PARTIAL OKOUNKOV BODIES AND DUISTERMAAT–HECKMAN MEASURES OF NON-ARCHIMEDEAN METRICS

MINGCHEN XIA

Abstract. Let X be a smooth projective variety. We construct partial Okounkov bodies associated to Hermitian pseudo-effective line bundles (L,ϕ) on X. We show that partial Okounkov bodies are universal invariants of the singularity of ϕ. As an application, we generalize the theorem of Boucksom–Chen and construct Duistermaat–Heckman measures associated to finite energy metrics on the Berkovich analytification of an ample line bundle.

Contents

1. Introduction

1.1. **Background.** Let X be an irreducible smooth projective variety of dimension n. Let L be a big holomorphic line bundle on X. Given any admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ on X (see Definition 2.7 for the precise definition), one can associate a natural convex body $\Delta(L)$ of dimension n to L, generalizing the classical Newton polytope construction in toric geometry. This construction was first considered by Lazarsfeld–Mustață [LM09] and Kaveh–Khovanskii [KK12] and $\Delta(L)$ is known as the *Okounkov body* or *Newton–Okounkov body* associated of L (with respect to the given flag). We briefly recall its definition: given any non-zero $s \in \mathrm{H}^0(X, L^k)$, let $\nu_1(s)$ be the vanishing order of s along Y_1. Then s can be regarded as a section of $\mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-\nu_1(s)Y_1))$ after possible shrinking X around the point Y_n. It follows that $s_1 := s|_{Y_1}$ is a non-zero section of $L|_{Y_1}^k \otimes \mathcal{O}_X(-\nu_1(s)Y_1)|_{Y_1}$. We can then repeat the same procedure with s_1, Y_2 in place of s, Y_1. Repeating this construction, we end up with $\nu(s) = (\nu_1(s), \dots, \nu_n(s)) \in \mathbb{N}^n$. In fact, ν extends naturally to a rank n valuation on $\mathbb{C}(X)$ of rational rank n. Consider the semigroup

$$\Gamma(L) := \left\{ (a,k) \in \mathbb{Z}^{n+1} : k \in \mathbb{N}, a = \nu(s) \text{ for some } s \in \mathrm{H}^0(X, L^k)^\times \right\}.$$

Keywords—Okounkov bodies, pseudo-effective line bundles, convex bodies, pluri-subharmonic functions
MSC: 14M25, 32U05
Mingchen Xia, Department of Mathematics, Chalmers Tekniska Högskola, Göteborg
Email address, xiam@chalmers.se
Homepage, http://www.math.chalmers.se/~xiam/.

Then $\Delta(L)$ is the intersection of the closed convex cone in $\mathbb{R}^{n+1}$ generated by $\Gamma(L)$ and $\{(x,1) : x \in \mathbb{R}^n\}$. A key property of $\Delta(L)$ is that the Lebesgue volume of $\Delta(L)$ is proportional to the volume of the line bundle L:

$$\operatorname{vol}\Delta(L) = \frac{1}{n!}\langle L^n\rangle\,. \tag{1.1}$$

Here $\langle\bullet\rangle$ denotes the movable intersection product in the sense of [BDPP13; BFJ09].

In [LM09], Lazarsfeld–Mustață showed moreover that $\Delta(L)$ depends only on the numerical class of L. Conversely, it is shown by Jow [Jow10] that the information of all Okounkov bodies with respect to various flags actually determine the numerical class of L. In other words, Okounkov bodies can be regarded as universal numerical invariants of big line bundles.

This paper concerns a similar problem. Assume that L is equipped with a singular positively-curved Hermitian metric ϕ. We will construct universal invariants of the singularity type of ϕ. We call these universal invariants the *partial Okounkov bodies* of (L,ϕ).

1.2. **Main results.** Let us explain more details about the construction of partial Okounkov bodies. Recall that any admissible flag on X induces a rank n valuation on $\mathbb{C}(X)$ of rational rank n. We will work more generally with such valuations, not necessarily coming from admissible flags on X. We define a set

$$\Gamma(L,\phi) := \left\{(a,k)\in\mathbb{Z}^{n+1} : k\in\mathbb{N}, a=\nu(s) \text{ for some } s\in\mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\phi))^\times\right\}$$

similar to $\Gamma(L)$. Here $\mathcal{I}(\bullet)$ denotes the multiplier ideal sheaf in the sense of Nadel. However, a key difference here is that $\Gamma(L,\phi)$ is not a semigroup in general. Thus, the constructions in both [LM09] and [KK12] break down. We will show that in this case, there is still a canonical construction of Okounkov bodies.

Before stating our main theorem, let us recall the definition of volume. The volume of (L,ϕ) is defined as

$$\operatorname{vol}(L,\phi) := \lim_{k\to\infty}\frac{1}{k^n}h^0(X,L^k\otimes\mathcal{I}(k\phi))\,.$$

The existence of this limit is proved in [DX21].

Theorem A. *Let (L,ϕ) be as above. Assume that $\operatorname{vol}(L,\phi)>0$. Then there is a canonical convex body $\Delta(L,\phi)\subseteq\Delta(L)$ associated to (L,ϕ) satisfying*

$$\operatorname{vol}\Delta(L,\phi) = \operatorname{vol}(L,\phi)\,. \tag{1.2}$$

Moreover, $\Delta(L,\phi)$ is continuous in ϕ if $\int_X(\mathrm{dd^c}\phi)^n>0$. Here the set of ϕ is endowed with the d_S-pseudo-metric in the sense of [DDNL21] and the set of convex bodies is endowed with the Hausdorff metric.

Define

$$\Gamma_k := \left\{k^{-1}\nu(s)\in\mathbb{R}^n : s\in\mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\phi))^\times\right\}$$

and let Δ_k denote the convex hull of Γ_k. Assume that ϕ has analytic singularities, then Δ_k converges to $\Delta(L,\phi)$ with respect to the Hausdorff metric.

Observe that an assignment $(L,\phi)\mapsto\Delta(L,\phi)$ satisfying Theorem A is unique, as a consequence of [DX21, Theorem 3.8]. The convex body $\Delta(L,\phi)$ is called the *partial Okounkov body* of (L,ϕ) with respect to the given valuation. We will also extend the definition to the case $\operatorname{vol}(L,\phi)=0$ in Section 5.5, at the expense of losing continuity in ϕ.

Observe that (1.2) bears strong resemblance with (1.1). In fact, when ϕ has minimal singularities, $\Delta(L,\phi)=\Delta(L)$ and (1.2) just reduces to (1.1).

The second main result says that partial Okounkov bodies uniquely determine the $\mathcal{I}$-singularity type of ϕ.

Theorem B. *Let L be a big line bundle on X. Let ϕ, ϕ' be two positively-curved singular Hermitian metrics on L. Then the following are equivalent:*

(1) $\phi \sim_{\mathcal{I}} \phi'$.
(2) $\Delta(L,\phi) = \Delta(L,\phi')$ for all valuations on $\mathbb{C}(X)$ of rank n and rational rank n.

Recall that $\phi \sim_{\mathcal{I}} \phi'$ means $\mathcal{I}(k\phi) = \mathcal{I}(k\phi')$ for all real $k > 0$. This relation is studied in detail in [DX20; DX21]. It captures a lot of important information about the singularity of a positively-curved metric. Theorem B should be regarded as a metric analogue of Jow's theorem.

The upshot of Theorem B is that it allows us to talk about Okounkov bodies of automorphic line bundles on Shimura varieties. In fact, under some extra conditions, it can be shown that the natural extensions of the metrics on the toroidal compatification of arithmetic quotients are all $\mathcal{I}$-equivalent. In this case, computing the partial Okounkov bodies helps us to understand the non-good singularities (in the sense of Mumford) occuring in the mixed Shimura case.

As a byproduct of our proof of Theorem B, we find a formula computing the generic Lelong numbers of currents of minimal singularities in $c_1(L)$, which seems to be new:

Theorem 1.1 (=Corollary 5.21)**.** *Let L be a big line bundle on X. Consider a current $T_{\min}$ of minimal singularity in $c_1(L)$. Then for any prime divisor E over X, we have*

$$\nu(T_{\min}, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathrm{H}^0(X, L^k)\,. \tag{1.3}$$

Here $\nu(T_{\min}, E)$ denotes the generic Lelong number of $T_{\min}$ along E.

This result generalizes [Bou02, Theorem 5.4]. As a consequence, we find a new formula computing the multiplier ideal sheaf $\mathcal{I}(T_{\min})$ in Corollary 5.22.

The third main result is an analogue of [WN14]. Given any continuous metric ψ on L, one can naturally associate a convex function $c[\psi]$ on $\operatorname{Int}\Delta(L)$, known as the *Chebyshev transform* of ψ. The main property of $c[\psi]$ is that given another continuous metric ψ' on L, we have

$$\int_{\Delta(L)} (c[\psi] - c[\psi'])\,\mathrm{d}\lambda = \operatorname{vol}(\psi,\psi')\,, \tag{1.4}$$

where $\operatorname{vol}(\psi,\psi')$ is the relative volume as studied in [BB10; BBWN11] and $\mathrm{d}\lambda$ is the Lebesgue measure on $\mathbb{R}^n$. In our setup, we also associate a convex function: $c_{[\phi]}[\psi] : \operatorname{Int}\Delta(L,\phi) \to \mathbb{R}$. Moreover,

Theorem C. *Assume that the valuation ν is induced by an admissible flag on X. Let ψ, ψ' be two continuous metrics on L, then*

$$\int_{\Delta(L,\phi)} \left(c_{[\phi]}[\psi] - c_{[\phi]}[\psi']\right)\mathrm{d}\lambda = \mathcal{E}^{\theta}_{[\phi]}(\psi) - \mathcal{E}^{\theta}_{[\phi]}(\psi')\,, \tag{1.5}$$

where $\mathcal{E}^{\theta}_{[\phi]}$ is the partial equilibrium energy functional defined in (6.1).

In addition, if $\mathrm{dd^c}\phi$ is a Kähler current, the function $c_{[\phi]}[\psi]$ is canonically associated to ϕ and ψ.

Theorem A, Theorem B and Theorem C together give geometric interpretations of the main results of [DX20; DX21]. These results also provide us with a geometric approach to the study of psh singularities.

As an application of our theory, we prove a generalization of Boucksom–Chen theorem (Theorem 7.9). Recall the theorem Boucksom–Chen [BC11] says that given a *multiplicative* filtration $\mathscr{F}$ on the section ring $R(X,L)$, one can naturally associate a probability measure on $\mathbb{R}$, known as the *Duistermaat–Heckman measure.* Moreover, the Duistermaat–Heckman measure is the weak limit of a sequence of discrete measures μ_k associated to the filtration $\mathscr{F}$ on $\mathrm{H}^0(X, L^k)$. We show that this construction can be generalized to all $\mathcal{I}$-model test curves, not necessarily coming from filtrations. Here we only prove the generalized Boucksom–Chen theorem for filtrations on the full graded linear series, which suffices for our purpose. It is, however, easy to see that the techniques applies to more general situations. In a forthcoming, we will apply these techniques in the setting of arithmetic varieties.

More generally, we introduce the notion of an Okounkov test curve (Definition 7.2) and generalize Duistermaat–Heckman measures to this setting.

When L is ample, this construction allows us to associate to each element η in the non-Archimedean space $\mathcal{E}^1(L^{\mathrm{an}})$ in the sense of [BJ21] a measure $\mathrm{DH}(\eta)$ on $\mathbb{R}$, see Definition 7.13. We prove that $\mathrm{DH}(\eta)$ depends continuously on η.

Theorem 1.2. *The Duistermaat–Heckman measure construction of test configurations admits a unique continuous extension* $\mathrm{DH} : \mathcal{E}^1(L^{\mathrm{an}}) \to \mathcal{M}(\mathbb{R})$*. Here* $\mathcal{M}(\mathbb{R})$ *is the set of Radon measures on* $\mathbb{R}$*.*

In Theorem 7.16, we will furthermore prove that $\mathrm{DH}(\eta)$ contains a lot of interesting information of η. Theorem 1.2 gives an intrinsic definition of the non-Archimedean $\mathcal{E}^2(L^{\mathrm{an}})$:

$$\mathcal{E}^2(L^{\mathrm{an}}) := \left\{ \phi \in \mathcal{E}^1(L^{\mathrm{an}}) : \int_{\mathbb{R}} |x|^2 \,\mathrm{DH}(\phi)(x) < \infty \right\} ,$$

The detailed construction together with an intrinsic construction of the d_2-metric will appear in a joint paper with Rémi Reboulet.

In the last section, we interpret the partial Okounkov bodies in the toric setting. We prove the following results:

Theorem 1.3. *Let X be a smooth toric variety of dimension n. Let (L, ϕ) be a toric invariant Hermitian psef line bundle on X. Assume that ϕ has $\mathcal{I}$-model singularities and has positive volume. Fix a toric invariant admissible flag on X. Recall that upon choosing a toric invariant rational section of L, ϕ can be identified with a convex function $\phi_{\mathbb{R}}$ on $\mathbb{R}^n$. Then the partial Okounkov body $\Delta(L, \phi)$ is naturally identified with the closure of the image of $\nabla \phi_{\mathbb{R}}$.*

For the definition of $\mathcal{I}$-model singularities of ϕ, see Definition 2.21.

Theorem D. *Let (L_i, ϕ_i) $(i = 1, \ldots, n)$ be toric invariant Hermitian psef line bundles on X of positive volumes. Assume that ϕ_i has $\mathcal{I}$-model singularities. If the toric invariant flag $(Y_\bullet)$ satisfies the additional condition: Y_n is not contained in the polar locus of any ϕ_i, then the following quantities are equal:*

(1) $\int_X \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_n$.
(2) The mixed mass of $\phi_1, \ldots, \phi_n$ in the sense of Cao [Cao14].
(3) $n! \operatorname{vol}(\Delta(L_1, \phi_1), \ldots, \Delta(L_n, \phi_n))$.

In fact, we prove that the equality between (1) and (2) holds in non-toric setting as well, see Corollary 4.5.

An unpublished result of Yi Yao says that in the toric setting, two invariant potentials ϕ, ϕ' are $\mathcal{I}$-equivalent if and only if $\overline{\nabla \phi_{\mathbb{R}}(\mathbb{R}^n)} = \overline{\nabla \phi'_{\mathbb{R}}(\mathbb{R}^n)}$. In particular, in Theorem 1.3 and Theorem D, the assumptions of having $\mathcal{I}$-model singularities are in fact unnecessary.

It is of interest to generalize Theorem D to the non-toric setting as well. As shown by Example 8.7, the non-toric generalization has to involve all valuations instead of just one.

Lastly, let us mention that our generalization of Boucksom–Chen theorem has important consequences in Archimedean pluripotential theory as well. When applied to *generalized deformation to the normal cone* in the sense of [Xia20], it gives a number of interesting equidistribution results of the jumping numbers of multiplier ideal sheaves. As a detailed investigation would lead us too far away, we do not include these results in this paper.

1.3. **Strategy of the proofs.** We will sketch the proof of these theorems.

The proof of Theorem A. In general, the graded linear space

$$W(L,\phi) := \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\phi))$$

is not an algebra. Thus, one can not directly apply the theory of graded linear series or the theory of semigroups as in [LM09] and [KK12]. A key observation here is that although $W(L,\phi)$ is not a graded linear series, it is not too far away from being one.

To make this precise, we introduce a pseudo-metric on the space $\mathrm{LS}(L)$ of graded linear subspaces of $R(X,L) := \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X, L^k)$:

$$d(W,W') := \varlimsup_{k\to\infty} k^{-n} \left(2\dim(W_k + W'_k) - \dim W_k - \dim W'_k\right) .$$

Let $\sim$ be the equivalence relation defined by d. The classical Okounkov body construction associates to each graded linear series a convex body: $\Delta : \mathrm{GLS}(L) \to \mathcal{K}_n$. Here $\mathrm{GLS}(L) \subseteq \mathrm{LS}(L)$ is the set of graded linear series and $\mathcal{K}_n$ is the space of convex bodies in $\mathbb{R}^n$. As we will prove in Theorem 3.21, this map factorizes through the $\sim$-equivalence classes, and it extends continuously to the positive volume part of the completion of $\mathrm{GLS}(L)$. We call the latter space $\overline{\mathrm{GLS}(L)}_{>0}$. In order to define the Okounkov body of (L,ϕ), we will actually show that

$$W(L,\phi) \in \overline{\mathrm{GLS}(L)}_{>0} . \tag{1.6}$$

Thus, we could simply define

$$\Delta(L,\phi) := \Delta(W(L,\phi)) .$$

The proof follows the same pattern as the proof in [DX21]. We proceed by approximations. We first consider the case where ϕ has analytic singularities. In this case, after taking a suitable resolution, we can easily see that $W(L,\phi)$ can be approximated by graded linear series both from above and from below. Hence, (1.6) follows. In the case of a more singular ϕ with $\mathrm{dd^c}\phi$ being a Kähler current, we make use of analytic approximations as in [DPS01; Cao14]. More precisely, take a quasi-equisingular approximation ϕ^j of ϕ. Based on the convergence theorems proved in [DX21], we can show that $W(L,\phi^j)$ converges to $W(L,\phi)$, which enables us to conclude (1.6) in this case. Finally, in the general case, a trick discovered in [DDNL21] and [DX21] enables us to reduce to the previous case.

Along the lines of the proof, we actually find that $W(L,\phi)$ satisfies a stronger property than (1.6):

$$\Delta_k(L,\phi) \to \Delta(L,\phi)$$

with respect to the Hausdorff metric if ϕ has analytic singularities. This property is essential to the proof of Theorem B, we call it the *Hausdorff convergence property*. Note that in contract to (1.6), the Hausdorff convergence property depends on the choice of the valuation.

The proof of Theorem B. Recall that in the classical setting, we can read information about the asymptotic base loci of L from the Okounkov body $\Delta(L)$ directly. In our setup, the analogue says that the Okounkov body $\Delta(L,\phi)$ gives information about the generic Lelong numbers of ϕ. We will prove a qualitative version of Theorem B:

Theorem 1.4. *Let E be a prime divisor over X. Let $\pi : Z \to X$ be a birational model of X such that E is a divisor on Z. Take an admissible flag $(Y_\bullet)$ on Z with $Y_1 = E$, then*

$$\nu(\phi,E) = \min_{x\in\Delta(\pi^*L,\pi^*\phi)} x_1 .$$

Here $\nu(\phi,E)$ is the generic Lelong number of $\pi^*\phi$ along E. The proof of Theorem 1.4 again follows the same pattern as in the proof of Theorem A. With some efforts, we can reduce the problem to the case where ϕ has analytic singularities along some normal crossing (nc) $\mathbb{Q}$-divisor on X and $\mathrm{dd^c}\phi$ is a Kähler current. In this case, the desired result follows from a result proved in [Xia20].

The proofs of Theorem C and Theorem D. The proofs roughly follow the same pattern as above. Namely, we first handle the case of analytic singularities and then conclude the general case by suitable approximations. We will not repeat the details here.

As explained above, our approach to general psh singularities requires a number of approximations, this motivates the study of the metric geometry of the space of psh singularity types. We prove the continuity of mixed masses under d_S-approximations:

Theorem 1.5 (= Theorem 4.2)**.** *Let θ_i $(i = 1, \dots, n)$ be smooth real $(1,1)$-forms representing pseudo-effective classes on a connected compact Kähler manifold X of dimension n. Let $\varphi_i^k, \varphi_i \in \mathrm{PSH}(X, \theta_i)$ $(i = 1, \dots, n,\ k \in \mathbb{N})$. Assume that $\varphi_i^k \xrightarrow{d_{S,\theta_i}} \varphi_i$ as $k \to \infty$. Then*

$$\lim_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}\,. \tag{1.7}$$

Here the Monge–Ampère operators are taken in the non-pluripolar sense.

This theorem and its various consequences are indispensable in all of our proofs. They are of independent interests as well. Note that Theorem 1.5 together with [DDNL18b, Theorem 2.3] gives a rather strong convergence theorem of non-pluripolar measures. We plan to explore these aspects more explicitly in a separate paper.

1.4. **Structure of the paper.** In Section 2, we collect a few preliminaries.

In Section 3, we study the closure of the space of graded linear series and define a general Okounkov body.

In Section 4, we further develop the theory of d_S-pseudo-metrics on the space of singularity types initiated in [DDNL21].

In Section 5, we define partial Okounkov bodies associated to Hermitian pseudo-effective line bundles and prove a number of properties.

In Section 6, we define and study Chebyshev transforms of continuous metrics.

In Section 7, we generalize the theory of Boucksom–Chen and study the non-Archimedean Duistermaat–Heckman measures.

In Section 8, we give an explicit description of partial Okounkov bodies construction in terms of the moment polytope in the toric situation.

1.5. **Conventions.** In this paper, Monge–Ampère operators θ_φ^n refers to the non-pluripolar product in the sense of [BEGZ10]. The group $\mathbb{Z}^n$ is always endowed with the lexicographic order. A line bundle always refers to a holomorphic line bundle. We do not distinguish a line bundle and the associated invertible sheaf. When talking about a birational modification (resolution) $\pi : Y \to X$, we always assume that Y is smooth and π is projective. We follow the convention that $\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}$.

1.6. **Acknowledgements.** I benefited a lot from discussions with David Witt Nyström, Chen Jiang, Yi Yao, Jian Xiao, Tamás Darvas, Kewei Zhang, Longke Tang and Rémi Reboulet. I would like to thank especially Yi Yao for explaining his computations in the toric setting to me, Chen Jiang for providing Example 8.7 and Yaxiong Liu for pointing out a number of typos in the arXiv version.

2. Preliminaries

2.1. **Hausdorff metric of convex bodies.** In this section, we recall the theory of Hausdorff metrics on the set of convex bodies following [Sch14, Section 1.8]. Fix $n \in \mathbb{N}$. Recall that a convex body in $\mathbb{R}^n$ is a non-empty compact convex subset of $\mathbb{R}^n$. Let $\mathcal{K}_n$ denote the set of convex bodies in $\mathbb{R}^n$. We will fix the Lebesgue measure $\mathrm{d}\lambda$ on $\mathbb{R}^n$, normalized so that the unit cube has volume 1.

