Unbounded Operators in Unitary Conformal Field Theory

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1 Methods of unbounded operators

1.1 Strong commutativity and cores

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

We always denote by $\mathcal{H}$ a complex Hilbert space. We assume unbounded operators are always densely defined, unless otherwise stated. If an unbounded operator A from $\mathcal{H}_1$ to $\mathcal{H}_2$ is continuous (with respect to the norms of $\mathcal{H}_1, \mathcal{H}_2$), we do not assume the dense domain $\mathscr{D}(A)$ of A is $\mathcal{H}_1$. If A is both continuous and everywhere-defined on $\mathcal{H}_1$, then A is called bounded.

The set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$. We set $\mathrm{End}(\mathcal{H}) = \mathrm{Hom}(\mathcal{H}, \mathcal{H})$.

If A is preclosed, then $\overline{A} = A^{**}$ denotes the closure of A.

It is a routine check that if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is preclosed and $A_1 \in \mathrm{End}(\mathcal{H}_1), A_2 \in \mathrm{End}(\mathcal{H}_2)$, then

$$A_2 T \subset T A_1 \qquad \Longrightarrow \qquad A_2 \overline{T} \subset \overline{T} A_1.$$

If H is a self-adjoint (closed) operator on $\mathcal{H}$, then (cf. [G-Sp, Sec. 10])

$$\{H\}'' = \{e^{\mathbf{i}tH} : t \in \mathbb{R}\}''.$$

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We refer the readers to [G-Sp, Sec. 6] for basic properties of strong commutativity. If S, T are preclosed operators on a Hilbert space $\mathcal{H}$, then the **strong commutativity** of S, T means that their closures $\overline{S}, \overline{T}$ commute strongly in the sense of [G-Sp, Sec. 6]. Also, recall that if A is normal and $f : \mathbb{C} \to \mathbb{C}$ is Borel, then $\{f(A)\} \subset \{A\}''$ (cf. [G-Sp, Sec. 9]). The following is from [G-Sp, Sec. 8]

Proposition 1.1.1. *Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a closed operator, and let A be a bounded operator on $\mathcal{H}_1$. Assume TA has dense domain.*

1. *TA is closed.*
2. *If the linear map $TA : \mathscr{D}(TA) \to \mathcal{H}_2$ is continuous, then TA is an (everywhere defined and) bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. In particular, $A\mathcal{H}_1 \subset \mathscr{D}(T)$.*

Remark 1.1.2. If T is closed on $\mathcal{H}$, A is bounded on $\mathcal{H}$, and T commutes strongly with A, then $AT \subset TA$ implies that TA has dense domain (containing $\mathscr{D}(T)$). Thus, by part 1 of Prop. 1.1.1, TA is closed, and is bounded when it is continuous.

For instance, suppose that S is a normal operator on $\mathcal{H}$, $f, g : \mathbb{C} \to \mathbb{C}$ are Borel functions, and both g and fg are bounded. By the properties of Borel functional calculus, $f(S)g(S) \subset (fg)(S)$ and $(fg)(S)$ is bounded. Thus, as $g(S)$ is also bounded and as $g(S)$ commutes strongly with $f(S)$, we conclude that $f(S)g(S)$ is bounded. In particular, $f(S)g(S) = (fg)(S)$.

Theorem 1.1.3. *Let H, K be self-adjoint closed operators on $\mathcal{H}$, and assume that K is affiliated with $\{H\}''$, the (abelian) von Neumann algebra generated by H. Suppose that $\mathscr{D}_0$ is a dense subspace of $\mathcal{H}$, that $\mathscr{D}_0 \subset \mathscr{D}(K)$, and that $e^{\mathbf{i}tH}\mathscr{D}_0 \subset \mathscr{D}_0$ for any $t \in \mathbb{R}$. Then $\mathscr{D}_0$ is a core for K.*

A typical case where this theorem applies is when $K = f(H)$ for some Borel function $f : \mathbb{R} \to \mathbb{R}$.

Proof. Let $T_0 = K|_{\mathscr{D}_0}$ and $T = \overline{T_0}$. Then $T \subset K$. We shall show $\mathscr{D}(K) \subset \mathscr{D}(T)$.

Since K commutes strongly with H, it also commutes strongly with each $e^{\mathbf{i}tH}$. Since $e^{\mathbf{i}tH}\mathscr{D}_0 \subset \mathscr{D}_0$ and $e^{\mathbf{i}tH}K \subset Ke^{\mathbf{i}tH}$, for each $\xi \in \mathscr{D}_0$, we have $e^{\mathbf{i}tH}T_0\xi = e^{\mathbf{i}tH}K\xi = Ke^{\mathbf{i}tH}\xi$, which equals $T_0e^{\mathbf{i}tH}\xi$ since $e^{\mathbf{i}tH}\xi \in \mathscr{D}_0$. Thus $e^{\mathbf{i}tH}T_0 \subset T_0e^{\mathbf{i}tH}$. So T commutes strongly with each $e^{\mathbf{i}tH}$, and hence with K. Let $p_t = \chi_{(-t,t)}(H)$, which commutes strongly with T. Thus, by Rem. 1.1.2, Tp_t has dense domain.

Since $Tp_t \subset Kp_t$ and Kp_t is continuous, Tp_t is continuous. So by Prop. 1.1.1, we have

$$p_t\mathcal{H} \subset \mathscr{D}(T).$$

Choose any $\xi \in \mathscr{D}(K)$. Then $p_t\xi \in \mathscr{D}(T)$, and as $t \to +\infty$, we have $p_t\xi \to \xi$ and $Tp_t\xi = Kp_t\xi = p_tK\xi \to K\xi$. Thus $\xi \in \mathscr{D}(T)$ and $T\xi = K\xi$. $\square$

