Bisognano-Wichmann property for rigid categorical extensions and non-local extensions of conformal nets

BIN GUI

Abstract

Given an (irreducible) Möbius covariant net $\mathcal{A}$, we prove a Bisognano-Wichmann theorem for its categorical extension $\mathscr{E}^{\mathrm{f}}$ associated to the braided C^*-tensor category $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ of dualizable Möbius covariant $\mathcal{A}$-modules. As a closely related result, we prove a (modified) Bisognano-Wichmann theorem for any (possibly) non-local extension of $\mathcal{A}$ obtained by a C^*-Frobenius algebra Q in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. As an application, we discuss the relation between the domains of modular operators and the preclosedness of certain unbounded operators in $\mathscr{E}^{\mathrm{f}}$.

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0 Introduction

The notion of categorical extensions of conformal nets was introduced in [Gui18] to understand the relations between the tensor categories of conformal nets and vertex operator algebras (VOA's). A categorical extension $\mathscr{E}$ of an irreducible conformal (or Möbius covariant) net $\mathcal{A}$ is the Haag-Kastler net of bounded charged fields (intertwiners) associated to $\mathcal{A}$. $\mathscr{E}$ satisfies a list of axioms similar to those of $\mathcal{A}$, including, most importantly, the *locality* axiom, which says that bounded charged fields supported in disjoint open intervals commute adjointly. One main observation in [Gui18] is that, in order to relate the tensor category of a unitary VOA V with the one of the corresponding conformal net $\mathcal{A}_V$ (assuming $\mathcal{A}_V$ exists), it suffices to show that the (usually) *unbounded* smeared intertwining operators of V give rise to *bounded* intertwiners satisfying the axioms of a categorical extension, especially the locality. Similar to the construction in [CKLW18] of $\mathcal{A}_V$ from V, proving the locality axiom is the most difficult step, which amounts to proving the strong commutativity of certain adjointly commuting unbounded closed operators.[1]

The Bisognano-Wichmann (B-W) theorem [BW75] is a powerful tool for proving the locality of the conformal net $\mathcal{A}_V$ associated to a unitary VOA V. In [CKLW18], Carpi-Kawahigashi-Longo-Weiner used this theorem to show that very often, one only needs the strong commutativity of a small amount of smeared vertex operators (which "generate V") to prove the strong commutativity of *all* smeared vertex operators supported in disjoint intervals. [2] The main motivation of our present article is to generalize this result to intertwining operators (charged fields) of VOA's.

Let us first recall the B-W theorem in (algebraic) chiral conformal field theory [BGL93, GF93, FJ96]. Let $\mathcal{A}$ be an (irreducible) Möbius covariant net with vacuum representation $\mathcal{H}_0$ and vacuum vector Ω. The representation of $\mathrm{PSU}(1,1)$ on $\mathcal{H}_0$ is denoted by U. By Reeh-Schlieder property, Ω is a cyclic and separating vector of $\mathcal{A}(I)$ where I is any open (non-dense non-empty) interval on the unit circle $\mathbb{S}^1$. Thus, one can associate to the pair $(\mathcal{A}(I),\Omega)$ the modular operator Δ_I and modular conjugation $\mathfrak{J}_I$ satisfying the Tomita-Takesaki theorem. Now, the B-W theorem for $\mathcal{A}$ says that:

- (Geometric modular theorem) $\Delta_I^{\mathbf{i}t} = \delta_I(-2\pi t)$, where δ_I is the dilation subgroup of the Möbius group $\mathrm{PSU}(1,1)$ associated to the interval I (see section 5 for more details).
- (PCT theorem) The antiunitary map $\Theta := \mathfrak{J}_{\mathbb{S}^1_+}$ (which is an involution by Tomita-Takesaki theory) is a PCT operator for $\mathcal{A}$, where $\mathbb{S}^1_+$ is the upper semi-circle. More precisely, if we set $\mathfrak{r}:\mathbb{S}^1\to\mathbb{S}^1, z\mapsto\overline{z}$, then we have $\Theta\mathcal{A}(I)\Theta=\mathcal{A}(\mathfrak{r}I)$ and $\Theta U(g)\Theta = U(\mathfrak{r}g\mathfrak{r})$ for any $g\in\mathrm{PSU}(1,1)$.

More generally, one has the B-W theorem for Fermi conformal nets [ALR01, CKL08] and irreducible finite-index non-local extensions of conformal nets [LR04].

[1] Two closed operators A and B on a Hilbert space $\mathcal{H}$ are said to commute adjointly if $[A,B] = [A^*,B] = 0$ when acting on suitable vectors; they are said to commute strongly if the von Neumann algebras generated by A and by B commute. Strong commutativity implies adjoint commutativity; the converse may not hold by the famous counterexample of Neilson [Nel59].

[2] For general quantum field theories, a similar result was proved in [DSW86].

To derive a B-W theorem for categorical extensions of $\mathcal{A}$, we first need to define the modular S and F operators for them. Before explaining the definition, we first recall what are categorical extensions.

Let $\mathbb{S}^1_-$ be the lower semi-circle. If $\mathcal{H}_i, \mathcal{H}_j$ are $\mathcal{A}$-modules, then $\mathcal{H}_j$ is a left $\mathcal{A}(\mathbb{S}^1_+)$ module, and $\mathcal{H}_i$ is a right $\mathcal{A}(\mathbb{S}^1_+)$ module defined by the action $x \in \mathcal{A}(\mathbb{S}^1_+) \mapsto \Theta x^* \Theta$. Then the fusion product $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is the Connes-Sauvageot relative tensor product of $\mathcal{H}_i$ and $\mathcal{H}_j$ over $\mathcal{A}(\mathbb{S}^1_+)$. $\mathcal{A}(\mathbb{S}^1_+)$ and $\mathcal{A}(\mathbb{S}^1_-)$ act naturally on $\mathcal{H}_i \boxtimes \mathcal{H}_j$ by acting respectively on the left and the right components, and can be extended to a representation of $\mathcal{A}$ on $\mathcal{H}_i \boxtimes \mathcal{H}_j$ using "path continuations". (See section A or [Gui18] chapter 2 for details.) Now, one can define a dense vector space $\mathcal{H}_i(I) = \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)\Omega$, where I' is the interior of the complement of I. $\mathcal{H}_j(I)$ is defined similarly. Then we know that $\mathcal{H}_i \boxtimes \mathcal{H}_j$ has a dense subspace spanned by vectors of the form $\xi \otimes \eta$ where $\xi \in \mathcal{H}_i(\mathbb{S}^1_+)$ and $\eta \in \mathcal{H}_j(\mathbb{S}^1_-)$. We then have bounded operators

$$L(\xi) \in \mathrm{Hom}_{\mathcal{A}(\mathbb{S}^1_-)}(\mathcal{H}_j, \mathcal{H}_i \boxtimes \mathcal{H}_j), \qquad R(\eta) \in \mathrm{Hom}_{\mathcal{A}(\mathbb{S}^1_+)}(\mathcal{H}_i, \mathcal{H}_i \boxtimes \mathcal{H}_j)$$

defined by $L(\xi)\phi = \xi \otimes \phi$ and $R(\eta)\psi = \psi \otimes \eta$ for any $\phi \in \mathcal{H}_j(\mathbb{S}^1_-), \psi \in \mathcal{H}_i(\mathbb{S}^1_+)$. We understand $L(\xi), R(\eta)$ as operators acting on any possible $\mathcal{A}$-modules. This means that when $\chi \in \mathcal{H}_k$, we have $L(\xi)\chi \in \mathcal{H}_i \boxtimes \mathcal{H}_k, R(\eta)\chi \in \mathcal{H}_k \boxtimes \mathcal{H}_j$.

The L and R operators defined above should be understood as supported in $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$ respectively. We would like to have them supported in any interval I, so that we have nets of sets of L operators and R operators. It turns out that in general, such nets can be defined not on $\mathbb{S}^1$ but on its universal cover. So one should consider the L and R operators localized not in intervals, but in arg-valued intervals. If I is an interval of $\mathbb{S}^1$, then one can choose a continuous argument function $\arg_I$. Then the pair $\widetilde{I} = (I, \arg_I)$ is called an arg-valued interval. We choose $\widetilde{\mathbb{S}^1_+}$ and $\widetilde{\mathbb{S}^1_-}$ such that $\arg_{\mathbb{S}^1_+}(e^{\mathbf{i}t}) = t$ $(0 < t < \pi)$, and that $\arg_{\mathbb{S}^1_-}(e^{\mathbf{i}t}) = t$ $(-\pi < t < 0)$. Then one can define consistently the L and R operators localized in any given arg-valued interval $\widetilde{I}$. To be more precise, for any $\mathcal{A}$-modules $\mathcal{H}_i, \mathcal{H}_k$ and any $\xi \in \mathcal{H}_i(I)$, one can define

$$L(\xi, \widetilde{I}) \in \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_k, \mathcal{H}_i \boxtimes \mathcal{H}_k), \qquad R(\xi, \widetilde{I}) \in \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_k, \mathcal{H}_k \boxtimes \mathcal{H}_i).$$

Moreover, when $\widetilde{I} = \widetilde{\mathbb{S}^1_+}$ we have $L(\xi, \widetilde{I}) = L(\xi)$; when $\widetilde{I} = \widetilde{\mathbb{S}^1_-}$ we have $R(\xi, \widetilde{I}) = R(\xi)$. These L and R operators form a categorical extension of $\mathcal{A}$.

We now focus on dualizable $\mathcal{A}$-modules $\mathcal{H}_i, \mathcal{H}_j, \mathcal{H}_k$, etc. Since $\mathcal{H}_i$ is dualizable, we have an $\mathcal{A}$-module $\mathcal{H}_{\bar{i}}$ (the dual object) and evaluations $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i \boxtimes \mathcal{H}_{\bar{i}}, \mathcal{H}_0)$ and $\mathrm{ev}_{\bar{i},i} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar{i}} \boxtimes \mathcal{H}_i, \mathcal{H}_0)$ satisfying the conjugate equations

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{\bar{i},i}) = \mathbf{1}_i = (\mathbf{1}_i \otimes \mathrm{ev}_{\bar{i},i})(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_i),$$
$$(\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{i}} \otimes \mathrm{coev}_{i,\bar{i}}) = \mathbf{1}_{\bar{i}} = (\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{i,\bar{i}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}}),$$

where we set $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}^*_{i,\bar{i}}, \mathrm{coev}_{\bar{i},i} = \mathrm{ev}^*_{\bar{i},i}$. Moreover, we may and do assume that the ev and coev are standard, which means $d_i = \|\mathrm{ev}_{i,\bar{i}}\|^2$ equals $d_{\bar{i}} = \|\mathrm{ev}_{\bar{i},i}\|^2$ and are the smallest possible values. (Equivalently, ev and coev satisfy (2.18); cf. [LR97].) Now, for any $\widetilde{I}$, we define the categorical S and F operators $S_{\widetilde{I}}, F_{\widetilde{I}}$. For any dualizable $\mathcal{H}_i$, we have

$$S_{\widetilde{I}}, F_{\widetilde{I}} : \mathcal{H}_i \to \mathcal{H}_{\bar{i}}$$

with common domain $\mathcal{H}_i(I)$ defined by

$$S_{\widetilde{I}}\xi = L(\xi, \widetilde{I})^* \mathrm{coev}_{i,\bar{i}}\Omega, \qquad F_{\widetilde{I}}\xi = R(\xi, \widetilde{I})^* \mathrm{coev}_{\bar{i},i}\Omega.$$

These two operators are indeed preclosed. Moreover, they are related by the (unitary) twist operator ϑ (proposition 4.5):

$$F_{\widetilde{I}} = \vartheta S_{\widetilde{I}}.$$

We can thus define the modular operator $\Delta_{\widetilde{I}}$ and modular conjugation $\mathfrak{J}_{\widetilde{I}}$ by the polar decompositions:

$$S_{\widetilde{I}} = \mathfrak{J}_{\widetilde{I}} \cdot \Delta_{\widetilde{I}}^{\frac{1}{2}}, \qquad F_{\widetilde{I}} = \vartheta\mathfrak{J}_{\widetilde{I}} \cdot \Delta_{\widetilde{I}}^{\frac{1}{2}},$$

where, for each $\mathcal{H}_i$, $\Delta_{\widetilde{I}}$ is a positive closed operator on $\mathcal{H}_i$, and $\mathfrak{J}_{\widetilde{I}} : \mathcal{H}_i \to \mathcal{H}_{\bar{i}}$ is antiunitary. Indeed, $\mathfrak{J}_{\widetilde{I}} : \mathcal{H}_i \to \mathcal{H}_{\bar{i}}$ is an involution, i.e., $\mathfrak{J}_{\widetilde{I}}^2 = \mathbf{1}$. Note that $\mathfrak{J}_{\widetilde{I}}$ depends on the choice of dual objects and standard evaluations because $S_{\widetilde{I}}$ do. This is related to the important fact that $\mathfrak{J}_{\widetilde{I}}$ *implements the conjugations of morphisms* which depend on dual objects and standard evaluations. Indeed, for any morphism $G \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H}_j)$ one can define its conjugate $\overline{G} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar{i}}, \mathcal{H}_{\bar{j}})$ to be the adjoint of the transpose $G^\vee$, where $G^\vee$ is the unique morphism in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar{j}}, \mathcal{H}_{\bar{i}})$ satisfying

$$\mathrm{ev}_{j,\bar{j}}(G \otimes \mathbf{1}_{\bar{j}}) = \mathrm{ev}_{i,\bar{i}}(\mathbf{1}_i \otimes G^\vee).$$

We will prove that

$$\overline{G} = \mathfrak{J}_{\widetilde{I}} \cdot G \cdot \mathfrak{J}_{\widetilde{I}}$$

(see proposition 4.13), which suggests that $\mathfrak{J}_{\widetilde{I}}$ is the correct modular conjugation for categorical extensions.

On the other hand, $\Delta_{\widetilde{I}}$ is independent of dual objects and standard evaluations. (It even does not depend on $\arg_I$, which means that we can write $\Delta_{\widetilde{I}}$ as Δ_I.) Moreover, the action of $\Delta_{\widetilde{I}}$ on any $\mathcal{H}_i$ can be interpreted as a Connes spatial derivative (see remark 4.14). Indeed, our definition and treatment of $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are deeply motivated by the matrix algebra approach to Connes fusion products and Connes spatial derivatives in [Fal00] and [Tak02] section IX.3. Those matrices of von Neumann algebras are described in our article by the C^*-Frobenius algebra $Q = (\mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}}, \mu, \iota)$ in the representation category $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ of dualizable Möbius covariant $\mathcal{A}$-modules, where $\iota \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_0, \mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}})$ is $\mathrm{coev}_{k,\bar{k}}$, and $\mu \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}} \boxtimes \mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}}, \mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}})$ is $\mathrm{ev}_{k,\bar{k}}(\mathbf{1}_k \otimes \mathrm{ev}_{\bar{k},k} \otimes \mathbf{1}_{\bar{k}})$. As we will show, $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are closely related to the S and F operators of non-local extensions of $\mathcal{A}$ constructed from C^*-Frobenius algebras. Thus, using the Tomita-Takesaki theory for those non-local extensions, we are able to show that $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are always preclosed, and that $\Delta_{\widetilde{I}}$ and $\mathfrak{J}_{\widetilde{I}}$ satisfy similar algebraic relations as those in Tomita-Takesaki theory (see for example proposition 4.10). The idea here is the same as in [Fal00] and [Tak02].

We emphasize that the categorical extensions and the non-local finite index extensions (by C^*-Frobenius algebras) of a conformal net $\mathcal{A}$ are closely related. So are their modular theories. As we will see, the proof of the B-W theorem for categorical extensions relies on that for non-local extensions, and vice versa: As indicated in [ALR01] and [LR04], for a non-local extension $\mathcal{B}$ constructed from the C^*-Frobenius algebra Q, $z(t) = \Delta_I^{\mathrm{i}t}\delta_I(2\pi t)$ is a one-parameter group independent of I. To show that $z(t) = 1$ *when Q is standard*, we have to make use of the relation $F_{\widetilde{I}} = \vartheta S_{\widetilde{I}}$ in the modular theory of categorical extensions. On the other hand, to prove the categorical B-W theorem, we need the non-local B-W theorem in the case that Q is standard; to prove the (modified) non-local B-W theorem for non-necessarily standard Q, one needs the categorical B-W theorem. These two B-W theorems are the main results of our paper, which are stated in details in theorems 5.3 and 5.10. Roughly speaking, the categorical B-W theorem says:

Theorem 0.1 (Categorical B-W theorem)**.** *We have*

$$\Delta_{\widetilde{I}}^{\mathrm{i}t} = \delta_I(-2\pi t) \tag{0.1}$$

when acting on any dualizable $\mathcal{H}_i$. Moreover, $\Theta := \mathfrak{J}_{\widetilde{\mathbb{S}^1_+}}$ is a PCT-operator for the (rigid) categorical extension.

Let $Q = (\mathcal{H}_a, \mu, \iota)$ be a C^*-Frobenius algebra in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, where $\iota \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_0, \mathcal{H}_a)$ and $\mu \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_a \boxtimes \mathcal{H}_a, \mathcal{H}_a)$. Choose dual object $\mathcal{H}_{\overline{a}}$ and standard evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$. Let ϵ be the unique invertible morphism in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_a, \mathcal{H}_{\overline{a}})$ satisfying

$$\mathrm{ev}_{\overline{a},a}(\epsilon \otimes \mathbf{1}_a) = \iota^* \mu.$$

(ϵ will be called reflection operator in our paper.) We remark that $\epsilon^*\epsilon$ is independent of dual objects and standard evaluations. Then we have:

Theorem 0.2 (Modified non-local B-W theorem)**.** *Let $\mathcal{B}$ be the non-local extension of $\mathcal{A}$ obtained through Q. For any $\widetilde{I} \in \widetilde{\mathcal{J}}$, Let $D_{\widetilde{I}}$ and $\mathfrak{J}^Q_{\widetilde{I}}$ be the modular operator and conjugation associated to $(\mathcal{B}(\widetilde{I}), \iota\Omega)$. Then*

$$D_{\widetilde{I}}^{\mathrm{i}t} = (\epsilon^*\epsilon)^{\mathrm{i}t}\delta_I(-2\pi t), \tag{0.2}$$

and $\Theta^Q := \mathfrak{J}^Q_{\widetilde{\mathbb{S}^1_+}}$ is a PCT operator for $\mathcal{B}$ and its "clockwise dual net" $\mathcal{B}'$.

Some remarks on these two theorems:

- Equivalent forms of equation (0.1) already appeared in [FRS92, Jörß96] and in [Lon97]. In [FRS92, Jörß96], the S operators are defined for reduced field bundles, which are an alternative model for charged fields (intertwining operators) of conformal nets. For our purpose (see the beginning of the introduction), categorical extensions might be more convenient than reduce field bundles. In [Lon97], Longo showed that the dilation group δ_I is related to Connes Radon-Nikodym derivatives, which are in turn related to Connes spatial derivatives and hence related to our $\Delta_{\widetilde{I}}$ (see remark 4.14).

- Similar to [Jörß96, GL96], the conformal spin-statistics theorem $\vartheta = e^{2\mathrm{i}\pi L_0}$ is a consequence of the PCT theorem for (rigid) categorical extensions (see theorem 5.7).

- The C^*-Frobenius algebra Q is standard if and only if ϵ is unitary. Thus, by (0.2), for the non-local extension $\mathcal{B}$ of $\mathcal{A}$ obtained by Q, the standard geometric modular theorem $D_{\widetilde{I}}^{\mathbf{i}t} = \delta_I(-2\pi t)$ holds if and only if Q is standard.
- When Q is irreducible (as a left Q-module), theorem 0.2 was proved by [LR04] proposition 3.5-(ii).

This article is organized as follows. In section 1 we review the definitions of Möbius covariant nets and conformal nets and their representations. In section 2 we review the definition and basic properties of categorical extensions of conformal nets. Construction of non-local extensions of conformal nets via C^*-Frobenius algebras (or Q-systems) was first studied in [LR95] using endomorphisms of von Neumann algebras. A parallel construction using bimodules and Connes fusions was given in [Mas97]. In section 3, we use categorical extensions as a new method to realize such construction of non-local extensions. Our method emphasizes the close relation between the charged field operators of a conformal net and the field operators of its non-local extensions, and explains the slogan "non-local extensions are subquotients of categorical extensions" proposed in [Gui18]. In section 4 we define the S and F operators for rigid categorical extensions, and prove basic properties of these operators by relating them with the S and F operators of non-local extensions. We also prove that the conjugations of morphisms are implemented by the modular conjugations of categorical extensions. In section 5 we prove the main results of this article, namely theorems 0.1 and 0.2. In section 6 we use the modular theory of categorical extensions to study the preclosedness of certain unbounded charged fields of conformal nets. Although our main motivation of this article is to study the functional analytic properties of these field operators, here we do not give a systematic study of this topic but leave it to future works.

Categorical extensions of conformal nets are closely related to Connes fusion. In section A we briefly explain this relation for the convenience of the readers who are not familiar with this relation. We hope that this appendix section would help them understand the axioms in the definition of categorical extensions. In section B we prove that the rigid categorical extensions of Möbius covariant nets are Möbius covariant. This result parallels the conformal covariance of the categorical extensions of conformal (covariant) nets proved in [Gui18] section 2.4 and theorem 3.5. Indeed, our proof of the Möbius covariance in this article can be adapted to give a simpler proof of the conformal covariance in [Gui18]; see the end of section B.

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1 Backgrounds

Let $\mathcal{J}$ be the set of all non-empty non-dense open intervals in the unit circle $\mathbb{S}^1$. If $I \in \mathcal{J}$, then I' denotes the interior of the complement of I, which is also an element in $\mathcal{J}$. The group $\mathrm{Diff}^+(\mathbb{S}^1)$ of orientation-preserving diffeomorphisms of $\mathbb{S}^1$ contains the subgroup $\mathrm{PSU}(1,1)$ of Möbius transforms of $\mathbb{S}^1$. If $I \in \mathcal{J}$, we let $\mathrm{Diff}(I)$ be the subgroup of all $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ acting as identity on I'.

