Genus-zero Permutation-twisted Conformal Blocks for Tensor Product Vertex Operator Algebras: The Tensor-separable Case

BIN GUI

Abstract

Let $\mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}(n)$ be a vertex operator algebra (VOA), let E be a finite set, and let G be a subgroup of the permutation group $\mathrm{Perm}(E)$ which acts on $\mathbb{V}^{\otimes E}$ in a natural way. For each $g \in G$, the g-twisted $\mathbb{V}^{\otimes E}$-moduls were first constructed and characterized by [BDM02] when $g \curvearrowright E$ has one orbit. In general, if one writes g as the product of disjoint cycles $g = g_1 \ldots g_k$ where k is the number of g-orbits of E, then a direct sum of tensor products of g_i-permutation twisted modules is a g-twisted module. We call such module $\otimes$-separable. It is known that all g-twisted modules are $\otimes$-separable if $\mathbb{V}$ is rational [BDM02].

In this article, we use the main result of [Gui21b] to construct an explicit isomorphism from the space of genus-0 conformal blocks associated to the $\otimes$-separable G-twisted $\mathbb{V}^{\otimes E}$-modules to the space of conformal blocks associated to the untwisted $\mathbb{V}$-modules and a branched covering C of the Riemann sphere $\mathbb{P}^1$. When $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational, we use the above result, the (untwisted) factorization property [DGT19b], and the Riemann-Hurwitz formula to completely determine the fusion rules among G-twisted $\mathbb{V}^{\otimes E}$-modules.

Furthermore, assuming $\mathbb{V}$ is as above, we prove that the sewing/factorization of genus-0 G-twisted $\mathbb{V}^{\otimes E}$-conformal blocks holds, and corresponds to the sewing/factorization of untwisted $\mathbb{V}$-conformal blocks associated to the branched coverings of $\mathbb{P}^1$. This proves, in particular, the operator product expansion (i.e., associativity) of G-twisted $\mathbb{V}^{\otimes E}$-intertwining operators (a key ingredient of the G-crossed braided tensor category $\mathrm{Rep}^G(\mathbb{V}^{\otimes E})$ of the G-twisted $\mathbb{V}^{\otimes E}$-modules) without assuming that the fixed point subalgebra $(\mathbb{V}^{\otimes E})^G$ is C_2-cofinite (and rational), a condition known so far only when G is solvable and remains a conjecture in the general case. More importantly, this result implies that besides the fusion rules, the associativity isomorphism of $\mathrm{Rep}^G(\mathbb{V}^{\otimes E})$ is also characterized by the higher genus data of untwisted $\mathbb{V}$-conformal blocks, which gives a new insight into the category $\mathrm{Rep}^G(\mathbb{V}^{\otimes E})$.

We also discuss the applications to conformal nets, which are indeed the original motivations for the author to study the subject of this paper.

1

Contents

2

0 Introduction

0.1 Motivations from fusion rule calculations

An important problem that has long attracted people in orbifold conformal field theory is the following: given a nice (say, CFT-type, C_2-cofinite, rational) vertex operator algebra (VOA) $\mathbb{V}$ (or a completely rational conformal net $\mathcal{A}$) with a finite automorphism group G, once we know the tensor category $\mathrm{Rep}(\mathbb{V})$ of untwisted $\mathbb{V}$-modules, what do we know about the category $\mathrm{Rep}^G(\mathbb{V})$ of G-twisted $\mathbb{V}$-modules?

The most studied examples of this problem are permutation orbifold VOAs, namely, the permutation action of G on $\mathbb{U} = \mathbb{V}^{\otimes E}$ where E is a finite set and G is a subgroup of the permutation group $\mathrm{Perm}(E)$. As shown in [BHS98, BDM02] (cf. [LX04, KLX05] for the conformal net version), all G-twisted $\mathbb{U}$-modules can be explicitly constructed from (untwisted) $\mathbb{V}$-modules. The n-fold covering maps $z \mapsto z^n$ for $\mathbb{P}^1$ (for VOAs) or S^1 (for conformal nets) play a central role in these constructions. Recently, [DXY21] gave a construction using Zhu's algebras.

Fusion rules

Fusion rules among G-twisted $\mathbb{U}$-modules have also been investigated by physicists [BHS98, Ban98, Ban02] and mathematicians. On the mathematics side, the fusion rules among $\mathbb{Z}_2$-twisted modules of $\mathbb{V}^{\otimes 2}$ or $\mathcal{A}^{\otimes 2}$, or among two twisted modules and an untwisted one, have been completely determined in [LX04, KLX05, DLXY19]. It turns out that the structure theory of completely rational conformal nets relies essentially on the idea of cyclic permutation orbifolds [KLM01, LX04].

On the abstract tensor category/modular functor level, the computation of fusion rules of $\mathrm{Rep}^G(\mathbb{U})$ (or any modular category) is more or less complete. [BS11] used topological methods to construct a G-crossed braided weakly fusion category $\mathscr{C}$ extending $\mathrm{Rep}(\mathbb{V})^{\boxtimes E} \simeq \mathrm{Rep}(\mathbb{U})$; the fusion rules of $\mathscr{C}$ can be described easily in terms of the higher genus fusion rules of $\mathrm{Rep}(\mathbb{V})$. [BS11] gives us a hint on what the fusion ring of $\mathrm{Rep}^G(\mathbb{U})$ looks like: If the expected property that $\mathscr{C}$ is rigid were proved ([BS11] proved this only for the $\mathbb{Z}_2$-permutation), then a result of [ENO10] would imply that $\mathscr{C}$ and $\mathrm{Rep}^G(\mathbb{U})$ have equivalent fusion rings, even though they are not necessarily equivalent as fusion categories (cf. [Bis20, EG18]). More recently, [BJ19] and [Del19] provided rigorous and complete algorithms for computing the fusion ring of $\mathrm{Rep}^G(\mathbb{U})$ (or any (spherical) crossed extension of a modular category). In particular, in the case that G is generated by a one cycle permutation, [BJ19] explicitly calculated the fusion rings, which agree with [BS11].

However, the important progress mentioned in the previous paragraph does not mean that the task of computing permutation fusion rules for VOAs or conformal nets is complete. Indeed, when applied to the VOA/conformal net context, the above categorical results do not directly give us the fusion rule among three twisted $\mathbb{V}$- or $\mathcal{A}$-modules constructed explicitly in [BDM02, LX04, KLX05]. In other words, *we still need to identify the objects constructed categorically in [BS11, BJ19, Del19] with those constructed explicitly in the VOA/conformal net context*. On the other hand, the objects in [LX04, KLX05, DLXY19] are explicit, but their results on the fusion rules are far from complete

even in the special case of cyclic permutations.

Why higher genus data appear in the fusion rules

We should do more than just calculate the fusion rules. The (genus-0) fusion rules describe the dimensions of the spaces of conformal blocks associated to $\mathbb{P}^1$ with three marked points. However, the expression of permutation-twisted fusion rules (as calculated in the previously mentioned literature) involves dimensions of untwisted conformal blocks on higher genus Riemann surfaces with arbitrary numbers of marked points. *We should develop a theory that explains this higher genus phenomenon; the complete computation of permutation-twisted fusion rules should follow as a consequence.*

A rough explanation of this phenomenon is this: the permutation-twisted conformal blocks associated to a pointed compact Riemann surface X correspond to untwisted conformal blocks associated to a branched covering of X. Bantay already noticed this fact in [Ban98, Ban02]. He used this idea to compute the modular data of $\mathrm{Rep}^G(\mathbb{U})$, and to calculate the fusion rules indirectly using these modular data. But the coverings used by Bantay are mainly the unbranched coverings of tori, which are also tori. From Bantay's work, it is not clear what the correct branched coverings (called the **permutation coverings** in this article) are in general.

In [BS11], Barmeier and Schweigert have made important progress on this problem by constructing the *topological* permutation coverings. Thus, to complete the story, we should not only equip the permutation covering C with a complex structure (which is a standard process), but also determine the locations and the local coordiates of the marked points, and find the correspondence between these marked points and the untwisted $\mathbb{V}$-modules. This task is nontrivial, and we will briefly explain our answer in the following section of the Introduction.

0.2   Main result: the twisted/untwisted correspondence

Positively N-pathed Riemann spheres with local coordinates

We are mainly interested in the case that the (complex analytic) permutation covering C is the branched covering of $\mathbb{P}^1$, but in fact many discussions also apply to other compact Riemann surfaces.

To determine the covering $\varphi : C \to \mathbb{P}^1$ and the marked points of C with the correct labels, we need to add not only distinct marked points $x_1, \ldots, x_N \in C$ and local coordinates η_j at each x_j (i.e., an analytic injective function from a neighborhood W_j of x_j to $\mathbb{C}$ sending x_j to 0), but also paths $\gamma_1, \ldots, \gamma_N$ in $\mathbb{P}^1 \backslash \mathbf{S}$ (where $\mathbf{S} = \{x_1, \ldots, x_N\}$) with common end point $\gamma_\bullet(1)$. We assume that each neighborhood W_j is an open disc contain only x_j among $x_1, \ldots, x_N$. We assume that $\gamma_j(0)$ is in the punctured disc $W_j \backslash \{x_j\}$ and satisfies

$$\eta_j(\gamma_j(0)) > 0. \tag{0.2.1}$$

Then the data

$$\mathfrak{P} = (\mathbb{P}^1; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N; \gamma_1, \ldots, \gamma_N) = (\mathbb{P}_1; x_\bullet; \eta_\bullet; \gamma_\bullet) \tag{0.2.2}$$

is called a **positively N-pathed Riemann sphere with local coordinates**. See Figure 0.2.1 for an example.
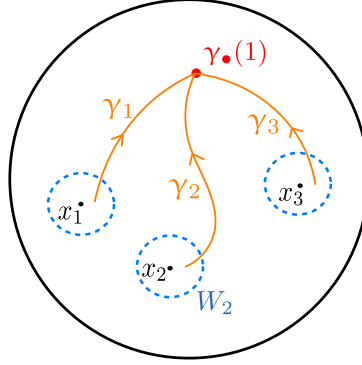

Figure 0.2.1. A positively 3-pathed Riemann sphere with local coordinates

Indeed, in the main body of this article, we assume one more condition on $\mathfrak{P}$ for the sake of convenience. The fundamental group

$$\Gamma = \pi_1(\mathbb{P}^1\backslash\mathbf{S}, \gamma_\bullet(1))$$

is free with $N-1$ free generators. For each j, let ϵ_j be an anticlockwise circle in $W_j\backslash\{x_j\}$ from and to $\gamma_j(0)$, and let

$$\alpha_j = \gamma_j^{-1}\epsilon_j\gamma_j.$$

Our assumption is that Γ is generated by the homotopy classes $[\alpha_1], \ldots, [\alpha_N]$. Thus, the monodromy on $\mathbb{P}^1\backslash\mathbf{S}$ is determined by that around the marked points.

Now, fix a homomorphism $\Gamma \to \mathrm{Perm}(E)$, namely, an action of Γ on the finite set E. We let g_j be the action of $[\alpha_j]$. Then these $g_\bullet$ determine the action of Γ, and we call the $g_\bullet$ arising from the actions of Γ *admissible* (with respect to $\mathfrak{P}$). These group elements determine the types of twisted modules associated to the marked points.

Conformal blocks

Let us pause for a moment and discuss the meaning of conformal blocks. This will also motivate the definition of permutation coverings. [FB04] defined an (infinite rank) vector bundle $\mathscr{U}_C$ (called **sheaf of VOA** for $\mathbb{U}$ in this article) over any Riemann surface C whose fibers are equivalent to the VOA $\mathbb{U}$, and whose transition functions are described by Huang's change of coordinate formula [Hua97]. If we associate untwisted $\mathbb{U}$-modules $\mathcal{W}_1, \ldots, \mathcal{W}_N$ to the marked points $x_1, \ldots, x_N$, then a conformal block was defined by [FB04] to be a linear functional $\psi : \mathcal{W}_\bullet = \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_N \to \mathbb{C}$ such that for each $w_\bullet = w_1 \otimes \cdots \otimes w_N$, the following condition holds: for all j, the expressions

$$u \in \mathbb{U} \mapsto \psi(w_1 \otimes \cdots \otimes Y(u, z)w_j \otimes \cdots \otimes w_N) \tag{0.2.3}$$

converge absolutely as a series of z when $|z|$ is reasonably small, and (assuming the a trivialization $\mathcal{U}_\varrho(\eta_j) : \mathscr{U}_{W_j} \xrightarrow{\simeq} \mathbb{U} \otimes_\mathbb{C} \mathscr{O}_{W_i}$ defined by the local coordinate η_j) can be extended to the same $\mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$-module morphism $\wr\psi(\cdot, w_\bullet) : \mathscr{U}_{\mathbb{P}^1\backslash\mathbf{S}} \to \mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$. When $\mathbb{P}^1$ is

replaced by any compact Riemann surface, one can use the same definition to define conformal blocks. See Sec. 1.2 for details.

We can see that in order to define untwisted conformal blocks, we do not need to choose paths in pointed Riemann surfaces (with local coordinates). This is not true for untwisted conformal blocks: $Y(u, z)$ is now multivalued over z, so (0.2.3) can never be extended to the same morphism.

The correct definition is as follows. Associate a g_j-twisted $\mathbb{U}$-module $\mathcal{W}_j$ (whose vertex operation Y^{g_j} is temporarily also denoted by Y) to the marked point x_j, where $g_1, \ldots, g_N$ arise from an action $\Gamma \curvearrowright E$. Due to the positivity condition (0.2.1), when z is close to $\eta_j(\gamma_j(0))$, we may uniquely determine $Y(u, z)w_j$ by the fact that $\arg z$ is close to 0. Then we require that (0.2.3) converges absolutely when z is near $\eta_j(\gamma_j(0))$, that for different j_1, j_2, the expression (0.2.3) with $j = j_1$ can be analytically continued to (0.2.3) with $j = j_2$ along the path $\gamma_{j_1}\gamma_{j_2}^{-1}$, and can furthermore be extended to a "multivalued" morphism $\mathscr{U}_{\mathbb{P}^1\backslash \mathbf{S}} \to \mathscr{O}_{\mathbb{P}^1\backslash \mathbf{S}}$. Cf. Subsec. 1.3.3.

We remark that the above definition relies on the previously mentioned assumption that Γ is generated by all $[\alpha_\bullet]$. If $\mathbb{P}^1$ is replaced by a higher genus X, then this condition is never satisfied. In this case, we just need to add one more condition relying on the data $\Gamma \curvearrowright E$. We do not explain this condition in the Introduction, and refer the readers to Rem. 2.1.5 for details.

Permutation coverings

We now describe the permutation covering of $\mathfrak{P}$ associated to the admissible elements $g_\bullet$ (equivalently, associated to the action $\Gamma \curvearrowright E$). The permutation covering $\varphi : C \to \mathbb{P}^1$ is unbranched outside $\mathbf{S}$, and is determined by the restriction $\varphi : C\backslash\varphi^{-1}(\mathbf{S}) \to \mathbb{P}^1\backslash\mathbf{S}$. By algebraic topology, a (resp. finite) *connected* covering of $\mathbb{P}^1\backslash\mathbf{S}$ is described by either of the following two equivalent objects:

(1) A conjugacy class of (resp. cofinite) subgroups of the fundamental group Γ.

(2) A transitive (i.e., single-orbit) action of Γ on a (resp. finite) set.

For our purpose, it is more convenient to use (2) to describe $\varphi : C\backslash\varphi^{-1}(\mathbf{S}) \to \mathbb{P}^1\backslash\mathbf{S}$. Then each connected component C^Ω of $C\backslash\varphi^{-1}(\mathbf{S})$ corresponds to an orbit Ω of E (under the action of Γ). The precise description is the following elegant statement (cf. Thm. 2.1.8):

There exists a Γ-covariant bijection $\Psi_{\gamma_\bullet(1)} : E \to \varphi^{-1}(\gamma_\bullet(1))$.

By "Γ**-covariant**", we mean that for every $e \in E$ and every closed path in $\mathbb{P}^1\backslash\mathbf{S}$ from and to $\gamma_\bullet(1)$, the lift of μ to $C\backslash\varphi^{-1}(\mathbf{S})$ ending at $\Psi_{\gamma_\bullet(1)}(e)$ must start from $\Psi_{\gamma_\bullet(1)}([\mu]e)$. (See Def. 2.1.7.) Then any two branched coverings of $\mathbb{P}^1$ with such Γ-covariant bijections are equivalent (Thm. 2.1.14).

The set of marked points of our permutation covering C is just $\varphi^{-1}(\mathbf{S})$. The untwisted $\mathbb{V}$-conformal blocks will be defined for C and these marked points. In order to associate the correct $\mathbb{V}$-module to each marked point, it is important to *label the marked points with certain orbits of E*. Note that $\varphi^{-1}(\mathbf{S})$ is the disjoint union of all $\varphi^{-1}(x_j)$. The the labeling of the elements of $\varphi^{-1}(x_j)$ is given by a bijection

$$\Upsilon_j : \{\langle g_j\rangle\text{-orbits in } E\} \longrightarrow \varphi^{-1}(x_j)$$

described as follows. We abbreviate Υ_j as Υ.

First of all, for each path λ in $\mathbb{P}^1 \backslash \mathbf{S}$ from a point x to $\gamma_\bullet(1)$, we define a bijection

$$\Psi_\lambda : E \to \varphi^{-1}(x)$$

sending each $e \in E$ to the initial point of the lift of λ to $C \backslash \varphi^{-1}(\mathbf{S})$ ending at $\Psi_{\gamma_\bullet(1)}(e)$. Recall that W_j is an open disc centered at x_j and contains $\gamma_j(0)$. For each g_j-orbit $\langle g_j \rangle e$, we can find a unique connected component $\widetilde{W}_j$ of $\varphi^{-1}(W_j)$ whose intersection with $\varphi^{-1}(\gamma_j(0))$ is exactly the set of points

$$\Psi_{\gamma_j}\big(\langle g_j \rangle e\big) := \Big\{ \Psi_{\gamma_j}(g_j^k e) : k \in \mathbb{Z} \Big\}$$

evenly located around the center $\widetilde{W}_j \cap \varphi^{-1}(x_j)$ of $\widetilde{W}_j$. (The set $\widetilde{W}_j \cap \varphi^{-1}(x_j)$ indeed has only one element.) The size of $\Psi_{\gamma_j}\big(\langle g_j \rangle e\big)$ equals the size $k = |\langle g_j \rangle e|$ of the orbit and also equals the branching index of φ at $\widetilde{W}_j \cap \varphi^{-1}(x_j)$. (See Prop. 2.1.11.) Then $\Upsilon(\langle g_j \rangle e)$ is defined to be the unique point of $\widetilde{W}_j \cap \varphi^{-1}(x_j)$.

We see that the covering $\varphi : C \to \mathbb{P}^1$ depends only $\Gamma \curvearrowright E$, but the labeling of the marked points of C depends also on the paths $\gamma_\bullet$.

Finally, we choose local coordinates of C at each marked point as follows. For each g_j-orbit we fix a distinct point, called the **marked point** of that orbit. We let

$$E(g_j) = \{\text{marked points of } g_j\text{-orbits}\}.$$

Recall that η_j is a local coordinate defined on W_j sending x_j to 0. For each marked point $\Upsilon(\langle g_j \rangle \mathcalligra{x})$ of C (where $\mathcalligra{x} \in E(g_j)$) which is contained in a unique connected component $\widetilde{W}_{j,\mathcalligra{x}}$ of $\varphi^{-1}(W_j)$, there are $k = |\langle g_j \rangle \mathcalligra{x}|$ different injective analytic functions on $\widetilde{W}_{j,\mathcalligra{x}}$ whose k-th power equals $\eta_j \circ \varphi$. The local coordinate we choose at $\Upsilon(\langle g_j \rangle \mathcalligra{x})$ and denote by $\widetilde{\eta}_{j,\mathcalligra{x}}$ is the one of them that satisfies

$$\widetilde{\eta}_{j,\mathcalligra{x}}\big(\Psi_{\gamma_j}(\mathcalligra{x})\big) > 0,$$

which exists because of $\Psi_{\gamma_j}(\mathcalligra{x}) \in \varphi^{-1}(\gamma_j(0))$ and the positivity condition (0.2.1).

The above branched covering $\varphi : C \to \mathbb{P}^1$ (or just C), together with the labeled marked points and local coordinates, is denoted by $\mathfrak{X}$ and called the **permutation covering** of $\mathfrak{P}$ (see (0.2.2)) associated to the action $\Gamma \curvearrowright E$ and the set $E(g_\bullet)$ of marked points of $g_\bullet$-orbits.

The correspondence of twisted/untwisted conformal blocks

If $g \in \mathrm{Perm}(E)$ has only one cycle, the g-twisted $\mathbb{U}$-modules are completely determined by [BDM02]. When g is a product of disjoint cycles $h_1 h_2 \cdots h_l$ where l is the number of g-orbits, by taking the tensor product of h_j-twisted modules, one obtains a g-twisted $\mathbb{U}$-module. We call any direct sum of such twisted modules to be $\otimes$**-separable**. If $\mathbb{V}$ is rational, then any g-twisted $\mathbb{V}$-module is $\otimes$-separable [BDM02, Thm. 6.4].

For each g_j-orbit $\langle g_j \rangle \mathcalligra{x}$ of E, we choose a $\mathbb{V}$-module $\mathbb{W}_{j,\mathcalligra{x}}$. Then by [BDM02], the vector space $\mathcal{W}_j = \otimes_{\mathcalligra{x} \in E(g_j)} \mathbb{W}_{j,\mathcalligra{x}}$ is naturally equipped with a ($\otimes$-separable) g_j-twisted

$\mathbb{U}$-module structure (see also Subsec. 2.2). It should be reminded that this twisted module structure depends not only on the g_j-orbits but also on their marked points. For example, when $E = \{1, \dots, n\}$ and g_j is the cycle $g = (12 \cdots n)$, then for each $u = v_1 \otimes \cdots \otimes v_n \in \mathbb{V}^{\otimes n}$, setting $\mathfrak{o}_n = e^{-2\pi\mathbf{i}/n}$, the twisted vertex operation $Y^g(u, z)$ at $z = 1$ (with $\arg z = 0$) is expressed by the untwisted ones $Y(\square v_i, \mathfrak{o}_n^{i-1})$ (for all i) where $\square$ is a suitable operator. This corresponds to the marked point 1. If we choose another marked point l, then $Y^g(u, 1)$ should be expressed by all $Y(\square v_i, \mathfrak{o}_n^{i-l})$. There is no reason to assume that 1 is superior than any other marked point.

We associate each $\mathbb{V}$-module $\mathbb{W}_{j,\mathfrak{x}}$ to the marked point $\Upsilon(\langle g_j \rangle_\mathfrak{x})$ of $\mathfrak{X}$, and associate the twisted $\mathbb{U}$-module $\mathcal{W}_j$ to the marked point x_j of $\mathfrak{P}$. Then, our first main result is:

Theorem A. *(Cf. Thm. 2.2.5) A linear functional*

$$\psi : \bigotimes_{1 \leqslant j \leqslant N} \mathcal{W}_j = \bigotimes_{1 \leqslant j \leqslant N} \bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} \to \mathbb{C}$$

is a conformal block associated to $\mathfrak{P}$ and the associated twisted $\mathbb{U} = \mathbb{V}^{\otimes E}$-modules if and only if it is a conformal block associated to $\mathfrak{X}$ and the associated untwisted $\mathbb{V}$-modules.

In particular, we have constructed an explicit isomorphism between the two spaces of conformal blocks. The proof of this theorem relies on the main results of [Gui21b].

Sewing and factorization

We can relate not only the permutation-twisted and untwisted conformal blocks, but also their sewing and factorization. If we have two positively pathed Riemann spheres with local coordinates $\mathfrak{P}^a$ and $\mathfrak{P}^b$, we can sew these two spheres by removing one disc from $\mathfrak{P}^a$ around one of its marked point x_0, removing another one around y_0 from $\mathfrak{P}^b$, and gluing the remaining part. Similarly, we can sew their permutation coverings $\mathfrak{X}^a, \mathfrak{X}^b$. Corresponding to this geometric sewing, we have the well-known sewing of conformal blocks. (See Subsec. 1.2.2.)

The product and the iterate of (twisted) intertwining operators can be viewed as sewing (twisted) conformal blocks associated to $\mathfrak{P}^a$ and $\mathfrak{P}^b$. These two types of sewing are equivalent and related by the **operator product expansion (OPE)** (i.e., associativity) of intertwining operators. Such relation defines the associativity isomorphisms of the crossed-braided tensor category $\mathrm{Rep}^G(\mathbb{U})$ of G-twisted $\mathbb{U}$-modules (where G is any subgroup of $\mathrm{Perm}(E)$). (Cf. [Hua95, McR21].) Our second main result is this:

Theorem B. *The following are true.*

1. *(Thm. 3.1.3) There is a suitable sewing $\mathfrak{X}^{a\#b}$ of $\mathfrak{X}^a$ and $\mathfrak{X}^b$ which is isomorphic to the permutation covering of the sewing $\mathfrak{P}^{a\#b}$ of $\mathfrak{P}^a$ and $\mathfrak{P}^b$.*

2. *(Thm. 3.2.1) Assume $\mathbb{V}$ is C_2-cofinite. If ψ^a, ψ^b are permutation-twisted $\mathbb{U}$-conformal blocks associated respectively to $\mathfrak{P}^a, \mathfrak{P}^b$ (equivalently, $\mathbb{V}$-conformal blocks associated to $\mathfrak{X}^a, \mathfrak{X}^b$), then their sewing as permutation-twisted $\mathbb{U}$-conformal blocks agree with that as $\mathbb{V}$-conformal blocks, and the result of this sewing converges absolutely to a $\mathbb{U}$-conformal block associated to $\mathfrak{P}^{a\#b}$ (equivalently, a $\mathbb{V}$-conformal block associated to $\mathfrak{X}^{a\#b}$).*

3. (Thm. 3.2.3) *If $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational, then any permutation-twisted $\mathbb{U}$-conformal block associated to $\mathfrak{P}^{a\#b}$ can be expressed in a unique way (understood in a suitable sense) as the sewing of those associated to $\mathfrak{P}^a$ and $\mathfrak{P}^b$.*

The first part of this theorem says that sewing and taking permutation coverings are commuting procedures. The second part says that sewing permutation-twisted $\mathbb{U}$-conformal blocks amounts to sewing untwisted $\mathbb{V}$-conformal blocks. The third part is the genus-0 sewing/factorization property for permutation-twisted conformal blocks. Its proof relies in particular on the untwisted factorization property [DGT19b].

Thus, if we assume $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational, then by the above theorem, *the OPE of permutation-twisted $\mathbb{U}$-intertwining operators can be interpreted as an associativity of (possibly) higher genus untwisted $\mathbb{V}$-conformal blocks*. (Here, the associativity means the equivalence of two different ways of factoring a possibly higher genus untwisted $\mathbb{V}$-conformal block into the sewing of two $\mathbb{V}$-conformal blocks). Therefore, we may use the higher genus data in the representation theory of $\mathbb{V}$ to study $\mathrm{Rep}^G(\mathbb{U})$, and vice versa. See Sec. 4.2 for a rigorous description.
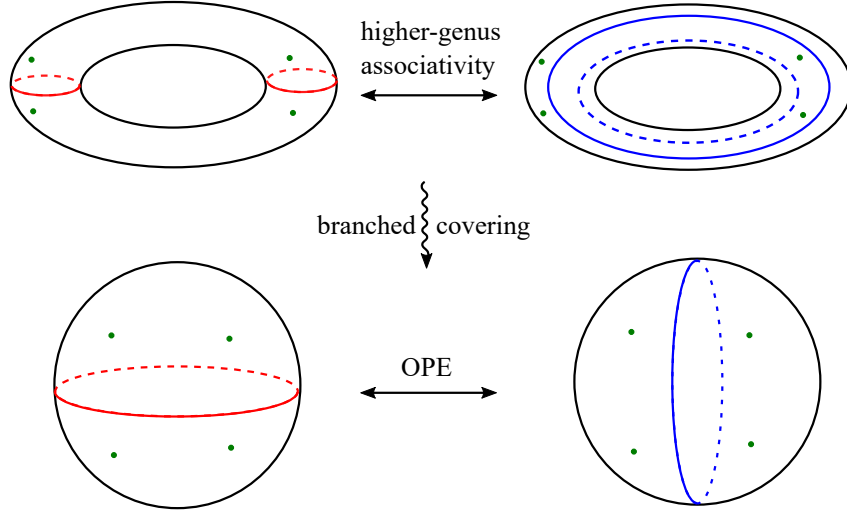
Figure 0.2.2. The permutation-twisted OPE is equivalent to an associativity of (possibly) higher genus conformal blocks via the permutation covering.

An example for $E = \{1, 2\}$ is illustrated in Figure 0.2.2. The bottom of this fugure shows two ways of factoring a 4-pointed (more precisely, positively 4-pathed) sphere that correspond to the product and the iterate of $\mathbb{Z}_2$-permutation twisted intertwining operators. All the four marked points are associated with $(1, 2)$-twisted $\mathbb{U} = \mathbb{V}^{\otimes 2}$-modules. The horizontal red circles and the vertical blue circles are associated with untwisted $\mathbb{V}^{\otimes 2}$-modules. All the four marked points are branched points with index 2, and the permutation covering is genus 1. In this example, the OPE of $\mathbb{Z}_2$-permutation twisted $\mathbb{V}^{\otimes 2}$-intertwining operators amounts to an associativity of genus-1 untwisted $\mathbb{V}$-conformal blocks.

From Figure 0.2.2, the readers may notice the striking fact that *this genus-1 associativity is the one that describes modular invariance*! Technically, the geometry of this associativity is *topologically but not complex-analytically equivalent* to that of the modular invariance studied in [Zhu96, Miy04, Hua05] and many other VOA articles: the

genus-1 sewing in Figure 0.2.2 is very different from sewing the boundary circles of *standard* complex annuli. Nevertheless, there is good reason to believe that these two approaches will provide many of the same results. For instance, the S-matrices defined originally by the invariance of genus-1 conformal blocks under the modular transform $\tau \mapsto -1/\tau$ should also be described by the genus-1 associativity in Figure 0.2.2, since it should be described by the braid/fusion matrices in the category $\mathrm{Rep}^{\mathbb{Z}_2}(\mathbb{V}^{\otimes 2})$ of $\mathbb{Z}_2$-twisted $\mathbb{V}^{\otimes 2}$-modules [LX19].

0.3 Motivations from conformal nets

Multi-interval Connes fusion and higher genus CFT

The Haag-Kastler/conformal net theory (namely, the operator-algebraic approach to conformal field theory (CFT)) describes quantum fields on the Minkowski space $\mathbb{R}^{1,1}$, in contrast to the VOA approach where the field operators are defined on the Euclidean space $\mathbb{R}^2 = \mathbb{C}$. The Euclidean CFT can be defined on higher genus Riemann surfaces by gluing genus-0 ones, since the Euclidean conformal symmetry is local. The $\mathbb{R}^2$-CFT (i.e., the genus-0 VOA theory) can be translated to the $\mathbb{R}^{1,1}$-CFT by the Wick rotation.[1] But it was not clear how to translate higher genus Euclidean CFT to Minkowskian CFT.

From this perspective, it is rather surprising that higher genus CFT data could be understood in the conformal net framework, which was indeed achieved by the multi-interval Connes fusion/Jones-Wassermann subfactors/Doplicher-Haag-Roberts superselection theory. (The single interval version describes genus-0 data; see e.g. [FRS89, FRS92, Was98].) This observation was already made in [Was94]. Later, it was proved in [KLM01, LX04] that if the index of multi-interval Jones-Wasserman subfactor (called the μ-index) is finite, then the category of semi-simple representations of the conformal net $\mathcal{A}$ is a modular tensor category; in particular, the S-matrices defined by the Hopf link is non-degenerate. Moreover, if the μ-index is one, then $\mathcal{A}$ is holomorphic, i.e., it has only one irreducible representation: the vacuum representation.

If one notices the fact that a holomorphic VOA is described by the modular invariance of the one-point genus-1 conformal block defined by the q-trace of the vacuum vertex operation, one is surprised that holomorphic conformal nets are characterized by μ index $= 1$, i.e., by the (multi-interval) Haag duality. Indeed, in full and boundary CFT, there are also equivalences of *modular invariance and "multi-region" Haag duality* (called strong Haag duality in [Hen14]). (For full CFT, compare [BKL15, Prop. 6.6] with [Kong08, Thm. 5.7] and [KR09a, Thm. 3.4]. For boundary CFT, see [KR09b, Sec. 4] and the reference therein, and note that a question posed after Thm. 4.8 was solved in [BKL15, Prop. 4.18].)

A main motivation of this article is to give a conceptual explanation of why multi-interval/multi-region Connes fusion is related to higher genus CFT. We now know the answer: it is well-known that multi-interval/multi-region Connes fusion is closely related to permutation orbifold CFT [LX04, KLX05], and from the main results of this

---

[1]Strictly speaking, the CFT one gets on $\mathbb{R}^{1,1}$ is a Wightmann field theory. To get Haag-Kastler CFT, one should take the smeared fields and consider certain von Neumann/C^*-algebras generated by them. See [CKLW18].

article, we know that the latter has a higher genus CFT interpretation. (See [BDH17] or [LX19] for different explanations.)

Higher genus unitary VOA/conformal net correspondence

A systematic study of the relationships between unitary VOAs and conformal nets was initiated by [CKLW18]. In particular, for many unitary VOAs, the corresponding conformal nets were constructed by taking the smeared vertex operators. A different approach was given by [Ten19a]. The methods in these two papers were generalized to relate the representation categories $\mathrm{Rep}(\mathbb{V}), \mathrm{Rep}(\mathcal{A}_{\mathbb{V}})$ of a (strongly-rational) unitary VOA $\mathbb{V}$ and the corresponding conformal net $\mathcal{A}_{\mathbb{V}}$; see [Ten19b, Ten19c] and [Gui21a, Gui20a].

In [Gui21a, Gui20a], smeared intertwining operators are crucial ingredients relating $\mathrm{Rep}(\mathbb{V})$ and $\mathrm{Rep}(\mathcal{A}_{\mathbb{V}})$: similar to the construction of smeared vertex operators in [CKLW18], we integrate $\mathcal{Y}(w,z)f(z)$ over the unit circle I, where $\mathcal{Y}$ is an intertwining operator of $\mathbb{V}$, w is a vector of a $\mathbb{V}$-module, I is an open non-dense interval of $\mathbb{S}^1$, and f is a smooth function supported in I. Since intertwining operators are 3-pointed genus-0 conformal blocks, our method of comparing the tensor categories $\mathrm{Rep}(\mathbb{V})$ and $\mathrm{Rep}(\mathcal{A}_{\mathbb{V}})$ are essentially genus-0.

It is natural to think about relating higher genus aspects of unitary VOAs and conformal nets by integrating higher genus conformal blocks. However, an arbitrary integral will not give us the correct results. Note that in the genus-0 case, intertwining operators can only be integrated on an interval I of $\mathbb{S}^1$; otherwise the operators one gets will not have nice analytic properties. In arbitrary genus, one should integrate the conformal blocks on a correct real path inside the moduli space of pointed compact Riemann surfaces with (analytic) local coordinates.

Now, the twisted/untwisted correspondence proved in this article suggests how to find the correct paths and integrals: if we take $\mathcal{Y}$ to be a permutation-twisted $\mathbb{U} = \mathbb{V}^{\otimes E}$-intertwining operator, take $I \subset \mathbb{S}^1$ to be an open interval, and consider the family of genus-0 permutation-twisted $\mathbb{U}$-conformal blocks $z \in I \mapsto \mathcal{Y}(\cdot, z)$ as a family of untwisted (possibly higher-genus) $\mathbb{V}$-conformal blocks, then the integral $\int_I \mathcal{Y}(w,z)f(z)dz$ is a correct integral of $\mathbb{V}$-conformal blocks. The path in the moduli space, considered as a family of possibly higher-genus pointed compact Riemann surfaces with local coordinates, is the permutation covering of the family of 3-pointed spheres $z \in I \mapsto (\mathbb{P}^1; 0, z, \infty)$ with local coordinates $\zeta, \zeta - z, \zeta^{-1}$ (where ζ is the standard coordinate of of $\mathbb{C}$).

This observation can be summarized by the slogan: *the genus-0 permutation-twisted VOA/conformal net correspondence (as in [CKLW18] or [Ten19a]) is a higher-genus untwisted VOA/conformal net correspondence.*

This observation shows the importance of relating the VOA and the conformal net tensor categories for orbifold CFT, especially for permutation orbifolds, in the framework of [CKLW18] or [Ten19a]. For instance, the equivalence of tensor categories discussed in [Gui21a, Gui20a] should be generalized to orbifold CFT. We plan to explore this problem in future works.

0.4 Future directions

⊗-nonseparable permutation twisted/untwisted correspondence.

When $\mathbb{V}$ is not rational, many twisted or untwisted $\mathbb{U} = \mathbb{V}^{\otimes E}$-modules are not $\otimes$-separable. For such modules, we need to relate the $\mathbb{U}$-conformal blocks with certain generalized $\mathbb{V}$-conformal blocks (which are not yet studied in the literature). A generalized $\mathbb{V}$-conformal block associated to a N-pointed compact Riemann surface with local coordinates is a linear functional $\psi : \mathcal{M} \to \mathbb{C}$ satisfying a similar invariant condition as ordinary conformal blocks, where $\mathcal{M}$ is a $\mathbb{V}^{\otimes N}$-module. We call such conformal blocks ⊗-**nonseparable**.

We plan to study $\otimes$-nonseparable conformal blocks in the future. We also expect that they will play an essential role in the geometric theory of logarithmic CFT.

Higher genus twisted conformal blocks

This article only discusses the $\mathbb{V}$-conformal blocks that correspond to the genus-0 permutation-twisted $\mathbb{U}$-conformal blocks. Higher genus twisted-conformal blocks are also defined (see Rem. 2.1.5). A natural problem is to generalize the results of this article to higher genus permutation-twisted conformal blocks. We expect that Thm. A can be generalized in a straightforward way. The generalization of Thm. B (the correspondence for sewing and factorization) would be more subtle and requires careful study.

Analogous results for conformal nets

Let $\mathcal{A}$ be a completely-rational conformal net. Conformal blocks for finite-index (untwisted) representations of $\mathcal{A}$ were defined and studied in [BDH17]. In particular, the factorization property was proved in that framework. Let $\mathcal{H}_1, \dots, \mathcal{H}_N, \mathcal{H}_{N+1}$ be finite-index permutation-twisted $\mathcal{A}^{\otimes E}$-representations. Construct an explicit isomorphism from $\mathrm{Hom}_{\mathcal{A}^{\otimes E}}(\mathcal{H}_1 \boxtimes \cdots \boxtimes \mathcal{H}_N, \mathcal{H}_{N+1})$ to a space of $\mathcal{A}$-conformal blocks defined in [BDH17]. Here, $\boxtimes$ is the Connes fusion product.

