TRANSCENDENTAL B-DIVISORS

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Abstract. We study transcendental b-divisors over compact Kähler manifolds. In particular, we establish their intersection theory, answering a question of Dang–Favre.

Contents

1. Introduction

In this paper, we study the transcendental b-divisors. In particular, we extend the intersection theory of b-divisors developed in [DF22] to the transcendental setting.

Let X be a connected compact Kähler manifold of dimension n. Recall that an algebraic b-divisor (class) is an assignment $(\pi\colon Y \to X) \mapsto \mathbb{D}_Y \in \mathrm{NS}^1(Y)_{\mathbb{R}}$ (the real vector space spanned by the Néron–Severi group of Y), where π runs over all modifications of X. These data are assumed to be compatible under push-forwards. An example is a so-called Cartier b-divisor, where we start with a modification $\pi\colon Y \to X$ and a class α on Y, the value of $\mathbb{D}_Z$ on any modification $Z \to X$ dominating π is the pull-back of α to Z. The Cartier b-divisor is called nef if α can be taken as nef. In general, an algebraic b-divisor is nef if it can be approximated by nef Cartier b-divisors.

B-divisors generalize divisors while incorporating bimeromorphic twists. It is of interest to understand their intersection theory. When X is projective, Dang–Favre [DF22] established an intersection theory for nef b-divisors, which has been applied in dynamical systems [DF21] and K-stability [Xia22]. Roughly speaking, they proved that in this case, a nef b-divisor can always be approximated by a *decreasing* sequence of nef Cartier b-divisors. This result reduces the general intersection theory to that of Cartier b-divisors, which is essentially the same as the classical intersection theory as in [Ful98].

In the same paper, Dang–Favre asked the question of whether one can develop a similar theory for transcendental b-divisors, namely, when X is not necessarily projective and when the $\mathbb{D}_Y$'s are just classes in $\mathrm{H}^{1,1}(Y,\mathbb{R})$. We give an affirmative answer in this paper.

The idea of the proof is already contained in the author's previous papers [Xia22; Xia24]. Let us content ourselves to the algebraic setting for the moment. In this case, the two papers give an analytic approach to the Dang–Favre intersection theory. Suppose that $L_1,\dots,L_n$ are a big line bundle on X. Then for any singular Hermitian metrics $h_1,\dots,h_n$ on $L_1,\dots,L_n$, we can construct natural algebraic nef b-divisors $\mathbb{D}_1,\dots,\mathbb{D}_n$ on X using Siu's decomposition. The key results in these papers show that the Dang–Favre intersection of $\mathbb{D}_1,\dots,\mathbb{D}_n$ coincides with the mixed volume of $h_1,\dots,h_n$.

Conversely, up to some technical details, any nef b-divisor essentially arises from Siu's decomposition of currents. In the algebraic setting, this was already known since due to [BJ22, Theorem 6.40] and [DXZ23, Theorem 1.1], nef b-divisors and the $\mathcal{I}$-equivalence classes of (non-divisorial) currents are both in bijection with the same class of homogeneous non-Archimedean metrics. A version of this fact was first explicitly written down in [Tru24], using a very technical and *ad hoc* definition of b-divisors. We prove the transcendental version in Theorem 4.11:

Theorem 1.1. *Let α be a modified nef cohomology class on X. There is a natural bijection (via Siu's decomposition) between the following sets:*

(1) The set of $\mathcal{I}$-equivalence classes of non-divisorial closed positive $(1,1)$-currents in α with positive volumes;
(2) the set of nef and big b-divisors $\mathbb{D}$ over X with $\mathbb{D}_X = \alpha$.

The precise definitions of the terms in this theorem will be explained later on. See Definition 2.6, Definition 2.11, Definition 4.2 and Definition 4.6.

Although Theorem 1.1 bears some resemblance with [Tru24, Proposition 3.1], the contents of these theorems are quite different. In fact, Trusiani worked with b-divisors with possibly infinitely many components without passing to the numerical classes. It is a key observation in this paper that the numerical classes suffice to fully determine the singularities of a current modulo $\mathcal{I}$-equivalence, explaining the neatness of our theorem.

Based on Theorem 1.1, in the transcendental setting, we use these analytic tools to conversely define the intersection theory of b-divisors. We show that the intersection products behave in the expected manner in Section 5.

There is a technical subtlety here: There are at least two candidates for the analytic theory: The $\mathcal{I}$-volumes developed in [DX24b; DX22] and the mixed volumes in the sense of Cao [Cao14]. We will show in Section 3 that these theories agree and satisfy the desired properties.

The idea of extending Dang–Favre's theory using analytic methods is well known to experts. The author knew this idea when he wrote [Xia22] 5 years ago. What hinders the appearance of this paper is the slow development of the analytic theory. The necessary tools were developed over years in [DDNL21b; DX22; DX24b; Xia24; Xia21] and systematically summarized and extended in [Xia].

Theorem 1.1 sheds light on the algebraic theory as well. The powerful analytic machinery can be translated back to the algebraic theory, giving new insights even in the purely algebraic theory. As an example, in Section 6 and Section 7, we will study the two functorial operations of nef b-divisors: Along a smooth morphism, nef b-divisors can be pulled back; given a subvariety, a nef b-divisor can be restricted under a mild assumption. Both operations are easily understood using the analytic language. Moreover, both operations can be defined for nef algebraic b-divisors over general base fields, which have not appeared in the literature so far.

Finally, all results in this paper hold for manifolds in Fujiki's class $\mathcal{C}$ as well. But for simplicity, we always restrict our discussion to Kähler manifolds.

Further directions. In [DF22], Dang–Favre suggested that defining the intersection theory may rely on the transcendental Morse inequality conjectured in [BDPP13]. We expect that conversely, our theory should be helpful for understanding this conjecture. Less ambitiously, our theory should be helpful when trying to understand a particular consequence of the transcendental Morse inequality, as conjectured in [CT22].

Finally, there is a notion of Okounkov bodies of transcendental nef b-divisors, exactly as the algebraic case studied in [Xia, Section 11.3]. Over $\mathbb{C}$, the theory is well-understood using the analytic theory. However, it is not clear how to work out similar results based on the theory of Dang–Favre over general base fields.

Acknowledgments. The author would like to thank Nicholas McCleerey and Antonio Trusiani for their comments on the draft. Part of the work was carried out during the author's visit to Yunnan Normal University in 2024, the author would like to thank Prof. Zhipeng Yang for his hospitality and the support of Yunnan Key Laboratory of Modern Analytical Mathematics and Applications (No. 202302AN360007). The author is supported by the Knut och Alice Wallenbergs Stiftelse KAW 2024.0273.

2. Preliminaries

Let X be a connected compact Kähler manifold of dimension n.

2.1. **Modifications and cones.** In this paper, we use the word *modification* in a very non-standard sense.

Definition 2.1. A *modification* of X is a bimeromorphic morphism $\pi\colon Y \to X$, which is a finite composition of blow-ups with smooth centers.

Note that π is necessarily projective and Y is always a Kähler manifold.

Definition 2.2. We say a modification $\pi'\colon Z \to X$ *dominates* another $\pi\colon Y \to X$ if there is a morphism $g\colon Z \to Y$ making the following diagram commutative:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X. & \end{tikzcd} \tag{2.1}$$

The modifications of X together with the domination relation form a directed set $\operatorname{Modif}(X)$. Given classes $\alpha, \beta \in \mathrm{H}^{1,1}(X, \mathbb{R})$, we say $\alpha \leq \beta$ if $\beta - \alpha$ is pseudoeffective.

Proposition 2.3. *Let $\pi\colon Y \to X$ be a blow-up with* connected *smooth center of codimension at least* 2 *with exceptional divisor E. Then there is a natural identification*

$$\mathrm{H}^{1,1}(Y, \mathbb{R}) = \mathrm{H}^{1,1}(X, \mathbb{R}) \oplus \mathbb{R}\{E\}. \tag{2.2}$$

See [RYY19] for a much more general result. In general, the pseudoeffective cone of Y does not admit any simple descriptions.

Fix a reference Kähler form ω on X. Recall that a class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$ is *modified nef* if for any $\epsilon > 0$, we can find a closed $(1,1)$-current $T \in \alpha$ such that

(1) $T + \epsilon\omega \geq 0$;
(2) $\nu(T + \epsilon\omega, D) = 0$ for any prime divisor D on X.

This definition is independent of the choice of ω. Here $\nu(\bullet, D)$ denote the generic Lelong number along D.

These classes are called *nef en codimension* 1 in Boucksom's thesis [Bou02], where they were introduced for the first time. Modified nef classes form a closed convex cone in $\mathrm{H}^{1,1}(X, \mathbb{R})$. Note that a modified nef class is necessarily pseudoeffective. A nef class is obviously modified nef.

Recall the multiplicity of a cohomology class as defined in [Bou02, Section 2.1.3].

Definition 2.4. Let $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$ be a pseudoeffective class and D be a prime divisor on X. We define the Lelong number $\nu(\alpha, D)$ as follows:

(1) When α is big, define $\nu(\alpha, D) = \nu(T, D)$ for any closed positive $(1,1)$-current $T \in \alpha$ with minimal singularities.
(2) In general, define
$$\nu(\alpha, D) \coloneqq \lim_{\epsilon \to 0+} \nu(\alpha + \epsilon\{\omega\}, D).$$

When α is big, (2) is compatible with (1) and the definition is independent of the choice of ω. By definition, a pseudoeffective class α is modified nef if and only if $\nu(\alpha, D) = 0$ for all prime divisors D on X.

Let us recall the behavior of several cones under modifications.

Proposition 2.5. *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y.*

(1) For any nef class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$, $\pi^*\alpha$ is nef.
(2) For any modified nef class $\beta \in \mathrm{H}^{1,1}(Y, \mathbb{R})$, $\pi_*\beta$ is modified nef.

(3) *For any big class* $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$, $\pi^*\alpha$ *is big. Moreover,* $\operatorname{vol}\pi^*\alpha = \operatorname{vol}\alpha$.
(4) *For any big class* $\beta \in \mathrm{H}^{1,1}(Y,\mathbb{R})$, $\pi_*\beta$ *is big. Moreover,* $\operatorname{vol}\pi_*\beta \geq \operatorname{vol}\beta$.

Proof. Only (2) requires a proof. Fix a Kähler class γ. Replacing β by $\beta + \epsilon\gamma$ for $\epsilon \in (0,1)$, we reduce immediately to the case where β is big as well. Let T (resp. S) be a current with minimal singularities in $\pi_*\beta$ (resp. in β) and D be a prime divisor on X, it suffices to show that

$$\nu(T,D) = 0,$$

by Lemma 2.8 below, $\nu(\pi_*S,D) = 0$, so our assertion follows. □

Let T be a closed positive $(1,1)$-current on X. Then we define the *regular part* $\operatorname{Reg}T$ of T as the regular part of T with respect to Siu's decomposition. In other words, we write

$$T = \operatorname{Reg}T + \sum_i c_i[E_i], \tag{2.3}$$

where E_i is a countable collection of prime divisors on X and $c_i = \nu(T,E_i) > 0$; the regular part $\operatorname{Reg}T$ is a closed positive $(1,1)$-current whose generic Lelong number along each prime divisor on X is 0.

Definition 2.6. We say a closed positive $(1,1)$-current T on X is *non-divisorial* (resp. *divisorial*) if $T = \operatorname{Reg}T$ (resp. $\operatorname{Reg}T = 0$).

Note that the cohomology class of a non-divisorial current is always modified nef. Conversely, a current with minimal singularities in a *big* and modified nef class is always non-divisorial.
There is a closely related notion introduced in [McC21]:

Definition 2.7. We say a closed positive $(1,1)$-current S on X is *non-pluripolar* (resp. *pluri-supported*) if $S = \langle S\rangle$ (resp. $\langle S\rangle = 0$).

Here $\langle S\rangle$ denotes the non-pluripolar polar part of S, namely the non-pluripolar product of S itself.
Clearly, a divisorial current is necessarily pluri-supported, and a non-pluripolar current is necessarily non-divisorial. But these notions are not equivalent in general. However, within the class of $\mathcal{I}$-good singularities, these notions turn out to be equivalent. Since we are not in need of the latter result in this paper, we omit the proof.

Lemma 2.8. *Let* $\pi\colon Y \to X$ *be a proper bimeromorphic morphism from Kähler manifold* Y*. Let* T *be a non-divisorial current on* Y*, then* π_*T *is non-divisorial.*

Conversely, if S is a non-divisorial current on X, π^*S is could have divisorial part. As a simple example, consider S on $\mathbb{P}^2$, whose local potential near $0 \in \mathbb{C}^2_{z,w}$ looks like $\log(|z|^2+|w|^2)$.