In this article, we always let $\mathcal{A}$ be an (irreducible) **Möbius covariant net**, which means that for each $I \in \mathcal{J}$ there is a von Neumann algebra $\mathcal{A}(I)$ acting on a fixed separable Hilbert space $\mathcal{H}_0$, such that the following conditions hold:
(a) (Isotony) If $I_1 \subset I_2 \in \mathcal{J}$, then $\mathcal{A}(I_1)$ is a von Neumann subalgebra of $\mathcal{A}(I_2)$.
(b) (Locality) If $I_1, I_2 \in \mathcal{J}$ are disjoint, then $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute.
(c) (Möbius covariance) We have a strongly continuous unitary representation U of $\mathrm{PSU}(1,1)$ on $\mathcal{H}_0$ such that for any $g \in \mathrm{PSU}(1,1), I \in \mathcal{J}$,,

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

(d) (Positivity of energy) The generator L_0 of the rotation subgroup ϱ is positive.
(e) There exists a unique (up to scalar) $\mathrm{PSU}(1,1)$-invariant unit vector $\Omega \in \mathcal{H}_0$. Moreover, Ω is cyclic under the action of $\bigvee_{I \in \mathcal{J}} \mathcal{M}(I)$ (the von Neumann algebra generated by all $\mathcal{M}(I)$).

We say that $\mathcal{A}$ is a **conformal (covariant) net** if the representation U of $\mathrm{PSU}(1,1)$ on $\mathcal{H}_0$ can be extended to a strongly continuous projective unitary representation U of $\mathrm{Diff}^+(\mathbb{S}^1)$ on $\mathcal{H}_0$, such that for any $g \in \mathrm{Diff}^+(\mathbb{S}^1), I \in \mathcal{J}$, and any representing element $V \in \mathcal{U}(\mathcal{H}_0)$ of $U(g)$,

$$V\mathcal{A}(I)V^* = \mathcal{A}(gI).$$

Moreover, if $g \in \mathrm{Diff}(I)$ and $x \in \mathcal{A}(I')$, then

$$VxV^* = x.$$

Let $\mathcal{H}_i$ be a separable Hilbert space. Recall that a (normal) representation $(\mathcal{H}_i, \pi_i)$ of $\mathcal{A}$ (also called an $\mathcal{A}$-module) associates to each $I \in \mathcal{J}$ a unital *-representation $\pi_{i,I}: \mathcal{A}(I) \to B(\mathcal{H}_i)$, such that for any $I_1, I_2 \in \mathcal{J}$ satisfying $I_1 \subset I_2$, and any $x \in \mathcal{A}(I_1)$, we have $\pi_{i,I_1}(x) = \pi_{i,I_2}(x)$. We write $\pi_{i,I}(x)$ as $\pi_i(x)$ or just x when no confusion arises.

Let $\mathscr{G}$ be the universal covering of $\mathrm{Diff}^+(\mathbb{S}^1)$. The corresponding projective representation of $\mathscr{G}$ on $\mathcal{H}_0$ is also denoted by U. Then $\mathscr{G}$ has a central extension

$$1 \to U(1) \to \mathscr{G}_{\mathcal{A}} \to \mathscr{G} \to 1$$

associated to the projective representation of $\mathrm{Diff}^+(\mathbb{S}^1)$ on $\mathcal{H}_0$. In other words, we set

$$\mathscr{G}_{\mathcal{A}} = \{(g, V) \in \mathscr{G} \times \mathcal{U}(\mathcal{H}_0) | V \text{ is a representing element of } U(g)\}.$$

Then the projective representation $\mathrm{Diff}^+(\mathbb{S}^1) \curvearrowright \mathcal{H}_0$ gives rise to an actual unitary representation of $\mathscr{G}_{\mathcal{A}}$ of $\mathcal{H}_0$, also denoted by U. For each I, we let $\mathscr{G}(I)$ be the preimage of $\mathrm{Diff}(I)$ under the covering map $\mathscr{G} \to \mathrm{Diff}^+(\mathbb{S}^1)$. Similarly, let $\mathscr{G}_{\mathcal{A}}(I)$ be the preimage of $\mathscr{G}(I)$ under $\mathscr{G}_{\mathcal{A}} \to \mathscr{G}$. If $\mathcal{A}$ is conformal covariant, then any $\mathcal{A}$-module $\mathcal{H}_i$ is **conformal covariant**, which means that there is a unique representation U_i of $\mathscr{G}_{\mathcal{A}}$ on $\mathcal{H}_i$ such that for any $I \in \mathcal{J}$ and $g \in \mathscr{G}_{\mathcal{A}}(I)$,

$$U_i(g) = \pi_i(U(g)). \tag{1.1}$$

This is proved in [AFK04] (only for irreducible representations) and in [Hen19] theorem 11. Moreover, the generator of the rotation subgroup acting on $\mathcal{H}_i$ is positive by [Wei06] theorem 3.8. From (1.1) and the fact that $\mathscr{G}_\mathcal{A}$ is algebraically generated by $\{\mathscr{G}_\mathcal{A}(I) : I \in \mathcal{J}\}$ proved in [Hen19] Lemma 17-(ii) (see also [Gui18] proposition 2.2), it is clear that any homomorphism of conformal net modules is also a homomorphism of representations of $\mathscr{G}_\mathcal{A}$. Moreover, for any $g \in \mathscr{G}_\mathcal{A}$ and $x \in \mathcal{A}(I)$ one has

$$U_i(g)\pi_{i,I}(x)U_i(g)^* = \pi_{i,gI}(U(g)xU(g)^*). \tag{1.2}$$

Very often, we will write $U(g)$ and $U_i(g)$ as g for short.

Let $\widetilde{\mathrm{PSU}}(1,1)$ be the universal cover of $\mathrm{PSU}(1,1)$, regarded as a subgroup of $\mathscr{G}$. By [Bar54], the restriction of any strongly continuous projective representation $\mathscr{G}$ to $\widetilde{\mathrm{PSU}}(1,1)$ can be lifted to a unique strongly continuous unitary representation of $\widetilde{\mathrm{PSU}}(1,1)$. Thus $\widetilde{\mathrm{PSU}}(1,1)$ is also a subgroup of $\mathscr{G}_\mathcal{A}$. Note that the action of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_0$ also preserves Ω. We say that an $\mathcal{A}$-module $\mathcal{H}_i$ is **Möbius covariant** if there is a strongly continuous unitary representation U_i of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_i$ such that (1.2) holds for any $g \in \widetilde{\mathrm{PSU}}(1,1)$ and $I \in \mathcal{J}$.

2 Rigid categorical extensions

Let $\mathrm{Rep}(\mathcal{A})$ be the C^*-category of $\mathcal{A}$-modules whose objects are denoted by $\mathcal{H}_i, \mathcal{H}_j, \mathcal{H}_k, \dots$. Then one can equip $\mathrm{Rep}(\mathcal{A})$ with a structure of braided C^*-tensor category either via Doplicher-Haag-Roberts (DHR) superselection theory [FRS89, FRS92], or via Connes fusion [BDH15, BDH17, Gui18]. These two constructions are equivalent by [Gui18] chapter 6. The unit of $\mathrm{Rep}(\mathcal{A})$ is $\mathcal{H}_0$. We write the tensor (fusion) product of two $\mathcal{A}$-modules $\mathcal{H}_i, \mathcal{H}_j$ as $\mathcal{H}_i \boxtimes \mathcal{H}_j$. We assume without loss of generality that $\mathrm{Rep}(\mathcal{A})$ is strict, which means that we will not distinguish between $\mathcal{H}_0, \mathcal{H}_0 \boxtimes \mathcal{H}_i, \mathcal{H}_i \boxtimes \mathcal{H}_0$, or $(\mathcal{H}_i \boxtimes \mathcal{H}_j) \boxtimes \mathcal{H}_k$ and $\mathcal{H}_i \boxtimes (\mathcal{H}_j \boxtimes \mathcal{H}_k)$ (abbreviated to $\mathcal{H}_i \boxtimes \mathcal{H}_j \boxtimes H_k$). In the following, we review the definition and the basic properties of closed vector-labeled categorical extensions of $\mathcal{A}$ (abbreviated to "categorical extensions" for short) introduced in [Gui18].

To begin with, if $\mathcal{H}_i, \mathcal{H}_j$ are $\mathcal{A}$-modules and $I \in \mathcal{J}$, then $\mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_i, \mathcal{H}_j)$ denotes the vector space of bounded linear operators $T : \mathcal{H}_i \to \mathcal{H}_j$ such that $T\pi_{i,I'}(x) = \pi_{j,I'}(x)T$ for any $x \in \mathcal{A}(I')$. We then define $\mathcal{H}_i(I) = \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)\Omega$, which is a dense subspace of $\mathcal{H}_i$. Note that $I \subset J$ implies $\mathcal{H}_i(I) \subset \mathcal{H}_i(J)$. Moreover, if $G \in \mathrm{Hom}_\mathcal{A}(\mathcal{H}_i, \mathcal{H}_j)$, then $G\mathcal{H}_i(I) \subset \mathcal{H}_j(I)$.

If $I \in \mathcal{J}$, an arg-function of I is, by definition, a continuous function $\arg_I : I \to \mathbb{R}$ such that for any $e^{it} \in I$, $\arg_I(e^{it}) - t \in 2\pi\mathbb{Z}$. $\widetilde{I} = (I, \arg_I)$ is called an **arg-valued interval**. Equivalently, $\widetilde{I}$ is a branch of I in the universal cover of $\mathbb{S}^1$. We let $\widetilde{\mathcal{J}}$ be the set of arg-valued intervals. If $\widetilde{I} = (I, \arg_I)$ and $\widetilde{J} = (J, \arg_J)$ are in $\widetilde{\mathcal{J}}$, we say that $\widetilde{I}$ and $\widetilde{J}$ are disjoint if I and J are so. Suppose moreover that for any $z \in I, \zeta \in J$ we have $\arg_J(\zeta) < \arg_I(z) < \arg_J(\zeta) + 2\pi$, then we say that $\widetilde{I}$ is **anticlockwise** to $\widetilde{J}$ (equivalently, $\widetilde{J}$ is **clockwise** to $\widetilde{I}$). We write $\widetilde{I} \subset \widetilde{J}$ if $I \subset J$ and $\arg_J|_I = \arg_I$. Given $\widetilde{I} \in \widetilde{\mathcal{J}}$, we also define $\widetilde{I}' = (I', \arg_{I'}) \in \widetilde{\mathcal{J}}$ such that $\widetilde{I}$ is anticlockwise to $\widetilde{I}'$. We say that $\widetilde{I}'$ is the **clockwise complement** of $\widetilde{I}$.

Definition 2.1. A (closed and vector-labeled) **categorical extension** $\mathscr{E} = (\mathcal{A}, \mathrm{Rep}(\mathcal{A}), \boxtimes, \mathcal{H})$ of $\mathcal{A}$ associates, to any $\mathcal{H}_i, \mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$ and any $\widetilde{I} \in \widetilde{\mathcal{J}}, \xi \in \mathcal{H}_i(I)$, bounded linear operators

$$L(\xi, \widetilde{I}) \in \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_k, \mathcal{H}_i \boxtimes \mathcal{H}_k),$$
$$R(\xi, \widetilde{I}) \in \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_k, \mathcal{H}_k \boxtimes \mathcal{H}_i),$$

such that the following conditions are satisfied:

(a) (Isotony) If $\widetilde{I}_1 \subset \widetilde{I}_2 \in \widetilde{\mathcal{J}}$, and $\xi \in \mathcal{H}_i(I_1)$, then $L(\xi, \widetilde{I}_1) = L(\xi, \widetilde{I}_2)$, $R(\xi, \widetilde{I}_1) = R(\xi, \widetilde{I}_2)$ when acting on any $\mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$.

(b) (Functoriality) If $\mathcal{H}_i, \mathcal{H}_k, \mathcal{H}_{k'} \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, $F \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_k, \mathcal{H}_{k'})$, the following diagrams commute for any $\widetilde{I} \in \widetilde{\mathcal{J}}, \xi \in \mathcal{H}_i(I)$.

$$\begin{tikzcd} \mathcal{H}_k \arrow[r, "F"] \arrow[d, "L(\xi,\widetilde{I})"'] & \mathcal{H}_{k'} \arrow[d, "L(\xi,\widetilde{I})"'] \\ \mathcal{H}_i \boxtimes \mathcal{H}_k \arrow[r, "\mathbf{1}_i \otimes F"] & \mathcal{H}_i \boxtimes \mathcal{H}_{k'} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_k \arrow[r, "R(\xi,\widetilde{I})"] \arrow[d, "F"'] & \mathcal{H}_k \boxtimes \mathcal{H}_i \arrow[d, "F \otimes \mathbf{1}_i"'] \\ \mathcal{H}_{k'} \arrow[r, "R(\xi,\widetilde{I})"] & \mathcal{H}_{k'} \boxtimes \mathcal{H}_i \end{tikzcd} . \tag{2.1}$$

(c) (State-field correspondence[3]) For any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, under the identifications $\mathcal{H}_i = \mathcal{H}_i \boxtimes \mathcal{H}_0 = \mathcal{H}_0 \boxtimes \mathcal{H}_i$, the relation

$$L(\xi, \widetilde{I})\Omega = R(\xi, \widetilde{I})\Omega = \xi \tag{2.2}$$

holds for any $\widetilde{I} \in \widetilde{\mathcal{J}}, \xi \in \mathcal{H}_i(I)$. It follows immediately that when acting on $\mathcal{H}_0$, $L(\xi, \widetilde{I})$ equals $R(\xi, \widetilde{I})$ and is independent of $\arg_I$.

(d) (Density of fusion products) If $\mathcal{H}_i, \mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A})), \widetilde{I} \in \widetilde{\mathcal{J}}$, then the set $L(\mathcal{H}_i(I), \widetilde{I})\mathcal{H}_k$ spans a dense subspace of $\mathcal{H}_i \boxtimes \mathcal{H}_k$, and $R(\mathcal{H}_i(I), \widetilde{I})\mathcal{H}_k$ spans a dense subspace of $\mathcal{H}_k \boxtimes \mathcal{H}_i$.

(e) (Locality) For any $\mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, disjoint $\widetilde{I}, \widetilde{J} \in \widetilde{\mathcal{J}}$ with $\widetilde{I}$ anticlockwise to $\widetilde{J}$, and any $\xi \in \mathcal{H}_i(I), \eta \in \mathcal{H}_j(J)$, the following diagram (2.3) commutes adjointly.

$$\begin{tikzcd} \mathcal{H}_k \arrow[r, "R(\eta,\widetilde{J})"] \arrow[d, "L(\xi,\widetilde{I})"'] & \mathcal{H}_k \boxtimes \mathcal{H}_j \arrow[d, "L(\xi,\widetilde{I})"'] \\ \mathcal{H}_i \boxtimes \mathcal{H}_k \arrow[r, "R(\eta,\widetilde{J})"] & \mathcal{H}_i \boxtimes \mathcal{H}_k \boxtimes \mathcal{H}_j \end{tikzcd} \tag{2.3}$$

Here, the **adjoint commutativity** of diagram (2.3) means that $R(\eta, \widetilde{J})L(\xi, \widetilde{I}) = L(\xi, \widetilde{I})R(\eta, \widetilde{J})$ when acting on $\mathcal{H}_k$, and $R(\eta, \widetilde{J})L(\xi, \widetilde{I})^* = L(\xi, \widetilde{I})^* R(\eta, \widetilde{J})$ when acting on $\mathcal{H}_i \boxtimes \mathcal{H}_k$.

(f) (Braiding) There is a unitary linear map $\mathbb{B}_{i,j} : \mathcal{H}_i \boxtimes \mathcal{H}_j \to \mathcal{H}_j \boxtimes \mathcal{H}_i$ for any $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, such that

$$\mathbb{B}_{i,j} L(\xi, \widetilde{I})\eta = R(\xi, \widetilde{I})\eta \tag{2.4}$$

whenever $\widetilde{I} \in \widetilde{\mathcal{J}}, \xi \in \mathcal{H}_i(I), \eta \in \mathcal{H}_j$.

[3] For general (i.e., non-necessarily closed or vector-labeled) categorical extensions, this axiom is replaced by the neutrality and the Reeh-Schlieder property; see [Gui18] section 3.1.

Note that $\mathbb{B}_{i,j}$ is unique by the density of fusion products. Moreover, $\mathbb{B}_{i,j}$ commutes with the actions of $\mathcal{A}$, and is the same as the braid operator of $\mathrm{Rep}(\mathcal{A})$; see [Gui18] sections 3.2, 3.3. The existence of $\mathscr{E}$ is also proved in [Gui18] sections 3.2.[4]

Remark 2.2. We see that $L(\xi,\widetilde{I})$ and $R(\xi,\widetilde{I})$ can act on any object in $\mathrm{Rep}(\mathcal{A})$. If we want to emphasize that they are acting on a specific object $\mathcal{H}_k$, we write $L(\xi,\widetilde{I})|_{\mathcal{H}_k}$ and $R(\xi,\widetilde{I})|_{\mathcal{H}_k}$. It is noteworthy that for any $x\in\mathcal{A}(I)$,

$$L(x\Omega,\widetilde{I})|_{\mathcal{H}_k}=R(x\Omega,\widetilde{I})|_{\mathcal{H}_k}=\pi_{k,I}(x). \tag{2.5}$$

See the end of [Gui18] section 3.1. By the locality and the state-field correspondence, it is also easy to see that

$$L(\xi,\widetilde{I})\eta=R(\eta,\widetilde{J})\xi \tag{2.6}$$

whenever $\xi\in\mathcal{H}_i(I)$, $\eta\in\mathcal{H}_j(J)$, and $\widetilde{I}$ is anticlockwise to $\widetilde{J}$.

Another useful fact is that if $F\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i,\mathcal{H}_{i'})$, $G\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_j,\mathcal{H}_{j'})$, $\xi\in\mathcal{H}_i(I)$, and $\eta\in\mathcal{H}_j$, then

$$(F\otimes G)L(\xi,\widetilde{I})\eta=L(F\xi,\widetilde{I})G\eta,\qquad (G\otimes F)R(\xi,\widetilde{I})\eta=R(F\xi,\widetilde{I})G\eta. \tag{2.7}$$

This was proved in [Gui18] section 3.3 using Connes fusion, but it also follows directly from the axioms of categorical extensions. To prove the first equation, it suffices to assume that $\eta\in\mathcal{H}_j(J)$ where $\widetilde{J}$ is clockwise to $\widetilde{I}$. Then, by the functoriality and relation (2.6),

$$\begin{aligned}&(F\otimes G)L(\xi,\widetilde{I})\eta=(\mathbf{1}\otimes G)(F\otimes\mathbf{1})L(\xi,\widetilde{I})\eta=(\mathbf{1}\otimes G)(F\otimes\mathbf{1})R(\eta,\widetilde{J})\xi\\=&(\mathbf{1}\otimes G)R(\eta,\widetilde{J})F\xi=(\mathbf{1}\otimes G)L(F\xi,\widetilde{I})\eta=L(F\xi,\widetilde{I})G\eta.\end{aligned}$$

The second relation follows from the first one and (2.6).

We now prove some fusion relations for the L and R operators of $\mathscr{E}$.

Proposition 2.3. *Let $\mathcal{H}_i,\mathcal{H}_j,\mathcal{H}_k\in\mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, $\widetilde{I}\in\widetilde{\mathcal{J}}$, and $\xi\in\mathcal{H}_i(I)$.*
(a) If $\eta\in\mathcal{H}_j(I)$, then $L(\xi,\widetilde{I})\eta\in(\mathcal{H}_i\boxtimes\mathcal{H}_j)(I)$, $R(\xi,\widetilde{I})\eta\in(\mathcal{H}_j\boxtimes\mathcal{H}_i)(I)$, and

$$L(\xi,\widetilde{I})L(\eta,\widetilde{I})|_{\mathcal{H}_k}=L(L(\xi,\widetilde{I})\eta,\widetilde{I})|_{\mathcal{H}_k}, \tag{2.8}$$

$$R(\xi,\widetilde{I})R(\eta,\widetilde{I})|_{\mathcal{H}_k}=R(R(\xi,\widetilde{I})\eta,\widetilde{I})|_{\mathcal{H}_k}. \tag{2.9}$$

(b) If $\psi\in(\mathcal{H}_i\boxtimes H_j)(I)$ and $\phi\in(\mathcal{H}_j\boxtimes H_i)(I)$, then $L(\xi,\widetilde{I})^*\psi\in\mathcal{H}_j(I)$, $R(\xi,\widetilde{I})^*\phi\in\mathcal{H}_j(I)$, and

$$L(\xi,\widetilde{I})^*L(\psi,\widetilde{I})|_{\mathcal{H}_k}=L(L(\xi,\widetilde{I})^*\psi,\widetilde{I})|_{\mathcal{H}_k}, \tag{2.10}$$

$$R(\xi,\widetilde{I})^*R(\phi,\widetilde{I})|_{\mathcal{H}_k}=R(R(\xi,\widetilde{I})^*\phi,\widetilde{I})|_{\mathcal{H}_k}. \tag{2.11}$$

[4] In [Gui18] we assume $\mathcal{A}$ to be conformal covariant for the simplicity of discussions. Most results in that article (for example, the construction of Connes categorical extensions, the uniqueness of braided C^*-tensor categories, the uniqueness of vector-labeled closed categorical extensions, etc.) do not rely on the conformal covariance and are also true for Möbius covariant nets. The only exception is the conformal covariance of categorical extensions, which should be replaced by Möbius covariance when the $\mathcal{A}$ is only Möbius covariant; see theorem 2.5 and section B.

