A Appendix: basic sheaf theory

The language of sheaves of modules is inevitable in the theory of conformal blocks for the following reason. The spaces of conformal blocks are expected to form a vector bundle (equivalently, locally free sheaves). This result is highly nontrivial. Moreover, we need to formulate the notion of "forming a vector bundle" in a precise way. To accomplish this goal, we need to expand the concept of vector bundles to that of sheaves of modules.

The goal of this appendix section is to get familiar with the basic language of sheaves. The key points are the following: The equivalence of holomorphic vector bundles and locally free sheaves, the description of dual vector bundles in terms of $\mathscr{O}_X$-module morphisms, the fibers of $\mathscr{O}_X$-modules and their relationship to the fibers of vector bundles.

A.1 (Pre)sheaves and stalks

By definition, a **presheaf** of (complex) vector spaces $\mathscr{F}$ associated to a topological space X consists of the following data: for each open $U \subset X$ there is a vector space $\mathscr{F}(U)$, and for each open $V \subset U$, there is a linear map $\mathscr{F}(U) \to \mathscr{F}(V), s \mapsto s|_V$ called the **restriction map** such that $s|_U = s$, and $(s|_V)|_W = s|_W$ for all $s \in \mathscr{F}(U)$ if $W \subset V$ is open. Elements in $\mathscr{F}(U)$ are called **sections**.

A presheaf $\mathscr{F}$ is called a **sheaf** if it satisfies:

- (Locality) If $U \subset X$ is a union $U = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$ of open subsets, and if $s \in \mathscr{F}(U)$ satisfies that $s|_{U_\alpha} = 0$ for each $\alpha \in \mathfrak{A}$, then $s = 0$.

- (Gluing) If $U \subset X$ is a union $U = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$ of open subsets, and if for each α there is an element $s_\alpha \in \mathscr{F}(U_\alpha)$ such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in \mathfrak{A}$, then there exists $s \in \mathscr{F}(U)$ whose restriction to each U_α is s_α.

We also write

$$H^0(X, \mathscr{F}) = \mathscr{F}(X), \tag{A.1}$$

regarding the space of global sections of $\mathscr{F}$ as the 0-th cohomology group of $\mathscr{F}$.

If Y is an open subset of X, then the set of all $\mathscr{F}(U)$ (where $U \subset Y$) form naturally a presheaf, which we denote by $\mathscr{F}_Y$ or $\mathscr{F}|_Y$.

Let $\mathscr{F}$ be a presheaf. For each $x \in X$, we let $\mathscr{F}_x$ be the set of all sections $s \in \mathscr{F}$ defined on a neighborhood of x, mod the equivalence relation that two elements s, t of $\mathscr{F}_x$ are regarded as equal iff s equals t on a possibly smaller neighborhood of x inside the open sets on which s, t are defined. $\mathscr{F}_x$ is called the **stalk** of $\mathscr{F}$ at x, and elements in $\mathscr{F}_x$ are called **germs**. For each $s \in \mathscr{F}$ defined near x, the corresponding germ at x is denoted by s_x.

Remark A.1. It is easy to see that the presheaf $\mathscr{F}$ satisfies locality iff the following holds: for every open $U \subset X$ and section $s \in \mathscr{F}(U)$, $s = 0$ iff $s_x = 0$ for all $x \in U$.

A.2 Sheafification

We are not interested in presheaves that are not sheaves. And each presheaf $\mathscr{F}_0$ can be made a sheaf $\mathscr{F}$ through the following procedure called **sheafification**:

For each open $U \subset X$, let $\mathscr{F}_1(U)$ be the set of all $s := (s_\alpha)_{\alpha\in\mathfrak{A}}$ where $(U_\alpha)_{\alpha\in\mathfrak{A}}$ is an open cover of U, and $s_{\alpha_1,x} = s_{\alpha_2,x}$ for all $\alpha_1, \alpha_2 \in \mathfrak{A}$ and $x \in U_{\alpha_1} \cap U_{\alpha_2}$. $\mathscr{F}(U)$ is $\mathscr{F}_1(U)$ mod the following relation: let $(V_\beta)_{\beta\in\mathfrak{B}}$ be another open cover. Then $s := (s_\alpha)_{\alpha\in\mathfrak{A}}$ and $t := (t_\beta)_{\beta\in\mathfrak{B}}$ are regarded equal iff $s_{\alpha,x} = t_{\beta,x}$ for all $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$, and $x \in U_\alpha \cap V_\beta$. The linear combinations of s and t can be defined easily by replacing $(U_\alpha)_{\alpha\in\mathfrak{A}}$ and $(V_\beta)_{\beta\in\mathfrak{B}}$ by a common finer open cover, e.g. $(U_\alpha \cap V_\beta)_{\alpha\in\mathfrak{A},\beta\in\mathfrak{B}}$.