Lemma 1.1.4. *Let T be a closed operators on the Hilbert space $\mathcal{H}$, and $\mathfrak{X}$ a locally compact Hausdorff space. Let $W : \mathfrak{X} \to \mathrm{End}(\mathcal{H})$ be a continuous function (where $\mathrm{End}(\mathcal{H})$ is given the strong-operator topology), such that*

$$\sup_{t \in \mathfrak{X}} \|W(t)\| < +\infty.$$

Assume that $\mathscr{D}_{\mathfrak{X}}$ is a (non-necessarily dense) subspace of $\mathscr{D}(T)$, and that for any $t \in \mathfrak{X}$ we have

$$W(t)T\big|_{\mathscr{D}_{\mathfrak{X}}} \subset TW(t). \tag{1.1.1}$$

Then for any Radon measure μ on $\mathfrak{X}$, and any Borel function $f \in L^1(\mathfrak{X}, \mu)$, the bounded operator

$$W(f) = \int_{\mathfrak{X}} f(t)W(t)d\mu(t) \tag{1.1.2}$$

satisfies

$$W(f)T\big|_{\mathscr{D}_{\mathfrak{X}}} \subset TW(f). \tag{1.1.3}$$

Note that (1.1.2) means that for each $\eta \in \mathcal{H}$, $W(f)\eta = \int_{\mathfrak{X}} f(t)W(t)\eta d\mu(t)$ is the vector whose evaluation with any $\psi \in \mathcal{H}$ is

$$\langle W(f)\eta|\psi\rangle = \int_{\mathfrak{X}} f(t)\langle W(t)\eta|\psi\rangle d\mu(t).$$

From this expression, it is clear that

$$\|W(f)\| \leqslant \|f\|_{L^1(\mathfrak{X},\mu)} \cdot \sup_{t\in\mathfrak{X}} \|W(t)\|. \tag{1.1.4}$$

Also, note that (1.1.1) means that $W(t)\mathscr{D}_{\mathfrak{X}} \subset \mathscr{D}(T)$, and that $TW(t)\xi = W(t)T\xi$ for each $\xi \in \mathscr{D}_{\mathfrak{X}}$. (1.1.3) can be understood in the same way.

Proof. First we assume that f is continuous, and has compact support in $\mathfrak{X}$. Choose any $\xi \in \mathscr{D}_{\mathfrak{X}}$. Then for any $\varepsilon > 0$, we can easily find $t_1, \dots, t_n \in \mathfrak{X}$ and $c_1, \dots, c_n \in \mathbb{C}$, such that the operator $W_\varepsilon = \sum_{i=1}^n c_i W(t_i)$ satisfies $\|W_\varepsilon\xi - W(f)\xi\| < \varepsilon$ and $\|W_\varepsilon T\xi - W(f)T\xi\| < \varepsilon$. Note that $TW_\varepsilon\xi = W_\varepsilon T\xi$, we therefore have $\|TW_\varepsilon\xi - W(f)T\xi\| < \varepsilon$. If $\varepsilon \to 0$, then $W_\varepsilon\xi \to W(f)\xi$ and $TW_\varepsilon\xi \to W(f)T\xi$. Thus, as T is closed, we conclude that $W(f)\xi \in \mathscr{D}(T)$ and $TW(f)\xi = W(f)T\xi$. This proves (1.1.3)

Now for a general L^1 function f, we can choose a sequence of continuous functions f_n with compact supports, such that $\|f - f_n\|_{L^1(\mathfrak{X},\mu)} \to 0$ as $n \to \infty$ ([Rud-R, Thm. 3.14]). Then by (1.1.4), $W(f_n) \to W(f)$ in the norm topology. An argument similar to the previous paragraph shows (1.1.3). □

The following is [CKLW18, Lemma 7.2]. We present a proof whose structure is similar to that of Thm. 1.1.2.

Theorem 1.1.5. *Let H be a self-adjoint (closed) operator on $\mathcal{H}$, and $k \in \mathbb{Z}_{\geqslant 0}$. Let $\mathscr{D}$ be a dense subspace of $\mathscr{D}(H^k)$. If there exists $\delta > 0$ and a dense subspace $\mathscr{D}_\delta \subset \mathscr{D}$, such that $e^{\mathbf{i}tH}\mathscr{D}_\delta \subset \mathscr{D}$ for any $t \in (-\delta, \delta)$, then $\mathscr{D}$ is a core for H^k.*

Proof. Let $T_0 = H^k|_{\mathscr{D}}$ and $T = \overline{T_0}$. Then $T \subset H^k$. We shall show $\mathscr{D}(H^k) \subset \mathscr{D}(T)$.

Since H^k commutes strongly with H, for any $t \in (-\delta, \delta)$ and $\xi \in \mathscr{D}_\delta$, we have $e^{\mathbf{i}tH}\xi \in \mathscr{D} \subset \mathscr{D}(T)$ and (as $e^{\mathbf{i}tH}$ commutes strongly with H^k) $Te^{\mathbf{i}tH}\xi = H^k e^{\mathbf{i}tH}\xi = e^{\mathbf{i}tH}H^k\xi = e^{\mathbf{i}tH}T\xi$. We conclude

$$e^{\mathbf{i}tH}T\big|_{\mathscr{D}_\delta} \subset Te^{\mathbf{i}tH}.$$

Choose a positive function $h \in C_c^\infty((-\delta, \delta))$ such that $\int_{\mathbb{R}} h(t)dt = 1$. Then, by lemma 1.1.4, the operator $\widehat{h}(H) = \int_{\mathbb{R}} h(t)e^{-\mathbf{i}tH}dt$ (where $\widehat{h}(s) := \int_{\mathbb{R}} h(t)e^{-\mathbf{i}ts}dt$) satisfies

$$\widehat{h}(H)T\big|_{\mathscr{D}_\delta} \subset T\widehat{h}(H).$$

This proves that $T\widehat{h}(H)$ has dense domain. $\widehat{h}(H)$ will play the role of p_t in the proof of Thm. 1.1.3.

