Lectures on Vertex Operator Algebras and Conformal Blocks

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Subsections marked with $\star$ can be skipped on first reading.

Notations

- $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_+ = \{1, 2, \dots\}$.
- $\mathbf{i} = \sqrt{-1}$, $\mathbb{S}^1$ =unit circle, $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.
- $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$, $\mathbb{D}_r^{\mathrm{cl}} = \{z \in \mathbb{C} : |z| \leqslant r\}$
- $\mathscr{O}(X)$ (resp. $\mathscr{O}_X$) is the space (resp. sheaf) of holomorphic functions on a complex manifold X. $\mathscr{O}_{X,x}$ is the stalk of $\mathscr{O}_X$ at x.
- Configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}$.
- z and ζ could mean either points, or the standard coordinate of $\mathbb{C}$, or formal variables. We will give their meanings when the context is unclear.
- All vector spaces are over $\mathbb{C}$, unless otherwise stated. If W is a vector space equipped with a Hermitian form $\langle\cdot|\cdot\rangle$, we let $|\cdot\rangle$ be the linear variable and $\langle\cdot|$ be the antilinear (i.e. conjugate linear) one, following physicists' convention.
- If W, W' are vector spaces, then $\mathrm{Hom}(W, W')$ denote the space of linear operators from W to W'. We let $\mathrm{End}(W) = \mathrm{Hom}(W, W)$.
- We use symbols $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$ to denote bilinear forms (i.e., linear on both variables).
- Given a vector space W and a formal variable z,

$$W[z] = \{\text{polynomials of } z \text{ whose coefficients are elements of } W\}$$
$$W[[z]] = \Big\{\sum_{n\in\mathbb{N}} w_n z^n : w_n \in W\Big\}$$
$$W((z)) = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W, \text{ and } w_n = 0 \text{ when } n \text{ is sufficiently negative}\Big\}$$
$$W[[z^{\pm1}]] = \Big\{\sum_{n\in\mathbb{Z}} w_n z^n : w_n \in W\Big\}.$$

  Each line is a subspace of the subsequent line. In case there are several formal variables, the spaces are defined in a similar way, expect $W((\cdots))$. For instance,

$$W[[z, \zeta^{\pm1}]] := W[[z]][[\zeta^{\pm1}]] = W[[\zeta^{\pm1}]][[z]]$$

  consists of $\sum_{m\in\mathbb{N}, n\in\mathbb{Z}} w_{m,n} z^m \zeta^n$ where each $w_{m,n} \in W$. However, note that $W((z))((\zeta))$ and $W((\zeta))((z))$ are not equal. (For instance, $\sum_{m\geqslant -n}\sum_{n\geqslant -1} z^m\zeta^n$ belongs to $\mathbb{C}((z))((\zeta))$ but not $\mathbb{C}((\zeta))((z))$.)

  Elements in $W[[z^{\pm1}]]$ are called **formal Laurent series** of z.

- We let

$$W((z_1, \dots, z_N)) = \Big\{ \sum_{n_1,\dots,n_N \geqslant L} w_{n_1,\dots,n_N} z_1^{n_1} \cdots z_N^{n_N} \text{ for some } L \in \mathbb{Z} \Big\}.$$

  Then $W((z_1, z_2))$ is a proper subspace of both $W((z_1))((z_2))$ and $W((z_2))((z_1))$.

- We set

$$\mathrm{Res}_{z=0} \sum_{n \in \mathbb{Z}} w_n z^n dz = w_{-1}. \tag{0.2}$$

  This is in line with the complex analytic residue.

- A vector of $W_1 \otimes \cdots \otimes W_N$ writen as $w_\bullet$ means that it is of the form $w_1 \otimes \cdots \otimes w_N$ where each $w_i \in W_i$. Depending on the context, $w_\bullet$ will also mean a tuple $(w_1, \dots, w_N)$. Similarly, $W_\bullet$ may mean $W_1 \otimes \cdots \otimes W_N$ or $(W_1, \dots, W_N)$ depending on the context.

- Unless otherwise stated, by a manifold, we mean one *without* boundaries. Also, ”with boundaries” means ”possibly with boundaries”.

1 Segal's picture of 2d CFT; motivations of VOAs and conformal blocks

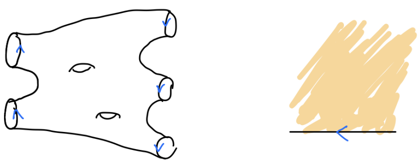
1.1

Vertex operator algebras (VOAs) are mathematical objects defined to understand and construct 2-dimensional conformal field theory (CFT for short). A CFT describes propagations and interactions of strings. The are two types of strings: closed strings $\simeq \mathbb{S}^1$ and open strings $\simeq [0,1]$. In this course, we will mainly focus on closed strings.

Let me explain how mathematicians understand CFT. Just like any quantum field theory (QFT), in CFT we must have a Hilbert space $\mathcal{H}$. The vectors in $\mathcal{H}$ are called "states", but unlike ordinary QFT, a vector $\xi \in \mathcal{H}$ is not a state of a particle, but a state of a closed string $\mathbb{S}^1$.

The most important and non-trivial part in CFT is to define/understand string interactions. According to Segal's picture [Seg88], an interaction is uniquely determined by a compact Riemann surface Σ with boundaries $\partial\Sigma$, where $\partial\Sigma$ is a disjoint union of some circles (strings). Each string is called either an incoming string or an outcomming one. Suppose $\partial\Sigma$ has N incoming strings and M outgoing ones, then this picture describes an interaction where N strings are going inside, and M strings are going outside.

Moreover, the boundary $\partial\Sigma$ must be **parametrized**. This means that to each connected component $\partial\Sigma_i$ a diffeomorphism $\eta_i : \partial\Sigma_i \xrightarrow{\simeq} \mathbb{S}^1$ is associated. The orientation on $\partial\Sigma_i$ defined by pulling back the one of $\mathbb{S}^1$ along η_i is assumed to be the opposite of the one defined in Stokes' theorem, shown as follows



(1.1)

1.2

Unless otherwise stated, we assume that the boundary parametrization is also **analytic**. Roughly speaking, this means that Σ can be obtained by removing some open discs from a compact Riemann surface C (without boundary) such that the parametrizations of $\partial\Sigma$ are given by local holomorphic functions of C.

Here is a more rigorous explanation. By a **local coordinate** η of C at $x \in C$, we mean η is a holomorphic injective function on a neighborhood U of x such that $\eta(x) = 0$. So η is a biholomorphism between U and a neighborhood $\eta(U)$ of 0. Now, suppose we have local coordinates $\eta_1, \dots, \eta_N$ at distinct points $x_1, \dots, x_N \in C$. The data

$$\mathfrak{X} := (C; x_\bullet; \eta_\bullet) = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N) \tag{1.2}$$

is called an N**-pointed compact Riemann surface with local coordinates**.

Let each η_i be defined on a neighborhood $U_i \ni x_i$. We assume moreover the following

Assumption 1.1. $U_i \cap U_j = \varnothing$ if $i \neq j$ (indeed, $\eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}}) \cap \eta_j^{-1}(\mathbb{D}_1^{\mathrm{cl}}) = \varnothing$ is sufficient), and $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$ for each i. Here $\mathbb{D}_1^{\mathrm{cl}}$ is the closed unit disc.

By removing all $\eta_i^{-1}(\mathbb{D}_1)$, we get Σ with boundary strings $\eta_i^{-1}(\partial\mathbb{D}_1^{\mathrm{cl}}) = \eta_i^{-1}(\mathbb{S}^1)$ whose parametrizations are η_i.

1.3

Any Σ as above determines uniquely an interaction of strings. Suppose it has N incoming strings and M outgoing ones. Then mathematically, such an interaction is described by a bounded linear map $T = T_\Sigma : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes M}$. (The boundedness is automatic thanks to the uniform boundedness principle. But this is not an important point in this course.) Given $\xi_\bullet = \xi_1 \otimes \cdots \otimes \xi_N \in \mathcal{H}^{\otimes N}$ and $\eta_\bullet = \eta_1 \otimes \cdots \otimes \eta_M \in \mathcal{H}^{\otimes M}$, the value

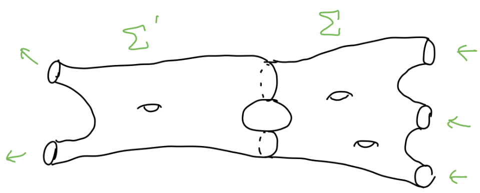
$$\langle \eta_\bullet | T\xi_\bullet \rangle \tag{1.3}$$

describes the probability amplitude that the N incoming closed strings with states $\xi_1, \dots, \xi_N$ become $\eta_1, \dots, \eta_M$ after interaction.

The word "conformal" in conformal field theory reflects the fact that T depends only on the complex structure of Σ and its parametrization, but not on the metric for instance. Thus, a CFT is more rigid than a topological quantum field theory (TQFT): in the latter case, T depends only on the topological structures of the manifolds.

1.4

Suppose we have another interaction $S : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes L}$ corresponding to the parametrized surface Σ', then the composition of them $S \circ T : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes L}$ corresponds to the **sewing** $\Sigma \# \Sigma'$ of Σ and Σ', where the j-th outgoing string $\partial_+\Sigma_j$ of Σ is sewn with the j-th incoming one $\partial_-\Sigma'_j$ of Σ'.



It is important to specify how $\partial_+\Sigma_j$ (with parametrization η_j) is identified with $\partial_-\Sigma'_j$ (with parametrization η'_j). Pick $x \in \partial_+\Sigma_j$ and $y \in \partial_-\Sigma'_j$. Then

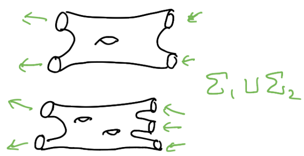
$$x = y \quad \Longleftrightarrow \quad \eta_j(x)\eta'_j(y) = 1. \tag{1.4}$$

It is clear from the picture that the orientations of $\partial_+\Sigma_j$ and $\partial_-\Sigma_j$ are opposite to each other. This is related to the fact that our rule for sewing is $\eta_j(x) = 1/\eta_j'(y)$ but not (say) $\eta_j(x) = \eta_j'(y)$.

Recall we assume that the parametrizations are analytic. We leave it to the readers to check that the sewing of Σ and Σ', a priori only a topological surface, has a natural complex analytic structure.

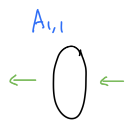
1.5

Suppose $T_1 : \mathcal{H}^{\otimes N_1} \to \mathcal{H}^{\otimes M_1}$ corresponds to Σ_1 and $T_2 : \mathcal{H}^{\otimes N_2} \to \mathcal{H}^{\otimes M_2}$ to Σ_2, then $T_1 \otimes T_2 : \mathcal{H}^{\otimes(N_1+N_2)} \to \mathcal{H}^{\otimes(M_1+M_2)}$ corresponds to the disjoint union $\Sigma_1 \sqcup \Sigma_2$.



1.6

Consider an annulus $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$ obtained by removing two open discs from the compact Riemann sphere $\mathbb{P}^1$ via the local coordinate $\eta_1(z) = z/r$ at 0 and $\eta_2(z) = R/z$ at ∞. We call such $A_{r,R}$ (with the given boundary parametrization) a **standard annlus**. Let $r \nearrow 1, R \searrow 1$. The limit of this annulus is a "degenerate" Riemann surface with 1 incoming boundary circle and 1 outing one. Both circles are $\mathbb{S}^1$. The incoming one has parametrization $z \mapsto z$ and the outgoing one $z \mapsto z^{-1}$. We call this annulus the **standard thin annulus** and denote it by $A_{1,1}$. *The map $T : \mathcal{H} \to \mathcal{H}$ associated to $A_{1,1}$ is the identity map.* This reflects the fact that sewing any Σ with a disjoint union of $A_{1,1}$ gives Σ.



1.7

We give a fancy way to summarize what we have so far: Let $\mathscr{C}$ be the monoidal category of compact 1-dimensional smooth manifolds such that a morphism from an object S_1 to another S_2 is a compact Riemann surface with incomming parametrized boundary $\simeq S_1$ and outgoing one $\simeq S_2$, that the identity morphism for a union of N circles is a disjoint union of N pieces of $A_{1,1}$, that the unit object is the empty set, and that the tensor product of objects and morphisms are respectively the disjoint unions of strings and Riemann surfaces. Then a CFT is a monoidal functor from $\mathscr{C}$ to the monoidal category of Hilbert spaces. So, roughly speaking, a CFT is a representation of $\mathscr{C}$.

Since we choose Hilbert spaces as our underlying spaces, we should expect that the representation of $\mathscr{C}$ is unitary. Technically, the functor mentioned above should be a $*$-functor: this means that for each morphism Σ from N strings to M strings, we should define its adjoint morphism Σ^* from M strings to N ones whose corresponding map is the adjoint $T^* : \mathcal{H}^{\otimes M} \to \mathcal{H}^{\otimes N}$ of T. Σ^* is defined simply to be the **complex conjugate** $\overline{\Sigma}$ of Σ:

Definition 1.2. $\overline{\Sigma}$ consists of points $\overline{x}$ where $x \in \Sigma$; the local holomorphic functions on $\overline{\Sigma}$ are η^* where η is a locally defined holomorphic function on Σ and

$$\eta^*(\overline{x}) = \overline{\eta(x)} \tag{1.5}$$

whenever η is defined on $x \in \Sigma$; similarly, boundary parametrizations are given by η_j^*. Note that if Σ is obtained by removing open discs from an N pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$, then $\overline{\Sigma}$ is obtained by removing discs from

$$\overline{\mathfrak{X}} := (\overline{C}; \overline{x}_1, \dots, \overline{x}_N; \eta_1^*, \dots, \eta_N^*) \tag{1.6}$$

η^* should not be confused with $\overline{\eta}$ defined on Σ by

$$\overline{\eta}(x) = \overline{\eta(x)}.$$

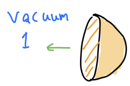
In the present context, we should assume that an incoming (resp. outgoing) string of Σ becomes an outgoing (resp. incoming) one of $\overline{\Sigma}$ via the conjugate map $\mathfrak{C} : x \in \Sigma \mapsto \overline{x} \in \overline{\Sigma}$. In the future, we will often consider all strings as incoming ones if necessary (cf. 1.9). In that case, we shall also assume all the boundary strings of $\overline{\Sigma}$ as incoming.

We should point out that although unitarity is a very important condition, there are important non-unitary CFTs, for instance, the logarithmic CFTs. (In such cases, $\mathcal{H}$ is a vector space without inner products.) Also, many VOA results and techniques do not rely on the unitarity. Nevertheless, assuming unitarity will often reasonably simply discussions or give motivations.

Example 1.3. Let $\mathfrak{X} = (\mathbb{P}^1; 0; \lambda\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$ and $\lambda \in \mathbb{C}^\times$. We can identify the conjugate of $\mathbb{P}^1$ with $\mathbb{P}^1$ by letting $x \in \mathbb{P}^1 \mapsto \overline{x}$ be the standard conjugate of $\mathbb{C}$: $z \mapsto \overline{z}$. Then $(\lambda\zeta)^*(\overline{z}) = \overline{\lambda\zeta(z)} = \overline{\lambda} \cdot \overline{z} = \overline{\lambda}\zeta(\overline{z})$. So the conjugate of $\mathfrak{X}$ is isomorphic to $\overline{\mathfrak{X}} = (\mathbb{P}^1; 0; \overline{\lambda}\zeta)$.

1.8

An interaction process could have no incoming or outgoing strings. *The Hilbert space for the empty string $\varnothing$ is $\mathbb{C}$.* The most elementary and important example with no incoming boundary is the closed unit disc $\mathbb{D}_1^{\mathrm{cl}}$ with 1 outgoing boundary parametrized by $z \mapsto z^{-1}$. The corresponding map $\mathbb{C} \to \mathcal{H}$ can be identified with its value at 1. This element in $\mathcal{H}$ is denoted by $\mathbf{1}$ and called the **vacuum vector**.



(1.7)

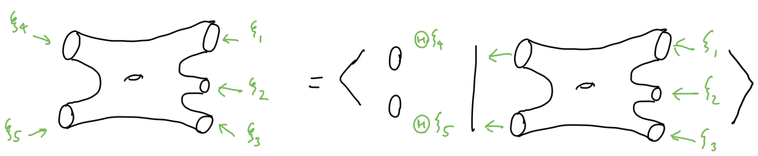
Assume as before that out theory is unitary. Then conjugate of the above disk is the same disk and boundary parametrization, but the original outgoing string is now the incoming one. The corresponding map $\mathcal{H} \to \mathbb{C}$ is, according to 1.7, the linear functional $\langle \mathbf{1} | \cdot \rangle$.

1.9

In general, one may wonder what the interaction $T : \mathcal{H}^{\otimes N} \to \mathbb{C}$ means physically for a surface Σ with N incoming strings but no outgoing ones. Choose $0 < M < N$, and make M of the N strings of $\partial\Sigma$ be outgoing strings. Then the corresponding interaction is a map $\widetilde{T} : \mathcal{H}^{\otimes(N-M)} \to \mathcal{H}^{\otimes M}$. In unitary CFT, T can be related to $\widetilde{T}$ by a anti-unitary (i.e. conjugate-unitary) map Θ on $\mathcal{H}$, called the **CPT operator**, such that for $\xi_1, \ldots, \xi_N \in \mathcal{H}$ (where the last M vectors are associated to the outgoing strings), we have

$$T(\xi_1 \otimes \cdots \otimes \xi_N) = \langle \Theta\xi_{N-M+1} \otimes \cdots \otimes \Theta\xi_N | \widetilde{T}(\xi_1 \otimes \cdots \otimes \xi_{N-M}) \rangle, \tag{1.8}$$

interpreted pictorially as



The operator Θ is an involution, i.e., $\Theta^2 = \mathbf{1}_{\mathcal{H}}$.

Such a linear functional T corresponding to an interaction with no outgoing strings is called a **correlation function** (or an N**-point function**). These functions are the central objects in CFT (and indeed, in any quantum field theory). Relation (1.8) teaches us that: (1) correlation functions can be interpreted as probability amplitudes in string interactions with the help of Θ, and (2) to study arbitrary interactions, it suffices to study those with no outgoing strings.

Let me close this subsection by mentioning an important fact: suppose the complex structure of Σ and the (assumed analtytic) boundary parametrizations are parametrized holomorphically by some complex variables $\tau_\bullet = (\tau_1, \ldots, \tau_k)$, then the value of $T(\xi_\bullet)$ is now a *real analytic function* of $\tau_\bullet$, i.e., it is locally a power series of $\tau_1, \ldots, \tau_k$ and their conjugates. Actually, the word "function" in "correlation function" means a function of $\tau_\bullet$, but not of $\xi_\bullet$.

1.10

You must be curious what CPT means. Indeed, Θ is responsible for the simultaneous symmetry of charge conjugation (C), parity transformation (P), and time reversal (T). P+T together means an *anti-biholomorphism* $\Sigma \to \Sigma'$. Now we have arrived at a point that we missed previously: since anti-holomorphic maps are also conformal maps, should we expect that the interaction maps (or the correlation functions) for anti-biholomorphic surfaces are equal? The answer is no. (Namely, P+T are not preserved.) Indeed, if we let Σ have N incomes and no outcomes, let $\overline{\Sigma}$ be its complex conjugate (cf. 1.7) but still with N incomes, and let $T_\Sigma, T_{\overline{\Sigma}}$ be the correlation functions associated to them. Then from 1.7 and relation (1.8), we have

$$T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N) = \overline{T_{\overline{\Sigma}}(\Theta\xi_1 \otimes \cdots \otimes \Theta\xi_N)}. \tag{1.9}$$

Proof. By the description in Subsec. 1.7, the interaction map $\widetilde{T}_{\overline{\Sigma}}$ associated $\overline{\Sigma}$ with no input and N outputs is $T_\Sigma^* : \mathbb{C} \to \mathcal{H}^{\otimes N}$, the adjoint of T_Σ. By $\Theta^2 = 1$, we have

$$
\begin{aligned}
&T_\Sigma(\xi_1 \otimes \cdots \xi_N) = \langle 1 | T_\Sigma(\xi_1 \otimes \cdots \otimes \xi_N) \rangle = \langle T_\Sigma^* 1 | \xi_1 \otimes \cdots \otimes \xi_N \rangle \\
=&\overline{\langle \xi_1 \otimes \cdots \otimes \xi_N | \widetilde{T}_{\overline{\Sigma}} 1 \rangle} \xlongequal{(1.8)} \overline{T_{\overline{\Sigma}}(\Theta \xi_1 \otimes \cdots \otimes \Theta \xi_N)}.
\end{aligned}
$$

Note that mathematically, the point of formula (1.9) is to translate (using (1.8)) the relation $\widetilde{T}_{\overline{\Sigma}} = T_\Sigma^*$ (regarding all the strings of $\overline{\Sigma}$ as outgoing) to the case that all the strings of $\overline{\Sigma}$ are incoming. $\square$

Formula (1.9) explains CPT symmetry: the symmetries of charge (taking complex conjugate of the values of correlation functions) and parity+time (the conjugate biholomophism $\complement : \Sigma \to \overline{\Sigma}$) are preserved, and the operator realizing this simultaneous symmetry is Θ.

Note that mathematically, charge conjugation C is related to taking complex conjugate of numbers (but not of Σ). Physically, it means making a string into its "antistring", or (in general QFT) making a particle (e.g. an election with negative charge) to its anti-particle (e.g. an antielectron with positive charge).

1.11

The CFT we have described so far is actually very special: it has no conformal anomaly. There are indeed no nontrivial CFTs which are both unitary and without anomaly. In this course, we will be mainly interested in CFTs with conformal anomaly. Technically, the conformal anomaly is determined by a complex number c (positive for unitary CFT), called **central charge**. To describe such CFT, we modify the previous descriptions as follows: The map (or the correlation function) T_Σ for Σ is only up to a positive scalar multiplication depending on Σ. $T_{\Sigma_1} \circ T_{\Sigma_2} = \lambda T_{\Sigma_1 \# \Sigma_2}$ where $\lambda > 0$. (The constants are not necessarily positive in non-unitary CFT.) If Σ is parametrized holomorphically by some complex variables $\tau_\bullet$, then by shrinking the domain of $\tau_\bullet$, we can choose T_Σ depending real analytically on $\tau_\bullet$.

There are many important cases where a real analytic (or even a holomorphic) T_Σ can be chosen globally for $\tau_\bullet$. This will be studied later in details.

Unless otherwise stated, a CFT always means one with (possible) conformal anomaly. Using the fancy language of 1.7, one can say that a unitary CFT is a *projective* monoidal $*$-functor from the category $\mathscr{C}$ in 1.7 to the category of Hilbert spaces. Namely, it is a projective unitary representation of $\mathscr{C}$.

1.12

To study the representations of a topological group G, one must first understand very well the topological and the algebraic structures of G. Similarly, the study of CFTs relies heavily on the geometric and analytic structures of compact Riemann surfaces. However, from what we have discussed, there is a huge obstacle for studying CFTs: the correlation functions are real analytic, but not complex analytic (i.e. holomorphic) functions of the parameters $\tau_\bullet$. Thus, in order to study CFTs using the powerful tools of complex analysis (residue theorem, for instance), we make the following Ansatz: A correlation function T is a sum : $T_\Sigma = \sum_j \Phi^j_\Sigma \cdot \Psi^j_{\overline{\Sigma}}$, where each Φ^j and Ψ^j relies holomorphically on Σ and $\overline{\Sigma}$ respectively (so $\Psi^j_{\overline{\Sigma}}$ relies anti-holomorphically on Σ).

This Ansatz is very vague. Let me explain it in more details. Consider the annulus $A_{r,R}$ with boundary parametrization as in 1.6. We move the inside circle to another one centered at z (where $z \in A_{r,R}$ is reasonably small), still with radius r. The new eccentric annulus $A_{z,r,R}$ has larger outgoing string parametrized by R/ζ and the smaller incoming one parametrized by $(\zeta - z)/r$, where ζ is the standard coordinate of $\mathbb{P}^1$. Namely, it is determined by the data

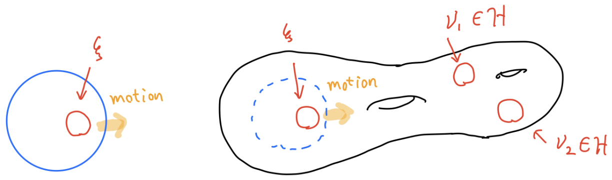
$$(\mathbb{P}^1; z, \infty; (\zeta - z)/r, R/\zeta). \tag{1.10}$$

Let $T_z : \mathcal{H} \to \mathcal{H}$ be the corresponding map. As we have said, for general vectors $\xi, \eta \in \mathcal{H}$, the expression $\langle \eta | T_z \xi \rangle = \langle \Theta\eta, T_z \xi \rangle$ can be chosen to be real analytic with respect to z. We now let

$$\begin{aligned} \mathbb{V} = \{\xi \in \mathcal{H} : &\text{ For all } r, R, \text{ the map } T \text{ can be chosen such that} \\ &z \mapsto \langle \nu | T_z \xi \rangle \text{ is holomorphic for all } \nu \in \mathcal{H}, \text{ and} \\ &\xi \text{ has "finite energy"}\} \end{aligned} \tag{1.11}$$

"Finite energy" is a minor condition to be explained later. (See 2.8.)

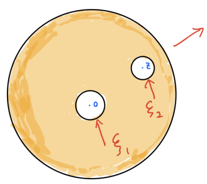
We can sew $A_{z,r,R}$ with any Σ, and the motion of the smaller string inside the annulus becomes, after sewing, the motion of a boundary string of Σ:



(1.12)

Therefore, if a vector $\xi \in \mathbb{V}$ is assigned to an incoming string of Σ with (analytic) boundary parametrization η_i, then, when translating this parametrized string with respect to η_i, the correlation function $T_\Sigma(\xi \otimes \cdots)$ should be holomorphic with respect to the motion, whatever states we assign to the other strings. We can therefore study $\mathbb{V}$ with the help of complex analysis. $\mathbb{V}$ is called a **vertex operator algebra** (VOA).

We have only described $\mathbb{V}$ as a vector space. But in which sense is $\mathbb{V}$ an algebra? An obvious candidate is as follows: consider $\mathbb{P}^1$ with three marked points $0, z, \infty$ and usual coordinates, e.g. $\eta_0 = \zeta/r_1, \eta_z = (\zeta - z)/r_2, \eta_\infty = R/\zeta$ at $0, z, \infty$ where $r_1, r_2 > 0$ are small and $R > 0$ is large, and ζ is again the standard coordinate of $\mathbb{C}$. We assume the strings around 0 and z are ingoing and that around ∞ outgoing. If we assign $\xi_1, \xi_2 \in \mathbb{V}$ to the incoming strings, then the outcome can be viewed as a product of ξ_1 and ξ_2.



Although this product does not have finite energy, it does satisfy the statement before the last line in (1.11). Thus, this product is almost a vector in $\mathbb{V}$. By modifying this product suitably, we can ensure that the products of vectors in $\mathbb{V}$ are always in $\mathbb{V}$. Details will be give in later sections.

Similarly to (1.11), we define $\widehat{\mathbb{V}} \subset \mathcal{H}$ to be the set of finite energy vectors ξ such that $\langle \nu | T_z \xi \rangle$ is anti-holomorphic over z. The vacuum vector $\mathbf{1}$ belongs to $\mathbb{V} \cap \widehat{\mathbb{V}}$: The result of gluing the unit disc into the inside of $A_{z,r,R}$ is just the disc with radius R and parametrization R/ζ, which is independent of z. So $T_z \mathbf{1}$ and hence $\langle \nu | T_z \mathbf{1} \rangle$ are constant over z, and hence both holomorphic and anti-holomorphic over z.

1.13

Now we can give a more detailed presentation of our Ansatz. We let $\mathcal{H}^{\mathrm{fin}}$ be the (indeed dense) subspace of vectors in $\mathcal{H}$ with "finite energy", which is acted on by $\mathbb{V} \otimes \widehat{\mathbb{V}}$. Ansatz:

1. $\mathcal{H}^{\mathrm{fin}}$ as a $\mathbb{V} \otimes \widehat{\mathbb{V}}$-module has decomposition

$$\mathcal{H}^{\mathrm{fin}} = \bigoplus_{i \in \mathfrak{I}} \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i \qquad \supset \mathbb{V} \otimes \widehat{\mathbb{V}} \tag{1.13}$$

   where each $\mathbb{W}_i, \widehat{\mathbb{W}}_i$ are respectively irreducible $\mathbb{V}$-modules and $\widehat{\mathbb{V}}$-modules. $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are (according to their definition cf. (1.11)) subspaces of $\mathcal{H}^{\mathrm{fin}}$ by identifying them with $\mathbb{V} \otimes \mathbf{1}$ and $\mathbf{1} \otimes \widehat{\mathbb{V}}$ respectively. The vacuum vector $\mathbf{1}$ of $\mathcal{H}$ is identified with $\mathbf{1} \otimes \mathbf{1}$ (which belongs to $\mathbb{V} \otimes \widehat{\mathbb{V}}$).

2. For some Σ without outgoing boundaries, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the corresponding map. Then, corresponding to the above direct sum decomposition, we have

$$T_\Sigma \Big|_{(\mathcal{H}^{\mathrm{fin}})^{\otimes N}} = \sum_{i_1, \dots, i_N \in \mathfrak{I}} \Phi_{\Sigma, i_\bullet} \otimes \Psi_{\overline{\Sigma}, i_\bullet} \tag{1.14}$$

   where

$$\begin{aligned} \Phi_{\Sigma, i_\bullet} &: \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \to \mathbb{C}, \\ \Psi_{\overline{\Sigma}, i_\bullet} &: \widehat{\mathbb{W}}_{i_1} \otimes \cdots \otimes \widehat{\mathbb{W}}_{i_N} \to \mathbb{C} \end{aligned}$$

   are linear. Moreover, when the complex structure and boundary parametrization are parametrized analytically by complex variables $\tau_\bullet$, then locally (with respect to the domain of $\tau_\bullet$), $T_\Sigma, \Psi_{\Sigma, i_\bullet}, \Psi_{\overline{\Sigma}, i_\bullet}$ can be chosen such that $\Psi_{\Sigma, i_\bullet}$ is holomorphic over $\tau_\bullet$ (for all input vectors), and $\Psi_{\overline{\Sigma}, i_\bullet}$ holomorphic over $\overline{\tau}_\bullet$. $\Phi_{\Sigma, i_\bullet}$ and $\Psi_{\overline{\Sigma}, i_\bullet}$ are called **conformal blocks** associated to Σ (resp. $\overline{\Sigma}$) and $\mathbb{V}$ (resp. $\widehat{\mathbb{V}}$).

In part one, $\bigoplus$ could be finite (our main focus in this course), infinite but discrete, or continuous.

The second part can be summarized by saying that the CFT is separated into the **chiral halves** (those Φ or $\mathbb{W}_i$) and the **anti-chiral halves** (those Ψ or $\widehat{\mathbb{W}}_i$). Here, "chiral"="holomorphic".

When physicists say a CFT is **rational**, they usually mean that the above direct sum is finite, and each $\mathbb{W}_{i_k}, \widehat{\mathbb{W}}_{i_k}$ are semi-simple (hence, by further decomposition, can be irreducible). So far, the mathematical theory of conformal blocks is complete almost only for rational CFTs. These will be the main examples of this course. For non-rational logarithmic CFTs, even the above Ansatz needs to be modified. (So far, it is not even clear how to do it.)