Note that the stalk $(\mathscr{F}_0)_x$ can be naturally identified with that of the sheafification $\mathscr{F}_x$.

A.3 (Pre)sheaves of modules and morphisms

We now let X be a complex manifold. Then all $\mathscr{O}(U)$ (where $U \subset X$ is open) form the sheaf $\mathscr{O}_X$ of holomorphic functions on X, called the **structure sheaf** of X.

Example A.2. Let $U \subset \mathbb{C}^m$ be open. Then the stalk $\mathscr{O}_{U,0} = \mathscr{O}_{\mathbb{C}^m,0}$ can be identified with the $\mathbb{C}$-subalgebra of elements of $\mathbb{C}[[z_1, \dots, z_m]]$ converging absolutely on an open ball centered at 0.

A **(pre)sheaf of $\mathscr{O}_X$-modules** $\mathscr{F}$ is a (pre)sheaf such that each $\mathscr{F}(U)$ is an $\mathscr{O}(U)$-module, and that for each open $V \subset U$, the restriction map $s \in \mathscr{F}(U) \mapsto s|_V \in \mathscr{F}(V)$ intertwines the actions of $\mathscr{O}(U)$, i.e., $(fs)|_V = f|_V \cdot s|_V$ for all $f \in \mathscr{O}(U)$. A sheaf of $\mathscr{O}_X$-modules is simply called an **$\mathscr{O}_X$-module**.

A **morphism of (resp. presheaves of) $\mathscr{O}_X$-modules** $\varphi : \mathscr{E} \to \mathscr{F}$ gives each open $U \subset X$ an $\mathscr{O}(U)$-module morphism $\varphi_U : \mathscr{E}(U) \to \mathscr{F}(U)$ that is compatible with the restriction to open subsets: if $V \subset U$ is open and $s \in \mathscr{E}(U)$ then $\varphi_U(s)|_V = \varphi_V(s|_V)$.

Convention A.3. We abbreviate each $\varphi_U(s)$ to $\varphi(s)$. So $\varphi(s|_V) = \varphi(s)|_V$.

Remark A.4. Note that the stalk $\mathscr{O}_{X,x}$ of $\mathscr{O}_X$ at x is a $\mathbb{C}$-algebra. A morphism $\varphi : \mathscr{E} \to \mathscr{F}$ naturally gives an $\mathscr{O}_{X,x}$-module morphism $\varphi_x : \mathscr{E}_x \to \mathscr{F}_x$.

Also, there is a natural $\mathscr{O}_X$-module morphism $\mathscr{E}^s \to \mathscr{F}^s$ where $\mathscr{E}^s$ and $\mathscr{F}^s$ are the sheafifications of $\mathscr{E}$ and $\mathscr{F}$. The corresponding stalk morphism $\varphi_x : \mathscr{E}^s_x \to \mathscr{F}^s_x$ agrees with $\varphi_x : \mathscr{E}_x \to \mathscr{F}_x$.

Example A.5. Any (holomorphic) vector bundle $\mathscr{F}$ [1] over X is an $\mathscr{O}_X$-module.

Example A.6. If W is a finite dimensional vector space, let $W \otimes_{\mathbb{C}} \mathscr{O}_X$ be the presheaf whose space of sections on each open $U \subset X$ is $W \otimes_{\mathbb{C}} \mathscr{O}(U)$. Then $W \otimes_{\mathbb{C}} \mathscr{O}_X$ is naturally a sheaf, and hence an $\mathscr{O}_X$-module. It is regarded as the trivial vector bundle with fiber W. We often suppress the subscript $\mathbb{C}$ in $W \otimes_{\mathbb{C}} \mathscr{O}_X$.