Recall the definition of the Hausdorff metric between $K_1, K_2 \in \mathcal{K}_n$:

$$d_n(K_1, K_2) := \max\left\{ \sup_{x_1\in K_1} \inf_{x_2\in K_2} |x_1 - x_2|, \sup_{x_2\in K_2} \inf_{x_1\in K_1} |x_1 - x_2| \right\}.$$

We extend d_n to an extended metric on $\mathcal{K}_n \cup \{\emptyset\}$ by setting

$$d_n(K, \emptyset) = \infty$$

for all $K \in \mathcal{K}_n$.

Theorem 2.1. *The metric space $(\mathcal{K}_n, d_n)$ is complete.*

Theorem 2.2 (Blaschke selection theorem)**.** *Every bounded sequence in $\mathcal{K}_n$ has a convergent subsequence.*

Theorem 2.3. *The Lebesgue volume $\mathrm{vol} : \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ is continuous.*

Theorem 2.4. *Let $K_i, K \in \mathcal{K}_n$ ($i \in \mathbb{N}$). Then $K_i \xrightarrow{d_n} K$ if and only if the following conditions hold*

(1) Each point $x \in K$ is the limit of a sequence $x_i \in K_i$.

(2) The limit of any convergent sequence $(x_{i_j})_{j \in \mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ lies in K, where i_j is a subsequence of $1, 2, \ldots$.

The proofs of all these results can be found in [Sch14, Section 1.8].

Lemma 2.5. *Let $K_0, K_1 \in \mathcal{K}_n$. Assume that $K_0 \subseteq K_1$ and*

$$\mathrm{vol}\, K_0 = \mathrm{vol}\, K_1 > 0\,.$$

Then $K_0 = K_1$.

Proof. In fact, if $K_1 \neq K_0$, then $K_1 \setminus K_0$ is a non-empty open subset of K_1. As $\mathrm{vol}\, K_1 > 0$, $(K_1 \setminus K_0) \cap \mathrm{Int}\, K_1 \neq \emptyset$. Thus, $\mathrm{vol}\, K_1 > \mathrm{vol}\, K_0$, which is a contradiction. □

Let $K \in \mathcal{K}_n$ be a convex body with positive volume. For $\delta > 0$ small enough, let $K^\delta := \{x \in K : d(x, \partial K) \geq \delta\}$. Then $K_\delta \in \mathcal{K}_n$ for δ small enough.

The following lemma is clear.

Lemma 2.6. *Let $K \in \mathcal{K}_n$ be a convex body with positive volume. Let $K' \in \mathcal{K}_n$. Assume that for a large enough $k \in \mathbb{Z}_{>0}$, K' contains $K \cap (k^{-1}\mathbb{Z})^n$, then $K' \supseteq K^{n^{1/2}k^{-1}}$.*

Given a sequence of convex bodies K_i ($i \in \mathbb{N}$), we set

$$\varliminf_{i \to \infty} K_i = \overline{\bigcup_{i=0}^{\infty} \bigcap_{j \geq i} K_j}\,, \quad \varlimsup_{i \to \infty} K_i = \bigcap_{i=0}^{\infty} \overline{\mathrm{Conv} \bigcup_{j \geq i} K_j}\,.$$

Here Conv denotes the convex hull.

Suppose K is the limit of a subsequence of K_i and $\mathrm{vol}\, K > 0$, we have

$$\varliminf_{i \to \infty} K_i \subseteq K \subseteq \varlimsup_{i \to \infty} K_i\,. \tag{2.1}$$

2.2. **Admissible flags and valuations.** Let X be an irreducible normal projective variety of dimension n.

Definition 2.7. An *admissible flag* $(Y_\bullet)$ on X is a flag of subvarieties

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

such that Y_i is irreducible of codimension i and smooth at the point Y_n.

Let $\mathbf{F}$ be an irreducible component of the moduli space of admissible flags. Then $\mathbf{F}$ can be realized as a subscheme of products of Hilbert schemes. So $\mathbf{F}$ has a natural quasi-projective scheme structure.

Given any flag $(Y_\bullet) \in \mathbf{F}$, we can define a rank n valuation $\nu_{(Y_\bullet)} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ as in [LM09]. Here we consider $\mathbb{Z}^n$ as a totally ordered Abelian group with the lexicographic order. We recall the definition: let $s \in \mathbb{C}(X)^\times$. Let $\nu_1(s) = \operatorname{ord}_{Y_1} s$. After localization around Y_n, we can take a local defining equation t^1 of Y_1, set $s_1 = (s(t^1)^{-\nu_1(s)})|_{Y_1}$. Then $s_1 \in \mathbb{C}(Y_1)$. We can repeat this construction with Y_2 in place of Y_1 to get $\nu_2(s)$, s_2. Repeating this construction n times, we get $\nu_{(Y_\bullet)}(s) = \nu(s) = (\nu_1(s), \nu_2(s), \dots, \nu_n(s)) \in \mathbb{Z}^n$. It is easy to verify that ν is indeed a rank n valuation with rational rank n.

Conversely, by a theorem of Abhyankar, any valuation of $\mathbb{C}(X)$ of rank n and rational rank n is equivalent to a valuation taking value in $\mathbb{Z}^n$, see [FK18, Theorem 6.5.2]. As shown in [CFKL+17, Theorem 2.9], any such valuation is induced by an admissible flag on a birational modification of X.

2.3. **Model potentials.** Let X be a connected compact Kähler manifold of dimension n and θ be a smooth real $(1,1)$-form representing a $(1,1)$-cohomology class $[\theta]$. Define $V_\theta := \sup\{\varphi \in \operatorname{PSH}(X,\theta) : \varphi \leq 0\}$. For any two $\varphi, \psi \in \operatorname{PSH}(X,\theta)$, we say φ is *more singular* than ψ and write $[\varphi] \preceq [\psi]$ if there is a constant C such that $\varphi \leq \psi + C$. We write $\theta_\varphi = \theta + \mathrm{dd^c}\varphi$.

Definition 2.8. Let $\varphi \in \operatorname{PSH}(X,\theta)$. Define

$$C^\theta[\varphi] := \sup{}^* \left\{ \psi \in \operatorname{PSH}(X,\theta) : [\varphi] \preceq [\psi], \psi \leq 0, \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\psi^k \wedge \theta_{V_\theta}^{n-k}, \forall k \right\}. \tag{2.2}$$

If $C^\theta[\varphi] = \varphi$, we say φ is a *model potential*. We omit θ from the notation if there is no risk of confusion.

Here and in the sequel the Monge–Ampère type operators are taken in the non-pluripolar sense [BEGZ10].

Proposition 2.9 ([DDNL21, Proposition 2.6])**.** *For any $\varphi \in \operatorname{PSH}(X,\theta)$, $C^\theta[\varphi]$ is a model potential in $\operatorname{PSH}(X,\theta)$. When $\int_X \theta_\varphi^n > 0$ we have*

$$C^\theta[\varphi] = P^\theta[\varphi],$$

where

$$P^\theta[\varphi] := \sup{}^* \{\psi \in \operatorname{PSH}(X,\theta) : [\psi] \preceq [\varphi], \psi \leq 0\}. \tag{2.3}$$

In general, we only have

$$C^\theta[\varphi] = \lim_{\epsilon \to 0+} P^\theta[(1-\epsilon)\varphi + \epsilon V_\theta]. \tag{2.4}$$

We omit θ from the notation $P^\theta[\varphi]$ if there is no risk of confusion.

Proposition 2.10 ([DX20, Proposition 2.18])**.** *For any $\varphi \in \operatorname{PSH}(X,\theta)$ the potential $P[\varphi]_{\mathcal{I}}$ is $\mathcal{I}$-model.*

Typical model potentials are not $\mathcal{I}$-model, however, the converse is true:

Proposition 2.11. *If $\psi \in \operatorname{PSH}(X,\theta)$ is an $\mathcal{I}$-model potential then it is model.*

Proof. We need to show that $\psi \sim_{\mathcal{I}} C[\psi]$. Let $\pi : Z \to X$ be a birational modification. Let $z \in Z$. As $\psi \leq C[\psi] + C$ for some constant C, it suffices to show that

$$\nu(C[\psi], z) \geq \nu(\psi, z).$$

Here $\nu(\psi, z)$ denotes the Lelong number of $\pi^*\psi$ at z. By (2.4) and the upper semi-continuity of Lelong numbers, we find

$$\nu(C[\psi], z) \geq \lim_{\epsilon \to 0+} \nu(P[(1-\epsilon)\psi + \epsilon V_\theta], z) = \lim_{\epsilon \to 0+} \nu((1-\epsilon)\psi + \epsilon V_\theta, z) = \nu(\psi, z).$$

We conclude our assertion. $\square$

2.4. **Potentials with analytic singularities.**

Definition 2.12. A quasi-plurisubharmonic function (quasi-psh) φ on X is said to have *analytic singularities* if for each $x \in X$, there is a neighborhood $U_x \subseteq X$ of x in the Euclidean topology, such that on U_x,

$$\varphi = c \log \left(\sum_{j=1}^{N_x} |f_j|^2 \right) + \psi \,,$$

where $c \in \mathbb{Q}_{\geq 0}$, f_j are analytic functions on U_x, $N_x \in \mathbb{Z}_{>0}$ is an integer depending on x, $\psi \in C^\infty(U_x)$.

Definition 2.13. Let D be an effective normal crossing (nc) $\mathbb{R}$-divisor on X. Let $D = \sum_i a_i D_i$ with D_i being prime divisors and $a_i \in \mathbb{R}_{>0}$. We say that $\varphi \in \mathrm{PSH}(X, \omega)$ has *analytic singularities along D* if locally (in the Euclidean topology),

$$\varphi = \sum_i a_i \log |s_i|_h^2 + \psi \,,$$

where s_i is a local section of L that defines D_i, ψ is a smooth function.

In the sequel, when we talk about an nc divisor, we always assume that it is effective.

Note that a potential with analytic singularities along an nc $\mathbb{Q}$-divisor has analytic singularities in the sense of Definition 2.12.

Definition 2.14. A *birational model* of X is a projective birational morphism $\pi : Y \to X$ from a *smooth* projective variety Y to X.

For any quasi-psh function φ on X, there is always a birational model $\pi : Y \to X$ such that $\pi^*\varphi$ has analytic singularities along an nc $\mathbb{Q}$-divisor on Y. See [MM07, Lemma 2.3.19] for example.

2.5. **Quasi-equisingular approximations.** We recall the concept of quasi-equisingular approximations in the sense of [Cao14; DPS01].

Let X be a connected compact Kähler manifold of dimension n and θ (resp. θ_i, $i = 1, \dots, n$) be a smooth real $(1,1)$-form representing a pseudo-effective $(1,1)$-cohomology class $[\theta]$ (resp. $[\theta_i]$). Take a Kähler form ω on X.

Definition 2.15. Let $\varphi \in \mathrm{PSH}(X, \theta)$. A *quasi-equisingular approximation* is a sequence $\varphi^j \in \mathrm{PSH}(X, \theta + \epsilon_i \omega)$ with $\epsilon_j \to 0$ such that

(1) $\varphi^j \to \varphi$ in L^1.
(2) φ^j has analytic singularities.
(3) $\varphi^{j+1} \leq \varphi^j$.
(4) For any $\delta > 0$, $k > 0$, there is $j_0 > 0$ such that for $j \geq j_0$,

$$\mathcal{I}(k(1+\delta)\varphi^j) \subseteq \mathcal{I}(k\varphi) \subseteq \mathcal{I}(k\varphi^j) \,.$$

The existence of a quasi-equisingular approximation follows from the arguments in [Cao14; Dem15; DPS01].

Definition 2.16. Let $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ $(i = 1, \dots, n)$. The *mixed mass* of the φ_i's in the sense of Cao is defined as follows: take quasi-equisingular approximations φ_i^j $(j \in \mathbb{N})$ of φ_i. Then

$$\langle \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \rangle := \lim_{j \to \infty} \int_X (\theta_1 + \mathrm{dd^c} \varphi_1^j)_{\mathrm{ac}} \wedge \cdots \wedge (\theta_n + \mathrm{dd^c} \varphi_n^j)_{\mathrm{ac}} \,.$$

Here the index ac denotes the absolutely continuous part of a current.

It is shown in [Cao14] that the mixed mass does not depend on the choice of quasi-equisingular approximations.

2.6. **Volumes of Hermitian pseudo-effective line bundles.** Let X be a smooth irreducible projective variety of dimension n.

Definition 2.17. A *Hermitian pseudo-effective(psef) line bundle* on X is a pair (L,ϕ), where L is a pseudo-effective line bundle on X and ϕ is a psh metric on L.

Let (L,ϕ) be a Hermitian psef line bundle on X. In this section, we recall the main results in [DX20; DX21] concerning the volume of (L,ϕ). Recall that $\mathcal{I}(\phi)$ denotes the multiplier ideal sheaf of ϕ in the sense of Nadel, namely the coherent subsheaf of $\mathcal{O}_X$ consisting of functions f such that $|f|^2_\phi$ is locally integrable.

Definition 2.18. The *volume* of (L,ϕ) is defined as

$$\operatorname{vol}(L,\phi) := \lim_{k\to\infty}\frac{1}{k^n}h^0\left(X, L^k\otimes\mathcal{I}(k\phi)\right).$$

The existence of the limit is one part of the main results of [DX20; DX21].

We take a smooth Hermitian metric h on L. Set $\theta = c_1(L,h)$. Then we can identify ϕ with a θ-psh function φ.

Definition 2.19. Let φ,ψ be two quasi-psh functions, we say $\varphi \preceq_{\mathcal{I}} \psi$ if the following equivalent conditions are satisfied:

(1) $\mathcal{I}(k\varphi)\subseteq\mathcal{I}(k\psi)$ for all real $k>0$.
(2) $\mathcal{I}(k\varphi)\subseteq\mathcal{I}(k\psi)$ for all integer $k>0$.
(3) For any birational model $\pi: Y\to X$ and any $y\in Y$, we have $\nu(\pi^*\varphi,y)\geq\nu(\pi^*\psi,y)$.

We say $\varphi\sim_{\mathcal{I}}\psi$ if $\varphi\preceq_{\mathcal{I}}\psi$ and $\psi\preceq_{\mathcal{I}}\varphi$.

Given any $\varphi\in\operatorname{PSH}(X,\theta)$, we define

$$P[\varphi]_{\mathcal{I}} := \sup\left\{\psi\in\operatorname{PSH}(X,\theta) : \psi\preceq_{\mathcal{I}}\varphi, \psi\leq 0\right\}.$$

It is shown in [DX20] that $P[\varphi]_{\mathcal{I}}\in\operatorname{PSH}(X,\theta)$ and $\varphi\sim_{\mathcal{I}}P[\varphi]_{\mathcal{I}}$. We can also talk about the $\sim_{\mathcal{I}}$ relation of two positive metric on L in the obvious manner.

Theorem 2.20. *Under the above assumptions,*

$$\operatorname{vol}(L,\phi) = \frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}}.$$

Definition 2.21. We say that ϕ has *$\mathcal{I}$-model singularities* if

$$\operatorname{vol}(L,\phi) = \frac{1}{n!}\int_X (\mathrm{dd^c}\phi)^n.$$

We argue that vol deserves the name *volume* by proving that it satisfies the Brunn–Minkowski inequality.

Corollary 2.22. *Let (L,ϕ), (L,ϕ') be two Hermitian psef line bundles on X. Then*

$$\operatorname{vol}(L+L',\phi+\phi')^{1/n}\geq\operatorname{vol}(L,\phi)^{1/n}+\operatorname{vol}(L',\phi')^{1/n}. \tag{2.5}$$

Proof. Fix a smooth Hermitian metric h' on L' with $\theta' = c_1(L',h')$. We identify ϕ' with $\varphi'\in\operatorname{PSH}(X,\theta')$. By Theorem 2.20, (2.5) is equivalent to

$$\left(\int_X\left(\theta+\theta'+\mathrm{dd^c}P^{\theta+\theta'}[\varphi+\varphi']_{\mathcal{I}}\right)^n\right)^{1/n}\geq\left(\int_X\theta^n_{P^\theta[\varphi]_{\mathcal{I}}}\right)^{1/n}+\left(\int_X\theta'^n_{P^{\theta'}[\varphi']_{\mathcal{I}}}\right)^{1/n}.$$

Observe that

$$P^{\theta+\theta'}[\varphi+\varphi']_{\mathcal{I}}\geq P^\theta[\varphi]_{\mathcal{I}}+P^{\theta'}[\varphi']_{\mathcal{I}}.$$

Thus, by the monotonicity theorem of [WN19], it suffices to show that

$$\left(\int_X\left(\theta+\theta'+\mathrm{dd^c}P^\theta[\varphi]_{\mathcal{I}}+\mathrm{dd^c}P^{\theta'}[\varphi']_{\mathcal{I}}\right)^n\right)^{1/n}\geq\left(\int_X\theta^n_{P^\theta[\varphi]_{\mathcal{I}}}\right)^{1/n}+\left(\int_X\theta'^n_{P^{\theta'}[\varphi']_{\mathcal{I}}}\right)^{1/n}.$$

This follows from [DDL21, Theorem 6.1]. □

3. The closure of the space of linear series

Let X be a smooth irreducible projective variety of dimension n. Let D be a big $\mathbb{R}$-divisor on X. Define

$$R(X,D) := \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X, \mathcal{O}_X(kD))\,.$$

Recall that $\mathcal{O}_X(kD)$ means $\mathcal{O}_X(\lfloor kD \rfloor)$.

3.1. Graded linear subspaces.

Definition 3.1. A *graded linear subspace* of $R(X,D)$ is a linear space

$$W = \bigoplus_{k=0}^{\infty} W_k\,, \quad W_k \subseteq \mathrm{H}^0(X, \mathcal{O}_X(kD))\,.$$

Let $\mathrm{LS}(D)$ be the set of graded linear subspaces of $R(X,D)$. When D is integral and $L = \mathcal{O}_X(D)$, we also write $\mathrm{LS}(L)$ for $\mathrm{LS}(D)$.

Given any $W, W' \in \mathrm{LS}(D)$, we define

$$(W+W')_k := W_k + W'_k\,, \quad W+W' := \bigoplus_{k=0}^{\infty} (W+W')_k\,,$$

$$(W\cap W')_k := W_k \cap W'_k\,, \quad W\cap W' := \bigoplus_{k=0}^{\infty} (W\cap W')_k\,.$$

For $W, W' \in \mathrm{LS}(D)$, define

$$\begin{aligned} d(W,W') &:= \varlimsup_{k\to\infty} k^{-n}\left(2\dim(W_k+W'_k) - \dim W_k - \dim W'_k\right) \\ &= \varlimsup_{k\to\infty} k^{-n}\left(\dim W_k + \dim W'_k - 2\dim(W_k\cap W'_k)\right)\,. \end{aligned} \tag{3.1}$$

Lemma 3.2. *d is a pseudo-metric on* $\mathrm{LS}(D)$.

Proof. We only need to verify the triangle inequality. Let $W, W', W'' \in \mathrm{LS}(D)$. We claim that for any $k \in \mathbb{N}$,

$$\begin{aligned} \dim W_k + \dim W'_k - 2\dim(W_k\cap W'_k) + \dim W''_k + \dim W'_k - 2\dim(W'_k\cap W''_k) \\ \geq \dim W_k + \dim W''_k - 2\dim(W_k \cap W''_k)\,. \end{aligned} \tag{3.2}$$

From this the triangle inequality follows. In fact, (3.2) is equivalent to

$$\dim W'_k - \dim(W_k\cap W'_k) \geq \dim(W'_k \cap W''_k) - \dim(W_k\cap W''_k)\,.$$

It is elementary that

$$\dim W'_k - \dim(W_k\cap W'_k) \geq \dim(W'_k \cap W''_k) - \dim(W_k\cap W'_k \cap W''_k)\,.$$

The desired inequality follows. □

For $W, W' \in \mathrm{LS}(D)$, we say $W \sim W'$ if $d(W,W') = 0$. This is an equivalence relation by Lemma 3.2.

Remark 3.3. Let H be a fixed effective $\mathbb{R}$-divisor on X. Then we have a natural inclusion $R(X,D) \to R(X,D+H)$. Thus, $\mathrm{LS}(D) \subseteq \mathrm{LS}(D+H)$. Moreover, this inclusion is an isometric embedding.

Similarly, if we replace X by a birational model $\pi : Y \to X$ and D by π^*D, we can identify $\mathrm{LS}(D)$ and $\mathrm{LS}(\pi^*D)$. The metrics on them are also naturally identified.

We will use this remark implicitly.

Lemma 3.4. *For any $W, W', W'' \in \mathrm{LS}(D)$, we have*

$$d(W\cap W'', W'\cap W'') \leq d(W,W')\,,\quad d(W+W'', W'+W'') \leq d(W,W')\,.$$

Proof. Observe that

$$\dim(W_k\cap W''_k) - \dim(W_k\cap W'_k\cap W''_k) \leq \dim W_k - \dim(W_k\cap W'_k)$$

and the same holds if we interchange W_k and W'_k, thus,

$$\dim(W_k\cap W''_k)+\dim(W'_k\cap W''_k)-2\dim(W_k\cap W'_k\cap W''_k) \leq \dim W_k+\dim W'_k - 2\dim(W_k\cap W'_k)\,.$$

From this the first assertion follows. The second is similar. □

Corollary 3.5. *Let $W^i, V^i \in \mathrm{LS}(D)$ be two sequences. Assume that $W^i \to W$, $V^i \to V$. Then*

$$W^i\cap V^i \to W\cap V\,,\quad W^i+V^i\to W+V$$

Proof. In fact, by Lemma 3.4,

$$d(W^i\cap V^i, W\cap V) \leq d(W^i\cap V^i, W\cap V^i) + d(W\cap V^i, W\cap V) \leq d(W^i,W)+d(V^i,V)\,.$$

Hence, the first assertion follows. The second is similar. □

Lemma 3.6. *Let $W^i, W'^i \in \mathrm{LS}(D)$ ($i\in\mathbb{N}$). Assume that*

(1) W^i is increasing in i and W'^i is decreasing in i.
(2) For any i, $W'^i \supseteq W^i$.
(3) $d(W^i, W'^i)\to 0$ as $i\to\infty$.

Then $\sum_j W^j \sim \bigcap_j W'^j$ and

$$W^i \to \sum_j W^j\,,\quad W'^i \to \bigcap_j W'^j$$

as $i\to\infty$.