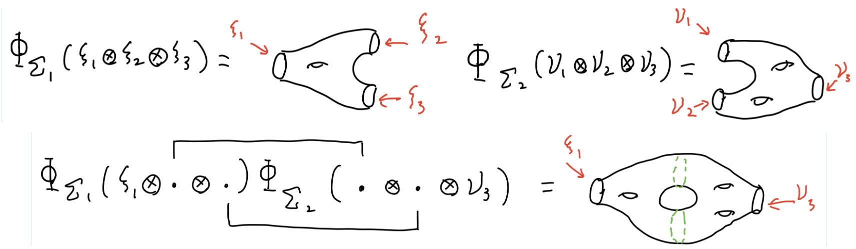
Physicists more or less consider the above description as the definition of conformal blocks. We mathematicians should do the opposite: define conformal blocks in a different way, and use them to *construct* CFTs following the above Ansatz.

1.14

You may notice that to make this Ansatz compatible with 1.4 and 1.5, it is necessarily to assume that

1. The tensor product of conformal blocks $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ associated to Σ_1, Σ_2 respectively should be a conformal block associated to $\Sigma_1 \sqcup \Sigma_2$.

2. The composition of $\Phi_{\Sigma_1}, \Phi_{\Sigma_2}$ (or more precisely, their contractions) should be conformal blocks associated to the sewings of Σ_1 and Σ_2, where the pair of $\mathbb{V}$-modules to be contracted must be dual to each other.



(A side note on linear algebra: If $V^\vee$ is the dual space (or a suitable dense subspace of the dual space) of a vetor space V, we choose a basis $\{v_\alpha\}_{\alpha\in\mathfrak{A}}$ labeled by elements of $\mathfrak{A}$, and choose a dual basis $\{v_\alpha^\vee\}_{\alpha\in\mathfrak{A}}$ of $V^\vee$ (i.e. the one determined by $\langle v_\alpha, v_\beta\rangle = \delta_{\alpha,\beta}$), then taking contraction means substituting $\sum_{\alpha\in\mathfrak{A}} v_\alpha \otimes v_\alpha^\vee$ inside the linear functional on a tensor product of vector spaces such that $V, V^\vee$ are tensor components.)

After we define conformal blocks rigorously, we will see that the first point is obvious, while the second one is a non-trivial theorem.

We briefly explain the meaning of "dual", and why the dual modules appear in $\mathcal{H}$. For instance, in the above picture, the unitary $\mathbb{V}$-module containing ξ_2 is dual to the one containing η_1. As vector spaces, they are "graded" dual spaces of each other. (It is a dense subspace of the full dual space, the subspace of "finite energy" linear functionals. We will talk about this in future sections.) In unitary CFTs, all $\mathbb{V}$ and $\widehat{\mathbb{V}}$ modules are unitary, and $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i)$ is equivalent to $\mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$ where $\mathbb{W}_i'$ is a $\mathbb{V}$-module dual to $\mathbb{W}_i$, and $\widehat{\mathbb{W}}_i'$ a $\widehat{\mathbb{V}}$-module dual to $\widehat{\mathbb{W}}_i$. The formal name for dual module is **contragredient module**, to be defined rigorously in later sections.

1.15

Let us describe the equivalence $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i) \simeq \mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$ in more details.

For each $w_i \otimes \widehat{w}_i \in \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, the vector $\Theta(w_i \otimes \widehat{w}_i)$ is regarded as a linear functional on $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ in the following way. Let the (clearly symmetric) bilinear form $\langle \cdot, \cdot \rangle : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ be the correlation function $T_{A_{1,1}}$ for the standard thin annulus $A_{1,1}$ (with two inputs and no outputs). Note that by (1.8), for each $\xi, \nu \in \mathcal{H}$, we have

$$\langle \Theta\xi, \nu \rangle = \langle \xi | \nu \rangle. \tag{1.15}$$

Then $\Theta(w_i \otimes \widehat{w}_i)$ is equivalent to the linear functional

$$\langle \Theta(w_i \otimes \widehat{w}_i), \cdot \rangle = \langle w_i \otimes \widehat{w}_i | \cdot \rangle \tag{1.16}$$

restricted onto $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$.

A conformal block with $M+N$ inputs $\Phi_\Sigma : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \otimes \mathbb{W}_{j_1} \otimes \cdots \otimes \mathbb{W}_{j_N} \to \mathbb{C}$ can be regarded as one with N inputs and M outputs $\Phi_\Sigma : \mathbb{W}_{j_1} \otimes \cdots \otimes \mathbb{W}_{j_N} \to \mathcal{H}_{i_1}' \otimes \cdots \otimes \mathcal{H}_{i_M}'$ where $\mathcal{H}_{i_k}'$ is the Hilbert space completion of $\mathbb{W}_{i_k}'$ and $\mathbb{W}_{i_k}'$ is the contragredient $\mathbb{V}$-module of $\mathbb{W}_{i_k}$. Using (1.15), it is not hard to show that taking compositions of conformal blocks with outputs is equivalent to taking contractions for conformal blocks without outputs.

2 Virasoro relations; change of boundary parametrizations; strings vs. punctures

2.1

The goal of this section is to understand conformal blocks associated to 2-pointed Riemann spheres, equivalently, genus-0 surfaces with two boundary strings. We simply call them **annuli**, although their complex structures and boundary parametrizations are not necessarily the standard ones as in 1.6.

Let us first consider some degenerate examples whose boundary parametrizations are not necessarily analyic. Let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the topological group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. For each $g \in \mathrm{Diff}^+(\mathbb{S}^1)$, we let $A_{1,1}^g$ be the thin annulus whose incoming and outgoing strings are both $\mathbb{S}^1$ with parametrizations

$$\text{Incoming} : z \mapsto z, \qquad \text{Outgoing} : z \mapsto 1/g(z).$$

Lemma 2.1. *If $h \in \mathrm{Diff}^+(\mathbb{S}^1)$, then $A_{1,1}^{gh}$ is obtained by gluing the incoming circle of $A_{1,1}^g$ with the outgoing one of $A_{1,1}^h$.*

Proof. By (1.4), a point $z \in A_{1,1}^h$ is glued with $\zeta \in A_{1,1}^g$ iff $\zeta \cdot 1/h(z) = 1$, i.e., $\zeta = h(z)$. Now, a point z of $A_{1,1}^h$ becomes the point $h(z)$ of $A_{1,1}^g$ after gluing, which is sent by the outgoing parametrization of $A_{1,1}^g$ to $1/g(h(z))$. □

This proof is not rigorous since we are considering degenerate annuli. A rigorous one would be approximating $A_{1,1}^g$ and $A_{1,1}^h$ by genuine annuli, identifying the sewn annuli, and then taking the limit. This proof is not easy, unless when g and h are real-analytic (e.g., rotations). Nevertheless, we only need this lemma to motivate our following discussions.

2.2

Thus, we may consider $\mathrm{Diff}^+(\mathbb{S}^1)$ as the group of thin annuli whose product is the sewing. The merit of this viewpoint is that it convinces us to *consider the semi-group* $\mathbf{Ann}$ *of annuli as the complexification of* $\mathrm{Diff}^+(\mathbb{S}^1)$. The multiplication A_1A_2 of $A_1, A_2 \in \mathbf{Ann}$ is the sewing of A_1, A_2 defined by gluing the inside of A_1 with the outside of A_2 using their parametrizations.

As an example, consider $\mathbb{P}^1$ with marked points $0, \infty$ and local coordinates $\eta_0(z) = z, \eta_\infty(z) = e^{-\mathbf{i}\tau}/z$, which gives a thin annulus corresponding to the rotation $z \mapsto e^{\mathbf{i}\tau}z$ when τ is real. Now consider τ as a complex variable $\tau = s + \mathbf{i}t$. Then the outgoing circle is the one with radius e^t. This gives a genuine annulus whenever $t > 0$.

The Ansatz in 1.13 should be expanded to include the following point: for each annulus $A \in \mathbf{Ann}$, the comformal block decomposition of the interaction $T_A : \mathcal{H} \to \mathcal{H}$ (with one income and one outcome) with respect to $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ is of the form

$$T_A = \sum_i \pi_i(A) \otimes \widehat{\pi}_i(\overline{A}) \tag{2.1}$$

where $\pi_i(A)$ is a bounded linear operator on the Hilbert space completion $\mathcal{H}_i$ of $\mathbb{W}_i$, and $\widehat{\pi}_i(\overline{A})$ is one on the completion $\widehat{\mathcal{H}}_i$ of $\widehat{\mathbb{W}}_i$. ($\overline{A}$ is the complex conjugate of A; see Def. 1.2. We assume the conjugate of the incomming string of $\overline{A}$ is the incoming of A, and similarly for the outcoming strings.) The choice of $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are unique up to scalar multiplications, and if A vary holomorphically over some complex variable $\tau_\bullet$, then locally $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ can be chosen to vary holomorphically with respect to $\tau_\bullet$ and $\overline{\tau}_\bullet$ respectively. Finally, if $A_1, A_2 \in \mathbf{Ann}$, then $\pi_i(A_1A_2)$ equals $\pi_i(A_1)\pi_i(A_2)$ up to scalar multiplication, and a similar thing can be said about $\widehat{\pi}_i$.

Namely, each π_i is a projective representation of $\mathbf{Ann}$ on $\mathcal{H}_i$, and so is $\widehat{\pi}_i$ on $\widehat{\mathcal{H}}_i$. They should be the analytic extensions of projective unitary representations of $\mathrm{Diff}^+(\mathbb{S}^1)$.

We emphasize that $\pi_i(A)$ and $\widehat{\pi}_i(\overline{A})$ are conformal blocks associated to A and $\overline{A}$ respectively. Roughly speaking, π_i describes the conformal symmetries of chiral halves and $\widehat{\pi}_i$ the anti-chiral halves. A and $\overline{A}$ have to act jointly on the full space $\mathcal{H}$.

2.3

Thus, the study of CFT interactions for annuli reduces to that of the projective representations of $\mathbf{Ann}$. Our goal is to describe such representations in terms of Lie algebras.

Let $\mathrm{Vec}(\mathbb{S}^1)$ be the Lie algebra of smooth real vector fields of $\mathbb{S}^1$, whose elements are of the form $f\partial_\theta$ where ∂_θ is the pushforward of the standard unit vector of the real line under the map $\theta \mapsto e^{\mathbf{i}\theta}$, and $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$. The action of $f\partial_\theta$ on $h \in C^\infty(\mathbb{S}^1, \mathbb{R})$ is the negative of the usual one, $-f(e^{\mathbf{i}\theta}) \cdot \frac{\partial}{\partial\theta}h(e^{\mathbf{i}\theta})$. This is because the action of $g \in \mathrm{Diff}^+(\mathbb{S}^1)$ on h should be $h \circ g^{-1}$ in order to respect the order of group multiplication. Therefore, the Lie bracket in $\mathrm{Vec}(\mathbb{S}^1)$ is the negative of the usual one:

$$[f_1\partial_\theta, f_2\partial_\theta]_{\mathrm{Vec}(\mathbb{S}^1)} = (-f_1\partial_\theta f_2 + f_2\partial_\theta f_1)\partial_\theta. \tag{2.2}$$

2.4

A projective unitary representation π of $\mathrm{Vec}(\mathbb{S}^1)$ and the corresponding one π of $\mathrm{Diff}^+(\mathbb{S}^1)$ (if exists) are related as follows. (Here unitary means that for each vector field $f\partial_\theta$, we have $\pi(f\partial_\theta)^\dagger = -\pi(f\partial_\theta)$, where $\dagger$ is the adjoint, or "formal adjoint" when the underlying inner product space is not Cauchy-complete.)

Let $t \in (-\epsilon, \epsilon) \mapsto g_t \in \mathrm{Diff}^+(\mathbb{S}^1)$ be a smooth family of diffeomorphisms satisfying $g_0 = \mathbf{1}$. Then up to addition by a number of $\mathbf{i}\mathbb{R}$,

$$\frac{d}{dt}\pi(g_t)\Big|_{t=0} = \pi(\partial_t g_0) \tag{2.3}$$

where $\partial_t g_0 \in \mathrm{Vec}(\mathbb{S}^1)$, the derivative of g at t_0, is the vector field determined by

$$(\partial_t g_0)(h) = \frac{d}{dt}(h \circ g_t)\Big|_{t=0} \tag{2.4}$$

for all smooth function h on $\mathbb{S}^1$.

Let now $t \in \mathbb{R} \mapsto \exp(tf\partial_\theta) \in \mathrm{Diff}^+(\mathbb{S}^1)$ be the flow generated by $f\partial_\theta \in \mathrm{Vec}(\mathbb{S}^1)$. So its derivative at $t = 0$ is $f\partial_\theta$, and $\exp((t_1 + t_2)f\partial_\theta) = \exp(t_1 f\partial_\theta) \circ \exp(t_2 f\partial_\theta)$. Then (2.4) implies that up to $\mathbb{S}^1$-multiplication,

$$\pi(\exp(tf\partial_\theta)) = e^{t\pi(f\partial_\theta)}, \tag{2.5}$$

since the derivative of $\pi(\exp(tf\partial_\theta))e^{-t\pi(f\partial_\theta)}$ is $\pi(\exp(tf\partial_\theta))(\pi(f\partial_\theta) - \pi(f\partial_\theta))e^{-t\pi(f\partial_\theta)} = 0$.

2.5

The Witt algebra $\mathrm{Span}_{\mathbb{C}} = \{l_n : n \in \mathbb{Z}\}$ is a complex dense Lie subalgebra of the complexification $\mathrm{Vec}(\mathbb{S}^1) \otimes_{\mathbb{R}} \mathbb{C}$. Here,

$$l_n = z^{n+1}\partial_z = -\mathbf{i}e^{\mathbf{i}n\theta}\partial_\theta \tag{2.6}$$

where $z = e^{\mathbf{i}\theta}$ and $\partial_z = \frac{1}{\mathbf{i}e^{\mathbf{i}\theta}}\partial_\theta$. (We use the chain rule to "define" ∂_z.) One checks

$$[l_m, l_n] = (m - n)l_{m+n} \tag{2.7}$$

where the bracket is the negative of the usual one for vector fields.

Let us assume for simplicity that the CFT is unitary. In the decomposition $\mathcal{H}^{\mathrm{fin}} = \bigoplus_i \mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$, each $\mathbb{W}_i$ is a projective unitary representation π_i of $\{l_n\}$, and similarly $\widehat{\mathbb{W}}_i$ is one $\widehat{\pi}_i$ of $\{l_n\}$. We know that the choice of $\pi_i(l_n)$ is unique up to $\mathbf{i}\mathbb{R}$-scalar addition. Here is a well-known fact about projective representations of Witt algebra (cf. for instance [Was10, Sec. IV.1]): one can make a particular choice of $\pi_i(l_n)$ (for each n), denoted by L_n, such that the **Virasoro relation**

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m + 1)m(m - 1)\delta_{m,-n} \tag{2.8}$$

holds and $c \in \mathbb{C}$ is called the **central charge**. In the case that π_i is projectively unitary, L_n can be chosen such that $L_n^\dagger = L_{-n}$ also holds.

We have abused the notation by writing the actions of l_n on all $\mathbb{V}$-modules $\mathbb{W}_i$ (as chiral halves of the CFT) as L_n. We are justified to do so because, as we will see later, the actions of l_n come from those of $\mathbb{V}$. Technically: Virasoro algebra is inside the VOA. So the action of $\{l_n\}$ on $\mathbb{W}_i$ is the restriction of that of $\mathbb{V}$. In particular, all chiral halves $\mathbb{W}_i$ share the same central charge c.

Similarly, we write the actions of l_n on all $\widehat{\mathbb{W}}_i$ as $\overline{L}_n$. (The bar over L_n reflects the fact that $\overline{L}_n$ describes the conformal symmetries of the anti-chiral halves of the CFT. $\overline{L}_n$ is not related with L_n by the CPT operator Θ.) The central charge $\widehat{c}$ for $\{\overline{L}_n\}$ is independent of $\widehat{\mathbb{W}}_i$ and in general could be different from the one c of $\{L_n\}$, although in most important cases they are equal. (E.g., when the CFT contains both closed and open strings.)

2.6

We shall generalize (2.5) to complex vector fields. First of all, we consider an element

$$f(z)\partial_z=\sum_{n\in\mathbb{Z}}a_nz^{n+1}\partial_z$$

where the sum could be infinite. We treat $f(z)=\sum_n a_nz^{n+1}$ as a Laurent series. Let us now assume that $f(z)$ *is a holomorphic function on a neighborhood* $U\subset\mathbb{C}$ *of* $\mathbb{S}^1$.

$f\partial_z$ is a complex holomorphic vector field of U, which (after shrinking U) gives a **holomorphic flow** $\tau\in\Delta\mapsto\exp(\tau f\partial_z)\in\mathscr{O}(U)$ where $\Delta\subset\mathbb{C}$ is a neighborhood of 0. (Recall from the notation section that $\mathscr{O}(U)$ is the space of holomorphic functions on U.) This means:

(1) $(\tau,z)\in\Delta\times U\mapsto\exp(\tau f\partial_z)(z)$ is holomorphic whose restriction to each slice $\tau\times U$ is injective (and hence, a biholomorphism onto its image).

(2) $\exp(0f\partial_z)(z)=z$.

(3) $\exp((\tau_1+\tau_2)f\partial_z)=\exp(\tau_1 f\partial_z)\circ\exp(\tau_2 f\partial_z)$ on an open subset of U containing $\mathbb{S}^1$.

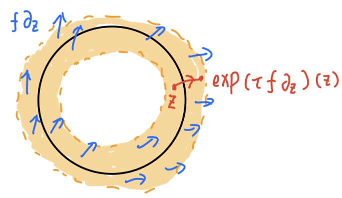
(4) For any holomorphic function h defined on an open set inside U,

$$f\partial_z h=\frac{\partial}{\partial\tau}h\circ\exp(\tau f\partial_z)\Big|_{\tau=0}. \tag{2.9}$$

(Compare (2.4).) This condition is equivalent to

$$\frac{\partial}{\partial\tau}\exp(\tau f\partial_z)\Big|_{\tau=0}=f. \tag{2.10}$$

(To see the equivalence, set $h(z)=z$ for one direction, and use chain rule for the other one.)



Remark 2.2. A caveat: The notations $f\partial_z$ and $\exp(\tau f\partial_z)$ are not compatible with those in the real case. Indeed, if we assume that τ only takes real values $\tau = t$, then by taking the real and the imaginary parts of (2.10), we see that σ_t is a real flow on the real surfaces U generated by the real vector field $\mathrm{Re} f \cdot \partial_x + \mathrm{Im} f \cdot \partial_y$. Writing $\partial_x = \partial_z + \partial_{\overline{z}}, \partial_y = \mathbf{i}(\partial_z - \partial_{\overline{z}})$, we see that this vector field $f\partial_z$ should more precisely be written as $f\partial_z + \overline{f}\partial_{\overline{z}}$ where $\overline{f}(x) = \overline{f(x)}$.

This point is also justified by the fact that if k is antiholomorphic, then

$$\overline{f}\partial_{\overline{z}}k = \frac{\partial}{\partial\overline{\tau}}k \circ \exp(\tau f\partial_z)\Big|_{\tau=0}. \tag{2.11}$$

(Proof: take $k = \overline{h}$ in (2.10).) Thus, a more precise notation for $\exp(\tau f\partial_z)$ should be $\exp(\tau f\partial_z + \overline{\tau}\,\overline{f}\partial_{\overline{z}})$. But we prefer to suppress the term $\overline{\tau}\,\overline{f}\partial_{\overline{z}}$ to keep the notations shorter.

2.7

One way to find the expression of $\sigma_\tau = \exp(\tau f\partial_z)$ is to solve the holomorphic nonlinear differential equation with initial condition:

$$\begin{aligned} \frac{\partial}{\partial\tau}\sigma_\tau(z) &= f(\sigma_\tau(z)), \\ \sigma_0(z) &= z. \end{aligned} \tag{2.12}$$

This is due to (2.10) and $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1} \circ \sigma_{\tau_2}$. (Indeed, the existence of holomorphic flows is due to that of the solutions of such equations.)

Alternatively, one may calculate the flow by brutal force using the formula

$$\begin{aligned} \exp(f\partial_z)(z) &= \sum_{k\in\mathbb{N}} \frac{1}{k!}(f(z)\partial_z)^k z \\ &= \sum_{k\in\mathbb{N}} \frac{1}{k!} \underbrace{f(z)\partial_z\Big(f(z)\partial_z\big(\cdots f(z)\partial_z}_{k \text{ times}}\, z\big)\cdots\Big)\Big). \end{aligned} \tag{2.13}$$

(One may treat this formula as a formal sum if one worries about the convergence issue.) To see why this formula is valid, check that such defined $\exp(\tau f\partial_z)(z) =: \sigma_\tau(z)$ satisfies that $\sigma_{\tau_1+\tau_2} = \sigma_{\tau_1} \circ \sigma_{\tau_2}$, that $\partial_\tau\sigma_\tau|_{\tau=0} = f$, and that $\sigma_0(z) = z$. This is easy.

2.8

Example 2.3. $\sigma_\tau(z) = e^\tau z$ is the holomorphic flow generated by the vector field $l_0 = z\partial_z$ since $\frac{\partial}{\partial\tau}e^\tau z|_{\tau=0} = z$. Namely,

$$\exp(\tau z\partial_z)(z) = e^\tau z.$$

Set $\lambda = e^\tau$. In view of the $A^g_{1,1}$ in 2.1, we consider the 2-pointed sphere $\mathfrak{X} = (\mathbb{P}^1; 0, \infty; \zeta, \lambda^{-1}\zeta^{-1})$ where $\zeta : z \mapsto z$ is the standard coordinate of $\mathbb{C}$. Then, when

$|\lambda| \leqslant 1$, $\mathfrak{X}$ defines an annulus A, either genuine or thin, whose incoming circle has radius 1 and outcoming $1/|\lambda|$. Thus, the conformal block $\pi_i(A)$ associated to this annulus, which is a linear operator on the Hilbert space completion $\mathcal{H}_i$, should be $e^{\tau L_0} = \lambda^{L_0}$ (by replacing $z\partial_z$ with L_0).

It is easy to check that $\overline{A}$ is isomorphic to the annulus defined by $(\mathbb{P}^1; 0, \infty; \zeta, \overline{\lambda^{-1}}\zeta^{-1})$. So the corresponding conformal block should be $\widehat{\pi}_i(\overline{A}) = \overline{\lambda}^{\overline{L}_0}$. Therefore, the interaction map $T_A : \mathcal{H} \to \mathcal{H}$ is determined by

$$T_A\big|_{\mathcal{H}_i \otimes \widehat{\mathcal{H}}_i} = \lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0}. \tag{2.14}$$

In a unitary CFT, L_0 and $\overline{L}_0$ (or more precisely, their closures) are self-adjoint operators so that λ^{L_0} and $\overline{\lambda}^{\overline{L}_0}$ can be defined and are unitary when $|\lambda| = 1$. Moreover, in a unitary CFT:

Assumption 2.4 (Positive energy)**.** The spectra of L_0 and $\overline{L}_0$ are both positive (i.e. $\geqslant 0$). In these notes, we are mainly interested in the case that the spectra are discrete. We identify L_0 with $L_0 \otimes \mathbf{1}$ and $\overline{L}_0$ with $\mathbf{1} \otimes \overline{L}_0$ so that $L_0, \overline{L}_0$ are commuting diagonalizable operators on $\mathcal{H}^{\mathrm{fin}}$ with $\geqslant 0$ eigenvalues.

Now we can explain what we meant by finite energy: A vector ξ of $\mathcal{H}$ has **finite energy** if ξ is a finite sum of eigenvectors of both L_0 and $\overline{L}_0$. (In general, a vector of $\mathcal{H}$ is an l^2-convergent sum, either finite or infinite, of eigenvectors.)

2.9

Example 2.5. Let $n \neq 0$. To understand the geometric meanings of $e^{\tau L_{-n}}$ and $e^{\overline{\tau}\overline{L}_{-n}}$, we find the expression of $\sigma_\tau = \exp(\tau z^{-n+1}\partial_z)$ by solving the differential equation $\partial_\tau \sigma_\tau = (\sigma_\tau)^{-n+1}$ with initial condition $\sigma_0(z) = z$ (cf. (2.12)). The solution is

$$\exp(\tau z^{-n+1}\partial_z)(z) = (z^n + n\tau)^{\frac{1}{n}}. \tag{2.15}$$

□

Unfortunately, this flow does not give us any annulus in the usual sense. Take $n = 1$ for instance. Then the flow is just the translation by τ. However, the circle after a small translation will intersect the original one. So there is no annulus whose outgoing circle is the translation of the incoming one. In fact, in most cases, $\exp(f\partial_z)$ is not the action of an annulus. We have to pursue another way of understanding this operator.

2.10

There are two ways to look at a group action $G \curvearrowright X$: (1) The action of $g \in G$ on X is a transformation. So $gx \neq x$ in general. (2) gx and x are different expressions (under different coordinates) of the same element. The rule for change of coordinate is given by the action of G. We shall take the second viewpoint.

Let $\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ be an N-pointed compact Riemann surface with local coordinates satisfying Assumption 1.1. Assume the setting of 2.6. Write $\sigma_\tau = \exp(\tau f \partial_z)$ and $f(z) = \sum_{n \in \mathbb{Z}} a_n z^{n+1}$ be defined on $U \supset \mathbb{S}^1$. Let $\tau \in \Delta$ be close to 0.

Remark 2.6. In case you want to know the precise meaning of "close": for the local coordinate η_i we are to discuss in the following, we choose $\epsilon > 0$ such that $\sigma_\tau(U \cap \mathrm{Rng}(\eta_i))$ contains $\mathbb{S}^1$ for all $\tau \in \mathbb{D}_\epsilon$, where the open set $\mathrm{Rng}(\eta_i)$ is the range of η_i.

Principle 2.7 (Change of boundary parametrizations)**.** Suppose that the local coordinate η_i at x_i is changed to the boundary parametrization $\sigma_\tau \circ \eta_i$ and the boundary string $\eta_i^{-1} \circ (\mathbb{S}^1)$ is gradually changed (with respect to the change of τ) to $\eta_i^{-1}\big(\sigma_\tau^{-1}(\mathbb{S}^1)\big)$. Then, in the expressions of conformal blocks and correlation functions (without outputs), each $w_i \in \mathbb{W}_i$ is replaced by $e^{\tau \sum_n a_n L_n} w_i$, and each $\widehat{w}_i \in \widehat{\mathbb{W}}_i$ by $e^{\overline{\tau} \sum \overline{a_n} \overline{L}_n} \widehat{w}_i$.

To be more precise, let $T_\Sigma : \mathcal{H}^{\otimes N} \to \mathbb{C}$ be the correlation function where Σ is obtained from $\mathfrak{X}$. Assume $i = 1$ for simplicity. Changing the local coordinate η_1 to $\sigma_\tau \circ \eta_1$ gives a new surface with parametrized boundary Σ'. Then up to scalar multiplication, $T_{\Sigma'}$ and T_Σ are related by

$$T_\Sigma(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N) = T_{\Sigma'}\Big(\big(e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \overline{L}_n}\big)\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_N\Big) \tag{2.16}$$

for all $\xi_1, \dots, \xi_N$. Similarly, if $\Phi_\Sigma : \mathbb{W}_{i_1} \otimes \cdots \otimes \mathbb{W}_{i_N} \to \mathbb{C}$ is a conformal block for Σ, then $\Phi_{\Sigma'}$ defined by

$$\Phi_\Sigma(w_1 \otimes w_2 \otimes \cdots \otimes w_N) = \Phi_{\Sigma'}\big(e^{\tau \sum_n a_n L_n} w_1 \otimes w_2 \otimes \cdots \otimes w_N\big) \tag{2.17}$$

is one for Σ'.

2.11

The geometric intuition in the above subsection is the following: ξ_1 in the η_1-parametrization is the same (up to scalar multiplication) vector as $(e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \cdot \overline{L}_n})\xi_1$ in the $\sigma_\tau \circ \eta_1$-parametrization. We call this same "abstract" vector $\widetilde{\xi}_1$, which is unique up to scalar multiplication. We write $\xi_1 = (\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*))\widetilde{\xi}_1$, understanding $\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*)$ as the map sending an abstract vector to its concrete expression under the boundary parametrization η_1. Namely, $\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*)$ is a vector bundle trivialization. The transition function from the η_1-parametrization to the $\sigma_\tau \circ \eta_1$-parametrization is

$$\big(\mathcal{U}(\sigma_\tau \circ \eta_1) \otimes \mathcal{U}((\sigma_\tau \circ \eta_1)^*)\big)\big(\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*)\big)^{-1} = e^{\tau \sum_n a_n L_n} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \cdot \overline{L}_n}. \tag{2.18}$$

We have a parametrization independent T (more precisely, independent of a small change of parametrizations) whose expressions under the concrete boundary parametrizations are (up to scalar multiplications)

$$\begin{aligned} T(\widetilde{\xi}_1 \otimes \cdots) &= T_\Sigma\Big(\big(\mathcal{U}(\eta_1) \otimes \mathcal{U}(\eta_1^*)\big)^{-1} \widetilde{\xi}_1 \otimes \cdots\Big) \\ &= T_{\Sigma'}\Big(\big(\mathcal{U}(\sigma_\tau \circ \eta_1) \otimes \mathcal{U}((\sigma_\tau \circ \eta_1)^*)\big)^{-1} \widetilde{\xi}_1 \otimes \cdots\Big). \end{aligned}$$

2.12

Let us do an example to see how the change of parametrization formula works.

Example 2.8. Let $\mathfrak{X} = (\mathbb{P}^1; 1/3, \infty; 2(\zeta - 1/3), \zeta^{-1})$ where $\zeta : z \mapsto z$ is the standard coordinate of $\mathbb{C}$. We choose $1/3$ to be the input point, and ∞ the outgoing one. The associated boundary parametrized surface Σ is an annulus whose incoming circle $\{z : |2(z-1/3)| = 1\}$ has center $1/3$ and radius $1/2$, and whose outgoing circle is $\mathbb{S}^1$. Let us find an expression for $T_\Sigma : \mathcal{H} \to \mathcal{H}$.

We know that the map for the standard thin annulus $A_{1,1}$ is $T_{A_{1,1}} = \mathbf{1}_{\mathcal{H}}$. Let $\mathfrak{X}_1 = (\mathbb{P}^1; 0, \infty; 2\zeta, \zeta^{-1})$, which gives an annlus Σ_1 with incoming string $\frac{1}{2}\mathbb{S}^1$ and outgoing one $\mathbb{S}^1$. $A_{1,1}$ is changed to Σ_1 by changing the incoming boundary parametrization ζ to 2ζ. By Ex. 2.3, $2\zeta = \exp(\log 2 \cdot z\partial_z)$. So, as $e^{\log 2 L_0} = 2^{L_0}$ and similarly $e^{\log 2 \overline{L}_0} = 2^{\overline{L}_0}$, by (2.16), T_{Σ_1} could be $(1/2)^{L_0} \otimes (1/2)^{\overline{L}_0}$.

