THE SKODA LOCALIZATIONS OF $(1,1)$-CURRENTS

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Abstract. We prove that given two closed positive $(1,1)$-currents T and S on a complex manifold with T more singular than S, the Skoda localization of T to any complete pluripolar set dominates that of S.

This manuscript is not intended to be published. As pointed out by T. Darvas, the main theorem in this paper is already proved in [McC21] using a different method. Results in Section 4 still seem new. Conjecture 4.2 is still of interest.

Contents

1. Introduction

Let X be a complex manifold. Consider a closed positive $(1,1)$-current T on X and a complete pluripolar set $A \subseteq X$. The classical Skoda–El Mir extension theorem [Sko82; EM84; Sib85] implies that when A is closed, the current $\mathbb{1}_A T$ is also closed. The same result holds when A is not necessarily closed as established in [BEGZ10]. We call the closed positive $(1,1)$-current $\mathbb{1}_A T$ the *Skoda localization* of T along A.

In this paper, we initiate the study of the following problem:

Problem. *What kind of information of T do the Skoda localizations contain?*

To understand the situation, let us first restrict ourselves to the classical case where A is a prime divisor. In this case, it is well-known that

$$\mathbb{1}_A T = \nu(T, A)[A],$$

where $\nu(T, A)$ is the generic Lelong number of T along A, and $[A]$ is the current of integration along A. See [Dem12, Proposition 8.16] for a proof.

Next we consider slightly more generally the Skoda localizations to all prime divisors over X: Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism from a complex manifold Y and $A \subseteq Y$ be a prime divisor on Y. Then we find

$$\mathbb{1}_A \pi^* T = \nu(T, A)[A].$$

In [DX22], we introduced the $\mathcal{I}$-equivalence relation: If S is another closed positive $(1,1)$-current on X, we say $T \sim_{\mathcal{I}} S$ if for any prime divisor E over X, we have $\nu(T, E) = \nu(S, E)$. By [BFJ08], this is equivalent to $\mathcal{I}(\lambda T) = \mathcal{I}(\lambda S)$ for all real $\lambda > 0$, where $\mathcal{I}$ is Nadel's multiplier ideal sheaf.

We end up with the conclusion that the Skoda localizations to all prime divisors over X determine the $\mathcal{I}$-equivalence class of a current.

The above discussions motivate us to restate the problem in a more precise form:

Problem. *Let T, S be closed positive $(1,1)$-currents on X. Assume that for any proper bimeromorphic morphism $\pi \colon Y \to X$ from a complex manifold Y, and any complete pluripolar set $A \subseteq Y$, we have $\mathbb{1}_A T = \mathbb{1}_A S$, then what are the relations between T and S?*

In general, $\mathcal{I}$-equivalent currents may have different Skoda localizations to complete pluripolar sets. A concrete example can be found in [BBJ21, Example 6.10]. Therefore, the Skoda localizations contain strictly more information compared with the $\mathcal{I}$-equivalence class.

This paper is an first attempt to solve this problem. Our main theorem is the following:

Theorem 1.1 (Corollary 2.2)**.** *Let T, S be closed positive $(1,1)$-currents on X. Assume that $T \preceq S$. Let A be a complete pluripolar set on X. Then*

$$\mathbb{1}_A T \geq \mathbb{1}_A S.$$

Here $T \preceq S$ means that T is more singular than S. Namely, on an open set $U \subseteq X$ where $T = \mathrm{dd^c}\varphi$, $S = \mathrm{dd^c}\psi$ for some plurisubharmonic functions φ and ψ, we have $\varphi \leq \psi + C$ on any compact subset of U. In the compact Kähler setting, we can prove a stronger version Theorem 4.1.

As a special case,

Corollary 1.2. *Let T, S be closed positive $(1,1)$-currents on X. If T and S have the same singularity type, then $\mathbb{1}_A T = \mathbb{1}_A S$.*

This special case is surprising in the sense that *a priori* we do not have any precise control of the difference of T and S, since the only information at hand is that their local potentials have locally bounded difference.

Theorem 1.1 follows from a more or less standard computation if A is closed, since we can then find a well-controlled plurisubharmonic function u with A as the polar locus. For example, we may require that u be smooth outside A, see [Dem12, Lemma 2.2]. When A is not closed, the proof becomes substantially more difficult.

The proof of Corollary 1.2 is a standard application of integration by parts. We approximate $\mathbb{1}_A$ by some nice functions as in [BEGZ10] and make explicit computations afterwards.

