr{O}_X/\mathcal{I}) \restriction_{N(\mathcal{I})} \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.12}$$

Remark 1.10.5. If $\iota : Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \to X$ is an embedding of closed complex subspace, one may view an $\mathscr{O}_Y$-module $\mathscr{F}$ as the corresponding $\mathscr{O}_X$-module $\iota_* \mathscr{F}$. A more precise statement is that the functor ι_* from the category of $\mathscr{O}_Y$-modules to the category of $\mathscr{O}_X$-modules annihilated by the multiplication of $\mathcal{I}$, sending each morphism φ to $\iota_* \varphi$, is an equivalence of categories. (Cf. Thm. 1.6.2 or Thm. 2.2.2 for the precise meaning.) An inverse functor can be chosen to be ι^*. In particular, we have a canonical equivalence $\mathscr{F} \simeq \iota^* \iota_* \mathscr{F}$ for any $\mathscr{O}_Y$-module $\mathscr{F}$ and $\mathscr{E} \simeq \iota_* \iota^* \mathscr{E}$ for any $\mathscr{O}_X$-module $\mathscr{E}$ annihilated by $\mathcal{I}$ (so that $\mathscr{E} = \mathscr{E}/\mathcal{I}\mathscr{E} \simeq \mathscr{E} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathcal{I})$). These equivalences are the identity maps at the level of stalks.

Moreover, the functor ι_* is an equivalence of tensor categories. Namely, we have functorial isomorphisms

$$\iota_*(\mathscr{F}_1 \otimes_{\mathscr{O}_Y} \mathscr{F}_2) \simeq (\iota_* \mathscr{F}_1) \otimes_{\mathscr{O}_X} (\iota_* \mathscr{F}_2).$$

Note that since $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is surjective (if $y \in Y$), we have

$$\mathscr{F}_{1,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{F}_{2,y} \simeq \mathscr{F}_{1,y} \otimes_{\mathscr{O}_{X,y}} \mathscr{F}_{2,y}. \tag{1.10.13}$$

If $\mathscr{E}$ is an $\mathscr{O}_X$-module, we also have a natural isomorphism

$$\iota_*(\mathscr{E}|_Y) \simeq (\mathscr{O}_X/\mathcal{I}) \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.14}$$

Thus, the study of the restriction $\mathscr{E}|_Y$ can be turned into the study of an $\mathscr{O}_X$-module. $\square$

1.11 Fiber products

Definition 1.11.1. Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. A **fiber product** of these two maps is a complex space $X \times_S Y$ together with holomorphic maps $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$ satisfying:

(1) $\varphi \circ \mathrm{pr}_X = \psi \circ \mathrm{pr}_Y$.

(2) For each complex space Z and holomorphic maps $\alpha : Z \to X$ and $\beta : Z \to Y$ satisfying $\varphi\circ\alpha = \psi\circ\beta$ there is a unique holomorphic map $\alpha\vee\beta : Z \to X\times_S Y$ such that $\alpha = \mathrm{pr}_X \circ (\alpha \vee \beta)$ and that $\beta = \mathrm{pr}_Y \circ (\alpha \vee \beta)$.

$$\begin{tikzcd}
 & & Z \arrow[lld, bend right, "\alpha"'] \arrow[ld, dashed, "\alpha\vee\beta"'] \arrow[ldd, bend left, "\beta"] \\
X \arrow[d, "\varphi"'] & X\times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & \\
S & Y \arrow[l, "\psi"] &
\end{tikzcd} \tag{1.11.1}$$

The commutative square diagram above involving $S, X, Y, X \times_S Y$ is called a **Cartesian square**. $\mathrm{pr}_Y : X \times_S Y \to Y$ is called the **pullback/base change** of $\varphi : X \to S$ along $\psi : Y \to S$. $\square$

The following is easy to check:

Proposition 1.11.2. *In Def. 1.11.1, let $\gamma : Z' \to Z$ be a holomorphic map. Then*

$$(\alpha \vee \beta) \circ \gamma = (\alpha \circ \gamma) \vee (\beta \circ \gamma) : Z' \to X \times_S Y. \tag{1.11.2}$$

Fiber products are clearly unique up to isomorphisms. The following is easy to check.

Remark 1.11.3. Suppose that the following two small commuting square diagrams are both Cartesian, then the largest rectangular square is also Cartesian.

$$\begin{tikzcd}
X \arrow[d] & X\times_S Y \arrow[l] \arrow[d] & (X\times_S Y)\times_Y Z \arrow[l] \arrow[d] \\
S & Y \arrow[l] & Z \arrow[l]
\end{tikzcd}$$

Namely, $(X \times_S Y) \times_Y Z$, together with its maps to X and Z, is a pullback of $X \to S$ along $Z \to S$. This can be generalized to more complicated situations. For instance, if the following 4 small cells are Cartesian squares, then so is the largest square diagram.

$$\begin{tikzcd}
X_1 \arrow[d] & Z_1 \arrow[l] \arrow[d] & Z_3 \arrow[l] \arrow[d] \\
X \arrow[d] & Z \arrow[l] \arrow[d] & Z_2 \arrow[l] \arrow[d] \\
S & Y \arrow[l] & Y_1 \arrow[l]
\end{tikzcd}$$

Example 1.11.4. Let U, V be open subsets of a complex space X. Then $U \cap V$ is a fiber product $U \times_X V$: we have Cartesian square

$$\begin{tikzcd}
U & U \cap V \arrow[l, hook'] \arrow[d, hook] \\
X \arrow[u, hookleftarrow] & V \arrow[l, hook']
\end{tikzcd}$$

Definition 1.11.5. Let $\varphi : X \to S$, $\psi : Y \to S$, $\alpha : X' \to X$, $\beta : Y' \to Y$ be holomorphic maps of complex spaces. Assume $X \times_S Y$ exists. Assume we have a fiber product $X' \times_S Y'$ of $\varphi \circ \alpha : X' \to S$ and $\psi \circ \beta : Y' \to S$. Then

$$\alpha \times \beta : X' \times_S Y' \to X \times_S Y \tag{1.11.3}$$

denotes $(\alpha \circ \mathrm{pr}_{X'}) \vee (\beta \circ \mathrm{pr}_{Y'})$, the unique holomorphic map making the following diagram commute.

$$\begin{tikzcd}
{} & {} & X' \arrow[dll, "\alpha"'] & {} & X' \times_S Y' \arrow[ll, "\mathrm{pr}_{X'}"'] \arrow[dlll, dashed, "\alpha\times\beta"'] \arrow[d, "\mathrm{pr}_{Y'}"] \\
X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & {} & {} & Y' \arrow[dlll, "\beta"] \\
S & Y \arrow[l, "\psi"] & & &
\end{tikzcd} \tag{1.11.4}$$

The following is easy to check:

Proposition 1.11.6. *In Def. 1.11.5, let $\mu : Z \to X'$, $\nu : Z \to Y'$ be holomorphic maps of complex spaces such that $\varphi \circ \alpha \circ \mu = \psi \circ \beta \circ \nu$. Then we have equality*

$$(\alpha \times \beta) \circ (\mu \vee \nu) = (\alpha \circ \mu) \vee (\beta \circ \nu) : Z \to X \times_S Y. \tag{1.11.5}$$

Let $\widetilde{\alpha} : X'' \to X'$, $\widetilde{\beta} : Y'' \to Y'$ be holomorphic maps of complex spaces, and assume that a fiber product $X'' \times_S Y''$ exists for $\varphi \circ \alpha \circ \widetilde{\alpha} : X'' \to S$ and $\psi \circ \beta \circ \widetilde{\beta} : Y'' \to S$. Then

$$(\alpha \times \beta) \circ (\widetilde{\alpha} \times \widetilde{\beta}) = (\alpha \circ \widetilde{\alpha}) \times (\beta \circ \widetilde{\beta}) : X'' \times_S Y'' \to X \times_S Y. \tag{1.11.6}$$

Remark 1.11.7. There are no canonical fiber products of give holomorphic $\varphi : X \to S$, $\psi : Y \to S$. But suppose that a fiber product $X \times_S Y$ exists and is fixed. Then for each open $U \subset X$ and $X \subset Y$, there is a unique (open) **fiber product** $U \times_S V$ **inside** $X \times_S Y$. which is the open complex subspace

$$U \times_S V := \mathrm{pr}_X^{-1}(U) \cap \mathrm{pr}_Y^{-1}(V)$$

of $X \times_S Y$. The projections $\mathrm{pr}_U : U \times_S V \to U$ and $\mathrm{pr}_V : U \times_S V \to V$ are defined respectively by the restrictions of $\mathrm{pr}_X, \mathrm{pr}_Y$.

Moreover, assume that $\alpha : X' \to X$, $\beta : Y' \to Y$ are holomorphic, and a fiber product $X' \times_S Y'$ is fixed. Let $U' \subset X'$ and $V' \subset Y'$ be open such that $\alpha(U') \subset U$, $\beta(V') \subset V$. Let $U' \times_S V'$ be the fiber product inside $X' \times_S Y'$. The we have a commutative diagram

$$\begin{tikzcd} X' \times_S Y' \arrow[r, "\alpha\times\beta"] & X \times_S Y \\ U' \times_S V' \arrow[u, hook] \arrow[r, "\alpha|_{U'}\times\beta|_{V'}"] & U \times_S V \arrow[u, hook] \end{tikzcd} \tag{1.11.7}$$

□

Proof. Show that the inclusion $U \times_S V \hookrightarrow X \times_S Y$ is the product of $U \hookrightarrow X$ and $V \hookrightarrow Y$ and $U' \times_S V' \hookrightarrow X' \times_S Y'$ similarly. Then apply Prop. 1.11.6. □

With the help of the above observation, we can prove:

Lemma 1.11.8 (Gluing fiber products). *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. Let $(U_i)_{i\in\mathfrak{I}}$ and $(V_t)_{t\in\mathfrak{T}}$ be open covers of X and Y respectively. Suppose that for each $i \in \mathfrak{I}$ and $t \in \mathfrak{T}$ there exists a fiber product $U_i \times_S V_t$. Then a fiber product $X \times_S Y$ exists.*

Proof. It suffices to assume (V_t) has only one member, which is Y. So each $U_i \times_S Y$ exists. To simplify notations, for each $i, j, k \in \mathfrak{I}$ we set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. We let $U_{ij} \times_i Y$ and $U_{ijk} \times_i Y$ denote the corresponding open fiber products inside $U_i \times_S Y$. So $U_{ij} \times_i Y$ and $U_{ij} \times_j Y$ are isomorphism but not identical.

We now apply the gluing construction Rem. 1.3.7 to construct $X \times Y$ by gluing all $U_i \times Y$ together. As gluing of topological spaces the process is trivial. To glue the structures of complex spaces, we must assign an isomorphism $\pi_{j,i} : U_{ij} \times_i Y \xrightarrow{\simeq} U_{ij} \times_j Y$ for all i, j. This is chosen to be $\mathbf{1}_{U_{ij}} \times_{j,i} \mathbf{1}_Y$ defined by Def. 1.11.5. (Note that this is not an identity map since the source does not equal the target. The symbol $\times_{j,i}$ reflects the fact that this product relies on both i and j.)

Clearly $\pi_{i,i}$ is the identity. To finish checking the cocycle condition, we must show that the holomorphic maps $\pi_{k,i}$ and $\pi_{k,j} \circ \pi_{j,i}$ are equal when restricted to open subsets $U_{ijk} \times_i Y \to U_{ijk} \times_k Y$. By Rem. 1.11.7, $\pi_{k,i}$ restricts to $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$, and $\pi_{k,j} \circ \pi_{j,i}$ restricts to $(\mathbf{1}_{U_{ijk}} \times_{k,j} \mathbf{1}_Y) \circ (\mathbf{1}_{U_{ijk}} \times_{j,i} \mathbf{1}_Y)$, which equals $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$ by Prop. 1.11.6.

Thus the complex space $X \times_S Y$ is constructed. We leave it to the readers to define pr_X and pr_Y. □

1.12 Fiber products and inverse images of subspaces

Proposition 1.12.1. *Let $\varphi : X \to S$ be a holomorphic map of complex spaces, and let $\mathscr{J}$ be a finite type ideal of $\mathscr{O}_S$. Then we have a Cartesian square*

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & \varphi^{-1}(S_0) := \mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X) \arrow[l, hook'] \arrow[d, "\tilde{\varphi}"'] \\ S & S_0 := \mathrm{Specan}(\mathscr{O}_S/\mathscr{J}) \arrow[l, hook'] \end{tikzcd} \tag{1.12.1}$$

*where $\mathscr{J}\mathscr{O}_X$ is the (necessarily unique) finite-type ideal of $\mathscr{O}_X$ whose stalks $(\mathscr{J}\mathscr{O}_X)_x$ are generated by $\mathscr{J}_{\varphi(x)}$ (more precisely, by $\varphi^\#(\mathscr{J}_{\varphi(x)})$, cf. (1.10.2)). $\varphi^{-1}(S_0) := \mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X)$ is called the **inverse image of** S_0 along φ.*

Proof. If $V \subset S$ is open and $\mathscr{J}|_V$ is generated by finitely many $g_1, g_2, \dots \in \mathscr{J}(V)$, then $(\mathscr{J}\mathscr{O}_X)|_{\varphi^{-1}(V)}$ is defined to be the ideal of $\mathscr{O}_X|_{\varphi^{-1}(V)}$ generated by $\varphi^\#(g_1), \varphi^\#(g_2), \dots$. Clearly the stalks of $(\mathscr{J}\mathscr{O}_X)|_{\varphi^{-1}(V)}$ satisfy the requirement. Thus, these ideals are compatible for different V, and can be glued together and form the desired ideal $\mathscr{J}\mathscr{O}_X$. To check that (1.12.1) is Cartesian one uses Thm. 1.4.8. □

Remark 1.12.2. Using the explicit construction of $\mathscr{J}$ in the proof of Prop. 1.12.1, one sees that the underlying set of $\varphi^{-1}(S_0)$ is the usual preimage of S_0, i.e., $\{x \in X : \varphi(x) \in S_0\}$.

Remark 1.12.3. As an $\mathscr{O}_X$-module, $\mathscr{O}_{\varphi^{-1}(S_0)}$ has a natural equivalence

$$\mathscr{O}_{\varphi^{-1}(S_0)} = \mathscr{O}_X/\mathscr{J}\mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathscr{J}) = \varphi^*(\mathscr{O}_{S_0}). \tag{1.12.2}$$

Proof. Using the right exactness of $\mathscr{O}_X \otimes_{\mathscr{O}_S} -$, we have

$$\begin{aligned} &\mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathscr{J}) = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathrm{Coker}(\mathscr{J} \hookrightarrow \mathscr{O}_S) \\ \simeq &\mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{J} \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S) \simeq \mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{J} \to \mathscr{O}_X) \end{aligned}$$

which equals $\mathscr{O}_X/\mathscr{J}\mathscr{O}_X$ since the term insider the last Coker is the multiplication map. (Compare Rem. 1.9.3.) □

Example 1.12.4. Let $\mathscr{I}, \mathscr{J}$ be finite-type ideals of $\mathscr{O}_S$. Using Thm. 1.4.8 again, one easily checks that there is a Cartesian square that breaks into two commuting triangles.

$$\begin{tikzcd} X = \mathrm{Specan}(\mathscr{O}_S/\mathscr{I}) \arrow[d, hook] & X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathscr{I}+\mathscr{J})) \arrow[l] \arrow[d] \arrow[dl, hook, dashed] \\ S & Y = \mathrm{Specan}(\mathscr{O}_S/\mathscr{J}) \arrow[l, hook'] \end{tikzcd} \tag{1.12.3}$$

Thus, the inverse image of Y along X is naturally equivalent to the closed subspace $X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathcal{I}+\mathcal{J}))$ of S, called the **intersection of** X **and** Y. (Compare this with Exp. 1.7.5.) In view of this equivalence, we shall view $X \cap Y$ as closed subspaces of X and Y in the future.

Proposition 1.12.5. *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic, and let X_0 and Y_0 be complex subspaces of X, Y respectively. Assume that $X \times_S Y$ is a fiber product of φ and ψ. Recall $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$. Then the intersection*

$$X_0 \times_S Y_0 := \mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$$

is a fiber product of $X_0 \hookrightarrow X \xrightarrow{\varphi} S$ and $Y_0 \hookrightarrow Y \xrightarrow{\psi} S$, called the ***(closed) fiber product inside*** *$X \times_S Y$. The projections of $\mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$ to X_0 and Y_0 are respectively the restrictions of pr_X and pr_Y. Moreover, the inclusion $X_0 \times_S Y_0 \hookrightarrow X \times_S Y$ equals the product of $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$.*

Proof. The four cells are Cartesian squares. So is the largest one (Rem. 1.11.3).

$$\begin{tikzcd}
X_0 & \mathrm{pr}_X^{-1}(X_0) \arrow[l] \arrow[d, hook] & X_0 \times_S Y_0 \arrow[l, hook'] \arrow[d, hook] \\
X \arrow[from=u, hook] \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"'] & \mathrm{pr}_Y^{-1}(Y_0) \arrow[l, hook'] \arrow[d] \\
S & Y \arrow[l, "\psi"'] & Y_0 \arrow[l, hook']
\end{tikzcd} \tag{1.12.4}$$

The claim about inclusions is obvious. □

Remark 1.12.6. The closed fiber product $X_0 \times_S Y_0 \subset X \times_S Y$ can be written more explicitly. Choose finite-type ideals $\mathcal{I} \subset \mathscr{O}_X$ and $\mathcal{J} \subset \mathscr{O}_Y$ defining X_0, Y_0 respectively. Then $X_0 \times_S Y_0$ is defined by the ideal $\mathcal{K} \subset \mathscr{O}_{X\times_S Y}$ generated by $\mathrm{pr}_X^\#(\mathcal{I})$ and $\mathrm{pr}_Y^\#(\mathcal{J})$. More precisely: each stalk $\mathcal{K}_{x\times y}$ is generated by $\mathrm{pr}_X^\#(\mathcal{I}_x)$ and $\mathrm{pr}_Y^\#(\mathcal{J}_y)$.

In practice, we may assume X and Y are small enough such that $\mathcal{I}$ is generated by $f_1, \dots, f_m \in \mathscr{O}(X)$ and $\mathcal{J}$ is generated by $g_1, \dots, g_n \in \mathscr{O}(Y)$. Then all $\mathrm{pr}_X^\#(f_i)$ and $\mathrm{pr}_Y^\#(g_j)$ generate $\mathcal{K}$. □

Remark 1.12.7. Similar to Rem. 1.11.7, suppose we have holomorphic $\alpha : X' \to X$, $\beta : Y' \to Y$, $\varphi : X \to S$, $\psi : Y \to S$. Let $X_0 \subset X, Y_0 \subset Y, X_0' \subset X', Y_0' \subset Y'$ be closed subspaces such that α restricts to $\alpha : X_0' \to X_0$ and β restricts to $\beta : Y_0' \to Y_0$ (in the sense of Thm. 1.4.8). Then for the closed fiber products $X_0 \times_S Y_0 \subset X \times_S Y$ and X_0', the following diagram commutes.

$$\begin{tikzcd}
X' \times_S Y' \arrow[r, "\alpha\times\beta"] & X \times_S Y \\
X_0' \times_S Y_0' \arrow[u, hook] \arrow[r, "\alpha|_{X_0'} \times \beta|_{Y_0'}"] & X_0 \times_S Y_0 \arrow[u, hook]
\end{tikzcd} \tag{1.12.5}$$

1.13 Fiber products, direct products, and equalizers

Definition 1.13.1. Let X, Y be complex spaces. A **direct product** of X, Y, or simply a **product** of X, Y, is a fiber product of $X \to 0$ and $Y \to 0$ and denoted by $X \times Y$ (together with the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$).

To spell out the definition: For each complex space Z and holomorphic $\alpha : Z \to X, \beta : Z \to Y$, there is a unique holomorphic map $\alpha \vee \beta : Z \to X \times Y$ such that the following diagram commute.

$$\begin{tikzcd}
 & Z \arrow[dl, "\alpha"'] \arrow[d, "\alpha\vee\beta"'] \arrow[dr, "\beta"] & \\
X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y
\end{tikzcd}$$

If $f \in \mathscr{O}_X$ and $g \in \mathscr{O}_Y$, we write

$$f \otimes 1 := \mathrm{pr}_X^\#(f), \qquad 1 \otimes g := \mathrm{pr}_Y^\#(g), \qquad f \otimes g := \mathrm{pr}_X^\#(f)\mathrm{pr}_Y^\#(g).$$

If $x \in X$ and $y \in Y$, we define the **completed tensor product**

$$\mathscr{O}_{X,x}\widehat{\otimes}\mathscr{O}_{Y,y} := \mathscr{O}_{X\times Y,x\times y}$$

which is well-defined up to isomorphisms by Cor. 1.6.3. □

Remark 1.13.2. One can also view $\mathscr{O}_{X\times_S Y,x\times y}$ as $\mathscr{O}_{X,x}\widehat{\otimes}_{\mathscr{O}_{S,s}}\mathscr{O}_{Y,y}$ (if $s = \varphi(x) = \psi(y)$), a completed tensor product over $\mathscr{O}_{S,s}$. In the case that either φ or ψ is "finite", the stalk $\mathscr{O}_{X\times_S Y,x\times y}$ is actually equal to the usual tensor product $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$. This will be studied in the next chapter.

Example 1.13.3. $\mathbb{C}^{m+n}$ is naturally a product of $\mathbb{C}^m$ and $\mathbb{C}^n$.

Proof. Apply Thm. 1.4.1. □

Lemma 1.13.4. *For every complex spaces X, Y there is a product $X \times Y$.*

Proof. We know this is true when X, Y are number spaces, and hence when X, Y are open subspaces of number spaces (cf. Exp. 1.11.7), and hence if X, Y are model spaces (due to Prop. 1.12.5), and hence for all complex spaces (by Lemma 1.11.8). □

Remark 1.13.5. If X and Y are model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ and $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open, $\mathcal{I}$ is generated by $f_1, f_2, \dots \in \mathcal{I}(U)$, and $\mathcal{J}$ is generated by $g_1, g_2, \dots \in \mathcal{J}(V)$, then $X \times Y$ as a closed direct product inside $U \times V$ can be written down explicitly with the help of Rem. 1.12.6: it is the model space $\mathrm{Specan}(\mathscr{O}_{U\times V}/\mathcal{K})$ where $\mathcal{K}$ is the ideal generated by all $f_i \otimes 1$ and $1 \otimes g_j$.

In the following, we give two proofs that fiber products always exist. We need the following notion:

Proposition 1.13.6. *Let $\varphi : X \to Y$ be a holomorphic map. Then $\mathbf{1}_X \vee \varphi : X \to X \times Y$ is an equalizer:*

$$X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y \overset{\varphi\circ\mathrm{pr}_X}{\underset{\mathrm{pr}_Y}{\rightrightarrows}} Y \tag{1.13.1}$$

*The canonical equalizer $\mathfrak{G}(\varphi)$ of $X \times Y \rightrightarrows Y$ (which is a closed subspace of $X \times Y$) is called the **graph of** φ.*

Proof. Let Z be a complex space. Any holomorphic map $Z \to X \times Y$ is $\alpha \vee \beta$ for some $\alpha : Z \to X$ and $\beta : Z \to Y$. Suppose that the compositions of $\alpha \vee \beta$ with $\varphi \circ \mathrm{pr}_X$ and with pr_Y are equal. Then $\varphi \circ \alpha = \beta$. Then we may find a holomorphic map $Z \to X$ such that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d] \arrow[dr, "\alpha\vee\beta"] & \\ X \arrow[r, "\mathbf{1}\vee\varphi"] & X \times Y \end{tikzcd}$$

Indeed, we can choose this map to be α. Then by Prop. 1.11.2, $(\mathbf{1} \vee \varphi) \circ \alpha = \alpha \vee (\varphi \circ \alpha) = \alpha \vee \beta$. On the other hand, if we have another such holomorphic map $\psi : Z \to X$. Composing the above triangle with $\mathrm{pr}_X : X \times Y \to X$ shows that $\psi = \mathrm{pr}_X \circ (\mathbf{1} \vee \varphi) \circ \psi$ equals $\mathrm{pr}_X \circ (\alpha \vee \beta) = \alpha$. This proves the uniqueness of such ψ. □

Remark 1.13.7. Using Thm. 1.8.2, one can give a more explicit description of the graph of $\varphi : X \to Y$. We write it as $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ for a finite-type ideal $\mathscr{J}$. Let $x \in X, y \in Y$. If $y \neq \varphi(x)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y,x\times y}$. If $y = \varphi(x)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y,x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes f \tag{1.13.2}$$

for all $f \in \mathscr{O}_{Y,y}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{Y,y}$ analytically). The underlying topological space of the graph is $\{x \times y \in X \times Y : y = \varphi(x)\}$.

Remark 1.13.8. The graph construction shows that every holomorphic map $\varphi : X \to Y$ is the composition of a closed embedding $X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y$ (cf. Rem. 1.8.3) and a projection of direct product $X \times Y \xrightarrow{\mathrm{pr}_Y} Y$. Thus, very often, the study of general holomorphic maps reduces to the studies of these two special types of maps. As an application of this observation, we prove:

Theorem 1.13.9. *For any holomorphic maps of complex spaces $\varphi : X \to S, \psi : Y \to S$, there exists a fiber product $X \times_S Y$.*

Proof. We want to show that the pullback of φ along ψ exists. We know it exists when ψ is a closed embedding due to Prop. 1.12.1. It also exists when ψ is a projection $S \times Y_1 \to S$: in that case $X \times_S Y$ is given by the Cartesian square

$$\begin{tikzcd} X & X \times Y_1 \arrow[l] \arrow[d, "\varphi\times\mathbf{1}"'] \\ S & S \times Y_1 \arrow[l] \end{tikzcd} \qquad \text{with } X \arrow[d, "\varphi"'] \tag{1.13.3}$$

(We leave it to the readers to check that this commutative diagram is indeed Cartesian.) The general case follows from Rem. 1.13.8 and the fact that the pullback of a pullback is a pullback (Rem. 1.11.3). $\square$

We now give another way of constructing fiber products. This construction is very explicit when X and Y are model spaces.

Proposition 1.13.10. *Let $\varphi : X \to S, \varphi : Y \to S$ be holomorphic maps of complex spaces. Let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ be the projections of $X \times Y$. Then the canonical equalizer E of the following double arrow is a fiber product $X \times_S Y$:*

$$E \xhookrightarrow{\iota} X \times Y \overset{\varphi\circ\mathrm{pr}_X}{\underset{\psi\circ\mathrm{pr}_Y}{\rightrightarrows}} S \tag{1.13.4}$$

The projections of E to X, Y are $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ respectively. We call E the (closed) ***fiber product of** X, Y **inside the direct product** $X \times Y$.*

Proof. That E is an equalizer means that $\varphi \circ (\mathrm{pr}_X \circ \iota) = \psi \circ (\mathrm{pr}_Y \circ \iota)$, and that for every holomorphic $\alpha \vee \beta : Z \to X \times Y$ whose compositions with $\varphi \circ \mathrm{pr}_X$ and with $\psi \circ \mathrm{pr}_Y$ are the same (namely, $\varphi \circ \alpha = \psi \circ \beta$) there is a unique holomorphic $\gamma : Z \to E$ such that $\iota \circ \gamma = \alpha \vee \beta$ (namely, $(\mathrm{pr}_X \circ \iota) \circ \gamma = \alpha$ and $(\mathrm{pr}_Y \circ \iota) \circ \gamma = \beta$). This means precisely that E equipped with $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ is a fiber product. $\square$

Remark 1.13.11. Using Thm. 1.8.2, we can describe the fiber product $X \times_S Y$ inside a given $X \times Y$ easily: It is $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ where $\mathscr{J}$ is a finite-type ideal. Let $x \in X, y \in Y$. If $\varphi(x) \neq \psi(y)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y, x\times y}$. If $\varphi(x) = \psi(y)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y, x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes (f \circ \psi) \tag{1.13.5}$$

for all $f \in \mathscr{O}_{S,\varphi(x)}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{S,\varphi(x)}$ analytically). The underlying topological space of $X \times_S Y$ is $\{x \times y \in X \times Y : \varphi(x) = \psi(y)\}$.

