

ON LIU MORPHISMS IN NON-ARCHIMEDEAN GEOMETRY

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ABSTRACT. We define Liu morphisms and quasi-Liu morphisms between Berkovich analytic spaces. We show that Liu morphisms and quasi-Liu morphisms behave as affine morphisms and quasi-affine morphisms of schemes in many aspects.

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1. INTRODUCTION

1.1. Motivation. In classical algebraic geometry, the theories of affine morphisms and quasi-affine morphisms play a prominent role. In the non-Archimedean world, it is highly desirable to have analogous results as well. However, there are two principal difficulties in the non-Archimedean setting:

- (1) First of all, there is no satisfactory theory of quasi-coherent sheaves in non-Archimedean geometry. There is indeed an *ad hoc* notion of quasi-coherent sheaves in rigid geometry defined by Conrad in [Con06]: A quasi-coherent sheaf is a sheaf of modules which can be expressed as a filtered colimit of coherent sheaves locally. However, Conrad’s notion of quasi-coherent sheaves does not behave as expected: On an affinoid space, the higher cohomologies of a quasi-coherent sheaf do not vanish in general. This makes it hard to handle affine morphisms in terms of quasi-coherent sheaves of algebras. The same problem persists in Berkovich geometry.
- (2) Secondly, a more severe problem was proposed by Liu [Liu88], [Liu90]. It is shown that there is a quasi-compact, separated non-affinoid rigid space X , a morphism $f : X \rightarrow Y$ to an affinoid space Y , an admissible affinoid covering $\{U_i\}$ of Y such that $f^{-1}U_i$ is affinoid for each i . See [Liu90, Proposition 3.3 and Section 5]. This means that the property that the inverse image of an affinoid domain is affinoid is *not* G-local.

Recall that in classical algebraic geometry, we have the celebrated *Serre’s criterion* ([EGA II, Théorème 5.2.1]): Affine schemes can be characterized by cohomological triviality among quasi-compact separated schemes. Similarly, in non-Archimedean setting, we replace the usual local notion of affinoid spaces by cohomologically trivial spaces. Such spaces are studied by Maculan–Poineau in [MP21] under the name of *Liu spaces*, we follow their terminology.

Definition 1.1 (c.f. Definition 3.1). Let k be a complete non-Archimedean valued field. A quasi-compact, separated k -analytic space X (in the sense of Berkovich) is said to be *Liu* if for any analytic extension k'/k , any coherent sheaf \mathcal{F} on $X_{k'}$ is acyclic.

On the morphism level, we define a *Liu morphism* as a morphism under which the inverse image of a Liu domain is a Liu space, see [Definition 4.1](#). Similarly, we have a notion of quasi-Liu morphisms analogous to the classical notion of quasi-affine morphisms:

Definition 1.2 (c.f. [Definition 5.2](#)). Let $f : X \rightarrow Y$ be a morphism of k -analytic spaces. We say f is *quasi-Liu* if for any Liu domain Z in Y , $f^{-1}Z$ can be embedded in a Liu k -analytic space as a compact analytic domain and $H^0(f^{-1}Z, \mathcal{O}_X)$ is a Liu k -algebra ([Definition 3.4](#)).

Similar to the situation in classical algebraic geometry, we prove a cohomological criterion of Liu morphisms when Y is Liu in [Theorem 4.4](#).

Unfortunately, as pointed out by Marco Maculan, contrary to the assertion in the previous versions of this paper, the notion of Liu morphisms is not G -local on the target, see an example due to Scholze–Weinstein in [Example 4.1](#).

As for (1), due to the progress made by Ben-Bassat–Kremnizer in [\[BBK17\]](#), it is by far clear that the natural notion on a non-Archimedean analytic space is not that of the quasi-coherent sheaves, but the derived category of quasi-coherent sheaves instead. However, as we will see, in the special case of sheaves of Liu algebras studied below, the derived notion reduces to a *bona fide* notion of quasi-coherence at the non-derived level. In particular, on a separated space, there is a global notion of quasi-coherent sheaves of Liu algebras, see [Definition 4.2](#).

1.2. Main results. We fix a complete non-Archimedean valued field k . We allow the valuation on k to be trivial. We work in the framework of Berkovich spaces as in [\[Ber93\]](#).

The main result says that Liu morphisms and quasi-coherent sheaves of Liu k -algebras are essentially equivalent:

Theorem 1.1 (=Corollary 4.7). *Let X be a separated k -analytic space. Then the functor*

$$\underline{\mathrm{Sp}}_X : \mathcal{LiuAlg}_{X,k}^{\mathrm{QCoh}} \rightarrow \mathcal{Liu}_{\rightarrow X,k}$$

is an anti-equivalence of categories.

Here $\mathcal{LiuAlg}_{X,k}^{\mathrm{QCoh}}$ is the category of quasi-coherent sheaves of Liu k -algebras on X , $\mathcal{Liu}_{\rightarrow X,k}$ is the category of Liu morphisms $Y \rightarrow X$. The functor $\underline{\mathrm{Sp}}_X$ is the relative spectrum functor defined in [Definition 4.3](#). This result is analogous to the classical result on affine morphisms and quasi-coherent sheaves of algebras ([\[EGA II, Proposition 1.2.7, Proposition 1.3.1\]](#)).

1.3. Structure of the paper. In [Section 2](#), we recall some basic results about Berkovich analytic spaces and the language developed by Ben-Bassat and Kremnizer ([\[BBK17\]](#)). Due to the lack of references, we also prove a representability theorem ([Theorem 2.1](#)) about presheaves on the category of analytic spaces.

In [Section 3](#), we recall the basic theory of Liu spaces and Liu algebras. We prove that Liu algebras behave very similar to affinoid algebras in many aspects.

In [Section 4](#), we introduce Liu morphisms and study their relation to quasi-coherent sheaves of Liu algebras.

In [Section 5](#), we introduce and study quasi-Liu morphisms.

In [Section 6](#), we give a list of unsolved problems related to this work.

We collect results from [\[BBK17\]](#) in [Appendix A](#).

1.4. Conventions. Let k be a complete non-Archimedean valued field. An *analytic extension* of k is a complete non-Archimedean valued field k' containing k such that the restriction of the valuation on k' to k coincides with the given valuation on k . We denote the spectrum of a Banach algebra A by $\mathrm{Sp} A$ instead of the more common notation $\mathcal{M}(A)$.

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2. PRELIMINARIES

Let k be a complete non-Archimedean valued field.

2.1. Analytic spaces. In this paper, by a k -analytic space, we mean a k -analytic space in the sense of [Ber93]. The category of k -analytic spaces is denoted by \mathbf{An}_k . For each k -analytic space X , we endow X with the G -topology as in [Ber93]. The corresponding site is still denoted by X . There is a natural sheaf of rings \mathcal{O}_X making X a ringed site. We always omit the subindex G and write \mathcal{O}_X instead. The category of coherent sheaves on X is denoted by \mathcal{Coh}_X .

Strict k -analytic spaces are defined as in [Ber93]. Recall that by a celebrated result of Temkin [Tem04], strict k -analytic spaces form a full subcategory of the category of k -analytic spaces if k is non-trivially valued. The category of k -affinoid spaces is denoted by \mathbf{Aff}_k , see [Ber12]. The category of k -affinoid algebras is denoted by \mathbf{AffAlg}_k . There is an equivalence between \mathbf{Aff}_k and \mathbf{AffAlg}_k , given by the functor of global sections $X \mapsto H^0(X, \mathcal{O}_X)$ and the functor of Berkovich spectrum $A \mapsto \mathrm{Sp} A$.

2.2. A representability theorem. The following result is analogous to [EGA I-new, Proposition 4.5.4].

Theorem 2.1. *Let F be a presheaf on \mathbf{An}_k . Assume that*

- (1) *F satisfies the sheaf property for the G -topology, namely, for any k -analytic space X , any G -covering $\{U_i\}$ of X , $F(X)$ is the equalizer of*

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

- (2) *There is a family $\{F_i\}_i$ of subfunctors of F such that*
- (a) *Each F_i is representable by a k -analytic space X_i .*
 - (b) *Each $F_i \rightarrow F$ is representable by a closed (resp. open) analytic domain. In particular, after base change to X_j , $F_i \rightarrow F$ is represented by a closed (resp. open) analytic domain U_{ji} . In the closed case, we assume furthermore that for each i , the collection of j such that $U_{ij} \neq \emptyset$ is finite.*
 - (c) *The collection F_i covers F .*

Then F is representable.