As a special case, we see that if $\xi \in \mathcal{H}_i(I)$ and $x \in \mathcal{A}(I)$, then $x\xi \in \mathcal{H}_i(I)$, and

$$L(x\xi, \widetilde{I}) = xL(\xi, \widetilde{I}), \qquad R(x\xi, \widetilde{I}) = xR(\xi, \widetilde{I}). \tag{2.12}$$

Proof. We only prove the first equation of part (b); the second one follows similarly. Part (a) follows either from a similar argument or from [Gui18] proposition 3.6. Since $L(\xi, \widetilde{I})^*\psi = L(\xi, \widetilde{I})^*L(\psi, \widetilde{I})\Omega$, we clearly have $L(\xi, \widetilde{I})^*\psi \in \mathcal{H}_j(I)$. Choose any $\chi \in \mathcal{H}_k(\widetilde{I}')$. Then, by the adjoint commutativity of left and right operators,

$$\begin{aligned} &L(\xi, \widetilde{I})^*L(\psi, \widetilde{I})\chi = L(\xi, \widetilde{I})^*L(\psi, \widetilde{I})R(\chi, \widetilde{I}')\Omega = R(\chi, \widetilde{I}')L(\xi, \widetilde{I})^*L(\psi, \widetilde{I})\Omega \\ =&R(\chi, \widetilde{I}')L(\xi, \widetilde{I})^*\psi = R(\chi, \widetilde{I}')L(L(\xi, \widetilde{I})^*\psi)\Omega = L(L(\xi, \widetilde{I})^*\psi)R(\chi, \widetilde{I}')\Omega \\ =&L(L(\xi, \widetilde{I})^*\psi)\chi. \end{aligned}$$

□

Next, we discuss the conformal covariance of $\mathscr{E}$. For any $\widetilde{I} = (I, \arg_I) \in \widetilde{\mathcal{J}}$ and $g \in \mathscr{G}_{\mathcal{A}}$, we have gI defined by the action of $\mathrm{Diff}^+(\mathbb{S}^1)$ on $\mathbb{S}^1$. We now set $g\widetilde{I} = (gI, \arg_{gI})$, where $\arg_{gI}$ is defined as follows. Choose any map $\gamma : [0,1] \to \mathscr{G}_{\mathcal{A}}$ satisfying $\gamma(0) = 1, \gamma(1) = g$ such that γ descends to a (continuous) path in $\mathscr{G}$. Then for any $z \in I$ there is a path $\gamma_z : [0,1] \to \mathbb{S}^1$ defined by $\gamma_z(t) = \gamma(t)z$. The argument $\arg_I(z)$ of z changes continuously along the path γ_z to an argument of gz, whose value is denoted by $\arg_{gI}(gz)$.

Theorem 2.4 ([Gui18] theorem 3.13)**.** *If $\mathcal{A}$ is conformal covariant, then $\mathscr{E} = (\mathcal{A}, \mathrm{Rep}(\mathcal{A}), \boxtimes, \mathcal{H})$ is **conformal covariant**, which means that for any $g \in \mathscr{G}_{\mathcal{A}}, \widetilde{I} \in \widetilde{\mathcal{J}}, \mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A})), \xi \in \mathcal{H}_i(I)$, there exists an element $g\xi g^{-1} \in \mathcal{H}_i(gI)$ such that*

$$L(g\xi g^{-1}, g\widetilde{I}) = gL(\xi, \widetilde{I})g^{-1}, \qquad R(g\xi g^{-1}, g\widetilde{I}) = gR(\xi, \widetilde{I})g^{-1} \tag{2.13}$$

when acting on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$.

It's clear that we have

$$g\xi g^{-1} = gL(\xi, \widetilde{I})g^{-1}\Omega = gR(\xi, \widetilde{I})g^{-1}\Omega. \tag{2.14}$$

In particular, when $g \in \widetilde{\mathrm{PSU}}(1,1)$ one has $g\xi g^{-1} = g\xi$ by the state-field correspondence and that $g\Omega = \Omega$. Therefore

$$L(g\xi, g\widetilde{I}) = gL(\xi, \widetilde{I})g^{-1}, \qquad R(g\xi, g\widetilde{I}) = gR(\xi, \widetilde{I})g^{-1} \qquad (\forall g \in \widetilde{\mathrm{PSU}}(1,1)). \tag{2.15}$$

The above property is called the **Möbius covariance** of $\mathscr{E}$.

In the remaining part of this paper, we will be interested in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, the C^*-tensor category of dualizable Möbius covariant representations of $\mathcal{A}$. (Recall that when $\mathcal{A}$ is conformal covariant, the conformal covariance and hence the Möbius covariance of dualizable representations are automatic.) Then $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ is a rigid C^*-tensor category.[5] Recall that a representation $\mathcal{H}_i$ of $\mathcal{A}$ is called dualizable if there exists an object $\mathcal{H}_{\bar{i}} \in$

[5]That $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ is closed under fusion product $\boxtimes$ is known to experts. In section B we give a proof of this fact.

$\mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$ (called **dual object**) and **evaluations** $\mathrm{ev}_{i,\bar{i}} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i \boxtimes \mathcal{H}_{\bar{i}}, \mathcal{H}_0)$ and $\mathrm{ev}_{\bar{i},i} \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar{i}} \boxtimes \mathcal{H}_i, \mathcal{H}_0)$ satisfying the conjugate equations

$$(\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{\bar{i},i}) = \mathbf{1}_i = (\mathbf{1}_i \otimes \mathrm{ev}_{\bar{i},i})(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_i), \tag{2.16}$$

$$(\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}})(\mathbf{1}_{\bar{i}} \otimes \mathrm{coev}_{i,\bar{i}}) = \mathbf{1}_{\bar{i}} = (\mathbf{1}_{\bar{i}} \otimes \mathrm{ev}_{i,\bar{i}})(\mathrm{coev}_{\bar{i},i} \otimes \mathbf{1}_{\bar{i}}), \tag{2.17}$$

where we set $\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}^*_{i,\bar{i}}, \mathrm{coev}_{\bar{i},i} = \mathrm{ev}^*_{\bar{i},i}$. Note that in each of (2.16) and (2.17), the first equation is equivalent to the second one by taking adjoint. Note also that $\mathrm{ev}_{\bar{i},i}$ is uniquely determined by $\mathrm{ev}_{i,\bar{i}}$ since $\mathrm{coev}_{\bar{i},i}$ is so. Moreover, one can choose the evaluations to be **standard**, which means that besides the conjugate equations, we also have

$$\mathrm{ev}_{i,\bar{i}}(F \otimes \mathbf{1}_{\bar{i}})\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}} \otimes F)\mathrm{coev}_{\bar{i},i} \tag{2.18}$$

for any $F \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_i)$. Then there exist positive numbers $d_i = d_{\bar{i}}$ satisfying $\mathrm{ev}_{i,\bar{i}}\mathrm{coev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i}\mathrm{coev}_{\bar{i},i} = d_i\mathbf{1}_0 = d_{\bar{i}}\mathbf{1}_0$, called the **quantum dimensions** of $\mathcal{H}_i$ and $\mathcal{H}_{\bar{i}}$. Standard evaluations exist and are unique up to unitaries, which means that if $u \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_i)$ is unitary, then $\widetilde{\mathrm{ev}}_{i,\bar{i}} := \mathrm{ev}_{i,\bar{i}}(u \otimes \mathbf{1}_{\bar{i}})$ and $\widetilde{\mathrm{ev}}_{\bar{i},i} := \mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}} \otimes u)$ are also standard, and any pair of standard evaluations arises in this way. We refer the reader to [LR97] or [Yam04] for more details. We remark that $\mathcal{H}_{\bar{i}}$ is also Möbius covariant by [GL96] theorem 2.11. Therefore $\mathcal{H}_{\bar{i}} \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ if $\mathcal{H}_i$ is so.

We can now restrict $\mathscr{E}$ to $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ to obtain a (closed, vector-labeled) **rigid categorical extension** $\mathscr{E}^{\mathrm{f}} = (\mathcal{A}, \mathrm{Rep}^{\mathrm{f}}(\mathcal{A}), \boxtimes, \mathcal{H})$, which is also conformal covariant when $\mathcal{A}$ is so. This means that when $\mathcal{A}$ is conformal covariant, definition 2.1 and theorem 2.4 hold verbatim for $\mathscr{E}^{\mathrm{f}}$, except that $\mathrm{Rep}(\mathcal{A})$ should be replaced by $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. When $\mathcal{A}$ is only Möbius covariant, these are also true except that theorem 2.4 should be replaced by Möbius covariance. Note first of all that for any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, the strongly continuous unitary representations of $\widetilde{\mathrm{PSU}}(1,1)$ making $\mathcal{H}_i$ Möbius covariant are unique by [GL96] proposition 2.2. It is easy to see that any morphism in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ intertwines the actions of $\widetilde{\mathrm{PSU}}(1,1)$ (see lemma B.1). The following is proved in section B.

Theorem 2.5. *$\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ is closed under $\boxtimes$. Moreover, $\mathscr{E} = (\mathcal{A}, \mathrm{Rep}^{\mathrm{f}}(\mathcal{A}), \boxtimes, \mathcal{H})$ is **Möbius covariant**, which means that for any $g \in \widetilde{\mathrm{PSU}}(1,1), \widetilde{I} \in \widetilde{\mathcal{J}}, \mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})), \xi \in \mathcal{H}_i(I)$, we have*

$$L(g\xi, g\widetilde{I}) = gL(\xi, \widetilde{I})g^{-1}, \qquad R(g\xi, g\widetilde{I}) = gR(\xi, \widetilde{I})g^{-1} \tag{2.19}$$

when acting on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$.

3 Non-local extensions

Q-systems were introduced by R.Longo [Lon94] and are powerful tools for studying local and non-local extensions of conformal and Möbius covariant nets [LR95, KL04, BKLR15]. In this section, we give a construction of non-local extensions by Q-systems under the framework of categorical extensions. We shall work with a general C^*-Frobenius algebra Q in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, and construct a non-local extension $\mathcal{B}$ of $\mathcal{A}$ via Q.

Recall that $Q = (\mathcal{H}_a, \mu, \iota)$ is called a C^***-Frobenius algebra** in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ if $\mathcal{H}_a \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})), \mu \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_a \boxtimes \mathcal{H}_a, \mathcal{H}_a), \iota \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_0, \mathcal{H}_a)$, and the following conditions are satisfied:

- (Unit) $\mu(\iota \otimes \mathbf{1}_a) = \mathbf{1}_a = \mu(\mathbf{1}_a \otimes \iota)$.
- (Associativity) $\mu(\mu \otimes \mathbf{1}_a) = \mu(\mathbf{1}_a \otimes \mu)$.
- (Frobenius relation) $(\mathbf{1}_a \otimes \mu)(\mu^* \otimes \mathbf{1}_a) = \mu^* \mu$.

Note that the associativity and the Frobenius relation are equivalent to the *adjoint commutativity* of the following diagram

$$\begin{tikzcd} \mathcal{H}_a \boxtimes \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "\mathbf{1}_a \otimes \mu"] \arrow[d, "\mu \otimes \mathbf{1}_a"'] & \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[d, "\mu"'] \\ \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "\mu"] & \mathcal{H}_a \end{tikzcd} . \tag{3.1}$$

Let us fix a C^*-Frobenius algebra Q. For any $\xi \in \mathcal{H}_a(I)$, we define bounded linear operators on $\mathcal{H}_a$:

$$A(\xi, \widetilde{I}) = \mu \cdot L(\xi, \widetilde{I})|_{\mathcal{H}_a}, \qquad B(\xi, \widetilde{I}) = \mu \cdot R(\xi, \widetilde{I})|_{\mathcal{H}_a}.$$

Definition 3.1. For any $\widetilde{I} \in \widetilde{\mathcal{J}}$, $\mathcal{B}(\widetilde{I})$ (resp. $\mathcal{B}'(\widetilde{I})$) is defined to be the set of all $A(\xi, \widetilde{I})$ (resp. $B(\xi, \widetilde{I})$) where $\xi \in \mathcal{H}_a(I)$.

We shall show that $\mathcal{B} : \widetilde{I} \in \widetilde{\mathcal{J}} \mapsto \mathcal{B}(\widetilde{I})$ and $\mathcal{B}' : \widetilde{I} \in \widetilde{\mathcal{J}} \mapsto \mathcal{B}'(\widetilde{I})$ are two nets of von Neumann algebras extending $\mathcal{A}$, and that the Haag duality $\mathcal{B}(\widetilde{I})' = \mathcal{B}'(\widetilde{I}')$ is satisfied. First, notice that $\mathcal{A}(I)$ is also acting on $\mathcal{H}_a$. We also denote by $\mathcal{A}(I)$ the image of $\mathcal{A}(I)$ under $\pi_{a,I}$. The following lemma shows that $\mathcal{B}$ and $\mathcal{B}'$ are extensions of $\mathcal{A}$.

Proposition 3.2. *We have $\mathcal{A}(I) \subset \mathcal{B}(\widetilde{I})$ and $\mathcal{A}(I) \subset \mathcal{B}'(\widetilde{I})$.*

Proof. Choose any $x \in \mathcal{A}(I)$. Then one has

$$A(\iota x \Omega, \widetilde{I}) = \pi_{a,I}(x) = B(\iota x \Omega, \widetilde{I}). \tag{3.2}$$

Indeed, for any $\eta \in \mathcal{H}_a$,

$$A(\iota x \Omega, \widetilde{I})\eta = \mu \cdot L(\iota x \Omega, \widetilde{I})\eta = \mu(\iota \otimes \mathbf{1}_a)L(x\Omega, \widetilde{I})\eta = L(x\Omega, \widetilde{I})\eta = x\eta,$$

where we have used (2.7), the unit property, and (2.5). The other relation is proved in a similar manner. □

Proposition 3.3. *If $\widetilde{I}$ is anticlockwise to $\widetilde{J}$, then for any $\xi \in \mathcal{H}_a(I)$ and $\eta \in \mathcal{H}_a(J)$, $A(\xi, \widetilde{I})$* ***commutes adjointly*** *with $B(\eta, \widetilde{J})$, which means that $A(\xi, \widetilde{I})B(\eta, \widetilde{J}) = B(\eta, \widetilde{J})A(\xi, \widetilde{I})$ and $A(\xi, \widetilde{I})^* B(\eta, \widetilde{J}) = B(\eta, \widetilde{J})A(\xi, \widetilde{I})^*$.*

Proof. Consider the following matrix of diagrams.

$$\begin{tikzcd}
\mathcal{H}_a \arrow[r, "R(\eta,\widetilde{J})"] \arrow[d, "L(\xi,\widetilde{I})"'] & \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "\mu"] \arrow[d, "L(\xi,\widetilde{I})"'] & \mathcal{H}_a \arrow[d, "L(\xi,\widetilde{I})"'] \\
\mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "R(\eta,\widetilde{J})"] \arrow[d, "\mu"'] & \mathcal{H}_a \boxtimes \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "\mathbf{1}_a\otimes\mu"] \arrow[d, "\mu\otimes\mathbf{1}_a"'] & \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[d, "\mu"'] \\
\mathcal{H}_a \arrow[r, "R(\eta,\widetilde{J})"] & \mathcal{H}_a \boxtimes \mathcal{H}_a \arrow[r, "\mu"] & \mathcal{H}_a
\end{tikzcd} \tag{3.3}$$

The $(1,1)$-diagram commutes adjointly by the locality of $\mathscr{E}^{\mathrm{f}}$. The $(2,1)$- and $(1,2)$-diagrams commute adjointly by the functoriality of $\mathscr{E}^{\mathrm{f}}$. The $(2,2)$-diagram is just (3.1), which we know is commuting adjointly by the associativity and the Frobenius property of Q. Thus the largest diagram commutes adjointly, which is exactly the adjoint commutativity of $A(\xi,\widetilde{I})$ and $B(\eta,\widetilde{J})$. □

Definition 3.4. If $\mathfrak{S}$ is a set of bounded linear operators on a Hilbert space $\mathcal{H}$, its **commutant** $\mathfrak{S}'$ is defined to be the set of bounded linear operators on $\mathcal{H}$ which commute adjointly with the operators in $\mathfrak{S}$. Then $\mathfrak{S}'$ is a von Neumann algebra. The double commutant $\mathfrak{S}''$ is called the von Neumann algebra generated by $\mathfrak{S}$.

Proposition 3.5. *For any $\widetilde{I}\in\widetilde{\mathcal{J}}$ we have $\mathcal{B}(\widetilde{I})'=\mathcal{B}'(\widetilde{I}')$ and $\mathcal{B}'(\widetilde{I}')'=\mathcal{B}(\widetilde{I})$. As a consequence, $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I}')$ are von Neumann algebras acting on $\mathcal{H}_a$.*

We remind the reader that $\widetilde{I}'$ is the clockwise complement of $\widetilde{I}$.

Proof. We only prove $\mathcal{B}(\widetilde{I})'=\mathcal{B}'(\widetilde{I}')$ as the other relation can be proved in a similar way. Note that by the previous proposition, we have $\mathcal{B}(\widetilde{I})'\supset\mathcal{B}'(\widetilde{I}')$. To prove $\mathcal{B}(\widetilde{I})'\subset\mathcal{B}'(\widetilde{I}')$, we choose any $Y\in\mathcal{B}(\widetilde{I})'$ and show that $Y\in\mathcal{B}'(\widetilde{I}')$.

Set $\eta=Y\iota\Omega$. Since $\iota\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_0,\mathcal{H}_a)$ and $Y\in\mathrm{Hom}_{\mathcal{A}(I)}(\mathcal{H}_a,\mathcal{H}_a)$ (by proposition 3.2), one has $Y\iota\in\mathrm{Hom}_{\mathcal{A}(I)}(\mathcal{H}_0,\mathcal{H}_a)$. Therefore $\eta\in\mathcal{H}_a(I')$. Choose any $\xi\in\mathcal{H}_a(I)$. Then, by proposition 3.6, we have

$$\begin{aligned}
&Y\xi=YA(\xi,\widetilde{I})\iota\Omega=A(\xi,\widetilde{I})Y\iota\Omega=A(\xi,\widetilde{I})\eta=A(\xi,\widetilde{I})B(\eta,\widetilde{I}')\iota\Omega\\
=&B(\eta,\widetilde{I}')A(\xi,\widetilde{I})\iota\Omega=B(\eta,\widetilde{I}')\xi.
\end{aligned}$$

This shows $Y=B(\eta,\widetilde{I}')$ and hence that $Y\in\mathcal{B}'(\widetilde{I}')$. □

Proposition 3.6. *For any $\widetilde{I}\in\widetilde{\mathcal{J}}$ and $\xi\in\mathcal{H}_i(I)$,*

$$A(\xi,\widetilde{I})\iota\Omega=\xi=B(\xi,\widetilde{I})\iota\Omega. \tag{3.4}$$

Proof. We shall prove the following more general relations:

$$A(\xi,\widetilde{I})\iota=L(\xi,\widetilde{I})|_{\mathcal{H}_0},\qquad B(\xi,\widetilde{I})\iota=R(\xi,\widetilde{I})|_{\mathcal{H}_0}. \tag{3.5}$$

Again, we only prove the first one as the second one can be argued similarly. We compute that

$$A(\xi,\widetilde{I})\iota=\mu\cdot L(\xi,\widetilde{I})\cdot\iota=\mu(\mathbf{1}_a\otimes\iota)L(\xi,\widetilde{I})|_{\mathcal{H}_0}=L(\xi,\widetilde{I})|_{\mathcal{H}_0}$$

where we have used the functoriality of $\mathscr{E}^{\mathrm{f}}$ and the unit property of Q. □

Finally, if $\mathcal{A}$ is conformal covariant, we notice that for any $g \in \mathscr{G}_{\mathcal{A}}$, we have $g\mathcal{B}(\widetilde{I})g^{-1} = \mathcal{B}(g\widetilde{I})$ and $g\mathcal{B}'(\widetilde{I})g^{-1} = \mathcal{B}'(g\widetilde{I})$. Indeed, we notice that the actions of g commute with μ (see the discussion after (1.1)). Therefore the conformal covariance of $\mathscr{E}^{\mathrm{f}}$ implies the two equations. If $\mathcal{A}$ is only Möbius covariant, we also have similar relations for $g \in \widetilde{\mathrm{PSU}}(1,1)$. We summarize the above results as follows. (Note that (3.6) follows from lemma 3.6.)

Theorem 3.7. *$\mathcal{B} : \widetilde{I} \in \widetilde{\mathcal{J}} \mapsto \mathcal{B}(\widetilde{I})$ and $\mathcal{B}' : \widetilde{I} \in \widetilde{\mathcal{J}} \mapsto \mathcal{B}'(\widetilde{I})$ are families of von Neumann algebras satisfying the following properties for any $\widetilde{I}, \widetilde{J} \in \widetilde{\mathcal{J}}$.*
(a) (Extension property) $\mathcal{A}(I) \subset \mathcal{B}(\widetilde{I}) \cap \mathcal{B}'(\widetilde{I})$.
(b) (Isotony) If $\widetilde{I} \subset \widetilde{J}$, then $\mathcal{B}(\widetilde{I}) \subset \mathcal{B}(\widetilde{J})$ and $\mathcal{B}'(\widetilde{I}) \subset \mathcal{B}'(\widetilde{J})$.
(c) (Reeh-Schlieder property) $\mathcal{B}(\widetilde{I})\iota\Omega$ and $\mathcal{B}'(\widetilde{I})\iota\Omega$ are dense subspaces of $\mathcal{H}_a$. Indeed, we have

$$\mathcal{B}(\widetilde{I})\iota\Omega = \mathcal{B}'(\widetilde{I})\iota\Omega = \mathcal{H}_a(I). \tag{3.6}$$

(d) (Haag duality) $\mathcal{B}(\widetilde{I})' = \mathcal{B}'(\widetilde{I}')$.
(e) (Möbius/conformal covariance) For any $g \in \widetilde{\mathrm{PSU}}(1,1)$ one has

$$g\mathcal{B}(\widetilde{I})g^{-1} = \mathcal{B}(g\widetilde{I}), \qquad g\mathcal{B}'(\widetilde{I})g^{-1} = \mathcal{B}'(g\widetilde{I}). \tag{3.7}$$

When $\mathcal{A}$ is conformal covariance, the above relations are also true when $g \in \mathscr{G}_{\mathcal{A}}$.

We say that $\mathcal{B}$ and $\mathcal{B}'$ are the **non-local extensions of** $\mathcal{A}$ associated to the C^*-Frobenius algebra Q, and that $\mathcal{B}'$ is the **clockwise dual net** of $\mathcal{B}$.