Proof. Let D be a prime divisor on X. It follows from Zariski's main theorem ([Dem85, Théorème 1.7]) that D is not contained in the exceptional locus of π. Let D' be the strict transform of D. Thanks to Siu's semicontinuity theorem, we have

$$\nu(\pi_*T,D) = \nu(T,D') = 0.$$

Hence π_*T is non-divisorial. □

2.2. **The convergences of quasi-plurisubharmonic functions.** We first recall the notions of P and $\mathcal{I}$-equivalences. The latter is introduced in [DX22] based on [BFJ08]. The former was introduced in [Xia] based on [RWN14].

Definition 2.9. Let φ,ψ be quasi-plurisubharmonic functions on X. We say $\varphi \sim_P \psi$ (resp. $\varphi \preceq_P \psi$) if there is a closed smooth real $(1,1)$-form θ on X such that $\varphi,\psi \in \operatorname{PSH}(X,\theta)_{>0}$ and

$$P_\theta[\varphi] = P_\theta[\psi] \quad (\text{resp.} P_\theta[\varphi] \leq P_\theta[\psi]).$$

Here $\mathrm{PSH}(X,\theta)$ denotes the space of θ-plurisubharmonic functions on X and $\mathrm{PSH}(X,\theta)_{>0}$ denotes the subset consisting of $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\varphi^n > 0$, with $\theta_\varphi = \theta + \mathrm{dd^c}\varphi$. Here and in the sequel, the Monge–Ampère type product θ_φ^n is always understood in the non-pluripolar sense of [BT87; GZ07; BEGZ10]. The envelope P_θ is defined as follows:

$$P_\theta[\varphi] \coloneqq \sup_{C\in\mathbb{R}}{}^*(\varphi + C) \wedge 0,$$

where $(\varphi + C) \wedge 0$ is the maximal element in $\mathrm{PSH}(X,\theta)$ dominated by both $\varphi + C$ and 0.

Given a closed smooth real $(1,1)$-form θ on X so that $\varphi,\psi \in \mathrm{PSH}(X,\theta)$, we also say $\theta_\varphi \sim_P \theta_\psi$ (resp. $\theta_\varphi \preceq_P \theta_\psi$) if $\varphi \sim_P \psi$ (resp. $\varphi \preceq_P \psi$). The same convention applies also to the $\mathcal{I}$-partial order introduced later.

The main interest of the P-partial order lies in the following monotonicity theorem.

Theorem 2.10. *Let $\theta_1,\dots,\theta_n$ be closed real smooth $(1,1)$-forms on X. Let $\varphi_i,\psi_i \in \mathrm{PSH}(X,\theta_i)$ for $i = 1,\dots,n$. Assume that $\varphi_i \preceq_P \psi_i$ for each i. Then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \leq \int_X \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{n,\psi_n}.$$

This result is a consequence of the monotonicity theorem of Witt Nyström [WN19; DDNL18]. See [Xia, Proposition 6.1.4] for the proof.

Definition 2.11. Let φ,ψ be quasi-plurisubharmonic functions on X. We say $\varphi \sim_{\mathcal{I}} \psi$ (resp. $\varphi \preceq_{\mathcal{I}} \psi$) if $\mathcal{I}(\lambda\varphi) = \mathcal{I}(\lambda\psi)$ (resp. $\mathcal{I}(\lambda\varphi) \subseteq \mathcal{I}(\lambda\psi)$) for all real $\lambda > 0$.

Here $\mathcal{I}$ denotes the multiplier ideal sheaf in the sense of Nadel.

If θ is a closed smooth real $(1,1)$-form such that $\varphi,\psi \in \mathrm{PSH}(X,\theta)$, then $\varphi \preceq_{\mathcal{I}} \psi$ if and only if

$$P_\theta[\varphi]_{\mathcal{I}} \leq P_\theta[\psi]_{\mathcal{I}},$$

where

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\left\{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq 0, \mathcal{I}(\lambda\varphi) \supseteq \mathcal{I}(\lambda\eta) \text{ for all } \lambda > 0\right\}.$$

Equivalently, we may replace $\supseteq$ by $=$ in this equation.

Another equivalent formulation of Definition 2.11 is that for any prime divisor E over X, we have

$$\nu(\varphi,E) = \nu(\psi,E) \quad \text{resp. } \nu(\varphi,E) \geq \nu(\psi,E).$$

Here ν denotes the generic Lelong number. We refer to [Xia, Section 3.2.1] for the details.

Given any $\varphi \in \mathrm{PSH}(X,\theta)$, we have

$$\varphi - \sup_X \varphi \leq P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}.$$

See [DX22, Proposition 2.18] or [Xia, Proposition 3.2.8].

For later use, let us recall the following:

Lemma 2.12. *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y. Given two quasi-plurisubharmonic functions φ,ψ on X, then the following are equivalent:*

- *$\varphi \preceq_{\mathcal{I}} \psi$;*
- *$\pi^*\varphi \preceq_{\mathcal{I}} \pi^*\psi$.*

Proof. (1) $\implies$ (2). Just observe that each prime divisor over Y is also a prime divisor over X.

(2) $\implies$ (1). This follows from the well-known formula:

$$\pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(\lambda\pi^*\varphi)\right) = \mathcal{I}(\lambda\varphi), \quad \lambda > 0,$$

where $\omega_{Y/X}$ is the relative dualizing sheaf. See [Dem12b, Proposition 5.8]. □

The operation $P_\theta[\bullet]_{\mathcal{I}}$ is idempotent. We say $\varphi \in \mathrm{PSH}(X,\theta)$ is *$\mathcal{I}$-model* if $P_\theta[\varphi]_{\mathcal{I}} = \varphi$. Similarly, on the subset $\mathrm{PSH}(X,\theta)_{>0}$ consisting of $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\varphi^n > 0$, the operation $P_\theta[\bullet]$ is also idempotent, see [DDNL18, Theorem 3.12]. We say $\varphi \in \mathrm{PSH}(X,\theta)$ is *model* if $P_\theta[\varphi] = \varphi$.

It is shown in [DDNL21b] that there is a pseudometric d_S on $\mathrm{PSH}(X,\theta)$ satisfying the following inequality: For any $\varphi,\psi\in\mathrm{PSH}(X,\theta)$, we have

$$
\begin{aligned}
d_S(\varphi,\psi) \leq & \frac{1}{n+1}\sum_{j=0}^{n}\left(2\int_X \theta^j_{\varphi\vee\psi}\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}-\int_X\theta^j_\psi\wedge\theta^{n-j}_{V_\theta}\right)\\
\leq & C_n d_S(\varphi,\psi),
\end{aligned}
\tag{2.4}
$$

where $C_n=3(n+1)2^{n+2}$. Here $V_\theta=\max\{\varphi\in\mathrm{PSH}(X,\theta):\varphi\leq 0\}$. Moreover, $d_S(\varphi,\psi)=0$ if and only if $\varphi\sim_P\psi$. See [Xia, Proposition 6.2.2]. In particular, the d_S-pseudometric descends to a pseudometric (still denoted by d_S) on the space of closed positive $(1,1)$-currents in $\{\theta\}$.

Given a net of closed positive $(1,1)$-currents T_i in $\{\theta\}$, and another closed positive $(1,1)$-current T in $\{\theta\}$. It is shown in [Xia21, Section 4] and [Xia, Corollary 6.2.8] that $T_i\xrightarrow{d_S}T$ if and only if $T_i+\omega\xrightarrow{d_S}T+\omega$ for any Kähler form ω on X.

In general, given closed positive $(1,1)$-currents T_i and T on X, we say $T_i\xrightarrow{d_S}T$ if we can find Kähler forms ω_i and ω on X such that the $T_i+\omega_i$'s and $T+\omega$ represent the same cohomology class and $T_i+\omega_i\xrightarrow{d_S}T+\omega$. This definition is independent of the choices of the ω_i's and ω.

We introduce a stronger notion in this paper:

Definition 2.13. Let $(T_i)_i$ be a net of closed positive $(1,1)$-current on X and T be a closed positive $(1,1)$-current on X. We say $T_i\implies T$ if

(1) $T_i\xrightarrow{d_S}T$;
(2) $\{T_i\}\to\{T\}$.

A quasi-plurisubharmonic function φ on X is called *$\mathcal{I}$-good* if there is a closed smooth real $(1,1)$-form θ on X such that $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$ and

$$P_\theta[\varphi]=P_\theta[\varphi]_{\mathcal{I}}.$$

For any closed smooth real $(1,1)$-form θ' on X so that $\theta'+\mathrm{dd^c}\varphi\geq 0$, we also say the current θ'_φ is $\mathcal{I}$-good. These notions are independent of the choices, as proved in [Xia24, Lemma 1.7]. See also [Xia, Section 7.1]. As a simple example, an $\mathcal{I}$-model potential with positive non-pluripolar mass is always $\mathcal{I}$-good.

A key result proved in [DX22; DX24b] is the following:

Theorem 2.14. *A closed positive $(1,1)$-current T on X is $\mathcal{I}$-good if and only if there is a sequence of closed positive $(1,1)$-currents $(T_j)_j$ on X with analytic singularities and $T_j\implies T$. In fact, $(T_j)_j$ can be taken as any quasi-equisingular approximation of T.*

Here we say a closed positive $(1,1)$-current T has analytic singularities if locally T can be written as $\mathrm{dd^c}f$, where f is a plurisubharmonic function of the following form:

$$c\log\left(|f_1|^2+\cdots+|f_N|^2\right)+R,$$

where $c\in\mathbb{Q}_{\geq 0}$, $f_1,\dots,f_N$ are holomorphic functions on X and R is a bounded function. A few subtleties of this notion are discussed in [DRWN+23, Remark 2.7]. When we write $T=\theta+\mathrm{dd^c}\varphi$ for some smooth closed real $(1,1)$-form θ and $\varphi\in\mathrm{PSH}(X,\theta)$, we also say φ has analytic singularities.

As a particular case, if D is an effective $\mathbb{Q}$-divisor on X, we say a closed positive $(1,1)$-current T has *log singularities* along D if $T-[D]$ is positive has locally bounded potentials. It is easy to see that T has analytic singularities. Conversely, if we begin with T with analytic singularities, there is always a modification $\pi\colon Y\to X$ so that π^*T has log singularities along an effective $\mathbb{Q}$-divisor on Y. See [MM07, Page 104].

Let θ be a smooth closed real $(1,1)$-form on X and $\eta\in\mathrm{PSH}(X,\theta)$. We say a sequence $(\eta^j)_j$ of quasi-plurisubharmonic functions is a *quasi-equisingular approximation* of η if the following are satisfied:

(1) η^j has analytic singularities;
(2) $(\eta^j)_j$ is decreasing with limit η;

(3) for each $\lambda' > \lambda > 0$, we can find $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}(\lambda' \eta^j) \subseteq \mathcal{I}(\lambda \eta);$$

(4) There is a decreasing sequence $(\epsilon_j)_j$ in $\mathbb{R}_{\geq 0}$ with limit 0 so that

$$\eta^j \in \mathrm{PSH}(X, \theta + \epsilon_j \omega)$$

for each $i = 1, \ldots, n$ and $j > 0$.

The existence of quasi-equisingular approximations is guaranteed by [DPS01]. We also say $(\theta + \mathrm{dd^c}\eta^j)_j$ is a quasi-equisingular approximation of $\theta + \mathrm{dd^c}\eta$.

The class $\mathcal{I}$-good singularities is closed under many natural operations.

Proposition 2.15. *The sum and maximum of two $\mathcal{I}$-good quasi-plurisubharmonic functions are still $\mathcal{I}$-good. If θ is a closed real smooth $(1,1)$-form on X and $(\varphi_i)_i$ is a non-empty bounded from above family of $\mathcal{I}$-good θ-psh functions, then $\sup^*_i \varphi_i$ is also $\mathcal{I}$-good.*

See [Xia, Section 7.2] for the proofs.

3. Mixed volumes

Let X be a connected compact Kähler manifold of dimension n. Let $T_1, \ldots, T_n$ be closed positive $(1,1)$-currents on X. Let $\theta_1, \ldots, \theta_n$ be closed real smooth $(1,1)$-forms on X in the cohomology classes of $T_1, \ldots, T_n$ respectively. Consider $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ so that $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for each $i = 1, \ldots, n$. Fix a reference Kähler form ω on X.

3.1. **The different definitions.** We first recall the notion of quasi-equisingular approximations.

For each $i = 1, \ldots, n$, let $(\varphi_i^j)_j$ be a quasi-equisingular approximation of φ_i.