Proof. Let $V=\sum_i W^i$ and $V'=\bigcap_i W'^i$. Clearly $V\subseteq V'$. For the first assertion, it suffices to prove that

$$\varlimsup_{k\to\infty} k^{-n}(\dim V'_k - \dim V_k) \leq 0\,.$$

In fact,

$$\varlimsup_{k\to\infty} k^{-n}(\dim V'_k - \dim V_k) \leq \varlimsup_{k\to\infty} k^{-n}(\dim W'^i_k - \dim W^i_k) = o(1)\,.$$

Our claim follows.

As for the second, it suffices to observe that

$$d(W^i, V)\leq d(W^i, W'^i)\,,\quad d(W'^i, V')\leq d(W^i,W'^i)\,.$$

□

3.2. Graded linear series.

Definition 3.7. A *graded linear series* on X relative to D is an element $W\in\mathrm{LS}(D)$ such that

(1) $W_0=\mathbb{C}$.
(2) $W_a\cdot W_b\subseteq W_{a+b}$ for all $a,b\in\mathbb{N}$.

Let $\mathrm{GLS}(D)$ be the set of graded linear series with respect to D. When D is integral and $L=\mathcal{O}_X(D)$, we also write $\mathrm{GLS}(L)$ for $\mathrm{GLS}(D)$.

Remark 3.8. In the whole paper, we do not really need to add W_k for different k. So it seems more natural to use to notion of *annéloïde* introduced in [Duc21] instead. As the latter notion is not yet widely adopted in the literature, we stick to the more traditional notion here for the convenience of readers.

Given any $W \in \mathrm{GLS}(D)$, define its *volume* as

$$\operatorname{vol} W := \lim_{k\to\infty} (ka)^{-n} \dim W_{ka}\,, \tag{3.3}$$

where a is a sufficiently divisible positive integer. The existence of the limit and its independence from a both follow from [KK12, Theorem 2] and [LM09, Lemma 1.4] after choosing an arbitrary admissible flag.

Remark 3.9. We cannot always take $a = 1$ in (3.3) as it can happen that $W_k \neq 0$ only when k is a multiple of a.

Lemma 3.10. *Let $W, W' \in \mathrm{GLS}(D)$. Then*

$$|\operatorname{vol} W - \operatorname{vol} W'| \leq d(W, W')\,.$$

Proof. For $W, W' \in \mathrm{GLS}(D)$, by definition, we have

$$d(W, W') \geq \operatorname{vol} W + \operatorname{vol} W' - 2\operatorname{vol}(W \cap W')\,. \tag{3.4}$$

It follows that

$$\operatorname{vol} W - \operatorname{vol} W' \leq d(W, W')\,, \quad \operatorname{vol} W' - \operatorname{vol} W \leq d(W, W')\,.$$

□

Let $\overline{\mathrm{LS}(D)}$ be a metric completion of $\mathrm{LS}(D)$. Let $\overline{\mathrm{GLS}(D)}$ be the metric completion of $\mathrm{GLS}(D)$ inside $\overline{\mathrm{LS}(D)}$. As the $\operatorname{vol} : \mathrm{GLS}(D) \to \mathbb{R}$ is 1-Lipschitz Lemma 3.10, it extends uniquely to a 1-Lipschitz function $\operatorname{vol} : \overline{\mathrm{GLS}(D)} \to \mathbb{R}$. By Lemma 3.4, intersection and plus naturally extend to bi-1-Lipschitz operators on $\overline{\mathrm{LS}(D)}$.

Let $\overline{\mathrm{GLS}(D)}_{>0}$ (resp. $\mathrm{GLS}(D)_{>0}$) denote the subspace of $\overline{\mathrm{GLS}(D)}$ (resp. $\mathrm{GLS}(D)$) consisting of elements W with $\operatorname{vol} W > 0$.

3.3. **Okounkov bodies.** Let $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ be rank n valuation with rational rank n. By [CFKL+17, Theorem 2.9], ν is the valuation induced by an admissible flag on some birational model $\pi : Z \to X$ of X.

Definition 3.11. Let $W \in \mathrm{LS}(D)$ be a graded linear subspace. We set

$$\Gamma_k(W) := \left\{ k^{-1}\nu(s) \in (k^{-1}\mathbb{N})^n : s \in W_k^\times \right\}\,, \quad \Delta_k(W) = \operatorname{Conv}(\Gamma_k(W)) \subseteq \mathbb{R}^n\,.$$

Let

$$\Gamma(W) = \left\{ (ka, k) \in \mathbb{N}^{n+1} : a \in \Gamma_k(W), k \in \mathbb{N} \right\}\,.$$

Let $C(W) \subseteq \mathbb{R}^{n+1}$ be the closed convex cone generated by $\Gamma(W)$.

It can happen that $\Gamma_k(W) = \emptyset$ and $\Delta_k(W) = \emptyset$ for some k.

Definition 3.12. For any $W \in \mathrm{GLS}(D)$, we let $\Delta(W)$ be the Okounkov body of $\Gamma(W)$ in the sense of [KK12]. Recall that this means

$$\Delta(W) = C(W) \cap \{x_{n+1} = 1\} \in \mathcal{K}_n\,.$$

Here we write the coordinates on $\mathbb{R}^{n+1}$ as $(x_1, \dots, x_n, x_{n+1})$.

Theorem 3.13. *For each $W \in \mathrm{GLS}(D)$ with $\operatorname{vol} W > 0$,*

$$\operatorname{vol} W = \operatorname{vol} \Delta(W)\,. \tag{3.5}$$

There is an integer $a > 0$ so that

$$\Delta_{ak}(W) \xrightarrow{d_n} \Delta(W)\,, \quad \text{as } k \to \infty\,. \tag{3.6}$$

If $\Gamma(W)$ generates $\mathbb{Z}^{n+1}$, then we can take $a = 1$.

This is essentially proved in [WN14, Lemma 4.8], taking into account of Lemma 2.6.

Proof. The equality (3.5) follows from [KK12, Theorem 2]. As we have assumed $\operatorname{vol} W > 0$, it follows that $\Gamma(W)$ generates a full rank lattice. Up to replacing $\mathbb{Z}^{n+1}$ by a smaller lattice, we can now assume that $\Gamma(W)$ generates $\mathbb{Z}^{n+1}$.

Claim: Fix $M > 0$. Let $K \subseteq \Delta_M(W)$ be a compact subset. Then there is $C_K > 0$ such that for any $\alpha \in K \cap (k^{-1}\mathbb{Z})^n$ satisfying $d(\alpha, \partial K) > k^{-1}C_K$, we have $\alpha \in \Delta_k(W)$.

To see this claim, consider the semigroup Γ generated by $(M\beta, M)$ for all $\beta \in \Gamma_M(W)$ and some unit simplex in $\Gamma(W)$. We conclude by [WN14, Lemma 2.3].

In particular, for any compact set $K \subseteq \operatorname{Int} \Delta(W)$, there is $k_0 > 0$ such that for any $k \geq k_0$, $\alpha \in K \cap (k^{-1}\mathbb{Z})^n$ implies that $\alpha \in \Delta_k(W)$.

In particular, taking $K = \Delta(W)^\delta$ for any $\delta > 0$ and applying Lemma 2.6, we find

$$d_n(\Delta(W), \Delta_k(W)) \leq n^{1/2}k^{-1} + \delta$$

when k is large enough. This implies (3.6). $\square$

Lemma 3.14. *Let $W, W' \in \operatorname{GLS}(D)$. Assume that $\operatorname{vol} W > 0$ and $d(W, W') = 0$, then $\Delta(W) = \Delta(W')$.*

Proof. By Lemma 3.4, we can replace W' by $W \cap W'$ and assume that $W \supseteq W'$. Then clearly $\Delta(W) \supseteq \Delta(W')$. By (3.5),

$$\operatorname{vol} \Delta(W) = \operatorname{vol} \Delta(W').$$

Thus, $\Delta(W) = \Delta(W')$ by Lemma 2.5. $\square$

Definition 3.15. For any $W \in \overline{\operatorname{GLS}(D)}_{>0}$, if there exists $W' \in \operatorname{GLS}(D)$ so that $d(W, W') = 0$, we define

$$\Delta(W) := \Delta(W') \in \mathcal{K}_n.$$

This is independent of the choice of W' by Lemma 3.14. Observe that

$$\operatorname{vol} W = \operatorname{vol} \Delta(W). \tag{3.7}$$

Definition 3.16. We say $W \in \operatorname{LS}(D)$ satisfy the *Hausdorff convergence property* if $W \in \overline{\operatorname{GLS}(D)}_{>0}$, there exists $W' \in \operatorname{GLS}(D)$ such that $W \sim W'$ and $\Delta_k(W) \xrightarrow{d_n} \Delta(W)$.

We emphasize that this definition depends on the choice of the valuation.

Also observe that if $W \in \operatorname{LS}(D)$ satisfies the Hausdorff convergence property, then

$$\operatorname{vol} W = \lim_{k \to \infty} k^{-n} \dim W_k.$$

This is a consequence of [KK12, Theorem 2].

Lemma 3.17. *Let $W \in \operatorname{LS}(D)$. Assume that there are $W' \in \operatorname{LS}(D)$, $W''^i \in \operatorname{GLS}(D)$ ($i \in \mathbb{N}$) satisfying the Hausdorff convergence property such that*

(1) $W'_k \subseteq W_k \subseteq W''^i_k$ for large enough k (depending on i).
(2) W''^i is decreasing in i and $W' \subseteq W''^i$ for each i.
(3) $\operatorname{vol} \Delta(W') > 0$.
(4) $\operatorname{vol} \Delta(W''^i) - \operatorname{vol} \Delta(W') = o(1)$ as $i \to \infty$.

Then W also satisfies the Hausdorff convergence property and

$$\Delta(W) = \bigcap_{i=0}^{\infty} \Delta(W''^i) = \Delta(W'). \tag{3.8}$$

Proof. By Lemma 3.6, $W''^i \to \bigcap_j W''^j$ and thus,

$$d(W', \bigcap_j W''^j) = 0.$$

On the other hand, $d(W,W') \leq d(W',W''^i) = o(1)$. We find $d(W,W')=0$. Hence, $W \in \overline{\mathrm{GLS}(D)}_{>0}$ and (3.8) holds.

It remains to prove that $\Delta_k(W)$ converges to $\bigcap_i \Delta(W''^i)$. By Theorem 2.2, we may assume that $\Delta_k(W)$ converges to some $\Delta \in \mathcal{K}_n$. It is clear that $\Delta \supseteq \Delta(W')$ and for each i, $\Delta \subseteq \Delta(W''^i)$. We conclude that $\Delta = \Delta(W)$ by (3.8). □

3.4. **Continuous extension of the Okounkov body map.** First we extend the definition of Okounkov bodies to a decreasing limit of elements in $\mathrm{GLS}(D)$.

Lemma 3.18. *Let $W \in \overline{\mathrm{GLS}(D)}_{>0}$. Assume that there exists a decreasing sequence $W^i \in \mathrm{GLS}(D)$ converging to W, then*

$$\Delta(W) := \bigcap_{i=0}^{\infty} \Delta(W^i)$$

does not depend on the choice of W^i. Moreover,

$$\operatorname{vol} W = \operatorname{vol} \Delta(W)\,. \tag{3.9}$$

In particular, this definition is compatible with Definition 3.15.

Proof. Observe that (3.9) follows immediately from Theorem 2.3.

Let U^i be another decreasing sequence in $\mathrm{GLS}(D)$ converging to W. We want to show that

$$\Delta(W) = \bigcap_{i=0}^{\infty} \Delta(U^i)\,.$$

Observe that $U^i \cap W^i$ is also a decreasing sequence converging to W, we may assume that $U^i \subseteq W^i$. Thus, the $\supseteq$ direction of the above equality is clear. On the other hand, both sides have the same volume by (3.9), we conclude by Lemma 2.5. □

Let $W^i \in \mathrm{GLS}(D)$ be a Cauchy sequence converging to $W \in \overline{\mathrm{GLS}(D)}_{>0}$. We assume that

$$d(W^i, W^{i+1}) < 2^{-i}\,.$$

For $i,j \geq 0$, let $V^{i,j} = W^i \cap \cdots \cap W^{i+j}$. Then by Lemma 3.4,

$$d(V^{i,j}, V^{i,j+1}) \leq 2^{-i-j}\,.$$

Thus, $V^{i,j}$ is a Cauchy sequence in j. Let $V^i \in \overline{\mathrm{GLS}(D)}$ be its limit. Observe that V^i is also a Cauchy sequence. Moreover, $V^i \to W$. Hence, for i large enough, $V^i \in \overline{\mathrm{GLS}(D)}_{>0}$. Thus, $\Delta(V^i)$ is well-defined by Lemma 3.18. Observe that $\Delta(V^i)$ is an increasing sequence. We define

$$\Delta(W) := \overline{\bigcup_i \Delta(V^i)}\,.$$

In other words, we have defined

$$\Delta(W) := \varliminf_{i\to\infty} \Delta(W^i)\,.$$

It follows from (3.9) that

$$\operatorname{vol} W = \operatorname{vol} \Delta(W)\,.$$

If Δ is the limit of a subsequence of $\Delta(W^i)$, then $\Delta(W) \subseteq \Delta$ by (2.1). Comparing the volumes, we find $\Delta(W) = \Delta$, hence

$$\Delta(W) = \lim_{i\to\infty} \Delta(W^i)\,. \tag{3.10}$$

Lemma 3.19. *The definition $\Delta(W)$ does not on the choices we made.*

Proof. Let $W'^i \in \mathrm{GLS}(D)$ be another sequence converging to W such that $d(W'^i, W'^{i+1}) < 2^{-i}$. Let $U^i = W^i \cap W'^i$. Then
$$d(U^i, U^{i+1}) < 2^{1-i}\,.$$
Clearly
$$\varinjlim_{i\to\infty} \Delta(U^i) \subseteq \varinjlim_{i\to\infty} \Delta(W^i)\,.$$
But both sides have the same volume $\operatorname{vol} W > 0$ as argued above, so they are indeed equal. Similarly,
$$\varinjlim_{i\to\infty} \Delta(U^i) = \varinjlim_{i\to\infty} \Delta(W'^i)\,.$$
□

Definition 3.20. Let $W \in \overline{\mathrm{GLS}(D)}_{>0}$. The convex body $\Delta(W)$ constructed above is called the *Okounkov body* of W.

Theorem 3.21. *The Okounkov body map*
$$\Delta : \overline{\mathrm{GLS}(D)}_{>0} \to \mathcal{K}_n$$
is continuous. Moreover, for any $W \in \overline{\mathrm{GLS}(D)}_{>0}$, we have
$$\operatorname{vol} W = \operatorname{vol} \Delta(W)\,. \tag{3.11}$$

Proof. The formula (3.11) is already proved during the construction of Δ.

Let $W^i \in \overline{\mathrm{GLS}(D)}_{>0}$ be a sequence converging to $W \in \overline{\mathrm{GLS}(D)}_{>0}$. We want to show that
$$\Delta(W^i) \xrightarrow{d_n} \Delta(W)\,. \tag{3.12}$$

Step 1. We reduce to the case where all W^i's are in $\mathrm{GLS}(D)_{>0}$.

By (3.10), for each i, we can choose $V^i \in \mathrm{GLS}(D)_{>0}$ with $d(V^i, W^i) < 2^{-i}$ and $d_n(\Delta(V^i), \Delta(W^i)) < 2^{-i}$. If we have shown $\Delta(V^i) \xrightarrow{d_n} \Delta(W)$, then (3.12) follows immediately.

Step 2. We reduce to the case $d(W^i, W^{i+1}) < 2^{-i}$.

In order to prove (3.12), it suffices to show that each subsequence of $\Delta(W^i)$ admits a subsequence that converges to $\Delta(W)$. Hence, we easily reduce to the required case.

Step 3. Finally, after all these reductions, (3.12) is already proved in (3.10). □

Lemma 3.22. *Let $W, W' \in \mathrm{LS}(D)$. Assume that they both lie in $\overline{\mathrm{GLS}(D)}_{>0}$. If $W \subseteq W'$, then $\Delta(W) \subseteq \Delta(W')$.*

Proof. Let $V^i \in \mathrm{GLS}(D)_{>0}$ be a sequence converging to W'. Then $V^i \cap W$ also converges to W. By Theorem 3.21, we may therefore assume that $W' \in \mathrm{GLS}(D)_{>0}$.

Let $U^i \in \mathrm{GLS}(D)_{>0}$ be a sequence converging to W. Then $U^i \cap W'$ also converges to W. In particular, we may assume that $U^i \subseteq W'$. It suffices to prove the result with U^i in place of W. Thus, we are reduced to the case $W, W' \in \mathrm{GLS}(D)_{>0}$. In this case $\Delta(W) \subseteq \Delta(W')$ follows from Theorem 3.13. □

Remark 3.23. Suppose $\pi : Y \to X$ is a birational resolution. Then $\pi^* : \mathrm{GLS}(D) \to \mathrm{GLS}(\pi^* D)$ is an isometric isomorphism. Thus, it extends to an isometric isomorphism $\pi^* : \overline{\mathrm{GLS}(D)}_{>0} \to \overline{\mathrm{GLS}(\pi^* D)}_{>0}$. For any $W \in \overline{\mathrm{GLS}(D)}$, $\Delta(W) = \Delta(\pi^* W)$.

4. The metric on the space of singularity types

Let X be a connected compact Kähler manifold of dimension n and θ (resp. θ_i, $i = 1, \dots, n$) be a smooth real $(1,1)$-form representing a big $(1,1)$-cohomology class $[\theta]$ (resp. $[\theta_i]$). Let ω be a Kähler form on X.

In this section, we develop further the metric geometry on the space of singularity types of quasi-psh functions, initiated in [DDNL21] and studied further in [DX20].

As explained in [DDNL21, Section 3], one can introduce a pseudo-metric d_S on the set of singularity types of functions in $\mathrm{PSH}(X,\theta)$. In particular, d_S lifts to a pseudo-metric on $\mathrm{PSH}(X,\theta)$ as well. We do not recall the precise definition, as the following double inequality from [DDNL21, Proposition 3.5] will be enough for us. For any $\varphi,\psi\in\mathrm{PSH}(X,\theta)$ we have

$$d_S(\varphi,\psi)\leq\sum_{i=0}^{n}\left(2\int_X\theta^i_{\max\{\varphi,\psi\}}\wedge\theta^{n-i}_{V_\theta}-\int_X\theta^i_{\varphi}\wedge\theta^{n-i}_{V_\theta}-\int_X\theta^i_{\psi}\wedge\theta^{n-i}_{V_\theta}\right)\leq C_0 d_S(\varphi,\psi)\,, \tag{4.1}$$

where $C_0>1$ is a constant depending only on n. In addition, $d_S(\varphi,\psi)=0$ if and only if

$$C[\varphi]=C[\psi]\,.$$

When there is a risk of confusion, we write $d_{S,\theta}$ instead of d_S.

Lemma 4.1. *Let $\varphi_i\in\mathrm{PSH}(X,\theta_i)$ $(i=1,\dots,n)$. Then*

$$\int_X\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}=\int_X\theta_{1,C[\varphi_1]}\wedge\cdots\wedge\theta_{n,C[\varphi_n]}\,.$$

Proof. Since $[u]\preceq[C[v]]$, the $\preceq$ direction is obvious. For the reverse direction, recall that $C[\varphi_i]=\lim_{\epsilon\to0+}P[(1-\epsilon)\varphi_i+\epsilon V_{\theta_i}]$. Thus, for $\epsilon\in(0,1)$,

$$\int_X\theta_{1,C[\varphi_1]}\wedge\cdots\wedge\theta_{n,C[\varphi_n]}\geq(1-\epsilon)^n\int_X\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,.$$

Letting $\epsilon\to0+$, we conclude. □

Theorem 4.2. *Let $\varphi_i^k,\varphi_i\in\mathrm{PSH}(X,\theta_i)$ $(i=1,\dots,n,\ k\in\mathbb{N})$. Assume that $\varphi_i^k\xrightarrow{d_{S,\theta_i}}\varphi_i$ as $k\to\infty$. Then*

$$\lim_{k\to\infty}\int_X\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}=\int_X\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,. \tag{4.2}$$

Proof. By Lemma 4.1, we may assume that φ_i^k and φ^k are model potentials.

Step 1. We assume that there is a constant $\delta>0$, such that for all i and k,

$$\int_X\theta^n_{i,\varphi_i^k}>\delta\,.$$

In order to show (4.2), it suffices to prove that any subsequence of $\int_X\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}$. Thus, by [DDNL21, Theorem 5.6], we may assume that for each fixed i, φ_i^k is either increasing or decreasing. We may assume that for $i\leq i_0$, the sequence is decreasing and for $i>i_0$, the sequence is increasing.

Recall that in (4.2) the $\geq$ inequality always holds [DDNL18b, Theorem 2.3], it suffices to prove

$$\varlimsup_{k\to\infty}\int_X\theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\leq\int_X\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,. \tag{4.3}$$

By Witt Nyström's monotonicity theorem [WN19; DDNL18b], in order to prove (4.3), we may assume that for $j>i_0$, the sequences φ_j^k are constant. Thus, we are reduced to the case where for all i, φ_i^k are decreasing.

In this case, for each i we may take an increasing sequence $b_i^k>1$, tending to ∞, such that

$$(b_i^k)^n\int_X\theta^n_{i,\varphi_i}\geq\left((b_i^k)^n-1\right)\int_X\theta^n_{i,\varphi_i^k}\,.$$

Let ψ_i^k be the maximal θ_i-psh function, such that

$$(b_i^k)^{-1}\psi_i^k+\left(1-(b_i^k)^{-1}\right)\varphi_i^k\leq\varphi_i\,,$$

whose existence is guaranteed by [DDNL21, Lemma 4.3].

Then by Witt Nyström's monotonicity theorem [WN19; DDNL18b] again,

$$\prod_{i=1}^n \left(1-(b_i^k)^{-1}\right)\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\leq \int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\,.$$

Let $k\to\infty$, we conclude (4.3).

Step 2. Now we deal with the general case.