By the basic properties of Borel functional calculus, $H^k\widehat{h}(H)$ is preclosed and its closure equals $((-\mathbf{i}\frac{\partial}{\partial t})^k h)\widehat{}(H)$, which is a bounded operator because $(-\mathbf{i}\frac{\partial}{\partial t})^k h$ is bounded. As $T \subset H^k$, we conclude that $T\widehat{h}(H)$ is continuous, and hence bounded by Prop. 1.1.1. So

$$\widehat{h}(H)\mathcal{H} \subset \mathscr{D}(T).$$

Choose any $\xi \in \mathscr{D}(H^k)$. Then $\widehat{h}(H)\xi \in \mathscr{D}(T)$. If we let such h approach the δ-function at 0 (for instance, we fix one such h and consider the sequence $h_n(t) = nh(nt)$), then $\widehat{h}(H) \to 1$ strongly, which implies $\widehat{h}(H)\xi \to \xi$ and $T\widehat{h}(H)\xi = H^k\widehat{h}(H)\xi = \widehat{h}(H)H^k\xi \to H^k\xi$. This proves that $\xi \in \mathscr{D}(T)$ and $T\xi = H^k\xi$. $\square$

Example 1.1.6 (Nelson's counterexample)**.** We use Thm. 1.1.3 to construct an example (cf. [Nel65]) of self-adjoint operators A, B on a Hilbert space, together with a core $\mathscr{D}$ for both A and B, such that $A\mathscr{D} \subset \mathscr{D}$, $B\mathscr{D} \subset \mathscr{D}$, that $AB\xi = BA\xi$ for every $\xi \in \mathscr{D}$, and that A does *not* commute strongly with B.

Let $M = \mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ and $\varphi : M \to \mathbb{C}^\times$ be the covering map $\varphi(z) = z^2$. We define a Borel measure μ on M to be the pullback of the Lebesgue measure on $\mathbb{C}^\times$ along φ. Define vector fields X, Y on M as follows: X resp. Y is the pullback of $\frac{\partial}{\partial x}$ resp. $\frac{\partial}{\partial y}$ along φ. Namely, since locally φ is a diffeomorphism, it transports $\frac{\partial}{\partial x}$ locally to X and $\frac{\partial}{\partial y}$ locally to Y.

X, Y generate one parameter groups of diffeomorphisms $\sigma_X(t), \sigma_Y(t)$. These flows preserve μ since the flows generated by $\frac{\partial}{\partial x}$ and by $\frac{\partial}{\partial y}$ preserve the Lebesgue measure. Set $\mathcal{H} = L^2(M, \mu)$. Thus, for each $f \in L^2(M, \mu)$ and $t \in \mathbb{R}$, we can define unitary operators $U(t), V(t)$ such that

$$U(t)f = f \circ \sigma_X(-t), \qquad V(t)f = f \circ \sigma_Y(-t).$$

It is a routine check that $U(t), V(t)$ are strongly continuous one-parameter groups, and that when f is smooth the derivatives of $U(t)f, V(t)f$ at $t = 0$ is $-Xf, -Yf$. Thus, by Stone's theorem, there exist self-adjoint A, B such that $e^{\mathbf{i}tA} = U(t)$, $e^{\mathbf{i}tB} = V(t)$, and $\mathbf{i}Af = -Xf, \mathbf{i}Bf = -Yf$ when f is smooth. Let $\mathscr{D}$ be the subspace of smooth functions with compact supports on M. Then for any $f \in \mathscr{D}$ we have $ABf = BAf$ because $XYf = YXf$.

We now show that $\mathscr{D}$ is a core for both A and B, and $U(t)V(s) \neq V(s)U(t)$ for some s, t (which implies that A and B do not commute strongly). Consider M as the gluing of two $\mathbb{C}^\times$ along $(0, +\infty)$: the $(0, +\infty)$ of the first $\mathbb{C}^\times$ is glued from below (resp. from above) to the $(0, +\infty)$ of the second $\mathbb{C}^\times$ from above (resp. from below). Then the $-1 - \mathbf{i}$ of the first $\mathbb{C}^\times$ is sent by $\sigma_X(1)\sigma_Y(1)$ to the $1 + \mathbf{i}$ of the second $\mathbb{C}^\times$, and sent by

$\sigma_Y(1)\sigma_X(1)$ to the $1+\mathbf{i}$ of the first $\mathbb{C}^\times$. So, for some smooth f supported in a small neighborhood of the $-1-\mathbf{i}$ of the first $\mathbb{C}^\times$, we have $U(1)V(1)f \neq V(1)U(1)f$. Now, let $\mathscr{D}_x$ be the set of smooth functions supported on the union of the two $\mathbb{C}^\times\backslash\mathbb{R}$. Then $\mathscr{D}_x$ is a dense subspace of $\mathcal{H}$ invariant under $U(t)$ for all t. Thus, by Thm. 1.1.3 (or by Thm. 1.1.5), $\mathscr{D} \supset \mathscr{D}_x$ is a core for A. A similar argument shows that $\mathscr{D}$ is a core for B.

1.2 A criterion on self-adjointness

In this section, we introduce a classical criterion on self-adjointness. Recall that an unbounded operator T on $\mathcal{H}$ is called **symmetric** if $T \subset T^*$, or equivalently, $\langle T\xi|\eta\rangle = \langle\xi|T\eta\rangle$ for every $\xi,\eta \in \mathscr{D}(T)$. A symmetric operator is necessarily preclosed.

Theorem 1.2.1. *Assume H is a closed operator on $\mathcal{H}$ such that $H-a$ is positive for some $a>0$. (Namely, $\mathrm{Sp}(H)\subset[a,+\infty)$.) Let $\mathscr{D}_0 \subset \mathscr{D}(H)$ be a (dense) core for H. Assume T is a closed symmetric operator on $\mathcal{H}$, $\mathscr{D}_0 \subset \mathscr{D}(T)$, and there exists $C>0$ such that for every $\xi,\eta \in \mathscr{D}_0$ we have*

$$\|T\xi\| \leqslant C\|H\xi\|, \tag{1.2.1}$$

$$\big|\langle T\xi|H\eta\rangle - \langle H\xi|T\eta\rangle\big| \leqslant C\|H\xi\|\cdot\|\eta\|. \tag{1.2.2}$$

Then $\mathscr{D}_0$ is a core for T, and $T=T^*$.

Roughly speaking, this theorem tells us that if T is symmetric, and if both T and the commutator $[H,T]$ are bounded by $C\cdot H$, then T is self-adjoint. This property (as well as the following strong-commutativity criterion) can be presented in many different ways which assume different conditions, cf. [Nel72, Prop. 2], [FL74], [DF77], [GJ, Thm. 19.4.3]. Our proof follows the approach in [FL74].