Proof. Let $\xi_i \in F_i(X_i)$ be the universal family of the presheaf F_i . By assumption a morphism of k -analytic spaces $T \rightarrow X_i$ factors through U_{ij} iff $\xi_i|_T \in F_j(T)$. In particular, $\xi_i|_{U_{ij}} \in F_j(U_{ij})$. So we get a morphism $f_{ij} : U_{ij} \rightarrow X_j$ such that $f_{ij}^* \xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji} , we know that f_{ij} factors through U_{ji} . Now observe that $(f_{ij} \circ f_{ji})^* \xi_j = f_{ji}^* \xi_i = \xi_j$, we conclude that $f_{ij} \circ f_{ji} = \mathrm{id}_{U_{ji}}$. In particular, all f_{ij} are isomorphisms. It is formal to see that the glueing conditions are satisfied by the f_{ij} 's, hence we can glue the X_i 's together to get a k -analytic space X by [Ber93, Proposition 1.3.3]. It is formal to check that X together with the glueing ξ of ξ_i represents F . We refer to [Stacks, Tag 01JJ] for the omitted details. \square

2.3. Polyradii.

Definition 2.1. A *polyradius* is an element $r \in \mathbb{R}_{>0}^n$ for some $n \in \mathbb{N}$. A polyradius r is *k -free* if the components of r are linearly independent as elements in the \mathbb{Q} -vector space $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{R}_{>0}/|k^*|)$.

For any k -polyradius $r \in \mathbb{R}_{>0}^n$, define k_r as the k -affinoid algebra of formal series

$$\left\{ \sum_{\alpha \in \mathbb{Z}^n} a_{\alpha} T^{\alpha} \in k[[T_1, \dots, T_n]] \mid a_{\alpha} \in k, |a_{\alpha}| r^{\alpha} \rightarrow 0 \text{ when } |\alpha| \rightarrow \infty \right\}$$

endowed with the multiplicative norm $\sum_{\alpha \in \mathbb{Z}^n} a_{\alpha} T^{\alpha} \mapsto \max_{\alpha \in \mathbb{Z}^n} |a_{\alpha}| r^{\alpha}$.

When r is k -free, k_r is a field.

For a given k -free polyradius r , a given Banach k -algebra A , for any Banach A -module M , we write $A_r = A \hat{\otimes}_k k_r$, $M_r = M \hat{\otimes}_k k_r$. Note that A_r is a Banach k_r -algebra and M_r is a Banach A_r -module. Similarly, given any k -analytic space, write $X_r := X \times_{\mathrm{Sp} k} \mathrm{Sp} k_r$.

2.4. The category of Banach modules. We briefly summarize a few results in [BBK17]. For the basic theory of quasi-Abelian categories, see [Sch99].

Let $\mathcal{B}an_k$ be the category of Banach k -modules, where morphisms are bounded homomorphisms. Recall that $\mathcal{B}an_k$ is a closed symmetric monoidal quasi-Abelian category with all finite limits and finite colimits, where the \otimes operator is given by the completed tensor product $\hat{\otimes}$. Moreover, finite products and finite coproducts coincide. The category $\mathcal{B}an_k$ has enough projectives. All projective objects in $\mathcal{B}an_k$ are flat in the sense of [BBB16]. We have derived categories $D^*(\mathcal{B}an_k)$, where $*$ means $+$, $-$, b or empty. Let $\mathcal{B}anAlg_k$ be the category of Banach k -algebras, which is also the category of algebras in the symmetric monoidal category $\mathcal{B}an_k$ in the abstract sense. Let $A \in \mathcal{B}anAlg_k$ be a Banach k -algebra. Let $\mathcal{B}anMod_A$ be the category of Banach A -modules, which is also the category of A -modules in the symmetric monoidal category $\mathcal{B}an_k$ in the abstract sense. Recall that $\mathcal{B}anMod_A$ is also a closed symmetric monoidal quasi-Abelian category with all finite limits and finite colimits, where the \otimes operator is also given by $\hat{\otimes}$. We write $D^*(A) = D^*(\mathcal{B}anMod_A)$.

Definition 2.2. Let $f : A \rightarrow B$ be a morphism in $\mathcal{B}anAlg_k$. Let $M \in \mathcal{B}anMod_A$. We say that M is *transversal* to f if the natural morphism

$$M \hat{\otimes}_A^{\mathbb{L}} B \rightarrow M \hat{\otimes}_A B$$

in $D^-(A)$ is an isomorphism.

Proposition 2.2 ([Ber12, Proposition 2.1.2]). *For any k -free polyradius, the Banach k -module k_r is flat in $\mathcal{B}an_k$: for any admissible exact short sequence $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ in $\mathcal{B}an_k$, the following sequence is also admissible and exact:*

$$0 \rightarrow E_r \rightarrow F_r \rightarrow G_r \rightarrow 0.$$

3. LIU SPACES AND LIU ALGEBRAS

Let k be a complete non-Archimedean valued field.

3.1. Liu spaces. In this section, we recall the basic theory of Liu k -analytic spaces following [MP21] and [Liu90].

Definition 3.1 ([MP21, Definition 1.9]). A k -analytic space X is called *Liu* if

- (1) X is quasi-compact, separated.
- (2) X is holomorphically separable: for any $x, y \in X$, $x \neq y$, there is $f \in H^0(X, \mathcal{O}_X)$ such that $|f(x)| \neq |f(y)|$.
- (3) \mathcal{O}_X is universally acyclic: for any analytic extension k'/k , $H^i(X_{k'}, \mathcal{O}_{X_{k'}}) = 0$ for any $i > 0$.

A morphism of Liu k -analytic spaces is a morphism of the underlying k -analytic spaces. We denote the category of Liu k -analytic spaces by $\mathcal{L}iu_k$.

Example 3.1. A k -affinoid space is a Liu k -analytic space. But the converse fails in general. We refer to [Liu90, Section 5] for details. In fact, the theory of non-Archimedean pinching in [Tem21] gives plenty of such examples.

Definition 3.2. Let X be a k -analytic space. An analytic domain Z of X is called a *Liu domain* if Z is a Liu k -analytic space.

Definition 3.3. Let X be a k -analytic space. We say X is *cohomologically Stein* if for any coherent sheaf of \mathcal{O}_X -modules \mathcal{F} ,

$$H^i(X, \mathcal{F}) = 0, \quad i > 0.$$

We say X is *universally cohomologically Stein* if for any analytic extension k'/k , $X_{k'}$ is cohomologically Stein.

Theorem 3.1 ([MP21, Theorem 1.11], [Liu90, Théorème 2]). *Let X be a separated, quasi-compact k -analytic space. Then the following are equivalent:*

- (1) X is Liu.
- (2) X is universally cohomologically Stein.
- (3) X is holomorphically separable and \mathcal{O}_X is universally acyclic.

Moreover, if k is non-trivially valued and X is strict, then the conditions are equivalent to

- (4) X is rig-holomorphically separable and \mathcal{O}_X is acyclic.

Note that in (4), we only need acyclicity of \mathcal{O}_X instead of universal acyclicity as explained in [MP21]. For the definition of rig-holomorphically separability, we refer to [MP21, Definition 1.5].

Theorem 3.2 ([MP21, Corollary 1.16]). *Let $f : Y \rightarrow X$ be a finite morphism of k -analytic spaces. Then*

- (1) *If X is Liu, then so is Y .*
(2) *If Y is Liu and f is surjective, then X is Liu.*

Theorem 3.3 ([MP21, Corollary 1.15, Corollary 1.17]). *Let X be a k -analytic space. Then*

- (1) *For any analytic extension k'/k , $X_{k'}$ is Liu iff X is Liu.*
(2) *Assume that X is separated. Then X is Liu iff X^{red} is.*
(3) *Assume that X is separated. Then X is Liu iff each irreducible component of X is.*

Proof. We only have to make the following remark to (1): X is separated iff $X_{k'}$ is. This follows from [CT21, Theorem 1.2]. \square

Proposition 3.4. *Let $f : Y \rightarrow X$, $g : X' \rightarrow X$ be morphisms in \mathcal{Liu}_k . Then $Y' := Y \times_X X' \in \mathcal{Liu}_k$.*

Proof. We have the following Cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \times X' \\ (f,g) \downarrow & \square & \downarrow f \times g \\ X & \xrightarrow{\Delta_X} & X \times X \end{array}$$

As X is separated, Δ_X is a closed immersion, so is the morphism $Y' \rightarrow Y \times X'$. By Theorem 3.2, in order to show that Y' is Liu, it suffices to show that $Y \times X'$ is Liu. This follows from [MP21, Theorem A.6]. \square

Corollary 3.5. *Let X be a separated k -analytic space. Let Y_1, Y_2 be Liu domains in X , then $Y_1 \cap Y_2$ is also a Liu domain.*

3.2. Liu algebras.

Definition 3.4. A *Liu k -algebra* is a Banach k -algebra A such that there is a Liu k -analytic space such that $A \cong H^0(X, \mathcal{O}_X)$, where the isomorphism is an isomorphism of Banach k -algebras. A Liu k -algebra is said to be *strict* if there is a strict Liu k -analytic space with $A \cong H^0(X, \mathcal{O}_X)$ in \mathcal{BanAlg}_k .