Σ_1 is changed to Σ by adding 2ζ by $-2/3$. According to Ex. 2.5, $\exp(-2/3\partial_z)(z) = z - 2/3$. Therefore, up to a scalar multiplication, $T_{\Sigma_1}(\xi) = T_\Sigma((e^{-\frac{2}{3}L_{-1}} \otimes e^{-\frac{2}{3}\overline{L}_{-1}})\xi)$. Thus, the answer is

$$T_\Sigma = \big((1/2)^{L_0} \otimes (1/2)^{\overline{L}_0}\big) \cdot \big((e^{\frac{2}{3}L_{-1}} \otimes e^{\frac{2}{3}\overline{L}_{-1}})\big) = \big((1/2)^{L_0} e^{\frac{2}{3}L_{-1}}\big) \otimes \big((1/2)^{\overline{L}_0} e^{\frac{2}{3}\overline{L}_{-1}}\big).$$

$(1/2)^{L_0} e^{\frac{2}{3}L_{-1}}$ is a conformal block for Σ. □

2.13

What is the change of parametrization formula for T_Σ (and hence Φ_Σ) when some output strings are involved? Recall from Subsec. 1.15 that the correlation function $T_{A_{1,1}} : \mathcal{H}^{\otimes 2} \to \mathbb{C}$ is a symmetric bilinear form $\langle \xi, \nu \rangle = \langle \nu, \xi \rangle = \langle \Theta\nu | \xi \rangle$. With respect to this form, we actually have

$$(L_n \otimes \mathbf{1})^{\mathrm{t}} = L_{-n} \otimes \mathbf{1}, \qquad (\mathbf{1} \otimes \overline{L}_n)^{\mathrm{t}} = \mathbf{1} \otimes \overline{L}_{-n}. \tag{2.19}$$

More precisely, for each $\xi, \nu \in \mathcal{H}^{\mathrm{fin}}$, we have

$$\langle (L_n \otimes \mathbf{1})\xi, \nu \rangle = \langle \xi, (L_{-n} \otimes \mathbf{1})\nu \rangle$$

and a similar relation for $\overline{L}_n$. Rewrite the above relation in terms of $\langle \cdot | \cdot \rangle$, we have $\langle \Theta(L_n \otimes \mathbf{1})\xi | \nu \rangle = \langle \Theta\xi | (L_{-n} \otimes \mathbf{1})\nu \rangle$, and noticing the unitarity property $L_n^\dagger = L_{-n}$, we get

$$\Theta(L_n \otimes \mathbf{1}) = (L_n \otimes \mathbf{1})\Theta, \qquad \Theta(\mathbf{1} \otimes \overline{L}_n) = (\mathbf{1} \otimes \overline{L}_n)\Theta. \tag{2.20}$$

These relations truly hold, not just up to scalar addition or multiply.

From this, we see that for the maps $T_\Sigma, T_{\Sigma'} : \mathcal{H}^{\otimes(N-1)} \to \mathcal{H}$ with $N-1$ inputs and 1 output,

$$T_\Sigma = \Big(e^{\tau \sum_n a_n L_{-n}} \otimes e^{\overline{\tau} \sum_n \overline{a_n} \overline{L}_{-n}}\Big) \circ T_{\Sigma'}. \tag{2.21}$$

You can easily generalize this formula to the case of more than one outputs.

Proof. Let $\xi_\bullet \in \mathcal{H}^{\otimes(N-1)}$ and $\nu \in \mathcal{H}$. By (1.8), the correlation function (with N-inputs and no outputs) for Σ and Σ' are $\langle\Theta\cdot|T_\Sigma\cdot\rangle$ and $\langle\Theta\cdot|T_{\Sigma'}\cdot\rangle$ respectively. So by (2.16),

$$\begin{aligned}
&\langle\Theta\nu|T_\Sigma(\xi_\bullet)\rangle = \langle\Theta(e^{\tau\sum_n a_nL_n}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\overline{L}_n})\nu|T_{\Sigma'}(\xi_\bullet)\rangle\\
\overset{(2.20)}{=\!=\!=}&\langle(e^{\overline{\tau}\sum_n\overline{a_n}L_n}\otimes e^{\tau\sum_n a_n\overline{L}_n})\Theta\nu|T_{\Sigma'}(\xi_\bullet)\rangle\\
\overset{\text{unitarity}}{=\!=\!=\!=}&\langle\Theta\nu|(e^{\tau\sum_n a_nL_{-n}}\otimes e^{\overline{\tau}\sum_n\overline{a_n}\overline{L}_{-n}})T_{\Sigma'}(\xi_\bullet)\rangle.
\end{aligned}$$

□

Exercise 2.9. Show that the formula (2.14) in Example 2.3 follows from (2.21).

2.14

In case you want to know why $(L_{-n}\otimes\mathbf{1}) = (L_n\otimes\mathbf{1})^{\mathrm{t}}$, we give a geometric explanation below, in which we pretend to ignore the issue of the uniqueness up to scalar additions/multiplications.

Proof. Let $\mathfrak{X} = (\mathbb{P}^1;0,\infty;z,z^{-1})$ where z is the standard coordinate of $\mathbb{C}$, which gives the standard thin annulus $A_{1,1}$. Assume the two strings are incoming. We know the correlation function is $\langle\xi,\nu\rangle$, where we assume ξ is associated to the string around 0 and ν the one around ∞.

Change the local coordinate z at 0 to σ_τ, and keep the other data of $\mathfrak{X}$. This changes $A_{1,1}$ to a new weird annulus A. By (2.16), the correlation function for A is

$$T_A(\xi\otimes\nu) = \langle(e^{-\tau\sum_n a_nL_n}\otimes e^{-\overline{\tau}\sum_n\overline{a_n}\cdot\overline{L}_n})\xi,\nu\rangle.$$

Note that if we set $\zeta = \sigma_\tau(z)$, then $z^{-1} = 1/\sigma_\tau^{-1}(\zeta)$, which equals $1/\sigma_{-\tau}(\zeta)$ by the definition of flows. Namely, A is equivalent to the weird annulus whose incoming boundary parametrization is z and outcoming $1/\sigma_{-\tau}(z)$. To compute the correlation function for this choice of boundary parametrization, we note that the original $1/z$ at ∞ is changed to $1/\sigma_{-\tau}(z)$. Therefore, if we let $\gamma_\tau(z) = 1/\sigma_{-\tau}(1/z)$ which is a holomorphic flow generated by some $\sum_n b_nz^{n+1}$, then the expression for T_A is

$$T_A(\xi\otimes\nu) = \langle\xi,(e^{-\tau\sum_n b_nL_n}\otimes e^{-\overline{\tau}\sum_n\overline{b_n}\cdot\overline{L}_n})\nu\rangle.$$

For the two expressions of T_A, we take the holomorphic derivative $-\partial_\tau$ at $\tau=0$ to get

$$\sum a_n\langle(L_n\otimes\mathbf{1})\xi,\nu\rangle = \sum b_n\langle\xi,(L_n\otimes\mathbf{1})\nu\rangle.$$

To finish the proof, it suffices to prove $b_n = a_{-n}$.

Recall $\sum a_nz^{n+1} = \partial_\tau\sigma_\tau|_{\tau=0}$. Similarly, $\sum b_nz^{n+1} = \partial_\tau\gamma_\tau|_{\tau=0}$, which is

$$\begin{aligned}
&\partial_\tau(1/\sigma_{-\tau}(1/z))\big|_{\tau=0} = -\frac{1}{\sigma_0(1/z)^2}\cdot\partial_\tau(\sigma_{-\tau}(1/z))\big|_{\tau=0}\\
=&z^2\cdot\sum a_n(1/z)^{n+1} = \sum a_nz^{-n+1} = \sum a_{-n}z^{n+1}.
\end{aligned}$$

□

2.15

As an easy application of our change of parametrization formula, we are able to describe the map $T_A : \mathcal{H} \to \mathcal{H}$ for an analytic annulus $A \in \mathbf{Ann}$ obtained from $(\mathbb{P}^1; 0, \infty; \eta_0, \eta_\infty)$ where η_0 and η_∞ are local coordinates at $0, \infty$ respectively. Set $\varpi = 1/z$. One can write

$$\eta_0(z) = \exp\Big(\sum_{n\in\mathbb{N}} a_n z^{n+1}\partial_z\Big)(z), \qquad \eta_\infty(\varpi) = \exp\Big(\sum_{n\in\mathbb{N}} b_n \varpi^{n+1}\partial_\varpi\Big)(\varpi),$$

where the coefficients a_n, b_n can be determined using (2.13). (We will say more about determining the coefficients in the future.) When A is the standard thin annulus (i.e., when $\eta_0 : z \mapsto z, \eta_\infty : z \mapsto z^{-1}$), we know $T_A = \mathbf{1}$. Thus, in general, by (2.16) and (2.19), the map T_A is (up to scalar multiplications)

$$T_A = \Big(e^{\sum_{n\in\mathbb{N}} -b_n L_{-n}} \otimes e^{\sum_{n\in\mathbb{N}} -\overline{b_n}\cdot\overline{L}_{-n}}\Big) \cdot \Big(e^{\sum_{n\in\mathbb{N}} -a_n L_n} \otimes e^{\sum_{n\in\mathbb{N}} -\overline{a_n}\cdot\overline{L}_n}\Big).$$

The reason that only $n \in \mathbb{N}$ are involved is because η_0 and η_∞ can be defined near 0 and send 0 to 0. Indeed, for $f(z) = \sum_{n\in\mathbb{Z}} a_n z^{n+1}$, assume that $\exp(\tau f\partial_z)(z)$ is defined near 0 and sends 0 to 0 for all small τ. Then its derivative over τ at $z = 0$, which is $f(\exp(\tau f\partial_z)(0)) = f(0)$ by (2.14), should also be 0. So f must be of the form $\sum_{n\geqslant 0} a_n z^{n+1}$.

2.16

We call those in 2.10 and 2.11 **change of (boundary) parametrizations** in general, and those in 2.15 **change of (local) coordinates**. The former contains the latter.

When changing the boundary parametrizations, the standard coordinate z could be changed to σ_τ not necessarily defined at 0, or more generally, a local coordinate (say) η_1 of an N-pointed $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ is changed to $\sigma_\tau \circ \eta_1$. This changes the boundary-parametrized Riemann surface Σ to Σ'. Note that this process does not violate our definition of *analytic* boundary parametrizations in 1.2: The new surface Σ' is obtained from a new N-pointed one $\mathfrak{X}' = (C'; x_\bullet; \sigma_\tau \circ \eta_1, \eta_1, \dots, \eta_N)$ where C' is a new compact Riemann surface, which is defined by gluing Σ with N pieces of unit discs $\mathbb{D}_1$ using the maps $\sigma_\tau \circ \eta_1, \eta_2, \dots, \eta_N$. (If you use the maps $\eta_1, \dots, \eta_N$ instead, you simply get C.) Thus, *for the change of boundary parametrizations in general, the underlying compact Riemann surfaces C could be changed*.

By change of coordinates, we mean $\mathfrak{X}$ is changed to $\mathfrak{X}' = (C; x_\bullet; \eta'_\bullet)$ with the same underlying compact Riemann surface C and the same marked points $x_\bullet$ as the original ones but different local coordiates at these marked points. As mentioned in 2.15, in this process, only $L_0, L_1, L_2, \dots$ (and also $\overline{L}_0, \overline{L}_1, \overline{L}_2, \dots$) are involved, while in the change of boundary parametrizations, all L_n are involved.

In the previous discussions, almost all formulas hold only up to scalar multiplications or additions. However, when only $L_{-1}, L_0, L_1, L_2, \dots$ are involved, the interaction maps T_Σ can indeed be chosen such that all the formulas truly hold, not just up to scalar multiplications or additions. This is because the conformal anomaly is due to the central term $c \cdot (m^3 - m)\delta_{m,-n}/12$ in the Virasoro relation (2.8), which vanishes when $m, n \geqslant -1$. Note that L_{-1} is responsible for translation. Thus:

Principle 2.10. T_Σ can be chosen to have no ambiguity when changing the local coordinates, or when translating a marked point x_i with respect to its local coordinate η_i.

To be more precise: We fix a compact Riemann surface C. Then for each choice of N marked points $x_\bullet$ and local coordinates $\eta_\bullet$, we can choose the correlation function $T_{\mathfrak{X}}:\mathcal{H}^{\otimes N}\to\mathbb{C}$ associated to the boundary parametrized surface associated to $\mathfrak{X}=(C;x_\bullet;\eta_\bullet)$ such that

- For another choice of N-pointed $\mathfrak{X}'=(C;x_\bullet;\eta'_\bullet)$ with the same marked points and different local coordinates $\eta'_\bullet$, $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}'}$ are related by (2.16).
- If $\mathfrak{X}'=(C;x'_1,x_2,\dots,x_N;\eta'_1,\eta_2,\dots,\eta_N)$ where $\eta'_1=\eta_1-\eta_1(x'_1)$, and if x'_1 is inside an open disc U_1 centered at x_1 on which η_1 is holomorphically defined (more precisely, this means $\eta_1(U_1)$ is an open disc centered at $\eta_1(x_1)=0$), then $T_{\mathfrak{X}}$ and $T_{\mathfrak{X}'}$ are related by (2.16), namely, (noticing (2.15) for $n=1$)

$$T_{\mathfrak{X}}(\xi_1\otimes\cdots\otimes\xi_N)=T_{\mathfrak{X}'}\Big(\big(e^{-\eta_1(x'_1)L_{-1}}\otimes e^{-\overline{\eta_1(x'_1)}\cdot\overline{L}_{-1}}\big)\xi_1\otimes\xi_2\otimes\cdots\otimes\xi_N\Big). \tag{2.22}$$

A similar principle holds when $T_{\mathfrak{X}}$ has output strings. □

Recall the geometric picture described in 2.11. We see that when changing local coordinates, everything in 2.11 truly holds, not just up to scalar multiplications. In particular, the abstract vector $\widetilde{\xi}_1$ is uniquely determined when only the change of local coordinates are allowed.

2.17

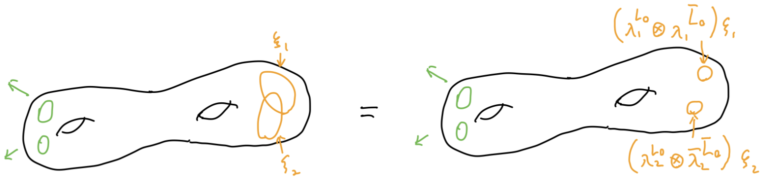
Assumption 2.11. We drop Assumption 1.1 for the incoming strings when we associate only finite energy vectors (i.e., vectors of $\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$, $\mathbb{V}\otimes\widehat{\mathbb{V}}$, etc.) to the incoming strings. Instead, we only assume that the (distinct) incoming points are outside the outgoing strings.

In this course, we will be mainly interested in finite energy vectors. Therefore, we do not assume that that each $\eta_i(U_i)$ contains $\mathbb{D}_1^{\mathrm{cl}}$, or that U_i and U_j are disjoint for different i and j. In the latter case, the two boundary strings $\eta_i^{-1}(\mathbb{S}^1)$ and $\eta_j^{-1}(\mathbb{S}^1)$ possibly overlap. What does this picture actually mean?

Note that multiplying η_i by $\lambda\eta_i$ amounts to shrinking the size of the string $\eta_i^{-1}(\mathbb{S}^1)$ by $|\lambda|$ and then rotating the string. If $\lambda>0$ then there is only shrinking but not rotating. Thus, *for an local coordinated N-pointed $\mathfrak{X}=(C;x_\bullet;\eta_\bullet)$, we can find $\lambda_1,\dots,\lambda_N\in\mathbb{C}^\times$ with large enough absolute values such that the new data $\mathfrak{X}'=(C;x_\bullet;\lambda_1\eta_1,\dots,\lambda_N\eta_N)$ satisfies Assumption 1.1.* Then for finite energy vectors $\xi_1,\dots,\xi_N\in\mathcal{H}^{\mathrm{fin}}=\bigoplus_i\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$, $T_{\mathfrak{X}}(\xi_1\otimes\cdots\otimes\xi_N)$ is understood as

$$T_{\mathfrak{X}}(\xi_1\otimes\cdots\otimes\xi_N):=T_{\mathfrak{X}'}\Big(\big(\lambda_1^{L_0}\otimes\overline{\lambda_1}^{\overline{L}_0}\big)\xi_1\otimes\cdots\otimes\big(\lambda_N^{L_0}\otimes\overline{\lambda_N}^{\overline{L}_0}\big)\xi_N\Big). \tag{2.23}$$

This definition is independent of the choice of sufficiently large $\lambda_1, \ldots, \lambda_N$. And each $\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}$ acts diagonally on $\mathcal{H}^{\mathrm{fin}}$ since $L_0 \otimes \overline{L}_0$ does. (Recall Assumption 2.4.)



In the spirit of the previous subsection, you should view the finite energy vectors ξ_j and $\big(\lambda_j^{L_0} \otimes \overline{\lambda_j}^{\overline{L}_0}\big)\xi_j$ not as different vectors, but as two coordinate representations of the same vector $\widetilde{\xi}_j$. When $|\lambda_j|$ becomes infinitely large, the string for ξ_j shrinks to an infinitesimal one around x_j, i.e., it shrinks to x_j as a **puncture**. It is very useful to view the abstract finite energy vector $\widetilde{\xi}_j$ not associated to any particular string, but associated to that puncture x_j. Thus, the marked points $x_\bullet$ of $\mathfrak{X}$ are also called punctures.

Remark 2.12. A side note: When we do local coordinate changes, finite energy vectors are changed to finite energy ones.

Therefore, in the above discussion, we don't have to stick to change of coordinates of the form $\eta_j \mapsto \lambda_j \eta_j$: any local coordinate change is valid. We will prove the above claim in later sections.

2.18

Let us choose $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$ inside $\mathcal{H}^{\mathrm{fin}}$. According to Assumption 2.4, the eigenvalues of the diagonalizable operators L_0 (on $\mathbb{W}_i$) and $\overline{L}_0$ (on $\widehat{\mathbb{W}}_i$) are $\geqslant 0$. Now choose eigenvectors $w \in \mathbb{W}_i$ and $\widehat{w} \in \widehat{\mathbb{W}}_i$ with $L_0 w = \Delta w, \overline{L}_0 \widehat{w} = \widehat{\Delta}\widehat{w}$ where $\Delta, \widehat{\Delta} \geqslant 0$.

Here is an important point about the two eigenvalues. They are not necessarily integers, which means that $\lambda^{L_0} w$ and $\overline{\lambda}^{\overline{L}_0}\widehat{w}$ might be *multivalued with respect to* λ, i.e., they may also depend on the choice of argument $\arg\lambda$. However, according to the No-Ambiguity Principle 2.10, the expression

$$\big(\lambda^{L_0} \otimes \lambda^{\overline{L}_0}\big)(w \otimes \widehat{w}) = \lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} \cdot w \otimes \widehat{w}$$

must be single-valued with respect to λ, namely, it does not rely on the choice of $\arg\lambda$. As $\lambda = |\lambda| e^{\mathbf{i}\arg\lambda}$ and hence $\lambda^{\Delta}\overline{\lambda}^{\widehat{\Delta}} = |\lambda|^{\Delta+\widehat{\Delta}} e^{\mathbf{i}(\Delta-\widehat{\Delta})\arg\lambda}$, we conclude that

$$\Delta - \widehat{\Delta} \in \mathbb{Z}. \tag{2.24}$$

This gives a constraint on the possible $\mathbb{V} \otimes \widehat{\mathbb{V}}$-submodules of $\mathcal{H}^{\mathrm{fin}}$.

That $\lambda^{L_0} w$ could be multivalued is a crucial property in CFT, and it is not related to conformal anomaly. Indeed, it is related to the non-uniqueness of decomposing T_Σ into conformal blocks. Thus, *the No-Ambiguity Principle 2.10 does not hold for conformal blocks*.

3 Definition of VOAs, I

3.1

We first give the rigorous definition of vertex operators algebras and a slightly weaker version, graded vertex algebras. Then we explain the meanings of the axioms.

Definition 3.1. A **graded vertex algebra** is a (complex) vector space $\mathbb{V}$ together with a diagonalizable operator L_0 acting on $\mathbb{V}$ whose eigenvalues are inside $\mathbb{N}$. We write the L_0-grading of $\mathbb{V}$ as $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$. Any eigenvector v of L_0 (including 0) is called (L_0)-**homogeneous**, and if $v \in \mathbb{V}(n)$ (i.e. $L_0 v = nv$), we write $\mathrm{wt}v = n$ and call $\mathrm{wt}v$ the **weight** of v. Moreover, we have a linear map

$$\begin{aligned} \mathbb{V} &\rightarrow \big(\mathrm{End}(\mathbb{V})\big)[[z^{\pm 1}]] \\ u &\mapsto Y(u,z) \equiv \sum_{n\in\mathbb{Z}} Y(u)_n z^{-n-1} \end{aligned} \tag{3.1}$$

where each $Y(u)_n \in \mathrm{End}(\mathbb{V})$ is called a **(Fourier) mode**. Here, z is treated as a formal variable. Thus $Y(u,z)v \in \mathbb{V}[[z^{\pm 1}]]$ for each $v \in \mathbb{V}$. The reason for associating z^{-n-1} to $Y(u)_n$ is because we could have (recalling (0.2))

$$\mathrm{Res}_{z=0}\, Y(u,z) z^n dz = Y(u)_n. \tag{3.2}$$

$Y(u,z)$ is called a **vertex operator**.

Moreover, the following axioms are satisfied:

- There is a distinguished vector $\mathbf{1} \in \mathbb{V}(0)$ called **vacuum vector** such that
$$Y(\mathbf{1},z) = \mathbf{1}_{\mathbb{V}}.$$
Namely $Y(\mathbf{1})_{-1} = \mathbf{1}_{\mathbb{V}}$ and $Y(\mathbf{1})_n = 0$ if $n \neq -1$.

- **Creation property**: For each $v \in \mathbb{V}$, $Y(v,z)\mathbf{1} = v + \bullet z + \bullet z^2 + \cdots$ where each $\bullet$ is in $\mathbb{V}$. Namely,
$$Y(v)_{-1}\mathbf{1} = v, \tag{3.3}$$
and $Y(v)_n\mathbf{1} = 0$ for all $n > -1$. This property is abbreviated to
$$\lim_{z\rightarrow 0} Y(v,z)\mathbf{1} = v.$$

- **Grading property**: For each $v \in \mathbb{V}$,
$$[L_0, Y(v,z)] = Y(L_0 v, z) + z\frac{d}{dz}Y(v,z). \tag{3.4}$$

- **Translation property**: There is a distinguished linear operator L_{-1} on $\mathbb{V}$ such that
$$L_{-1}\mathbf{1} = 0, \tag{3.5}$$
and that for each $v \in \mathbb{V}$,
$$[L_{-1}, Y(v,z)] = \frac{d}{dz}Y(v,z). \tag{3.6}$$

- **Jacobi identity**: This is the most crucial yet complicated axiom. We postpone its definition to the next section. (See Def. 4.5.)

We say that $\mathbb{V}$ is a **vertex operator algebra** (VOA) if L_0, L_{-1} can be extended to a sequence of linear operators $(L_n)_{n\in\mathbb{Z}}$ on $\mathbb{V}$ satisfying the Virasoro relation (2.8) for some central charge $c \in \mathbb{C}$, and if there is a distinguished vector $\mathbf{c} \in \mathbb{V}$, called the **conformal vector**, such that

$$Y(\mathbf{c})_n = L_{n-1}, \tag{3.7}$$

or equivalently,

$$Y(\mathbf{c}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{3.8}$$

□

You may wonder why the right hand side of (3.7) is not L_n or L_{n-a} for some constant $a \neq 1$. Indeed, if it were not L_{n-1}, then the Virasoro relation would be not compatible with the Jacobi identity. We will explain this in more details after defining the Jacobi identity. (See Exercise 5.4.)

We warn the readers that our definitions of graded vertex algebras and VOAs are slightly stronger than the usual ones in the VOA literature, which do not require L_0 to have non-negative eigenvalues. This positivity condition $L_0 \geqslant 0$ is very mild and satisfied by most interesting examples including all unitary ones. Since assuming this condition will simply proofs, we keep it in our definition.

Also, in most interesting cases, each $\mathbb{V}(n)$ is finite-dimensional. We do not include this in our definition of VOA here, but we will assume this fact in later sections.

Most VOA textbooks and articles use either ω or ν to denote the conformal vector $\mathbf{c}$. In our notes, ω and ν are reserved for other meanings and hence do not denote conformal vectors in order to avoid conflicts of notations.

The reason why we should assume that $\sum L_n z^{-n-2}$ can be written as $Y(\mathbf{c}, z)$ for some $\mathbf{c} \in \mathbb{V}$ will not be explained in this section. We will explain it in Subsec. 5.4.

There is a notion of **unitary VOA** which we do not define in this course (although our motivations are mainly from unitary CFTs). We refer the readers to [CKLW18, DL14] for details.

3.2

Before we give the motivations for these axioms, let us first derive some useful facts.

Expand the series (3.4) and take the coefficients before each z^{-n-1}. This gives us the following equivalent form of grading property:

$$[L_0, Y(v)_n] = Y(L_0 v)_n - (n+1)Y(v)_n. \tag{3.9}$$

To be more concrete, assuming that v is homogeneous, then

$$[L_0, Y(v)_n] = (\mathrm{wt}v - n - 1)Y(v)_n. \tag{3.10}$$

Namely: *$Y(v)_n$ raises the weights by* $\mathrm{wt}v - n - 1$. It is useful to keep in mind that in the VOA theory, $Y(v)_n$ raises weights when n is sufficiently negative, and lowers weights when n is sufficiently positive. As a related fact, as

$$[L_0, L_n] = -nL_n \tag{3.11}$$

by the Virasoro relation (2.8), L_{-n} raises (resp. L_n lowers) the weights by n.

Remark 3.2. As an application of (3.11), we compute $L_n\mathbf{c}$ when $n \geqslant 0$. Since

$$\mathbf{c} = Y(\mathbf{c})_{-1}\mathbf{1} = L_{-2}\mathbf{1}, \tag{3.12}$$

and since L_{-2} raises the weights by 2, we see that

$$L_0\mathbf{c} = 2\mathbf{c}. \tag{3.13}$$

By $[L_1, L_{-2}] = 3L_{-1}$, $[L_2, L_{-2}] = 4L_0 + \frac{1}{2}c$, and that $L_n\mathbf{1} = 0$ whenever $n > 0$ (since its weight is < 0), we have

$$L_1\mathbf{c} = 0, \qquad L_2\mathbf{c} = \frac{c}{2}\mathbf{1}. \tag{3.14}$$

3.3

By (3.10), for each $u, v \in \mathbb{V}$, we know that $Y(u)_n v$ vanishes when n is sufficiently large. Equivalently, we have

$$Y(u,z)v \in \mathbb{C}((z)). \tag{3.15}$$

This important fact is called the **lower truncation property**. It allows us to use meromorphic functions to study VOAs.

In the definition of graded vertex algebras, if the grading property is replaced by the lower truncation property, and if in particular the diagonalizable L_0 is not introduced, then $\mathbb{V}$ is called a **vertex algebra**. We will not address this most general notion in our notes.

3.4

We let

$$\mathbb{V}' = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)^*$$

where $\mathbb{V}(n)^*$ is the dual space of $\mathbb{V}$. $\mathbb{V}'$ is called the **graded dual space** of $\mathbb{V}$. We let L_0 act on $\mathbb{V}'$ such that $L_0 v' = nv'$ whenever $v' \in \mathbb{V}(n)^*$. Then $L_0^{\mathrm{t}} = L_0$. As before, a **homogeneous** vector of $\mathbb{V}'$ is either 0 or an eigenvector of L_0. From our definition, it is clear that the evaluation between $\mathbb{V}'(m) = \mathbb{V}(m)^*$ and $\mathbb{V}(n)$ vanishes if $m \neq n$.

Proposition 3.3. *For each $u, v \in \mathbb{V}, v' \in \mathbb{V}'$, $\langle v', Y(u,z)v\rangle := \sum_{n\in\mathbb{Z}}\langle v', Y(u)_n v\rangle z^{-n-1}$ is a **Laurent polynomial** of z, i.e.,*

$$\langle v', Y(u,z)v\rangle \in \mathbb{C}[z^{\pm 1}].$$

Thus, when evaluating between **finite energy vectors** (i.e., vectors of $\mathbb{V}$ and $\mathbb{V}'$), $Y(u,z)$ is not only a formal series, but a meromorphic function of $\mathbb{P}^1$ with poles at $0,\infty$.

Proof. We must show that $\sum_{n\in\mathbb{Z}}\langle v',Y(u)_n v\rangle z^{-n-1}$ is a finite sum. By linearity, it suffices to assume that u,v,v' are homogeneous. Then $Y(u)_n v$ is homogeneous with weight $\mathrm{wt}u+\mathrm{wt}v-n-1$. So $\langle v',Y(u)_n v\rangle$ is non-zero only if $\mathrm{wt}v'=\mathrm{wt}u+\mathrm{wt}v-n-1$. Thus

$$\langle v',Y(u,z)v\rangle=\langle v',Y(u)_{\mathrm{wt}u+\mathrm{wt}v-\mathrm{wt}v'-1}\cdot v\rangle\cdot z^{\mathrm{wt}v'-\mathrm{wt}u-\mathrm{wt}v}.$$

□

Remark 3.4. The formula $\lim_{z\to 0}Y(u,z)\mathbf{1}$ can now be understood in an analytic sense: By the creation property, for each $v'\in\mathbb{V}$, $\langle v',Y(u,z)\mathbf{1}\rangle$ is a polynomial of z since it has no negative powers of z. So

$$\lim_{z\to 0}\langle v',Y(u,z)\mathbf{1}\rangle=\langle v',u\rangle \tag{3.16}$$

where the left hand side is the limit of a polynomial function.

3.5

The grading and the translation properties were presented in the "derivative form". We shall present them in the integral form. To prepare for this task, we introduce

$$\mathbb{V}^{\mathrm{cl}}:=\prod_{n\in\mathbb{N}}\mathbb{V}(n)=\big\{(v_0,v_1,v_2,\dots):v_n\in\mathbb{V}(n)\big\}, \tag{3.17}$$

called the **algebraic completion** of $\mathbb{V}$. $\mathbb{V}^{\mathrm{cl}}$ is a naturally a subspace of the dual space $(\mathbb{V}')^*$ of $\mathbb{V}'$. (Indeed, we are mostly interested in the case that each $\mathbb{V}(n)$ is finite dimensional. In such case, one checks easily that $\mathbb{V}^{\mathrm{cl}}=(\mathbb{V}')^*$.) We let

$$P_n:\mathbb{V}^{\mathrm{cl}}\to\mathbb{V}(n),\qquad (v_0,v_1,v_2,\dots)\mapsto v_n \tag{3.18}$$

be the canonical projection onto the n-th component. Then for each $z\in\mathbb{C}^\times=\mathbb{C}\backslash\{0\}$, we have

$$Y(u,z)v\in\mathbb{V}^{\mathrm{cl}}$$

whose projection onto $\mathbb{V}(\mathrm{wt}u+\mathrm{wt}v-n-1)$ is $Y(u)_n v\cdot z^{-n-1}$.

Note that L_0 and λ^{L_0} act on $\mathbb{V}^{\mathrm{cl}}$ in an obvious way:

$$L_0(v_n)_{n\in\mathbb{N}}=(nv_n)_{n\in\mathbb{N}},\qquad \lambda^{L_0}(v_n)_{n\in\mathbb{N}}=(\lambda^n v_n)_{n\in\mathbb{N}}.$$

3.6

Proposition 3.5 (Scale covariance). *For each $\lambda\in\mathbb{C}^\times$, we have*

$$\lambda^{L_0}Y(u,z)\lambda^{-L_0}v=Y(\lambda^{L_0}u,\lambda z)v \tag{3.19}$$

on the level of $\mathbb{V}^{\mathrm{cl}}$. We drop the symbol v and simply write the above relation as

$$\lambda^{L_0}Y(u,z)\lambda^{-L_0}=Y(\lambda^{L_0}u,\lambda z).$$

The method in the following proof will appear repeatedly in our notes.

Proof. Recall $L_0^{\mathrm{t}} = L_0$. Fix $z \in \mathbb{C}^\times$. We prove that for each homogeneous u, v, v',

$$\langle \lambda^{L_0} v', Y(u,z)\lambda^{-L_0} v\rangle = \langle v', Y(\lambda^{L_0}u, \lambda z)v\rangle. \tag{3.20}$$

The left hand side f is a scalar times $\lambda^{\mathrm{wt}v' - \mathrm{wt}v}$, and the right hand side g is a Laurent polynomial of λ. So both are holomorphic functions on $\mathbb{C}^\times$. Clearly these two expressions are equal when $\lambda = 1$. Let us prove that they are equal for all $\lambda \neq 0$ by showing that they satisfy the same differential equation.