Proof. Suppose that we can prove $T=T^*$ whenever $\mathscr{D}_0$ is a core for T. Then, for a general T satisfying the requirements of this theorem, we let $\widetilde{T}$ be the closure of $T|_{\mathscr{D}_0}$. Then, as $\widetilde{T}$ is symmetric and also satisfies (1.2.1) and (1.2.2), we conclude $\widetilde{T}=\widetilde{T}^*$. Since $\widetilde{T}\subset T\subset T^*\subset\widetilde{T}^*$, we must have $\widetilde{T}=T=T^*$, which shows that $\mathscr{D}_0$ is a core for T and $T=T^*$. Thus, in the following, we assume $\mathscr{D}_0$ is a core for T.

Step 1. We claim that $\mathscr{D}(H)\subset\mathscr{D}(T)$, and that (1.2.1) and (1.2.2) hold for all $\xi,\eta\in\mathscr{D}(H)$. This result will imply that we can assume $\mathscr{D}_0=\mathscr{D}(H)$.

Choose any $\xi\in\mathscr{D}(H)$. Since $\mathscr{D}_0$ is a core for $\mathscr{D}(H)$, we can find a sequence $\xi_n\in\mathscr{D}_0$ converging to ξ such that $H\xi_n\to H\xi$. Apply (1.2.1) to ξ_n, we conclude that $T\xi_n$ is a Cauchy sequence, which must converge. Since T is closed, we have $\xi\in\mathscr{D}(T)$ and $T\xi_n\to T\xi$. Since (1.2.1) and (1.2.2) hold for ξ_n and for all $\eta\in\mathscr{D}_0$, they hold for all $\xi\in\mathscr{D}(H)$ and $\eta\in\mathscr{D}_0$.

Now assume $\xi\in\mathscr{D}(H)$ and $\eta\in\mathscr{D}(H)$. We choose a sequence $\eta_n\in\mathscr{D}_0$ satisfying $\eta_n\to\eta$ and $H\eta_n\to H\eta$. By similar reasoning, $T\eta_n\to T\eta$. So, as (1.2.2) holds for ξ and η_n, it holds for ξ and η.

Step 2. Recall the well-known fact that T, as a symmetric closed operator, is self-adjoint if and only if $T+\mathbf{i}$ and $T-\mathbf{i}$ have dense ranges (cf. [G-Sp, Sec. 10]). Thus, if for some $\lambda>0$ we can prove that $T\pm\lambda\mathbf{i}$ have dense ranges, then $\lambda^{-1}T$ (and hence T) is self-adjoint.

By the spectral theorem for H, we know that H^{-1} is bounded, $HH^{-1} = \mathbf{1}_{\mathcal{H}}$, and

$$\mathrm{Rng}(H^{-1}) = \mathscr{D}(H) \subset \mathscr{D}(T).$$

(Alternatively, one may use the result in [G-Sp, Sec. 4] on the relation between H and its inverse.) Choose any $\xi \in \mathcal{H}$ orthogonal to $\mathrm{Rng}(T + \lambda\mathbf{i})$ where $\lambda \in \mathbb{R}\backslash\{0\}$. Then

$$\mathrm{Im}\langle(T + \lambda\mathbf{i})H^{-2}\xi|\xi\rangle = 0,$$

namely,

$$2\lambda\langle H^{-1}\xi|H^{-1}\xi\rangle = -\langle TH^{-2}\xi|\xi\rangle + \langle\xi|TH^{-2}\xi\rangle.$$

Let us use (1.2.2) to show that $[H^{-2}, T] \leqslant 2CH^{-2}$. More precisely, we compute that

$$\langle TH^{-2}\xi|\xi\rangle = \langle TH^{-2}\xi|HH^{-1}\xi\rangle \tag{1.2.3}$$
$$=C_1 + \langle HH^{-2}\xi|TH^{-1}\xi\rangle = C_1 + \langle TH^{-1}\xi|HH^{-2}\xi\rangle \tag{1.2.4}$$
$$=C_1 + C_2 + \langle HH^{-1}\xi|TH^{-2}\xi\rangle = C_1 + C_2 + \langle\xi|TH^{-2}\xi\rangle \tag{1.2.5}$$

where

$$C_1 = \langle TH^{-2}\xi|HH^{-1}\xi\rangle - \langle HH^{-2}\xi|TH^{-1}\xi\rangle,$$
$$C_2 = \langle TH^{-1}\xi|HH^{-2}\xi\rangle - \langle HH^{-1}\xi|TH^{-2}\xi\rangle.$$

By (1.2.2), we have $|C_1|, |C_2| \leqslant C\langle H^{-1}\xi|H^{-1}\xi\rangle$.

It now follows that $2\lambda\langle H^{-1}\xi|H^{-1}\xi\rangle \leqslant 2C\langle H^{-1}\xi|H^{-1}\xi\rangle$. Thus, by choosing $\lambda = \pm 2C$, we conclude that $H^{-1}\xi = 0$ and hence $\xi = HH^{-1}\xi = 0$. So $T \pm 2C\mathbf{i}$ have dense ranges. This proves $T = T^*$. □

Example 1.2.2. Let $\mathcal{H} = L^2(\mathbb{R}, m)$ where m is the Lebesgue measure. Set $\partial_x = \frac{d}{dx}$. Let $\mathscr{D}_0$ be the space of rapid decreasing functions. Then by Fourier transform (which preserves $\mathscr{D}_0$ and transforms ∂_x to the multiplication of $\mathbf{i}x$), we have a positive operator $-\partial_x^2$ with core $\mathscr{D}_0$, and whose action on $\mathscr{D}_0$ is understood in the usual way.

Let V be a real valued function on $\mathbb{R}$ (the potential function) such that V'' exists, and that V, V', V'' are continuous and uniformly bounded on $\mathbb{R}$ by $C > 0$. Let

$$T = -\partial_x^2 + V$$

with domain $\mathscr{D}_0$, where V is the multiplication of the function V. Then $[-\partial_x^2, T] = -\partial_x \cdot V' - V' \cdot \partial_x = -V'' - 2V' \cdot \partial_x$. Using Fourier transform or Spectral theorem, it is easy to see that for each $\xi \in \mathscr{D}_0$,

$$\|\partial_x\xi\|^2 = \langle -\partial_x^2\xi|\xi\rangle \leqslant \langle(1 - \partial_x^2)^2\xi|\xi\rangle = \|(1 - \partial_x^2)\xi\|^2.$$

Thus $\|[-\partial_x^2, T]\xi\| \leqslant C\|\xi\| + 2C\|(1 - \partial_x^2)\xi\| \leqslant 3C\|(1 - \partial_x^2)\xi\|$. Set $H = -\partial_x^2 + 1 + C$, which is a positive operator with core $\mathscr{D}_0$, we have

$$\|T\xi\| \leqslant \|H\xi\|, \qquad \|[H, T]\xi\| \leqslant 3C\|H\xi\|$$

for every $\xi \in \mathscr{D}_0$. By Thm. 1.2.1, $\overline{T}$ is self-adjoint.