Geometry of genus-1 CFT: tori vs. elliptic curves

Conventionally, the genus-1 properties of Euclidean CFT were studied using the *torus model*: taking the q-trace of vertex operators corresponds geometrically to sewing a standard annulus $\{a < |z| < b\}$ along the two boundaries. Our results in this article suggest that genus-1 properties can also be studied using the *elliptic curve model*: elliptic curves arise naturally not as the sewing of annuli, but as the branched coverings of $\mathbb{P}^1$. This geometric picture corresponds to the permutation-twisted/untwisted correspondence in the VOA world.

It would be interesting to explore the genus-1 aspects of VOA via the elliptic curve model. For instance, when $\mathbb{V}$ is strongly rational, using the genus-0 theory for permutation orbifolds, one may try to understand or give new proofs for the rigidity and the modularity of $\mathrm{Rep}(\mathbb{V})$ (originally proved in [Hua08a, Hua08b] using the torus model), and to understand the full and boundary Euclidean CFT. (It should be sufficient to

just consider $\mathbb{Z}_2$-permutations.) We also hope that this new framework will shed some light on proving the rigidity and understanding the modularity when $\mathbb{V}$ is not rational.

0.5 Outline

In Chapter 1, we review the definition and the basic properties of (untwisted) conformal blocks. We also define a notion of conformal blocks for twisted VOA modules, and prove some elementary facts. For twisted conformal blocks, we focus mainly on the genus-0 case, although a definition for higher genus ones is also given (see Rem. 2.1.5).

We remark that a notion of conformal blocks for twisted modules already exists in the algebro-geometric approach to VOA (cf. [FS04]). Assuming genus-0 for simplicity, the main difference between that notion and ours is that [FS04] considered (single-valued) functions/ morphisms on a finite Galois branched covering of $\mathbb{P}^1$, while we consider multi-valued functions/morphisms on $\mathbb{P}^1$. These two definitions should be equivalent, although a translation between them may take quite a few pages. We choose our definition because it is not difficult to relate to the twisted intertwining operators in the VOA literature [Hua18, McR21]. Note also that the permutation coverings of $\mathbb{P}^1$ are not necessarily Galois, thus they are in general not the same as the coverings considered in [FS04].

In Chapter 2, we describe the permutation coverings in details, and relate the permutation-twisted genus-0 conformal blocks and untwisted conformal blocks. In particular, we prove Main Thm. A.

In Chapter 3, we show that the sewing procedure commutes with taking permutation coverings. We then show the permutation-twisted/untwisted correspondence for the sewing and factorization of conformal blocks. Namely, we prove Main Thm. B.

In Chapter 4, we relate our definition of twisted conformal blocks with twisted intertwining operators. We then prove the OPE for permutation-twisted intertwining operators, and show that they correspond to the associativity of possibly higher genus untwisted conformal blocks (as indicated in Figure 0.2.2).

Acknowledgement

1 General results

1.1 The setting

We set $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. Let $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. For each $r > 0$, we let $\mathcal{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathcal{D}_r^\times = \mathcal{D}_r\backslash\{0\}$. For any topological space X, we define the configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_N) \in X^n : x_i \neq x_j \; \forall 1 \leqslant i < j < n\}$.

For a complex manifold X, $\mathscr{O}_X$ denotes the sheaf of (germ of) holomorphic functions on X, and $\mathscr{O}(X) = \mathscr{O}_X(X)$ is the space of (global) holomorphic functions on

X.

Given a Riemann surface C and a point $x \in C$, a **local coordinate** $\eta \in \mathscr{O}(U)$ of C at x means that η is a holomorphic injective function on a neighborhood $U \subset C$ of x, and that $\eta(x) = 0$.

For any ($\mathbb{C}$-)vector space W, we define four spaces of formal series

$$W[[z]] = \left\{ \sum_{n \in \mathbb{N}} w_n z^n : \text{each } w_n \in W \right\},$$

$$W[[z^{\pm 1}]] = \left\{ \sum_{n \in \mathbb{Z}} w_n z^n : \text{each } w_n \in W \right\},$$

$$W((z)) = \left\{ f(z) : z^k f(z) \in W[[z]] \text{ for some } k \in \mathbb{Z} \right\},$$

$$W\{z\} = \left\{ \sum_{n \in \mathbb{C}} w_n z^n : \text{each } w_n \in W \right\}.$$

If X is a locally compact Hausdorff space, and if $\sum_{n \in \mathbb{C}} f_n$ is a series of continuous functions f_n on X, we say this series **converges absolutely and locally uniformly (a.l.u.)** on X, if for each compact subset $K \subset X$,

$$\sup_{x \in K} \sum_{n \in \mathbb{C}} |f_n(x)| < +\infty. \tag{1.1.1}$$

Let $\mathbb{V}$ be a VOA with vertex operator $Y(v, z) = \sum_{n \in \mathbb{Z}} Y(v)_n z^{-n-1}$, vacuum vector $\mathbf{1}$, and Virasoro operators $L_n = Y(\mathbf{c})_{n+1}$. We assume throughout this article that a VOA $\mathbb{V}$ has L_0-grading $\mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}(n)$ where each $\mathbb{V}(n)$ is finite-dimensional.

As in [Gui21b], a $\mathbb{V}$-module $\mathbb{W}$ always means a **finitely-admissible $\mathbb{V}$-module**. This means that $\mathbb{W}$ is a weak $\mathbb{V}$-module in the sense of [DLM97], that $\mathbb{W}$ is equipped with a diagonalizable operator $\widetilde{L}_0$ satisfying

$$[\widetilde{L}_0, Y_{\mathbb{W}}(v)_n] = Y_{\mathbb{W}}(L_0 v)_n - (n+1)Y_{\mathbb{W}}(v)_n, \tag{1.1.2}$$

that the eigenvalues of $\widetilde{L}_0$ are in $\mathbb{N}$, and that each eigenspace $\mathbb{W}(n)$ is finite-dimensional. Let

$$\mathbb{W} = \bigoplus_{n \in \mathbb{N}} \mathbb{W}(n)$$

be the grading given by $\widetilde{L}_0$. We set

$$\mathbb{W}^{\leqslant n} = \bigoplus_{0 \leqslant k \leqslant n} \mathbb{W}(k)$$

We choose

$$\widetilde{L}_0 = L_0 \qquad \text{on } \mathbb{V}.$$

Any $\widetilde{L}_0$-eigenvector of $\mathbb{W}$ is called $\widetilde{L}_0$-**homogeneous**.

14

We can define the **contragredient $\mathbb{V}$-module** $\mathbb{W}'$ of $\mathbb{W}$ as in [FHL93]. We choose $\widetilde{L}_0$-grading to be

$$\mathbb{W}' = \bigoplus_{n \in \mathbb{N}} \mathbb{W}'(n), \qquad \mathbb{W}'(n) = \mathbb{W}(n)^*.$$

Therefore, if we let $\langle \cdot, \cdot \rangle$ be the pairing between $\mathbb{W}$ and $\mathbb{W}'$, then $\langle \widetilde{L}_0 w, w' \rangle = \langle w, \widetilde{L}_0 w' \rangle$ for each $w \in \mathbb{W}, w' \in \mathbb{W}'$. $\mathbb{W}$ is the contragredient of $\mathbb{W}'$. The vertex operation $Y_{\mathbb{W}'}$ will be described in Example 1.1.1.

The vertex operation $Y_{\mathbb{W}}$ of $\mathbb{W}$ will be denoted by Y when no confusion arises. Moreover, it can be extended $\mathbb{C}((z))$-linearly to a map

$$Y : \Big(\mathbb{V} \otimes \mathbb{C}((z))\Big) \otimes \mathbb{W} \to \mathbb{W} \otimes \mathbb{C}((z)), \qquad v \otimes w \to Y(v, z)w, \qquad (1.1.3)$$

and similarly

$$Y : \Big(\mathbb{V} \otimes \mathbb{C}((z))dz\Big) \otimes \mathbb{W} \to \mathbb{W} \otimes \mathbb{C}((z))dz, \qquad v \otimes w \to Y(v, z)wdz \qquad (1.1.4)$$

so that we can take residue $\mathrm{Res}_{z=0}Y(v, z)wdz$. Let $(\mathbb{W}')^*$ be the dual space of $\mathbb{W}'$. Then for each $n \in \mathbb{N}$ we can define a projection

$$P_n : (\mathbb{W}')^* = \prod_{n \in \mathbb{N}} \mathbb{W}(n) \to \mathbb{W}(n). \qquad (1.1.5)$$

If $v \in \mathbb{V}$ and $w \in \mathbb{W}$ are homogeneous with $\widetilde{L}_0$-weights $\mathrm{wt}v, \widetilde{\mathrm{wt}}w$ respectively, then by (1.1.2),

$$P_n Y(v, z)w = Y(v)_{-n-1+\mathrm{wt}v+\widetilde{\mathrm{wt}}w} w \cdot z^{n-\mathrm{wt}v-\widetilde{\mathrm{wt}}w}. \qquad (1.1.6)$$

A **family of transformations** over a complex manifold X is by definition a holomorphic function ρ on a neighborhood of $0 \times X \subset \mathbb{C} \times X$ sending each (z, x) to $\rho_x(z)$, such that $\rho_x(0) = 0$ and $(\partial_z \rho_x)(0) \neq 0$ for all $x \in X$. Let $c_0, c_1, \cdots \in \mathscr{O}(X)$ be determined by

$$\rho_x(z) = c_0(x) \cdot \exp \Big(\sum_{n>0} c_n(x) z^{n+1} \partial_z \Big) z$$

on the level of $\mathscr{O}(X)[[z]]$. Then we necessarily have $c_0(x) = (\partial_z \rho_x)(0)$. On each $\mathbb{W}$, we set

$$\mathcal{U}(\rho) = (\partial_z \rho)(0)^{\widetilde{L}_0} \cdot \exp \Big(\sum_{n>0} c_n L_n \Big) n$$

as an automorphism of $\mathbb{W} \otimes_{\mathbb{C}} \mathscr{O}_X$, i.e., an "$\mathrm{End}(\mathbb{W})$-valued holomorphic function" on X.

As an example, if X is a Riemann surface, and if $\eta, \mu \in \mathscr{O}(X)$ are both locally injective (equivalenly, if $d\eta, d\mu$ vanish nowhere), we can define a family of transformations $\varrho(\eta|\mu)$ on X defined by

$$\eta(y) - \eta(x) = \varrho(\eta|\mu)_x \Big(\mu(y) - \mu(x) \Big) \qquad (1.1.7)$$

15

for any $x \in X$ and any y close to x.

For any two families of transformations ρ_1, ρ_2, if we let ρ_1, ρ_2 be their pointwise multiplication, then (cf. [Hua97, Sec. 4.2])

$$\mathcal{U}(\rho_1 \rho_2) = \mathcal{U}(\rho_1)\mathcal{U}(\rho_2).$$

Example 1.1.1. Consider $\zeta^{-1} \in \mathscr{O}(\mathbb{C}^\times)$ where ζ is the standard coordinate of $\mathbb{C}$. Then the value of $\gamma := \varrho(\zeta^{-1}|\zeta)$ at each $z \in \mathbb{C}^\times$ is

$$\gamma_z(t) = \frac{1}{z+t} - \frac{1}{z}.$$

On any $\mathbb{V}$-module $\mathbb{W}$ we have (cf. for instance [Gui20b, Ex. 1.4])

$$\mathcal{U}(\gamma_z) \equiv \mathcal{U}(\gamma)_z = e^{zL_1}(-z^{-2})^{\tilde{L}_0}. \tag{1.1.8}$$

The vertex operator for the contragredient module $\mathbb{W}'$ is determined by the fact that for each $v \in \mathbb{V}, w \in \mathbb{W}, w' \in \mathbb{W}'$,

$$\langle Y(v,z)w, w' \rangle = \langle w, Y(\mathcal{U}(\gamma_z)v, z^{-1})w' \rangle.$$

The **sheaf of VOA** $\mathscr{V}_C$ associated to $\mathbb{V}$ and a (non-necessarily compact) Riemann surface C is an $\mathscr{O}$-module which associates to each connected open $U \subset C$ with locally injective $\eta \in \mathscr{O}(U)$ a trivialization (i.e., an $\mathscr{O}_U$-module isomorphism)

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_C|_U \xrightarrow{\simeq} \mathbb{V} \otimes_\mathbb{C} \mathscr{O}_U \tag{1.1.9}$$

such that for another similar $V \subset \mathcal{C}, \mu \in \mathscr{O}(V)$, the transition function is

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} = \mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V} \otimes_\mathbb{C} \mathscr{O}_{U \cap V} \xrightarrow{\simeq} \mathbb{V} \otimes_\mathbb{C} \mathscr{O}_{U \cap V}. \tag{1.1.10}$$

For each $n \in \mathbb{N}$, this transition function restricts to an automorphism of $\mathbb{V}^{\leq n} \otimes_\mathbb{C} \mathscr{O}_{U \cap V}$. Thus, we have a finite-rank locally free sheaf $\mathscr{V}_C^{\leq n} \subset \mathscr{V}_C$ having trivialization

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_C^{\leq n}|_U \xrightarrow{\simeq} \mathbb{V}^{\leq n} \otimes_\mathbb{C} \mathscr{O}_U$$

being the restriction of (1.1.9). $\mathscr{V}_C$ can be regarded as an infinite-rank (holomorphic) vector bundle which is the direct limit of the finite ones $\mathscr{V}_C^{\leq n}$.

The vacuum section $\mathbf{1} \in \mathscr{V}_C(C)$ denotes the one that is sent under any trivialization (1.1.9) to the constant vacuum vector $\mathbf{1} \in \mathbb{V} \subset \mathbb{V} \otimes_\mathbb{C} \mathscr{O}(U)$.

1.2 Conformal blocks for untwisted modules

1.2.1 Conformal blocks and propagation

By an N-**pointed compact Riemann surface with local coordinates**, we mean the following data

$$\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N) \tag{1.2.1}$$

where C is a (non-necessarily connected) compact Riemann surface, $x_1, \dots, x_N$ are distinct points on C such that each connected component of C contains at least one of these points, and $\eta_1, \dots, \eta_N$ are local coordinates at $x_1, \dots, x_N$ respectively. We set

$$S_{\mathfrak{X}} = \{x_1, \dots, x_N\}.$$

For each i, choose a neighborhood W_i of x_i on which η_i is defined, such that $W_i \cap S_{\mathfrak{X}} = \{x_i\}$.

Suppose $\mathbb{V}$-modules $\mathbb{W}_1, \dots, \mathbb{W}_N$ are associated to the marked points $x_1, \dots, x_N$ respectively. Write

$$\mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N.$$

By $w \in \mathbb{W}_\bullet$, we mean a vector in $\mathbb{W}_\bullet$; by $w_\bullet \in \mathbb{W}_\bullet$, we mean a vector of the form

$$w_\bullet = w_1 \otimes \cdots \otimes w_N$$

where each $w_i \in \mathbb{W}_i$. Depending on the context, sometimes we also understand $\mathbb{W}_\bullet$ as the tuple $(\mathbb{W}_1, \dots, \mathbb{W}_N)$.

Let ω_C be the (holomorphic) cotangent bundle for C. Then $\mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}}) = \varinjlim_{n \in \mathbb{N}} \mathscr{V}_C \otimes \omega_C(n S_{\mathfrak{X}})$ is the sheaf of meromorphic sections of $\mathscr{V}_C \otimes \omega_C$ whose poles are only in $S_{\mathfrak{X}}$. The space of global sections is $H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}}))$, which acts on $\mathbb{W}_\bullet$ in the following way. For each v in this space, the restriction $v|_{W_i}$ can be regarded as a section of $\mathbb{V} \otimes_{\mathbb{C}} \omega_{W_i}$ through the trivialization $\mathcal{U}_\varrho(\eta_i) : \mathscr{V}_C|_{W_i} \xrightarrow{\simeq} \mathbb{V} \otimes \omega_{W_i}$. By taking series expansion with respect to the variable η_i at x_i, $v|_{W_i}$ can furthermore be regarded as an element of $\mathbb{V} \otimes \mathbb{C}((z))dz$. Recall (1.1.4), we define the action of v on each $w_\bullet \in \mathbb{W}_\bullet$ to be

$$v \cdot w_\bullet = \sum_{i=1}^N w_1 \otimes \cdots \otimes \mathrm{Res}_{z=0} Y(v, z) w_i \otimes \cdots \otimes w_N.$$

A **conformal block** ϕ associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$ is a linear functional

$$\phi : \mathbb{W}_\bullet \to \mathbb{C}$$

vanishing on $v \cdot w_\bullet$ for all $w_\bullet \in \mathbb{W}_\bullet$ and $v \in H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}}))$.

We need the following propagation property of conformal blocks. Cf. [Gui21b, Sec. 8]. For each open $U_1, \dots, U_n \subset C$, we set

$$\mathrm{Conf}(U_\bullet \backslash S_{\mathfrak{X}}) = (U_1 \times \cdots \times U_n) \cap \mathrm{Conf}^n(C \backslash S_{\mathfrak{X}}).$$

Theorem 1.2.1. *For each $n \in \mathbb{N}$, we have the n-**propagation** of ϕ, which associates to each open $U_1, \dots, U_n \subset C$ a linear functional*

$$\wr^n \phi : \mathscr{V}_C(U_1) \otimes \cdots \otimes \mathscr{V}_C(U_n) \otimes \mathbb{W}_\bullet \to \mathscr{O}(\mathrm{Conf}(U_\bullet \backslash S_{\mathfrak{X}}))$$
$$v_1 \otimes \cdots \otimes v_n \otimes w \mapsto \wr^n \phi(v_1, \dots, v_n, w)$$

which is compatible with restriction to open subsets; namely, if $V_1 \subset U_1, \dots, V_n \subset U_n$ are open, then

$$\wr^n \phi(v_1|_{V_1}, \dots, v_n|_{V_n}, w) = \wr^n \phi(v_1, \dots, v_n, w)\big|_{\mathrm{Conf}(V_\bullet \backslash S_{\mathfrak{X}})}.$$

$\wr^n \phi$ *intertwines the actions of* $\mathcal{O}_C$, *namely, for each* $f_1 \in \mathcal{O}(U_1), \ldots, f_n \in \mathcal{O}(U_n)$,

$$\wr^n \phi(f_1 v_1, \ldots, f_n v_n, w) = (f_1 \circ \mathrm{pr}_1) \cdots (f_n \circ \mathrm{pr}_n) \wr^n \phi(v_1, \ldots, v_n, w)$$

where $\mathrm{pr}_i : C^n \to C$ *is the projection onto the* i-*th component.*

Choose any $w_\bullet \in \mathbb{W}_\bullet$. *For each* $1 \leqslant i \leqslant n$, *choose an open subset* U_i *of* C *equipped with an injective* $\mu_i \in \mathcal{O}(U_i)$. *Identify*

$$\mathscr{V}_C\big|_{U_i} = \mathbb{V} \otimes_\mathbb{C} \mathcal{O}_{U_i} \qquad \text{via } \mathcal{U}_\varrho(\mu_i).$$

Choose $v_i \in \mathscr{V}_C(U_i) = \mathbb{V} \otimes_\mathbb{C} \mathcal{O}(U_i)$, *and choose* $(y_1, \ldots, y_n) \in \mathrm{Conf}(U_\bullet \backslash S_\mathfrak{x})$. *Then the following are true.*

(1) *If* $U_1 = W_j$ *(where* $1 \leqslant j \leqslant N$*) and contains only* y_1, x_j *of all the points* $x_\bullet, y_\bullet$, *if* $\mu_1 = \eta_j$, *and if* U_1 *contains the closed disc with center* x_j *and radius* $|\eta_j(y_1)|$ *(under the coordinate* η_j*), then*

$$\wr^n \phi(v_1, v_2, \ldots, v_n, w_\bullet)\big|_{y_1, y_2, \ldots, y_n}$$
$$= \wr^{n-1} \phi\big(v_2, \ldots, v_n, w_1 \otimes \cdots \otimes Y(v_1, z) w_j \otimes \cdots \otimes w_N\big)\big|_{y_2, \ldots, y_n}\big|_{z=\eta_j(y_1)} \qquad (1.2.2)$$

where the series of z *on the right hand side converges absolutely, and* v_1 *is considered as an element of* $\mathbb{V} \otimes \mathbb{C}((z))$ *by taking Taylor series expansion with respect to the variable* η_j *at* x_j *(cf.* (1.1.3)*).*

(2) *If* $U_1 = U_2$ *and contains only* y_1, y_2 *of all the points* $x_\bullet, y_\bullet$, *if* $\mu_1 = \mu_2$, *and if* U_2 *contains the closed disc with center* y_2 *and radius* $|\mu_2(y_1) - \mu_2(y_2)|$ *(under the coordinate* μ_2*), then*

$$\wr^n \phi(v_1, v_2, v_3, \ldots, v_n, w_\bullet)\big|_{y_1, y_2, \ldots, y_n}$$
$$= \wr^{n-1} \phi\big(Y(v_1, z) v_2, v_3, \ldots, v_n, w_\bullet\big)\big|_{y_2, \ldots, y_n}\big|_{z=\mu_2(y_1)-\mu_2(y_2)} \qquad (1.2.3)$$

where the series of z *on the right hand side converges absolutely, and* v_1 *is considered as an element of* $\mathbb{V} \otimes \mathbb{C}((z))$ *by taking Taylor series expansion with respect to the variable* $\mu_2 - \mu_2(y_2)$ *at* y_2.

(3) *We have*

$$\wr^n \phi(\mathbf{1}, v_2, v_3, \ldots, v_n, w_\bullet) = \wr^{n-1} \phi(v_2, \ldots, v_n, w_\bullet). \qquad (1.2.4)$$

(4) *For any permutation* π *of the set* $\{1, 2, \ldots, n\}$, *we have*

$$\wr^n \phi\big(v_{\pi(1)}, \ldots, v_{\pi(n)}, w_\bullet\big)\big|_{y_{\pi(1)}, \ldots, y_{\pi(n)}} = \wr^n \phi(v_1, \ldots, v_n, w_\bullet)\big|_{y_1, \ldots, y_n}. \qquad (1.2.5)$$

In the above theorem, $\wr^0 \phi$ is understood as ϕ.

The existence of $\wr \phi$ that is compatible with restriction to open subsets, that intertwines the actions of $\mathcal{O}_C$ and that satisfies condition (1) of Theorem 1.2.1, can be regarded as an equivalent definition of a conformal block $\phi : \mathbb{W}_\bullet \to \mathbb{C}$; see [FB04, 10.1.2]. We present the precise statement in a form that is closely related to the definition of twisted conformal blocks in Sec. 1.3.

Theorem 1.2.2. *A linear functional* $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ *is a conformal block associated to* $\mathfrak{X}$ *and* $\mathbb{W}_\bullet$ *if and only if the following are satisfied*

1. *For each* $v \in \mathbb{V}$ *and* $1 \leqslant j \leqslant N$, *the series*

$$\phi(w_1 \otimes \cdots \otimes Y(v, z)w_j \otimes \cdots \otimes w_N)$$
$$:= \sum_{n \in \mathbb{N}} \phi(w_1 \otimes \cdots \otimes P_n Y(v, z)w_j \otimes \cdots \otimes w_N) \qquad (1.2.6)$$

   of functions of z *converges a.l.u. on* $\eta_j(W_j \backslash \{x_j\})$ *in the sense of* (1.1.1).

2. *There exists an operation* $\wr\phi$ *which associates to each* $w_\bullet \in \mathbb{W}_\bullet$ *an* $\mathscr{O}_{C \backslash S_{\mathfrak{X}}}$-*module morphism* $\wr\phi : \mathscr{V}_{C \backslash S_{\mathfrak{X}}} \to \mathscr{O}_{C \backslash S_{\mathfrak{X}}}$ *satisfying the following conditions:*
   For each $1 \leqslant j \leqslant N$, *identify*

$$\mathscr{V}_{W_i} = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{W_i} \qquad via \; \mathcal{U}_\varrho(\eta_j).$$

   Choose any $v \in \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(W_i \backslash \{x_i\})$. *Then the function* $\wr\phi(v, w_\bullet) \in \mathscr{O}(W_i \backslash \{x_i\})$ *satisfies*

$$\wr\phi(v, w_\bullet)_x = \phi\big(w_1 \otimes \cdots \otimes Y\big(v(x), \eta_j(x)\big)w_j \otimes \cdots \otimes w_N\big) \qquad (1.2.7)$$

   for each $x \in U$, *where the right hand side is understood as the limit of the series* (1.2.6) *by replacing* v *by* $v(x)$ *and substituting* $z = \eta_j(x)$.

Recall that the projection P_n was defined in (1.1.5). Also, note that by linearity, in the second condition it suffices to verify (1.2.7) when v is a constant section, i.e., $v \in \mathbb{V} \subset \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(W_i \backslash \{x_i\})$.

Proof. "Only if": Assume ϕ is a conformal block, and let $\wr\phi$ be its propagation. Then Thm. 1.2.1 implies the two conditions of this theorem whenever v, w_j are homogeneous: indeed, by (1.1.6), the convergences of the right hand side of (1.2.6) and that of (1.2.2) are in the same sense. The general case follows from $\mathbb{C}$- or $\mathscr{O}(W_i)$-linearity and the triangle inequality.

"If": Let $\wr\phi$ be as in Condition 2 of this theorem. Choose $v \in H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}})) \subset H^0(C \backslash S_{\mathfrak{X}}, \mathscr{V}_{C \backslash S_{\mathfrak{X}}} \otimes \omega_{C \backslash S_{\mathfrak{X}}})$. Consider $\wr\phi(\cdot, w_\bullet) \otimes 1 : \mathscr{V}_{C \backslash S_{\mathfrak{X}}} \otimes \omega_{C \backslash S_{\mathfrak{X}}} \to \omega_{C \backslash S_{\mathfrak{X}}}$, also written as $\wr\phi(\cdot, w_\bullet)$ for simplicity. Then $\wr\phi(v, w_\bullet)$ is a global holomorphic 1-form on $C \backslash S_{\mathfrak{X}}$. By Stokes theorem (or residue theorem), if we choose a small circle around each x_j, then the sum of the integrals of $\wr\phi(v, w_\bullet)$ along each circle is 0. Substituting (1.2.7) into this equality, we see $\phi(v \cdot w_\bullet) = 0$. This proves that ϕ is a conformal block. $\qquad \square$

1.2.2 Sewing and propagation

Let $N, M \in \mathbb{Z}_+$. Let

$$\mathfrak{X} = (C; x_1, \ldots, x_N; x_1', \ldots, x_M'; x_1'', \ldots, x_M'') \qquad (1.2.8)$$

be an $(N + 2M)$-pointed compact Riemann surface with local coordinates η_i at x_i, $\xi_j \in \mathscr{O}(W_j')$ at x_j', and $\varpi_j \in \mathscr{O}(W_j'')$ at x_j''. We assume that each connected component of C contains at least one of $S_{\mathfrak{X}} = \{x_1, \ldots, x_N\}$. We assume that

$$\xi_j(W_j') = \mathcal{D}_{r_j}, \qquad \varpi_j(W_j'') = \mathcal{D}_{\rho_j}$$

where

$$r_j \rho_j > 1.$$

Moreover, we shall also assume that:

Assumption 1.2.3. The open sets $W'_1, \ldots, W'_M, W''_1, \ldots, W''_M$ are mutually disjoint and are disjoint from $S_{\mathfrak{X}}$. (Thus, W'_j resp. W''_j contains only x'_j resp. x''_j among the $N + 2M$ marked points.)

W'_j, W''_j are the discs to be sewn. So the above assumption says that the marked points left after sewing (namely, $x_1, \ldots, x_N$) should be away from the discs to be sewn.

The **sewing of $\mathfrak{X}$ along the pairs** x'_j, x''_j (for all j) is an N-pointed compact Riemann surface with local coordinates

$$\mathscr{S}\mathfrak{X} = (\mathscr{S}C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N) \tag{1.2.9}$$

constructed as follows. Remove the closed subsets

$$F'_j = \{x \in W'_j : |\xi_j(x)| \leqslant 1/\rho_j\}, \qquad F''_j = \{x \in W''_j : |\varpi_j(x)| \leqslant 1/r_j\}$$

(for all $1 \leqslant j \leqslant M$) from C. Glue the remaining part of C by gluing all $x \in W'_j \backslash F'_j$ and $y \in W''_j \backslash F''_j$ satisfying

$$\xi_j(x)\varpi_j(y) = 1.$$

This gives us a new compact Riemann surface $\mathscr{S}C$. Clearly, $C \backslash \bigcup_j (F'_j \cup F''_j)$ can be identified with an open subset of $\mathscr{S}C$. In particular, each x_i is also a point of $\mathscr{S}C$, and η_i can be regarded as a local coordinate of $\mathscr{S}C$ at x_i.

The above sewing procedure is unchanged if for each j we choose $\lambda_j > 0$ and replace ξ_j, ϖ_j by $\lambda_j \xi_j, \lambda^{-1} \varpi_j$, or if for each j we replace W'_j, W''_j by new neighborhoods $\widetilde{W}'_j \ni x'_j, \widetilde{W}''_j \ni x''_j$ on which ξ_j, ϖ_j are defined respectively, such that $\xi_j(\widetilde{W}'_j) = \mathcal{D}_{\tilde{r}_j}, \varpi_j(\widetilde{W}''_j) = \mathcal{D}_{\tilde{\rho}_j}$ and $\tilde{r}_j \tilde{\rho}_j > 1$.

Corresponding to this geometric sewing, we can define, for every conformal block

$$\phi : \mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N \otimes \mathbb{M}_1 \otimes \mathbb{M}'_1 \otimes \cdots \otimes \mathbb{M}_M \otimes \mathbb{M}'_M \to \mathbb{C}$$

associated to $\mathfrak{X}$ and the chosen local coordinates, the **sewing** $\mathscr{S}\phi$ as follows. For each $1 \leqslant j \leqslant M, n \in \mathbb{N}$, set

$$\bowtie_{j,n} \in \mathbb{M}_j(n) \otimes \mathbb{M}'_j(n) \tag{1.2.10}$$

which, considered as an element of $\mathrm{End}(\mathbb{M}_j(n))$, is the identity operator. (Recall that $\dim \mathbb{M}_j(n) < +\infty$ and $\mathbb{M}'_j(n)$ is the dual space of $\mathbb{M}_j(n)$.) Then $\mathscr{S}\phi$ associates to each $w \in \mathbb{W}_\bullet$ an infinite sum

$$\mathscr{S}\phi(w) = \sum_{n_1, \ldots, n_M \in \mathbb{N}} \phi(w \otimes \bowtie_{1,n_1} \otimes \cdots \otimes \bowtie_{M,n_M}). \tag{1.2.11}$$

Definition 1.2.4. We say the sewing $\mathscr{S}\phi$ **converges q-absolutely**, if there exist $R_1, \ldots, R_M > 1$ such that for each $w \in \mathbb{W}_\bullet$, the infinite series of functions of $(q_1, \ldots, q_M) \in \mathcal{D}_{R_\bullet} := \mathcal{D}_{R_1} \times \cdots \times \mathcal{D}_{R_M}$:

$$\sum_{n_1, \ldots, n_M \in \mathbb{N}} \phi(w \otimes \bowtie_{1, n_1} \otimes \cdots \otimes \bowtie_{M, n_M}) q_1^{n_1} \cdots q_M^{n_M}$$

converges a.l.u. on $\mathcal{D}_{R_\bullet}$ in the sense of (1.1.1); equivalently, there exist $R_1, \ldots, R_M > 1$ such that for each $w \in \mathbb{W}_\bullet$,

$$\sum_{n_1, \ldots, n_M \in \mathbb{N}} \left| \phi(w \otimes \bowtie_{1, n_1} \otimes \cdots \otimes \bowtie_{M, n_M}) \right| \cdot R_1^{n_1} \cdots R_M^{n_M} < +\infty.$$

When $\mathscr{S}\phi$ converges q-absolutely, it can be regarded as a linear functional on $\mathbb{W}_\bullet$ sending each $w \in \mathbb{W}_\bullet$ to the limit of (1.2.11).

Theorem 1.2.5 ([Gui20b, Thm. 11.3]). *If $\mathscr{S}\phi$ converges q-absolutely, then $\mathscr{S}\phi$ is a conformal block associated to $\mathscr{S}\mathfrak{X}$ and $\mathbb{W}_\bullet$.*

It is not known whether this theorem still holds if we assume the weaker condition that the series (1.2.11) converges absolutely for each w. However, in practice, whenever we can show the absolute convergence of sewing, we can also show the q-absolute convergence.

Theorem 1.2.6 ([Gui20b, Thm. 13.1]). *If $\mathbb{V}$ is C_2-cofinite, and if all the modules $\mathbb{W}_1, \ldots, \mathbb{W}_N, \mathbb{M}_1, \ldots, \mathbb{M}_M$ are finitely-generated, then $\mathscr{S}\phi$ converges q-absolutely.*

Let $\mathcal{E}$ be a complete list of irreducible $\mathbb{V}$-modules, namely, every irreducible $\mathbb{V}$-module is equivalent to a unique element of $\mathcal{E}$. If $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational, then $\mathcal{E}$ is a finite set. Let

$$\mathrm{CB}_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet) \qquad \text{resp.} \qquad \mathrm{CB}_{\mathscr{S}\mathfrak{X}}(\mathbb{W}_\bullet)$$

denote the space of conformal blocks associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet$ (resp. $\mathscr{S}\mathfrak{X}$ and $\mathbb{W}_\bullet$). Fix finitely-generated $\mathbb{W}_1, \ldots, \mathbb{W}_N$. Then, by Thm. 1.2.5 and 1.2.6, we have a linear map

$$\begin{aligned} \mathscr{S} : \bigoplus_{\mathbb{M}_1, \ldots, \mathbb{M}_M \in \mathcal{E}} \mathrm{CB}_{\mathfrak{X}}(\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet) &\to \mathrm{CB}_{\mathscr{S}\mathfrak{X}}(\mathbb{W}_\bullet) \\ \bigoplus_{\mathbb{M}_1, \ldots, \mathbb{M}_M \in \mathcal{E}} \phi_{\mathbb{M}_\bullet} &\mapsto \textstyle\sum_{\mathbb{M}_1, \ldots, \mathbb{M}_M \in \mathcal{E}} \mathscr{S}\phi_{\mathbb{M}_\bullet} \end{aligned} \qquad (1.2.12)$$

where $\mathbb{M}_\bullet$ denotes also the tuple $(\mathbb{M}_1, \ldots, \mathbb{M}_M)$.

Theorem 1.2.7. *If $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational, and if $\mathbb{W}_1, \ldots, \mathbb{W}_N$ are semi-simple $\mathbb{V}$-modules, then the linear map $\mathscr{S}$ defined by (1.2.12) is bijective.*

Proof. When $M = 1$, this follows from [Gui20b, Thm. 12.1]. (Note that the surjectivity of $\mathscr{S}$ follows from the remarkable factorization property proved by [DGT19b].) Apply this result inductively, we can prove this theorem for a general M. $\qquad\square$

The following theorem is crucial to the main result of this article. Cf. [Gui21b, Thm. 9.1].

Theorem 1.2.8. *Assume $\mathscr{S}\phi$ converges q-absolutely. Let $U_1, \ldots U_n \subset C$ be open and disjoint from W_j', W_j'' (for all $1 \leqslant j \leqslant N$), which can also be viewed as open subsets of $\mathscr{S}C$. Then there exist $R_1, \ldots, R_M > 1$ such that for each $v_i \in \mathscr{V}_C(U_i) = \mathscr{V}_{\mathscr{S}C}(U_i)$ and $w \in \mathbb{W}_\bullet$, the following infinite series*

$$\widetilde{\mathscr{S}} \wr^n \phi(v_1, \ldots, v_n, w) = \sum_{n_1, \ldots, n_M \in \mathbb{N}} q_1^{n_1} \cdots q_M^{n_M} \cdot \wr^n \phi(v_1, \ldots, v_n, w \otimes \bowtie_{1, n_1} \otimes \cdots \otimes \bowtie_{M, n_M})$$

of holomorphic functions on $\mathcal{D}_{R_1} \times \cdots \times \mathcal{D}_{R_M} \times \mathrm{Conf}(U_\bullet \backslash S_{\mathfrak{x}})$ converges a.l.u. in the sense of (1.1.1). Moreover, let $\mathscr{S} \wr^n \phi$ be the limit of the above series at $q_1 = \cdots = q_M = 1$. Then

$$\mathscr{S} \wr^n \phi(v_1, \ldots, v_n, w) = \wr^n \mathscr{S} \phi(v_1, \ldots, v_n, w). \tag{1.2.13}$$

1.3 Conformal blocks for twisted modules

Let $\mathbb{U} = \bigoplus_{n \in \mathbb{N}} \mathbb{U}(n)$ $(\dim \mathbb{U}(n) < +\infty)$ be a VOA. An automorphism g of $\mathbb{U}$ is a linear map preserving the vacuum and the conformal vector of $\mathbb{U}$, and satisfying $gY(u)_n v = Y(gu)_n gv$ for each $u, v \in \mathbb{U}, n \in \mathbb{Z}$.

We let G be a finite group of automorphisms of $\mathbb{U}$.

1.3.1 Twisted modules

For any $g \in G$ with order $|g|$, a (finitely-admissible) g-**twisted** $\mathbb{U}$-**module** is a vector space $\mathcal{W}$ together with a diagonalizable operator $\widetilde{L}_0^g$, and an operation

$$Y^g : \mathbb{U} \otimes \mathcal{W} \to \mathcal{W}[[z^{\pm 1/k}]]$$
$$u \otimes w \mapsto Y^g(u, z)w = \sum_{n \in \frac{1}{|g|}\mathbb{Z}} Y^g(u)_n w \cdot z^{-n-1}$$

satisfying the following conditions:

1. $\mathcal{W}$ has $\widetilde{L}_0^g$-grading $\mathcal{W} = \bigoplus_{n \in \frac{1}{|g|}\mathbb{N}} \mathcal{W}(n)$, each eigenspace $\mathcal{W}(n)$ is finite-dimensional, and for any $u \in \mathbb{U}$ we have

$$[\widetilde{L}_0^g, Y^g(u)_n] = Y^g(L_0 u)_n - (n+1)Y^g(u)_n. \tag{1.3.1}$$

   In particular, for each $w \in \mathcal{W}$ the lower truncation condition follows: $Y^g(u)_n w = 0$ when n is sufficiently small.

2. $Y^g(\mathbf{1}, z) = \mathbf{1}_{\mathcal{W}}$.