A morphism of Liu k -algebras is a bounded homomorphism of the underlying Banach k -algebras.

The category of Liu k -algebras is denoted by \mathcal{LiuAlg}_k . It is a full subcategory of \mathcal{BanAlg}_k .

Proposition 3.6. *Let A be a Liu k -algebra. Then*

- (1) *A is Noetherian and all of its ideals are closed.*
(2) *Suppose that k is non-trivially valued and A is strict. For any maximal ideal \mathfrak{m} of A , A/\mathfrak{m} is finite dimensional as a vector space over k .*
(3) *We have*

$$\bigcap_{\mathfrak{m} \in \text{Max}(A)} \bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0.$$

Proof. (1) That A is noetherian follows from [MP21, Proposition 2.6(3), Remark 2.7]. When k is non-trivially valued, all ideals are closed by [BGR84, Proposition 3.7.2.2]. In general, this follows from a base field extension argument, see [Ber12, Proposition 2.1.3].

(2) By [Liu90, Proposition 1.3], there is a rigid point $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_{\text{Sp } A, x}$. Take a strictly affinoid domain $\text{Sp } B$ of $\text{Sp } A$ containing x . Then x is also rigid in $\text{Sp } B$. It is well-known that $B/\mathfrak{m}_{\text{Sp } B, x}$ is finite dimensional, hence so is A/\mathfrak{m} .

(3) Take an element $a \in A$ that lies in the intersection of all \mathfrak{m}^n for any $\mathfrak{m} \in \text{Max } A$, $n \geq 1$. Then By Krull's intersection theorem, for each $\mathfrak{m} \in \text{Max } A$, there is an element $m \in \mathfrak{m}$ such that $(1 - m)a = 0$. Thus the annihilator of a does not lie in any maximal ideal of A , hence $a = 0$. \square

Corollary 3.7. *Let A be a Liu k -algebra. All k -algebra homomorphisms from a Banach k -algebra to A are bounded. In particular, the Liu k -algebra structure of A is uniquely determined by the underlying algebraic structure.*

Proof. When k is non-trivially valued and A is strict, this follows from [Proposition 3.6](#) and [\[BGR84, Proposition 3.7.5.2\]](#).

In general, this follows from the change of base argument. \square

Theorem 3.8 (Liu). *The functor of global sections gives an anti-equivalence $\mathcal{L}iu_k \rightarrow \mathcal{L}iuAlg_k$. The inverse functor is denoted by $\mathrm{Sp} A$. Moreover, for any k -analytic space Y , any Liu k -analytic space X , the canonical map*

$$\mathrm{Hom}_{\mathrm{An}_k}(Y, X) \rightarrow \mathrm{Hom}_{\mathrm{Alg}_k}(H^0(X, \mathcal{O}_X), H^0(Y, \mathcal{O}_Y))$$

is bijective.

Remark 3.1. The space $\mathrm{Sp} A$ as a topological space coincides with the spectrum in the sense of Berkovich [\[Ber12, Section 1.2\]](#). See [\[MP21, Corollary 3.17\]](#) for example.

Proof. The latter statement is a formal consequence of the former.

When k is non-trivially valued, by [\[Liu90, Proposition 3.2\]](#) and [\[Ber93, Theorem 1.6.1\]](#), we know that the global section functor is an anti-equivalence from the category of strict Liu k -analytic spaces to the category of strict Liu k -algebras.

In general, let X, Y be Liu k -analytic spaces. Let $A = H^0(X, \mathcal{O}_X)$, $B = H^0(Y, \mathcal{O}_Y)$. Let $F : A \rightarrow B$ be a homomorphism of k -algebras. We want to construct a morphism $Y \rightarrow X$, whose induced map on global sections is given by F . We may assume that Y is affinoid. Take an analytic field extension k'/k , so that k' is non-trivially valued, $A_{k'}$ and $B_{k'}$ become strict Liu k -algebras. We may assume that $k' = k_r$ for some k -free polyradius. Then there is a unique morphism $g : Y_{k'} \rightarrow X_{k'}$ inducing $F_{k'}$. We claim that there is a unique morphism $f : Y \rightarrow X$ such that $g = f_{k'}$. Note that it is automatic that f induces F on global sections by [\[Ber12, Proposition 2.1.2\]](#).

By [\[MP21, Proposition 3.13\]](#), there is a k -affinoid space Z , a locally closed immersion $h : X \rightarrow Z$ such that there is a finite covering Z_1, \dots, Z_m of Z by rational domains such that $h^{-1}(Z_i) \rightarrow Z_i$ is a Runge immersion for each i :

$$\begin{array}{ccccc} Y_{k'} & \xrightarrow{g} & X_{k'} & \xrightarrow{h_{k'}} & Z_{k'} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{\quad f \quad} & X & \xrightarrow{h} & Z \\ & \searrow w & & & \end{array}$$

Now observe that the composition of maps on global sections

$$H^0(Z_{k'}, \mathcal{O}_{Z_{k'}}) \rightarrow H^0(X_{k'}, \mathcal{O}_{X_{k'}}) \rightarrow H^0(Y_{k'}, \mathcal{O}_{Y_{k'}})$$

is the same as the base extension of the map of k -algebras

$$H^0(Z, \mathcal{O}_Z) \rightarrow A \xrightarrow{F} B.$$

Thus if we denote by $w : Y \rightarrow Z$ the morphism of k -analytic spaces corresponding to this latter map, we have $w_{k'} = h_{k'} \circ g$. Replacing Y by $w^{-1}(Z_i)$, X by $h^{-1}Z_i$ and Z by Z_i and applying [\[Ber93, Proposition 1.3.2\]](#) and [\(3.1\)](#), we may assume that $X \rightarrow Z$ is a Runge immersion. In particular X is affinoid. We can take f to be the morphism corresponding to F . Moreover, such f (such that $g = f_{k'}$) is clearly unique. We conclude. \square

Lemma 3.9. *Let A be a Liu k -algebra. Let B, C be Liu k -algebras over A , then $B \hat{\otimes}_A C$ is Liu. In particular, for any k -free polyradius r , A_r is a Liu k -algebra.*

Proof. Let $Z = \mathrm{Sp} B \times_{\mathrm{Sp} A} \mathrm{Sp} C$. By [Proposition 3.4](#), it suffices to prove

$$(3.1) \quad H^0(Z, \mathcal{O}_Z) = B \hat{\otimes}_A C.$$

Firstly, we consider the morphism

$$\Delta_{\mathrm{Sp} A} : \mathrm{Sp} A \rightarrow \mathrm{Sp} A \times \mathrm{Sp} A.$$

It is easy to see that this is a closed immersion, corresponding to the closed ideal J in $A \hat{\otimes} A$ generated by $1 \otimes a - a \otimes 1$ for $a \in A$. Also by [PP, Corollary 3.30], we have

$$H^0(\mathrm{Sp} B \times \mathrm{Sp} C, \mathcal{O}_{\mathrm{Sp} B \times \mathrm{Sp} C}) = B \hat{\otimes} C.$$

Hence the closed immersion $Z \rightarrow \mathrm{Sp} B \times \mathrm{Sp} C$ corresponds to the closed ideal of $B \hat{\otimes} C$ generated by $J(B \hat{\otimes} C)$. In particular, (3.1) holds. \square

Definition 3.5. Let A be a Liu k -algebra. A Banach A -module M is *finite* if there is an admissible epimorphism $A^n \rightarrow M$.

Let $\mathrm{Mod}^{\mathrm{fin}}(A)$ be the category of finite A -modules.

Proposition 3.10. *Let A be a Liu k -algebra. The forgetful functor from the category of finite Banach A -modules (with bounded A -algebra homomorphisms as morphisms) to $\mathrm{Mod}^{\mathrm{fin}}(A)$ is an equivalence.*

Proof. The functor is fully faithful. In fact, we prove more generally that for any finite Banach A -module M , any Banach A -module N , any A -linear map $F : M \rightarrow N$ is bounded. In fact, taking an admissible epimorphism $A^n \rightarrow M$, we may assume that $M = A^n$. In this case, the claim is clear.