1.3 Taylor's theorem for $e^{\mathrm{i}\overline{T}}\xi$ and strong commutativity

The main reference of this section is [TL99].

Throughout this section, we assume H is a closed operator on a Hilbert space $\mathcal{H}$ such that $H - a$ is positive for some $a > 0$. $H - a$ will play the role of Hamiltonian L_0 in conformal field theory.

We set

$$\mathscr{D}(H^\infty) := \bigcap_{n\in\mathbb{N}} \mathscr{D}(H^n) = \bigcap_{s\in[0,+\infty)} \mathscr{D}(H^s).$$

Remark 1.3.1. $\mathscr{D}(H^\infty)$ is a core for H^n (for every $n \in \mathbb{Z}$) and for every $f(H)$ where $f : \mathbb{C} \to \mathbb{C}$ is bounded and Borel, since $\mathscr{D}(H^\infty)$ contains the range of $\chi_{(-a,a)}(H)$ for all $a > 0$.

Let T be a preclosed operator on $\mathcal{H}$ with **invariant domain** $\mathscr{D}(H^\infty)$, which means that $T\mathscr{D}(H^\infty) \subset \mathscr{D}(H^\infty)$. Note that $H^n\mathscr{D}(H^m) \subset \mathscr{D}(H^{n+m})$ shows that $\mathscr{D}(H^\infty)$ is H^n-invariant for every $n \in \mathbb{Z}$.

Definition 1.3.2. We say that T satisfies H**-bounds of order** r (where $r \geqslant 0$) if for each $n \in \mathbb{N}$ there exists a constant $|T|_n \geqslant 0$ (the n**-th bounding constant**) such that for every $\xi \in \mathscr{D}(H^\infty)$ we have

$$\|H^nT\xi\| \leqslant |T|_n \cdot \|H^{n+r}\xi\|. \tag{1.3.1}$$

H-bounds of order 1 are called **linear** H**-bounds**.

Remark 1.3.3. Note that if $0 \leqslant a < b$, then there is $C > 0$ such that $\|H^a\xi\| \leqslant C\|H^b\xi\|$ for all $\xi \in \mathscr{D}(H^\infty)$. It follows that if $0 \leqslant r_1 < r_2$, and if T satisfies H-bounds of order r_1, then it satisfies H-bounds of order r_2.

Lemma 1.3.4. *Choose $n \in \mathbb{N}$. Assume (1.3.1) holds for all $\xi \in \mathscr{D}(H^\infty)$. Then*

$$\mathscr{D}(H^{n+r}) \subset \mathscr{D}(\overline{T}), \qquad \overline{T}\mathscr{D}(H^{n+r}) \subset \mathscr{D}(H^n),$$

and (1.3.1) holds for all $\xi \in \mathscr{D}(H^{n+r})$ if T is replaced by $\overline{T}$.

Proof. Let $\xi \in \mathscr{D}(H^{n+r}) \subset \mathscr{D}(H^r)$. Choose $\xi_k = \chi_{(-k,k)}(H)\xi$, which is in $\mathscr{D}(H^\infty)$. Since $H^r\xi_k$ converges to $H^r\xi$, by (1.3.1) (where $n = 0$), $T\xi_k$ is a Cauchy sequence. So, since $\xi_k \to \xi$, we conclude $\xi \in \mathscr{D}(\overline{T})$ and $T\xi_k \to \overline{T}\xi$.

Similarly, since $H^{n+r}\xi_k \to H^{n+r}\xi$, by (1.3.1), we conclude that $H^nT\xi_k$ converges to a vector whose norm is bounded by $|T|_n\|H^{n+r}\xi\|$. Thus, because $T\xi_k \to \overline{T}\xi$, we have $\overline{T}\xi \in \mathscr{D}(H^n)$ and $H^n\overline{T}\xi$ has norm bounded by $|T|_n\|H^{n+r}\xi\|$. □

Lemma 1.3.5. *Assume T is preclosed with invariant domain $\mathscr{D}(H^\infty)$. Assume also that T satisfies linear H-bounds. Then for every $n \in \mathbb{Z}$ there exists a bounding constant $|T|_n \geqslant 0$ such that for every $\xi \in \mathscr{D}(H^\infty)$ we have*

$$\|H^nT\xi\| \leqslant |T|_n \cdot \|H^{n+1}\xi\|.$$

Proof. We know this is true when $n \geqslant 0$. Now assume $n < 0$ and let $m = -n$. Then for every $\xi, \eta \in \mathscr{D}(H^\infty)$,

$$\begin{aligned}&\big|\langle H^{-m}T\xi|H^m\eta\rangle\big| = \big|\langle \xi|T\eta\rangle\big| = \big|\langle H^{-m+1}\xi|H^{m-1}T\eta\rangle\big| \\ \leqslant& \|H^{-m+1}\xi\|\cdot|T|_{m-1}\|H^m\eta\|.\end{aligned}$$

Since $H^m\mathscr{D}(H^\infty) \supset \mathscr{D}(H^\infty)$ because $\mathscr{D}(H^\infty)$ is H^{-m}-invariant, $H^m\mathscr{D}(H^\infty)$ is dense. Therefore $\|H^{-m}T\xi\| \leqslant |T|_{m-1}\|H^{-m+1}\xi\|$. □

Theorem 1.3.6. *Assume T is symmetric with dense invariant domain $\mathscr{D}(T) = \mathscr{D}(H^\infty)$. Assume that both T and $[H,T]$ satisfy linear H-bounds. Then $\overline{T}$ is self-adjoint. Moreover, for any $n \in \mathbb{N}$, $\mathscr{D}(H^n)$ is $e^{\mathbf{i}t\overline{T}}$-invariant, and there exists a constant $C_n \geqslant 0$ such that for every $\xi \in \mathscr{D}(H^n)$ and $t \in \mathbb{R}$ we have*

$$\|H^n e^{\mathbf{i}t\overline{T}}\xi\| \leqslant e^{C_n|t|}\cdot\|H^n\xi\|. \tag{1.3.2}$$

It follows that $\mathscr{D}(H^\infty)$ is $e^{\mathbf{i}t\overline{T}}$-invariant, and $e^{\mathbf{i}t\overline{T}}$ satisfies H-bounds of order 0.