We claim that if $t\in(0,1]$, $(1-t)\varphi_i^k+tV_{\theta_i}\xrightarrow{d_S}(1-t)\varphi_i+tV_{\theta_i}$ as $k\to\infty$ for each i. From this and Step 1, we find that for $t_i\in(0,1]$,

$$\lim_{k\to\infty}\int_X \theta_{1,(1-t_1)\varphi_1^k+t_1V_{\theta_1}}\wedge\cdots\wedge\theta_{n,(1-t_n)\varphi_n^k+t_nV_{\theta_n}}=\int_X \theta_{1,(1-t_1)\varphi_1+t_1V_{\theta_1}}\wedge\cdots\wedge\theta_{n,(1-t_n)\varphi_n+t_nV_{\theta_n}}\,.$$

Thus, (4.2) follows, after letting $t_i\searrow 0$.

It remains to prove the claim. For simplicity, let us suppress the i indices momentarily. We need to argue that

$$2\int_X \theta^j_{\max\{(1-t)\varphi^k+tV_\theta,(1-t)\varphi+tV_\theta\}}\wedge\theta_{V_\theta}^{n-j}-\int_X\theta^j_{(1-t)\varphi^k+tV_\theta}\wedge\theta_{V_\theta}^{n-j}-\int_X\theta^j_{(1-t)\varphi+tV_\theta}\wedge\theta_{V_\theta}^{n-j}\to 0\,.$$

Note that the above expression is a linear combination of terms of the following type:

$$2\int_X \theta^r_{\max\{\varphi^k,\varphi\}}\wedge\theta_{V_\theta}^{n-r}-\int_X\theta^r_{\varphi^k}\wedge\theta_{V_\theta}^{n-r}-\int_X\theta^r_{\varphi^k}\wedge\theta_{V_\theta}^{n-r}\,.$$

All these expressions tend to 0 as $k\to\infty$ since $\varphi^k\xrightarrow{d_S}\varphi$, which proves our claim. □

Corollary 4.3. *Let $\varphi^k,\varphi\in\mathrm{PSH}(X,\theta)$ ($k\in\mathbb{N}$). Let ω be a Kähler form on X. Assume that $\varphi^k\xrightarrow{d_{S,\theta}}\varphi$. Then $\varphi^k\xrightarrow{d_{S,\theta+\omega}}\varphi$.*

Proof. It suffices to show that for each $j=0,\dots,n$, we have

$$2\int_X(\theta+\omega)^j_{\max\{\varphi^k,\varphi\}}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^j_{\varphi^k}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^j_{\varphi}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\to 0$$

as $k\to\infty$. Note that this quantity is a linear combination of terms of the following form:

$$2\int_X\theta^r_{\max\{\varphi^k,\varphi\}}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X\theta^r_{\varphi^k}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X\theta^r_{\varphi}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\,,$$

where $r=0,\dots,j$. By Theorem 4.2, it suffices to show that $\max\{\varphi,\varphi^k\}\xrightarrow{d_S}\varphi$. But this follows from [DDNL21, Proposition 3.5]. □

Corollary 4.4. *Let $\varphi\in\mathrm{PSH}(X,\theta)$ be an $\mathcal{I}$-model potential of positive mass. Let ω be a Kähler form on X. Then $P^{\theta+\omega}[\varphi]$ is $\mathcal{I}$-model.*

Proof. Take a sequence φ^j with analytic singularities such that $\varphi^j\xrightarrow{d_{S,\theta}}\varphi$. Then $\varphi^j\xrightarrow{d_{S,\theta+\omega}}\varphi$ by Corollary 4.3. Thus, $P^{\theta+\omega}[\varphi]$ is $\mathcal{I}$-model. □

Corollary 4.5. *Let $\varphi_i\in\mathrm{PSH}(X,\theta_i)$ ($i=1,\dots,n$). Assume that all $P[\varphi_i]_{\mathcal{I}}$'s have positive masses. Then*

$$\int_X\theta_{1,P[\varphi_1]_{\mathcal{I}}}\wedge\cdots\wedge\theta_{n,P[\varphi_n]_{\mathcal{I}}}=\langle\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\rangle\,.\tag{4.4}$$

The right-hand side is the mixed mass in the sense of Cao, see Definition 2.16.

Proof. By Corollary 4.4, for any $\epsilon>0$,

$$\int_X(\theta_1+\epsilon\omega)_{P^{\theta_1+\epsilon\omega}[\varphi_1]_{\mathcal{I}}}\wedge\cdots\wedge(\theta_n+\epsilon\omega)_{P^{\theta_n+\epsilon\omega}[\varphi_n]_{\mathcal{I}}}=\int_X(\theta_1+\epsilon\omega)_{P^{\theta_1}[\varphi_1]_{\mathcal{I}}}\wedge\cdots\wedge(\theta_n+\epsilon\omega)_{P^{\theta_n}[\varphi_n]_{\mathcal{I}}}\,.$$

In particular,

$$\lim_{\epsilon\to 0+}\int_X (\theta_1+\epsilon\omega)_{P^{\theta_1+\epsilon\omega}[\varphi_1]_{\mathcal{I}}}\wedge\cdots\wedge(\theta_n+\epsilon\omega)_{P^{\theta_n+\epsilon\omega}[\varphi_n]_{\mathcal{I}}}=\int_X \theta_{1,P[\varphi_1]_{\mathcal{I}}}\wedge\cdots\wedge\theta_{n,P[\varphi_n]_{\mathcal{I}}}\,. \tag{4.5}$$

Similarly, by definition,

$$\lim_{\epsilon\to 0+}\langle(\theta_1+\epsilon\omega)_{\varphi_1}\wedge\cdots\wedge(\theta_n+\epsilon\omega)_{\varphi_n}\rangle=\langle\theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}\rangle\,.$$

Thus, it suffices to prove (4.4) under the additional assumption that θ_{i,φ_i} is a Kähler current. In this case, take quasi-equisingular approximations φ_i^j ($j\in\mathbb{N}$) of φ_i, by Theorem 4.2 and the definition of Cao's masses, we find that both sides of (4.4) are limits of

$$\int_X \theta_{1,\varphi_1^j}\wedge\cdots\wedge\theta_{n,\varphi_n^j}\,.$$

□

Remark 4.6. In general, we expect Corollary 4.5 to fail when some $P[\varphi_i]_{\mathcal{I}}$ has zero mass, namely when some φ_i does not have full numerical dimension. We do not yet have any examples.

We also note that following corollaries of Theorem 4.2, which are of independent interest.

Corollary 4.7. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Assume that φ and ψ are both $\mathcal{I}$-model and of positive masses, then for any $t\in[0,1]$, $C[t\varphi+(1-t)\psi]$ is also $\mathcal{I}$-model.*

Proof. Let φ^j, ψ^j be potentials with analytic singularities converging to φ and ψ with respect to d_S. Then by [DX21, Theorem 3.8], it is known that $\varphi^j\xrightarrow{d_S}\varphi$ and $\psi^j\xrightarrow{d_S}\psi$. We claim that $t\varphi^j+(1-t)\psi^j\xrightarrow{d_S}t\varphi+(1-t)\psi$. This implies the desired result by [DX21, Theorem 3.8] again.

To prove the claim, it suffices to show that for each $r=0,\dots,n$,

$$2\int_X\theta^r_{\max\{t\varphi^j+(1-t)\psi^j,t\varphi+(1-t)\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{t\varphi^j+(1-t)\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{t\varphi+(1-t)\psi^j}\wedge\theta^{n-r}_{V_\theta}\to 0\,.$$

Note that the left-hand side is non-negative and as

$$\max\{t\varphi^j+(1-t)\psi^j,t\varphi+(1-t)\psi\}\leq t\max\{\varphi^j,\varphi\}+(1-t)\max\{\psi^j,\psi\}\,,$$

it suffices to show that

$$2\int_X\theta^r_{t\max\{\varphi^j,\varphi\}+(1-t)\max\{\psi^j,\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{t\varphi^j+(1-t)\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{t\varphi+(1-t)\psi^j}\wedge\theta^{n-r}_{V_\theta}\to 0\,.$$

The left-hand side is a linear combination of terms of the form

$$2\int_X\theta^a_{\max\{\varphi^j,\varphi\}}\wedge\theta^{r-a}_{\max\{\psi^j,\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^a_{\varphi^j}\wedge\theta^{r-a}_{\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^a_{\varphi}\wedge\theta^{r-a}_{\psi}\wedge\theta^{n-r}_{V_\theta}\,,$$

where $a=0,\dots,r$. Observe that $\max\{\varphi^j,\varphi\}\xrightarrow{d_S}\varphi$ and $\max\{\psi^j,\psi\}\xrightarrow{d_S}\psi$ by [DDNL21, Proposition 3.5], each term tends to 0 by Theorem 4.2. □

Corollary 4.8. *Let $\varphi^j,\varphi\in\mathrm{PSH}(X,\theta_1),\psi^j,\psi\in\mathrm{PSH}(X,\theta_2)$ ($j\in\mathbb{N}$). Assume that $\varphi^j\xrightarrow{d_{S,\theta_1}}\varphi$, $\psi^j\xrightarrow{d_{S,\theta_2}}\psi$. Then*

$$\varphi^j+\psi^j\xrightarrow{d_{S,\theta_1+\theta_2}}\varphi+\psi\,.$$

Proof. Let $\theta=\theta_1+\theta_2$. It suffices to show that for each $r=0,\dots,n$,

$$2\int_X\theta^r_{\max\{\varphi^j+\psi^j,\varphi+\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{\varphi^j+\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{\varphi+\psi}\wedge\theta^{n-r}_{V_\theta}\to 0\,.$$

Observe that

$$\max\{\varphi^j+\psi^j,\varphi+\psi\}\leq\max\{\varphi^j,\varphi\}+\max\{\psi^j,\psi\}\,.$$

Thus, it suffices to show that

$$2\int_X \theta^r_{\max\{\varphi^j,\varphi\}+\max\{\psi^j,\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{\varphi^j+\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^r_{\varphi+\psi}\wedge\theta^{n-r}_{V_\theta}\to 0\,.$$

The left-hand side is a linear combination of

$$2\int_X \theta^a_{1,\max\{\varphi^j,\varphi\}}\wedge\theta^{r-a}_{2,\max\{\psi^j,\psi\}}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^a_{1,\varphi^j}\wedge\theta^{r-a}_{2,\psi^j}\wedge\theta^{n-r}_{V_\theta}-\int_X\theta^a_{1,\varphi}\wedge\theta^{r-a}_{2,\psi}\wedge\theta^{n-r}_{V_\theta}$$

with $a=0,\ldots,r$. Observe that $\max\{\varphi^j,\varphi\}\xrightarrow{d_S}\varphi$ and $\max\{\psi^j,\psi\}\xrightarrow{d_S}\psi$ by [DDNL21, Proposition 3.5], each term tends to 0 by Theorem 4.2. □

Finally, we prove the continuity of $P[\bullet]_{\mathcal{I}}$.

Theorem 4.9. *The map* $\mathrm{PSH}(X,\theta)_{>0}\to\mathrm{PSH}(X,\theta)_{>0}$ *given by* $\varphi\mapsto P[\varphi]_{\mathcal{I}}$ *is continuous with respect to the* d_S*-pseudometric.*

Here $\mathrm{PSH}(X,\theta)_{>0}$ denotes the subset of $\mathrm{PSH}(X,\theta)$ consisting of φ with $\int_X\theta^n_\varphi>0$.

Proof. Let $\varphi_i,\varphi\in\mathrm{PSH}(X,\theta)_{>0}$, $\varphi_i\xrightarrow{d_S}\varphi$. We want to show that

$$P[\varphi_i]_{\mathcal{I}}\xrightarrow{d_S}P[\varphi]_{\mathcal{I}}\,. \tag{4.6}$$

We may assume that the φ_i's and φ are all model potentials. By [DDNL21, Theorem 5.6], we may assume that φ_i is either increasing or decreasing. Both cases follow from [DX20, Lemma 2.21] and [DDNL21, Proposition 4.8]. □

5. Partial Okounkov bodies

Let X be an irreducible smooth complex projective variety of dimension n and D be a big divisor on X. Let $L=\mathcal{O}_X(D)$. Take a singular positive Hermitian metric ϕ on L. We assume that $\mathrm{vol}(L,\phi)>0$. Let h be a smooth metric on L. Let $\theta=c_1(L,h)$. Then we can identify ϕ with a function $\varphi\in\mathrm{PSH}(X,\theta)$. We will use interchangeably the notations (θ,φ) and (L,ϕ). Let

$$W_k(\theta,\varphi):=\mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))\,,\quad W(\theta,\varphi)=\bigoplus_{k=0}^\infty W_k(\theta,\varphi)\,.$$

We omit (θ,φ) form our notations when there is no risk of confusion.

Fix a rank n valuation $\nu:\mathbb{C}(X)^\times\to\mathbb{Z}^n$ of rational rank n.

5.1. Construction of partial Okounkov bodies.

5.1.1. *The case of analytic singularities.* Assume that φ has analytic singularities and $\int_X\theta^n_\varphi>0$. We set

$$W_k(\theta,\varphi):=\mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))\,,\quad W(\theta,\varphi)=\bigoplus_{k=0}^\infty W_k(\theta,\varphi)\,.$$

Lemma 5.1. *Under the above assumptions,* $W(\theta,\varphi)$ *satisfies the Hausdorff convergence property. If we set*

$$\Delta(\theta,\varphi)=\Delta(W(\theta,\varphi))\,.$$

Then

$$\mathrm{vol}\,\Delta(\theta,\varphi)=\frac{1}{n!}\int_X\theta^n_\varphi\,. \tag{5.1}$$

Proof. For any $\epsilon \geq 0$, we define

$$W_k^\epsilon = W_k^\epsilon(\theta,\varphi) := \left\{ s \in \mathrm{H}^0(X, L^k) : |s|^2_{h^k} e^{-k(1-\epsilon)\varphi} \text{ is bounded} \right\} . \tag{5.2}$$

Then

$$W^\epsilon := \bigoplus_{k=0}^\infty W_k^\epsilon \in \mathrm{GLS}(D) . \tag{5.3}$$

For any $\epsilon \in \mathbb{Q}_{>0}$, it is a consequence of Ohsawa–Takegoshi theorem that

$$W_k^0 \subseteq W_k \subseteq W_k^\epsilon$$

for k large enough depending on ϵ. See [DX21, Remark 2.9] for details.

Let $\pi : Y \to X$ be a resolution such that $\pi^*\varphi$ has analytic singularities along an nc $\mathbb{Q}$-divisor E. Then we have a natural identification for k large

$$W_k^\epsilon \cong \mathrm{H}^0(Y, \pi^* L^k \otimes \mathcal{O}_Y(-(1-\epsilon)kE)) .$$

On the other hand,

$$W_k^0 \cong \mathrm{H}^0(Y, \pi^* L^k \otimes \mathcal{O}_Y(-kE)) \subseteq \mathrm{H}^0(Y, \pi^* L^k) .$$

We compute the volumes,

$$\operatorname{vol} \Delta(W^\epsilon) = \frac{1}{n!} \int_X \theta^n_{(1-\epsilon)\varphi} , \quad \operatorname{vol} \Delta(W^0) = \frac{1}{n!} \int_X \theta^n_\varphi . \tag{5.4}$$

Thus, we conclude by Lemma 3.17. ☐

Remark 5.2. It follows from the proof that if $W^0(\theta,\varphi)$ is defined as in (5.2) and (5.3):

$$W_k^0(\theta,\varphi) := \left\{ s \in \mathrm{H}^0(X, L^k) : |s|^2_{h^k} e^{-k\varphi} \text{ is bounded} \right\} ,$$

then

$$\Delta(W^0(\theta,\varphi)) = \Delta(\theta,\varphi) . \tag{5.5}$$

If we assume furthermore that φ has analytic singularity along some nc $\mathbb{Q}$-divisor E on X, then $\Delta(\theta,\varphi)$ is just the translation of $\Delta(L-E)$ by $\nu(E)$.

5.1.2. *The case of Kähler currents.* Now assume that θ_φ is Kähler current. Let $\varphi^j \in \mathrm{PSH}(X,\theta)$ be a quasi-equisingular approximation of φ.

In this case, we claim that

$$W(\theta,\varphi^j) \to W(\theta,\varphi) . \tag{5.6}$$

In fact, by Theorem 2.20, we have

$$\begin{aligned} d(W(\theta,\varphi^j), W(\theta,\varphi)) &= \operatorname{vol}(W(\theta,\varphi^j)) - \lim_{k\to\infty} k^{-n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) \\ &= \frac{1}{n!}\int_X \theta^n_{\varphi^j} - \frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} . \end{aligned}$$

Let $j \to \infty$, we conclude (5.6) by [DX21, Theorem 3.8].

Thus, $W(\theta,\varphi) \in \overline{\mathrm{GLS}(D)}_{>0}$. So we can define

$$\Delta(\theta,\varphi) = \Delta(W(\theta,\varphi)) .$$

By Theorem 3.21, we find that

$$\Delta(\theta,\varphi) = \bigcap_{j=0}^\infty \Delta(\theta,\varphi^j) . \tag{5.7}$$

In particular,

$$\operatorname{vol} \Delta(\theta,\varphi) = \frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} . \tag{5.8}$$

5.1.3. *General case.* Now we consider general φ with the assumption that $\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} > 0$. We may replace φ with $P[\varphi]_{\mathcal{I}}$ and then assume that the non-pluripolar mass of φ is positive. Take a potential $\psi \in \mathrm{PSH}(X,\theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. The existence of ψ is proved in [DX21, Proposition 3.6]. For each $\epsilon \in \mathbb{Q} \cap (0,1]$, let

$$\varphi_\epsilon = (1-\epsilon)\varphi + \epsilon\psi \,.$$

Then we have

$$W(\theta,\varphi_\epsilon) \subseteq W(\theta,\varphi) \,.$$

By (5.8),

$$\operatorname{vol} \Delta(\theta,\varphi_\epsilon) = \frac{1}{n!}\int_X \theta^n_{P[\varphi_\epsilon]_{\mathcal{I}}} \,.$$

By [DX21, Proposition 2.7], the sequence converges to $\frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}}$. Thus, by Theorem 2.20,

$$W(\theta,\varphi_\epsilon) \to W(\theta,\varphi)$$

as $\epsilon \to 0+$. Thus, $W(\theta,\varphi) \in \overline{\mathrm{GLS}(D)}_{>0}$. Define

$$\Delta(\theta,\varphi) = \Delta(W(\theta,\varphi)) \,.$$

By Theorem 3.21,

$$\Delta(\theta,\varphi) = \overline{\bigcup_{\epsilon>0} W(\theta,\varphi_\epsilon)}$$

and

$$\operatorname{vol} \Delta(\theta,\varphi) = \frac{1}{n!}\lim_{\epsilon\to 0+}\int_X \theta^n_{P[\varphi_\epsilon]_{\mathcal{I}}} = \frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} \,.$$

Definition 5.3. Assume that $\varphi \in \mathrm{PSH}(X,\theta)$, $\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} > 0$. We call $\Delta(\theta,\varphi)$ the *partial Okounkov body* of (L,ϕ) or of (θ,φ) with respect to ν. When ν is induced by an admissible flag $(Y_\bullet)$ on X, we also say that $\Delta(\theta,\varphi)$ the *partial Okounkov body* of (L,ϕ) or of (θ,φ) with respect to $(Y_\bullet)$.

We use interchangeably the notations $\Delta(\theta,\varphi)$ and $\Delta(L,\phi)$.

Remark 5.4. We do not know if the Hausdorff convergence property holds for a general $W(\theta,\varphi)$. By induction on dimension, we can show that this holds if the flag is very general.

Remark 5.5. We have assumed X to be smooth only for simplicity. All of our constructions work equally well when X is normal, based on the pluripotential theory in these settings developed in [Xia21].

Remark 5.6. It is of interest to see if the construction of partial Okounkov bodies can be generalized to transcendental classes as well. In general, we expect that the approximation techinques developed in [DX20; DX21] are still applicable, based on the constructions of [Den17] in the case without singularities. However, a key difficulty lies in the lack of extension theorems of Kähler currents in general, making the computation of volumes of Okounkov bodies extremely difficult. See [Den17, Conjecture 1.4].

5.2. **Basic properties of partial Okounkov bodies.** We first show that $\Delta(\theta,\varphi)$ does not depend on the explicit choice of L, h and φ, it just depends on $\mathrm{dd^c}\phi$.

Lemma 5.7. *Let L' be another line bundle. Let h' be a smooth Hermitian metric on L' with $c_1(L,h) = c_1(L',h')$. Then $\Delta(\theta,\varphi)$ defined with respect to (L,h) is the same as the one defined with respect to (L',h').*

Proof. From our construction, we may assume that θ_φ is a Kähler current and φ has analytic singularities. After taking a birational resolution, it suffices to deal with the case where φ has analytic singularities along nc $\mathbb{Q}$-divisors E. By rescaling, we may also assume that E is a divisor. By Remark 5.2, we further reduce to the case without the singular potential ϕ.

In this case, the assertion is essentially proved in [LM09, Proposition 4.1]. We reproduce the proof for the convenience of readers. Observe that $P := L' - L$ is numerically trivial. We can find a divisor B on X so that $B + kP$ is very ample for all $k \in \mathbb{Z}$. Choose $a \in \mathbb{N}$ large enough so that $aL - B$ is linearly equivalent to some effective divisor F. Then $(k+a)(L+P)$ is linearly equivalent to $kL + (aL - B) + (B + (k+a)P)$. As $B + (k+a)P$ is very ample, replacing it by a linearly equivalent divisor that does not pass through the center of ν on X, it does not affect the valuation. Thus,

$$\Delta_{k+a}(W^0_L(\theta,\varphi)) + \nu(F) \subseteq \Delta_k(W^0_{L'}(\theta,\varphi))\,.$$

Here the sub-indicies L and L' denote the corresponding objects defined using L and L'. Thus,

$$\Delta(W^0_L(\theta,\varphi)) \subseteq \Delta(W^0_{L'}(\theta,\varphi))\,.$$

The converse can be proved in exactly the same way. □

Lemma 5.8. *Let h' be another metric on L. Set $\theta' = c_1(L,h')$. Take $\mathrm{dd^c} f = \theta - \theta'$ Let $\varphi' = \varphi + f \in \mathrm{PSH}(X,\theta')$. Then*

$$\Delta(\theta,\varphi) = \Delta(\theta',\varphi')\,. \tag{5.9}$$

Proof. This is obvious as $W(\theta,\varphi) = W(\theta',\varphi')$. □

Corollary 5.9. *The partial Okounkov body $\Delta(L,\phi)$ depends only on $\mathrm{dd^c}\phi$ not on the explicit choices of L,ϕ,h.*

Proof. This is a direct consequence of Lemma 5.7 and Lemma 5.8. □

Let $\mathrm{PSH}(X,\theta)_{>0}$ denote the subset of $\mathrm{PSH}(X,\theta)$ consisting of potentials φ such that $\int_X \theta_\varphi^n > 0$.