When W is infinite-dimensional, the above defined presheaf is not a sheaf since the gluing property does not hold when considering an open subset $U \subset X$ that has infinitely many connected components. We let $W \otimes_{\mathbb{C}} \mathscr{O}_X$ denote the sheafification of

[1] Unless otherwise stated, all vector bundles are holomorphic with finite ranks.

this presheaf. Then $(W \otimes_{\mathbb{C}} \mathscr{O}_X)(U)$ equals $W \otimes \mathscr{O}(U)$ if U is connected, or more generally, iff U has finitely many connected components. Thus, we have a natural isomorphism of $\mathscr{O}_{X,x}$-modules

$$(W \otimes \mathscr{O}_X)_x \simeq W \otimes \mathscr{O}_{X,x}.$$

Note that when U is connected, elements of $W \otimes \mathscr{O}(U)$ can be viewed as holomorphic functions from U to a finite-dimensional subspace of W. We shall call such sections **W-valued holomorphic functions**. □

Convention A.7. The space of $\mathscr{O}_X$-module morphisms $\varphi : \mathscr{E} \to \mathscr{F}$ form a vector space, which is clearly an $\mathscr{O}(X)$-module such that $f \in \mathscr{O}(X)$ times φ is $f\varphi$, sending each $s \in \mathscr{E}(U)$ (where $U \subset X$ is open) to $f|_U \cdot \varphi(s)$. We denote this $\mathscr{O}(X)$-module by $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.

Example A.8. Let V, W be finite dimensional vector spaces. A morphism

$$\varphi : V \otimes \mathscr{O}_X \to W \otimes \mathscr{O}_X$$

is equivalently a $\mathrm{Hom}(V, W)$-valued holomorphic function Φ on X. Indeed, choose basis $\{e_i\}$ of V and $\{f_j\}$ of W^*. Identify each vector of W as a constant section of $W \otimes \mathscr{O}(X)$. Then $\varphi(e_i) \in W \otimes \mathscr{O}(X)$, and Φ is a matrix-valued holomorphic function whose (j, i)-component is the function $x \mapsto \langle f_j, \varphi(e_i)(x)\rangle$.

To summarize, we have a canonical isomorphism of $\mathscr{O}(X)$-modules

$$\mathrm{Hom}_{\mathscr{O}_X}(V \otimes \mathscr{O}_X, W \otimes \mathscr{O}_X) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W) \otimes \mathscr{O}(X).$$

□

A.4 Injectivity, surjectivity, isomorphisms

An $\mathscr{O}_X$-module morphism $\varphi : \mathscr{E} \to \mathscr{F}$ is called **injective** resp. **surjective** if for each $x \in X$ the corresponding stalk morphism $\varphi_x : \mathscr{E}_x \to \mathscr{F}_x$ is injective resp. surjective.

Exercise A.9. Show that φ is injective iff $\varphi : \mathscr{E}(U) \to \mathscr{F}(U)$ is injective for all open $U \subset X$. Show that φ is surjective iff for each $x \in X$ and each section $t \in \mathscr{F}$ on a neighborhood U of X, by shrinking U to a smaller neighborhood $V \ni x$, we can find $s \in \mathscr{E}(V)$ such that $\varphi(s) = t$ when restricted to V.

(Warning: surjectivity does not mean that each x is contained in a neighborhood U such that $\varphi : \mathscr{E}(U) \to \mathscr{F}(U)$ is surjective. Thus, surjectivity of sheaves is defined both locally and sectionwisely!)

Remark A.10. Let $\mathscr{E}, \mathscr{F}$ be presheaves of $\mathscr{O}_X$-modules. Suppose that each $\mathscr{E}(U)$ is an $\mathscr{O}(U)$-submodule of $\mathscr{F}(U)$, and the inclusion maps $\mathscr{E}(U) \hookrightarrow \mathscr{F}(U)$ are compatible with the restriction maps of sheaves. Then there is a natural morphism $\iota : \mathscr{E} \to \mathscr{F}$ such that ι_U is the inclusion $\mathscr{E}(U) \hookrightarrow \mathscr{F}(U)$. We say that $\mathscr{E}$ is a **sub-presheaf of $\mathscr{O}_X$-modules** of $\mathscr{F}$. If both $\mathscr{E}, \mathscr{F}$ are sheaves, we say $\mathscr{E}$ is an **$\mathscr{O}_X$-submodule** of $\mathscr{F}$.