Idea of the proof. We know from Thm. 1.1.5 that $\overline{T}$ is self-adjoint. Take $N = H^{2n}$. Using the fact that $[H,T]$ satisfies linear H-bounds, it is not hard to check that $[N,T]$ satisfies H-bounds of order $2n$, and hence satisfies linear N-bounds. From this, one shows that $\mathbf{i}[N,T] \leqslant cN$ for $c > 0$, namely, $\mathbf{i}\langle [N,T]\eta|\eta\rangle \leqslant c\langle N\eta|\eta\rangle$ where $\eta \in \mathscr{D}(H^\infty)$.

We "integrate" the inequality $\mathbf{i}[N,T] \leqslant cN$ for $t \geqslant 0$, which gives $e^{-\mathbf{i}t\overline{T}}Ne^{\mathbf{i}t\overline{T}} \leqslant e^{ct}N$ evaluate in $\langle\cdot\xi|\xi\rangle$ for every "nice" vector ξ. This shows $\|H^n e^{\mathbf{i}t\overline{T}}\xi\|^2 \leqslant e^{ct}\|H^n\xi\|^2$. To be more precise about "integrating" the inequality, we consider the function $f_\xi(t) = e^{-ct}\langle e^{-\mathbf{i}t\overline{T}}Ne^{\mathbf{i}t\overline{T}}\xi|\xi\rangle$. Then $f'_\xi(t) = \mathbf{i}\langle e^{-\mathbf{i}t\overline{T}}[N,T]e^{\mathbf{i}t\overline{T}}\xi|\xi\rangle - c\langle Ne^{\mathbf{i}t\overline{T}}\xi|e^{\mathbf{i}t\overline{T}}\xi\rangle \leqslant 0$. So $f_\xi(t) \leqslant f_\xi(0)$, which shows (1.3.2) when $t \geqslant 0$. In the case $t \leqslant 0$, we replace t by $-t$ and T by $-T$, to obtain again (1.3.2).

The problem with this argument is that we don't know if every $\xi \in \mathscr{D}(\mathcal{H}^n)$ is good or not. To overcome this difficulty, we replace N by the bounded operator $N_\epsilon = N(1+\epsilon N)^{-1}$ so that all the expressions in the above paragraph can be defined. One shows again that $\mathbf{i}\langle[N_\epsilon, T]\eta|\eta\rangle \leqslant c\langle N_\epsilon\eta|\eta\rangle$ where $\eta \in \mathscr{D}(H^\infty)$, and by approximation, $\mathbf{i}\langle N_\epsilon\overline{T}\eta|\eta\rangle - \mathbf{i}\langle N_\epsilon\eta|\overline{T}\eta\rangle \leqslant c\langle N_\epsilon\eta|\eta\rangle$ when $\eta \in \mathscr{D}(\overline{T})$. Using the argument in the previous paragraph, we obtain (for $t \geqslant 0$) $e^{-\mathbf{i}t\overline{T}}N_\epsilon e^{\mathbf{i}t\overline{T}} \leqslant e^{ct}N_\epsilon$ evaluated in $\langle\cdot\xi|\xi\rangle$ whenever $\xi \in \mathscr{D}(\overline{T})$, and hence whenever $\xi \in \mathcal{H}$. Assume $\xi \in \mathscr{D}(H^n)$. Then the limit of this inequality when $\epsilon \searrow 0$ yields the desired result. □

Proof. By symmetry, it suffices to prove the claim for $t \geqslant 0$. By Thm. 1.1.5, $\overline{T}$ is self-adjoint. Set $N = T^{2n}$.

Step 1. For each $k \in \mathbb{Z}$, let $|T|_k$ be a k-th bounding constant for both T and $[H,T]$. (Cf. Lemma 1.3.5.) Then, when acting on $\mathscr{D}(H^\infty)$,

$$[N,T] = \sum_{j=0}^{2n-1} H^j[H,T]H^{2n-1-j}.$$

Set $c = \sum_{j=0}^{2n-1}|T|_{j-n}$. For each $\eta \in \mathscr{D}(H^\infty)$,

$$\mathbf{i}\langle[N,T]\eta|\eta\rangle \leqslant \|H^n\eta\|\cdot\|H^{-n}[N,T]\eta\|$$

$$
\begin{aligned}
&=\sum_{j=0}^{2n-1}\|H^n\eta\|\cdot\|H^{j-n}[H,T]H^{2n-1-j}\eta\|\leqslant\sum_{j=0}^{2n-1}|T|_{j-n}\|H^n\eta\|^2\\
&\leqslant c\langle N\eta|\eta\rangle.
\end{aligned}\tag{1.3.3}
$$

Step 2. In general, $[S^{-1},T]=-S^{-1}[S,T]S^{-1}$ if S and its inverse S^{-1} is defineable on $\mathscr{D}(H^\infty)$. Choose $\epsilon>0$. Then, when acting on $\mathscr{D}(H^\infty)$, we have

$$
[(1+\epsilon N)^{-1},T]=-\epsilon(1+\epsilon N)^{-1}[N,T](1+\epsilon N)^{-1},
$$

and hence

$$
\begin{aligned}
&[N(1+\epsilon N)^{-1},T]=-N\cdot\epsilon(1+\epsilon N)^{-1}[N,T](1+\epsilon N)^{-1}+[N,T](1+\epsilon N)^{-1}\\
=&(1+\epsilon N)^{-1}[N,T](1+\epsilon N)^{-1}.
\end{aligned}
$$