From the form of f, it is clear that $\partial_\lambda f(\lambda) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} f(\lambda)$. To compute $\partial_\lambda g$, we first compute an easier derivative $\partial_\lambda \langle v', Y(u, \lambda z)v\rangle$. By the chain rule, we have

$$\frac{\partial}{\partial\lambda}\langle v', Y(u,\lambda z)v\rangle = z\frac{d}{d\zeta}\langle v', Y(u,\zeta)v\rangle\Big|_{\zeta=\lambda z},$$

which, due to the grading property, equals

$$\begin{aligned} &\lambda^{-1}\Big\langle v', \big([L_0, Y(u,\lambda z)] - Y(L_0 u, \lambda z)\big)v\Big\rangle \\ =&(\mathrm{wt}v' - \mathrm{wt}v - \mathrm{wt}u)\lambda^{-1}\big\langle v', Y(u,\lambda z)v\big\rangle. \end{aligned}$$

So

$$\partial_\lambda g(\lambda) = \partial_\lambda \big\langle v', Y(\lambda^{L_0}u, \lambda z)v\big\rangle = \partial_\lambda\big(\lambda^{\mathrm{wt}u}\big\langle v', Y(u,\lambda z)v\big\rangle\big) = (\mathrm{wt}v' - \mathrm{wt}v)\lambda^{-1} g(\lambda).$$

□

Informally, the integral form (3.19) (i.e., the scale covariance) also implies the derivative form (3.9) by taking partial derivative over λ. Thus, on a non-rigorous level, these two forms are equivalent. But the integral form has a clearer geometric meaning, which we shall give later.

In the above proof, we have done our first serious VOA calculation. You should be so familiar with these computations that you can "immediately see" the equivalence of the two forms.

The integral form of $[L_{-1}, Y(u,z)] = \partial_z Y(u,z)$ is

$$e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} = Y(u, z+\tau),$$

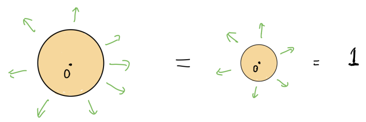
called the **translation covariance**. You may give an informal proof yourself by checking that both sides satisfy the same "linear differential equation". A rigorous treatment is more difficult than the scale covariance. So we leave it to the end of this section.

3.7

We now explain the motivations behind the definition of VOAs. Namely, we shall explain how the axioms are natural assumptions from the point of view of the previous two sections. The following explanations are heuristic and non-rigorous.

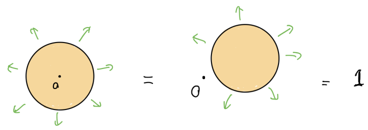
Recall the non-rigorous "definition" of $\mathbb{V}$ in (1.11). We know that $\mathbb{V}$ and $\widehat{\mathbb{V}}$ are subspaces of $\mathcal{H}^{\mathrm{fin}}$, and the decomposition of $\mathcal{H}^{\mathrm{fin}}$ into $\mathbb{V} \otimes \widehat{\mathbb{V}}$-submodules contains a piece $\mathbb{V}\otimes\widehat{\mathbb{V}}$, which furthermore contains $\mathbb{V}\simeq\mathbb{V}\otimes\mathbf{1}$ and $\widehat{\mathbb{V}}\simeq\mathbf{1}\otimes\widehat{\mathbb{V}}$. The vacuum vector is $\mathbf{1}\simeq\mathbf{1}\otimes\mathbf{1}$.

We have said in Subsection 1.8 that the standard unit closed disc $\mathbb{D}_1^{\mathrm{cl}}$ with no input and whose boundary $\mathbb{S}^1$ is parametrized by $z\mapsto z^{-1}$ produces from nothing the vacuum vector $\mathbf{1}\otimes\mathbf{1}$. Namely, the vacuum vector comes from the data $(\mathbb{P}^1;\infty;\zeta^{-1})$ where ζ is the standard coordinate. This data is equivalent to $(\mathbb{P}^1;\infty;\lambda^{-1}\zeta^{-1})$ (where $\lambda\in\mathbb{C}^\times$) via the biholomorphism $z\in\mathbb{P}^1\mapsto\lambda z\in\mathbb{P}^1$. By the change of local coordinate formula (Principle 2.10), the later geometric data produces uniquely the vector $(\lambda^{L_0}\otimes\overline{\lambda}^{\overline{L}_0})\mathbf{1}$, which is equal to $\mathbf{1}$ by the equivalence of the two geometric data. Apply ∂_λ and $\partial_{\overline{\lambda}}$ to $(\lambda^{L_0}\otimes\overline{\lambda}^{\overline{L}_0})\mathbf{1}=\mathbf{1}$, we see that $L_0\mathbf{1}=\overline{L}_0\mathbf{1}=0$. This explain $\mathbf{1}\in\mathbb{V}(0)$ in Def. 3.1.



Consequently, by (2.24), the eigenvalues of L_0 are integers, and hence $\geqslant 0$ integers by the positive energy Assumption 2.4. This explains $\mathrm{Spec}(L_0)\subset\mathbb{N}$.

Similarly, the standard disc $\mathbb{D}_1^{\mathrm{cl}}$ is equivalent to its translation by some $\tau\in\mathbb{C}$. So we must have $(e^{\tau L_{-1}}\otimes e^{\overline{\tau}\overline{L}_{-1}})\mathbf{1}=\mathbf{1}$ and hence, similarly, $L_{-1}\mathbf{1}=\overline{L}_{-1}\mathbf{1}=0$. This explains part of the translation property.



3.8

Recall

$$[L_0,L_n]=-nL_n,\qquad[\overline{L}_0,\overline{L}_n]=-n\overline{L}_n.\tag{3.21}$$

As the L_0 and $\overline{L_0}$ spectral are $\geqslant 0$, and since $\mathbf{1}$ is a zero eigenvectors of them, we must have

$$L_n\mathbf{1}=\overline{L}_n\mathbf{1}=0\qquad(n\geqslant -1).\tag{3.22}$$

From (3.22), we see that for each $v\in\mathbb{V}$, if the change of boundary parametrization does not involve $L_{-2},L_{-3},\dots$ and $\overline{L}_{-2},\overline{L}_{-3},\dots$, then all $\overline{L}_n$ can be ignored:

$$\big(e^{\sum_{n\geqslant -1}a_nL_n}\otimes e^{\sum_{n\geqslant -1}\overline{a_n}\overline{L}_n}\big)v=e^{\sum_{n\geqslant -1}a_nL_n}v.\tag{3.23}$$

To see this, identify v with $v\otimes\mathbf{1}\in\mathbb{V}\otimes\widehat{\mathbb{V}}\subset\mathcal{H}$ and note that $\mathbf{1}$ is fixed by $e^{\sum_{n\geqslant -1}\overline{a_n}\overline{L}_n}$.

Thus, we conclude: *The translation of the change of local coordinates formula for vectors of $\mathbb{V}$ does not involve $\overline{L}_n$.* In particular, note that the right hand side of (3.23) is almost a vector of $\mathbb{V}$. It is a genuine vector of $\mathbb{V}$ when it has finite energy. Thus, *the change of local coordinates and the translation almost preserve* $\mathbb{V}$. Indeed, the change of local coordinates truly preserve $\mathbb{V}$, as we will see in later sections.

A general change of *boundary parametrization* does not necessarily preserve $\mathbb{V}$ in any weak sense.

3.9

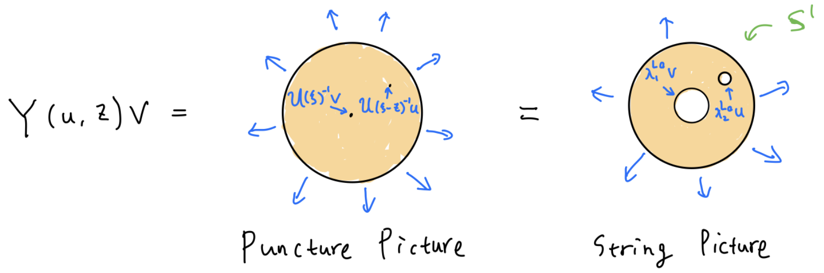
Let us describe the meaning of $Y(u,z)v$. For each $z \in \mathbb{C}^\times$, we define a local-coordinated 3-pointed sphere

$$\mathfrak{P}_z = \{\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}\} \tag{3.24}$$

where ζ is the standard coordinate of $\mathbb{C}$.

Let us regard $0, z$ as incoming punctures and ∞ outgoing. Roughly speaking, $Y(u,z)v$ is just $T_{\mathfrak{P}_z}(v \otimes u)$ where v is associate to 0 and u to z, understood in a suitable way by change of coordinates. Assume first of all that $0 < |z| < 1$. After scaling ζ and $\zeta - z$ to $\lambda_1\zeta, \lambda_2(\zeta - z)$ and hence shrinking the two incoming strings, Assumption 1.1 is satisfied. Let the new N-pointed sphere by denoted by $\mathfrak{P}_z^{\lambda_1,\lambda_2}$. Note that v in the ζ coordinate becomes $(\lambda_1^{L_0} \otimes \overline{\lambda_1}^{\overline{L}_0})v = \lambda_1^{L_0} v$ in the $\lambda_1\zeta$ coordinate. Similarly, u becomes $\lambda_2^{L_0}u$ in the new coordinate. Then $Y(u,z)v$ is (physically) defined as $T_{\mathfrak{P}_z^{\lambda_1,\lambda_2}}(\lambda_1^{L_0}v \otimes \lambda_2^{L_0}u)$.

As in Subsec. 2.17, we can use the *puncture picture* to view u and v as the states associated to the punctures $0, z$ with respect to the local coordinates $\zeta, \zeta - z$. Or moreover, formulated in a coordinate independent way as in Subsec. 2.11, we associate the abstract vector $\mathcal{U}(\zeta)^{-1}v$ (the one whose explicit expression under the coordinate ζ is v) to the puncture 0 and $\mathcal{U}(\zeta - z)^{-1}v$ to z. Then:



(3.25)

According to the notation in Subsec. 2.11, the abstract vectors should be written as $\big(\mathcal{U}(\zeta) \otimes \mathcal{U}(\zeta^*)\big)^{-1}v$ and $\big(\mathcal{U}(\zeta - z) \otimes \mathcal{U}((\zeta - z)^*)\big)^{-1}u$. Here we suppress the second tensor component because, by (3.23), the change of local coordinates for vectors of $\mathbb{V}$ does not involve $\overline{L}_n$.

3.10

In the string picture of (3.25), setting u to be $\mathbf{1}$ means filling the hole around z using the solid disc. The result we get is an annulus $\mathcal{A}_{\lambda_1,1}$ with inside parametrization $\lambda_1\zeta$ and outside one ζ^{-1}.[1] According to the change of coordinate formula, the interaction map $\mathcal{H} \to \mathcal{H}$ for this annulus satisfies $T_{\mathcal{A}_{\lambda_1,1}}(\lambda_1^{L_0}v) = T_{A_{1,1}}v = v$. This explains $Y(\mathbf{1}, z)v = v$.

If we set $v = \mathbf{1}$ instead, then we fill the hold around 0 with the solid disc. The result we get is an eccentric annulus $\mathcal{A}_{z,\lambda_2,1}$ with inside boundary parametrization $\lambda_2(\zeta - z)$ and outside one ζ^{-1}. Let $T_{\mathcal{A}_{z,\lambda_2,1}} : \mathcal{H} \to \mathcal{H}$ be the interaction map. Then, by (3.25), $Y(u,z)\mathbf{1} = T_{\mathcal{A}_{z,\lambda_2,1}}(\lambda_2^{L_0}u)$. Let $z \to 0$. Then $\mathcal{A}_{z,\lambda_2,1}$ converges to $\mathcal{A}_{0,\lambda_2,1}$, which is just the concentric annulus $\mathcal{A}_{\lambda_2,1}$. We have $T_{\mathcal{A}_{\lambda_2,1}}(\lambda_2^{L_0}u) = u$. This explains $\lim_{z\to 0} Y(u,z)\mathbf{1} = u$.

[1] We have previously defined an annulus $A_{r,R}$ with incoming string $|z| = r$ and outgoing $|z| = R$. According to that definition, $\mathcal{A}_{r^{-1},1} = A_{r,1}$.

3.11

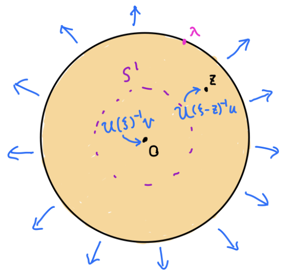
For a general $z \in \mathbb{C}^\times$, in the string picture, we must also shrink the outgoing string in order to get a true surface Σ. We thus choose $\lambda \in \mathbb{C}^\times$ with $|\lambda| > 1$. Let

$$\mathfrak{P}_z^{\lambda_1,\lambda_2,\lambda} = \{\mathbb{P}^1; 0, z, \infty; \lambda_1\zeta, \lambda_2(\zeta - z); \lambda\zeta^{-1}\}.$$

Then $Y(u, z)v$ is physically "defined" to be

$$Y(u, z)v = \lambda^{L_0} T_{\mathfrak{P}_z^{\lambda_1,\lambda_2,\lambda}}(\lambda_1^{L_0} v \otimes \lambda_2^{L_0} u). \tag{3.26}$$

In the puncture picture, it is

$Y(u,z)v = \lambda^{L_0} \cdot$ 

The meaning of $Y(u)_n$ is clear:

$$\langle v', Y(u)_n v\rangle = \oint_0 \langle v', Y(u, z)v\rangle z^n \cdot \frac{dz}{2i\pi} = \mathrm{Res}_{z=0}\, \langle v', Y(u, z)v\rangle z^n dz.$$

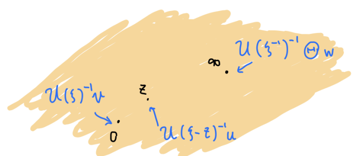
where the subscript under $\oint$ means that the integral is over any loop around 0.

3.12

If we prefer not to scale ζ^{-1}, we can make the output point ∞ input. To do this, note that from Subsec. 1.14 and 1.15, we know that each $\Theta(\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i)$ is equivalent to $\mathbb{W}_i' \otimes \widehat{\mathbb{W}}_i'$, the space of finite energy dual vectors on $\mathbb{W}_i \otimes \widehat{\mathbb{W}}_i$. In the case of $\mathbb{V}$, we get an equivalence

$$\Theta\mathbb{V} \xrightarrow{\simeq} \mathbb{V}', \qquad \Theta w \mapsto \langle \Theta w, \cdot\rangle\big|_{\mathbb{V}} = \langle w|\cdot\rangle\big|_{\mathbb{V}}$$

where $\langle\cdot,\cdot\rangle$ is the correlation function associated to $A_{1,1}$. (From $\langle w|\cdot\rangle\big|_{\mathbb{V}}$ you can see why this linear map is an isomorphism. Here, you may assume each $\mathbb{V}(n)$ is finite-dimensional, or even pretend that $\mathbb{V}$ is finite dimensional.) Then in the puncture picture, the vertex operator and the correlation function of $\mathfrak{P}_z$ (restricted to a linear functional on $\mathbb{V} \otimes \mathbb{V} \otimes \Theta\mathbb{V} \simeq \mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V}'$) are related by

$$\langle \Theta w, Y(u, z)v\rangle = \langle w|Y(u, z)v\rangle \overset{(1.8)}{=\!=}$$ 

for all $u, v, w \in \mathbb{V}$ and hence $\Theta w \in \Theta\mathbb{V} \simeq \mathbb{V}'$.

3.13

We actually have

$$\Theta\mathbb{V} = \mathbb{V} \tag{3.27}$$

and similarly $\Theta\widehat{\mathbb{V}} = \widehat{\mathbb{V}}$. An explanation is as follows:

Proof. First of all, Θ maps finite energy vectors to finite energy ones since Θ commutes with the energy operators $L_0 \otimes \mathbf{1}, \mathbf{1} \otimes \overline{L}_0$. (See Subsec. 2.13.) By the physical definition of $\mathbb{V}$ in (1.11), for each $u \in \mathbb{V}$, the correlation function T_z associated to $\mathfrak{P}_{z,r,R} = (\mathbb{P}^1; z, \infty; (\zeta - z)/r, R/\zeta)$ varies holomorphically if u is associated to the puncture z. Namely, $T_z(u \otimes \nu)$ is holomorphic for all $\nu \in \mathcal{H}$. It is easy to see that the conjugate of $\mathfrak{P}_{z,r,R}$ is equivalent (via the standard conjugation of the complex plane) to $\mathfrak{P}_{\overline{z},r,R} = (\mathbb{P}^1; \overline{z}, \infty; (\zeta - \overline{z})/r, R/\zeta)$, whose correlation function is $T_{\overline{z}}$. Thus, by (1.9)

$$T_z(\Theta u \otimes \nu) = \overline{T_{\overline{z}}(u \otimes \Theta\nu)},$$

which is also holomorphic over z. This proves $\Theta u \in \mathbb{V}$ if $u \in \mathbb{V}$. $\square$

Consequently, $\mathbb{V} = \Theta\mathbb{V} \simeq \mathbb{V}'$. The equivalence is given by

$$\mathbb{V} \xrightarrow{\simeq} \mathbb{V}', \qquad u \mapsto \langle u, \cdot \rangle \tag{3.28}$$

Due to this equivalence, we call the VOA $\mathbb{V}$ to be **self-dual**.

So, in all unitary CFTs (and indeed, also in many non-unitary CFTs), the VOAs are self-dual. We remark that there is a mathematically rigorous definition of self-dualness, which plays an important role in the tensor categories of $\mathbb{V}$-modules. However, the definition of a general VOA does not require self-dualness, because many properties can be derived without assuming self-dualness.

3.14

Let ζ be the standard coordinate of $\mathbb{C}$ as usual. For each $\lambda \neq 0$, we have an equivalence

$$(\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}) \simeq (\mathbb{P}^1; 0, \lambda z, \infty; \lambda^{-1}\zeta, \lambda^{-1}\zeta - z, \lambda\zeta^{-1}) \tag{3.29}$$

realized by the biholomorphism $\gamma \mapsto \lambda\gamma$ of $\mathbb{P}^1$. (You should check that the pull-back of the local coordinates on the right hand side equal those on the left.) The correlation function for the left hand side, evaluating on $v \otimes u \otimes w \in \mathbb{V}^{\otimes 3}$, is $\langle w, Y(u, z)v \rangle$. The right hand side of (3.29) is obtained by scaling the local coordinates of $(\mathbb{P}^1; 0, \lambda z, \infty; \zeta, \zeta - \lambda z, \zeta^{-1})$ (whose correlation function on $\mathbb{V}^{\otimes 3}$ takes the form $\langle w, Y(u, \lambda z)v \rangle$) by $\lambda^{-1}, \lambda^{-1}, \lambda$ respectively. By the change of coordinate formula, the correlation function for the right hand side of (3.29), denoted temporarily by ω, must satisfy

$$\langle w, Y(u, \lambda z)v \rangle = \omega(\lambda^{-L_0}v \otimes \lambda^{-L_0}u \otimes \lambda^{L_0}w),$$

namely, ω should be $\langle \lambda^{-L_0}w, Y(\lambda^{L_0}u, \lambda z)\lambda^{L_0}v \rangle$. This last equation must equal $\langle w, Y(u, z)v \rangle$ due to the equivalence (3.29). This explains the scale covariance.

3.15

Similarly, for each $\tau \in \mathbb{C}$, consider the equivalence

$$(\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1}) \simeq \Big(\mathbb{P}^1; \tau, z+\tau, \infty; \zeta - \tau, \zeta - z - \tau, \frac{1}{\zeta - \tau}\Big) \tag{3.30}$$

induced by the biholomorphism $\gamma \mapsto \gamma + \tau$ of $\mathbb{P}^1$. The right hand side is a change of parametrization from $(\mathbb{P}^1; 0, z+\tau, \infty; \zeta, \zeta - z - \tau, \zeta^{-1})$ (whose correlation function is $\langle w, Y(u, z+\tau)v\rangle$), where ζ is changed to $\zeta - \tau$ (which is a translation), and $\eta := \zeta^{-1}$ is changed to $1/(\eta^{-1} - \tau)$. The translation corresponds to $e^{-\tau L_{-1}}$. The second change of coordinate is $\exp(\tau z^2 \partial_z)$ due to Ex. 2.5, which gives $e^{\tau L_1}$.

Let ω now be the correlation function (restricted to $\mathbb{V}^{\otimes 3}$) of the right hand side. Then we have

$$\langle w, Y(u, z+\tau)v\rangle = \omega(e^{-\tau L_{-1}} v \otimes u \otimes e^{\tau L_1} w).$$

So ω is $\langle e^{-\tau L_1} w, Y(u, z+\tau) e^{\tau L_1} v\rangle = \langle w, e^{-\tau L_{-1}} Y(u, z+\tau) e^{\tau L_1} v\rangle$, which must equal $\langle w, Y(u,z)v\rangle$ due to the equivalence (3.30). This explains the translation covariance.

Exercise 3.6. Find a geometric explanation of $Y(u, z+\tau) = Y(e^{\tau L_{-1}} u, z)$.

There is a another shorter geometric explanation of translation covariance: $e^{\tau L_{-1}} Y(u,z)v$ amounts to moving the outgoing large string in the string picture in (3.25) by $-\tau$. This is the same as fixing the outgoing string and translating the two incoming strings by τ. Translating the one around 0 changes v to $e^{\tau L_{-1}} v$, and translating the one around z just changes z to $z + \tau$.

This second explanation is however less rigorous than the first one. But the first one is not rigorous anyway. So why should we care about the issue of rigor here? Well, our first geometric explanation for translation covariance, as well as the one in Subsec. 3.14 for rotation covariance, is much more rigorous in the sense that you can easily get the correct formulas using this method. You may try and give a short explanation for rotation covariance using our second method. Then you will realize that it is not easy to get the correct formula since the change of local coordinates is not so easy to visualize.

3.16

Now we return to rigorous mathematics. We are going to prove translation covariance rigorously. For that purpose, we need to generalize the differential equation method in the proof of scale covariance to the following vector-valued form:

Lemma 3.7. *Let W be a (non-necessarily finite dimensional) vector space, and $f \in W[[z]]$. Suppose that $\frac{d}{dz} f(z) = A f(z)$ for some $A \in \mathrm{End}(W)$. Suppose also that $f(0) = 0$, namely, the constant term in the power series $f(z)$ is 0. Then $f = 0$.*

Proof. Write $f(z) = \sum_{n \in \mathbb{N}} f_n z^n$ where each $f_n \in W$. The assumptions say that $f_0 = 0$ and

$$\sum_{n \in \mathbb{N}} n f_n z^{n-1} = \sum_{n \in \mathbb{N}} A f_n z^n.$$

So $n f_n = A f_{n-1}$ where $n > 0$. This proves that all f_n are 0. □

3.17

We have said that the integral form of $[L_{-1}, Y(u,z)] = \partial_z Y(u,z)$ is

$$\big\langle v', e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} v \big\rangle = \big\langle v', Y(u, z+\tau) v \big\rangle. \tag{3.31}$$

This relation is more difficult to address than the scale covariance since both sides actually involve infinite sums of powers of τ. Our goal is to understand: on which domain does this relation hold? Certainly we need $\tau \neq -z$. But this condition is far from enough.

Let us first understand the two sides as infinite series of τ and z. Assume without loss of generality that u, v, v' are homogeneous. The right hand side is of the form $a(z+\tau)^m$ for some $a \in \mathbb{C}, m \in \mathbb{Z}$. Certainly this expression makes sense as a rational function, but we shall first regard it as a formal series of τ, z by expanding $(z+\tau)^m$ on the domain $|\tau| < |z|$, namely $(z+\tau)^m = \sum_{k \in \mathbb{N}} \binom{m}{k} z^{m-k} \tau^k$. Thus, the right hand side of (3.31), as an element of $\mathbb{C}[z^{\pm 1}][[\tau]]$, is understood as

$$\big\langle v', Y(u, z+\tau) v \big\rangle = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \binom{-n-1}{k} \big\langle v', Y(u)_n v \big\rangle \cdot z^{-n-1-k} \tau^k.$$

Here, the sum over $n \in \mathbb{Z}$ is finite, and when the vectors are homogeneous, there is only one possibly non-zero summand.

But why do we expand $(z+\tau)^m$ on $|\tau| < |z|$? Why not $|z| < |\tau|$? Well, this will give us $\sum_{k \in \mathbb{N}} \binom{m}{k} z^k \tau^{m-k}$ which contains negative powers of τ. But the left hand side of (3.31) actually has only non-negative powers of τ.

So let us turn to the left hand side of (3.31). It would be easier to first understand why

$$\big\langle v', e^{\lambda L_{-1}} Y(u,z) e^{-\mu L_{-1}} v \big\rangle \tag{3.32}$$

is an element of $\mathbb{C}[z^{\pm 1}][[\lambda, \mu]]$. We first want to move $e^{\lambda L_{-1}}$ to the left hand side of the bracket. In general, if L_n is defined on $\mathbb{V}$, we define L_{-n} on $\mathbb{V}'$ to be the transpose of L_n: $L_{-n} = L_n^{\mathrm{t}}$, or more precisely,

$$\langle L_{-n} v', v \rangle := \langle v', L_n v \rangle. \tag{3.33}$$

In case you doubt why this transpose exists, we can write the definition even more precisely: Assume $v' \in \mathbb{V}'(m)$. Then $L_{-n} v'$ is a linear functional on $\mathbb{V}(m+n)$ (so L_{-n} raises the weights by n) whose value at any $v \in \mathbb{V}(m+n)$ is $\langle v', L_n v \rangle$. (Recall that L_n lowers the weights by n so $L_n v \in \mathbb{V}(m)$.) And $L_{-n} v'$ vanishes on $\mathbb{V}(a)$ if $a \neq m+n$.

Now, (3.32) equals

$$f(z, \lambda, \mu) := \big\langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} v \big\rangle = \sum_{n, l \in \mathbb{N}} \frac{\lambda^n (-\mu)^l}{n! l!} \big\langle L_1^n v', Y(u,z) L_{-1}^l v \big\rangle. \tag{3.34}$$

This is in $\mathbb{C}[z^{\pm 1}][[\lambda, \mu]]$. Indeed, it is in $\mathbb{C}[z^{\pm 1}][[\mu]][\lambda]$ since $L_1^n v'$ lowers the weight by n, and hence vanishes when $n > \mathrm{wt} v'$. But we will not need this fact here.

Now, the left hand side of (3.31) can be understood as $f(z, \tau, \tau)$, noting the following fact:

Lemma 3.8. *Let W be a vector space. If $\varphi(z_1,\dots,z_N) \in W[[z_1,\dots,z_N]]$, then $\varphi(z,\dots,z)$ naturally makes sense as an element of $W[[z]]$.*

Proof. Write $\varphi(z_\bullet) = \sum a_{n_1,\dots,n_N} z_1^{n_1}\cdots z_N^{n_N}$. Then

$$\varphi(z,\dots,z) = \sum_{n\in\mathbb{N}} \sum_{n_1+\cdots+n_N=n} a_{n_1,\dots,n_N} z^n$$

where the inside sum is clearly finite. □

3.18

Proposition 3.9 (Translation covariance). *For each $u, v \in \mathbb{V}$, $v' \in \mathbb{V}'$, the following equation holds on the level of $\mathbb{C}[z^{\pm 1}][[\tau]]$:*

$$\big\langle v', e^{\tau L_{-1}} Y(u,z) e^{-\tau L_{-1}} v \big\rangle = \big\langle v', Y(u, z+\tau) v \big\rangle. \tag{3.35}$$

Here, the right hand side, which is a priori a Laurent polynomial of $z+\tau$, is expanded as if $|\tau| < |z|$.

Proof. Let $f_z(\tau)$ and $g_z(\tau)$ be the left and the right hand sides of (3.35), considered as formal power series of τ whose coefficients are elements of $\mathbb{C}[z^{\pm 1}]$. Then clearly $f_z(0) = g_z(0)$ as polynomials of $z^{\pm 1}$. So, it suffices to prove that f_z and g_z satisfy the same linear differential equation. The left hand side is $f_z(\tau,\tau)$ where

$$f_z(\lambda,\mu) = \big\langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} v \big\rangle \qquad \in \mathbb{C}[z^{\pm 1}][[\lambda,\mu]].$$

As a general result about multivariable formal power series, we have chain rule

$$\partial_\tau f_z(\tau,\tau) = (\partial_\lambda + \partial_\mu) f_z(\lambda,\mu)\big|_{\lambda=\mu=\tau}.$$

(It is reasonable to believe that this is true. But you can also give a rigorous proof by expanding the two series and check that their coefficients agree!) So, as

$$\begin{aligned}\partial_\lambda f_z(\lambda,\mu) &= \big\langle e^{\lambda L_1} L_1 v', Y(u,z) e^{-\mu L_{-1}} v \big\rangle,\\ \partial_\mu f_z(\lambda,\mu) &= -\big\langle e^{\lambda L_1} v', Y(u,z) e^{-\mu L_{-1}} L_{-1} v \big\rangle,\end{aligned}$$

we have

$$\partial_\tau f_z(\tau) = \big\langle e^{\tau L_1} L_1 v', Y(u,z) e^{-\tau L_{-1}} v \big\rangle - \big\langle e^{\tau L_1} v', Y(u,z) e^{-\tau L_{-1}} L_{-1} v \big\rangle.$$

This expression is not a differential equation of the $\mathbb{C}[z^{\pm 1}]$-coefficients power series f_z. But we can make it an ODE by fixing u, varying v, v', and view f_z as a $\mathcal{V} := \mathrm{Hom}(\mathbb{V}\otimes\mathbb{V}', \mathbb{C}[z^{\pm 1}])$-valued power series of τ. Then $\partial_\tau f_z = A f_z$ where $A \in \mathrm{End}\mathcal{V}$ is defined by sending each $\Phi : \mathbb{V}\otimes\mathbb{V}' \to \mathbb{C}[z^{\pm 1}]$ to

$$A\Phi = \Phi \circ (\mathbf{1}\otimes L_1 - L_{-1}\otimes \mathbf{1}).$$

Now, we compute (noting that the following sum is finite for each fixed u, v)

$$\partial_\tau g_z(\tau) = \partial_\tau \big\langle v', Y(u, z+\tau) v \big\rangle = \partial_\tau \Big(\sum_n a_n (z+\tau)^n \Big)$$

$$= \sum_n n a_n (z+\tau)^{n-1} = \partial_\zeta \Big(\sum_n a_n \zeta^n \Big)\Big|_{\zeta=z+\tau} = \partial_\zeta \langle v', Y(u,\zeta)v \rangle\big|_{\zeta=z+\tau}.$$

By the translation property, the above equals

$$\partial_\tau g_z(\tau) = \langle v', [L_{-1}, Y(u,\zeta)]v \rangle\big|_{\zeta=z+\tau},$$

which also equals $Ag_z(\tau)$ if we now vary v, v' and regard g as $\mathcal{V}$-valued. Therefore, $f_z(\tau) = g_z(\tau)$ due to Lemma 3.7. □

3.19

Let us consider a useful variant of Prop. 3.9. Notice that (3.35) holds if v' is replaced by L_1^n and also both sides are multiplied by τ^n. Thus, (3.35) holds on the level of $\mathbb{C}[z^{\pm 1}][[\tau]]$ if v' is replaced by $e^{-\tau L_1} v'$. Namely:

$$\langle v', Y(u,z) e^{-\tau L_{-1}} v \rangle = \langle e^{-\tau L_1} v', Y(u, z+\tau) v \rangle. \tag{3.36}$$

Remark 3.10. The left hand sides of (3.35) and (3.36) converges absolutely when $|\tau| < |z|$ since the right hand side does. These right hand sides are linear combinations of $(z+\tau)^m$ for some $m \in \mathbb{Z}$, whose expansion $\sum_{j,k\in\mathbb{Z}} a_{j,k} z^j \tau^k := \sum_{n\in\mathbb{N}} \binom{m}{n} z^{m-n}\tau^n$ clearly satisfies

$$\sup_{(z,\tau)\in K} \sum_{j,k\in\mathbb{Z}} |a_{j,k} z^j \tau^k| < +\infty \tag{3.37}$$

on every compact subset K of $\{(z,\tau) : |\tau| < |z|\}$. Thus, the same convergence property holds for the left hand sides of (3.35) and (3.36). We call this property the **absolute and locally uniform converge**, which will be the focus of our study in this course.