Given $Q = (\mathcal{H}_a, \mu, \iota)$ and the associated non-local extensions $\mathcal{B}, \mathcal{B}'$, we define $Q' = (\mathcal{H}_a, \mu', \iota)$, where $\mu' = \mu\mathbb{B}_{a,a}$.

Proposition 3.8. *The non-local extensions of $\mathcal{A}$ associated to the C^*-Frobenius algebra Q' are $\mathcal{B}'$ and $\mathcal{B}''$, where $\mathcal{B}''$ is the clockwise commutant of $\mathcal{B}'$.*

Proof. Operators in $\mathcal{B}'(\widetilde{I})$ are written as $\mu' L(\xi, \widetilde{I})|_{\mathcal{H}_a}$ where $\xi \in \mathcal{H}_a(I)$. By the braiding axiom of $\mathscr{E}^{\mathrm{f}}$, $\mu' L(\xi, \widetilde{I})|_{\mathcal{H}_a} = \mu\mathbb{B}L(\xi, \widetilde{I})|_{\mathcal{H}_a} = \mu R(\xi, \widetilde{I})|_{\mathcal{H}_a}$ which is inside $\mathcal{B}'(\widetilde{I})$. □

We describe the relation between $\mathcal{B}$ and its clockwise double dual net $\mathcal{B}''$. Let $\widetilde{I}''$ be the clockwise complement of $\widetilde{I}'$. Then we have $I'' = I$ and $\arg_{I''} = \arg_I - 2\pi$.

Proposition 3.9. *$\mathcal{B}''(\widetilde{I}'') = \mathcal{B}(\widetilde{I})$ for any $\widetilde{I} \in \widetilde{\mathcal{J}}$.*

Proof. We have $\mathcal{B}(\widetilde{I}) = \mathcal{B}'(\widetilde{I}')'$ and, similarly, $\mathcal{B}'(\widetilde{I}') = \mathcal{B}''(\widetilde{I}'')'$. □

4 Categorical modular operators and conjugations

We first recall the Tomita-Takesaki theory for von Neumann algebras associated with cyclic seperating vectors; details can be found in [Tak02] or [Tak70]. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and assume that $\Omega \in \mathcal{H}$ is a cyclic and separating vector of $\mathcal{M}$. (We do not require $\|\Omega\| = 1$.) One defines unbounded antilinear operators $S:\mathcal{M}\Omega\to\mathcal{M}\Omega$ and $F:\mathcal{M}'\Omega\to\mathcal{M}'\Omega$ such that for any $x\in\mathcal{M}, y\in\mathcal{M}'$,

$$S x\Omega = x^*\Omega, \qquad F y\Omega = y^*\Omega.$$

S and F are indeed preclosed operators, whose closures are also denoted by the same symbols S and F respectively. Moreover, $S^* = F$. Let $S = \mathfrak{J}\Delta^{\frac{1}{2}}$ be the polar decomposition of S, where the positive operator $\Delta = S^*S$ is called the modular operator, and the antiunitary map $\mathfrak{J}$ is called the modular conjugation. We have $\Delta^{\mathbf{i}t}\Omega = \mathfrak{J}\Omega = \Omega, S = S^{-1}, \mathfrak{J}^2 = \mathbf{1}, S = \mathfrak{J}\Delta^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}\mathfrak{J}$. Let $\mathbf{i} = \sqrt{-1}$. For any $t\in\mathbb{R}$, we have

$$\Delta^{\mathbf{i}t}\mathcal{M}\Delta^{-\mathbf{i}t} = \mathcal{M}, \qquad \mathfrak{J}\mathcal{M}\mathfrak{J} = \mathcal{M}'.$$

Tomita-takesaki theory can be applied to non-local extensions without difficulty. This will be used to derive a categorical Tomita-takesaki theory in this section. We first choose a **system of dual objects and standard evaluations** for $\mathscr{E}^{\mathrm{f}}$. This means that for each $\mathcal{H}_i\in\mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, we choose a dual object $\mathcal{H}_{\bar i}$ and *standard* evaluations $\mathrm{ev}_{i,\bar i}, \mathrm{ev}_{\bar i,i}$ satisfying (2.16) and (2.17). For the vacuum representation $\mathcal{H}_0$, its dual object is fixed to be $\mathcal{H}_0$, and the standard evaluations are chosen in an obvious way. The dual object of $\mathcal{H}_{\bar i}$ is fixed to be $\mathcal{H}_i$. Namely, we set $\mathcal{H}_{\bar{\bar i}} = \mathcal{H}_i$ and $\mathrm{ev}_{\bar i,\bar{\bar i}} = \mathrm{ev}_{\bar i,i}, \mathrm{ev}_{\bar{\bar i},\bar i} = \mathrm{ev}_{i,\bar i}$. (This is necessary for the categorical S and F operators to be involutions.) Finally, if the dual objects and the standard evaluations of $\mathcal{H}_i,\mathcal{H}_j\in\mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ are chosen, then the dual object of $\mathcal{H}_i\boxtimes\mathcal{H}_j$ is assumed to be $\mathcal{H}_{\bar j}\boxtimes\mathcal{H}_{\bar i}$, and the standard evaluations are

$$\mathrm{ev}_{i\boxtimes j,\bar j\boxtimes\bar i} = \mathrm{ev}_{i,\bar i}(\mathbf{1}_i\otimes\mathrm{ev}_{j,\bar j}\otimes\mathbf{1}_{\bar i}), \qquad \mathrm{ev}_{\bar j\boxtimes\bar i,i\boxtimes j} = \mathrm{ev}_{\bar j,j}(\mathbf{1}_{\bar j}\otimes\mathrm{ev}_{\bar i,i}\otimes\mathbf{1}_j). \tag{4.1}$$

This last condition will be used in the PCT theorem of $\mathscr{E}^{\mathrm{f}}$ (see (5.8)).

Recall that we set $\mathrm{coev}_{i,\bar i} = \mathrm{ev}^*_{i,\bar i}$ and $\mathrm{coev}_{\bar i,i} = \mathrm{ev}^*_{\bar i,i}$. Then for any $\widetilde I\in\widetilde{\mathcal{J}}$ we define unbounded linear operators $S_{\widetilde I}, F_{\widetilde I}:\mathcal{H}_i\to\mathcal{H}_{\bar i}$ with domains $\mathcal{H}_i(I)$ such that for any $\xi\in\mathcal{H}_i(I)$,

$$S_{\widetilde I}\xi = L(\xi,\widetilde I)^*\mathrm{coev}_{i,\bar i}\Omega, \qquad F_{\widetilde I}\xi = R(\xi,\widetilde I)^*\mathrm{coev}_{\bar i,i}\Omega \tag{4.2}$$

Note that $S_{\widetilde I}$ and $F_{\widetilde I}$ depend not only on $\widetilde I$ but also on the choice of dual objects and evaluations. We also understand $S_{\widetilde I}$ and $F_{\widetilde I}$ as categorical operators, which means that they can act on any object of $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. We write $S_{\widetilde I}$ as $S_{\widetilde I}|_{\mathcal{H}_i}$ (and similarly $F_{\widetilde I}$ as $F_{\widetilde I}|_{\mathcal{H}_i}$) if we want to emphasize that $S_{\widetilde I}$ is acting on the object $\mathcal{H}_i$. We first discuss the dependence of $S_{\widetilde I}$ and $F_{\widetilde I}$ on duals and evaluations.

Proposition 4.1. *For each $\mathcal{H}_i\in\mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, we chose another dual object $\mathcal{H}_{\hat i}$ and standard evaluations $\mathrm{ev}'_{i,\hat i}, \mathrm{ev}'_{\hat i,i}$, and define the corresponding S and F operators $S'_{\widetilde I}, F'_{\widetilde I}$. Then there exists a unitary $u_i\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar i},\mathcal{H}_{\hat i})$ such that*

$$S'_{\widetilde I} = u_i S_{\widetilde I}, \qquad F'_{\widetilde I} = u_i F_{\widetilde I}.$$

Proof. By the uniqueness up to unitaries of standard evaluations, there exists a unitary $u_i\in\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar i},\mathcal{H}_{\hat i})$ such that $\mathrm{ev}'_{i,\hat i} = \mathrm{ev}_{i,\bar i}(\mathbf{1}_i\otimes u_i^{-1})$ and $\mathrm{ev}'_{\hat i,i} = \mathrm{ev}_{\bar i,i}(u_i^{-1}\otimes\mathbf{1}_i)$. Using the functoriality of $\mathscr{E}^{\mathrm{f}}$ one obtains the desired equations. □

Proposition 4.2. *If $\xi \in \mathcal{H}_i(I)$, then $S_{\widetilde{I}}\xi \in \mathcal{H}_{\bar{i}}(I), F_{\widetilde{I}}\xi \in \mathcal{H}_{\bar{i}}(I)$. Moreover, for any $\mathcal{H}_k \in$ $\mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$ we have*

$$L(S_{\widetilde{I}}\xi, \widetilde{I})|_{\mathcal{H}_k} = L(\xi, \widetilde{I})^*(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_k), \tag{4.3}$$

$$R(F_{\widetilde{I}}\xi, \widetilde{I})|_{\mathcal{H}_k} = R(\xi, \widetilde{I})^*(\mathbf{1}_k \otimes \mathrm{coev}_{\bar{i},i}). \tag{4.4}$$

Note that in the above two equations, $L(\xi, \widetilde{I})^*$ is a bounded linear operator from $\mathcal{H}_i \boxtimes \mathcal{H}_{\bar{i}} \boxtimes \mathcal{H}_k$ to $\mathcal{H}_{\bar{i}} \boxtimes \mathcal{H}_k$, and $R(\xi, \widetilde{I})^*$ from $\mathcal{H}_k \boxtimes \mathcal{H}_{\bar{i}} \boxtimes \mathcal{H}_i$ to $\mathcal{H}_k \boxtimes \mathcal{H}_{\bar{i}}$.

Proof. That $S_{\widetilde{I}}\xi$ and $F_{\widetilde{I}}\xi$ are inside $\mathcal{H}_i(I)$ is obvious from the definition of $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$. For any $\chi \in \mathcal{H}_k$, we use proposition 2.3 to compute that

$$\begin{aligned} &L(S_{\widetilde{I}}\xi, \widetilde{I})\chi = L\big(L(\xi, \widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega, \widetilde{I}\big)\chi = L(\xi, \widetilde{I})^* L\big(\mathrm{coev}_{i,\bar{i}}\Omega, \widetilde{I}\big)\chi \\ \overset{(2.7)}{=\!=}\, &L(\xi, \widetilde{I})^*(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_k)L\big(\Omega, \widetilde{I}\big)\chi \overset{(2.5)}{=\!=} L(\xi, \widetilde{I})^*(\mathrm{coev}_{i,\bar{i}} \otimes \mathbf{1}_k)\chi. \end{aligned}$$

The other equation is proved similarly. □

We now show that $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are involutions. This fact is closely related to the conjugate equations (2.16) and (2.17).

Proposition 4.3. *For any $\xi \in \mathcal{H}_i(I)$, we have $S_{\widetilde{I}}^2\xi = F_{\widetilde{I}}^2\xi = \xi$.*

Notice that the actions of $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ on $\mathcal{H}_{\bar{i}}$ are defined respectively by $\mathrm{coev}_{\bar{i},i}$ and $\mathrm{coev}_{i,\bar{i}}$, which are the same coevaluations as those in (4.2), (4.3), (4.4).

Proof. We compute

$$\begin{aligned} S_{\widetilde{I}}^2\xi \overset{(4.2)}{=\!=} &L(S_{\widetilde{I}}\xi, \widetilde{I})^*\mathrm{coev}_{\bar{i},i}\Omega \overset{(4.3)}{=\!=} (\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)L(\xi, \widetilde{I})\mathrm{coev}_{\bar{i},i}\Omega \\ =&(\mathrm{ev}_{i,\bar{i}} \otimes \mathbf{1}_i)(\mathbf{1}_i \otimes \mathrm{coev}_{\bar{i},i})L(\xi, \widetilde{I})\Omega \overset{(2.16)}{=\!=} L(\xi, \widetilde{I})\Omega = \xi. \end{aligned}$$

Similarly, we may use (2.17) to show $F_{\widetilde{I}}^2\xi = \xi$. □

The above two propositions imply immediately the following:

Corollary 4.4. *For any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})), \mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A})), \xi \in \mathcal{H}_i(I), \phi \in \mathcal{H}_i \boxtimes \mathcal{H}_k, \psi \in \mathcal{H}_k \boxtimes \mathcal{H}_i$,*

$$L(\xi, \widetilde{I})^*\phi = (\mathrm{ev}_{\bar{i},i} \otimes \mathbf{1}_k)L(S_{\widetilde{I}}\xi, \widetilde{I})\phi, \tag{4.5}$$

$$R(\xi, \widetilde{I})^*\psi = (\mathbf{1}_k \otimes \mathrm{ev}_{i,\bar{i}})R(F_{\widetilde{I}}\xi, \widetilde{I})\psi. \tag{4.6}$$

Next, we relate $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$. To begin with, recall that we can define the **twist operator** ϑ_i on any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ to be the unique operator in $\mathrm{End}_{\mathcal{A}}(\mathcal{H}_i)$ such that

$$\mathrm{ev}_{i,\bar{i}} = \mathrm{ev}_{\bar{i},i}ß_{i,\bar{i}}(\vartheta_i \otimes \mathbf{1}_{\bar{i}}), \tag{4.7}$$

where we recall that ß is the braid operator of $\mathrm{Rep}(\mathcal{A})$, and the evaluations are assumed to be standard. Then, by [Müg00], ϑ_i is a unitary operator independent of the choice of standard evaluations, and that the actions of ϑ on all $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ (as ϑ_i) define a ribbon structure compatible with the braided C^*-tensor structure of $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. (Indeed, ϑ_i is unitary if and only if the evaluations are standard.) This means, among other things, that ϑ commutes with homomorphisms, that for any $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$,

$$\text{ß}_{i,j}^2 = \vartheta_{i\boxtimes j}(\vartheta_i^{-1} \otimes \vartheta_j^{-1}), \tag{4.8}$$

$$\mathrm{ev}_{i,\bar{i}}(\vartheta_i \otimes \mathbf{1}_{\bar{i}}) = \mathrm{ev}_{i,\bar{i}}(\mathbf{1}_i \otimes \vartheta_{\bar{i}}), \qquad \mathrm{ev}_{\bar{i},i}(\vartheta_{\bar{i}} \otimes \mathbf{1}_i) = \mathrm{ev}_{\bar{i},i}(\mathbf{1}_{\bar{i}} \otimes \vartheta_i). \tag{4.9}$$

and (hence) that

$$(\mathbf{1}_i \otimes \vartheta_{\bar{i}})\mathrm{coev}_{i,\bar{i}} = \text{ß}_{\bar{i},\bar{i}}^{-1}\mathrm{coev}_{\bar{i},i}. \tag{4.10}$$

Proposition 4.5. *We have*

$$F_{\widetilde{I}} = \vartheta S_{\widetilde{I}}. \tag{4.11}$$

More precisely, for any $\xi \in \mathcal{H}_i(I)$ we have $F_{\widetilde{I}}\xi = \vartheta_{\bar{i}} S_{\widetilde{I}}\xi$.

Proof. By the braiding axiom of $\mathscr{E}^{\mathrm{f}}$ we have $R(\xi, \widetilde{I})|_{\mathcal{H}_{\bar{i}}} = \text{ß}_{i,\bar{i}} L(\xi, \widetilde{I})|_{\mathcal{H}_{\bar{i}}}$. Therefore

$$\begin{aligned} F_{\widetilde{I}}\xi &= R(\xi, \widetilde{I})^*\mathrm{coev}_{\bar{i},i}\Omega = L(\xi, \widetilde{I})^*\text{ß}_{i,\bar{i}}^{-1}\mathrm{coev}_{\bar{i},i}\Omega \xlongequal{(4.10)} L(\xi, \widetilde{I})^*(\mathbf{1}_i \otimes \vartheta_{\bar{i}})\mathrm{coev}_{i,\bar{i}}\Omega \\ &= \vartheta_{\bar{i}} L(\xi, \widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega = \vartheta_{\bar{i}} S_{\widetilde{I}}\xi. \end{aligned}$$

□

We will see later that $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are preclosed. Therefore $S_{\widetilde{I}}^* S_{\widetilde{I}} = F_{\widetilde{I}}^* F_{\widetilde{I}}$, which will be denoted by $\Delta_{\widetilde{I}}$. This fact is crucial for proving the geometric modular theorems.

We now show the Möbius covariance of $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$.

Proposition 4.6. *For any $g \in \widetilde{\mathrm{PSU}}(1,1)$,*

$$gS_{\widetilde{I}}g^{-1} = S_{g\widetilde{I}}, \qquad gF_{\widetilde{I}}g^{-1} = F_{g\widetilde{I}}. \tag{4.12}$$

Proof. The domain of $S_{g\widetilde{I}}|_{\mathcal{H}_i}$ is $\mathcal{H}_i(gI)$, whereas the domain of $gS_{\widetilde{I}}g^{-1}|_{\mathcal{H}_i}$ is $g\mathcal{H}_i(I)$. From (1.2) one clearly has $g\mathcal{H}_i(I) = \mathcal{H}_i(gI)$. Now choose any $\xi \in \mathcal{H}_i(gI)$. Then $g^{-1}\xi \in \mathcal{H}_i(I)$, and $L(\xi, g\widetilde{I}) = gL(g^{-1}\xi, \widetilde{I})g^{-1}$ by the Möbius covariance of $\mathscr{E}^{\mathrm{f}}$. Notice that Ω is $\widetilde{\mathrm{PSU}}(1,1)$-invariant. Therefore

$$\begin{aligned} S_{g\widetilde{I}}\xi &= L(\xi, g\widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega = gL(g^{-1}\xi, \widetilde{I})^*g^{-1}\mathrm{coev}_{i,\bar{i}}\Omega \\ &= gL(g^{-1}\xi, \widetilde{I})^*\mathrm{coev}_{i,\bar{i}}g^{-1}\Omega = gL(g^{-1}\xi, \widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega = gS_{\widetilde{I}}g^{-1}\xi. \end{aligned}$$

□

To prove further properties of $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$, we have to relate them with the S and F operators of non-local extensions of $\mathcal{A}$. First of all, if $Q = (\mathcal{H}_a, \mu, \iota)$ is a C^*-Frobenius algebra in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, then $\mathcal{H}_a$ is self dual, and $\mathrm{ev}_{a,a} := \iota^*\mu$ defines an evaluation satisfying the conjugate equation

$$(\mathrm{ev}_{a,a} \otimes \mathbf{1}_a)(\mathbf{1}_a \otimes \mathrm{coev}_{a,a}) = \mathbf{1}_a = (\mathbf{1}_a \otimes \mathrm{ev}_{a,a})(\mathrm{coev}_{a,a} \otimes \mathbf{1}_a).$$

We say that Q is **standard** if $\mathrm{ev}_{a,a}$ is a standard evaluation.

In the remaining part of this section, we shall always assume that Q is a standard C^*-Frobenius algebra in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. Note that we have already chosen a dual object and a pair of standard evaluations for each object in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. In particular, we have fixed for the object $\mathcal{H}_a$ a dual $\mathcal{H}_{\overline{a}}$ and standard evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$. By the uniqueness up to unitaries of standard evaluations, we have a unitary $\epsilon \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_a, \mathcal{H}_{\overline{a}})$ satisfying

$$\mathrm{ev}_{\overline{a},a}(\epsilon \otimes \mathbf{1}_a) = \iota^* \mu = \mathrm{ev}_{a,\overline{a}}(\mathbf{1}_a \otimes \epsilon), \tag{4.13}$$

called the **reflection operator** of Q (with respect to the dual object $\mathcal{H}_a$ and the standard evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$).

Let $\mathcal{B}$ and $\mathcal{B}'$ be the pair of non-local extensions of $\mathcal{A}$ obtained by Q. Then for each $\widetilde{I} \in \widetilde{\mathcal{J}}$, $\iota\Omega$ is a cyclic separating vector for $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$.

Proposition 4.7. *Assume that Q is standard. Then $\epsilon^{-1}S_{\widetilde{I}}|_{\mathcal{H}_a}$ and $\epsilon^{-1}F_{\widetilde{I}}|_{\mathcal{H}_a}$ are respectively the (preclosed) S operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$ with respect to $\iota\Omega$. More precisely, for any $X \in \mathcal{B}(\widetilde{I})$ and $Y \in \mathcal{B}'(\widetilde{I})$,*

$$\epsilon^{-1}S_{\widetilde{I}} \cdot X\iota\Omega = X^*\iota\Omega, \qquad \epsilon^{-1}F_{\widetilde{I}} \cdot Y\iota\Omega = Y^*\iota\Omega. \tag{4.14}$$

Proof. We prove the second equation. Choose any $\xi \in \mathcal{H}_a(I)$. We want to show that $F_{\widetilde{I}}B(\xi,\widetilde{I})\iota\Omega = \epsilon B(\xi,\widetilde{I})^*\iota\Omega$. By proposition 3.6 we have $F_{\widetilde{I}}B(\xi,\widetilde{I})\iota\Omega = F_{\widetilde{I}}\xi$. On the other hand,

$$\begin{aligned}
&\epsilon B(\xi,\widetilde{I})^*\iota\Omega = \epsilon R(\xi,\widetilde{I})^*\mu^*\iota\Omega \xlongequal{(4.6)} \epsilon(\mathbf{1}_a \otimes \mathrm{ev}_{a,\overline{a}})R(F_{\widetilde{I}}\xi,\widetilde{I})\mu^*\iota\Omega\\
=&\epsilon(\mathbf{1}_a \otimes \mathrm{ev}_{a,\overline{a}})(\mu^*\iota \otimes \mathbf{1}_{\overline{a}})R(F_{\widetilde{I}}\xi,\widetilde{I})\Omega = \epsilon(\mathbf{1}_a \otimes \mathrm{ev}_{a,\overline{a}})(\mu^*\iota \otimes \mathbf{1}_{\overline{a}}) \cdot F_{\widetilde{I}}\xi.
\end{aligned}$$

By (4.13) we have $(\epsilon \otimes \mathbf{1}_a)\mu^*\iota = \mathrm{coev}_{\overline{a},a}$. Therefore

$$\begin{aligned}
&\epsilon(\mathbf{1}_a \otimes \mathrm{ev}_{a,\overline{a}})(\mu^*\iota \otimes \mathbf{1}_{\overline{a}}) = (\epsilon \otimes \mathrm{ev}_{a,\overline{a}})(\mu^*\iota \otimes \mathbf{1}_{\overline{a}})\\
=&(\mathbf{1}_{\overline{a}} \otimes \mathrm{ev}_{a,\overline{a}})(\epsilon \otimes \mathbf{1}_a \otimes \mathbf{1}_{\overline{a}})(\mu^*\iota \otimes \mathbf{1}_{\overline{a}}) = (\mathbf{1}_{\overline{a}} \otimes \mathrm{ev}_{a,\overline{a}})((\epsilon \otimes \mathbf{1}_a)\mu^*\iota \otimes \mathbf{1}_{\overline{a}})\\
=&(\mathbf{1}_{\overline{a}} \otimes \mathrm{ev}_{a,\overline{a}})(\mathrm{coev}_{\overline{a},a} \otimes \mathbf{1}_{\overline{a}}) = \mathbf{1}_{\overline{a}}.
\end{aligned}$$

This proves the second equation. A similar argument proves the first one. □

Remark 4.8. Since ϑ commutes with homomorphisms, it commutes in particular with ϵ. Therefore the S operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$ differ by a trist: $\epsilon^{-1}F_{\widetilde{I}} = \vartheta \cdot \epsilon^{-1}S_{\widetilde{I}}$.