Now suppose $\mathscr{F}$ is an $\mathscr{O}_X$-modules and $\mathscr{E}$ is a sub-presheaf of $\mathscr{O}_X$-modules of $\mathscr{F}$. Then the sheafification of $\mathscr{E}$ can be viewed as an $\mathscr{O}_X$-submodule of $\mathscr{F}$. Its spaces of sections are all $s \in \mathscr{F}(U)$ such that $s_x \in \mathscr{E}_x$ for every $x \in U$. □

We say that an $\mathscr{O}_X$-module morphism $\varphi : \mathscr{E} \to \mathscr{F}$ is an **isomorphism of $\mathscr{O}_X$-modules** if it is bijective (i.e. injective+surjective).

Exercise A.11. Show that φ is an isomorphism if and only if for each open subset $U \subset X$, $\varphi_U : \mathscr{E}(U) \to \mathscr{F}(U)$ is an isomorphism of $\mathscr{O}(U)$-modules. (Indeed, the only nontrivial part is to show that φ being an isomorphism implies the surjectivity of φ_U. Surprisingly, to prove this part we need the injectivity!)

A.5 Kernals, cokernels, images

Let $\varphi : \mathscr{E} \to \mathscr{F}$ be an $\mathscr{O}_X$-module morphism. The **kernel** $\mathrm{Ker}(\varphi)$ is the presheaf whose space of sections on any open subset U is the kernel of $\varphi : \mathscr{E}(U) \to \mathscr{F}$. It is clear that $\mathrm{Ker}(\varphi)$ is a sheaf and is an $\mathscr{O}_X$-module. Clearly $\mathrm{Ker}(\varphi)_x$ is the kernel of the stalk map $\varphi : \mathscr{E}_x \to \mathscr{F}_x$.

The **image** $\varphi(\mathscr{E}) = \mathrm{Im}(\varphi)$ is the sheafification of the presheaf whose space of sections on each U is $\varphi(\mathscr{E}(U))$.

The **cokeral** $\mathrm{Coker}(\varphi)$ is the sheafification of the presheaf whose space of sections on each U is $\mathscr{F}(U)/\varphi(\mathscr{E}(U))$. Equivalently, $\mathrm{Coker}(\varphi)$ is the sheafification of the presheaf whose space of sections on each U is $\mathscr{F}(U)/\varphi(\mathscr{E})(U)$. Thus, we also say that $\mathrm{Coker}(\varphi)$ is the **quotient** of the sheaves $\mathscr{F}$ and $\varphi(\mathscr{E})$, and write

$$\mathscr{F}/\varphi(\mathscr{E}) = \mathrm{coker}(\varphi). \tag{A.2}$$

Remark A.12. Show that we have natural equivalences

$$\varphi(\mathscr{E})_x \simeq \varphi(\mathscr{E}_x), \tag{A.3}$$
$$\mathrm{Coker}(\varphi)_x \simeq \mathscr{F}_x/\varphi(\mathscr{E}_x). \tag{A.4}$$

A.6 Locally free sheaves

Let I be an index set. Let $\mathbb{C}^I$ be the direct sum of $|I|$ copies of $\mathbb{C}$ indexed by elements of I. Then $\mathbb{C}^I$ has basis $\{e_i\}_{i\in I}$ where e_i is the vector whose only non-zero component is the i-th one, which is 1.

Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. A collection of sections $(s_i)_{i\in I} \subset \mathscr{E}(X)$ is said to **generate** (resp. **generate freely**) $\mathscr{E}$ if the natural $\mathscr{O}_X$-module $\psi : \mathbb{C}^I \otimes \mathscr{O}_X \to \mathscr{E}$ sending each e_i (regarded as a constant section $e_i \otimes 1$) to s_i is surjective (resp. bijective).

Equivalently, $(s_i)_{i\in I}$ generates (resp. generates freely) $\mathscr{E}$ iff for each $x \in X$, each $t \in \mathscr{E}_x$ can be written as a (resp. unique) $\mathscr{O}_{X,x}$-linear combination of the germs $(s_{i,x})_{i\in I}$.

If $U \subset X$ is open, we say $(s_i)_{i\in I}$ generates (resp. freely) $\mathscr{E}_U$ if $(s_i|_U)_{i\in I}$ does.

We say that $\mathscr{E}$ is **locally free** if each $x \in X$ is contained in a neighborhood U such that the following equivalent conditions hold:

- $\mathscr{E}_U$ is generated freely by finitely many sections $s_1, \dots, s_n \in \mathscr{E}(U)$. ($s_\bullet$ play the role of basis of a vector space.)