Definition 3.1. The mixed volume of $T_1, \ldots, T_n$ in the sense of Cao is defined as follows:

$$\langle T_1, \ldots, T_n \rangle_C \coloneqq \lim_{j \to \infty} \int_X \left(\theta_1 + \epsilon_j \omega + \mathrm{dd^c}\varphi_1^j\right) \wedge \cdots \wedge \left(\theta_n + \epsilon_j \omega + \mathrm{dd^c}\varphi_n^j\right),$$

where $(\epsilon_j)_j$ is a decreasing sequence with limit 0 such that $\varphi_i^j \in \mathrm{PSH}(X, \theta_i + \epsilon_j \omega)$ for each $i = 1, \ldots, n$.

It is shown in [Cao14] Section 2 that this definition is independent of the choices of the θ_i's, the ϵ_j's, the φ_i's, the φ_i^j's and ω.

A different definition relies on the $\mathcal{I}$-envelope technique studied in [DX24b; DX22]. Recall that the volume of a current is defined in [Xia, Definition 3.2.3]:

$$\mathrm{vol}(\theta + \mathrm{dd^c}\varphi) = \int_X \left(\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}}\right)^n.$$

It depends only on the current $\theta + \mathrm{dd^c}\varphi$, not on the choice of the choices of θ and φ. In general, as shown in [DX22; DX24b],

$$\mathrm{vol}(\theta + \mathrm{dd^c}\varphi) \geq \int_X (\theta + \mathrm{dd^c}\varphi)^n.$$

If furthermore the right-hand side is positive, then the equality holds if and only if φ is $\mathcal{I}$-good. We refer to [Xia, Section 7.1] for the details.

Definition 3.2. Assume that $\mathrm{vol}\, T_i > 0$ for all $i = 1, \ldots, n$. The mixed volume of $T_1, \ldots, T_n$ in the sense of Darvas–Xia is defined as follows:

$$\mathrm{vol}(T_1, \ldots, T_n) = \int_X \left(\theta_1 + \mathrm{dd^c} P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c} P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right). \tag{3.1}$$

In general, define

$$\mathrm{vol}(T_1, \ldots, T_n) = \lim_{\epsilon \to 0+} \mathrm{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega). \tag{3.2}$$

This definition is again independent of the choices of ω, the θ_i's and the φ_i's, using the same proof as [Xia, Proposition 3.2.7].

The mixed volume can be regarded as generalizations of the movable intersection theory. In fact, when each T_i has minimal singularities, the mixed volume is exactly the movable intersection of corresponding cohomology classes.

When $\operatorname{vol} T_i > 0$ for all $i = 1, \dots, n$, the definition (3.2) is compatible with (3.1), as from the $\mathcal{I}$-goodness of $P_{\theta_i}[\varphi_i]_{\mathcal{I}}$, we have

$$P_{\theta_i+\epsilon\omega}\left[P_{\theta_i}[\varphi_i]_{\mathcal{I}}\right] = P_{\theta_i+\epsilon\omega}[\varphi_i]_{\mathcal{I}}.$$

Hence (3.2) reduces to (3.1) as a consequence of Theorem 2.10.

When $T_1 = \cdots = T_n = T$, the above definition is compatible with pure case:

Proposition 3.3. *We always have*

$$\operatorname{vol}(T, \dots, T) = \operatorname{vol} T.$$

Proof. Write $T = \theta_\varphi$. In more concrete terms, we need to show that

$$\lim_{\epsilon\to 0+} \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

We may replace φ by $P_\theta[\varphi]_{\mathcal{I}}$ and assume that φ is $\mathcal{I}$-model in $\mathrm{PSH}(X, \theta)$. Then we claim that

$$\varphi = \inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}.$$

From this, our assertion follows from [Xia, Proposition 3.1.9].

The $\leq$ direction is clear. For the converse, it suffices to show that for each prime divisor E over X, we have

$$\nu(\varphi, E) \leq \nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right).$$

We simply compute

$$\nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) \geq \sup_{\epsilon>0} \nu\left(P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) = \nu(\varphi, E).$$

□

Proposition 3.4. *Both volumes are symmetric. The mixed volume in the sense of Cao is multi-$\mathbb{Q}_{\geq 0}$-linear, while the mixed volume in the sense of Darvas–Xia is multi-$\mathbb{R}_{\geq 0}$-linear.*

The multi-$\mathbb{Q}_{\geq 0}$-linearity means two things:

(1) For each $\lambda \in \mathbb{Q}_{\geq 0}$, we have

$$\langle \lambda T_1, T_2, \dots, T_n \rangle_C = \lambda \langle T_1, T_2, \dots, T_n \rangle_C.$$

(2) If T_1' is anther closed positive $(1,1)$-current, then

$$\langle T_1 + T_1', T_2, \dots, T_n \rangle_C = \langle T_1, T_2, \dots, T_n \rangle_C + \langle T_1', T_2, \dots, T_n \rangle_C. \tag{3.3}$$

Multi-$\mathbb{R}_{\geq 0}$-linearity is defined similarly.

Proof. We first handle the mixed volumes in the sense of Cao. Only the property (3.3) needs a proof. But this follows from the fact that the sum of two quasi-equisingular approximations is again a quasi-equisingular approximation. See [Xia, Theorem 6.2.2, Corollary 7.1.2].

Next we handle the case of mixed volumes in the sense of Darvas–Xia. We only need to show that

$$\operatorname{vol}(T_1 + T_1', T_2, \dots, T_n) = \operatorname{vol}(T_1, T_2, \dots, T_n) + \operatorname{vol}(T_1', T_2, \dots, T_n). \tag{3.4}$$

Thanks to the definition (3.2), we may assume that $\operatorname{vol} T_i > 0$ for each i and $\operatorname{vol} T_1' > 0$. Write $T_1' = \theta_1' + \mathrm{dd^c}\varphi_1'$. Then thanks to Proposition 2.15,

$$P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_1'}[\varphi_1']_{\mathcal{I}} \sim_P P_{\theta_1+\theta_1'}[\varphi_1 + \varphi_1']_{\mathcal{I}}.$$

Therefore, (3.4) follows from Theorem 2.10. □

Theorem 3.5. *We have*

$$\langle T_1, \dots, T_n \rangle_C = \operatorname{vol}(T_1, \dots, T_n). \tag{3.5}$$

In particular, we no longer need the notation $\langle T_1, \dots, T_n \rangle_C$.

Proof. **Step 1**. We reduce to the case where $T_1 = \cdots = T_n$.

Suppose this special case has been proved. Let $\lambda_1, \dots, \lambda_n \in \mathbb{Q}_{>0}$ be some numbers. Then

$$\langle \sum_{i=1}^n \lambda_i T_i, \dots, \sum_{i=1}^n \lambda_i T_i \rangle_C = \operatorname{vol}(\sum_{i=1}^n \lambda_i T_i).$$

It follows from Proposition 3.4 that both sides are polynomials in the λ_i's. Comparing the coefficients of $\lambda_1 \cdots \lambda_n$, we conclude (3.5).

From now on, we assume that $T_1 = \cdots = T_n = T$. Write $T = \theta_\varphi$.

Step 2. We reduce to the case where T is a Kähler current. For this purpose, it suffices to show that

$$\lim_{\epsilon \to 0+} \langle T_1 + \epsilon\omega, \dots, T_n + \epsilon\omega \rangle_C = \langle T_1, \dots, T_n \rangle_C,$$

which is obvious by definition.

Step 3. Let $(\varphi^j)_j$ be a quasi-equisingular approximation of φ in $\operatorname{PSH}(X, \theta)$. We need to show that

$$\lim_{j \to \infty} \int_X (\theta + \mathrm{dd^c} \varphi^j)^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

This follows from [DX24b, Corollary 3.4], see also [Xia, Corollary 7.1.2]. ☐

3.2. Properties of mixed volumes.

Proposition 3.6. *Let $S_1, \dots, S_n$ be closed positive $(1,1)$-currents on X. Assume that for each $i = 1, \dots, n$,*

(1) $T_i \preceq_{\mathcal{I}} S_i$;
(2) $\{T_i\} = \{S_i\}$.

Then

$$\operatorname{vol}(T_1, \dots, T_n) \le \operatorname{vol}(S_1, \dots, S_n). \tag{3.6}$$

Proof. Let ω be a Kähler form on X. It suffices to show that for each $\epsilon > 0$, we have

$$\operatorname{vol}(T_1 + \epsilon\omega, \dots, T_n + \epsilon\omega) \le \operatorname{vol}(S_1 + \epsilon\omega, \dots, S_n + \epsilon\omega).$$

In particular, we reduce to the case where $\operatorname{vol} T_i > 0$, $\operatorname{vol} S_i > 0$ for each $i = 1, \dots, n$.

In this case, (3.6) is a consequence of Theorem 2.10. ☐

Proposition 3.7. *We have*

$$\operatorname{vol}(T_1, \dots, T_n) \ge \prod_{i=1}^n (\operatorname{vol} T_i)^{1/n}.$$

Proof. We may assume that $\operatorname{vol} T_i > 0$ for each $i = 1, \dots, n$ since there is nothing to prove otherwise. In this case, we need to show that

$$\int_X (\theta_1 + \mathrm{dd^c} P_{\theta_1}[\varphi_1]_{\mathcal{I}}) \wedge \cdots \wedge (\theta_n + \mathrm{dd^c} P_{\theta_1}[\varphi_n]_{\mathcal{I}}) \ge \prod_{i=1}^n \left(\int_X (\theta_i + \mathrm{dd^c} P_{\theta_i}[\varphi_i]_{\mathcal{I}})^n \right)^{1/n}.$$

This is a special case of the main theorem of [DDNL21a]. ☐

Proposition 3.8. *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X, then*

$$\operatorname{vol}(\pi^* T_1, \dots, \pi^* T_n) = \operatorname{vol}(T_1, \dots, T_n).$$

Proof. As in the proof of Proposition 3.6, we may easily reduce to the case where $\operatorname{vol} T_i > 0$ for each $i = 1, \ldots, n$. By [Xia, Proposition 3.2.5], we know that if we write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$, then $\pi^* P_{\theta_i}[\varphi_i] = P_{\pi^*\theta_i}[\pi^*\varphi_i]$. In particular,

$$\operatorname{vol} \pi^* T_i = \operatorname{vol} T_i > 0.$$

Our assertion follows from the obvious bimeromorphic invariance of the non-pluripolar product. □

Lemma 3.9. *Let ω be a Kähler form on X. Then there is a constant $C > 0$ depending only on $X, \omega, \{\theta_1\}, \ldots, \{\theta_n\}$ such that*

$$0 \leq \operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega) - \operatorname{vol}(T_1, \ldots, T_n) \leq C\epsilon$$

for any $\epsilon \in [0, 1]$.

Proof. By linearity, we can write

$$\operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega) - \operatorname{vol}(T_1, \ldots, T_n)$$

as a linear combination of the mixed volumes between the T_i's and ω with coefficients ϵ^j for some $j \geq 1$. The mixed volumes are clearly bounded by a constant. □

Proposition 3.10. *Let $(T_i^j)_{j\in J}$ be nets of closed positive $(1,1)$-current on X. Assume that for each $i = 1, \ldots, n$, we have*

$$T_i^j \implies T_i.$$

Then

$$\lim_{j\in J} \int_X T_1^j \wedge \cdots \wedge T_n^j = \int_X T_1 \wedge \cdots \wedge T_n, \tag{3.7}$$

and

$$\lim_{j\in J} \operatorname{vol}\left(T_1^j, \ldots, T_n^j\right) = \operatorname{vol}\left(T_1, \ldots, T_n\right). \tag{3.8}$$

Recall that $\implies$ is defined in Definition 2.13.

Proof. Let ω be a Kähler form on X. For each $\epsilon > 0$, we can find $j_0 \in J$ so that for $j \geq j_0$, the following classes are Kähler:

$$\{T_i\} + 2^{-1}\epsilon\{\omega\} - \{T_i^j\}, \quad i = 1, \ldots, n.$$

Take a Kähler form ω_i^j in the class $\{T_i\} + \epsilon\{\omega\} - \{T_i^j\}$. Then observe that for $i = 1, \ldots, n$,

$$T_i^j + \omega_i^j \xrightarrow{d_S} T_i + \epsilon\omega.$$

Since these currents are now in the same cohomology class, it follows from [Xia21, Theorem 4.2] (see also [Xia, Theorem 6.2.1]) that

$$\lim_{j\in J} \int_X (T_1^j + \omega_1^j) \wedge \cdots \wedge (T_n^j + \omega_n^j) = \int_X (T_1 + \epsilon\omega) \wedge \cdots \wedge (T_n + \epsilon\omega). \tag{3.9}$$

Note that we can find a constant $C > 0$ independent of $j \geq j_0$ so that for any $j \geq j_0$, we have

$$\int_X (T_1^j + \omega_1^j) \wedge \cdots \wedge (T_n^j + \omega_n^j) - \int_X T_1^j \wedge \cdots \wedge T_n^j \leq C\epsilon,$$
$$\int_X (T_1 + \epsilon\omega) \wedge \cdots \wedge (T_n + \epsilon\omega) - \int_X T_1 \wedge \cdots \wedge T_n \leq C\epsilon.$$

Hence (3.7) follows.