By contrast the proof of Theorem 1.1 is much more involved. As pointed out by Bo Berndtsson, a crucial step in the proof of Corollary 1.2 fails in general. Our approach to Theorem 1.1 is by approximation. We find a sequence of currents S_j with the same singularity type as S approximating T. Then we need to show that the Skoda localizations are somewhat upper semicontinuous. One can easily reduce the general case to the 1-dimensional case. In Section 3, we prove some potential-theoretic results, eventually leading to the key theorem Theorem 3.3.

In Section 4, we give a precise conjecture about our original problem when X is compact Kähler and establish the 1-dimensional case.

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2. The Skoda localizations

Let X be a connected complex manifold of dimension n. Let Ω denote the unit disk in $\mathbb{C}$.

Given any closed positive $(1,1)$-current T on X and any complete pluripolar set $A \subseteq X$, we define the *Skoda localization* of T along A as the current $\mathbb{1}_A T$. Recall that by the Skoda–El Mir theorem proved in [BEGZ10, Remark 1.10], the current $\mathbb{1}_A T$ is also closed. It is obviously positive.

We first observe that if T is a smooth form, then $\mathbb{1}_A T = 0$ since A is a Lebesgue null set. We will use this result implicitly in the sequel.

Theorem 2.1. *Let T, S be closed positive $(1,1)$-currents on X. Assume that $T \sim S$. Let A be a complete pluripolar set on X. Then*

$$\mathbb{1}_A T = \mathbb{1}_A S.$$

Here $T \sim S$ means that T and S have the same singularity type. That is, $T \preceq S$ and $S \preceq T$. See the introduction part for the precise definition.

Proof. The problem is local, so we may assume that $X=\Omega^n$, $T=\mathrm{dd^c}\varphi$, $S=\mathrm{dd^c}\psi$, where $\varphi,\psi\in\mathrm{PSH}(\Omega^n)$ with
$$\varphi\leq\psi\leq\varphi+C.$$
After possibly shrinking Ω^n, we could find $u\in\mathrm{PSH}(\Omega^n)$, such that $A=\{u=-\infty\}$ and $u\leq 0$.

Let ω denote the standard Kähler form on Ω^n. Take $f\in C^\infty_c(\Omega^n)$ with $f\geq 0$. By symmetry, it suffices to show that
$$\int_{\Omega^n} f\cdot\mathbb{1}_A\left(\mathrm{dd^c}\varphi\wedge\omega^{n-1}-\mathrm{dd^c}\psi\wedge\omega^{n-1}\right)\geq 0. \tag{2.1}$$
Take a smooth strictly increasing convex function $\chi\colon\mathbb{R}\to\mathbb{R}$ so that $\chi(t)=0$ for $t\leq 1/2$ and $\chi(1)=1$. Let
$$\gamma_k=\chi\left(\mathrm{e}^{u/k}\right).$$
Let $\theta\in C^\infty([0,1])$ be a decreasing function such that

(1) $\theta=1$ on $[0,1/3]$;
(2) $\theta=0$ on $[2/3,1]$.

In particular, we can write $\mathbb{1}_A$ as a pointwise decreasing limit
$$\mathbb{1}_A=\lim_{k\to\infty}\theta\circ\gamma_k.$$
For the sequel, it is crucial to note that the convergence holds everywhere, not only almost everywhere.

Consider a *smooth* psh function γ on Ω^n taking value in $[0,1]$. Take $\eta\in C^\infty_c(\Omega^n)$ with $0\leq\eta\leq 1$ and $\eta|_{\mathrm{Supp}\,f}=1$. By integration by parts, we have
$$\begin{aligned}
&\int_{\Omega^n} f\cdot(\theta\circ\gamma)\left(\mathrm{dd^c}\varphi\wedge\omega^{n-1}-\mathrm{dd^c}\psi\wedge\omega^{n-1}\right)\\
=&\int_{\Omega^n}(\varphi-\psi)\mathrm{dd^c}\left(f\cdot(\theta\circ\gamma)\right)\wedge\omega^{n-1}\\
=&\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma)\mathrm{dd^c}f\wedge\omega^{n-1}+\int_{\Omega^n}f(\varphi-\psi)\theta''(\gamma)\mathrm{d}\gamma\wedge\mathrm{d^c}\gamma\wedge\omega^{n-1}\\
&+\int_{\Omega^n}f(\varphi-\psi)\theta'(\gamma)\mathrm{dd^c}\gamma\wedge\omega^{n-1}\\
&+\int_{\Omega^n}(\varphi-\psi)\theta'(\gamma)\mathrm{d}f\wedge\mathrm{d^c}\gamma\wedge\omega^{n-1}+\int_{\Omega^n}(\varphi-\psi)\theta'(\gamma)\mathrm{d}\gamma\wedge\mathrm{d^c}f\wedge\omega^{n-1}\\
\geq&\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma)\mathrm{dd^c}f\wedge\omega^{n-1}+D\int_{\Omega^n}f(\varphi-\psi)\mathrm{d}\gamma\wedge\mathrm{d^c}\gamma\wedge\omega^{n-1}\\
&-B\left(\int_{\Omega^n}|\varphi-\psi|\mathrm{d}f\wedge\mathrm{d^c}f\wedge\omega^{n-1}\right)^{1/2}\left(\int_{\Omega^n}\eta|\varphi-\psi|\mathrm{d}\gamma\wedge\mathrm{d^c}\gamma\wedge\omega^{n-1}\right)^{1/2},
\end{aligned}$$
where $D=\sup_{[0,1]}\theta''$, $B=2\sup_{[0,1]}(-\theta')$.