Proposition 5.10. *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$. Assume that $\varphi \preceq_{\mathcal{I}} \psi$, then*

$$\Delta(\theta,\varphi) \subseteq \Delta(\theta,\psi)\,. \tag{5.10}$$

In particular,

$$\Delta(\theta,\varphi) \subseteq \Delta(L)\,.$$

Proof. This follows from Lemma 3.22. □

Proposition 5.11. *Let $\pi : Y \to X$ be a birational resolution. Let (L,ϕ) be a Hermitian psef line bundle on X with positive volume, then*

$$\Delta(\pi^*L,\pi^*\phi) = \Delta(L,\phi)\,.$$

Proof. Observe that $\pi^*W(\theta,\varphi) = W(\pi^*\theta,\pi^*\varphi)$. Our assertion follows from Remark 3.23. □

Theorem 5.12. *The Okounkov body map*

$$\Delta(\theta,\bullet) : \mathrm{PSH}(X,\theta)_{>0} \to \mathcal{K}_n$$

is continuous, where on $\mathrm{PSH}(X,\theta)_{>0}$ we endow the pseudo-metric d_S.

Proof. Let $\varphi_j \to \varphi$ be a converging sequence in $\mathrm{PSH}(X,\theta)_{>0}$. We want to show that

$$\Delta(\theta,\varphi_j) \xrightarrow{d_n} \Delta(\theta,\varphi)\,. \tag{5.11}$$

We may assume that all φ_j's and φ are model potentials. By Theorem 2.2 and [DDNL21, Theorem 5.6], we may assume that φ_j is either decreasing or increasing. In both cases, we have $W(\theta,\varphi_j) \to W(\theta,\varphi)$ by Theorem 4.9 and Theorem 2.20. Hence, (5.11) follows from Theorem 3.21. □

In particular, the proof of Theorem A is now complete.

Next we prove the Brunn–Minkowski inequality.

Theorem 5.13. *Let (L,ϕ), (L',ϕ') be Hermitian psef line bundles on X of positive volumes. Then*

$$(\operatorname{vol}\Delta(L+L',\phi+\phi'))^{1/n} \geq (\operatorname{vol}\Delta(L,\phi))^{1/n} + (\operatorname{vol}\Delta(L',\phi'))^{1/n}.$$

Proof. This follows from Corollary 2.22. ☐

Theorem 5.14. *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$. Then for any $t\in[0,1]$,*

$$\Delta(\theta, t\varphi+(1-t)\psi) \supseteq t\Delta(\theta,\varphi)+(1-t)\Delta(\theta,\psi)\,. \tag{5.12}$$

Proof. We may assume that t is rational as a consequence of Theorem 5.12. Similarly, by [DX21, Theorem 3.8], we could reduce to the case where both φ and ψ has analytic singularities. Taking a resolution, we may assume that φ (resp. ψ) has analytic singularities along an nc $\mathbb{Q}$-divisor E (resp. E'). In this case, let $N>0$ be an integer such that Nt is an integer. Then for any $s\in W^0_k(\theta,\varphi)$, $t\in W^0_k(\theta,\psi)$, we have

$$\left(s^t t^{1-t}\right)^N \in W^0_{Nk}(\theta, t\varphi+(1-t)\psi)\,.$$

By the Hausdorff convergence property, (5.12) follows. ☐

Proposition 5.15. *Let (L',ϕ') be another Hermitian psef line bundle on X with positive volume. Then*

$$\Delta(L,\phi)+\Delta(L',\phi') \subseteq \Delta(L\otimes L',\phi\otimes\phi')\,.$$

Proof. Take a smooth metric h' on L', let $\theta' = c_1(L',h')$. We identify ϕ' with $\varphi'\in\mathrm{PSH}(X,\theta')$. Then we need to show

$$\Delta(\theta,\varphi)+\Delta(\theta',\varphi') \subseteq \Delta(\theta+\theta',\varphi+\varphi')\,. \tag{5.13}$$

By [DX21, Theorem 3.8], we can find $\varphi^j\in\mathrm{PSH}(X,\theta)$, $\varphi'^j\in\mathrm{PSH}(X,\theta')$ such that

(1) φ^j and φ'^j both has analytic singularities.
(2) $\varphi^j \xrightarrow{d_S} \varphi$, $\varphi'^j \xrightarrow{d_S} \varphi'$.

Then $\varphi^j+\varphi'^j \in \mathrm{PSH}(X,\theta+\theta')$ and $\varphi^j+\varphi'^j \xrightarrow{d_S} \varphi+\varphi'$ by Corollary 4.8. Thus, by Theorem 5.12, we may assume that φ and ψ both have analytic singularities. Taking a birational resolution, we may further assume that they have analytic singularities along some nc divisors. By Remark 5.2, we reduce to the case without singularities, in which case the result is well-known. ☐

Proposition 5.16. *For any integer $a>0$,*

$$\Delta(a\theta,a\varphi) = a\Delta(\theta,\varphi)\,.$$

Proof. By Theorem 5.12, it suffices to treat the case where φ has analytic singularities. Taking a birational resolution, we may assume that φ has analytic singularities along an nc $\mathbb{Q}$-divisor E. By Remark 5.2, we reduce to the case without the singularity φ, which is already proved in [LM09]. ☐

We also need the following perturbation. Let A be an ample line bundle on X. Fix a smooth Hermitian metric h_A on A such that $\omega := c_1(A,h_A)$ is a Kähler form on X. Then for any $\delta\in\mathbb{Q}_{>0}$, we can define

$$\Delta(\theta+\delta\omega,\varphi) := \Delta(C\theta+C\delta\omega, C\varphi)\,,$$

where $C\in\mathbb{N}_{>0}$ is any integer so that $C\delta\in\mathbb{N}$. By Proposition 5.16, $\Delta(\theta+\delta\omega,\varphi)$ is independent of the choice of C.

Proposition 5.17. *Under the assumptions above, as $\delta\in\mathbb{Q}_{>0}$ decreases to 0, $\Delta(\theta+\delta\omega,\varphi)$ is decreasing with Hausdorff limit $\Delta(\theta,\varphi)$.*

Proof. Let $0 \leq \delta < \delta'$ be two rational numbers. Take $C \in \mathbb{N}_{>0}$ divisible enough, so that $C\delta$ and $C\delta'$ are both integers. Then by Proposition 5.15,

$$\Delta(C\theta + C\delta\omega, C\varphi) \subseteq \Delta(C\theta + C\delta'\omega, C\varphi)\,.$$

It follows that

$$\Delta(\theta + \delta\omega, \varphi) \subseteq \Delta(\theta + \delta'\omega, \varphi)\,.$$

On the other hand,

$$\operatorname{vol}\Delta(\theta + \delta\omega, \varphi) = \frac{1}{n!}\int_X (\theta + \delta\omega)^n_{P[\varphi]_{\mathcal{I}}}\,.$$

By [DX21, Corollary 3.5], as $\delta \to 0+$, the right-hand side converges to

$$\operatorname{vol}\Delta(\theta, \varphi) = \frac{1}{n!}\int_X \theta^n_{P[\varphi]_{\mathcal{I}}}\,.$$

It follows that

$$\Delta(\theta, \varphi) = \bigcap_{\delta \in \mathbb{Q}_{>0}} \Delta(\theta + \delta\omega, \varphi)\,.$$

□

Theorem 5.18. *The Okounkov body $\Delta(L, \phi)$ is independent of the choice of a very general flag in a family of admissible flags.*

Proof. By our construction, it suffices to prove this when ϕ has analytic singularities. In particular, the Hausdorff convergence property is satisfied in this case.

It suffices to show that $\Delta_k(W(\theta, \varphi))$ is independent of the choice of a very general flag. For this purpose, we may assume that $k = 1$.

Let T be an irreducible component of the moduli space of admissible flags. Let

$$X \times T = \mathcal{Y}_0 \supseteq \cdots \supseteq \mathcal{Y}_n$$

be the universal flag. The Hermitian line bundle (L, ϕ) pullbacks to $(\mathcal{L}, \Phi)$ on $X \times T$. We denote quantities at the fiber at $t \in T$ by a sub-index t.

We claim that for each $\sigma \in \mathbb{N}^n$, there is a proper Zarski closed set $\Sigma \subsetneq T$, so that

$$\dim \mathrm{H}^0(X_t, L_t \otimes \mathcal{I}(\phi_t))^{\geq \sigma}$$

are constants for $t \in T \setminus \Sigma$, where $\mathrm{H}^0(X_t, L_t \otimes \mathcal{I}(\phi_t))^{\geq \sigma}$ denotes the space of sections in $\mathrm{H}^0(X_t, L_t \otimes \mathcal{I}(\phi_t))$ with valuations no less that σ.

Let $\mathcal{L}^{\geq \sigma}$ be the coherent subsheaf of $\mathcal{L}$ introduced in [LM09, Remark 1.6]. After possibly shrinking T, we may guarantee that $\mathcal{L}^{\geq \sigma} \otimes \mathcal{I}(\Phi)$ is flat over T. By further shrinking T, we may guarantee that

$$\dim \mathrm{H}^0(X_t, (\mathcal{L}^{\geq \sigma} \otimes \mathcal{I}(\Phi))|_{X_t})$$

is constant. Observe that

$$(\mathcal{L}^{\geq \sigma} \otimes \mathcal{I}(\Phi))|_{X_t}) = L_t^{\geq \sigma} \otimes \mathcal{I}(\phi)\,.$$

Thus, our claim follows.

From this claim, it follows that the images of $\Gamma_k(W(L, \phi))$ are independent of the choice of a very general flag $(Y_\bullet)$ as [LM09, Proof of Theorem 5.1]. Thus, $\Delta(W(L, \phi))$ is independent of the choice of a very general flag. □

5.3. **Recover Lelong numbers from partial Okounkov bodies.**

Lemma 5.19. *Let E be a prime divisor on X. Let $(Y_\bullet)$ be an admissible flag with $E = Y_1$. Then*

$$\nu(\varphi, E) = \min_{x \in \Delta(\theta,\varphi)} x_1 \,. \tag{5.14}$$

Here x_1 denotes the first component of x.

Proof. We first reduce to the case where θ_φ is a Kähler current. Let $\psi \leq \varphi$, θ_ψ is a Kähler current. Then by (5.14) applied to $\varphi_\epsilon := (1-\epsilon)\varphi + \epsilon\psi$, we have

$$\nu(\varphi_\epsilon, E) = \min_{x \in \Delta(\theta,\varphi_\epsilon)} x_1 \,.$$

Let $\epsilon \to 0+$ using Theorem 5.12, we conclude (5.14).

Similarly, taking a quasi-equisingular approximation of φ and applying [Xia20, Lemma 2.2]*, we easily reduce to the case where φ also has analytic singularities. Replacing X by a birational model, we may assume that φ has analytic singularities along an nc $\mathbb{Q}$-divisor F. Perturbing L by an ample $\mathbb{Q}$-line bundle by Proposition 5.17, we may assume that θ_φ is a Kähler current. Finally, by rescaling, we may assume that F is a divisor and L is a line bundle and $L - F$ is ample by [Xia20, Lemma 2.4].

By the Hausdorff convergence property, we know that

$$\min_{x \in \Delta(\theta,\varphi)} x_1 = \lim_{k\to\infty} \min_{x \in \Delta_k(\theta,\varphi)} x_1 \,.$$

By definition,

$$\min_{x \in \Delta_k(\theta,\varphi)} x_1 = k^{-1} \operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) \,.$$

It remains to show that

$$\lim_{k\to\infty} k^{-1} \operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \lim_{k\to\infty} k^{-1} \operatorname{ord}_E \mathcal{I}(k\varphi) \,. \tag{5.15}$$

The $\geq$ direction is trivial, we prove the converse. Observe that

$$\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) = \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-kF)) \,, \quad \mathcal{I}(k\varphi) = \mathcal{O}(-kF) \,.$$

As $L - F$ is ample,

$$\operatorname{ord}_E \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-kF)) = \operatorname{ord}_E(kF) \,.$$

Thus, (5.15) is clear. □

Corollary 5.20. *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. If*

$$\Delta(\pi^*\theta, \pi^*\varphi) \subseteq \Delta(\pi^*\theta, \pi^*\psi)$$

for all birational models $\pi : Y \to X$ and all admissible flags on Y, then $\varphi \preceq_{\mathcal{I}} \psi$.

In particular, Theorem B is proved. This corollary is similar to [Jow10]. It suggests that $\Delta(\theta,\varphi)$ is a universal invariant of the singularities of φ.

We also observe the following corollary, which relates the current with minimal singularities and the asymptotic base locus of L.

Corollary 5.21. *Let E be a prime divisor over X. Then*

$$\nu(V_\theta, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathrm{H}^0(X, L^k) \,. \tag{5.16}$$

Proof. This follows from Lemma 5.19 and the fact that $\Delta(\theta, V_\theta) = \Delta(L)$. □

We write

$$\operatorname{ord}_E \|L\| := \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathrm{H}^0(X, L^k) \,.$$

*The lemma is stated only when θ is a Kähler form, but the proof works in the general case as well.

Corollary 5.22. *We have*

$$\mathcal{I}(V_\theta) = \{f \in \mathcal{O}_X : \exists \epsilon > 0 \ s.t. \ \operatorname{ord}_E(f) \geq (1+\epsilon)\operatorname{ord}_E \|L\| - A_X(E) \ \forall prime\ E\ over\ X\} ,$$

where $A_X(E)$ is the log discrepancy of E.

Proof. This follows from [Bou17, Corollary 10.17] and Corollary 5.21. □

5.4. **Okounkov bodies induced by filtrations.** Assume that L is ample.

Definition 5.23. A *filtration* on $R(X,L)$ is a decreasing, left continuous, multiplicative $\mathbb{R}$-filtration $\mathscr{F}^\bullet$ on the ring $R(X,L)$ which is linearly bounded in the sense that there is $C>0$, so that

$$\mathscr{F}^{-k\lambda}\mathrm{H}^0(X,L^k) = \mathrm{H}^0(X,L^k)\,, \quad \mathscr{F}^{k\lambda}\mathrm{H}^0(X,L^k) = 0\,,$$

when $\lambda > C$.

A filtration $\mathscr{F}$ is called a $\mathbb{Z}$-*filtration* if $\mathscr{F}^\lambda = \mathscr{F}^{\lfloor\lambda\rfloor}$ for any $\lambda\in\mathbb{R}$.

A $\mathbb{Z}$-filtration $\mathscr{F}$ is called *finitely generated* if the bigraded algebra

$$\bigoplus_{\lambda\in\mathbb{Z},k\in\mathbb{Z}_{\geq 0}} \mathscr{F}^\lambda\mathrm{H}^0(X,L^k)$$

is finitely generated over $\mathbb{C}$.

Let $\mathscr{F}^\bullet$ be a multiplicative filtration on $R(X,L)$. Then we can associate a test curve $\psi_\bullet$ as in [RWN14; Xia20].

$$\psi_\tau := \sup_{k\in\mathbb{Z}_{>0}}{}^{*}\, k^{-1}\sup{}^{*}\left\{\log|s|^2_{h^k} : s\in\mathscr{F}^{k\tau}\mathrm{H}^0(X,L^k), \sup_X |s|_{h^k}\leq 1\right\}. \tag{5.17}$$

Here $\sup^*$ denotes the upper-semicontinuous regularized supremum. By [DX20, Theorem 3.11], ψ_τ is $\mathcal{I}$-model or $-\infty$ for each $\tau\in\mathbb{R}$.

Theorem 5.24. *Let $\mathscr{F}^\bullet$ be a finitely generated $\mathbb{Z}$-filtration on $R(X,L)$. Let $\psi_\bullet$ be the test curve associated to $\mathscr{F}$. For any $\tau<\tau^+$,*

$$\Delta\left(\bigoplus_{k=0}^{\infty}\mathscr{F}^{k\tau}\mathrm{H}^0(X,L^k)\right) = \Delta(\theta,\psi_\tau)\,.$$

Proof. Observe that $\mathscr{F}^{k\tau}\mathrm{H}^0(X,L^k)\subseteq \mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\psi_\tau))$ for any $k\in\mathbb{N}$. Thus, by Lemma 3.22,

$$\Delta\left(\bigoplus_{k=0}^{\infty}\mathscr{F}^{k\tau}\mathrm{H}^0(X,L^k)\right) \subseteq \Delta(\theta,\psi_\tau)\,.$$

On the other hand, the two sides have the same volume by [Xia20, Lemma 4.5]. Thus, equality holds. □

5.5. **Limit partial Okounkov bodies.** Let $\varphi\in\mathrm{PSH}(X,\theta)$, not necessarily of positive volume. Take an ample effective divisor H on X and a Kähler form $\omega\in c_1(H)$. Then we just set

$$\Delta(\theta,\varphi) := \bigcap_{\epsilon\in\mathbb{Q}_{>0}}\Delta(\theta+\epsilon\omega,\varphi)\,.$$

Here $\Delta(\theta+\epsilon\omega,\varphi)$ is defined in the same way as when φ has positive volume. Clearly, this definition does not depend on the choice of H and ω. As in [CPW18], we cannot expect $\Delta(\theta,\varphi)$ to be continuous along decreasing sequences of φ. Note that Lemma 5.19, Corollary 5.20 and Proposition 5.10 extend to this setup without changes.

We propose the following conjecture:

Conjecture 5.25. *Under the above assumptions,*

$$\dim\Delta(\theta,\varphi) = \mathrm{nd}(\theta,\varphi)\,.$$

For the definition of the analytic numerical dimension, we refer to [Cao14].

We expect that this conjecture follows from the arguments in [CPW18] together with the numerical criterion of [Cao14].

6. Chebyshev transform

Let X be an irreducible smooth complex projective variety of dimension n and L be a big line bundle on X. Let h be a fixed smooth Hermitian metric on L and $\theta = c_1(L,h)$. Consider a singular positive Hermitian metric ϕ on L corresponding to $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that $\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} > 0$.

Let $v \in C^0(X)$ corresponding to a continuous metric $he^{-v/2}$ on L. We do not distinguish v and $he^{-v/2}$. Fix a valuation $\nu = (\nu_1,\ldots,\nu_n) : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ of rank n and rational rank n. Assume that ν is defined by an admissible flag $(Y_\bullet)$ on X.

The whole section is devoted to the proof of Theorem C.

6.1. **Equilibrium energy.** Let $\mathcal{E}^\infty(X,\theta;P[\varphi]_{\mathcal{I}})$ denote the set of $\psi \in \mathrm{PSH}(X,\theta)$ such that ψ and $P[\varphi]_{\mathcal{I}}$ have the same singularity types.

Let $E^\theta_{[\varphi]} : \mathcal{E}^\infty(X,\theta;P[\varphi]_{\mathcal{I}}) \to \mathbb{R}$ be the relative Monge–Ampère energy:

$$E^\theta_{[\varphi]}(\psi) := \frac{1}{n+1}\sum_{i=0}^n \int_X (\psi - P[\varphi]_{\mathcal{I}})\, \theta^i_\psi \wedge \theta^{n-i}_{P[\varphi]_{\mathcal{I}}}\,.$$

Define the equilibrium energy $\mathcal{E}^\theta_{[\varphi]} : C^0(X) \to \mathbb{R}$:

$$\mathcal{E}^\theta_{[\varphi]}(v) := E^\theta_{[\varphi]}(P[\varphi]_{\mathcal{I}}(v))\,. \tag{6.1}$$

Note that this definition is different from the energy defined in [DX21], so we choose a different notation.

Theorem 6.1. *The Gateaux differential of $\mathcal{E}^\theta_{[\varphi]}$ at $v \in C^0(X)$ is given by $\theta^n_{P[\varphi]_{\mathcal{I}}(v)}$. In other words, for any $f \in C^0(X)$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{E}^\theta_{[\varphi]}(v+tf) = \int_X f\, \theta^n_{P[\varphi]_{\mathcal{I}}(v)}\,. \tag{6.2}$$

Proof. This is not exactly [DX21, Proposition 5.9] because we are using $P[\bullet]_{\mathcal{I}}$ projections instead of $P[\bullet]$ projections, but the proofs are identical. ☐

The metric $he^{-v/2}$ induces an L^∞-type norm $\|\bullet\|_{L^\infty(kv)}$ on $\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))$:

$$\|s\|_{L^\infty(kv)} := \sup_X |s|_{h^k} e^{-kv/2}\,.$$

In particular, $\det \|\bullet\|_{L^\infty(kv)}$ is a Hermitian metric on $\det \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))$.

Theorem 6.2. *Let $v, v' \in C^0(X)$,*

$$\lim_{k\to\infty} \frac{n!}{k^{n+1}} \log\left(\det\|\bullet\|_{L^\infty(kv)} / \det\|\bullet\|_{L^\infty(kv')}\right) = \mathcal{E}^\theta_{[\varphi]}(v) - \mathcal{E}^\theta_{[\varphi]}(v')\,. \tag{6.3}$$

Remark 6.3. When $\varphi = V_\theta$, the left-hand side of (6.3) is known as the *relative volume* between the two metrics $he^{-v/2}$ and $he^{-v'/2}$.

This theorem partially generalizes [BB10, Theorem A]. We remind the readers that our conventions of multiplier ideal sheaves are different from those in [BB10] and [BBWN11], which explains the difference between our coefficients and theirs.

For the definition of Bernstein–Markov property, see [BB10, Definition 2.3].