The functor is essentially surjective. Take an A -linear epimorphism $\pi : A^n \rightarrow M$, then $\ker \pi$ is closed by Proposition 3.6 (1) and [BGR84, Proposition 3.7.2.2], so we can endow M with the residue Banach norm. \square

Proposition 3.11. *Let A be a Liu k -algebra. Let r be a k -free polyradius. Let M be Banach A -module. Then M is a finite Banach A -module iff M_r is a finite Banach A_r -module.*

Proof. This follows *verbatim* from [Ber12, Proof of Proposition 2.1.11]. \square

Theorem 3.12. *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let r be a k -free polyradius. Consider a descent datum (M_r, φ) of Banach modules over A_r . Then the descent datum is effective with respect to the natural morphism $\mathrm{Sp} A_r \rightarrow \mathrm{Sp} A$. Moreover, if M_r is finitely generated as A_r -module, then the descent M is finitely generated as A -module.*

Proof. The first part follows *verbatim* from [Day21, Proof of Proposition 3.3]. The second part follows from Proposition 3.11. \square

3.3. Coherent sheaves on Liu k -analytic spaces.

Definition 3.6. Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let M be a finite A -module. Then we define a sheaf \widetilde{M} on X as the sheafification of the presheaf $\mathrm{Sp} B \mapsto M \otimes_A B$, where $\mathrm{Sp} B$ runs over the set of affinoid domains in X .

Proposition 3.13 ([MP21, Lemma 2.4]). *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let M be a finite A -module. Then \widetilde{M} is a coherent sheaf on X . Moreover, for each affinoid domain $\mathrm{Sp} B$ in X ,*

$$(3.2) \quad H^0(\mathrm{Sp} B, \widetilde{M}) = M \otimes_A B.$$

Now we recall the theory of coherent sheaves on Liu k -analytic spaces. The following is the analogue of Cartan's Theorem A.

Theorem 3.14 ([MP21, Proposition 2.1]). *Assume that k is non-trivially valued. Let X be a Liu k -analytic space. For each coherent sheaf \mathcal{F} on X and each $x \in X$, $H^0(X, \mathcal{F})$ generates \mathcal{F}_x as an $\mathcal{O}_{X,x}$ -module.*

In the rigid setting, Cartan's Theorem A and Theorem B are due to Kiehl [Kie67] and Tate [Tat71] respectively.

As explained in [Kie67], Theorem A and Theorem B together imply the following result:

Theorem 3.15. *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Then the category of coherent sheaves on X is equivalent to the category of finite A -modules. The functors are given by $\mathcal{F} \mapsto H^0(X, \mathcal{F})$ and $M \mapsto \widetilde{M}$ respectively.*

Proof. This result was proved in [MP21, Proposition 2.6] under the assumption that k is non-trivially valued. When k is trivially valued, take a k -free polyradius r with at least one component. By [Day21, Théorème 3.13], the category of coherent sheaves on X is equivalent to the category of descent data of coherent sheaves on X_r with respect to $X_r \rightarrow X$. The latter category is equivalent to the category of

descent data of finite A_r -modules with respect to $A \rightarrow A_r$, which is then equivalent to the category of finite A -modules by [Theorem 3.12](#). It is easy to see that the composition of these functors is exactly the one given in the theorem. The functors in the proof are summarized in the following diagram:

$$\begin{array}{ccc} \mathcal{D}\text{es}(\mathcal{C}\text{oh}, X_r \rightarrow X) & \longrightarrow & \mathcal{D}\text{es}(\text{Mod}^{\text{fin}}, A \rightarrow A_r) \\ \downarrow & & \downarrow \\ \mathcal{C}\text{oh}(X) & \longrightarrow & \text{Mod}^{\text{fin}}(A) \end{array} .$$

□

In particular, [Theorem 3.14](#) holds even when k is trivially valued.

3.4. Quasi-coherent sheaves on Liu spaces.

Definition 3.7. Let $f : A \rightarrow B$ be a morphism in $\mathcal{L}\text{iuAlg}_k$. We say f is a *homotopy epimorphism* if the corresponding morphism $\text{Sp } B \rightarrow \text{Sp } A$ of Liu k -spaces identifies $\text{Sp } B$ with a Liu domain in $\text{Sp } A$.

Definition 3.8. Let A be a Liu k -algebra. A Banach A -module M is called *transversal* if M is transversal to all homotopy epimorphisms from A : for all homotopy epimorphism $A \rightarrow B$ to a Liu k -algebra B , the natural morphism

$$M \hat{\otimes}_A^{\mathbb{L}} B \rightarrow M \hat{\otimes}_A B$$

is an isomorphism.

The following result will be proved in [Appendix A](#).

Theorem 3.16. Let A be a Liu k -algebra. Let B, C be Liu k -algebras over A such that $\text{Sp } C \rightarrow \text{Sp } A$ is a Liu domain. Then the natural morphism

$$C \hat{\otimes}_A^{\mathbb{L}} B \rightarrow C \hat{\otimes}_A B$$

is an isomorphism.

Definition 3.9. Let A be a Liu k -algebra. Let M be a transversal Banach A -module. Write $X = \text{Sp } A$. We define a sheaf of \mathcal{O}_X -modules \widetilde{M} as the sheafification of the presheaf

$$\text{Sp } B \mapsto M \hat{\otimes}_A B$$

on X , where $\text{Sp } B$ runs over the set of affinoid domains in X . We call \widetilde{M} the sheaf associated to M .

An \mathcal{O}_X -module \mathcal{M} is *quasi-coherent* if there is a transversal A -module M such that $\mathcal{M} = \widetilde{M}$.

Example 3.2. Let X be a Liu k -analytic space. Then all coherent sheaves on X are quasi-coherent. See for example [\[MP21, Proof of Proposition 2.6\(1\)\]](#). To be more precise, the same proof shows that for any Liu domain $\text{Sp } B \rightarrow \text{Sp } A = X$, B is a flat A -algebra. Let M be a finite A -module. Consider a presentation

$$A^{\oplus S} \rightarrow A^{\oplus N} \rightarrow M \rightarrow 0.$$

We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A^{\oplus S} \hat{\otimes}_A^{\mathbb{L}} B & \longrightarrow & A^{\oplus N} \hat{\otimes}_A^{\mathbb{L}} B & \longrightarrow & M \hat{\otimes}_A^{\mathbb{L}} B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{\oplus S} \otimes_A B & \longrightarrow & A^{\oplus N} \otimes_A B & \longrightarrow & M \otimes_A B & \longrightarrow & 0 \end{array} .$$

In order to show that M is transversal, it suffices to show that A is, which is obvious.

Theorem 3.17 (Tate acyclicity theorem). Let $X = \text{Sp } A$ be a Liu k -analytic space. Let $\text{Sp } A_1, \dots, \text{Sp } A_n$ be a finite G -covering of X by Liu domains. Let M be a transversal Banach A -module, then the following sequence is admissible and exact

$$(3.3) \quad 0 \rightarrow M \rightarrow \prod_{i_1} M \hat{\otimes}_A A_{i_1} \rightarrow \prod_{i_1 < i_2} M \hat{\otimes}_A A_{i_1} \hat{\otimes}_A A_{i_2} \rightarrow \cdots \rightarrow M \hat{\otimes}_A A_1 \hat{\otimes}_A A_2 \hat{\otimes}_A \cdots \hat{\otimes}_A A_n \rightarrow 0.$$

Proof. It follows from the same proof as [BBK17, Lemma 5.34 and Remark 5.35]. We give a sketch for the convenience of the readers. When $M = A$, we can prove (3.3) exactly as in the affinoid setting, namely it suffices to treat the case where the covering is given by $\{A\{f\}, A\{f^{-1}\}\}$ for some $f \in A$. Then the acyclicity follows from a direct computation. See [BGR84, Chapter 8] for details. For a general M , taking derived tensor product with (3.3) for $M = A$ and apply the transversality condition, we get (3.3) for M . \square

Corollary 3.18. *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let \mathcal{M} be a quasi-coherent sheaf on X . Let $M = H^0(X, \mathcal{M})$. Then for any Liu domain $\mathrm{Sp} B$ in X , we have*

$$H^0(\mathrm{Sp} B, \mathcal{M}) = M \hat{\otimes}_A B.$$

Corollary 3.19. *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let \mathcal{M} be a quasi-coherent sheaf on X . Then*

$$H^i(X, \mathcal{M}) = 0, \quad i > 0.$$

Proof. This follows from [Stacks, Tag 01EW]* and Theorem 3.17. \square

Definition 3.10. Let X be a k -analytic space. Let \mathcal{F} be a sheaf of \mathcal{O}_X -modules (resp. \mathcal{O}_X -algebras). A *Banach structure* on \mathcal{F} is the following data: given any Liu domain $\mathrm{Sp} A$ in X , $\mathcal{F}(\mathrm{Sp} A)$ is topologized so that it forms a Banach A -module (resp. Banach A -algebra). We assume that the following condition holds: if $\mathrm{Sp} A, \mathrm{Sp} B$ are Liu domains in X such that $\mathrm{Sp} A$ is an analytic domain of $\mathrm{Sp} B$, then the natural morphism of A -modules (resp. A -algebras) $\mathcal{F}(\mathrm{Sp} B) \hat{\otimes}_B A \rightarrow \mathcal{F}(\mathrm{Sp} A)$ is bounded.