From this, it is clear that given a fiber product $X \times_S Y$, if $x \in X, y \in Y$ and $\varphi(x) = \psi(y)$, then there is a unique point of $X \times_S Y$, denoted by (x, y) or $x \times y$, whose projections to X, Y are x, y respectively. Moreover, all points of $X \times_S Y$ are in this form. $\square$

Exercise 1.13.12. Show that the pullback of $\varphi \times \psi : X \times Y \to S \times S$ along the **diagonal map** Δ_S defined by $\mathbf{1}_S \vee \mathbf{1}_S : S \to S \times S$ is a fiber product $X \times_S Y$.

We have seen that fiber products can be constructed from equalizers. Conversely, equalizers can also be viewed as special cases of fiber products:

Proposition 1.13.13. *Let $\varphi, \psi : X \to Y$ be holomorphic maps, and let $\Delta_Y : Y \to Y \times Y$ be the diagonal map of Y with image $\widetilde{Y}$ being a closed subspace of $Y \times Y$, called the* ***diagonal of*** *$Y \times Y$. Then the inverse image E of $\widetilde{Y}$ along $\varphi \vee \psi : X \to Y \times Y$ is the canonical equalizer of* $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$.

Proof. Write $\widetilde{Y}$ as $\mathrm{Specan}(\mathscr{O}_{Y\times Y}, \mathscr{J})$. Then by Rem. 1.13.7, $\mathscr{J}_{y,y'} = \mathscr{O}_{Y\times Y, y\times y'}$ if $y \neq y'$, and $\mathscr{J}_{y,y'}$ is generated by all $f \otimes 1 - 1 \otimes f$ where $f \in \mathscr{O}_{Y,y}$.

Write E as $\mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then by Prop. 1.12.1, if $\varphi(x) \neq \psi(x)$ then $\mathcal{I}_x$ equals $\mathscr{O}_{X,x}$ (since $\mathscr{J}_{\varphi(x),\psi(x)} = \mathscr{O}_{Y\times Y, \varphi(x)\times\psi(x)}$); if $\varphi(x) = \psi(x)$ then $\mathcal{I}_x$ is generated by $(f \otimes 1 - 1 \otimes f) \circ (\varphi \vee \psi)$ (i.e. by $f \circ \varphi - f \circ \psi$) for all $f \in \mathscr{O}_{Y,\varphi(x)}$. Comparing this description with Thm. 1.8.2, we see that E is the canonical equalizer. $\square$

Chapter 2

Finite holomorphic maps and coherence

2.1 Coherent sheaves

We fix a $\mathbb{C}$-ringed space X.

Definition 2.1.1. An $\mathscr{O}_X$-module $\mathscr{E}$ is called **coherent** if the following conditions are satisfied:

1. $\mathscr{E}$ is of finite-type.

2. For every open set $U \subset X$, any $n \in \mathbb{N}$, and any $\mathscr{O}_U$-module morphism $\varphi : \mathscr{O}_U^n \to \mathscr{E}|_U$, the kernel $\mathrm{Ker}\varphi$ is a finite-type $\mathscr{O}_U$-module.

Set $s_1 = \varphi(1, 0, \cdots, 0), \ldots, s_n = \varphi(0, 0, \ldots, 1)$. Then $\mathrm{Ker}\varphi$ is called the **sheaf of relations of** $s_1, \ldots, s_n$ and denoted by $\mathscr{Rel}(s_\bullet) = \mathscr{Rel}(s_1, \ldots, s_n)$.

In other words, $\mathscr{Rel}(s_\bullet)$ is the sheaf of all $(f_1, \ldots, f_n) \in \mathscr{O}_U^n$ such that $f_1 s_1 + \cdots + f_n s_n = 0$. A coherent $\mathscr{O}_X$-module is a finite-type $\mathscr{O}_X$-module such that any sheaf of relations is finite-type.

Remark 2.1.2. It is clear that a finite type submodule of a coherent $\mathscr{O}_X$-module is coherent.

Theorem 2.1.3. *Let $0 \to \mathscr{E} \to \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to 0$ be an exact sequence of $\mathscr{O}_X$-modules. If two of the three sheaves are coherent, then the remaining one is also coherent.*

We view $\mathscr{E}$ as a subsheaf of $\mathscr{F}$.

Proof of $\mathscr{E}, \mathscr{F}$ coherent $\Rightarrow \mathscr{G}$ coherent. Since $\mathscr{F}$ is finite-type and φ is surjective, $\mathscr{G}$ is finite-type. Choose any $x \in X$, any neighborhood $U \ni x$, and any $t_1, \ldots, t_n \in \mathscr{G}(U)$.

We shall show that $\mathscr{Rel}(t_\bullet)$ is generated by finitely many global sections after shrinking U to a smaller neighborhood of x.

Shrink U so that we can find $s_1,\dots,s_n \in \mathscr{F}(U)$ sent to $t_1,\dots,t_n$ by φ, and that $\mathscr{E}|_U$ is generated by some elements $e_1,\dots,e_k \in \mathscr{E}(U)$. As $\mathscr{F}$ is coherent, $\mathscr{Rel}(e_\bullet, s_\bullet)$ is finite-type. So we can further shrink U so that $\mathscr{Rel}(e_\bullet, s_\bullet)$ is generated by $(f_1^l,\dots,f_k^l,g_1^l,\dots,g_n^l) \in \mathscr{O}(U)^{k+n}$ for finitely many l.

Clearly $(g_1^l,\dots,g_n^l) \in \mathscr{O}(U)^n$ are in $\mathscr{Rel}(t_\bullet)$. We claim that they generate $\mathscr{Rel}(t_\bullet)$. Choose any $y \in U$ and $h_1,\dots,h_n \in \mathscr{O}_{X,y}$ such that $h_1t_1+\cdots+h_nt_n=0$ in $\mathscr{G}_x$. So $h_1s_1+\cdots+h_ns_n \in \mathscr{E}_y$. So $\mu_1e_1+\cdots+\mu_ke_k+h_1s_1+\cdots+h_ns_n=0$ in $\mathscr{F}_y$ for some $\mu_1,\dots,\mu_k \in \mathscr{O}_{X,y}$. So $(\mu_\bullet,h_\bullet) \in \mathscr{Rel}(e_\bullet,s_\bullet)_y$. So $(\mu_\bullet,h_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(f_\bullet^l,g_\bullet^l)$. Hence $(h_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(g_\bullet^l)$. □

Proof of $\mathscr{F},\mathscr{G}$ coherent $\Rightarrow$ $\mathscr{E}$ coherent. As $\mathscr{E}$ is a subsheaf of $\mathscr{F}$ and $\mathscr{F}$ is coherent, the sheaves of relations of $\mathscr{E}$ are clearly finite-type. Let us prove that $\mathscr{E}$ is finite-type. Choose $x \in X$ and a neighborhood $U \ni x$ such that $\mathscr{F}|_U$ is generated by $s_1,\dots,s_n \in \mathscr{F}(U)$. Then each $t_i=\varphi(s_i)$ is in $\mathscr{G}(U)$. Since $\mathscr{G}$ is coherent, $\mathscr{Rel}(t_\bullet)$ is finite-type. Thus, after shrinking U to a smaller neighborhood, $\mathscr{Rel}(t_\bullet)$ is generated by $(f_1^l,\dots,f_n^l) \in \mathscr{O}(U)^n$ for finitely many l.

Let $e^l=f_1^ls_1+\cdots+f_n^ls_n$. Then $\varphi(e^l)=0$, and hence $e^l \in \mathscr{E}(U)$. We claim that $e^1,e^2,\dots$ generate $\mathscr{E}|_U$. Choose any $y \in U$ and $\sigma \in \mathscr{E}_y$. Then $\varphi(\sigma)=0$ and $\sigma=g_1s_1+\cdots+g_ns_n$ for some $g_1,\dots,g_n \in \mathscr{O}_{X,y}$. So $(g_\bullet) \in \mathscr{Rel}(t_\bullet)_y$. Hence $(g_\bullet)$ is an $\mathscr{O}_{X,y}$-linear combination of $(f_\bullet^1),(f_\bullet^2),\dots$. So σ is the same $\mathscr{O}_{X,y}$-linear combination of $e^1,e^2,\dots$. □

Proof of $\mathscr{E},\mathscr{G}$ coherent $\Rightarrow$ $\mathscr{F}$ coherent. Step 1. We prove that $\mathscr{F}$ is finite-type. Choose $x \in X$ and a neighborhood $U \ni x$. Shrink U so that we can find $s_1,\dots,s_n \in \mathscr{F}(U)$ such that $t_1=\varphi(s_1),\dots,t_n=\varphi(s_n)$ generate $\mathscr{G}|_U$, and that there are $e_1,\dots,e_k \in \mathscr{E}(U)$ generating $\mathscr{E}|_U$. Then for each $y \in U$ and $\sigma \in \mathscr{E}_y$, $\varphi(\sigma)=f_1t_1+\cdots+f_nt_n$ for some $f_1,\dots,f_n \in \mathscr{O}_{X,y}$. So $\sigma-f_1s_1-\cdots-f_ns_n$ belongs to $\mathscr{E}_y$, which is an $\mathscr{O}_{X,y}$-linear combination of $e_1,\dots,e_k$. This shows that $s_1,\dots,s_n,e_1,\dots,e_k$ generate $\mathscr{F}|_U$.

Step 2. We prove that all sheaves of relations of $\mathscr{F}$ are finite-type. Again we choose $x \in X$ and a neighborhood $U \ni x$. Choose any $s_1,\dots,s_n \in \mathscr{F}(U)$, and let $t_\bullet=\varphi(s_\bullet)$. Since $\mathscr{Rel}(t_\bullet)$ is finite-type, we may shrink U to a smaller neighborhood such that we can find $G \in \mathscr{O}(U)^{n\times k}$ (i.e. an $\mathscr{O}(U)$-valued $n\times k$ matrix) such that the columns $G_{\bullet,1},\dots,G_{\bullet,k} \in \mathscr{O}(U)^n$ generate $\mathscr{Rel}(t_\bullet)$. Set

$$(e_1,\dots,e_k)=(s_1,\dots,s_n)G \qquad \in \mathscr{F}(U)^k,$$

namely, $e_j=\sum_{i=1}^n s_iG_{i,j}$. Then $e_1,\dots,e_n$ are killed by φ, i.e. they are in $\mathscr{E}(U)$. As $\mathscr{Rel}(e_\bullet)$ is finite-type, we may shrink U and find a $k\times m$ matrix $E \in \mathscr{O}(U)^{k\times m}$ whose columns generate $\mathscr{Rel}(e_\bullet)$. Let $F=GE$ (which is in $\mathscr{O}(U)^{n\times m}$). We claim that the columns of F generate $\mathscr{Rel}(s_\bullet)$.

Choose any $y \in U$ and an element of $\mathscr{Rel}(s_\bullet)_y$, written as an $n \times 1$ matrix $A \in \mathscr{O}_{X,x}^{n\times 1}$. So $(s_1, \dots, s_n)A = 0$. Hence $(t_1, \dots, t_n)A = 0$. So $A \in \mathscr{Rel}(t_\bullet)_y$. Since $G_{\bullet,1}, \dots, G_{\bullet,k}$ generate $\mathscr{Rel}(t_\bullet)_y$, we may write $A = GB$ for some $B \in \mathscr{O}_{X,y}^{k\times 1}$. So $(e_1, \dots, e_k)B = 0$. Thus, as $E_{\bullet,1}, \dots, E_{\bullet,m}$ generate $\mathscr{Rel}(e_\bullet)_y$, we may write $B = EC$ for some $C \in \mathscr{O}_{X,y}^{m\times 1}$. Thus $A = FC$. So A is an $\mathscr{O}_{X,y}$-linear combination of columns of F. □

Corollary 2.1.4. *$\mathscr{E}_1, \mathscr{E}_2$ are coherent $\mathscr{O}_X$-modules if and only if $\mathscr{E}_1 \oplus \mathscr{E}_2$ is coherent.*

Proof. The exactness of $0 \to \mathscr{E}_1 \to \mathscr{E}_1 \oplus \mathscr{E}_2 \to \mathscr{E}_2 \to 0$ shows that "$\mathscr{E}_1, \mathscr{E}_2$ coherent" $\Rightarrow$ "$\mathscr{E}_1 \oplus \mathscr{E}_2$ coherent", and that if $\mathscr{E}_1 \oplus \mathscr{E}_2$ is coherent then $\mathscr{E}_2$ is finite type and the sheaves of relations of $\mathscr{E}_1$ are finite-type. Exchanging the roles of $\mathscr{E}_1, \mathscr{E}_2$ shows that "$\mathscr{E}_1 \oplus \mathscr{E}_2$ coherent" $\Rightarrow$ "$\mathscr{E}_1, \mathscr{E}_2$ coherent". □

Corollary 2.1.5. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of coherent $\mathscr{O}_X$-modules. Then* $\mathrm{Im}\varphi, \mathrm{Ker}\varphi, \mathrm{Coker}\varphi$ *are coherent.*

Proof. $\mathrm{Im}\varphi$ is finite-type since $\mathscr{F} \to \mathrm{Im}\varphi$ is surjective and $\mathscr{F}$ is finite-type. The sheaves of relations of $\mathrm{Im}\varphi$ are finite-type because $\mathscr{G}$ is coherent and $\mathrm{Im}\varphi$ is its $\mathscr{O}_X$-submodule. So $\mathrm{Im}\varphi$ is coherent. That $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\varphi$ are coherent follows from Thm. 2.1.3 and the exact sequences $0 \to \mathrm{Ker}\varphi \to \mathscr{F} \to \mathrm{Im}\varphi \to 0$ and $0 \to \mathrm{Im}\varphi \to \mathscr{G} \to \mathrm{Coker}\varphi \to 0$. □

Corollary 2.1.6. *If $\mathscr{E}, \mathscr{F}$ are coherent $\mathscr{O}_X$-submodules of a coherent $\mathscr{O}_X$-module $\mathscr{G}$, then $\mathscr{E} + \mathscr{F}$ and $\mathscr{E} \cap \mathscr{F}$ are coherent.*

Note that the **intersection sheaf** $\mathscr{E} \cap \mathscr{F}$ is defined to be the sheaf of all sections of $\mathscr{G}$ whose germ at each $x \in X$ belongs to $\mathscr{E}_x \cap \mathscr{F}_x$. It is easy to check that $(\mathscr{E} \cap \mathscr{F})_x$ is canonically equivalent to $\mathscr{E}_x \cap \mathscr{F}_x$.

Proof. Clearly $\mathscr{E} + \mathscr{F}$ is finite-type and hence coherent. So by Cor. 2.1.5, $\mathscr{E}/(\mathscr{E} \cap \mathscr{F}) \simeq (\mathscr{E} + \mathscr{F})/\mathscr{F}$ is coherent, and hence $\mathscr{E} \cap \mathscr{F}$ is coherent. □

Theorem 2.1.7. *Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module. Then an $\mathscr{O}_X$-module $\mathscr{E}$ is coherent if and only if for each $x \in X$ there is a neighborhood $U \ni x$ such that $\mathscr{E}|_U$ is isomorphic to* $\mathrm{Coker}\varphi$ *for some morphism of free $\mathscr{O}_U$-modules $\varphi : \mathscr{O}_U^m \to \mathscr{O}_U^n$ (where $m, n \in \mathbb{N}$).*

Indeed, the "only if" part does not need $\mathscr{O}_X$ to be coherent.

Proof. "If": Since $\mathscr{O}_U$ is coherent, $\mathscr{O}_U^m$ and $\mathscr{O}_U^n$ are coherent. So $\mathrm{Coker}\varphi$ is coherent by Cor. 2.1.5.

"Only if": Let $\mathscr{E}$ be coherent. Choose $x \in X$. Since $\mathscr{E}$ is finite-type, we may find a neighborhood U such that there is a surjective $\psi : \mathscr{O}_U^n \to \mathscr{E}|_U$. Since $\mathscr{E}$ is coherent, $\mathrm{Ker}\psi$ is finite-type. Thus, after shrinking U, we may find a surjective $\pi : \mathscr{O}_U^m \to \mathrm{Ker}\psi$. Then $\mathscr{E}|_U \simeq \mathrm{Coker}(\iota \circ \pi)$ where $\iota : \mathrm{Ker}\psi \to \mathscr{O}_U^n$ is the inclusion. □

Corollary 2.1.8. *For any coherent $\mathscr{O}_X$-modules $\mathscr{E}, \mathscr{F}$, the tensor product $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ is coherent.*

Proof. Choose any $x \in X$. By Thm. 2.1.7, we may shrink X to a neighborhood of x such that $\mathscr{E} \simeq \mathrm{Coker}\varphi$ where $\varphi : \mathscr{O}_X^m \to \mathscr{O}_X^n$ is a morphism. By the right exactness of $- \otimes \mathscr{F}$ (cf. Prop. 1.9.5), $\mathscr{E} \otimes \mathscr{F}$ is equivalent to $\mathrm{Coker}(\mathscr{O}_X^m \otimes \mathscr{F} \to \mathscr{O}_X^n \otimes \mathscr{F})$, which is $\mathrm{Coker}(\mathscr{F}^m \to \mathscr{F}^n)$. By Cor. 2.1.4, $\mathscr{F}^m, \mathscr{F}^n$ are coherent. So the cokernel is coherent by Cor. 2.1.5. □

We end this section with some more criteria on coherence.

Proposition 2.1.9. *Let $\varphi : X \to S$ be a morphism of $\mathbb{C}$-ringed spaces, and let $\mathscr{E}$ be a finite-type $\mathscr{O}_S$-module. Then $\varphi^*\mathscr{E}$ is a finite type $\mathscr{O}_X$-module. If moreover $\mathscr{E}$ is $\mathscr{O}_S$-coherent and $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent, then $\varphi^*\mathscr{E}$ is a coherent $\mathscr{O}_X$-module.*

Proof. If $\mathscr{E}$ is finite-type, then for each $x \in X$, we may shrink X to a neighborhood of x such that $\mathscr{E}$ is generated by finitely many $s_1, s_2, \dots \in \mathscr{E}(X)$. So $\varphi^*\mathscr{E} = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{E}$ is generated by all $\varphi^* s_i = 1 \otimes s_i$. So $\varphi^*\mathscr{E}$ is finite-type.

Now assume $\mathscr{E}$ is $\mathscr{O}_S$-coherent and $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent. By Thm. 2.1.7, we may shrink X so that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_S^m \to \mathscr{O}_S^n)$. Then

$$\begin{aligned} &\varphi^*\mathscr{E} \simeq \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathrm{Coker}(\mathscr{O}_S^m \to \mathscr{O}_S^n) \simeq \mathrm{Coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S^m \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S^n) \\ \simeq &\mathrm{Coker}(\mathscr{O}_X^m \to \mathscr{O}_X^n) \end{aligned}$$

which is $\mathscr{O}_X$-coherent by Thm. 2.1.7 □

Proposition 2.1.10 (Extension principle). *Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ be a closed complex subspace of a complex space X where $\mathcal{I}$ is finite-type. Let $\iota : Y \to X$ be the inclusion, and let $\mathscr{E}$ be an $\mathscr{O}_Y$-module. Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module. Then $\mathscr{E}$ is a coherent $\mathscr{O}_Y$-module if and only if $\iota_*\mathscr{E}$ is a coherent $\mathscr{O}_X$-module.*

Extension principle is an important special case of Finite mapping Thm. 2.7.1 which we will prove later.

Proof. We identify $\mathscr{E}$ with $\iota_*\mathscr{E}$ and $\mathscr{O}_Y$ with $\iota_*\mathscr{O}_Y = \mathscr{O}_X/\mathcal{I}$. (Cf. Rem. 1.10.5.) Clearly $\mathcal{I}$ is $\mathscr{O}_X$-coherent. So $\mathscr{O}_Y$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5.

Assume $\iota_*\mathscr{E}$ is $\mathscr{O}_X$-coherent. Then by Prop. 2.1.9, $\mathscr{E} \simeq \iota^*\iota_*\mathscr{E}$ is a finite-type $\mathscr{O}_Y$-module. Suppose that after shrinking X we have a morphism $\alpha : \mathscr{O}_Y^n \to \mathscr{E}$. Since $\mathscr{O}_Y^n$ is $\mathscr{O}_X$-coherent, $\mathrm{Ker}\alpha$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5. So $\mathrm{Ker}\alpha$ (or more precisely, $\iota_*(\mathrm{Ker}\alpha)$) is a finite-type $\mathscr{O}_X$-module. So by Prop. 2.1.9, it is a finite-type $\mathscr{O}_Y$-module.

Assume $\mathscr{E}$ is $\mathscr{O}_Y$-coherent. Then by Thm. 2.1.7, $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_Y^m \to \mathscr{O}_Y^n)$ after shrinking X to a neighborhood of $x \in Y \subset X$. Since $\mathscr{O}_Y$ is $\mathscr{O}_X$-coherent, by Cor. 2.1.4, $\mathscr{O}_Y^m, \mathscr{O}_Y^n$ are $\mathscr{O}_X$-coherent. So $\mathscr{E}$ is $\mathscr{O}_X$-coherent by Cor. 2.1.5. □

Corollary 2.1.11. *Let Y be a closed complex subspace of X. Assume $\mathscr{O}_X$ is $\mathscr{O}_X$-coherent. Then $\mathscr{O}_Y$ is $\mathscr{O}_Y$-coherent.*

Proof. Write $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ where $\mathcal{I}$ is a finite-type ideal of $\mathscr{O}_X$. So $\mathcal{I}$ is $\mathscr{O}_X$-coherent. Hence $\mathscr{O}_Y = \mathscr{O}_X/\mathcal{I}$ is $\mathscr{O}_X$-coherent, and hence $\mathscr{O}_Y$-coherent by Extension principle. □

Thus, if we can show that $\mathscr{O}_{\mathbb{C}^n}$ is coherent for any n, then all model spaces, and hence all complex spaces have coherent structure sheaves.

2.2 Germs of coherent sheaves; coherence of hom sheaves

Let X be a $\mathbb{C}$-ringed space.

An important reason for studying coherent sheaves is that germs of coherent sheaves are equivalent to finitely-generated modules of local analytic $\mathbb{C}$-algebras, just as germs of complex spaces are equivalent to local analytic $\mathbb{C}$-algebras (Thm. 1.6.2). Let us be more precise.

Definition 2.2.1. Let X be a $\mathbb{C}$-ringed space and $x \in X$. The **category of germs of coherent modules at** x is the category whose objects are coherent $\mathscr{O}_U$-modules $\mathscr{E}_U$ where $U \ni x$ is open. If $V \subset U$ is a neighborhood of x, then $\mathscr{E}_U$ and $\mathscr{E}_V := \mathscr{E}_U|_V$ are viewed as the same object.

A **morphism** between two objects $\mathscr{E}_U, \mathscr{F}_U$ is an element $\varphi \in \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{F}_V)$ for a possibly smaller neighborhood $V \ni x$. Two morphisms are regarded as equal if then agree when restricted to a possibly smaller neighborhood of x on which both are defined. Compositions of morphisms are defined in the obvious way. Thus, in this category the set of morphisms from $\mathscr{E}_U$ to $\mathscr{F}_U$ is precisely the stalk $\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x$ of $\mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)$. □

Theorem 2.2.2. *Let X be a $\mathbb{C}$-ringed space and $x \in X$. Assume that $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module, and $\mathscr{O}_{X,x}$ is Noetherian. Then the functor $\mathfrak{F}$ from the category of germs of coherent modules at x to the category of finitely-generated $\mathscr{O}_{X,x}$-modules, sending $\mathscr{E}_U$ to the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$ and sending each $\varphi \in \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x$ (namely, each $\varphi \in \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{F}_V)$ for a possibly smaller neighborhood $V \ni x$) to the corresponding stalk map $\mathscr{E}_x \to \mathscr{F}_x$, is an **equivalence of categories**. Namely, the following two statements hold:*

(1) For each objects $\mathscr{E}_U, \mathscr{F}_U$, the following $\mathscr{O}_{X,x}$-module morphism is bijective:

$$\mathfrak{F} : \mathscr{H}om_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)_x \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \tag{2.2.1}$$

(2) Each finitely-generated $\mathscr{O}_{X,x}$-module is isomorphic to $\mathfrak{F}(\mathscr{E}_U)$ for some object $\mathscr{E}_U$. Namely, it is isomorphic to $\mathscr{E}_{U,x}$.

Remark 2.2.3. If only (1) resp. (2) is satisfied, we say $\mathfrak{F}$ is **fully-faithful** resp. **essentially surjective**. These names also apply to contravariant functors.

From the proof, we shall see that the $\mathfrak{F}$ in (2.2.1) is an isomorphism even without assuming that $\mathscr{O}_X, \mathscr{F}_U$ are coherent or $\mathscr{O}_{X,x}$ is Noetherian.

Proof of (2). Choose any finitely generated $\mathscr{O}_{X,x}$-module $\mathcal{M}$. Then we have a surjective morphism $\alpha : \mathscr{O}_{X,x}^n \to \mathcal{M}$. $\mathrm{Ker}\alpha$ is an $\mathscr{O}_{X,x}$-submodule of $\mathscr{O}_{X,x}^n$, which is finitely-generated since $\mathscr{O}_{X,x}$ is Noetherian. Thus we have a surjective $\beta : \mathscr{O}_{X,x}^m \to \mathrm{Ker}\alpha$. Let $\gamma : \mathscr{O}_{X,x}^m \to \mathscr{O}_{X,x}^n$ be the composition of β and the inclusion $\iota : \mathrm{Ker}\alpha \to \mathscr{O}_{X,x}^n$. Then $\mathcal{M} \simeq \mathrm{Coker}\gamma$.

We can extend γ to an $\mathscr{O}_U$-module morphism $\varphi : \mathscr{O}_U^m \to \mathscr{O}_U^n$ for some neighborhood $U \ni x$. Namely, the stalk map of φ at x is γ. (For instance, choose U such that $s_1 = \gamma(1, 0, \dots, 0), \dots, s_n = \gamma(0, 0, \dots, 1) \in \mathscr{O}_{X,x}^n$ can be defined on U. Then φ is defined to be the $\mathscr{O}_U$-module morphism sending $(1, 0, \dots, 0) \in \mathscr{O}(U)^m$ to $s_1 \in \mathscr{O}(U)^n$, etc., and $(0, 0, \dots, 1)$ to s_n.) Then $\mathrm{Coker}\varphi$ is a coherent $\mathscr{O}_U$-module (Cor. 2.1.4 and 2.1.5) whose stalk at x is $\mathrm{Coker}\gamma \simeq \mathcal{M}$. $\square$

Proof of (1). Choose an $\mathscr{O}_U$-module morphism $\varphi : \mathscr{E}_U \to \mathscr{F}_U$ such that $\mathfrak{F}(\varphi) = 0$. So the stalk map $\varphi : \mathscr{E}_{U,x} \to \mathscr{F}_{U,x}$ is zero. Since $\mathscr{E}_U$ is finite-type, $\mathscr{E}_{U,x}$ is finitely-generated. So we may choose $s_1, \dots, s_n \in \mathscr{E}_{U,x}$ generating $\mathscr{E}_{U,x}$. We may find a neighborhood $V \ni x$ in U such that $s_1, \dots, s_n \in \mathscr{E}(V)$, that $\varphi(s_1) = \cdots = \varphi(s_n) = 0$ in $\mathscr{F}(V)$, and that (by Rem. 1.2.16 and that $\mathscr{E}_U$ is finite-type) $s_1, \dots, s_n$ generate $\mathscr{E}_V$. So φ sends all sections of $\mathscr{E}_V$ to 0. This proves that $\mathfrak{F}$ is injective and uses only the condition that $\mathscr{E}_U$ is finite-type.

