

HAUSDORFF CONVERGENCE PROPERTY OF PARTIAL OKOUNKOV BODIES

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1. INTRODUCTION

This note is a refinement of [Xia21, Theorem A]. We prove the Hausdorff convergence property in full generality.

This note is motivated by a discussion with Sébastien Boucksom.

2. HAUSDORFF CONVERGENCE PROPERTY

Let X be a connected smooth projective manifold of dimension n . Let (L, h) be a Hermitian pseudo-effective line bundle on X with L big. Fix $\nu : \mathbb{C}(X)^\times \rightarrow \mathbb{Z}^n$ a valuation of rank n and rational rank n . Take a smooth Hermitian metric h_0 on L and set $\theta = c_1(L, h_0)$. We can then identify h with $\varphi \in \text{PSH}(X, \theta)$.

For each $k \in \mathbb{Z}_{>0}$, we introduce

$$\Delta_\nu^k(\theta, \varphi) := \text{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the closed convex hull.

For later use, we introduce a twisted version as well. If T is a holomorphic line bundle on X , we introduce

$$\Delta_\nu^{k,T}(\theta, \varphi) := \text{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, T \otimes L^k \otimes \mathcal{I}(h^k)) \right\} \subseteq \mathbb{R}^n.$$

We also write

$$\Delta_\nu^{k,T}(L) := \text{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, T \otimes L^k) \right\} \subseteq \mathbb{R}^n$$

and

$$\Delta_\nu^k(L) := \text{Conv} \left\{ k^{-1} \nu(f) : f \in H^0(X, L^k) \right\} \subseteq \mathbb{R}^n$$

We write $\mathcal{I}_\infty(\varphi) = \mathcal{I}_\infty(h)$ for the ideal sheaf on X locally consisting of holomorphic functions f such that $|f|_h$ is locally bounded.

We first extend [Xia21, Theorem 3.13] to the twisted case.

Proposition 2.1. *For any holomorphic line bundle T on X ,*

$$\Delta_\nu^{k,T}(L) \rightarrow \Delta_\nu(L)$$

as $k \rightarrow \infty$.

Here and later on, we endow the space of convex bodies with the Hausdorff metric.

Proof. As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

- (1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0 ;
- (2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1 .

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$H^0(X, L^{k-k_0}) \xrightarrow{\times s_1} H^0(X, T \otimes L^k) \xrightarrow{\times s_0} H^0(X, T \otimes L^{k+k_0}).$$

It follows that

$$(k - k_0)\Delta_\nu^{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_\nu^{k,T}(L) \subseteq (k + k_0)\Delta_\nu^{k+k_0}(L) - \nu(s_0).$$

By [Xia21, Theorem 3.13], we conclude. \square

Lemma 2.2. *Let T be a holomorphic line bundle on X . Assume that φ has analytic singularities and θ_φ is a Kähler current, then*

$$\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi).$$

Proof. Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along a nc \mathbb{Q} -divisor D . Take $\epsilon \in (0, 1) \cap \mathbb{Q}$. In this case, by Ohsawa–Takegoshi theorem, for any $k \in \mathbb{Z}_{>0}$ we have

$$H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \subseteq H^0(X, T \otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{k,T}(\theta, \varphi))_k$, say the sequence defined by the indices k_1, k_2, \dots . We want to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

There exists $t \in \{0, 1, \dots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i , up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i . Write $k_i = Ng_i + t$.

Now we have

$$\Delta_\nu^{g_i, T \otimes L^t}(NL - ND) + N\nu(D) \subseteq N\Delta_\nu^{k_i, T}(\theta, \varphi) \subseteq \Delta_\nu^{g_i, T \otimes L^t}(NL - N(1-\epsilon)D) + N(1-\epsilon)\nu(D).$$

By Proposition 2.1,

$$\Delta_\nu(L - D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L - (1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Let $\epsilon \rightarrow 0+$, we find that

$$\Delta_\nu(L - D) + \nu(D) = \Delta'.$$

It follows from Blanschke selection theorem that

$$\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(L - D) + \nu(D) = \Delta_\nu(\theta, \varphi)$$

as $k \rightarrow \infty$. \square

Lemma 2.3. *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \rightarrow 0+$, we have*

$$\Delta_\nu((1-\beta)\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi).$$

Proof. By [Xia21, Proposition 5.15], we have

$$\Delta_\nu((1-\beta)\theta, \varphi) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta, \varphi).$$

In particular, if Δ' is a limit of a subsequence of $(\Delta_\nu((1-\beta)\theta, \varphi))_\beta$, then

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi).$$

But

$$\text{vol } \Delta' = \lim_{\beta \rightarrow 0+} \Delta_\nu((1-\beta)\theta, \varphi) = \lim_{\beta \rightarrow 0+} \int_X ((1-\beta)\theta + \text{dd}^c P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n.$$

We claim that

$$\lim_{\beta \rightarrow 0+} \int_X ((1-\beta)\theta + \text{dd}^c P^{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n.$$

Note that this finishes the proof as $\text{vol } \Delta_\nu(\theta, \varphi)$ is exactly equal to the right-hand side.

