HAUSDORFF CONVERGENCE PROPERTY OF PARTIAL OKOUNKOV BODIES

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Contents

1. Introduction

This note is a refinement of [Xia21, Theorem A]. We prove the Hausdorff convergence property in full generality.

This note is motivated by a discussion with Sébastien Boucksom.

2. Hausdorff convergence property

Let X be a connected smooth projective variety of dimension n. Let (L,h) be a Hermitian pseudo-effective line bundle on X with $\int_X (\mathrm{dd^c} h)^n > 0$. Fix $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ a valuation of rank n and rational rank n. Take a smooth Hermitian metric h_0 on L and set $\theta = c_1(L,h_0)$. We can then identify h with $\varphi \in \mathrm{PSH}(X,\theta)$.

For each $k \in \mathbb{Z}_{>0}$, we introduce

$$\Delta^k_\nu(\theta,\varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the closed convex hull.

For later use, we introduce a twisted version as well. If T is a holomorphic line bundle on X, we introduce

$$\Delta^{k,T}_\nu(\theta,\varphi) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T \otimes L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

We also write

$$\Delta^{k,T}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, T \otimes L^k) \right\} \subseteq \mathbb{R}^n$$

and

$$\Delta^{k}_\nu(L) := \mathrm{Conv}\left\{ k^{-1}\nu(f) : f \in H^0(X, L^k) \right\} \subseteq \mathbb{R}^n$$

We write $\mathcal{I}_\infty(\varphi) = \mathcal{I}_\infty(h)$ for the ideal sheaf on X locally consisting of holomorphic functions f such that $|f|_h$ is locally bounded.

We first extend [Xia21, Theorem 3.13] to the twisted case.

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Proposition 2.1. *For any holomorphic line bundle T on X,*

$$\Delta_\nu^{k,T}(L) \to \Delta_\nu(L)$$

as $k \to \infty$.

Here and later on, we endow the space of convex bodies with the Hausdorff metric.

Proof. As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0;
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$H^0(X, L^{k-k_0}) \xrightarrow{\times s_1} H^0(X, T \otimes L^k) \xrightarrow{\times s_0} H^0(X, L^{k+k_0}).$$

It follows that

$$(k-k_0)\Delta_\nu^{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_\nu^{k,T}(L) \subseteq (k+k_0)\Delta_\nu^{k+k_0}(L) - \nu(s_0).$$

By [Xia21, Theorem 3.13], we conclude. □

Lemma 2.2. *Let T be a holomorphic line bundle on X. Assume that φ has analytic singularities and φ has positive mass, then*

$$\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(\theta, \varphi)$$

as $k \to \infty$.

Proof. Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along a nc $\mathbb{Q}$-divisor D. Take $\epsilon \in (0,1) \cap \mathbb{Q}$. In this case, by Ohsawa–Takegoshi theorem, for any $k \in \mathbb{Z}_{>0}$ we have

$$H^0(X, T\otimes L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq H^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, T\otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta,\varphi))_k$, say the sequence defined by the indices $k_1, k_2, \ldots$. We want to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

There exists $t \in \{0, 1, \ldots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$.

Now we have

$$\Delta_\nu^{g_i, T\otimes L^t}(NL - ND) + N\nu(D) \subseteq N\Delta_\nu^{k,T}(\theta,\varphi) \subseteq \Delta_\nu^{g_i, T\otimes L^t}(NL - N(1-\epsilon)D) + N(1-\epsilon)\nu(D).$$

By Proposition 2.1,

$$\Delta_\nu(L-D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L - (1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Let $\epsilon \to 0+$, we find that

$$\Delta_\nu(L-D) + \nu(D) = \Delta'.$$

It follows from Blanschke selection theorem that

$$\Delta_\nu^{k,T}(\theta,\varphi) \to \Delta_\nu(L-D) + \nu(D) = \Delta_\nu(\theta,\varphi)$$

as $k \to \infty$. □

Lemma 2.3. *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \to 0+$, we have*

$$\Delta_\nu((1-\beta)\theta, \varphi) \to \Delta_\nu(\theta, \varphi).$$

Proof. By [Xia21, Proposition 5.15], we have

$$\Delta_\nu((1-\beta)\theta, \varphi) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, if Δ' is a limit of a subsequence of $(\Delta_\nu((1-\beta)\theta, \varphi))_\beta$, then