An \mathcal{O}_X -module (resp. \mathcal{O}_X -algebra) with a given Banach structure is called a *sheaf of Banach modules* (resp. *sheaf of Banach algebras*) on X .

A morphism $\mathcal{F} \rightarrow \mathcal{G}$ of sheaves of Banach modules (resp. sheaves of Banach algebras) on X is a morphism of the underlying sheaves of modules (resp. sheaves of algebras) such that for each Liu domain $\mathrm{Sp} B$ in X , $\mathcal{F}(\mathrm{Sp} B) \rightarrow \mathcal{G}(\mathrm{Sp} B)$ is bounded.

The category of sheaves of Banach modules on X is denoted by $\mathcal{B}\mathrm{anMod}_X$.

Proposition 3.20. *Let $X = \mathrm{Sp} A$ be a Liu k -analytic space. Let \mathcal{M} be a quasi-coherent sheaf on X . Let $M = H^0(X, \mathcal{M})$. Let \mathcal{F} be a sheaf of Banach \mathcal{O}_X -modules. Then*

$$\mathrm{Hom}_{\mathcal{B}\mathrm{anMod}_X}(\mathcal{M}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{B}\mathrm{anMod}_A}(M, H^0(X, \mathcal{F})).$$

Proof. Given a morphism $f : \mathcal{M} \rightarrow \mathcal{F}$ in $\mathcal{B}\mathrm{anMod}_X$, by taking global sections, we get $H^0(f) : M \rightarrow H^0(X, \mathcal{F})$. Conversely, given a bounded homomorphism $F : M \rightarrow H^0(X, \mathcal{F})$, we construct the morphism of sheaves $f : \mathcal{M} \rightarrow \mathcal{F}$ as follows: for any affinoid domain $\mathrm{Sp} B$ in X , define $f(\mathrm{Sp} B) : M \hat{\otimes}_A B \rightarrow H^0(\mathrm{Sp} B, \mathcal{F})$ as the natural homomorphism of Banach B -modules induced by the homomorphism of Banach A -modules:

$$M \xrightarrow{F} H^0(X, \mathcal{F}) \rightarrow H^0(\mathrm{Sp} B, \mathcal{F}).$$

By the obvious functoriality, this is a morphism of Banach \mathcal{O}_X -modules. It is easy to verify that these maps are inverse to each other. \square

Theorem 3.21. *Let $f : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$ be a morphism in $\mathcal{L}\mathrm{i}u_k$. Let \mathcal{M} be a quasi-coherent sheaf on $\mathrm{Sp} B$. Then $f_* \mathcal{M}$ is a quasi-coherent sheaf on $\mathrm{Sp} A$ associated to the transversal A -module $H^0(\mathrm{Sp} B, \mathcal{M})$.*

Proof. Let $F : A \rightarrow B$ be the corresponding homomorphism of Liu k -algebras. Let $M = H^0(\mathrm{Sp} B, \mathcal{M})$. We claim that M is transversal as Banach A -module.

This is proved in [BBK17, Lemma 4.48], we reproduce the argument: let $\mathrm{Sp} D \rightarrow \mathrm{Sp} A$ be a Liu domain. We need to show that

$$M \hat{\otimes}_A^{\mathbb{L}} D = M \hat{\otimes}_A D.$$

Observe that

$$M \hat{\otimes}_A^{\mathbb{L}} D = M \hat{\otimes}_B^{\mathbb{L}} (B \hat{\otimes}_A^{\mathbb{L}} D) = M \hat{\otimes}_B^{\mathbb{L}} (B \hat{\otimes}_A D) = M \hat{\otimes}_B (B \hat{\otimes}_A D) = M \hat{\otimes}_A D,$$

where for the second equality, we have applied Theorem 3.16; for the third we used Lemma 3.9 and the transversality of M . This concludes the claim.

In order to prove the theorem, it suffices to show $\widetilde{M^A} = f_* \mathcal{M}$. Here M^A is M regarded as a Banach A -module. To prove this, it suffices to take an affinoid domain $\mathrm{Sp} C$ in $\mathrm{Sp} A$ and show that

$$(3.4) \quad M \hat{\otimes}_A C = \mathcal{M}(f^{-1} \mathrm{Sp} C).$$

*This result is only stated for a ringed space, but it is easy to check that the proof works in the current situation as well.

By Lemma 3.9, $f^{-1} \mathrm{Sp} C$ is a Liu domain in $\mathrm{Sp} B$ and $f^{-1} \mathrm{Sp} C = \mathrm{Sp}(B \hat{\otimes}_A C)$. Hence (3.4) follows from Corollary 3.18. \square

Lemma 3.22. *Let A be a Liu k -algebra. Consider an admissible exact sequence*

$$0 \rightarrow F \rightarrow G \rightarrow H$$

in BanMod_A . Assume that G, H are both transversal, then so is F .

This is clear by definition.

Corollary 3.23. *Let $f : Y \rightarrow X$ be a quasi-compact and quasi-separated morphism of k -analytic spaces. Assume that $X = \mathrm{Sp} A$ is Liu. Let \mathcal{F} be a Banach sheaf of \mathcal{O}_Y -modules such that for each affinoid domain $\mathrm{Sp} C$ in Y , $\mathcal{F}|_{\mathrm{Sp} C}$ is quasi-coherent. Then $f_* \mathcal{F}$ is quasi-coherent on X .*

Proof. Let $\{U_i = \mathrm{Sp} B_i\}$ be a finite affinoid covering of Y . For each i, j , let $U_{ij} = U_i \cap U_j$, take a finite affinoid covering $\{U_{ijk}\}$ of U_{ij} . Let f_i (resp. f_{ijk}) be the restriction of f to U_i (resp. U_{ijk}). Then $f_{i*} \mathcal{F}$ (resp. $f_{ijk*} \mathcal{F}$) is the quasi-coherent sheaf associated to $\mathcal{F}(U_i)$ (resp. $\mathcal{F}(U_{ijk})$) by Theorem 3.21. In particular, $\mathcal{F}(U_i)$ (resp. $\mathcal{F}(U_{ijk})$) is a transversal Banach A -module.

There is an admissible exact sequence

$$0 \rightarrow \mathcal{F}(Y) \rightarrow \prod_i \mathcal{F}(U_i) \rightarrow \prod_{i,j,k} \mathcal{F}(U_{ijk}).$$

Thus $\mathcal{F}(Y)$ is a transversal Banach A -modules by Lemma 3.22. In particular, for any affinoid domain $\mathrm{Sp} B$ in X , we have an admissible exact sequence

$$0 \rightarrow \mathcal{F}(Y) \hat{\otimes}_A B \rightarrow \prod_i \mathcal{F}(U_i) \hat{\otimes}_A B \rightarrow \prod_{i,j,k} \mathcal{F}(U_{ijk}) \hat{\otimes}_A B.$$

By our assumption and Corollary 3.18, this sequence can be rewritten as

$$0 \rightarrow \mathcal{F}(Y) \hat{\otimes}_A B \rightarrow \prod_i \mathcal{F}(U_i \cap f^{-1}(\mathrm{Sp} B)) \rightarrow \prod_{i,j,k} \mathcal{F}(U_{ijk} \cap f^{-1}(\mathrm{Sp} B)).$$

It is now clear that $\widetilde{\mathcal{F}(Y)^A} = f_* \mathcal{F}$ and $f_* \mathcal{F}$ is quasi-coherent. \square

Theorem 3.24. *Let $f : Y = \mathrm{Sp} B \rightarrow X = \mathrm{Sp} A$ be a morphism in Liu_k . Let \mathcal{F} be a quasi-coherent sheaf on X . Let $F = H^0(X, \mathcal{F})$. Assume that \mathcal{F} is transversal to f :*

$$F \hat{\otimes}_A^{\mathbb{L}} C = F \hat{\otimes}_A C$$

for all Liu domains $\mathrm{Sp} C$ in $\mathrm{Sp} B$. Then the left adjoint f^ of $f_* : \mathrm{BanMod}_Y \rightarrow \mathrm{BanMod}_X$ at \mathcal{F} exists and $f^* \mathcal{F}$ is the quasi-coherent sheaf associated to $F \hat{\otimes}_A B$.*

Proof. We claim that $F \hat{\otimes}_A B$ is a transversal Banach B -module.