After slightly shrinking Ω^n, we can take a decreasing sequence of smooth psh functions γ_k^j taking value in $[0,1]$ converging to γ_k. For example, we could use the standard Friedrichs mollifier technique. Then
$$\begin{aligned}
\lim_{j\to\infty}\int_{\Omega^n}f(\theta\circ\gamma_k^j)\mathrm{dd^c}\varphi\wedge\omega^{n-1}&=\int_{\Omega^n}f(\theta\circ\gamma_k)\mathrm{dd^c}\varphi\wedge\omega^{n-1},\\
\lim_{j\to\infty}\int_{\Omega^n}f(\theta\circ\gamma_k^j)\mathrm{dd^c}\psi\wedge\omega^{n-1}&=\int_{\Omega^n}f(\theta\circ\gamma_k)\mathrm{dd^c}\psi\wedge\omega^{n-1},\\
\lim_{j\to\infty}\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma_j^k)\mathrm{dd^c}f\wedge\omega^{n-1}&=\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma_j)\mathrm{dd^c}f\wedge\omega^{n-1}
\end{aligned}$$
by the dominated convergence theorem. While
$$\begin{aligned}
\lim_{j\to\infty}\int_{\Omega^n}f(\varphi-\psi)\mathrm{d}\gamma_k^j\wedge\mathrm{d^c}\gamma_k^j\wedge\omega^{n-1}&=\int_{\Omega^n}f(\varphi-\psi)\mathrm{d}\gamma_k\wedge\mathrm{d^c}\gamma_k\wedge\omega^{n-1},\\
\lim_{j\to\infty}\int_{\Omega^n}\eta(\varphi-\psi)\mathrm{d}\gamma_k^j\wedge\mathrm{d^c}\gamma_k^j\wedge\omega^{n-1}&=\int_{\Omega^n}\eta(\varphi-\psi)\mathrm{d}\gamma_k\wedge\mathrm{d^c}\gamma_k\wedge\omega^{n-1}
\end{aligned}$$

by [BT87, Theorem 3.2].

It follows that

$$\begin{aligned}
&\int_{\Omega^n} f\cdot(\theta\circ\gamma_k)\left(\mathrm{dd^c}\varphi\wedge\omega^{n-1}-\mathrm{dd^c}\psi\wedge\omega^{n-1}\right)\\
\geq&\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma_k)\mathrm{dd^c}f\wedge\omega^{n-1}+D\int_{\Omega^n}f(\varphi-\psi)\mathrm{d}\gamma_k\wedge\mathrm{d^c}\gamma_k\wedge\omega^{n-1}\\
&-B\left(\int_{\Omega^n}|\varphi-\psi|\mathrm{d}f\wedge\mathrm{d^c}f\wedge\omega^{n-1}\right)^{1/2}\left(\int_{\Omega^n}\eta|\varphi-\psi|\mathrm{d}\gamma_k\wedge\mathrm{d^c}\gamma_k\wedge\omega^{n-1}\right)^{1/2},
\end{aligned}$$

Now

$$\begin{aligned}
\lim_{k\to\infty}\int_{\Omega^n}(\varphi-\psi)(\theta\circ\gamma_k)\mathrm{dd^c}f\wedge\omega^{n-1}=&0,\\
\lim_{k\to\infty}\int_{\Omega^n}f(\theta\circ\gamma_k)\left(\mathrm{dd^c}\varphi\wedge\omega^{n-1}-\mathrm{dd^c}\psi\wedge\omega^{n-1}\right)=&\int_A f\left(\mathrm{dd^c}\varphi\wedge\omega^{n-1}-\mathrm{dd^c}\psi\wedge\omega^{n-1}\right)
\end{aligned}$$

by the dominated convergence theorem, while

$$\lim_{k\to\infty}\int_{\Omega^n}\eta|\varphi-\psi|\mathrm{d}\gamma_k\wedge\mathrm{d^c}\gamma_k\wedge\omega^{n-1}=0$$

by [BT87, Corollary 3.3]. So (2.1) follows. □

The inequality version of Theorem 2.1 is much more difficult. The proof below depends on some non-trivial potential theory developed in Section 3.