Next we prove our claim. We make use of the b-divisors introduced in [Xia20; Xia22]. By [Xia22, Theorem 0.6], the claim is equivalent to

$$\lim_{\beta \rightarrow 0+} \text{vol } \mathbb{D}((1-\beta)\theta, \varphi) = \text{vol } \mathbb{D}(\theta, \varphi).$$

This is a special case of [Xia22, Theorem 9.6] \square

Theorem 2.4. *Let T be a holomorphic line bundle on X . As $k \rightarrow \infty$, $\Delta_\nu^{k,T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$.*

Proof. Fix a Kähler form $\omega \geq \theta$ on X .

Step 1. We first handle the case where $\text{dd}^c h$ is a Kähler current, say $\text{dd}^c h \geq \beta_0 \omega$ for some $\beta_0 \in (0, 1)$.

Take a decreasing quasi-equisingular approximation φ_j of φ . Up to replacing β_0 by $\beta_0/2$, we may assume that

$$\theta_{\varphi_j} \geq \beta_0 \omega$$

for all $j \geq 1$.

Take $\beta \in (0, \beta_0) \cap \mathbb{Q}$. Write $\beta = p/q$ with $p, q \in \mathbb{Z}_{>0}$. Fix $t \in \{0, 1, \dots, q-1\}$.

By [DX21, Lemma 4.2], we can find $k_0 \in \mathbb{Z}_{>0}$ such that for all $k \geq k_0$, there is $v_{\beta,k} \in \text{PSH}(X, \theta)$ satisfying

(1)

$$P[\varphi]_{\mathcal{I}} \geq (1-\beta)\varphi_k + \beta v_{\beta,k};$$

(2) $v_{\beta,k}$ has positive mass.

Observe that for any $j \geq 1$,

$$\theta_{\varphi_j} \geq \beta \omega \geq \beta \theta.$$

Namely, $\varphi_j \in \text{PSH}(X, (1-\beta)\theta)$.

Fix $k \geq k_0$. Let $\pi : Y \rightarrow X$ be a log resolution of the singularities of φ_k . By the proof of [DX21, Proposition 4.3], there is $j_0 = j_0(\beta, k) \in \mathbb{Z}_{>0}$ such that for any $j \geq j_0$, we can find a non-zero section $s_j \in H^0(Y, \pi^* L^{pj} \otimes \mathcal{I}(jp\pi^* v_{\beta,k}))$ such that we get an injective linear map

$$H^0(Y, \pi^* T \otimes \pi^* L^t \otimes K_{Y/X} \otimes \pi^* L^{(q-p)j} \otimes \mathcal{I}(jq\pi^* \varphi_k)) \xrightarrow{\times s_j} H^0(X, T \otimes L^t \otimes L^{jq} \otimes \mathcal{I}(jq\varphi)).$$

It follows that

$$\Delta_\nu^{j, \pi^* T \otimes \pi^* L^t \otimes K_{Y/X}}((1-\beta)q\pi^*\theta, q\pi^*\varphi_k) + j^{-1}\nu(s_j) \subseteq q\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi).$$

We observe that $j^{-1}\nu(s_j)$ is bounded as the right-hand side is bounded when j varies.

Let Δ' be the limit of a subsequence of $(\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi))_j$, then by [Lemma 2.2](#), there is a vector $v'_k \in \mathbb{R}_{\geq 0}^n$ such that

$$\Delta_\nu((1-\beta)\pi^*\theta, \pi^*\varphi_k) + v'_k \subseteq \Delta'.$$

By the birational invariance of the partial Okounkov bodies,

$$\Delta_\nu((1-\beta)\theta, \varphi_k) + v'_k \subseteq \Delta'.$$

Let $k \rightarrow \infty$, by [\[Xia21, Theorem A\]](#),

$$\Delta_\nu((1-\beta)\theta, \varphi) + v' \subseteq \Delta'$$

for some vector $v' \in \mathbb{R}_{\geq 0}^n$ depending on β .

Let $\beta \rightarrow 0+$, by [Lemma 2.3](#), we have

$$\Delta_\nu(\theta, \varphi) + v'' \subseteq \Delta'$$

for some vector $v'' \in \mathbb{R}_{\geq 0}^n$.