By Tomita-Takesaki theory, we know that $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are preclosed on $\mathcal{H}_a$, and hence on any subobject of $\mathcal{H}_a$ by proposition 4.1. (Note that it is in general not true that the restriction of $S_{\widetilde{I}}|_{\mathcal{H}_a}$ to a sub-module $\mathcal{H}_j$ of $\mathcal{H}_a$ is $S_{\widetilde{I}}|_{\mathcal{H}_j}$, since the dual objects and the evaluations of $\mathcal{H}_a$ and of $\mathcal{H}_j$ are not related a priori.) To show that $S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are preclosed on any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, we need to show that any $\mathcal{H}_i$ is a subobject of $\mathcal{H}_a$ for some standard C^*-Frobenius algebra $Q = (\mathcal{H}_a, \mu, \iota)$. For this purpose we review a well known construction of Q in the following.

First, assume $\mathcal{H}_k \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ with a dual object $\mathcal{H}_{\overline{k}}$ and standard evaluations $\mathrm{ev}_{k,\overline{k}}, \mathrm{ev}_{\overline{k},k}$. Then $Q = (\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}}, \mu, \iota)$ is a standard C^*-Frobenius algebra, where

$$\iota = \mathrm{coev}_{k,\overline{k}}, \qquad \mu = \mathbf{1}_k \otimes \mathrm{ev}_{\overline{k},k} \otimes \mathbf{1}_{\overline{k}}. \tag{4.15}$$

By our definition of a system of dual objects and standard evaluations, $\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}}$ is the dual object of itself, and we have $\mathrm{ev}_{k\boxtimes\overline{k},k\boxtimes\overline{k}} = \iota^*\mu$ by (4.1). Thus the reflection operator $\epsilon = \mathbf{1}_{k\boxtimes\overline{k}}$.

Now assume that $\{\mathcal{H}_i : i \in \mathcal{E}\}$ is a family of objects in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ where $\mathcal{E}$ is a finite set. Let $\mathcal{H}_k = \bigoplus_{i\in\mathcal{E}} \mathcal{H}_i$, and assume that the dual object and standard evaluations are

$$\mathcal{H}_{\overline{k}} = \bigoplus_{i\in\mathcal{E}} \mathcal{H}_{\overline{i}}, \qquad \mathrm{ev}_{k,\overline{k}} = \sum_{i\in\mathcal{E}} \mathrm{ev}_{i,\overline{i}}, \qquad \mathrm{ev}_{\overline{k},k} = \sum_{i\in\mathcal{E}} \mathrm{ev}_{\overline{i},i}, \tag{4.16}$$

where $\mathrm{ev}_{i,\overline{i}} : \mathcal{H}_i\boxtimes\mathcal{H}_{\overline{i}} \to \mathcal{H}_0$ is naturally extended to $\mathcal{H}_k\boxtimes\mathcal{H}_{\overline{k}} \to \mathcal{H}_0$ and $\mathrm{ev}_{\overline{i},i}$ to $\mathcal{H}_{\overline{k}}\boxtimes\mathcal{H}_k \to \mathcal{H}_0$. (We can always assume this by slightly adjusting the system of dual objects and standard evaluations without affecting the other results under discussions.) Let Q be the corresponding standard C^*-Frobenius algebra for $\mathcal{H}_k\boxtimes\mathcal{H}_{\overline{k}}$. Then we have a natural unitary equivalence

$$\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}} \simeq \bigoplus_{i,j\in\mathcal{E}} \mathcal{H}_j \boxtimes \mathcal{H}_{\overline{j}}. \tag{4.17}$$

Moreover, for each $i, j, l, m \in \mathcal{E}$, the restriction of μ to $(\mathcal{H}_i \boxtimes \mathcal{H}_{\overline{j}}) \boxtimes (\mathcal{H}_l \boxtimes \mathcal{H}_{\overline{m}})$ is

$$\mu|_{\mathcal{H}_i\boxtimes\mathcal{H}_{\overline{j}}\boxtimes\mathcal{H}_l\boxtimes\mathcal{H}_{\overline{m}}} = \delta_{j,l}(\mathbf{1}_i \otimes \mathrm{ev}_{\overline{j},j} \otimes \mathbf{1}_{\overline{m}}), \tag{4.18}$$

and we have

$$\mathrm{ev}_{k\boxtimes\overline{k},k\boxtimes\overline{k}}|_{\mathcal{H}_i\boxtimes\mathcal{H}_{\overline{j}}\boxtimes\mathcal{H}_j\boxtimes\mathcal{H}_{\overline{i}}} = \mathrm{ev}_{i\boxtimes\overline{j},j\boxtimes\overline{i}}. \tag{4.19}$$

In the case that $0 \in \mathcal{E}$, i.e., $\{\mathcal{H}_i : i \in \mathcal{E}\}$ contains the vacuum module, notice that $\mathcal{H}_{\overline{0}} = \mathcal{H}_0$. Then for each $i, j \in \mathcal{E}$,

$$\mu|_{\mathcal{H}_i\boxtimes\mathcal{H}_0\boxtimes\mathcal{H}_0\boxtimes\mathcal{H}_{\overline{j}}} = \mathbf{1}_i \otimes \mathbf{1}_{\overline{j}}, \tag{4.20}$$

$$\mathrm{ev}_{k\boxtimes\overline{k},k\boxtimes\overline{k}}|_{\mathcal{H}_i\boxtimes\mathcal{H}_0\boxtimes\mathcal{H}_0\boxtimes\mathcal{H}_{\overline{i}}} = \mathrm{ev}_{i,\overline{i}}, \qquad \mathrm{ev}_{k\boxtimes\overline{k},k\boxtimes\overline{k}}|_{\mathcal{H}_0\boxtimes\mathcal{H}_{\overline{i}}\boxtimes\mathcal{H}_i\boxtimes\mathcal{H}_0} = \mathrm{ev}_{\overline{i},i} \tag{4.21}$$

Consider $\mathcal{H}_i \simeq \mathcal{H}_i \boxtimes \mathcal{H}_0$ and $\mathcal{H}_{\overline{i}} \simeq \mathcal{H}_0 \boxtimes \mathcal{H}_{\overline{i}}$ as subspaces of $\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}}$. Then by (4.21), the restrictions of $S_{\widetilde{I}}|_{\mathcal{H}_k\boxtimes\mathcal{H}_{\overline{k}}}$ to the sub-modules $\mathcal{H}_i, \mathcal{H}_{\overline{i}}$ are $S_{\widetilde{I}}|_{\mathcal{H}_i}$ and $S_{\widetilde{I}}|_{\mathcal{H}_{\overline{i}}}$ respectively.

Now, for any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, we set $\mathcal{H}_k = \mathcal{H}_0 \bigoplus \mathcal{H}_i$. Then $\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}}$ defines a standard C^*-Frobenius algebra. By (4.17), $\mathcal{H}_i \boxtimes \mathcal{H}_0$ is a sub-representation of $\mathcal{H}_k \boxtimes \mathcal{H}_{\overline{k}}$. Thus we conclude:

Proposition 4.9. *$S_{\widetilde{I}}$ and $F_{\widetilde{I}}$ are preclosed operators on any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$.*

In the following, we will always let $S_{\widetilde{I}}, F_{\widetilde{I}}$ denote the closures of the preclosed operators in (4.2). We have a positive closed operator $\Delta_{\widetilde{I}} := S_{\widetilde{I}}^* S_{\widetilde{I}} = F_{\widetilde{I}}^* F_{\widetilde{I}}$ definable on any object $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$. (We will write $\Delta_{\widetilde{I}}$ as Δ_I after showing that $\Delta_{\widetilde{I}}$ is independent of the choice of $\arg_I$.) So $\Delta_{\widetilde{I}}|_{\mathcal{H}_i}$ is a positive closed operator on $\mathcal{H}_i$. We call $\Delta_{\widetilde{I}}$ the **modular operator** of $\mathscr{E}^{\mathrm{f}}$. Note that by proposition 4.1, $\Delta_{\widetilde{I}}$ is independent of the choice of dual objects and standard evaluations. We define the categorical domain $\mathscr{D}(\Delta_{\widetilde{I}}^{\frac{1}{2}})$ which associates to each $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ the dense subspace $\mathscr{D}(\Delta_{\widetilde{I}}^{\frac{1}{2}}|_{\mathcal{H}_i})$ of $\mathcal{H}_i$. Then $\mathscr{D}(\Delta_{\widetilde{I}}^{\frac{1}{2}}) = \mathscr{D}(S_{\widetilde{I}}) = \mathscr{D}(F_{\widetilde{I}})$. We also have (categorical) polar decompositions

$$S_{\widetilde{I}} = \mathfrak{J}_{\widetilde{I}} \cdot \Delta_{\widetilde{I}}^{\frac{1}{2}}, \qquad F_{\widetilde{I}} = \vartheta\mathfrak{J}_{\widetilde{I}} \cdot \Delta_{\widetilde{I}}^{\frac{1}{2}} \tag{4.22}$$

where $\mathfrak{J}_{\widetilde{I}}$, when restricted to any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, is an anti-unitary operator

$$\mathfrak{J}_{\widetilde{I}}|_{\mathcal{H}_i} : \mathcal{H}_i \to \mathcal{H}_{\bar{i}}.$$

We call $\mathfrak{J}_{\widetilde{I}}$ the **(left) modular conjugation** of $\mathscr{E}^{\mathrm{f}}$. (The right modular conjugation is $\vartheta\mathfrak{J}_{\widetilde{I}}$ for the obvious reason.) The following are some easy consequences of Tomita-Takesaki theorem, some of which can also be proved using results in the next section.

Proposition 4.10. *The following are true.*

$$\mathfrak{J}_{\widetilde{I}}^2 = \mathbf{1}. \tag{4.23}$$

$$\Delta_{\widetilde{I}'} = \Delta_{\widetilde{I}}^{-1}, \qquad \vartheta\mathfrak{J}_{\widetilde{I}'} = \mathfrak{J}_{\widetilde{I}}. \tag{4.24}$$

$$\mathfrak{J}_{\widetilde{I}}\Delta_{\widetilde{I}}^{\frac{1}{2}} = \Delta_{\widetilde{I}}^{-\frac{1}{2}}\mathfrak{J}_{\widetilde{I}}. \tag{4.25}$$

$$\vartheta\mathfrak{J}_{\widetilde{I}} = \mathfrak{J}_{\widetilde{I}}\vartheta^{-1}, \qquad \vartheta\Delta_{\widetilde{I}} = \Delta_{\widetilde{I}}\vartheta. \tag{4.26}$$

Proof. For any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, consider $\mathcal{H}_k = \mathcal{H}_0 \oplus \mathcal{H}_i$ and the standard C^*-Frobenius algebra $Q = (\mathcal{H}_a, \mu, \iota)$ where $\mathcal{H}_a = \mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}}$. Let $\mathcal{B}$ and $\mathcal{B}'$ be the non-local extensions of $\mathcal{A}$ associated to Q as usual. Then $\Delta_{\widetilde{I}}|_{\mathcal{H}_a}$ and $\Delta_{\widetilde{I}'}|_{\mathcal{H}_a}$ are respectively the modular operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I}')$. Since $\mathcal{B}(\widetilde{I})$ is the commutant of $\mathcal{B}'(\widetilde{I}')$, by Tomita-Takesaki theory we have $\Delta_{\widetilde{I}}|_{\mathcal{H}_a} = \Delta_{\widetilde{I}'}^{-1}|_{\mathcal{H}_a}$. Since $\mathcal{H}_i$ is a submodule of $\mathcal{H}_a$, we have $\Delta_{\widetilde{I}'} = \Delta_{\widetilde{I}}^{-1}$ when acting on $\mathcal{H}_i$.

Since the reflection operator ϵ of Q equals $\mathbf{1}$, the modular conjugations of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I}')$ are $\mathfrak{J}_{\widetilde{I}}|_{\mathcal{H}_a}$ and $\vartheta\mathfrak{J}_{\widetilde{I}'}|_{\mathcal{H}_a}$ respectively. Thus $(\mathfrak{J}_{\widetilde{I}}|_{\mathcal{H}_a})^2 = \mathbf{1}_a$. Since we know that the restriction of $S_{\widetilde{I}}|_{\mathcal{H}_a}$ to $\mathcal{H}_i \simeq \mathcal{H}_i \boxtimes \mathcal{H}_0$ is $S_{\widetilde{I}}|_{\mathcal{H}_i}$, the same is true for the modular conjugations. Therefore $\mathfrak{J}_{\widetilde{I}}^2 = \mathbf{1}$ when acting on $\mathcal{H}_i$. By Tomita-Takesaki theory, we have $\vartheta\mathfrak{J}_{\widetilde{I}'} = \mathfrak{J}_{\widetilde{I}}$ and $\mathfrak{J}_{\widetilde{I}}\Delta_{\widetilde{I}}^{\frac{1}{2}} = \Delta_{\widetilde{I}}^{-\frac{1}{2}}\mathfrak{J}_{\widetilde{I}}$ when acting on $\mathcal{H}_a$. Thus they are also true when acting on $\mathcal{H}_i$.

Since $\vartheta\mathfrak{J}_{\widetilde{I}}$ is the modular conjugation for $\mathcal{B}'(\widetilde{I})$, equations (4.23) and (4.25) also hold if $\mathfrak{J}_{\widetilde{I}}$ is replaced by $\vartheta\mathfrak{J}_{\widetilde{I}}$. This proves (4.26). $\square$

Corollary 4.11. *We have $F_{\widetilde{I}'} = S_{\widetilde{I}}^*$ when acting on any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$.*

Proof. We compute

$$S_{\widetilde{I}}^* = (\mathfrak{J}_{\widetilde{I}}\Delta_{\widetilde{I}}^{\frac{1}{2}})^* \overset{(4.23)}{=\!=} \Delta_{\widetilde{I}}^{\frac{1}{2}}\mathfrak{J}_{\widetilde{I}} \overset{(4.25)}{=\!=} \mathfrak{J}_{\widetilde{I}}\Delta_{\widetilde{I}}^{-\frac{1}{2}} \overset{(4.24)}{=\!=} \vartheta\mathfrak{J}_{\widetilde{I}'}\Delta_{\widetilde{I}'}^{\frac{1}{2}} = F_{\widetilde{I}'}.$$

Of course, this can also be proved using the construction of Q. $\square$

Suppose that $G \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H}_j)$ where $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$. Then there exists a unique $G^\vee \in \mathrm{Hom}(\mathcal{H}_{\bar{j}}, \mathcal{H}_{\bar{i}})$ satisfying

$$\mathrm{ev}_{j,\bar{j}}(G \otimes \mathbf{1}_{\bar{j}}) = \mathrm{ev}_{i,\bar{i}}(\mathbf{1}_i \otimes G^\vee), \tag{4.27}$$

called the transpose of G. $G^\vee$ is independent of the choice standard evaluations, and we have $G^{\vee\vee} = G$. Thus

$$\overline{G} := (G^\vee)^* = (G^*)^\vee \tag{4.28}$$

which is in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_{\bar{i}}, \mathcal{H}_{\bar{j}})$ and called the **conjugation** of G. See for example [Yam04]. It is easy to see that $\overline{\overline{G}} = G$. We now show that conjugations of morphisms are implemented by the modular conjugation $\mathfrak{J}_{\widetilde{I}}$. Recall that two closed operators A and B on a Hilbert space $\mathcal{H}$ are said to **commute strongly** if the von Neumann algebras generated by A and by B commute.[6] We now recall the definition of strongly commuting diagrams of closed operators:

Definition 4.12. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$ be Hilbert spaces, and $A : \mathcal{P} \to \mathcal{R}, B : \mathcal{Q} \to \mathcal{S}, C : \mathcal{P} \to \mathcal{Q}, D : \mathcal{R} \to \mathcal{S}$ be unbounded closed operators. By saying that the diagram of closed operators

$$\begin{tikzcd} \mathcal{P} \arrow[r, "C"] \arrow[d, "A"'] & \mathcal{Q} \arrow[d, "B"'] \\ \mathcal{R} \arrow[r, "D"] & \mathcal{S} \end{tikzcd} \tag{4.29}$$

commutes strongly, we mean the following: Let $\mathcal{H} = \mathcal{P} \oplus \mathcal{Q} \oplus \mathcal{R} \oplus \mathcal{S}$. Define closed operators R, S on $\mathcal{H}$ with domains $\mathscr{D}(R) = \mathscr{D}(A) \oplus \mathscr{D}(B) \oplus \mathcal{R} \oplus \mathcal{S}$, $\mathscr{D}(S) = \mathscr{D}(C) \oplus \mathcal{Q} \oplus \mathscr{D}(D) \oplus \mathcal{S}$, such that

$$\begin{aligned} R(\xi \oplus \eta \oplus \chi \oplus \varsigma) &= 0 \oplus 0 \oplus A\xi \oplus B\eta \qquad (\forall \xi \in \mathscr{D}(A), \eta \in \mathscr{D}(B), \chi \in \mathcal{R}, \varsigma \in \mathcal{S}), \\ S(\xi \oplus \eta \oplus \chi \oplus \varsigma) &= 0 \oplus C\xi \oplus 0 \oplus D\chi \qquad (\forall \xi \in \mathscr{D}(C), \eta \in \mathcal{Q}, \chi \in \mathscr{D}(D), \varsigma \in \mathcal{S}). \end{aligned}$$

(Such construction is called the **extension** from A, B to R, and from C, D to S.) Then R and S commute strongly. In the case that A and B are preclosed antilinear operators, we choose anti-unitary operators U_1 on $\mathcal{R}$ and U_2 on $\mathcal{S}$. We say that (4.29) commutes strongly if the following diagram of closed linear operators commutes strongly:

$$\begin{tikzcd} \mathcal{P} \arrow[r, "C"] \arrow[d, "U_1 A"'] & \mathcal{Q} \arrow[d, "U_2 B"'] \\ \mathcal{R} \arrow[r, "U_2 D U_1^{-1}"] & \mathcal{S} \end{tikzcd} . \tag{4.30}$$

This definition is independent of the choice of U_1, U_2.

Proposition 4.13. *For any $G \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H}_j)$, the following equation holds when acting on $\mathcal{H}_{\bar{j}}$.*

$$\overline{G} = \mathfrak{J}_{\widetilde{I}} \cdot G \cdot \mathfrak{J}_{\widetilde{I}}. \tag{4.31}$$

Proof. For any $\xi \in \mathcal{H}_i(I)$, we have $L(G\xi, \widetilde{I}) = (G \otimes \mathbf{1})L(\xi, \widetilde{I})$ by (2.7). Therefore

$$\begin{aligned} S_{\widetilde{I}} G\xi &= L(G\xi, \widetilde{I})^* \mathrm{coev}_{j,\bar{j}} \Omega = L(\xi, \widetilde{I})^* (G^* \otimes \mathbf{1}_{\bar{j}}) \mathrm{coev}_{j,\bar{j}} \Omega \\ &\xlongequal{(4.27)} L(\xi, \widetilde{I})^* (\mathbf{1}_i \otimes \overline{G}) \mathrm{coev}_{i,\bar{i}} \Omega = \overline{G} L(\xi, \widetilde{I})^* \mathrm{coev}_{i,\bar{i}} \Omega = \overline{G} S_{\widetilde{I}} \xi. \end{aligned}$$

[6]The von Neumann algebra generated by A is the one generated by U and all $e^{\mathrm{i}tH}$ where $A = UH$ is the polar decomposition of A.