Corollary 2.2. *Let T,S be closed positive $(1,1)$-currents on X. Assume that $T\preceq S$. Let A be a complete pluripolar set on X. Then*

$$\mathbb{1}_A T\geq\mathbb{1}_A S.$$

Proof. The problem is local, so we may assume that $X=\Omega^n$, $T=\mathrm{dd^c}\varphi$, $S=\mathrm{dd^c}\psi$, where $\varphi,\psi\in\mathrm{PSH}(\Omega^n)$ with

$$\varphi\leq\psi.$$

From Theorem 2.1, we know that for any $C>0$,

$$\mathbb{1}_A\mathrm{dd^c}\left(\varphi\vee(\psi-C)\right)=\mathbb{1}_A\mathrm{dd^c}\psi.$$

Fix $f\in C_c^\infty(\Omega^n)$ and $f\geq 0$, we want to show (2.1), or equivalently

$$\int_{\Omega^n}f\cdot\mathbb{1}_A\mathrm{dd^c}\varphi\wedge\omega^{n-1}\geq\lim_{C\to\infty}\int_{\Omega^n}f\cdot\mathbb{1}_A\mathrm{dd^c}\left(\varphi\vee(\psi-C)\right)\wedge\omega^{n-1}.$$

By Fubini's theorem, this reduces immediately to the 1-dimensional case. Then the result follows from Theorem 3.3. □

Theorem 2.3. *Let $\varphi\in\mathrm{PSH}(X)$. Then*

$$\langle\mathrm{dd^c}\varphi\rangle=\mathbb{1}_{\{\varphi>-\infty\}}\mathrm{dd^c}\varphi.$$

Here $\langle\mathrm{dd^c}\varphi\rangle$ denotes the non-pluripolar product in the sense of Bedford–Taylor [BT87; BEGZ10]. This theorem is stated in [BEGZ10] right after Definition 1.1 without proof. As pointed out by David Witt Nyström, the statement is not obvious unless the set $\{\varphi=-\infty\}$ is closed.

Proof. The problem is local, we may assume that $X=\Omega^n$. Let $f\in C_c^\infty(\Omega^n)$, $f\geq 0$. We need to show the following:

$$\lim_{k\to\infty}\int_{\Omega^n}f\mathbb{1}_{\{\varphi>-k\}}\mathrm{dd^c}\left(\varphi\vee(-k)\right)\wedge\omega^{n-1}=\int_{\Omega^n}\mathbb{1}_{\{\varphi>-\infty\}}f\mathrm{dd^c}\varphi\wedge\omega^{n-1},$$

where ω is the standard Kähler form on $\mathbb{C}^n$. By the dominated convergence theorem, this reduces to the 1-dimension case:

$$\lim_{k\to\infty}\int_\Omega f\mathbb{1}_{\{\varphi>-k\}}\Delta\varphi_k=\int_\Omega\mathbb{1}_{\{\varphi>-\infty\}}f\Delta\varphi,$$

where $\varphi_k=\varphi\vee(-k)$, which is a special case of [EK24, Proposition 3.3]. □

Remark 2.4. There are no circular arguments here. The proof of [EK24, Proposition 3.3] relies on an earlier result of the same author [EK23]. Both papers are independent of the results in [BEGZ10] and they are independent of the wrong statement [BT87, Proposition 4.4].

In particular, any closed positive $(1,1)$-current T on X admits a canonical decomposition:

$$T = \langle T\rangle + \mathbb{1}_{\mathrm{Pol}\,T} T,$$

where $\mathrm{Pol}\,T \subseteq X$ is the polar locus of T. Namely, on an open set U where $T|_U = \mathrm{dd^c}\varphi$ for some plurisubharmonic function φ, we have $\mathrm{Pol}\,T \cap U = \{\varphi = -\infty\}$.

In particular, for any complete pluripolar set $A \subseteq X$, we have

$$\mathbb{1}_A T = \mathbb{1}_{A\cap \mathrm{Pol}\,T} T.$$

3. Some potential theory

We omit the standard Lebesgue measure in the integrals in this section to save space.

As a technical comment, for a general subharmonic function v in real dimension 2, we know that v lies in the Sobolev space $W^{1,p}_{\mathrm{loc}}$ for any $1 \le p < 2$. See [GZ17, Theorem 1.48]. When v is bounded, it lies in $W^{1,2}_{\mathrm{loc}}$ (and hence has locally bounded mean oscillation) due to the Chern–Levine–Nirenberg estimate, see [GZ17, Theorem 3.9]. Even when v is bounded, it does not necessarily lie in any $W^{1,p}_{\mathrm{loc}}$ for $p > 2$, as by Morrey's inequality, any such v is necessarily Hölder continuous. We shall use these facts without further comments in the sequel.