On the other hand, take $j \geq 1$, as $\varphi \leq \varphi_j$,

$$\Delta_\nu^{k, T}(\theta, \varphi) \subseteq \Delta_\nu^{k, T}(\theta, \varphi_j).$$

By [Lemma 2.2](#),

$$\Delta' \subseteq \Delta_\nu(\theta, \varphi_j).$$

So

$$\Delta_\nu(\theta, \varphi) + v'' \subseteq \Delta_\nu(\theta, \varphi_j).$$

Let $j \rightarrow \infty$, we find that $v'' = 0$. Namely,

$$(2.1) \quad \Delta_\nu(\theta, \varphi) \subseteq \Delta'$$

Next we compute

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(\theta, \varphi_j) = \int_X \theta_{\varphi_j}^n.$$

Let $j \rightarrow \infty$, we find

$$\text{vol } \Delta' \leq \int_X \theta_{P[\varphi]_{\mathcal{I}}}^n = \text{vol } \Delta_\nu(\theta, \varphi).$$

It follows that equality holds in [\(2.1\)](#). Namely,

$$\Delta_\nu^{qj, T \otimes L^t}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$$

as $j \rightarrow \infty$. It follows that

$$(qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{qj+t} \otimes \mathcal{I}(qj\varphi))\} \rightarrow \Delta_\nu(\theta, \varphi)$$

as $j \rightarrow \infty$. Observe that

$$\begin{aligned} & (qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{qj+t} \otimes \mathcal{I}(qj\varphi))\} \supseteq \Delta_\nu^{qj+t, T}(\theta, \varphi) \\ & \supseteq (qj+t)^{-1} \text{Conv}\{\nu(s) : s \in H^0(X, T \otimes L^{-q} \otimes L^{q(j+1)+t} \otimes \mathcal{I}(q(j+1)\varphi))\}. \end{aligned}$$

It follows that $\Delta_\nu^{qj+t, T}(\theta, \varphi) \rightarrow \Delta_\nu(\theta, \varphi)$ as $j \rightarrow \infty$. As t is arbitrary, we conclude.

Step 2. Next we handle the general case.

Take $\psi \in \text{PSH}(X, \theta)$ such that

- (1) θ_ψ is a Kähler current;
- (2) $\psi \leq \varphi$.

The existence of ψ is proved in [DX21, Proposition 3.6].

Then for any $\epsilon \in \mathbb{Q} \cap (0, 1)$,

$$\Delta_\nu^{k,T}(\theta, \varphi) \supseteq \Delta_\nu^{k,T}(\theta, (1 - \epsilon)\varphi + \epsilon\psi)$$

for all k . It follows from Step 1 that for any limit Δ' of any subsequence of $\{\Delta_\nu^{k,T}(\theta, \varphi)\}_k$, we have

$$\Delta' \supseteq \Delta_\nu(\theta, (1 - \epsilon)\varphi + \epsilon\psi).$$

For later use, we denote the indices defining the subsequence as k_1, k_2, \dots

Letting $\epsilon \rightarrow 0$ and applying [Xia21, Theorem A], we have

$$\Delta' \supseteq \Delta_\nu(\theta, \varphi).$$

We claim that

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(\theta, \varphi).$$

From this claim, the theorem follows.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in H^0(X, H)$. Take $N \in \mathbb{Z}_{>0}$ then at least one element among $\{0, \dots, N-1\}$ occurs infinitely many times as the residues of k_i modulo N for various i . Up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ for all i , where $t \in \{0, \dots, N-1\}$. Up to changing T to $T \otimes L^{-t}$, we may assume that $t = 0$.

We have an injective linear map

$$H^0(X, T \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)) \xrightarrow{\times s^k} H^0(X, T \otimes H^k \otimes L^{kN} \otimes \mathcal{I}(kN\varphi)).$$

In particular, for each $i \geq 1$,

$$k_i \Delta_\nu^{k_i, T}(N\theta, N\varphi) + k_i \nu(s) \subseteq k_i \Delta_\nu^{k_i, T}(N\theta + \omega, N\varphi).$$

Let $i \rightarrow \infty$, by Step 1,

$$N\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + N\theta, N\varphi).$$

So

$$\text{vol } \Delta' \leq \text{vol } \Delta_\nu(N^{-1}\omega + \theta, \varphi) = \int_X (N^{-1}\omega + \theta + \text{dd}^c P^{N^{-1}\omega + \theta}[\varphi]_{\mathcal{I}})^n.$$

By [Xia21, Corollary 4.4], the right-hand side is equal to

$$\int_X (N^{-1}\omega + \theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n.$$

Let $N \rightarrow \infty$, we find

$$\text{vol } \Delta' \leq \int_X (\theta + \text{dd}^c P^\theta[\varphi]_{\mathcal{I}})^n = \text{vol } \Delta_\nu(\theta, \varphi).$$

Our claim holds. \square

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