3. (g-equivariance) For each $u \in \mathbb{U}$,

$$Y^g(gu, z) = Y^g(u, e^{-2\mathbf{i}\pi}z) := \sum_{n \in \frac{1}{|g|}\mathbb{Z}} Y^g(u)_n w \cdot e^{2(n+1)\mathbf{i}\pi} z^{-n-1}. \tag{1.3.2}$$

4. (Jacobi identity) For each $u, v \in \mathbb{U}, w \in \mathcal{W}, w' \in \mathcal{W}'$, and for each $z \neq \mathfrak{z}$ in $\mathbb{C}^\times$ with chosen $\arg z, \arg \mathfrak{z}$, the following series of single-valued functions of $\log z, \log z, \log(z - \mathfrak{z})$

$$\langle Y^g(u, z)Y^g(v, \mathfrak{z})w, w' \rangle := \sum_{n \in \frac{1}{|g|}\mathbb{N}} \langle Y^g(u, z)P_n^g Y^g(v, \mathfrak{z})w, w' \rangle \tag{1.3.3}$$

$$\langle Y^g(v, \mathfrak{z})Y^g(u, z)w, w' \rangle := \sum_{n \in \frac{1}{|g|}\mathbb{N}} \langle Y^g(v, \mathfrak{z})P_n^g Y^g(u, z)w, w' \rangle \tag{1.3.4}$$

$$\langle Y^g(Y(u, z - \mathfrak{z})v, \mathfrak{z})w, w' \rangle := \sum_{n \in \mathbb{N}} \langle Y^g(P_n Y(u, z - \mathfrak{z})v, \mathfrak{z})w, w' \rangle \tag{1.3.5}$$

(where $\mathfrak{z}$ is fixed) converge a.l.u. on $|z| > |\mathfrak{z}|, |z| < |\mathfrak{z}|, |z - \mathfrak{z}| < |\mathfrak{z}|$ respectively (in the sense of (1.1.1)). Moreover, for any fixed $\mathfrak{z} \in \mathbb{C}^\times$ with chosen argument $\arg \mathfrak{z}$, let $R_{\mathfrak{z}}$ be the ray with argument $\arg \mathfrak{z}$ from 0 to ∞, but with $0, \mathfrak{z}, \infty$ removed. Any point on $R_{\mathfrak{z}}$ is assumed to have argument $\arg \mathfrak{z}$. Then the above three expressions, considered as functions of z defined on $R_{\mathfrak{z}}$ satisfying the three mentioned inequalities respectively, can be analytically continued to the same holomorphic function on the open set

$$\Delta_{\mathfrak{z}} = \mathbb{C} \backslash \{\mathfrak{z}, -t\mathfrak{z} : t \geqslant 0\},$$

which can furthermore be extended to a multivalued holomorphic function $f_{\mathfrak{z}}(z)$ on $\mathbb{C}^\times \backslash \{\mathfrak{z}\}$ (i.e., a holomorphic function on the universal cover of $\mathbb{C}^\times \backslash \{\mathfrak{z}\}$).

In the above Jacobi identity, let $\mathcal{W}' = \bigoplus_{n \in \frac{1}{|g|}\mathbb{N}} \mathcal{W}(n)^*$, then P_n^g is defined to be the projection

$$P_n^g : (\mathcal{W}')^* = \prod_{n \in \frac{1}{|g|}\mathbb{N}} \mathcal{W}(n) \to \mathcal{W}(n). \tag{1.3.6}$$

The above vector space $\mathcal{W}'$ can be equipped with a g^{-1}-twisted $\mathbb{U}$-module structure with $\widetilde{L}_0$-grading $\mathcal{W}' = \bigoplus_{n \in \frac{1}{|g|}\mathbb{N}} \mathcal{W}'(n) = \bigoplus_{n \in \frac{1}{|g|}\mathbb{N}} \mathcal{W}(n)^*$. Then for each $w \in \mathcal{W}, w' \in \mathcal{W}'$,

$$\begin{aligned}
\langle Y^{g^{-1}}(v, z)w', w \rangle &= \langle w', Y^g(\mathcal{U}(\gamma_z)v, z^{-1})w \rangle \\
&= \langle w', Y^g(e^{zL_{-1}}(-z^{-2})^{L_0}v, z^{-1})w \rangle.
\end{aligned} \tag{1.3.7}$$

$\mathcal{W}'$ is called the **contragredient module** of $\mathcal{W}$. Cf. [Hua18, Prop. 3.3]. Using the easy fact that $\gamma_{z^{-1}} \cdot \gamma_z = 1$ (and hence $\mathcal{U}(\gamma_{z^{-1}}) \cdot \mathcal{U}(\gamma_z) = 1$), one sees that $\mathcal{W}$ is the contragredient of $\mathcal{W}'$. By choosing v to be the conformal vector $\mathbf{c}$ and setting

$$L_n = Y^g(\mathbf{c})_{n+1}$$

when acting on $\mathcal{W}$, we conclude

$$\langle L_n w, w' \rangle = \langle w, L_{-n}w' \rangle. \tag{1.3.8}$$

The above Jacobi identity is equivalent to its well-known algebraic form (cf. [Gui21b, Sec. 10]). It follows easily from that algebraic Jacobi identity that (1.3.1) holds if $\widetilde{L}_0^g$ is replaced by L_0. Thus, $\widetilde{L}_0^g - L_0$ commutes with the actions of vertex operators. In particular, if $\mathcal{W}$ is an irreducible twisted $\mathbb{U}$-module, then $\widetilde{L}_0^g - L_0$ is a constant.

1.3.2 $\mathfrak{P}$ is a positively N-pathed Riemann spheres with local coordinates

Let

$$\mathfrak{P} = (\mathbb{P}^1; x_1, \dots, x_N; \eta_1, \dots, \eta_N; \gamma_1, \dots, \gamma_N) = (\mathbb{P}_1; x_\bullet; \eta_\bullet; \gamma_\bullet) \qquad (1.3.9)$$

where $(\mathbb{P}^1; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ is an N-pointed Riemann sphere with local coordinates. Each η_j is defined analytically on an open disc W_j centered at x_j, i.e., $\eta_j(W_j)$ is an open disc centered at 0. We assume $W_j \cap \{x_1, \dots, x_N\} = x_j$. We assume the (continuous) paths $\gamma_1, \dots, \gamma_N : [0, 1] \to \mathbb{P}^1$ have common end point in $\mathbb{P}^1 \backslash \{x_1, \dots, x_N\}$ (denoted by $\gamma_\bullet(1)$), and the initial point $\gamma_j(0)$ for each $1 \leqslant j \leqslant N$ satisfies $\gamma_j(0) \in W_j$. (See Figure 0.2.1.)

Convention 1.3.1. Unless otherwise stated, for any path λ in $\mathbb{P}^1 \backslash \mathbf{S}$, its homotopy class $[\lambda]$ denotes the class of all paths $\tilde{\lambda}$ in $\mathbb{P}^1 \backslash \mathbf{S}$ having the same initial and end points as λ, and is homotopic (assuming the initial and end points are always fixed) in $\mathbb{P}^1 \backslash \mathbf{S}$ to λ.

For each $x \in \mathbb{P}^1 \backslash \mathbf{S}$, let

$$\Lambda_x = \left\{ \text{continuous maps } \lambda : [0, 1] \to \mathbb{P}^1 \backslash \mathbf{S}, \lambda(0) = x, \lambda(1) = \gamma_\bullet(1) \right\}.$$

Namely, Λ_x is the set of all paths in $\mathbb{P}^1 \backslash \mathbf{S}$ going from x to $\gamma_\bullet(1)$.
For each j, we let

$$\epsilon_j : [0, 1] \to W_j \backslash \{x_j\}$$

be the anticlockwise circle (defined using η_j) centered at x_j whose initial and end point is $\gamma_\bullet(0)$. We write

$$\Gamma := \pi_1(\mathbb{P}^1 \backslash \mathbf{S}, \gamma_\bullet(1))$$

(the fundamental group of $\mathbb{P}^1 \backslash \mathbf{S}$ with basepoint $\gamma_\bullet(1)$). Equivalently, $\Gamma = \{[\mu] : \mu \in \Lambda_{\gamma_\bullet(1)}\}$. Then Γ is isomorphic to the free group $\mathbb{F}_{N-1}$. Set

$$\alpha_j := \gamma_j^{-1} \epsilon_j \gamma_j. \qquad (1.3.10)$$

Then the homotopy class $[\alpha_j]$ (in $\mathbb{P}^1 \backslash \mathbf{S}$) belongs to Γ. We assume

$$\Gamma = \langle [\alpha_1], \dots, [\alpha_N] \rangle, \qquad (1.3.11)$$

namely, these N elements generate Γ.
Such data $\mathfrak{P}$ is called an N-**pathed Riemann sphere with local coordinates**. If, moreover, for each j we have

$$\begin{aligned} \eta_j \circ \gamma_j(0) \in (0, +\infty) \\ \arg \eta_j \circ \gamma_j(0) = 0 \end{aligned} \qquad (1.3.12)$$

We say $\mathfrak{P}$ is **positively N-pathed**. We set

$$\mathbf{S} = \{x_1, \dots, x_N\} \qquad \subset \mathbb{P}^1. \qquad (1.3.13)$$

1.3.3 Conformal blocks

In this subsection, we define genus-0 twisted conformal blocks. For a comparison of our definition with that in algebraic geometry (cf. [FS04]), see Introduction-Outline. We let

$$\mathcal{P} = \text{the universal cover of } \mathbb{P}^1 \backslash \mathbf{S}.$$

Let $\mathscr{U}_C$ be the sheaf of VOA for $\mathbb{U}$ associated to any Riemann surface C. Then $\mathscr{U}_{\mathcal{P}}$ can be identified naturally with the pullback of $\mathscr{U}_{\mathbb{P}^1 \backslash \mathbf{S}}$ along the covering map $\mathcal{P} \to \mathbb{P}^1 \backslash \mathbf{S}$.

Lemma 1.3.2. *There is a one-to-one correspondence between:*

1. *An $\mathscr{O}_{\mathcal{P}}$-module morphism $\natural : \mathscr{U}_{\mathcal{P}} \to \mathscr{O}_{\mathcal{P}}$.*

2. *An operation ψ which associates to each simply-connected open subset $U \subset \mathbb{P}^1 \backslash \mathbf{S}$, each path $\lambda \subset \mathbb{P}^1 \backslash \mathbf{S}$ from a point of U to $\gamma_{\bullet}(1)$, and each section $v \in \mathscr{U}_{\mathbb{P}^1}(U)$, an element $\psi(\lambda, v) \in \mathscr{O}(U)$ satisfying the following properties:*

   (a) *If $V \subset U$ is open, simply-connected, and contains $\lambda(0)$, then $\psi(\lambda, v|_V) = \psi(\lambda, v)|_V$.*

   (b) *If $f \in \mathscr{O}(U)$ then $\psi(\lambda, fv) = f\psi(\lambda, v)$.*

   (c) *If λ' is another path in $\mathbb{P}^1 \backslash \mathbf{S}$ with the same initial and end points as λ, and if $[\lambda] = [\lambda']$, then $\psi(\lambda, v) = \psi(\lambda', v)$. Therefore, we may write $\psi(\lambda, v)$ as $\psi([\lambda], v)$.*

   (d) *If l is a path in U ending at the initial point of λ, then $\psi(\lambda, v) = \psi(l\lambda, v)$.*

We call any ψ in part 2 a **multivalued ($\mathscr{O}_{\mathbb{P}^1 \backslash \mathbf{S}}$-module) morphism** $\mathscr{U}_{\mathbb{P}^1 \backslash \mathbf{S}} \to \mathscr{O}_{\mathbb{P}^1 \backslash \mathbf{S}}$.

Proof. Fix a lift $p \in \mathcal{P}$ of $\gamma_{\bullet}(1)$. If $\natural$ is as in part 1, we lift λ to a path $\tilde{\lambda}$ in $\mathcal{P}$ ending at p, and lift U to a unique simply-connected $\tilde{U}_{\lambda}$ containing the initial point of the $\tilde{\lambda}$. Identify $\tilde{U}_{\lambda}$ with U via the covering map. Then $\psi(\lambda, v) := \natural(v)$ defines ψ.

Conversely, choose ψ as in part 2. For each open simply-connected $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ and any lift $\tilde{U}$, we choose a path $\lambda \subset \mathbb{P}^1 \backslash \mathbf{S}$ from a point of U to $\gamma_{\bullet}(1)$ such that $\tilde{U} = \tilde{U}_{\lambda}$. Identify $\tilde{U}$ with U via the covering map, and set $\natural(v) = \psi(\lambda, v)$ for each $v \in \mathscr{U}_{\tilde{U}}(\tilde{U})$. This defines $\natural$ on any simply connected open set, and can be easily extended onto all open subsets of $\mathcal{P}$. $\square$

Now, for each j, choose $g_j \in G$, and choose a g_j-twisted $\mathbb{U}$-module $\mathcal{W}_j$ associated to the marked point x_j. Let $\mathcal{W}_{\bullet} = \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_N$. We are ready to define genus 0 conformal blocks for twisted modules.

Definition 1.3.3. A **conformal block** associated to $\mathfrak{P}$ and $\mathcal{W}_{\bullet}$ is a linear functional $\psi : \mathcal{W}_{\bullet} \to \mathbb{C}$ satisfying the following conditions:

1. For each $u \in \mathbb{U}$ and each $1 \leqslant j \leqslant N$, the series

$$\psi(w_1 \otimes \cdots \otimes Y^{g_j}(u, z)w_j \otimes \cdots \otimes w_N)$$
$$:= \sum_{n \in \frac{1}{|g_j|}\mathbb{N}} \psi(w_1 \otimes \cdots \otimes P_n^{g_j} Y^{g_j}(u, z)w_j \otimes \cdots \otimes w_N) \qquad (1.3.14)$$

of single-valued functions of $\log z$ converges a.l.u. on $\exp^{-1}\big(\eta_j(W_j\backslash\{x_j\})\big)$ in the sense of (1.1.1).

2. There exists $\wr\psi$ associating to each $w_\bullet \in \mathcal{W}_\bullet$ a multivalued $\mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$-module morphism

$$\wr\psi(\cdot,\cdot,w_\bullet) : \mathscr{U}_{\mathbb{P}^1\backslash\mathbf{S}} \to \mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$$

satisfying the following condition:

For each $1 \leqslant j \leqslant N$, identify

$$\mathscr{U}_{W_j} = \mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}_{W_j} \qquad \text{via } \mathcal{U}_\varrho(\eta_j). \tag{1.3.15}$$

Choose any open simply-connected subset $U \subset W_j\backslash\{x_j\}$ containing $\gamma_j(0)$ and equipped with a continuous $\arg$ function on $\eta_j(U)$ whose value at $\eta_j \circ \gamma_j(0)$ is 0. Choose any $u \in \mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}(U)$. Then the function $\wr\psi(\gamma_j, u, w_\bullet) \in \mathscr{O}(U)$ satisfies

$$\wr\psi(\gamma_j, u, w_\bullet)_x = \psi\big(w_1 \otimes \cdots \otimes Y^{g_j}\big(u(x), \eta_j(x)\big)w_j \otimes \cdots \otimes w_N\big) \tag{1.3.16}$$

for each $x \in U$, where the right hand side is understood as the limit of the series (1.3.14) by replacing u by $u(x)$, substituting $z = \eta_j(x)$, and defining $\arg \eta_j(x)$ using the $\arg$ function on $\eta_j(U)$.

Note that in (1.3.16), we understand u as an $\mathbb{U}$-valued function whose value at each $x \in U$ is $u(x) \in \mathbb{U}$.

We give some comments on this definition.

Remark 1.3.4. It suffices to check (1.3.16) for any constant section $u \in \mathbb{U} \simeq \mathbb{U} \otimes_{\mathbb{C}} \mathbf{1} \subset \mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}(U)$.

Remark 1.3.5. If for some open simply-connected $U \subset W_j$, the relation (1.3.16) holds for any $x \in U$, then due to the uniqueness of analytic continuation, for every open simply-connected $U \subset W_j$ this is also true.

Moreover, given one simply-connected open $U \subset W_j$, if I is subset of U with at least one accumulation point in U, then by complex analysis, (1.3.16) holds for all $x \in U$ if it holds for all $x \in I$.

Remark 1.3.6. Suppose that u is $\widetilde{L}_0 = L_0$-homogeneous with weight $\mathrm{wt}u$, and each w_j is $\widetilde{L}_0$-homogeneous with weight $\widetilde{\mathrm{wt}}w_j$. Then by (1.3.1),

$$P_n^{g_j} Y^{g_j}(u, z)w_j = Y^{g_j}(u)_{-n-1+\mathrm{wt}u+\widetilde{\mathrm{wt}}w_j} w_j \cdot z^{n-\mathrm{wt}u-\widetilde{\mathrm{wt}}w_j}.$$

It follows that for homogeneous vectors, the following statements are equivalent:

(1) (1.3.14) converges a.l.u. as a function of $\log z$ on $\exp^{-1}\big(\eta_j(W_j\backslash\{x_j\})\big)$.

(2) The laurent series

$$\sum_{n \in \frac{1}{|g_j|}\mathbb{Z}} \psi(w_1 \otimes \cdots \otimes Y^{g_j}(u)_n w_j \otimes \cdots \otimes w_N)z^{-n-1} \tag{1.3.17}$$

of $z^{1/|g_j|}$ (which is a power series when multiplied by a power of $z^{1/|g_j|}$) converges absolutely for any $z^{1/|g_j|}$ on the punctured disc $\mathcal{D}^\times_{r_j^{1/|g_j|}}$ (if $\eta_j(W_j)$ has radius r_j).

(3) For each $z \in \eta_j(W_j \backslash \{x_j\})$ and every argument $\arg z$, (1.3.14) converges absolutely.

(4) For each $z \in \eta_j(W_j \backslash \{x_j\})$ with one argument $\arg z$, (1.3.14) converges absolutely.

Moreover, by linearity and triangle inequality, any of these four statements holds for all vectors provided that it holds for homogeneous vectors.

Remark 1.3.7. It follows from the previous two remarks that the definition for $\psi : \mathcal{W}_\bullet \to \mathbb{C}$ to be a conformal block is independent of (the sizes of) the open discs W_j under η_j.

Indeed, suppose we can verify the two conditions for W_j with radius r_j. Let $\widehat{W}_j$ be a larger one with radius R_j centered at x_j on which η_j is still defined. Then condition 2 for W_j implies that (1.3.17), which converges to a holomorphic function of $z^{1/|g_j|}$ on $\mathcal{D}^\times_{r_j^{1/|g_j|}}$, can be analytically continued (by $\wr\phi(\gamma_j, u, w)$) to one on $\mathcal{D}^\times_{R_j^{1/|g_j|}}$. Thus (1.3.17), whose coefficients are given by those of the series expansion of $\wr\phi(\gamma_j, u, w)$, converges absolutely on this larger punctured disc, i.e., condition 1 holds on $\widehat{W}_j$. By Rem. 1.3.5, condition 2 also holds on $\widehat{W}_j$.

Remark 1.3.8. Assume a positively N-pathed $\mathfrak{P}'$ is $\mathbb{P}^1$ with the same marked points $x_\bullet$ and local coordinates $\eta_\bullet$, but with different paths $\gamma'_1, \ldots, \gamma'_N$ (in $\mathbb{P}^1 \backslash \mathbf{S}$ ending at a common point $\gamma'_\bullet(1)$). We say that $\gamma_\bullet$ and $\gamma'_\bullet$ are **equivalent (positive) paths** (or that $\mathfrak{P}$ a and $\mathfrak{P}'$ are equivalent) if there exists a path σ in $\mathbb{P}^1 \backslash \mathbf{S}$ with initial point $\gamma_\bullet(1)$, and for each $1 \leqslant j \leqslant N$ there is a path l_j in W_j from $\gamma'(0)$ to $\gamma_j(0)$ satisfying

$$\mathrm{Range}(\eta_j \circ l_j) \subset (0, +\infty), \tag{1.3.18}$$
$$[\gamma'_j] = [l_j \gamma_j \sigma]. \tag{1.3.19}$$

If $\gamma_\bullet$ and $\gamma'_\bullet$ are equivalent, then it is clear that a conformal block associated to $\mathfrak{P}$ and $\mathcal{W}_\bullet$ is also one associated to $\mathfrak{P}'$ and $\mathcal{W}_\bullet$.

Lemma 1.3.9. *If ψ is a conformal block, then the $\wr\psi$ satisfying condition 2 of Def. 1.3.3 are unique.*

Proof. For two such operations $\wr_1\psi, \wr_2\psi$ (considered as morphisms $\mathcal{U}_\mathcal{P} \to \mathcal{O}_\mathcal{P}$), let Ω be the open set of all $y \in \mathcal{P}$ on a neighborhood of which these two agree (for all $w_\bullet$). By (1.3.16), Ω intersects a lift of W_j in $\mathcal{P}$. If U is a simply-connected open subset of $\mathbb{P}^1 \backslash \mathbf{S}$ such that there is an injective $\eta \in \mathcal{O}(V)$, and if $\widetilde{U}$ is a lift of U in $\mathcal{P}$ (namely, $\widetilde{U}$ is represented by U and a path γ from inside U to $\gamma_\bullet(1)$), then $\mathcal{U}_{\widetilde{U}} = \mathcal{U}_U \simeq \mathbb{U} \otimes_\mathbb{C} \mathcal{O}_U$. Hence, by complex analysis, if $\widetilde{U}$ intersects Ω, then $\widetilde{U} \subset \Omega$. So Ω is closed, and hence must be $\mathcal{P}$. $\square$

The following results are not expected to hold for higher genus conformal blocks.

Lemma 1.3.10. *There exists a set $\mathfrak{U}$ of elements of $\mathcal{U}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \mathbf{S})$ that generates freely the $\mathcal{O}_{\mathbb{P}^1 \backslash \mathbf{S}}$-module $\mathcal{U}_{\mathbb{P}^1 \backslash \mathbf{S}}$, namely, any section of $\mathcal{U}_{\mathbb{P}^1 \backslash \mathbf{S}}$ on an open set $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ can be written in a unique way as a finite sum $f_1 v_1 + f_2 v_2 + \cdots$ where $v_1, v_2, \cdots \in \mathfrak{U}$ and $f_1, f_2, \cdots \in \mathcal{O}(U)$.*

Proof. Assume with out loss of generality that $x_N \in \mathbf{S}$ is ∞. Let $\mathfrak{U}_0$ be a basis of $\mathbb{U}$. Let ζ be the standard coordinate of $\mathbb{C}$. Then one can set $\mathfrak{U} = \{\mathcal{U}_\varrho(\zeta)^{-1} u : u \in \mathfrak{U}_0\}$. $\square$

Proposition 1.3.11. *Let $\mathfrak{U}$ be a set of elements of $\mathscr{U}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \mathbf{S})$ generating freely $\mathscr{U}_{\mathbb{P}^1 \backslash \mathbf{S}}$. Then a linear functional $\psi : \mathcal{W}_\bullet \to \mathbb{C}$ is a conformal block if and only if the following are true:*

1. *Condition 1 of Def. 1.3.3 is satisfied.*

2. *There is $\wr\psi$ associating to each $w_\bullet \in \mathcal{W}_\bullet$ and $u \in \mathfrak{U}$ a multivalued holomorphic function $\wr\psi(\cdot, u, w_\bullet)$ on $\mathbb{P}^1 \backslash \mathbf{S}$ (single-valued on any simply-connected subset U if a path λ from inside U to $\gamma_\bullet(1)$ is specified) satisfying the following condition:*

   For each $1 \leqslant j \leqslant N$, identify $\mathscr{U}_{W_i}$ with $\mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}_{W_i}$ through $\mathcal{U}_\varrho(\eta_i)$. Choose any open simply-connected subset $U \subset W_j \backslash \{x_j\}$ containing $\gamma_j(0)$ and equipped with a continuous $\arg$ function on $\eta_j(U)$ whose value at $\eta_j \circ \gamma_j(0)$ is 0. Then the function $\wr\psi(\gamma_j, u, w_\bullet) \in \mathscr{O}(U)$ satisfies (1.3.16) for all $x \in U$.

Proof. The "only if" part is obvious. Now assume $\wr\psi$ satisfies the two conditions of the present proposition. Then $\wr\psi(\cdot, \cdot, w_\bullet)$ can be extended uniquely to a multivalued homomorphism $\mathscr{U}_{\mathbb{P}^1 \backslash \mathbf{S}} \to \mathscr{O}_{\mathbb{P}^1 \backslash \mathbf{S}}$, which satisfies (1.3.16) for all $u \in \mathfrak{U}$, and hence all $u \in \mathscr{U}_U(U)$. $\qquad\square$

2 Relating untwisted and permutation-twisted conformal blocks

2.1 Permutation branched coverings of $\mathbb{P}^1$

Let C be a (non-necessarily connected) compact Riemann surface. A **branched covering** $\varphi : C \to \mathbb{P}^1$ is by definition a holomorphic map which is non-constant on each connected component of C. Then φ is surjective on each component of C since the image of φ is both open and compact. By complex analysis, $y \in C$ has a neighborhood V such that φ on V is equivalent to the holomorphic map $z \mapsto z^n$. n is called the **branching index** of φ at y, which is 0 precisely when $d\varphi$ is not 0 at y. The (necessarily finite) set of **branch points** is $\Sigma := \{x \in C : d\varphi = 0\}$, which is also the set of points with non-zero branching indexes. Let $\Delta = \varphi(\Sigma)$ be the **critical locus**. Then the restricted map $\varphi : C \backslash \varphi^{-1}(\Delta) \to \mathbb{P}^1 \backslash \Delta$ is a finite (unbranched) covering map, since φ is proper (cf. [Don, Sec. 4.2.1]).

2.1.1 Actions $\Gamma = \pi_1(\mathbb{P}^1 \backslash \mathbf{S}, \gamma_\bullet(1)) \curvearrowright E$ and admissible group elements

Let $\mathfrak{P} = (1.3.9)$ be an N-pathed Riemann spheres with local coordinates. We use the notations in Subsection 1.3.2. We do not assume (1.3.12).

Let E be a finite set. Let $\mathrm{Perm}(E)$ be the permutation group of E. An action of Γ on E is equivalently a homomorphism $\Gamma \to \mathrm{Perm}(E)$.

Definition 2.1.1. We say that $g_1, \ldots, g_N \in \mathrm{Perm}(E)$ are **admissible (with respect to $\mathfrak{P}$)** if there is a (necessarily unique) action $\Gamma \curvearrowright E$ sending

$$[\alpha_j] \mapsto g_j \tag{2.1.1}$$

for each $j = 1, \ldots, N$. The action $\Gamma \curvearrowright E$ is called **the action arising from** $g_1, \ldots, g_N$.

Assume the setting of Def. 1.3.3, in which $\psi : \mathcal{W}_\bullet \to \mathbb{C}$ is a conformal block. We say ψ is **separating** on $\mathcal{W}_j$, if the only $w_j \in \mathcal{W}_j$ satisfying $\psi(w_1 \otimes \cdots \otimes w_N) = 0$ for all $w_1 \in \mathcal{W}_1, \ldots, w_{j-1} \in \mathcal{W}_{j-1}, w_{j+1} \in \mathcal{W}_{j+1}, \ldots, w_N \in \mathcal{W}_N$ is $w_j = 0$. The following proposition is similar to [Hua18, Thm. 4.7]. By this proposition, it is reasonable to consider only twisted conformal blocks associated to $g_1, \ldots, g_N$-twisted modules where $g_1, \ldots, g_N$ are admissible. And we will actually consider in this section twisted conformal blocks only of these types, which correspond well to untwisted conformal blocks associated to the permutation covering of $\mathfrak{P}$. (However, the other results of this article do not logically rely on the following proposition.)

Proposition 2.1.2. *Assume that ψ is separating on $\mathcal{W}_i$ for some $1 \leqslant i \leqslant N$, and the map $u \in \mathbb{U} \mapsto Y^{g_j}(u, z) \in \mathrm{End}(\mathcal{W}_j)[[z]]$ is injective. Then $g_1, \ldots, g_N$ are admissible with respect to $\mathfrak{P}$.*

Proof. Let $\mathfrak{U}$ be as in Prop. 1.3.11. Then for each $u \in \mathfrak{U}$, $\wr\psi(\cdot, u, w_\bullet)$ is a multivalued holomorphic function on $\mathbb{P}^1 \backslash \mathbf{S}$. Then for each $\lambda \in \Lambda_{\gamma_\bullet(1)}$, (1.3.16) and the g_j-equivariance (1.3.2) show that for a small open disc U centered at $\gamma_j(0)$,

$$\wr\psi(\epsilon_j^{\pm 1}\gamma_j, u, w_\bullet)|_U = \wr\psi(\gamma_j, g_j^{\pm 1}u, w_\bullet)|_U$$

since the left hand side is the analytic continuation of $\wr\psi(\gamma_j, u, w_\bullet)$ from U along $\epsilon_j^{\mp 1}$ to U (i.e., multiplying z by $e^{\mp 2\mathbf{i}\pi}$ in the expression $\psi(w_1 \otimes \cdots \otimes Y^{g_j}(u, z)w_j \otimes \cdots \otimes w_N)$). By analytic continuation along γ_j from U to a simply-connected neighborhood V of $\gamma_\bullet(1)$, we have (noticing (1.3.10))

$$\wr\psi(\alpha_j^{\pm 1}, u, w_\bullet)|_V = \wr\psi(\gamma_\bullet(1), g_j^{\pm 1}u, w_\bullet)|_V \tag{2.1.2}$$

where $\gamma_\bullet(1) = \gamma_j^{-1}\gamma_j$ denotes the constant path pointing at $\gamma_\bullet(1)$. Apply this relation successively, we get

$$\wr\psi(\gamma_\bullet(1), f(g_1, \ldots, g_N)u, w_\bullet)|_V = \wr\psi(f(\alpha_1, \ldots, \alpha_N), u, w_\bullet)|_V,$$

for any word f.

Now suppose $g_1, \ldots, g_N$ are not admissible. Then there exists a word f such that $f([\alpha_1], \ldots, [\alpha_N]) = 1$ but $g := f(g_1, \ldots, g_N) \neq 1$. Thus, the above equation implies

$$\wr\psi(\gamma_\bullet(1), gu, w_\bullet)|_V = \wr\psi(\gamma_\bullet(1), u, w_\bullet)|_V.$$

Consider $\wr\psi(\gamma_\bullet(1), gu - u, w_\bullet)$ as a multivalued holomorphic functions on $\mathbb{P}^1 \backslash \mathbf{S}$, i.e. single-valued on $\mathcal{P}$. Since $g \neq 1$, we can choose $v = gu - u \neq 0$. Then $\psi(w_1 \otimes \cdots \otimes Y^{g_j}(v, z)w_j \otimes \cdots \otimes w_N) = 0$ for some nonzero $u \in \mathbb{U}$ and all $w_1, \ldots, w_N$. Since ψ is separating on $\mathcal{W}_j$, we conclude $Y^{g_j}(v, z)w_j = 0$ for all $w_j \in \mathcal{W}_j$, and hence $v = 0$. This gives a contradiction. $\square$

Remark 2.1.3. Note that if a g-twisted $\mathcal{W}$ ($g \in G$) is non-trivial, then $u \in \mathbb{U} \mapsto Y^g(u, z)$ is injective whenever $\mathbb{U}$ is simple (as a $\mathbb{U}$-module). Indeed, assume $Y^g(u, z) = 0$. Choose any $v \in \mathbb{U}$, and assume without loss of generality that $gv = e^{2\mathbf{i}a\pi/k}v$. Then the algebraic

Jacobi identity for Y^g (cf. for instance [Gui21b, Sec. 10]) shows that for each $m, n \in \mathbb{Z}$ we have

$$\sum_{l \in \mathbb{N}} \binom{\frac{a}{k} + m}{l} Y^g \big(Y(v)_{n+l} u, z \big) \cdot z^{\frac{a}{k} + m - l} = 0.$$

Since $Y(v)_n u = 0$ for sufficiently large n, one can easily show, by induction on n starting from sufficiently large numbers, that $Y^g(Y(v)_n u, z) = 0$ for all n. A similar argument shows $Y^g(Y(v_1)_{n_1} \cdots Y(v_l)_{n_l} u, z) = 0$. If $u \neq 0$, as $\mathbb{U}$ is simple, we have $Y^g(v, z) = 0$ for all $v \in \mathbb{U}$. This is impossible.

Remark 2.1.4. Suppose $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ is open and simply-connected, and λ is a path in $\mathbb{P}^1 \backslash \mathbf{S}$ from inside U to $\gamma_\bullet(1)$. Then, by analytic continuation of (2.1.2) from V to U along λ, we see

$$\wr\psi(\lambda \alpha_j^{\pm 1}, u, w_\bullet)\big|_U = \wr\psi(\lambda, g_j^{\pm 1} u, w_\bullet)\big|_U \tag{2.1.3}$$

for all $u \in \mathfrak{U}$, and hence all $u \in \mathscr{U}_{\mathbb{P}^1}(U)$. Formula (2.1.3) will be used in Subsec. 2.4.

Remark 2.1.5. Formula (2.1.3) suggests a definition of twisted conformal blocks when $\mathfrak{P}$ is replaced by an arbitrary positively N-pathed compact Riemann surface $\mathfrak{X} = (X; x_\bullet; \eta_\bullet; \gamma_\bullet)$, but $\Gamma = \pi_1(X \backslash \mathbf{S}, \gamma_\bullet(1))$ (where $\Gamma = \pi_1(X \backslash \mathbf{S}, \gamma_\bullet(1))$) is not necessarily generated by $[\alpha_1], \ldots, [\alpha_N]$. (This will happen even if $X = \mathbb{P}^1$.) Fix an action of Γ on E, and assume each $\mathcal{W}_j$ associated to x_j is g_j-twisted where g_j is the image of $[\alpha_j]$. Then a linear functional $\psi : \mathcal{W}_\bullet \to \mathbb{C}$ is called a conformal block if it satisfies, in addition to the conditions in Def. 1.3.3, that for each path δ in $X \backslash \mathbf{S}$ from and to $\gamma_\bullet(1)$, the relation

$$\wr\psi(\delta, u, w)\big|_U = \wr\psi(\gamma_\bullet(1), [\delta]u, w)\big|_U \tag{2.1.4}$$

holds for any open simply-connected $U \subset X \backslash \mathbf{S}$ containing $\gamma_\bullet(1)$, any $u \in \mathscr{U}_X(U)$, and any $w \in \mathcal{W}_\bullet$.

2.1.2 The permutation covering of $\mathbb{P}^1$ associated to $\Gamma \curvearrowright E$

Fix an action $\Gamma \curvearrowright E$, and let $g_j \in \mathrm{Perm}(E)$ be the image of $[\alpha_j]$. Then E is the disjoint union of orbits of Γ. We choose one element for each orbit, called the marked point of that Γ-orbit. The set of all these marked points of Γ-orbits is denoted by $E(\Gamma)$, which is a subset of E. Then

$$E = \coprod_{\mathbf{e} \in E(\Gamma)} \Gamma \mathbf{e}.$$

The following is well-known; see [Don, Sec. 4.2.2, Thm. 2] or [Ful, Sec. 19b].

Proposition 2.1.6. *There is a compact Riemann surface $C = \coprod_{\mathbf{e} \in E(\Gamma)} C^{\mathbf{e}}$ whose connected components $\{C^{\mathbf{e}} : \mathbf{e} \in E(\Gamma)\}$ are in one-to-one correspondence with the points of $E(\Gamma)$, and a branched covering $\varphi : C \to \mathbb{P}^1$ whose restriction $\varphi : C \backslash \varphi^{-1}(\mathbf{S}) \to \mathbb{P}^1 \backslash \mathbf{S}$ is unbranched, such that the following condition is satisfied: (Note that $\varphi^{-1}(\mathbf{S})$ must be a discrete and hence finite subset of C.)*

For each $\mathbf{e} \in E(\Gamma)$, there is an element $p^{\mathbf{e}} \in C^{\mathbf{e}} \cap \varphi^{-1}(\gamma_\bullet(1))$ satisfying the following condition: for any path $\nu \in \Lambda_{\gamma_\bullet(1)}$ in $\mathbb{P}^1 \backslash \mathbf{S}$, if $\tilde{\nu}$ is its lift to C ending at $p^{\mathbf{e}}$, then the initial point of $\tilde{\nu}$ is $p^{\mathbf{e}}$ if and only if $[\nu]\mathbf{e} = \mathbf{e}$.

Proof. For each $\mathbf{e} \in E(\Gamma)$, the existence of such a topological (and hence analytic) unbranched covering $\varphi : C^{\mathbf{e}}\backslash\varphi^{-1}(\mathbf{S}) \to \mathbb{P}^1\backslash\mathbf{S}$ follows from basic algebraic topology. This covering is finite, since for any $x \in \mathbb{P}^1\backslash\mathbf{S}$ the set $C^{\mathbf{e}} \cap \varphi^{-1}(y)$ is bijective to $\Gamma\mathbf{e}$. One checks easily that for any compact set K, every sequence of $\varphi^{-1}(K)$ has a subsequence converging to a point in $C^{\mathbf{e}} \cap \varphi^{-1}(\mathbf{S})$. Namely, φ is proper on $C^{\mathbf{e}}\backslash\varphi^{-1}(\mathbf{S})$.

We now extend it to $C^{\mathbf{e}}$. The (finite) covering $\varphi : \varphi^{-1}(W_j\backslash\{x_j\}) \to W_j\backslash\{x_j\}$ restricts to a map $\varphi : V \to W_j\backslash\{x_j\}$ for each connected component V of $\varphi^{-1}(W_j\backslash\{x_j\})$, which is (easy to see) surjective and proper, hence a covering map. This covering map is (topologically and hence analytically) equivalent to $\mathcal{D}_r^\times \xrightarrow{z^n} \mathcal{D}_r^\times$ for some $r > 0$. We may thus add a point y to V such that $\varphi : V \cup \{y\} \to W_j$ is analytically equivalent to $\mathcal{D}_r \xrightarrow{z^n} \mathcal{D}_r$. By adding all such y, we get a new Riemann surface $C^{\mathbf{e}}$ and a holomorphic $\varphi : C^{\mathbf{e}} \to \mathbb{P}^1$. It is clear that φ is proper on each $\varphi^{-1}(W_j)$. Thus it is proper on C. In particular, C is compact. $\qquad\square$

Let us formulate the above proposition in a way independent of $E(\Gamma)$.

Definition 2.1.7. Let $\varphi : C \to \mathbb{P}^1$ be a branched covering which is unbranched outside **S**. A map $\Psi_{\gamma_\bullet(1)} : E \to \varphi^{-1}(\gamma_\bullet(1))$ is called Γ**-covariant** if for every $e \in E$ and $\mu \in \Lambda_{\gamma_\bullet(1)}$, the lift of μ to C (or more precisely, in $C\backslash\varphi^{-1}(\mathbf{S})$) ending at $\Psi_{\gamma_\bullet(1)}(e)$ must start from $\Psi_{\gamma_\bullet(1)}([\mu]e)$.

Theorem 2.1.8. *There exists a compact Riemann surface C, a branched covering $\varphi : C \to \mathbb{P}^1$ unbranched outside **S**, and a Γ-covariant bijection $\Psi_{\gamma_\bullet(1)} : E \to \varphi^{-1}(\gamma_\bullet(1))$.*

Proof. Let $\varphi : C \to \mathbb{P}^1$ be as in Thm. 2.1.6. For each $e \in E$, we may find $\mu \in \Lambda_{\gamma_\bullet(1)}$ and $\mathbf{e} \in E(\Gamma)$ such that $e = [\mu]\mathbf{e}$. Define $\Psi_{\gamma_\bullet(1)}(e)$ to be the initial point of the lift of μ ending at $p^{\mathbf{e}}$, which is clearly inside the connected component of C containing $p^{\mathbf{e}} = \Psi_{\gamma_\bullet}(\mathbf{e})$.