Let Ω denote the unit disk in $\mathbb{C}$.

Lemma 3.1. *Let φ_j be a decreasing sequence of subharmonic functions on Ω with limit $\varphi \not\equiv -\infty$. Then for any bounded subharmonic function ψ on Ω, we have*

$$\psi\Delta\varphi_j \xrightarrow{D} \psi\Delta\varphi.$$

Here $\xrightarrow{D}$ denotes the convergence as currents. In our setting, it is weaker than the weak convergence of measures.

Proof. We may assume that $\psi \le 0$.

Fix $f \in C^\infty_c(\Omega)$, $f \ge 0$. We need to show that

$$\lim_{j\to\infty} \int_\Omega f\psi\Delta\varphi_j = \int_\Omega f\psi\Delta\varphi.$$

We claim that we can do integration by parts on φ, namely,

$$\int_\Omega f\psi\Delta\varphi = \int_\Omega \varphi f\Delta\psi + \int_\Omega \varphi\psi\Delta f + \int_\Omega \varphi\nabla f\cdot\nabla\psi. \tag{3.1}$$

This is obvious if ψ is smooth. In general, after slightly shrinking Ω, we approximate ψ by a decreasing sequence of smooth subharmonic functions $\psi^j \le 0$.

Then

$$\lim_{j\to\infty} \int_\Omega f\psi^j\Delta\varphi = \int_\Omega f\psi\Delta\varphi$$

by the monotone convergence theorem,

$$\lim_{j\to\infty} \int_\Omega \varphi\psi^j\Delta f = \int_\Omega \varphi\psi\Delta f$$

by the dominated convergence theorem, while

$$\lim_{j\to\infty} \int_\Omega \varphi f\Delta\psi^j = \int_\Omega \varphi f\Delta\psi$$

by [GZ17, Theorem 4.29]. Finally, since $\nabla\psi^j \to \nabla\psi$ in $L^{3/2}_{\mathrm{loc}}$ (see [GZ17, Theorem 1.48]) and $\varphi \in L^3_{\mathrm{loc}}$, we conclude that

$$\lim_{j\to\infty} \int_\Omega \varphi\nabla f\cdot\nabla\psi^j = \int_\Omega \varphi\nabla f\cdot\nabla\psi.$$

In particular, (3.1) is justified.

Similarly, for any j, we have

$$\int_\Omega f\psi\Delta\varphi_j = \int_\Omega \varphi_j f\Delta\psi + \int_\Omega \varphi_j\psi\Delta f + \int_\Omega \varphi_j \nabla f\cdot\nabla\psi.$$

Observe that

$$\lim_{j\to\infty}\int_\Omega \varphi_j f\Delta\psi = \int_\Omega \varphi f\Delta\psi$$

by the monotone convergence theorem,

$$\lim_{j\to\infty}\int_\Omega \varphi_j \psi\Delta f = \int_\Omega \varphi\psi\Delta f$$

by the dominated convergence theorem, while

$$\lim_{j\to\infty}\int_\Omega \varphi_j \nabla f\cdot\nabla\psi. = \int_\Omega \varphi\nabla f\cdot\nabla\psi$$

since $\nabla\psi\in L^{3/2}_{\mathrm{loc}}$ and $\varphi_j\to\varphi$ in L^3_{loc}. □

Next we establish the key integration by parts formula.

Corollary 3.2. *Let $\theta\in C^\infty([0,1])$, γ be a subharmonic function on Ω with value in $[0,1]$, φ be a subharmonic function on Ω. Then for any $f\in C^\infty_c(\Omega)$, $f\geq 0$, we have*

$$\int_\Omega f(\theta\circ\gamma)\Delta\varphi = \int_\Omega \varphi\theta(\gamma)\Delta f + \int_\Omega \varphi f\theta''(\gamma)|\nabla\gamma|^2 + \int_\Omega \varphi\theta'(\gamma)\nabla f\cdot\nabla\gamma + \int_\Omega \varphi f\theta'(\gamma)\Delta\gamma. \tag{3.2}$$

Note that all terms in (3.2) are well-defined and finite. The finiteness of the second and the fourth term on the right-hand side follows from Chern–Levine–Nirenberg estimate.

Proof. Adding a constant to θ, we may assume that $\theta\geq 0$ on $[0,1]$.