Note that (by Rem. 1.3.1) $\mathscr{D}(H^\infty)$ is invariant under $T,N,(1+\epsilon N)^{-1}$. It follows that for every $\eta\in\mathscr{D}(H^\infty)$, we have (by (1.3.3))

$$
\begin{aligned}
&\mathbf{i}\langle[N(1+\epsilon N)^{-1},T]\eta|\eta\rangle=\mathbf{i}\langle[N,T](1+\epsilon N)^{-1}\eta|(1+\epsilon N)^{-1}\eta\rangle\\
\leqslant&c\langle(1+\epsilon N)^{-1}N(1+\epsilon N)^{-1}\eta|\eta\rangle.
\end{aligned}
$$

By Borel functional calculus, $N(1+\epsilon N)^{-1}-N(1+\epsilon N)^{-2}$ is positive (since its closure is $g(N)$ where the function $g(x)=x(1+\epsilon x)^{-1}-x(1+\epsilon x)^{-2}$ is positive). Therefore

$$
\mathbf{i}\langle[N(1+\epsilon N)^{-1},T]\eta|\eta\rangle\leqslant c\langle N(1+\epsilon N)^{-1}\eta|\eta\rangle\tag{1.3.4}
$$

for all $\eta\in\mathscr{D}(H^\infty)$.

Now suppose $\eta\in\mathscr{D}(\overline{T})$. Since $\mathscr{D}(H^\infty)=\mathscr{D}(T)$ is a core for $\overline{T}$, we may choose a sequence $\eta_n\in\mathscr{D}(H^\infty)$ converging to η such that $T\eta_n$ converges to $T\eta$. Note that $N(1+\epsilon N)^{-1}$ is bounded by Rem. 1.1.2. Therefore, since each η_n satisfies (1.3.4), we obtain

$$
\mathbf{i}\langle N(1+\epsilon N)^{-1}\overline{T}\eta|\eta\rangle-\mathbf{i}\langle N(1+\epsilon N)^{-1}\eta|\overline{T}\eta\rangle\leqslant c\langle N(1+\epsilon N)^{-1}\eta|\eta\rangle.\tag{1.3.5}
$$

Step 3. By Rem. 1.1.2, $N^{\frac12}(1+\epsilon N)^{-\frac12}$ and $N(1+\epsilon N)^{-1}$ are bounded. For any $\xi\in\mathcal{H}$, we set

$$
\begin{aligned}
\xi_t&=e^{\mathbf{i}t\overline{T}}\xi,\\
f_{\epsilon,\xi}(t)&=e^{-ct}\big\|N^{\frac12}(1+\epsilon N)^{-\frac12}\xi_t\big\|^2=e^{-ct}\langle N(1+\epsilon N)^{-1}\xi_t|\xi_t\rangle.
\end{aligned}
$$

Assume $\xi\in\mathscr{D}(\overline{T})$. Then $\xi_t\in\mathscr{D}(\overline{T})$ and $\frac{d}{dt}\xi_t=\mathbf{i}\overline{T}\xi_t$. So the derivative $f'_{\epsilon,\xi}(t)=\frac{d}{dt}f_{\epsilon,\xi}(t)$ exists for all $t\in\mathbb{R}$. We compute

$$
f'_{\epsilon,\xi}(t)=-cf_{\epsilon,\xi}(t)+\mathbf{i}\langle N(1+\epsilon N)^{-1}\overline{T}\xi_t|\xi_t\rangle-\mathbf{i}\langle N(1+\epsilon N)^{-1}\xi_t|\overline{T}\xi_t\rangle.\tag{1.3.6}
$$

By (1.3.5),

$$
f'_{\epsilon,\xi}(t)\leqslant-cf_{\epsilon,\xi}(t)+c\langle N(1+\epsilon N)^{-1}\xi_t|\xi_t\rangle=0.
$$

Therefore, when $t \geqslant 0$, we have $f_{\epsilon,\xi}(t) \leqslant f_{\epsilon,\xi}(0)$, namely,

$$e^{-ct}\big\|N^{\frac{1}{2}}(1+\epsilon N)^{-\frac{1}{2}}\xi_t\big\|^2 \leqslant \big\|N^{\frac{1}{2}}(1+\epsilon N)^{-\frac{1}{2}}\xi\big\|^2. \tag{1.3.7}$$

By approximation, (1.3.7) holds for all $\xi \in \mathcal{H}$.

By spectral theorem, for each $\eta \in \mathcal{H}$, $a_\epsilon := \big\|N^{\frac{1}{2}}(1+\epsilon N)^{-\frac{1}{2}}\xi\big\|^2$ increases as ϵ decreases; $\lim_{\epsilon\to 0} a_\epsilon$ converges if and only if $\eta \in \mathscr{D}(N^{\frac{1}{2}}) = \mathscr{D}(H^n)$; if it converges, then it must converge to $\|H^n\eta\|^2$.

We now assume $\xi \in \mathscr{D}(H^n)$. Let $\epsilon \to 0$. Then, by the previous paragraph, the right hand side of (1.3.7) converges to $\|N^{\frac{1}{2}}\xi\|^2 = \|H^n\xi\|^2$. So the left hand side of (1.3.7) (which increases as $\epsilon \searrow 0$) must converge to $e^{-ct}\|H^n\xi_t\|^2$ where we have $\xi_t \in \mathscr{D}(H^n)$. This proves $\|H^n\xi_t\|^2 \leqslant e^{2C_n t}\|H^n\xi\|^2$ when $t \geqslant 0$, if we set $C_n = c/2$. □

Definition 1.3.7. For each $n \in \mathbb{N}$, we let $o_H(h)$ be the set of $\mathscr{D}(H^\infty)$-valued functions $\psi = \psi(h)$ where each ψ is defined on a neighborhood of $0 \in \mathbb{R}$ and satisfies for all $m \in \mathbb{N}$ that

$$\lim_{h\to 0} \frac{\|H^m\psi(h)\|}{h^n} = 0. \tag{1.3.8}$$

Remark 1.3.8. If S is preclosed on $\mathcal{H}$ with invariant domain $\mathscr{D}(H^\infty)$, and if S satisfies H-bounds of some order r, then for all $n \in \mathbb{N}$ it is clear that

$$S \cdot o_H(h^n) \subset o_H(h^n).$$

Moreover, by Thm. 1.3.6, we also have for all $t \in \mathbb{R}$ that

$$e^{\mathbf{i}t\overline{S}} o_H(h) \subset o_H(h), \qquad e^{\mathbf{i}(t+h)\overline{S}} o_H(h) \subset o_H(h).$$

By Taylor series expansion, if f is a smooth function on $(a,b) \subset \mathbb{R}$, then for any $t, t+h \in (a,b)$ and $n \in \mathbb{N}$, we have (cf. [Apo, Thm. 9.29])

$$f(t+h) = \sum_{k=0}^n \frac{f^{(k)}(t)}{k!}h^k + \frac{1}{n!}\int_t^{t+h}(t-s)^n f^{(n+1)}(s)ds. \tag{1.3.9}$$

We are now ready prove the Taylor theorem for $e^{\mathbf{i}t\overline{T}}\xi$.