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

But

$$\operatorname{vol} \Delta' = \lim_{\beta \to 0+} \Delta_\nu((1-\beta)\theta, \varphi) = \lim_{\beta \to 0+} \int_X ((1-\beta)\theta + \mathrm{dd^c} P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n.$$

We claim that

$$\lim_{\beta \to 0+} \int_X ((1-\beta)\theta + \mathrm{dd^c} P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c} P^{\theta}[\varphi]_{\mathcal{I}})^n.$$

Note that this finishes the proof as $\operatorname{vol} \Delta_\nu(\theta, \varphi)$ is exactly equal to the right-hand side.

Next we prove our claim. We make use of the b-divisors introduced in [Xia22b; Xia22a]. By [Xia22a, Theorem 0.6], the claim is equivalent to

$$\lim_{\beta \to 0+} \operatorname{vol} \mathbb{D}((1-\beta)\theta, \varphi) = \operatorname{vol} \mathbb{D}(\theta, \varphi).$$

This is a special case of [Xia22a, Theorem 9.6] ☐

Theorem 2.4. *Let T be a holomorphic line bundle on X. As $k \to \infty$, $\Delta_\nu^{k,T}(\theta, \varphi) \to \Delta_\nu(\theta, \varphi)$.*

Proof. Fix a Kähler form $\omega \geq \theta$ on X.

Step 1. We first handle the case where $\mathrm{dd^c} h$ is a Kähler current, say $\mathrm{dd^c} h \geq \beta_0 \omega$ for some $\beta_0 \in (0,1)$.

Take a decreasing quasi-equisingular approximation φ_j of φ. Up to replacing β_0 by $\beta_0/2$, we may assume that $\theta_{\varphi_j} \geq \beta_0 \omega$ for all $j \geq 1$.

Let Δ' be a limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$. Let us say the indices of the subsequence are $k_1 < k_2 < \cdots$. By Blaschke selection theorem, it suffices to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

As $\varphi \leq \varphi_j$ for each $j \geq 1$, we have

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi_j)$$

by Lemma 2.2. Let $j \to \infty$, we find

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, it suffices to prove that

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu(\theta, \varphi).$$

Take $\beta \in (0, \beta_0) \cap \mathbb{Q}$. Write $\beta = p/q$ with $p, q \in \mathbb{Z}_{>0}$. Observe that for any $j \geq 1$,

$$\theta_{\varphi_j} \geq \beta\omega \geq \beta\theta.$$

Namely, $\varphi_j \in \mathrm{PSH}(X, (1-\beta)\theta)$. Similarly, $\varphi \in \mathrm{PSH}(X, (1-\beta)\theta)$. By Lemma 2.3, it suffices to argue that

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu((1-\beta)\theta, \varphi). \tag{2.1}$$

For this purpose, we are free to replace k_i's by a subsequence, so we may assume that $k_i \equiv a$ modulo q for all $i \geq 1$, where $a \in \{0, 1, \dots, q-1\}$. We write $k_i = g_i q + a$. Observe that for each $i \geq 1$,

$$H^0(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi)) \supseteq H^0(X, T \otimes L^{-q+a} \otimes L^{g_i q+q} \otimes \mathcal{I}((g_i q + q)\varphi)).$$

Up to replacing T by $T \otimes L^{-q+a}$, we may therefore assume that $a = 0$.

By [DX21, Lemma 4.2], we can find $k' \in \mathbb{Z}_{>0}$ such that for all $k \geq k'$, there is $v_{\beta,k} \in \mathrm{PSH}(X, \theta)$ satisfying

(1)
$$P[\varphi]_{\mathcal{I}} \geq (1-\beta)\varphi_k + \beta v_{\beta,k};$$

(2) $v_{\beta,k}$ has positive mass.

Fix $k \geq k'$. It suffices to show that

$$\Delta_\nu((1-\beta)\theta, \varphi_k) + v' \subseteq \Delta' \tag{2.2}$$

for some $v' \in \mathbb{R}^n_{\geq 0}$. In fact, if this is true, we have

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu((1-\beta)\theta, \varphi_k).$$

Let $k \to \infty$, by [Xia21, Theorem A], we conclude (2.1).