We first assume that φ is bounded. In this case, if we assume further that γ is smooth, then (3.2) is obvious. After possibly shrinking Ω, we can take a decreasing sequence of smooth subharmonic functions γ^j taking value in $[0,1]$ converging to γ. Then we know that

$$\int_\Omega f(\theta\circ\gamma^j)\Delta\varphi = \int_\Omega \varphi\theta(\gamma^j)\Delta f + \int_\Omega \varphi f\theta''(\gamma^j)|\nabla\gamma^j|^2 + \int_\Omega \varphi\theta'(\gamma^j)\nabla f\cdot\nabla\gamma^j + \int_\Omega \varphi f\theta'(\gamma^j)\Delta\gamma^j.$$

Note that

$$\lim_{j\to\infty}\int_\Omega f(\theta\circ\gamma^j)\Delta\varphi = \int_\Omega f(\theta\circ\gamma)\Delta\varphi$$

by the bounded convergence theorem,

$$\lim_{j\to\infty}\int_\Omega \varphi\theta(\gamma^j)\Delta f = \int_\Omega \varphi\theta(\gamma)\Delta f$$

by the dominated convergence theorem. We claim that

$$\lim_{j\to\infty}\int_\Omega \varphi f\theta''(\gamma^j)|\nabla\gamma^j|^2 = \int_\Omega \varphi f\theta''(\gamma)|\nabla\gamma|^2. \tag{3.3}$$

Since $\nabla\gamma^j\to\nabla\gamma$ in L^1_{loc}, we may assume that $\nabla\gamma^j\to\nabla\gamma$ almost everywhere. Take a constant $B>0$ so that $-B\leq\theta''\leq B$ on $[0,1]$. By Fatou's lemma,

$$\int_\Omega \varliminf_{j\to\infty}\varphi f(\theta''(\gamma^j)-B)|\nabla\gamma^j|^2 \leq \varliminf_{j\to\infty}\int_\Omega \varphi f(\theta''(\gamma^j)-B)|\nabla\gamma^j|^2.$$

In other words,

$$\int_\Omega \varphi f(\theta''(\gamma)-B)|\nabla\gamma|^2 \leq \varliminf_{j\to\infty}\int_\Omega \varphi f(\theta''(\gamma^j)-B)|\nabla\gamma^j|^2.$$

Since

$$\lim_{j\to\infty}\int_\Omega \varphi f|\nabla\gamma^j|^2 = \int_\Omega \varphi f|\nabla\gamma|^2$$

by [BT87, Theorem 3.2](Here we used the fact that φ is bounded.), we conclude that

$$\int_\Omega \varphi f\theta''(\gamma)|\nabla\gamma|^2 \leq \varliminf_{j\to\infty}\int_\Omega \varphi f\theta''(\gamma^j)|\nabla\gamma^j|^2.$$

Similarly, by Fatou's lemma,

$$\int_\Omega \varliminf_{j\to\infty}(-\varphi)f(\theta''(\gamma^j)+B)|\nabla\gamma^j|^2 \le \varliminf_{j\to\infty}\int_\Omega(-\varphi)f(\theta''(\gamma^j)+B)|\nabla\gamma^j|^2.$$

So

$$\int_\Omega \varphi f\theta''(\gamma)|\nabla\gamma|^2 \ge \varlimsup_{j\to\infty}\int_\Omega \varphi f\theta''(\gamma^j)|\nabla\gamma^j|^2.$$

Therefore, (3.3) follows.

Since θ' is Lipschitz, we know that $\theta'(\gamma^j)\to\theta'(\gamma)$ in L^6_{loc}. Since $\nabla\gamma^j\to\nabla\gamma$ in $L^{3/2}_{\mathrm{loc}}$, we conclude that

$$\lim_{j\to\infty}\int_\Omega \varphi\theta'(\gamma^j)\nabla f\cdot\nabla\gamma^j = \int_\Omega \varphi\theta'(\gamma)\nabla f\cdot\nabla\gamma.$$

Finally,

$$\lim_{j\to\infty}\int_\Omega \varphi f\theta'(\gamma^j)\Delta\gamma^j = \int_\Omega \varphi f\theta'(\gamma)\Delta\gamma$$

by [BT87, Theorem 3.2]. In particular, (3.2) follows if φ is bounded.

Next we handle the general case. First note that θ could be written as $\theta_1-\theta_2$, where θ_1 and θ_2 are both smooth increasing convex functions. For example, θ_2 can be taken as $x\mapsto C\exp(x)$ for some large C.