- $\mathscr{E}_U$ is isomorphic to $\mathbb{C}^n \otimes \mathscr{O}_U$ for some $n \in \mathbb{N}$.

Remark A.13. It is an important fact that locally free $\mathscr{O}_X$-modules are the same as holomorphic vector bundles. Indeed, the sections of vector bundles clearly form a locally free module. Conversely, suppose $\mathscr{E}$ is locally free, then we can get a vector bundle whose transition functions are $\psi \circ \varphi^{-1} : W \otimes \mathscr{O}_U \xrightarrow{\simeq} W \otimes \mathscr{O}_U$ (considered as $\mathrm{End}W$-valued holomorphic functions) where $\varphi, \psi : \mathscr{E} \xrightarrow{\simeq} W \otimes \mathscr{O}_U$ are trivializations. Equivalently, if $s_1, \dots, s_n$ and $t_1, \dots, t_n$ both generate freely $\mathscr{E}_U$, then there is a unique invertible $M_{n\times n}(\mathbb{C})$-valued holomorphic function A such that $t_i(x) = \sum_j A_{i,j}(x)s_j(x)$. Then A gives a transition function.

A.7 Sheaves of morphisms, dual modules

If $\mathscr{E}, \mathscr{F}$ are $\mathscr{O}_X$-modules, we have a presheaf $\mathscr{G}$ whose space of sections on each open $U \subset X$ is $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)$. There is an obvious restriction map from $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}_U, \mathscr{F}_U)$ to $\mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{F}_V)$ if $V \subset U$ is open. $\mathscr{G}$ is clearly a presheaf of $\mathscr{O}_X$-modules. It is a routine check that $\mathscr{G}$ is a sheaf. We denote this sheaf of $\mathscr{O}_X$-modules by

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F}).$$

Exercise A.14. Find a natural equivalence $\mathscr{F} \xrightarrow{\simeq} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{F})$.

Example A.15. In the setting of Example A.8, we have a natural $\mathscr{O}_X$-module isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(V \otimes \mathscr{O}_X, W \otimes \mathscr{O}_X) \simeq \mathrm{Hom}_{\mathbb{C}}(V, W) \otimes \mathscr{O}_X. \tag{A.5}$$

We define

$$\mathscr{E}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X),$$

called the **dual $\mathscr{O}_X$-module** of $\mathscr{E}$. Then by (A.5), if $\mathscr{E}$ is locally free (i.e., a vector bundle), then so is $\mathscr{E}^\vee$, and they have the same rank. We regard $\mathscr{E}$ as the **dual vector bundle** of $\mathscr{E}$.

Exercise A.16. Let $\mathscr{E}$ be an $\mathscr{O}_X$-submodule of $\mathscr{F}$. Show that $(\mathscr{F}/\mathscr{E})^\vee$ is the sheaf whose sections over any open $U \subset X$ are the $\mathscr{O}_U$-module morphisms $\mathscr{F}_U \to \mathscr{O}_U$ vanishing on the stalks of $\mathscr{E}_U$.

Convention A.17. If $U, V \subset X$ are open, $\varphi \in \mathrm{Hom}_{\mathscr{O}_V}(\mathscr{E}_V, \mathscr{O}_V)$ and $s \in \mathscr{E}(U)$, we set

$$\langle \varphi, s \rangle = \varphi(s|_{U\cap V}) \qquad (\in \mathscr{O}(U \cap V)).$$

Remark A.18. If $\mathscr{E}$ is a vector bundle, then the transition functions of $\mathscr{E}^\vee$ are the inverses of those of $\mathscr{E}$. To see this, choose $s_1, \dots, s_n \in \mathscr{E}(U)$ generating freely $\mathscr{E}_U$. Then by $\mathscr{E}_U \simeq \mathbb{C}^n \otimes \mathscr{O}_U$, we can easily find $\check{s}_1, \dots, \check{s}_n \in \mathscr{E}^\vee(U)$ generating freely $\mathscr{E}_U^\vee$ such that $\langle s_j, s_i \rangle$ is the constant section $\delta_{i,j}$. $\check{s}_1, \dots, \check{s}_n$ are regarded as the dual basis of $s_1, \dots, s_n$.