As for (3.8), it suffices to replace (3.9) by

$$\lim_{j\in J} \operatorname{vol}\left(T_1^j + \omega_1^j, \ldots, T_n^j + \omega_n^j\right) = \operatorname{vol}\left(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega\right),$$

which follows from [Xia21, Theorem 4.2, Theorem 4.6] (see also [Xia, Theorem 6.2.1, Theorem 6.2.3]). □

Next we establish a semicontinuity property of the mixed volumes.

Theorem 3.11. *Let $(\varphi_i^j)_{j\in J}$ $(i=1,\dots,n)$ be nets in* $\mathrm{PSH}(X,\theta_i)$*. Assume that for each prime divisor E over X, we have*

$$\lim_{j\in J}\nu(\varphi_i^j,E)=\nu(\varphi_i,E).$$

Then

$$\varlimsup_{j\in J}\mathrm{vol}\left(\theta_1+\mathrm{dd^c}\varphi_1^j,\dots,\theta_n+\mathrm{dd^c}\varphi_n^j\right)\leq\mathrm{vol}\left(\theta_1+\mathrm{dd^c}\varphi_1,\dots,\theta_n+\mathrm{dd^c}\varphi_n\right).$$

Proof. **Step 1**. We first assume that $\mathrm{vol}(\theta_i+\mathrm{dd^c}\varphi_i^j)>0$ and $\mathrm{vol}(\theta_i+\mathrm{dd^c}\varphi_i)>0$ for all $i=1,\dots,n$ and $j\in J$.

Without loss of generality, we may assume that φ_i^j and φ_i are $\mathcal{I}$-model for all $i=1,\dots,n$ and $j\in J$. Our assertion becomes

$$\varlimsup_{j\in J}\int_X\left(\theta_1+\mathrm{dd^c}\varphi_1^j\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}\varphi_n^j\right)\leq\int_X\left(\theta_1+\mathrm{dd^c}\varphi_1\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}\varphi_n\right). \tag{3.10}$$

For each $j\in J$, define

$$\psi_i^j:=\sup_{k\geq j}{}^*P_{\theta_i}[\varphi_i^k],\quad i=1,\dots,n.$$

Observe that ψ_i^j is $\mathcal{I}$-good thanks to Proposition 2.15. It follows from [Xia, Corollary 1.4.1] and our assumption that

$$\lim_{j\in J}\nu\left(\psi_i^j,E\right)=\nu\left(\varphi_i,E\right)$$

for $i=1,\dots,n$. For each $i=1,\dots,n$, we define

$$\psi_i=\inf_{j\in J}P_{\theta_i}[\psi_i^j].$$

Due to [DX22, Lemma 2.21] (see also [Xia, Proposition 3.2.12]), ψ_i is $\mathcal{I}$-model. Thanks to [Xia, Proposition 3.1.9], we know

$$\nu(\psi_i,E)=\nu(\varphi_i,E)$$

for any $i=1,\dots,n$ and any prime divisor E over X. In other words, $\psi_i\sim_{\mathcal{I}}\varphi_i$. But both φ_i and ψ_i are $\mathcal{I}$-good, therefore,

$$\psi_i\sim_P\varphi_i.$$

By Theorem 2.10, we have

$$\int_X(\theta_1+\mathrm{dd^c}\psi_1)\wedge\cdots\wedge(\theta_n+\mathrm{dd^c}\psi_n)=\int_X(\theta_1+\mathrm{dd^c}\varphi_1)\wedge\cdots\wedge(\theta_n+\mathrm{dd^c}\varphi_n).$$

Next by Theorem 2.10 again,

$$\varlimsup_{j\in J}\int_X\left(\theta_1+\mathrm{dd^c}\varphi_1^j\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}\varphi_n^j\right)\leq\varlimsup_{j\in J}\int_X\left(\theta_1+\mathrm{dd^c}\psi_1^j\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}\psi_n^j\right).$$

On the other hand, due to [DDNL21b, Proposition 4.8], for each $i=1,\dots,n$, we have

$$\psi_i^j\xrightarrow{d_S}\psi_i.$$

We conclude from Proposition 3.10 that

$$\varlimsup_{j\in J}\int_X\left(\theta_1+\mathrm{dd^c}\psi_1^j\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}\psi_n^j\right)=\int_X(\theta_1+\mathrm{dd^c}\psi_1)\wedge\cdots\wedge(\theta_n+\mathrm{dd^c}\psi_n).$$

Putting these equations together, (3.10) follows.

Step 2. Next we handle the general case.

Fix a Kähler form ω on X. For any $\epsilon\in(0,1]$, from Step 1, we know that

$$\varlimsup_{j\in J}\mathrm{vol}\left(\theta_1+\epsilon\omega+\mathrm{dd^c}\varphi_1^j,\dots,\theta_n+\epsilon\omega+\mathrm{dd^c}\varphi_n^j\right)\leq\mathrm{vol}\left(\theta_1+\epsilon\omega+\mathrm{dd^c}\varphi_1,\dots,\theta_n+\epsilon\omega+\mathrm{dd^c}\varphi_n\right).$$

Using Lemma 3.9, we have

$$
\begin{aligned}
&\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1+\mathrm{dd^c}\varphi_1^j,\dots,\theta_n+\mathrm{dd^c}\varphi_n^j\right)\\
\leq&\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1+\epsilon\omega+\mathrm{dd^c}\varphi_1^j,\dots,\theta_n+\epsilon\omega+\mathrm{dd^c}\varphi_n^j\right)\\
\leq&\operatorname{vol}\left(\theta_1+\epsilon\omega+\mathrm{dd^c}\varphi_1,\dots,\theta_n+\epsilon\omega+\mathrm{dd^c}\varphi_n\right)\\
\leq&\operatorname{vol}\left(\theta_1+\mathrm{dd^c}\varphi_1,\dots,\theta_n+\mathrm{dd^c}\varphi_n\right)+C\epsilon.
\end{aligned}
$$

But since ϵ is arbitrary, our assertion follows. □

Lemma 3.12. *Let $\pi\colon Y\to X$ be a proper bimeromorphic morphism from a Kähler manifold Y. Then for any non-divisorial closed positive $(1,1)$-current T on X, we have*

$$
\pi^*\pi_*T=T+\sum_{i=1}^N c_i[E_i]
$$

for finitely many π-exceptional divisors E_i and $c_i>0$.

In particular, if S is a closed positive $(1,1)$-current on X, we have

$$
\pi^*\langle S\rangle=\langle \pi^*S\rangle+\sum_{i=1}^N c_i[E_i].
$$

Proof. Let E be the exceptional locus of π. Then

$$
T=\mathbb{1}_{Y\setminus E}\pi^*\pi_*T.
$$

Therefore,

$$
\pi^*\pi_*T-T=\mathbb{1}_E\pi^*\pi_*T,
$$

which has the stated form, due to the support theorems, see [Dem12a, Section 8]. □

It turns out that the mixed volume depends only on the regular parts of the currents.

Theorem 3.13. *We have*

$$
\operatorname{vol}\left(T_1,\dots,T_n\right)=\operatorname{vol}\left(\operatorname{Reg}T_1,\dots,\operatorname{Reg}T_n\right).
$$

Recall that Reg is defined in (2.3).

Remark 3.14. In general, it is not true that the mixed volume depends only on the non-pluripolar parts of the currents. This even fails for the pure volume, see [BBJ21, Example 6.10] for an example.

Proof. **Step 1**. We first prove the assertion when $T_1=\cdots=T_n=T$ and $\operatorname{vol}T>0$. We want to show that

$$
\operatorname{vol}T=\operatorname{vol}\operatorname{Reg}T.
$$

We decompose T as in (2.3). When the collection of the E_i's is finite, our assertion follows from [Xia, Proposition 7.2.3]. So we may assume that the index i runs over all positive integers. From the same proposition, we know that for any $N\geq 0$,

$$
\operatorname{vol}T=\operatorname{vol}\left(T-\sum_{i=1}^N c_i[E_i]\right).
$$

Fix a Kähler form ω on X. Thanks to Proposition 3.10 and [Xia, Theorem 6.2.2], it suffices to show that

$$
\sum_{i=1}^N c_i[E_i]\implies\sum_{i=1}^\infty c_i[E_i] \tag{3.11}
$$

as $N\to\infty$.

We can find $N_0>0$ so that for any $N\geq N_0$, the class of

$$
\omega+\sum_{i=N+1}^\infty c_i[E_i]
$$

is Kähler. Take a Kähler form ω_N in this class. Then the currents

$$\sum_{i=1}^{N} c_i[E_i] + \omega_N, \quad \sum_{i=1}^{\infty} c_i[E_i] + \omega$$

all lie in the same cohomology class. So our problem is reduced to

$$\sum_{i=1}^{N} c_i[E_i] + \omega_N \xrightarrow{d_S} \sum_{i=1}^{\infty} c_i[E_i] + \omega.$$

In fact, it suffices to show the convergence of the non-pluripolar masses, due to [Xia, Corollary 6.2.5]. In other words, we need to show that

$$\lim_{N\to\infty} \int_X \omega_N^n = \int_X \omega^n,$$

which follows from the convergence $\{\omega_N\} \to \{\omega\}$.

Step 2. We handle the general case. Fix a Kähler form ω on X, by Step 1, for any $d_1, \dots, d_n > 0$ and $\epsilon > 0$, we have

$$\operatorname{vol}\left(\sum_{i=1}^{n} d_i(T_i + \epsilon\omega)\right) = \operatorname{vol}\left(\epsilon\omega + \sum_{i=1}^{n} d_i(\operatorname{Reg} T_i + \epsilon\omega)\right).$$

Since both sides are polynomials in $d_1, \dots, d_n$, we conclude that

$$\operatorname{vol}(T_1 + \epsilon\omega, \dots, T_n + \epsilon\omega) = \operatorname{vol}(\operatorname{Reg} T_1 + \epsilon\omega, \dots, \operatorname{Reg} T_n + \epsilon\omega).$$

Letting $\epsilon \to 0+$, our assertion follows. □

Corollary 3.15. *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Then*

$$\operatorname{vol}(T_1, \dots, T_n) = \operatorname{vol}(\pi_* T_1, \dots, \pi_* T_n). \tag{3.12}$$

Proof. Observe that we may assume that $T_i = \operatorname{Reg} T_i$ for all $i = 1, \dots, n$. In fact, clearly the pushforward of the divisorial part of T_i is divisorial as well, hence by Theorem 3.13, they do not contribute to the volumes.

Now by Proposition 3.8, it remains to show that

$$\operatorname{vol}(T_1, \dots, T_n) = \operatorname{vol}(\pi^*\pi_* T_1, \dots, \pi^*\pi_* T_n).$$

By Lemma 3.12, the difference $\pi^*\pi_* T_i - T_i$ is divisorial, hence our desired equality follows from Theorem 3.13. □

A particular corollary of Corollary 3.15 is of independent interest as well.

Corollary 3.16. *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Assume that T is an $\mathcal{I}$-good closed positive $(1,1)$-current on X, then so is $\pi_* T$.*

Proof. We may assume that $\int_X T^n > 0$. Then by Corollary 3.15,

$$\operatorname{vol} \pi_* T = \operatorname{vol} T > 0$$

as well. Since T is $\mathcal{I}$-good, we have

$$\operatorname{vol} T = \int_X T^n.$$

But $\int_X T^n = \int_Z (\pi_* T)^n$, so

$$\operatorname{vol} \pi_* T = \int_Z (\pi_* T)^n > 0.$$

It follows that $\pi_* T$ is $\mathcal{I}$-good. □

Lemma 3.17. *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Consider non-divisorial closed positive $(1,1)$ currents T, S on X in the same cohomology class. Assume that $T \preceq_{\mathcal{I}} S$, then $\pi_* T \preceq_{\mathcal{I}} \pi_* S$.*

Proof. We may assume that π is a modification thanks to Hironaka's Chow lemma [Hir75, Corollary 2] and Lemma 2.12.