By linearity it suffices to handle θ_1 and θ_2 separately. We may assume that θ is non-negative smooth increasing and convex. In particular, $\theta\circ\gamma$ is subharmonic. By Lemma 3.1, we know that

$$\lim_{j\to\infty}\int_\Omega f(\theta\circ\gamma)\Delta\varphi_j = \int_\Omega f(\theta\circ\gamma)\Delta\varphi,$$

where $\varphi_j=\varphi\vee(-j)$. On the other hand, by the special case that we have established, we have

$$\int_\Omega f(\theta\circ\gamma)\Delta\varphi_j = \int_\Omega \varphi_j\theta(\gamma)\Delta f + \int_\Omega \varphi_j f\theta''(\gamma)|\nabla\gamma|^2 + \int_\Omega \varphi_j\theta'(\gamma)\nabla f\cdot\nabla\gamma + \int_\Omega \varphi_j f\theta'(\gamma)\Delta\gamma.$$

We have

$$\lim_{j\to\infty}\int_\Omega \varphi_j\theta(\gamma)\Delta f = \int_\Omega \varphi\theta(\gamma)\Delta f,$$
$$\lim_{j\to\infty}\int_\Omega \varphi_j f\theta''(\gamma)|\nabla\gamma|^2 = \int_\Omega \varphi f\theta''(\gamma)|\nabla\gamma|^2,$$
$$\lim_{j\to\infty}\int_\Omega \varphi_j f\theta'(\gamma)\Delta\gamma = \int_\Omega \varphi f\theta'(\gamma)\Delta\gamma$$

by the dominated convergence theorem, while

$$\lim_{j\to\infty}\int_\Omega \varphi_j\theta'(\gamma)\nabla f\cdot\nabla\gamma = \int_\Omega \varphi\theta'(\gamma)\nabla f\cdot\nabla\gamma$$

since $\nabla\gamma\in L^{3/2}_{\mathrm{loc}}$ and $\varphi_j\to\varphi$ in L^3_{loc}. Therefore, (3.2) follows. $\square$

Theorem 3.3. *Let φ_j be a decreasing sequence of subharmonic functions on Ω with limit $\varphi\not\equiv-\infty$, then for any complete polar set $A\subseteq\Omega$, and any $f\in C^\infty_c(\Omega)$, $f\ge 0$ we have*

$$\varlimsup_{j\to\infty}\int_A f\Delta\varphi_j \le \int_A f\Delta\varphi. \tag{3.4}$$

This result is obvious if A is closed since $f\Delta\varphi_j$ converges weakly to $f\Delta\varphi$ as measures. The whole section is devoted to establish the case where A is not necessarily closed.

Proof. We define the functions γ_k and θ as in the proof of Theorem 2.1. Then

$$\mathbb{1}_A = \lim_{k\to\infty}\theta\circ\gamma_k.$$

Note that $\mathbb{1}_A\le\theta\circ\gamma_k$ for any k. Fix k. Then

$$\int_A f\Delta\varphi_j \le \int_\Omega f(\theta\circ\gamma_k)\Delta\varphi_j.$$

Applying Corollary 3.2, we have

$$\int_A f\Delta\varphi_j \leq \int_\Omega \varphi_j\theta(\gamma_k)\Delta f+\int_\Omega \varphi_j f\theta''(\gamma_k)|\nabla\gamma_k|^2+\int_\Omega \varphi_j\theta'(\gamma_k)\nabla f\cdot\nabla\gamma_k+\int_\Omega \varphi_j f\theta'(\gamma_k)\Delta\gamma_k. \tag{3.5}$$

Letting $j\to\infty$, as in the proof of Corollary 3.2, we conclude that

$$\varlimsup_{j\to\infty}\int_A f\Delta\varphi_j \leq \int_\Omega \varphi\theta(\gamma_k)\Delta f+\int_\Omega \varphi f\theta''(\gamma_k)|\nabla\gamma_k|^2+\int_\Omega \varphi\theta'(\gamma_k)\nabla f\cdot\nabla\gamma_k+\int_\Omega \varphi f\theta'(\gamma_k)\Delta\gamma_k.$$

Applying Corollary 3.2 once more, we find that

$$\varlimsup_{j\to\infty}\int_A f\Delta\varphi_j \leq \int_X f(\theta\circ\gamma_k)\Delta\varphi.$$

Letting $k\to\infty$, by the dominated convergence theorem, we conclude (3.4). □

With an almost identical proof, we also find:

Theorem 3.4. *Let φ_j be an increasing sequence of locally bounded from above subharmonic functions on Ω. Let $\varphi=\sup^*\varphi_j$. Then for any complete polar set $A\subseteq\Omega$, and any $f\in C_c^\infty(\Omega)$, $f\geq 0$, we have*

$$\varlimsup_{j\to\infty}\int_A f\Delta\varphi_j \leq \int_A f\Delta\varphi. \tag{3.6}$$

4. The global setting

Let X be a connected compact Kähler manifold of dimension n.