We now prove that $\mathfrak{F}$ is surjective. Choose any $\eta \in \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x)$. By Thm. 2.1.7, there is a neighborhood $V \ni x$ inside U and an $\mathscr{O}_V$-module morphism $\alpha : \mathscr{O}_V^m \to \mathscr{O}_V^n$ such that $\mathscr{E}_V = \mathrm{Coker}(\alpha)$. Let $\pi_x : \mathscr{O}_{V,x}^n \to \mathscr{E}_x = \mathrm{Coker}(\alpha_x : \mathscr{O}_{V,x}^m \to \mathscr{O}_{V,x}^n)$ be the quotient map. Let η' be $\mathscr{O}_{V,x}^n \xrightarrow{\pi_x} \mathscr{E}_x \xrightarrow{\eta} \mathscr{F}_x$. Then as argued in the proof of part (2), the stalk map η' can be extended to an $\mathscr{O}_V$-module morphism $\widetilde{\eta}' : \mathscr{O}_V^n \to \mathscr{F}_V$ after shrinking V. $\widetilde{\eta}' \circ \alpha : \mathscr{O}_V^m \to \mathscr{F}_V$ has stalk map $\eta \circ \pi_x \circ \alpha_x$ at x, which is 0. So by the injectivity of $\mathfrak{F}$, we may shrink V so that $\widetilde{\eta}' \circ \alpha = 0$. So $\widetilde{\eta}'$ equals $\mathscr{O}_V^n \xrightarrow{\pi} \mathscr{E}_V = \mathrm{Coker}(\alpha) \xrightarrow{\widetilde{\eta}} \mathscr{F}_V$ for some $\mathscr{O}_V$-module morphism $\widetilde{\eta}$. Then $\widetilde{\eta}_x = \eta$, i.e. $\mathfrak{F}(\widetilde{\eta}) = \eta$. $\square$

Let us emphasize the following crucial special case of Thm. 2.2.2:

Corollary 2.2.4. *Let X be a $\mathbb{C}$-ringed space and $x \in X$. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{O}_X$-modules. Then the canonical $\mathscr{O}_{X,x}$-module morphism*

$$\mathfrak{F} : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x, \mathscr{F}_x) \tag{2.2.2}$$

is injective if $\mathscr{E}$ is finite-type, and is bijective if $\mathscr{E}$ is coherent.

Corollary 2.2.5. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module.*

1. *The contravariant functor $\mathscr{Hom}_{\mathscr{O}_X}(-,\mathscr{F})$ on the category of coherent $\mathscr{O}_X$-modules is left exact, where the contravariant functor sends each $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_1,\mathscr{E}_2)$ to $\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}_2,\mathscr{F}) \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}_1,\mathscr{F}), \psi \mapsto \psi \circ \varphi$.*

2. *Assume that $\mathscr{F}$ is coherent. Then the functor $\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},-)$ on the category of $\mathscr{O}_X$-modules is left exact, where the functor sends each $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}_1,\mathscr{E}_2)$ to $\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}_1) \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}_2), \psi \mapsto \varphi \circ \psi$.*

Note that these two exactness is equivalent to saying that we have equivalences

$$\mathscr{Hom}_{\mathscr{O}_X}\big(\mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2),\mathscr{F}\big) \simeq \mathrm{Ker}\big(\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}_2,\mathscr{F}) \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}_1,\mathscr{F})\big) \tag{2.2.3a}$$

$$\mathscr{Hom}_{\mathscr{O}_X}\big(\mathscr{F},\mathrm{Ker}(\mathscr{E}_1 \to \mathscr{E}_2)\big) \simeq \mathrm{Ker}\big(\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}_1) \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}_2)\big) \tag{2.2.3b}$$

induced by the obvious inclusions

$$\mathscr{Hom}_{\mathscr{O}_X}\big(\mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2),\mathscr{F}\big) \hookrightarrow \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}_2,\mathscr{F}),$$
$$\mathscr{Hom}_{\mathscr{O}_X}\big(\mathscr{F},\mathrm{Ker}(\mathscr{E}_1 \to \mathscr{E}_2)\big) \hookrightarrow \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{F},\mathscr{E}_1).$$

Proof. Let $\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules. Then we have $0 \to \mathscr{Hom}(\mathscr{F},\mathscr{E}_3) \to \mathscr{Hom}(\mathscr{F},\mathscr{E}_2) \to \mathscr{Hom}(\mathscr{F},\mathscr{E}_1)$ which, thanks to Cor. 2.2.4, gives stalk maps $0 \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{E}_{3,x}) \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{E}_{2,x}) \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x,\mathscr{E}_{1,x})$ at each $x \in X$ which is exact by Rem. 1.9.6. This proves part 1. Part 2 is proved in a similar way. □

Corollary 2.2.6. *Assume that $\mathscr{E},\mathscr{F}$ are coherent $\mathscr{O}_X$-modules. Then $\mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ is coherent. So $\mathscr{E}^\vee$ is coherent if $\mathscr{E},\mathscr{O}_X$ are coherent.*

Proof. If $\mathscr{E} = \mathscr{O}_X^n$ then $\mathscr{Hom}(\mathscr{E},\mathscr{F}) \simeq \mathscr{F}^n$ is coherent by Cor. 2.1.4. In the general case, choose $x \in X$. Then by Thm. 2.1.7 we may shrink X to a neighborhood of x such that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{E}_1 \to \mathscr{E}_2)$ where $\mathscr{E}_1,\mathscr{E}_2$ are free $\mathscr{O}_X$-modules. The coherence of $\mathscr{Hom}(\mathscr{E},\mathscr{F})$ follows from (2.2.3a) and Cor. 2.1.5. □

2.3 Supports and annihilators of coherent sheaves; image spaces

In this section, we assume X, Y are complex spaces.

From Rem. 1.10.5, we know that if $\mathcal{I}$ is a finite-type ideal of $\mathscr{O}_X$ annihilating an $\mathscr{O}_X$-module $\mathscr{E}$, then the study of $\mathscr{E}$ is equivalent to the study of the $\mathscr{O}_Y$-module $\mathscr{E}|_Y$ where $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. A natural question is whether we can find a largest such $\mathcal{I}$, i.e., a smallest such Y. To study this problem, we introduce:

Definition 2.3.1. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the **annihilator sheaf** of $\mathscr{E}$, written as $\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})$ or simply $\mathscr{Ann}(\mathscr{E})$, is the ideal sheaf of $\mathscr{O}_X$ defined to be the kernel of the $\mathscr{O}_X$-module morphism $\mathscr{O}_X \to \mathscr{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) =: \mathscr{End}_{\mathscr{O}_X}(\mathscr{E})$ sending each $f \in \mathscr{O}_X$ to the multiplication of f on $\mathscr{E}$. So we have an exact sequence

$$0 \to \mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E}) \to \mathscr{O}_X \to \mathscr{End}_{\mathscr{O}_X}(\mathscr{E}). \tag{2.3.1}$$

If $\mathscr{E}$ and $\mathscr{O}_X$ are coherent then so is $\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})$ (due to Cor. 2.1.5 and 2.2.6).

Similarly, if A is a commutative ring and $\mathcal{M}$ an A-module, then the **annihilator** $\mathrm{Ann}_A(\mathcal{M})$ is defined to be the kernel of $A \to \mathrm{End}_A(\mathcal{M})$. □

Remark 2.3.2. (2.3.1) gives an exact sequence of stalk maps at each x. Assume that $\mathscr{E}$ is $\mathscr{O}_X$-coherent. Then by Prop. 2.2.4, $\mathscr{End}_{\mathscr{O}_X}(\mathscr{E})_x \simeq \mathrm{End}_{\mathscr{O}_{X,x}}(\mathscr{E}_x)$. This shows that we have a canonical equivalence of $\mathscr{O}_{X,x}$-modules

$$\mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})_x \simeq \mathrm{Ann}_{\mathscr{O}_{X,s}}(\mathscr{E}_x) \tag{2.3.2}$$

if $\mathscr{E}$ is coherent.

Definition 2.3.3. Assume $\mathscr{O}_X$ is coherent. Given a coherent $\mathscr{O}_X$-module $\mathscr{E}$, we define the **support of** $\mathscr{E}$, written as $\mathrm{Supp}(\mathscr{E})$, to be the complex space

$$\mathrm{Supp}(\mathscr{E}) = \mathrm{Specan}\big(\mathscr{O}_X / \mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})\big). \tag{2.3.3}$$

Remark 2.3.4. $\mathrm{Ann}(\mathscr{E}_x) = \mathscr{O}_{X,x}$ iff $1 \in \mathrm{Ann}(\mathscr{E}_x)$ iff 1 annihilates $\mathscr{E}_x$ iff $\mathscr{E}_{X,x} = 0$. This shows that the underlying topological space of $\mathrm{Supp}(\mathscr{E})$ defined above (i.e. the set of all x such that $\mathscr{O}_{X,x}/\mathscr{Ann}(\mathscr{E})_x \neq 0$) agrees with the usual one (i.e. the set of all x such that $\mathscr{E}_x \neq 0$) when $\mathscr{E}$ is coherent.

Remark 2.3.5. We know that the support (as a set) of a finite-type $\mathscr{O}_X$-module is a closed subset of X (Cor. 1.2.17). Now we know that if $\mathscr{E}, \mathscr{O}_X$ are coherent, then $\mathrm{Supp}(\mathscr{E})$ as a set is an **analytic subset** of X, which means that it is $N(\mathcal{I})$ for a finite-type ideal $\mathcal{I}$.

Convention 2.3.6. If $\mathscr{E}, \mathscr{O}_X$ are coherent, we understand $\mathrm{Supp}(\mathscr{E})$ as a complex subspace of X. Otherwise we understand it as only a subset of X.

Exercise 2.3.7. Show that if $\mathcal{I}$ is a finite-type (and hence coherent) ideal of $\mathscr{O}_X$, then

$$\mathrm{Supp}(\mathscr{O}_X/\mathcal{I}) = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}). \tag{2.3.4}$$

Definition 2.3.8. Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Assume that $\mathscr{O}_Y, \varphi_*\mathscr{O}_X$ are coherent $\mathscr{O}_Y$-modules and $\mathrm{Im}(\varphi) = \{\varphi(x) : x \in X\}$ is a closed subset of Y. We define the **image space** $\varphi(X)$ of φ to be

$$\varphi(X) = \mathrm{Supp}(\varphi_*\mathscr{O}_X) = \mathrm{Specan}\big(\mathscr{O}_Y / \mathscr{Ann}_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)\big). \tag{2.3.5}$$

The notation $\varphi(X)$ and the name "image space" is justified by the following lemma.

Lemma 2.3.9. *The underlying topological space of $\varphi(X)$ is the usual one $\mathrm{Im}(\varphi) = \{\varphi(x) : x \in X\}$. In particular, $\mathrm{Im}(\varphi)$ is an analytic subset of Y.*

Proof. Choose $y \in Y$. We show that $(\varphi_*\mathscr{O}_X)_y = 0$ iff $y \notin \mathrm{Im}(\varphi)$. First assume $(\varphi_*\mathscr{O}_X)_y = 0$. Choose a neighborhood V of y. The non-zero element $1 \in (\varphi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\varphi^{-1}(V))$ becomes 0 in $(\varphi_*\mathscr{O}_X)_y$, which means that we may shrink V so that $1 = 0$ in $\mathscr{O}_X(\varphi^{-1}(V))$. So $\varphi^{-1}(V) = \varnothing$. Hence $y \notin \mathrm{Im}(\varphi)$. Conversely, suppose $y \notin \mathrm{Im}(\varphi)$. Since $\mathrm{Im}(\varphi)$ is closed, we may find a small enough neighborhood $V \ni y$ such that $\varphi^{-1}(V) = \varnothing$. So $(\varphi_*\mathscr{O}_X)_y = 0$. □

Remark 2.3.10. In the setting of Def. 2.3.8, using (2.3.2), it is easy to see that we have a canonical equivalence of $\mathscr{O}_{Y,y}$-modules

$$\mathscr{A}nn_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)_y \simeq \mathrm{Ker}\big(\varphi^{\#} : \mathscr{O}_{Y,y} \to (\varphi_*\mathscr{O}_X)_y\big). \tag{2.3.6}$$

To study a coherent sheaf $\mathscr{E}$ one can restrict the underlying complex space to $\mathrm{Supp}(\mathscr{E})$. Likewise, to study φ when $\varphi_*\mathscr{O}_X$ and $\mathscr{O}_Y$ are coherent and $\mathrm{Im}(\varphi)$ is closed, one can restrict the codomain of φ to $\varphi(X)$:

Proposition 2.3.11. *Let $\varphi : X \to Y$ be holomorphic. Assume that $\mathscr{O}_Y, \varphi_*\mathscr{O}_X$ are coherent $\mathscr{O}_Y$-modules and $\mathrm{Im}(\varphi)$ is closed in Y. Then there is a unique holomorphic map $\widetilde{\varphi} : X \to \varphi(Y)$ (the restriction of φ) such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}"'] & & Y \\ & \varphi(X) \arrow[ur, hook] & \end{tikzcd}$$

Proof. This follows immediately from Thm. 1.4.8. □

Let us give another application of supports of coherent sheaves. Recall that if A is a commutative ring and $\mathcal{M}$ is an A-module, an element $a \in A$ is called a **zero divisor of** $\mathcal{M}$ if $a\xi = 0$ for a non-zero $\xi \in \mathcal{M}$. Equivalently a is a zero divisor iff $\mathrm{Ker}(\mathcal{M} \xrightarrow{\times a} \mathcal{M})$ is non-zero. If a is not a zero divisor of $\mathcal{M}$, we call it a **non zero-divisor of** $\mathcal{M}$, not to be confused with a **non-zero zero divisor**, which is by definition a zero divisor which itself is not zero. Finally, a zero divisor means a zero divisor of A.

In the following we assume $\mathscr{O}_X$ is coherent, which is redundant after Oka's coherence theorem is proved.

Proposition 2.3.12. *Let X be a complex space, $\mathscr{E}$ a coherent $\mathscr{O}_X$-module, and choose $f \in \mathscr{O}(X)$. Then*

$$Z = \{x \in X : \text{The germ of } f \text{ at } x \text{ is a zero divisor of } \mathscr{E}_x\}$$

is an analytic subset of X. In particular, the set of $x \in X$ such that f is a non zero-divisor of $\mathscr{E}_x$ is open in X.

Proof. Let $\mathscr{K} = \mathrm{Ker}(\mathscr{E} \xrightarrow{\times f} \mathscr{E})$, which is coherent by Cor. 2.1.5. Then $\mathrm{Supp}(\mathscr{K})$ is a complex subspace of X. A point $x \in X$ belongs to $\mathrm{Supp}(\mathscr{K})$ iff $\mathscr{K}_x = \mathrm{Ker}(\mathscr{E}_x \xrightarrow{\times f} \mathscr{E}_x)$ is non-zero iff f is a zero divisor of $\mathscr{E}_x$. This shows that Z equals $\mathrm{Supp}(\mathscr{K})$ as sets. $\square$

2.4 Finite maps and proper maps

The proof of coherence of the structure sheaves of complex spaces is closely related to the study of finite holomorphic maps $\varphi : X \to Y$ and the coherence of $\varphi_*\mathscr{O}_X$. In this section, we discuss finite maps in the purely topological setting.

We assume X, Y are topological spaces. Recall that a continuous map $\varphi : X \to Y$ is called **closed** if φ sends closed subsets of X to closed subsets of Y.

Proposition 2.4.1. *Let $\varphi : X \to Y$ be a continuous map. Then the following are equivalent.*

(1) φ is a closed map.

(2) For each $y \in Y$,

$$\{\varphi^{-1}(V) : V \subset Y \text{ is a neighborhood of } y\}$$

*is a **basis of neighborhoods of** $\varphi^{-1}(y)$, which means that for each open $U \subset X$ containing $\varphi^{-1}(y)$ there is a neighborhood $V \ni y$ such that $\varphi^{-1}(V) \subset U$.*

Proof. Assume (1). For each open $U \subset X$ containing $\varphi^{-1}(y)$, let $V \subset Y$ be defined by $Y\backslash V = \varphi(X\backslash U)$ where $\varphi(X\backslash U)$ is closed because φ is closed. So V is open and clearly contains y. Since $V \cap \varphi(X\backslash U) = \emptyset$, $\varphi^{-1}(V) \cap (X\backslash U) = \emptyset$. So $\varphi^{-1}(V) \subset U$. This proves (2).

Assume (2). Choose any closed subset $E \subset X$. We shall show that $\varphi(E)$ is closed in Y. Choose any $y \in Y\backslash\varphi(E)$. Then $X\backslash E$ is a neighborhood of $\varphi^{-1}(y)$. So we can choose a neighborhood $V \subset Y$ of y such that $\varphi^{-1}(V) \subset X\backslash E$. So $\varphi^{-1}(V) \cap E = \emptyset$, and hence $V \cap \varphi(E) = \emptyset$. This proves that y is an interior point of $Y\backslash\varphi(E)$. So $Y\backslash\varphi(E)$ is open, and (1) is proved. $\square$

Remark 2.4.2. The above proposition shows that closedness is a local property (with respect to the base Y): If Y has an open cover $(V_\alpha)_\alpha$ such that for each α, the restriction $\varphi : \varphi^{-1}(V_\alpha) \to V_\alpha$ is closed. Then $\varphi : X \to Y$ is closed.

Definition 2.4.3. A continuous map $\varphi : X \to Y$ is called **finite** if it is a closed map and if $\varphi^{-1}(y)$ is a finite set for all $y \in Y$. The composition of two finite maps is clearly finite. If $\varphi : X \to Y$ is a holomorphic map of complex spaces which is finite as a continuous map of topological spaces, we say φ is a **finite holomorphic map**.

Remark 2.4.4. A main reason that we require finite maps to be closed is the following: Suppose φ is finite. Given $y \in Y$, choose mutually disjoint neighborhoods $U_x \subset X$ for all $x \in \varphi^{-1}(y)$. Then by Prop. 2.4.1, there is a sufficiently small neighborhood $V \subset Y$ of y such that

$$\varphi^{-1}(V) = \coprod_{x \in \varphi^{-1}(y)} \varphi^{-1}(V) \cap U_x. \tag{2.4.1}$$

In other words, we can shrink each U_x to a smaller neighborhood of x such that

$$\varphi^{-1}(V) = \coprod_{x \in \varphi^{-1}(y)} U_x. \tag{2.4.2}$$

From this it is clear that the restriction $\varphi|_{U_x} : U_x \to Y$ is finite.

As applications of this observation, we prove several important facts about direct images.

Proposition 2.4.5. *Let $\varphi : X \to Y$ be a finite continuous map, and let $\mathscr{E}$ be an X-sheaf. Then for each $y \in Y$, we have an isomorphism of abelian groups*

$$\Phi : (\varphi_*\mathscr{E})_y \xrightarrow{\simeq} \bigoplus_{x \in \varphi^{-1}(y)} \mathscr{E}_x \tag{2.4.3}$$

defined componentwisely by the obvious restriction maps.

If φ is a morphism of $\mathbb{C}$-ringed spaces and $\mathscr{E}$ is an $\mathscr{O}_Y$-module, then Φ is clearly an isomorphism of $\mathscr{O}_{Y,y}$-modules. Moreover, Φ *is an isomorphism of* $(\varphi_*\mathscr{O}_X)_y$*-modules* if we let $(\varphi_*\mathscr{O}_X)_y \simeq \bigoplus_{x \in \varphi^{-1}(y)} \mathscr{O}_{X,x}$ act on the codomain of Φ componentwisely.

Proof. Ψ is defined by passing to the direct limit of the map

$$\Phi_V : \mathscr{E}(\varphi^{-1}(V)) \to \bigoplus_{x \in \varphi^{-1}(y)} \mathscr{E}_x \tag{2.4.4}$$

over all open $V \ni y$. If $s \in \mathscr{E}(\varphi^{-1}(V))$ and $\Phi_V(s) = 0$, then we may find disjoint neighborhoods $U_x \ni x$ such that $s|_{U_x} = 0$. After shrinking V such that (2.4.1) holds, we have $s = 0$. So Φ is injective.

On the other hand, choose $s_x \in \mathscr{E}_x$ for each $x \in \varphi^{-1}(y)$. Then we may choose small enough neighborhoods $U_x \ni x$ and $V \ni y$ such that $s_x \in \mathscr{E}(U_x)$ and (2.4.2) holds. Let $s \in \mathscr{E}(\varphi^{-1}(V))$ be s_x when restricted to U_x. Then $\Phi_V(s) = s_x$. So Φ is surjective. □

Recall that for an arbitrary continuous map φ, the functor φ_* is left exact.

Corollary 2.4.6. *Let $\varphi : X \to Y$ be a finite continuous map. Then φ_* is an **exact functor** (i.e. a left and right exact functor) from the category of X-sheaves to that of Y-sheaves. Namely: if a sequence of maps of X-sheaves*

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0, \tag{2.4.5}$$

is exact, then the following is exact:

$$0 \to (\varphi_*\mathscr{E})_y \to (\varphi_*\mathscr{F})_y \to (\varphi_*\mathscr{G})_y \to 0. \tag{2.4.6}$$

Indeed, (2.4.5) is exact if and only if (2.4.6) is exact.

Proof. By Prop. 2.4.5, (2.4.6) is the same as

$$0 \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{E}_x \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{F}_x \to \bigoplus_{x\in\varphi^{-1}(y)} \mathscr{G}_x \to 0.$$

The equivalence of the exactness of (2.4.5) and (2.4.6) follows. □

Proposition 2.4.7 (Base change proposition). *Let $\pi : X \to S$ be a finite continuous map. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module and $\mathscr{M}$ an $\mathscr{O}_S$-module. Then we have a (clearly functorial) $\mathscr{O}_S$-module isomorphism*

$$\begin{gathered}\Upsilon : (\pi_*\mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{M} \xrightarrow{\simeq} \pi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}) \\ \sigma \otimes \mu \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_S(W)} \mathscr{M}(W) \quad \mapsto \quad \sigma \otimes \mu \in (\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})(\pi^{-1}(W))\end{gathered} \tag{2.4.7}$$

for all open $W \subset S$.

Note that the stalk map of Φ at any $t \in S$ is the canonical morphism

$$\Upsilon : (\pi_*\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \longrightarrow \pi_*(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_t \tag{2.4.8}$$

Proof. By Prop. 2.4.5, the stalk map (2.4.8) can be extended to a commutative diagram

$$\begin{tikzcd}
(\pi_*\mathscr{E})_t \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \arrow[r, "\Upsilon"] \arrow[d, "\simeq"'] & \pi(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_t \arrow[d, "\simeq"] \\
\left(\bigoplus_{x\in\pi^{-1}(t)} \mathscr{E}_x\right) \otimes_{\mathscr{O}_{S,t}} \mathscr{M}_t \arrow[r, "\simeq"] & \bigoplus_{x\in\pi^{-1}(t)} (\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_x
\end{tikzcd} \tag{2.4.9}$$

where the other three morphisms of $\mathscr{O}_{S,t}$-modules are isomorphisms. So (2.4.8) is an isomorphism. □

Lemma 2.4.8. *Let $\varphi : X \to Y$ be a finite continuous map. Assume that $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module. Then each $y \in Y$ is contained in neighborhood $V \subset Y$ such that $\mathscr{E}|_{\pi^{-1}(V)}$ is the cokernel of a morphism of free $\mathscr{O}_{\pi^{-1}(V)}$-modules.*

Proof. Choose V such that (2.4.2) holds and U_x is a small enough neighborhood of $x \in \varphi^{-1}(y)$ such that $\mathscr{E}|_{U_x}$ is equivalent to $\mathrm{Coker}(\mathscr{O}_{U_x}^m \to \mathscr{O}_{U_x}^n)$. The natural numbers m, n might initially depend on x, but we can enlarge m, n so that they do not. Then $\mathscr{E}|_{\pi^{-1}(V)}$ is clearly the cokernel of a morphism $\mathscr{O}_{\pi^{-1}(V)}^m \to \mathscr{O}_{\pi^{-1}(V)}^n$. □

Definition 2.4.9. A continuous map $\varphi : X \to Y$ is called **proper** if for each compact subset $K \subset Y$, $\varphi^{-1}(K)$ is compact.

Finite maps are special cases of proper maps as shown by the following proposition. Indeed, a deep theorem by Grauert says that if φ is a proper holomorphic map then $\varphi_*\mathscr{E}$ is $\mathscr{O}_Y$-coherent whenever $\mathscr{E}$ is $\mathscr{O}_X$-coherent. In particular, $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent. So we can study $f(X)$. In the special case that φ is finite, the study of the coherence of $\varphi_*\mathscr{O}_X$ is crucial to the proof of coherence of all structure sheaves of complex spaces.

Proposition 2.4.10. *Let $\varphi : X \to Y$ be a continuous map of topological spaces. If X and Y are locally compact and Y is Hausdorff, then the following are equivalent.*

(1) φ is proper.

(2) φ is closed, and $\varphi^{-1}(y)$ is compact for each $y \in Y$.

Thus, a finite map is precisely a proper map whose fibers $\varphi^{-1}(y)$ are all discrete sets.

Note: To prove (1)⇒(2) we don't need X to be locally compact. To prove (2)⇒(1) we don't need Y to be Hausdorff.

Proof. Assume (1). Let us prove that φ is closed by proving part (2) of Prop. 2.4.1. Choose $y \in Y$ and any neighborhood $U \supset \varphi^{-1}(y)$. Since Y is locally compact, we can fix a precompact neighborhood $V_0 \subset Y$ of y. Then $E := (X\backslash U) \cap \varphi^{-1}(V_0^{\mathrm{cl}})$ is compact by the properness of φ. Let $\mathfrak{V}$ be the set of all precompact open subsets of V_0 containing y. Then $\bigcap_{V\in\mathfrak{V}} V^{\mathrm{cl}} = \{y\}$ since Y is Hausdorff, and hence $E \cap \bigcap_{V\in\mathfrak{V}} \varphi^{-1}(V^{\mathrm{cl}}) = \varnothing$. So by the compactness of E, there is $V \in \mathfrak{V}$ such that $E \cap \varphi^{-1}(V^{\mathrm{cl}}) = 0$. So $\varphi^{-1}(V^{\mathrm{cl}}) \subset U$.