We have $\Psi_{\gamma_\bullet(1)}([\mu_1]\mathbf{e}) = \Psi_{\gamma_\bullet(1)}([\mu_2]\mathbf{e})$ iff the lifts of μ_1, μ_2 ending at $p^{\mathbf{e}}$ have the same initial point, iff the lift of $\mu_2^{-1}\mu_1$ ending at $p^{\mathbf{e}}$ must start at $p^{\mathbf{e}}$, iff (by the statements in Thm. 2.1.6) $[\mu_2^{-1}\mu_1]\mathbf{e} = \mathbf{e}$, iff $[\mu_1]\mathbf{e} = [\mu_2]\mathbf{e}$. This proves that $\Psi_{\gamma_\bullet(1)}$ is well-defined.

Suppose $e_1, e_2 \in E$ and $\Psi_{\gamma_\bullet(1)}(e_1) = \Psi_{\gamma_\bullet(1)}(e_2)$. Write $e_1 = [\mu_1]\mathbf{e}_1, e_2 = [\mu_2]\mathbf{e}_2$ for some $\mu_1, \mu_2 \in \Lambda_{\gamma_\bullet(1)}$ and $\mathbf{e}_1, \mathbf{e}_2 \in E(\Gamma)$. Since $\Psi_{\gamma_\bullet(1)}(e_1)$ belongs to $C^{\mathbf{e}_1}$ and $\Psi_{\gamma_\bullet(1)}(e_2)$ belongs to $C^{\mathbf{e}_2}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are equal, which we denote by $\mathbf{e}$. Then the above paragraph shows $[\mu_1]\mathbf{e} = [\mu_2]\mathbf{e}$, i.e. $e_1 = e_2$. This proves that $\Psi_{\gamma_\bullet(1)}$ is injective. The Γ-covariance is obvious.

Finally, for each $x \in \varphi^{-1}(\gamma_\bullet(1))$, we choose $\mathbf{e} \in E(\Gamma)$ such that $x \in C^{\mathbf{e}}$. Choose a curve $\widetilde{\mu}$ in $C^{\mathbf{e}}\backslash\mathbf{S}$ from x to $\mathbf{e}$, and let $\mu = \varphi \circ \widetilde{\mu}$. Then we clearly have $\Psi_{\gamma_\bullet(1)}([\mu]\mathbf{e}) = x$. So $\Psi_{\gamma_\bullet(1)}$ is surjective. $\qquad\square$

We explore some properties of this branched covering.

Theorem 2.1.9. *For each $x \in \mathbb{P}^1\backslash\mathbf{S}$ and $\lambda \in \Lambda_x$, there is a unique bijection*

$$\Psi_\lambda : E \longrightarrow \varphi^{-1}(x),$$

satisfying the following properties (a) and (b):

  (a) By considering $\gamma_\bullet(1)$ as the constant path at this point, $\Psi_{\gamma_\bullet(1)}$ is the Γ-covariant bijection given in Thm. 2.1.8.

31

(b) Suppose λ_1, λ_2 are paths in $\mathbb{P}^1 \backslash \mathbf{S}$, λ_1 ends at the initial point of λ_2, and λ_2 ends at $\gamma_\bullet(1)$. Let $\widetilde{\lambda}_1$ be the lift to C of λ_1 ending at $\Psi_{\lambda_2}(e)$. Then

$$\widetilde{\lambda}_1 \text{ goes from } \Psi_{\lambda_1\lambda_2}(e) \text{ to } \Psi_{\lambda_2}(e). \tag{2.1.5}$$

Ψ_λ depends only on the homotopy class $[\lambda]$. Moreover, for each $\mu \in \Lambda_{\gamma_\bullet(1)}$ and $e \in E$ we have

$$\Psi_\lambda([\mu]e) = \Psi_{\lambda\mu}(e). \tag{2.1.6}$$

We call Ψ **the trivialization of** $\varphi : C \to \mathbb{P}^1$ **determined by** $\Psi_{\gamma_\bullet(1)}$. This name is justified by the fact that, by varying x in a simply-connected open set $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ and multiplying λ from the left by a curve l in U ending at the initial point of λ, we obtain (using Ψ) an equivalence between the projection $E \times U \to U$ and the covering $\varphi : \varphi^{-1}(U) \to U$.

Proof. Uniqueness: $\Psi_{\gamma_\bullet(1)}$ is unique. By (b), $\Psi_\lambda(e)$ is the initial point of the lift of λ to C ending at $\Psi_{\gamma_\bullet(1)}(e)$, which is unique.

Existence: We already have $\Psi_{\gamma_\bullet(1)}$. For each $x \in \mathbb{P}^1 \backslash \mathbf{S}, \lambda \in \Lambda_x$, we define the map Ψ_λ sending each e to the initial point of the lift of λ to C ending at $\Psi_{\gamma_\bullet(1)}(e)$. It is clear that Ψ satisfies (a) and (b).

Obviously, Ψ_λ relies only on $[\lambda]$; it is a bijection since $\Psi_{\gamma_\bullet(1)}$ is so. By Γ-covariance, $\Psi_{\gamma_\bullet(1)}([\mu]e) = \Psi_\mu(e)$. Denote this point by x. Then by (b), the left and the right of (2.1.6) are both the initial point of the lift of λ to C ending at x. This proves (2.1.6). $\square$

We shall investigate the local behavior of φ near $\varphi^{-1}(\mathbf{S})$. Recall that W_i is a disc centered at x_i (with respect to the coordinate η_i) which does not contain any other points of $\mathbf{S}$.

Proposition 2.1.10. *Each connected component $\widetilde{W}_j$ of $\varphi^{-1}(W_j)$ contains exactly one point of $\varphi^{-1}(x_j)$. Let $R > 0$ such that the disc $\eta_j(W_j)$ equals $\mathcal{D}_R$. Then there is $n \in \mathbb{Z}_+$ and a bi-holomorphic function $\widetilde{\eta}_j : \widetilde{W}_j \to \mathcal{D}_r$ (where $r = \sqrt[n]{R}$) such that the following diagram commutes:*

$$
\begin{array}{ccc}
\widetilde{W}_j & \xrightarrow[\simeq]{\widetilde{\eta}_j} & \mathcal{D}_r \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle z^n} \\
W_j & \xrightarrow[\simeq]{\eta_j} & \mathcal{D}_{r^n}
\end{array}
\tag{2.1.7}
$$

Proof. Since φ is locally equivalent to $z \mapsto z^n$ for some $n \in \mathbb{Z}_+$, it is clear that $\varphi(\widetilde{W}_j)$ contains at least one point x of $W_j \backslash \mathbf{S} = W_j \backslash \{x_j\}$. For any $x' \in W_j \backslash \{x_j\}$, we can lift a path in $W_j \backslash \{x_j\}$ from x to x' into C (and hence in $\varphi^{-1}(W_j)$), and the end point of that lifted path must be in $\varphi^{-1}(x')$. Since $\widetilde{W}_j$ is a connected component, the end point must be in $\widetilde{W}_j$. So the locally biholomorphic map $\varphi : \widetilde{W}_j \backslash \varphi^{-1}(\mathbf{S}) \to W_j \backslash \{x_j\} = W_j \backslash \mathbf{S}$ is surjective. We show that it is proper, and therefore a covering map. Let $K \subset W_j \backslash \{x_j\}$ be compact, and let (y_k) be a sequence in $\widetilde{W}_j \cap \varphi^{-1}(K) \subset \widetilde{W}_j \backslash \varphi^{-1}(\mathbf{S})$. By passing to a subsequence, y_k converges to $y \in C$. Since $\varphi(y_k) \to \varphi(y)$, we have $\varphi(y) \in K \subset W_j$ and hence $y \in \varphi^{-1}(W_j)$. Since any connected component of $\varphi^{-1}(W_j)$ is its closed subset, we have $y \in \widetilde{W}_j$ and hence $y \in \widetilde{W}_j \cap \varphi^{-1}(K)$. This shows $\widetilde{W}_j \cap \varphi^{-1}(K)$ is compact.

Since W_j is biholomorphic to $\mathcal{D}_R$ via η_j, we identify these two spaces via η_j. In particular, $W_j\backslash\{x_j\}$ equals $\mathcal{D}_R^\times$, and hence we have a holomorphic covering $\varphi : \widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S}) \to \mathcal{D}_R^\times$. Since all connected topological (and hence analytic) coverings of $\mathcal{D}_R^\times$ are equivalent topologically (and hence analytically) to $\mathcal{D}_r^\times \xrightarrow{z^n} \mathcal{D}_R^\times$ where $n \in \mathbb{Z}_+$ and $r = \sqrt[n]{R}$, we conclude that there is a biholomorphic $\tilde{\eta}_j : \widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S}) \to \mathcal{D}_r^\times$ such that (2.1.7) commutes when restricted to $\widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S})$.

We set $\tilde{\eta}_j$ to be 0 on $\widetilde{W}_j\cap\varphi^{-1}(\mathbf{S})$, which is a discrete and hence finite subset of $\widetilde{W}_j$. To check that $\tilde{\eta}_j$ is analytic, by Morera's theorem, it suffices to check that $\tilde{\eta}_j$ is continuous at any $y \in \widetilde{W}_j \cap \varphi^{-1}(\mathbf{S}) = \widetilde{W}_j \cap \varphi^{-1}(x_j)$. Choose a sequence $y_k \in \widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S})$ converging to y. Then $(\tilde{\eta}_j(y_k))^n = \eta_j \circ \varphi(y_k) \to \eta_j \circ \varphi(y) = \eta_j(x_j) = 0$. This proves the continuity.

We have proved that the diagram (2.1.7) commutes. To finish the proof, we shall show that $\widetilde{W}_j \cap \varphi^{-1}(\mathbf{S})$ contains precisely one element of $\varphi^{-1}(\mathbf{S})$. This will also imply that $\eta_j : \widetilde{W}_j \to \mathcal{D}_r$ is bijective.

A similar argument as above shows that $\varphi : \widetilde{W}_j \to W_j$ is proper. Thus, if we choose a sequence y_k in $W_j\backslash\varphi^{-1}(\mathbf{S})$ such that $\varphi(y_k) \to x_j$, by passing to a subsequence, we see $y_k \to y \in W_j$ and $\varphi(y) = x_j$. So $\widetilde{W}_j$ contains at least one point of $\varphi^{-1}(x_j)$.

Now assume $y_1, y_2 \in \widetilde{W}_j \cap \varphi^{-1}(x_j)$. For each $i = 1, 2$, we have proved that $\tilde{\eta}_j$ is holomorphic near y_i, and sends y_i to 0. Since $\tilde{\eta}_j$ sends nearby points of y_i to $\mathcal{D}_r^\times$, it is not constant near y_i, and hence it is open near y_i. Therefore, if $y_1 \neq y_2$, we may find $y_1', y_2' \in \widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S})$ close to y_1 and y_2 respectively, such that $\tilde{\eta}_j(y_1') = \tilde{\eta}_j(y_2')$. This is impossible, since we know that $\tilde{\eta}_j : \widetilde{W}_j\backslash\varphi^{-1}(\mathbf{S}) \to \mathcal{D}_r^\times$ is bijective. $\qquad\square$

The branching index n in the previous Proposition can be calculated explicitly:

Proposition 2.1.11. *Let $\widetilde{W}_j$ be a connected component of $\varphi^{-1}(W_j)$, and let n be the branching index in Prop. 2.1.10. Then $\widetilde{W}_j \cap \varphi^{-1}(\gamma_j(0))$ has precisely n elements, and there exists $e \in E$ such that*

$$\widetilde{W}_j \cap \varphi^{-1}(\gamma_j(0)) = \Psi_{\gamma_j}\big(\langle g_j\rangle e\big)$$
$$:= \Big\{\Psi_{\gamma_j}(g_j^k e) : k \in \mathbb{Z}\Big\}.$$

In particular, for any $e \in E$ such that $\Psi_{\gamma_j}(e) \in \widetilde{W}_j \cap \varphi^{-1}(\gamma_j(0))$,

$$n = \big|\langle g_j\rangle e\big|,$$

the number of elements in the orbit $\langle g_j\rangle e$.

Recall that the action of $[\alpha_j]$ on E equals that of g_j. $\langle g_j\rangle$ is the cyclic subgroup generated by g_j.

Proof. By Prop. 2.1.10, we identify $\varphi : \widetilde{W}_j \to W_j$ with $\mathcal{D}_r \xrightarrow{z^n} \mathcal{D}_{r^n}$. Since we have assumed $\gamma_j(0) \in W_j\backslash\{x_j\} = \mathcal{D}_{r^n}^\times$, there are n elements in $\widetilde{W}_j \cap \varphi^{-1}(\gamma_j(0))$. By Thm. 2.1.9, any of them is of the form $\Psi_{\gamma_j}(e)$ for some $e \in E$. Hence $\widetilde{W}_j \cap \varphi^{-1}(\gamma_j(0)) = \{e^{2k\mathbf{i}\pi/n}\Psi_{\gamma_j}(e) : 0 \leqslant k \leqslant n-1\}$.

Recall ϵ_j is an anticlockwise circle in W_j from and to $\gamma_j(0)$. By Thm. 2.1.9-(b), $\Psi_{\gamma_j}(e)$ is the initial point of the lift of γ_j in $C\backslash\varphi^{-1}(\mathbf{S})$ ending at $\Psi_{\gamma_\bullet(1)}(e)$. Thus, for each $k \in \mathbb{Z}$, the lift of $\epsilon_j^k\gamma_j = \gamma_j\alpha_j^k$ (recall (1.3.10)) ending at $\Psi_{\gamma_\bullet(1)}(e)$ and the lift of ϵ_j^k ending at $\Psi_{\gamma_j}(e)$ have the same initial point, which is $e^{-2k\mathbf{i}\pi/n}\Psi_{\gamma_j}(e)$. Thus, we conclude $\Psi_{\gamma_j}(g_j^k e) = \Psi_{\gamma_j}([\alpha_j^k]e) = e^{-2k\mathbf{i}\pi/n}\Psi_{\gamma_j}(e)$. This finishes the proof. (The formula for n follows immediately since Ψ_{γ_j} is one-to-one.) $\qquad\square$

The above propositions immediately show:

Corollary 2.1.12. *For each $1 \leqslant j \leqslant N$, there is a (necessarily unique) bijective map*

$$\Upsilon : \{\langle g_j\rangle\text{-orbits in } E\} \longrightarrow \varphi^{-1}(x_j)$$

such that for each $e \in E$, the point $\Upsilon(\langle g_j\rangle e)$ and the set $\Psi_{\gamma_j}(\langle g_j\rangle e)$ are contained in the same connected component $\widetilde{W}_j$ of $\varphi^{-1}(W_j)$. In particular, the domain and the codomain of Υ are both bijective to the set of connected components of $\varphi^{-1}(W_j)$.

Pictorially, $\Upsilon(\langle g_j\rangle e)$ is the center of the disc $\widetilde{W}_j$, and $\Psi_{\gamma_j}(\langle g_j\rangle e)$ is a set of n points on $\widetilde{W}_j$ surrounding that center.

Remark 2.1.13. We have suppressed the subscript j and write Υ_j as Υ for simplicity. This means that if $i \neq j$, Υ_i and Υ_j are different maps. In particular, even if (say) g_i equals g_j, $\Upsilon(\langle g_i\rangle e)$ and $\Upsilon(\langle g_j\rangle e)$ are allowed to be different.

Theorem 2.1.14. *Any two branched coverings satisfying Thm. 2.1.8 are equivalent. More precisely, assume $\varphi : C \to \mathbb{P}^1$ and $\varphi' : C' \to \mathbb{P}^1$ are branched coverings with Γ-covariant bijections $\Psi_{\gamma_\bullet(1)}, \Psi_{\gamma'_\bullet(1)}$ as in Thm. 2.1.8. Then there is a unique holomorphic map $F : C \to C'$ such that the following diagrams commute:*

$$
\begin{array}{ccc}
C & \xrightarrow[\simeq]{F} & C' \\
& \searrow{\scriptstyle\varphi} \quad \swarrow{\scriptstyle\varphi'} & \\
& \mathbb{P}^1 &
\end{array}
\tag{2.1.8}
$$

$$
\begin{array}{ccc}
& E & \\
{\scriptstyle\Psi_{\gamma_\bullet(1)}}\swarrow & & \searrow{\scriptstyle\Psi'_{\gamma'_\bullet(1)}} \\
\varphi^{-1}(\gamma_\bullet(1)) & \xrightarrow[\simeq]{F} & (\varphi')^{-1}(\gamma'_\bullet(1))
\end{array}
\tag{2.1.9}
$$

F is a bi-holomorphism. Moreover, for each path λ in $\mathbb{P}^1\backslash\mathbf{S}$ ending at $\gamma_\bullet(1)$, $e \in E$, and $1 \leqslant j \leqslant N$, F satisfies

$$F\big(\Psi_\lambda(e)\big) = \Psi'_\lambda(e),$$
$$F\big(\Upsilon(\langle g_j\rangle e)\big) = \Upsilon'(\langle g_j\rangle e).$$

Note that Ψ'_λ is the trivialization for φ' defined by $\Psi'_{\gamma'_\bullet(1)}$, which defines Υ' defined for φ' as in Cor. 2.1.12.

Proof. Uniqueness: By basic algebraic topology, if we fix a point for each connected component of C, the continuous maps F satisfying (2.1.8) are uniquely determined by their values at these points. Therefore, the holomorphic F satisfying the two commuting diagrams are unique when restricted to $C \backslash \varphi^{-1}(\mathbf{S})$. Since $\varphi^{-1}(\mathbf{S})$ is discrete, the values of F on $\varphi^{-1}(\mathbf{S})$ are also unique.

Existence: Define a map $F : C \backslash \varphi^{-1}(\mathbf{S}) \to C' \backslash (\varphi')^{-1}(\mathbf{S}), \Psi_\lambda(e) \mapsto \Psi'_\lambda(e)$ for each path λ in $\mathbb{P}^1 \backslash \mathbf{S}$ and each $e \in E$. Once we have shown that F is well-defined, then F clearly satisfies (2.1.8) and (2.1.9) (outside $\varphi^{-1}(\mathbf{S})$), and is a local homeomorphism by Thm. 2.1.9-(b) (applied to the two trivializations).

Suppose $\Psi_{\lambda_1}(e_1) = \Psi_{\lambda_2}(e_2)$. Let x denote their image under φ. Then $\lambda_1, \lambda_2 \in \Lambda_x$, and hence $\mu := \lambda_2^{-1} \lambda_1$ belongs to $\Lambda_{\gamma_\bullet(1)}$. By (2.1.6), $\Psi_{\lambda_2}(e_2) = \Psi_{\lambda_1 \mu}(e_2) = \Psi_{\lambda_1}([\mu]e_2)$. So $\Psi_{\lambda_1}(e_1) = \Psi_{\lambda_2}(e_2)$ iff $\Psi_{\lambda_1}(e_1) = \Psi_{\lambda_1}([\mu]e_2)$ iff $e_1 = [\mu]e_2$. Similarly, $\Psi'_{\lambda_1}(e_1) = \Psi'_{\lambda_2}(e_2)$ iff $e_1 = [\mu]e_2$. So F is well-defined and is a bijection.

We now extend F to $\varphi^{-1}(\mathbf{S})$: We let F send each $\Upsilon(\langle g_j \rangle e)$ to $\Upsilon'(\langle g_j \rangle e)$ (for all $e \in E$). Now we have a bijective $F : C \to C'$, and (2.1.8) commutes. To finish the proof, it remains to show that F is holomorphic at each point of the finite set $\varphi^{-1}(\mathbf{S})$. By Morera's theorem, it suffices to check the continuity.

Let $\widetilde{W}_j$ be the connected component of $\varphi^{-1}(W_j)$ containing $y = \Upsilon(\langle g_j \rangle e)$, and also containing a set $\Psi_{\gamma_j}(\langle g_j \rangle e)$ of n points surrounding $\Upsilon(\langle g_j \rangle e)$. By (2.1.7), $\widetilde{W}_j \backslash \varphi^{-1}(\mathbf{S}) = \widetilde{W}_j \backslash \{y\}$ is the set of all $\Psi_{l\gamma_j}(e)$ where l is a path in $W_j \backslash \{x_j\}$ ending at $\gamma_j(0)$. (Note that $\Psi_{l\gamma_j}(e)$ is the initial point of the lift of l ending at $\Psi_{\gamma_j}(e)$.) Similarly, if we let $\widetilde{W}'_j$ be the connected component of $(\varphi')^{-1}(W_j)$ containing $y' = \Upsilon'(\langle g_j \rangle e)$ and hence containing the set $\Psi'_{\gamma_j}(\langle g_j \rangle e)$, then $\widetilde{W}'_j \backslash \{y'\}$ is the set of all $\Psi'_{l\gamma_j}(e)$. This shows that F restricts to a holomorphic map from $\widetilde{W}_j \backslash \{y\}$ to $\widetilde{W}'_j \backslash \{y'\}$. Choose any sequence y_k in $\widetilde{W}_j \backslash \{y\}$ converging to y. Then $\varphi'(Fy_k) = \varphi(y_k) \to \varphi(y) = x_j$. By Prop. 2.1.10, $\varphi' : \widetilde{W}'_j \to W_j$ is equivalent to $\mathcal{D}_r \xrightarrow{z^n} \mathcal{D}_{r^n}$, and x_j is equivalent to $0 \in \mathcal{D}_{r^n}$. This shows that Fy_k converges to $0 \in \mathcal{D}_r$, namely, converges to $y' = Fy$. This proves the continuity. $\qquad \square$

We call the data

$$\left(\varphi : C \to \mathbb{P}^1; \Psi_{\gamma_\bullet(1)} \right)$$

the **permutation covering** of $\mathbb{P}^1$ associated to the permutations $g_1, \ldots, g_N$ (equivalently, the action $\Gamma \curvearrowright E$). Its isomorphism class (in the sense of Thm. 2.1.14) depends only on the action $\Gamma \curvearrowright E$ (note that Γ also contains the information $\gamma_\bullet(1)$) but not on the other information of $\gamma_\bullet$. However, Υ does rely on the homotopy class of each γ_j.

Remark 2.1.15. The Γ-orbits in E are in one-to-one correspondence with the connected components of C: given a Γ-orbit $\Omega \subset E$, the corresponding connected component C^Ω is the one containing the set $\Psi_{\gamma_\bullet(1)}(\Omega)$, and hence containing $\Psi_\lambda(\Omega)$ for any path λ in $C \backslash \varphi^{-1}(\mathbf{S})$ ending at $\gamma_\bullet(1)$. $C^\Omega \backslash \varphi^{-1}(\mathbf{S})$ is an $|\Omega|$-fold covering of $\mathbb{P}^1 \backslash \mathbf{S}$ where $|\Omega|$ is the cardinality of Ω. It is clear that for each g_j-orbit $\omega = \langle g_j \rangle e$, the branched point $\Upsilon(\langle g_j \rangle e)$ is on C^Ω if and only if $\omega \subset \Omega$. We know that the branching index at $\Upsilon(\omega)$ is $|\omega|$. Therefore, we may use the Riemann-Hurwitz formula [For, 17.14] to conclude:

Corollary 2.1.16. *For each Γ-orbit Ω, let $\mathrm{Orb}_\Omega(g_j)$ be the set of g_j-orbits inside Ω. Then the genus $g(C^\Omega)$ of C^Ω equals*

$$g(C^\Omega) = 1 - |\Omega| + \frac{1}{2} \sum_{j=1}^{N} \sum_{\omega \in \mathrm{Orb}_\Omega(g_j)} (|\omega| - 1). \tag{2.1.10}$$

In this formula, the sum $\sum |\omega|$ clearly equals $|\Omega|$. Therefore

$$g(C^\Omega) = 1 + \left(\frac{N}{2} - 1\right) |\Omega| - \frac{1}{2} \sum_{j=1}^{N} |\mathrm{Orb}_\Omega(g_j)|. \tag{2.1.11}$$

When $N = 3$, this formula agrees with [BS11, Lemma 8].

Remark 2.1.17. Suppose that we have new paths $\gamma'_1, \dots, \gamma'_N$ with common end point satisfying that $[\gamma'_j] = [l_j \gamma_j \sigma]$, l_j is a path in W_j from $\gamma'_j(0)$ to $\gamma_j(0)$, and σ is a path in $\mathbb{P}^1 \backslash \mathbf{S}$ with end point $\gamma_\bullet(1)$. (In particular, we assume $\gamma'_j(0) = l_j(0)$ and $\gamma'_j(1) = \sigma(1)$.)
  (Namely, we assume $\gamma'_\bullet$ satisfy the conditions in Rem. 1.3.8 except (1.3.18).)
  Similar to (1.3.10), we define $\alpha'_j = (\gamma'_j)^{-1} \epsilon'_j \gamma'_j$ where ϵ'_j is an anticlockwise loop in W_j from and to $\gamma'_j(0)$. This defines an action of $\Gamma' = \pi_1(\mathbb{P}^1 \backslash \mathbf{S}, \gamma'_\bullet(1))$ by sending each $[\alpha'_j]$ to g_j. We then have an isomorphism $\Gamma \xrightarrow{\simeq} \Gamma'$ defined by $[\mu] \mapsto [\sigma^{-1} \mu \sigma]$, and this isomorphism sends $[\alpha_j]$ to $[\alpha'_j]$. This isomorphism intertwines the actions of Γ, Γ' on E, namely, we have commuting diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow[\simeq]{[\mu] \mapsto [\sigma^{-1} \mu \sigma]} & \Gamma' \\ & \searrow \quad \swarrow & \\ & \mathrm{Aut}(E) & \end{array} \tag{2.1.12}$$

The following proposition allows us to change the base point $\gamma_\bullet(1)$.

Proposition 2.1.18. *Let $(\varphi : C \to \mathbb{P}^1; \Psi_{\gamma_\bullet(1)})$ be a branched covering associated to $\Gamma \curvearrowright E$ and the paths $\gamma_\bullet$. Assume the setting of Rem. 2.1.17. Let Ψ be the trivialization determined by $\Psi_{\gamma_\bullet(1)}$, which defines Υ as in Cor. 2.1.12. Then the map*

$$\begin{aligned} \Psi'_{\gamma'_\bullet(1)} &: E \to \varphi^{-1}(\gamma'_\bullet(1)), \\ \Psi'_{\gamma'_\bullet(1)}(e) &= \Psi_{\sigma^{-1}}(e) \end{aligned} \tag{2.1.13}$$

is a Γ'-covariant bijection, and $(\varphi : C \to \mathbb{P}^1; \Psi'_{\gamma'_\bullet(1)})$ is a branched covering associated to $\Gamma' \curvearrowright E$ and the paths $\gamma'_\bullet(1)$.
  Moreover, let Ψ' be the trivialization defined by $\Psi'_{\gamma'_\bullet(1)}$, which defines Υ' as in Cor. 2.1.12. Then

$$\Upsilon = \Upsilon',$$

and for each $e \in E$ and each path λ in $\mathbb{P}^1 \backslash \mathbf{S}$ ending at $\gamma_\bullet(1)$,

$$\Psi_\lambda(e) = \Psi'_{\lambda\sigma}(e). \tag{2.1.14}$$

Proof. For each path λ' in $\mathbb{P}^1 \backslash \mathbf{S}$ ending at $\gamma'_\bullet(1)$, define a bijection $\Psi'_{\lambda'} = \Psi_{\lambda'\sigma^{-1}} : E \to \varphi^{-1}(\lambda'(0))$. Then Ψ' satisfies conditions (a) and (b) of Thm. 2.1.9. Let us show that, by considering $\gamma'_\bullet(1)$ as a constant path, $\Psi'_{\gamma'_\bullet(1)} = \Psi_{\sigma^{-1}}$ is Γ'-covariant. Choose any path μ' in $\mathbb{P}^1 \backslash \mathbf{S}$ from and to $\gamma'_\bullet(1)$. For each $e \in E$, by (2.1.12), $[\mu']e = [\sigma\mu'\sigma^{-1}]e$. Therefore

$$\Psi'_{\gamma'_\bullet(1)}([\mu']e) = \Psi_{\sigma^{-1}}([\sigma\mu'\sigma^{-1}]e) \xLeftrightarrow{(2.1.6)} \Psi_{\mu'\sigma^{-1}}(e),$$

which, by Thm. 2.1.9-(b), is the initial point of the lift of μ' ending at $\Psi_{\sigma^{-1}}(e) = \Psi'_{\gamma'_\bullet(1)}(e)$.

We have proved that $\Psi'_{\gamma'_\bullet(1)}$ is a Γ'-covariant bijection, that the above defined Ψ' is the corresponding trivialization, and that (2.1.14) holds. We now show $\Upsilon = \Upsilon'$. We know that $\Psi_{\gamma_j}(\langle g_j \rangle e)$ are points around $\Upsilon(\langle g_j \rangle e)$, and that they are on the same connected component of $\varphi^{-1}(W_j)$. The same can be said for $\Psi'_{\gamma'_j}(\langle g_j \rangle e)$ and $\Upsilon'(\langle g_j \rangle e)$. Since $\gamma'_j = l_j \gamma_j \sigma$, from (2.1.14), $\Psi'_{\gamma'_j}(\langle g_j \rangle e) = \Psi_{\gamma'_j \sigma^{-1}}(\langle g_j \rangle e) = \Psi_{l_j \gamma_j}(\langle g_j \rangle e)$. Since $\Psi_{l_j \gamma_j}(g_j^k e)$ is the initial point of the lift of l_j ending at $\Psi_{\gamma_j}(g_j^k e)$, and since l_j is in W_j, $\Psi_{l_j \gamma_j}(g_j^k e)$ and $\Psi_{\gamma_j}(g_j^k e)$ are in the same connected component of $\varphi^{-1}(W_j)$. The same can be said for $\Upsilon(\langle g_j \rangle e)$ and $\Upsilon'(\langle g_j \rangle e)$. Therefore, we must have $\Upsilon(\langle g_j \rangle e) = \Upsilon'(\langle g_j \rangle e)$. $\qquad\square$

2.1.3 The permutation covering of $\mathfrak{P}$ associated to $\Gamma \curvearrowright E$ and $E(g_\bullet)$

We now assume $\mathfrak{P}$ is positively N-pathed, namely, (1.3.12) is also true. We have constructed the permutation covering $(C; \varphi^{-1}(\mathbf{S}))$ of $(\mathbb{P}^1; \mathbf{S})$. In this subsection, we shall add local coordinates to C at these marked points. These local coordinates should be compatible with those of $\mathfrak{P}$ and the branched covering map φ. Moreover, to uniquely determine these local coordinates, we have to fix an element $\mathfrak{x}$ for each g_j-orbit, called the **marked point** of that g_j-orbit. We let

$$E(g_j) = \{\text{all marked points } \mathfrak{x} \text{ of the } g_j\text{-orbits of } E\}.$$

For each point of $\varphi^{-1}(x_j)$, necessarily represented by $\Upsilon(\langle g_j \rangle \mathfrak{x})$ where $\mathfrak{x} \in E(g_j)$, we let $\widetilde{W}_{j,\mathfrak{x}}$ be the unique connected component of $\varphi^{-1}(W_j)$ containing this point (equivalently, containing $\Psi_{\gamma_j}(\langle g_j \rangle \mathfrak{x})$). By Prop. 2.1.10, we can choose a local coordinate $\widetilde{\eta}_{j,\mathfrak{x}} \in \mathscr{O}(\widetilde{W}_{j,\mathfrak{x}})$ of C at $\Upsilon(\langle g_j \rangle \mathfrak{x})$ such that (for $k = |\langle g_j \rangle \mathfrak{x}|$ and for some $r_i > 0$) the diagram

$$
\begin{array}{ccc}
\widetilde{W}_{j,\mathfrak{x}} & \xrightarrow[\simeq]{\widetilde{\eta}_{j,\mathfrak{x}}} & \mathcal{D}_{\sqrt[k]{r_j}} \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle z^k} \\
W_j & \xrightarrow[\simeq]{\eta_j} & \mathcal{D}_{r_j}
\end{array}
\tag{2.1.15}
$$

commutes.

Such $\widetilde{\eta}_{j,\mathfrak{x}}$ is unique up to multiplication by a power of

$$\omega_k = e^{-\frac{2i\pi}{k}}. \tag{2.1.16}$$

However, since η_j sends $\gamma_j(0)$ to a positive number (by (1.3.12)), we may (and shall) choose $\widetilde{\eta}_{j,\mathfrak{x}}$ such that

$$\widetilde{\eta}_{j,\mathfrak{x}} \circ \Psi_{\gamma_j}(\mathfrak{x}) > 0. \tag{2.1.17}$$

Then

$$\mathfrak{X} = \big(C; \Upsilon(\langle g_j \rangle \mathfrak{x}); \widetilde{\eta}_{j,\mathfrak{x}} \in \mathscr{O}(\widetilde{W}_{j,\mathfrak{x}}) \text{ for all } 1 \leqslant j \leqslant N, \mathfrak{x} \in E(g_j)\big) \qquad (2.1.18)$$

is a pointed compact Riemann surface with local coordinates. The data

$$\big(\mathfrak{X}, \varphi : C \to \mathbb{P}^1, \Psi_{\gamma_\bullet(1)}\big)$$

(or simply just $\mathfrak{X}$) is called the **permutation covering** of the positively N-pathed

$$\mathfrak{P} = (\mathbb{P}^1; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N; \gamma_1, \ldots, \gamma_N)$$

associated to the action $\Gamma \curvearrowright E$ and the set $E(g_j)$ (for all j) of marked points of g_j-orbits. $\widetilde{\eta}_{j,\mathfrak{x}}$ is the local coordinate at $\Upsilon(\langle g_j \rangle \mathfrak{x})$ defined on an open disc $\widetilde{W}_{j,\mathfrak{x}}$. The set of all marked points is $\varphi^{-1}(\mathbf{S})$.

Remark 2.1.19. Suppose we have positively N-pathed $\mathfrak{P}' = (\mathbb{P}^1; x_\bullet; \eta_\bullet; \gamma_\bullet')$ where $\gamma_1', \ldots, \gamma_N'$ are equivalent to $\gamma_\bullet$ (cf. Rem. 1.3.8). Define the action of $\Gamma' = \pi_1(\mathbb{P}^1 \backslash \mathbf{S}, \gamma_\bullet'(1))$ on E as in Rem. 2.1.17. Let $\mathfrak{X}$ be (2.1.18). Define a Γ'-covariant bijection $\Psi_{\gamma_\bullet'(1)}'$ using (2.1.13), which defines a trivialization Ψ. Noting (1.3.19), for each $\mathfrak{x} \in E(g_j)$ we have $\Psi_{\gamma_j'}'(\mathfrak{x}) = \Psi_{l_j \gamma_j}(\mathfrak{x})$, whose value under $\widetilde{\eta}_{j,\mathfrak{x}}$ is positive. This fact, together with Prop. 2.1.18, implies that

$$\big(\mathfrak{X}, \varphi : C \to \mathbb{P}^1, \Psi_{\gamma_\bullet'(1)}'\big)$$

is a permutation covering of $\mathfrak{P}'$ associated to $\Gamma' \curvearrowright E$ and $E(g_\bullet)$.

2.2 Permutation-twisted modules; Main Theorem

We now let $\mathbb{U} = \mathbb{V}^{\otimes E}$, a tensor product of $|E|$ pieces of $\mathbb{V}$ where $|E|$ is the cardinality of the finite set E. $\mathbb{U}$ is spanned by vectors of the form

$$\mathbf{v} = \bigotimes_{e \in E} \mathbf{v}(e) \qquad (\text{each } \mathbf{v}(e) \in \mathbb{V}).$$

Namely, $\mathbf{v}(e)$ is the e-th tensor component of $\mathbf{v}$. $\mathrm{Perm}(E)$ acts faithfully on $\mathbb{U}$: for each $g \in \mathrm{Perm}(E)$,

$$(g \cdot \mathbf{v})(e) = \mathbf{v}(g^{-1}e). \qquad (2.2.1)$$

Namely, $g \cdot \mathbf{v}$ is a tensor product of vectors whose e-th component is $v_{g^{-1}e}$. The conformal vector $\mathbf{c}_\mathbb{U}$ of $\mathbb{U}$ is $\sum_{\epsilon \in E} \mathbf{c}_\epsilon$ where $\mathbf{c}_\epsilon(e)$ equals $\mathbf{c}_\mathbb{V}$ (the conformal vector of $\mathbb{V}$) when $\epsilon = e$, and equals $\mathbf{1}$ otherwise.

Recall $g_1, \ldots, g_N \in \mathrm{Perm}(E)$. For each g_j, we shall construct g_j-twisted $\mathbb{U}$-modules. For each g_j-orbit $\langle g_j \rangle \mathfrak{x}$ we choose a $\mathbb{V}$-module $\mathbb{W}_{j,\mathfrak{x}}$.

Let ζ be the standard coordinate of $\mathbb{C}$. Fix any $\mathfrak{x} \in E(g_j)$. We set a 2-pointed Riemann sphere with local coordinates

$$(\mathbb{P}^1_{j,\mathfrak{x}}; 0, \infty; \zeta, \zeta^{-1}) \qquad (2.2.2)$$

where $\mathbb{P}^1_{j,\mathfrak{x}} = \mathbb{P}^1$. We associate $\mathbb{W}_{j,\mathfrak{x}}$ and its contragredient $\mathbb{W}'_{j,\mathfrak{x}}$ to $0, \infty$ respectively. Then the standard pairing

$$\varphi_{j,\mathfrak{x}} : \mathbb{W}_{j,\mathfrak{x}} \otimes \mathbb{W}'_{j,\mathfrak{x}} \to \mathbb{C}, \qquad w \otimes w' \mapsto \langle w, w' \rangle \tag{2.2.3}$$

is a conformal block associated to (2.2.2) and the two $\mathbb{V}$-modules. Set

$$k = |\langle g_j \rangle \mathfrak{x}|. \tag{2.2.4}$$

Let $\mathbb{C}_{j,\mathfrak{x}} = \mathbb{P}^1_{j,\mathfrak{x}} \backslash \{\infty\}$ and $\mathbb{C}^\times_{j,\mathfrak{x}} = \mathbb{P}^1_{j,\mathfrak{x}} \backslash \{0, \infty\}$. Identify

$$\boxed{\mathscr{V}_{\mathbb{C}^\times_{j,\mathfrak{x}}} = \mathbb{V} \otimes_\mathbb{C} \mathscr{O}_{\mathbb{C}^\times_{j,\mathfrak{x}}} \qquad \text{via } \mathcal{U}_\varrho(\zeta^k)} \tag{2.2.5}$$

where $\zeta^k : z \mapsto z^k$. By Thm. 1.2.1, we have

$$\wr^k \varphi_{j,\mathfrak{x}} : \left(\mathbb{V} \otimes \mathscr{O}(\mathbb{C}^\times_{j,\mathfrak{x}})\right)^{\otimes k} \otimes \mathbb{W}_{j,\mathfrak{x}} \otimes \mathbb{W}'_{j,\mathfrak{x}} \to \mathscr{O}\left(\mathrm{Conf}^k(\mathbb{C}^\times_{j,\mathfrak{x}})\right)$$

where all $\otimes$ are over $\mathbb{C}$. For each $z \in \mathbb{C}^\times$ with argument $\arg z$, $\sqrt[k]{z}$ is assumed to have argument $\frac{1}{k} \arg z$. Let

$$\omega_k^{\bullet-1} z^{1/k} = \left(z^{1/k}, \omega_k z^{1/k}, \omega_k^2 z^{1/k}, \dots, \omega_k^{k-1} z^{1/k}\right) \qquad \in \mathrm{Conf}^k(\mathbb{C}^\times_{j,\mathfrak{x}}).$$

Then for each $\mathbf{v} \in \mathbb{U}$, $w \in \mathbb{W}_{j,\mathfrak{x}}$, $w' \in \mathbb{W}'_{j,\mathfrak{x}}$, and $z \in \mathbb{C}^\times$, we set

$$\boxed{\begin{aligned}
&\mathcal{L}_{j,\mathfrak{x}}(\mathbf{v}, w, w', z) \\
&:= \wr^k \varphi_{j,\mathfrak{x}}\left(\mathbf{v}(\mathfrak{x}), \mathbf{v}(g_j\mathfrak{x}), \dots, \mathbf{v}(g_j^{k-1}\mathfrak{x}), w \otimes w'\right)_{\omega_k^{\bullet-1} z^{1/k}}
\end{aligned}} \tag{2.2.6}$$

We now set

$$\mathcal{W}_j = \bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} \tag{2.2.7}$$

with $\tilde{L}_0$-action

$$\tilde{L}_0^{g_j}\left(\bigotimes_\mathfrak{x} w_{j,\mathfrak{x}}\right) = \left(\sum_\mathfrak{x} \frac{\widetilde{\mathrm{wt}} w_{j,\mathfrak{x}}}{|\langle g_j \rangle \mathfrak{x}|}\right)\left(\bigotimes_\mathfrak{x} w_{j,\mathfrak{x}}\right) \tag{2.2.8}$$

if each $w_{j,\mathfrak{x}} \in \mathbb{W}_{j,\mathfrak{x}}$ is $\tilde{L}_0$-homogeneous with eigenvalue $\widetilde{\mathrm{wt}} w_{j,\mathfrak{x}}$.