We briefly recall the $\preceq_P$ relation introduced in [Xia]. Let φ,ψ be quasi-plurisubharmonic functions on X. We say $\varphi\preceq_P\psi$ if for any Kähler form ω on X so that φ,ψ become ω-psh functions with positive non-pluripolar masses, we have

$$P_\omega[\varphi]\leq P_\omega[\psi].$$

Here

$$P_\omega[\varphi]=\sup_{C\in\mathbb{R}}{}^*(\varphi+C)\wedge 0, \tag{4.1}$$

where $(\varphi+C)\wedge 0$ is the largest ω-psh function lying below both $\varphi+C$ and 0. It is shown in [Xia, Lemma 6.1.1] that the condition $\varphi\preceq_P\psi$ is independent of the choice of ω. In particular, $\preceq_P$ is a (non-strict) partial order.

Similarly, given closed positive $(1,1)$-currents T and S on X, we say $T\preceq_P S$ if when we write $T=\theta_\varphi$, $S=\theta'_{\varphi'}$, we have $\varphi\preceq\varphi'$. Here θ,θ' are closed smooth real $(1,1)$-forms on X and φ is θ-psh, φ' is θ'-psh. The notation θ_φ means $\theta+\mathrm{dd^c}\varphi$. This definition is independent of the choices of $\theta,\theta',\varphi,\varphi'$.

Theorem 4.1. *Let T, S be closed positive $(1,1)$-currents on X and $A\subseteq X$ be a complete pluripolar set. Assume that $T\preceq_P S$, then*

$$\mathbb{1}_A T\geq \mathbb{1}_A S.$$

Proof. We may assume that T,S lie in the same Kähler class. Take a Kähler form ω and ω-psh functions φ,ψ such that $T=\omega_\varphi$ and $S=\omega_\psi$. We may assume that T and S have positive non-pluripolar masses. In this case, $T\preceq_P S$ means

$$P_\omega[\varphi]\leq P_\omega[\psi].$$

In view of Corollary 2.2, it suffices to show that

$$\mathbb{1}_A\omega_\varphi\leq \mathbb{1}_A\omega_{P_\omega[\varphi]}.$$

Recall that $P_\omega[\varphi]$ is defined in (4.1). Observe that $(\varphi+C)\wedge 0\sim\varphi$ for any $C\in\mathbb{R}$. It follows from Theorem 2.1 that

$$\mathbb{1}_A\left(\omega+\mathrm{dd^c}\left((\varphi+C)\wedge 0\right)\right)=\mathbb{1}_A\omega_\varphi.$$

So our assertion follows from Theorem 3.4. □

We conjecture that the converse holds as well.

Conjecture 4.2. *Let T, S be two closed positive $(1,1)$-currents on X. Then the following are equivalent:*

(1) $T \preceq_P S$*;*
(2) for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold Y and any complete pluripolar set $A \subseteq Y$, we have

$$\mathbb{1}_A \pi^* T \geq \mathbb{1}_A \pi^* S.$$

Note that (1) implies (2) is a consequence of Corollary 2.2 and the obvious bimeromorphic invariance of the P-partial order.

The converse holds in dimension 1. In fact, assume that $n = 1$ and (2) holds. In this case, the morphism π is necessarily an isomorphism. Therefore, Condition (2) can be equivalently reformulated as follows: For any complete pluripolar set $A \subseteq X$, we have

$$\mathbb{1}_A T \geq \mathbb{1}_A S. \tag{4.2}$$

Then we know that

$$T \geq \mathbb{1}_{\operatorname{Pol} S} T \geq \mathbb{1}_{\operatorname{Pol} S} S.$$

In particular, $T' = T - \mathbb{1}_{\operatorname{Pol} S} S$ is also a closed positive $(1,1)$-current. Define $S' = S - \mathbb{1}_{\operatorname{Pol} S} S$. It follows from Theorem 2.3 that $S' = \langle S \rangle$. The condition (4.2) is also satisfied with T' and S' in place of T and S. It suffices to show that $T' \preceq_P S'$, since then Condition (1) follows from [Xia, Proposition 6.1.5].

Now we have reduced to the case where $S = \langle S \rangle$. We may assume that S represents a Kähler cohomology class. Take a Kähler form ω cohomologous to S and an ω-subharmonic function φ so that $S = \omega_\varphi$. Now the condition $S = \langle S \rangle$ means that φ is in the full mass class $\mathcal{E}(X, \omega)$. See [BEGZ10, Definition 1.21]. In particular, $\varphi \sim_P 0$. See [Xia, Proposition 3.1.11] for example. Therefore, Condition (1) follows.

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