Now, if $t_1, \dots, t_n \in \mathscr{E}(U)$ also generates freely $\mathscr{E}_U$, then by Rem. A.13, the matrix valued holomorphic function $A \in M_{n\times n} \otimes \mathscr{O}(U)$ such that $t_i = \sum_j A_{i,j}s_j$ is a transition function of $\mathscr{E}$. Let $A^{-1} \in M_{n\times n} \otimes \mathscr{O}(U)$ be the function sending $x \in U$ to $A(x)^{-1}$. Then $\check{t}_i = \sum_j (A^{-1})_{i,j}\check{s}_j$. □

A.8 Fibers

One can recover the fibers from a locally free sheaf in the following way. Let us consider a general $\mathscr{O}_X$-module $\mathscr{E}$. For each $x \in X$, let $\mathfrak{m}_{X,x}$ (or simply $\mathfrak{m}_x$) be the ideal of $\mathscr{O}_{X,x}$ consisting of all $s \in \mathscr{O}_{X,x}$ whose values at x vanish. Then $\mathfrak{m}_x\mathscr{E}_x$ is an $\mathscr{O}_{X,x}$-submodule of $\mathscr{E}_x$, and so is the **fiber**

$$\mathscr{E}|x \equiv \mathscr{E}|_x = \frac{\mathscr{E}_x}{\mathfrak{m}_x\mathscr{E}_x}. \tag{A.6}$$

where the $\mathrm{Span}_{\mathbb{C}}$ is suppressed in the notation $\mathfrak{m}_x\mathscr{E}_x$. The equivalence class of $s \in \mathscr{E}_x$ in $\mathscr{E}|x$ is denoted by $s(x)$, called the value of s on the fiber $\mathscr{E}|x$.

If $\varphi : \mathscr{E} \to \mathscr{F}$ is an $\mathscr{O}_X$-module morphism and $x \in X$, then $\varphi : \mathscr{E}_x \to \mathscr{F}_x$ descends to a linear map

$$\varphi : \mathscr{E}|x \to \mathscr{F}|x \tag{A.7}$$

since $\varphi(\mathfrak{m}_x\mathscr{E}_x) = \mathfrak{m}_x\varphi(\mathscr{E}_x) \subset \mathfrak{m}_x\mathscr{F}_x$.

Example A.19. Let $U \ni 0$ be an open subset of $\mathbb{C}^m$. Then $\mathfrak{m}_{U,0}$ is the set of all series $\sum_{n_1,\dots,n_m\in\mathbb{N}} a_{n_1,\dots,n_m} z_1^{n_1}\cdots z_m^{n_m}$ converging absolutely near 0 such that $a_{0,\dots,0} = 0$. Equivalently,

$$\mathfrak{m}_{\mathbb{C}^m,0} = z_1\mathscr{O}_{\mathbb{C}^m,0} + \cdots + z_m\mathscr{O}_{\mathbb{C}^m,0}.$$

Exercise A.20. Let W be a vector space, and let $\mathscr{E} = W \otimes \mathscr{O}_U$ where $U \subset \mathbb{C}^m$. Let $x \in U$. Show that the evaluation map

$$(W \otimes \mathscr{O}_U)_x \to W, \qquad w \otimes f \mapsto f(x)w. \tag{A.8}$$

descends to an isomorphism of vector spaces $(W \otimes \mathscr{O}_X)\big|x \simeq W$.

A.9 A criterion on local freeness

This subsection is needed only in the section on the local freeness of sheaves of coinvariants (covacua) and conformal blocks.

Let X be a complex manifold and $\mathscr{E}$ an $\mathscr{O}_X$-module. We say that $\mathscr{E}$ is of **finite type** (also called **finitely generated**) if each $x \in X$ is contained in a neighborhood $U \subset X$ such that there exist finitely many sections $s_1, \dots, s_n \in \mathscr{E}(U)$ generating $\mathscr{E}_U$. Equivalently, each x is contained in a neighborhood U such that there is a surjective $\mathscr{O}_U$-module morphism $\mathbb{C}^n \otimes \mathscr{O}_U \to \mathscr{E}_U$.

Warning: knowing that $\mathscr{E}(U)$ is a finitely generated $\mathscr{O}(U)$-module is not enough to show that $\mathscr{E}_U$ is generated by finitely many elements of $\mathscr{E}(U)$.