Since $\mathcal{H}_i(I)$ is a core for $S_{\widetilde{I}}|_{\mathcal{H}_i}$, we conclude $\overline{G}S_{\widetilde{I}} \subset S_{\widetilde{I}}G$. Since $\overline{G^*} = G^\vee = \overline{G}^*$, we also have $G^*S_{\widetilde{I}} \subset S_{\widetilde{I}}\overline{G}^*$. Therefore the following diagram of closed operators commute[7]:

$$\begin{tikzcd} \mathcal{H}_i \arrow[r, "G"] \arrow[d, "S_{\widetilde{I}}"'] & \mathcal{H}_j \arrow[d, "S_{\widetilde{I}}"'] \\ \mathcal{H}_{\bar{i}} \arrow[r, "\overline{G}"] & \mathcal{H}_{\bar{j}} \end{tikzcd} . \tag{4.32}$$

If we take the polar decomposition of the two vertical $S_{\widetilde{I}}$, then its phase commutes adjointly with the horizontal G and $\overline{G}$, i.e., the following diagram commutes adjointly

$$\begin{tikzcd} \mathcal{H}_i \arrow[r, "G"] \arrow[d, "\mathfrak{J}_{\widetilde{I}}"'] & \mathcal{H}_j \arrow[d, "\mathfrak{J}_{\widetilde{I}}"'] \\ \mathcal{H}_{\bar{i}} \arrow[r, "\overline{G}"] & \mathcal{H}_{\bar{j}} \end{tikzcd} . \tag{4.33}$$

□

We remark that in the above proof, the second half can be simplified after proving the geometric modular theorem, since we can use the fact that $\Delta_{\widetilde{I}}$ commutes strongly with any homomorphism. However, if we deal with bimodules of von Neumann algebras instead of conformal net modules, the above is the only way of proving the strong commutativity of modular conjugations and homomorphisms.

Remark 4.14. We close this section with a brief discussion of the relations between $\Delta_{\widetilde{I}}$ and Connes spatial derivatives [Con80]. Fix $I \in \mathcal{J}$. Choose a non-empty $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ and let $\mathcal{N} = \pi_{i,I'}(\mathcal{A}(I'))$. The (normalized) state on $\mathcal{A}(I')$ defined by $\langle \cdot \Omega | \Omega \rangle$ is transported through the isomorphism $\pi_{i,I'}$ to a state φ on $\mathcal{N}$. Let $\mathcal{M} = \mathcal{N}'$ be the commutant of $\mathcal{N}$ (acting on $\mathcal{H}_i$). Then $\mathcal{M}$ can be described by the left representation of $Q = (\mathcal{H}_i \boxtimes \mathcal{H}_{\bar{i}}, \mu, \iota)$ on $\mathcal{H}_i$ as follows: Let $\mathcal{B}$ be the non-local extension associated to Q. Then any $X \in \mathcal{B}(\widetilde{I})$ can be expressed as $A(\chi, \widetilde{I}) = \mu L(\chi, \widetilde{I})$ for some $\chi \in (\mathcal{H}_i \boxtimes \mathcal{H}_{\bar{i}})(I)$. We then define a representation of $\mathcal{B}(\widetilde{I})$ on $\mathcal{H}_i$ by defining the action of $A(\chi, \widetilde{I})$ on any $\xi \in \mathcal{H}_i$ to be $(\mathbf{1}_i \otimes \mathrm{ev}_{\bar{i},i}) L(\chi, \widetilde{I})\xi$. This representation is indeed faithful (since $\mathcal{B}(\widetilde{I})$ is indeed a (type III) factor). Moreover, the image of $\mathcal{B}(\widetilde{I})$ under this representation is exactly $\mathcal{M}$. The state of $\mathcal{B}(\widetilde{I})$ defined by $\langle \cdot \iota\Omega | \iota\Omega \rangle$ is transported through this representation to a state ψ of $\mathcal{M}$. Then we actually have

$$\Delta_{\widetilde{I}}|_{\mathcal{H}_i} = \frac{d\psi}{d\varphi}. \tag{4.34}$$

Note that ψ is in general not normalized: By the fact that $\iota = \mathrm{coev}_{i,\bar{i}}$ we have $\psi(1) = d_i$ where d_i is the quantum dimension of $\mathcal{H}_i$. We give another description of ψ: Let $\mathcal{E} : \mathcal{M} \to \pi_{i,I}(\mathcal{A}(I))$ be the minimal conditional expectation of the subfactor $\pi_{i,I}(\mathcal{A}(I)) \subset \mathcal{M}$. Transport the state $\langle \cdot \Omega | \Omega \rangle$ of $\mathcal{A}(I)$ to $\pi_{i,I}(\mathcal{A}(I))$ and denote it by ψ_0. Then the normalized state $d_i^{-1}\psi$ equals $\psi_0 \circ \mathcal{E}$.

[7] To prove that a closed operator A commutes strongly with a bounded operator x, it suffices to check that $x \cdot A|_{\mathscr{D}} \subset A|_{\mathscr{D}} \cdot x$ and $x^* \cdot A|_{\mathscr{D}} \subset A|_{\mathscr{D}} \cdot x^*$ where $\mathscr{D}$ is a core for A. See for example [Gui19] Appendix B.1.

5 Categorical and non-local Bisognano-Wichmann theorems

Let $\mathfrak{r} : z \in \mathbb{S}^1 \mapsto \overline{z} \in \mathbb{S}^1$ be the reflection. Then $\mathfrak{r} = \mathfrak{r}^{-1}$, and $g \in \mathrm{PSU}(1,1) \mapsto \mathfrak{r}g\mathfrak{r} \in \mathrm{PSU}(1,1)$ is an automorphism of $\mathrm{PSU}(1,1)$. Recall that any element in $\mathrm{PSU}(1,1)$ takes the form $g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$ where $|\alpha|^2 - |\beta|^2 = 1$. Then we have $\mathfrak{r}g\mathfrak{r} = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ \beta & \alpha \end{pmatrix}$. We lift this automorphism to $\widetilde{\mathrm{PSU}}(1,1)$ and also denote it by $\mathfrak{r}(\cdot)\mathfrak{r}$. For $\widetilde{I} = (I, \arg I) \in \widetilde{\mathcal{J}}$, we define $\mathfrak{r}\widetilde{I} = (\mathfrak{r}I, \arg_{\mathfrak{r}I})$ where $\arg_{\mathfrak{r}I}(z) = -\arg_I(\overline{z})$ for any $z \in \mathfrak{r}I$.

Define $\mathbb{S}^1_+ = \{a + \mathrm{i}b \in \mathbb{S}^1 : b > 0\}$ be the upper semi-circle. Define $\widetilde{\mathbb{S}^1_+}$ such that $\arg_{\mathbb{S}^1_+}$ takes values in $(0, \pi)$. Then $\widetilde{\mathbb{S}^1_-} := \mathfrak{r}\widetilde{\mathbb{S}^1_+}$ is the lower semi-circle with arg values in $(-\pi, 0)$. Note that $\widetilde{\mathbb{S}^1_+}$ is the clockwise complement of $\widetilde{\mathbb{S}^1_-}$. We write $\Delta_{\widetilde{\mathbb{S}^1_+}}, \Delta_{\widetilde{\mathbb{S}^1_-}}$, as Δ_+, Δ_- respectively. We also define $\Theta = \mathfrak{J}_{\widetilde{\mathbb{S}^1_+}}$, called the **PCT operator** of $\mathscr{E}^{\mathrm{f}}$. Note that Θ is an involution. Also, if $g \in \widetilde{\mathrm{PSU}}(1,1)$ and $\widetilde{I} = g\widetilde{\mathbb{S}^1_+}$, then by proposition 4.6,

$$\mathfrak{J}_{\widetilde{I}} = g\Theta g^{-1}. \tag{5.1}$$

The following noteworthy result is just proposition 4.13.

Theorem 5.1. *For any morphism G of objects in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, we have*

$$\overline{G} = \Theta \cdot G \cdot \Theta.$$

Consider the rotation subgroup $\varrho(t) = \begin{pmatrix} e^{\frac{\mathrm{i}t}{2}} & 0 \\ 0 & e^{\frac{-\mathrm{i}t}{2}} \end{pmatrix}$ and dilation subgroup $\delta(t) = \begin{pmatrix} \cosh\frac{t}{2} & -\sinh\frac{t}{2} \\ -\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}$ of $\mathrm{PSU}(1,1)$. For each $I \in \mathcal{J}$, define $\delta_I(t) = g\delta(t)g^{-1}$ where $g \in \mathrm{PSU}(1,1)$ and $g\mathbb{S}^1_+ = I$. Then δ_I is well defined, and $\delta(t) = \delta_{\mathbb{S}^1_+}(t)$. We lift ϱ and δ to one-parameter subgroups of $\widetilde{\mathrm{PSU}}(1,1)$ and denote them by the same symbols.

Let $Q = (\mathcal{H}_a, \mu, \iota)$ be a standard C^*-Frobenius algebra in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, and let $\mathcal{B}, \mathcal{B}'$ be the pair of non-local extensions of $\mathcal{A}$ associated to Q. Let $\epsilon : \mathcal{H}_a \to \mathcal{H}_{\overline{a}}$ be the reflection operator. By proposition 4.7, for any $\widetilde{I} \in \widetilde{\mathcal{J}}$, $\Delta_{\widetilde{I}}|_{\mathcal{H}_a}$ is the modular operator of both $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$, and $\epsilon^{-1}\mathfrak{J}_{\widetilde{I}}|_{\mathcal{H}_a}$ is the modular conjugation of $\mathcal{B}(\widetilde{I})$. In particular, $\epsilon^{-1}\Theta|_{\mathcal{H}_a}$ is the modular operator of both $\mathcal{B}(\widetilde{\mathbb{S}^1_+})$ and its commutant $\mathcal{B}'(\widetilde{\mathbb{S}^1_-})$. Then we have the non-local Bisognano-Wichmann Theorem:

Theorem 5.2. *Let Q be standard. For any $\widetilde{I} \in \widetilde{\mathcal{J}}$, the following are true when acting on $\mathcal{H}_a$.*

(a) (Geometric modular theorem) For any $t \in \mathbb{R}$,

$$\Delta^{\mathrm{i}t}_{\widetilde{I}} = \delta_I(-2\pi t). \tag{5.2}$$

(b) (PCT theorem) For any $g \in \widetilde{\mathrm{PSU}}(1,1)$,

$$\epsilon^{-1}\Theta \cdot g \cdot \Theta\epsilon = \mathfrak{r}g\mathfrak{r} \tag{5.3}$$

$$\epsilon^{-1}\Theta \cdot \mathcal{B}(\widetilde{I}) \cdot \Theta\epsilon = \mathcal{B}'(\mathfrak{r}\widetilde{I}). \tag{5.4}$$

Note that in equation (5.3), $\mathfrak{r}g\mathfrak{r}$ is in $\widetilde{\mathrm{PSU}}(1,1)$ and is acting on $\mathcal{H}_a$.

Proof. The restriction of $\widetilde{\mathrm{PSU}}(1,1) \curvearrowright \mathcal{H}_a$ to its rotation subgroup has positive generator by [GL96] proposition 2.14. Therefore the generators of translation subgroups are also positive by [Wei06] lemma 3.1. As in [ALR01] theorem 2.1 or [LR04] proposition 3.2, one may use Borchers theorem ([Bor92] theorem 2.9) to show (5.3) and show that $z(t) := \delta_I(2\pi t)\Delta_{\widetilde{I}}^{\mathrm{i}t}$ is a one-parameter group independent of $\widetilde{I}$. Thus $\delta(2\pi t)\Delta_+^{\mathrm{i}t} = \delta_-(2\pi t)\Delta_-^{\mathrm{i}t}$ where $\delta_- = \delta_{\mathbb{S}^1_-}$. By (4.24), we have $\Delta_-^{\mathrm{i}t} = \Delta_+^{-\mathrm{i}t}$. We also have $\delta_-(2\pi t) = \varrho(\pi)\delta(2\pi t)\varrho(-\pi)$, which equals $\delta(-2\pi t)$ by an easy calculation. Thus $z(t) = z(-t)$, which forces $z(t)$ to be $\mathbf{1}$. This proves the geometric modular theorem. By the non-local Haag-duality (theorem 3.7), $\mathcal{B}'(\widetilde{\mathbb{S}^1_-})$ is the commutant of $\mathcal{B}(\widetilde{\mathbb{S}^1_+})$. Thus, by Tomita-Takesaki theorem, (5.4) holds in the special case that $\widetilde{I} = \widetilde{\mathbb{S}^1_+}$. The general case follows from the special case, the Möbius covariance of $\mathcal{B}$, and equation (5.3). □

Theorem 5.3. *For any $\widetilde{I} \in \widetilde{\mathcal{J}}$, the following are true when acting on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$.*
(a) (Geometric modular theorem) For any $t \in \mathbb{R}$,

$$\Delta_{\widetilde{I}}^{\mathrm{i}t} = \delta_I(-2\pi t). \tag{5.5}$$

(b) (PCT theorem) For any $g \in \widetilde{\mathrm{PSU}}(1,1)$,

$$\Theta \cdot g \cdot \Theta = \mathfrak{r}g\mathfrak{r}. \tag{5.6}$$

Moreover, for any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ and $\xi \in \mathcal{H}_i(I)$, we have

$$\Theta \cdot \mathcal{H}_i(I) = \mathcal{H}_{\bar{i}}(\mathfrak{r}I), \tag{5.7}$$

$$\Theta \cdot L(\xi, \widetilde{I}) \cdot \Theta = R(\Theta\xi, \mathfrak{r}\widetilde{I}). \tag{5.8}$$

Proof. Let $Q = (\mathcal{H}_a, \mu, \iota)$ where $\mathcal{H}_a = \mathcal{H}_k \boxtimes \mathcal{H}_{\bar{k}}$ and $\mathcal{H}_k = \mathcal{H}_0 \oplus \mathcal{H}_j$. Then $\epsilon = 1$. By the non-local Bisognano-Wichmann theorem, (5.5) and (5.6) are true when acting on $\mathcal{H}_a$. Thus they are also true when acting on the submodule $\mathcal{H}_j \simeq \mathcal{H}_j \boxtimes \mathcal{H}_0$ by (4.21).

Now we take $\mathcal{H}_k = \mathcal{H}_0 \oplus \mathcal{H}_i \oplus \mathcal{H}_j$ and define Q in the same way. Consider $\mathcal{H}_i \simeq \mathcal{H}_i \boxtimes \mathcal{H}_0$ and $\mathcal{H}_j \simeq \mathcal{H}_j \boxtimes \mathcal{H}_0$ as submodules of $\mathcal{H}_a$. Recall that $\epsilon = 1$. So $\Theta|_{\mathcal{H}_a}$ is the modular conjugation for $\mathcal{B}(\widetilde{\mathbb{S}^1_+})$ with respect to $\iota\Omega$. Therefore $\Theta\iota\Omega = \iota\Omega$. Choose any $\xi \in \mathcal{H}_i(I)$. Then

$$\Theta A(\xi, \widetilde{I})\Theta\iota\Omega = \Theta A(\xi, \widetilde{I})\iota\Omega \xlongequal{(3.4)} \Theta\xi.$$

By the non-local PCT theorem, $\Theta A(\xi, \widetilde{I})\Theta$ is inside $\mathcal{B}'(\mathfrak{r}\widetilde{I})$. So there exists $\eta \in \mathcal{H}_a(\mathfrak{r}\widetilde{I})$ such that $\Theta A(\xi, \widetilde{I})\Theta = B(\eta, \mathfrak{r}\widetilde{I})$. Again, by (3.4), we must have $\eta = \Theta\xi$. Therefore $\Theta\xi \in \mathcal{H}_a(\mathfrak{r}I)$. Since Θ maps $\mathcal{H}_i \simeq \mathcal{H}_i \boxtimes \mathcal{H}_0$ to $\mathcal{H}_{\bar{i}} \simeq \mathcal{H}_0 \boxtimes \mathcal{H}_{\bar{i}}$, both considered as submodules of $\mathcal{H}_a$, we have $\Theta\xi \in \mathcal{H}_{\bar{i}}$. Therefore $\Theta\xi \in \mathcal{H}_{\bar{i}}(\mathfrak{r}I)$. Thus $\Theta\mathcal{H}_i(I) \subset \mathcal{H}_{\bar{i}}(\mathfrak{r}I)$. Similarly, $\Theta\mathcal{H}_{\bar{i}}(\mathfrak{r}I) \subset \mathcal{H}_i(I)$. Therefore (5.7) is true.

We have proved that $\Theta A(\xi, \widetilde{I})\Theta = B(\Theta\xi, \mathfrak{r}\widetilde{I})$. Recall that $\Theta\xi \in \mathcal{H}_{\bar{i}} \simeq \mathcal{H}_0 \boxtimes \mathcal{H}_{\bar{i}}$. By (4.20), the restriction of $A(\xi, \widetilde{I})$ to $\mathcal{H}_{\bar{j}} \simeq \mathcal{H}_0 \boxtimes \mathcal{H}_{\bar{j}}$ is $L(\xi, \widetilde{I})|_{\mathcal{H}_{\bar{j}}}$, and the restriction of $B(\Theta\xi, \mathfrak{r}\widetilde{I})$ to $\mathcal{H}_j \simeq \mathcal{H}_j \boxtimes \mathcal{H}_0$ is $R(\Theta\xi, \mathfrak{r}\widetilde{I})|_{\mathcal{H}_j}$. Therefore (5.8) is true. □

Convention 5.4. By (5.5), $\Delta_{\widetilde{I}}$ depends only on I but not on $\arg_I$. Thus we will write $\Delta_{\widetilde{I}}$ as Δ_I in the future.

Corollary 5.5 ([GL96] Thm.2.11)**.** *For any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$, $I \in \mathcal{J}$, and $x \in \mathcal{A}(I)$, we have*

$$\Theta \pi_{i,I}(x) \Theta = \pi_{\bar{i},\mathfrak{r}I}(\Theta x \Theta). \tag{5.9}$$

Notice that $\Theta x \Theta \in \mathcal{A}(\mathfrak{r}I)$ by the PCT theorem for $\mathcal{A}$.

Proof. Choose an $\arg_I$. Then, by (2.5), we have $\pi_{i,I}(x) = L(x\Omega, \widetilde{I})|_{\mathcal{H}_i}$ and $\pi_{\bar{i},\mathfrak{r}I}(\Theta x \Theta) = R(\Theta x \Theta \Omega, \mathfrak{r}\widetilde{I})|_{\mathcal{H}_{\bar{i}}} = R(\Theta x \Omega, \mathfrak{r}\widetilde{I})|_{\mathcal{H}_{\bar{i}}}$. We may now apply (5.8) to prove the desired equation. □

Remark 5.6. The above corollary gives an explicit construction of dual representation of any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$. Namely, we choose any seperable Hilbert space $\mathcal{H}_{\bar{i}}$, and choose an arbitrary anti-unitary map $\complement : \mathcal{H}_i \to \mathcal{H}_{\bar{i}}$. Define a representation $\pi_{\bar{i}}$ of $\mathcal{A}$ on $\mathcal{H}_{\bar{i}}$ such that for any $I \in \mathcal{J}$ and $x \in \mathcal{A}(I)$,

$$\pi_{\bar{i},I}(x) = \complement \cdot \pi_{i,\mathfrak{r}I}(\Theta x \Theta) \cdot \complement^{-1}. \tag{5.10}$$

Then $(\mathcal{H}_{\bar{i}}, \pi_{\bar{i}})$ is a dual object of $(\mathcal{H}_i, \pi_i)$.

The conformal spin-statistics theorem is also an easy consequence of the categorical PCT theorem:

Theorem 5.7 ([GL96] Thm. 3.13, [Jörß96] Sect. 4.1)**.** *On any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ we have*

$$\vartheta = \varrho(2\pi). \tag{5.11}$$

Proof. By (4.24) we have $\Theta = \mathfrak{J}_{\widetilde{\mathbb{S}^1_+}} = \vartheta \mathfrak{J}_{\widetilde{\mathbb{S}^1_-}}$. So $\mathfrak{J}_{\widetilde{\mathbb{S}^1_-}} = \vartheta^{-1}\Theta$. Since $\widetilde{\mathbb{S}^1_-} = \varrho(-\pi)\widetilde{\mathbb{S}^1_+}$, by proposition 4.6 we also have $S_{\widetilde{\mathbb{S}^1_-}} = \varrho(-\pi) S_{\widetilde{\mathbb{S}^1_+}} \varrho(\pi)$, and hence $\mathfrak{J}_{\widetilde{\mathbb{S}^1_-}} = \varrho(-\pi)\Theta\varrho(\pi)$. So $\vartheta^{-1}\Theta = \varrho(-\pi)\Theta\varrho(\pi)$. By the categorical PCT theorem, $\Theta\varrho(t) = (\mathfrak{r}\varrho(t)\mathfrak{r})\Theta = \varrho(-t)\Theta$. Therefore $\vartheta^{-1}\Theta = \varrho(-2\pi)\Theta$ and hence $\vartheta = \varrho(2\pi)$. □

We now want to generalize theorem 5.2 to any (non-necessarily standard) C^*-Frobenius algebra $Q = (\mathcal{H}_a, \mu, \iota)$ in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$. Notice that $\iota^*\mu$ is an evaluation of the self-dual object $\mathcal{H}_a$. Thus, by the uniqueness of evaluations up to multiplications by invertible morphisms, there is a unique invertible $\epsilon \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_a, \mathcal{H}_{\bar{a}})$, also called **reflection operator**, such that

$$\mathrm{ev}_{\bar{a},a}(\epsilon \otimes \mathbf{1}_a) = \iota^*\mu = \mathrm{ev}_{a,\bar{a}}(\mathbf{1}_a \otimes (\epsilon^{-1})^*). \tag{5.12}$$

(Recall that we assume $\mathrm{ev}_{a,\bar{a}}, \mathrm{ev}_{\bar{a},a}$ to be standard.) Thus, by (4.27), we have

$$\epsilon^\vee = (\epsilon^{-1})^* \quad \text{and hence} \quad \epsilon^{-1} = \bar{\epsilon}. \tag{5.13}$$

From this one easily shows that $\mathrm{coev}_{\bar{a},a} = (\epsilon^\vee \otimes \mathbf{1}_a)\mu^*\iota$ and $\mathrm{coev}_{a,\bar{a}} = (\mathbf{1}_a \otimes \epsilon)\mu^*\iota$. Using these two equations, the following can be proved in essentially the same way as proposition 4.7.