Thus, we have actually proved our first convergence result in this course. The method used here is standard in the VOA theory: we show that a formal power series converges by identifying it with the power series expansion of a holomorphic function, which can be achieved with the help of linear differential equations.

3.20

Let us choose $v = \mathbf{1}$ in the formula (3.35). Then, as $L_{-1}\mathbf{1} = 0$, we obtain

$$\langle v', e^{\tau L_{-1}} Y(u,z)\mathbf{1} \rangle = \langle v', Y(u, z+\tau)\mathbf{1} \rangle \tag{3.38}$$

on the level of $\mathbb{C}[z,\tau]$, since, by Rem. 3.4, the right hand side is a polynomial of $z+\tau$. As $z \to 0$, the left hand side converges to $\langle e^{\tau L_1} v', u \rangle = \langle v', e^{\tau L_{-1}} u \rangle$ by (3.16). So we conclude:

Corollary 3.11. *For each $u \in \mathbb{V}, v \in \mathbb{V}'$, the equation*

$$\langle v', e^{\tau L_{-1}} u \rangle = \langle v', Y(u,\tau)\mathbf{1} \rangle$$

holds as polynomials of τ. Equivalently, the equation

$$e^{\tau L_{-1}}u = Y(u,\tau)\mathbf{1}$$

holds on the level of $\mathbb{V}[[\tau]]$, which is equivalent to that for each $n \in \mathbb{N}$,

$$Y(u)_{-n-1}\mathbf{1} = \frac{1}{n!}L_{-1}^n u. \tag{3.39}$$

We leave it to the reader to find a geometric explanation of $e^{\tau L_{-1}}u = Y(u,\tau)\mathbf{1}$.

4 Definition of VOAs, II: Jacobi Identity

4.1

Principle 4.1. When gluing Riemann spheres to get new spheres, the formula $T_{\Sigma_1} \circ T_{\Sigma_2} = T_{\Sigma_1 \# \Sigma_2}$ truely holds if the local coordinates at the points for sewing are Möbius transformations, i.e. of the form $z \mapsto \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

A rough reason for this No-Ambiguity Principle is that only $L_0, L_{\pm 1}$ are involved in the change of coordinate formulas between Möbius transformations, and the Lie bracket relations between them do not involve the central charge.

4.2

We shall give motivations for the Jacobi identity.

We first remark on the sewing of compact Riemann surfaces in Subsec. 1.4. Suppose we have data $\mathfrak{X} = (C; x_\bullet; \eta_\bullet)$ and $\mathfrak{X}' = (C'; y_\bullet; \eta'_\bullet)$ and we sew them along x_1 and x'_1. For simplicity, we set $\xi = \eta_1, \varpi = \eta'_1$. From (1.4), we know that the gluing law is that any $x \in \xi^{-1}(\mathbb{S}^1)$ (recall that $\xi^{-1}(\mathbb{S}^1)$ is a boundary string of the corresponding surface Σ for $\mathfrak{X}$) and any $y \in \varpi^{-1}(\mathbb{S}^1)$ are identified following the rule

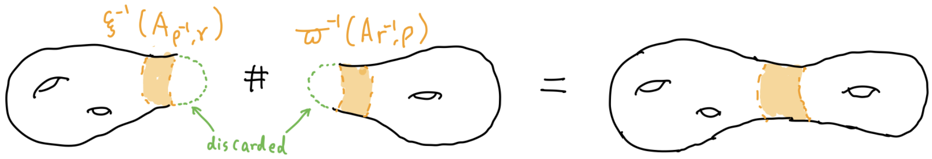
$$x = y \qquad \Longleftrightarrow \qquad \xi(x)\varpi(y) = 1. \tag{4.1}$$

This definition of gluing is topological, but not complex analytic. Analytically, we are actually gluing a neighborhood of $\xi^{-1}(\mathbb{S}^1)$ and one of $\varpi^{-1}(\mathbb{S}^1)$ using the rule (4.1) for all x in the first neighborhood and y in the second one. It is clear that a (locally defined) function on the first neighborhood is holomorphic if and only if it is so on the second one. This defines the complex analytic structure on $C\#C'$.

Remark 4.2. Let us be more precise on the shape of the neighborhoods. Let ξ and ϖ be defined (and injective) on U, U' respectively. Choose $r > 1, \rho > 1$ such that $\xi(U) \supset \mathbb{D}_r, \varpi(U') \supset \mathbb{D}_\rho$. Then the following neighborhoods of $\xi^{-1}(\mathbb{S}^1)$ and $\varpi^{-1}(\mathbb{S}^1)$ are glued via the relation (4.1):

$$\begin{array}{c} \xi^{-1}(A_{\rho^{-1},r}) = \{x \in U : \rho^{-1} < |\xi(x)| < r\} \\ \Big\updownarrow \text{identified via (4.1)} \\ \varpi^{-1}(A_{r^{-1},\rho}) = \{y \in U' : r^{-1} < |\varpi(y)| < \rho\} \end{array} \tag{4.2}$$

The parts $\{x \in U : |\xi(x)| \leqslant \rho^{-1}\}$ and $\{y \in U' : |\varpi(y)| \leqslant r^{-1}\}$ are discarded when gluing.



(4.3)

4.3

As pointed out before, when we associate finite energy vectors to the incoming strings/points, we may scale their local coordinates. However, for the local coordinates at the output points and the points to be sewn, an arbitrary scaling is not allowed. We thus assume that Assumption 1.1 holds after scaling (by some λ with arbitrarily large $|\lambda|$) the local coordinates at the incoming points. This amounts to the following

Assumption 4.3. If x_i is either an outgoing point or a point to be sewn with another point, then the local coordinate η_i at x_i defined on a neighborhood $U_i \ni x_i$ satisfies that $\eta_i(U_i) \supset \mathbb{D}_1^{\mathrm{cl}}$, that $\eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}}) \cap \eta_j^{-1}(\mathbb{D}_1^{\mathrm{cl}}) = \varnothing$ if x_j is either outgoing or a point to be sewn, and that $x_j \in \eta_i^{-1}(\mathbb{D}_1^{\mathrm{cl}})$ if x_j is incoming and not to be sewn.

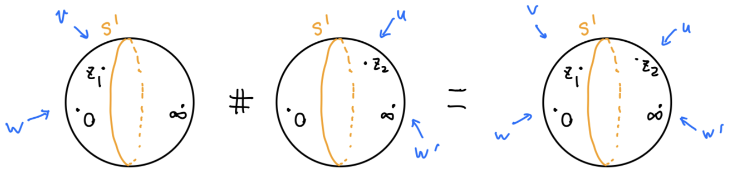
Remark 4.4. There is indeed one way we can slightly loosen the above assumption. Using the notation of (4.1). Then we may assume that Assumption 4.3 after scaling ξ by some $\lambda \in \mathbb{C}^\times$ and ϖ by λ^{-1}. Then the rule for gluing (4.1) is not changed. On the side of interaction maps T_Σ, the change $\xi \rightsquigarrow \lambda\xi$ adds a factor $\lambda^{-L_0} \otimes (\overline{\lambda})^{-\overline{L}_0}$ to one tensor component in T_Σ, and $\xi \rightsquigarrow \lambda^{-1}\xi$ adds a factor $\lambda^{L_0} \otimes \overline{\lambda}^{\overline{L}_0}$. These two are canceled after taking contraction or composition.

4.4

We want to understand the product $\langle w', Y(u, z_2)Y(v, z_1)w\rangle$. Let ζ be the standard coordinate of $\mathbb{C}$. By the sewing property in Segal's picture, this expression should correspond to the sewing of

$$\mathfrak{P}_{z_1} = (\mathbb{P}_1^1; 0, z_1, \infty; \zeta, \zeta - z_1, \zeta^{-1}), \qquad \mathfrak{P}_{z_2} = (\mathbb{P}_2^1; 0, z_2, \infty; \zeta, \zeta - z_2, \zeta^{-1})$$

along the points ∞ of $\mathfrak{P}_{z_1}$ and 0 of $\mathfrak{P}_{z_2}$. (Here, both $\mathbb{P}_1^1$ and $\mathbb{P}_2^1$ are $\mathbb{P}^1$. We assume the two ∞ are outgoing before sewing.) Assumption 4.3 is satisfied when $0 < |z_1| < 1 < |z_2| < +\infty$ if we consider all the points not for sewing as incoming. The sewing rule is that $\gamma_1 \in \mathbb{P}_1^1, 0 < |\gamma_1^{-1}| < +\infty$ is identified with $\gamma_2 \in \mathbb{P}_2^1, 0 < |\gamma_2| < +\infty$ if and only if $\gamma_1^{-1} \cdot \gamma_2 = 1$, namely $\gamma_1 = \gamma_2$. (Here, we set $r = \rho = \infty$ in order to apply Rem. 4.2. The discarded points are the ∞ of $\mathbb{P}_1^1$ and the 0 of $\mathbb{P}_2^1$.) Thus, the sewing is just placing the first sphere onto the second one.



The result of sewing is

$$\mathfrak{P}_{z_1,z_2} = (\mathbb{P}^1; 0, z_1, z_2, \infty, \zeta - z_1, \zeta - z_2, \zeta^{-1}) \tag{4.4}$$

Assuming all the points of $\mathfrak{P}_{z_1,z_2}$ as incoming, for each $u, v, w, w' \in \mathbb{V}$,

$$T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, w') = \langle w', Y(u, z_2)Y(v, z_1)w\rangle \qquad (\text{if } 0 < |z_1| < |z_2| < +\infty). \tag{4.5}$$

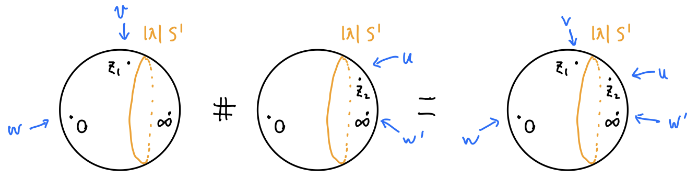
The reason why the conditions $|z_1| < 1$ and $1 < |z_2|$ can be dropped is explained below.

4.5

We explain why (4.5) holds provided $0 < |z_1| < |z_2| < +\infty$.

Pick $\lambda \in \mathbb{C}$ such that $|z_1| < |\lambda| < |z_2|$. Following the guide of Rem. 4.4, we replace the local coordinate ζ^{-1} of $\mathfrak{P}_{z_1}$ by $\lambda\zeta^{-1}$ and the one ζ of $\mathfrak{P}_{z_2}$ by ζ/λ. Then Assumption 4.3 is again satisfied. In particular, the outgoing string of $\mathbb{P}^1_1$ around ∞ and the incoming one of $\mathbb{P}^1_2$ around 0 are both $|\lambda|\mathbb{S}^1$.

The interaction map $T_{\mathfrak{P}_{z_1}} : \mathcal{H}^{\otimes 2} \to \mathcal{H}$ acting on $w \otimes v$ is $\lambda^{-L_0}Y(v, z_1)w$. $T_{\mathfrak{P}_{z_2}}$ sends $u \otimes \_ \in \mathbb{V} \otimes \mathbb{V}$ to $Y(u, z_2)\lambda^{L_0}\_$. The composition of these two expressions, evaluated with $w' \in \mathbb{V}$, is again the right hand side of (4.5). And the result of sewing is again $\mathfrak{P}_{z_1,z_2}$. So (4.5) holds in general.



4.6

According to the physical definition of $\mathbb{V}$ in Subsec. 1.12 as well as the No-Ambiguity Principle 2.10, we know that when the vectors of $\mathbb{V}$ are inserted, the correlation functions change holomorphically with respect to the translation of the marked points and their local coordinates. Thus $T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, v')$ is a holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times) = \{(z_1, z_2) \in \mathbb{C}^\times : z_1 \neq z_2\}$. Since, similar to (4.5), we also have

$$T_{\mathfrak{P}_{z_1,z_2}}(w, v, u, w') = \langle w', Y(v, z_1)Y(u, z_2)w\rangle \qquad (\text{if } 0 < |z_2| < |z_1| < +\infty), \tag{4.6}$$

we conclude that $\langle w', Y(u, z_2)Y(v, z_1)w\rangle$ defined on $0 < |z_1| < |z_2|$ and $\langle w', Y(v, z_1)Y(u, z_2)w\rangle$ defined on $0 < |z_2| < |z_1|$ can be continued to the same holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times)$. That this fact is true for all $w, w' \in \mathbb{V}$ (or more generally, all $w \in \mathbb{V}, w' \in \mathbb{V}'$ if $\mathbb{V} \simeq \mathbb{V}'$ is not assumed) is simply written as

$$Y(u, z_2)Y(v, z_1) \sim Y(v, z_1)Y(u, z_2). \tag{4.7}$$

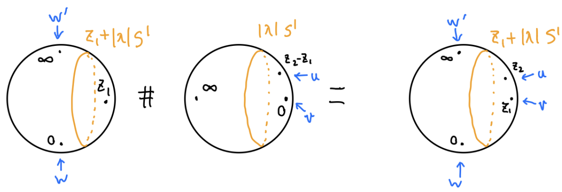
This property is called **commutativity**.

4.7

We now consider the sewing of

$$\mathfrak{P}_{z_1} = (\mathbb{P}^1_1; 0, z_1, \infty; \zeta, \zeta - z_1, \zeta^{-1}), \qquad \mathfrak{P}_{z_2 - z_1} = (\mathbb{P}^1_{21}; 0, z_2 - z_1, \infty; \zeta, \zeta - z_2 + z_1, \zeta^{-1})$$

(where $\mathbb{P}^1_{21} = \mathbb{P}^1$) along the points $z_1 \in \mathbb{P}^1_1$ and $\infty \in \mathbb{P}^1_{21}$. We assume $0 < |z_2 - z_1| < |z_1| < +\infty$. Choose $\lambda \in \mathbb{C}$ satisfying $|z_2 - z_1| < |\lambda| < |z_1|$. Replace the local coordinate $\zeta - z_1$ of $\mathfrak{P}_{z_1}$ by $\lambda^{-1}(\zeta - z_1)$ and the one ζ^{-1} of $\mathfrak{P}_{z_2 - z_1}$ by $\lambda\zeta^{-1}$. Then Assumption 4.3 is satisfied. The rule for sewing is identifying $\gamma_1 \in \mathbb{P}^1_1, 0 < |\lambda^{-1}(\gamma_1 - z_1)| < +\infty$ with $\gamma_{21} \in \mathbb{P}^1_{21}, 0 < |\lambda\gamma_{21}^{-1}| < +\infty$ if and only if $(\gamma_1 - z_1) = \gamma_{21}$. Thus, gluing $\mathfrak{P}_{z_1 - z_1}$ to $\mathfrak{P}_{z_1}$ amounts to translating $\mathfrak{P}_{z_2 - z_1}$ to $\mathfrak{P}_{z_2}$. After sewing, the points 0 and $z_2 - z_1$ of $\mathfrak{P}_{z_2 - z_1}$ become z_1 and z_2. (The points z_1 of $\mathfrak{P}_{z_1}$ and ∞ of $\mathfrak{P}_{z_2 - z_1}$ are discarded.)



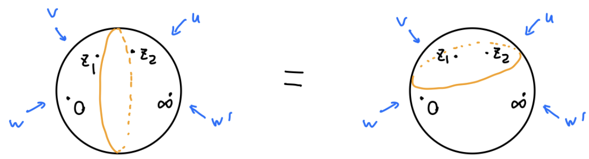
This sewing picture gives

$$T_{\mathfrak{P}_{z_1, z_2}}(w, v, u, w') = \langle w', Y(Y(u, z_2 - z_1)v, z_1)w \rangle \quad (\text{if } 0 < |z_2 - z_1| < |z_1| < +\infty). \tag{4.8}$$

We therefore have the **associativity** property

$$\begin{gathered} \langle w', Y(u, z_2)Y(v, z_1)w \rangle = \langle w', Y(Y(u, z_2 - z_1)v, z_1)w \rangle \\ \text{if } 0 < |z_2 - z_1| < |z_1| < |z_2|. \end{gathered} \tag{4.9}$$

Geometrically, it means the equivalence of sewing spheres in the following way:



4.8

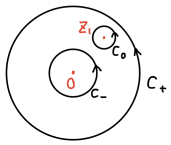
The fact that for all $u, v, w \in \mathbb{V}, w' \in \mathbb{V}'$, (4.5), (4.6), and (4.8) can be defined as holomorphic functions of z_1, z_2 on the given domain (the precise meaning will be given later), and that these three expressions can be extended to the same holomorphic function (namely $T_{\mathfrak{P}_{z_1, z_2}}(w \otimes v \otimes u \otimes w')$) on $\mathrm{Conf}^2(\mathbb{C}^\times)$ is called the **Jacobi identity** in the complex analytic form. Roughly speaking,

$$\text{Jacobi identity} = \text{Commutativity} + \text{Associativity}. \tag{4.10}$$

For the moment, we will derive an algebraic version, and use it as the formal definition of Jacobi identity in Def. 3.1.

Write $f(z_1,z_2)=T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$. Fix $z_1\in\mathbb{C}^\times$, and consider f as a holomorphic function of z_2 on $\mathbb{C}^\times\backslash\{z_1\}$. (Moreover, from (4.5), (4.6), (4.8), and by the lower truncation property (3.15), it is easy to see that f has finite poles at $z_1=0,z_2,\infty$. So f is a meromorphic function.) By the residue theorem, for each meromorphic 1-form μ on $\mathbb{P}^1$ with possible poles only at $0,z_1,\infty$, we must have $(\mathrm{Res}_{z_2=0}+\mathrm{Res}_{z_2=z_1}+\mathrm{Res}_{z_2=\infty})f\mu=0$. It is easy to see that such μ are linear combinations of those of the form $z_2^m(z_2-z_1)^n dz_2$.

Equivalently, choose C_+ to be a circle around 0 whose radius is $>|z_1|$, C_- is one around 0 whose radius is $<|z_1|$, and C_0 a small circle around z_1 between C_+ and C_-.



Let f_+,f_-,f_0 be respectively the right hand sides of (4.5), (4.6), (4.8). Then, when z_2 is on C_+,C_-,C_0 respectively, f equals f_+,f_-,f_0. Then the fact that f_+,f_-,f_0 defined on their domains extend to the same meromophic function on $\mathbb{P}^1$ with poles $0,z_1,\infty$ implies for any $m,n\in\mathbb{Z}$ and $\mu=z_2^m(z_2-z_1)^n dz_2$ that

$$\oint_{C_+}\frac{f_+\mu}{2\mathbf{i}\pi}-\oint_{C_-}\frac{f_-\mu}{2\mathbf{i}\pi}=\oint_{C_0}\frac{f_0\mu}{2\mathbf{i}\pi}. \tag{4.11}$$

Indeed, the latter one also implies the previous one. This is guaranteed by the so called *strong residue theorem*, which will be discussed in later sections. The strong residue theorem will imply that the analytic form and the algebraic form of Jacobi identity are equivalent.

Recall the general formula $\oint_C Y(u,z)z^k\frac{dz}{2\mathbf{i}\pi}=Y(u)_k$ if C is a circle around the origin. When $z_2\in C_+$, μ has absolutely convergent expansion $\mu=\sum_{l\in\mathbb{N}}\binom{n}{l}(-z_1)^l z_2^{m+n-l}dz_2$. So

$$\begin{aligned}&\oint_{C_+}\frac{f_+\mu}{2\mathbf{i}\pi}=\sum_{l\in\mathbb{N}}\oint_{C_+}\binom{n}{l}(-z_1)^l z_2^{m+n-l}\langle w',Y(u,z_2)Y(v,z_1)w\rangle\frac{dz_2}{2\mathbf{i}\pi}\\ =&\sum_{l\in\mathbb{N}}\binom{n}{l}(-z_1)^l\langle w',Y(u)_{m+n-l}Y(v,z_1)w\rangle=:a(z_1)\end{aligned}$$

When $z_2\in C_-$,

$$\begin{aligned}&\oint_{C_-}\frac{f_-\mu}{2\mathbf{i}\pi}=\sum_{l\in\mathbb{N}}\oint_{C_+}\binom{n}{l}(-z_1)^{n-l}z_2^{m+l}\langle w',Y(v,z_1)Y(u,z_2)w\rangle\frac{dz_2}{2\mathbf{i}\pi}\\ =&\sum_{l\in\mathbb{N}}\binom{n}{l}(-z_1)^{n-l}\langle w',Y(v,z_1)Y(u)_{m+l}w\rangle=:b(z_1)\end{aligned}$$

When $z_2\in C_0$, since $0<|z_2-z_1|<|z_1|$, we have the absolutely convergent expansion $\mu=(z_1+(z_2-z_1))^m(z_2-z_1)^n dz_2=\sum_{l\in\mathbb{N}}\binom{m}{l}z_1^{m-l}(z_2-z_1)^{n+l}dz_2$. So

$$\oint_{C_0}\frac{f_0\mu}{2\mathbf{i}\pi}=\sum_{l\in\mathbb{N}}\oint_{C_0}\binom{m}{l}z_1^{m-l}(z_2-z_1)^{n+l}\langle w',Y(Y(u,z_2-z_1)v,z_1)w\rangle\frac{dz_2}{2\mathbf{i}\pi}$$

$$=\sum_{l\in\mathbb{N}}\binom{m}{l}z_1^{m-l}\langle w', Y(Y(u)_{n+l}v, z_1)w\rangle := c(z_1)$$

Now we have $c(z_1) = a(z_1) - b(z_1)$. We vary z_1. For each $k \in \mathbb{Z}$, multiply both sides by $z_1^k \frac{dz_1}{2i\pi}$ and apply the residue at $z_1 = 0$. We then get (by suppressing w' and w)

Definition 4.5 (Jacobi identity (algebraic version)**).** For each $u, v, w \in \mathbb{V}$, and each $m, n, k \in \mathbb{Z}$, we have

$$\begin{aligned}&\sum_{l\in\mathbb{N}}\binom{m}{l}Y\big(Y(u)_{n+l}v\big)_{m+k-l}\\ =&\sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l}Y(u)_{m+n-l}Y(v)_{k+l}-\sum_{l\in\mathbb{N}}(-1)^{n+l}\binom{n}{l}Y(v)_{n+k-l}Y(u)_{m+l}.\end{aligned}\tag{4.12}$$

This completes Definition 3.1.

In the above three terms, when acting on every $w \in \mathbb{V}$, each sum over $l \in \mathbb{N}$ is finite thanks to the lower truncation property.

5 Consequences of Jacobi identity; reconstruction theorem

5.1

The algebraic form of Jacobi identity is very complicated. Very few people can write down exactly the right formula without checking the references or reproving this formula using the long argument in Subsec. 4.8. But we shall try our best to explain how to use this formula and what this formula implies.

First of all, if (4.12) holds whenever $m = 0$ or $n = 0$, then it holds in general. We will not give a rigorous proof for this statement. But, since (4.12) is derived from (4.11) for all $\mu = z_2^m(z_2 - z_1)^n dz_2$, the readers can be convinced of this statement by the following elementary fact:

Exercise 5.1. Show that $z_2^m(z_2 - z_1)^n$ is a $\mathbb{C}[z_1^{\pm 1}]$-linear combination of z_2^k and $(z_2 - z_1)^l$ where $k, l \in \mathbb{Z}$ and $l < 0$. (Hint: Assume without loss of generality that $m, n < 0$. Prove the statement by induction on $|m|$ and $|n|$.)

Thus, we may understand (4.12) by restricting to the special cases $m = 0, n < 0$ or $n = 0$.

5.2

We now return to rigorous mathematics. Consider the case that $n = 0$, i.e., $\mu = z_2^m dz_2$. Then (4.12) reads

$$\big[Y(u)_m, Y(v)_k\big] = \sum_{l\in\mathbb{N}}\binom{m}{l}Y\big(Y(u)_l v\big)_{m+k-l}.\tag{5.1}$$

This is a Lie bracket relation. Interestingly, this general formula does not come from Lie groups, but from the residue theorem. However, in many concrete examples, such Lie bracket relations do have Lie-theoretic origins.

Let me take this chance to say a few words about the similarity and the difference between the VOA theory and the Lie theory. In the VOA theory, the residue theorem is the standard way of passing from the complex analytic world to the algebraic world. The opposite direction is through the strong residue theorem. This is strikingly different from the Lie theory, in which one passes from the differential geometric formulation (i.e. Lie groups) to the algebraic one (i.e. Lie algebras) by taking derivatives, and vice versa by taking exponentiation/integral. Thus, although Lie brackets do appear in VOAs, it is not always fruitful to think of VOAs as generalizations of Lie algebras. These two mathematical objects have very different geometric intuitions. Also, if we view VOAs in the complex analytic way, then by (4.10), VOAs are more like commutative algebras. Thus, VOAs can be viewed as a quantum version of both the Lie algebras and the commutative algebras.

5.3

Take u to be the conformal vector $\mathbf{c}$ in (5.1) and recall that $Y(\mathbf{c})_{m+1} = L_m$. We obtain

$$\begin{aligned}[L_m, Y(v)_k] &= \sum_{l\in\mathbb{N}} \binom{m+1}{l} Y(L_{l-1}v)_{m+k+1-l} \\ &= Y(L_{-1}v)_{m+k+1} + \sum_{l\in\mathbb{N}} \binom{m+1}{l+1} Y(L_l v)_{m+k-l}. \end{aligned} \tag{5.2}$$

Multiply z^{-k-1} to both sides and take the sum over all $k \in \mathbb{Z}$, we obtain

$$[L_m, Y(v,z)] = z^{m+1}Y(L_{-1}v,z) + \sum_{l\in\mathbb{N}} \binom{m+1}{l+1} z^{m-l} Y(L_l v, z) \tag{5.3}$$

either on the level of $\mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$, or as Laurent polynomials of z when evaluating between any $w \in \mathbb{V}$ and $w' \in \mathbb{V}'$. Then the cases $m = -1$ and $m = 0$ imply

$$[L_{-1}, Y(v,z)] = Y(L_{-1}v,z) \tag{5.4a}$$
$$[L_0, Y(v,z)] = zY(L_{-1}v,z) + Y(L_0 v,z). \tag{5.4b}$$

Note that these two equations follow solely from the Jacobi identity. By the translation property, we have

$$Y(L_{-1}v,z) = \frac{d}{dz}Y(v,z). \tag{5.5}$$

Equivalently, by applying $\mathrm{Res}_{z=0}(\cdot)z^n dz$, we get a crucial relation

$$Y(L_{-1}v)_n = -nY(v)_{n-1}. \tag{5.6}$$

(The quickest way to get the formula on the right hand side is integration by parts.)

Exercise 5.2. Show that (3.39) follows from (5.6) and the creation property.

Exercise 5.3. Assume that $\mathbb{V}$ satisfies the lower truncation property (3.15) and all the axioms of VOAs in Def. 3.1 except the grading and the translation property. Use (5.4) to prove that the following conditions are equivalent.

1. The grading property.
2. $Y(L_{-1}v,z) = \partial_z Y(v,z)$ for all $v \in \mathbb{V}$.
3. The translation property.
4. The translation property without assuming $L_{-1}\mathbf{1} = 0$.

Thus, we may use the lower truncation property and any of these four conditions to replace the grading and the translation properties in the definition of VOAs.

Exercise 5.4. In (5.2), set $v = \mathbf{c}$, and show that this formula is compatible with the Virasoro relation.

5.4

We see that (5.3) for $m = 0, -1$ (together with (5.5)) means the grading and the translation properties, which integrate to the rotation and the translation covariance. For general m, (5.3) also has a geometric explanation. To simplify discussions, we give such an explanation by assuming that v is primary.

Definition 5.5. A vector $v \in \mathbb{V}$ is called a **primay vector** if it is homogeneous and $L_n v = 0$ for all $n > 0$.

Some important VOAs (affine VOAs for instance) are generated by primary vectors. And many important formulas in CFT were first proved by physics who assumed that their theories are generated by primary vectors in the following sense:

Definition 5.6. We say that a VOA $\mathbb{V}$ is **generated** by a subset $E \subset \mathbb{V}$ if $\mathbb{V}$ is spanned by vectors of the form $Y(v_1)_{n_1} \cdots Y(v_k)_{n_k}\mathbf{1}$ where $k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{Z}$, and $v_1, \dots, v_k \in E$.

Indeed, formula (5.3) for any primary vector v is one such example, which (combined with (5.5)) reads

$$[L_m, Y(v,z)] = z^{m+1}\partial_z Y(v,z) + (m+1)\mathrm{wt}v \cdot z^m Y(v,z). \tag{5.7}$$

This is called by physicists (or more precisely, is equivalent to what physicists call) the **conformal Ward identity**.

Choose a holomorphic vector field $f(z)\partial_z = \sum_{n\in\mathbb{Z}} a_n z^{n+1}\partial_z$ on a neighborhood of $\mathbb{S}^1$. Let $\sigma_\tau = \exp(\tau f \partial_z)$ be the holomorphic flow. Then (5.7) (with L_m, z^m replaced by $\sum_m a_m L_m, \sum_m a_m z_m$) integrates to

$$e^{\tau\sum_{n\in\mathbb{Z}} a_n L_n} Y(v,z) e^{-\tau\sum_{n\in\mathbb{Z}} a_n L_n} = \big(\partial_z\sigma_\tau(z)\big)^{\mathrm{wt}v} Y\big(v, \sigma_\tau(z)\big), \tag{5.8}$$

called **conformal covariance**. For now, we do not treat this formula in a rigorous way. But the readers can convince themselves by checking that both sides satisfy the same linear differental equation over τ.

The right hand side of (5.8) looks familiar to us. Set $\tau = 1$, $\sigma = \sigma_1$, and $\Delta = \mathrm{wt}v$. Then formula (5.8) resembles the change of variable formula $\big(\partial(\varphi \circ \sigma)\big)^\Delta = \big(\partial\varphi \circ \sigma\big)^\Delta \cdot (\partial_z\sigma)^\Delta$ for a function $\varphi = \varphi(z)$ and ∂ is the standard holomorphic derivative. Indeed, the primary field $Y(v, z)$ can be viewed as the quantization of $(\partial\varphi)^\Delta$, or more generally, of $\partial\varphi_1 \cdots \partial\varphi_\Delta$. It is also interesting to write (5.8) in the form

$$e^{\sum a_n L_n}\big(Y(v, z)dz^\Delta\big)e^{-\sum a_n L_n} = Y(v, \sigma)d\sigma^\Delta. \tag{5.9}$$

Conformal covariance (5.8) can be interpreted in a similar geometric way as we did for rotation and translation covariance in Subsec. 3.14 and 3.15. (We will give this explanation in the future assuming $f = \sum_{n\geqslant 0} a_n z^{n+1}\partial_z$.) So, from the CFT point of view, this formula follows naturally from our change of parametrization formula in Sec. 2 and the physical definition of the vertex operator $Y(v, z)$ in Sec. 3 (if we ignore the issue of uniqueness up to scalar multiplications). In particular, the geometric intuition we are using for formula (5.7) is Lie theoretic, because the relationship between Virasoro algebras and change of parametrization formula is the one between the representations of Lie algebras and Lie groups. But we have also derived (5.7) from the Jacobi identity, whose geometric intuition relies on the residue theorem. How should we view this coincidence of the two geometric pictures?