Theorem 1.3.9. *Let T be as in Thm. 1.3.6. Then for every $\xi \in \mathscr{D}(H^\infty)$ and $n \in \mathbb{N}$, we have*

$$e^{\mathbf{i}(t+h)\overline{T}}\xi = \sum_{k=0}^n \frac{(\mathbf{i}T)^k}{k!} e^{\mathbf{i}t\overline{T}}\xi + R(h)\xi \tag{1.3.10}$$

where each summand is in $\mathscr{D}(H^\infty)$, and $R(h)\xi \in o_H(h^n)$.

Proof. Apply (1.3.9) to $f(t) = \langle e^{\mathbf{i}(t+h)\overline{T}}\xi|\eta\rangle$ for every $\eta \in \mathcal{H}$, we obtain

$$R(h)\xi = \frac{1}{n!}\int_t^{t+h}(t-s)^n T^{n+1} e^{\mathbf{i}s\overline{T}}\xi ds.$$

Thus, for any $\eta \in \mathcal{H}$, by (1.3.1) and Thm. 1.3.6, there exist $\lambda, C > 0$ such that

$$\begin{aligned}
|\langle H^m R(h)\xi|\eta\rangle| \leqslant \frac{h^{n+1}}{n!} \sup_{t\leqslant s\leqslant t+h} \left|\langle H^m T^{n+1} e^{\mathbf{i}s\overline{T}}\xi|\eta\rangle\right| \\
\leqslant \frac{h^{n+1}}{n!}\lambda \cdot \sup_{t\leqslant s\leqslant t+h} \|H^{m+n+1}e^{\mathbf{i}s\overline{T}}\xi\|\cdot\|\eta\| \\
\leqslant \frac{h^{n+1}}{n!}\lambda e^{C|h|}\cdot \|H^{m+n+1}e^{\mathbf{i}t\overline{T}}\xi\|\cdot\|\eta\|.
\end{aligned}$$

This proves $R(h)\xi \in o_H(h^n)$. □

Theorem 1.3.10. *Let S, T be preclosed operators on $\mathcal{H}$ with common (S- and T-)invariant domain $\mathscr{D}(S) = \mathscr{D}(T) = \mathscr{D}(H^\infty)$. Assume T is symmetric, T and $[H, T]$ satisfies linear H-bounds, T satisfies H-bounds of some order $r \geqslant 0$, and $ST\xi = TS\xi$ for every $\xi \in \mathscr{D}(H^\infty)$. Then S commutes strongly with T.*

Proof. By Thm. 1.3.6, $\overline{T}$ is self-adjoint, and $e^{\mathbf{i}t\overline{T}}$ leaves $\mathscr{D}(H^\infty)$ invariant. Since $\{\overline{T}\}'' = \{e^{\mathbf{i}t\overline{T}} : t \in \mathbb{R}\}$, we need to show $e^{\mathbf{i}t\overline{T}}\overline{S} = Te^{\mathbf{i}t}$ for all t, which follows if we can show $e^{\mathbf{i}t\overline{T}}S = Se^{\mathbf{i}t\overline{T}}$.

Let us choose any $\xi \in \mathscr{D}(H^\infty)$, and let

$$\Xi(t) = e^{\mathbf{i}t\overline{T}}Se^{-\mathbf{i}t\overline{T}}\xi.$$

If we can show that the derivative $\Xi'(t)$ exists and is 0 everywhere, then $\Xi(t) = \Xi(0) = S$, which will finish the proof. Choose $h \in \mathbb{R}$. Then by Thm. 1.3.9,

$$\begin{aligned}
\Xi(t+h) = e^{\mathbf{i}(t+h)\overline{T}}Se^{-\mathbf{i}(t+h)\overline{T}}\xi \\
\in e^{\mathbf{i}(t+h)\overline{T}}S\big((1-\mathbf{i}hT)e^{-\mathbf{i}t\overline{T}}\xi + o_H(h)\big).
\end{aligned}$$

By Rem. 1.3.8, we have $e^{\mathbf{i}(t+h)\overline{T}}So_H(h) \subset o_H(h)$. So

$$\Xi(t+h) \in \ e^{\mathbf{i}(t+h)\overline{T}}S(1-\mathbf{i}hT)e^{-\mathbf{i}t\overline{T}}\xi + o_H(h).$$

By Thm. 1.3.9 and Rem. 1.3.9 again,

$$\begin{aligned}
\Xi(t+h) \in \ (1+\mathbf{i}hT)e^{\mathbf{i}t\overline{T}}S(1-\mathbf{i}hT)e^{-\mathbf{i}t\overline{T}}\xi + o_H(h) \\
= e^{\mathbf{i}t\overline{T}}(1+\mathbf{i}hT)S(1-\mathbf{i}hT)e^{-\mathbf{i}t\overline{T}}\xi + o_H(h) \\
= e^{\mathbf{i}t\overline{T}}(S+\mathbf{i}[T,S]-h^2T^2)e^{-\mathbf{i}t\overline{T}}\xi + o_H(h) \\
= e^{\mathbf{i}t\overline{T}}(S+\mathbf{i}[T,S])e^{-\mathbf{i}t\overline{T}}\xi + o_H(h).
\end{aligned}$$

Since $TS = ST$ on $\mathscr{D}(H^\infty)$, we have

$$\Xi(t+h) \in \ e^{\mathbf{i}t\overline{T}}Se^{-\mathbf{t}\mathbf{i}T}\xi + o_H(h) = \Xi(t) + o_H(h).$$

This shows that $\lim_{h\to 0} h^{-1}(\Xi(t+h) - \Xi(t)) = 0$ for all t. We are done. □

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