Theorem 2.2.1. *There is a (necessarily unique) vertex operation Y^{g_j} which makes the $\tilde{L}_0^{g_j}$-graded vector space $\mathcal{W}_j$ a g_j-twisted $\mathbb{U}$-module and satisfies the following condition: for each $\mathbf{v} \in \mathbb{U}$, $\mathbf{w}_j = \bigotimes_\mathfrak{x} w_{j,\mathfrak{x}} \in \otimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} = \mathcal{W}_j$, and $\mathbf{w}'_j = \bigotimes_\mathfrak{x} w'_{j,\mathfrak{x}} \in \otimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}'_{j,\mathfrak{x}} = \mathcal{W}'_j$,*

$$\left\langle Y^{g_j}(\mathbf{v}, z)\mathbf{w}_j, \mathbf{w}'_j \right\rangle = \prod_{\mathfrak{x} \in E(g_j)} \mathcal{L}_{j,\mathfrak{x}}(\mathbf{v}, w_{j,\mathfrak{x}}, w'_{j,\mathfrak{x}}, z). \tag{2.2.9}$$

Note that Y^{g_j} depends not only on g_j and the $\mathbb{V}$-modules associated to the g_j-orbits, but also on the set $E(g_j)$.

Proof. When there is only one g_j-orbit, this was proved in [Gui21b, Sec. 10]. In the general case, this follows from the fact that if for each $1 \leqslant l \leqslant m$, h_l is an automorphism of a VOA $\mathbb{V}_l$ with finite order, and if $\mathcal{M}_l$ is a h_l-twisted $\mathbb{V}_l$-module with vertex operation Y^{h_l}, then $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m$ is an $h = h_1 \otimes \cdots \otimes h_m$-twisted $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_m$-module with vertex operation $v_1 \otimes \cdots \otimes v_m \mapsto Y^{h_1}(v_1, z) \otimes \cdots \otimes Y^{h_m}(v_m, z)$ and $\widetilde{L}_0^h$-grading defined by the sum over all $1 \leqslant l \leqslant m$ of the $\widetilde{L}_0^{h_l}$-weight on the $\mathcal{M}_l$-component. $\square$

The following Proposition is [BDM02, Thm. 7.10-(1)]

Proposition 2.2.2. *If for each $\varkappa \in E(g_j)$, $\mathbb{W}_{j,\varkappa}$ is an irreducible $\mathbb{V}$-module, then $\mathcal{W}_j$ is an irreducible g_j-twisted $\mathbb{U}$-module.*

We now relate contragredient untwisted and twisted modules. Set

$$h_j = g_j^{-1}, \qquad E(h_j) = E(g_j). \tag{2.2.10}$$

For each h_j-orbit $\langle h_j \rangle \varkappa$ where $\varkappa \in E(h_j)$, we choose $\mathbb{W}'_{j,\varkappa}$, the contragredient module of $\mathbb{W}_{j,\varkappa}$. Then $\mathcal{W}'_j$ equals $\bigotimes_{\varkappa \in E(h_j)} \mathbb{W}'_{j,\varkappa}$ as $\widetilde{L}_0$-graded vector spaces. We use these data to construct a vertex operation Y^{h_j} for $\mathcal{W}'_j$ as in Thm. 2.2.1. Recall the definition of contragredient twisted modules in (1.3.7). The following theorem will not be used until Chapter 3.

Theorem 2.2.3. *$(\mathcal{W}'_j, Y^{h_j})$ is the contragredient twisted module of $(\mathcal{W}_j, Y^{g_j})$.*

Proof. Consider $\varphi_{j,\varkappa}$ defined by (2.2.3) as a conformal block associated to $(\mathbb{P}^1; \infty, 0; \zeta^{-1}, \zeta)$, which is equivalent to $(\mathbb{P}^1_{j,\varkappa}; 0, \infty; \zeta, \zeta^{-1})$ via the map $z \mapsto z^{-1}$. This equivalence sends the standard coordinate ζ to ζ^{-1}, and sends any $\omega_k^l z^{1/k}$ to $\omega_k^{-l} z^{-1/k}$ where $z \in \mathbb{P}, k \in \mathbb{Z}_+, l \in \mathbb{Z}$. Thus, assuming the identification $\mathscr{V}_{\mathbb{C}^\times} = \mathbb{V} \otimes_\mathbb{C} \mathscr{O}_{\mathbb{C}^\times}$ via $\mathcal{U}_\varrho(\zeta^{-k_{j,\varkappa}})$, by the definition of Y^{h_j}, we have (noticing Thm. 1.2.1-(4))

$$\big\langle \mathbf{w}_j, Y^{h_j}(\mathbf{v}, z)\mathbf{w}'_j \big\rangle$$
$$= \prod_{\varkappa \in E(h_j)} \iota^{k_{j,\varkappa}} \varphi_{j,\varkappa}\Big(\mathbf{v}(\varkappa), \mathbf{v}(g_j^{-1}\varkappa), \ldots, \mathbf{v}(g_j^{-k_{j,\varkappa}+1}\varkappa), w \otimes w' \Big)_{\omega_{k_{j,\varkappa}}^{-\bullet+1} z^{-1/k_{j,\varkappa}}}$$
$$= \prod_{\varkappa \in E(h_j)} \iota^{k_{j,\varkappa}} \varphi_{j,\varkappa}\Big(\mathbf{v}(\varkappa), \mathbf{v}(g_j\varkappa), \ldots, \mathbf{v}(g_j^{k_{j,\varkappa}-1}\varkappa), w \otimes w' \Big)_{\omega_{k_{j,\varkappa}}^{\bullet-1} z^{-1/k_{j,\varkappa}}}$$

where for each $k \in \mathbb{Z}_+$ we set

$$\omega_k^{-\bullet+1} z^{-1/k} = \big(z^{-1/k}, \omega_k^{-1} z^{-1/k}, \ldots, \omega_k^{-k+1} z^{-1/k} \big).$$

We now do not assume the identification. Then

$$\big\langle \mathbf{w}_j, Y^{h_j}\big(\mathcal{U}(\gamma_z)\mathbf{v}, z^{-1}\big)\mathbf{w}'_j \big\rangle$$
$$= \prod_{\varkappa \in E(h_j)} \iota^{k_{j,\varkappa}} \varphi_{j,\varkappa}\Big(\mathcal{U}_\varrho(\zeta^{-k_{j,\varkappa}})^{-1} \mathcal{U}(\gamma_z)\mathbf{v}(\varkappa), \ldots,$$
$$\mathcal{U}_\varrho(\zeta^{-k_{j,\varkappa}})^{-1} \mathcal{U}(\gamma_z)\mathbf{v}(g_j^{k_{j,\varkappa}-1}\varkappa), w \otimes w' \Big)_{\omega_{k_{j,\varkappa}}^{\bullet-1} z^{1/k_{j,\varkappa}}}.$$

40

By (1.1.10), for each $k \in \mathbb{Z}_+$, $\mathcal{U}_\varrho(\zeta^{-k})\mathcal{U}_\varrho(\zeta^k)^{-1} = \mathcal{U}(\varrho(\zeta^{-k}|\zeta^k))$. By Example 1.1.1, the value of $\varrho(\zeta^{-k}|\zeta^k)$ at each $z \in \mathbb{C}^\times$ is γ_{z^k}. Therefore, its value at $\mathfrak{o}_k^l z^{1/k}$ (for each $l \in \mathbb{Z}$) is γ_z. Thus, the value at $\mathfrak{o}_k^l z^{1/k}$ of $\mathcal{U}_\varrho(\zeta^{-k})\mathcal{U}_\varrho(\zeta^k)^{-1}$ is $\mathcal{U}(\gamma_z)$. It follows that

$$
\left\langle \mathbf{w}_j, Y^{h_j}\big(\mathcal{U}(\gamma_z)\mathbf{v}, z^{-1}\big)\mathbf{w}'_j \right\rangle
$$
$$
= \prod_{\mathfrak{x} \in E(h_j)} \wr^{k_{j,\mathfrak{x}}} \wp_{j,\mathfrak{x}}\Big(\mathcal{U}_\varrho(\zeta^{k_{j,\mathfrak{x}}})^{-1}\mathbf{v}(\mathfrak{x}), \ldots, \mathcal{U}_\varrho(\zeta^{k_{j,\mathfrak{x}}})^{-1}\mathbf{v}(g_j^{k_{j,\mathfrak{x}}-1}\mathfrak{x}), w \otimes w'\Big)_{\mathfrak{o}_{k_{j,\mathfrak{x}}}^{-1} z^{1/k_{j,\mathfrak{x}}}},
$$

which is just $\left\langle Y^{g_j}(\mathbf{v}, z)\mathbf{w}_j, \mathbf{w}'_j \right\rangle$. $\qquad\square$

The following theorem is due to [BDM02, Thm. 6.4].

Theorem 2.2.4. *Suppose $\mathbb{V}$ is rational (i.e., any admissible $\mathbb{V}$-module is completely reducible). Then any g_j-twisted $\mathbb{U}$-module is a direct sum of those of the form* (2.2.7) *(whose module structures are described in Thm. 2.2.1), where each $\mathbb{W}_{j,\mathfrak{x}}$ is an irreducible $\mathbb{V}$-module.*

Let $\mathfrak{P}$ be a positively N-pathed Riemann sphere with local coordinates, together with a permutation covering $(\mathfrak{X}, \varphi, \Psi_{\gamma_\bullet(1)})$ as in Subsection 2.1.3.

Associate $\mathcal{W}_1, \ldots, \mathcal{W}_N$ to the marked points $x_1, \ldots, x_N$ of $\mathfrak{P}$. Any marked point of $\mathfrak{X}$ is of the form $\Upsilon(\langle g_j \rangle \mathfrak{x})$ for some $1 \leqslant j \leqslant N$ and $\mathfrak{x} \in E(g_j)$, to which we associate the $\mathbb{V}$-module $\mathbb{W}_{j,\mathfrak{x}}$. By (2.2.7), we have

$$
\mathcal{W}_\bullet := \bigotimes_{1 \leqslant j \leqslant N} \mathcal{W}_j = \bigotimes_{\substack{1 \leqslant j \leqslant N \\ \mathfrak{x} \in E(g_j)}} \mathbb{W}_{j,\mathfrak{x}} =: \mathbb{W}_{\bullet,\bullet}.
$$

Consider a linear functional $\phi : \mathbb{W}_{\bullet,\bullet} \to \mathbb{C}$. This same map can be regarded as a linear functional $\psi : \mathcal{W}_\bullet \to \mathbb{C}$. The reason for using two different symbols for the same linear functional is due to the following reason: we can ask whether ϕ is a conformal block associated to $\mathfrak{X}$, and whether ψ is a conformal block associated to $\mathfrak{P}$; even when both are true, their propagations $\wr\phi$ and $\wr\psi$ will have different meanings.

The theorem below is the first major result of this article, and its proof is left to the following two sections.

Theorem 2.2.5 (Main Theorem). *ϕ is a conformal block associated to $\mathfrak{X}$ and the $\mathbb{V}$-modules $\mathbb{W}_{\bullet,\bullet}$ if and only if ψ is a conformal block associated to $\mathfrak{P}$ and the twisted $\mathbb{V}^E$-modules $\mathcal{W}_\bullet$.*

Consequently, the spaces of the two types of conformal blocks are isomorphic.

2.3 From untwisted to permutation-twisted conformal blocks

In this section, we prove the "only if" part of Thm. 2.2.5. Let us make some preparations. First, note that:

Remark 2.3.1. $\mathscr{U}_{\mathbb{P}^1}$ can be identified with a tensor product (indexed by E) over $\mathscr{O}_{\mathbb{P}^1}$ of $\mathscr{V}_{\mathbb{P}^1}$ such that for any open $V \subset \mathbb{P}^1$ and locally injective $\mu \in \mathscr{O}(V)$, the trivialization $\mathcal{U}_\varrho(\eta) : \mathscr{U}_V \xrightarrow{\simeq} \mathbb{U} \otimes_\mathbb{C} \mathscr{O}_V = \mathbb{V}^{\otimes E} \otimes_\mathbb{C} \mathscr{O}_V$ agrees with $\mathcal{U}_\varrho(\eta)^{\otimes E} : \mathscr{V}_V^{\otimes E} \xrightarrow{\simeq} \mathbb{V}^{\otimes E} \otimes_\mathbb{C} \mathscr{O}_V$. To see this, one checks that the two transition functions agree. This is easy from the definition of $\mathbf{c}_{\mathbb{U}}$. It follows that for each connected open $U \subsetneq \mathbb{P}^1$, we have

$$
\mathscr{U}_{\mathbb{P}^1}(U) = \mathscr{V}_{\mathbb{P}^1}(U)^{\otimes E} \tag{2.3.1}
$$

where the tensor product is over $\mathscr{O}(U)$.

Another useful fact is:

Remark 2.3.2. For any open and simply-connected $U \subset \mathbb{P}^1 \backslash \mathbf{S}$, since $\varphi^{-1}(U)$ is a disjoint union of some open sets biholomorphic to U, we can use the pull back $\varphi^* : \mathscr{O}(U) \to \mathscr{O}(\varphi^{-1}(U)), f \mapsto f \circ \varphi$ to define a natural pullback map $\varphi^* : \mathscr{V}_{\mathbb{P}^1}(U) \to \mathscr{V}_C(\varphi^{-1}(U))$, i.e. the one compatible with the restriction to open subsets of U and intertwines the actions of $\mathscr{O}(U)$, and in the case that there exists any locally injective $\mu \in \mathscr{O}(U)$, the following diagram commutes

$$
\begin{array}{ccc}
\mathscr{V}_{\mathbb{P}^1}(U) & \xrightarrow{\ \varphi^*\ } & \mathscr{V}_C(\varphi^{-1}(U)) \\
{\scriptstyle \mathcal{U}_\varrho(\mu)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \mathcal{U}_\varrho(\mu\circ\varphi)} \\
\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(U) & \xrightarrow{\ 1\otimes\varphi^*\ } & \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(\varphi^{-1}(U))
\end{array}
\tag{2.3.2}
$$

We now begin to prove the "only if" part of Thm. 2.2.5. Assume $\phi : \mathbb{W}_{\bullet,\bullet} \to \mathbb{C}$ is a conformal block associated to $\mathfrak{X}$, which equals ψ as a linear functional. ζ always denotes the standard coordinate of a complex plane.

Step 1. Let us construct the $\wr\psi$ that will satisfy Condition 2 of Def. 1.3.3. Choose any $\mathbf{w} = \bigotimes_{j,\mathfrak{r}} w_{j,\mathfrak{r}} \in \mathcal{W}_\bullet$. For any open simply-connected $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ and a path λ in $\mathbb{P}^1 \backslash \mathbf{S}$ from inside U to $\gamma_\bullet(1)$, we define an $\mathscr{O}(U)$-module homomorphism $\wr\psi(\lambda, \cdot, \mathbf{w}) :$ $\mathscr{U}_{\mathbb{P}^1}(U) \to \mathscr{O}(U)$ as follows. By Rem. 2.3.1, we have identification $\mathscr{U}_{\mathbb{P}^1}(U) = \mathscr{V}_{\mathbb{P}^1}(U)^{\otimes E}$. This space is spanned by $\mathbf{v} = \bigotimes_{e \in E} \mathbf{v}(e)$ where each $\mathbf{v}(e) \in \mathscr{V}_{\mathbb{P}^1}(U)$. We understand

$$
\varphi^* \mathbf{v} \in \mathscr{V}_C(\varphi^{-1}(U))^E
$$

as the $|E|$-tuple labeled by E (i.e. a function from E to $\mathscr{V}_C(\varphi^{-1}(U))$) whose e-component (for each $e \in E$) is $\varphi^* \mathbf{v}(e) = \varphi^*(\mathbf{v}(e))$ defined by Rem. 2.3.2.

Let $\mathrm{Conf}^E(\varphi^{-1}(U))$ be the subset of all points $\mathbf{y}$ of $\varphi^{-1}(U)^E$ (i.e. any function $\mathbf{y} : E \to \varphi^{-1}(U)$) such that any two components $\mathbf{y}(e_1), \mathbf{y}(e_2)$ are different if $e_1 \neq e_2$. For each $y \in U$, we can set

$$
\varphi^*_\lambda y \in \mathrm{Conf}^E(\varphi^{-1}(U)),
$$
$$
\varphi^*_\lambda y(e) = \Psi_{l\lambda}(e) \qquad (l \text{ is a path in } U \text{ from } y \text{ to } \lambda(0)). \tag{2.3.3}
$$

Then $y \in U \mapsto \varphi^*_\lambda y$ is holomorphic, and $\varphi^*_\lambda y \in \varphi^{-1}(y)$.

Consider the E-propagation $\wr^E \phi$ of ϕ, i.e., the $|E|$-propagation whose components are labeled by E. Then

$$
\wr^E \phi(\varphi^* \mathbf{v}, \mathbf{w}) \in \mathrm{Conf}^E(\varphi^{-1}(U)).
$$

We can then define $\wr\psi(\lambda, \mathbf{v}, \mathbf{w}) \in \mathscr{O}(U)$ such that for each $y \in U$,

$$
\boxed{\wr\psi(\lambda, \mathbf{v}, \mathbf{w})_y = \wr^E \phi(\varphi^* \mathbf{v}, \mathbf{w})_{\varphi^*_\lambda y}}
\tag{2.3.4}
$$

One checks easily that $\wr\psi(\cdot, \cdot, \mathbf{w})$ is a multivalued $\mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$-module morphism from $\mathscr{U}_{\mathbb{P}^1\backslash\mathbf{S}}$ to $\mathscr{O}_{\mathbb{P}^1\backslash\mathbf{S}}$ (cf. Lemma 1.3.2).

We remark that $\varphi^* \mathbf{v}$ is not uniquely determined by $\mathbf{v}$: it depends on how $\mathbf{v}$ is factored into the tensor product over $\mathscr{O}(U)$ of elements of $\mathscr{V}_{\mathbb{P}^1}(U)$. However, if any tensor component of $\mathbf{v}$ is multiplied by some $f \in \mathscr{O}(U)$, then $\wr^E \phi(\varphi^*\mathbf{v}, \mathbf{w})_{\varphi_\lambda^* y}$ is multiplied $f(y)$ since $\wr^E \phi$ intertwines the actions of $\mathscr{O}_C$ (cf. Thm. 1.2.1). Therefore (2.3.4) is $\mathscr{O}(U)$-multilinear with respect to the components of $\mathbf{v}$, and hence $\wr^E \phi(\varphi^*\mathbf{v}, \mathbf{w})_{\varphi_\lambda^* y}$ is uniquely determined. $\qquad\square$

***Step* 2.** Let us give an explicit expression of $\varphi_{\gamma_j}^* y$ when y is near $\gamma_j(0)$. This will help us relate $\wr \psi$ and the expression (2.2.6).

For each j, identify

$$\widetilde{W}_{j,\mathcal{x}} = \widetilde{\eta}_{j,\mathcal{x}}(\widetilde{W}_{j,\mathcal{x}}) \qquad \text{via } \widetilde{\eta}_{j,\mathcal{x}},$$

which is an open disc inside $\mathbb{C}_{j,\mathcal{x}}$ centered at 0. Likewise, we identify

$$W_j = \eta_j(W_j) \qquad \text{via } \eta_j,$$

which is an open disc inside $\mathbb{C}_j := \mathbb{C}$ centered at 0. Then by (2.1.15), the branched covering $\varphi : \widetilde{W}_{j,\mathcal{x}} \to W_j$ is equal to $\zeta^{k_{j,\mathcal{x}}} : z \mapsto z^{k_{j,\mathcal{x}}}$ where

$$k_{j,\mathcal{x}} = |\langle g_j \rangle \mathcal{x}|.$$

(Recall ζ is the standard coordinate of $\mathbb{C}_j = \mathbb{C}$).

Choose an open simply-connected $U \subset W_j \backslash \{x_j\} \subset \mathbb{C}_j^\times$ containing $\gamma_j(0)$ together with a continuous $\arg$-function as in Condition 2 of Def. 1.3.3. We prove in this step that for each $z \in U, m \in \mathbb{Z}, \mathcal{x} \in E(g_j)$,

$$\varphi_{\gamma_j}^* z(g_j^m \mathcal{x}) = \mathrm{o}_k^m z^{\frac{1}{k}} \in \widetilde{W}_{j,\mathcal{x}} \qquad \text{where } k = k_{j,\mathcal{x}}. \tag{2.3.5}$$

We note that both sides of this relation rely continuously on z and is sent by the covering map φ to z. Namely, both sides are lifts of the inclusion $U \hookrightarrow W_j \backslash \{x_j\} = W_j \backslash \{0\}$ to $C \backslash \varphi^{-1}(\mathbf{S})$. Moreover, if $z = \gamma_j(0)$, by (2.3.3), $\varphi_{\gamma_j}^* z(g_j^m \mathcal{x}) = \Psi_{\gamma_j}(g_j^m \mathcal{x}) \in \widetilde{W}_{j,\mathcal{x}}$. Therefore, both sides of (2.3.5) are lifts of $U \hookrightarrow W_j \backslash \{0\}$ to $\widetilde{W}_{j,\mathcal{x}} \backslash \{0\}$. Thus, it suffices to prove (2.3.5) for one point in U. Note that, since the left hand side of (2.3.5) is a lift of z through $\varphi = \zeta^k$, it must be a k-th root of z.

Let us prove (2.3.5) for $z = \gamma_j(0)$. Then

$$\varphi_{\gamma_j}^* z(g_j^m \mathcal{x}) \overset{(2.3.3)}{=\!=\!=\!=} \Psi_{\gamma_j}(g_j^m \mathcal{x}) \overset{(2.1.1)}{=\!=\!=\!=} \Psi_{\gamma_j}([\alpha_j]^m \mathcal{x}) \overset{(2.1.6)}{=\!=\!=\!=} \Psi_{\gamma_j \alpha_j^m}(\mathcal{x}) \overset{(1.3.10)}{=\!=\!=\!=} \Psi_{\epsilon_j^m \gamma_j}(\mathcal{x}).$$

By (1.3.12), $z = \gamma_j(0)$ is positive with zero argument. When $m = 0$, $\Psi_{\epsilon_j^m \gamma_j}(\mathcal{x})$, which is a k-th root of z, is also positive due to (2.1.17). So it must be $z^{\frac{1}{k}}$. This proves (2.3.5) when $m = 0$. For a general $m \in \mathbb{Z}$, by the definition of Ψ in Thm. 2.1.9, $\Psi_{\epsilon_j^m \gamma_j}(\mathcal{x})$ is the initial point of the lift of the path ϵ_j^m through $\varphi = \zeta^k$ into the punctured disc $\widetilde{W}_{j,\mathcal{x}} \backslash \{0\}$ ending at $\Psi_{\gamma_j}(\mathcal{x}) = z^{\frac{1}{k}}$. It must be $\mathrm{o}_k^m z^{\frac{1}{k}}$, since ϵ_j is the anticlockwise circle around the origin (whose lift under ζ^k goes anticlockwisely by $2\pi/k$). $\qquad\square$

Step 3. Assume the identifications in Step 2. Let $\mathbf{w} \in \mathcal{W}_\bullet$ be $\bigotimes_{1 \leqslant j \leqslant N} \mathbf{w}_j \in \bigotimes_j \mathcal{W}_j$ where $\mathbf{w}_j = \bigotimes_{\mathfrak{x} \in E(g_j)} w_{j,\mathfrak{x}}$. Choose $\mathbf{v} = \otimes_{e \in E} \mathbf{v}(e) \in \mathbb{U}$. Choose any $z \in W_j \backslash \{0\}$.

For each $n \in \mathbb{N}$, let $\{m_{j,\mathfrak{x}}(n, \alpha) : \alpha \in \mathcal{A}_{j,\mathfrak{x},n}\}$ be a finite set of basis of $\mathbb{W}_{j,\mathfrak{x}}(n)$ whose dual basis $\{\widecheck{m}_{j,\mathfrak{x}}(n, \alpha) : \alpha \in \mathcal{A}_{j,\mathfrak{x},n}\}$ is a basis of $\mathbb{W}'_{j,\mathfrak{x}}(n) = \mathbb{W}_{j,\mathfrak{x}}(n)^*$. For each $\mathbf{n} \in \mathbb{N}^{E(g_j)}$ (i.e., a function $E(g_j) \to \mathbb{N}$), and for each $\boldsymbol{\alpha}$ sending each $\mathfrak{x} \in E(g_j)$ to an element $\boldsymbol{\alpha}(\mathfrak{x}) \in \mathcal{A}_{j,\mathfrak{x},\mathbf{n}(\mathfrak{x})}$ (the set of all such $\boldsymbol{\alpha}$ is denoted by $\mathfrak{A}_{j,\mathbf{n}}$), set

$$\mathbf{m}_j(\mathbf{n}, \boldsymbol{\alpha}) = \bigotimes_{\mathfrak{x} \in E(g_j)} m_{j,\mathfrak{x}}(\mathbf{n}(\mathfrak{x}), \boldsymbol{\alpha}(\mathfrak{x})) \qquad \in \bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} = \mathcal{W}_j,$$

$$\widecheck{\mathbf{m}}_j(\mathbf{n}, \boldsymbol{\alpha}) = \bigotimes_{\mathfrak{x} \in E(g_j)} \widecheck{m}_{j,\mathfrak{x}}(\mathbf{n}(\mathfrak{x}), \boldsymbol{\alpha}(\mathfrak{x})) \qquad \in \bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}'_{j,\mathfrak{x}} = \mathcal{W}'_j.$$

Then the following infinite series of $n \in \frac{1}{|g_j|}\mathbb{N}$

$$\psi(\mathbf{w}_1 \otimes \cdots \otimes Y^{g_j}(\mathbf{v}, z)\mathbf{w}_j \otimes \cdots \otimes \mathbf{w}_N), \tag{2.3.6}$$

understood in the sense of (1.3.14), converges absolutely provided that the following multi-series of $\mathbf{n} \in \mathbb{N}^{E(g_j)}$ converges absolutely:

$$\sum_{\mathbf{n} \in \mathbb{N}^{E(g_j)}} \sum_{\boldsymbol{\alpha} \in \mathfrak{A}_{j,\mathbf{n}}} \psi\big(\mathbf{w}_1 \otimes \cdots \otimes \mathbf{m}_j(\mathbf{n}, \boldsymbol{\alpha}) \otimes \cdots \otimes \mathbf{w}_N\big) \cdot \big\langle Y^{g_j}(\mathbf{v}, z)\mathbf{w}_j, \widecheck{\mathbf{m}}_j(\mathbf{n}, \boldsymbol{\alpha})\big\rangle$$

$$= \sum_{\mathbf{n} \in \mathbb{N}^{E(g_j)}} \sum_{\boldsymbol{\alpha} \in \mathfrak{A}_{j,\mathbf{n}}} \phi\big(\mathbf{w}_1 \otimes \cdots \otimes \mathbf{m}_j(\mathbf{n}, \boldsymbol{\alpha}) \otimes \cdots \otimes \mathbf{w}_N\big)$$

$$\cdot \prod_{\mathfrak{x} \in E(g_j)} \wr^{k_{j,\mathfrak{x}}} \wp_{j,\mathfrak{x}}\Big(\mathbf{v}(\mathfrak{x}), \mathbf{v}(g_j\mathfrak{x}), \ldots, \mathbf{v}(g_j^{k_{j,\mathfrak{x}}-1}\mathfrak{x}), w_{j,\mathfrak{x}} \otimes \widecheck{m}_{j,\mathfrak{x}}(\mathbf{n}(\mathfrak{x}), \boldsymbol{\alpha}(\mathfrak{x}))\Big)_{\varpi_{k_{j,\mathfrak{x}}}^{\bullet-1} z^{1/k_{j,\mathfrak{x}}}}.$$

$$\tag{2.3.7}$$

(We have used (2.2.9) and (2.2.6).) In that case, (2.3.6) converges absolutely to (2.3.7).

Note that inside $\wr^{k_{j,\mathfrak{x}}}\wp_{j,\mathfrak{x}}$ (for each $\mathfrak{x} \in E(g_j)$), we have used the identification (2.2.5) by setting $k = k_{i,\mathfrak{x}}$.

According to (1.2.11), we know that (2.3.7) is the sewing of a propagation. By Thm. 1.2.8, it converges absolutely to the propagation of the sewing, provided that the sewing of the (unpropagated) conformal block converges q-absolutely, and that the marked points and the points of propagation are away from the discs to be sewn (Assumption 1.2.3 and the statement "...disjoint from W'_j, W''_j..." in Thm. 1.2.8).

The unpropagated pointed Riemann surface with local coordinates is

$$\mathfrak{Y} = \mathfrak{X} \sqcup \coprod_{\mathfrak{x} \in E(g_j)} \mathfrak{Q}_{j,\mathfrak{x}}$$

where $\mathfrak{Q}_{j,\mathfrak{x}} = (\mathbb{P}^1_{j,\mathfrak{x}}; 0, \infty; \zeta, \zeta^{-1})$. The two marked points $0, \infty$ of $\mathfrak{Q}_{j,\mathfrak{x}}$ are associated with $\mathbb{V}$-modules $\mathbb{W}_{j,\mathfrak{x}}, \mathbb{W}'_{j,\mathfrak{x}}$ respectively. The (unpropagated) conformal block associated to $\mathfrak{Y}$ is (recall (2.2.3))

$$\chi := \phi \otimes \Big(\bigotimes_{\mathfrak{x} \in E(g_j)} \wp_{j,\mathfrak{x}}\Big) : \mathcal{W}_\bullet \otimes \Big(\bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} \otimes \mathbb{W}'_{j,\mathfrak{x}}\Big) \to \mathbb{C}.$$

44

We sew $\mathfrak{Y}$ along each pair $\Upsilon(\langle g_j \rangle \mathfrak{x})$ (i.e. the center of $\widetilde{W}_{j,\mathfrak{x}}$) and $\infty \in \mathbb{P}^1_{j,\mathfrak{x}}$ (for every $\mathfrak{x} \in E(g_j)$) using their local coordinates. More precisely, we remove a small disc inside $\widetilde{W}_{j,\mathfrak{x}}$ and another disc inside

$$M_{j,\mathfrak{x}} := \{ z \in \mathbb{P}^1_{j,\mathfrak{x}} : |z^{-1}| < R^{1/|E|} \}$$

for sufficiently large $R > 1$, and glue the remaining part using the relation $\widetilde{\eta}_{j,\mathfrak{x}} \cdot \zeta^{-1} = 1$.

The sewn data $\mathscr{S}\mathfrak{Y}$ is equal to $\mathfrak{X}$. Moreover, our sewing is compatible with the identifications in Step 2. Namely: this sewing process is just removing $\widetilde{W}_{j,\mathfrak{x}}$ (for each $\mathfrak{x}$) from C, and filling into the holes the equivalent open disc $\widetilde{W}_{j,\mathfrak{x}} \subset \mathbb{P}^1_{j,\mathfrak{x}}$ (associated to $\mathfrak{Q}_{j,\mathfrak{x}}$). It is clear that $\mathscr{S}\chi$ converges q-absolutely to ϕ. Also, when $0 < |z| < R^{-1}$, $\omega^\bullet_{k_{j,\mathfrak{x}}-1} z^{1/k_{j,\mathfrak{x}}}$ is disjoint from $M_{j,\mathfrak{x}}$. So, by Thm. 1.2.8, (2.3.7) converges absolutely. This proves Condition 1 of Def. 1.3.3 for all $0 < |z| < R^{-1}$ (with any choice of $\arg z$), thanks to Rem. 1.3.6. $\square$

Step 4. Assume the identifications in Step 2. Let $U = \{ z \in \mathbb{C}^\times_j : |z| < R^{-1} \} \setminus (-\infty, 0]$ with $\arg$ function ranging in $(-\pi, \pi)$. Note $U \subset W_j \subset \mathbb{C}_j$. Choose $z \in U$. After sewing, the point $\omega^m_{k_{j,\mathfrak{x}}} z^{1/k_{j,\mathfrak{x}}}$ originally in $\mathfrak{Q}_{j,\mathfrak{x}}$ becomes the same point in $\widetilde{W}_{j,\mathfrak{x}}$, which is $\varphi^*_{\gamma_j} z(g^m_j \mathfrak{x})$ by (2.3.5).

Note that both Rem. 2.3.1 and Rem. 2.3.2 are considered in the definition of $\wr\psi$ in Step 1. In view of Rem. 2.3.2, we assume two more identifications

$$\mathscr{V}_{\mathbb{P}^1}(U) = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(U) \qquad \text{via } \mathcal{U}_\varrho(\zeta) \tag{2.3.8}$$
$$\mathscr{V}_C(\varphi^{-1}(U)) = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(\varphi^{-1}(U)) \qquad \text{via } \mathcal{U}_\varrho(\zeta \circ \varphi). \tag{2.3.9}$$

Since

$$\zeta \circ \varphi = \zeta^{k_{j,\mathfrak{x}}} \qquad \text{on } \varphi^{-1}(U) \cap \widetilde{W}_{j,\mathfrak{x}},$$

after sewing, the identification (2.2.5) (where $k = k_{j,\mathfrak{x}}$ for each $\mathfrak{x}$) used in the definition of $\wr^{k_{j,\mathfrak{x}}}\mathfrak{Q}_{j,\mathfrak{x}}$ is compatible with the identification (2.3.9). Also, if we take Rem.2.3.1 into account, then (2.3.8) yields the identification

$$\mathscr{U}_{\mathbb{P}^1}(U) = \mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}(U) \qquad \text{via } \mathcal{U}_\varrho(\zeta). \tag{2.3.10}$$

Under these identifications, for each $\mathbf{v} \in \mathbb{V}^{\otimes E} = \mathbb{U} \subset \mathbb{U} \otimes_{\mathbb{C}} \mathscr{O}(U)$, $\varphi^* \mathbf{v} \in \mathbb{V}^E \subset (\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(\varphi^{-1}(U)))^E$ is a tuple of constant sections whose component at each $e = g^m_j \mathfrak{x}$ (where $m \in \mathbb{Z}, \mathfrak{x} \in E(g_j)$) is $\mathbf{v}(g^m_j \mathfrak{x})$. Thus (2.3.7), which is the sewing of propagation, converges absolutely to the propagation of the sewing $\mathscr{S}\chi = \phi$ (by Thm. 1.2.8), which is $\wr^E \phi(\varphi^* \mathbf{v}, \mathbf{w})$ at the point of $\mathrm{Conf}^E(\varphi^{-1}(U))$ whose $g^m_j \mathfrak{x}$-component is $\varphi^*_{\gamma_j} z(g^m_j \mathfrak{x})$ (according to the first paragraph). This proves that (2.3.7) (and hence (2.3.6)) converge absolutely to $\wr^E \phi(\varphi^* \mathbf{v}, \mathbf{w})_{\varphi^*_{\gamma_j} z}$, which is $\wr\psi(\gamma_j, \mathbf{v}, \mathbf{w})_z$ by (2.3.4). Thus, by Rem. 1.3.6, the two conditions of Def. 1.3.3 hold for possibly smaller discs $W_1, \ldots, W_N$; the choice of $\arg z$ is not important for Condition 1, due to part (2) of that remark; the choice of the simply-connected subset U can be arbitrary, due to Rem. 1.3.5. By Rem. 1.3.7, these two conditions hold for the original discs. $\square$

We are done with the proof of the "only if" part of Thm. 2.2.5.

2.4 From permutation-twisted to untwisted conformal blocks

We prove the "if" part of Thm. 2.2.5. Assume $\psi : \mathcal{W}_\bullet \to \mathbb{C}$ is a conformal block associated to $\mathfrak{P}$. We have $\wr\psi$ as described in Def. 1.3.3.

Step 1. To show that the same linear functional $\phi : \mathcal{W}_{\bullet,\bullet} \to \mathbb{C}$ is a conformal block associated to $\mathfrak{X}$, we shall construct its propagation $\wr\phi$.

Choose any simply-connected open $U \subset \mathbb{P}^1\backslash \mathbf{S}$ together with a path λ in $\mathbb{P}^1\backslash \mathbf{S}$ from inside U to $\gamma_\bullet(1)$. For each $e \in E$, let $\widetilde{U}_{\lambda,e}$ be the connected component of $\varphi^{-1}(U)$ containing $\Psi_\lambda(e)$. Then all such sets form a basis of the topology of C. Define

$$\varphi_* : \mathscr{V}_C(\widetilde{U}_{\lambda,e}) \xrightarrow{\simeq} \mathscr{V}_{\mathbb{P}^1}(U)$$

to be the inverse of φ^* defined in Rem. 2.3.2. Recall $\mathscr{U}_{\mathbb{P}^1}(U) = \mathscr{V}_{\mathbb{P}^1}(U)^{\otimes_{\mathscr{O}(U)}E}$ by Rem. 2.3.1. For each $v \in \mathscr{V}_C(\widetilde{U}_{\lambda,e})$, we let

$$(\varphi_* v)_e \otimes \mathbf{1} \in \mathscr{U}_{\mathbb{P}^1}(U)$$

be a tensor product of elements of $\mathscr{V}_{\mathbb{P}^1}(U)$ such that the e-component is $\varphi_* v$, and that the other tensor components are the vacuum section $\mathbf{1}$.