If $x \in U$ and $s_1, \dots, s_n \in \mathscr{E}(U)$ generate $\mathscr{E}_U$, then they clearly generate $\mathscr{E}_x$, and hence $s_1(x), \dots, s_n(x)$ span the fiber $\mathscr{E}|x$. In particular, $\mathscr{E}|x$ is finite-dimensional. Conversely, we have:

Proposition A.21 (Nakayama's lemma). *Suppose $\mathscr{E}$ is of finite type. Choose $x \in X$ and a neighborhood $U \ni x$. Let $s_1, \dots, s_n \in \mathscr{E}(U)$ such that $s_1(x), \dots, s_N(x)$ span the fiber $\mathscr{E}|x$. Then there exists a neighborhood $V \subset U$ of x such that $s_1|_V, \dots, s_n|_V$ generate $\mathscr{E}|_V$.*

Proof. By shrinking U, we may expand the list $s_1, \ldots, s_n$ to $s_1, \ldots, s_N \in \mathscr{E}(U)$ (where $N \geqslant n$) such that they generate $\mathscr{E}_U$. If $N = n$ then there is nothing to prove.

Suppose $N > n$. Since $s_1(x), \ldots, s_n(x)$ span $\mathscr{E}|x = \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$, every element of $\mathscr{E}_x$, and in particular s_N, can be written as

$$s_N = a_1 s_1 + \cdots + a_n s_n + \sigma \qquad \in \mathscr{E}_x$$

where $a_1, \ldots, a_n \in \mathbb{C}$ and $\sigma \in \mathfrak{m}_x\mathscr{E}_x$. Since $s_1, \ldots, s_N$ generate the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$, we have $\sigma = f_1 s_1 + \cdots + f_N s_N$ in $\mathscr{E}_x$ where $f_1, \ldots, f_N \in \mathfrak{m}_x$.So

$$s_N = g_1 s_1 + \cdots + g_N s_N$$

in $\mathscr{E}_x$ where $g_1, \ldots, g_N \in \mathscr{O}_{X,x}$ and $g_{n+1}(x) = \cdots = g_N(x) = 0$. Since $g_N(x) = 0$, $1 - g_N$ is invertible in $\mathscr{O}_{X,x}$. So

$$s_N = (1 - g_N)^{-1} \sum_{i=1}^{N-1} g_i s_i$$

in $\mathscr{E}_x$. So, after shrinking U to a smaller neighborhood of x on which $g_1, \ldots, g_N, (1 - g_N)^{-1}$ are holomorphic, the above equation holds in $\mathscr{E}(U)$. This shows that s_N is an $\mathscr{O}(U)$-linear combination of $s_1, \ldots, s_{N-1}$. So $s_1, \ldots, s_{N-1}$ generate $\mathscr{E}_U$. By repeating this argument, we see that $s_1, \ldots, s_n$ generated $\mathscr{E}_U$ for a smaller U. $\square$

Theorem A.22. *Assume that $\mathscr{E}$ is of finite type. Then the* ***rank function***

$$r : X \to \mathbb{N}, \qquad x \mapsto r(x) = \dim \mathscr{E}|x \tag{A.9}$$

is upper semicontinuous. (So $r(x) \geqslant r(y)$ for all y in a neighborhood of x.) Moreover, if the rank function is locally constant, then $\mathscr{E}$ is locally free.

Proof. Let $n = r(x)$. Choose $s_1, \ldots, s_n \in \mathscr{E}(U)$ (where $U \ni x$) such that $s_1(x), \ldots, s_n(x)$ form a basis of $\mathscr{E}|x$. Then by Nakayama's Lemma, after shrinking U, $s_1, \ldots, s_n$ generate $\mathscr{E}|_U$, and hence span $\mathscr{E}|y$ for all $y \in U$. This proves the upper semicontinuity.

Now suppose r is constantly n on U. Then, as $s_1(y), \ldots, s_n(y)$ span $\mathscr{E}|y$, and since $\dim \mathscr{E}|y = n$, $s_1(y), \ldots, s_n(y)$ are linearly independent. Let us show that $s_1, \ldots, s_n$ generate freely $\mathscr{E}_U$ by showing that they are $\mathscr{O}_U$-linearly independent. Choose any open $V \subset U$ and $f_1, \ldots, f_n \in \mathscr{O}(V)$ such that $f_1 s_1 + \cdots + f_n s_n = 0$. Then for each $y \in V$, $\sum_{i=1}^n f(y) s_n(y)$ equals 0 in $\mathscr{E}|y$. So $f_1(y) = \cdots = f_n(y) = 0$ by the linear independence of $s_1(y), \ldots, s_n(y)$. So $f_1 = \cdots = f_n = 0$. $\square$