By Lemma 3.12,

$$\pi^*\pi_*T = T + \sum_{i=1}^{N} c_i[E_i],$$

where $c_i > 0$ and the E_i's are π-exceptional divisors. It follows that

$$T + \sum_{i=1}^{N} c_i[E_i] \preceq_{\mathcal{I}} S + \sum_{i=1}^{N} c_i[E_i].$$

Replacing T and S by $T + \sum_{i=1}^{N} c_i[E_i]$ and $S + \sum_{i=1}^{N} c_i[E_i]$ respectively, we may assume that $T = \pi^*\pi_*T$. In particular, S and $\pi^*\pi_*S$ lie in the same cohomology class, and hence $S = \pi^*\pi_*S$. Our assertion then follows from Lemma 2.12. □

4. Transcendental b-divisors

Let X be a connected compact Kähler manifold of dimension n.

4.1. The definitions.

The b-divisors defined in this section are sometimes known as b-divisor classes. We always omit the word *classes* to save space.

Definition 4.1. A *(Weil) b-divisor* $\mathbb{D}$ over X is an assignment $(\mathbb{D}_\pi)_{\pi\colon Y\to X}$, where $\pi\colon Y \to X$ runs over all modifications of X such that

1. $\mathbb{D}_\pi \in \mathrm{H}^{1,1}(Y,\mathbb{R})$;
2. The classes are compatible under push-forwards: If $\pi'\colon Z \to X$ and $\pi\colon Y \to X$ are both in $\mathrm{Modif}(X)$ and π' dominates π through $g\colon Z \to Y$ (namely, g makes the diagram (2.1) commutative), then $g_*\mathbb{D}_{\pi'} = \mathbb{D}_\pi$.

We also write $\mathbb{D}_Y = \mathbb{D}_\pi$ if there is no risk of confusion.

Given two Weil b-divisors $\mathbb{D}$ and $\mathbb{D}'$ over X, we say $\mathbb{D} \leq \mathbb{D}'$ if for each $\pi \in \mathrm{Modif}(X)$, we have $\mathbb{D}_\pi \leq \mathbb{D}'_\pi$. Recall that by definition, this means the class $\mathbb{D}'_\pi - \mathbb{D}_\pi$ is pseudoeffective.

The class $\mathbb{D}_X$ is called the *root* of $\mathbb{D}$.

Definition 4.2. The *volume* of a Weil b-divisor $\mathbb{D}$ over X is

$$\operatorname{vol}\mathbb{D} \coloneqq \lim_{\pi\colon Y\to X} \operatorname{vol}\mathbb{D}_Y.$$

The right-hand side is a decreasing net due to Proposition 2.5, hence the limit always exists.

We say $\mathbb{D}$ is *big* if $\operatorname{vol}\mathbb{D} > 0$.

Lemma 4.3. *Let $(\mathbb{D}_i)_i$ be a net of b-divisors converging to $\mathbb{D}$. Then*

$$\varlimsup_i \operatorname{vol}\mathbb{D}_i \leq \operatorname{vol}\mathbb{D}. \tag{4.1}$$

If the net is decreasing, then

$$\lim_i \operatorname{vol}\mathbb{D}_i = \operatorname{vol}\mathbb{D}.$$

Here we say $\mathbb{D}_i$ converges to $\mathbb{D}$ if for any modification $\pi\colon Y \to X$, we have $\mathbb{D}_{i,Y} \to \mathbb{D}_Y$.

In general, we cannot expect equality in (4.1), as shown by [DF22, Example 3.3].

Proof. Let $\pi\colon Y \to X$ be a modification. Then

$$\operatorname{vol}\mathbb{D}_Y = \lim_i \operatorname{vol}\mathbb{D}_{i,Y} \geq \varlimsup_i \operatorname{vol}\mathbb{D}_i.$$

The inequality follows. As for the decreasing case, it suffices to observe that both sides can be written as

$$\inf_i \inf_{\pi\colon Y\to X} \operatorname{vol}\mathbb{D}_{i,Y}.$$

□

Definition 4.4. A *Cartier b-divisor* $\mathbb{D}$ over X is a Weil b-divisor $\mathbb{D}$ over X such that there exists a modification $\pi\colon Y \to X$ and a class $\alpha_Y \in \mathrm{H}^{1,1}(Y,\mathbb{R})$ so that for each $\pi'\colon Z \to X$ dominating π, the class $\mathbb{D}_Z$ is the pull-back of α_Y. Any such (π, α_Y) is called a *realization* of $\mathbb{D}$.

By abuse of language, we also say (Y, α_Y) is a realization of $\mathbb{D}$. The realization is not unique in general.

Definition 4.5. A Cartier b-divisor $\mathbb{D}$ over X is *nef* if there exists a realization $(\pi\colon Y \to X, \alpha_Y)$ of $\mathbb{D}$ such that α_Y is nef.

Definition 4.6. A Weil b-divisor $\mathbb{D}$ over X is *nef* if there is a net of nef Cartier b-divisors $(\mathbb{D}_i)_i$ over X such that for each modification $\pi\colon Y \to X$, we have $\mathbb{D}_{i,Y} \to \mathbb{D}_Y$.

Note that thanks to Proposition 2.5, each $\mathbb{D}_Y$ is necessarily modified nef, but it is not nef in general.

A priori, for a Cartier b-divisor, nefness could mean two different things, either defined by Definition 4.5 or by Definition 4.6. We will show in Corollary 4.13 that they are actually equivalent. Before that, by a nef Cartier b-divisor, we always mean in the sense of Definition 4.5.

Our definition Definition 4.6 amounts defining the set of Weil b-divisors as the closure of the set of Cartier b-divisors in $\varprojlim_\pi \mathrm{H}^{1,1}(Y,\mathbb{R})$ with respect to the projective limit topology. In particular, the limit of a converging net of nef b-divisors is still nef.

4.2. **The b-divisors of currents.** Let T be a closed positive $(1,1)$-current on X.

Given any modification $\pi\colon Y \to X$, we define

$$\mathbb{D}(T)_Y \coloneqq \{\operatorname{Reg} \pi^* T\} \in \mathrm{H}^{1,1}(Y,\mathbb{R}). \tag{4.2}$$

The b-divisor $\mathbb{D}(T)$ was firstly explicitly introduced in [Xia22] in 2020. The paper received very little attention and the same object was re-introduced in [BBGHdJ22] and [Tru24] later on.

Lemma 4.7. *Let T be a closed positive $(1,1)$-current on X. Then $\mathbb{D}(T)$ is nef. Moreover,*

$$\operatorname{vol} T = \operatorname{vol} \mathbb{D}(T). \tag{4.3}$$

Note that when T has analytic singularities, $\mathbb{D}(T)$ is Cartier.

Proof. Let ω be a Kähler form on X. Then $\mathbb{D}(\omega)$ is the Cartier b-divisor realized by $(X, \{\omega\})$. We could always approximate $\mathbb{D}(T)$ by $\mathbb{D}(T + \epsilon\omega) = \mathbb{D}(T) + \epsilon\mathbb{D}(\omega)$. Hence we may assume that T is a Kähler current.

Next, we take a closed smooth real $(1,1)$-form θ cohomologous to T and write $T = \theta_\varphi$ for some $\varphi \in \mathrm{PSH}(X,\theta)$. Let $(\varphi_j)j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. Then it is easy to see that $\mathbb{D}(\theta + \mathrm{dd^c}\varphi_j) \to \mathbb{D}(\theta + \mathrm{dd^c}\varphi)$. See [Xia24, Theorem 9.6] Step 2 for the details. As a consequence,

$$\operatorname{vol} \mathbb{D}(\theta + \mathrm{dd^c}\varphi_j) \to \operatorname{vol} \mathbb{D}(\theta + \mathrm{dd^c}\varphi),$$

thanks to Lemma 4.3.

So we may assume that T has analytic singularities. Let $\pi\colon Y \to X$ be a modification so that

$$\pi^* T = [D] + R,$$

where D is an effective $\mathbb{Q}$-divisor on Y and R is a closed positive $(1,1)$-current with locally bounded potentials. Then $\mathbb{D}(T)$ is the nef Cartier b-divisor realized by $(\pi, \{R\})$. Note that (4.3) is obvious in this case. □

Remark 4.8. There is a different possibility: Replace Reg by the non-pluripolar part. Given T as above, we define

$$\mathbb{D}'(T)_\pi \coloneqq [\langle \pi^* T \rangle].$$

But thanks to Lemma 3.12, we have

$$\mathbb{D}'(T) = \mathbb{D}(\langle T \rangle).$$

Conversely, we want to realize nef b-divisors as $\mathbb{D}(T)$. We first prove a continuity result.

Proposition 4.9. *Let $(T_i)_{i\in I}$ be a net of closed positive $(1,1)$-currents on X and T be a closed positive $(1,1)$-current on X. Assume that $T_i \implies T$, then*

$$\mathbb{D}(T_i) \to \mathbb{D}(T).$$

Recall that $\implies$ is defined in Definition 2.13.

Proof. When the cohomology classes $\{T_i\}$ and $\{T\}$ are all the same, the proof is the same as that in the algebraic case. See [Xia24, Theorem 9.8], which we omit.

In general, fix a Kähler form ω on X. Then we can find $i_0 \in I$ so that for $i \geq i_0$, the class

$$\omega + \{T\} - \{T_i\}$$

is Kähler, and we can find a Kähler form ω_i in this class. It follows that

$$T_i + \omega_i, \quad T + \omega$$

are all in the same cohomology class, and hence

$$\mathbb{D}(T_i + \omega_i) \to \mathbb{D}(T + \omega).$$

But clearly

$$\mathbb{D}(\omega_i) \to \mathbb{D}(\omega),$$

so our assertion follows. □

Proposition 4.10. *Each big and nef b-divisor $\mathbb{D}$ over X can be realized as $\mathbb{D}(T)$ for some $T \in \mathbb{D}_X$. Furthermore, we may always assume that T is $\mathcal{I}$-good.*

Note that T is unique up to $\mathcal{I}$-equivalence. The current T is necessarily non-divisorial.

Proof. Fix a big and nef b-divisor $\mathbb{D}$ over X.

For each $\pi\colon Y \to X$, we take a current with minimal singularities T_Y in $\mathbb{D}_Y$. We claim that $\mathbb{D}(\pi_* T_Y)$ coincides with $\mathbb{D}$ up to the level of Y: For any modification $\pi'\colon Z \to X$ dominated by π through a morphism $g\colon Y \to Z$, we have

$$\mathbb{D}_Z = \mathbb{D}(\pi_* T_Y)_Z.$$

After unfolding the definitions, this means

$$\operatorname{Reg}(\pi'^* \pi_* T_Y) \in \mathbb{D}_Z.$$

Note that

$$\operatorname{Reg}(\pi'^* \pi_* T_Y) = \operatorname{Reg}(\pi'^* \pi'_* g_* T_Y).$$

Due to Proposition 2.5, we know that $\mathbb{D}_Y$ is modified nef and big. In particular, T_Y is non-divisorial, hence so is $g_* T_Y$ by Lemma 2.8. It follows from Lemma 3.12 that

$$\operatorname{Reg}(\pi'^* \pi'_* g_* T_Y) = \operatorname{Reg}(g_* T_Y) = g_* T_Y \in \mathbb{D}_Z.$$

Note that

$$\operatorname{vol} T_Y \geq \operatorname{vol} \mathbb{D} > 0.$$

Now $(\pi_* T_Y)_Y$ is a net in the cohomology class $\mathbb{D}_X$ and the P-singularity types are decreasing, as a consequence of Lemma 3.17 and Corollary 3.16. Thanks to the completeness result [DDNL21b, Theorem 1.1], we could find a closed positive $(1,1)$-current $T \in \mathbb{D}_X$ such that

$$\pi_* T_Y \xrightarrow{d_S} T.$$

It follows from Proposition 4.9 that

$$\mathbb{D}(\pi_* T_Y) \to \mathbb{D}(T).$$

Therefore, we conclude that

$$\mathbb{D}(T) = \mathbb{D}.$$

Thanks to Lemma 4.7, $\operatorname{vol} T > 0$. Write $T = \theta + \mathrm{dd^c}\varphi$ for some smooth closed real $(1,1)$-form θ and $\varphi \in \operatorname{PSH}(X,\theta)$, then

$$T' := \theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}}$$

is $\mathcal{I}$-good non-divisorial and $\mathbb{D}(T') = \mathbb{D}(T)$. □

Let α be a modified nef class on X. We write $\mathcal{G}(\alpha)$ for the set of closed positive $(1,1)$-currents T on X with $T = \operatorname{Reg} T \in \alpha$ and $\operatorname{vol} T > 0$.