My answer is that we should regard the Lie theoretic explanation as the fundamental one for conformal covariance/Ward identity. In fact, to use the Jacobi identity to obtain (5.7), we have assumed that $\sum L_n z^{-n-2}$ is the vertex operator of a vector of $\mathbb{V}$, namely the conformal vector $\mathbf{c}$. But the reason that this assumption should be included in the definition of VOA was not explained in Sec. 3. Here we give a short explanation: we will see later (cf. the reconstruction Thm. 5.12 and Rem. 5.13) that if the Fourier modes $A_m \in \mathrm{End}(\mathbb{V})$ of a field $A(z)$ satisfy the correct Jacobi identity (such as (5.1) or (5.7)) with the modes $Y(v)_k$ for v inside a generating subset $E \subset \mathbb{V}$, then $A(z)$ must be $Y(u, z)$ for some $u \in \mathbb{V}$. Thus, (in my opinion) the better point of view is that we use the conformal Ward identity (whose geometric intuition relies on the change of parametrization formula and the physical meaning of $Y(u, z)$) and the Jacobi identity to explain the fact that $\sum L_n z^{-n-2}$ is represented by a vector $\mathbf{c}$ in $\mathbb{V}$, but not that we explain the Ward identity using the VOA Jacobi identity.

5.5

We say that $\mathbb{V}$ is of **CFT-type** if $\dim \mathbb{V}(n) < +\infty$ for each n, and $\mathbb{V}(0) = \mathbb{C}\mathbf{1}$. The CFT-type condition is a very natural and mild one satisfied by all the examples in our notes. It says that the only quantum states with zero energy are the vacuum.

In this subsection, we assume $\mathbb{V}$ is CFT-type, and study (5.1) for vectors in $\mathbb{V}(1)$. For each $u \in \mathbb{V}(1)$, we write $Y(u)_m$ as u_m for short. By (3.10), u_l lowers the weights by l. Then (5.1) says $[u_m, v_n] = (u_0 v)_{m+n} + m(u_1 v)_{m+n-1}$, where $u_l v$ vanishes when $l > 1$ since its weight is $1 - l$. Since $u_1 v \in \mathbb{V}(0) \in \mathbb{C}$, we may write

$$u_1 v = (u, v)\mathbf{1} \tag{5.10}$$

where $(\cdot,\cdot)$ is a bilinear form on $\mathbb{V}(1)$. Thus $(u_1 v)_{m+n-1} = (u,v)\delta_{m,-n}$ since $Y(\mathbf{1},z) = \mathbf{1}$. Set

$$[u,v] := u_0 v. \tag{5.11}$$

Then

$$[u_m, v_n] = [u,v]_{m+n} + m(u,v)\delta_{m,-n}. \tag{5.12}$$

Proposition 5.7. *$[\cdot,\cdot]$ defines a Lie algebra structure on $\mathbb{V}(1)$, and $(\cdot,\cdot)$ is an invariant symmetric bilinear form, namely, $(u,v) = (v,u)$ and $([w,u],v) = -(u,[w,v])$.*

Proof. $w \in \mathbb{V}(1) \mapsto w_{-1}$ is injective since $w_{-1}\mathbf{1} = w$ by the creation property. By (5.12), $[u,v]_{-1} = [u_0, v_{-1}] = -[v_{-1}, u_0] = -[v,u]_{-1}$. This proves $[u,v] = -[v,u]$. By calculating $[u_1, v_{-1}]$ and $[v_{-1}, u_1]$ using (5.12), we obtain $(u,v) = (v,u)$. (5.12) implies

$$[w_k, [u_m, v_n]] = [w,[u,v]]_{k+m+n} + k(w,[u,v])\delta_{k+m+n,0}.$$

Apply the Jacobi identity for the Lie bracket of linear operators, we obtain the Jacobi identity for $[\cdot,\cdot]$ on $\mathbb{V}(1)$ if we set $k=-1, m=n=0$, and we obtain the invariance of $(\cdot,\cdot)$ if we set $k=0, m=1, n=-1$. $\square$

The vector space $\mathrm{Span}_{\mathbb{C}}\{v_n, \mathbf{1}_{\mathbb{V}} : n \in \mathbb{Z}\}$ is a Lie algebra whose bracket is the standard one for linear operators. Since it satisfies (5.12), we call it an **affine Lie algebra** associated to the finite-dimensional complex Lie algebra $\mathbb{V}(1)$. When $\mathbb{V}$ is generated by $\mathbb{V}(1)$, we say $\mathbb{V}$ is an **affine VOA**.

We are mostly interested in the case that $(\cdot,\cdot)$ is non-degenerate. This is always true when the CFT (or the VOA) is unitary, since $(\cdot,\cdot)$ is indeed the negative of the correlation function $\langle\cdot,\cdot\rangle = \langle\Theta\cdot|\cdot\rangle$ of $A_{1,1}$ restricted to $\mathbb{V}^{\otimes 2}$. Moreover, a unitary affine VOA $\mathbb{V}$ is indeed uniquely determined by its Lie subalgebra $\mathbb{V}(1)$, where $\mathbb{V}(1)$ is a direct sum of an abelian Lie algebra and a semisimple one. (We refer the readers to [Gui19, Sec. 1 and 2] for a detailed account of the relationship between unitary VOAs and their "unitary" Lie subalgebras $\mathbb{V}(1)$.) Affine Lie algebras and affine VOAs in the strict sense are those such that $\mathbb{V}(1)$ are simple Lie algebras. If on the other hand $\mathbb{V}(1)$ is abelian, then $\mathbb{V}$ is called a **free boson VOA** or a **Heisenberg VOA**.

If $\mathbb{V}$ is generated by $\mathbf{c}$, we call $\mathbb{V}$ a **Virasoro VOA**.

5.6

We now turn to the case $m=0, n<0$ in the VOA Jacobi identity (4.12). First consider $n=-1$. Then (4.12) reads

$$Y\big(Y(u)_{-1}v\big)_k = \sum_{l\in\mathbb{N}} Y(u)_{-1-l}Y(v)_{k+l} + \sum_{l\in\mathbb{N}} Y(v)_{k-1-l}Y(u)_l. \tag{5.13}$$

This formula can be written in a compact way. For a general series $f(z) = \sum_{l\in\mathbb{Z}} a_l z^{-l-1} \in W[[z^{\pm1}]]$ where W is a vector space, we let

$$f(z)_+ = \sum_{l\in\mathbb{N}} a_l z^{-1-l}, \qquad f(z)_- = \sum_{l\in\mathbb{N}} a_{-l-1} z^l \tag{5.14}$$

(so we have $f(z) = f(z)_+ + f(z)_-$). Define the **normal-ordered product**

$$:Y(u,z)Y(v,z): = Y(u,z)_- Y(v,z) + Y(v,z)Y(u,z)_+ \tag{5.15}$$

which is non-commutative in general. Then (5.13) can be abbreviated to

$$Y\big(Y(u)_{-1}v, z\big) = :Y(u,z)Y(v,z): \tag{5.16}$$

By (5.6) we have

$$Y(u)_{-j-1} = \frac{1}{j!}Y(L_{-1}^j u)_{-1} \tag{5.17}$$

when $j \geqslant 0$. Combine this with $Y(L_{-1}^j u, z) = \partial_z^j Y(u,z)$, we obtain

$$Y\big(Y(u)_{-j-1}v, z\big) = \frac{1}{j!}:\big(\partial_z^j Y(u,z)\big)Y(v,z): \tag{5.18}$$

where the normal-ordered product is defined in a similar way using the positive and the negative parts of $\partial_z^j Y(u,z)$. We leave it to the readers to check that this formula agrees with the Jacobi identity (4.12) when $m = 0, n < 0$.

Thus, once we know how $Y(u,z)$ looks like for all u in a small generating subset E of $\mathbb{V}$, we can write down the formula of $Y(w,z)$ for any $w \in \mathbb{V}$ using the formula

$$Y\big(Y(u_1)_{-j_1-1}\cdots Y(u_k)_{-j_k-1}v, z\big) = \frac{1}{j_1!\cdots j_k!}:\partial_z^{j_1}Y(u_1,z)\cdots\partial_z^{j_k}Y(u_k,z)\cdot Y(v,z): \tag{5.19}$$

where the normal-ordered product for several operators is defined inductively by

$$:A_1A_2\cdots A_n: = :A_1(:A_2\cdots A_n:): \tag{5.20}$$

5.7

One can also write down the explicit formula of $Y(Y(u)_n v, z)$ for $n \geqslant 0$ using (4.12) where $m = 0, n \geqslant 0$. But as I said, (4.12) is determined by the special cases $m = 0, n < 0$ and $n = 0$. So we hope that $Y(Y(u)_n v, z), n \geqslant 0$ can be calculated using (5.1). This is true.

Write (5.1) in the equivalent form

$$\big[Y(u)_m, Y(v,z)\big] = \sum_{l\in\mathbb{N}}\binom{m}{l}z^{m-l}Y\big(Y(u)_l v, z\big). \tag{5.21}$$

Thus, for $m \geqslant 0$, $Y(Y(u)_m v, z)$ can be computed inductively by

$$\begin{aligned} Y\big(Y(u)_0 v, z\big) &= \big[Y(u)_0, Y(v,z)\big] \\ Y\big(Y(u)_m v, z\big) &= \big[Y(u)_m, Y(v,z)\big] - \sum_{l=0}^{m-1}\binom{m}{l}z^{m-l}Y\big(Y(u)_l v, z\big). \end{aligned} \tag{5.22}$$

We now see the close relation between the Lie brackets of vertex operators and the data $Y\big(Y(u)_m v, z\big), m \geqslant 0$. The latter plays a very different role from

$Y(Y(u)_m v, z), m < 0$. To understand this relation better, we write the associativity relation (4.9) as

$$Y(u, z_2)Y(v, z_1) = \sum_{m\in\mathbb{Z}} (z_2 - z_1)^{-m-1} Y(Y(u)_m v, z_1) \tag{5.23}$$

when $0 < |z_2 - z_1| < |z_1|$. Here, we understand $Y(u, z_2)Y(v, z_1)$ as $Y(v, z_1)Y(u, z_2)$ when $0 < |z_1| < |z_2|$ or more generally, as a linear functional on $\mathbb{V}^{\otimes 2}$ sending $w \otimes w'$ to $T_{\mathfrak{P}_{z_1,z_2}}(w \otimes v \otimes u \otimes w')$ (the correlation function associated to (4.4)) for all $(z_1, z_2) \in \mathrm{Conf}^2(\mathbb{C}^\times)$. Then the part $m \geqslant 0$ in (5.23) accounts for the poles of $T_{\mathfrak{P}_{z_1,z_2}}(w\otimes v\otimes u\otimes w')$ at $z_2 = z_1$.

The summand in (5.23) vanishes for sufficiently positive m. In physics, a series expansion of the form

$$A(z_2)B(z_1) = \sum_{m\geqslant -N} (z_2 - z_1)^m C^m(z_1)$$

is called the **operator product expansion (OPE)** of the fields $A(z_2), B(z_1)$. Thus, in the VOA context, *OPEs are just the associativity property* (4.9). OPE is useful to physicists because it allows them to reduce the calculation of 4-point correlations functions to that of 3-point ones, or in general, N-point to $(N-1)$-point.

We split the right hand side of (5.23) into two parts: $m \geqslant 0$, which is called the **regular terms** since it has no poles at $z_2 = z_1$, and $m < 0$ called the **singular terms**. Thus

$$Y(u, z_2)Y(v, z_1) = \frac{Y(Y(u)_{N-1}v, z_1)}{(z_2 - z_1)^N} + \cdots + \frac{Y(Y(u)_0 v, z_1)}{(z_2 - z_1)} + \text{regular terms},$$

or, written in physics language,

$$Y(u, z_2)Y(v, z_1) \sim \frac{Y(Y(u)_{N-1}v, z_1)}{(z_2 - z_1)^N} + \cdots + \frac{Y(Y(u)_0 v, z_1)}{(z_2 - z_1)}. \tag{5.24}$$

Thus, to summarize, (5.1) *establishes a close relationship between the Lie brackets of vertex operators, the finite poles of the correlation function $T_{\mathfrak{P}_{z_1,z_2}}$ at $z_1 = z_2$, and the finitely may singular terms in the OPE of vertex operators*. As a special case, from (5.21) and (5.22) one sees that two vertex operators $Y(u, z_2), Y(v, z_1)$ commute (namely, their Fourier modes $Y(u)_m, Y(v)_k$ commute) iff there are no singular terms in the OPE of $Y(u, z_2)Y(v, z_1)$, iff $T_{\mathfrak{P}_{z_1,z_2}}(\cdot \otimes v \otimes u \otimes \cdot)$ is holomorphic on a neighborhood of $z_2 = z_1$.

5.8

In the previous subsection, we derived the relationship from the definition of VOAs (in particular, from the VOA Jacobi identity). So one may ask this natural question: does this relationship rely on the full Jacobi identity? For instance, does it rely on (5.18)?

The answer is no. In a very vague sense, any of the following three implies the others without assuming the full Jacobi identity.

1. Suitable Lie bracket relations hold for a pair of field operators $A(z_2), B(z_1)$.
2. The finite poles of (the analytic continuation of) $\langle w', A(z_2)B(z_1)w\rangle$ at $z_2 = z_1$.
3. The finitely many singular terms in the OPE of $A(z_2)B(z_1)$ and, in particular, *the existence of such OPE*.

Clearly, the third one a priori implies the second one, since the second does not assume the existence of OPE. Thus, as we have said that OPEs are roughly the same as associativity, we see that the associativity (and indeed, the full Jacobi identity) can be derived from the first or the second statement above. This is called the **reconstruction theorem** because it allows us to build examples of VOAs by checking only a small part of the Jacobi identity, namely the Lie bracket relations. This theorem is the most important one for constructing examples of VOAs.

A rigorous and detailed discussion of the equivalence of the above three statements will be given in the later section on local fields. The first and the second statements correspond to three seeming different but indeed equivalent definitions of the **locality** of $A(z_2), B(z_1)$. (There are two ways to describe the second one, a formal variable way and a complex analytic way.) Here, we first state the rigorous definition of the first one.

5.9

We let $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$ be an $\mathbb{N}$-graded vector space, graded by a diagonalizable operator L_0. We do not assume that $\mathbb{V}$ and L_0 are from any graded vertex algebra.

Definition 5.8. An **(L_0-)homogeneous field (operator)** on $\mathbb{V}$ is an element

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$$

(where each A_n is in $\mathrm{End}(\mathbb{V})$) satisfying

$$[L_0, A(z)] = \Delta_A \cdot A(z) + z\partial_z A(z) \tag{5.25}$$

or equivalently,

$$[L_0, A_n] = (\Delta_A - n - 1)A_n \qquad (\forall n \in \mathbb{Z}). \tag{5.26}$$

Δ_A is called the **weight** of $A(z)$.

Clearly, a homogeneous field $A(z)$ satisfies the **lower truncation property** $A(z)w \in \mathbb{C}((z))$ (for all $w \in \mathbb{V}$).

Definition 5.9 (**Local fields** (Lie algebraic version))**.** Given homogeneous fields $A(z)$ and $B(z)$, we say $A(z)$ is **local** to $B(z)$ if there exist $C^j(z) = \sum_{n\in\mathbb{Z}} C^j_n z^{-n-1} \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$ (where $j = 0, 1, \ldots, N-1$ for some $N \in \mathbb{N}$) satisfying

$$[A_m, B_k] = \sum_{l=0}^{N-1} \binom{m}{l} C^l_{m+k-l} \tag{5.27}$$

for all $m, k \in \mathbb{Z}$. We consider the right hand side of (5.27) as 0 if $N = 0$.

Remark 5.10. $A(z)$ is local to $B(z)$ if and only if there exist $D^0(z), \dots, D^{N-1}(z) \in \mathrm{End}(\mathbb{V})[[z^{\pm 1}]]$ satisfying for all $m, k \in \mathbb{Z}$ that

$$[A_m, B_k] = \sum_{l=0}^{N-1} m^l D^l_{m+k}. \tag{5.28}$$

This is because $\widetilde{C}^l_j := C^l_{j-l}$ and D^l_j are related by $\widetilde{C}^l_j + \sum_{p=l+1}^{N-1} a_{p,l} \cdot \widetilde{C}^p_j = D^l_j$ where each $a_{p,l} \in \mathbb{R}$ is determined by $\binom{m}{p} = m^p + \sum_{l=1}^{p-1} a_{p,l} \cdot m^l$.

Exercise 5.11. Use (5.28) to show that if $A(z)$ is local to $B(z)$ then $B(z)$ is local to $A(z)$.

5.10

Roughly speaking, reconstruction theorem says that if we have a small set $\mathcal{E}$ of operators $A(z) \in \mathrm{End}(\mathbb{V})$ that generates $\mathbb{V}$ and satisfies all the axioms in the definition of graded vertex algebras/VOAs, except that the Jacobi identity is replaced by the weaker condition that the operators in $\mathcal{E}$ are mutually local, then the Jacobi identity is automatically satisfies, and hence $\mathbb{V}$ is a graded vertex algebra/VOA. This theorem will be proved in a later section.

Theorem 5.12 (Reconstruction theorem). *Let $\mathcal{E}$ be a set of L_0-homogeneous fields on $\mathbb{V}$. Assume that the following conditions are satisfied. Then $\mathbb{V}$ has a unique graded vertex algebra structure such that each $A(z) \in \mathcal{E}$ is a vertex operator (namely, is of the form $Y(u, z)$ for some $u \in \mathbb{V}$), and that the vacuum vector $\mathbf{1}$ and the operator L_{-1} are those described in the following.*

- *Creation property: There is a distinguished vector $\mathbf{1} \in \mathbb{V}(0)$ such that $A(z)\mathbf{1}$ has no negative powers of z for all $A(z) \in \mathcal{E}$.*
- *Translation property: There is a distinguished $L_{-1} \in \mathrm{End}(\mathbb{V})$ such that $L_{-1}\mathbf{1} = 0$, and that for each $A(z) \in \mathcal{E}$ we have $[L_{-1}, A(z)] = \partial_z A(z)$.*
- *Generating property: Vectors of the form $A^1_{n_1} \cdots A^k_{n_k}\mathbf{1}$ (where $k \in \mathbb{N}$, $A^1(z), \dots, A^k(z) \in \mathcal{E}$, and $n_1, \dots, n_k \in \mathbb{Z}$) span $\mathbb{V}$.*
- *Locality: Any two fields of $\mathcal{E}$ are local.*

Moreover, if L_0, L_{-1} can be extended to a sequence of operators $(L_n)_{n\in\mathbb{Z}}$ on $\mathbb{V}$ such that $\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ belongs to $\mathcal{E}$, and that the Virasoro relation (2.8) is satisfied for some $c \in \mathbb{C}$, then $\mathbb{V}$ is a VOA whose conformal vector $\mathbf{c}$ satisfies $Y(\mathbf{c}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$.

Note that the uniqueness of the graded vertex algebra/VOA structure follows directly from (5.19). The non-trivial part of this theorem is of course the existence of such structure.

Remark 5.13. The end of the reconstruction Thm. 5.12 means that in order to show that a graded vertex algebra $\mathbb{V}$ is a VOA, it suffices to show that L_0, L_{-1} can be extended to $(L_n)_{n\in\mathbb{Z}}$ satisfying the Virasoro relation, that $T(z) = \sum L_n z^{-n-2}$ satisfies the creation property (namely, $L_n\mathbf{1} = 0$ for all $n \geqslant -1$), and that $T(z)$ is local with any field in $\mathcal{E}$ (by showing for instance the conformal Ward identity $[L_m, A(z)] = z^{m+1}\partial_z A(z) + \Delta_A \cdot z^m A(z)$ for all $A(z) \in \mathcal{E}$ if one expects that all $A(z)$ are "primary"). The translation property is automatically satisfied due to the Virasoro relation $[L_{-1}, L_n] = -(n+1)L_{n-1}$.

6 Constructing examples of VOAs

6.1

In the previous section, we have mentioned some important examples of VOAs: affine VOAs and Virasoro VOAs. But we didn't explain why they exist. This is the task of this section. The standard references for this section are [LL, Chapter 6] and [Was10] (with emphasis on the unitarity aspect).

The style of this section is different from the previous ones: it has a strong flavor of Lie theory. The methods in this section will not be used in the future (except when we discuss examples of VOA modules). So the readers can safely skip this section if they do not want to bother with the existence issue. (But they should at least read Subsec. 6.17 on tensor product VOAs.)

Our first class of examples are Virasoro VOAs, namely, those generated by the conformal vector **c**. To begin with, the **Virasoro algebra** is a Lie algebra $\mathrm{Vir} = \mathrm{Span}_{\mathbb{C}}\{L_n, K : n \in \mathbb{Z}\}$ satisfying the bracket relation

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{K}{12}(m+1)m(m-1)\delta_{m,-n},$$
$$[K, L_n] = 0.$$

We know that any VOA must satisfy $L_n\mathbf{1} = 0$ for all $n \geqslant -1$. Motivated by this fact, we have:

Proposition 6.1. *Let $\mathbb{V}$ be a representation of Vir such that L_0 is diagonalizable and has $\mathbb{N}$-spectrum. Assume that $\mathbb{V}$ has a distinguished vector $\mathbf{1}$ killed by L_n for all $n \geqslant -1$, that vectors of the form $L_{n_1}\cdots L_{n_k}\mathbf{1}$ (where $k \in \mathbb{N}, n_1, \dots, n_k \in \mathbb{Z}$) span $\mathbb{V}$, and that K acts as a constant $c \in \mathbb{C}$. Then $\mathbb{V}$ has a unique natural structure of a Virasoro VOA. Its central charge is c.*

Proof. This follows immediately from the reconstruction Thm. 5.12. Note that by (5.28), $\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ is local to itself due to the Virasoro relation. □

6.2

Thus, it remains to construct Vir-modules satisfying the conditions in Prop. 6.1. Let us first find a "largest" such module. We expect that this module should have basis $L_{-n_1}\cdots L_{-n_k}\mathbf{1}$ where $n_1 \geqslant \cdots \geqslant n_k \geqslant 2$, because:

Exercise 6.2. Let $\mathbb{V}$ be as in Prop. 6.1. Prove by induction on k that $L_{n_1}\cdots L_{n_k}\mathbf{1}$ (for any $n_1, \dots, n_k$) can be written as a linear combination of $L_{-m_1}\cdots L_{-m_l}\mathbf{1}$ where $l \in \mathbb{N}, m_1, \dots, m_l \geqslant 2$. (Hint: if $n_j \leqslant -2$, move L_{n_j} to the rightmost by using the Virasoro relation.)

Now let us construct this largest module $V_{\mathrm{Vir}}(c, 0)$ for each $c \in \mathbb{C}$. Its basis consists of $(-n_1, \dots, -n_k)$ where $k \in \mathbb{N}$ and $n_1 \geqslant \cdots \geqslant n_k \geqslant 2$. The one with $k = 0$ is denoted by $\mathbf{1}$. If $n \geqslant n_1$, we simply define the action of L_{-n} on each $(-n_1, \dots, -n_k)$ to be $(-n, -n_1, \dots, -n_k)$. But we also want to define the action of L_n on $(-n_1, \dots, -n_k) = L_{-n_1}\cdots L_{-n_k}\mathbf{1}$ for all $n \in \mathbb{Z}$. In practice, we can write down the formula explicitly using the Virasoro relation. For instance: $L_0L_{-n_1}\cdots L_{-n_k}\mathbf{1} = (n_1+\cdots+n_k)L_{-n_1}\cdots L_{-n_k}\mathbf{1}$, and

$$\begin{aligned}&L_3L_{-4}L_{-3}\mathbf{1} = [L_3,L_{-4}]L_{-3}\mathbf{1} + L_{-4}[L_3,L_{-3}]\mathbf{1}\\ =&7L_{-1}L_{-3}\mathbf{1} + 6L_{-4}L_0\mathbf{1} + 2cL_{-4}\mathbf{1} = (14+2c)L_{-4}\mathbf{1}.\end{aligned} \tag{6.2}$$

There is a natural question about this approach: how do we verify that such defined action of Vir on $V_{\mathrm{Vir}}(c,0)$ preserves the Lie bracket relations of Vir?

6.3

The standard way to deal with is issue is to use the **Poincaré–Birkhoff–Witt (PBW)** theorem, which says the following: Let $\mathfrak{g}$ be a Lie algebra (over any field). Let $U(\mathfrak{g})$ be its universal enveloping algebra, i.e., the largest unital associative algebra containing and generated by the vector space $\mathfrak{g}$ such that $xy-yx=[x,y]$ for all $x,y\in\mathfrak{g}$. If E is a basis of $U(\mathfrak{g})$ with a total order $\leqslant$, then vectors of the form

$$x_1x_2\cdots x_k \qquad (k\in\mathbb{N}, x_1\geqslant x_2\geqslant\cdots\geqslant x_k\in E) \tag{6.3}$$

(when $k=0$, we understand this expression as 1) form a basis of $U(\mathfrak{g})$.

The remarkable point about the PBW theorem is that if we define a vector space V to have a basis of vectors as in (6.3), and if we define the action of $x\in\mathfrak{g}$ using the Lie bracket relations of $\mathfrak{g}$ (similar to the argument in (6.2)), then this gives a well defined action of $\mathfrak{g}$ on V preserving the bracket relations of $\mathfrak{g}$, i.e., this gives a well defined representation of $\mathfrak{g}$.

To apply the PBW theorem to our construction of VOAs, we need the following result:

Lemma 6.3. *Suppose $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ where $\mathfrak{g}_1,\mathfrak{g}_2$ are Lie subalgebras of $\mathfrak{g}$. Use the PBW theorem to show that there is an isomorphism of vector spaces $U(\mathfrak{g}_1)\otimes U(\mathfrak{g}_2)\to U(\mathfrak{g})$ sending each $x_1\cdots x_k\otimes y_1\cdots y_l$ to $x_1\cdots x_ky_1\cdots y_k$ where $x_\bullet\in\mathfrak{g}_1, y_\bullet\in\mathfrak{g}_2$.*

The proof is an easy application of the PBW theorem, which we leave to the readers.

6.4

Consider the following Lie subalgebras of Vir:

$$V_- = \mathrm{Span}\{L_n : n\leqslant -2\}, \qquad V_+ = \mathrm{Span}\{K, L_n; n\geqslant -1\}.$$

$\mathbb{C}_c=\mathbb{C}$ is a representation of V_+ if we let L_n act as 0 and K as c. So $\mathbb{C}_c$ is also a $U(V_+)$-module. Now $U(\mathrm{Vir})$ is clearly a right $U(V_+)$-module. So

$$\mathrm{Ind}_{U(V_+)}^{U(V)}\mathbb{C}_c := U(\mathrm{Vir})\otimes_{U(V_+)}\mathbb{C}_c$$

is a (left) $U(\mathrm{Vir})$-module, called the **induced representation** of $\mathbb{C}_c$. This is a Vir-module, and by Lemma 6.3, this vector space is isomorphic to $U(V_-)\otimes_{\mathbb{C}}U(V_+)\otimes_{(U(V_+))}\mathbb{C}_c\simeq U(V_-)$, which by the PBW theorem has a basis of vectors the form $L_{-n_1}\cdots L_{-n_k}\mathbf{1}$ where $\mathbf{1}$ is the unit 1 and $n_1\geqslant\cdots\geqslant n_k\geqslant 2$. So we can view $V_{\mathrm{Vir}}(c,0)$ as $\mathrm{Ind}_{U(V_+)}^{U(V)}\mathbb{C}_c$. In particular, this proves that $V_{\mathrm{Vir}}(c,0)$ carries a ntural structure of representation of Vir. Hence, by Prop. 6.1, $V_{\mathrm{Vir}}(c,0)$ is a Virasoro VOA with central charge c.

Exercise 6.4. Find an explicit expression of $Y(L_{-4}\mathbf{c}, z)$ on $V_{\mathrm{Vir}}(c, 0)$ in terms of the Virasoro operators L_n.

6.5

$V_{\mathrm{Vir}}(c, 0)$ is not always an irreducible Vir-module. But the irreducible cases are the most interesting one. For instance, every CFT-type unitary VOA is irreducible. (See [CKLW18].)

The method of getting irreducible examples is quite standard in Lie theory: We shall take the largest quotient of $V_{\mathrm{Vir}}(c, h)$. To be more precise, note that for any proper Vir-invariant subspace W of $V_{\mathrm{Vir}}(c, h)$, note that L_0 is diagonalizable on W,[2] i.e., W has a L_0-grading, whose lowest weight must not be 0 since otherwise it contains $\mathbf{1}$ and hence must be $V_{\mathrm{Vir}}(c, 0)$. Let I be the span of all such W, then I is the largest proper Vir-subspace since I has no non-zero weight-0 vectors. Then

$$L_{\mathrm{Vir}}(c, 0) := V_{\mathrm{Vir}}(c, 0)/I$$

is an irreducible Vir-module, which is also a Virasoro VOA of CFT type by Prop. 6.1.

6.6

One may wonder when $L_{\mathrm{Vir}}(c, 0)$ equals $V_{\mathrm{Vir}}(c, 0)$, i.e., when I is trivial. Indeed, I is non-trivial if and only if

$$c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \tag{6.4}$$

where $p, q \in \{2, 3, 4, \dots\}$ are relatively prime. (Cf. [LL, Rem. 6.1.13] and the reference therein.) In this case, $L_{\mathrm{Vir}}(c, 0)$ is called a **minimal model**. It has finitely many irreducible modules. Minimal models are an important class of "rational" VOAs. More precisely: **rational and C_2-cofinite** VOAs. We will give precise meanings of these terms in later sections. The theory of conformal blocks for such VOAs is well-established.

It is a deep result that $L_{\mathrm{Vir}}(c, 0)$ is a unitary Vir-module if and only if $c \geqslant 1$ or c satisfies (6.4) with $|p - q| = 1$, namely,

$$c = 1 - \frac{6}{m(m+1)} \tag{6.5}$$

for some integer $m \geqslant 2$. We refer the readers to [FMS, Chapter 8] and [Was10, Chapter IV] for details.

[2] In general, if D is a diagonalizable linear operator on a vector space M and W is an D-invariant subspace of M, then $D|_W$ is diagonalizable. To see this, choose any $w \in M$ which is a finite sum $w_1 + \cdots + w_k$ where each summand is an eigenvector of D in M, and they have distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Use polynomial interpolation to find a polynomial p such that $p(\lambda_j) = \delta_{1,j}\lambda_1$. So $w_1 = p(D)w \in M$.

6.7

We now turn to affine VOAs. We fix a finite dimensional complex Lie algebra $\mathfrak{g}$ together with a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$. (Indeed, we will not use the non-degeneracy until we define the Virasoro operators.) Recall that invariance means

$$([X,Y],Z) = -(Y,[X,Z]). \tag{6.6}$$

An **affine Lie algebra** is $\widehat{\mathfrak{g}}$ with basis X_n, K (where $X \in \mathfrak{g}, n \in \mathbb{Z}$) satisfying the Lie bracket relation

$$\begin{aligned}[X_m, Y_n] &= [X,Y]_{m+n} + m(X,Y)\delta_{m,-n}K,\\ [K, X_m] &= 0.\end{aligned}$$

It is more convenient to add a basis element D (which will be the L_0 of our VOA) to $\widehat{\mathfrak{g}}$ to get a slightly larger Lie algebra $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}D$ such that

$$[D, X_m] = -mX_m, \qquad [D,K] = 0.$$

$\widetilde{\mathfrak{g}}$ is also called an affine Lie algebra.