Assume (2). For each $y \in Y$, since $\varphi^{-1}(y)$ is compact and X is locally compact, we may find a precompact neighborhood $U \subset X$ of $\varphi^{-1}(y)$. By Prop. 2.4.1, we can find a neighborhood V of y such that $\varphi^{-1}(V) \subset U$. So $\varphi^{-1}(V)^{\mathrm{cl}}$ is compact since it is closed in U^{cl}. From this we conclude that any compact $K \subset Y$ can be covered by finitely many open sets $V_1, V_2, \dots$ such that $\varphi^{-1}(V_j)^{\mathrm{cl}}$ is compact. Then $\varphi^{-1}(K)$ as a closed subset of $\bigcup_j \varphi^{-1}(V_j)^{\mathrm{cl}}$ is compact. □

The following important fact says that properness and finiteness are preserved by base changes.

Proposition 2.4.11. *Let $\pi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. If π is proper resp. finite, then $\mathrm{pr}_Y : X \times_S Y \to Y$ is proper resp. finite.*

Proof. As a topological space, $X \times_S Y$ is the closed subset of all $x \times y \in X \times Y$ such that $\pi(x) = \psi(y)$ (Rem. 1.13.11). The relation $\mathrm{pr}_Y^{-1}(y) = \pi^{-1}(\psi(y)) \times y$ shows that the fibers of pr_Y are finite sets if those of π are finite. It also shows that if $K \subset Y$ is compact then $\mathrm{pr}_Y^{-1}(K)$ is a (clearly closed) subset of $\pi^{-1}(\psi(K)) \times K$ which is compact if π is proper. So pr_Y is proper if π is so. □

2.5 Weierstrass maps and Weierstrass preparation theorem

The goal of this section is to study an important class of finite holomorphic maps called Weierstrass maps.

Definition 2.5.1. Let S be a complex space. Let $k \in \mathbb{N}$. For each $i = 1, \dots, k$, we choose a polynomial of degree n_i

$$p_i(z_i) = 1 \otimes a_{i,0} + (1 \otimes a_{i,1}) z_i + \cdots + (1 \otimes a_{i,n_i}) z_i^{n_i} \qquad \in \mathscr{O}(\mathbb{C}^k \times S)[z_i]$$

where $a_{i,j} \in \mathscr{O}(S)$, $n_i \in \mathbb{Z}_+$, and $a_{i,n_i}(t) \neq 0$ for all $t \in S$. Consider p_i as an element of $\mathscr{O}(\mathbb{C}^k \times S)$. Let

$$X = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k \times S}/\mathcal{I}) \qquad \mathcal{I} = p_1 \mathscr{O}_{\mathbb{C}^k \times S} + \cdots + p_k \mathscr{O}_{\mathbb{C}^k \times S}. \tag{2.5.1}$$

Then the holomorphic map $\pi : X \to S$ defined by restricting the projection $\mathrm{pr}_S : \mathbb{C}^k \times S \to S$ is called a **Weierstrass map**.

Recall that by our notations, $1 \otimes a_{i,j}$ means $\mathrm{pr}_S^{\#} a_{i,j} = a_{i,j} \circ \mathrm{pr}_S$. We shall write $1 \otimes a_{i,j}$ as $a_{i,j}$ if no confusion arises.

Proposition 2.5.2. *Weierstrass maps are finite.*

Proof. Clearly each fiber of $\pi : X \to S$ is a finite set. To check that π is closed, by Rem. 2.4.2, it suffices to check it locally with respect to the base.

By Rem. 1.5.2 we can shrink S and find an open disc $D \subset \mathbb{C}$ such that for each $t \in S$ and each i, the polynomial $p_i(z_i, t)$ of z_i has n_i zeros in D counting multiplicities. So X (as a topological space, namely $N(\mathcal{I})$) is a closed subset of $(D^{\mathrm{cl}})^k \times S$. Therefore $\pi : X \to S$ is the restriction of the projection $(D^{\mathrm{cl}})^k \times S \to S$ to a closed subset, which is closed because the projection $(D^{\mathrm{cl}})^k \times S \to S$ is proper and hence closed (Prop. 2.4.10). $\square$

The next proposition says that a canonical pullback $Y \to T$ of a Weierstrass map $X \to S$ along a holomorphic map $\psi : T \to S$ is Weierstrass.

Proposition 2.5.3. *Assume the setting of Def. 2.5.1. Let $\psi : T \to S$ be a holomorphic map of complex spaces. Set*

$$\begin{gathered}\widetilde{a}_{i,j} = a_{i,j} \circ \psi \in \mathscr{O}(T)\\ \widetilde{p}_i(z_i) = 1 \otimes \widetilde{a}_{i,0} + (1 \otimes \widetilde{a}_{i,1}) z_i + \cdots + (1 \otimes \widetilde{a}_{i,n_i}) z_i^{n_i} \qquad \in \mathscr{O}(\mathbb{C}^k \times T)[z_i]\end{gathered}$$

and set

$$Y = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k \times T}/\mathcal{J}) \qquad \mathcal{J} = \widetilde{p}_1 \mathscr{O}_{\mathbb{C}^k \times T} + \cdots + \widetilde{p}_k \mathscr{O}_{\mathbb{C}^k \times T}. \tag{2.5.2}$$

Then the Cartesian square

$$\begin{tikzcd} \mathbb{C}^k \times S \arrow[d] & \mathbb{C}^k \times T \arrow[l, "\mathbf{1} \times \psi"'] \arrow[d] \\ S & T \arrow[l, "\psi"'] \end{tikzcd}$$

restricts to a Cartesian square

$$\begin{tikzcd} X \arrow[d, "\pi"'] & Y \arrow[l, "\widetilde{\psi}"'] \arrow[d, "\widetilde{\pi}"] \\ S & T \arrow[l, "\psi"'] \end{tikzcd}$$

Proof. By Prop. 1.12.1 we have a Cartesian square

$$\begin{tikzcd} X \arrow[d, hook] & Y \arrow[l] \arrow[d, hook] \\ \mathbb{C}^k \times S & \mathbb{C}^k \times T \arrow[l] \end{tikzcd}$$

which, together with Rem. 1.11.3, proves our proposition. $\square$

The following theorem is the first major result of this chapter. Many subsequent major results in this chapter are proved using this theorem.

Theorem 2.5.4 (Fundamental theorem of Weierstrass maps). *Assume the setting of Def. 2.5.1. Then*

$$\{z_1^{\nu_1}\cdots z_k^{\nu_k}:0\leqslant\nu_i\leqslant n_i-1 \text{ for all } 1\leqslant i\leqslant k\} \tag{2.5.3}$$

(or more precisely, these elements acted on by $\mathrm{Pr}^\#_{\mathbb{C}^k\times S\to\mathbb{C}^k}$) is a set of free generators of the $\mathscr{O}_S$-module $\pi_*\mathscr{O}_X$.

In particular, $\pi_*\mathscr{O}_X$ is a free $\mathscr{O}_S$-module of rank $n_1n_2\cdots n_k$.

Lemma 2.5.5. *If Thm. 2.5.4 holds when S is smooth, then Thm. 2.5.4 holds when S is any complex space.*

Proof. Note that Thm. 2.5.4 is local by nature since it can be checked at the level of stalks. So we may assume S is so small that it is a closed subspace of an open subset $\Omega\subset\mathbb{C}^m$, and that each $a_{i,j}$ is the restriction of an element of $\mathscr{O}(\Omega)$. Therefore, by Prop. 2.5.3, we have a Weierstrass map $Y\hookrightarrow\mathbb{C}^k\times\Omega\to\Omega$ (which we also denote by π) such that the following two squares are Cartesian.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\
\mathbb{C}^k\times S \arrow[r, hook] \arrow[d] & \mathbb{C}^k\times\Omega \arrow[d] \\
S \arrow[r, hook] & \Omega
\end{tikzcd}$$

In particular, $\pi:X\to S$ is the base change of $\pi:Y\to\Omega$ to S.

Write $S=\mathrm{Specan}(\mathscr{O}_\Omega/\mathcal{I})$. Then by Rem. 1.12.3, $\mathscr{O}_X$ is $\mathscr{O}_Y\otimes_{\mathscr{O}_\Omega}\mathscr{O}_S$ (if we regard $\mathscr{O}_X$ as an $\mathscr{O}_Y$-module and $\mathscr{O}_S$ as an $\mathscr{O}_\Omega$-module). By Base change Prop. 2.4.7, we have canonical isomorphisms of $\mathscr{O}_\Omega$-modules

$$\pi_*\mathscr{O}_X\simeq\pi_*(\mathscr{O}_Y\otimes_{\mathscr{O}_\Omega}\mathscr{O}_S)\simeq\pi_*\mathscr{O}_Y\otimes_{\mathscr{O}_\Omega}\mathscr{O}_S.$$

Equivalently, $\pi_*\mathscr{O}_X\simeq\pi_*\mathscr{O}_Y|_S$ as $\mathscr{O}_S$-modules. Since we assume that Thm. 2.5.4 holds for $\pi:Y\to\Omega$, we know that $\pi_*\mathscr{O}_Y$ is generated freely by (2.5.3). So $\pi_*\mathscr{O}_X$ is generated freely by (the restrictions to S of) (2.5.3). $\square$

Due to Lemma 2.5.5, we can assume that:

Convention 2.5.6. In the remaining part of this section, S is an open subset of $\mathbb{C}^m$. Let $t_\bullet=(t_1,\dots,t_m)$ be the variables of S.

To prepare for the proof, we let $N(p_i) \subset \mathbb{C} \times S$ be the subset of all $(z_i, t_\bullet)$ such that $p_i(z_i, t_\bullet) = 0$. For each $(t_\bullet) \in S$, define a subset of $\mathbb{C}$

$$N(p_i)_{t_\bullet} = N(p_i) \cap \mathrm{pr}^{-1}_{\mathbb{C}\times S\to S}(t_\bullet),$$

namely, the set of all z_i satisfying $p_i(z_i, t_\bullet) = 0$. Then by Prop. 2.4.5, we have an obvious isomorphism of $\mathscr{O}_{S,t_\bullet}$-modules

$$(\pi_*\mathscr{O}_X)_{t_\bullet} \simeq \bigoplus_{\substack{w_i\in N(p_i)_{t_\bullet}\\ 1\leqslant i\leqslant k}} \mathscr{O}_{\mathbb{C}^k\times S,(w_\bullet,t_\bullet)}/\mathcal{I}_{(w_\bullet,t_\bullet)} \tag{2.5.4}$$

where

$$\mathcal{I}_{(w_\bullet,t_\bullet)} = p_1\mathscr{O}_{\mathbb{C}^k\times S,(w_\bullet,t_\bullet)} + \cdots + p_k\mathscr{O}_{\mathbb{C}^k\times S,(w_\bullet,t_\bullet)}.$$

Our goal is to show that (2.5.3) generates (2.5.4) freely.

2.5.1 Proof of Thm. 2.5.4, I

In this subsection, we assume $(t_\bullet) = 0 \in S \subset \mathbb{C}^m$ for simplicity, and show that $z_1, \ldots, z_k$ generate $(\pi_*\mathscr{O}_X)_0$. We let $(\tau_\bullet)$ denote a set of general variables of S. (2.5.4) reads

$$(\pi_*\mathscr{O}_X)_0 \simeq \bigoplus_{\substack{w_i\in N(p_i)_0\\ 1\leqslant i\leqslant k}} \mathscr{O}_{\mathbb{C}^k\times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}. \tag{2.5.5}$$

Lemma 2.5.7. (2.5.3) *generates* $(\pi_*\mathscr{O}_X)_0$.

Proof-special case. We consider the special case that for each i, $N(p_i)_0$ is the single point 0. In this case, $p_i(z_i, \tau_\bullet)$ has order n_i in z_i (recall Def. 1.5.1). (Namely, p_i is, up to multiplication by a nowhere zero element of $\mathscr{O}(S)$, a *Weierstrass polynomial* of z_i.) Now (2.5.5) reads

$$(\pi_*\mathscr{O}_X)_0 \simeq \mathscr{O}_{\mathbb{C}^k\times S,(0,0)}/\mathcal{I}_{(0,0)}.$$

We explain the proof when $k = 2$. The general case follows from exactly the same argument.

Choose $f(z_1, z_2, \tau_\bullet) \in \mathscr{O}_{\mathbb{C}^2\times S,(0,0)}$. Then by WDT (Weierstrass division theorem),

$$f(z_1, z_2, \tau_\bullet) = \sum_{j=0}^{n_1-1} z_1^j \cdot g_j(z_2, \tau_\bullet) \qquad \mathrm{mod}\ \ p_1\mathscr{O}_{\mathbb{C}^2\times S,(0,0)}$$

where $g_j \in \mathscr{O}_{\mathbb{C}\times S,(0,0)}$. Apply WDT again, we have

$$g_j(z_2, \tau_\bullet) = \sum_{l=0}^{n_2-1} z_2^l \cdot h_l(\tau_\bullet) \qquad \mathrm{mod}\ \ p_2\mathscr{O}_{\mathbb{C}\times S,(0,0)}$$

where $h_l \in \mathscr{O}_{S,0}$. This finishes the proof. □

To prove the general case, for each $w_i \in N(p_i)_0$ we define integer

$$\mu_i(w_i) = \{\text{The multiplicity of the root } z_i = w_i \text{ of } p_i(z_i, 0)\}$$

So $0 < \mu_i(w_i) \leqslant n_i$.

Lemma 2.5.8. *For each i, choose $w_i \in N(p_i)_0$. Then there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q_1(z_\bullet, \tau_\bullet)$ of $z_1, \dots, z_n$ with multi-degree $\leqslant \big(n_1 - \mu_1(w_1), \dots, n_k - \mu_k(w_k)\big)$ satisfying the following conditions.*

(1) Its germ at $(w_\bullet, 0)$ is an invertible element of the ring $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}$.

(2) Its germ at $(\widetilde{w}_\bullet, 0)$ is 0 in the ring $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)}/\mathcal{I}_{(\widetilde{w}_\bullet,0)}$ for any $(\widetilde{w}_\bullet) = (\widetilde{w}_1, \dots, \widetilde{w}_k) \in \mathbb{C}^k$ such that $\widetilde{w}_i \in N(p_i)_0$ (for all i) and that $(\widetilde{w}_\bullet) \neq (w_\bullet)$.

This lemma can be viewed as a partition of unity of $(\pi_*\mathscr{O}_X)_0$. We postpone the proof of this lemma until after proving Lemma 2.5.7.

Proof of Lemma 2.5.7-general case. In view of (2.5.5), it suffices to show prove the following claim:

- Choose any $(w_\bullet) \in \mathbb{C}^k$ such that $w_i \in N(p_i)_0$, and choose any $f(z_\bullet, \tau_\bullet) \in (\pi_*\mathscr{O}_X)_0$ which is zero in $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)}/\mathcal{I}_{(\widetilde{w}_\bullet,0)}$ whenever $(\widetilde{w}_\bullet) \neq (w_\bullet)$. Then f belongs to the $\mathscr{O}_{S,0}$-submodule of $(\pi_*\mathscr{O}_X)_0$ generated by (2.5.3).
  - Namely, there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q(z_\bullet, \tau_\bullet)$ of $z_\bullet$ with multi-degree $\leqslant (n_1 - 1, \dots, n_k - 1)$ whose germ at $(w_\bullet, 0)$ is equal to the germ of f mod $\mathcal{I}_{(w_\bullet,0)}$, and whose germ at $(\widetilde{w}_\bullet, 0)$ (where $(\widetilde{w}_\bullet) \neq (w_\bullet)$) is in $\mathcal{I}_{(\widetilde{w}_\bullet,0)}$.

Let q_1 be as in Lemma 2.5.8, whose germ at $(w_\bullet, 0)$ is an invertible element of $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}$. Note that f/q_1 is in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}$ (but not in $(\pi_*\mathscr{O}_X)_0$). We now apply the proof of the special case to f/q_1. Then by WDT (noting that $p_i(z_i, \tau_\bullet)$ has order $\mu_i(w_i)$ in $z_i - w_i$), there is an $\mathscr{O}_{S,0}$-coefficients polynomial $q_2(z_\bullet, \tau_\bullet)$ of $z_\bullet$ with multi-degree $\leqslant (\mu_1(n_1) - 1, \dots, \mu_k(n_k) - 1)$ which equals f/q_1 in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}$. Then f and $q := q_1 q_2$ are clearly equal in $\mathscr{O}_{\mathbb{C}^k \times S,(w_\bullet,0)}/\mathcal{I}_{(w_\bullet,0)}$. They are also equal in $\mathscr{O}_{\mathbb{C}^k \times S,(\widetilde{w}_\bullet,0)}/\mathcal{I}_{(\widetilde{w}_\bullet,0)}$ since both are 0. □

We are done with the proof of Lemma 2.5.7.

2.5.2 Proof of Lemma 2.5.8

Definition 2.5.9. A polynomial $q(z, \tau_\bullet) \in \mathbb{C}\{\tau_\bullet\}[z]$ is called a **Weierstrass polynomial of z** if it is monic and the degree equals the order in z. Namely,

$$q(z, \tau_\bullet) = a_0(\tau_\bullet) + a_1(\tau_\bullet) z + \cdots + a_{n-1}(\tau_\bullet) z^{n-1} + z^n \tag{2.5.6}$$

where $a_0, \ldots, a_{n-1} \in \mathbb{C}\{\tau_\bullet\}$, and

$$a_0(0) = a_1(0) = \cdots = a_{n-1}(0) = 0.$$

Theorem 2.5.10 (Weierstrass preparation theorem (WPT)). *Choose $f(z, \tau_\bullet) \in \mathbb{C}\{z, \tau_\bullet\}$ with finite order n in z. Then there exist a unique invertible $u \in \mathbb{C}\{z, \tau_\bullet\}$ and a Weiertrass polynomial $q \in \mathbb{C}\{\tau_\bullet\}[z]$ of z such that in $\mathbb{C}\{z, \tau_\bullet\}$ we have*

$$f = uq.$$

Proof. Uniqueness: $f = uq$ can be written as $q = u^{-1}f$. Write $q(z, \tau_\bullet) = z^n - r$ where the polynomial $r \in \mathbb{C}\{\tau_\bullet\}[z]$ of z has degree $< n$. Then $z^n = u^{-1}f + r$ gives the unique Weierstrass division of z^n by f. So u, q are unique.

Existence: By WDT, we have $z^n = \alpha f + r$ where $\alpha \in \mathbb{C}\{z, \tau_\bullet\}$ and $r \in \mathbb{C}\{\tau_\bullet\}[z]$ has degree $< n$. Now, $z^n = \alpha(z, 0)f(z, 0) + r(z, 0)$ gives the unique Weierstrass division of z^n by $f(z, 0)$. Since f has order n in z, we may write $f(z, 0) = z^n h(z)$ where $h \in \mathbb{C}\{z\}$ is invertible. So $z^n = h(z)^{-1} \cdot f(z, 0)$ also gives a Weierstrass division. Therefore $r(z, 0) = 0$ and $\alpha(z, 0) = h(z)^{-1}$. So $\alpha(0, 0) \neq 0$, i.e. α is invertible in $\mathbb{C}\{z, \tau_\bullet\}$. We have $f = \alpha^{-1}q$ where $q = z^n - r$. $\square$

We are ready to prove Lemma 2.5.8.

Proof of Lemma 2.5.8. Recall the polynomials p_i in Def. 2.5.1. By WPT, for each $w_i \in N(p_i)_0$, in the ring $\mathbb{C}\{z_i - w_i, \tau_\bullet\}$, $p_i(z_i, \tau_\bullet)$ equals a unit times a Weierstrass polynomial $r_{i,w_i}(z_i, \tau_\bullet)$ of $z_i - w_i$. So $r_{i,w_i}(z_i, \tau_\bullet) \in \mathscr{O}_{S,0}[z_i]$ has degree $\mu_i(w_i)$ in z_i, and $r_{i,w_i}(z_i, 0) = (z_i - w_i)^{\mu_i(w_i)}$. So in the ring $\mathscr{O}_{\mathbb{C}^k \times S, (\widetilde{w}_\bullet, 0)}/\mathcal{I}_{(\widetilde{w}_\bullet, 0)}$, r_{i,w_i} is invertible when $\widetilde{w}_i \neq w_i$ (since $r_{i,w_i}(\widetilde{w}_i, 0) \neq 0$), and is 0 when $\widetilde{w}_i = w_i$. Thus

$$R_i := \prod_{\substack{\widetilde{w}_i \in N(p_i)_0 \\ \widetilde{w}_i \neq w_i}} r_{i,w_i}$$

is invertible in $\mathscr{O}_{\mathbb{C}^k \times S, (\widetilde{w}_\bullet, 0)/\mathcal{I}_{(\widetilde{w}_\bullet, 0)}}$ when $\widetilde{w}_i = w_i$ and is zero when $\widetilde{w}_i \neq w_i$. $R_i \in \mathscr{O}_{S,0}[z_i]$ has degree $n - \mu_i(w_i)$ in z_i. So $p_1 = \prod_{i=1}^k R_i$ gives the desired polynomial. $\square$

2.5.3 Proof of Thm. 2.5.4, II

Finishing the proof of Thm. 2.5.4. We have already shown that the set (2.5.3) (which has $n_1 \cdots n_k$ elements) generate $\pi_*\mathscr{O}_X$. In particular, $\pi_*\mathscr{O}_X$ is a finite-type $\mathscr{O}_S$-module. To show that (2.5.3) generates $\pi_*\mathscr{O}_X$ freely, by Prop. 1.3.14, it suffices to show that the fiber $(\pi_*\mathscr{O}_X)|y = (\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathfrak{m}_{S,y})$ has dimension $n_1 \cdots n_k$ for each $y \in S$.

By Base change Prop. 2.4.7, $(\pi_*\mathscr{O}_X)|y$ is canonically equivalent to

$$\pi_*(\mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathfrak{m}_{S,y})),$$

which equals $\pi_*\mathscr{O}_{X_y} = \mathscr{O}(X_y)$ (where $X_y = \pi^{-1}(y)$ is the inverse image of y and is a closed subspace of X) by Rem. 1.12.3. By Prop. 2.5.3, $\pi : \pi^{-1}(y) \to \{y\}$ is a Weierstrass map. It is the restriction of $\mathbb{C}^k \to \{y\}$ to the complex subspace of $\mathbb{C}^k$ defined by the ideal sheaf generated by $p_i(z_i, y) = a_{i,0}(y) + a_{i,1}(y)z_i + \cdots + a_{i,n_i}(y)z_i^{n_i}$ for all $1 \leqslant i \leqslant k$. Thus, it suffices to prove the following lemma. $\square$

Lemma 2.5.11. *Let $X = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k}/\mathcal{I})$ where $\mathcal{I}$ is the ideal sheaf generated by $p_1, \ldots, p_k$ where each $p_i(z_i) \in \mathbb{C}[z_i]$ has degree n_i. Then $\mathscr{O}(X)$ has dimension $n_1 \cdots n_k$.*

Proof. We are still in the setting of Def. 2.5.1, but assuming that S is a single point 0. So $N(p_i)_0 = N(p_i)$. By (2.5.5),

$$\mathscr{O}(X) \simeq \bigoplus_{\substack{w_i \in N(p_i) \\ 1 \leqslant i \leqslant k}} \mathscr{O}_{\mathbb{C}^k, w_\bullet}/\mathcal{I}_{w_\bullet}.$$

Clearly $\mathcal{I}_{w_\bullet}$ is the ideal generated by $(z_i - w_i)^{\mu_i(w_i)}$ for all $1 \leqslant i \leqslant k$. So

$$\left\{ \prod_{i=1}^{k} (z_i - w_i)^{\nu_i} : 0 \leqslant \nu_i \leqslant \mu_i(w_i) - 1 \right\}$$

is a basis of $\mathscr{O}_{\mathbb{C}^k, w_\bullet}/\mathcal{I}_{w_\bullet}$. This calculates the dimension of $\mathscr{O}(X)$. $\square$

2.6 Coherence of $\mathscr{O}_X$

The goal of this section is to prove that $\mathscr{O}_X$ is coherent for every complex space X. By Cor. 2.1.11, it suffices to prove that $\mathscr{O}_{\mathbb{C}^n}$ is coherent. The role that Thm. 2.5.4 plays in the proof of coherence of $\mathscr{O}_{\mathbb{C}^n}$ is similar to the role that WDT plays in the proof that $\mathscr{O}_{\mathbb{C}^n,0}$ is Noetherian.

Lemma 2.6.1. *Assume that X is an open subset of $\mathbb{C}^n$. Assume that for each open connected $U \subset X$ and each non-zero $h \in \mathscr{O}(U)$, $\mathscr{O}_U/h\mathscr{O}_U$ is a coherent $\mathscr{O}_U/h\mathscr{O}_U$-module. Then $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module.*

More precisely, our assumption is that the structure sheaf of $\mathrm{Specan}(\mathscr{O}_U/h\mathscr{O}_U)$ is coherent.

Proof. Choose any open connected $U \subset X$ and $h_1, \ldots, h_N \in \mathscr{O}(U)$. We want to show that $\mathscr{R}el(h_1, \ldots, h_N)$ is a finite-type $\mathscr{O}_U$-submodule of $\mathscr{O}_U^N$. We assume one of $h_1, \ldots, h_N$ is non-zero, say $h_1 \neq 0$; otherwise the proof is obvious. For each

$f \in \mathscr{O}_U$, we let $[f]$ denotes its residue class in $\mathscr{O}_Y = (\mathscr{O}_U/h_1\mathscr{O}_U) \restriction_{N(h_1\mathscr{O}_U)}$ where $Y = \mathrm{Specan}(\mathscr{O}_U/h_1\mathscr{O}_U)$.

Choose any $x \in U$. By assumption, $\mathscr{O}_Y$ is coherent. So $\mathscr{Rel}([h_2], \dots, [h_N])$ is a finite type $\mathscr{O}_Y$-submodule of $\mathscr{O}_Y^{N-1}$. Thus, after shrinking U to a smaller neighborhood of x, we may find $(s_2^i, \dots, s_N^i) \in \mathscr{O}(U)$ (for finitely many i) such that $([s_2^i], \dots, [s_N^i])$ generate $\mathscr{Rel}([h_2], \dots [h_N])$. This means:

(a) For each i, $s_2^i h_2 + \cdots + s_N^i h_N \in h_1\mathscr{O}_U$. (This can be checked at the level of stalks.)

(b) For each $y \in U$ and $(\sigma_2, \dots, \sigma_N) \in \mathscr{O}_{U,y}^{N-1}$ such that

$$\sigma_2 h_2 + \cdots + \sigma_N h_N \in h_1\mathscr{O}_{U,y},$$

there exist $f_i \in \mathscr{O}_{U,y}$ for all i and $g_2, \dots, g_N \in \mathscr{O}_{U,y}$ such that

$$(\sigma_2, \dots, \sigma_N) = \sum_i f_i(s_2^i, \dots, s_N^i) + h_1(g_2, \dots, g_N). \tag{2.6.1}$$

By (a), we may shrink U further so that for each i, we may find $s_1^i \in \mathscr{O}(U)$ such that $s_1^i h_1 + s_2^i h_2 + \cdots + s_N^i h_N = 0$. We claim that

$$(s_1^i, \dots, s_N^i)$$

for all i and

$$(-h_2, h_1, 0, \dots, 0), \quad (-h_3, 0, h_1, \dots, 0), \quad \dots, \quad (-h_N, 0, 0, \dots, h_1)$$

(which are clearly in $\mathscr{Rel}(h_1, h_2, \dots, h_N)$) generate $\mathscr{Rel}(h_1, h_2, \dots, h_N)$.