Theorem 4.11. *There is a natural bijection from $\mathcal{G}(\alpha)/\sim_{\mathcal{I}}$ to the set of big and nef b-divisors $\mathbb{D}$ over X with $\mathbb{D}_X = \alpha$.*

Proof. Given $T \in \mathcal{G}(\alpha)$, we associate the b-divisor $\mathbb{D}(T)$. It is big and nef due to Lemma 4.7. This map clearly descends to $\mathcal{G}(\alpha)/\sim_{\mathcal{I}}$.

This map is surjective by Proposition 4.10. Now we show that it is injective. Let $T, T' \in \mathcal{G}(\alpha)$. Assume that $\mathbb{D}(T) = \mathbb{D}(T')$, we want to show that $T \sim_{\mathcal{I}} T'$.

Let E be a prime divisor over X, it suffices to show that

$$\nu(T, E) = \nu(T', E). \tag{4.4}$$

We may assume that E is not a prime divisor on X, as otherwise both sides vanish.

Choose a sequence of blow-ups with smooth *connected* centers

$$Y \coloneqq X_k \to X_{k-1} \to \cdots \to X_0 \coloneqq X$$

so that E is a prime divisor on Y, exceptional with respect to $X_k \to X_{k-1}$. Denote the composition by $\pi\colon Y \to X$. Thanks to Proposition 2.3,

$$\mathrm{H}^{1,1}(X_k, \mathbb{R}) = \mathrm{H}^{1,1}(X_{k-1}, \mathbb{R}) \oplus \mathbb{R}\{E_k\},$$

where $E_k = E$ is the exceptional divisor of $X_k \to X_{k-1}$.

By induction,

$$\mathrm{H}^{1,1}(X_k, \mathbb{R}) = \mathrm{H}^{1,1}(X, \mathbb{R}) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\{E_i\},$$

where E_i is the exceptional divisor of $X_i \to X_{i-1}$. Now by Lemma 3.12,

$$\operatorname{Reg} \pi^* T = \pi^* T - \sum_{i=1}^{k} \nu(T, E_i)[E_i]. \tag{4.5}$$

In particular, the cohomology class of $\operatorname{Reg} \pi^* T$ determines $\nu(T, E)$. Hence, (4.4) follows. □

Corollary 4.12. *The set of nef b-divisors over X can be naturally identified with*

$$\varprojlim_{\omega} \left(\mathcal{G}(\alpha + \omega)/\sim_{\mathcal{I}}\right),$$

where ω runs over the directed set of Kähler forms on X (with respect to the partial order of reverse domination), and given two Kähler forms $\omega \leq \omega'$ the transition map

$$\mathcal{G}(\alpha + \omega)/\sim_{\mathcal{I}} \to \mathcal{G}(\alpha + \omega')/\sim_{\mathcal{I}}$$

is induced by the map $\mathcal{G}(\alpha + \omega) \to \mathcal{G}(\alpha + \omega')$ sending T to $T + \omega' - \omega$.

Corollary 4.13. *Let $\mathbb{D}$ be a Cartier b-divisor over X. Then $\mathbb{D}$ is nef in the sense of Definition 4.5 if and only if it is nef in the sense of Definition 4.6.*

This result is the transcendental version of [DF22, Theorem 2.8].

Proof. We only handle the non-trivial implication. Assume that $\mathbb{D}$ is nef in the sense of Definition 4.6. We want to show that $\mathbb{D}$ is nef in the sense of Definition 4.5. We may clearly assume that $\mathbb{D}$ is big. Take a non-divisorial closed positive $(1,1)$-current T on X such that $\mathbb{D} = \mathbb{D}(T)$.

Without loss of generality, we may also assume that $\mathbb{D}$ is realized by (X, α) for some cohomology class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$. Now $\mathbb{D} = \mathbb{D}(T)$ means that for each modification $\pi\colon Y \to X$, the current $\pi^* T$ is non-divisorial. In particular, T has vanishing generic Lelong number along each prime divisor over X, see (4.5). That means, T has vanishing Lelong number everywhere. It follows that $\alpha = \{T\}$ is nef. □

Corollary 4.14. *Let T and T' be non-divisorial closed positive $(1,1)$-currents on X. Suppose that $\mathbb{D}(T)_X = \mathbb{D}(T')_X$, then the following are equivalent:*

(1) $\mathbb{D}(T) \leq \mathbb{D}(T')$;
(2) $T \preceq_{\mathcal{I}} T'$.

Proof. This follows from (4.5). □

In particular, we obtain the transcendental analogue of [DF22, Theorem A].

Corollary 4.15. *Let $\mathbb{D}$ be a nef b-divisor over X. Then there is a decreasing sequence of nef and big b-divisors $\mathbb{D}^i$ over X with limit $\mathbb{D}$.*

Proof. Take a Kähler form ω on X. By Proposition 4.10, for each $i > 0$, we can find a non-divisorial Kähler current $T_i \in \mathbb{D}_X + i^{-1}\{\omega\}$ such that

$$\mathbb{D}(T_i) = \mathbb{D} + i^{-1}\mathbb{D}(\omega).$$

We observe that

$$T_{i+1} \sim_{\mathcal{I}} T_i.$$

This follows from applying Corollary 4.14 to T_i and $T_{i+1} + (i - (i+1)^{-1})\omega$. Let $(T_i^j)_j$ be quasi-equisingular approximations of T_i such that

(1) $T_i^j \in \mathbb{D}_X + i^{-1}\{\omega\}$ and is a Kähler current for $j \geq j_0(i)$, and
(2) the singularity types of $(T_i^j)_i$ is constant.

Note that (2) is possible by the using the Bergman kernel construction of the equisingular approximations.

It suffices to take $\mathbb{D}^i = \mathbb{D}(T_{j_i}^i)$, where j_i is a strictly increasing sequence of positive integers with $j_i \geq j_0(i)$. □

5. The intersection theory

Let X be a connected compact Kähler manifold of dimension n. We will define the intersection numbers of nef b-divisors and show that they satisfy the same properties as their algebraic analogues, *c.f.* [DF22, Theorem 3.2].

Definition 5.1. Let $\mathbb{D}_1, \ldots, \mathbb{D}_n$ be big and nef b-divisors over X. Then we define their intersection as

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) \coloneqq \operatorname{vol}(T_1, \ldots, T_n),$$

where $T_1, \ldots, T_n$ are closed positive $(1,1)$-currents in $\mathbb{D}_{1,X}, \ldots, \mathbb{D}_{n,X}$ respectively such that $\mathbb{D}(T_i) = \mathbb{D}_i$.

In general, if the $\mathbb{D}_i$'s are only nef, we define

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) \coloneqq \lim_{\epsilon \to 0+} (\mathbb{D}_1 + \epsilon\mathbb{D}(\omega), \ldots, \mathbb{D}_n + \epsilon\mathbb{D}(\omega)),$$

where ω is a Kähler form on X.

The definition makes sense thanks to Proposition 4.10. It does not depend on the choices of $T_1, \ldots, T_n$ since they are uniquely defined up to $\mathcal{I}$-equivalence.

We first note that even when the T_i's have vanishing volumes, the two intersection products still agree.

Proposition 5.2. *Let $T_1, \ldots, T_n$ be a closed positive $(1,1)$-currents on X. Then*

$$(\mathbb{D}(T_1), \ldots, \mathbb{D}(T_n)) = \operatorname{vol}(T_1, \ldots, T_n).$$

This is a trivial consequence of the definitions.

Proposition 5.3. *The product in Definition 5.1 is symmetric and multi-$\mathbb{R}_{\geq 0}$-linear.*

Proof. The multi-linearity follows immediately from Proposition 3.4. The symmetry is immediate. □

Proposition 5.4. *The product in Definition 5.1 is monotonically increasing in each variable.*

Proof. Let $\mathbb{D}_1,\dots,\mathbb{D}_n$ and $\mathbb{D}'$ be nef b-divisors over X so that $\mathbb{D}_1\leq\mathbb{D}'$. We want to show that

$$(\mathbb{D}_1,\dots,\mathbb{D}_n)\leq(\mathbb{D}',\mathbb{D}_2,\dots,\mathbb{D}_n)\,.$$

We can easily reduce to the case where $\mathbb{D}_1,\dots,\mathbb{D}_n$ and $\mathbb{D}'$ are all big. In this case, take $\mathcal{I}$-good non-divisorial closed positive $(1,1)$-currents $T_1,\dots,T_n$ and T' so that $\mathbb{D}(T_i)=\mathbb{D}_i$ for all $i=1,\dots,n$ and $\mathbb{D}(T')=\mathbb{D}'$. Furthermore, we may assume that the T_i's and T' are Kähler currents by the same perturbation argument.

Let $(T_i^j)_j$ be a quasi-equisingular approximation of T_i for $i=2,\dots,n$. It follows from Proposition 3.10 that

$$\int_X T_1\wedge\cdots\wedge T_n=\lim_{j\to\infty}\int_X T_1\wedge T_2^j\wedge\cdots\wedge T_n^j.$$

It suffices to show that for all $j\geq 1$,

$$\int_X T_1\wedge T_2^j\wedge\cdots\wedge T_n^j\leq\int_X T'\wedge T_2^j\wedge\cdots\wedge T_n^j.$$

Therefore, we have reduced to the case where $T_2,\dots,T_n$ have analytic singularities. After a resolution, we may assume that they have log singularities along $\mathbb{Q}$-divisors. By Theorem 3.13, we can further reduce to the case where $T_2,\dots,T_n$ have bounded local potentials. Perturb $T_2,\dots,T_n$ by a Kähler form, we may further assume that $\{T_2\},\dots,\{T_n\}$ are Kähler classes. By Proposition 3.6, we finally reduce to the case where $T_2,\dots,T_n$ are Kähler forms. In this case, our assertion is obvious. □

Lemma 5.5. *Let ω be a Kähler form on X. Fix a compact set $K\subseteq\mathrm{H}^{1,1}(X,\mathbb{R})$. Let $\mathbb{D}_1,\dots,\mathbb{D}_n$ be nef b-divisors over X such that $\mathbb{D}_{i,X}\in K$ for each $i=1,\dots,n$. Then there is a constant C depending only on $X,K,\{\omega\}$ such that for any $\epsilon>0$, we have*

$$0\leq(\mathbb{D}_1+\epsilon\mathbb{D}(\omega),\dots,\mathbb{D}_n+\epsilon\mathbb{D}(\omega))-(\mathbb{D}_1,\dots,\mathbb{D}_n)\leq C\epsilon.$$

Proof. This is a simple consequence of the linearity Proposition 5.3. □

We first make a consistency check.

Proposition 5.6. *Suppose that $\mathbb{D}$ is a nef b-divisor over X, then*

$$(\mathbb{D},\dots,\mathbb{D})=\operatorname{vol}\mathbb{D}.$$

Proof. Using Lemma 5.5, we may easily reduce to the case where $\mathbb{D}$ is nef and big. In this case, take a non-divisorial closed positive $(1,1)$-current T in $\mathbb{D}_X$ such that $\mathbb{D}(T)=\mathbb{D}$. Then we need to show that

$$\lim_{\pi\colon Y\to X}\operatorname{vol}\mathbb{D}(T)_Y=\operatorname{vol}T.$$

In fact, for any modification $\pi\colon Y\to X$, we have

$$\operatorname{vol}\mathbb{D}(T)_Y=\operatorname{vol}T.$$

This is a consequence of Theorem 3.13. □

Proposition 5.7. *Let $\mathbb{D}_1,\dots,\mathbb{D}_n$ be nef b-divisors over X. Then*

$$(\mathbb{D}_1,\dots,\mathbb{D}_n)\geq\prod_{i=1}^n(\operatorname{vol}\mathbb{D}_i)^{1/n}\,.$$

Proof. We may assume that $\operatorname{vol}\mathbb{D}_i>0$ for each $i=1,\dots,n$ since there is nothing to prove otherwise. In this case, our assertion follows from Proposition 3.7. □

Proposition 5.8. *The product in Definition 5.1 is upper semicontinuous in the following sense. Suppose that $(\mathbb{D}_i^j)_{j\in J}$ are nets of nef b-divisors over X with limits $\mathbb{D}_i$ for each $i=1,\dots,n$. Then*

$$\varlimsup_{j\in J}\left(\mathbb{D}_1^j,\dots,\mathbb{D}_n^j\right)\leq(\mathbb{D}_1,\dots,\mathbb{D}_n)\,.$$

Proof. **Step 1**. We first assume that the $\mathbb{D}_i^j$'s and the $\mathbb{D}_i$'s are all big.

Take $\mathcal{I}$-good non-divisorial closed positive $(1,1)$-currents T_i^j and T_i so that $\mathbb{D}(T_i^j) = \mathbb{D}_i^j$ and $\mathbb{D}(T_i) = \mathbb{D}_i$. Note that by our assumption and the proof of Theorem 4.11, for any prime divisor E over X, we have

$$\lim_{j\in J} \nu(T_i^j, E) = \nu(T_i, E).$$

So our assertion follows from Theorem 3.11.