Proposition 5.8. *$\epsilon^{-1}S_{\widetilde{I}}|_{\mathcal{H}_a}$ and $\epsilon^* F_{\widetilde{I}}|_{\mathcal{H}_a}$ are respectively the S operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$ with respect to $\iota\Omega$.*

Hence we have:

Proposition 5.9. *The modular conjugations and operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$ with respect to $\iota\Omega$ are described by the following polar decompositions:*

$$\epsilon^{-1}S_{\widetilde{I}}\big|_{\mathcal{H}_a} = \big(\mathfrak{J}_{\widetilde{I}}\epsilon(\epsilon^*\epsilon)^{-\frac{1}{2}}\big)\cdot\big(\Delta_I^{\frac{1}{2}}(\epsilon^*\epsilon)^{\frac{1}{2}}\big)\Big|_{\mathcal{H}_a}, \tag{5.14}$$

$$\epsilon^* F_{\widetilde{I}}\Big|_{\mathcal{H}_a} = \big(\vartheta\mathfrak{J}_{\widetilde{I}}\epsilon^\vee(\epsilon^*\epsilon)^{\frac{1}{2}}\big)\cdot\big(\Delta_I^{\frac{1}{2}}(\epsilon^*\epsilon)^{-\frac{1}{2}}\big)\Big|_{\mathcal{H}_a}. \tag{5.15}$$

Proof. Recall the polar decompositions (4.22). Note also that Δ_I commutes with any homomorphism. By proposition 4.13, we have $\epsilon^{-1}\mathfrak{J}_{\widetilde{I}}\Delta_I^{\frac{1}{2}} = \mathfrak{J}_{\widetilde{I}}\overline{\epsilon^{-1}}\Delta_I^{\frac{1}{2}} = \mathfrak{J}_{\widetilde{I}}\epsilon\Delta_I^{\frac{1}{2}}$. This proves (5.14). Similarly we have $\epsilon^*\vartheta\mathfrak{J}_{\widetilde{I}}\Delta_I^{\frac{1}{2}} = \vartheta\mathfrak{J}_{\widetilde{I}}(\epsilon^*)^{-1}\Delta_I^{\frac{1}{2}} = \vartheta\mathfrak{J}_{\widetilde{I}}\epsilon^\vee\Delta_I^{\frac{1}{2}}$. In addition, using (5.13) we have

$$(\epsilon^\vee)^*\epsilon^\vee = \epsilon^{-1}\epsilon^\vee = \epsilon^{-1}(\epsilon^*)^{-1} = (\epsilon^*\epsilon)^{-1}. \tag{5.16}$$

This proves the polar decomposition (5.15). □

We now prove the following modified Bisognano-Wichmann theorem. Note that although ϵ depends on the choice of $\mathcal{H}_{\overline{a}}$ and standard evaluations $\mathrm{ev}_{a,\overline{a}}, \mathrm{ev}_{\overline{a},a}$, its absolute value $(\epsilon^*\epsilon)^{\frac{1}{2}}$ does not.

Theorem 5.10. *Let $Q = (\mathcal{H}_a, \mu, \iota)$ be a (non-necessarily standard) C^*-Frobenius algebra in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, let $\mathcal{B}$ and $\mathcal{B}'$ be the associated non-local extensions of $\mathcal{A}$, and let ϵ be the reflection operator. Choose any $\widetilde{I} \in \widetilde{\mathcal{J}}$. The following are true when acting on $\mathcal{H}_a$.*

(a) (Modified geometric modular theorem) Let $D_{\widetilde{I}}$ and $D'_{\widetilde{I}}$ be respectively the modular operators of $\mathcal{B}(\widetilde{I})$ and $\mathcal{B}'(\widetilde{I})$ with respect to $\iota\Omega$. Then for any $t \in \mathbb{R}$,

$$D_{\widetilde{I}}^{\mathrm{i}t} = (\epsilon^*\epsilon)^{\mathrm{i}t}\delta_I(-2\pi t), \qquad (D'_{\widetilde{I}})^{\mathrm{i}t} = (\epsilon^*\epsilon)^{-\mathrm{i}t}\delta_I(-2\pi t). \tag{5.17}$$

(b) (PCT theorem) Let $\Theta^Q = \big(\mathfrak{J}_{\widetilde{\mathbb{S}^1_+}}\epsilon(\epsilon^*\epsilon)^{-\frac{1}{2}}\big)\big|_{\mathcal{H}_a}$ be the modular conjugation of $\mathcal{B}(\widetilde{\mathbb{S}^1_+})$ with respect to $\iota\Omega$. Then for any $g \in \widetilde{\mathrm{PSU}}(1,1)$,

$$\Theta^Q\cdot g\cdot\Theta^Q = \mathfrak{r}g\mathfrak{r} \tag{5.18}$$

$$\Theta^Q\cdot\mathcal{B}(\widetilde{I})\cdot\Theta^Q = \mathcal{B}'(\mathfrak{r}\widetilde{I}). \tag{5.19}$$

Proof. The proof of PCT theorem is exactly the same as in the standard case. Equations (5.17) follow from proposition 5.9 and the geometric modular theorem for $\mathscr{E}^{\mathrm{f}}$. □

Corollary 5.11. *For the non-local extension $\mathcal{B}$ obtained by a C^*-Frobenius algebra Q in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$, the standard geometric modular theorem (5.2) holds if and only if Q is standard.*

Proof. Q is standard if and only if the invertible homomorphism ϵ is unitary, if and only if $\epsilon^*\epsilon = \mathbf{1}_a$. □

6 Unbounded field operators in rigid categorical extensions

In this section, we discuss the relation between the domain of $\Delta_I^{\frac{1}{2}}$ and the preclosedness of certain unbounded operators in $\mathscr{E}^{\mathrm{f}}$. First, we recall the following well-known fact (cf. [Tak02] section VI.1). A proof is included for the reader's convenience.

Proposition 6.1. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with commutant $\mathcal{M}'$, and let Ω be a cyclic separating vector of $\mathcal{M}$. Let $\Delta, \mathfrak{J}$ be the modular operator and conjugation of $(\mathcal{M},\Omega)$, and set $S = \mathfrak{J}\Delta^{\frac{1}{2}}$. For any $\xi \in \mathcal{H}$, define an unbounded operator $\mathscr{L}(\xi)$ with domain $\mathcal{M}'\Omega$ such that $\mathscr{L}(\xi)y\Omega = y\xi$ for any $y \in \mathcal{M}'$. Then the following two conditions are equivalent.*

(a) $\Omega \in \mathscr{D}(\mathscr{L}(\xi)^*)$.

(b) $\xi \in \mathscr{D}(\Delta^{\frac{1}{2}})$.

If either (a) or (b) is true, then $\mathscr{L}(\xi)$ is preclosed, and $S\xi = \mathscr{L}(\xi)^*\Omega$.

Proof. Let $\mathfrak{J}$ be the modular conjugation. Recall that $S := \mathfrak{J}\Delta^{\frac{1}{2}}$ has core $\mathcal{M}\Omega$, $F = S^* = \mathfrak{J}\Delta^{-\frac{1}{2}}$ has core $\mathcal{M}'\Omega$, and $Sx\Omega = x^*\Omega, Fy\Omega = y^*\Omega$ for any $x \in \mathcal{M}, y \in \mathcal{M}'$.

First, we assume that (a) is true. Then, for any $y \in \mathcal{M}'$, we compute

$$\langle S^*y\Omega|\xi\rangle = \langle y^*\Omega|\xi\rangle = \langle \Omega|y\xi\rangle = \langle\Omega|\mathscr{L}(\xi)y\Omega\rangle = \langle\mathscr{L}(\xi)^*\Omega|y\Omega\rangle,$$

which shows that $\xi \in \mathscr{D}(S) = \mathscr{D}(\Delta^{\frac{1}{2}})$ and $S\xi = \mathscr{L}(\xi)^*\Omega$.

Next, assume that (b) is true. Choose any $y_1, y_2 \in \mathcal{M}'$. Then

$$\begin{aligned}&\langle\mathscr{L}(S\xi)y_1\Omega|y_2\Omega\rangle = \langle y_1S\xi|y_2\Omega\rangle = \langle S\xi|y_1^*y_2\Omega\rangle = \langle Fy_1^*y_2\Omega|\xi\rangle = \langle y_2^*y_1\Omega|\xi\rangle\\ =&\langle y_1\Omega|y_2\xi\rangle = \langle y_1\Omega|\mathscr{L}(\xi)y_2\Omega\rangle,\end{aligned}$$

which shows $\mathscr{L}(S\xi) \subset \mathscr{L}(\xi)^*$. Thus $\Omega \in \mathscr{D}(\mathscr{L}(S\xi)) \subset \mathscr{D}(\mathscr{L}(\xi)^*)$, and $S\xi = \mathscr{L}(S\xi)\Omega = \mathscr{L}(\xi)^*\Omega$. Since $\mathscr{L}(S\xi)$ has dense domain, so does $\mathscr{L}(\xi)^*$. Therefore $\mathscr{L}(\xi)$ is preclosed. □

We would like to generalize the above proposition to $\mathscr{E}^{\mathrm{f}}$. For any $\widetilde{I} \in \widetilde{\mathcal{J}}$, recall that $\widetilde{I}'$ is the clockwise complement of $\widetilde{I}$. We define $`\widetilde{I} \in \widetilde{\mathcal{J}}$ such that $(`\widetilde{I})' = \widetilde{I}$, and call $`\widetilde{I}$ the **anticlockwise complement of** $\widetilde{I}$. Choose $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$. (We do not assume $\mathcal{H}_i$ to be dualizable.) For any $\xi \in \mathcal{H}_i(I)$, we let $\mathscr{L}(\xi,\widetilde{I})$ (resp. $\mathscr{R}(\xi,\widetilde{I})$) act on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$ as an unbounded operator $\mathcal{H}_j \to \mathcal{H}_i \boxtimes \mathcal{H}_j$ (resp. $\mathcal{H}_j \to \mathcal{H}_j \boxtimes \mathcal{H}_i$) with domain $\mathcal{H}_j(I')$ such that for any $\eta \in \mathcal{H}_j(I')$,

$$\mathscr{L}(\xi,\widetilde{I})\eta = R(\eta,\widetilde{I}')\xi, \quad \text{resp.} \quad \mathscr{R}(\xi,\widetilde{I})\eta = L(\eta,`\widetilde{I})\xi. \tag{6.1}$$

It is clear that Ω is inside the domains of $\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_0}$ and $\mathscr{R}(\xi,\widetilde{I})|_{\mathcal{H}_0}$, and the state-field correspondence

$$\mathscr{L}(\xi,\widetilde{I})\Omega = \mathscr{R}(\xi,\widetilde{I})\Omega = \xi \tag{6.2}$$

is satisfied. We also have that

$$\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_0} = \mathscr{R}(\xi,\widetilde{I})|_{\mathcal{H}_0}, \tag{6.3}$$

and that they depend only on I but not on the choice of $\arg_I$. Indeed, both operators send any $y\Omega \in \mathcal{A}(I')\Omega$ to $y\xi$.

Definition 6.2. For any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$ and $I \in \mathcal{J}$, $\mathcal{H}_i^{\mathrm{pr}}(I)$ is the set of all $\xi \in \mathcal{H}_i$ such that $\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_0} = \mathscr{R}(\xi, \widetilde{I})|_{\mathcal{H}_0}$ is precloded. It is clear that $\mathcal{H}_i(I) \subset \mathcal{H}_i^{\mathrm{pr}}(I)$.

It turns out that for any $\xi \in \mathcal{H}_i^{\mathrm{pr}}(I)$, $\mathscr{L}(\xi, \widetilde{I})$ is preclosed on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$. To prove this, we first need a lemma.

Lemma 6.3. *Let $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, $\widetilde{I}, \widetilde{J} \in \widetilde{\mathcal{J}}$, and assume that $\widetilde{J}$ is clockwise to $\widetilde{I}$. If $\xi \in \mathcal{H}_i$, $\xi_0 \in \mathscr{D}((\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_0})^*)$, and $\eta, \eta_0 \in \mathcal{H}_j(J)$, then*

$$\langle \mathscr{L}(\xi, \widetilde{I})\eta | R(\eta_0, \widetilde{J})\xi_0 \rangle = \langle R(\eta_0, \widetilde{J})^*\eta | \mathscr{L}(\xi, \widetilde{I})^*\xi_0 \rangle. \tag{6.4}$$

Proof. Choose ξ, ξ_0, η, η_0 as in the lemma. Recall that by proposition 2.3, we have $R(\eta_0, \widetilde{J})^*\eta \in \mathcal{H}_0(J)$ and

$$R(R(\eta_0, \widetilde{J})^*\eta, \widetilde{I}')|_{\mathcal{H}_i} = R(R(\eta_0, \widetilde{J})^*\eta, \widetilde{J})|_{\mathcal{H}_i} = R(\eta_0, \widetilde{J})^* R(\eta, \widetilde{J})|_{\mathcal{H}_i}.$$

Thus

$$\begin{aligned} &\langle \mathscr{L}(\xi, \widetilde{I})\eta | R(\eta_0, \widetilde{J})\xi_0 \rangle = \langle R(\eta, \widetilde{J})\xi | R(\eta_0, \widetilde{J})\xi_0 \rangle = \langle R(\eta_0, \widetilde{J})^* R(\eta, \widetilde{J})\xi | \xi_0 \rangle \\ =&\langle R(R(\eta_0, \widetilde{J})^*\eta, \widetilde{I}')\xi | \xi_0 \rangle = \langle \mathscr{L}(\xi, \widetilde{I}) R(\eta_0, \widetilde{J})^*\eta | \xi_0 \rangle = \langle R(\eta_0, \widetilde{J})^*\eta | \mathscr{L}(\xi, \widetilde{I})^*\xi_0 \rangle. \end{aligned}$$

□

Theorem 6.4. *Choose any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, $\widetilde{I} \in \widetilde{\mathcal{J}}$, and $\xi \in \mathcal{H}_i^{\mathrm{pr}}(I)$. Then $\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_j}$ and $\mathscr{R}(\xi, \widetilde{I})|_{\mathcal{H}_j}$ are preclosed for any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$.*

Proof. Let $\widetilde{J} = \widetilde{I}'$. Assume that η_n is a sequence of vectors in $\mathcal{H}_j(J)$ converging to 0 such that $\mathscr{L}(\xi, \widetilde{I})\eta_n$ converges to $\chi \in \mathcal{H}_i \boxtimes \mathcal{H}_j$. We shall show that $\chi = 0$. Since $\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_0}$ is preclosed, $\mathcal{W} := \mathscr{D}((\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_0})^*)$ is a dense subspace of $\mathcal{H}_i$. Since $R(\eta_0, \widetilde{J})$ is bounded for any $\eta_0 \in \mathcal{H}_j(J)$, we conclude that $R(\mathcal{H}_j(J), \widetilde{J})\mathcal{W}$ is dense in $R(\mathcal{H}_j(J), \widetilde{J})\mathcal{H}_i$ which spans a dense subspace of $\mathcal{H}_i \boxtimes \mathcal{H}_j$ by the density axiom of $\mathscr{E}$. Therefore, it suffices to show that $\langle \chi | R(\eta_0, \widetilde{J})\xi_0 \rangle = 0$ for any $\xi_0 \in \mathcal{W}$ and $\eta_0 \in \mathcal{H}_j(J)$. We notice that $\langle \chi | R(\eta_0, \widetilde{J})\xi_0 \rangle$ is the limit of

$$\langle \mathscr{L}(\xi, \widetilde{I})\eta_n | R(\eta_0, \widetilde{J})\xi_0 \rangle \xlongequal{(6.4)} \langle R(\eta_0, \widetilde{J})^*\eta_n | \mathscr{L}(\xi, \widetilde{I})^*\xi_0 \rangle,$$

which converges to 0 since $R(\eta_0, \widetilde{J})$ is bounded. This proves that $\mathscr{L}(\xi, \widetilde{I})|_{\mathcal{H}_j}$ is precloded. As for $\mathscr{R}(\xi, \widetilde{I})|_{\mathcal{H}_j}$, the argument is similar. □

We now relate the preclosedness of $\mathscr{L}(\xi, \widetilde{I})$ with the domain of $\Delta_I^{\frac{1}{2}}$.

Theorem 6.5. *Let $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$ and $\widetilde{I} \in \widetilde{\mathcal{J}}$. Then for any $\xi \in \mathcal{H}_i$, the following are equivalent:*

(a) Ω is in the domain of $\mathscr{L}(\xi, \widetilde{I})^* \mathrm{coev}_{i,\bar{i}}$.

(a′) Ω is in the domain of $\mathscr{R}(\xi,\widetilde{I})^*\mathrm{coev}_{\bar{i},i}$.

(b) ξ is in the domain of $\Delta_I^{\frac{1}{2}}|_{\mathcal{H}_i}$.

If any of them are true, then $\xi\in\mathcal{H}_i^{\mathrm{pr}}(I)$, and

$$S_{\widetilde{I}}\xi=\mathscr{L}(\xi,\widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega,\qquad F_{\widetilde{I}}\xi=\mathscr{R}(\xi,\widetilde{I})^*\mathrm{coev}_{\bar{i},i}\Omega. \tag{6.5}$$

Here, as usual, $\mathcal{H}_{\bar{i}}$ is a dual object of $\mathcal{H}_i$, and $\mathrm{ev}_{i,\bar{i}},\mathrm{ev}_{\bar{i},i}$ are standard evaluations with adjoints $\mathrm{coev}_{i,\bar{i}},\mathrm{coev}_{\bar{i},i}$.

Proof. Choose any $\xi\in\mathcal{H}_i$. Let $\mathcal{H}_k=\mathcal{H}_0\oplus\mathcal{H}_i$, let $\mathcal{H}_a=\mathcal{H}_k\boxtimes\mathcal{H}_{\bar{k}}$, and consider the standard C^*-Frobenius algebra $Q=(\mathcal{H}_a,\mu,\iota)$. Let $\mathcal{B}$ and $\mathcal{B}'$ be the associated non-local extensions of $\mathcal{A}$, and let $\mathcal{M}=\mathcal{B}(\widetilde{I})$. Consider $\mathcal{H}_i\simeq\mathcal{H}_i\boxtimes\mathcal{H}_0$ as a submodule of $\mathcal{H}_a$. Then $\xi\in\mathcal{H}_a$, and we can define an unbounded operator $\mathscr{L}^Q(\xi)$ on $\mathcal{H}_a$ with domain $\mathcal{M}'\iota\Omega$ such that $\mathscr{L}^Q(\xi)y\iota\Omega=y\xi$ for any $y\in\mathcal{M}'=\mathcal{B}'(\widetilde{I}')$. Note that $\mathcal{M}'\iota\Omega=\mathcal{H}_a(I')$ by (3.6), and that elements in $\mathcal{M}'$ are of the form $B(\eta,\widetilde{I}')=\mu R(\eta,\widetilde{I}')|_{\mathcal{H}_a}$ where $\eta\in\mathcal{H}_a(I')$. We compute that for any $\eta\in\mathcal{H}_a(I')$,

$$\mathscr{L}^Q(\xi)\eta\xlongequal{(3.4)}\mathscr{L}^Q(\xi)B(\eta,\widetilde{I}')\iota\Omega=B(\eta,\widetilde{I}')\xi=\mu R(\eta,\widetilde{I}')\xi=\mu\mathscr{L}(\xi,\widetilde{I})\eta,$$

which shows

$$\mathscr{L}^Q(\xi)=\mu\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_a} \tag{6.6}$$

with common domain $\mathcal{H}_a(I')$.

Note that

$$\begin{aligned}\mathcal{H}_a\simeq&(\mathcal{H}_0\boxtimes\mathcal{H}_0)\oplus(\mathcal{H}_i\boxtimes\mathcal{H}_0)\oplus(\mathcal{H}_0\boxtimes\mathcal{H}_{\bar{i}})\oplus(\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}})\\ \simeq&\mathcal{H}_0\oplus\mathcal{H}_i\oplus\mathcal{H}_{\bar{i}}\oplus(\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}).\end{aligned}$$

Then, by (4.18) and (6.6), $\mathscr{L}^Q(\xi)$ acts trivially on $\mathcal{H}_i$ and $\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}$. It acts as $\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_0}$ on $\mathcal{H}_0$, and as $\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_{\bar{i}}}$ on $\mathcal{H}_{\bar{i}}$. Consequently, $\mathscr{L}^Q(\xi)^*$ acts trivially on $\mathcal{H}_0$ and $\mathcal{H}_{\bar{i}}$; it acts as $\mathscr{L}(\xi,\widetilde{I})^*|_{\mathcal{H}_i}$ on $\mathcal{H}_i$, and as $\mathscr{L}(\xi,\widetilde{I})^*|_{\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}}$ on $\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}$. Thus, $\mathscr{L}^Q(\xi)$ is preclosed if and only if both $\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_0}$ and $\mathscr{L}(\xi,\widetilde{I})|_{\mathcal{H}_{\bar{i}}}$ are preclosed, if and only if $\xi\in\mathcal{H}_i^{\mathrm{pr}}(I)$ by theorem 6.4. By (4.15) and (4.16),

$$\iota\Omega=\mathrm{coev}_{k,\bar{k}}\Omega=\mathrm{coev}_{0,0}\Omega\oplus\mathrm{coev}_{i,\bar{i}}\Omega=\Omega\oplus\mathrm{coev}_{i,\bar{i}}\Omega\in\mathcal{H}_0\oplus(\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}).$$

Thus $\iota\Omega$ is in the domain of $\mathscr{L}^Q(\xi)^*$ if and only if $\mathrm{coev}_{i,\bar{i}}\Omega$ is in the domain of $\mathscr{L}(\xi,\widetilde{I})^*|_{\mathcal{H}_i\boxtimes\mathcal{H}_{\bar{i}}}$. If this is true then $\mathscr{L}^Q(\xi)^*\iota\Omega=\mathscr{L}(\xi,\widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega$. We thus conclude that (a) holds if and only if $\iota\Omega\in\mathscr{D}(\mathscr{L}^Q(\xi)^*)$. On the other hand, since the reflection operator ϵ of Q equals $\mathbf{1}_a$, by proposition 4.7 we know that $\Delta_I|_{\mathcal{H}_a}$ and $S_{\widetilde{I}}|_{\mathcal{H}_a}$ are respectively the modular operator and the S operator for $(\mathcal{M},\iota\Omega)$. By (4.21) and the sentences thereafter, the restrictions of $\Delta_I|_{\mathcal{H}_a}$ and $S_{\widetilde{I}}|_{\mathcal{H}_a}$ to $\mathcal{H}_i$ are $\Delta_I|_{\mathcal{H}_i}$ and $S_{\widetilde{I}}|_{\mathcal{H}_i}$ respectively. Thus (b) holds if and only if ξ is in the domain of the square root of the modular operator for $(\mathcal{M},\iota\Omega)$. Therefore, by proposition 6.1, (a) and (b) are equivalent; when (a) or (b) is true, we have $\xi\in\mathcal{H}_i^{\mathrm{pr}}(I)$ and $S_{\widetilde{I}}\xi=\mathscr{L}^Q(\xi)^*\iota\Omega=\mathscr{L}(\xi,\widetilde{I})^*\mathrm{coev}_{i,\bar{i}}\Omega$.