We now define $\wr\phi$ on $\widetilde{U}_{\lambda,e}$. Choose any $\mathbf{w} = \bigotimes_{j,\mathfrak{x}} w_{j,\mathfrak{x}} \in \mathcal{W}_\bullet$. For each $\widetilde{y} \in \widetilde{U}_{\lambda,e}$ and $v \in \mathscr{V}_C(\widetilde{U}_{\lambda,e})$, we define

$$\boxed{\wr\phi\big(v, \mathbf{w}\big)_{\widetilde{y}} = \wr\psi\big(\lambda, (\varphi_* v)_e \otimes \mathbf{1}, \mathbf{w}\big)_{\varphi(\widetilde{y})}} \tag{2.4.1}$$

We need to show that this definition is independent of λ and e. Suppose λ' is another path from inside U to $\gamma_\bullet(1)$, $e' \in E$, and $\widetilde{U}_{\lambda,e} = \widetilde{U}_{\lambda',e'}$. The above definition is clearly unchanged if $e' = e$ and $\lambda' = l\lambda$ where l is a path in U from $\lambda'(0)$ to $\lambda(0)$. Thus, we may assume λ and λ' have the same initial point (and end point). Choose $\mu \in \Lambda_{\gamma_\bullet(1)}$ such that

$$\lambda' = \lambda\mu.$$

We compute

$$\Psi_\lambda(e) = \Psi_{\lambda'}(e') = \Psi_{\lambda\mu}(e') \x鈥攛rightarrow{(2.1.6)} \Psi_\lambda([\mu]e').$$

Therefore, as Ψ_λ is one-to-one (cf. Thm. 2.1.9), we have

$$e = [\mu]e'.$$

The map $[\alpha_j] \mapsto g_j$ defines an action of the fundamental group Γ on $\mathbb{U}$ and hence on $\mathscr{U}_{\mathbb{P}^1}$. It is easy to check that

$$[\mu]\big((\varphi_* v)_e \otimes \mathbf{1}\big) = (\varphi_* v)_{[\mu]e} \otimes \mathbf{1}.$$

Therefore, by (2.1.3),

$$\wr\psi\big(\lambda', (\varphi_* v)_{e'} \otimes \mathbf{1}, \mathbf{w}\big) = \wr\psi\big(\lambda\mu, (\varphi_* v)_{e'} \otimes \mathbf{1}, \mathbf{w}\big)$$
$$= \wr\psi\big(\lambda, (\varphi_* v)_{[\mu]e'} \otimes \mathbf{1}, \mathbf{w}\big) = \wr\psi\big(\lambda, (\varphi_* v)_e \otimes \mathbf{1}, \mathbf{w}\big).$$

Thus $\wr\phi(\cdot, \mathbf{w})$ is a well-defined $\mathscr{O}_{C\backslash\varphi^{-1}(\mathbf{S})}$-module morphism $\mathscr{V}_{C\backslash\varphi^{-1}(\mathbf{S})} \to \mathscr{O}_{C\backslash\varphi^{-1}(\mathbf{S})}$. $\quad\square$

In Step 2, we verify the two conditions of Thm. 1.2.2.

Step 2. We write $\mathbf{w}_j = \bigotimes_{x \in E(g_j)} w_{j,x} \in \mathcal{W}_j$ so that $\mathbf{w} = \bigotimes_j \mathbf{w}_j$. Let $U \subset W_j \backslash \{x_j\}$ be open and simply-connected. According to the notations in Step 1,

$$\Psi_{\gamma_j}(x) \in \tilde{U}_{\gamma_j,x} \subset \widetilde{W}_{j,x}.$$

(Recall that $\widetilde{W}_{j,x}$ is a disc with center $\Upsilon(\langle g_j \rangle x)$, and hence contains the set of points $\Psi_{\gamma_j}(\langle g_j \rangle x)$ surrounding the center.) Choose any $v \in \mathbb{V} \subset \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(\tilde{U}_{\gamma_j,x})$, and set

$$v_x \otimes \mathbf{1} \in \mathbb{U} = \mathbb{V}^{\otimes E}$$

to be a tensor product of vectors of $\mathbb{V}$ whose x-component is v and the other components are $\mathbf{1}$. Then by (2.4.1) and Condition 2 of Def. 1.3.3 (notice the identification there), for each $\tilde{y} \in \tilde{U}_{\gamma_j,x}$,

$$\wr \phi \big(\mathcal{U}_\varrho (\eta_j \circ \varphi)^{-1} v, \mathbf{w} \big)_{\tilde{y}}$$
$$= \psi \big(\mathbf{w}_1 \otimes \cdots \otimes Y^{g_j} \big(v_x \otimes \mathbf{1}, \eta_j \circ \varphi(\tilde{y}) \big) \mathbf{w}_j \otimes \cdots \otimes \mathbf{w}_N \big)$$

where the right hand side converges a.l.u. in the sense of Def. 1.3.3. Using the notations of Step 3 in Sec. 2.3, we have

$$\wr \phi \big(\mathcal{U}_\varrho (\eta_j \circ \varphi)^{-1} v, \mathbf{w} \big)_{\tilde{y}}$$
$$= \sum_{\mathbf{n} \in \mathbb{N}^{E(g_j)}} \sum_{\boldsymbol{\alpha} \in \mathfrak{A}_{j,\mathbf{n}}} \psi \big(\mathbf{w}_1 \otimes \cdots \otimes \mathbf{m}_j(\mathbf{n}, \boldsymbol{\alpha}) \otimes \cdots \otimes \mathbf{w}_N \big)$$
$$\cdot \big\langle Y^{g_j} \big(v_x \otimes \mathbf{1}, \eta_j \circ \varphi(\tilde{y}) \big) \mathbf{w}_j, \check{\mathbf{m}}_j(\mathbf{n}, \boldsymbol{\alpha}) \big\rangle.$$

Let us write the above $\langle Y^{g_j} \cdots \rangle$ in terms of $\wr \wp_{j,x}$. Let

$$k = k_{j,x} = |\langle g_j \rangle x|.$$

Recall that ζ is the standard coordinate of $\mathbb{C}$. By (2.2.6) and Thm. 1.2.1-(3,4), if $\mathbf{v}(g_j x) = \mathbf{v}(g_j^2 x) = \cdots = \mathbf{1}$ then

$$\wr_{j,x}(\mathbf{v}, w, w', z) = \wr \wp_{j,x} \big(\mathcal{U}_\varrho (\zeta^k)^{-1} \mathbf{v}(x), w \otimes w' \big)_{z^{1/k}};$$

if also $\mathbf{v}(x) = \mathbf{1}$ then $\wr_{j,x}(\mathbf{v}, w, w', z) = \langle w, w' \rangle$. For each $m_{j,x} \in \mathbb{W}_{j,x}$ we set

$$\mathbf{w}_{\backslash j,x} \otimes m_{j,x} \in \mathbb{W}_{\bullet,\bullet}$$

be a tensor product of vectors whose (j, x)-component is $m_{j,x}$ and the other components agree with the corresponding ones of $\mathbf{w}$. By (2.1.15), we have

$$(\eta_j \circ \varphi(\tilde{y}))^{1/k} = \tilde{\eta}_{j,x}(\tilde{y}),$$

where the argument of $\eta_j \circ \varphi(\tilde{y})$ is defined such that if $\tilde{y}$ changes continuously to $\Psi_{\gamma_j}(x)$ then the argument changes continuously to 0 (notice (2.1.17)). Then, using the construction of Y^{g_j} in Sec. 2.2 (especially, pay attention to the identification (2.2.5) there), we have

$$\wr \phi \big(\mathcal{U}_\varrho (\eta_j \circ \varphi)^{-1} v, \mathbf{w} \big)_{\tilde{y}}$$

47

$$= \sum_{n\in\mathbb{N}} \sum_{\alpha\in\mathcal{A}_{j,\mathcal{x},n}} \psi\big(\mathbf{w}_{\backslash j,\mathcal{x}} \otimes m_{j,\mathcal{x}}(n,\alpha)\big) \cdot \wr\wp_{j,\mathcal{x}}\Big(\mathcal{U}_\varrho(\zeta^k)^{-1}v, w_{j,\mathcal{x}} \otimes \check{m}_{j,\mathcal{x}}(n,\alpha)\Big)_{\tilde{\eta}_{j,\mathcal{x}}(\tilde{y})} \qquad (2.4.2)$$

where the right hand side converges absolutely as a series of n.

We now identify

$$\widetilde{W}_{j,\mathcal{x}} = \tilde{\eta}_{j,\mathcal{x}}(\widetilde{W}_{j,\mathcal{x}}) \qquad \text{via } \tilde{\eta}_{j,\mathcal{x}},$$

considered as an open disc in $\mathbb{C}_{j,\mathcal{x}} = \mathbb{C}$. Then $\tilde{\eta}_{j,\mathcal{x}}$ is the standard coordinate ζ, $\eta_j \circ \varphi = \zeta^k$, and $\tilde{y} = \tilde{\eta}_{j,\mathcal{x}}(\tilde{y})$. Therefore, (2.4.2) holds if we replace $\tilde{\eta}_{j,\mathcal{x}}(\tilde{y})$ by $\tilde{y}$, and (by $\mathscr{O}(\tilde{U}_{j,\mathcal{x}})$-linearity) replace $\mathcal{U}_\varrho(\eta_j \circ \varphi)^{-1}v = \mathcal{U}_\varrho(\zeta^k)^{-1}v$ by any element of $\mathscr{V}_C(\tilde{U}_{\gamma_j,\mathcal{x}})$. Thus, if we fix identification

$$\mathscr{V}_{\tilde{U}_{j,\mathcal{x}}} = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\tilde{U}_{j,\mathcal{x}}} \qquad \text{via } \mathcal{U}_\varrho(\tilde{\eta}_{j,\mathcal{x}}) = \mathcal{U}_\varrho(\zeta),$$

then for each $z \in \tilde{U}_{j,\mathcal{x}} \subset \mathbb{C}_{j,\mathcal{x}}$ and $v \in \mathbb{V} \subset \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\tilde{U}_{j,\mathcal{x}}}$,

$$\begin{aligned}
&\wr\phi(v,\mathbf{w})_z \\
&= \sum_{n\in\mathbb{N}} \sum_{\alpha\in\mathcal{A}_{j,\mathcal{x},n}} \psi\big(\mathbf{w}_{\backslash j,\mathcal{x}} \otimes m_{j,\mathcal{x}}(n,\alpha)\big) \cdot \wr\wp_{j,\mathcal{x}}\Big(v, w_{j,\mathcal{x}} \otimes \check{m}_{j,\mathcal{x}}(n,\alpha)\Big)_z \\
&= \sum_{n\in\mathbb{N}} \sum_{\alpha\in\mathcal{A}_{j,\mathcal{x},n}} \phi\big(\mathbf{w}_{\backslash j,\mathcal{x}} \otimes m_{j,\mathcal{x}}(n,\alpha)\big) \cdot \big\langle Y(v,z)w_{j,\mathcal{x}}, \check{m}_{j,\mathcal{x}}(n,\alpha) \big\rangle \\
&= \sum_{n\in\mathbb{N}} \phi\big(\mathbf{w}_{\backslash j,\mathcal{x}} \otimes P_n Y(v,z)w_{j,\mathcal{x}}\big),
\end{aligned}$$

where the rightmost part converges absolutely. Similar to the argument in Rem. 1.3.6, the above equation holds and the series of n converges a.l.u. for $z \in \widetilde{W}_{j,\mathcal{x}} \backslash \{x_j\}$. So ϕ satisfies the two conditions of Thm. 1.2.2. $\qquad\square$

2.A Dimension of the space of permutation-twisted conformal blocks

In this subsection, we assume $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational. Let $\mathcal{E}$ be a (finite) complete list of irreducible $\mathbb{V}$-modules (cf. the paragraph above Thm. 1.2.7). By Main Theorem 2.2.5, the calculation of the dimension of the space of permutation-twisted conformal blocks is reduced to that of untwisted ones. For the reader's convenience, we explicitly write down the steps of calculating such dimension.

If C is a connected compact Riemann surface of genus g, together with N marked points, local coordinates, and semi-simple $\mathbb{V}$-modules $\mathbb{W}_1, \dots, \mathbb{W}_N$, we let

$$\mathbf{N}(g; \mathbb{W}_\bullet) = \mathbf{N}(g; \mathbb{W}_1, \dots, \mathbb{W}_N)$$

be the dimension of the space of conformal blocks associated to these data. By [DGT19a, DGT19b], this number finite, and is independent of the complex structure of C, the position of N marked points, and the local coordinates. Moreover,

$\mathbf{N}(g; \mathbb{W}_1, \dots, \mathbb{W}_N)$ is unchanged if we rearrange the order $\mathbb{W}_1, \dots, \mathbb{W}_N$. In the special case that $g = 0$ and $N = 3$,

$$\mathbf{N}_{\mathbb{W}_1, \mathbb{W}_2}^{\mathbb{W}_3} := \mathbf{N}(0; \mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3')$$

is the **fusion rule** among $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3$.

By the factorization property proved by [DGT19b] (cf. also Thm. 1.2.7), if $g \geqslant 1$, we have

$$\mathbf{N}(g; \mathbb{W}_1, \dots, \mathbb{W}_N) = \sum_{\mathbb{M} \in \mathcal{E}} \mathbf{N}(g - 1; \mathbb{W}_1, \dots, \mathbb{W}_N, \mathbb{M}, \mathbb{M}'). \qquad (2.A.1)$$

If $g = g_1 + g_2$ where $g_1, g_2 \geqslant 0$, and if $1 \leqslant L < N$, then

$$\mathbf{N}(g; \mathbb{W}_1, \dots, \mathbb{W}_N) = \sum_{\mathbb{M} \in \mathcal{E}} \mathbf{N}(g_1; \mathbb{W}_1, \dots, \mathbb{W}_L, \mathbb{M}) \cdot \mathbf{N}(g_2; \mathbb{W}_{L+1}, \dots, \mathbb{W}_N, \mathbb{M}'). \qquad (2.A.2)$$

These two formulas allow us to calculate any $\mathbf{N}(g; \mathbb{W}_\bullet)$ using the fusion rules.

Now we assume the setting of Main Theorem 2.2.5. Namely, $\mathfrak{P} = (1.3.9)$ is a positively N-pathed Riemann sphere with local coordinates, and to each marked point x_j we associate a g_j-twisted $\mathbb{U} = \mathbb{V}^{\otimes E}$-module $\mathcal{W}_j$ defined in Thm. 2.2.1. Note that $E(g_j)$ is the set of marked points of g_j-orbits. We assume $g_1, \dots, g_N$ are admissible, i.e., there is an action $\Gamma \curvearrowright E$ where each $[\alpha_j]$ acts as g_j. So Γ-orbits in E are precisely $\langle g_1, \dots, g_N \rangle$-orbits. We let $\mathrm{Orb}(\Gamma)$ be the set of Γ-orbits in E. As usual, for $\Omega \in \mathrm{Orb}(\Gamma)$, $|\Omega|$ denotes its cardinality.

Now, by the Main Theorem 2.2.5, we have:

Corollary 2.A.1. *For each $\Omega \in \mathrm{Orb}(\Gamma)$, let $(\mathbb{W})_\Omega$ be the list of all $\mathbb{W}_{j,\mathfrak{x}}$ where $j = 1, \dots, N$, and $\mathfrak{x} \in E(g_j)$ is such that the g_j-orbit $\langle g_j \rangle \mathfrak{x}$ is contained in Ω. Then the dimension of the space of conformal blocks associated to $\mathfrak{P}$ and the twisted modules $\mathcal{W}_\bullet$ equals*

$$\prod_{\Omega \in \mathrm{Orb}(\Gamma)} \mathbf{N}(g(C^\Omega); (\mathbb{W})_\Omega)$$

where $g(C^\Omega)$ is given by (2.1.10) or equivalently (2.1.11).

3 Relating untwisted and permutation-twisted sewing and factorization

3.1 Sewing Riemann spheres and their permutation coverings

3.1.1 The setting

Let $\mathbb{P}_a^1 = \mathbb{P}^1, \mathbb{P}_b^1 = \mathbb{P}^1$. Choose two positively pathed Riemann spheres with local coordinates

$$\mathfrak{P}^a = \big(\mathbb{P}^1; x_0, x_1, \dots, x_N; \eta_0, \eta_1, \dots, \eta_N; \gamma_0, \gamma_1, \dots, \gamma_N\big),$$
$$\mathfrak{P}^b = \big(\mathbb{P}^1; y_0, y_1, \dots, y_K; \varpi_0, \varpi_1, \dots, \varpi_K; \delta_0, \delta_1, \dots, \delta_K\big)$$

where $N, K \geqslant 1$. So each η_j (resp. ϖ_l) is a local coordinate at x_j (resp. y_l). For each $1 \leqslant j \leqslant N, 1 \leqslant k \leqslant K$, we assume $\eta_j \in \mathcal{O}(W_j)$ (resp. $\varpi_l \in \mathcal{O}(M_l)$) where $W_j \subset \mathbb{P}^1$ (resp. $M_l \subset \mathbb{P}^1$) is open, and $\eta_j(W_j)$ (resp. $\varpi_l(M_l)$) is an open disc centered at 0 with radius r_j (resp. ρ_l). We choose $r_0, \rho_0 > 0$ such that

$$\eta_0(W_0) = \mathcal{D}_{r_0}, \qquad \varpi_0(M_0) = \mathcal{D}_{\rho_0}.$$

We assume

$$r_0 \rho_0 > 1.$$

We assume that W_0 does not contain $x_1, \ldots, x_N$, and M_0 does not contain $y_1, \ldots, y_K$. Let

$$F^a = \{x \in W_0 : |\eta_0(x)| \leqslant 1/\rho_0\}, \qquad F^b = \{y \in M_0 : |\varpi_0(y)| \leqslant 1/r_0\}.$$

We assume[2]

$$\boxed{\begin{array}{l} F^a \text{ is disjoint from } W_1, \ldots, W_N, \gamma_0, \gamma_1, \ldots, \gamma_N \\ F^b \text{ is disjoint from } M_1, \ldots, M_K, \delta_0, \delta_1, \ldots, \delta_K \end{array}} \tag{3.1.1}$$

Note that every $\eta_j(\gamma_j(0))$ and every $\varpi_l(\delta_l(0))$ are positive by (1.3.12). We also assume

$$\boxed{\eta_0\big(\gamma_0(0)\big) \varpi_0\big(\delta_0(0)\big) = 1} \tag{3.1.2}$$

so that $\gamma_0(0)$ and $\delta_0(0)$ can be glued to the same point.

We let

$$\mathbf{S}^a = \{x_0, x_1, \ldots, x_N\}, \qquad \mathbf{S}^b = \{y_0, y_1, \ldots, y_K\}.$$

Note that $\gamma_0, \ldots, \gamma_N$ (resp. $\delta_0, \ldots, \delta_K$) have a common end point $\gamma_\bullet(1)$ (resp. $\delta_\bullet(1)$). We set

$$\Gamma^a = \pi_1\big(\mathbb{P}^1_a \backslash \mathbf{S}^a, \gamma_\bullet(1)\big), \qquad \Gamma^b = \pi_1\big(\mathbb{P}^1_b \backslash \mathbf{S}^b, \delta_\bullet(1)\big).$$

Similar to the setting of Sec. 1.3, we let ϵ^a_j (resp. ϵ^b_l) be a small anticlockwise circle in W_j (resp. M_l) around the center x_j (resp. y_l) from and to $\gamma_j(0)$ (resp. $\delta_l(0)$), and let

$$\alpha_j = \gamma_j^{-1} \epsilon^a_j \gamma_j, \qquad \beta_l = \delta_l^{-1} \epsilon^b_l \delta_l. \tag{3.1.3}$$

We assume

$$\boxed{\Gamma^a = \langle [\alpha_1], \ldots, [\alpha_N] \rangle, \quad \Gamma^b = \langle [\beta_1], \ldots, [\beta_K] \rangle} \tag{3.1.4}$$

Recall that E is a finite set. We fix an action of Γ^a on E and another one Γ^b on E, namely, we fix homomorphisms $\Gamma^a, \Gamma^b \to \operatorname{Perm}(E)$. We assume these two homomorphisms send each $[\alpha_j] \in \Gamma^a$ and each $[\gamma_l] \in \Gamma^b$ to

$$[\alpha_j] \mapsto g_j, \qquad [\beta_l] \mapsto h_l$$

where $g_0, g_1, \ldots, g_N, h_0, h_1, \ldots, h_M \in \operatorname{Perm}(E)$, and

$$\boxed{g_0 h_0 = 1} \tag{3.1.5}$$

---

[2]Some assumptions in this chapter are similar to those in the previous chapters and are reviewed here for the readers' convenience. As for some other assumptions that were not mentioned in the previous chapters, we enclose them in boxes.

3.1.2 $\mathfrak{P}^{a\#b}$ is the sewing of $\mathfrak{P}^a$ and $\mathfrak{P}^b$

We define a positively $(N+K)$-pathed Riemann sphere with local coordinates

$$\mathfrak{P}^{a\#b} = \big(\mathbb{P}^1_{a\#b}; x_1, \ldots, x_N, y_1, \ldots, y_K; \eta_1, \ldots, \eta_N, \varpi_1, \ldots, \varpi_K;$$
$$\gamma_1\gamma_0^{-1}, \ldots, \gamma_N\gamma_0^{-1}, \delta_1\delta_0^{-1}, \ldots, \delta_K\delta_0^{-1}\big).$$

as follows. Without the paths, $\mathfrak{P}^{a\#b}$ is the sewing of $\mathfrak{P}^a \sqcup \mathfrak{P}^b$ along the pair of points x_0, y_0 using the local coordinates η_0, ϖ_0. (Cf. Sec. 1.2.) Here are the details: we remove F^a from the $\mathbb{P}^1_a$, remove F^b from $\mathbb{P}^1_b$, and glue the remaining parts by identifying $W_0\backslash F^a$ and $M_0\backslash F^b$ using the relation

$$x \in W_0\backslash F^a \text{ equals } y \in M_0\backslash F^b$$
$$\Updownarrow$$
$$\eta_0(x)\varpi_0(y) = 1.$$

This gives us a new Riemann surface $\mathbb{P}^1_{a\#b} \simeq \mathbb{P}^1$. Since, by our assumption, $x_1, \ldots, x_N$, $W_1, \ldots, W_N$, and $\gamma_0, \gamma_1, \ldots, \gamma_N$ are disjoint from F^a, they can be regarded as sets/open subsets/paths of $\mathbb{P}^1_{a\#b}$. The same can be said about $y_1, \ldots, y_K$, $M_1, \ldots, M_K$, and $\delta_0, \delta_1, \ldots, \delta_K$. Moreover, for each $1 \leqslant j \leqslant N, 1 \leqslant k \leqslant K$, $\eta_j \in \mathscr{O}(W_j)$ and $\varpi_l \in \mathscr{O}(M_l)$ can be viewed as local coordinates of $\mathbb{P}^1_{a\#b}$ at x_j, y_l respectively.

By (3.1.2), $\gamma_0(0)$ and $\delta_0(0)$ become the same point of $\mathbb{P}^1_{a\#b}$ after gluing, which we denote by $\bigstar$. We record this definition:

$$\boxed{\bigstar = \gamma_0(0) = \delta_0(0) \in \mathbb{P}^1_{a\#b}}$$

Set

$$\mathbf{S}^{a\#b} = \{x_1, \ldots, x_N, y_1, \ldots, y_K\} \subset \mathbb{P}^1_{a\#b}.$$

Then $\bigstar$ is the common end point of the paths $\gamma_1\gamma_0^{-1}, \ldots, \gamma_N\gamma_0^{-1}, \delta_1\delta_0^{-1}, \ldots, \delta_K\delta_0^{-1}$ in $\mathbb{P}^1_{a\#b}\backslash\mathbf{S}^{a\#b}$.

We define

$$\Gamma^{a\#b} = \pi_1\big(\mathbb{P}^1_{a\#b}\backslash\mathbf{S}^{a\#b}, \bigstar\big).$$

For each $1 \leqslant j \leqslant N, 1 \leqslant k \leqslant K$, we let

$$\alpha_j^\# := \gamma_0\alpha_j\gamma_0^{-1} = \gamma_0\gamma_j^{-1}\epsilon_j^a\gamma_j\gamma_0^{-1}, \qquad \beta_l^\# := \delta_0\beta_l\delta_0^{-1} = \delta_0\delta_l^{-1}\epsilon_j^b\delta_l\delta_0^{-1}, \qquad (3.1.6)$$

and let $[\alpha_j^\#], [\beta_l^\#] \in \Gamma^{a\#b}$ be their homotopy classes (on $\mathbb{P}^1_{a\#b}\backslash\mathbf{S}^{a\#b}$). By (3.1.4), (3.1.5), and Van Kampen Theorem, $\Gamma^{a\#b}$ is generated by $[\alpha_j^\#], [\beta_l^\#]$ for all $l, k \geqslant 1$ (note that this is required in the definition of positive pathed Riemann spheres; see (1.3.11)), and we can define a unique action

$$\boxed{\Gamma^{a\#b} \curvearrowright E : [\alpha_j^\#] \mapsto g_j, [\beta_l^\#] \mapsto h_l} \qquad (3.1.7)$$

3.1.3 $\mathfrak{X}^a$ and $\mathfrak{X}^b$ are the permutation coverings of $\mathfrak{P}^a$ and $\mathfrak{P}^b$ associated to $\Gamma^a, \Gamma^b \curvearrowright E$

As in Sec. 2.2, we can construct permutation coverings

$$\left(\mathfrak{X}^a, \varphi^a : C^a \to \mathbb{P}^1_a, \Psi^a_{\gamma_\bullet(1)}\right), \qquad \left(\mathfrak{X}^b, \varphi^b : C^b \to \mathbb{P}^1_b, \Psi^b_{\delta_\bullet(1)}\right)$$

of $\mathfrak{P}^a$ and $\mathfrak{P}^b$ respectively, where

$$\mathfrak{X}^a = (C^a; \Upsilon^a(\langle g_j \rangle \mathfrak{x}); \widetilde{\eta}_{j,\mathfrak{x}} \in \mathscr{O}(\widetilde{W}_{j,\mathfrak{x}}) \text{ for all } 0 \leqslant j \leqslant N, \mathfrak{x} \in E(g_j)),$$

$$\mathfrak{X}^b = (C^b; \Upsilon^b(\langle h_l \rangle \mathfrak{x}); \widetilde{\varpi}_{l,\mathfrak{x}} \in \mathscr{O}(\widetilde{M}_{l,\mathfrak{x}}) \text{ for all } 0 \leqslant l \leqslant K, \mathfrak{x} \in E(h_l)).$$

We recall some details for the readers' convenience. As in Thm. 2.1.9 and Cor. 2.1.12, we define Ψ^a, Υ^a for $\mathfrak{P}^a$ and $\mathfrak{X}^a$, and define Ψ^b, Υ^b for $\mathfrak{P}^b$ and $\mathfrak{X}^b$. For each $0 \leqslant j \leqslant N$ (resp. $0 \leqslant k \leqslant K$), $E(g_j)$ (resp. $E(h_l)$) is the set marked points of g_j-orbits (resp. h_l-orbits). For each $\mathfrak{x} \in E(g_j)$ (resp. $\mathfrak{x} \in E(h_l)$), we let $\widetilde{W}_{j,\mathfrak{x}}$ (resp. $\widetilde{M}_{l,\mathfrak{x}}$) be the connected component of W_j (resp. M_l) containing $\Upsilon^a(\langle g_j \rangle \mathfrak{x})$ (resp. $\Upsilon^b(\langle h_l \rangle \mathfrak{x})$). Let

$$k_{j,\mathfrak{x}} = |\langle g_j \rangle \mathfrak{x}| \qquad \text{resp.} \qquad \Bbbk_{l,\mathfrak{x}} = |\langle h_l \rangle \mathfrak{x}|.$$

Then, as in (2.1.15), $\widetilde{\eta}_{j,\mathfrak{x}}$ (resp. $\widetilde{\varpi}_{l,\mathfrak{x}}$) is determined by the fact that the diagram

$$
\begin{array}{ccc}
\widetilde{W}_{j,\mathfrak{x}} \xrightarrow{\ \widetilde{\eta}_{j,\mathfrak{x}}\ }_{\simeq} \mathcal{D}_{(r_j)^{1/k_{j,\mathfrak{x}}}} & & \widetilde{M}_{l,\mathfrak{x}} \xrightarrow{\ \widetilde{\varpi}_{j,\mathfrak{x}}\ }_{\simeq} \mathcal{D}_{(\rho_l)^{1/\Bbbk_{l,\mathfrak{x}}}} \\
\varphi^a \downarrow \quad \downarrow z^{k_{j,\mathfrak{x}}} & \text{resp.} & \varphi^b \downarrow \quad \downarrow z^{\Bbbk_{l,\mathfrak{x}}} \\
W_j \xrightarrow{\ \eta_j\ }_{\simeq} \mathcal{D}_{r_j} & & M_l \xrightarrow{\ \varpi_l\ }_{\simeq} \mathcal{D}_{\rho_l}
\end{array}
\qquad (3.1.8)
$$

commutes for some $r_j > 0$ (resp. $\rho_l > 0$), and that (similar to (2.1.17))

$$\widetilde{\eta}_{j,\mathfrak{x}} \circ \Psi^a_{\gamma_j}(\mathfrak{x}) > 0 \qquad \text{resp.} \qquad \widetilde{\varpi}_{l,\mathfrak{x}} \circ \Psi^b_{\delta_l}(\mathfrak{x}) > 0. \qquad (3.1.9)$$

In addition to the above conditions (which we have assumed before), for the purpose of sewing $\mathfrak{X}^a$ and $\mathfrak{X}^b$, we also assume that g_0-orbits and h_0-orbits have the same set of marked points, i.e.,

$$\boxed{E(g_0) = E(h_0)} \qquad (3.1.10)$$

This is possible, since we have assumed $g_0 = h_0^{-1}$ (Eq. (3.1.5)). Thus, for each $\mathfrak{x} \in E(g_0) = E(h_0)$, we have

$$k_{0,\mathfrak{x}} = \Bbbk_{0,\mathfrak{x}}.$$

3.1.4 $\mathfrak{X}^{a\#b}$ is the sewing of $\mathfrak{X}^a$ and $\mathfrak{X}^b$

We now sew $\mathfrak{X}^a \sqcup \mathfrak{X}^b$ along all the pairs $\Upsilon^a(\langle g_0 \rangle \mathfrak{x}), \Upsilon^b(\langle h_0 \rangle \mathfrak{x})$ (for all $\mathfrak{x} \in E(g_0) = E(h_0)$) to obtain

$$\mathfrak{X}^{a\#b} = \big(C^{a\#b}; \Upsilon^a(\langle g_j \rangle \mathfrak{x}), \Upsilon^b(\langle h_l \rangle \mathfrak{x}'); \widetilde{\eta}_{j,\mathfrak{x}} \text{ and } \widetilde{\varpi}_{l,\mathfrak{x}'}$$
$$\text{for all } 1 \leqslant j \leqslant N, 1 \leqslant l \leqslant K, \mathfrak{x} \in E(g_j), \mathfrak{x}' \in E(h_l)\big).$$

Thus, for each $\mathfrak{x} \in E(g_0) = E(h_0)$, we remove closed subsets

$$\widetilde{F}_{\mathfrak{x}}^a := \big\{\widetilde{x} \in \widetilde{W}_{0,\mathfrak{x}} : |\widetilde{\eta}_{0,\mathfrak{x}}(\widetilde{x})|^{k_{0,\mathfrak{x}}} \equiv |\eta_0 \circ \varphi^a(\widetilde{x})| \leqslant 1/\rho_0\big\}$$

from $\widetilde{W}_{0,\mathfrak{x}}$ and

$$\widetilde{F}_{\mathfrak{x}}^b := \big\{\widetilde{y} \in \widetilde{M}_{0,\mathfrak{x}} : |\widetilde{\varpi}_{0,\mathfrak{x}}(\widetilde{y})|^{k_{0,\mathfrak{x}}} \equiv |\varpi_0 \circ \varphi^b(\widetilde{y})| \leqslant 1/r_0\big\},$$

from $\widetilde{M}_{0,\mathfrak{x}}$, and glue the remaining parts such that

$$\widetilde{x} \in \widetilde{W}_{0,\mathfrak{x}} \backslash \widetilde{F}_{\mathfrak{x}}^a \text{ equals } \widetilde{y} \in \widetilde{M}_{0,\mathfrak{x}} \backslash F_{\mathfrak{x}}^b$$
$$\Updownarrow$$
$$\widetilde{\eta}_{0,\mathfrak{x}}(\widetilde{x})\widetilde{\varpi}_{0,\mathfrak{x}}(\widetilde{y}) = 1.$$

Thus we obtained $C^{a\#b}$ as the sewing of C^a and C^b.

This gluing process is compatible with φ^a and φ^b. Thus, we obtain a holomorphic surjective

$$\varphi^{a\#b} : C^{a\#b} \to \mathbb{P}^1_{a\#b}$$

such that

$$\varphi^{a\#b} = \varphi^a \text{ when restricted to } C^a \Big\backslash \bigcup_{\mathfrak{x}\in E(g_0)} \widetilde{F}_{\mathfrak{x}}^a,$$
$$\varphi^{a\#b} = \varphi^b \text{ when restricted to } C^b \Big\backslash \bigcup_{\mathfrak{x}\in E(h_0)} \widetilde{F}_{\mathfrak{x}}^b.$$

Since $\Upsilon^a(\langle g_j \rangle \mathfrak{x}), \Upsilon^b(\langle h_l \rangle \mathfrak{x}')$ (for all $j \geqslant 1, k \geqslant 1, \mathfrak{x} \in E(g_j), \mathfrak{x}' \in E(h_j)$) are outside any $\widetilde{F}_e^a, \widetilde{F}_e^b$ (where $e \in E(g_0)$), we can define these points as marked points of $\mathfrak{X}^{a\#b}$. Thus, we can also view $\widetilde{\eta}_{j,\mathfrak{x}}, \widetilde{\varpi}_{l,\mathfrak{x}'}$ as the local coordinates of $\mathfrak{X}^{a\#b}$ at these two points points, defined on $\widetilde{W}_{j,\mathfrak{x}}, \widetilde{M}_{l,\mathfrak{x}'} \subset C^{a\#b}$.

3.1.5 $\mathfrak{X}^{a\#b}$ is the permutation covering of $\mathfrak{P}^{a\#b}$ associated to $\Gamma^{a\#b} \curvearrowright E$ and $E(g_\bullet), E(h_\bullet)$

Since $\gamma_0(0)$ and $\delta_0(0)$ are outside F^a, F^b (by (3.1.1)), their preimages $\Psi_{\gamma_0}^a(e)$ and $\Psi_{\delta_0}^b(e)$ (for all $e \in E$) are outside $\widetilde{F}_{\mathfrak{x}}^a, \widetilde{F}_{\mathfrak{x}}^b$ (for every $\mathfrak{x} \in E(g_0) = E(h_0)$). Moreover:

Lemma 3.1.1. $\Psi_{\gamma_0}^a(e)$ and $\Psi_{\delta_0}^b(e)$ are the same point on $C^{a\#b}$.

Proof. Write $e = g_0^m \mathfrak{x} = h_0^{-m} \mathfrak{x}$ for some $m \in \mathbb{Z}, \mathfrak{x} \in E(g_0) = E(h_0)$. Let $\widetilde{x} = \Psi_{\gamma_0}^a(e)$ and $\widetilde{y} = \Psi_{\delta_0}^b(e)$. Set $k = k_{0,\mathfrak{x}}$. We have

$$\widetilde{x} = \Psi_{\gamma_0}^a([\alpha_0]^m \mathfrak{x}) \xupdownequal{(2.1.6)} \Psi_{\gamma_0 \alpha_0^m}^a(\mathfrak{x}) \xupdownequal{(3.1.3)} \Psi_{(\epsilon_0^a)^m \gamma_0}^a(\mathfrak{x}).$$

So $\widetilde{x}$ is the initial point of the lift of $(\epsilon_0^a)^m$ to C^a ending at $\Psi_{\gamma_0}^a(\mathfrak{x})$. Since $\eta_0 \circ \epsilon_0^a$ is an anticlockwise circle going by 2π, $\widetilde{\eta}_{0,\mathfrak{x}}$ sends the lift of ϵ_0^a to the anticlockwise arc going by $2\pi/k$. Therefore

$$\widetilde{\eta}_{0,\mathfrak{x}}(\widetilde{x}) = \omega_k^m \cdot \widetilde{\eta}_{0,\mathfrak{x}}\big(\Psi_{\gamma_0}^a(\mathfrak{x})\big).$$

A similar argument shows

$$\widetilde{\varpi}_{0,\mathfrak{x}}(\widetilde{y}) = \mathfrak{o}_k^{-m} \cdot \widetilde{\varpi}_{0,\mathfrak{x}}\big(\Psi_{\delta_0}^b(\mathfrak{x})\big).$$

Set $d = \eta_0(\gamma_0(0)) = \varpi_0(\delta_0(0))^{-1}$ (cf. (3.1.2)), which is positive. Since $\Psi_{\gamma_0}^a(\mathfrak{x}) \in (\varphi^a)^{-1}(\gamma_0(0))$, by (3.1.8),

$$\widetilde{\eta}_{0,\mathfrak{x}}\big(\Psi_{\gamma_0}^a(\mathfrak{x})\big)^k = \eta_0 \circ \varphi^a\big(\Psi_{\gamma_0}^a(\mathfrak{x})\big) = \eta_0 \circ \gamma_0(0) = d.$$

Thus, by (3.1.9), we have $\widetilde{\eta}_{0,\mathfrak{x}}\big(\Psi_{\gamma_0}^a(\mathfrak{x})\big) = d^{1/k}$. A similar argument shows $\widetilde{\varpi}_{0,\mathfrak{x}}\big(\Psi_{\delta_0}^b(\mathfrak{x})\big) = d^{-1/k}$. Therefore $\widetilde{\eta}_{0,\mathfrak{x}}(\widetilde{x}) = 1/\widetilde{\varpi}_{0,\mathfrak{x}}(\widetilde{y})$, which shows that $\widetilde{x}$ and $\widetilde{y}$ are identical after sewing. $\qquad\square$

It follows that the preimage of $\bigstar = \gamma_0(0) = \delta_0(0)$ under $\varphi^{a\#b}$ consists of

$$\Psi_\bigstar^{a\#b}(e) := \Psi_{\gamma_0}^a(e) = \Psi_{\delta_0}^b(e) \tag{3.1.11}$$

for all $e \in E$. This gives us a bijection

$$\Psi_\bigstar^{a\#b} : E \to (\varphi^{a\#b})^{-1}(\bigstar).$$

The following lemma is an easy consequence of Thm. 2.1.9-(b).

Lemma 3.1.2. *Let λ be a path in $\mathbb{P}_{a\#b}^1 \backslash \mathbf{S}^{a\#b}$ ending at $\bigstar$. Choose $e \in E$, and let $\widetilde{\lambda}$ be the lift of λ to $C^{a\#b}$ ending at $\Psi_\bigstar^{a\#b}(e)$. If λ is in $\mathbb{P}_a^1 \backslash F^a$ (resp. in $\mathbb{P}_b^1 \backslash F^b$), then the initial point of $\widetilde{\lambda}$ is $\Psi_{\lambda\gamma_0}^a(e)$ (resp. $\Psi_{\lambda\delta_0}^b(e)$).*

The main result of this section is the following theorem.