6.8

$\widetilde{\mathfrak{g}}$ decomposes into Lie subalgebras $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_- \oplus \widetilde{\mathfrak{g}}_+$ where

$$\widetilde{\mathfrak{g}}_- = \mathrm{Span}\{X_n : X \in \mathfrak{g}, n < 0\}, \qquad \widetilde{\mathfrak{g}}_+ = \mathrm{Span}\{X_n, K, D : X \in \mathfrak{g}, n \geqslant 0\}.$$

Then $U(\widetilde{\mathfrak{g}}) \simeq U(\widetilde{\mathfrak{g}}_-) \otimes U(\widetilde{\mathfrak{g}}_+)$ by Lemma 6.3. For each $l \in \mathbb{C}$ called the **level**, we let $\mathbb{C}_l = \mathbb{C}$ be an $\widetilde{\mathfrak{g}}_+$-module such that K acts as l and X_n, D act trivially. We are interested in two types of associated VOAs:

$$V_{\mathfrak{g}}(l,0) := \mathrm{Ind}_{U(\widetilde{\mathfrak{g}}_+)}^{U(\widetilde{\mathfrak{g}})}\mathbb{C}_l = U(\widetilde{\mathfrak{g}})_{\otimes U(\widetilde{\mathfrak{g}}_+)}\mathbb{C}_l \tag{6.7}$$

which as a vector space is naturally equivalent to $U(\widetilde{\mathfrak{g}}_-)$. Let $\mathbf{1}$ be the $1 \otimes 1$ in $U(\widetilde{\mathfrak{g}})_{\otimes U(\widetilde{\mathfrak{g}}_+)}\mathbb{C}_l$. Then $V_{\mathfrak{g}}(l,0)$ has a basis of vectors

$$X^{i_1}_{-n_1} \cdots X^{i_k}_{-n_k}\mathbf{1}$$

(which has D-weight $n_1 + \cdots + n_k$) written in the lexicon order where $\{X^1, X^2, \dots\}$ is a basis of $\mathfrak{g}$ and $n_1, \dots, n_k > 0$. Thus, D is diagonaizable on $V_{\mathfrak{g}}(l,0)$ with non-negative spectrum, and each eigenspace is finite dimensional. Similar to the argument in Subsec. 6.5, we can take a simple quotient

$$L_{\mathfrak{g}}(l,0) = V_{\mathfrak{g}}(l,0)/I \tag{6.8}$$

where I is the largest proper $\widetilde{\mathfrak{g}}$-submodule.

$V_{\mathfrak{g}}(0,0)$ and $L_{\mathfrak{g}}(0,0)$ are never equal, because:

Exercise 6.5. Show that $L_{\mathfrak{g}}(0,0)$ is spanned by $\mathbf{1}$. Equivalently, show that if $l = 0$, then I contains all D-eigenvectors with eigenvalues > 0.

In the following, we discuss how to make $L_{\mathfrak{g}}(l,0)$ a VOA since $L_{\mathfrak{g}}(l,0)$ is our main interest. The same method applies to $V_{\mathfrak{g}}(l,0)$.

For each $X \in \mathfrak{g}$, X_n acts on $L_{\mathfrak{g}}(l,0)$ in an obvious way. We define $X(z) \in \mathrm{End}(L_{\mathfrak{g}}(l,0))[[z^{\pm 1}]]$ to be

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}.$$

It is a homogeneous field (with respect to D) with weight 1 since $[D, X_n] = -nX_n$. One checks easily that these fields satisfy the creation property and locality, and that they generate $L_{\mathfrak{g}}(l,0)$. So it remains to construct L_{-1} and verify the translation property. We shall actually construct all L_n in a uniform way.

6.9

Choose a basis E of $\mathfrak{g}$, which gives a dual basis $\{\check{e} : e \in E\}$, namely, for each $e, f \in E$, $(e, \check{f}) = \delta_{e,f}$ with respect to the given non-degenerate symmetric bilinear form $(\cdot,\cdot)$. By linear algebra,

$$\sum_{e\in E} e \otimes \check{e} \in \mathfrak{g}\otimes\mathfrak{g} \tag{6.9}$$

is independent of the choice of basis E. As an immediate consequence, we have

$$\sum_{e\in E} \check{e}\otimes e = \sum_{e\in E} e\otimes \check{e}. \tag{6.10}$$

With the help of $\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, $X\otimes Y \mapsto [X,Y]$, this shows $\sum[\check{e}, e] = \sum[e,\check{e}] = -\sum[\check{e}, e]$, i.e.,

$$\sum_{e\in E}[\check{e}, e] = 0. \tag{6.11}$$

Lemma 6.6. *For each $X \in \mathfrak{g}$, we have*

$$\sum_{e\in E} \check{e}\otimes [e, X] = -\sum_{e\in E}[\check{e}, X]\otimes e. \tag{6.12}$$

Proof. Evaluate both sides by $Y\otimes Z$ using $(\cdot,\cdot)$, and use the invariance condition (6.6) to show that both sides equal $(Y, [X, Z])$. □

Thus, on each $\mathfrak{g}$-module V, we have $\sum \check{e}[e,X] + \sum[\check{e}, X]e = 0$, namely,

$$\sum_e [\check{e}e, \mathfrak{g}] = 0. \tag{6.13}$$

So when V is finite dimensional and is either irreducible or trivial, $\Omega = \sum \check{e}e \in \mathrm{End}(V)$ is a constant by Schur's lemma, called **Casimir element**. The operator Ω in general gives the nagative Laplactian of the Lie group action.

Assumption 6.7. We assume that for the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}, X \mapsto [X, \cdot]$, the Casimir element is a constant $2h^\vee \in \mathbb{C}$, i.e.,

$$\sum_{e \in E} [\check{e}, [e, \cdot]] = 2h^\vee \mathbf{1}_\mathfrak{g}. \tag{6.14}$$

This is always true when $\mathfrak{g}$ is abelian (in which case $h^\vee = 0$) or simple. We assume

$$l + h^\vee \neq 0.$$

6.10

We define the Virasoro operator "as if" the conformal vector is

$$\mathbf{c} = \gamma^{-1} \sum_e \check{e}_{-1} e_{-1} \mathbf{1} \qquad \big(\text{where } \gamma = 2(l + h^\vee)\big). \tag{6.15}$$

Thus, using (5.13) and $L_m = Y(\mathbf{c})_{m+1}$, and noting that $\check{e}_i e_j = e_i \check{e}_j$ by (6.10), we write down the definition

$$L_m = \gamma^{-1} \sum_e \Big(\sum_{k \leqslant -1} \check{e}_k e_{m-k} + \sum_{k \geqslant 0} \check{e}_{m-k} e_k \Big) \tag{6.16}$$

acting on $L_\mathfrak{g}(l, 0)$. This is called **Sugawara construction**. One checks that this sum is finite when acting on any vector.

To use the reconstruction theorem, we need the following crucial fact:

Proposition 6.8. *For each $m, n \in \mathbb{Z}$ and $X \in \mathfrak{g}$,*

$$[L_m, X_n] = -n X_{m+n}. \tag{6.17}$$

(Note that if we assume the existence of the VOA structure, then (6.17) can be derived from the conformal Ward identity (5.7) and the fact that $X_{-1}\mathbf{1}$ is indeed primary.)

Convention 6.9. In the remaining part of this section, we suppress $\sum_e$ if possible.

From this proposition, we know that $T(z) = \sum_m L_m z^{-m-2}$ and $X(z)$ are local, and $X(z)$ satisfies the translation property. To use the reconstruction theorem, we need to check the following facts:

Lemma 6.10. *The following are true.*

(a) $T(z)$ satisfies the creation property, namely, $L_n \mathbf{1} = 0$ if $n \geqslant -1$.

(b) L_0 agrees with D.

(c) $\{L_n\}$ satisfy the Virasoro relation.

Proof. (a) Assume $m \geqslant -1$. $\sum_{k\geqslant 0} \breve{e}_{m-k}e_k\mathbf{1}$ is 0 since all $X_0\mathbf{1}$ are zero by our construction. $\sum_{k\leqslant -1} \breve{e}_k e_{m-k}\mathbf{1}$ is 0 because $m-k \geqslant m+1 \geqslant 0$.

(b) Since $L_0\mathbf{1} = 0$ and $[L_0, X_n] = -nX_n = [D, X_n]$, L_0 and D act the same on any $X^1_{n_1}\cdots X^k_{n_k}\mathbf{1}$. So $L_0 = D$.

(c) By the reconstruction theorem, $L_{\mathfrak{g}}(l,0)$ is a graded vertex algebra. Clearly $L_m = Y(\mathbf{c})_{m+1}$ by our definition of L_m and $\mathbf{c}$. We can use (5.1) or (5.2) to show

$$[L_m, L_n] = Y(L_{-1}\mathbf{c})_{m+n+2} + \sum_{l\geqslant 0} \binom{m+1}{l+1} Y(L_l\mathbf{c})_{m+n+1-l}. \tag{6.18}$$

By the expression $\mathbf{c}$, clearly $L_0\mathbf{c} = D\mathbf{c} = 2\mathbf{c}$. Also, from the Sugawara construction, we clearly have $[D, L_m] = -mL_m$, i.e., $[L_0, L_m] = -mL_m$. So $L_l\mathbf{c} = 0$ if $l > 2$. To find $[L_m, L_n]$, we need to find $L_1\mathbf{c}$ and $L_2\mathbf{c}$.

Using (6.17), we calculate that $\gamma L_1\mathbf{c}$ equals

$$L_1\breve{e}_{-1}e_{-1}\mathbf{1} = [L_1, \breve{e}_{-1}]e_{-1}\mathbf{1} + \breve{e}_{-1}[L_1, e_{-1}]\mathbf{1} = \breve{e}_0 e_{-1}\mathbf{1} + \breve{e}_{-1}e_0\mathbf{1} = \breve{e}_0 e_{-1}\mathbf{1}.$$

And $\breve{e}_0 e_{-1}\mathbf{1} = [\breve{e}_0, e_{-1}]\mathbf{1} = [\breve{e}, e]_{-1}\mathbf{1}$ equals 0 by (6.11). Recall K acts as l on $L_{\mathfrak{g}}(l,0)$. Then $\gamma L_2\mathbf{c}$ equals

$$\begin{aligned}
&L_2\breve{e}_{-1}e_{-1}\mathbf{1} = [L_2, \breve{e}_{-1}]e_{-1}\mathbf{1} + \breve{e}_{-1}[L_2, e_{-1}]\mathbf{1} = \breve{e}_1 e_{-1}\mathbf{1} + \breve{e}_{-1}e_1\mathbf{1}\\
=&\breve{e}_1 e_{-1}\mathbf{1} = [\breve{e}_1, e_{-1}]\mathbf{1} = [\breve{e}, e]_0\mathbf{1} + l(\breve{e}, e)\mathbf{1},
\end{aligned}$$

which equals $l\cdot\dim\mathfrak{g}\cdot\mathbf{1}$. Therefore, using (5.6), we find that (6.18) becomes the Virasoro relation where $\frac{c}{2} = \gamma^{-1}l\cdot\dim\mathfrak{g}$. □

Thus, by the reconstruction Thm. 5.12, we conclude:

Theorem 6.11. *For $l \neq -h^\vee$, $V_{\mathfrak{g}}(l,0)$ and $L_{\mathfrak{g}}(l,0)$ are VOAs satisfying $Y(X_{-1}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1}$ (for all $X \in \mathfrak{g}$) if we define the conformal vector $\mathbf{c}$ as in (6.15). The central charge is $\frac{l\dim\mathfrak{g}}{l+h^\vee}$.*

6.11 ⋆

It remains to prove Prop. 6.8. Recall Convention 6.9 that we are suppressing $\sum_e$. The following discussions focus on $L_{\mathfrak{g}}(l,0)$, though the same argument works for $V_{\mathfrak{g}}(l,0)$.

Lemma 6.12. *For all $i,j,n \in \mathbb{Z}$, on $L_{\mathfrak{g}}(l,0)$ we have $[\breve{e}_i e_j, X_n] = A_{i,j,n} + B_{i,j,n}$ where*

$$A_{i,j,n} = \breve{e}_i[e, X]_{j+n} - \breve{e}_{i+n}[e, X]_j \tag{6.19a}$$
$$B_{i,j,n} = -nl(\delta_{j,-n}X_i + \delta_{i,-n}X_j). \tag{6.19b}$$

In particular, $B_{i,j,n} = B_{j,i,n}$.

Proof. We compute

$$[\breve{e}_i e_j, X_n] = \breve{e}_i[e_j, X_n] + [\breve{e}_i, X_n]e_j = A_{i,j,n} + B_{i,j,n}$$

where

$$A_{i,j,n} = \breve{e}_i[e,X]_{j+n} + [\breve{e},X]_{i+n}e_j$$
$$B_{i,j,n} = -nl\delta_{j,-n}\cdot\breve{e}_i(e,X) - nl\delta_{i,-n}(\breve{e},X)e_j.$$

$B_{i,j,n}$ clearly equals (6.19b) by the basic property of (dual) basis. Note that in general, for all $i,j\in\mathbb{Z}$, by Lemma 6.6 and the map $\mathfrak{g}\otimes\mathfrak{g}\to\mathrm{End}(L_{\mathfrak{g}}(l,0))$ sending $Y\otimes Z$ to Y_iZ_j, we have

$$[\breve{e},X]_ie_j = -\breve{e}_i[e,X]_j. \tag{6.20}$$

This proves that $A_{i,j,n}$ equals (6.19a). □

Proof of Prop. 6.8. We compute

$$\begin{aligned}[\gamma L_m, X_n] &= \sum_{k\leqslant -1}[\breve{e}_k e_{m-k}, X_n] + \sum_{k\geqslant 0}[\breve{e}_{m-k}e_k, X_n]\\ &= \sum_{k\leqslant -1}(A_{k,m-k,n}+B_{k,m-k,n}) + \sum_{k\geqslant 0}(A_{m-k,k,n}+B_{m-k,k,n}).\end{aligned}$$

By Lemma 6.12, the sum of the two B is

$$\sum_{k\in\mathbb{Z}}B_{k,m-k,n} = -nl\sum_{k\in\mathbb{Z}}(\delta_{m-k,-n}X_k + \delta_{k,-n}X_{m-k}) = -2nlX_{m+n}.$$

Also,

$$\sum_{k\geqslant 0}A_{m-k,k,n} = \sum_{k\geqslant 0}\breve{e}_{m-k}[e,X]_{k+n} - \sum_{k\geqslant 0}\breve{e}_{m+n-k}[e,X]_k$$

where the two sums are both finite when acting on any vector. But the first summand is just (setting $j=k+n$) $\sum_{j\geqslant n}\breve{e}_{m+n-j}[e,X]_j$. So

$$\sum_{k\geqslant 0}A_{m-k,k,n} = -\big(\breve{e}_{m+n}[e,X]_0 + \breve{e}_{m+n-1}[e,X]_1 + \cdots + \breve{e}_{m+1}[e,X]_{n-1}\big). \tag{6.21}$$

Simiarly, setting $i=m-k$,

$$\begin{aligned}\sum_{k\leqslant -1}A_{k,m-k,n} &= \sum_{i\geqslant m+1}\breve{e}_{m-i}[e,X]_{i+n} - \sum_{i\geqslant m+1}\breve{e}_{m+n-i}[e,X]_i\\ &= -\big(\breve{e}_{n-1}[e,X]_{m+1} + \cdots + \breve{e}_0[e,X]_{m+n}\big).\end{aligned} \tag{6.22}$$

By Lemma 6.13, the sum of (6.21) and (6.22) is $-2nh^\vee X_{m+n}$. This finishes the proof. □

Lemma 6.13. *For each $i,j\in\mathbb{Z}$ and $X\in\mathfrak{g}$,*

$$\breve{e}_i[e,X]_j + \breve{e}_j[e,X]_i = 2h^\vee X_{i+j}. \tag{6.23}$$

This is the only place we use the definition of $h^\vee$ (cf. Assumption 6.7).

Proof. By (6.20),

$$\breve{e}_i[e,X]_j + \breve{e}_j[e,X]_i = \breve{e}_i[e,X]_j - [\breve{e},X]_je_i,$$

which, according to (6.10) and the map $\mathfrak{g}\otimes\mathfrak{g}\to\mathrm{End}(L_{\mathfrak{g}}(l,0))$, $Y\otimes Z\mapsto [Y,X]_jZ_i$, is

$$\breve{e}_i[e,X]_j - [e,X]_j\breve{e}_i = [\breve{e}_i,[e,X]_j] = [\breve{e},[e,X]]_{i+j} + il\delta_{i,-j}(\breve{e},[e,X]).$$

Now, by the invariance of $(\cdot,\cdot)$, $(\breve{e},[e,X]) = ([\breve{e},e],X)$, which equals 0 by (6.11). By the definition of $h^\vee$, $[\breve{e},[e,X]] = 2h^\vee X$. We are done with the proof. □

6.12

We now discuss the unitarity problem for affine VOAs. We first look at Heisenberg VOAs, namely, we assume $\mathfrak{g}$ is abelian. We assume that $\mathfrak{g}$ is equipped with an inner product $(\cdot|\cdot)$ (antilinear on the first variable) and an anti-unitary involution $X \in \mathfrak{g} \mapsto X^* \in \mathfrak{g}$. Recall that "anti-unitary" means that $*$ is conjugate linear, bijective, and satisfies

$$(X^*|Y^*) = (Y|X).$$

Involution means $X^{**} = X$. By considering $\mathfrak{g}$ as an (abelian) **unitary Lie algebra**, we regard $*$ and $(\cdot|\cdot)$ as part of the data of $\mathfrak{g}$.

Exercise 6.14. Show that $\mathfrak{g}$ is unitarily isomorphic to $\mathbb{C}^n$ with the standard inner product, where the involution is $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$, the unique one fixing $\mathbb{R}^n$. (Hint: First find an real isomorphism from $\{X \in \mathfrak{g} : X^* = X\}$ to $\mathbb{R}^n$ preserving the inner products.)

It is easy to check that the bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ defined by

$$(X, Y) = (X^*|Y) \tag{6.24}$$

is symmetric. (It is obviously invariant.) We define $V_{\mathfrak{g}}(l, 0)$ using this bilinear form.

Proposition 6.15. *$l > 0$ if and only if there exists an inner product $\langle\cdot|\cdot\rangle$ on $V_{\mathfrak{g}}(l, 0)$ satisfying $\langle\mathbf{1}|\mathbf{1}\rangle = 1$ such that the representation of $\widetilde{\mathfrak{g}}$ on $V_{\mathfrak{g}}(l, 0)$ is unitary, namely, for each $X \in \mathfrak{g}, u, v \in V_{\mathfrak{g}}(l, 0), n \in \mathbb{Z}$,*

$$\langle u|X_n v\rangle = \langle (X^*)_{-n} u|v\rangle, \qquad \langle u|Kv\rangle = \langle Ku|v\rangle, \qquad \langle u|Dv\rangle = \langle Dv|u\rangle,$$

or simply $(X_n)^\dagger = X^*_{-n}$, $K^\dagger = K$, $D^\dagger = D$ for short. Such $\langle\cdot|\cdot\rangle$ is unique if it exists.

The if part is easy to explain: We compute that $\langle X_{-1}\mathbf{1}|X_{-1}\mathbf{1}\rangle = \langle \mathbf{1}|X_1^* X_{-1}\mathbf{1}\rangle = \langle \mathbf{1}|[X^*, X]_0\mathbf{1}\rangle + l(X^*, X) = l(X|X)$. So if $\langle\cdot|\cdot\rangle$ is an inner product, then for each $X \neq 0$, $l(X|X)$ is > 0. So $l > 0$. We now explain the only if part. To simplify discussions, by scaling $(\cdot|\cdot)$ and hence $(\cdot, \cdot)$ by l and K by l^{-1}, it suffices to assume $l = 1$. (Indeed, people usually just assume $l = 1$ when discussing Heisenberg VOAs.)

6.13 ⋆

Assume $l = 1$. The uniqueness of $\langle\cdot|\cdot\rangle$ is easy to prove:

$$\langle X^1_{n_1} \cdots X^k_{n_k}\mathbf{1}|Y^1_{m_1} \cdots Y^l_{m_l}\mathbf{1}\rangle = \langle \mathbf{1}|(X^k)^*_{-n_k} \cdots (X^1)^*_{-n_1} Y^1_{m_1} \cdots Y^l_{m_l}\mathbf{1}\rangle =: \langle \mathbf{1}|w\rangle.$$

If $n_1 + \cdots + n_k = m_1 + \cdots + m_l$, then w has D-weight 0. But the weight-0 homogeneous vectors are $\mathbb{C}\mathbf{1}$. So $w = \lambda\mathbf{1}$, and λ uniquely determined by the Lie bracket relations. If $n_1 + \cdots + n_k \neq m_1 + \cdots + m_l$, then the weight of w is not 0. So $w = 0$ since $\langle D\mathbf{1}|w\rangle = \langle \mathbf{1}|Dw\rangle$.

The existence part follows from the general construction of symmetric Fock spaces. Let W be a (complex) inner product space together with an antiunitary involution $*$.

Note that for each $N \in \mathbb{N}$, $W^{\otimes N}$ is naturally an inner product space. We assume W has an orthonormal basis $\{e_i : i \in \mathfrak{I}\}$ (which spans W algebraically). Let $\mathfrak{S}_N$ be the set of permutations on $\{1, \dots, N\}$. For each $v_1, \dots, v_N \in W$, we define

$$v_1 \cdots v_N := \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathfrak{S}_N} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)},$$

and let $S^N(W) \subset W^{\otimes N}$ be spanned by all such vectors. We understand $S^0(W)$ to be the standard one dimensional inner product space $\mathbb{C}$. In particular, it has a unit vector 1. $S^N(W)$ has an orthonormal basis consisting of vectors

$$\frac{(e_{i_1})^{m_1} \cdots (e_{i_k})^{m_k}}{\sqrt{m_1! \cdots m_k!}} \qquad (\text{where } i_1, \dots, i_k \in \mathfrak{I} \text{ are distinct and } \sum_{j=1}^{k} m_j = N). \tag{6.25}$$

Define an inner product space

$$S^\bullet(W) = \bigoplus_{N \in \mathbb{N}} S^N(W), \tag{6.26}$$

called the **symmetric Fock space** associated to W. For each $v \in W$, define linear maps $a^+(v), a^-(v)$ on $S^\bullet(W)$ determined by

$$a^+(v)1 = v, \qquad a^+(v)v_1 \cdots v_N = v v_1 \cdots v_N. \tag{6.27a}$$

$$a^-(v)1 = 0, \qquad a^-(v)v_1 \cdots v_N = \sum_{j=1}^{N} \langle v^* | v_j \rangle \cdot v_1 \cdots v_{j-1} v_{j+1} \cdots v_N. \tag{6.27b}$$

The maps $a^\pm(v)$ are well-defined, thanks to the basis (6.25).

Exercise 6.16. Prove the following relations.

1. $a^+(v)^\dagger = a^-(v^*)$, namely, $\langle \xi | a^+(v) \nu \rangle = \langle a^-(v^*) \xi | \nu \rangle$ for all $\xi, \nu \in S^\bullet(W)$. (Hint: write ξ, ν, v in terms of the previously mentioned orthonormal basis vectors.)
2. $[a^-(u), a^+(v)] = \langle u^* | v \rangle \mathbf{1}_{S^\bullet(W)}$. This is called the **canonical commutation relation (CCR)**.

Now let $W = t^{-1} \cdot \mathfrak{g}[t^{-1}]$ with inner product

$$\langle X t^{-m} | Y t^{-n} \rangle = m (X|Y) \delta_{m,n}$$

for all $m, n \in \mathbb{Z}_+$. The involution is defined to be $(Xt^{-m})^* = X^* t^{-m}$. According to the description of the basis of $S^\bullet(W)$, $V_{\mathfrak{g}}(1,0)$ is linearly equivalent to $S^\bullet(W)$ by identifying $\mathbf{1}$ with 1 and

$$X^1_{-n_1} \cdots X^k_{-n_k} \mathbf{1} \qquad \text{with} \qquad X^1 t^{-n_1} \cdots X^k t^{-n_k}. \tag{6.28}$$

We use the inner product on $S^\bullet(W)$ to define the one on $V_{\mathfrak{g}}(1,0)$. Using CCR, it is not hard to check that the action of X_n on $V_{\mathfrak{g}}(1,0) \simeq S^\bullet(W)$ is

$$X_n = \begin{cases} a^+(Xt^{-|n|}) & \text{if } n < 0, \\ 0 & \text{if } n = 0, \\ a^-(Xt^{-n}) & \text{if } n > 0. \end{cases} \tag{6.29}$$

Thus, the representation of $\widetilde{\mathfrak{g}}$ on $V_{\mathfrak{g}}(1,0)$ is unitary.

6.14

When $l > 0$, $L_{\mathfrak{g}}(l,0)$ and $V_{\mathfrak{g}}(l,0)$ share the same unitarity property, because:

Proposition 6.17. *If $l \in \mathbb{C}^\times$, then $V_{\mathfrak{g}}(l,0)$ is an irreducible $\widetilde{\mathfrak{g}}$-module, i.e., $V_{\mathfrak{g}}(l,0) = L_{\mathfrak{g}}(l,0)$.*

Proof. We assume $l > 0$ and prove the irreducibility using the unitarity. Choose any non-zero $\widetilde{\mathfrak{g}}$-submodule W of $V_{\mathfrak{g}}(l,0)$. We shall show $W = V_{\mathfrak{g}}(l,0)$.

Since W is a D-invariant subspace, D is diagonalizable on W. So W has D-grading $W = \bigoplus_{n \geqslant a} W(n)$ where a is the smallest eigenvalue of D on W. We claim that $a = 0$. Then, as the D-weight 0 subspace of $V_{\mathfrak{g}}(l,0)$ is clearly spanned by $\mathbf{1}$, we must have $\mathbf{1} \in W$. From this one sees that $W = V_{\mathfrak{g}}(l,0)$.

Suppose $a > 0$. We choose a non-zero $w \in W(a)$, which must be a sum of vectors of the form $X^1_{-n_1} \cdots X^k_{-n_k}\mathbf{1}$ where the sum of the positive integers $n_1, \dots, n_k$ is a. Then by the unitarity, $\langle w|w\rangle$ (which is non-zero) is a sum of $\langle \mathbf{1}|(X^k)^*_{n_k} \cdots (X^1)^*_{n_1} w\rangle$. So for some $X^1_{n_1}$, the vector $v = (X^1)^*_{n_1} w$ must be nonzero. But v has D-weight $a - n_1 < a$, and clearly $v \in W$. This is a contradiction.

Now, for a general $l = |l|e^{\mathbf{i}\theta} \in \mathbb{C}^\times$, we may replace $(\cdot,\cdot)$ by $e^{\mathbf{i}\theta}(\cdot,\cdot)$ and K by $e^{-\mathbf{i}\theta}K$. Then $(\cdot|\cdot)$ and the new $(\cdot,\cdot)$ are related by $(X|Y) = (e^{\mathbf{i}\theta}X^*, Y)$, and $X \mapsto e^{\mathbf{i}\theta}X^*$ is clearly an antiunitary involution. So $V_{\mathfrak{g}}(l,0)$ becomes $V_{\mathfrak{g}}(|l|,0)$ under the new involution and bilinear form, and the latter has been proved irreducible. □

6.15

In general, we say a finite-dimensional (complex) Lie algebra $\mathfrak{g}$ is **unitary** if it is equipped with an inner product $(\cdot|\cdot)$ and an antiunitary involution $*$ satisfying the following conditions:

1. $[X,Y]^* = [Y^*, X^*]$.
2. The inner product is **invariant**, namely, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is unitary:
$$([X,Y]|Z) = (Y|[X^*, Z]).$$

Then $(X,Y) := (X^*|Y)$ defines a symmetric invariant bilinear form on $\mathfrak{g}$.

Exercise 6.18. Let $\mathfrak{k}$ be an $\mathfrak{g}$-invariant and $*$-invariant (i.e. $[\mathfrak{g},\mathfrak{k}] \subset \mathfrak{k}$, $\mathfrak{k}^* = \mathfrak{k}$) subspace of $\mathfrak{g}$. Let $\mathfrak{k}^\perp$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$.

1. Show that $\mathfrak{k}^\perp$ is also $\mathfrak{g}$-invariant and $*$-invariant.
2. Show that $[\mathfrak{k},\mathfrak{k}^\perp] = 0$ and hence $[\mathfrak{g},\mathfrak{k}] = [\mathfrak{k},\mathfrak{k}]$. Consequently, if $\mathfrak{k}$ is an irreducible $\mathfrak{g}$-submodule, then $\mathfrak{k}$ is an irreducible $\mathfrak{k}$-module, which is (by definition) a simple Lie algebra if moreover $[\mathfrak{k},\mathfrak{k}] \neq 0$.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, which is clearly $\mathfrak{g}$- and $*$-invariant. Let $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{z}^\perp$ so that $\mathfrak{g} = \mathfrak{z} \oplus^\perp \mathfrak{g}_{\mathrm{ss}}$. Then the adjoint representation $\mathfrak{g} \curvearrowright \mathfrak{g}_{\mathrm{ss}}$ (equivalently, $\mathfrak{g}_{\mathrm{ss}} \curvearrowright \mathfrak{g}_{\mathrm{ss}}$) has orthogonal irreducible $*$-invariant decomposition $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}_1 \oplus^\perp \cdots \oplus^\perp \mathfrak{g}_N$. Each $\mathfrak{g}_j$ is a simple unitary Lie algebra, which is classified by the type A-G Dynkin diagrams.

Conversely, suppose $\mathfrak{g}$ is a complex simple Lie algebra, which is the complexification of $\mathfrak{g}_{\mathbb{R}}$ which is the real Lie algebra of a finite dimensional compact real Lie group G. It is well known in Lie theory that the real vector space $\mathfrak{g}_{\mathbb{R}}$ has a unique up to $\mathbb{R}_{>0}$-scalar multiplication G-invariant (equivalently, $\mathfrak{g}_{\mathbb{R}}$-invariant) inner product, which extends to a complex invariant inner product $(\cdot|\cdot)$ on $\mathfrak{g}$ thanks to the real direct sum $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \mathbf{i}\mathfrak{g}_{\mathbb{R}}$. The antiunitary involution on $\mathfrak{g}$ is defined to be the unique one fixing $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Thus $\mathfrak{g}$ is unitary.

Therefore, in general, if $\mathfrak{z}$ is abelian and $\mathfrak{g}_1, \dots, \mathfrak{g}_N$ are simple, then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ is naturally a unitary Lie algebra. So the study of unitary affine VOAs for unitary Lie algebras reduces to that of the abelian case (which we have finished) and the simple case.

6.16

When $\mathfrak{g}$ is simple, the unitarity properties of $V_{\mathfrak{g}}(l,0)$ and $L_{\mathfrak{g}}(l,0)$ are very different from the abelian case. Indeed, in the abelian case, scaling the inner product does not change the unitary equivalence class of abelian unitary Lie algebras. (This is because scaling the vectors by a non-zero constant is an isomorphism of abelian Lie algebras.) But this is no longer true for non-abelian Lie algebras. Also, it turns out that for a simple $\mathfrak{g}$, $V_{\mathfrak{g}}(l,0)$ is never a unitary $\widetilde{\mathfrak{g}}$-module, and $L_{\mathfrak{g}}(l,0)$ is unitary for a discrete set of levels l if one fixes the invariant inner product, or for a discrete set of invariant inner product if one fixes the level l.

Assume $\mathfrak{g}$ is a simple Lie algebra with compact form decomposition $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \oplus \mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Let $*$ be the unique involution fixing $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. As we have said, the invariant bilinear forms on $\mathfrak{g}_{\mathbb{R}}$ (and hence on $\mathfrak{g}$) are unique up to scalar multiplication. So it would be better to fix one. The one that people usually choose is:

Convention 6.19. We choose the invariant inner product on $\mathfrak{g}$ (under which $*$ is antiunitary) to be the unique one such that the longest roots of $\mathfrak{g}$ have length $\sqrt{2}$.