To prove the equivalence of (a′) and (b) and the second equation of (6.5), construct $Q=(\mathcal{H}_{\bar{k}}\boxtimes\mathcal{H}_k,\mu,\iota)$ where $\mathcal{H}_k=\mathcal{H}_0\oplus\mathcal{H}_i$ and use similar arguments. $\square$

A Connes categorical extensions

In this appendix section, we sketch the constuction of categorical extensions via Connes fusion. Details can be found in [Gui18]. We do not assume the representations to be dualizable in this section.

Recall that we set $\mathcal{H}_i(I) = \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)\Omega$. For any $\xi \in \mathcal{H}_i(I)$, we define $Z(\xi, \widetilde{I})$ to be the unique element in $\mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)$ satisfying $\xi = Z(\xi, \widetilde{I})\Omega$. If J is disjoint from I, we define a (degenerate) inner product $\langle \cdot | \cdot \rangle$ (antilinear on the second variable) on the algebraic tensor product $\mathcal{H}_i(I) \otimes \mathcal{H}_j(J)$ such that for any $\xi_1, \xi_2 \in \mathcal{H}_i(I)$ and $\eta_1, \eta_2 \in \mathcal{H}_j(J)$,

$$\langle \xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2 \rangle = \langle Z(\eta_2, J)^* Z(\eta_1, J) Z(\xi_2, I)^* Z(\xi_1, I)\Omega | \Omega \rangle. \tag{A.1}$$

This is nothing but the formula of Connes relative tensor product. We let $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J)$ be the Hilbert space completion of $\mathcal{H}_i(I) \otimes \mathcal{H}_j(J)$ under this inner product. Note that $\mathcal{A}(I)$ and $\mathcal{A}(J)$ can naturally act on $\mathcal{H}_i(I) \otimes \mathcal{H}_j(J)$ by acting on the first resp. second component of the tensor product.

If $I_0 \subset I$ and $J_0 \subset J$, then $\mathcal{H}_i(I_0)$ and $\mathcal{H}_j(J_0)$ are dense in $\mathcal{H}_i(I)$ and $\mathcal{H}_j(J)$. It is not hard to check that $\mathcal{H}_i(I_0) \otimes \mathcal{H}_j(J_0)$ is also dense in $\mathcal{H}_i(I) \otimes \mathcal{H}_j(J)$ under the above inner product. Thus we have a natural unitary map $\mathcal{H}_i(I_0) \boxtimes \mathcal{H}_j(J_0) \xrightarrow{\simeq} \mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J)$ induced by inclusion of intervals. Its adjoint $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J) \xrightarrow{\simeq} \mathcal{H}_i(I_0) \boxtimes \mathcal{H}_j(J_0)$ is the natural unitary map induced by restriction of intervals. Both maps commute with the actions of $\mathcal{A}(I_0)$ and $\mathcal{A}(J_0)$.

Let I_1, I_2 be disjoint respectively from J_1, J_2. First, assume that these two pairs of intervals "overlap properly", which means that $I_1 \cap I_2, J_1 \cap J_2 \in \mathcal{J}$. Then

$$\mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j(J_1) \xrightarrow{\simeq} \mathcal{H}_i(I_1 \cap I_2) \boxtimes \mathcal{H}_j(J_1 \cap J_2) \xrightarrow{\simeq} \mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j(J_2) \tag{A.2}$$

defines a natural unitary map $\mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j(J_1) \xrightarrow{\simeq} \mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j(J_2)$. In general, we need to choose a path $\gamma : [0,1] \to \mathrm{Conf}_2(\mathbb{S}^1)$, where $\mathrm{Conf}_2(\mathbb{S}^1) = \{(z,w) \in \mathbb{S}^1 : z \neq w\}$. We assume that $\gamma(0) \in I_1 \times J_1$ and $\gamma(1) \in I_2 \times J_2$. Then we can define a natural unitary map $\gamma^\bullet : \mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j(J_1) \xrightarrow{\simeq} \mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j(J_2)$ by covering γ by a chain of pairs of open intervals $(K_1, L_1), \dots (K_n, L_n)$, such that $K_1 = I_1, L_1 = J_1, K_n = I_2, L_n = J_2$, and that for each $l = 1, 2, \dots, n$, (K_{l-1}, L_{l-1}) and (K_l, K_l) overlap properly. Then we can use a chain of unitary maps induced by restriction and inclusion of intervals (as in (A.2)) to define $\gamma^\bullet$. We say that $\gamma^\bullet$ is the path continuation induced by γ. If we have paths γ_1, γ_2, then $(\gamma_1 * \gamma_2)^\bullet = \gamma_1^\bullet \gamma_2^\bullet$. Moreover, $\gamma^\bullet$ depends only on the homotopy class of γ. Then we can transport the actions of $\mathcal{A}(I_1), \mathcal{A}(J_1)$ from $\mathcal{H}_i(I_1) \boxtimes \mathcal{H}_j(I_1)$ to $\mathcal{H}_i(I_2) \boxtimes \mathcal{H}_j(I_2)$ through the map $\gamma^\bullet$. Indeed, the result of transportation is independent of the choice of γ (but not just its homotopy class). It turns out that we have a well defined action of $\mathcal{A}$ on any Connes fusion $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J)$ so that it restricts to the standard actions of $\mathcal{A}(I), \mathcal{A}(J)$ on $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J)$, and that the actions of $\mathcal{A}$ commute with all path continuations. Thus $\mathcal{H}_i(I) \boxtimes \mathcal{H}_j(J)$ becomes a representation of $\mathcal{A}$, and the path continuations are unitary isomorphisms of $\mathcal{A}$-modules.

In the construction of the tensor category $\mathrm{Rep}(\mathcal{A})$, we let $\mathcal{H}_i \boxtimes \mathcal{H}_j$ be $\mathcal{H}_i(\mathbb{S}^1_+) \boxtimes \mathcal{H}_j(\mathbb{S}^1_-)$, where $\mathbb{S}^1_+, \mathbb{S}^1_- \in \mathcal{J}$ are respectively the upper and lower semi-circle. Choose $\arg_{\mathbb{S}^1_+}$ (resp. $\arg_{\mathbb{S}^1_-}$) to be the one whose values are inside $(0, \pi)$ (resp. $(-\pi, 0)$). This defines $\widetilde{\mathbb{S}^1_+}$ and

$\widetilde{\mathbb{S}^1_-}$. Now, for each $\widetilde{I} = (I, \arg_I) \in \widetilde{\mathcal{J}}$ and $\xi \in \mathcal{H}_i(I)$, we describe the operator $L(\xi, \widetilde{I})$. Recall that $\widetilde{I}'$ is the clockwise complement of $\widetilde{I}$, which means that I' is (the interior of) the complement of I, and $\widetilde{I}'$ is clockwise to $\widetilde{I}$. We write $\widetilde{I}' = (I', \arg_{I'})$. The action $Z(\xi, I) : \eta \in \mathcal{H}_j(I') \mapsto \xi \otimes \eta \in \mathcal{H}_i(I) \boxtimes \mathcal{H}_j(I')$ is a bounded operator which intertwines the actions of $\mathcal{A}(I')$. Thus $Z(\xi, I) \in \mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_j, \mathcal{H}_i(I) \boxtimes \mathcal{H}_j(I'))$. We now choose a path $\gamma : [0,1] \to \mathrm{Conf}_2(\mathbb{S}^1)$ from $I \times I'$ to $\mathbb{S}^1_+ \times \mathbb{S}^1_-$ such that the arguments of $\widetilde{I}$ and $\widetilde{I}'$ are changing continuously to those of $\widetilde{\mathbb{S}^1_+}$ and $\widetilde{\mathbb{S}^1_-}$ respectively along γ. Then for each $\eta \in \mathcal{H}_j$, $L(\xi, \widetilde{I})\eta$ is defined to be

$$L(\xi, \widetilde{I})\eta = \gamma^\bullet Z(\xi, I)\eta \in \mathcal{H}_i \boxtimes \mathcal{H}_j. \tag{A.3}$$

Define a path $\rho : [0,1] \to \mathrm{Conf}_2(\mathbb{S}^1)$ from $\mathbb{S}^1_+ \times \mathbb{S}^1_-$ to $\mathbb{S}^1_- \times \mathbb{S}^1_+$ by $\rho(t) = (e^{\mathbf{i}\pi(\frac{1}{2}-t)}, e^{\mathbf{i}\pi(-\frac{1}{2}-t)})$. Then

$$\rho^\bullet : \mathcal{H}_i \boxtimes \mathcal{H}_j = \mathcal{H}_i(\mathbb{S}^1_+) \boxtimes \mathcal{H}_j(\mathbb{S}^1_-) \to \mathcal{H}_i(\mathbb{S}^1_-) \boxtimes \mathcal{H}_j(\mathbb{S}^1_+) \simeq \mathcal{H}_j \boxtimes \mathcal{H}_i \tag{A.4}$$

is the braiding $\mathbb{B}_{i,j}$. We define $R(\xi, \widetilde{I})\eta = \mathbb{B}_{i,j} L(\xi, \widetilde{I})\eta$. That these operators define a categorical extension $\mathscr{E} = (\mathcal{A}, \mathrm{Rep}(\mathcal{A}), \boxtimes, \mathcal{H})$ of $\mathcal{A}$ (called Connes categorical extension) was proved in [Gui18].

B Möbius covariance of categorical extensions

Let $\mathscr{E}^{\mathrm{f}} = (\mathcal{A}, \mathrm{Rep}^{\mathrm{f}}(\mathcal{A}), \boxtimes, \mathcal{H})$ be the rigid (vector-labeled and closed) categorical extension of the Möbius covariant net $\mathcal{A}$. Recall that we assume objects in $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ (which are dualizable) to be Möbius covariant, which means that (1.2) holds for any $g \in \widetilde{\mathrm{PSU}}(1,1)$ and $I \in \mathcal{J}$. In this section we prove theorem 2.5, namely, that the fusion of two Möbius covariant representations is also Möbius covariant, and that $\mathscr{E}^{\mathrm{f}}$ is Möbius covariant. We remark that the arguments in this section can also be used to show that $\mathscr{E}$ is conformal covariant when $\mathcal{A}$ is so; see the end of the section.

We first notice the following easy fact:

Lemma B.1. *Any morphism in* $\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ *commutes with the actions of* $\widetilde{\mathrm{PSU}}(1,1)$.

Proof. Let $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Rep}^{\mathrm{f}}(\mathcal{A})$ and $G \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_i, \mathcal{H}_j)$. Let $\mathcal{H}_k = \mathcal{H}_i \oplus \mathcal{H}_j$. Then the unique representation U_k of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_k$ is described by $U_k(g) = \mathrm{diag}(U_i(g), U_j(g))$ for any $g \in \widetilde{\mathrm{PSU}}(1,1)$. We regard G as an endomorphism of $\mathcal{H}_k$ by acting trivially on $\mathcal{H}_j$. Then it suffices to show that any endomorphism of $\mathcal{H}_k$ commutes with the action of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_k$. By linearity, it suffices to prove this for any unitary $V \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_k)$. Then, by the uniqueness of the representation of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_k$, we have $U_k(g) = VU_k(g)V^*$ for any $g \in \widetilde{\mathrm{PSU}}(1,1)$. Therefore V commutes with the action of $\widetilde{\mathrm{PSU}}(1,1)$. □

Choose any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A})), \widetilde{I} \in \widetilde{\mathcal{J}}$, and define a unitary representation $V_{\widetilde{I}}$ of $\widetilde{\mathrm{PSU}}(1,1)$ on $\mathcal{H}_i \boxtimes \mathcal{H}_j$ by setting

$$V_{\widetilde{I}}(g) L(\xi, \widetilde{I})\eta = L(g\xi, g\widetilde{I})g\eta \tag{B.1}$$

for any $\xi \in \mathcal{H}_i(I), \eta \in \mathcal{H}_j(I'), g \in \widetilde{\mathrm{PSU}}(1,1)$. Note that we have $g\mathcal{H}_i(I) = \mathcal{H}_i(gI)$ since, by (1.2), we have $g\mathrm{Hom}_{\mathcal{A}(I')}(\mathcal{H}_0, \mathcal{H}_i)g^{-1} = \mathrm{Hom}_{\mathcal{A}(gI')}(\mathcal{H}_0, \mathcal{H}_i)$. Thus we have

$$L(g\xi, g\widetilde{I}) = gL(\xi, \widetilde{I})g^{-1}$$

when acting on $\mathcal{H}_0$. Now, we choose any $\xi_1, \xi_2 \in \mathcal{H}_i(I), \eta_1, \eta_2 \in \mathcal{H}_j(I')$, and use the locality of $\mathscr{E}^{\mathrm{f}}$ and the fact that $g\Omega = \Omega$ to compute that

$$\begin{aligned}
&\langle L(g\xi_1, g\widetilde{I})g\eta_1 | L(g\xi_2, g\widetilde{I})g\eta_2\rangle = \langle L(g\xi_1, g\widetilde{I})R(g\eta_1, g\widetilde{I}')\Omega | L(g\xi_2, g\widetilde{I})R(g\eta_2, g\widetilde{I}')\Omega\rangle \\
=&\langle L(g\xi_2, g\widetilde{I})^*L(g\xi_1, g\widetilde{I})\Omega | R(g\eta_1, g\widetilde{I}')^*R(g\eta_2, g\widetilde{I}')\Omega\rangle \\
=&\langle gL(\xi_2, \widetilde{I})^*L(\xi_1, \widetilde{I})\Omega | gR(\eta_1, \widetilde{I}')^*R(\eta_2, \widetilde{I}')\Omega\rangle = \langle L(\xi_2, \widetilde{I})^*L(\xi_1, \widetilde{I})\Omega | R(\eta_1, \widetilde{I}')^*R(\eta_2, \widetilde{I}')\Omega\rangle \\
=&\langle L(\xi_1, \widetilde{I})\eta_1 | L(\xi_2, \widetilde{I})\eta_2\rangle.
\end{aligned} \tag{B.2}$$

This proves the well-definedness and the unitarity of $V_{\widetilde{I}}(g)$.

Notice that $V_{\widetilde{I}}$ is independent of $\widetilde{I}$, namely, $V_{\widetilde{I}} = V_{\widetilde{I}_0}$ when $\widetilde{I}, \widetilde{I}_0 \in \widetilde{\mathcal{J}}$. Indeed, it suffices to check this when $\widetilde{I}_0 \subset \widetilde{I}$. In that case, the actions of $V_{\widetilde{I}}(g)$ and $V_{\widetilde{I}_0}(g)$ on $L(\mathcal{H}_i(I_0), \widetilde{I})\mathcal{H}_j(I')$ are clearly the same. So they must be equal. We write $V_{\widetilde{I}}$ as V for short. From our definition (B.1), it is clear that $V(gh) = V(g)V(h)$ for any $g, h \in \widetilde{\mathrm{PSU}}(1,1)$. Thus V is a representation of $\widetilde{\mathrm{PSU}}(1,1)$.

We choose $\widetilde{I}_0 \subset \widetilde{I}$ such that $I_0 \subset\subset I$. To check the continuity of the representation V, we need to show that for any sequence of elements g_n in $\widetilde{\mathrm{PSU}}(1,1)$ converging to 1, $L(g_n\xi, g_n\widetilde{I}_0)g_n\eta$ converges to $L(\xi, \widetilde{I}_0)\eta$ for any $\xi \in \mathcal{H}_i(I_0)$ and $\eta \in \mathcal{H}_j$. Assume without loss of generality that $g_nI_0 \subset I$ for any n. Since $g_n\eta$ converges to η, it suffices to show that $L(g_n\xi, g_n\widetilde{I}_0)|_{\mathcal{H}_j} = L(g_n\xi, \widetilde{I})|_{\mathcal{H}_j}$ converge strongly to $L(\xi, \widetilde{I})$ and are uniformly bounded over n. Using the locality of $\mathscr{E}^{\mathrm{f}}$, it is easy to see that $L(g_n\xi, \widetilde{I})\chi$ converges to $L(\xi, \widetilde{I})\chi$ for any $\chi \in \mathcal{H}_j(I')$. Set $x = L(\xi, \widetilde{I})^*L(\xi, \widetilde{I})|_{\mathcal{H}_0} \in \mathcal{A}(I)$. Then $x_n := g_nxg_n^*$ equals $L(g_n\xi, g_n\widetilde{I})^*L(g_n\xi, g_n\widetilde{I})|_{\mathcal{H}_0}$. So $x_n\Omega = L(g_n\xi, g_n\widetilde{I})^*g_n\xi$. Hence, by proposition 2.3,

$$\begin{aligned}
&\|L(g_n\xi, \widetilde{I})|_{\mathcal{H}_j}\|^2 = \|L(g_n\xi, \widetilde{I})^*L(g_n\xi, \widetilde{I})|_{\mathcal{H}_j}\| = \|L(L(g_n\xi, \widetilde{I})^*g_n\xi, \widetilde{I})|_{\mathcal{H}_j}\| \\
=&\|L(x_n\Omega, g_n\widetilde{I})|_{\mathcal{H}_j}\| = \|\pi_{j,g_nI}(x_n)\| \leqslant \|x_n\| = \|x\|.
\end{aligned}$$

This shows that $\|L(g_n\xi, \widetilde{I})|_{\mathcal{H}_j}\|$ is uniformly bounded over all n. Thus $L(g_n\xi, \widetilde{I})$ converges strongly to $L(\xi, \widetilde{I})$.

To show that V makes $\mathcal{H}_i \boxtimes \mathcal{H}_j$ Möbius covariant, we need to check that

$$V(g)\pi_{i\boxtimes j,J}(x) = \pi_{i\boxtimes j,gJ}(gxg^{-1})V(g)$$

for any $g \in \widetilde{\mathrm{PSU}}(1,1), J \in \mathcal{J}, x \in \mathcal{A}(J)$. It suffices to verify this equation when both sides act on $L(\mathcal{H}_i(I), \widetilde{I})\mathcal{H}_j(J)$ where I is disjoint from J. This is easy. Therefore $\mathcal{H}_i \boxtimes \mathcal{H}_j$ is Möbius covariant, and the unique representation of $\widetilde{\mathrm{PSU}}(1,1)$ is described by (B.1). From (B.1) it is clear that $L(g\xi, g\widetilde{I}) = gL(\xi, \widetilde{I})g^{-1}$ is always true on any $\mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{f}}(\mathcal{A}))$. By the braiding axiom of $\mathscr{E}^{\mathrm{f}}$ and lemma B.1, we also have $R(g\xi, g\widetilde{I}) = gR(\xi, \widetilde{I})g^{-1}$. This proves the Möbius covariance of $\mathscr{E}^{\mathrm{f}}$.

Finally, we explain how the above arguments can be adapted to show the conformal covariance of $\mathscr{E}$ when $\mathcal{A}$ is conformal covariant. Let $\mathcal{A}$ be conformal covariant. Recall that for any $\mathcal{H}_i \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, $\xi \in \mathcal{H}_i(I)$, and $g \in \mathscr{G}_{\mathcal{A}}$, we have $g\xi g^{-1} \in \mathcal{H}_i(gI)$ where $g\xi g^{-1} := gL(\xi, \widetilde{I})g^{-1}\Omega = gR(\xi, \widetilde{I})g^{-1}\Omega$. Thus, for any $\mathcal{H}_i, \mathcal{H}_j \in \mathrm{Obj}(\mathrm{Rep}(\mathcal{A}))$, one can define an action of $\mathscr{G}_{\mathcal{A}}$ on $\mathcal{H}_i \boxtimes \mathcal{H}_j$ by setting

$$V_{\widetilde{I}}(g)L(\xi, \widetilde{I})\eta = L(g\xi g^{-1}, g\widetilde{I})g\eta \tag{B.3}$$

for any $\xi \in \mathcal{H}_i(I), \eta \in \mathcal{H}_j(I'), g \in \mathscr{G}_{\mathcal{A}}$. Note that (B.3) also equals

$$L(g\xi g^{-1}, g\widetilde{I})gR(\eta, \widetilde{I}')g^{-1}g\Omega = L(g\xi g^{-1}, g\widetilde{I})R(g\eta g^{-1}, g\widetilde{I}')g\Omega$$

since we have $gR(\eta, \widetilde{I}')g^{-1} = R(g\eta g^{-1}, g\widetilde{I}')$ when acting on $\mathcal{H}_0$. Using this relation and the calculations as in (B.2), one checks that $V_{\widetilde{I}}(g)$ is well-defined and unitary. Similar arguments as in the above paragraphs show that $V_{\widetilde{I}}$ is independent of $\widetilde{I}$, that V respects the group multiplication of $\mathscr{G}_{\mathcal{A}}$ (which follows clearly from the definition of $V_{\widetilde{I}}(g)$[8]), that $V : \mathscr{G}_{\mathcal{A}} \curvearrowright \mathcal{H}_i \boxtimes \mathcal{H}_j$ descends to a continuous projective representation of $\mathscr{G}$, and that (1.1) holds. Thus V is the unique representation of $\mathscr{G}_{\mathcal{A}}$ making $\mathcal{H}_i \boxtimes \mathcal{H}_j$ conformal covariant. The relations (2.13) follow easily from the definition (B.3) of V.

[8]In [Gui18], the fact that V preserves the group multiplications is proved in lemma 2.16, which is long and complicated. Here, our proof is much simplified and follows directly from the definition of V.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, USA.
E-mail: bin.gui@rutgers.edu   binguimath@gmail.com