It remains to prove (2.2). Let $\pi : Y \to X$ be a log resolution of the singularities of φ_k. By the proof of [DX21, Proposition 4.3], there is $j_0 = j_0(\beta, k) \in \mathbb{Z}_{>0}$ such that for any $j \geq j_0$, we can find a non-zero section $s_j \in H^0(Y, \pi^* L^{pj} \otimes \mathcal{I}(jp\pi^* v_{\beta,k}))$ such that we get an injective linear map

$$H^0(Y, \pi^*T \otimes K_{Y/X} \otimes \pi^* L^{(q-p)j} \otimes \mathcal{I}(jq\pi^*\varphi_k)) \xrightarrow{\times s_j} H^0(X, T \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)).$$

In particular, when $j = k_i$ for some i large enough, we then find

$$\Delta_\nu^{k_i, \pi^*T \otimes K_{Y/X}}((1-\beta)q\pi^*\theta, q\pi^*\varphi_k) + (k_i)^{-1}\nu(s_{k_i}) \subseteq q\Delta_\nu^{k_i,T}(\theta, \varphi).$$

We observe that $(k_i)^{-1}\nu(s_{k_i})$ is bounded as the right-hand side is bounded when i varies. Then by Lemma 2.2, there is a vector $v' \in \mathbb{R}^n_{\geq 0}$ such that

$$\Delta_\nu((1-\beta)\pi^*\theta, \pi^*\varphi_k) + v' \subseteq \Delta'.$$

By the birational invariance of the partial Okounkov bodies, we find (2.2).

Step 2. Next we handle the general case.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$, say the subsequence with indices $k_1 < k_2 < \cdots$. By Blaschke selection theorem, it suffices to prove that $\Delta' = \Delta_\nu(\theta, \varphi)$.

Take $\psi \in \mathrm{PSH}(X, \theta)$ such that

(1) θ_ψ is a Kähler current;

(2) $\psi \leq \varphi$.

The existence of ψ is proved in [DX21, Proposition 3.6].

Then for any $\epsilon \in \mathbb{Q} \cap (0, 1)$,

$$\Delta_\nu^{k,T}(\theta, \varphi) \supseteq \Delta_\nu^{k,T}(\theta, (1-\epsilon)\varphi + \epsilon\psi)$$

for all k. It follows from Step 1 that

$$\Delta' \supseteq \Delta_\nu(\theta, (1-\epsilon)\varphi + \epsilon\psi).$$

Letting $\epsilon \to 0$ and applying [Xia21, Theorem A], we have

$$\Delta' \supseteq \Delta_\nu(\theta, \varphi).$$

It remains to establish that

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(\theta,\varphi). \tag{2.3}$$

For this purpose, we are free to replace $k_1 < k_2 < \cdots$ by a subsequence. Fix $q > 0$, we may then assume that $k_i \equiv a$ modulo q for all $i \geq 1$ for some $a \in \{0, 1, \ldots, q-1\}$. We write $k_i = g_i q + a$. Observe that

$$H^0(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi)) \subseteq H^0(X, T \otimes L^a \otimes L^{g_i q} \otimes \mathcal{I}(g_i q\varphi)).$$

Up to replacing T by $T \otimes L^a$, we may assume that $a = 0$.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in H^0(X, H)$.

We have an injective linear map

$$H^0(X, T \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)) \xrightarrow{\times s^j} H^0(X, T \otimes H^j \otimes L^{jq} \otimes \mathcal{I}(jq\varphi))$$

for each $j \geq 1$. In particular, for each $i \geq 1$,

$$k_i \Delta_\nu^{k_i, T}(q\theta, q\varphi) + k_i \nu(s) \subseteq k_i \Delta_\nu^{k_i, T}(\omega + q\theta, q\varphi).$$

Let $i \to \infty$, by Step 1,

$$q\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + q\theta, q\varphi).$$

So

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(q^{-1}\omega + \theta, \varphi) = \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P^{q^{-1}\omega+\theta}[\varphi]_{\mathcal{I}})^n.$$

By [Xia21, Corollary 4.4],

$$\operatorname{vol}\Delta' \leq \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P^{\theta}[\varphi]_{\mathcal{I}})^n.$$

Let $q \to \infty$, we conclude (2.3). $\square$

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