Step 2. Next we handle the general case.

Take a Kähler form ω on X. Then by Lemma 5.5, for any $\epsilon \in (0,1]$, we have

$$\begin{aligned}\varlimsup_{j\in J} \left(\mathbb{D}_1^j, \ldots, \mathbb{D}_n^j\right) \leq& \varlimsup_{j\in J} \left(\mathbb{D}_1^j + \epsilon\mathbb{D}(\omega), \ldots, \mathbb{D}_n^j + \epsilon\mathbb{D}(\omega)\right)\\ \leq& \left(\mathbb{D}_1 + \epsilon\mathbb{D}(\omega), \ldots, \mathbb{D}_n + \epsilon\mathbb{D}(\omega)\right)\\ \leq& \left(\mathbb{D}_1, \ldots, \mathbb{D}_n\right) + C\epsilon.\end{aligned}$$

Since ϵ is arbitrary, our assertion follows. □

Proposition 5.9. *The product in Definition 5.1 is continuous along decreasing nets in each variable. In other words, if $(\mathbb{D}_i^j)_{j\in J}$ $(i = 1, \ldots, n)$ are decreasing nets of nef b-divisors over X with limits $\mathbb{D}_i$. Then*

$$\lim_{j\in J} \left(\mathbb{D}_1^j, \ldots, \mathbb{D}_n^j\right) = \left(\mathbb{D}_1, \ldots, \mathbb{D}_n\right).$$

Proof. This is a straightforward consequence of Proposition 5.4 and Proposition 5.8. □

Remark 5.10. As shown in [Xia22; Xia24], this intersection theory coincides with the Dang–Favre theory if X is projective and $\mathbb{D}_1, \ldots, \mathbb{D}_n$ are algebraic.

To be more precise, these papers handled the case where the cohomology class $\{T_1\}, \ldots, \{T_n\}$ lie in the Néron–Severi group $\mathrm{NS}^1(X)$. By scaling, the same holds if $\{T_1\}, \ldots, \{T_n\}$ lie in the $\mathbb{Q}$-span of $\mathrm{NS}^1(X)$. Finally, by Proposition 5.9, the same holds in general.

6. Smooth pull-backs of b-divisors

Let X be a connected compact Kähler manifold of dimension n. Consider a smooth morphism $f\colon Y \to X$ of relative dimension m from another connected compact Kähler manifold Y. Given a nef b-divisor $\mathbb{D}$ over X, we shall define a functorial pull-back $f^*\mathbb{D}$ over Y. In the next section, we will need a special case of this construction, where Y is a projective bundle on X.

We first assume that $\mathbb{D}$ is big and nef. Thanks to Theorem 4.11, we can find a non-divisorial closed positive $(1,1)$-current T in $\mathbb{D}_X$ such that $\mathbb{D}(T) = \mathbb{D}$. Moreover, T is unique up to $\mathcal{I}$-equivalence.

We can therefore define

$$f^*\mathbb{D} := \mathbb{D}(f^*T).$$

Note that thanks to [Xia, Proposition 1.4.5], the $\mathcal{I}$-equivalence class of f^*T is independent of the choices of T. Hence $f^*\mathbb{D}$ is a well-defined nef b-divisor over Y, independent of the choice of T.

Observe that f^*T is non-divisorial since this is the case if f is either a projection or étale. In particular, $(f^*\mathbb{D})_Y = f^*\mathbb{D}_X$.

In general, if $\mathbb{D}$ is not necessarily nef, we take a Kähler form ω on X and define

$$f^*\mathbb{D} := \lim_{\epsilon\to 0+} f^*\left(\mathbb{D} + \epsilon\mathbb{D}(\omega)\right).$$

Note that $f^*\left(\mathbb{D} + \epsilon\mathbb{D}(\omega)\right)$ is increasing with respect to $\epsilon > 0$, so the limit makes sense. It is clear that this definition is independent of the choice of ω. Observe that

$$(f^*\mathbb{D})_Y = f^*\mathbb{D}_X. \tag{6.1}$$

Proposition 6.1. *The pull-back f^* defined above is $\mathbb{R}_{\geq 0}$-linear. Moreover, for any closed positive $(1,1)$-current T on X, we have*

$$\mathbb{D}(f^*T) = f^*\mathbb{D}(T). \tag{6.2}$$

Proof. The $\mathbb{R}_{\geq 0}$-linearity is obvious.

We prove the latter part. Fix a Kähler form ω on X. It suffices to handle two cases separately: T is either non-divisorial or divisorial. In the first case, by definition,

$$f^*\mathbb{D}(T) = \lim_{\epsilon \to 0+} \mathbb{D}\left(f^*(T + \epsilon\omega)\right) = \mathbb{D}(f^*T).$$

Next we assume that T is divisorial, say $T = \sum_i c_i[E_i]$. In this case, by (3.11) and Proposition 4.9, we may assume that T has finitely many components. By linearity, we reduce to the case where $T = [E]$ for some prime divisor E on X. In this case, we have $f^*T = [f^{-1}E]$. Hence both sides of (6.2) vanish. □

The pull-back is functorial as expected.

Proposition 6.2. *Let $g\colon Z \to Y$ be another smooth morphism from a connected compact Kähler manifold Z. Then for any nef b-divisor $\mathbb{D}$ over X, we have*

$$(f \circ g)^*\mathbb{D} = g^* f^*\mathbb{D}. \tag{6.3}$$

Proof. We may assume that $\mathbb{D}$ is big. Then there is a non-divisorial closed positive $(1,1)$-current $T \in \mathbb{D}_X$ so that $\mathbb{D}(T) = \mathbb{D}_X$. Note that both sides of (6.3) have the same root thanks to (6.1).

Thanks to Proposition 6.1, both sides of (6.3) are equal to $\mathbb{D}(g^* f^* T)$. □

Proposition 6.3. *Let $\pi\colon X' \to X$ be a modification. Consider the Cartesian diagram,*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X. \end{tikzcd} \tag{6.4}$$

Then for any nef b-divisor $\mathbb{D}$ over X, we have

$$(f^*\mathbb{D})_{Y'} = f'^*\mathbb{D}_{X'}. \tag{6.5}$$

Thanks to the smoothness of f, π_Y is also a modification, so the left-hand side of (6.5) makes sense.

Proof. We may assume that $\mathbb{D}$ is big. Take a non-divisorial closed positive $(1,1)$-current T in $\mathbb{D}_X$ so that $\mathbb{D} = \mathbb{D}(T)$. Since f'^* preserves non-divisorial currents and divisorial currents, we have

$$f'^* \operatorname{Reg} \pi^* T = \operatorname{Reg} \pi_Y^*(f^*T).$$

Therefore,

$$\{\operatorname{Reg} \pi_Y^*(f^*T)\} = f'^*\{\operatorname{Reg} \pi^* T\}.$$

Our assertion follows. □

Proposition 6.4. *Let $\mathbb{D}, \mathbb{D}'$ be nef b-divisors over X with $\mathbb{D}_X = \mathbb{D}'_X$. Then the following are equivalent:*

(1) $\mathbb{D} \leq \mathbb{D}'$;
(2) $f^*\mathbb{D} \leq f^*\mathbb{D}'$.

Proof. We may assume that $\mathbb{D}$ and $\mathbb{D}'$ are both big. Take non-divisorial closed positive $(1,1)$-currents T and T' in $\mathbb{D}_X$ such that $\mathbb{D} = \mathbb{D}(T)$ and $\mathbb{D}' = \mathbb{D}(T')$.

(1) $\implies$ (2). Assume (1). It follows from Corollary 4.14 that $T \preceq_{\mathcal{I}} T'$. By [Xia, Proposition 1.4.5], we have $f^*T \preceq_{\mathcal{I}} f^*T'$, hence by (6.1) and Corollary 4.14 again, we find $f^*\mathbb{D} \leq f^*\mathbb{D}'$.

(2) $\implies$ (1). Assume (2). Fix a prime divisor E over X. It suffices to show that

$$\nu(T, E) \geq \nu(T', E). \tag{6.6}$$

Take a modification $\pi\colon X' \to X$ so that E is a prime divisor on X'. Form the Cartesian diagram (6.4). Then by Corollary 4.14,

$$\nu(\pi^*T, f'^{-1}E) \geq \nu(\pi^*T', f'^{-1}E).$$

Thanks to [Bou02, Proposition 1.1.7], this is just (6.6). □

Proposition 6.5. *Let θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Let $(\varphi_i)_{i\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi\in\mathrm{PSH}(X,\theta)$. Assume that $\varphi_i \xrightarrow{d_S} \varphi$, then*
$$f^*\varphi_i \xrightarrow{d_S} f^*\varphi.$$

Proof. Since $\mathrm{PSH}(X,\theta)$ is a pseudometric space, we may assume that $(\varphi_i)_i$ is a sequence. Replacing θ by $\theta+\omega$ for some Kähler form ω on X, we may assume that the non-pluripolar masses of the φ_i's are bounded from below by a positive constant. Then it follows from [DDNL18, Proposition 4.2] and [Xia, Corollary 6.2.11] that we may assume without loss of generality that $(\varphi_i)_i$ is either increasing or decreasing.

The increasing case follows from [Xia, Corollary 6.2.3]. We assume that (φ_i) is a decreasing sequence. Fix a Kähler form Ω on Y. By [Xia, Corollary 6.2.5], it remains to argue that
$$\lim_{i\to\infty}\int_Y \left(f^*\theta+\Omega+\mathrm{dd^c} f^*\varphi_i\right)^{n+m} = \int_Y \left(f^*\theta+\Omega+\mathrm{dd^c} f^*\varphi\right)^{n+m}.$$
After a binomial expansion, it suffices to show that for any $a=0,\dots,n$, we have
$$\lim_{i\to\infty}\int_Y \left(f^*\theta+\mathrm{dd^c} f^*\varphi_i\right)^{a}\wedge\Omega^{n+m-a} = \int_Y \left(f^*\theta+\mathrm{dd^c} f^*\varphi\right)^{a}\wedge\Omega^{n+m-a},$$
or equivalently,
$$\lim_{i\to\infty}\int_X \left(\theta+\mathrm{dd^c}\varphi_i\right)^{a}\wedge f_*\Omega^{n+m-a} = \int_X \left(\theta+\mathrm{dd^c}\varphi\right)^{a}\wedge f_*\Omega^{n+m-a}.$$
Since f is smooth, the form $f_*\Omega^{n+m-a}$ is smooth as well. Our assertion then follows from [Xia24, Theorem 1.9]. □

Corollary 6.6. *Let $(T_i)_{i\in I}$ be a net of closed positive $(1,1)$-currents on X and T be a closed positive $(1,1)$-current on X. Assume that $T_i \implies T$, then $f^*T_i \implies f^*T$.*

Proof. This is an immediate consequence of Proposition 6.5. □

7. The trace operator of b-divisors

Let X be a connected compact Kähler manifold of dimension n and Z be a smooth irreducible analytic set of dimension m in X. Let $\mathbb{D}$ be a nef b-divisor over X.

We will study the problem of restricting nef b-divisors over X to Z in this section. This problem has been studied in the analytic setting in [DX24a]. We shall follow the slightly different approach as studied in [Xia, Chapter 8], which is better behaved in the zero mass case.

7.1. The analytic theory.
Let T be a closed positive $(1,1)$-current on X representing a cohomology class α. Assume that $\nu(T,Z)=0$.

Consider a quasi-equisingular approximation $(T_j)_j$ of T, where the currents T_j are not necessarily in α. Then $\nu(T_j,Z)=0$ and hence $T_j|_Z$ makes sense. We then define $\mathrm{Tr}_Z T$ as any closed positive $(1,1)$-current on Z such that $T_j|_Z \implies \mathrm{Tr}_Z T$.

One can show that $\mathrm{Tr}_Z T$ is always well-defined modulo P-equivalence and is independent of the choice of the sequence $(T_j)_j$.

If furthermore, T is a Kähler current, then $\mathrm{Tr}_Z T$ can be represented by Kähler current in $\alpha|_Z$.

The details can be found in [Xia, Chapter 8].

7.2. The codimension 1 case.
We assume that Z is a divisor so that $m=n-1$.

For the moment, let us assume that $\mathbb{D}$ is a Cartier nef b-divisor. Let $(\pi\colon Y\to X,\alpha)$ be a realization of $\mathbb{D}$.