Proof. We may assume that $v'=0$. Let ν be a smooth volume form on X. Then recall that ν satisfies the Bernstein–Markov property with respect to tv for all $t\in[0,1]$ [BB10, Theorem 2.4]. We may replace the L^∞-norm on the left-hand side with the $L^2(\nu)$-norm by [DX21, Lemma 6.5]. Recall that the definition of the partial Bergman kernel:

$$B^k_{tv,\varphi,\nu}(x):=\sup\left\{|s|^2_{h^k}e^{-kv}(x):\int_X |s|^2_{h^k}e^{-tv}\leq 1\,,s\in \mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))\right\}$$

and

$$\beta^k_{tv,\varphi,\nu}:=\frac{n!}{k^n}B^k_{tv,\varphi,\nu}\,\mathrm{d}\nu\,,$$

where $k\in\mathbb{Z}_{>0}$.

By [DX21, Theorem 1.2],

$$\beta^k_{tv,\varphi,\nu}\rightharpoonup\theta^n_{P_X[\varphi]_{\mathcal{I}}(tv)}$$

as $k\to\infty$ for all $t\in[0,1]$. By dominant convergence theorem,

$$\lim_{k\to\infty}\int_0^1\int_X v\,\beta^k_{tv,\varphi,\nu}\,\mathrm{d}t=\int_0^1\int_X v\,\theta^n_{P_X[\varphi]_{\mathcal{I}}(tv)}\,\mathrm{d}t$$

and (6.3) follows. □

Proposition 6.4. *Let $\varphi\in\mathrm{PSH}(X,\theta)$ such that θ_φ is a Kähler current. Let $(\varphi^j)_{j\in\mathbb{N}}$ be a quasi-equisingular approximation of φ. Then*

$$\lim_{\epsilon\to 0+}\mathcal{E}^\theta_{[\varphi^j]}(v)=\mathcal{E}^\theta_{[\varphi]}(v)\,. \tag{6.4}$$

Proof. By Theorem 6.1, for $j\in\mathbb{N}$,

$$\mathcal{E}^\theta_{[\varphi^j]}(v)=\int_0^1\int_X v\,\theta^n_{P[\varphi^j]_{\mathcal{I}}(tv)}\,\mathrm{d}t\,,$$
$$\mathcal{E}^\theta_{[\varphi]}(v)=\int_0^1\int_X v\,\theta^n_{P[\varphi]_{\mathcal{I}}(tv)}\,\mathrm{d}t\,.$$

It follows from [DX21, Proposition 3.3] and [DDNL18b, Theorem 1.2] that as $j\to\infty$,

$$\theta^n_{P[\varphi^j]_{\mathcal{I}}(tv)}\rightharpoonup\theta^n_{P[\varphi]_{\mathcal{I}}(tv)}\,.$$

By dominant convergence theorem, (6.4) follows. □

Proposition 6.5. *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Assume that $\psi\leq\varphi$. Set $\varphi_\epsilon=(1-\epsilon)\varphi+\epsilon\psi$ for any $\epsilon\in[0,1]$. Then*

$$\lim_{\epsilon\to 0+}\mathcal{E}^\theta_{[\varphi_\epsilon]}(v)=\mathcal{E}^\theta_{[\varphi]}(v)\,. \tag{6.5}$$

Proof. The proof is similar to that of Proposition 6.4. We just replace [DX21, Proposition 3.3] by [DX21, Proposition 2.7]. □

We finally recall a technical lemma.

Lemma 6.6 ([WN14, Corollary 3.4])**.** *Let $C\subseteq\mathbb{R}^{n+1}$ be an open convex cone. Let F be a subadditive function on $C\cap\mathbb{Z}^{n+1}$ defined outside a compact set. Then for any sequence $\alpha_k\in C\cap\mathbb{Z}^{n+1}$ tending to infinity such that $\alpha_k/|\alpha_k|$ converges to some point $p\in C$. Then the limit*

$$c[F](p):=\lim_{k\to\infty}\frac{F(\alpha_k)}{|\alpha_k|}$$

exists and depends only on p and F. Moreover, $c[F]$ is a convex function on $C\cap\{x_{n+1}=1\}$.

Here $|\alpha_k|$ denotes the absolute value of the last component of α_k.

Recall that a real-valued function F defined on a semigroup Γ is said to be *sub-additive* if for any $x,y\in\Gamma$, $F(x+y)\leq F(x)+F(y)$.

6.2. **The case of analytic singularities.** Assume that φ has analytic singularities.

Let $\pi : Y \to X$ be a resolution such that $\pi^*\varphi$ has analytic singularity along an nc $\mathbb{Q}$-divisor E. We define as before

$$W_k^0 = \mathrm{H}^0(Y, \pi^* L^k \otimes \mathcal{O}_Y(-kE)) \subseteq \mathrm{H}^0(X, L^k)\,.$$

Fix $a \in \Gamma_k(W^0)$. Let p be the center of ν on X. Let $z = (z_1, \dots, z_n)$ be a regular sequence in $\mathcal{O}_{X,p}$ such that $(Y_i)_x$ is the zero locus of $z_1, \dots, z_i$. Fix a local trivialization of L near p. Define

$$A_k^a := \left\{ s \in W_k^0 : \nu(s) \geq ka\,, s = z^{ka} + \text{higher order terms near } p \right\}.$$

Define

$$F[v](ka, k) = \inf_{s \in A_{a,k}} \log |s|_{L^\infty(kv)}\,.$$

Lemma 6.7. *$F[v]$ is subadditive on $\Gamma(W^0)$.*

Proof. Let $(ka, k), (k'a', k') \in W^0$. Let

$$\gamma := \frac{ka + k'a'}{k + k'}\,.$$

Then $(ka, k) + (k'a', k') = ((k + k')\gamma, k + k')$. Observe that

$$A_k^a \cdot A_{k'}^{a'} \subseteq A_{k+k'}^{\gamma}\,.$$

On the other hand, for $s \in W_k^0$, $s' \in W_{k'}^0$,

$$\|ss'\|_{L^\infty((k+k')v)} \leq \|s\|_{L^\infty(kv)} \|s\|_{L^\infty(k'v)}$$

Thus

$$F[v]((k + k')\gamma, k + k') \leq F[v](ka, k) + F[v](k'a', k')\,.$$

□

Lemma 6.8. *There is $C > 0$, so that for any $(ka, k) \in \Gamma(W^0)$,*

$$F[v](ka, k) \geq C|(ka, k)|\,.$$

Proof. It suffices to apply [WN14, Lemma 5.4]. □

Let $c_{[\varphi]}[v] : \operatorname{Int}\Delta(\theta, \varphi) \to \mathbb{R}$ be the convex function $c[F[v]]$ defined by Lemma 6.6.

Theorem 6.9. *We have*

$$\int_{\Delta(W(\theta,\varphi))} \left(c_{[\varphi]}[v] - c_{[\varphi]}[0] \right) \mathrm{d}\lambda = \mathcal{E}^{\theta}_{[\varphi]}(v)\,.$$

Proof. The proof follows *verbatim* from that of [WN14, Theorem 6.2], taking in account Theorem 6.2. □

Observe that

$$\sup_{\operatorname{Int}\Delta(W(\theta,\varphi))} \left| c_{[\varphi]}[v] - c_{[\varphi]}[0] \right| \leq \|v\|_{C^0(X)}/2\,. \tag{6.6}$$

The following results are obvious:

Lemma 6.10. *For any $(ka, k) \in \Gamma(W^0)$, $s \in A_k^a$,*

$$F[v](ka, k) \leq \log \|s\|_{L^\infty(kv)}\,.$$

Corollary 6.11. *Let $s \in W_k(\theta, \varphi)$, locally written as z^{ka} plus higher order terms near p, we have*

$$c_{[\varphi]}[v](a) \leq k^{-1} \log \|s\|_{L^\infty(kv)}\,.$$

Lemma 6.12. *Let $\varphi, \varphi' \in \mathrm{PSH}(X, \theta)$ be potentials with analytic singularities. If $[\varphi] \preceq [\varphi']$, then*

$$c_{[\varphi]}[v] \geq c_{[\varphi']}[v]$$

when restricted to $\operatorname{Int}\Delta(\theta, \varphi)$.

6.3. **The case of Kähler currents.** Assume that θ_φ is a Kähler current. Let φ^j be a quasi-equisingular approximation of φ. Then $c_{[\varphi^j]}[v]$ restricted to $\operatorname{Int}\Delta(W(\theta,\varphi))$ is an increasing sequence. Thus, we can define $c_{[\varphi]}[v] : \operatorname{Int}\Delta(\theta,\varphi) \to \mathbb{R}\cup\{\infty\}$ by

$$c_{[\varphi]}[v] := \lim_{j\to\infty} c_{[\varphi^j]}[v]\,.$$

Lemma 6.13. *Let $s \in W_k(\theta,\varphi)$, locally written as z^{ka} plus higher order terms near p. Then*

$$c_{[\varphi]}[v](a) \leq k^{-1}\log \|s\|_{L^\infty(kv)}\,.$$

Proof. This follows from Corollary 6.11. ☐

By convexity, $c_{[\varphi]}[v]$ is takes value in $\mathbb{R}$.

It follows that (6.6) still holds in this case. By dominant convergence theorem, Proposition 6.4 and the previous case we find

$$\int_{\Delta(\theta,\varphi)} \Big(c_{[\varphi]}[v] - c_{[\varphi]}[0]\Big)\,\mathrm{d}\lambda = \mathcal{E}^\theta_{[\varphi]}(v)\,.$$

It follows from Lemma 6.12 that our definition of $c_{[\varphi]}(v)$ is independent of the choice of φ^j.

Lemma 6.14. *Let $\varphi,\varphi' \in \operatorname{PSH}(X,\theta)$ be potentials such that θ_φ and $\theta_{\varphi'}$ are both Kähler currents. If $[\varphi] \preceq_{\mathcal{I}} [\varphi']$, then*

$$c_{[\varphi]}[v] \geq c_{[\varphi']}[v]$$

when restricted to $\operatorname{Int}\Delta(\theta,\varphi)$.

Proof. This follows from Lemma 6.12. ☐

6.4. **General case.** Let $\varphi \in \operatorname{PSH}(X,\theta)$ such that $\int_X \theta^n_{P[\varphi]_{\mathcal{I}}} > 0$. We may replace φ with $P[\varphi]_{\mathcal{I}}$ and therefore assume that the non-pluripolar mass of φ is positive.

Let $\eta \in \operatorname{PSH}(X,\theta)$ be a potential so that θ_η is a Kähler current. Define $\varphi_\epsilon := (1-\epsilon)\varphi + \epsilon\eta$. Then we define $c_{[\varphi]}[v] : \operatorname{Int}\Delta(\theta,\varphi) \to \mathbb{R}\cup\{-\infty\}$ as

$$c_{[\varphi]}[v] := \lim_{\epsilon\to 0+} c_{[\varphi_\epsilon]}[v]\,.$$

This is a decreasing limit by Lemma 6.14. On the other hand, $c_{[\varphi]}[v] \geq c_{[V_\theta]}[v]$, the latter is finite by [WN14]. Thus, $c_{[\varphi]}[v]$ is real-valued. (6.6) extends to this situation. By dominant convergence theorem and Proposition 6.5 again

$$\int_{\Delta(\theta,\varphi)} (c_{[\varphi]}[v] - c_{[\varphi]}[0])\,\mathrm{d}\lambda = \mathcal{E}^\theta_{[\varphi]}(v)\,.$$

We do not know if $c_{[\varphi]}[v]$ is independent of the choice of η.

7. A generalization of Boucksom–Chen theorem

In this section, let X be an irreducible smooth projective variety of dimension n. Let L be a big line bundle on X. Take a smooth Hermitian metric h on L with $\theta = c_1(L,h)$.

Fix a rank n valuation $\nu : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ of rational rank n.

7.1. **The theory of test curves.** Let $V = \langle L^n\rangle$.

Definition 7.1. A *test curve (of finite energy)* with respect to (X,θ) is a map $\psi = \psi_\bullet : \mathbb{R} \to \operatorname{PSH}(X,\theta)\cup\{-\infty\}$, such that

(1) $\psi_\bullet$ is concave in $\bullet$.
(2) ψ_τ is a model potential or $-\infty$ for any τ.
(3) ψ is usc as a function in the $\mathbb{R}$-variable.
(4) $\lim_{\tau\to-\infty}\psi_\tau = V_\theta$ in L^1.
(5) $\psi_\tau = -\infty$ for τ large enough.

(6)

$$\mathbf{E}(\psi_\bullet) := \tau^+ V + \int_{-\infty}^{\tau^+} \left(\int_X \theta_{\psi_\tau}^n - V \right) \mathrm{d}\tau > -\infty\,. \tag{7.1}$$

Here $\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$. The set of test curves of finite energy with respect to (X, θ) is denoted by $\mathcal{TC}^1(X, \theta)$. We say ψ is *normalized* if $\tau^+ = 0$. The test curve is called *bounded* if $\psi_\tau = V_\theta$ for τ small enough. Let $\tau^- := \sup\{\tau \in \mathbb{R} : \psi_\tau = V_\theta\}$ in this case. The set of bounded test curves is denoted by $\mathcal{TC}^\infty(X, \theta)$.

We say a test curve is *$\mathcal{I}$-model* if each ψ_τ is either $\mathcal{I}$-model or $-\infty$. The set of $\mathcal{I}$-model test curves is denoted by $\mathcal{TC}^1_{\mathcal{I}}(X, \theta)$.

7.2. **Okounkov test curves.** Let $\Delta \in \mathcal{K}^n$. Assume that $V = n! \operatorname{vol} \Delta > 0$.

Definition 7.2. An *Okounkov test curve* relative to Δ is an assignment $(\Delta_\tau)_{\tau \leq \tau^+}$ ($\tau^+ \in \mathbb{R}$):

(1) Δ_τ is a decreasing assignment of convex bodies in $\mathbb{R}^n$ for $\tau \leq \tau^+$.
(2) Δ_τ converges to Δ as $\tau \to -\infty$.
(3) Δ_τ is concave in the τ variable.
(4) The energy is finite:

$$\mathbf{E}(\Delta_\bullet) := \tau^+ V + V \int_{-\infty}^{\tau^+} \left(\frac{n!}{V} \operatorname{vol} \Delta_\tau - 1 \right) \mathrm{d}\tau > -\infty\,.$$

(5) Continuity holds at τ^+:

$$\Delta_{\tau^+} = \bigcap_{\tau < \tau^+} \Delta_\tau\,.$$

Proposition 7.3. *Any Okounkov test curve $(\Delta_\tau)_{\tau \leq \tau^+}$ relative to Δ is continuous for $\tau < \tau^+$.*

Proof. We first claim that $\operatorname{vol} \Delta_{\tau'} > 0$ for all $\tau' < \tau^+$. By Condition (2) and Theorem 2.3, we know that $\operatorname{vol} \Delta_{\tau''} > 0$ when τ'' is small enough. Fix one such τ''. Any $\tau' < \tau^+$ can be written as a convex combination of τ^+ and τ'', thus $\Delta_{\tau'}$ has positive volume by Condition (3).

Next we claim that $\operatorname{vol} \Delta_\tau$ is continuous for $\tau < \tau^+$. In fact, by Condition (3) and the Minkowski inequality, we know that $\log \operatorname{vol} \Delta_\tau$ is concave for $\tau < \tau^+$. The continuity follows.

Next we show that

$$\Delta_\tau = \bigcap_{\tau' < \tau} \Delta_{\tau'}\,.$$

The $\supseteq$ direction is obvious. By the continuity of the volume, both sides have the same volume and the volume is positive, hence, equality holds by Lemma 2.5.

Similarly, we have

$$\Delta_\tau = \overline{\bigcup_{\tau' > \tau} \Delta_{\tau'}}\,.$$

The continuity of Δ_τ at $\tau < \tau^+$ is proved. $\square$

Definition 7.4. A *test function* on Δ is a function $F : \Delta \to [-\infty, \infty)$ such that

(1) F is concave.
(2) F is finite on $\operatorname{Int} \Delta$.
(3) F is usc.
(4) Let $\tau^+ = \sup_\Delta F$, then

$$\mathbf{E}(F) := n! \int_\Delta F \, \mathrm{d}\lambda > -\infty\,. \tag{7.2}$$

Observe that

$$\mathbf{E}(F) = \tau^+ V + V \int_{-\infty}^{\tau^+} \left(\frac{n!}{V} \operatorname{vol}\{F \geq \tau\} - 1 \right) \mathrm{d}\tau \,. \tag{7.3}$$

Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. Let $\Delta \in \mathcal{K}^n$. We define the *Legendre transform* of $\Delta_\bullet$ as

$$G[\Delta_\bullet] : \Delta \to [-\infty, \infty) \,, \quad a \mapsto \sup \left\{ \tau < \tau^+ : a \in \Delta_\tau \right\} \,.$$

Conversely, a test function F on Δ, set $\tau^+ = \sup_\Delta F$. We define the *inverse Legendre transform* of F as

$$\Delta[F] : (-\infty, \tau^+] \to \mathcal{K}_n \,, \quad \Delta[F]_\tau = \{F \geq \tau\} \,.$$

Theorem 7.5. *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between the set of Okounkov test curves relative to Δ and test functions on Δ. Moreover, if $\Delta_\bullet$ is an Okounkov test curve relative to Δ, then*

$$\mathbf{E}(\Delta_\bullet) = \mathbf{E}\left(G[\Delta_\bullet]\right) \,. \tag{7.4}$$

Proof. Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We prove that $G[\Delta_\bullet]$ is a test function on Δ.

Firstly $G[\Delta_\bullet]$ is concave by Condition (1) and Condition (3) in Definition 7.2. More precisely, take $a, b \in \Delta$. We want to prove that for any $t \in (0,1)$,

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b) \,. \tag{7.5}$$

There is nothing to prove if $G[\Delta_\bullet](a)$ or $G[\Delta_\bullet](b)$ is $-\infty$. So we assume that both are finite. Take $\epsilon > 0$, then $a \in \Delta_{G[\Delta_\bullet](a)-\epsilon}$ and $b \in \Delta_{G[\Delta_\bullet](b)-\epsilon}$. Thus,

$$ta + (1-t)b \in t\Delta_{G[\Delta_\bullet](a)-\epsilon} + (1-t)\Delta_{G[\Delta_\bullet](b)-\epsilon} \subseteq \Delta_{tG[\Delta_\bullet](a)+(1-t)G[\Delta_\bullet](b)-\epsilon} \,.$$

We deduce that

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b) - \epsilon \,.$$

Since $\epsilon > 0$, (7.5) follows.

Next $G[\Delta_\bullet]$ is finite on $\operatorname{Int}\Delta$ by Condition (2). In fact, as Δ_τ is increasing and converges to Δ as $\tau \to -\infty$, we have

$$\Delta = \overline{\bigcup_\tau \Delta_\tau} \,.$$

Hence, by [Sch14, Theorem 1.1.15] and the assumption that $\operatorname{vol}\Delta > 0$, $\bigcup_\tau \Delta_\tau$ contains $\operatorname{Int}\Delta$.

Thirdly, we show that $G[\Delta_\bullet]$ is usc. Let $a_i \in \Delta$ with $a_i \to a \in \Delta$. Define $\tau_i = G[\Delta_\bullet](a_i)$. Let $\tau = \overline{\lim}_i \tau_i$. We need to show that

$$G[\Delta_\bullet](a) \geq \tau \,. \tag{7.6}$$

There is nothing to prove if $\tau = -\infty$. We assume that it is not this case. Up to subtracting a subsequence we may assume that $\tau_i \to \tau$. In particular, we can assume that $\tau_i \neq -\infty$ for all i. Fix $\epsilon > 0$, then $a_i \in \Delta_{\tau_i - \epsilon}$. Observe that $\Delta_{\tau_i-\epsilon} \xrightarrow{d_n} \Delta_{\tau-\epsilon}$. By Theorem 2.4 it follows that $a \in \Delta_{\tau-\epsilon}$. Thus,(7.6) follows since $\epsilon > 0$ is arbitrary.

Finally, (7.4) follows from (7.3), and it follows that $\mathrm{E}(G[\Delta_\bullet]) > -\infty$.

Conversely, if $F : \Delta \to [-\infty, \infty)$ is a test function on Δ. Let $\Delta[F]$ be the inverse Legendre transform of F. Then one can similarly show that $\Delta[F]$ is an Okounkov test curve.

Firstly, for each $\tau < \tau^+ := \sup_\Delta F$, $\Delta[F](\tau)$ is a convex body as F is concave and usc. Moreover, $\Delta[F]_\tau$ is clearly decreasing in τ. Hence, $\Delta[F]_{\tau^+}$ is also a convex body.

Secondly, for each $a \in \Delta$, we can write $a = \lim_i a_i$ with $a_i \in \operatorname{Int}\Delta$. By assumption, F is finite at a_i. Thus,

$$a \in \overline{\{F > -\infty\}} = \overline{\bigcup_\tau \Delta[F]_\tau} \,.$$

By Theorem 2.4, $\Delta[F]_\tau \xrightarrow{d_n} \Delta$ as $\tau \to -\infty$.

Thirdly, $\Delta[F]$ is concave. To see, take $\tau, \tau' \leq \tau^+$, we need to prove that for any $t \in (0,1)$,

$$\Delta[F]_{t\tau+(1-t)\tau'} \supseteq t\Delta[F]_\tau + (1-t)\Delta[F]_{\tau'} \,. \tag{7.7}$$

Let $a \in \Delta[F]_\tau$ and $b \in \Delta[F]_{\tau'}$. We have $F(a) \geq \tau$ and $F(b) \geq \tau'$. As F is concave, we have $F(ta + (1-t)b) \geq t\tau + (1-t)\tau'$. Thus,

$$ta + (1-t)b \in \Delta[F]_{t\tau+(1-t)\tau'}$$

and (7.7) follows.

Fourthly, (7.2) follows immediately from (7.3).

Finally, we show that $\Delta[F]_\bullet$ is continuous at τ^+. This amounts to

$$\{F \geq \tau^+\} = \bigcap_{\tau<\tau^+} \{F \geq \tau\} \,,$$

which is obvious.

To see that these two operations are inverse to each other, observe that by definition for any Okounkov test curve $\Delta_\bullet$, any $a \in \Delta$ and any $\tau \leq \tau^+$, $G[\Delta_\bullet](a) \geq \tau$ if and only if $a \in \Delta_{\tau-\epsilon}$ for any $\epsilon > 0$. By Proposition 7.3, this happens if and only if $a \in \Delta_\tau$, that is,

$$\{G[\Delta_\bullet] \geq \tau\} = \Delta_\tau \,.$$

Conversely, for any test function $F : \Delta \to [-\infty, \infty)$, any $\tau \leq \tau^+$, by definition,

$$\{F \geq \tau\} = \Delta[F]_\tau \,.$$

□

Definition 7.6. Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) := G[\Delta_\bullet]_*(\mathrm{d}\lambda) \,.$$

It is Radon measure on $\mathbb{R}$.