Choose any $y \in U$ and $(\sigma_1, \dots, \sigma_N) \in \mathscr{O}_{U,y}^N$ in $\mathscr{Rel}(h_1, \dots, h_N)_y$, namely $\sigma_1 h_1 + \cdots + \sigma_N h_N = 0$. Then by (b), we can find $f_i, g_2, \dots, g_N \in \mathscr{O}_{U,y}$ such that the relation

$$\begin{aligned}(\sigma_1, \sigma_2, \dots, \sigma_N) =& \sum_i f_i(s_1^i, s_2^i, \dots, s_N^i) \\ &+ \sum_{j=2}^N g_j(-h_j, 0, \dots, \underset{\substack{\uparrow \\ i\text{-th component}}}{h_1}, \dots, 0)\end{aligned} \tag{2.6.2}$$

holds for the last $N-1$ components. To show that it holds also for the first component, we write the RHS of (2.6.2) as $(\widetilde{\sigma}_1, \sigma_2, \dots, \sigma_N)$, which is an element of $\mathscr{Rel}(h_1, \dots, h_N)_y$. So

$$\sigma_1 h_1 + \sigma_2 h_2 \cdots + \sigma_N h_N = \widetilde{\sigma}_1 h_1 + \sigma_2 h_2 \cdots + \sigma_N h_N = 0,$$

which shows $(\sigma_1 - \widetilde{\sigma}_1)h_1 = 0$. Since h_1 is a non-zero element of $\mathscr{O}(U)$, by the Identitätssatz 1.1.2, the germ of h_1 at $\mathscr{O}_{U,y}$ is non-zero. So $\sigma_1 = \widetilde{\sigma}_1$ since $\mathscr{O}_{U,y}$ is an integral domain. This proves (2.6.2). $\square$

Theorem 2.6.2 (Oka's coherence theorem). *For every complex space X, $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module.*

Proof. We prove the coherence of $\mathscr{O}_{\mathbb{C}^m}$ by induction on m. The case $m = 0$ is obvious. Assume that $\mathscr{O}_{\mathbb{C}^m}$ is coherent. Let us prove that $\mathscr{O}_{\mathbb{C}^{m+1}}$ is coherent.

By Lemma 2.6.1, it suffices to show that for each open connected $U \subset \mathbb{C}^{m+1}$ and non-zero $h \in \mathscr{O}(U)$, if we write $Y = \mathrm{Specan}(\mathscr{O}_U/h\mathscr{O}_U)$ then $\mathscr{O}_Y$ is a coherent $\mathscr{O}_Y$-module. Let $\mathscr{K}$ be the kernel of a morphism $\mathscr{O}_Y^N \to \mathscr{O}_Y$. Then we have an exact sequence of $\mathscr{O}_Y$-modules

$$0 \to \mathscr{K} \to \mathscr{O}_Y^N \to \mathscr{O}_Y.$$

We need to show that for each $x \in U$, say $x = 0$, after shrinking U to a neighborhood of x, $\mathscr{K}$ is $\mathscr{O}_U$-generated by finitely many elements of $\mathscr{K}(U)$.

The germ of h in $\mathscr{O}_{U,x}$ is non-zero by the Identitätssatz 1.1.2. Thus, by choosing a new set of coordinates $(z, t_1, \dots, t_m)$ of U such that $x = 0$, we may assume that the germ of h at 0, which is an element of $\mathbb{C}\{z, t_1, \dots, t_m\}$, has finite order n in z. (Cf. the proof of Thm. 1.5.5). Thus, by WPT, after shrinking U to a smaller neighborhood of 0 we may assume that $h \in \mathbb{C}\{t_\bullet\}[z]$ is a Weierstrass polynomial of degree=order n in z.

We assume $U = V \times W$ where $V \subset \mathbb{C}$ and $W \subset \mathbb{C}^m$ are neighborhoods of 0. By Rem. 1.5.2, we may assume that $N(h) = \{(z, t_\bullet) \in \mathbb{C} \times W : h(z, t_\bullet) = 0\}$ is like Fig. 1.5.1: for each $(t_\bullet) \in W$, the polynomial $h(z, t_\bullet)$ of z has n zeros in V counting multiplicities. Thus $N(h) \subset U$. Therefore

$$\mathscr{O}_U/h\mathscr{O}_U = \mathscr{O}_{\mathbb{C}\times W}/h\mathscr{O}_{\mathbb{C}\times W}.$$

So the projection of $\pi : Y \to W$ (inherited from $\mathbb{C} \times W \to W$) is a Weierstrass map. By the Fundamental Thm. 2.5.4 of Weierstrass maps, $\pi_*\mathscr{O}_Y$ and hence $\pi_*(\mathscr{O}_Y^N) = (\pi_*\mathscr{O}_Y)^N$ are $\mathscr{O}_W$-free. So they are $\mathscr{O}_W$-coherent *by our assumption that $\mathscr{O}_{\mathbb{C}^m}$ is coherent*. Therefore $\pi_*\mathscr{K}$ is $\mathscr{O}_W$-coherent by Cor. 2.1.5 and the exactness of

$$0 \to \pi_*\mathscr{K} \to \pi_*\mathscr{O}_Y^N \to \pi_*\mathscr{O}_Y.$$

So $\mathscr{K}$ is $\mathscr{O}_Y$-finite-type by the following lemma. □

Lemma 2.6.3. *Let $\pi : X \to S$ be a finite morphism of $\mathbb{C}$-ringed spaces, and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. If $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-finite-type, then $\mathscr{E}$ is $\mathscr{O}_X$-finite-type.*

Proof. Choose any $t \in S$. By shrinking S to a neighborhood of t (and shrinking X to $\pi^{-1}(S)$), we can find $\sigma_1, \dots, \sigma_k \in \mathscr{E}(X) = (\pi_*\mathscr{E})(S)$ which $\mathscr{O}_S$-generate $\pi_*\mathscr{E}$. For each $x \in X$, by Prop. 2.4.5, $\mathscr{E}_x$ is a direct summand of the $\mathscr{O}_{S,\pi(x)}$-module $(\pi_*\mathscr{E})_{\pi(x)}$. So $\mathscr{E}_x$ is $\mathscr{O}_{S,\pi(x)}$-generated (and hence $\mathscr{O}_{X,x}$-generated) by $\sigma_1, \dots, \sigma_k$. This proves that $\mathscr{E}$ is $\mathscr{O}_X$-generated by $\sigma_1, \dots, \sigma_k$. □

Corollary 2.6.4. *Let X be a complex space. An ideal of $\mathscr{O}_X$ is finite-type if and only if it is coherent.*

2.7 Finite mapping theorem

The following two theorems are the main results of this section.

Theorem 2.7.1 (Finite mapping theorem). *Let $\pi : X \to S$ be a finite holomorphic map of complex spaces, and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the following are equivalent.*

(1) $\mathscr{E}$ is $\mathscr{O}_X$-coherent.

(2) $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-coherent.

Theorem 2.7.2. *Let $\pi : X \to S$ be a holomorphic map of complex spaces. Let $t \in S$, and assume that $x \in \pi^{-1}(t)$ is an isolated point of $\pi^{-1}(t)$. Then there are neighborhoods $U \subset X$ of x and $W \subset S$ of $\pi(U)$ such that π restricts to a finite holomorphic map $\pi : U \to W$.*

Remark 2.7.3. It follows immediately from Thm. 2.7.2 that if $\pi : X \to S$ is holomorphic and if $t \in S$ is such that $\pi^{-1}(t)$ is a finite set, then there are neighborhoods $U \subset X$ of $\pi^{-1}(t)$ and $W \subset S$ of $\pi(U)$ such that the restriction $\pi : U \to W$ is finite.

2.7.1 Proof of the main results

We begin with the following preliminary lemma.

Lemma 2.7.4. *Given a finite holomorphic $\pi : X \to S$, if $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent, then for each coherent $\mathscr{O}_X$-module $\mathscr{E}$, $\pi_*\mathscr{E}$ is $\mathscr{O}_S$-coherent.*

Proof. Choose any $t \in S$. By Lemma 2.4.8, we can shrink S to a neighborhood of t and shrink X to $\pi^{-1}(S)$ so that $\mathscr{E} \simeq \mathrm{Coker}(\mathscr{O}_X^m \to \mathscr{O}_X^n)$ for a morphism $\mathscr{O}_X^m \to \mathscr{O}_X^n$. Thus, by the (right) exactness of π_* (Cor. 2.4.6), $\pi_*\mathscr{E} \simeq \mathrm{Coker}(\pi_*\mathscr{O}_X^m \to \pi_*\mathscr{O}_X^n)$, which is coherent since $\pi_*\mathscr{O}_X$ is coherent. $\square$

The crucial part of the proof is the following lemma.

Lemma 2.7.5. *Choose open subsets $R \subset \mathbb{C}^k$ and $S \subset \mathbb{C}^m$. Let $X = \mathrm{Specan}(\mathscr{O}_{R\times S}/\mathcal{I})$ where $\mathcal{I}$ is a coherent ideal of $\mathscr{O}_{R\times S}$. Let $\pi : X \to S$ be the holomorphic map restricted from the projection $R \times S \to S$. Let $t \in S$ and assume that $x \in \pi^{-1}(t)$ is an isolated point of $\pi^{-1}(t)$. Then there are neighborhoods $U \subset R$ of x and $W \subset S$ of $\pi(U)$ such that the restriction $\pi : (U \times W) \cap X \to W$ is finite, and that $\pi_*\mathscr{O}_{(U\times W)\cap X}$ is $\mathscr{O}_W$-coherent.*

We assume $x = 0_R$ and $t = 0_S$ for simplicity, and prove the lemma by induction on k.

Proof for the case $k=1$. Shrink R to a neighborhood of 0_R such that $\pi^{-1}(0_S)=(R\times 0_S)\cap N(\mathcal{I})$ is $\{0\}$. So we may shrink R further so that we can find $f\in\mathcal{I}(R\times S)$ such that $(R\times 0_S)\cap N(f)=\{0\}$. So f, as an element of $\mathbb{C}\{z,t_1,\dots,t_m\}$, has finite order in z. So we may shrink R,S further and replace f by a Weierstrass polynomial of z, which we still denote by f.

Let $\mathcal{J}=f\mathscr{O}_{R\times S}$ and $Y=\mathrm{Specan}(\mathscr{O}_{R\times S}/\mathcal{J})$. Let $\widetilde{\pi}:Y\to S$ be the restriction of $R\times S\to S$ to Y. As in the proof of Oka's coherence theorem, we may shrink R and S so that Fig. 1.5.1 holds, and hence that $\widetilde{\pi}$ is a Weierstrass map. So $\pi=\widetilde{\pi}\circ\iota_{X,Y}$ is finite since both $\widetilde{\pi}$ and the inclusion map $\iota=\iota_{X,Y}$ are finite.

By the Fundamental Thm. 2.5.4 of Weierstrass maps (and Oka's coherence theorem), $\widetilde{\pi}_*\mathscr{O}_Y$ is $\mathscr{O}_S$-coherent. So by Lemma 2.7.4, $\widetilde{\pi}_*$ sends coherent $\mathscr{O}_Y$-modules to coherent $\mathscr{O}_S$-modules. But $\iota_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent by Extension principle 2.1.10. So $\pi_*\mathscr{O}_X=\widetilde{\pi}_*\iota_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. □

Proof that case $k\Rightarrow$ case $k+1$. Assume that case k is true. Now assume R is an open subset of $\mathbb{C}^{k+1}$. By shrinking R to a neighborhood of 0_R we assume $R=U\times V$ where $U\subset\mathbb{C}$ and $V\subset\mathbb{C}^k$ are open subsets containing $0_{\mathbb{C}}$ and $0_{\mathbb{C}^k}$ respectively, and that $\pi^{-1}(0_S)=(U\times V\times 0_S)\cap N(\mathcal{I})$ equals $\{0\}$.

Let $\alpha:X\to V\times S$ be the restriction of the projection $U\times V\times S\to V\times S$. Then $\alpha^{-1}(0_{V\times S})=(U\times 0_V\times 0_S)\times N(\mathcal{I})$ is $\{0\}$. So by the case $k=1$, we may shrink U,V,S to smaller neighborhoods of $0_U,0_V,0_S$ respectively so that α is finite and $\alpha_*\mathscr{O}_X$ is $\mathscr{O}_{V\times S}$-coherent. By Def. 2.3.8, we can define the image space $\alpha(X)$ whose underlying topological space is $\mathrm{Im}(\alpha)$, and by Prop. 2.3.11, α factors as the composition of a holomorphic $\widetilde{\alpha}:X\to\alpha(X)$ and the inclusion $\alpha(X)\hookrightarrow V\times S$. We thus obtain a commutative diagram

$$\begin{tikzcd}
 & V\times S \arrow[dr, "\mathrm{pr}_S"] & \\
X=\mathrm{Specan}(\mathscr{O}_{U\times V\times S}/\mathcal{I}) \arrow[ur, "\alpha"] \arrow[dr, "\widetilde{\alpha}"'] & & S \\
 & \alpha(X) \arrow[uu, hook, "\iota"] \arrow[ur, "\widetilde{\pi}"'] &
\end{tikzcd}$$

where $\widetilde{\pi}$ is the restriction of pr_S to $\alpha(X)$. We have $\pi=\mathrm{pr}_S\circ\alpha=\widetilde{\pi}\circ\widetilde{\alpha}$.

Clearly $\widetilde{\pi}^{-1}(0_S)=\{0_{V\times S}\}$. Thus, by our assumption on case k, we may shrink V,S so that $\widetilde{\pi}$ is finite and (by Lemma 2.7.4) $\widetilde{\pi}_*$ sends coherent $\mathscr{O}_{\alpha(X)}$-modules to coherent $\mathscr{O}_S$-modules. Note that we still have that α is finite and $\iota_*\widetilde{\alpha}_*\mathscr{O}_X=\alpha_*\mathscr{O}_X$ is $\mathscr{O}_{V\times S}$-coherent after shrinking V,S (but not shrinking U). So $\widetilde{\alpha}$ is finite, and by Extension principle 2.1.10, $\widetilde{\alpha}_*\mathscr{O}_X$ is $\mathscr{O}_{\alpha(X)}$-coherent. So $\pi=\widetilde{\pi}\circ\widetilde{\alpha}$ is finite, and $\pi_*\mathscr{O}_X=\widetilde{\pi}_*\widetilde{\alpha}_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. We are done with the proof of Lemma 2.7.5. □

We are now ready to prove Thm. 2.7.2 and more:

Lemma 2.7.6. *Thm. 2.7.2 is true. Moreover, in Thm. 2.7.2, U and W can be chosen so that (besides that π is finite) $\pi_*\mathscr{O}_U$ is also $\mathscr{O}_W$-coherent.*

Proof. It suffices to assume that X is a model space, say a closed subspace of an open $R \subset \mathbb{C}^k$. We first assume S is an open subset of $\mathbb{C}^m$. Define $\varphi : X \to R \times S$ so that the following triangular diagram commutes

$$\begin{tikzcd} & X \times S \arrow[dr, "\iota_{X,R}\times\mathbf{1}"] & & \\ X \arrow[ur, "\mathbf{1}\vee\pi"] \arrow[rr, "\varphi"'] & & R \times S \arrow[r, "\mathrm{pr}_S"'] & S \end{tikzcd}$$

By Prop. 1.13.6 and Prop. 1.12.5, $\mathbf{1} \vee \pi$ and $\iota_{X,R} \vee \mathbf{1}$ are closed embeddings. So their composition φ is a closed embedding (Cor. 1.7.6). By Prop. 1.11.6,

$$\mathrm{pr}_S \circ \varphi = \mathrm{pr}_S \circ (\iota \times \mathbf{1}) \circ (\mathbf{1} \vee \pi) = \mathrm{pr}_S \circ (\iota \vee \pi) = \pi.$$

Thus, by identifying X with $\varphi(X)$, the assumptions of Lemma 2.7.5 are satisfied. The conclusions of Lemma 2.7.5 prove what we want to prove.

In the general case, we may shrink S (and shrink X accordingly) so that S is a closed subspace of an open $\Omega \subset \mathbb{C}^m$. Let $\iota : S \to \Omega$ be the inclusion. Then by shrinking X and Ω (and S accordingly) to neighborhoods of any given points, $\iota \circ \pi : X \to \Omega$ is finite and $\iota_*\pi_*\mathscr{O}_X$ is $\mathscr{O}_\Omega$-coherent. Clearly π is finite, and by Extension principle 2.1.10, $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-coherent. $\square$

Proof of Thm. 2.7.1, (1)⇒(2). Let us prove that $\pi_*\mathscr{O}_X$ is coherent. Choose any $t \in S$. By Lemma 2.7.6, for each $x \in \pi^{-1}(t)$ we can choose neighborhoods $U_x \ni x$ and $W_x \supset \pi(U_x)$ such that $\pi_*\mathscr{O}_{U_x}$ is $\mathscr{O}_{W_x}$-coherent, and that $U_x \cap U_{x'} = \emptyset$ if $x \neq x'$. So for each open $W \subset \bigcap_{x\in\pi^{-1}(t)} W_x$, we have that $\pi_*\mathscr{O}_{U_x\cap\pi^{-1}(W)}$ is $\mathscr{O}_W$-coherent. Therefore, if we set $U = \bigcup_{x\in\pi^{-1}(t)} U_x$, then

$$\pi_*\mathscr{O}_{U\cap\pi^{-1}(W)} \simeq \bigoplus_{x\in\pi^{-1}(t)} \pi_*\mathscr{O}_{U_x\cap\pi^{-1}(W)}$$

is $\mathscr{O}_W$-coherent.

Since $\pi : X \to S$ is finite, by Prop. 2.4.1, there is a neighborhood $W \ni t$ inside $\bigcap_{x\in\pi^{-1}(t)} W_x$ such that $\pi^{-1}(W) = U \cap \pi^{-1}(W)$. So $\pi_*\mathscr{O}_{\pi^{-1}(W)} = (\pi_*\mathscr{O}_X)|_W$ is $\mathscr{O}_W$-coherent. $\square$

The proof of (2)⇒(1) is similar to that of Oka's coherence Thm. 2.6.2:

Proof of Thm. 2.7.1, (2)⇒(1). Assume that $\pi_*\mathscr{E}$ is coherent. Then $\mathscr{E}$ is $\mathscr{O}_X$-finite-type by Lemma 2.6.3. Let us show that the sheaves of relations of $\mathscr{E}$ are finite-type. By Prop. 2.4.1 or Rem. 2.4.4, we have a neighborhood W of t such that

$$\pi^{-1}(W) = \coprod_{x\in\pi^{-1}(t)} U_x$$

where each U_x is a small enough neighborhood of y. Shrink Y to W and X to $\pi^{-1}(W)$. So we have an equivalence of $\mathscr{O}_W$-modules

$$\pi_*\mathscr{E} \simeq \bigoplus_{x\in\pi^{-1}(t)} \pi_*(\mathscr{E}|_{U_x}).$$

Suppose $\alpha : \mathscr{O}_{U_x}^N \to \mathscr{E}_{U_x}$ is a morphism of $\mathscr{O}_{U_x}$-modules. Let $\mathscr{K} = \mathrm{Ker}(\alpha)$ so that we have an exact

$$0 \to \mathscr{K} \to \mathscr{O}_{U_x}^N \to \mathscr{E}_{U_x}.$$

We regard $\mathscr{K}, \mathscr{O}_{U_x}, \mathscr{E}_{U_x}$ as $\mathscr{O}_X$-modules by identifying them with their direct images under $U_x \hookrightarrow X$. Clearly $\mathscr{O}_{U_x}$ is $\mathscr{O}_X$-coherent. So $\pi_*\mathscr{O}_{U_x}$ is $\mathscr{O}_S$-coherent. Also $\pi_*\mathscr{E}_{U_x}$ is $\mathscr{O}_S$-coherent since it is a direct summand of the coherent sheaf $\pi_*\mathscr{E}$ (cf. Cor. 2.1.4). Thus, the exact sequence of $\mathscr{O}_S$-modules

$$0 \to \pi_*\mathscr{K} \to \pi_*\mathscr{O}_{U_x}^N \to \pi_*\mathscr{E}_{U_x}$$

together with Cor. 2.1.5 show that $\pi_*\mathscr{K}$ is $\mathscr{O}_S$-coherent. Therefore, by Lemma 2.6.3, $\mathscr{K}$ is $\mathscr{O}_X$-finite-type. $\square$

We are done with the proofs of Thm. 2.7.1 and 2.7.2. In the following, we give some applications.

2.7.2 Applications

Corollary 2.7.7. *Let $\varphi : X \to Y$ be a holomorphic map of complex spaces. Then the following are equivalent.*

(1) φ is a closed embedding.

(2) φ is an immersion of complex spaces, and it is a closed and injective map of topological spaces.

Proof. (1)$\Rightarrow$(2) is obvious. Assume (2). Then as φ is finite, $\varphi_*\mathscr{O}_X$ is $\mathscr{O}_Y$-coherent. By (2.3.6), the coherent ideal

$$\mathscr{J} = \mathscr{A}nn_{\mathscr{O}_Y}(\varphi_*\mathscr{O}_X)$$

satisfies the assumptions in Prop. 1.7.3. Thus (1) follows from Prop. 1.7.3. $\square$

Rem. 1.13.8 tells us that any holomorphic map factors as the composition of a closed embedding and the projection of a direct product. When the holomorphic map is finite, such decomposition might not be useful because, although closed embeddings are finite, projections are usually not. The following proposition gives a refinement of this decomposition. It says that any finite holomorphic map locally factors as the composition of a closed embedding and a Weierstrass map. This result will be used e.g. in the proof of Base change Thm. 2.8.2.

Proposition 2.7.8. *Let $\pi : X \to S$ be a finite holomorphic map of complex spaces. Then each $t \in S$ is contained in a neighborhood $W \subset S$ such that the restriction $\pi : \pi^{-1}(W) \to W$ is equivalent to the restriction of a Weierstrass map. More precisely, there exist a Weierstrass map $\kappa : Y \to W$ and a closed embedding $\varphi : \pi^{-1}(W) \to Y$ such that the following diagram commutes.*

$$\begin{tikzcd} \pi^{-1}(W) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\ & W & \end{tikzcd} \tag{2.7.1}$$

Proof-Step 1. By Finite mapping theorem, $\pi_*\mathscr{O}_X$ is coherent. So we may shrink S to a neighborhood of t and shrink X accordingly (i.e. replace X by the new $\pi^{-1}(S)$) so that $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-generated by $f_1, \dots, f_k \in \mathscr{O}(X)$. Consider $F = (f_1, \dots, f_k)$ as a holomorphic map $F : X \to \mathbb{C}^k$ (Thm. 1.4.1). Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "F \vee \pi"] \arrow[dr, "\pi"'] & & \mathbb{C}^k \times S \arrow[dl, "\mathrm{pr}_S"] \\ & S & \end{tikzcd} \tag{2.7.2}$$

We want to show that $F \vee \pi$ is a closed embedding.

Since π is closed, one checks easily using (2.7.2) that $F \vee \pi$ is closed. To show that $F \vee \pi$ is injective, it suffices to show that F is injective when restricted to each fiber $\pi^{-1}(\tau)$ (where $\tau \in S$). By Prop. 2.4.5, we have

$$(\pi_*\mathscr{O}_X)_\tau \simeq \bigoplus_{x \in \pi^{-1}(\tau)} \mathscr{O}_{X,x} \tag{2.7.3}$$

which is $\mathscr{O}_{S,s}$-generated by $f_1, \dots, f_k$. If $x, x' \in \pi^{-1}(\tau)$ and $x \neq x'$, then an $\mathscr{O}_{S,\tau}$-linear combination of $f_1, \dots, f_k$ is 1 in $\mathscr{O}_{X,x}$ and 0 in $\mathscr{O}_{X,x'}$. So a $\mathbb{C}$-linear combination of $f_1, \dots, f_k$ takes value 1 at x and 0 at x'. So $F(x) \neq F(x')$. To show that $F \vee \pi$ is an immersion, note that by (2.7.3), the $\mathbb{C}$-algebra morphism

$$F^\# : \mathscr{O}_{\mathbb{C}^k, F(x)} \to \mathscr{O}_{X,x}$$

sends $z_1, \dots, z_k$ to (the germs at x of) $f_1, \dots, f_k$ respectively. So the morphism

$$(F \vee \pi)^\# : \mathscr{O}_{\mathbb{C}^k \times S, x \times \tau} = \mathscr{O}_{\mathbb{C}^k, x} \widehat{\otimes} \mathscr{O}_{S,\tau} \longrightarrow \mathscr{O}_{X,x}$$

sends $z_i \otimes h$ (where $h \in \mathscr{O}_{S,\tau}$) to $h \cdot f_i$. Thus, this morphism is surjective since $\mathscr{O}_{X,x}$ is $\mathscr{O}_{S,\tau}$-generated by $f_1, \dots, f_k$. So $F \vee \pi$ is an immersion. By Cor. 2.7.7, $F \vee \pi$ is a closed embedding. $\square$

Proof-Step 2. Since $(\pi_*\mathscr{O}_X)_t$ is a finitely generated module of the Noetherian ring $\mathscr{O}_{S,t}$, for each i, the $\mathscr{O}_{S,t}$-submodule of $(\pi_*\mathscr{O}_X)_t$ generated by all non-negative powers of f_i is finitely generated. So f_i is **integral over** $\mathscr{O}_{S,t}$. Namely, we may find $n_i \in \mathbb{Z}_+$ such that

$$a_{i,0} + a_{i,1}f_i + \cdots + a_{i,n_i-1}f_i^{n_i-1} + f_i^{n_i} = 0 \tag{2.7.4}$$

where each $a_{i,j} \in \mathscr{O}_{S,t}$.

Shrink S to a neighborhood of t (and shrink X to $\pi^{-1}(S)$) so that all $a_{i,j}$ are elements of $\mathscr{O}(S)$, and that (2.7.4) holds at the level of $\mathscr{O}(X)$. Then

$$p_i(z_i) = a_{i_0} + a_{i,1}z_i + \cdots + a_{i,n_i-1}z_i^{n_i-1} + z_i^{n_i}$$

is a monic polynomial of z_i, viewed as in $\mathscr{O}(\mathbb{C}^k \times S)$. Note that $F \vee \pi$ is still a closed embedding. We let $\mathcal{I}$ be the ideal of $\mathscr{O}_{\mathbb{C}^k\times S}$ generated by $p_1, \dots, p_k$, and let $Y = \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^k\times S}/\mathcal{I})$. Then $\mathrm{pr}_S : \mathbb{C}^k \times S \to S$ restricts to a Weierstrass map $\kappa : Y \to S$. By Thm. 1.4.8, $F \vee \pi : X \to \mathbb{C}^k \times S$ restricts to $\varphi : X \to Y$, which is clearly a closed embedding. And we clearly have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\ & S & \end{tikzcd}$$

This finishes the proof. $\square$

2.8 Base change theorem for finite holomorphic maps

In algebraic geometry, if X, Y, S are affine schemes, then $\mathscr{O}(X \times_S Y) \simeq \mathscr{O}(X) \otimes_{\mathscr{O}(S)} \mathscr{O}(Y)$. In complex analytic geometry, fiber products are in general related to completed tensor products. But in the case that one holomorphic map is finite, the usual (algebraic) tensor products are sufficient. The goal of this section is to explore the relationship between $X \times_S Y$ and tensor products in the analytic setting and at the level of stalks. This goal will be achieved in Cor. 2.8.4 which is crucial to the future proof that "flatness of holomorphic maps is preserved by base change". We shall prove Cor. 2.8.4 as a consequence of the Base change theorem of finite holomorphic maps.