Let Z_Y denote the strict transform of Z and $p_Y\colon Z_Y\to Z$ denotes the restriction of π. The notations are summarized in the commutative diagram:

$$\begin{tikzcd} Z_Y \arrow[r, hook] \arrow[d, "p_Y"'] & Y \arrow[d, "\pi"] \\ Z \arrow[r, hook] & X. \end{tikzcd} \tag{7.1}$$

After replacing π by a further modification, we may assume that Z_Y is smooth. This follows from the embedded resolution [BM97; Wło09]. In this case, we define the *trace* $\operatorname{Tr}_Z \mathbb{D}$ of $\mathbb{D}$ on Z as the nef Cartier b-divisor over Z realized by $(p_Y, \alpha|_{Z_Y})$. Note that we are slightly abusing our language since p_Y is not a modification in general. To be more precise, here we mean that for any modification $Z' \to Z$ dominating Z, $\operatorname{Tr}_Z \mathbb{D}$ is defined as the nef Cartier b-divisor over Z realized by $(Z' \to Z, \beta)$, where β is the pull-back of $\alpha|_{Z_Y}$.

Lemma 7.1. *Assume that $\mathbb{D}$ is a Cartier nef b-divisor, then $\operatorname{Tr}_Z \mathbb{D}$ defined above is independent of the choice of π.*

Proof. Given a different realization $(\pi' \colon Y' \to X, \alpha')$ of $\mathbb{D}$, we want to show that it defines the same $\operatorname{Tr}_Z \mathbb{D}$. We may assume that π' dominates π so that we have a commutative diagram:

$$\begin{tikzcd}
Z_{Y'} \arrow[r, hook] \arrow[d, "\tau"] \arrow[dd, bend right, "p_{Y'}"'] & Y' \arrow[d, "\sigma"'] \arrow[dd, bend left, "\pi'"] \\
Z_Y \arrow[r, hook] \arrow[d, "p_Y"] & Y \arrow[d, "\pi"'] \\
Z \arrow[r, hook] & X.
\end{tikzcd}$$

The notations $\tau, \sigma, p_{Y'}$ have the obvious meanings. We may assume that Z_Y and $Z_{Y'}$ are both smooth.

Our assertion becomes the following:

$$\tau^* \left(\alpha|_{Z_Y}\right) = \left(\sigma^* \alpha\right)|_{Z_{Y'}},$$

which is obvious since the upper square in the diagram commutes. ☐

Proposition 7.2. *Let $\mathbb{D}'$ be another nef Cartier b-divisor over X. Assume that $\mathbb{D}' \leq \mathbb{D}$, then $\operatorname{Tr}_Z \mathbb{D}' \leq \operatorname{Tr}_Z \mathbb{D}$.*

Proof. We take realizations $(\pi \colon Y \to X, \alpha')$ and (π, α) of $\mathbb{D}'$ and $\mathbb{D}$ on the same modification. Then by assumption $\alpha \geq \alpha'$. It follows that $\alpha|_{Z_Y} \geq \alpha'|_{Z_Y}$. Therefore, our assertion follows. ☐

In general, when $\mathbb{D}$ is not necessarily Cartier, by Corollary 4.15, we can find a decreasing sequence $(\mathbb{D}_i)_i$ of nef Cartier b-divisors converging to $\mathbb{D}$. We then define

$$\operatorname{Tr}_Z \mathbb{D} \coloneqq \lim_{i\to\infty} \operatorname{Tr}_Z \mathbb{D}_i.$$

This means, for any modification $p_Y \colon Z_Y \to Z$, we define

$$\left(\operatorname{Tr}_Z \mathbb{D}\right)_{Z_Y} \coloneqq \lim_{i\to\infty} \left(\operatorname{Tr}_Z \mathbb{D}_i\right)_{Z_Y}.$$

Note that p_Y can always be completed in to a diagram as in (7.1). Thanks to Proposition 7.2, the definition makes sense. The b-divisor $\operatorname{Tr}_Z \mathbb{D}$ is obviously nef.

Lemma 7.3. *The definition of $\operatorname{Tr}_Z \mathbb{D}$ is independent of the choice of the $\mathbb{D}_i$'s.*

Proof. Let $\mathbb{D}' \geq \mathbb{D}$ be a nef Cartier b-divisor realized by $(\pi \colon Y \to X, \alpha)$, it suffices to show that

$$\lim_{i\to\infty} \operatorname{Tr}_Z \mathbb{D}_i \leq \operatorname{Tr}_Z \mathbb{D}.$$

We use the notations in (7.1). We may assume that Z_Y is smooth by embedded resolution.

Thanks to Lemma 3.12, this is reduced immediately to

$$\lim_{i\to\infty} \left(\operatorname{Tr}_Z \mathbb{D}_i\right)_{Z_Y} \leq \alpha|_{Z_Y}.$$

We can easily reduce to the case where π is the identity. In this case, the assertion becomes the following: If $\alpha \geq \mathbb{D}_X$ is a nef class, then

$$\alpha|_Z \geq \lim_{i\to\infty} \left(\operatorname{Tr}_Z \mathbb{D}_i\right)_Z. \tag{7.2}$$

Fix a Kähler form ω, then for any $\epsilon > 0$, we can find $i_0 > 0$ so that for $i > i_0$,

$$\mathbb{D}_{i,X} \leq \alpha + \epsilon\{\omega\}.$$

It follows that

$$\mathbb{D}_i \leq \mathbb{D}(\alpha + \epsilon\{\omega\}),$$

where $\mathbb{D}(\alpha+\epsilon\{\omega\})$ is the Cartier b-divisor realized by $(X, \alpha+\epsilon\{\omega\})$. It follows from Proposition 7.2 that

$$\alpha|_Z + \epsilon\{\omega|_Z\} \geq \lim_{i\to\infty} (\operatorname{Tr}_Z \mathbb{D}_i)_Z.$$

Since $\epsilon > 0$ is arbitrary, (7.2) follows. □

Proposition 7.4. *The map* Tr_Z *is* $\mathbb{R}_{\geq 0}$*-linear and order preserving.*

Proof. The linearity is obvious by definition, while the latter property follows from the proof of Lemma 7.3. □

Theorem 7.5. *Let* T *be a closed positive* $(1,1)$*-current such that* $\operatorname{Tr}_Z(T - \nu(T,Z)[Z])$ *can be represented by a closed positive* $(1,1)$*-current in* $\{T - \nu(T,Z)[Z]\}|_Z$*. Take such a representative. Then*

$$\operatorname{Tr}_Z \mathbb{D}(T) = \mathbb{D}\left(\operatorname{Tr}_Z (T - \nu(T,Z)[Z])\right).$$

Proof. Replacing T by $T - \nu(T,Z)[Z]$, we may assume that $\nu(T,Z) = 0$. Then we need to show that

$$\operatorname{Tr}_Z \mathbb{D}(T) = \mathbb{D}\left(\operatorname{Tr}_Z T\right).$$

Here $\operatorname{Tr}_Z T$ is in $\{T\}|_Z$. After perturbing T by a Kähler form, we may assume that T is a Kähler current.

Let $(T_j)_j$ be a quasi-equisingular approximation of T in the same cohomology class of T. Then $T_j \xrightarrow{d_S} T$. Hence by Proposition 4.9, we have

$$\mathbb{D}(T_i) \to \mathbb{D}(T).$$

By definition and Proposition 4.9,

$$\operatorname{Tr}_Z \mathbb{D}(T) = \lim_{i\to\infty} \operatorname{Tr}_Z \mathbb{D}(T_i), \quad \mathbb{D}\left(\operatorname{Tr}_Z T\right) = \lim_{i\to\infty} \mathbb{D}(T_i|_Z).$$

Hence, it remains to show that

$$\operatorname{Tr}_Z \mathbb{D}(T_i) = \mathbb{D}(T_i|_Z). \tag{7.3}$$

Let $\pi\colon Y \to X$ be a modification so that

$$\pi^* T_i = [D] + R,$$

where D is an effective $\mathbb{Q}$-divisor and R is a closed positive $(1,1)$-current with locally bounded potential. We may assume that the strict transform Z_Y of Z is smooth. Then by definition, both sides of (7.3) are Cartier nef b-divisors realized by $(Z_Y, \{R\}|_{Z_Y})$. □

7.3. The higher codimension case.

Now assume that Z has codimension at least 2.

In this case, similar to the analytic theory, we cannot restrict a general nef b-divisor.

We consider the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "q"'] & \operatorname{Bl}_Z X \arrow[d, "p"] \\ Z \arrow[r, hook] & X, \end{tikzcd} \tag{7.4}$$

where $p\colon \operatorname{Bl}_Z X \to X$ is the blow-up of X along Z and E is the exceptional divisor. Note that $q\colon E \to Z$ can be naturally identified with the projectivized normal bundle of Z in X.

Let $\mathbb{D}$ be a nef b-divisor over X such that

$$\mathbb{D}_{\operatorname{Bl}_Z X} = p^* \mathbb{D}_X. \tag{7.5}$$

Assume (7.5), then the trace can be defined. To do so, we shall rely on the analytic theory.

Proposition 7.6. *Let $\mathbb{D}$ be a nef b-divisor over X satisfying (7.5). Then there is a unique nef b-divisor $\operatorname{Tr}_Z \mathbb{D}$ such that*

$$q^* \operatorname{Tr}_Z \mathbb{D} = \operatorname{Tr}_E \mathbb{D}, \tag{7.6}$$

where $\mathbb{D}$ is regarded as a nef b-divisor over $\mathrm{Bl}_Z X$ in the obvious way.

The pull-back q^* is defined in Section 6.

We first recall the following decomposition:

$$\mathrm{H}^{1,1}(E,\mathbb{R}) = \mathrm{H}^{1,1}(Z,\mathbb{R}) \oplus \mathbb{R}\zeta, \tag{7.7}$$

where ζ is the tautological class of the projective bundle q. See [RYY19, Proposition 3.3] for example. This decomposition also explains why we need to impose the condition (7.5).

Proof. Thanks to Proposition 6.3, (7.5) and (7.7), the root of $\operatorname{Tr}_Z \mathbb{D}$ is necessarily the first component of $(\operatorname{Tr}_E \mathbb{D})_E$ with respect to the decomposition (7.7). By Proposition 6.4, the nef b-divisor $\operatorname{Tr}_Z \mathbb{D}$ is unique if it exists.

Fix a Kähler form ω on X. It suffices to prove the existence of $\operatorname{Tr}_Z(\mathbb{D} + \epsilon\mathbb{D}(\omega))$ for any $\epsilon > 0$. In fact, if we have established these existence, then thanks to Proposition 6.4, we know that $\operatorname{Tr}_Z(\mathbb{D} + \epsilon\mathbb{D}(\omega))$ is increasing with respect to ϵ, hence defining

$$\operatorname{Tr}_Z \mathbb{D} \coloneqq \lim_{\epsilon\to 0+} \operatorname{Tr}_Z (\mathbb{D} + \epsilon\mathbb{D}(\omega))$$

would suffice.

Therefore, we may assume that there is a non-divisorial Kähler current T in $\mathbb{D}_X$ such that $\mathbb{D} = \mathbb{D}(T)$. Then (7.5) translates into $\nu(T, Z) = 0$. In particular, $\operatorname{Tr}_Z T$ is defined and can be represented by a Kähler current in $\{T\}|_Z$. We fix such a representative. We claim that in fact

$$\operatorname{Tr}_E \mathbb{D} = q^*\mathbb{D}(\operatorname{Tr}_Z T). \tag{7.8}$$

In fact, due to Proposition 6.1, we know that

$$q^*\mathbb{D}(\operatorname{Tr}_Z T) = \mathbb{D}(q^* \operatorname{Tr}_Z T).$$

Thanks to Theorem 7.5, (7.8) translates into

$$\mathbb{D}(q^* \operatorname{Tr}_Z T) = \mathbb{D}(\operatorname{Tr}_E(p^*T)). \tag{7.9}$$

Now Corollary 6.6 and Proposition 4.9 allow us to reduce to the case where T has analytic singularities, and (7.9) finally reduces to

$$\mathbb{D}\left(q^*(T|_Z)\right) = \mathbb{D}\left((p^*T)|_E\right),$$

which follows immediately from the commutativity of (7.4). $\square$

Definition 7.7. Let $\mathbb{D}$ be a nef b-divisor over X satisfying (7.5). Then $\operatorname{Tr}_Z \mathbb{D}$ is defined as the unique nef b-divisor over Z such that (7.6) holds.

One can easily deduce the basic properties of the trace $\operatorname{Tr}_Z \mathbb{D}$ from the analytic theory of trace operators. We omit these transparent translations.

The trace operator of b-divisors has a natural explanation in terms of non-Archimedean metrics, see [Xia25].

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