Observe that

$$\int_{\mathbb{R}} \mathrm{DH}(\Delta_\bullet) = \operatorname{vol} \Delta \,. \tag{7.8}$$

7.3. **Boucksom–Chen theorem.** Let $\psi_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X, \theta)$. Let $\tau^+ = \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$.

Lemma 7.7. *The curve*

$$\Delta[\psi_\bullet]_\tau := \begin{cases} \Delta(\theta, \psi_\tau) \,, & \tau < \tau^+ \,, \\ \bigcap_{\tau'<\tau^+} \Delta[\psi_\bullet]_{\tau'} \,, & \tau = \tau^+ \end{cases}$$

is an Okounkov test curve relative to $\Delta(L)$. Moreover,

$$\mathbf{E}(\psi_\bullet) = \mathbf{E}(\Delta[\psi_\bullet]_\bullet) \,. \tag{7.9}$$

Proof. We verify the conditions in Definition 7.2. Condition (1) follows from Proposition 5.10. Condition (2) follows from the fact

$$\lim_{\tau\to-\infty} \operatorname{vol} \Delta_\tau = \operatorname{vol} \Delta \,.$$

Condition (3) follows from Theorem 5.14 and Proposition 5.10. Condition (4) is a translation of (7.1). Condition (5) is obvious.

Finally, (7.9) follows from (7.1) and (1.2). □

Definition 7.8. Let $\psi_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X, \theta)$. Define the *Duistermaat–Heckman measure* of $\psi_\bullet$ as

$$\mathrm{DH}(\psi_\bullet) := \mathrm{DH}(\Delta[\psi_\bullet]_\bullet) \,.$$

We write

$$G[\psi_\bullet] = G[\Delta[\psi_\bullet]] .$$

Then

$$\mathrm{DH}(\psi_\bullet) = G[\psi_\bullet]_*(\mathrm{d}\lambda) .$$

Now consider the filtration:

$$\mathscr{F}^k_\tau \mathrm{H}^0(X, L^k) := \begin{cases} \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\psi_\tau)) , & \tau < \tau^+ , \\ 0 , & \tau \geq \tau^+ . \end{cases}$$

Let $e_j(\mathrm{H}^0(X, L^k), \mathscr{F}^k)$ be the jumping numbers of $\mathscr{F}^k$ listed in the decreasing order. In order words,

$$e_j\left(\mathrm{H}^0(X, L^k), \mathscr{F}^k\right) := \sup\left\{\tau \in \mathbb{R} : \dim \mathscr{F}^k_\tau \mathrm{H}^0(X, L^k) \geq j\right\} .$$

Let

$$\mu_k := \frac{1}{k^n} \sum_{j=1}^{h^0(X,L^k)} \delta_{e_j\left(\mathrm{H}^0(X,L^k),\mathscr{F}^k\right)} .$$

Theorem 7.9. *Let $\psi_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X, \theta)$. Then as $k \to \infty$, μ_k converges weakly to* $\mathrm{DH}(\psi_\bullet)$*.*

As explained in [RWN14; DX20; Xia20], $\mathcal{TC}^1_{\mathcal{I}}(X, \theta)$ is the completion of the space of filtrations, so this theorem indeed generalizes [BC11, Theorem A], in the case of full graded linear series.

Proof. It suffices to show the convergence holds as distributions. By our definition, μ_k is the distributional derivative of the function

$$h_k(\tau) := \begin{cases} k^{-n} h^0(X, L^k \otimes \mathcal{I}(k\psi_\tau)) , & \tau < \tau^+ , \\ 0 , & \tau \geq \tau^+ . \end{cases}$$

On the other hand, $\mathrm{DH}(\psi_\bullet)$ is the distributional derivative of $h(\tau) := \mathrm{vol}\{G[\Delta[\psi_\bullet]_\bullet] \geq \tau\} = \mathrm{vol}\,\Delta_\tau$ by Fubini–Tonelli theorem.

By [DX21, Theorem 1.1], $h_k(\tau) \to h(\tau)$ for all $\tau \neq \tau^+$. By dominant convergence theorem $h_k \to h$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. Hence, $\mu_k \rightharpoonup \mathrm{DH}(\psi_\bullet)$. □

Corollary 7.10. *For any $\psi_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X, \theta)$. The Duistermaat–Heckman measure* $\mathrm{DH}(\psi_\bullet)$ *is independent of the choice of the valuation ν.*

7.4. **Applications to non-Archimedean geometry.** Assume that L is ample and θ is a Kähler form. We write $\omega = \theta$ instead. Let $\mathcal{E}^1(X, \omega)$ denote the space of ω-psh functions with finite energy:

$$\mathcal{E}^1(X, \omega) := \left\{\varphi \in \mathrm{PSH}(X, \omega) : \int_X \omega_\varphi^n = \int_X \omega^n, \int_X |\varphi| \omega_\varphi^n < \infty\right\} .$$

See [Dar19] for a detailed introduction. Recall that $\mathcal{E}^1(X, \omega)$ admits a natural metric d_1: for $\varphi, \psi \in \mathcal{E}^1(X, \omega)$, given by

$$d_1(\varphi, \psi) := \mathrm{E}(\varphi) + \mathrm{E}(\psi) - 2\mathrm{E}(\varphi \wedge \psi) .$$

Here $\varphi \wedge \psi$ is the greatest ω-psh function lying below both φ and ψ; the Monge–Ampère energy functional $\mathrm{E} : \mathcal{E}^1(X, \omega) \to \mathbb{R}$ is defined as

$$\mathrm{E}(\varphi) = \frac{1}{n+1} \sum_{i=0}^{n} \int_X \varphi\, \omega_\varphi^i \wedge \omega^{n-i} .$$

In this case, let $\mathcal{R}^1(X, \omega)$ denote the set of geodesic rays in $\mathcal{E}^1(X, \omega)$ emanating from 0. For a detailed study of $\mathcal{R}^1(X, \omega)$, we refer to [DL20]. Here we only recall the definition of the metric on $\mathcal{R}^1(X, \omega)$. Given $\ell, \ell' \in \mathcal{R}^1(X, \omega)$, we define

$$d_1(\ell, \ell') := \lim_{t \to \infty} \frac{1}{t} d_1(\ell_t, \ell'_t) .$$

It is shown in [DL20; CC18] that d_1 is a well-defined and $(\mathcal{R}^1(X,\omega),d_1)$ is a complete metric space.

The following notion is introduced in [Xia21]:

Definition 7.11. A *rooftop metric space* is a tuple $(E,d,\wedge)$: (E,d) is a metric space and $\wedge : E\times E\to E$ is an associative, commutative binary operator on E satisfying

$$d(a\wedge c,b\wedge c)\leq d(a,b)$$

for any $a,b,c\in E$.

For $\ell,\ell'\in\mathcal{R}^1(X,\omega)$, define $\ell\wedge\ell'$ as the greatest geodesic in $\mathcal{R}^1(X,\omega)$ that lies below both ℓ and ℓ'. It is shown in [Xia21] that $\wedge$ is well-defined and $(\mathcal{R}^1(X,\omega),d_1,\wedge)$ is a complete rooftop metric space.

The energy functional $\mathbf{E}:\mathcal{R}^1(X,\omega)\to\mathbb{R}$ is defined as

$$\mathbf{E}(\ell):=\mathrm{E}(\ell_1)\,.$$

Recall that we have the following two maps: given any $\ell\in\mathcal{R}^1(X,\omega)$, its *inverse Legendre transform* is defined as

$$\hat{\ell}_\tau:=\inf_{t\geq 0}(\ell_t-t\tau)\,.$$

Conversely, given any $\psi_\bullet\in\mathcal{TC}^1(X,\omega)$, we define its *Legendre transform* by

$$\check{\psi}_t:=\sup_{\tau\in\mathbb{R}}(\psi_\tau+t\tau)\,.$$

Let X^{an} be the Berkovich analytification of X with respect to the trivial valuation on X. There is a natural morphism of ringed spaces $X^{\mathrm{an}}\to X$. Let L^{an} be the pull-back (along the induced morphism of ringed topoi) of L. For an introduction to the global pluripotential theory on L^{an}, see [BJ21]. We write $\mathcal{E}^1(L^{\mathrm{an}})$ for the space of finite energy potentials on L^{an}. Let $\mathrm{E}:\mathcal{E}^1(L^{\mathrm{an}})\to\mathbb{R}$ be the energy functional (c.f. [BJ21, Definition 5.6]):

$$E(\alpha):=\frac{1}{n+1}([L],\alpha)^{n+1}\,.$$

Here the parathesis denote the energy pairing of [BJ21, Section 5]. Note that our definition of E differs from the definition of [BJ21] by a multiple $\frac{1}{V}$.

Recall that by [BBJ21], there is a natural inclusion

$$\iota:\mathcal{E}^1(L^{\mathrm{an}})\hookrightarrow\mathcal{R}^1(X,\omega)\,.$$

The geodesics lying in the image of this map are known as *maximal geodesic rays* or *approximable geodesic rays*. Moreover,

$$\mathbf{E}(\iota(\alpha))=\mathrm{E}(\alpha) \tag{7.10}$$

for any $\alpha\in\mathcal{E}^1(L^{\mathrm{an}})$.

Theorem 7.12. *The Legendre transform is a bijection from $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$ (resp. $\mathcal{TC}^1(X,\omega)$) to $\iota(\mathcal{E}^1(L^{\mathrm{an}}))$ (resp. $\mathcal{R}^1(X,\omega)$), the converse is given by the inverse Legendre transform. Moreover, for any $\psi_\bullet\in\mathcal{TC}^1(X,\omega)$,*

$$\mathbf{E}(\psi_\bullet)=\mathbf{E}(\check{\psi})\,. \tag{7.11}$$

This is one of the main theorems of [DX20, Theorem 3.7, Theorem 3.17]. It is based on the previous works [RWN14; DDNL18a].

We can now define the Duistermaat–Heckman measure associated to a potential in $\mathcal{E}^1(L^{\mathrm{an}})$.

Definition 7.13. For any $\alpha\in\mathcal{E}^1(L^{\mathrm{an}})$, define the *Duistermaat–Heckman measure* of α as

$$\mathrm{DH}(\alpha):=\mathrm{DH}\left(\widehat{\iota(\alpha)}\right)\,.$$

We get a map $\mathrm{DH} : \mathcal{E}^1(L^{\mathrm{an}}) \to \mathcal{M}(\mathbb{R})$. Here $\mathcal{M}(\mathbb{R})$ denotes the space of Radon measures on $\mathbb{R}$.

For the proof of the next theorem, we need to recall several basic constructions of test curves.

The space $\mathcal{TC}^1(X,\omega)$ is a rooftop metric space. Its rooftop structures $(d_1, \wedge)$ are induced from the corresponding structures on $\mathcal{R}^1(X,\omega)$.

Corollary 7.14. *Let $\psi_\bullet, \varphi_\bullet, \eta_\bullet \in \mathcal{TC}^1(X,\omega)$.*

(1) The rooftop operator on $\mathcal{TC}^1(X,\omega)$ is given by

$$(\psi \wedge \varphi)_\tau = \psi_\tau \wedge \varphi_\tau \,. \tag{7.12}$$

It is the maximal element in $\mathcal{TC}^1(X,\omega)$ that lies below both $\psi_\bullet$ and $\varphi_\bullet$. In particular,

$$d_1((\psi\wedge\eta)_\bullet, (\varphi\wedge\eta)_\bullet) \leq d_1(\psi_\bullet, \varphi_\bullet)\,. \tag{7.13}$$

(2) The metric on $\mathcal{TC}^1(X,\omega)$ is given by

$$d_1(\psi_\bullet, \varphi_\bullet) := \mathbf{E}(\psi_\bullet) + \mathbf{E}(\varphi_\bullet) - 2\mathbf{E}((\psi\wedge\varphi)_\bullet)\,. \tag{7.14}$$

Proof. (1) Note that (7.13) is part of our definition of a rooftop structure.

Observe that the bijection $\mathcal{TC}^1(X,\omega) \to \mathcal{R}^1(X,\omega)$ is order-preserving. In order to prove our claim, it suffices to show that $(\varphi\wedge\psi)_\bullet$ defined by (7.12) is indeed in $\mathcal{TC}^1(X,\omega)$, which is obvious.

(2) This follows simply from (1) and (7.11). □

Now the d_1 metric on $\mathcal{TC}^1(X,\omega)$ restricts to a metric d_1 on $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$. The rooftop structure also restricts to a rooftop structure on $\mathcal{TC}^1_{\mathcal{I}}(X,\omega)$.

We need the following constructions on test curves.

(1) Increasing limit. Let $\psi^j_\bullet \in \mathcal{TC}^1(X,\omega)$ be an increasing sequence. Assume that $\tau^+_{\psi^j}$ is bounded from above. Define
$$\tilde{\psi}_\tau := C[\sup_j{}^* \psi^j_\tau]\,.$$
Let $\tau^+ = \inf\{\tau : \tilde{\psi}_\tau = -\infty\}$. We define
$$\psi_\tau = \begin{cases} \tilde{\psi}_\tau\,, & \tau \neq \tau^+\,; \\ \lim_{\sigma\to\tau^+-} \tilde{\psi}_\sigma\,, & \tau \neq \tau^+\,. \end{cases}$$
It is easy to verify that $\psi_\bullet \in \mathcal{TC}^1(X,\omega)$.

(2) Inf. Let $\psi^\alpha_\bullet, \eta_\bullet \in \mathcal{TC}^1(X,\omega)$. Here α lies in some index set A. Assume that $\psi^\alpha_\bullet \geq \eta_\bullet$ for all $\alpha \in A$. Define
$$(\inf\psi)_\tau := \inf_{\alpha\in A} \psi^\alpha_\tau\,.$$
Then if $(\inf_\psi)_\bullet$ is not identically $-\infty$, then $(\inf\psi)_\bullet \in \mathcal{TC}^1(X,\omega)$.

(3) Max. Let $\varphi_\bullet, \psi_\bullet \in \mathcal{TC}^1(X,\omega)$. There is the smallest test curve $(\varphi\vee\psi)_\bullet \in \mathcal{TC}^1(X,\omega)$ such that $(\varphi\vee\psi)_\bullet \geq \varphi_\bullet$, $(\varphi\vee\psi)_\bullet \geq \psi_\bullet$. In fact, we could simply define
$$(\varphi\vee\psi)_\tau := \inf\left\{\eta_\tau : \eta_\bullet \in \mathcal{TC}^1(X,\omega), \eta_\bullet \geq \varphi_\bullet, \eta_\bullet \geq \psi_\bullet\right\}\,.$$
In terms of the Legendre transform, $(\varphi\vee\psi)\check{}$ is the minimal geodesic ray lying above both $\check{\varphi}$ and $\check{\psi}$. We observe that
$$d_1(\varphi_\bullet, \psi_\bullet) \leq d_1(\varphi_\bullet, (\varphi\vee\psi)_\bullet) + d_1(\psi_\bullet, (\varphi\vee\psi)_\bullet) \leq C_0 d_1(\varphi_\bullet, \psi_\bullet) \tag{7.15}$$
for some $C_0(n) > 0$. See [DDNL21, Proposition 2.15] for the proof of the latter inequality. Moreover, if $\eta_\bullet \in \mathcal{TC}^1(X,\omega)$ and if $\varphi_\bullet \leq \psi_\bullet$, then
$$d_1((\varphi\vee\eta)_\bullet, (\psi\vee\eta)_\bullet) \leq d_1(\varphi_\bullet, \psi_\bullet)\,. \tag{7.16}$$
This follows from the corresponding inequality of geodesic rays, which in turn follows from [Xia21, Proposition 6.8].

We also observe that the operator $\vee$ is associative and commutative, hence, we could also define $\psi^1_\bullet \vee \cdots \vee \psi^k_\bullet$ in the obvious way.

Lemma 7.15. *Let $\psi^j_\bullet, \psi_\bullet \in \mathcal{TC}^1(X,\omega)$. Assume that one of the following conditions holds*

(1) $\psi^j_\bullet$ is increasing and $\psi_\bullet$ is the increasing limit of $\psi^j_\bullet$.

(2) $\psi^j_\bullet$ is decreasing and $\psi_\bullet = (\inf \psi)_\bullet$.

Then $\psi^j_\bullet \xrightarrow{d_1} \psi_\bullet$.

Proof. We assume that condition (2) holds, the other case is similar. First observe that $\tau^+_{\psi^j} \to \tau^+_\psi$. It suffices to observe that

$$d_1(\psi^j_\bullet, \psi_\bullet) = (\tau^+_{\psi^j} - \tau^+_\psi)\int_X \omega^n + \int_{-\infty}^{\infty}\left(\int_X \omega^n_{\psi^j_\tau} - \int_X \omega^n_{\psi_\tau}\right) \mathrm{d}\tau\,.$$

The assertion is a simple consequence of dominant convergence theorem. □

Theorem 7.16. *The map $\mathrm{DH} : \mathcal{E}^1(L^{\mathrm{an}}) \to \mathcal{M}(\mathbb{R})$ is continuous.*

For any $\alpha \in \mathcal{E}^1(L^{\mathrm{an}})$,

$$\int_{\mathbb{R}} x\,\mathrm{dDH}(\alpha)(x) = \mathrm{E}(\alpha) \tag{7.17}$$

and

$$\int_{\mathbb{R}} \mathrm{DH}(\alpha) = \frac{1}{n!}(L^n)\,. \tag{7.18}$$

Proof. We first prove the continuity of DH.

By dominant convergence theorem, it suffices to show that $G[\psi_\bullet](x)$ depends continuously on $\psi_\bullet$ for almost all $x \in \operatorname{Int}\Delta(L)$. To be more precise, let $\psi^j_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X,\omega)$ be a sequence converging to $\psi_\bullet$. We want to show that

$$G[\psi^j_\bullet](x) \to G[\psi_\bullet](x)$$

for almost all $x \in \operatorname{Int}\Delta(L)$. We will reduce to the case where $\psi^j_\bullet$ is either increasing or decreasing. In these cases, it suffices to show that $G[\psi^j_\bullet] \to G[\psi_\bullet]$ in L^1. By (7.4) and (7.9), this amounts to showing that $\mathbf{E}(\psi^j_\bullet) \to \mathbf{E}(\psi_\bullet)$. The latter follows from Lemma 7.15.

In order to make the reduction, we will prove that after passing to a subsequence, there exists an increasing sequence $\varphi^j_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X,\omega)$ and a decreasing sequence $\eta^j_\bullet \in \mathcal{TC}^1_{\mathcal{I}}(X,\omega)$ such that $\varphi^j_\bullet \le \psi^j_\bullet \le \eta^j_\bullet$ and $\varphi^j_\bullet \xrightarrow{d_1} \psi_\bullet$, $\eta^j_\bullet \xrightarrow{d_1} \psi_\bullet$. In fact, we can relax the requirement to $\varphi^j_\bullet, \eta^j_\bullet \in \mathcal{TC}^1(X,\omega)$, not necessarily $\mathcal{I}$-model. Then it suffices to replace both test curves by their pointwise $\mathcal{I}$-projections, which satisfy the same conditions by [DX20, Theorem 3.18].

Up to subtracting a subsequence, we may assume that for all j,

$$d_1(\psi^j_\bullet, \psi_\bullet) \le 2^{-j}\,.$$

For $k \ge j \ge 0$, we set

$$\eta^{j,k}_\bullet := \psi^j_\bullet \vee \cdots \vee \psi^k_\bullet \in \mathcal{TC}^1(X,\omega)\,.$$

Let $\eta^j_\bullet \in \mathcal{TC}^1(X,\omega)$ be the increasing limit of $\eta^{j,k}_\bullet$ as $k \to \infty$. We then have

$$\begin{aligned} d_1(\eta^{j,k}_\bullet, \psi_\bullet) \le & d_1(\psi_\bullet, (\psi \vee \psi^j)_\bullet) + d_1((\psi \vee \psi^j)_\bullet, (\psi \vee \psi^j \vee \psi^{j+1})_\bullet) + \cdots \\ & + d_1((\psi \vee \psi^j \vee \cdots \vee \psi^{k-1})_\bullet, (\psi \vee \psi^j \vee \cdots \vee \psi^k)_\bullet) \\ \le & d_1(\psi_\bullet, (\psi \vee \psi^j)_\bullet) + \cdots + d_1(\psi_\bullet, (\psi \vee \psi^k)_\bullet) \\ \le & C_0 \sum_{i=j}^{k} d_1(\psi_\bullet, \psi^i_\bullet) \\ \le & C_0 2^{1-j}\,. \end{aligned}$$

Here the second inequality follows from (7.16), the third inequality follows from (7.15). Then by Lemma 7.15, we find that $d_1(\eta^j_\bullet, \psi_\bullet) \leq C2^{1-j}$. Thus, $\eta^j_\bullet \xrightarrow{d_1} \psi_\bullet$.

Similarly, for $k \geq j \geq 0$, let

$$\varphi^{j,k}_\bullet := \psi^j_\bullet \wedge \cdots \wedge \psi^k_\bullet \in \mathcal{TC}^1(X, \omega)\,.$$

The same argument as above shows that for $k \geq j \geq 0$, $d_1(\varphi^{j,k}_\bullet, \psi_\bullet) \leq 2^{1-j}$. Let

$$\psi^j_\tau := \inf_{k \geq j} \varphi^{j,k}_\tau\,.$$

By monotone convergence theorem, we find that $\psi^j \in \mathcal{TC}^1(X, \omega)$. Thus, by Lemma 7.15, $d_1(\varphi^j_\bullet, \psi_\bullet) \leq 2^{1-j}$.

Next we prove (7.17). Let $\alpha \in \mathcal{E}^1(L^{\mathrm{an}})$. Let $\psi_\bullet$ be the test curve corresponding to α. We need to compute

$$\int_{\mathbb{R}} x\,\mathrm{DH}(\alpha)(x) = \int_{\Delta(L)} G[\psi_\bullet]\,\mathrm{d}\lambda\,.$$

By (7.3), (7.4) and (7.9), the right-hand side is just $\mathbf{E}(\psi_\bullet)$, which is equal to $\mathrm{E}(\alpha)$ by (7.10) and (7.11).