This is proved in [BBK17, Lemma 4.48], we reproduce their proof: let $\mathrm{Sp} C \rightarrow \mathrm{Sp} B$ be a Liu domain, we need to show

$$(F \hat{\otimes}_A B) \hat{\otimes}_B^{\mathbb{L}} C = (F \hat{\otimes}_A B) \hat{\otimes}_B C.$$

In fact,

$$(F \hat{\otimes}_A B) \hat{\otimes}_B^{\mathbb{L}} C = (F \hat{\otimes}_A^{\mathbb{L}} B) \hat{\otimes}_B^{\mathbb{L}} C = F \hat{\otimes}_A^{\mathbb{L}} C = F \hat{\otimes}_A C = (F \hat{\otimes}_A B) \hat{\otimes}_B C,$$

which concludes the claim.

By Proposition 3.20, for any sheaf of Banach \mathcal{O}_Y -modules \mathcal{G} ,

$$\mathrm{Hom}_{\mathrm{BanMod}_Y}(\widetilde{F \hat{\otimes}_A B}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{BanMod}_B}(F \hat{\otimes}_A B, H^0(Y, \mathcal{G})) = \mathrm{Hom}_{\mathrm{BanMod}_A}(F, H^0(Y, \mathcal{G})).$$

On the other hand, by Proposition 3.20, we have

$$\mathrm{Hom}_{\mathrm{BanMod}_X}(\mathcal{F}, f_* \mathcal{G}) = \mathrm{Hom}_{\mathrm{BanMod}_A}(F, H^0(Y, \mathcal{G})).$$

We conclude. \square

4. LIU MORPHISMS AND QUASI-COHERENT SHEAVES OF LIU ALGEBRAS

Let k be a complete non-Archimedean valued field.

4.1. Liu morphisms.

Definition 4.1. Let $f : X \rightarrow Y$ be a morphism in $\mathcal{A}n_k$. We say f is *Liu* if for any Liu domain Z of Y , $f^{-1}Z$ is a Liu domain.

For any k -analytic space Y , let $\mathcal{L}iu_{\rightarrow Y, k}$ denote the category of Liu morphisms $X \rightarrow Y$. A morphism between two Liu morphisms $X_1 \rightarrow Y$ and $X_2 \rightarrow Y$ is a morphism of in the over-category $\mathcal{A}n_k/Y$.

The following two propositions are obvious.

Proposition 4.1. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be morphisms in $\mathcal{A}n_k$. Assume that f, g are both Liu, then so is $g \circ f$.

Proposition 4.2. Let $f : X \rightarrow Y$ be a Liu morphism in $\mathcal{A}n_k$. Then f is separated and quasi-compact.

Lemma 4.3. Let $f : X \rightarrow Y$ be a morphism in $\mathcal{L}iu_k$. Let \mathcal{F} be a coherent sheaf on X . Then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.

Proof. The problem is local, so it suffices to show that $H^i(f^{-1}(\text{Sp } A), \mathcal{F}) = 0$ for any affinoid domain $\text{Sp } A$ of Y . This follows from the fact that $f^{-1}(\text{Sp } A)$ is Liu (Proposition 3.4). \square

Theorem 4.4. Let $f : X \rightarrow Y$ be a morphism in $\mathcal{A}n_k$. Assume that Y is Liu. Then the following are equivalent:

- (1) f is Liu.
- (2) f is quasi-compact and separated, for any analytic field extension k'/k , and coherent sheaf \mathcal{F} on $X_{k'}$,

$$R^i f_{k'*} \mathcal{F} = 0, \quad i > 0.$$
- (3) X is Liu.

Proof. (1) \implies (2): We may assume that $k' = k$ and it suffices to prove that for any affinoid domain $\text{Sp } A$ in Y , $H^i(f^{-1}(\text{Sp } A), \mathcal{F}) = 0$ for all $i > 0$, which is trivial as $f^{-1}(\text{Sp } A)$ is Liu.

(2) \implies (3): This follows from Leray's spectral sequence.

(3) \implies (1): This follows from Proposition 3.4. \square

Example 4.1. Recall [Day21, Définition 3.18]: A morphism $f : X \rightarrow Y$ in $\mathcal{A}n_k$ is said to be *almost affinoid* (presque affinoïde in French) if there is a G -covering of Y by affinoid domains $\{U_i\}$ such that $f^{-1}U_i$ is affinoid for each i .

Similarly, we could define an *almost Liu morphism* as follows: A morphism $f : X \rightarrow Y$ in $\mathcal{A}n_k$ is *almost Liu* if there is a G -covering of Y by Liu domains $\{U_i\}$ such that $f^{-1}U_i$ is Liu for each i .

An almost Liu morphism is not necessarily Liu even if the target is affinoid. See [SW20, Example 9.1.2] for a counterexample. I would like to thank Marco Maculan for pointing this out to me.

4.2. Quasi-coherent sheaves of Liu algebras.

Definition 4.2. Let X be a k -analytic space. A sheaf of Banach algebras \mathcal{F} on X is a *quasi-coherent sheaf of Liu k -algebras* if for each Liu domain $\text{Sp } A$ in X , $H^0(\text{Sp } A, \mathcal{F})$ is a Liu k -algebra and $\mathcal{F}|_{\text{Sp } A}$ is a quasi-coherent sheaf (in the sense of Definition 3.9). A morphism of quasi-coherent sheaves of Liu k -algebras on X is a homomorphism of the underlying sheaves of \mathcal{O}_X -algebras. We denote the category of quasi-coherent sheaves of Liu k -algebras on X by $\mathcal{L}iu\text{Alg}_{X, k}^{\text{QCoh}}$.

Remark 4.1. By Corollary 3.7, a sheaf of Liu k -algebras admits a natural Banach structure. Moreover, a morphism of quasi-coherent sheaves of Liu k -algebras on X is automatically a morphism in $\mathcal{B}an\text{Mod}_X$. Hence $\mathcal{L}iu\text{Alg}_{X, k}^{\text{QCoh}}$ is a full subcategory of $\mathcal{B}an\text{Mod}_X$.

Remark 4.2. We do not define a quasi-coherent sheaf on a k -analytic space. In fact, according to the philosophy of [BBK17], in the global setting, the correct notion to consider is the derived category of quasi-coherent sheaves.

Proposition 4.5. Let X be a separated k -analytic space. Let \mathcal{A} be a quasi-coherent sheaf of Liu k -algebras on X . Consider the presheaf F on $\mathcal{A}n_k$:

$$T \mapsto \{ (f, \varphi) : f \in \text{Hom}_{\mathcal{A}n_k}(T, X), \varphi \in \text{Hom}_{\mathcal{O}_T}(f^* \mathcal{A}, \mathcal{O}_T) \}.$$

Then F is representable.

Proof. Assume first that X is paracompact. It suffices to verify that the conditions of [Theorem 2.1](#) are satisfied.

(1) The sheaf condition follows from [\[Ber93, Proposition 1.3.2\]](#).

(2) Take a locally finite affinoid covering $\{U_i\}$ of X . Observe that each U_i is closed as X is separated. Take F_i to be the subfunctor of F consisting of pairs $(f : T \rightarrow S, \varphi)$ such that $f(T) \subseteq U_i$. Then F_i is represented by $\mathrm{Sp} \mathcal{A}(U_i)$. Thus 2(a) is satisfied. The conditions 2(b) and 2(c) follows from the choice of U_i .