Theorem 3.1.3. *The bijection $\Psi_\bigstar^{a\#b}$ is $\Gamma^{a\#b}$-covariant, and*

$$\big(\mathfrak{X}^{a\#b}, \varphi^{a\#b} : C^{a\#b} \to \mathbb{P}_{a\#b}^1, \Psi_\bigstar^{a\#b}\big)$$

is a permutation covering of $\mathfrak{P}^{a\#b}$ associated to the action $\Gamma^{a\#b} \curvearrowright E$ (defined by (3.1.7)) and $E(g_j), E(h_l)$ (for all $1 \leqslant j \leqslant N$ and $1 \leqslant l \leqslant K$).

Moreover, let $\Psi^{a\#b}$ be the trivilization determined by $\Psi_\bigstar^{a\#b}$, and define $\Upsilon^{a\#b}$ as in Cor. 2.1.12. Then for any path λ in $\mathbb{P}_{a\#b}^1 \backslash \mathbf{S}^{a\#b}$ ending at $\bigstar$, if λ is in $\mathbb{P}_a^1 \backslash F^a$ (resp. in $\mathbb{P}_b^1 \backslash F^b$), then

$$\Psi_\lambda^{a\#b}(e) = \Psi_{\lambda\gamma_0}^a(e) \qquad resp. \qquad \Psi_\lambda^{a\#b}(e) = \Psi_{\lambda\delta_0}^b(e). \tag{3.1.12}$$

For any $1 \leqslant j \leqslant N$ and $1 \leqslant l \leqslant K$, we have

$$\Upsilon^{a\#b}(\langle g_j \rangle e) = \Upsilon^a(\langle g_j \rangle e), \qquad \Upsilon^{a\#b}(\langle h_l \rangle e) = \Upsilon^b(\langle h_l \rangle e). \tag{3.1.13}$$

Proof. Note that for any $\alpha_j^\#$ and $\beta_l^\#$ defined by (3.1.6) (where $j, l \geqslant 1$), applying Lemma 3.1.2 to $\lambda = \alpha_j^\#, \beta_l^\#$, we see that the lift of $\alpha_j^\#$ (resp. $\beta_l^\#$) to $C^{a\#b}$ ending at $\Psi_\bigstar^{a\#b}(e)$ must start from $\Psi_{\gamma_0\alpha_j}^a(e) = \Psi_{\gamma_0}^a([\alpha_j]e) = \Psi_\bigstar^{a\#b}(g_j e)$ (resp. $\Psi_\bigstar^{a\#b}(h_l e)$). Thus, for any path μ in $\mathbb{P}_{a\#b}^1 \backslash \mathbf{S}^{a\#b}$ from and to $\bigstar$, which is homotopic to a product of powers of $\alpha_j^\#, \beta_l^\#$ (for all

$j, l \geqslant 1$), the initial point of the lift of μ to $C^{a\#b}$ ending at $\Psi_{\bigstar}^{a\#b}(e)$ is $\Psi_{\bigstar}^{a\#b}(ge)$, where g is the same product of powers of g_j, h_l, which is $[\mu]$. So $\Psi_{\bigstar}^{a\#b}$ is $\Gamma^{a\#b}$-covariant.

By the construction of local coordinates, it is clear that $\mathfrak{X}^{a\#b}$ is the permutation covering of $\mathfrak{P}^{a\#b}$ associated to the action of $\Gamma^{a\#b}$ and all $E(g_j), E(h_l)$ (where $j, l \geqslant 1$).

For any path λ in $\mathbb{P}_a^1 \backslash F^a$ ending at $\bigstar$, by Thm. 2.1.9-(b) (applied to $\Psi^{a\#b}$), we know that $\Psi_\lambda^{a\#b}(e)$ is the initial point of the lift $\tilde{\lambda}$ of λ to $C^{a\#b}$ ending at $\Psi_{\bigstar}^{a\#b}(e) = \Psi_{\gamma_0}^a(e)$. Since $\tilde{\lambda}$ is also the lift of λ to C^a ending at $\Psi_{\gamma_0}^a(e)$, $\Psi_\lambda^{a\#b}(e) = \tilde{\lambda}(0)$ must equal $\Psi_{\lambda\gamma_0}^a(e)$ by Thm. 2.1.9-(b). This proves the first half of (3.1.12). The second half follows from a similar argument.

By Cor. 2.1.12 and the definition of the paths of $\mathfrak{P}^{a\#b}$, $\Upsilon^{a\#b}(\langle g_j \rangle e)$ and $\Psi_{\gamma_j\gamma_0^{-1}}^{a\#b}(\langle g_j \rangle e)$ are in the same connected component of $(\varphi^{a\#b})^{-1}(W_j) = (\varphi^a)^{-1}(W_j)$. Similarly, $\Upsilon^a(\langle g_j \rangle e)$ and $\Psi_{\gamma_j}^a(\langle g_j \rangle e)$ (which equals $\Psi_{\gamma_j\gamma_0^{-1}}^{a\#b}(\langle g_j \rangle e)$ by (3.1.12)) are in the same connected component of $(\varphi^a)^{-1}(W_j)$. This proves the first half of (3.1.13). The second half can be proved in the same way. $\qquad\square$

3.2 Sewing and factorization of permutation-twisted conformal blocks

In this section, we assume $\mathbb{V}$ is C_2-cofinite. Then so is $\mathbb{U} = \mathbb{V}^{\otimes E}$. For each $0 \leqslant j \leqslant N, 0 \leqslant l \leqslant K$, and for each $\mathfrak{x} \in E(g_j)$ (resp. $\mathfrak{x} \in E(h_l)$), we associate a *finitely-generated* $\mathbb{V}$-module $\mathbb{W}_{j,\mathfrak{x}}$ (resp. $\mathbb{M}_{l,\mathfrak{x}}$) to the marked point $\Upsilon^a(\langle g_j \rangle \mathfrak{x})$ of $\mathfrak{X}^a$ (resp. $\Upsilon^b(\langle h_l \rangle \mathfrak{x})$ of $\mathfrak{X}^b$). Then we have a g_j- resp. h_l-twisted $\mathbb{U}$-module

$$\mathcal{W}_j = \bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}} \qquad \text{resp.} \qquad \mathcal{M}_l = \bigotimes_{\mathfrak{x} \in E(h_l)} \mathbb{M}_{l,\mathfrak{x}}$$

defined as in Thm. 2.2.1 associated to the marked point x_j of $\mathfrak{P}^a$ (resp. y_l of $\mathfrak{P}^b$). As usual, we set

$$k_{j,\mathfrak{x}} = |\langle g_j \rangle \mathfrak{x}|, \qquad \mathfrak{k}_{l,\mathfrak{x}} = |\langle h_l \rangle \mathfrak{x}|.$$

According to (2.2.8), the $\tilde{L}_0^{g_j}$- (resp. $\tilde{L}_0^{h_j}$-)grading and the $\tilde{L}_0$-grading are related by the fact that for each $n \in \frac{1}{|g_j|}\mathbb{N}$,

$$\begin{aligned}
\mathcal{W}_j(n) &= \bigoplus_{\sum_{\mathfrak{x} \in E(g_j)} n_{\mathfrak{x}}/k_{j,\mathfrak{x}} = n} \left(\bigotimes_{\mathfrak{x} \in E(g_j)} \mathbb{W}_{j,\mathfrak{x}}(n_{\mathfrak{x}}) \right) \\
\mathcal{M}_l(n) &= \bigoplus_{\sum_{\mathfrak{x} \in E(h_l)} n_{\mathfrak{x}}/\mathfrak{k}_{l,\mathfrak{x}} = n} \left(\bigotimes_{\mathfrak{x} \in E(h_l)} \mathbb{M}_{l,\mathfrak{x}}(n_{\mathfrak{x}}) \right)
\end{aligned} \tag{3.2.1}$$

where all $n_{\mathfrak{x}}$ are in $\mathbb{N}$.

We assume that for each $\mathfrak{x} \in E(g_0) = E(h_0)$,

$$\boxed{\mathbb{M}_{0,\mathfrak{x}} = \mathbb{W}_{0,\mathfrak{x}}'}$$

i.e., $\mathbb{W}_{0,\mathfrak{x}}$ and $\mathbb{M}_{0,\mathfrak{x}}$ are the contragredient $\mathbb{V}$-modules of each other. Then, by Thm. 2.2.3, $\mathcal{W}_0$ and $\mathcal{M}_0$ are the contragredient twisted $\mathbb{U}$-modules of each other.

Let

$$\mathcal{W}_\bullet = \bigotimes_{0 \leqslant j \leqslant N} \mathcal{W}_j, \qquad \mathcal{M}_\bullet = \bigotimes_{0 \leqslant j \leqslant N} \mathcal{M}_j,$$

$$\mathcal{W}_{\bullet \backslash 0} = \bigotimes_{1 \leqslant j \leqslant N} \mathcal{W}_j, \qquad \mathcal{M}_{\bullet \backslash 0} = \bigotimes_{1 \leqslant j \leqslant N} \mathcal{M}_j.$$

Let $\mathrm{CB}_{\mathfrak{P}^a}(\mathcal{W}_\bullet)$ resp. $\mathrm{CB}_{\mathfrak{P}^b}(\mathcal{M}_\bullet)$ be the space of conformal blocks associated to $\mathfrak{P}^a$ and the twisted $\mathbb{U}$-modules $\mathcal{W}_\bullet$ (resp. $\mathfrak{P}^b$ and $\mathcal{M}_\bullet$). For each $\psi^a \in \mathrm{CB}_{\mathfrak{P}^a}(\mathcal{W}_\bullet)$ and $\psi^b \in \mathrm{CB}_{\mathfrak{P}^b}(\mathcal{M}_\bullet)$, we define, for every $n \in \frac{1}{|g_0|}\mathbb{N} = \frac{1}{|h_0|}\mathbb{N}$, a linear functional

$$\psi_n^{a\#b} : \mathcal{W}_{\bullet \backslash 0} \otimes \mathcal{M}_{\bullet \backslash 0} \to \mathbb{C}$$

such that for each $w \in \mathcal{W}_{\bullet \backslash 0}$ and $m \in \mathcal{M}_{\bullet \backslash 0}$, $\psi_n^{a\#b}(w \otimes m)$ is the contraction of the linear functional (noting $\mathcal{W}_0'(n) = \mathcal{W}_0(n)^*$)

$$\psi^a(\cdot \otimes w)\psi^b(\cdot \otimes m)\Big|_{\mathcal{W}_0(n) \otimes \mathcal{W}_0(n)^*} : \mathcal{W}_0(n) \otimes \mathcal{W}_0(n)^* \to \mathbb{C}.$$

We say that **the sewing $\psi^{a\#b}$ converges q-absolutely** if there exists $R > 1$ such that for each $w \in \mathcal{W}_{\bullet \backslash 0}, m \in \mathcal{M}_{\bullet \backslash 0}$,

$$\sum_{n \in \frac{1}{|g_0|}\mathbb{N}} \big|\psi_n^{a\#b}(w \otimes m)\big| R^n < +\infty.$$

If so, we define

$$\psi^{a\#b} : \mathcal{W}_{\bullet \backslash 0} \otimes \mathcal{M}_{\bullet \backslash 0} \to \mathbb{C}$$
$$w \otimes m \mapsto \sum_{\frac{1}{|g_0|}\mathbb{N}} \psi_n^{a\#b}(w \otimes m).$$

We have $\mathcal{W}_\bullet = \mathbb{W}_{\bullet,\bullet}, \mathcal{M}_\bullet = \mathbb{M}_{\bullet,\bullet}, \mathcal{W}_{\bullet \backslash 0} = \mathbb{W}_{\bullet \backslash 0,\bullet}, \mathcal{M}_{\bullet \backslash 0} = \mathbb{M}_{\bullet \backslash 0,\bullet}$ where

$$\mathbb{W}_{\bullet,\bullet} = \bigotimes_{0 \leqslant j \leqslant N} \bigotimes_{x \in E(g_j)} \mathbb{W}_{j,x}, \qquad \mathbb{M}_{\bullet,\bullet} = \bigotimes_{0 \leqslant l \leqslant N} \bigotimes_{x \in E(h_l)} \mathbb{W}_{l,x},$$

$$\mathbb{W}_{\bullet \backslash 0,\bullet} = \bigotimes_{1 \leqslant j \leqslant N} \bigotimes_{x \in E(g_j)} \mathbb{W}_{j,x}, \qquad \mathbb{M}_{\bullet \backslash 0,\bullet} = \bigotimes_{1 \leqslant l \leqslant N} \bigotimes_{x \in E(h_l)} \mathbb{W}_{l,x}.$$

Let $\mathrm{CB}_{\mathfrak{X}^a}(\mathbb{W}_{\bullet,\bullet})$ resp. $\mathrm{CB}_{\mathfrak{X}^b}(\mathbb{M}_{\bullet,\bullet})$ be the space of conformal blocks associated to $\mathfrak{X}^a$ and the $\mathbb{V}$-modules $\mathbb{W}_{\bullet,\bullet}$ (resp. $\mathfrak{X}^b$ and $\mathbb{M}_{\bullet,\bullet}$). Then by Thm. 2.2.5,

$$\mathrm{CB}_{\mathfrak{X}^a}(\mathbb{W}_{\bullet,\bullet}) = \mathrm{CB}_{\mathfrak{P}^a}(\mathcal{W}_\bullet), \qquad \mathrm{CB}_{\mathfrak{X}^b}(\mathbb{M}_{\bullet,\bullet}) = \mathrm{CB}_{\mathfrak{P}^b}(\mathcal{M}_\bullet).$$

Consider ψ^a, ψ^b as elements of $\mathrm{CB}_{\mathfrak{P}^a}(\mathcal{W}_\bullet), \mathrm{CB}_{\mathfrak{P}^b}(\mathcal{M}_\bullet)$, which we denote by ϕ^a, ϕ^b in order to follow the notation in Thm. 2.2.5. Then $\phi^a \cdot \phi^b : \mathbb{W}_{\bullet,\bullet} \otimes \mathbb{M}_{\bullet,\bullet} \to \mathbb{C}$ is a conformal block associated to $\mathfrak{X}^a \sqcup \mathfrak{X}^b$. Corresponding to the geometric sewing process in Subsec. 3.1.4, we can define the algebraic sewing

$$\mathscr{S}(\phi^a \cdot \phi^b) : \mathbb{W}_{\bullet \backslash 0,\bullet} \otimes \mathbb{M}_{\bullet \backslash 0,\bullet} \to \mathbb{C}$$

56

as in Subsec. 1.2.2, which converges q-absolutely by Thm. 1.2.6. Moreover, let

$$\mathrm{CB}_{\mathfrak{X}^{a\#b}}(\mathbb{W}_{\bullet\backslash 0,\bullet} \otimes \mathbb{M}_{\bullet\backslash 0,\bullet})$$

be the space of conformal blocks associated to $\mathfrak{X}^{a\#b}$ and the corresponding $\mathbb{V}$-modules. For each $1 \leqslant j \leqslant N$ and $\mathscr{x} \in E(g_j)$, $\mathbb{W}_{j,\mathscr{x}}$ is associated to $\Upsilon^{a\#b}(\langle g_j \rangle \mathscr{x}) = \Upsilon^a(\langle g_j \rangle \mathscr{x})$, and for each $1 \leqslant l \leqslant K$ and $\mathscr{x} \in E(h_l)$, $\mathbb{M}_{l,\mathscr{x}}$ is associated to $\Upsilon^{a\#b}(\langle h_l \rangle \mathscr{x}) = \Upsilon^b(\langle h_l \rangle \mathscr{x})$. (Recall (3.1.13).) Then $\mathscr{S}(\phi^a \cdot \phi^b) \in \mathrm{CB}_{\mathfrak{X}^{a\#b}}(\mathbb{W}_{\bullet\backslash 0,\bullet} \otimes \mathbb{M}_{\bullet\backslash 0,\bullet})$ by Thm. 1.2.5.

Let $\mathrm{CB}_{\mathfrak{P}^{a\#b}}(\mathcal{W}_{\bullet\backslash 0} \otimes \mathcal{M}_{\bullet\backslash 0})$ denote the space of conformal blocks associated to $\mathfrak{P}^{a\#b}$ and the corresponding twisted $\mathbb{U}$-modules. $g_j, \mathcal{W}_j$ are associated to x_j, and $h_l, \mathcal{M}_j$ to y_l (where $1 \leqslant j \leqslant N, 1 \leqslant l \leqslant K$).

Theorem 3.2.1. *Assume the setting of Sec. 3.1. Assume that $\mathbb{V}$ is C_2-cofinite. Assume that $\mathbb{W}_{j,\mathscr{x}}$ (for each $0 \leqslant j \leqslant N, \mathscr{x} \in E(g_j)$) and $\mathbb{M}_{l,\mathscr{x}}$ (for each $0 \leqslant l \leqslant K, \mathscr{x} \in E(h_l)$) are finitely-generated $\mathbb{V}$-modules. Choose $\psi^a \in \mathrm{CB}_{\mathfrak{P}^a}(\mathcal{W}_\bullet)$ and $\psi^b \in \mathrm{CB}_{\mathfrak{P}^b}(\mathcal{W}_\bullet)$, where the same linear functionals are denoted by $\phi^a \in \mathrm{CB}_{\mathfrak{X}^a}(\mathbb{W}_{\bullet,\bullet})$ and $\phi^b \in \mathrm{CB}_{\mathfrak{X}^b}(\mathbb{M}_{\bullet,\bullet})$. Then*

(1) $\psi^{a\#b}$ converges q-absolutely to an element of $\mathrm{CB}_{\mathfrak{P}^{a\#b}}(\mathcal{W}_{\bullet\backslash 0} \otimes \mathcal{M}_{\bullet\backslash 0})$.

(2) $\psi^{a\#b}$ equals $\mathscr{S}(\phi^a \cdot \phi^b)$ as linear functionals.

Proof. By the definition of $\psi^{a\#b}$ and $\mathscr{S}(\phi^a \cdot \phi^b)$, and by (3.2.1), $\psi^{a\#b}$ converges q-absolutely to $\mathscr{S}(\phi^a \cdot \phi^b)$, which is an element of $\mathrm{CB}_{\mathfrak{X}^{a\#b}}(\mathbb{W}_{\bullet\backslash 0,\bullet} \otimes \mathbb{M}_{\bullet\backslash 0,\bullet})$. By Thm. 3.1.3, $\mathfrak{X}^{a\#b}$ is a permutation covering of $\mathfrak{P}^{a\#b}$ associated to the action (3.1.7) and the sets $E(g_j), E(h_l)$ (where $j, l > 0$) of marked points of g_j- and h_l-orbits. By Thm. 2.2.5, we have

$$\mathrm{CB}_{\mathfrak{X}^{a\#b}}(\mathbb{W}_{\bullet\backslash 0,\bullet} \otimes \mathbb{M}_{\bullet\backslash 0,\bullet}) = \mathrm{CB}_{\mathfrak{P}^{a\#b}}(\mathcal{W}_{\bullet\backslash 0} \otimes \mathcal{M}_{\bullet\backslash 0}). \tag{3.2.2}$$

This finishes the proof. $\qquad\square$

Remark 3.2.2. Relations (3.1.13) are necessary for the above results: they tell us that the association of $\mathbb{V}$-modules to the marked points of $\mathfrak{X}^{a\#b}$ determined by sewing $\mathfrak{X}^a, \mathfrak{X}^b$ agrees with the one determined by the permutation covering of $\mathfrak{X}^{a\#b}$.

As an application of Thm. 3.2.1, we prove a sewing-factorization theorem for genus-0 permutation-twisted conformal blocks.

Let $\mathcal{E}$ be a complete list of irreducible $\mathbb{V}$-modules. (Cf. the paragraph containing (1.2.12).) Define $\mathcal{E}_{0,\bullet}$ to be the set consisting of

$$\mathcal{X}_0 = \otimes_{\mathscr{x} \in E(g_0)} \mathbb{X}_{0,\mathscr{x}}$$

where each $\mathbb{X}_{0,\mathscr{x}}$ is in $\mathcal{E}$. Set

$$\mathcal{X}_0' = \otimes_{\mathscr{x} \in E(g_0)} \mathbb{X}_{0,\mathscr{x}}'.$$

Consider $\mathcal{X}_0$ and $\mathcal{X}_0'$ as mutually contragredient twisted $\mathbb{U}$-modules (cf. Thm. 2.2.3). Note that by Prop. 2.2.2 and Thm. 2.2.4, if $\mathbb{V}$ is also rational, then $\mathcal{E}_{0,\bullet}$ is a finite and complete list of irreducible g_0-twisted $\mathbb{U}$-modules.

Theorem 3.2.3. *Assume that $\mathbb{V}$ is CFT-type (i.e. $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$), C_2-cofinite, and rational. Choose finitely-generated $\mathbb{V}$-modules $\mathbb{W}_{j,\mathfrak{x}}$ (for each $1 \leqslant j \leqslant N, \mathfrak{x} \in E(g_j)$) and $\mathbb{M}_{l,\mathfrak{x}}$ (for each $1 \leqslant l \leqslant K, \mathfrak{x} \in E(h_l)$). Then the linear map*

$$\bigoplus_{\mathcal{X}_0 \in \mathcal{E}_{0,\bullet}} \mathrm{CB}_{\mathfrak{P}^a}(\mathcal{X}_0 \otimes \mathcal{W}_{\bullet \backslash 0}) \otimes \mathrm{CB}_{\mathfrak{P}^b}(\mathcal{X}'_0 \otimes \mathcal{M}_{\bullet \backslash 0}) \to \mathrm{CB}_{\mathfrak{P}^{a\#b}}(\mathcal{W}_{\bullet \backslash 0} \otimes \mathcal{M}_{\bullet \backslash 0})$$

$$\bigoplus_{\mathcal{X}_0} \psi^a_{\mathcal{X}_0} \otimes \psi^b_{\mathcal{X}_0} \mapsto \sum_{\mathcal{X}_0} \psi^{a\#b}_{\mathcal{X}_0}$$

(where $\psi^{a\#b}_{\mathcal{X}_0}$ is the sewing of $\psi^a_{\mathcal{X}_0}$ and $\psi^b_{\mathcal{X}_0}$) is bijective.

Proof. By Thm. 2.2.5 and 3.2.1, the above linear map is equivalent to a sewing map of spaces of untwisted conformal blocks. This linear map is bijective by Thm. 1.2.7. □

4 Applications

4.1 Twisted intertwining operators

In this section, we assume the setting at the beginning of Sec. 1.3, namely, G is a general finite automorphism automorphism group of a VOA $\mathbb{U}$. We assume that 1 has argument 0, and that in general $e^{\mathbf{i}t}$ has argument t. If the arguments of $z_1, z_2 \in \mathbb{C}^{\times}$ are chosen, we assume the argument of $z_1 z_2$ is $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

Definition 4.1.1. Let $g_1, g_2 \in G$ and $g_3 = g_1 g_2$. Let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ be respectively g_1-, g_2-, g_3-twisted $\mathbb{V}$-modules. A type $\binom{\mathcal{W}_3}{\mathcal{W}_1 \mathcal{W}_2}$-**intertwining operator** is an operation $\mathcal{Y}$ that associates to each $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w'_3 \in \mathcal{W}'_3$ a multivalued homolorphic function

$$z \in \mathbb{C}^{\times} \mapsto \langle \mathcal{Y}(w_1, z) w_2, w'_3 \rangle$$

(i.e. a holomorphic function which depends on z as well as its argument $\arg z$) depending linearly on w_1, w_2, w'_3, such that the following conditions are satisfied for every $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w'_3 \in \mathcal{W}'_3$. Consider $\mathcal{Y}(w_1, z)$ as a linear map from $\mathcal{W}_2$ to $(\mathcal{W}'_3)^*$.

1. (L_{-1}-derivative) For each $z \in \mathbb{C}^{\times}$,

$$\frac{d}{dz} \langle \mathcal{Y}(w_1, z) w_2, w'_3 \rangle = \langle \mathcal{Y}(L_{-1} w_1, z) w_2, w'_3 \rangle \tag{4.1.1}$$

2. (Jacobi identity) For each $u \in \mathbb{U}$, and for each $z \neq \mathfrak{z}$ in $\mathbb{C}^{\times}$ with chosen $\arg z, \arg \mathfrak{z}, \arg(z - \mathfrak{z})$, the following series of single-valued functions of $\log z, \log z, \log(z - \mathfrak{z})$

$$\langle Y^{g_3}(u, z) \mathcal{Y}(w_1, \mathfrak{z}) w_2, w'_3 \rangle := \sum_{n \in \frac{1}{|g_3|} \mathbb{N}} \langle Y^{g_3}(u, z) P^{g_3}_n \mathcal{Y}(w_1, \mathfrak{z}) w_2, w'_3 \rangle \tag{4.1.2}$$

$$\langle \mathcal{Y}(w_1, \mathfrak{z}) Y^{g_2}(u, z) w_2, w'_3 \rangle := \sum_{n \in \frac{1}{|g_2|} \mathbb{N}} \langle \mathcal{Y}(w_1, \mathfrak{z}) P^{g_2}_n Y^{g_2}(u, z) w_2, w'_3 \rangle \tag{4.1.3}$$

$$\langle \mathcal{Y}(Y^{g_1}(u, z - \mathfrak{z}) w_1, \mathfrak{z}) w_2, w'_3 \rangle := \sum_{n \in \frac{1}{|g_1|} \mathbb{N}} \langle \mathcal{Y}(P^{g_1}_n Y^{g_1}(u, z - \mathfrak{z}) w_1, \mathfrak{z}) w_2, w'_3 \rangle \tag{4.1.4}$$

converge a.l.u. on $|z| > |\mathfrak{z}|$, $|z| < |\mathfrak{z}|$, $|z - \mathfrak{z}| < |\mathfrak{z}|$ respectively (in the sense of (1.1.1)). Moreover, for any fixed $\mathfrak{z} \in \mathbb{C}^\times$ with chosen argument $\arg \mathfrak{z}$, let $R_{\mathfrak{z}} = \{t\mathfrak{z} : t \in (0, 1) \cup (1, +\infty)\}$. For any $z \in R_{\mathfrak{z}}$, we assume that

$$
\begin{aligned}
\arg z &= \arg \mathfrak{z}, \\
\arg(z - \mathfrak{z}) &= \arg \mathfrak{z} \text{ or } \arg \mathfrak{z} - \pi,
\end{aligned}
\tag{4.1.5}
$$

where second equality depends on whether $|z| > |\mathfrak{z}|$ or $|z| < |\mathfrak{z}|$. Then the above three expressions (4.1.2)-(4.1.4), considered as functions of z defined on $R_{\mathfrak{z}}$ satisfying the three mentioned inequalities respectively, can be analytically continued to the same holomorphic function on the simply-connected open set

$$
\Sigma_{\mathfrak{z}} = \mathbb{C} \backslash \{\mathbf{i}t\mathfrak{z}, \mathfrak{z} + \mathbf{i}t\mathfrak{z} : t \geqslant 0\},
$$

which can furthermore be extended to a multivalued holomorphic function $f_{\mathfrak{z}}(z)$ on $\mathbb{C}^\times \backslash \{\mathfrak{z}\}$ (i.e., a holomorphic function on the universal cover of $\mathbb{C}^\times \backslash \{\mathfrak{z}\}$).

Remark 4.1.2. In the above Jacobi identify, if we assume u is fixed by G, then $f_{\mathfrak{z}}(z)$ is single-valued. In particular, if we choose u to be the conformal vector of $\mathbb{U}$, and apply the residue theorem to $zf_{\mathfrak{z}}(z)dz$, we obtain

$$
\begin{aligned}
&\langle \mathcal{Y}(w_1, \mathfrak{z})w_2, L_0 w_3' \rangle - \langle \mathcal{Y}(w_1, \mathfrak{z})L_0 w_2, w_3' \rangle \\
=&\langle \mathcal{Y}(L_0 w_1, \mathfrak{z})w_2, w_3' \rangle + \mathfrak{z} \langle \mathcal{Y}(L_{-1}w_1, \mathfrak{z})w_2, w_3' \rangle.
\end{aligned}
$$

Thus, assuming the Jacobi identity, condition 1 of Def. 4.1.1 (the L_{-1}-derivative) is equivalent to

$$
\begin{aligned}
&\langle \mathcal{Y}(w_1, \mathfrak{z})w_2, L_0 w_3' \rangle - \langle \mathcal{Y}(w_1, \mathfrak{z})L_0 w_2, w_3' \rangle \\
=&\langle \mathcal{Y}(L_0 w_1, \mathfrak{z})w_2, w_3' \rangle + \mathfrak{z} \frac{d}{d\mathfrak{z}} \langle \mathcal{Y}(w_1, \mathfrak{z})w_2, w_3' \rangle.
\end{aligned}
\tag{4.1.6}
$$

If we apply the residue theorem to $f_{\mathfrak{z}}(z)dz$, we get

$$
\langle L_{-1}\mathcal{Y}(w_1, \mathfrak{z})w_2, w_3' \rangle - \langle \mathcal{Y}(w_1, \mathfrak{z})L_{-1}w_2, w_3' \rangle = \langle \mathcal{Y}(L_{-1}w_1, \mathfrak{z})w_2, w_3' \rangle.
\tag{4.1.7}
$$

Remark 4.1.3. If $\mathcal{W}$ is a g-twisted $\mathbb{U}$-module, then its vertex operator Y^g clearly defines a type $\binom{\mathcal{W}}{\mathbb{U}\mathcal{W}}$-intertwining operator.

Remark 4.1.4. Since $[L_0, \widetilde{L}_0^g] = 0$ by (1.3.1), L_0 preserves each $\mathcal{W}(n)$. Since $\mathcal{W}(n)$ is finite-dimensional, we have a decomposition $L_0 = L_{0,\mathrm{s}} + L_{0,\mathrm{n}}$ where $[L_{0,\mathrm{s}}, L_{0,\mathrm{n}}] = 0$, $L_{0,\mathrm{s}}$ is diagonal, and for every $w \in \mathcal{W}$, $L_{0,\mathrm{n}}^k w = 0$ for sufficiently large k. Thus we can define $z^{L_0} w \in \mathcal{W}[\log z]\{z\}$ by

$$
z^{L_0} w = z^{L_{0,\mathrm{s}}} \sum_{k \in \mathbb{N}} \frac{L_{0,\mathrm{n}}^k w}{k!} (\log z)^k.
$$

Proposition 4.1.5. *Let $\mathcal{Y}$ be as in Def. 4.1.1. Then for each $\xi, z \in \mathbb{C}^\times$ with chosen $\arg \xi, \arg z$, and for each $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w_3' \in \mathcal{W}_3'$,*

$$
\langle \mathcal{Y}(w_1, z\xi)w_2, w_3' \rangle = \langle \mathcal{Y}(z^{-L_0}w_1, \xi)z^{-L_0}w_2, z^{L_0}w_3' \rangle.
\tag{4.1.8}
$$

Proof. $f(z) = \langle \mathcal{Y}(z^{L_0}w_1, z\xi)z^{L_0}w_2, z^{-L_0}w_3' \rangle$ is a multivalued holomorphic function of $z \in \mathbb{C}^\times$. Using (4.1.6), it is easy to see that $\frac{d}{dz}f(z) = 0$. Thus, $f(z) = f(1)$, which is equivalent to (4.1.8). $\qquad\square$

Remark 4.1.6. Set $\xi = 1$ in (4.1.8). If we assume w_1, w_2, w_3' are eigenvectors of $L_{0,\mathrm{s}}$ with eigenvalues $\mathrm{wt}w_1, \mathrm{wt}w_2, \mathrm{wt}w_3'$, then (4.1.8) is clearly an element of $\mathbb{C} \cdot z^{\mathrm{wt}w_3' - \mathrm{wt}w_1 - \mathrm{wt}w_2}[\log z]$. Suppose that each eigenspace of $L_{0,\mathrm{s}}$ on $\mathcal{W}_3$ is finite-dimensional. Then for each $n \in \mathbb{C}, k \in \mathbb{N}$ we can find a unique $L_{0,\mathrm{s}}$-eigenvector $\mathcal{Y}(w_1)_{n,k}w_2$ of $\mathcal{W}_3$ with eigenvalue $\mathrm{wt}w_1 + \mathrm{wt}w_2 - n - 1$ such that for each $L_{0,\mathrm{s}}$-eigenvector $w_3' \in \mathcal{W}_3$ of $\mathcal{W}_3'$ with the same eigenvalue, $z^{n+1}\langle \mathcal{Y}(w_1, z)w_2, w_3' \rangle = \sum_{k \in \mathbb{N}}\langle \mathcal{Y}(w_1)_{n,k}w_2, w_3' \rangle(\log z)^k$.

It follows that for any $L_{0,\mathrm{s}}$-eigenvalues $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2$ (and hence, for any non-necessarily eigenvalues), we have expansion

$$\mathcal{Y}(w_1, z)w_2 = \sum_{n \in \mathbb{C}}\sum_{k \in \mathbb{N}} \mathcal{Y}(w_1)_{n,k}w_2 \cdot z^{-n-1}(\log z)^k,$$
$$\mathcal{Y}(w_1)_{n,k}w_2 \in \mathcal{W}_3[\log z]\{z\}, \qquad\qquad (4.1.9)$$

and we have

$$[L_{0,\mathrm{s}}, \mathcal{Y}(w_1)_{n,k}] = \mathcal{Y}(L_{0,\mathrm{s}}w_1)_{n,k} - (n+1)\mathcal{Y}(w_1)_{n,k}. \qquad (4.1.10)$$

Consequently, if we assume that each $L_{0,\mathrm{s}}$-eigenspace of $\mathcal{W}_3$ is finite-dimensional, and that the real parts of the eigenvalues of $L_{0,\mathrm{s}}$ on $\mathcal{W}_3$ are bounded below, then we have the expansion (4.1.9), and $\mathcal{Y}$ satisfies the lower truncation property: $\mathcal{Y}(w_1)_{n,k}w_2 = 0$ for sufficiently large n. Thus, $\mathcal{Y}$ is an intertwining operator of the fixed point subalgebra $\mathbb{U}^G$ in the usual sense as [HLZ10]. Note that by [Miy04, Lemma 2.4], our assumption on $\mathcal{W}_3$ automatically holds when $\mathbb{U}^G$ is C_2-cofinite and $\mathcal{W}_3$ is $\mathbb{U}^G$-generated by finitely many vectors (equivalently, $\mathcal{W}_3$ is a grading-restricted generalized $\mathbb{U}^G$-module [Hua09, Cor. 3.16]).

The following property says that in order to show that $\mathcal{Y}$ is an intertwining operator, it suffices to verify the Jacobi identity for $\mathfrak{z}$ in a small region. Therefore, at least in the case that $\mathbb{U}^G$ is C_2-cofinite and the modules are $\mathbb{U}^G$-finitely generated, our definition of intertwining operators agrees with that in [McR21] (cf. the paragraph above Rem. 4.16).

Proposition 4.1.7. *Let $U \subset \mathbb{C}^\times$ be simply-connected with a continuous $\arg$ function $\arg_U$. Let $\mathcal{Y}$ be as in Def. 4.1.1, but satisfies the two conditions only when $\mathfrak{z} \in U$ and $\arg z = \arg_U(z)$ is defined by that of U. Then $\mathcal{Y}$ is a type $\binom{\mathcal{W}_3}{\mathcal{W}_1\mathcal{W}_2}$ intertwining operator.*

Proof. Note that (4.1.8) applies to Y^{g_j} since Y^{g_j} is an intertwining operator. By the argument in Rem. 4.1.2, (4.1.6) holds for all $\mathfrak{z} \in U, \arg(\mathfrak{z}) = \arg_U(\mathfrak{z})$. Hence, it holds for any $\mathfrak{z} \in \mathbb{C}^\times$ and any compatible $\arg \mathfrak{z}$. Thus, the same argument as in Prop. 4.1.5 proves (4.1.8) for $\mathcal{Y}$ and any w_1, w_2, w_3'.

Fix $\mathfrak{z}_0 \in U$ with $\arg \mathfrak{z}_0 = \arg_U(\mathfrak{z}_0)$. For each $\mathfrak{z} \in \mathbb{C}^\times$, write $\mathfrak{z} = \xi\mathfrak{z}_0$ and, in particular, $\arg \mathfrak{z} = \arg \xi + \arg \mathfrak{z}_0$. Choose $\arg(\xi^{-1}) = -\arg \xi$. Since $[L_0, \tilde{L}_0^g] = 0$, we see that

L_0 commutes with $P_n^{g_j}$. Therefore ξ^{-L_0} commutes with each $P_n^{g_j}$. Using this fact and (4.1.8), we obtain three equations about series of n:

$$\langle Y^{g_3}(u,z)\mathcal{Y}(w_1,\mathfrak{z})w_2, w_3'\rangle = \langle Y^{g_3}(\xi^{-L_0}u,\xi^{-1}z)\mathcal{Y}(\xi^{-L_0}w_1,\mathfrak{z}_0)\xi^{-L_0}w_2, \xi^{L_0}w_3'\rangle$$

$$\langle \mathcal{Y}(w_1,\mathfrak{z})Y^{g_2}(u,z)w_2, w_3'\rangle = \langle \mathcal{Y}(\xi^{-L_0}w_1,\mathfrak{z}_0)Y^{g_2}(\xi^{-L_0}u,\xi^{-1}z)\xi^{-L_0}w_2, \xi^{L_0}w_3'\rangle$$

$$\langle \mathcal{Y}(Y^{g_1}(u,z-\mathfrak{z})w_1,\mathfrak{z})w_2, w_3'\rangle = \langle \mathcal{Y}(Y^{g_1}(\xi^{-L_0}u,\xi^{-1}(z-\mathfrak{z}))\xi^{-L_0}w_1,\mathfrak{z}_0)\xi^{-L_0}w_2, \xi^{L_0}w_3'\rangle$$

which are defined in the similar way as (4.1.2), (4.1.3), (4.1.4). Since the Jacobi identity holds for $\mathfrak{z}_0$, we know that: (a) The above three series converges a.l.u. when $|\xi^{-1}z| > |\mathfrak{z}_0|, |\xi^{-1}z| < |\mathfrak{z}_0|, |\xi^{-1}(z-\mathfrak{z})| < |\mathfrak{z}_0|$ respectively. (b) If we assume that $\xi^{-1}z \in R_{\mathfrak{z}_0}$, that $\arg(\xi^{-1}z) = \arg z - \arg\xi$ equals $\arg\mathfrak{z}_0$, and that $\arg(\xi^{-1}z - \mathfrak{z}_0) := \arg(\xi^{-1}(z-\mathfrak{z})) = \arg(z-\mathfrak{z}) - \arg\xi$ equals either $\arg\mathfrak{z}_0$ or $\arg\mathfrak{z}_0 - \pi$, then the above three functions of $\xi^{-1}z$ can be extended to the same holomorphic function on $\Sigma_{\mathfrak{z}_0}$ which can furthermore be extended to a multi-valued holomorphic function on $\mathbb{C}^\times\backslash\{\mathfrak{z}_0\}$. The Jacobi identity for $\mathfrak{z}$ follows immediately. $\qquad\square$

We now relate twisted intertwining operators and twisted conformal blocks. Choose any $r > 0$. Then

$$\Sigma_r = \mathbb{C}\backslash\{\mathbf{i}t, r+\mathbf{i}t : t \geqslant 0\}.$$

Define a positively 3-pathed Riemann sphere with local coordinates

$$\mathfrak{P}_r = \{\mathbb{P}^1; 0, r, \infty; \zeta, \zeta - r, 1/\zeta; \gamma_0, \gamma_r, \gamma_\infty\}$$

where ζ is the standard coordinate of $\mathbb{C}$, $\gamma_0, \gamma_r, \gamma_\infty$ are paths in Σ_r with common end point $\star$, and their initial points are on the real line satisfying

$$0 < \gamma_0(0) < r < \gamma_r(0) < 2r < \gamma_\infty(0).$$

The equivalence classes of $(\gamma_0, \gamma_r, \gamma_\infty)$ (in the sense of Rem. 1.3.8) subject to such condition are clearly unique. Thus, the twisted conformal blocks and the branched coverings associated to $\mathfrak{P}_r$ are independent of the choice of such paths, thanks to Rem. 1.3.8 and 2.1.19.