2.8.1 The setting

Consider a Cartesian square of holomorphic maps of complex spaces.

$$
\begin{tikzcd}
X \arrow[d, "\pi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"] \\
S & Y \arrow[l, "\psi"']
\end{tikzcd}
\tag{2.8.1}
$$

Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then we have an $\mathscr{O}_Y$-module morphism

$$\Psi : \psi^* \pi_* \mathscr{E} \longrightarrow \mathrm{pr}_{Y,*} \mathrm{pr}_X^* \mathscr{E}, \tag{2.8.2}$$

namely, a morphism

$$\Psi : (\pi_* \mathscr{E}) \otimes_{\mathscr{O}_S} \mathscr{O}_Y \longrightarrow \mathrm{pr}_{Y,*}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{X \times_S Y}) \tag{2.8.3}$$

such that for each open $V \subset Y$ and each open $W \subset S$ containing $\psi(V)$, Ψ sends

$$\sigma \otimes g \qquad \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_S(W)} \mathscr{O}_Y(V) \tag{2.8.4}$$

to

$$\sigma \otimes \mathrm{pr}_Y^\# g \qquad \in \mathscr{E}(\pi^{-1}(W)) \otimes_{\mathscr{O}_X(\pi^{-1}(W))} \mathscr{O}_{X \times_S Y}(\mathrm{pr}_Y^{-1}(V)). \tag{2.8.5}$$

(Note that $\mathrm{pr}_X(\mathrm{pr}_Y^{-1}(V)) \subset \pi^{-1}(W)$.) It is easy to see that Ψ is functorial. We call Ψ the **base change morphism**.

Remark 2.8.1. The stalk map of Ψ at each $y \in Y$ is the $\mathscr{O}_{Y,y}$-module morphism determined by

$$
\begin{aligned}
\Psi : (\pi_* \mathscr{E})_{\psi(y)} \otimes_{\mathscr{O}_{S,\psi(y)}} \mathscr{O}_{Y,y} &\longrightarrow \mathrm{pr}_{Y,*}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{X \times_S Y})_y \\
\sigma \otimes 1 \quad &\mapsto \quad \sigma \otimes 1
\end{aligned}
\tag{2.8.6}
$$

2.8.2 Base change theorem

The following theorem is the main result of this section. Note that in the Cartesian square (2.8.1), if π is finite then pr_Y is finite (Prop. 2.4.11).

Theorem 2.8.2 (Base change theorem). *In the setting of Subsec. 2.8.1, assume that $\pi : X \to S$ is finite and $\mathscr{E}$ is a coherent $\mathscr{O}_X$-module. Then the base change morphism Ψ (cf. (2.8.3)) is an isomorphism of $\mathscr{O}_Y$-modules.*

Note that this theorem is local by nature. Namely, in the proof we may shrink S to a neighborhood of any given point, and replace X by $\pi^{-1}(S)$ and Y by $\psi^{-1}(S)$.

In the special case that $\mathscr{E} = \mathscr{O}_X$, we have:

Corollary 2.8.3. *Let (2.8.1) be a Cartesian square of holomorphic maps of complex spaces. Assume that $\pi : X \to S$ is finite. Then we have an $\mathscr{O}_Y$-module isomorphism*

$$\Psi : (\pi_* \mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y \xrightarrow{\simeq} \mathrm{pr}_{Y,*} \mathscr{O}_{X \times_S Y} \tag{2.8.7}$$

whose stalk map at each $y \in Y$ is an $\mathscr{O}_{Y,y}$-module morphism determined by

$$\begin{aligned} \Psi : (\pi_* \mathscr{O}_X)_{\psi(y)} \otimes_{\mathscr{O}_{S,\psi(y)}} \mathscr{O}_{Y,y} &\longrightarrow \mathrm{pr}_{Y,*}(\mathscr{O}_{X \times_S Y})_y \\ f \otimes 1 \quad &\mapsto \quad \mathrm{pr}_X^\# f \end{aligned} \tag{2.8.8}$$

Corollary 2.8.4. *Let (2.8.1) be a Cartesian square, and assume that $\pi : X \to S$ is finite. Then for each $x \in X$ and $y \in Y$ such that $\pi(x)$ equals $t = \psi(y)$, there is an isomorphism of $\mathscr{O}_{S,t}$-modules*

$$\begin{aligned} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,t}} \mathscr{O}_{Y,y} &\xrightarrow{\simeq} \mathscr{O}_{X \times_S Y, x \times y} \\ f \otimes g \quad &\mapsto \quad \mathrm{pr}_X^\# f \cdot \mathrm{pr}_Y^\# g \end{aligned} \tag{2.8.9}$$

First Proof. By Thm. 2.7.2, we may shrink X and S to neighborhoods of x and t respectively, and shrink Y to $\psi^{-1}(S)$, so that $\pi^{-1}(t) = \{x\}$ (as sets) and π is still finite. Then in view of Prop. 2.4.5, we see that (2.8.8) becomes exactly (2.8.9). □

Second Proof. By Prop. 2.4.5, for each y and $t = \psi(y)$, (2.8.8) is precisely the direct sum of (2.8.9) over all $x \in \pi^{-1}(t) = \mathrm{pr}_Y^{-1}(y)$. □

The second proof shows that Cor. 2.8.3 and Cor. 2.8.4 are indeed equivalent.

2.8.3 Proof of Base change Thm. 2.8.2

Lemma 2.8.5. *Assume that Thm. 2.8.2 holds when $\mathscr{E} = \mathscr{O}_X$. Then Thm. 2.8.2 holds for any coherent $\mathscr{O}_X$-module $\mathscr{E}$.*

Proof. If Thm. 2.8.2 holds when $\mathscr{E} = \mathscr{O}_X$, then it holds when $\mathscr{E}$ is $\mathscr{O}_X$-free. Now in the general case, by Lemma 2.4.8 we can assume that S is so small that there is an exact sequence of $\mathscr{O}_X$-modules

$$\mathscr{F} \to \mathscr{G} \to \mathscr{E} \to 0$$

where $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-free. By the right exactness of ψ^* and π_* (Cor. 2.4.6), we have an exact sequence

$$\psi^* \pi_* \mathscr{F} \to \psi^* \pi_* \mathscr{G} \to \psi^* \pi_* \mathscr{E} \to 0.$$

Since the base change map Ψ is functorial, we have a commutative diagram

$$\begin{tikzcd}
\psi^*\pi_*\mathscr{F} \arrow[r] \arrow[d, "\Psi"', "\simeq"] & \psi^*\pi_*\mathscr{G} \arrow[r] \arrow[d, "\Psi"', "\simeq"] & \psi^*\pi_*\mathscr{E} \arrow[r] \arrow[d, "\Psi"'] & 0 \\
\mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{F} \arrow[r] & \mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{G} \arrow[r] & \mathrm{pr}_{Y,*}\mathrm{pr}_X^*\mathscr{E} \arrow[r] & 0
\end{tikzcd}$$

where the first two Ψ are isomorphisms by assumption. So the third Ψ is an isomorphism by Five Lemma. $\square$

Lemma 2.8.6. *Cor. 2.8.3 holds if $\pi : X \to S$ is a Weierstrass map.*

Proof. By Prop. 2.5.3, we may assume that $\mathrm{pr}_Y : X \times_S Y \to Y$ is a Weierstrass map. More precisely, we may assume that (2.8.1) factors as

$$\begin{tikzcd}
X \arrow[d, hook] & X \times_S Y \arrow[l] \arrow[d, hook] \\
\mathbb{C}^k \times S \arrow[d] & \mathbb{C}^k \times Y \arrow[l] \arrow[d] \\
S & Y \arrow[l]
\end{tikzcd}$$

where the two small squares are Cartesian. By the Fundamental Thm. 2.5.4 of Weierstrass maps, $\pi_*\mathscr{O}_X$ is $\mathscr{O}_S$-freely generated by (2.5.3), and so $(\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y$ is $\mathscr{O}_Y$-freely generated by (2.5.3) $\otimes\, 1$. Also, $\mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}$ is $\mathscr{O}_Y$-freely generated by (2.5.3). Using e.g. (2.8.8) one checks that Ψ sends the given free generators of $(\pi_*\mathscr{O}_X) \otimes_{\mathscr{O}_S} \mathscr{O}_Y$ bijectively to those of $\mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}$. So Ψ must be an isomorphism. $\square$

***Proof of Thm.* 2.8.2.** By Lemma 2.8.5, it suffices to prove Cor. 2.8.3. By Prop. 2.7.8, we may assume S is so small that $\pi : X \to S$ factors as $X \hookrightarrow Z \xrightarrow{\widetilde{\pi}} S$ where X is a closed subspace of Z and $\widetilde{\pi}$ is equivalent to a Weierstrass map. Thus, (2.8.1) factors as the combination of two Cartesian squares

$$\begin{tikzcd}
X \arrow[d, hook, "\iota"'] & X \times_S Y \arrow[l] \arrow[d, hook, "\iota\times\mathbf{1}"] \\
Z \arrow[d, "\widetilde{\pi}"'] & Z \times_S Y \arrow[l] \arrow[d, "\widetilde{\mathrm{pr}}_Y"] \\
S & Y \arrow[l, "\psi"']
\end{tikzcd} \tag{2.8.10}$$

where $\mathrm{pr}_Y : X \times_S Y \to Y$ is $\widetilde{\mathrm{pr}}_Y \circ (\iota \times \mathbf{1})$.

We have proved that Cor. 2.8.3 holds (and hence Thm. 2.8.2 holds, cf. Lemma 2.8.5) for the lower Cartesian square. Apply Thm. 2.8.2 to the lower square and the coherent $\mathscr{O}_Z$-module $\iota_*\mathscr{O}_X$: The domain of the isomorphism Ψ is

$$(\widetilde{\pi}_*\iota_*\mathscr{O}_X)\otimes_{\mathscr{O}_S}\mathscr{O}_Y=\pi_*\mathscr{O}_X\otimes_{\mathscr{O}_S}\mathscr{O}_Y$$

and the codomain is

$$\widetilde{\mathrm{pr}}_{Y,*}(\iota_*\mathscr{O}_X\otimes_{\mathscr{O}_Z}\mathscr{O}_{Z\times_S Y})\simeq\widetilde{\mathrm{pr}}_{Y,*}\big((\iota\times\mathbf{1})_*\mathscr{O}_{X\times_S Y}\big)=\mathrm{pr}_{Y,*}\mathscr{O}_{X\times_S Y}.$$

By checking stalkwisely with the help of (2.8.6) and (2.8.8) (and possibly Prop. 2.4.5), one sees that this morphism (i.e. the base change map for the lower square of (2.8.10) and the $\mathscr{O}_Z$-module $\iota_*\mathscr{O}_X$) agrees with the morphism Ψ in Cor. 2.8.3. So the latter must be an isomorphism. □

2.9 Analytic spectra Specan

We fix a complex space S.

2.9.1 Main results

Definition 2.9.1. A **morphism** from a finite holomorphic map $\pi : X \to S$ to a finite holomorphic $\kappa : Y \to S$ is a holomorphic map $\varphi : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\kappa"] \\ & S & \end{tikzcd} \tag{2.9.1}$$

The set of morphisms is denoted by $\mathrm{Mor}_S(X,Y)$. This defines the **category of finite holomorphic maps to** S.

Definition 2.9.2. An $\mathscr{O}_S$**-algebra** is an S-sheaf of $\mathbb{C}$-algebras $\mathscr{A}$ together with a morphism of sheaves of $\mathbb{C}$-algebras $\mathscr{O}_S \to \mathscr{A}$. Since $\mathscr{A}$ is an $\mathscr{A}$-module, it becomes an $\mathscr{O}_S$-module. We say that $\mathscr{A}$ is a **coherent** $\mathscr{O}_S$**-algebra** if it is an $\mathscr{O}_S$-algebra which is coherent as an $\mathscr{O}_S$-module.

A **morphism** of $\mathscr{O}_S$-algebras from $\mathscr{B}$ to $\mathscr{A}$ is by definition a morphism $\Phi : \mathscr{B} \to \mathscr{A}$ of sheaves of $\mathbb{C}$-algebras such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{A} & & \mathscr{B} \arrow[ll, "\Phi"'] \\ & \mathscr{O}_S \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{2.9.2}$$

The commutativity of (2.9.2) is equivalent to saying that the morphism of sheaves of $\mathbb{C}$-algebras Φ is also a morphism of $\mathscr{O}_S$-modules. The set of morphisms is denoted by $\mathrm{Mor}_{\mathscr{O}_S}(\mathscr{B},\mathscr{A})$. This defines the **category of coherent $\mathscr{O}_S$-algebras**. $\square$

We have avoided using the symbol $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{B},\mathscr{A})$, which is the set of $\mathscr{O}_S$-module morphisms but not $\mathscr{O}_S$-algebra morphisms.

Theorem 2.9.3. *The contravariant functor $\mathfrak{F}$ from the category of finite holomorphic maps to S to the category of coherent $\mathscr{O}_S$-algebras is an antiequivalence of categories. The functor $\mathfrak{F}$ sends each finite holomorphic map $\pi : X \to S$ to the coherent $\mathscr{O}_S$-algebra $\pi_*\mathscr{O}_X$. At the level of morphisms the functor is*

$$\mathfrak{F} : \mathrm{Mor}_S(X,Y) \to \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y,\pi_*\mathscr{O}_X), \qquad \varphi \mapsto \varphi^\#. \tag{2.9.3}$$

Thus, for each coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ there is, *up to isomorphisms*, a unique finite holomorphic map $\pi : X \to S$ such that $\pi_*\mathscr{O}_X = \mathscr{A}$. We write this map as $\mathrm{Specan}(\mathscr{A}) \to S$ and call this map (or simply call the complex space $\mathrm{Specan}(\mathscr{A})$) the **analytic spectrum** of $\mathscr{A}$.

Note that when $\mathscr{A} = \mathscr{O}_S/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathscr{O}_S$, as before, $\mathrm{Specan}(\mathscr{A})$ denotes the unique analytic spectrum as a closed subspace of S. For a general $\mathscr{A}$, $\mathrm{Specan}(\mathscr{A})$ is not unique.

Corollary 2.9.4. *Let $\psi : Z \to S$ be a holomorphic map of complex spaces. Then*

$$\mathrm{Specan}(\mathscr{A} \otimes_{\mathscr{O}_S} \mathscr{O}_Z) \simeq \mathrm{Specan}(\mathscr{A}) \times_S Z$$

Proof. This is just a rephrasing of Cor. 2.8.3. $\square$

2.9.2 Proof of Thm. 2.9.3

Proof that* (2.9.3) *is injective. Let $\varphi,\psi \in \mathrm{Mor}_S(X,Y)$ such that $\psi^*,\varphi^* : \kappa_*\mathscr{O}_Y \to \pi_*\mathscr{O}_X$ are equal. By Prop. 2.4.5, for each $t \in S$, $\varphi^\# : (\kappa_*\mathscr{O}_Y)_t \to (\pi_*\mathscr{O}_X)_t$ is an $\mathscr{O}_{S,t}$-module morphism of the form

$$\varphi^\# : \bigoplus_{y\in\kappa^{-1}(t)} \mathscr{O}_{Y,y} \to \bigoplus_{x\in\pi^{-1}(t)} \mathscr{O}_{X,x}$$

whose restriction to $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is non-zero iff $y = \varphi(x)$. A similar description holds for $\psi^\#$. It follows that φ and ψ must be equal, first of all as maps of sets, and then clearly as holomorphic maps. $\square$

Proof that* (2.9.3) *is surjective. Choose any $\Phi \in \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y,\pi_*\mathscr{O}_X)$. It suffices to show that Φ is locally realized by φ_W, i.e., that each $t \in S$ is contained in a neighborhood $W \subset S$ such that, after shrinking S to W, X to $\pi^{-1}(X)$, and Y

to $\kappa^{-1}(X)$, Φ equals $\varphi_W^\#$. Then by the injectivity of (2.9.3), φ_W and $\varphi_{W'}$ agree on $W \cap W'$. So these φ_W can be glued together to realize Φ globally.

To find φ locally, we first assume that κ is a Weierstrass map, which factors as $\kappa : Y \hookrightarrow \mathbb{C}^k \times S \xrightarrow{\mathrm{pr}_S} S$. Consider $z_1, \dots, z_k$ as elements of $\mathscr{O}(\mathbb{C}^k \times S)$ and also of $\mathscr{O}(Y) = (\kappa_*\mathscr{O}_Y)(S)$ by restriction. Let $f_i = \Phi(z_i)$, which is an element of $(\pi_*\mathscr{O}_X)(S) = \mathscr{O}(X)$. Regard $F = (f_1, \dots, f_k)$ as a holomorphic map $X \to \mathbb{C}^k$ (Thm. 1.4.1). Then by Thm. 1.4.8, the holomorphic map $F \vee \mathrm{pr}_S : X \to \mathbb{C}^k \times S$ restricts to a holomorphic $\varphi : X \to Y$. (This is similar to the Proof-Step 2 of Prop. 2.7.8. Note that one needs the commutativity of (2.9.2) to check condition (b) of Thm. 1.4.8!) Then (2.9.1) commutes because $\kappa \circ \varphi = \mathrm{pr}_S \circ (F \vee \pi) = \pi$. Both $\varphi^\#$ and Φ send each $z_i \in (\kappa_*\mathscr{O}_Y)(S)$ to f_i. So $\varphi^\# = \Phi$ because the powers of $z_1, \dots, z_k$ generate the $\mathscr{O}_S$-module $\kappa_*\mathscr{O}_Y$ by Thm. 2.5.4.

Now, in the general case, by Prop. 2.7.8 we may assume S is small enough such that κ factors as

$$\kappa : Y \hookrightarrow Z \xrightarrow{\varpi} S$$

where $\varpi : Z \to S$ is a isomorphic to a Weierstrass map and $Y = \mathrm{Specan}(\mathscr{O}_Z/\mathscr{J})$ is a closed subspace of Z. We have a sequence of morphisms of $\mathscr{O}_S$-algebras

$$\pi_*\mathscr{O}_X \xleftarrow{\Phi} \kappa_*\mathscr{O}_Y \xleftarrow{\iota^\#} \varpi_*\mathscr{O}_Z.$$

By the previous paragraph, there is $\psi \in \mathrm{Mor}_S(X, Z)$ such that $\psi^\# : \varpi_*\mathscr{O}_Z \to \pi_*\mathscr{O}_X$ equals $\Phi \circ \iota^\#$ and hence vanishes on $\varpi_*\mathscr{J}$. Thus, by Prop. 2.4.5, for each $x \in X$, $\psi^\# : \mathscr{O}_{Z,\psi(x)} \to \mathscr{O}_{X,x}$ vanishes on $\mathscr{J}_{\psi(x)}$. So Thm. 1.4.8 tells us that ψ restricts to a holomorphic $\varphi : X \to Y$. Namely $\psi = \iota \circ \varphi$. Clearly $\varphi \in \mathrm{Mor}_S(X, Y)$.

We have $\varphi^\# \circ \iota^\# = \psi^\# = \Phi \circ \iota^\#$. Thus, to show that $\varphi^\# = \Phi$, it suffices to show that $\iota^\# : \varpi_*\mathscr{O}_Z \to \kappa_*\mathscr{O}_Y$ is surjective. This is clear from Prop. 2.4.5 and the fact that Y is a closed subspace of Z. $\square$

The above two proofs together show that $\mathfrak{F}$ is fully faithful.

Proof that $\mathfrak{F}$ is essentially surjective. Given any coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, our goal is to find a finite holomorphic map $\pi : X \to S$ (for some complex space X) such that $\pi_*\mathscr{O}_X$ is equivalent to $\mathscr{A}$ as $\mathscr{O}_S$-algebras.

We first show that the construction of π is local by nature. Suppose that we have an open cover $(S_i)_{i \in I}$ of S such that for each i we have a finite holomorphic $\pi_i : X_i \to S_i$ such that there is an isomorphism of $\mathscr{O}_{S_i}$-algebras

$$\Phi_i : \pi_{i,*}\mathscr{O}_{X_i} \xrightarrow{\simeq} \mathscr{A}|_{S_i}.$$

Write $S_{ij} = S_i \cap S_j$, $X_{ij}^i = \pi_i^{-1}(S_{ij})$, and let $\pi_{ij}^i : X_{ij}^i \to S_{ij}$ be the restriction of π_i. Then by the full-faithfulness of $\mathfrak{F}$, there is a unique isomorphism $\gamma_{j,i} \in$

$\mathrm{Mor}_{S_{ij}}(X^i_{ij}, X^j_{ij})$ such that $\gamma^{\#}_{j,i} : \pi^j_{ij,*}\mathscr{O}_{X^j_{ij}} \to \pi^i_{ij,*}\mathscr{O}_{X^i_{ij}}$ equals $\Phi_i^{-1}|_{S_{ij}} \circ \Phi_j|_{S_{ij}}$. One checks easily that these $\gamma_{j,i}$ satisfy the cocycle condition so that they can be used as the gluing maps to glue all π_i together and form a desired $\pi : X \to S$.

Let us construct π locally. Choose $t \in S$. Using the methods in the proof of Prop. 2.7.8, one shows that if S is sufficiently small then there exist a Weierstrass map $\kappa : Y \to S$ and $\Phi : \mathrm{Mor}_{\mathscr{O}_S}(\kappa_*\mathscr{O}_Y, \mathscr{A})$ which is surjective as an $\mathscr{O}_S$-module morphism. $\mathcal{T} = \mathrm{Ker}(\Phi)$ is an ideal of $\kappa_*\mathscr{O}_Y$, i.e., an $\mathscr{O}_S$-submodule of $\kappa_*\mathscr{O}_Y$ whose stalks at each $\tau \in S$ is invariant under $(\kappa_*\mathscr{O}_Y)_\tau$. So $\mathcal{T}_\tau = \mathcal{T}_\tau \cdot (\kappa_*\mathscr{O}_Y)_\tau$. Thus, by Prop. 2.4.5, we have an $(\kappa_*\mathscr{O}_Y)_\tau$-module isomorphism

$$\kappa_*(\mathcal{T}\mathscr{O}_Y)_\tau \simeq \bigoplus_{y\in\kappa^{-1}(\tau)} (\mathcal{T}\mathscr{O}_Y)_y = \bigoplus_{y\in\kappa^{-1}(\tau)} \mathcal{T}_\tau\mathscr{O}_{Y,y} \simeq \mathcal{T}_\tau \cdot (\kappa_*\mathscr{O}_Y)_\tau = \mathcal{T}_\tau$$

such that each $\sigma \in \mathcal{T}_\tau$ corresponds to $\sigma \cdot 1$ on the LHS.

$\mathcal{T}\mathscr{O}_Y$ is a finite-type ideal of $\mathscr{O}_Y$ since $\mathcal{T}$ is $\mathscr{O}_S$-coherent by Cor. 2.1.5. Define $X = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{T}\mathscr{O}_Y)$, and let $\pi : X \to S$ be the restriction of κ. This gives the desired finite holomorphic map since, by the exactness of κ_*, we have an $\kappa_*\mathscr{O}_Y$-module isomorphism

$$\pi_*\mathscr{O}_X = \kappa_*(\mathscr{O}_Y/\mathcal{T}\mathscr{O}_Y) \simeq \kappa_*\mathscr{O}_Y/\kappa_*(\mathcal{T}\mathscr{O}_Y) \simeq \kappa_*\mathscr{O}_Y/\mathcal{T} \simeq \mathscr{A}.$$

(These isomorphisms are explicit at the level of stalks.) $\square$

2.10 Nullstellensatz

In this section, we give another application of Finite mapping Thm. 2.7.1 and Thm. 2.7.2: we prove the complex analytic version of Hilbert Nullstellensatz, called Rückert Nullstellensatz in [GR] and [GPR]. Nullstellensatz will be used in an essential way to prove that every complex space X has an associated reduced complex space X_{red}, and that if X is reduced at x then X is reduced near x.

2.10.1 Equivalent forms of Nullstellensatz

Theorem 2.10.1 (Nullstellensatz). *Let X be a complex space. If $f \in \mathscr{O}(X)$ satisfies that $f(x) = 0$ for all $x \in X$, then the germ of f at each $x \in X$ is a nilpotent element of $\mathscr{O}_{X,x}$.*

The converse is clearly true: If f is nilpotent at $\mathscr{O}_{X,x}$ for each x, then f a zero continuous function.

Recall that if I is an ideal of a commutative ring A, then its **radical** $\sqrt{I}$ is

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n \in \mathbb{Z}_+\}.$$

Similarly:

Definition 2.10.2. If X is a $\mathbb{C}$-ringed space and $\mathcal{I}$ is an ideal of $\mathscr{O}_X$, then the **radical of** $\mathcal{I}$ is the ideal $\sqrt{\mathcal{I}}$ of $\mathscr{O}_X$ defined by

$$\sqrt{\mathcal{I}}(U) = \{f \in \mathscr{O}(U) : f \in \sqrt{\mathcal{I}_x} \text{ for all } x \in U\}.$$

So $\mathcal{I}$ is determined by $(\sqrt{\mathcal{I}})_x = \sqrt{\mathcal{I}_x}$ for all $x \in X$.

Then there is an equivalent way of stating Nullstellensatz:

Theorem 2.10.3 (Nullstellensatz). *Let X be a complex space. Then the kernel of the reduction map* $\mathrm{red} : \mathscr{O}_X \to \mathscr{C}_X$ *(where $\mathscr{C}_X$ is the sheaf of germs of continuous functions) equals $\sqrt{0_X}$, the radical of the zero ideal of $\mathscr{O}_X$.*

We call $\sqrt{0_X}$ the **nilradical** of $\mathscr{O}_X$ (or of X).

Remark 2.10.4. There are some other equivalent statements of **Nullstellensatz**:

1. Let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$. Then $f \in \mathscr{O}(X)$ vanishes on the subset $N(\mathcal{I})$ if and only if $f \in \sqrt{\mathcal{I}}$.

2. Let $\mathscr{O}_{X,x}$ be an analytic local $\mathbb{C}$-algebra, and let I be an ideal. Then $f \in \mathscr{O}_{X,x}$ is an nilpotent element of I if and only if f vanishes on the $\mathrm{Specan}(\mathscr{O}_{X,x}/I)$, the **germ of complex subspace** of X defined by I.

3. If $\mathscr{E}$ is a coherent sheaf on a complex space X. Then $f \in \mathscr{O}(X)$ vanishes on the subset $\mathrm{Supp}(\mathscr{E})$ if and only if for each $x \in X$ there is $n \in \mathbb{Z}_+$ such that $f^n \mathscr{E}_x = 0$.

Proof. 1$\Leftrightarrow$Thm. 2.10.1: Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then $f \in \mathscr{O}_{X,x}$ belongs to $\sqrt{\mathcal{I}_x}$ iff the residue class of f in $\mathscr{O}_{Y,x} = \mathscr{O}_{X,x}/\mathcal{I}_x$ is nilpotent.

1$\Leftrightarrow$2: Obvious. 3$\Rightarrow$ 1: Take $\mathscr{E} = \mathscr{O}_X/\mathcal{I}$. 1$\Rightarrow$3: Take $\mathcal{I} = \mathscr{Ann}_{\mathscr{O}_X}(\mathscr{E})$. □

2.10.2 Proof of Nullstellensatz

We start by proving a special case.