In general, take a paracompact open covering $\{V_i\}$ of X as in the final step of [\[Ber93, Proof of Proposition 1.4.1\]](#). Repeat the same construction as in the previous step, with $\{V_i\}$ in place of $\{U_i\}$, we get subfunctors F_i of F . Again, it suffices to verify the conditions of 2(a), 2(b), 2(c) of [Theorem 2.1](#). The conditions 2(b), 2(c) follows from the choice of $\{V_i\}$, while the condition 2(a) follows from the special we just treated. \square

Remark 4.3. Of course, in [Proposition 4.5](#), one can weaken the separateness assumption to Hausdorff condition. It is not clear to the author if one can remove this condition.

Definition 4.3. Let X be a *separated* k -analytic space. Let \mathcal{A} be a quasi-coherent sheaf of Liu k -algebras on X . We define the *relative spectrum* $\underline{\mathrm{Sp}}_X \mathcal{A}$ as the k -analytic space representing the presheaf F in [Proposition 4.5](#). Note that there is a natural morphism $\pi : \underline{\mathrm{Sp}}_X \mathcal{A} \rightarrow X$. We sometimes call π the relative spectrum as well.

Proposition 4.6. *Let X be a separated k -analytic space. Let \mathcal{A} be a quasi-coherent sheaf of Liu k -algebras on X . Let $\pi : \underline{\mathrm{Sp}}_X \mathcal{A} \rightarrow X$ be the relative spectrum, then*

- (1) *For each Liu domain $\mathrm{Sp} A$ in X , the restriction of π to $\pi^{-1}(\mathrm{Sp} A) \rightarrow \mathrm{Sp} A$ is the same as $\mathrm{Sp} H^0(\mathrm{Sp} A, \mathcal{A}) \rightarrow \mathrm{Sp} A$.*
- (2) *For any morphism of separated k -analytic spaces $g : X' \rightarrow X$, $g^* \mathcal{A}$ is a quasi-coherent sheaf of Liu k -algebras and the natural morphism*

$$X' \times_X \underline{\mathrm{Sp}}_X \mathcal{A} \rightarrow \underline{\mathrm{Sp}}_{X'} g^* \mathcal{A}$$

is an isomorphism over X' .

- (3) *The universal map*

$$\mathcal{A} \rightarrow \pi_* \mathcal{O}_{\underline{\mathrm{Sp}}_X \mathcal{A}}$$

is an isomorphism of sheaves of Banach algebras on X .

We omit the straightforward proof. See [\[Stacks, Tag 01LQ\]](#) for example.

Corollary 4.7. *Let X be a separated k -analytic space. Then the functor*

$$\underline{\mathrm{Sp}}_X : \mathcal{LiuAlg}_{X,k}^{\mathrm{QCoh}} \rightarrow \mathcal{Liu}_{\rightarrow X,k}$$

is an anti-equivalence of categories. The quasi-inverse is given by $f \mapsto f_$.*

5. QUASI-LIU MORPHISMS

Let k be a complete non-Archimedean valued field.

Definition 5.1. A k -analytic space X is called *quasi-Liu* if the following conditions hold:

- (1) X is quasi-compact.
- (2) $H^0(X, \mathcal{O}_X)$ is a Liu k -algebra.
- (3) There is a Liu k -analytic space $\mathrm{Sp} B$ and a morphism $i : X \rightarrow \mathrm{Sp} B$, which realizes X as an analytic domain in $\mathrm{Sp} B$.

Proposition 5.1. *Let X be a quasi-Liu k -analytic space. Then the natural morphism $X \rightarrow \mathrm{Sp} H^0(X, \mathcal{O}_X)$ is an analytic domain embedding.*

Proof. Let $Y = \mathrm{Sp} B$ be a Liu k -analytic space such that there is a morphism $i : X \rightarrow Y$, which is an analytic domain embedding. Now we have a natural homomorphism $B \rightarrow A$ given by the restriction map $B = H^0(Y, \mathcal{O}_Y) \rightarrow A = H^0(X, \mathcal{O}_X)$. In particular, we get a factorization $X \rightarrow \mathrm{Sp} A \rightarrow Y$ of i by [Theorem 3.8](#). Now it remains to show that $X \rightarrow \mathrm{Sp} A$ is an analytic domain. Take $x \in X$. We can find rational domains V_1, \dots, V_m of Y contained in X such that $x \in \cap_i V_i$ and $\cup_i V_i$ is a neighborhood of x in Y . Let U_i be the rational domain of $\mathrm{Sp} A$ induced by V_i . We claim that $U_i \subseteq X$. Assuming this claim, then we

find that $x \in \cup_i U_i$ and $\cup_i U_i \subseteq X$ is a neighborhood of x in $\mathrm{Sp} A$. We conclude that $X \rightarrow \mathrm{Sp} A$ is indeed an analytic domain.

To prove the claim, we will fix some i and omit the indices from V_i, U_i . We write $V = \mathrm{Sp} B\{r^{-1}f/g\}$, where $f = (f_1, \dots, f_n)$ is a tuple of elements in B , $r = (r_1, \dots, r_n)$ is a tuple of positive real numbers and g is an element in B such that f_j, g do not have a common zero. Then $U = \mathrm{Sp} A\{r^{-1}f/g\}$. Let X' denote the analytic domain of X consisting of points where $|f_j| \leq r_j|g|$ for all $j = 1, \dots, n$. As $V \subseteq X$, we could identify X' with the analytic domain in Y defined by the same inequalities. In particular, X' is a Liu space. Take a finite affinoid covering $\mathrm{Sp} A_i$ of X , we know that A is the equalizer of $\prod_i A_i \rightrightarrows \prod_{i,j} A_{ij}$, where $\mathrm{Sp} A_{ij} = \mathrm{Sp} A_i \cap \mathrm{Sp} A_j$. By [Theorem 3.16](#), $A\{r^{-1}f/g\}$ is the equalizer of $\prod_i A_i\{r^{-1}f/g\} \rightrightarrows \prod_{i,j} A_{ij}\{r^{-1}f/g\}$. As $\mathrm{Sp} A_i\{r^{-1}f/g\}$ is an affinoid covering of X' , we find an isomorphism $H^0(X', \mathcal{O}_{X'}) \cong A\{r^{-1}f/g\}$. It induces an isomorphism $X' \rightarrow U$ by [Theorem 3.8](#), which is the inverse of the composition $U \rightarrow V \rightarrow X'$. In particular, we find that $U \rightarrow X$ is injective. \square

Lemma 5.2. *Let $f : X \rightarrow Y$ be a morphism in $\mathcal{A}n_k$. Assume that Y is Liu and X is quasi-Liu. Let $g : Y' \rightarrow Y$ be a Liu domain in Y . Then $X' := X \times_Y Y'$ is also quasi-Liu.*

Proof. Let $f' : X' \rightarrow Y'$ be the base change of f . It suffices to show that $H^0(X', \mathcal{O}_{X'})$ is a Liu k -algebra. By decomposing $X \rightarrow Y$ as in the proof of [Proposition 5.1](#), we have the commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow \\ Y' \times_Y \mathrm{Sp} H^0(X, \mathcal{O}_X) & \longrightarrow & \mathrm{Sp} H^0(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\quad} & Y \end{array} \cdot$$

Replacing Y by $\mathrm{Sp} H^0(X, \mathcal{O}_X)$ and Y' by $Y' \times_Y \mathrm{Sp} H^0(X, \mathcal{O}_X)$, we may assume that $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$ and f is the analytic domain embedding $X \rightarrow H^0(X, \mathcal{O}_X)$ in [Proposition 5.1](#).

We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & \square & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array} \cdot$$

Take a finite affinoid G-covering X_i of X , then we get an admissible exact sequence

$$0 \rightarrow H^0(Y, \mathcal{O}_Y) \rightarrow \prod_i H^0(X_i, \mathcal{O}_X) \rightarrow \prod_{i,j} H^0(X_{ij}, \mathcal{O}_X),$$

where $X_{ij} := X_i \cap X_j$. Taking the derived tensor $\hat{\otimes}_{H^0(Y, \mathcal{O}_Y)}^{\mathbb{L}} H^0(Y', \mathcal{O}_{Y'})$ and applying [Theorem 3.16](#) and [\(3.1\)](#), we get an admissible exact sequence

$$0 \rightarrow H^0(Y', \mathcal{O}_{Y'}) \rightarrow \prod_i H^0(g'^{-1}(X_i), \mathcal{O}_{X'}) \rightarrow \prod_{i,j} H^0(g'^{-1}(X_{ij}), \mathcal{O}_{X'}).$$

In particular,

$$H^0(Y', \mathcal{O}_{Y'}) = H^0(X', \mathcal{O}_{X'})$$

and this algebra is a Liu algebra. Also observe that the morphism $f' : X' \rightarrow Y'$ satisfies the assumption of [Definition 5.1\(3\)](#) and X' is quasi-Liu. \square

Definition 5.2. Let $f : X \rightarrow Y$ be a morphism of k -analytic spaces. We say f is *quasi-Liu* if for any Liu domain Z in Y , $f^{-1}Z$ is quasi-Liu.