Finally, (7.18) follows from (7.8). □

Remark 7.17. On the subspace $\mathcal{H}^{\mathrm{NA}}$, the Duistermaat–Heckman measure is the same as the one defined in [BHJ16]. This follows from Theorem 5.24 and [BC11, Theorem A].

In this case, another interpretation of the Duistermaat–Heckman measure is provided by [His16]. Hisamoto proved that the Duistermaat–Heckman measure of a test configuration $(\mathcal{X}, \mathcal{L})$ can be interpreted as follows: let ℓ be the Phong–Sturm geodesic ray associated to $(\mathcal{X}, \mathcal{L})$ in the sense of [PS07; PS10]. Then $\mathrm{DH}(\mathcal{X}, \mathcal{L}) = \dot{\ell}_{t*}\omega^n_{\ell_t}$ for any $t \geq 0$. These measures are also studied in [Ber18].

Using a different method, we can also prove that $\mathrm{DH}(\alpha) = \dot{\ell}_{0*}\omega^n$, where $\alpha \in \mathcal{E}^1(L^{\mathrm{an}})$ and ℓ is the geodesic ray associated to ℓ, which gives a different proof of Theorem 7.16.

8. Toric setting

This section is devoted to a toric interpretation of the partial Okounkov body construction.

8.1. **Technical lemmata.**

Lemma 8.1. *Let $\alpha, \beta_1, \ldots, \beta_m \in \mathbb{Z}^n$. Let Δ be the convex polytope generated by $\beta_1, \ldots, \beta_m$. Then the following are equivalent:*

(1)

$$|z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2\right)^{-1} \tag{8.1}$$

is a bounded function on $\mathbb{C}^{*n}$.

(2) $\alpha \in \Delta$.

Proof. (2) implies (1): Write $\alpha = \sum_i t_i\beta_i$, where $t_i \in [0,1]$, $\sum_i t_i = 1$. Then

$$|z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2\right)^{-1} = \prod_i |z^{\beta_i}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2\right)^{-1} \leq \prod_i \sum_j |z^{\beta_j}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2\right)^{-1} \leq 1\,.$$

(1) implies (2): Assume that $\alpha \notin \Delta$. Let H be a hyperplane that separates α and Δ. Say H is defined by $a_1x_1 + \cdots + a_nx_n = C$. Set

$$z(t) := (t^{a_1}, \ldots, t^{a_n})\,.$$

Then clearly (8.1) evaluated at $z(t)$ is not bounded. □

Lemma 8.2. *Let $\beta_1,\dots,\beta_m\in\mathbb{N}^n$ and $\beta\in\mathbb{R}^n$. Then the following are equivalent*

(1) $\log\sum_{i=1}^m e^{x\cdot\beta_i}-(x,\beta)$ *is bounded from below.*

(2) β *is in the convex hull of* β_i*.*

Proof. The proof follows the same pattern as Lemma 8.1. □

8.2. **Toric Okounkov bodies.** Let X be an n-dimensional smooth projective toric variety, corresponding to a smooth complete fan Σ in $N_{\mathbb{R}}\cong\mathbb{R}^n$. Let N be the lattice in $N_{\mathbb{R}}$, whose dual is the character lattice M. Let $T:=N\otimes_{\mathbb{Z}}\mathbb{R}$ be the corresponding torus. Define $M_{\mathbb{R}}=N_{\mathbb{R}}^{\vee}$. Given any T-invariant divisor D on X, let $P_D\subseteq M_{\mathbb{R}}$ be the polyhedron associated to D.

Let $D_1,\dots,D_s$ be the class of prime T-invariant divisors on X, each corresponding to a ray ρ_i in Σ. Let v_i be the primitive generator of ρ_i. Any T-invariant admissible flag $Y_\bullet$ has the following form after renumbering the D_i's:

$$Y_i=D_1\cap\cdots\cap D_i\,.$$

Now the v_i's induce an isomorphism $\Phi:M\to\mathbb{Z}^n$, $u\mapsto((u,v_i))_i$. Let $\Phi_{\mathbb{R}}:M_{\mathbb{R}}\to\mathbb{R}^n$ be the extension of ϕ to $M_{\mathbb{R}}$ and σ be the cone generated by the v_i's. Let U_σ be the corresponding orbit of T. Given any T-invariant line bundle, there is a unique T-invariant divisor D with $D|_{U_\sigma}=0$ such that $\mathcal{O}_X(D)=L$.

It is shown in [LM09, Proposition 6.1] that

$$\Gamma_k(L)=\Phi_{\mathbb{R}}\left((kP_D)\cap M\right) \tag{8.2}$$

for sufficiently divisible k. We will omit $\Phi_{\mathbb{R}}$ from out notations from now on.

Let T_c be the compact torus in T. Next consider a T_c-invariant metric ϕ on L. We assume that ϕ has $\mathcal{I}$-model singularities in the sense that

$$\operatorname{vol}(L,\phi)=\frac{1}{n!}\int_X(\mathrm{dd^c}\phi)^n\,.$$

Let U_0 be the maximal orbit of T. The basis (v_i) allows us to identify $U_0=\mathbb{C}^{*n}$. We denote the coordinates on $\mathbb{C}^{*n}$ by $(z_1,\dots,z_n)$, $z_i=x_i+\mathrm{i}y_i$. Fix a T-invariant section s_0 of L on U_0 corresponding to D. Then we can identify ϕ with a T_c-invariant function on U_0. Given the identification $U_0=\mathbb{C}^{*n}$, ϕ can be identified with a convex function $\phi_{\mathbb{R}}:\mathbb{R}^n\to\mathbb{R}$ such that $\nabla\phi_{\mathbb{R}}\subseteq P_D$. We let $P_{D,\phi}$ be the closure of the image of $\nabla\phi$. By [BB13, Lemma 2.5], $P_{D,\phi}$ corresponds to the closure of

$$Q_{D,\phi}:=\{y\in M_{\mathbb{R}}:\phi(x)-(x,y)\text{ is bounded from below}\}\ .$$

We will be more explicit at this point. Assume that

$$\phi=\log\sum_{i=1}^a|s_i|^2+\mathcal{O}(1)\,,$$

where $s_i\in\mathrm{H}^0(X,L)$. Let β_i be the lattice points in P_D corresponding to s_i. In this case, $Q_{D,\phi}$ is just the convex polytope generated by the β_i's by Lemma 8.2.

Consider $\alpha\in M\cap P_D$. It corresponds to a Laurent polynomial z^α on $\mathbb{C}^{*n}$. Observe that $\alpha\in Q_{D,\phi}$ if and only if $|z^\alpha|^2e^{-\phi}$ is bounded from above. This is just a reformulation of Lemma 8.1.

Thus, we find

$$\Gamma_k(W^0(L,\phi))=(kQ_{D,\phi})\cap M \tag{8.3}$$

when k is sufficiently divisible. Hence, $\Delta(L,\phi)\subseteq P_{D,\phi}$. Comparing the volumes, we find that equality holds.

Next we deal with T_c-invariant ϕ such that $\mathrm{dd^c}\phi$ is a Kähler current. We assume that ϕ has $\mathcal{I}$-model singularity type. Let ϕ^j be an equivariant quasi-equisingular approximation of ϕ constructed as in [Dem12, Corollary 13.23]. Then by definition,

$$\Delta(L,\phi)=\bigcap_j\Delta(L,\phi^j)\,.$$

On the other hand,

$$P_{D,\phi} \subseteq \bigcap_j P_{D,\phi^j}\,.$$

Hence, $P_{D,\phi} \subseteq \Delta(L,\phi)$. On the other hand, the volume of both sides agree, so they are indeed equal thanks to the assumption that ϕ has analytic singularities.

In general, if ϕ is T_c-invariant, of $\mathcal{I}$-model singularities and has positive volume. Let $\psi \leq \phi$ be a potential with $\mathrm{dd^c}\psi$ being a Kähler current. We may guarantee that ψ is T_c-invariant. Then by definition, if we set $\phi_\epsilon = (1-\epsilon)\phi + \epsilon\psi$, then

$$\Delta(L,\phi) = \overline{\bigcup_{\epsilon\in(0,1)} \Delta(L,\phi_\epsilon)}\,.$$

While

$$P_{D,\phi} \supseteq \overline{\bigcap_{\epsilon} P_{D,P[\phi_\epsilon]_{\mathcal{I}}}}\,.$$

Thus, $\Delta(L,\phi) \supseteq P_{D,\phi}$. Comparing the volumes, we find that these convex bodies are equal.

Theorem 8.3. *Let ϕ be a T_c-invariant psh metric on L of $\mathcal{I}$-model singularities with positive volume. Then*

$$\Delta(L,\phi) = P_{D,\phi}$$

under the identification $\Phi_{\mathbb{R}}$ as above.

Remark 8.4. According to an unpublished result of Yi Yao, two toric invariant potentials ϕ and ϕ' are $\mathcal{I}$-equivalent if and only if $P_{D,\phi} = P_{D,\phi'}$. Thus, in Theorem 8.3 the assumption that ϕ has $\mathcal{I}$-model singularities is in fact unnecessary.

8.3. **Mixed volumes of line bundles.** Let X, T be as in Section 8.2.

Lemma 8.5. *Let $L_1,\dots,L_n$ be big and nef T-invariant line bundles on X. Assume that the flag is T-invariant. Then*

$$\frac{1}{n!}(L_1,\dots,L_n) = \operatorname{vol}(\Delta(L_1),\dots,\Delta(L_n))\,. \tag{8.4}$$

Here vol denotes the mixed volume functional.

Remark 8.6. As pointed out by Rémi Reboulet, this result is already proved in [BGPS14, Proposition 3.4.3].

Proof. **Step 1**. We first assume that all L_i's are ample.

In this case, we know that for any $t_i \in \mathbb{N}$ ($i = 1,\dots,n$),

$$\Delta\left(\sum_{i=1}^n t_i L_i\right) = \sum_{i=1}^n t_i \Delta(L_i)$$

by [Kav11, Theorem 3.1]. Hence,

$$\operatorname{vol}\Delta\left(\sum_{i=1}^n t_i L_i\right) = \sum_{\alpha\in\mathbb{N}^n,|\alpha|=n} \binom{n}{\alpha} t^\alpha \operatorname{vol}\left(\Delta(L_1)^{\alpha_1},\dots,\Delta(L_n)^{\alpha_n}\right)\,.$$

On the other hand, by (1.2),

$$\operatorname{vol}\Delta\left(\sum_{i=1}^n t_i L_i\right) = \frac{1}{n!}\sum_{\alpha\in\mathbb{N}^n,|\alpha|=n} \binom{n}{\alpha} t^\alpha \left(L_1^{\alpha_1},\dots,L_n^{\alpha_n}\right)\,.$$

Comparing the coefficients, we find (8.4).

Step 2. General case.

The results of Step 1 generalize immediately to ample $\mathbb{Q}$-divisors. Hence, the nef case follows from a simple perturbation argument. ☐

The following example is due to Chen Jiang.

Example 8.7. *If the flag is not toric invariant, Lemma 8.5 fails. For example, consider $X = \mathbb{P}^1 \times \mathbb{P}^1$, $L_1 = \mathcal{O}(1,2)$ and $L_2 = \mathcal{O}(2,1)$. Take a flag $X = Y_0 \supseteq Y_1 \supseteq Y_2$ with Y_1 being the diagonal. In this case, (8.4) fails.*

In this case, $(L_1, L_2) = 5$. By a simple computation using [LM09, Theorem 6.4], we find $\Delta(L_1) = \Delta(L_2)$ is the trapezoid in shown in Fig. 1. In particular,

$$\mathrm{vol}(\Delta(L_1), \Delta(L_2)) = 2 < \frac{5}{2!}\,.$$

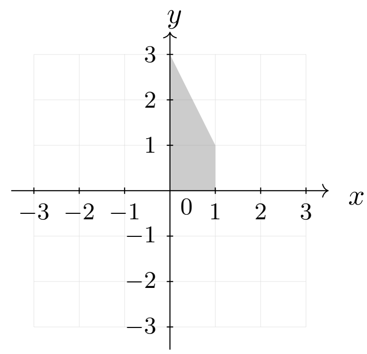

FIGURE 1. Okounkov body

For simplicity, we call (L, ϕ) a *T-invariant Hermitian psef line bundle* on X if (T, ϕ) is a Hermitian psef line bundle on X, L is T-invariant and ϕ is T_c-invariant.

Corollary 8.8. *Let (L_i, ϕ_i) $(i = 1, \dots, n)$ be T-invariant Hermitian psef line bundles on X with positive volumes. Assume that each ϕ_i has $\mathcal{I}$-model singularities. If the T-invariant flag satisfies that Y_n is not contained in any of the polar loci of ϕ_i, then*

$$\frac{1}{n!}\int_X \mathrm{dd^c}\phi_1 \wedge \cdots \wedge \mathrm{dd^c}\phi_n = \mathrm{vol}(\Delta(L_1, \phi_1), \dots, \Delta(L_n, \phi_n))\,. \tag{8.5}$$

Remark 8.9. As in Remark 8.4, the assumption that ϕ_i has $\mathcal{I}$-model singularities is in fact unnecessary.

Proof. According to (4.5) and Proposition 5.17, by perturbing L_i, we may assume that each $\mathrm{dd^c}\phi_i$ is a Kähler current.

Observe that both sides of (8.5) are continuous under d_S-approximations of ϕ_i: the left-hand side follows from Theorem 4.2 and the right-hand side follows from Theorem 5.12.

Hence, by [DX21, Lemma 3.7], we may assume that each ϕ_i has analytic singularities. Taking a birational resolution, we may assume that ϕ_i has analytic singularities along nc $\mathbb{Q}$-divisor E_i. By Remark 5.2, we reduce to the situation of Lemma 8.5. □

Together with Corollary 4.5, we finished the proof of Theorem D.

Corollary 8.10. *Let $L_1, \dots, L_n$ be big T-invariant line bundles on X. Assume that the flag $(Y_\bullet)$ is T-invariant and Y_n is not contained in the non-Kähler locus of any $c_1(L_i)$. Then*

$$\frac{1}{n!}\langle L_1, \dots, L_n\rangle = \mathrm{vol}(\Delta(L_1), \dots, \Delta(L_n))\,. \tag{8.6}$$

Here $\langle \bullet \rangle$ denotes the movable intersection in the sense of [BDPP13; BFJ09].

Proof. It suffices to apply Corollary 8.8 to the case where ϕ_i has minimal singularities. □

References

[BB10] R. J. Berman and S. Boucksom. Growth of balls of holomorphic sections and energy at equilibrium. *Inventiones mathematicae* 181.2 (2010), pp. 337–394.

[BB13] R. J. Berman and B. Berndtsson. Real Monge–Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. *Annales de la Faculté des sciences de Toulouse: Mathématiques* 22.4 (2013), pp. 649–711.

[BBJ21] R. J. Berman, S. Boucksom, and M. Jonsson. A variational approach to the Yau–Tian–Donaldson conjecture. *Journal of the American Mathematical Society* (2021).

[BBWN11] R. J. Berman, S. Boucksom, and D. Witt Nyström. Fekete points and convergence towards equilibrium measures on complex manifolds. *Acta Mathematica* 207.1 (2011), pp. 1–27.

[BC11] S. Boucksom and H. Chen. Okounkov bodies of filtered linear series. *Compositio Mathematica* 147.4 (2011), pp. 1205–1229.

[BDPP13] S. Boucksom, J.-P. Demailly, M. Paun, and T. Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *Journal of Algebraic Geometry* 22.2 (2013), pp. 201–248.

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi. Monge–Ampère equations in big cohomology classes. *Acta mathematica* 205.2 (2010), pp. 199–262.

[Ber18] B. Berndtsson. Probability measures associated to geodesics in the space of Kähler metrics. *Algebraic and analytic microlocal analysis*. Vol. 269. Springer Proc. Math. Stat. Springer, Cham, 2018, pp. 395–419.

[BFJ09] S. Boucksom, C. Favre, and M. Jonsson. Differentiability of volumes of divisors and a problem of Teissier. *Journal of Algebraic Geometry* 18.2 (2009), pp. 279–308.

[BGPS14] J. I. Burgos Gil, P. Philippon, and M. Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. 2014. arXiv: 1105.5584 [math.AG].

[BHJ16] S. Boucksom, T. Hisamoto, and M. Jonsson. Uniform K-stability and asymptotics of energy functionals in Kähler geometry. 2016. arXiv: 1603.01026 [math.DG].

[BJ21] S. Boucksom and M. Jonsson. Global pluripotential theory over a trivially valued field. 2021. arXiv: 1801.08229 [math.AG].

[Bou02] S. Boucksom. Higher dimensional Zariski decompositions. 2002. arXiv: 0204336 [math.AG].

[Bou17] S. Boucksom. Singularities of plurisubharmonic functions and multiplier ideals. http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf. 2017.

[Cao14] J. Cao. Numerical dimension and a Kawamata–Viehweg–Nadel-type vanishing theorem on compact Kähler manifolds. *Compositio Mathematica* 150.11 (2014), pp. 1869–1902.

[CC18] X. Chen and J. Cheng. On the constant scalar curvature Kähler metrics, general automorphism group. 2018. arXiv: 1801.05907 [math.DG].

[CFKL+17] C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, and C. Shramov. Newton–Okounkov bodies sprouting on the valuative tree. *Rendiconti del Circolo Matematico di Palermo Series 2* 66.2 (2017), pp. 161–194.

[CPW18] S. Choi, J. Park, and J. Won. Okounkov bodies associated to pseudoeffective divisors. *Journal of the London Mathematical Society* 97.2 (2018), pp. 170–195.

[Dar19] T. Darvas. Geometric pluripotential theory on Kähler manifolds. *Advances in complex geometry*. Vol. 735. Amer. Math. Soc., 2019, pp. 1–104.

[DDL21] T. Darvas, E. Di Nezza, and C. H. Lu. Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity. *Mathematische Annalen* 379.1-2 (2021), pp. 95–132.

[DDNL18a] T. Darvas, E. Di Nezza, and C. H. Lu. L^1 metric geometry of big cohomology classes. *Annales de l'institut Fourier* 68.7 (2018), pp. 3053–3086.

[DDNL18b] T. Darvas, E. Di Nezza, and C. H. Lu. Monotonicity of non-pluripolar products and complex Monge–Ampère equations with prescribed singularity. *Analysis & PDE* 11.8 (2018), pp. 2049–2087.

[DDNL21] T. Darvas, E. Di Nezza, and H. C. Lu. The metric geometry of singularity types. *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2021.771 (2021), pp. 137–170.

[Dem12] J.-P. Demailly. Analytic methods in algebraic geometry. International Press Somerville, MA, 2012.

[Dem15] J.-P. Demailly. On the cohomology of pseudoeffective line bundles. *Complex geometry and dynamics*. Springer, 2015, pp. 51–99.

[Den17] Y. Deng. Transcendental Morse Inequality and Generalized Okounkov Bodies. *Algebraic Geometry* 4.2 (2017), pp. 177–202.

[DL20] T. Darvas and C. H. Lu. Geodesic stability, the space of rays and uniform convexity in Mabuchi geometry. *Geometry & Topology* 24.4 (2020), pp. 1907–1967.

[DPS01] J.-P. Demailly, T. Peternell, and M. Schneider. Pseudo-effective line bundles on compact Kähler manifolds. *International Journal of Mathematics* 12.06 (2001), pp. 689–741.

[Duc21] A. Ducros. Réduction en famille d'espaces affinoïdes. 2021. arXiv: 1903.00942 [math.AG].

[DX20] T. Darvas and M. Xia. The closures of test configurations and algebraic singularity types. 2020. arXiv: 2003.04818.

[DX21] T. Darvas and M. Xia. The volume of pseudoeffective line bundles and partial equilibrium. 2021. arXiv: 2112.03827 [math.DG].

[FK18] K. Fujiwara and F. Kato. Foundations of Rigid Geometry I. European Mathematical Society Publishing House, 2018. ISBN: 9783037191354.

[His16] T. Hisamoto. On the limit of spectral measures associated to a test configuration of a polarized Kähler manifold. *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2016.713 (2016), pp. 129–148.

[Jow10] S.-Y. Jow. Okounkov bodies and restricted volumes along very general curves. *Advances in Mathematics* 223.4 (2010), pp. 1356–1371.

[Kav11] K. Kaveh. Note on Cohomology Rings of Spherical Varieties and Volume Polynomial. *Journal of Lie Theory* 21.2 (2011), pp. 263–283.

[KK12] K. Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Annals of Mathematics* (2012), pp. 925–978.

[LM09] R. Lazarsfeld and M. Mustaţă. Convex bodies associated to linear series. *Annales scientifiques de l'École normale supérieure* 42.5 (2009), pp. 783–835.

[MM07] X. Ma and G. Marinescu. Holomorphic Morse inequalities and Bergman kernels. Vol. 254. Springer Science & Business Media, 2007.

[PS07] D. H. Phong and J. Sturm. Test configurations for K-stability and geodesic rays. *Journal of Symplectic Geometry* 5.2 (2007), pp. 221–247.

[PS10] D. H. Phong and J. Sturm. Regularity of geodesic rays and Monge–Ampère equations. *Proceedings of the American Mathematical Society* 138.10 (2010), pp. 3637–3650.

[RWN14] J. Ross and D. Witt Nyström. Analytic test configurations and geodesic rays. *Journal of Symplectic Geometry* 12.1 (2014), pp. 125–169.

[Sch14] R. Schneider. Convex bodies: the Brunn–Minkowski theory. 151. Cambridge university press, 2014.

[WN14] D. Witt Nyström. Transforming metrics on a line bundle to the Okounkov body. *Annales scientifiques de l'École normale supérieure* 47.6 (2014), pp. 1111–1161.

[WN19] D. Witt Nyström. Monotonicity of non-pluripolar Monge–Ampère masses. *Indiana University Mathematics Journal* 68.2 (2019), pp. 579–591.

[Xia20] M. Xia. Pluripotential-theoretic stability thresholds. 2020. arXiv: 2012.12039 [math.DG].

[Xia21] M. Xia. Mabuchi geometry of big cohomology classes with prescribed singularities. 2021. arXiv: 1907.07234 [math.DG].