Lemma 2.10.5. *Let X be a neighborhood of $0 \in \mathbb{C}^{m+1}$ where $m \in \mathbb{N}$. Let $(z, w, t_2, \dots, t_m)$ be the standard coordinates of $\mathbb{C}^{m+1}$. Let $\mathcal{I}$ be a coherent ideal of $\mathscr{O}_X$ such that*

$$N(\mathcal{I}) \subset \{(z, w, t_\bullet) \in X : z = 0\}.$$

Then (the germ at 0 of) z is an element of $\sqrt{\mathcal{I}_0}$.

Proof. We prove by induction on $m \in \mathbb{N}$. The base case $m = 0$ is elementary and is hence omitted. Assume the lemma holds for $m-1$ where $m \geqslant 1$. Let us prove it for m. Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$.

We first assume that $\mathcal{I}_0$ contains

$$h(z,w,t_\bullet) = \sum_{n=0}^{\infty} a_n(w,t_\bullet)z^n \tag{2.10.1}$$

where $a_0 \neq 0$. Then as in the proof of Thm. 1.5.5, we may choose a new set of coordinates $(w,t_\bullet)$ for $\mathbb{C}^m$ such that $a_0(w,t_\bullet) = h(0,w,t_\bullet)$ has finite order in w, i.e. $a(w,0)$ is non-zero. So $0_{\mathbb{C}^{m+1}}$ is an isolated point of the fiber at $0_{\mathbb{C}^m}$ of the holomorphic map $\pi : Y \to \mathbb{C}^m$ defined by the restriction of $\mathbb{C}^{m+1} \to \mathbb{C}^m$, $(z,w,t_\bullet) \mapsto (z,t_\bullet)$. We shrink X to a neighborhood of 0 so that $0_{\mathbb{C}^{m+1}}$ is the only point of that fiber, and that (by Thm. 2.7.2) $\pi : Y \to V$ is finite where V is a neighborhood of $0 \in \mathbb{C}^m$. See Fig. 2.10.1.

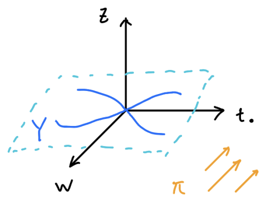

Figure 2.10.1

By Finite mapping Thm. 2.7.1, $\pi_*\mathscr{O}_Y$ is a coherent $\mathscr{O}_V$-module. By assumption, the Nullstellensatz holds for any coherent ideal $\mathcal{J}$ of $\mathscr{O}_V$ such that

$$N(\mathcal{J}) \subset \{(z,t_\bullet) \in V : z = 0\}.$$

Choose $\mathcal{J} = \mathscr{A}nn_{\mathscr{O}_V}(\pi_*\mathscr{O}_Y)$. Then the assumption tells us that there is $n \in \mathbb{Z}_+$ such that $z^n \in \mathscr{O}_{\mathbb{C}^m,0}$ kills the stalk $(\pi_*\mathscr{O}_Y)_0 \simeq \mathscr{O}_{Y,0}$ (Prop. 2.4.5). So $\pi^\# z^n$ (or simply z^n as an element of $\mathscr{O}(\mathbb{C}^{m+1})$) kills $\mathscr{O}_{Y,0} = \mathscr{O}_{\mathbb{C}^{m+1},0}/\mathcal{I}_0$. Therefore $z^n \in \mathcal{I}_0$.

Now, in the general case, note that it suffices to prove that z is nilpotent in $z^{-k}\mathcal{I}_0 = \{f \in \mathscr{O}_{\mathbb{C}^{m+1},0} : z^k f \in \mathcal{I}_0\}$ for some $k \in \mathbb{N}$. This statement is true if we can find k and $h \in z^{-k}\mathcal{I}_0$ whose series expansion as in (2.10.1) has non-zero constant term. This follows by choosing a non-zero $g \in \mathcal{I}_0$, letting k be the smallest power of z such that the series expansion of g in z has non-zero coefficient before z^k, and setting $h = z^{-k}g$. $\square$

Proof of Nullstellensatz. Let X be a complex space, and assume that $f \in \mathscr{O}(X)$ vanishes at every $x \in X$. We now fix $x \in X$ and show that f is nilpotent in $\mathscr{O}_{X,x}$. Consider the graph $\mathfrak{G}(f)$ of f, namely the image of the closed embedding $f \vee \mathbf{1} : X \to \mathbb{C} \times X$ (cf. Prop. 1.13.6). Assume X is a small enough neighborhood of x so that X is a closed subspace of an open $U \subset \mathbb{C}^m$ and $x = 0_{\mathbb{C}^m}$. Then $\mathfrak{G}(f)$ is a closed subspace of $\mathbb{C} \times U$.

As a set, $\mathfrak{G}(f)$ is contained in $0 \times U$. Let $z \in \mathscr{O}(\mathbb{C})$ be the standard coordinate of $\mathbb{C}$. Then by Lemma 2.10.5, $z \otimes 1 \in \mathscr{O}_{\mathbb{C}\times U, 0\times 0}$ is nilpotent in $\mathscr{O}_{\mathfrak{G}(f),0\times 0}$. But the restriction $f \vee \mathbf{1} : X \to \mathfrak{G}(f)$ is a biholomorphism, and it pulls $z \otimes 1 = \mathrm{pr}_{\mathbb{C}}^{\#} z$ (where $\mathrm{pr}_{\mathbb{C}} : \mathbb{C} \times U \to \mathbb{C}$ is the projection) back to $z \circ \mathrm{pr}_{\mathbb{C}} \circ (f \vee \mathbf{1}) = z \circ f = f$. So f is nilpotent in $\mathscr{O}_{X,0}$. $\square$

Chapter 3

Dimensions and local geometry of complex spaces

3.1 Prime decomposition

We fix a commutative ring $\mathcal{A}$. Recall that $\mathcal{A}$ is called **reduced** if $\mathcal{A}$ has no non-zero nilpotent elements. This is equivalent to saying that $\{0\} = \sqrt{\{0\}}$. If I is an ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is reduced iff $\sqrt{I} = I$.

Remark 3.1.1. Recall the general fact that for any ideals $I_1, \dots, I_k$ of $\mathcal{A}$ we have

$$\sqrt{I_1 \cdots I_k} = \sqrt{I_1 \cap \cdots \cap I_k} = \sqrt{I_1} \cap \cdots \cap \sqrt{I_k}. \tag{3.1.1}$$

In view of Nullstellensatz, the first equality says that "the zero sets defined by $I_1 \cdots I_k$ and defined by $I_1 \cap \cdots \cap I_k$ are equal" (namely, they are equal to the union of the zero sets of $I_1, \dots, I_k$). The second equality implies that if $I_i = \sqrt{I_i}$ for each i, then $I_1 \cap \cdots \cap I_k$ is its own radical.

Proof. The two equalities in (3.1.1) are clearly $\subset$. If $f \in \cap_i \sqrt{I_i}$ then $f^{n_i} \in I_i$ for some $n_i \in \mathbb{Z}_+$. Then $f^{n_1+\cdots+n_k} \in I_1 \cdots I_k$, and hence $f \in \sqrt{I_1 \cdots I_k}$. This proves (3.1.1). □

Lemma 3.1.2. *Let $\mathfrak{p}$ be an ideal of $\mathcal{A}$. Then the following are equivalent.*

(a) *$\mathfrak{p}$ is a prime ideal. Equivalently, $\mathcal{A}/\mathfrak{p}$ is an integral domain.*

(b) *$\mathfrak{p} = \sqrt{\mathfrak{p}}$. Moreover, if $\mathfrak{p} = I_1 \cap I_2$ where I_1, I_2 are ideals of $\mathcal{A}$, then $I_1 = \mathfrak{p}$ or $I_2 = \mathfrak{p}$.*

(c) *$\mathfrak{p} = \sqrt{\mathfrak{p}}$. Moreover, if $\mathfrak{p} = I_1 \cap I_2$ where I_1, I_2 are ideals of $\mathcal{A}$ satisfying $I_1 = \sqrt{I_1}$ and $I_2 = \sqrt{I_2}$, then $I_1 = \mathfrak{p}$ or $I_2 = \mathfrak{p}$.*

We leave it to the readers to figure out the geometric meaning of this lemma (in the case that $\mathcal{A}$ is an analytic $\mathbb{C}$-algebra).

Proof. By replacing $\mathcal{A}$ by $\mathcal{A}/\mathfrak{p}$, we may assume $\mathfrak{p} = \{0\}$. Clearly (b)$\Rightarrow$(c).

(a)$\Rightarrow$(b): Assume $\{0\}$ is prime. Then clearly $\{0\} = \sqrt{\{0\}}$. Suppose $\{0\} = I_1 \cap I_2$ and $I_1, I_2 \neq \{0\}$. Then we may choose non-zero $f_i \in I_i$. And $f_1 f_2 \in I_1 \cdot I_2 \subset I_1 \cap I_2 = \{0\}$. So $f_1 f_2 = 0$, contradicting that $\{0\}$ is prime. So (b) follows.

(c)$\Rightarrow$(a). Assume (c). Suppose that there are non-zero $f, g \in \mathcal{A}$ such that $fg \in \{0\}$, i.e. $fg = 0$. Then as $\mathcal{A}$ is reduced, $\{0\} = \sqrt{\{0\}} = \sqrt{f\mathcal{A} \cdot g\mathcal{A}} = \sqrt{f\mathcal{A}} \cap \sqrt{g\mathcal{A}}$. This contradicts (c). □

Theorem 3.1.3. *If $\mathcal{A}$ is Noetherian and reduced, then there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ of $\mathcal{A}$ such that*

$$\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N \tag{3.1.2}$$

and that for each $1 \leqslant i \leqslant N$,

$$\{0\} \neq \bigcap_{j \neq i} \mathfrak{p}_j. \tag{3.1.3}$$

*Moreover the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ satisfying (3.1.2) and (3.1.3) are unique. We call this unique decomposition the **prime decomposition**.*

The geometric meaning of (3.1.2) is that an element $f \in \mathcal{A}$ is zero iff f restricts to zero on $\mathcal{A}/\mathfrak{p}_i$ (i.e. "f vanishes on the zero set $N(\mathfrak{p}_i)$") for all i.

Note that if $\mathfrak{p} = \sqrt{\mathfrak{p}}$ is an ideal of a Noetherian ring $\mathcal{A}$, then Thm. 3.1.3 applied to $\mathcal{A}/\mathfrak{p}$ says that there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ of $\mathcal{A}$ such that

$$\mathfrak{p} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$$

and that for each $1 \leqslant i \leqslant N$,

$$\mathfrak{p} \neq \bigcap_{j \neq i} \mathfrak{p}_j.$$

Proof of the existence. We first note that if we can find prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_N$ satisfying (3.1.2), then by discarding some members of these ideals so that the intersection of the remaining ones is still $\{0\}$ until we cannot do this anymore, (3.1.3) is automatically satisfied. So we only need to find prime ideals satisfying (3.1.2).

Let $\mathfrak{A}$ be the set of all ideals $\mathfrak{a}$ not equal to $\mathcal{A}$ such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and that $\mathcal{A}/\mathfrak{a}$ has no prime decomposition (equivalently, $\mathfrak{a}$ is not a finite intersection of prime ideals). Note that if $\mathfrak{a} \in \mathfrak{A}$, then $\mathfrak{a} = \sqrt{\mathfrak{a}}$ and $\mathfrak{a}$ is not prime. So by Lemma 3.1.2, $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$ where the ideals $\mathfrak{b}, \mathfrak{c}$ are not $\mathfrak{a}$ and are the radicals of themselves. One of $\mathfrak{b}, \mathfrak{c}$ is not a finite intersection of prime ideals, otherwise $\mathfrak{a}$ is a finite intersection of prime ideals. So one of $\mathfrak{b}, \mathfrak{c}$ is in $\mathfrak{A}$.

The above argument shows that if $\mathfrak{a}_1 = \{0\}$ belongs to $\mathfrak{A}$, then we can construct a strictly increasing infinite chain of elements of $\mathfrak{A}$: $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \mathfrak{a}_3 \subsetneq \cdots$, contradicting that $\mathcal{A}$ is Noetherian. So $\{0\} \notin \mathfrak{A}$. □

Remark 3.1.4. In Thm. 3.1.3, (3.1.2) and (3.1.3) imply that

$$\bigcap_{j\neq i}\mathfrak{p}_j\Big\backslash\mathfrak{p}_i\neq\varnothing.$$

This means that we can find $f\in\mathcal{A}$ which is non-zero when restricted to $\mathcal{A}/\mathfrak{p}_i$ (i.e. "non-zero on $N(\mathfrak{p}_i)$") and zero in the other $\mathcal{A}/\mathfrak{p}_j$. Thus, by taking sums, we see that there always exists $f\in\mathcal{A}$ which is non-zero precisely when restricted to the given ones of $\mathcal{A}/\mathfrak{p}_1,\dots,\mathcal{A}/\mathfrak{p}_N$.

We remark that when $\mathcal{A}$ is not necessarily reduced, there is a generalization called **primary decomposition**, cf. [AM]. We will not use this notion in out notes.

To prove the uniqueness part of Thm. 3.1.3 we first need:

Lemma 3.1.5. *In Thm. 3.1.3, for each $f\in\mathcal{A}$, the annihilator $\mathrm{Ann}_{\mathcal{A}}(f)$ equals*

$$\mathrm{Ann}_{\mathcal{A}}(f)=\bigcap_{\substack{1\leqslant i\leqslant N\\ f\notin\mathfrak{p}_i}}\mathfrak{p}_i \tag{3.1.4}$$

Recall that $\mathrm{Ann}_{\mathcal{A}}(f)=\mathrm{Ann}_{\mathcal{A}}(f\mathcal{A})$ is the ideal of all $a\in\mathcal{A}$ satisfying $af=0$ (Def. 2.3.1). Then (3.1.4) says that $af=0$ iff a "vanishes on all $N(\mathfrak{p}_i)$ where f is non-zero on $N(\mathfrak{p}_i)$".

Proof. Suppose $a\in\mathcal{A}$ and $af=0$. Then af restricts to 0 on the integral domain $\mathcal{A}/\mathfrak{p}_i$. If $f\notin\mathfrak{p}_i$ then f is nonzero on $\mathcal{A}/\mathfrak{p}_i$. So a is 0 on $\mathfrak{p}_i$. Hence $a\in\mathfrak{p}_i$. Conversely, if $a\in\mathfrak{p}_i$ for all i such that $f\notin\mathfrak{p}_i$, then af belongs to $\mathfrak{p}_i$ for all $1\leqslant i\leqslant N$. So $af\in\cap_i\mathfrak{p}_i=\{0\}$. □

Note that f is a non zero-divisor iff $\mathrm{Ann}_{\mathcal{A}}(f)=\{0\}$. Therefore:

Corollary 3.1.6. *In Thm. 3.1.3, $f\in\mathcal{A}$ is a non zero-divisor if and only if $f\notin\mathfrak{p}_i$ for all $1\leqslant i\leqslant N$.*

Now the uniqueness of prime decomposition follows immediately from the following fact:

Proposition 3.1.7. *In Thm. 3.1.3, $\mathfrak{p}_1,\dots,\mathfrak{p}_N$ are precisely the* ***associated primes*** *of $\mathcal{A}$, i.e. prime ideals of the form $\mathrm{Ann}_{\mathcal{A}}(f)$ for some $f\in\mathcal{A}$.*

Proof. We first note that an intersection of more than one members of $\mathfrak{p}_1,\dots,\mathfrak{p}_N$ is not prime. This together with Lemma 3.1.5 would imply that $\mathrm{Ann}_{\mathcal{A}}(f)$ is prime only if $\mathrm{Ann}_{\mathcal{A}}(f)=\mathfrak{p}_i$ for some i, and hence that the associated primes are among $\mathfrak{p}_1,\dots,\mathfrak{p}_N$. To prove the claim, consider for instance $\mathfrak{p}=\mathfrak{p}_1\cap\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_k$ where $k>1$. Suppose $\mathfrak{p}$ is prime. Then by Lemma 3.1.2, either $\mathfrak{p}_1$ or $\mathfrak{p}_2\cap\cdots\cap\mathfrak{p}_k$ equals $\mathfrak{p}$, contradicting (3.1.3). So $\mathfrak{p}$ cannot be prime.

For each i, by Rem. 3.1.4 we can choose $f\in\mathcal{A}$ non-zero on $\mathcal{A}/\mathfrak{p}_i$ but zero on $\mathcal{A}/\mathfrak{p}_j$ whenever $j\neq i$. Then $\mathfrak{p}_i=\mathrm{Ann}_{\mathcal{A}}(f)$ by Lemma 3.1.5, which shows that $\mathfrak{p}_i$ must be an associated prime. □

3.2 Reduction $\mathrm{red}(X)$ and coherence of $\sqrt{\mathcal{I}}$

In this section we study the reduction of complex spaces. The main results Thm. 3.2.1 and equivalently Thm. 3.2.2 are originally due to Oka and H. Cartan. Some key ingredients of the proof are prime decomposition, Nullstellensatz, and the ranks of Jacobian matrices (which are a guise for embedding dimensions to be studied later). Our approach follows [GPR].

3.2.1 Main results and consequences

Theorem 3.2.1. *Let X be a complex space reduced at a point x. There X is reduced on a neighborhood U of x.*

This theorem is equivalent to:

Theorem 3.2.2. *Let X be a complex space. Then for each coherent ideal $\mathcal{I}$ of $\mathscr{O}_X$, its radical $\sqrt{\mathcal{I}}$ is coherent.*

Remark 3.2.3. Note that Thm. 3.2.2 is equivalent to the seemingly special case that for each complex space X, $\sqrt{0_X}$ is coherent. Indeed, if this special case is true, let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then $\sqrt{0_Y}$ is (or more precisely, $\iota_{Y,X,*}\sqrt{0_Y}$ is)

$$\sqrt{0_Y} = \sqrt{\mathscr{O}_X/\mathcal{I}} \simeq \sqrt{\mathcal{I}}/\mathcal{I}. \tag{3.2.1}$$

So $\sqrt{\mathcal{I}}/\mathcal{I}$ is coherent, and hence $\sqrt{\mathcal{I}}$ is coherent. Therefore Thm. 3.2.2 holds.

Proof that Thm. 3.2.1 and 3.2.2 are equivalent. Assume Thm. 3.2.2. Then $\sqrt{\mathcal{I}}$ is coherent and its stalk at x is 0. So its stalk is zero on a neighborhood U of x. Then X is reduced everywhere on U.

Assume Thm. 3.2.1. Choose any complex space X and coherent ideal $\mathcal{I}$. Choose $x \in X$. Since $\mathscr{O}_{X,x}$ is Noetherian, $\sqrt{\mathcal{I}_x}$ is generated by finitely many elements $f_1, f_2, \dots$. By shrinking X to a neighborhood of x, we assume $f_1, f_2, \dots \in \sqrt{\mathcal{I}}(X)$. Let $\mathcal{J}$ be the ideal generated by $f_1, f_2, \dots$. Then $\mathcal{J} \subset \sqrt{\mathcal{I}}$ and $\mathcal{J}_x = \sqrt{\mathcal{I}_x}$. This implies that $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{J})$ is reduced at x (since $\sqrt{0_{Y,x}} = \sqrt{\mathcal{J}_x}/\mathcal{J}_x$).

$\mathcal{J}_x = \sqrt{\mathcal{I}_x}$ also implies $\mathcal{I}_x \subset \mathcal{J}_x$. So by shrinking X we have $\mathcal{I} \subset \mathcal{J}$. We conclude that

$$\mathcal{I} \subset \mathcal{J} \subset \sqrt{\mathcal{I}} \subset \sqrt{\mathcal{J}}.$$

By Thm. 3.2.1, we may shrink X so that Y is reduced everywhere on X. This means $\mathcal{J} = \sqrt{\mathcal{J}}$, which proves that $\sqrt{\mathcal{I}}$ equals $\mathcal{J}$ and is therefore coherent. $\square$

Corollary 3.2.4. *Let X be a complex space. Then for each analytic subset A of X, the **ideal associated to** A defined by*

$$\mathscr{I}_A = \{f \in \mathscr{O}_X : f(x) = 0 \quad \forall x \in A\}$$

is coherent.

Proof. If locally $A = N(\mathcal{I})$ for some coherent ideal $\mathcal{I}$ then

$$\mathscr{I}_A = \sqrt{\mathcal{I}}. \tag{3.2.2}$$

□

Remark 3.2.5. For a reduced complex space X, we clearly have a bijection

$$\begin{gathered}\{\text{Analytic subsets of } X\} \xleftrightarrow{\simeq} \{\text{Coherent ideals } \mathcal{I} \subset \mathscr{O}_X \text{ satisfying } \mathcal{I} = \sqrt{\mathcal{I}}\} \\ A \mapsto \mathscr{I}_A \qquad N(\mathcal{I}) \leftarrow\!\shortmid \mathcal{I}\end{gathered} \tag{3.2.3}$$

If A, B are analytic subsets of X then clearly

$$A \subset B \iff \mathscr{I}_A \supset \mathscr{I}_B$$

$A \cap B$ and $A \cup B$ are both analytic subsets of X, and we indeed have

$$\mathscr{I}_{A\cap B} = \sqrt{\mathscr{I}_A + \mathscr{I}_B} \qquad \mathscr{I}_{A\cup B} = \mathscr{I}_A \cap \mathscr{I}_B = \sqrt{\mathscr{I}_A \cdot \mathscr{I}_B} \tag{3.2.4}$$

Proof. It is clear that the coherent ideals (cf. Cor. 2.1.6 for the coherence) $\mathscr{I}_A + \mathscr{I}_B$ has zero set $A \cap B$ and $\mathscr{I}_A \cdot \mathscr{I}_B$ has zero set $A \cup B$. And $\sqrt{\mathscr{I}_A \cdot \mathscr{I}_B} = \mathscr{I}_A \cap \mathscr{I}_B$ by Rem. 3.1.1. □

Remark 3.2.6. We often identify an analytic subset A with the corresponding reduced complex subspace $\mathrm{Specan}(\mathscr{O}_X/\mathscr{I}_A)$. In that case "analytic subsets" and "reduced complex subspaces" are synonymous. But there is one exception. The intersection of analytic subsets $A \cap B$ are usually not the intersection of two (reduced) complex spaces (as defined in Exp. 1.12.4): In the former case $A \cap B$ is determined by the ideal $\mathscr{I}_{A\cap B} = \sqrt{\mathscr{I}_A + \mathscr{I}_B}$ and the latter case $\mathscr{I}_A + \mathscr{I}_B$. *So we will make distinctions between analytic subsets and reduced complex subspaces when taking intersections.*

There is no such a problem when taking unions: We haven't defined unions for closed complex subspaces, since both $\mathcal{I}_1 \cap \mathcal{I}_2$ and $\mathcal{I}_1 \cdot \mathcal{I}_2$ are reasonable ideals for defining the union. Certainly, for analytic subspaces, $\mathscr{I}_{A\cap B}$ is the correct ideal defining the union. □

Corollary 3.2.7. *Let X be a complex space. Then the set of all non-reduced points of X is an analytic subset of X.*

Proof. $x \in X$ is not reduced iff $x \in \mathrm{Supp}(\sqrt{0_X})$. □

Definition 3.2.8. Let X be a complex space. Then the reduced space

$$\mathrm{red}(X) = \mathrm{Specan}(\mathscr{O}_X/\sqrt{0_X})$$

is called the **reduction** of X.

3.2.2 Proof of Thm. 3.2.1

Definition 3.2.9. We say that a complex space X is **irreducible at** x if $\mathscr{O}_{X,x}$ is an integral domain. (Note that if X is irreducible at x then X is reduced at x.) We say that X is **locally irreducible** if X is irreducible at every point of X. If X is not irreducible at x, we say that X is **reducible at** x. (Note that "reducible"≠"reduced"!)

Lemma 3.2.10. *Suppose that Thm. 3.2.1 holds whenever X is irreducible at x. Then Thm. 3.2.1 holds in general.*

Proof. Assume $\mathscr{O}_{X,x}$ is reduced. Apply prime decomposition (Thm. 3.1.3) to $A = \mathscr{O}_{X,x}$ to get $\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$. By shrinking X to a neighborhood of x we assume each $\mathfrak{p}_i$ is the stalk $\mathscr{P}_{i,x}$ of a coherent ideal $\mathscr{P}_i$ of $\mathscr{O}_X$. Let $Y_i = \mathrm{Specan}(\mathscr{O}_X/\mathscr{P}_i)$. Then Y_i is irreducible at x. Since $\bigcap_{i=1}^N \mathscr{P}_i$ is $\mathscr{O}_X$-coherent (Cor. 2.1.6), we may shrink X so that $\bigcap_i \mathscr{P}_{i,y} = \{0\}$ for all $y \in X$.

By assumption, we can shrink X further so that each Y_i is reduced everywhere. This means that for each $y \in X$ we have $\mathscr{P}_{i,y} = \sqrt{\mathscr{P}_{i,y}}$. Therefore by Rem. 3.1.1, the zero ideal of $\mathscr{O}_{X,y}$ is its own radical. So $\mathscr{O}_{X,y}$ is reduced. □

Lemma 3.2.11. *Let X be a model space irreducible at $0 \in X$. Then after shrinking X to a neighborhood of 0, there exists $\Delta \in \mathscr{O}(X)$ whose germ is non-zero at 0 such that X is smooth outside $N(\Delta)$.*

Proof of Thm. 3.2.1. By Lemma 3.2.10, it suffices to assume that X is a complex model space irreducible (and hence reduced) at 0. Assume that the statement in Lemma 3.2.11 holds. Since Δ is non-zero in the integral domain $\mathscr{O}_{X,0}$, Δ is a non zero-divisor of $\mathscr{O}_{X,0}$. Therefore, by Prop. 2.3.12, we may shrink X to a neighborhood of 0 so that Δ is a non zero-divisor of $\mathscr{O}_{X,x}$ for all $x \in X$.

Choose any open $V \subset X$ and $f \in \sqrt{0_X}(V)$. Since $X \backslash N(\Delta)$ is a complex manifold, $\sqrt{0_{X\backslash N(\Delta)}} = 0$. So the support of f, or more precisely $\mathrm{Supp}(f\mathscr{O}_V)$, is inside $N(\Delta)$. So Δ vanishes on $\mathrm{Supp}(f\mathscr{O}_V)$. Therefore, by Nullstellensatz (Rem. 2.10.4-3), for each $x \in V$ there is $n \in \mathbb{N}$ such that $f\Delta^n = 0$ in $\mathscr{O}_{X,x}$. This proves $f = 0$ in $\mathscr{O}_{X,x}$ because Δ is a non zero-divisor. Therefore $\sqrt{0_X} = 0$. □

The proof of Lemma 3.2.11 is postponed to Sec. 3.4. We first give a preliminary lemma which will aid in the proof of Lemma 3.2.11.

Lemma 3.2.12. *Let $(w_1, \dots, w_m, z_1, \dots, z_n)$ be the standard coordinates of $\mathbb{C}^{m+n}$. Let I be an ideal of $\mathscr{A} = \mathscr{O}_{\mathbb{C}^{m+n},0}$ such that $I \neq \mathscr{A}$. Then the following are equivalent.*

(1) $I \subset w_1\mathscr{A} + \cdots + w_m\mathscr{A}$.

(2) $\partial_{z_j} I \subset I$ for all $1 \leqslant j \leqslant n$.