Proposition 5.3. *Let $f : X \rightarrow Y$ be a quasi-Liu morphism in $\mathcal{A}n_k$. Then f is separated and quasi-compact.*

This is obvious.

Proposition 5.4. *Let $f : X \rightarrow Y$ be a morphism of k -analytic spaces. Assume that Y is separated. The following are equivalent:*

- (1) f is quasi-Liu.
- (2) $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of Liu k -algebras and the natural morphism $X \rightarrow \underline{\mathrm{Sp}}_Y f_*\mathcal{O}_X$ is quasi-compact and realizes X as an analytic domain.
- (3) $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of Liu k -algebras on Y and X can be realized as an analytic domain in $\underline{\mathrm{Sp}}_Y \mathcal{A}$ through a quasi-compact morphism $X \rightarrow \underline{\mathrm{Sp}}_Y \mathcal{A}$ over Y , where \mathcal{A} is a quasi-coherent sheaf of Liu k -algebras on Y .

Proof. It is clear that (2) \implies (3) \implies (1).

(1) \implies (2): Observe that $f_*\mathcal{O}_X$ is quasi-coherent by [Corollary 3.23](#). It is a quasi-coherent sheaf of Liu k -algebras by [Lemma 5.2](#). The last assertion follows from [Proposition 5.1](#). \square

Proposition 5.5. *Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be morphisms in $\mathcal{A}n_k$. If f is quasi-Liu and g is Liu, then $g \circ f$ is quasi-Liu.*

Proof. We need to show that the inverse image of a Liu domain U in Z by $g \circ f$ is quasi-Liu. But $g^{-1}(U)$ is Liu and we find that $f^{-1}(g^{-1}(U))$ is quasi-Liu by definition. \square

6. OPEN PROBLEMS

Let k be a complete non-Archimedean valued field.

We give a list of unsolved problems related to Liu k -algebras and Liu morphisms.

Question 6.1. *Is there a global version of Zariski's main theorem in non-Archimedean geometry?*

A local version is proved by Ducros in [[Duc07](#), Théorème 3.2] based on Temkin's graded reduction. This theorem roughly says that a quasi-finite morphism of separated k -analytic spaces can be written locally as the composition of an étale morphism, an analytic domain embedding and a finite morphism. This theorem, however, does not tell us much information about the global structure of a quasi-finite morphism, in contrast to the classical Zariski's main theorem ([[Stacks](#), [Tag 02LR](#)]).

We would like to know if the following holds:

Conjecture 6.1. *Let $f : X \rightarrow S$ be a quasi-finite morphism of quasi-compact, separated k -analytic spaces. Then we can decompose f into $h \circ i \circ g$, where $g : X \rightarrow Y$ is finite, $i : Y \rightarrow Z$ is a quasi-compact analytic domain embedding, $h : Z \rightarrow S$ is étale.*

We hope to find suitable extra conditions on f , which guarantee that i is a Liu domain embedding as well.

Question 6.2. *Are Liu k -algebras excellent?*

In the case of affinoid algebras, this is proved by Ducros [[Duc09](#)]. The author is not sure if Ducros' argument can be generalized to the current setting.

Question 6.3. *Can Liu morphisms be effectively descended with respect to fpqc (or Tate-flat) coverings?*

In a previous version of this paper, the author claimed a proof. But as pointed out by the referee, the proof contains a gap. By [[Day21](#), Théorème A], the essential difficulty is to treat the case of descending along a finite faithfully flat morphism of affinoid spaces.

APPENDIX A. RESULTS FROM BEN-BASSAT–KREMNIZER

We slightly generalize a few results in [[BBK17](#)].

Definition A.1. Let $f : A \rightarrow B$ be a morphism in $\mathcal{B}an\mathcal{A}lg_k$. We say f is a *homotopy epimorphism* if the following equivalent conditions are satisfied

- (1) $\mathbb{L}f_* : D^-(B) \rightarrow D^-(A)$ is fully faithful.
- (2) The natural morphism

$$\mathbb{L}f^* \circ \mathbb{L}f_* \rightarrow \mathrm{id}_{D^-(B)}$$

is a natural equivalence.

- (3) $B \hat{\otimes}_A^{\mathbb{L}} B = B$.

Definition A.2. Let $f : \mathrm{Sp} A \rightarrow \mathrm{Sp} B$ be a morphism in \mathcal{Liu}_k . We say f is a *homotopy monomorphism* if the corresponding morphism $B \rightarrow A$ in \mathcal{LiuAlg}_k is a homotopy epimorphism (Definition A.1).

Lemma A.1. Let $A \rightarrow B$ be a morphism in \mathcal{LiuAlg}_k . For any $r > 0$, $f \in A$, we have the natural isomorphisms in $D^-(A)$:

$$B \hat{\otimes}_A^{\mathbb{L}} A\{r^{-1}f\} \rightarrow B \hat{\otimes}_A A\{r^{-1}f\}, \quad B \hat{\otimes}_A^{\mathbb{L}} A\{rf^{-1}\} \rightarrow B \hat{\otimes}_A A\{rf^{-1}\}.$$

Proof. We only treat the former. As in the case of affinoid algebras ([BBK17, Lemma 5.13]), it suffices to prove that the morphism

$$T - f : A\{r^{-1}f\} \rightarrow A\{r^{-1}f\}$$

is a strict monomorphism. That this morphism is a monomorphism is well-known (and can be proved exactly as in the affinoid case).

To see $T - f$ is strict, by [Ber12, Proposition 2.1.2], we could assume that k is non-trivially valued. Then the image of $T - f$ is closed by Proposition 3.6. Hence $T - f$ is strict. \square

Lemma A.2. Let $A \rightarrow B$ be a morphism in \mathcal{LiuAlg}_k . Let $f_1, \dots, f_n, g \in A$ be elements that generate A . Let $r_1, \dots, r_n \in \mathbb{R}_{>0}$, Then we have the natural isomorphism in $D^-(A)$:

$$B \hat{\otimes}_A^{\mathbb{L}} A\{r_i^{-1}f_i/g\} \rightarrow B \hat{\otimes}_A A\{r_i^{-1}f_i/g\}.$$

The proof goes exactly as [BBK17, Lemma 5.14].

Lemma A.3. Let A be a Liu k -algebra. Let A_1, A_2, B be Liu k -algebras over A . Assume that

- (1) $\mathrm{Sp} A_i \rightarrow \mathrm{Sp} A$ ($i = 1, 2$) are Liu domains.
- (2) $\mathrm{Sp} A_1 \cup \mathrm{Sp} A_2$ is also a Liu domain in $\mathrm{Sp} A$ with Liu k -algebra C .
- (3) Let A_0 be the Liu k -algebra of the Liu domain $\mathrm{Sp} A_1 \cap \mathrm{Sp} A_2$ (c.f. Corollary 3.5). Then the following natural morphisms are isomorphisms

$$A_i \hat{\otimes}_A^{\mathbb{L}} B \rightarrow A_i \hat{\otimes}_A B$$

for $i = 0, 1, 2$.

Then we have a natural isomorphism

$$C \hat{\otimes}_A^{\mathbb{L}} B \rightarrow C \hat{\otimes}_A B.$$

This is obvious.

Theorem A.4. Let A be a Liu k -algebra. Let B, C be Liu k -algebras over A such that $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$ is a Liu domain. Then we have a natural isomorphism

$$C \hat{\otimes}_A^{\mathbb{L}} B \rightarrow C \hat{\otimes}_A B.$$

In particular, $\mathrm{Sp} C \rightarrow \mathrm{Sp} A$ is a homotopy monomorphism.

Proof. Having established the three preceding lemmas, the proof is the same as [BBK17, Proof of Theorem 5.16]. \square

Theorem A.5. Let $f : A \rightarrow B$ be a morphism in \mathcal{LiuAlg}_k . Then f is a homotopy epimorphism iff the corresponding morphism $\mathrm{Sp} B \rightarrow \mathrm{Sp} A$ is a Liu domain.

Proof. Same proof as [BBK17, Theorem 5.31]. \square

In terms of [BBK17], we have shown that \mathcal{LiuAlg}_k is a homotopy Zariski transversal subcategory of \mathcal{Ban}_k .

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