Set $\mathbf{S} = \{x_1, x_2, x_3\}$ as usual. For $x = 0, r, \infty$, let ϵ_x be anticlockwise circle (defined by the give local coordinate at x) from and to $\gamma_x(0)$, and set $\alpha_x = \gamma_x^{-1}\epsilon_x\gamma_x$ (cf. (1.3.10)). Then $[\alpha_0], [\alpha_r], [\alpha_\infty]$ generate $\Gamma = \pi_1(\mathbb{P}^1\backslash\mathbf{S}; \star)$ (which is required in the definition of positively pathed Riemann spheres) since any two of these three elements are clearly free generators of Γ. Recall $g_3 = g_1 g_2$

Lemma 4.1.8. g_1, g_2, g_3^{-1} *are admissible, i.e., there is an action* $\Gamma \curvearrowright E$ *sending*

$$[\alpha_r] \mapsto g_1, \quad [\alpha_0] \mapsto g_2, \quad [\alpha_\infty] \mapsto g_3^{-1}.$$

Proof. We have $[\alpha_r\alpha_0] = [\alpha_\infty^{-1}]$, and $[\alpha_0], [\alpha_r]$ are free generators of Γ. $\qquad\square$

We shall also choose open discs W_0, W_r, W_∞, each of which contains only one of $0, r, \infty$. We set

$$W_0 = \{z \in \mathbb{C} : |z| < r\}, \quad W_r = \{z \in \mathbb{C} : |z - r| < r\}, \quad W_\infty = \{z \in \mathbb{P}^1 : |z| > r\}.$$

Proposition 4.1.9. *Let $\mathcal{Y}(\cdot, r)$ denote a linear map*

$$\mathcal{W}_1 \otimes \mathcal{W}_2 \to (\mathcal{W}_3')^*, \qquad w_1 \otimes w_2 \mapsto \mathcal{Y}(w_1, r)w_2,$$

also regarded as a linear functional on $\mathcal{W}_1 \otimes \mathcal{W}_2 \otimes \mathcal{W}_3$. Then the following two statements are equivalent.

1. *$\mathcal{Y}(\cdot, r)$ satisfies the Jacobi identity in Def. 4.1.1 (for the g_1-, g_2-, and g_3-twisted modules $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$) in the special case that $\mathfrak{z} = r$ and $\arg \mathfrak{z} = 0$.*

2. *$\mathcal{Y}(\cdot, r)$ is a conformal block associated to $\mathfrak{P}_r$ and the g_1-, g_2-, and g_3^{-1}-twisted modules $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3'$.*

Proof. Step 1. Each of the two statements consists of two parts: the "convergence part" and the "extension" part. Let us first verify the equivalence of the "convergence parts". This is obvious "near 0 and r". So we only need to focus on the convergence near ∞, namely, the equivalence of convergence of (4.1.2) and of (1.3.14) (when $j = 3$). On the side of (1.3.14), noting Rem. 1.3.6, we have the absolute convergence of the series of $z^{1/|g_3|}$

$$\langle \mathcal{Y}(w_1, r)w_2, Y^{g_3^{-1}}(u, z)w_3' \rangle := \sum_{n \in \frac{1}{|g_3|}\mathbb{N}} \langle \mathcal{Y}(w_1, r)w_2, Y^{g_3^{-1}}(u)_n w_3' \rangle z^{-n-1}$$

on $0 < |z^{1/|g_3|}| < |r^{-1/|g_3|}|$. This is equivalent, by linearity, to the convergence of the series of $z^{-1/|g_3|}$

$$\langle \mathcal{Y}(w_1, r)w_2, Y^{g_3^{-1}}(\mathcal{U}(\gamma_z)u, z^{-1})w_3' \rangle$$
$$:= \sum_{n \in \frac{1}{|g_3|}\mathbb{N}} \langle \mathcal{Y}(w_1, r)w_2, Y^{g_3^{-1}}(e^{zL_1}(-z^{-2})^{L_0}u)_n w_3' \rangle z^{n+1}$$

on $0 < |z^{-1/|g_3|}| < |r^{-1/|g_3|}|$, where $\mathcal{U}(\gamma_z)$ equals $e^{zL_1}(-z^{-2})^{L_0}$ on $\mathbb{U}$. By (1.3.7), the above series is equivalent to the convergence of the series of (4.1.2) (by setting $\mathfrak{z} = r$ with zero arg) on the same domain, and we have

$$\langle \mathcal{Y}(w_1, r)w_2, Y^{g_3^{-1}}(\mathcal{U}(\gamma_z)u, z^{-1})w_3' \rangle = \langle Y^{g_3}(u, z)\mathcal{Y}(w_1, r)w_2, w_3' \rangle \tag{4.1.11}$$

where the right hand side is defined by (4.1.2). This proves the equivalence of the "convergence" part.

Step 2. Recall that ζ is the standard coordinate of $\mathbb{C}$. We set $\mathfrak{U} = \{\mathcal{U}_\varrho(\zeta)^{-1}u : u \in \mathbb{U}\}$, which is a subspace of $\mathscr{U}_{\mathbb{P}^1}(\mathbb{P}^1 \backslash \mathbf{S})$ containing a subset that generates freely $\mathscr{U}_{\mathbb{P}^1 \backslash \mathbf{S}}$ (cf. the proof of Lemma. 1.3.10). Set $\psi = \mathcal{Y}(\cdot, r)$. We shall verify the equivalence of: (a) the two conditions of Prop. 1.3.11 (stated for the linear functional $\mathcal{Y}(\cdot, r)$), (b) the Jacobi identity in Def. 4.1.1.

Assume (b). Part 1 of Prop. 1.3.11 is already proved. For each $u \in \mathbb{U}, w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w_3' \in \mathcal{W}_3'$, set $w_\bullet = w_1 \otimes w_2 \otimes w_3'$, let $f_{r,u,w_\bullet}$ be the multivalued holomorphic function $f_r = f_{\mathfrak{z}}$ in the Jacobi identity of Def. 4.1.1, which becomes single-valued on any open simply-connected $U \subset \mathbb{P}^1 \backslash \mathbf{S}$ if we specify a path λ in $\mathbb{P}^1 \backslash \mathbf{S}$ from inside U to $\star$. We define this function on U to be $\wr\psi(\lambda, \mathcal{U}_\varrho(\zeta)^{-1}u, w_\bullet)\big|_U$. Moreover, this multivalued function

can be chosen such that when $U = \Sigma_r$ and λ is any path in Σ_r with initial point $\star$, then $\wr\psi(\lambda, \mathcal{U}_\varrho(\zeta)^{-1}u, w_\bullet)\big|_{\Sigma_r}$ agrees with the common (single-valued) holomorphic function on Σ_r mentioned in the Jacobi identity of Def. 4.1.1. It is now easy to check that condition 2 of Prop. 1.3.11 holds for W_j being W_0 or W_r (notice Rem. 1.3.5). As for W_∞, condition 2 also holds due to (4.1.11) and the fact that $\mathcal{U}_\varrho(\zeta)^{-1}u$ equals $\mathcal{U}(\gamma_z)u$ under the trivialization $\mathcal{U}_\varrho(\zeta^{-1})$ (cf. Ex. 1.1.1). The proof of (a) is complete.

Assume (a). Let $\wr\psi$ be as in Prop. 1.3.11. Then it is easy to check that $\wr\psi(\lambda, \mathcal{U}_\varrho(\zeta^{-1})u, w_\bullet)\big|_{\Sigma_r}$ restricts to (4.1.2), (4.1.3), (4.1.4) in the required regions. This verifies the Jacobi identity in Def. 4.1.1, hence proves (b). $\qquad\square$

We are now ready to prove the main result of this subsection. Let

$$\mathcal{I}\begin{pmatrix} \mathcal{W}_3 \\ \mathcal{W}_1 \mathcal{W}_3 \end{pmatrix} = \left\{ \text{Type} \begin{pmatrix} \mathcal{W}_3 \\ \mathcal{W}_1 \mathcal{W}_2 \end{pmatrix} \text{ intertwining operators of } \mathbb{U} \right\}.$$

Recall that $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ are g_1-, g_2-, g_3-twisted $\mathbb{U}$-modules. Let $\mathrm{CB}_{\mathfrak{P}_r}(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3')$ be the space of conformal blocks associated to $\mathfrak{P}_r$ and $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3'$. For each $r > 0$, define the restriction map

$$\nu_r : \mathcal{I}\begin{pmatrix} \mathcal{W}_3 \\ \mathcal{W}_1 \mathcal{W}_2 \end{pmatrix} \to \mathrm{CB}_{\mathfrak{P}_r}(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3'),$$
$$\mathcal{Y} \mapsto \mathcal{Y}(\cdot, r).$$

That the range of ν_r is inside $\mathrm{CB}_{\mathfrak{P}_r}(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3')$ is due to Prop. 4.1.9.

Theorem 4.1.10. *For each $r > 0$, the linear map ν_r is bijective.*

Proof. By (4.1.1) and Lemma 4.1.11, ν_r is injective. To prove that ν_r is surjective, we choose any $\mathcal{Y}(\cdot, r)$ in the codomain of ν_r, which satisfies Jacobi identify (in Def. 4.1.1) for $\mathfrak{z} = r$ and $\arg \mathfrak{z} = \arg r = 0$. For a general $\mathfrak{z} \in \mathbb{C}^\times$ with argument, we set

$$\langle \mathcal{Y}(w_1, \mathfrak{z})w_2, w_3' \rangle = \langle \mathcal{Y}(\xi^{-L_0}w_1, r)\xi^{-L_0}w_2, \xi^{L_0}w_3 \rangle$$

where $\xi = r^{-1}\mathfrak{z}$ and $\arg \xi = \arg \mathfrak{z}$. We may use the same method as in the proof of Prop. 4.1.7 to prove the Jacobi identity for all $\mathfrak{z}$. Finally, it is straightforward to check that (4.1.6) holds, which is equivalent to condition 1 of Def. 4.1.1 due to Rem. 4.1.2. This proves that $\mathcal{Y}$ is an intertwining operator whose value at r is the given $\mathcal{Y}(\cdot, r)$. $\qquad\square$

Lemma 4.1.11. *Let $\mathcal{V}$ be a vector space, and let $f(t_1, \ldots, t_N) = \sum_{n_1, \ldots, n_N \in \mathbb{N}} f_{n_1, \ldots, n_N} t_1^{n_1} \cdots t_N^{n_N}$ be a formal power series of $t_1, \ldots, t_N$ where $f_{n_1, \ldots, n_N} \in \mathcal{V}$. Suppose that for each i there is a linear operator A_i on $\mathcal{V}$ such that $\partial_{t_i} f = A_i f$. Then f is determined by $f(0, \ldots, 0) = f_{0, \ldots, 0}$.*

Proof. $\partial_{t_i} f = A_i f$ shows $f_{n_1, \ldots, n_i+1, \ldots, n_N} = (n_i + 1)^{-1}A f_{n_1, \ldots, n_i, \ldots, n_N}$. Therefore all $f_{n_1, \ldots, n_N}$ are determined by $f_{0, \ldots, 0}$. $\qquad\square$

4.2 OPE for permutation-twisted intertwining operators

Choose $0 < r_2 < r_1$ satisfying $r_1 - r_2 < r_2$. Set $\arg r_1 = \arg r_2 = \arg(r_2 - r_1) = 0$. Set

$$\Sigma_{r_1, r_2} = \mathbb{C}\backslash\{\mathbf{i}t, r_1 + \mathbf{i}t, r_2 + \mathbf{i}t : t \geqslant 0\}.$$

We can then define a positively 4-pathed Riemann sphere

$$\mathfrak{P}_{r_1,r_2} = (\mathbb{P}^1; 0, r_1, r_2, \infty; \zeta, \zeta - r_1, \zeta - r_2, \zeta^{-1}; \sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

where the four paths $\sigma_1, \ldots, \sigma_4$ with common end point are all inside Σ_{r_1,r_2}. Then, by replacing $\sigma_\bullet$ by equivalent paths (cf. 1.3.8), $\mathfrak{P}_{r_1,r_2}$ has the following two decompositions:

(a) $\mathfrak{P}_{r_1,r_2}$ is the sewing of $\mathfrak{P}_{r_1}$ along the marked point 0 (and its local coordinate ζ), and $\mathfrak{P}_{r_2}^{(a)} \simeq \mathfrak{P}_{r_2}$ along the marked point ∞ (with local coordinate ζ^{-1}). To perform this sewing, we choose $r_2 < \rho_2 < \rho_1 < r_1$, remove a small closed disk from $\{z \in \mathbb{C} : |z| < \rho_1\}$ inside $\mathfrak{P}_{r_1}$, remove and one from $\{z \in \mathbb{P}^1 : |z|^{-1} < \rho_2^{-1}\}$ inside $\mathfrak{P}_{r_2}^{(a)}$ (the two discs play the role of W_0, M_0 in Sec. 4.1.1), and glue the remaining part to obtain $\mathfrak{P}_{r_1,r_2}$. After gluing, any point $|z| \geqslant r_1$ of $\mathfrak{P}_{r_1}$ and any point $|z| \leqslant r_2$ of $\mathfrak{P}_{r_2}^{(b)}$ become the point z of $\mathfrak{P}_{r_1,r_2}$.

(b) $\mathfrak{P}_{r_1,r_2}$ is the sewing of $\mathfrak{P}_{r_2}^{(b)} \simeq \mathfrak{P}_{r_2}$ along the marked point r_2 and $\mathfrak{P}_{r_1-r_2}$ along the marked point ∞. Similar to (a), one removes closed discs from two open discs and glue the remaining part to get $\mathfrak{P}_{r_1,r_2}$. After gluing, any point $|z| \geqslant r_2$ of $\mathfrak{P}_{r_2}^{(b)}$ becomes z of $\mathfrak{P}_{r_1,r_2}$, and any point $|z| \leqslant r_1 - r_2$ of $\mathfrak{P}_{r_1-r_2}$ becomes the point $z + r_2$ of $\mathfrak{P}_{r_1,r_2}$.

Note that in both (a) and (b), we need to replace the paths in $\mathfrak{P}_{r_1}, \mathfrak{P}_{r_1-r_2}, \mathfrak{P}_{r_2}^{(a)}, \mathfrak{P}_{r_2}^{(b)}$ by equivalent paths such that (3.1.1) and (3.1.2) hold. We record the result

$$\mathfrak{P}_{r_1,r_2} = \mathfrak{P}_{r_1} \# \mathfrak{P}_{r_2}^{(a)} = \mathfrak{P}_{r_2}^{(b)} \# \mathfrak{P}_{r_1-r_2}. \tag{4.2.1}$$

Assume E is a finite set. Choose $g_1, g_2, g_3 \in \text{Perm}(E)$ and set $g_4 = (g_1 g_2 g_3)^{-1}$, and assign group elements to the marked points as follows

$$\begin{aligned} \mathfrak{P}_{r_1} : g_2 g_3 &\rightsquigarrow 0, & g_1 &\rightsquigarrow r_1, & (g_1 g_2 g_3)^{-1} &\rightsquigarrow \infty \\ \mathfrak{P}_{r_2}^{(a)} : g_3 &\rightsquigarrow 0, & g_2 &\rightsquigarrow r_2, & (g_2 g_3)^{-1} &\rightsquigarrow \infty \end{aligned} \tag{4.2.2}$$

and also

$$\begin{aligned} \mathfrak{P}_{r_2}^{(b)} : g_3 &\rightsquigarrow 0, & g_1 g_2 &\rightsquigarrow r_2, & (g_1 g_2 g_3)^{-1} &\rightsquigarrow \infty \\ \mathfrak{P}_{r_1-r_2} : g_2 &\rightsquigarrow 0, & g_1 &\rightsquigarrow r_1 - r_2, & (g_1 g_2)^{-1} &\rightsquigarrow \infty \end{aligned} \tag{4.2.3}$$

In each of the four cases, the group elements are admissible (cf. Def. 2.1.1). We can use either of the above two sets of data to define an action of $\Gamma_{r_1,r_2} = \pi_1(\mathbb{P}^1 \backslash \{0, r_1, r_2, \infty\}, \bigstar)$ on E (where $\bigstar$ is the common end point of the four paths of $\mathfrak{P}_{r_1,r_2}$) as in Subsec. 3.1.2, and the results are the same: let ϵ_j be the circle around $0, r_1, r_2, \infty$ respectively when $j = 1, 2, 3, 4$, then $[\sigma_j^{-1} \epsilon_j \sigma_j]$ acts as g_j. (We set $g_4 = (g_1 g_2 g_3)^{-1}$.)

Now, let $\mathfrak{X}_{r_1}, \mathfrak{X}_{r_2}^{(a)}, \mathfrak{X}_{r_2}^{(b)}, \mathfrak{X}_{r_1-r_2}$ be respectively the permutation branched coverings of $\mathfrak{P}_{r_1}, \mathfrak{P}_{r_2}^{(a)}, \mathfrak{P}_{r_2}^{(b)}, \mathfrak{P}_{r_1-r_2}$ and the fundamental group actions defined by (4.2.2) and (4.2.3). Let $\mathfrak{X}_{r_1,r_2}$ be the permutation branched covering of $\mathfrak{P}_{r_1,r_2}$ defined by the action described previously. Then, by Thm. 3.1.3,

$$\mathfrak{X}_{r_1,r_2} \simeq \mathfrak{X}_{r_1} \# \mathfrak{X}_{r_2}^{(a)} \simeq \mathfrak{X}_{r_2}^{(b)} \# \mathfrak{X}_{r_1-r_2} \tag{4.2.4}$$

where the two sewings are defined with respect the sewings of $\mathfrak{P}_{r_1}$ with $\mathfrak{P}_{r_2}^{(a)}$ and $\mathfrak{P}_{r_2}^{(b)}$ with $\mathfrak{P}_{r_1-r_2}$ (cf. Subsec. 3.1.4).

Now we assume $\mathbb{V}$ is CFT-type, C_2-cofinite, and rational. Let $\mathbb{U} = \mathbb{V}^{\otimes E}$. Associate semi-simple permutation twisted $\mathbb{U}$-modules ("semi-simple" means that it is a finite direct sum of irreducible twisted $\mathbb{U}$-modules) to marked points

$$
\begin{aligned}
\mathfrak{P}_{r_1} &: \mathcal{W}_\pi \rightsquigarrow 0, \quad \mathcal{W}_1 \rightsquigarrow r_1, \quad \mathcal{W}_4' \rightsquigarrow \infty \\
\mathfrak{P}_{r_2}^{(a)} &: \mathcal{W}_3 \rightsquigarrow 0, \quad \mathcal{W}_2 \rightsquigarrow r_2, \quad \mathcal{W}_\pi' \rightsquigarrow \infty
\end{aligned}
\tag{4.2.5}
$$

and also

$$
\begin{aligned}
\mathfrak{P}_{r_2}^{(b)} &: \mathcal{W}_3 \rightsquigarrow 0, \quad \mathcal{W}_\iota \rightsquigarrow r_2, \quad \mathcal{W}_4' \rightsquigarrow \infty \\
\mathfrak{P}_{r_1-r_2} &: \mathcal{W}_2 \rightsquigarrow 0, \quad \mathcal{W}_1 \rightsquigarrow r_1 - r_2, \quad \mathcal{W}_\iota' \rightsquigarrow \infty
\end{aligned}
\tag{4.2.6}
$$

whose types correspond to the group elements in (4.2.2), (4.2.3). By Thm. 2.2.4, all these twisted $\mathbb{U}$-modules arise from untwisted semi-simple $\mathbb{V}$-modules as described in Thm. 2.2.5. The following is the main result of this section. We do not assume $0 < r_1 - r_2 < r_2 < r_1$.

Theorem 4.2.1. *The following are true.*

1. *For any $\mathcal{Y}_\alpha \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_1 \mathcal{W}_\pi}$, $\mathcal{Y}_\beta \in \mathcal{I}\binom{\mathcal{W}_\pi}{\mathcal{W}_2 \mathcal{W}_3}$, $\mathcal{Y}_\gamma \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_\iota \mathcal{W}_3}$, $\mathcal{Y}_\delta \binom{\mathcal{W}_\iota}{\mathcal{W}_1 \mathcal{W}_2}$, and for any $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w_3 \in \mathcal{W}_3, w_4' \in \mathcal{W}_4$, the series*

$$
\begin{aligned}
&\langle \mathcal{Y}_\alpha(w_1, r_1) \mathcal{Y}_\beta(w_2, r_2) w_3, w_4' \rangle \\
&:= \sum_{n \in \frac{1}{|g_2 g_3|} \mathbb{N}} \langle \mathcal{Y}_\alpha(w_1, r_1) P_n^{g_2 g_3} \mathcal{Y}_\beta(w_2, r_2) w_3, w_4' \rangle
\end{aligned}
\tag{4.2.7}
$$

and

$$
\begin{aligned}
&\langle \mathcal{Y}_\gamma(\mathcal{Y}_\delta(w_1, r_1 - r_2) w_2, r_2) w_3, w_4' \rangle \\
&:= \sum_{n \in \frac{1}{|g_1 g_2|} \mathbb{N}} \langle \mathcal{Y}_\gamma(P_n^{g_1 g_2} \mathcal{Y}_\delta(w_1, r_1 - r_2) w_2, r_2) w_3, w_4' \rangle
\end{aligned}
\tag{4.2.8}
$$

converge absolutely on $I_1 = \{(r_1, r_2) : 0 < r_2 < r_1\}$ and $I_2 = \{(r_1, r_2) : 0 < r_1 - r_2 < r_1\}$ respectively. Moreover, if we vary r_1, r_2 and assume the $\arg$ of $r_1, r_2, r_1 - r_2$ are all 0, then these two functions are real analytic functions of r_1, r_2, namely, they can be extended to holomorphic functions on neighborhoods of I_1 and I_2 respectively.

2. *Let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4$ be semi-simple $g_1, g_2, g_3, g_1 g_2 g_3$-twisted $\mathbb{U} = \mathbb{V}^{\otimes E}$-modules. Then for each semi-simple $g_2 g_3$-twisted module $\mathcal{W}_\pi$ (resp. $g_1 g_2$-twisted module $\mathcal{W}_\iota$) and each $\mathcal{Y}_\alpha \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_1 \mathcal{W}_\pi}$, $\mathcal{Y}_\beta \in \mathcal{I}\binom{\mathcal{W}_\pi}{\mathcal{W}_2 \mathcal{W}_3}$ (resp. $\mathcal{Y}_\gamma \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_\iota \mathcal{W}_3}$, $\mathcal{Y}_\delta \binom{\mathcal{W}_\iota}{\mathcal{W}_1 \mathcal{W}_2}$), there exists a $g_1 g_2$-twisted module $\mathcal{W}_\iota$ (resp. $g_2 g_3$-twisted module $\mathcal{W}_\pi$) and $\mathcal{Y}_\gamma \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_\iota \mathcal{W}_3}$, $\mathcal{Y}_\delta\binom{\mathcal{W}_\iota}{\mathcal{W}_1 \mathcal{W}_2}$ (resp. $\mathcal{Y}_\alpha \in \mathcal{I}\binom{\mathcal{W}_4}{\mathcal{W}_1 \mathcal{W}_\pi}$, $\mathcal{Y}_\beta \in \mathcal{I}\binom{\mathcal{W}_\pi}{\mathcal{W}_2 \mathcal{W}_3}$)) such that for any $w_1 \in \mathcal{W}_1, w_2 \in \mathcal{W}_2, w_3 \in \mathcal{W}_3, w_4' \in \mathcal{W}_4$, (4.2.7) and (4.2.8) agree on $I_1 \cap I_2$, assuming the $\arg$ of $r_1, r_2, r_1 - r_2$ are all 0.*

Note that if $g_1, g_2, g_3 \in G \leqslant \mathrm{Perm}(E)$ where G is solvable, $\mathbb{U}^G$ is C_2-cofinite and rational by [Miy15, CM16]. Then Thm. 4.2.1 follows from [McR21]. The C_2-cofiniteness of $\mathbb{U}^G$ is conjectured to be true for any finite group G. If this were proved, then [McR21] would imply that $\mathbb{U}^G$ is also rational. Then Thm. 4.2.1 would also follow from [McR21]. Here, we provide a proof without knowing $\mathbb{U}^G$ to be C_2-cofinite.

Proof. (4.2.7) and (4.2.8) are the sewing of permutation-twisted conformal blocks corresponding to the geometric sewing (4.2.1). Therefore, by Thm. 3.2.1, when r_1, r_2 are in I_1 or I_2 respectively, (4.2.7) or (4.2.8) converges absolutely to a conformal block associated to $\mathfrak{P}_{r_1, r_2}$ and the twisted modules $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4'$. Moreover, due to Thm. 3.2.3, for fixed r_1, r_2, any such conformal block can be expressed either as (4.2.7) or (4.2.8). Therefore, statement 2 holds for any fixed $(r_1, r_2) \in I_1 \cap I_2$.

Note that L_0 commutes with P_n^g (since $[L_0, \widetilde{L}_0^g] = 0$). Choose $\rho \in (0, 1)$. Then by (4.1.8), the series (4.2.7) equals $\sum_{n \in \frac{1}{|g_2 g_3|} \mathbb{N}} f_n(r_1, r_2)$ where

$$f_n(r_1, r_2) = \langle \mathcal{Y}_\alpha(r_1^{-L_0} w_1, 1) \Big(\frac{r_2}{r_1 \rho}\Big)^{L_0} P_n^{g_2 g_3} \mathcal{Y}_\beta((\rho/r_2)^{L_0} w_2, \rho)(\rho/r_2)^{L_0} w_3, r_1^{L_0} w_4' \rangle.$$

To prove the analyticity of (4.2.7), it suffices to restrict to each simple submodule of $\mathcal{W}_\pi$ such that on this submodule L_0 and $\widetilde{L}_0^{g_2 g_3}$ differ by a constant λ. It follows that $\Big(\frac{r_2}{r_1 \rho}\Big)^{L_0} P_n^{g_2 g_3} = \Big(\frac{r_2}{r_1 \rho}\Big)^{n+\lambda} P_n^{g_2 g_3}$. It is clear that for each (r_1, r_2) in a compact subset K of $O_1 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < |z_1|, \mathrm{Re}(z_1) > 0, \mathrm{Re}(z_2) > 0\}$, one can choose ρ greater than every r_2/r_1. Hence, by the absolute convergence proved in the first paragraph, we have $\sup_{(r_1, r_2) \in K} \sum_n |f_n(r_1, r_2)| < +\infty$. Therefore, as each $f_n(r_1, r_2)$ is analytic over r_1, r_2, the sum of f_n, which is just (4.2.7), must be analytic on O_1 and hence analytic on I_1. The same method proves that (4.2.8) is analytic on I_2.

Finally, we assume that (4.2.7) equals (4.2.8) for one $(r_1, r_2) \in I_1 \cap I_2$, and show that they are equal for all $(r_1 + t_1, r_2 + t_2) \in I_1 \cap I_2$. By (4.1.1) (the L_{-1}-derivative) and (4.1.7) (applied to $\mathcal{Y}_\delta$), if we write (4.2.7) (resp. (4.2.8)) as $f_i(t_1, t_2, w_1 \otimes w_2 \otimes w_3 \otimes w_4')$ where $i = 1$ (resp. $i = 2$), then

$$\partial_{t_1} f_i(t_1, t_2, w_1 \otimes w_2 \otimes w_3 \otimes w_4') = f_i(t_1, t_2, L_{-1} w_1 \otimes w_2 \otimes w_3 \otimes w_4'),$$
$$\partial_{t_2} f_i(t_1, t_2, w_1 \otimes w_2 \otimes w_3 \otimes w_4') = f_i(t_1, t_2, w_1 \otimes L_{-1} w_2 \otimes w_3 \otimes w_4').$$

The proof is thus finished by taking power series expansions of t_1, t_2 and applying Lemma 4.1.11. $\qquad \square$

Remark 4.2.2. By the Main Theorem 2.2.5, $\mathcal{Y}_\alpha, \mathcal{Y}_\beta, \mathcal{Y}_\gamma, \mathcal{Y}_\delta$ can be viewed as untwisted conformal blocks associated to $\mathfrak{X}_{r_1}, \mathfrak{X}_{r_2}^{(a)}, \mathfrak{X}_{r_2}^{(b)}, \mathfrak{X}_{r_1 - r_2}$ (and suitable $\mathbb{V}$-modules) respectively. Thus, Thm. 4.2.1 can be viewed as the equivalence of sewing untwisted conformal blocks for $\mathbb{V}$-modules associated to the two geometric sewing procedures described in (4.2.4). Namely, Thm. 4.2.1, which describes the operator product expansion (OPE) of permutation-twisted intertwining operators, describes equivalently the OPE of certain untwisted conformal blocks associated to permutation coverings of $\mathbb{P}^1$. Therefore, it relates the associativity isomorphism for tensor products in the crossed braided fusion category of $\mathrm{Perm}(E)$-twisted $\mathbb{V}^{\otimes E}$-modules and the OPE of untwisted $\mathbb{V}$-conformal blocks associated to (possibly non-zero genera) compact Riemann surfaces. See Figure 0.2.2 in the Introduction.

Index

References

[BDH17] Bartels, A., Douglas, C.L. and Henriques, A., 2017. Conformal nets II: Conformal blocks. Communications in Mathematical Physics, 354(1), pp.393-458.

[BDM02] Barron, K., Dong, C. and Mason, G., 2002. Twisted sectors for tensor product vertex operator algebras associated to permutation groups. Communications in mathematical physics, 227(2), pp.349-384.

[BHS98] Borisov, L., Halpern, M.B. and Schweigert, C., 1998. Systematic approach to cyclic orbifolds. International Journal of Modern Physics A, 13(01), pp.125-168.

[BJ19] Bischoff, M. and Jones, C., 2019. Computing fusion rules for spherical G-extensions of fusion categories. arXiv preprint arXiv:1909.02816.

[BKL15] Bischoff, M., Kawahigashi, Y. and Longo, R., 2015. Characterization of 2D rational local conformal nets and its boundary conditions: the maximal case. Documenta Math. 20 (2015) 1137-1184.

[BS11] Barmeier, T. and Schweigert, C., 2011. A geometric construction for permutation equivariant categories from modular functors. Transformation groups, 16(2), pp.287-337.

[Ban98] Bantay, P., 1998. Characters and modular properties of permutation orbifolds. Physics Letters B, 419(1-4), pp.175-178.

[Ban02] Bantay, P., 2002. Permutation orbifolds. Nuclear Physics B, 633(3), pp.365-378.

[Bis20] Bischoff, M., 2020. A remark about the anomalies of cyclic holomorphic permutation orbifolds. International Journal of Mathematics, 31(10), p.2050080.

[CKLW18] Carpi, S., Kawahigashi, Y., Longo, R. and Weiner, M., 2018. From vertex operator algebras to conformal nets and back (Vol. 254, No. 1213). Memoirs of the American Mathematical Society

[DGT19a] Damiolini, C., Gibney, A., and Tarasca, N., 2019. Conformal blocks from vertex algebras and their connections on $\overline{\mathcal{M}}_{g,n}$. To appear in Geometry & Topology, arXiv:1901.06981.

[DGT19b] Damiolini, C., Gibney, A. and Tarasca, N., 2019. On factorization and vector bundles of conformal blocks from vertex algebras. arXiv preprint arXiv:1909.04683.

[DLXY19] Dong, C., Li, H., Xu, F. and Yu, N., 2019. Fusion products of twisted modules in permutation orbifolds. arXiv preprint arXiv:1907.00094.

[DLM97] Dong, C., Li, H. and Mason, G., 1997. Regularity of Rational Vertex Operator Algebras. Advances in Mathematics, 132(1), pp.148-166.

[CM16] Carnahan, S. and Miyamoto, M., 2016. Regularity of fixed-point vertex operator subalgebras. arXiv preprint arXiv:1603.05645.

[DXY21] Dong, C., Xu, F. and Yu, N., 2021. Permutation orbifolds and associative algebras. Science China Mathematics, pp.1-10.

[Del19] Delaney, C., 2019. Fusion rules for permutation extensions of modular tensor categories. arXiv preprint arXiv:1909.03003.

[Don] Donaldson, S., 2011. Riemann surfaces. Oxford University Press.

[EG18] Evans, D.E. and Gannon, T., 2018. Reconstruction and local extensions for twisted group doubles, and permutation orbifolds. arXiv preprint arXiv:1804.11145.

[ENO10] Etingof, P., Nikshych, D. and Ostrik, V., 2010. Fusion categories and homotopy theory. Quantum topology, 1(3), pp.209-273.

[FB04] Frenkel, E. and Ben-Zvi, D., 2004. Vertex algebras and algebraic curves (No. 88). American Mathematical Soc..

[FHL93] Frenkel, I., Huang, Y. Z., & Lepowsky, J. (1993). On axiomatic approaches to vertex operator algebras and modules (Vol. 494). American Mathematical Soc..

[FRS89] Fredenhagen, K., Rehren, K.H. and Schroer, B., 1989. Superselection sectors with braid group statistics and exchange algebras. Communications in Mathematical Physics, 125(2), pp.201-226.

[FRS92] Fredenhagen, K., Rehren, K.H. and Schroer, B., 1992. Superselection sectors with braid group

statistics and exchange algebras II: Geometric aspects and conformal covariance. Reviews in Mathematical Physics, 4(spec01), pp.113-157.

[FS04]   Frenkel, E. and Szczesny, M., 2004. Twisted modules over vertex algebras on algebraic curves. Advances in Mathematics, 187(1), pp.195-227.

[Fio16]   Fiordalisi, F., 2016. Logarithmic intertwining operators and genus-one correlation functions. Communications in Contemporary Mathematics, 18(06), p.1650026.

[For]   Forster, O., 2012. Lectures on Riemann surfaces (Vol. 81). Springer Science & Business Media.

[Ful]   Fulton, W., 2013. Algebraic topology: a first course (Vol. 153). Springer Science & Business Media.

[Gui20a]   Gui, B., 2020. Unbounded Field Operators in Categorical Extensions of Conformal Nets, arXiv:2001.03095

[Gui20b]   Gui, B., 2020. Convergence of Sewing Conformal Blocks. arXiv preprint arXiv:2011.07450.

[Gui21a]   Gui, B., 2021. Categorical extensions of conformal nets. Comm. Math. Phys. 383, 763-839 (2021).

[Gui21b]   Gui, B., 2021. Sewing and Propagation of Conformal Blocks. arXiv: 2110.04774.

[HLZ10]   Huang, Y.Z., Lepowsky, J. and Zhang, L., 2010. Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators. arXiv preprint arXiv:1012.4196.

[Hen14]   Henriques, A., 2014. Course notes for Conformal field theory (math 290).

[Hua95]   Huang, Y.Z., 1995. A theory of tensor products for module categories for a vertex operator algebra, IV. Journal of Pure and Applied Algebra, 100(1-3), pp.173-216.

[Hua97]   Huang, Y.Z., 1997. Two-dimensional conformal geometry and vertex operator algebras (Vol. 148). Springer Science & Business Media.

[Hua05]   Huang, Y.Z., 2005. Differential equations, duality and modular invariance. Communications in Contemporary Mathematics, 7(05), pp.649-706.

[Hua08a]   Huang, Y.Z., 2008. Vertex operator algebras and the Verlinde conjecture. Communications in Contemporary Mathematics, 10(01), pp.103-154.

[Hua08b]   Huang, Y.Z., 2008. Rigidity and modularity of vertex tensor categories. Communications in contemporary mathematics, 10(supp01), pp.871-911.

[Hua09]   Huang, Y.Z., 2009. Cofiniteness conditions, projective covers and the logarithmic tensor product theory. Journal of Pure and Applied Algebra, 213(4), pp.458-475.

[Hua18]   Huang, Y.Z., 2018. Intertwining operators among twisted modules associated to not-necessarily-commuting automorphisms. Journal of Algebra, 493, pp.346-380.

[KLM01]   Kawahigashi, Y., Longo, R. and Müger, M., 2001. Multi-Interval Subfactors and Modularity of Representations in Conformal Field Theory. Communications in Mathematical Physics, 219(3), pp.631-669.

[KLX05]   Kac, V.G., Longo, R. and Xu, F., 2005. Solitons in affine and permutation orbifolds. Communications in mathematical physics, 253(3), pp.723-764.

[KR09a]   Kong, L. and Runkel, I., 2009. Cardy algebras and sewing constraints, I. Communications in Mathematical Physics, 292(3), pp.871-912.

[KR09b]   Kong, L. and Runkel, I., 2009. Algebraic structures in Euclidean and Minkowskian two-dimensional conformal field theory. arXiv preprint arXiv:0902.3829.

[Kong08]   Kong, L., 2008. Cardy condition for open-closed field algebras. Communications in mathematical physics, 283(1), pp.25-92.

[LX04]   Longo, R. and Xu, F., 2004. Topological sectors and a dichotomy in conformal field theory. Communications in mathematical physics, 251(2), pp.321-364.

[LX19]   Liu, Z. and Xu, F., 2019. Jones-Wassermann subfactors for modular tensor categories. Advances in Mathematics, 355, p.106775.

[McR21]   McRae, R., 2021. Twisted modules and G-equivariantization in logarithmic conformal field theory. Communications in Mathematical Physics, 383(3), pp.1939-2019.

[Miy04]   Miyamoto, M., 2004. Modular invariance of vertex operator algebras satisfying C2-

cofiniteness. Duke Mathematical Journal, 122(1), pp.51-91.

[Miy15]   Miyamoto, M., 2015. C_2-cofiniteness of Cyclic-Orbifold Models. Communications in Mathematical Physics, 335(3), pp.1279-1286.

[Ten19a]   Tener, J.E., 2019. Geometric realization of algebraic conformal field theories. Advances in Mathematics, 349, pp.488-563.

[Ten19b]   Tener, J.E., 2019. Representation theory in chiral conformal field theory: from fields to observables. Selecta Math. (N.S.) (2019), 25:76

[Ten19c]   Tener, J.E., 2019. Fusion and positivity in chiral conformal field theory. arXiv preprint arXiv:1910.08257.

[Was94]   Wassermann, A.J., 1995. Operator algebras and conformal field theory. In Proceedings of the International Congress of Mathematicians (pp. 966-979). Birkhäuser, Basel.

[Was98]   Wassermann, A., 1998. Operator algebras and conformal field theory III. Fusion of positive energy representations of $LSU(N)$ using bounded operators. Inventiones mathematicae, 133(3), pp.467-538.

[Zhu96]   Zhu, Y., 1996. Modular invariance of characters of vertex operator algebras. Journal of the American Mathematical Society, 9(1), pp.237-302.

YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA.

E-mail: binguimath@gmail.com   bingui@tsinghua.edu.cn