Proof. By taking power series expansions one sees immediately (1)⇒(2). Now let us assume (2) and prove (1). Note that $I \neq \mathscr{A}$ means that all elements of I vanish at 0. Now (2) implies that all higher partial derivatives over $z_1, \dots, z_n$ of $f \in I$ are in I, and hence vanish at 0. Therefore the restriction of f to $0_{\mathbb{C}^m} \times \mathbb{C}^n$ must be constantly zero, since its power series expansion at 0 is zero. But the ideal of elements of $\mathscr{A}$ vanishing on $0 \times \mathbb{C}^n$ is precisely $w_1\mathscr{A} + \cdots + w_m\mathscr{A}$. This proves (1). □

3.3 Local decomposition of reduced complex spaces

3.3.1 Local decomposition

Fix a complex space X. Suppose that X is reduced and $x \in X$. Then similar to Rem. 3.2.5, we have a bijection $A \mapsto I_A, N(I) \leftarrow I$:

(1) Germs of analytic subsets of X at x.

(2) Ideals $I \subset \mathscr{O}_{X,x}$ satisfying $I = \sqrt{I}$.

Indeed, (1) are precisely the germs of closed reduced complex subspaces of X passing through x, and (2) are precisely the germs of coherent ideals $\mathcal{I} \subset \mathscr{O}_X$ at x satisfying $\mathcal{I} = \sqrt{\mathcal{I}}$ (cf. Thm. 2.2.2).

Remark 3.3.1. To be more explicit, if a germ A in (1) is represented by an analytic subset A closed in a neighborhood U of x, then the stalk at x of $\mathscr{I}_A = \{f \in \mathscr{O}_U : f(y) = 0, \forall y \in A\}$ gives the corresponding ideal I_A in (2). Conversely, given an ideal I in (2) which is finitely generated because $\mathscr{O}_{X,x}$ is Noetherian, let $f_1, \dots, f_k \in I$ generated I, and choose a neighborhood $U \subset X$ of x such that $f_1, \dots, f_k \in \mathscr{O}_X(U)$. Then the germ at x of $N(f_1\mathscr{O}_U + \cdots + f_k\mathscr{O}_U)$ gives the germ $N(I)$ in (1).

Remark 3.3.2. We list some easy but useful facts about this correspondence. Let (X, x) be a germ of reduced complex space.

- $I_{A\cup B} = I_A \cap I_B = \sqrt{I_A \cdot I_B}$.
- By Lemma 3.1.2-(c), $\mathscr{O}_{X,x}$ is an integral domain if and only if (X, x) is an irreducible germ, namely if $(X, x) = (A, x) \cup (B, x)$ where $(A, x), (B, x)$ are germs of analytic subsets then $(A, x) = (X, x)$ or $(B, x) = (X, x)$.
  - More precisely, $\mathscr{O}_{X,x}$ is an integral domain iff for every neighborhood U of x written as $U = A \cup B$ where A, B are analytic subsets of U, one of A and B contains a neighborhood of $x \in X$.

Theorem 3.3.3. *Let X be a reduced complex space and $x \in X$. Then after shrinking X to a neighborhood of x, we have*

$$X = X_1 \cup \cdots \cup X_N \tag{3.3.1}$$

where each X_i is an analytic subset of X which is irreducible at x, and for each $1 \leqslant i \leqslant N$,

$$\bigcup_{j \neq i} X_j \quad \text{contains no neighborhoods of } x \in X. \tag{3.3.2}$$

Such decomposition of X is unique up to shrinking X to smaller neighborhoods of x. We call it the ***local decomposition of X at x****. Moreover, we have*

$$\{0\} = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x} \tag{3.3.3}$$

which gives the prime decomposition of $\mathscr{O}_{X,x}$.

Note that (assuming (3.3.1) then) (3.3.2) is equivalent to saying that

$$X \Big\backslash \bigcup_{j \neq i} X_j = X_i \Big\backslash \bigcup_{j \neq i} X_j \quad \text{intersects every neighborhood of } x \in X. \tag{3.3.4}$$

Proof. Uniqueness: Every local decomposition (3.3.1) clearly gives a prime decomposition (3.3.3), where the condition $\bigcap_{j \neq i} \mathscr{I}_{X_j,x} \neq 0$ corresponds precisely to (3.3.2). The uniqueness of prime decomposition implies the uniqueness of local decomposition.

Existence: Let $\{0\} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_N$ be the prime decomposition of $\mathscr{O}_{X,x}$. By shrinking X, for each i we may find a coherent ideal $\mathscr{P}_i$ whose stalk at x is $\mathfrak{p}_i$. Since $\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N$ is coherent (Cor. 2.1.6), we can shrink X further so that $\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N = 0_X$. So by Rem. 3.1.1,

$$X = N(0_X) = N(\mathscr{P}_1 \cap \cdots \cap \mathscr{P}_N) = N(\mathscr{P}_1 \cdots \mathscr{P}_N) = X_1 \cup \cdots \cup X_N.$$

This gives a local decomposition. □

Property (3.3.2) can be upgraded to the following form:

Theorem 3.3.4. *Let $X = X_1 \cup \cdots \cup X_N$ be a local decomposition of a reduced complex space X at x. Then after shrinking X to a neighborhood of x, for each $i \neq j$,*

$$X_i \cap X_j \ \text{ is nowhere dense in } X_i \tag{3.3.5}$$

In that case, X is reducible at each point of $X_i \cap X_j$ where $i \neq j$.

Note that (3.3.5) implies, for instance, that if $1 \leqslant k < N$ then $(X_1 \cup \cdots \cup X_k) \cap (X_{k+1} \cup \cdots \cup X_N)$ is nowhere dense in every X_i. Hence it is nowhere dense in any union of subclass of $X_1, \ldots, X_N$.

We will prove Thm. 3.3.4 in Subsec. 3.3.2. Note that (cf. Rem. 3.2.6) here $X_i \cap X_j$ means set-theoretic intersection (i.e. intersection of analytic subsets), but not intersection of complex spaces. But this is not really a big issue here; we are just reminding the readers of the conventions we set before.

3.3.2 Non zero-divisors and nowhere dense analytic subsets

As an application of local decomposition, we give a useful method for showing that an analytic subset is nowhere dense:

Proposition 3.3.5. *Let X be a reduced complex space and $x \in X$. Choose $f \in \mathscr{O}(X)$. Then the following are equivalent.*

(1) f is a non zero-divisor of $\mathscr{O}_{X,x}$.

(2) There is a neighborhood $U \subset X$ of x such that $N(f) \cap U$ is nowhere dense in U.

Proof. Assume (1) is true. Then by Prop. 2.3.12, after shrinking X to a neighborhood of x, f is a non-zero divisor of $\mathscr{O}_{X,x}$ for all $x \in X$. If $N(f)$ contains an open subset V of X, then f takes zero value everywhere on V. So $f|_V = 0$ because X is reduced, contradicting the fact that f is a non zero-divisor of $\mathscr{O}_{V,x}$ when $x \in V$. So (2) must be true.

Assume that (1) is not true. By shrinking X, we may find a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. By Cor. 3.1.6, the germ of f at x belongs to $\mathscr{I}_{X_i,x}$ for some i. Shrink X so that $f \in \mathscr{I}_{X_i}(X)$. Then f vanishes on X_i. Thus, after shrinking X to any neighborhood U of x, $N(f) \supset X_i$ contains an open subset $X \backslash \bigcup_{j \neq i} X_j$ of X which is non-empty by (2.6.2). So (2) is not true. □

Remark 3.3.6. Prop. 3.3.5 can be used in the following way.

- Suppose A is an analytic subset of a reduced space X. To show that A is nowhere dense, it suffices to prove that for each $x \in A$ there is a non zero-divisor $f \in \mathscr{O}_{X,x}$ vanishing on $A \cap U$ for a neighborhood U of x. Then after shrinking U, $N(f) \cap U$ is nowhere dense. So its subset $A \cap U$ is nowhere dense.

Actually, if A is expected to be nowhere dense, then one must be able to find such f due to the following generalization of Prop. 3.3.5:

Proposition 3.3.7. *Let A be an analytic subset of a reduced complex space X. The following are equivalent.*

(1) A is nowhere dense in X.

(2) For each $x \in X$, $\mathscr{I}_{A,x}$ contains a non zero-divisor of $\mathscr{O}_{X,x}$.

Proof. (2)⇒(1) is already explained in Rem. 3.3.6. Let us prove (1)⇒(2).

Assume that A is nowhere dense. By shrinking X to a neighborhood of x we may find a local decomposition $X = X_1 \cup \cdots \cup X_N$ at x. For each i, we have $(X_i, x) \not\subset (A, x)$, namely, we cannot find any neighborhood $U \subset X$ of x such that $X_i \cap U \subset A \cap U$: Otherwise, by (3.3.4), X_i contains an open subset (namely $X_i \backslash \bigcap_j X_j$) which intersects U, contradicting the fact that A is nowhere dense.

Therefore, we have $\mathscr{I}_{A,x} \not\subset \mathscr{I}_{X_i,x}$ for all i. The existence of a non zero-divisor follows from the next lemma. $\square$

Lemma 3.3.8. *Let $X = X^1 \cup \cdots \cup X^N$ be a decomposition of reduced complex space X into analytic subsets. Let $x \in X$, and assume that each X^j has a local decomposition at x:*

$$X^j = X_1^j \cup X_2^j \cup \cdots$$

Suppose that we have a linear subspace $\mathscr{W} \subset \mathscr{O}_{X,x}$ such that

$$\mathscr{W} \not\subset \mathscr{I}_{X_i^j,x} \qquad (\forall i,j)$$

Then there is an element of $\mathscr{W}$ which is a non zero-divisor of $\mathscr{O}_{X^1,x}, \dots, \mathscr{O}_{X^N,x}$.

Proof. Let $\mathscr{W} = \mathrm{Span}\mathfrak{A}$. Then $\mathscr{W} \not\subset \mathscr{I}_{X_i^j,x}$ for all i,j. Then $\mathscr{W} \cap \mathscr{I}_{X_i^j,x}$ is not the full space $\mathscr{W}$. Therefore the *finite* union $\bigcup_{i,j}(\mathscr{W} \cap \mathscr{I}_{X_i^j,x}) = \mathscr{W} \cap \big(\bigcup_{i,j} \mathscr{I}_{X_i^j,x}\big)$ is not $\mathscr{W}$. So there is an element $f \in \mathscr{W}$ which is not in $\bigcup_{i,j} \mathscr{I}_{X_i^j,x}$. By Cor. 3.1.6, f is a non zero-divisor of each $\mathscr{O}_{X^j,x}$. $\square$

Note that in the above proof we have used the fact that $\mathbb{C}$ is an infinite field. Over a finite field, a finite union of proper linear subspaces might be the full linear space.

We are now ready to prove Thm. 3.3.4.

Proof of Thm. 3.3.4. We set $A = X_i$, $B = X_j$ for simplicity. In view of Prop. 3.3.7, proving (3.3.5) means proving the following claim: After shrinking X to a neighborhood of x, for each $y \in A \cap B$, $\mathscr{I}_{A\cap B,y}$ contains a non zero-divisor of $\mathscr{O}_{A,y}$.

Note that $\mathscr{I}_{A\cap B} \supset \mathscr{I}_A + \mathscr{I}_B$. Since $\{0\} = \mathscr{I}_{X_1,x} \cap \cdots \cap \mathscr{I}_{X_N,x}$ is a prime decomposition, we have $\mathscr{I}_{B,x} \not\subset \mathscr{I}_{A,x}$. Therefore $(\mathscr{I}_{A,x} + \mathscr{I}_{B,x}) \backslash \mathscr{I}_{A,x}$ is non-empty. Choose any element f of this set. Then since $\mathscr{I}_{A,x}$ is prime, f is a non zero-divisor of $\mathscr{I}_{A,x}$. By shrinking X to a neighborhood of x, we have that $f \in (\mathscr{I}_A + \mathscr{I}_B)(U)$ and that (by Prop. 2.3.12) f is a non zero-divisor of $\mathscr{O}_{A,y}$ for all $y \in X$. This proves the claim.

Now assume that (3.3.5) holds for all $i \neq j$. Let us prove the last sentence of Thm. 3.3.4. Let $X_2' = X_2 \cup X_3 \cup \cdots \cup X_N$. Then $X_1 \cap X_2'$ is nowhere dense in X_1 and in X_2'. Therefore we have decomposition $X = X_1 \cup X_2'$, and for each $y \in X_1 \cap X_2 \subset X_1 \cap X_2'$, X_1 and X_2' contain no neighborhoods of y in X. So by Rem. 3.3.2, the germ (X,y) is not reducible when $y \in X_1 \cap X_2$, and similarly when $y \in X_i \cap X_j$ for all $i \neq j$. $\square$

3.4 Ranks of Jacobian matrices and singular loci

The goal of this section is to prove Lemma 3.2.11, a crucial ingredient in the proof that any complex space reduced at a point is reduced near that point (Thm. 3.2.1). Indeed, even if we assume that a complex space is reduced everywhere, this lemma still tells us something interesting: it says that if X is irreducible at 0 then, after shrinking X to a neighborhood of 0, X is smooth outside a nowhere dense analytic subset (due to Prop. 3.3.5).

The proof of Lemma 3.2.11 relies on Jacobian matrices, which are very useful for determining the singular locus of a complex space.

Definition 3.4.1. If X is a complex space, we define the **singular locus** of X to be the closed (cf. Cor. 1.6.5) subset

$$\mathrm{Sing}(X) = \{x \in X : X \text{ is not smooth at } x\}.$$

3.4.1 Jacobian matrices

Assume $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is a closed subspace of an open $U \subset \mathbb{C}^m$, where $\mathcal{I}$ is generated by $f^1, \dots, f^n \in \mathscr{O}(U)$. Let $(z_1, \dots, z_m)$ be the standard coordinates of $\mathbb{C}^m$, and consider the Jacobian matrix function

$$\partial_{z_\bullet}(f^\bullet) = \Big(\partial_{z_i} f^j\Big)_{1\leqslant i\leqslant m}^{1\leqslant j\leqslant n}$$

which is an $m \times n$ matrix valued function on U whose $i \times j$ entry is $\partial_{z_i} f^j$.

For each $k \in \mathbb{N}$, let

$$Z_k = \{x \in U : \mathrm{rank}\,\partial_{z_\bullet}(f^\bullet)(x) \leqslant k\}. \tag{3.4.1}$$

Then clearly

$$Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = Z_{m+1} = Z_{m+2} = \cdots = U. \tag{3.4.2}$$

Each Z_k is an analytic subset of U, because

$$Z_k = \bigcap_{\substack{1\leqslant i_1<\cdots<i_{k+1}\leqslant m\\ 1\leqslant j_1<\cdots<j_{k+1}\leqslant n}} N\left(\det\ \partial_{z_\bullet}(f^\bullet)\Big|_{i=i_1,\dots,i_{k+1}}^{j=j_1,\dots,j_{k+1}}\right) \tag{3.4.3}$$

3.4.2 Proof of Lemma 3.2.11

Proof-Step 1. Assume the setting of Subsec. 3.4.1, and assume $0 \in X$. In this first step, we construct Δ. Fix $r \in \mathbb{N}$ to be

$$r = \text{“the smallest number such that } (Z_r \cap X, 0) = (X, 0)\text{”}$$

where $(Z_r \cap X, 0), (X, 0)$ are germs of sets at 0. Namely, r is the smallest number such that $Z_r \cap X$ contains a neighborhood of $0 \in X$. Thus, we may shrink[1] U so that

$$X \subset Z_r$$

at the level of sets. More precisely, $N(\mathcal{I}) \subset Z_r$.

Since $Z_{r-1} \cap X$ containes no neighborhoods of $0 \in X$, by (3.4.3) we can choose an $r \times r$-submatrix, say the first r rows and the first r columns:

$$\partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} = \left(\partial_{z_i} f^j\right)_{1 \leqslant i \leqslant r}^{1 \leqslant j \leqslant r},$$

such that the zero set of its determinant

$$\varDelta = \det\ \partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} \qquad \in \mathscr{O}(U)$$

intersected with X contains no neighborhoods of $0 \in X$. (Note that $Z_{r-1} \subset N(\varDelta)$.) This implies that $\varDelta$ is non-zero in $\mathscr{O}_{X,0}$. Our goal is to show that $X \backslash N(\varDelta)$ is smooth. $\square$

***Proof-Step* 2.** Set

$$w_1 = f^1, \ldots, w_r = f^r, \qquad w_{r+1} = z_{r+1}, \ldots, w_m = z_m.$$

Then by inverse function theorem, each point $x \in U \backslash N(\varDelta)$ has a neighborhood on which $w_1, \ldots, w_m$ are a set of coordinates. Recall that $\mathcal{I}_x$ is generated by $w_1, \ldots, w_r$ and $f^{r+1}, \ldots, f^n$. If we can show for each $x \in X \backslash N(\varDelta)$ that $\mathcal{I}_x$ is generated by $w_1, \ldots, w_r$, then X is smooth at x, since X is near x the $(m-r)$-dimensional submanifold defined by $w_1 = \cdots = w_r = 0$. Thus $\mathrm{Sing}(X) \subset N(\varDelta)$.

- Claim: After possibly shrinking X to a neighborhood of 0, for each $x \in X \backslash N(\varDelta)$ we have

$$\partial_{w_i} f^j \in \mathcal{I}_x \qquad (\forall i, j > r)$$

If this is proved, then for each $i > r$, $\partial_{w_i} f^j$ belongs to $\mathcal{I}_x$ for all j since it is zero when $j \leqslant r$. Then $\partial_{w_i} \mathcal{I}_x \subset \mathcal{I}_x$. Thus by Lemma 3.2.12, $\mathcal{I}_x$ is generated by $w_1, \ldots, w_r$, finishing the proof. (We warn the reader that ∂_{w_i} is not equal to ∂_{z_i} even if $i > r$, and is not defined on $N(\varDelta)$.)

[1]This is the only place we shrink U in Step 1 and 2 of the proof.

Let us take a closer look at the relationship between the Jacobians of $(f^\bullet)$ over $z_\bullet$ and over $w_\bullet$. On $U\backslash N(\Delta)$ we have

$$\partial_{z_\bullet}(f^\bullet) = \left[\begin{array}{c|c} \partial_{z_\bullet}(f^\bullet)\Big|_{\leqslant r}^{\leqslant r} & 0 \\ \hline * & I_{(m-r)\times(m-r)} \end{array}\right] \cdot \partial_{w_\bullet}(f^\bullet) \tag{3.4.4}$$

and also

$$\partial_{w_\bullet}(f^\bullet) = \left[\begin{array}{c|c} I_{r\times r} & \clubsuit \\ \hline 0 & \partial_{w_\bullet}(f^\bullet)\Big|_{>r}^{>r} \end{array}\right] \tag{3.4.5}$$

where $* \in \mathscr{O}(U)$ and $\clubsuit \in \mathscr{O}(U\backslash N(\Delta))$. From these two relations we observe:

Ob 1. $\partial_{z_\bullet}(f^\bullet)\big|_{\leqslant r}^{\leqslant r} \cdot \clubsuit$ equals the upper right block of $\partial_{z_\bullet}(f^\bullet)$ which is holomorphic on U. So by inverse matrix formula/Cramer's rule, $\Delta \cdot \clubsuit$ can be extended to an element of $\mathscr{O}(U)$. So the same can be said about $\Delta \cdot \partial_{w_\bullet}(f^\bullet)\big|_{>r}^{>r}$. We conclude

$$\partial_{w_i} f^j = h_i^j/\Delta \qquad \text{for some } h_i^j \in \mathscr{O}(U) \qquad (\forall i, j > r)$$

Ob 2. At each $x \in X\backslash N(\Delta) \subset Z_r\backslash Z_{r-1}$, the rank of $\partial_{w_\bullet}(f^\bullet)$ equals that of $\partial_{z_\bullet}(f^\bullet)$, which is r. Therefore, by (3.4.5), for all $i, j > r$, $\partial_{w_i} f^j$ vanishes on $X\backslash N(\Delta)$, and hence h_i^j vanishes on $X\backslash N(\Delta)$.

□

Observation 2 shows that if we already know that X is reduced, then every holomorphic function vanishing on $X\backslash N(\Delta)$, in particular $\partial_{w_i} f^j$ where $i, j > r$, must be an element of $\mathcal{I}(X\backslash N(\Delta))$. Then the Claim in Step 2 follows and hence $\mathrm{Sing}(X) \subset N(\Delta)$. But since we cannot assume what we want to prove, we need a little more effort to prove the Claim.

In Step 1 and 2, we have not used the fact that X is irreducible at x. This condition enters Step 3 of the proof. Indeed, we only need the weaker condition that X is reduced at x.

Proof-Step 3. Assume that $\mathscr{O}_{X,0}$ is an integral domain, and hence reduced. For each $i, j > r$, the two observations in Step 2 show that the holomorphic function $\Delta \cdot h_i^j$ on U takes zero value at every point of X. So its germ at 0 is a nilpotent element of $\mathscr{O}_{X,0}$ by Nullstellensatz, and hence is zero. We can thus shrink U to a neighborhood of 0 so that $\Delta \cdot h_i^j$ is zero in $\mathscr{O}_X(X)$ for all $i, j > r$. If $x \in X\backslash N(\Delta)$, then $\Delta(x) \neq 0$ and hence Δ is invertible in $\mathscr{O}_{X,x}$. Therefore in $\mathscr{O}_{X,x}$ we have $h_i^j = 0$ and hence $\partial_{w_i} f^j = 0$ if $i, j > r$. This proves the claim in Step 2 that $\partial_{w_i} f^j$ is in $\mathcal{I}_x$. □

We are done with the proof of Lemma. 3.2.11.

3.4.3 Additional comments

Assume the setting of Subsec. 3.4.1, and assume moreover that X is reduced. Assume U is small enough so that $X \subset Z_r$. Then Proof-Step 1&2 show that $\mathrm{Sing}(X) \subset X \cap N(\Delta)$ (see the comments before Step 3), and that $X \backslash N(\Delta)$ is an $m - r$ dimensional complex manifold. Note that in the proof we take Δ to be the determinant of one $r \times r$ submatrix of $\partial_{z_\bullet} f^\bullet$, and we may well take other submatrices. By (3.4.3), Z_{r-1} is the intersection of $N(\Delta)$ where Δ runs through the determinants of all $k \times k$ submatrices of $\partial_{z_\bullet} f^\bullet$. Therefore $\mathrm{Sing}(X) \subset X \cap Z_{r-1}$.

It is natural to ask if we have $\mathrm{Sing}(X) = X \cap Z_{r-1}$. In Sec. 3.5, we will prove Lemma 3.4.2 saying that this is indeed true if $X \cap Z_{r-1}$ is nowhere dense in X. Note that if X is irreducible at 0, then Δ is non-zero in $\mathscr{O}_{X,0}$ and hence is a non zero-divisor. Thus, by Prop. 3.3.5, we can shrink X to a neighborhood of 0 so that $X \cap N(\Delta)$ and hence $X \cap Z_{r-1}$ are nowhere dense in X.

Lemma 3.4.2. *Assume the setting of Subsec. 3.4.1.*

(1) Assume that X is reduced, that $X \subset Z_r$, and that $X \cap Z_{r-1}$ is nowhere dense in X. Then

$$\mathrm{Sing}(X) = X \cap Z_{r-1}$$

and $X \backslash Z_{r-1}$ is an $(m - r)$-dimensional complex manifold.

(2) If the X in Subsec. 3.4.1 is irreducible at $0 \in X$, then we can shrink U to a neighborhood of $0 \in U$ (and replace X by $X \cap U$) so that the assumptions in (1) are satisfied for some $r \in \mathbb{N}$.

The only thing in Lemma 3.4.2 unproved so far is $\mathrm{Sing}(X) \supset X \cap Z_{r-1}$.

3.5 Embedding dimensions and singular loci

The rank of $\partial_{z_\bullet} f^\bullet$ in Subsec. 3.4.1 depends on how X is embedded into an open subset of a number space. Using Jacobi criterion, we can relate this rank to intrinsic numbers of X call embedding dimensions.

Definition 3.5.1. Let X be a complex space and $x \in X$. The **embedding dimension** of X at x, denoted by $\mathrm{emb}_x X$ or $\mathrm{emb}\mathscr{O}_{X,x}$, is the smallest n such that a neighborhood U of x can be closely embedded to an open subset of $\mathbb{C}^n$.

Equivalently (Prop. 1.7.2), $\mathrm{emb}_x X$ is the smallest n such that there is a neighborhood U of x and a holomorphic $f : U \to \mathbb{C}^n$ which is an immersion at x. □

Proposition 3.5.2. *For each complex space X and $x \in X$,*

$$\mathrm{emb}_x X = \mathrm{emb}\mathscr{O}_{X,x} = \dim_{\mathbb{C}} \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2. \tag{3.5.1}$$

Proof. If $\varphi : X \to \mathbb{C}^n$ is an immersion at x, then by Thm. 1.7.8, $n \geqslant \dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. We can choose n to be $\dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ by shrinking X to a neighborhood of x, and choosing $f_1, \ldots, f_n \in \mathscr{O}(X)$ generating $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}$. Then $\varphi = (f_1, \ldots, f_n)$ is an immersion at x due to Thm. 1.7.8. □

As an immediate consequence of Prop. 3.5.2, $\mathbb{C}^n$ has embedding dimension n everywhere. Thus, for complex manifolds, embedding dimensions agree with the usual dimensions.

Proposition 3.5.3 (Jacobi criterion). *Let U be an open subset of $\mathbb{C}^m$, let $\mathcal{I}$ be the ideal of $\mathscr{O}_U$ generated by $f^1, \ldots, f^n \in \mathscr{O}(U)$, and let $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$. Then for each $x \in X$,*

$$\mathrm{emb}_x X + \mathrm{rank}_x\big(\partial_{z_\bullet} f^\bullet\big) = m. \tag{3.5.2}$$

Proof. In the exact sequence of vector spaces

$$0 \to \frac{\mathfrak{m}_{\mathbb{C}^m,x}^2 + \mathcal{I}_x}{\mathfrak{m}_{\mathbb{C}^m,x}^2} \to \frac{\mathfrak{m}_{\mathbb{C}^m,x}}{\mathfrak{m}_{\mathbb{C}^m,x}^2} \to \frac{\mathfrak{m}_{\mathbb{C}^m,x}}{\mathfrak{m}_{\mathbb{C}^m,x}^2 + \mathcal{I}_x} \to 0$$

the middle one has dimension m and the right one has dimension $\dim \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = \mathrm{emb}_x X$ by (1.7.6). The left one is the image of $\mathcal{I}_x$ under the quotient map $d_x : \mathfrak{m}_{\mathbb{C}^m,x} \to \mathfrak{m}_{\mathbb{C}^m,x}/\mathfrak{m}_{\mathbb{C}^m,x}^2$ (the differential at x), whose dimension is easily checked to be $\mathrm{rank}_x\big(\partial_{z_\bullet} f^\bullet\big)$. □

Proof of Lemma 3.4.2. Under the assumptions of (1), we need to show that each $x \in X \cap Z_{r-1}$ is a singular point. If x is smooth, we can find a neighborhood $W \subset X$ of x which is a complex manifold. In particular, the embedding dimensions of W must be constant on W. Thus, by Jacobi criterion, the ranks of $\partial_{z_\bullet} f^\bullet$ are constant on W.

Notice the assumptions in (1) that $X \cap Z_{r-1}$ is nowhere dense in X. So $W \not\subset X \cap Z_{r-1}$. From the definition of $Z_\bullet$, we know that the ranks of $\partial_{z_\bullet} f^\bullet$ on Z_{r-1} (and in particular at $x \in W$) are $\leqslant r-1$, and that the rank on the non-empty set $W \backslash Z_{r-1}$ is r (since $X \subset Z_r$). This is impossible. So x is singular. □

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