It follows from the invariance of $(\cdot|\cdot)$ that $h^\vee$ (defined in Assumption 6.7) is a positive number. (To see this, one may choose E to be an orthonormal basis of $\mathfrak{g}$, and check that its dual basis $\{\breve{e} : e \in E\}$ satisfies $\breve{e} = e^*$.) The $h^\vee$ corresponding to the inner product in Convention 6.19 is called the **dual Coxeter number**. We have said that $L_{\mathfrak{g}}(l,0)$ and $V_{\mathfrak{g}}(l,0)$ are VOAs if $l \neq -h^\vee$. So this is true when $l \geqslant 0$.

Theorem 6.20. *$L_{\mathfrak{g}}(l,0)$ is unitary if and only if $l \in \mathbb{N}$. For such l, $L_{\mathfrak{g}}(l,0)$ is called a* ***Weiss-Zumino-Witten (WZW)*** *model.*

This is a highly non-trivial result whose proof relies on deep Lie theory. We refer the readers to [Was10, Chapter III, Sec. 2 and 10] for a proof. Moreover, just like minimal models, WZW models are C_2-cofinite and rational. So their representation categories are extremely nice. Due to these properties, WZW models are central objects in the study of CFT and VOAs. (However, Heisenberg VOAs are neither C_2-cofinite nor rational.)

6.17

We have shown the existence of affine VOAs when the unitary Lie algebra $\mathfrak{g}$ is abelian or simple. The general case can be addressed by tensor product VOAs.

Let $\mathbb{V}_1, \mathbb{V}_2$ be VOAs. We use the diagonalizable operator $L_0 \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_0$ to define the grading on $\mathbb{V}_1 \otimes \mathbb{V}_2$. The vacuum vector is $\mathbf{1} \otimes \mathbf{1}$. $\mathbb{V}_1 \otimes \mathbb{V}_2$ is clearly generated by $Y(v_1)_m \otimes \mathbf{1}_{\mathbb{V}_2}$ and $\mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2)_n$ where $v_j \in \mathbb{V}_j$, and $Y(v_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}$ is clearly local to $Y(u_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}$ (where $u_1 \in \mathbb{V}_1$) and $\mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2, z)$. One checks that $L_{-1} \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_{-1}$ satisfies the translation property. So $\mathbb{V} \otimes \mathbb{V}$ is naturally a graded vertex algebra by the reconstruction theorem. Its vertex operator satisfies

$$Y(v_1 \otimes \mathbf{1}, z) = Y(v_1, z) \otimes \mathbf{1}_{\mathbb{V}_2}, \qquad Y(\mathbf{1} \otimes v_2, z) = \mathbf{1}_{\mathbb{V}_1} \otimes Y(v_2, z). \tag{6.30}$$

Exercise 6.21. Use (5.13) or (5.16) to show

$$Y(v_1 \otimes v_2, z) = Y(v_1, z) \otimes Y(v_2, z). \tag{6.31}$$

Equivalently,

$$Y(v_1 \otimes v_2)_n = \sum_{n \in \mathbb{Z}} \sum_{n_1 + n_2 = n-1} Y(v_1)_{n_1} Y(v_2)_{n_2}. \tag{6.32}$$

When $\mathbb{V}_1, \mathbb{V}_2$ are VOAs with conformal vectors $\mathbf{c}_1, \mathbf{c}_2$ and central charges c_1, c_2, it is easy to check that $\mathbb{V}_1 \otimes \mathbb{V}_2$ is a VOA with conformal vector $\mathbf{c}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{c}_2$. In particular, its Virasoro operators are $Y(\mathbf{c}_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{c})_{n+1} = L_n \otimes \mathbf{1}_{\mathbb{V}_2} + \mathbf{1}_{\mathbb{V}_1} \otimes L_n$. We call $\mathbb{V}_1 \otimes \mathbb{V}_2$ the **tensor product VOA** of $\mathbb{V}_1$ and $\mathbb{V}_2$.

Exercise 6.22. Show that $\mathbb{V}_1 \otimes \mathbb{V}_2$ has central charge $c_1 + c_2$.

We remark that if $\mathbb{V}_1$ and $\mathbb{V}_2$ are unitary, then their tensor product is also unitary (cf. [DL14, CKLW18]).

Exercise 6.23. Let $\mathfrak{g}_1, \dots, \mathfrak{g}_N$ be either abelian or simple. Let $\mathbb{V} = L_{\mathfrak{g}_1}(l_1, 0) \otimes \cdots \otimes L_{\mathfrak{g}_N}(l_N, 0)$. Show that the weight-1 subspace $\mathbb{V}(1)$, as a Lie algebra (cf. Subsec. 5.5), is naturally isomorphic to $\mathfrak{g} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$. Show that $\mathbb{V}(1)$ generates $\mathbb{V}$.

Exercise 6.24. Show that $L_{\mathbb{C}^n}(1, 0) \simeq \underbrace{L_{\mathbb{C}}(1, 0) \otimes \cdots \otimes L_{\mathbb{C}}(1, 0)}_{n \text{ times}}$.

7 Local fields

7.1

Having explored some important examples, we now return to the general theory. The goal of this section is to understand the close relationship between the three statements in Subsec. 5.8. The precise formulation of statement 1 is the Lie bracket version of local fields, as defined in Def. 5.9 or Rem. 5.10. For statement 2 we give two rigorous descriptions: the complex analytic version and the formal variable version of local fields. We first give the complex analytic version, which is more intuitive.

We first need to define:

Definition 7.1. Let Ω be a locally compact Hausdorff space. A series of functions $\sum_n f_n$ is said to **converge absolutely and locally uniformly (a.l.u.) on** Ω if each $x_0 \in \Omega$ is contained in a neighborhood U such that

$$\sup_{x \in U} \sum_n |f_n(x)| < +\infty.$$

Equivalently, for each compact subset $K \subset \Omega$, we have $\sup_{x \in K} \sum_n |f_n(x)| < +\infty$

Clearly, if each $\sum f_n$ converges a.l.u. and each f_n is continuous (resp. holomorphic), then so is the limit $\sum f_n$.

7.2

Now let $\mathbb{V} = \bigoplus_{n \in \mathbb{N}} \mathbb{V}(n)$ be graded by a diagonalizable L_0. Recall the projection $P_n : \mathbb{V}^{\mathrm{cl}} = \prod_{m \in \mathbb{N}} \mathbb{V}(m) \to \mathbb{V}(n)$ (cf. (3.18)). Let $A(z) = \sum A_n z^{-n-1}$, $B(z) = \sum B_n z^{-n-1}$ be homogeneous fields with weights Δ_A, Δ_B (cf. Def. 5.8). For each $n \in \mathbb{N}$ and $v, v' \in \mathbb{V}$, we have

$$\langle v', A(z_1) P_n B(z_2) v \rangle \quad \in \mathscr{O}(\mathbb{C}^\times \times \mathbb{C}^\times) \tag{7.1}$$

since, when v, v' are homogeneous, this expression equals

$$\langle v', A_{n_1} B_{n_2} v \rangle z_1^{-n_1-1} z_2^{-n_2-1}$$

where n_2, n_1 are determined by $\Delta_B + \mathrm{wt} v - n_2 - 1 = n$ and $\Delta_A + n - n_1 - 1 = \mathrm{wt} v'$.

Definition 7.2 (Local fields (complex analytic version)**).** We say $A(z)$ and $B(z)$ are **local** to each other if for each $v \in \mathbb{V}$, $v' \in \mathbb{V}'$ the following hold.

1. The series

$$\langle v', A(z_1) B(z_2) v \rangle := \sum_{n \in \mathbb{N}} \langle v', A(z_1) P_n B(z_2) v \rangle \tag{7.2a}$$

$$\langle v', B(z_2) A(z_1) v \rangle := \sum_{n \in \mathbb{N}} \langle v', B(z_2) P_n A(z_1) v \rangle \tag{7.2b}$$

   converge a.l.u. respectively on the open sets $\Omega_1 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < |z_1|\}$ and $\Omega_2 = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < |z_2|\}$. So (7.2a) and (7.2b) are automatically holomorphic functions on Ω_1 and Ω_2.

2. (7.2a) and (7.2b) can be analytically continued to the same holomorphic function $f_{v,v'}$ on $\mathrm{Conf}^2(\mathbb{C}^\times)$. Moreover, there exists $N \in \mathbb{N}$ depending only on A, B but not on v, v' such that the function

$$(z_1 - z_2)^N f_{v,v'}(z_1, z_2) \tag{7.3}$$

   is holomorphic on $\mathbb{C}^\times \times \mathbb{C}^\times$.

Roughly speaking, this definition says that (7.2a) and (7.2b) converge a.l.u on Ω_1, Ω_2 and extend to the same holomorphic function on $\mathrm{Conf}^2(\mathbb{C}^\times)$ which has poles of order at most N at $z_1 = z_2$, where N is independent of v, v'.

7.3

The readers will immediately notice that there is another natural convergence condition on $A(z_1)B(z_2)$: that $\langle v', A(z_1)B(z_2)v\rangle$ as a formal Laurent series of z_1, z_2 converges a.l.u. on Ω_1. Or more precisely, the joint series

$$\sum_{m,n\in\mathbb{Z}} \langle v', A_m B_n v\rangle z_1^{-m-1} z_2^{-n-1} \tag{7.4}$$

converges a.l.u. on Ω. Is this equivalent to the convergence statement in Def. 7.2? The answer is yes. But people will easily overlook the need to justify this equivalence. And we need both versions of convergence since they are useful in different situations. For instance, to prove that formal variable implies complex analytic, it is easier to prove the a.l.u. convergence of the formal Laurent series; to prove the other direction, it is better to use the a.l.u. convergence of the RHS of (7.2a) and (7.2b).

There is (unfortunately) one more way to understand the convergence (7.2a): we regard the RHS as a series of formal Laurent series of z_1, z_2, which converges formally to the LHS also as a formal Laurent series in the following sense:

Definition 7.3. We say that a sequence (indexed by k)

$$f_k(z_1,\dots,z_M) = \sum_{n_1,\dots,n_M\in\mathbb{Z}} f_{k,n_\bullet} z_1^{n_1}\cdots z_M^{n_M}$$

of elements of $W[[z_1^{\pm1},\dots,z_M^{\pm1}]]$ **converges formally** to

$$f(z_1,\dots,z_M) = \sum_{n_1,\dots,n_M\in\mathbb{Z}} f_{n_\bullet} z_1^{n_1}\cdots z_M^{n_M}$$

if for each $n_\bullet$, the coefficient $f_{k,n_\bullet}$ equals $f_{n_\bullet}$ except for finitely many k.

Note that in applications, k can be in any countable set: $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^2$, etc.

We will show the equivalence of the two a.l.u. convergences with the help of the following obvious lemma.

Lemma 7.4. *Let X be a complex manifold. Let $f_k(x, z_\bullet)$ be a series of $\mathscr{O}(X)$ -coefficients monomials of $z_1^{\pm1},\dots,z_M^{\pm1}$, i.e., $f_k(x,z_\bullet) = g_k(x) z_1^{n_{k,1}}\cdots z_M^{n_{k,M}}$ where each $g_k \in \mathscr{O}(X)$ and $n_{k,j}\in\mathbb{Z}$. Assume that if $k\neq k'$ then $n_{k,j}\neq n_{k',j}$ for some $1\leqslant j\leqslant M$. Then $\sum_k f_k(x,z_\bullet)$ clearly converges formally to some $f\in\mathscr{O}(X)[[z_1^{\pm1},\dots,z_M^{\pm1}]]$. Namely, the following holds formally:*

$$f(x,z_\bullet) = \sum_k f_k(x,z_\bullet). \tag{7.5}$$

Moreover, let Ω be an open subset of $\mathbb{C}^M$. Then $f(x,z_\bullet)$ as an $\mathscr{O}(X)$-coefficients formal Laurent series of $z_\bullet$ (indexed by the powers of $z_\bullet$) converges a.l.u. on $X\times\Omega$ if and only if the series $\sum_k f_k(x,z_\bullet)$ (indexed by k) converges a.l.u. on $X\times\Omega$. If so, then the two limits are equal, i.e., (7.5) holds as holomorphic functions on $X\times\Omega$.

7.4

We now show that (7.2a) as an infinite sum over n converges a.l.u. on Ω_1 iff the LHS of (7.2a) as a formal Laurent series of z_1, z_2 converges a.l.u. on Ω_1. Note that both convergences are preserved by taking linear combinations. So it suffices to assume that v, v' are homogeneous.[3] Let us prove our claim by checking that the sum (7.2a) satisfies the assumption in Lemma 7.4:

Since $B(z_2)$ is homogeneous, similar to the proof of Prop. 3.5, we have the translation covariance

$$B(\lambda z_2) = \lambda^{-\Delta_B} \cdot \lambda^{L_0} B(z_2) \lambda^{-L_0}. \tag{7.6}$$

This shows

$$B(z_2) = z_2^{-\Delta_B} \cdot z_2^{L_0} B(1) z_2^{-L_0}. \tag{7.7}$$

A similar relation holds for $A(z_1)$. So for each $n \in \mathbb{N}$, we have (in the sense of $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$)

$$\begin{aligned}
&\langle v', A(z_1) P_n B(z_2) v \rangle = \langle v', z_1^{L_0 - \Delta_A} A(1) z_1^{-L_0} P_n z_2^{-\Delta_B + L_0} B(1) z_2^{-L_0} v \rangle \\
=& z_1^{\mathrm{wt} v' - \Delta_A} z_2^{-\Delta_B - \mathrm{wt} v} \cdot \Big(\frac{z_2}{z_1}\Big)^n \big\langle v', A(1) P_n B(1) v \big\rangle,
\end{aligned} \tag{7.8}$$

noting that $z_1^{-L_0} P_n = z_1^{-n} P_n$ and $P_n z_2^{L_0} = P_n z_2^n$.

Exercise 7.5. Let $\mathbb{V}$ be a graded vertex algebra. Choose $u, v \in \mathbb{V}$ and $v' \in \mathbb{V}'$. Use (3.36) and Lemma 7.4 to show that

$$\sum_{n \in \mathbb{N}} \big\langle v', Y(u,z) P_n e^{-\tau L_{-1}} v \big\rangle = \sum_{n \in \mathbb{N}} \big\langle v', e^{-\tau L_{-1}} P_n Y(u, z+\tau) v \big\rangle, \tag{7.9}$$

where both sides converge a.l.u. on $\{z \neq 0, |\tau| < |z|\}$ to the same function. (Note that the RHS is a finite sum.)

7.5

Definition 7.6 (Local fields (formal variable version)**).** There exists $N \in \mathbb{N}$ depending only on A and B such that the equation

$$(z_1 - z_2)^N [A(z_1), B(z_2)] = 0 \tag{7.10}$$

holds on the level of $\mathrm{End}(\mathbb{V})[[z_1^{\pm 1}, z_2^{\pm 1}]]$.

This version of local fields is the most common in the literature, partly because it is the most concise. Indeed, since locality implies Jacobi identity, many people use locality instead of Jacobi identity in the definition of VOAs. We do not take this approach because locality has its own limitation: in the definition of VOA modules and conformal blocks, we need the full Jacobi identity, but not just locality.

[3] We cannot directly apply Lemma 7.4 if v, v' are not homogeneous.

7.6

Almost everyone will have the following question when they first see this definition: doesn't (7.10) imply $[A(z_1), B(z_2)] = 0$? The answer is no: for a vector space W in general, it is possible that $fg = 0$ for some $f(z_1, z_2), g(z_1, z_2) \in W[[z_1^{\pm 1}, z_2^{\pm 1}]]$ although $f \neq 0, g \neq 0$. In other words, assuming $W = \mathbb{C}$ for simplicity, then $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$ (unlike $\mathbb{C}[[z_1, z_2]]$) has "zero divisors". (We put quotation marks here because $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$ is actually not a ring.)

Indeed, choose $N > 0$. Then $(z_1 - z_2)^{-N}$ can be expanded in two ways: $f = \sum_{j \geqslant 0} \binom{-N}{j} z_1^j (-z_2)^{-N-j}$ as if $|z_1| < |z_2|$, and $g = \sum_{j \geqslant 0} \binom{-N}{j} z_1^{-N-j} (-z_2)^j$ as if $|z_1| > |z_2|$. Then $f \neq g$, but $(z_1 - z_2)^N f = (z_1 - z_2)^N g = 1$. So $(z_1 - z_2)^N$ is a zero divisor. Similarly, one shows that $(1 + z)^N$ (where $N > 0$) is a zero divisor in $\mathbb{C}[[z^{\pm 1}]]$ by expanding $(1 + z)^{-N}$ as if $|z| < 1$ and as if $|z| > 1$.

This phenomenon is closely related to the fact that $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$ (and similarly $\mathbb{C}[[z^{\pm 1}]]$) is not a ring: the product of two arbitrary elements cannot be defined. This is in contrast to the following basic fact:

Lemma 7.7. *If $\mathbb{F}$ is a field, then $\mathbb{F}((z))$ is naturally a field. In particular, $\mathbb{F}((z))$ is closed under taking product and inverse (for non-zero elements).*

Exercise 7.8. Suppose $f(z) \in \mathbb{F}((z))$ is not zero. Find an algorithm of determining the inverse $1/f(z)$.

Thus, by taking $\mathbb{F} = \mathbb{C}((z_1))$, we see that $\mathbb{C}((z_1))((z_2))$ is also a field. This implies that $(z_1 - z_2)^N$ is not a zero divisor in $\mathbb{C}((z_1))((z_2))$: Suppose that $(z_1 - z_2)^N f(z_1, z_2) = 0$, and that $f \in \mathbb{C}((z_1))((z_2))$, i.e.,

$$f(z_1, z_2) = \sum_{\substack{n_2 \geqslant L \\ n_1 \geqslant K_{n_2}}} f_{n_1, n_2} z_1^{n_1} z_2^{n_2}$$

for some $L \in \mathbb{Z}$ and $K_{n_2} \in \mathbb{Z}$ for each n_2. Then $f = 0$ because $f = (z_1 - z_2)^{-N} (z_1 - z_2)^N f = 0$ where $(z_1 - z_2)^{-N}$ is the *inverse of* $(z_1 - z_2)^N$ *in* $\mathbb{C}((z_1))((z_2))$, which is $\sum_{j \geqslant 0} \binom{-N}{j} z_1^{-N-j} (-z_2)^j$. (If we expand $(z_1 - z_2)^{-N}$ as if $|z_1| < |z_2|$, we get the inverse of $(z_1 - z_2)^N$ in $\mathbb{C}((z_2))((z_1))$.)

If, however, $f \in \mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$ is neither in $\mathbb{C}((z_2))((z_1))$ nor in $\mathbb{C}((z_1))((z_2))$, then $(z_1 - z_2)^N f = 0$ does not imply $f = 0$ since we cannot multiply both sides by either inverse of $(z_1 - z_2)^N$. (There is no associativity law $(fg)h = f(gh)$ in $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$ even if both sides can be defined.)

7.7

Each of the three versions has its own advantage, and it is the goal of this section to prove the equivalence of them. This is a crucial step for proving the reconstruction theorem. Moreover, note that in each of these three versions there is a number N. *We can prove the equivalence of the three versions for the same N.*

The Lie algebraic version is the easiest to verify in concrete examples: we have already seen this in the previous section. In contrast, the complex analytic one is the most difficult to verify. But the complex analytic version is closest to how physicists understand local fields. So it allows us to prove results in a similar fashion as in physics literature. For instance: we will prove the existence of OPE using the complex analytic version of local fields. And with the help of OPE, we can prove that complex analytic implies Lie bracket version in the same way as deriving the algebraic Jacobi identity from the complex analytic one using residue theorem. Finally, to prove the complex analytic version from the Lie algebraic one, we need the help of the formal variable version. Also, using the formal variable version, we can generalize the statements in Def. 7.2 to more than two fields. This generalization is crucial for proving the reconstruction theorem.

$$
\begin{tikzcd}
\text{Lie algebraic} \arrow[r, "7.8"] & \text{Formal variable} \arrow[r, bend left, "7.9"] & \text{Complex analytic} \arrow[l, bend left, "7.10"] \arrow[dl, bend left, "7.11"] \\
& \text{OPE} \arrow[ul, bend left, "7.12"] &
\end{tikzcd}
$$

From the above chart, we see that a direct proof from complex analytic to formal variable is not necessary for proving the equivalence of the three versions. We will still give such a proof because: In the VOA theory, many definitions and properties can be stated in both algebraic (i.e., formal variable) and complex analytic language. It is important to learn how to translate between these two.

7.8

The proof that Lie algebraic implies formal variable is by brutal force. Assume the homogeneous fields $A(z), B(z)$ satisfies (5.27). Let us prove that $(z_1 - z_2)^N[A(z_1), B(z_2)] = 0$.

Proof. Showing $(z_1 - z_2)^N[A(z_1), B(z_2)] = 0$ amounts to showing that for all $m, n \in \mathbb{Z}$, the following expression vanishes:

$$
\begin{aligned}
&\mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0}\, z_1^m z_2^n \cdot (z_1 - z_2)^N [A(z_1), B(z_2)] dz_1 dz_2 \\
=& \sum_{j=0}^{N} \mathrm{Res}_{z_1=0}\mathrm{Res}_{z_2=0} \binom{N}{j} z_1^{m+j} z_2^{n+N-j} (-1)^{N-j} [A(z_1), B(z_2)] dz_1 dz_2 \\
=& \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} [A_{m+j}, B_{n+N-j}] \\
=& \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} \sum_{l=0}^{N-1} \binom{m+j}{l} C^l_{m+n+N-l}.
\end{aligned}
\tag{7.11}
$$

This expression vanishes because of the next lemma. □

Lemma 7.9. *For each $N \in \mathbb{Z}_+$, $m \in \mathbb{Z}$, and $l = 0, 1, \ldots, N-1$, we have*

$$
\sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} \binom{m+j}{l} = 0.
$$

Proof. The function $f(z) = (1+z)^m z^N$ is holomorphic on $\mathbb{D}_1$, and its power series expansion contains no less-than-N powers of z. But we can expand $f(z)$ in the following way:

$$f(z) = (1+z)^m(-1+1+z)^N = \sum_{j=0}^{N}(1+z)^m \cdot \binom{N}{j}(-1)^{N-j}(1+z)^j$$

$$= \sum_{j=0}^{N} \cdot \binom{N}{j}(-1)^{N-j}(1+z)^{m+j} = \sum_{j=0}^{N}\sum_{l\in\mathbb{N}}\binom{N}{j}(-1)^{N-j}\binom{m+j}{l}z^l.$$

The coefficient before z^l vanishes when $l < N$. This proves our formula. $\square$

7.9

Let us prove that formal variable implies complex analytic.

Proof. Assume $(z_1 - z_2)^N[A(z_1), B(z_2)] = 0$. Choose homogeneous $v \in \mathbb{V}, v' \in \mathbb{V}'$. Let

$$f(z_1, z_2) = \langle v', A(z_1)B(z_2)v\rangle, \qquad g(z_1, z_2) = \langle v', B(z_2)A(z_1)v\rangle$$

which are both in $\mathbb{C}[[z_1^{\pm 1}, z_2^{\pm 1}]]$. So is

$$\phi(z_1, z_2) = (z_1 - z_2)^N f(z_1, z_2) = (z_1 - z_2)^N g(z_1, z_2).$$

Step 1. We claim that ϕ is actually in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$. Note that

$$f(z_1, z_2) = \sum_{m,n\in\mathbb{Z}} \langle A_m^{\mathrm{t}} v', B_n v\rangle z_1^{-m-1} z_2^{-n-1}. \tag{7.12}$$

Since B_n increases the weights by $\Delta_B - n - 1$, we have $B_n v = 0$ for sufficiently positive n. A_m^{t} is the transpose of A sending each $u' \in \mathbb{V}'(k)$ to $u' \circ A_m$. One checks easily that A_m^{t} lowers the weights by $\Delta_A - m - 1$. So $A_m^{\mathrm{t}} v'$ vanishes for sufficiently negative m. Therefore, the coefficients of f vanish the if powers of z_2 are sufficiently negative or the powers of z_1 are sufficiently positive. The same can be said about $\phi = (z_1 - z_2)^N f$. Similarly, the coefficients of g vanishes when the powers of z_1 (resp. z_2) are sufficiently negative (resp. positive), and the same can be said about ϕ. Therefore ϕ has finitely many terms: $\phi(z_1, z_2) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$. In particular, $\phi \in \mathscr{O}(\mathbb{C}^\times \times \mathbb{C}^\times)$.

Step 2. From (7.12), it is clear that $f(z_1, z_2)$ is in $\mathbb{C}[z_1^{\pm 1}]((z_2)) \subset \mathbb{C}((z_1))((z_2))$. So $f(z_1, z_2) = (z_1 - z_2)^{-N}\phi(z_1, z_2)$ where $(z_1 - z_2)^{-N} \in \mathbb{C}((z_1))((z_2))$ is the inverse of $(z_1 - z_2)^N$ expanded in $|z_2| < |z_1|$ (cf. Subsec. 7.6). So the formal Laurent series $f(z_1, z_2)$ converges a.l.u. to the rational function $(z_1 - z_2)^{-N}\phi(z_1, z_2)$ on $0 < |z_2| < |z_1|$ since the series expansion of $(z_1 - z_2)^{-N}\phi(z_1, z_2)$ does. Similarly, $g(z_1, z_2)$ converges a.l.u. on $0 < |z_1| < |z_2|$ to $(z_1 - z_2)^{-N}\phi(z_1, z_2)$. This finishes the proof. $\square$

7.10

We now prove that complex analytic implies formal variable. To prepare for the proof, note that for any $k \in \mathbb{N}$, any $m, n \in \mathbb{Z}$, and any $R_1, R_2 > 0$,

$$\oint_{|z_1|=R_1}\oint_{|z_2|=R_2} z_1^m z_2^n \langle v', A(z_1)P_k B(z_2)v\rangle \frac{dz_1 dz_2}{(2\mathbf{i}\pi)^2} = \langle v', A_m P_k B_n v\rangle. \tag{7.13}$$

Indeed, this is obvious when $\mathbb{V}(k)$ is finite dimensional, in which case $\langle v', A(z_1)P_kB(z_2)v\rangle = \sum_e \langle v', A(z_1)e\rangle\langle \check{e}, B(z_2)v\rangle$ where $\{e\}$ is a basis of $\mathbb{V}(k)$ and $\{\check{e}\}$ is its dual basis. In the general case, we may first fix z_2 and integrate z_1 by considering $P_kB(z_2)v$ as a fixed vector in $\mathbb{V}(k)$, and then integrate z_2 by considering $\langle v', A_mP_k\cdot\rangle$ as an element of $\mathbb{V}'(k) = \mathbb{V}(k)^*$.

Proof. Assume the statements in Def. 7.2 hold. Let $f_{v,v'} \in \mathscr{O}(\mathrm{Conf}^2(\mathbb{C}^\times))$ be as in Def. 7.2. Since $\phi := (z_1 - z_2)^N f_{v,v'}$ belongs to $\mathscr{O}(\mathbb{C}^\times \times \mathbb{C}^\times)$, by complex analysis, for each $m, n \in \mathbb{Z}$ the value of

$$\Gamma := \oint_{|z_1|=R_1}\oint_{|z_2|=R_2} z_1^m z_2^n \phi(z_1, z_2)\frac{dz_1dz_2}{(2\mathbf{i}\pi)^2}$$

is independent of the specific values of R_1, R_2. (This is where we use the fact that ϕ has no poles at $z_1 = z_2$.)

We compute Γ in two ways. Assume $R_1 > R_2$. Then since $0 < |z_2| < |z_1|$, we have

$$\phi(z_1, z_2) = \sum_{k\in\mathbb{N}} (z_1 - z_2)^N\langle v', A(z_1)P_kB(z_2)v\rangle.$$

Thus, using (7.13), we can compute

$$\begin{aligned}
\Gamma &= \oint_{|z_1|=R_1}\oint_{|z_2|=R_2}\sum_{k\in\mathbb{N}} z_1^m z_2^n(z_1-z_2)^N\langle v', A(z_1)P_kB(z_2)v\rangle\frac{dz_1dz_2}{(2\mathbf{i}\pi)^2}\\
&= \sum_{k\in\mathbb{N}}\oint_{|z_1|=R_1}\oint_{|z_2|=R_2}\sum_{j=0}^N \binom{N}{j} z_1^{m+j}z_2^{n+N-j}(-1)^{N-j}\langle v', A(z_1)P_kB(z_2)v\rangle\frac{dz_1dz_2}{(2\mathbf{i}\pi)^2}\\
&= \sum_{k\in\mathbb{N}}\sum_{j=0}^N\binom{N}{j}(-1)^{N-j}\langle v', A_{m+j}P_kB_{n+N-j}v\rangle = \sum_{j=0}^N\binom{N}{j}(-1)^{N-j}\langle v', A_{m+j}B_{n+N-j}v\rangle
\end{aligned}$$

where $\sum_{k\in\mathbb{N}}$ commutes with the two contour integrals thanks to the a.l.u. convergence. Similarly, if we assume $R_1 < R_2$, then $\phi(z_1, z_2) = \sum_{k\in\mathbb{N}}(z_1 - z_2)^N\langle v', B(z_2)P_kA(z_1)v\rangle$, and hence

$$\Gamma = \sum_{j=0}^N\binom{N}{j}(-1)^{N-j}\langle v', B_{n+N-j}A_{m+j}v\rangle.$$

This shows $\sum_{j=0}^N\binom{N}{j}(-1)^{N-j}[A_{m+j}, B_{n+N-j}] = 0$. If we compare this with the first several lines of (7.11), we see that this is equivalent to $(z_1 - z_2)^N[A(z_1), B(z_2)] = 0$ in $\mathrm{End}\mathbb{V}[[z_1^{\pm 1}, z_2^{\pm 1}]]$. □

7.11

In this subsection, we assume the statements in Def. 7.2, and derive the OPE $A(z_1)B(z_2) = \sum_{k\in\mathbb{Z}}(z_1 - z_2)^{-k-1}(A_kB)(z_2)$ similar to $Y(u, z_1)Y(v, z_2) = \sum_{k\in\mathbb{Z}}(z_1 - z_2)^{-k-1}Y(Y(u)_kv, z_2)$ for some fields $(A_kB)(z)$. This is simply done by taking Laurent series expansions of $z_1 - z_2$ of the function $f_{v,v'}$ in Def. 7.2. Thus, the existence of OPE simply follows from complex analysis. Since we are treating multivariable holomorphic functions, to be serious about the domain of a.l.u. convergence, we provide some details below.

Definition 7.10. For each $k \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$, let $f_{v,v'} \in \mathscr{O}(\mathrm{Conf}^2(\mathbb{C}^\times))$ be as in Def. 7.2. We define the linear map

$$(A_k B)(z) : \mathbb{V}' \otimes \mathbb{V} \to \mathbb{C}, \qquad v' \otimes v \mapsto \langle v', (A_k B)(z) v \rangle$$

to be

$$\langle v', (A_k B)(z_2) v \rangle = \oint_{C(z_2)} (z_1 - z_2)^k f_{v,v'}(z_1, z_2) \frac{dz_1}{2\mathrm{i}\pi} \tag{7.14}$$

where $C(z_2)$ is any circle in $\mathbb{C}^\times$ surrounding z_2. Note that $(A_k B)(z)v$ is naturally an element of $(\mathbb{V}')^* = \prod_{n \in \mathbb{N}} \mathbb{V}(n)^{**}$, the (algebraic) dual space of $\mathbb{V}'$. Also, $\langle v', (A_k B)(z_2) v \rangle$ is clearly a holomorphic function of z_2 on $\mathbb{C}^\times$.

Lemma 7.11. *$A_k B = 0$ whenever $k \geqslant N$.*

Proof. When $k \geqslant N$, $(z_1 - z_2)^k f_{v,v'}$ has no poles at $z_1 = z_2$. So the RHS of (7.14) vanishes. □

Proposition 7.12. *For each $v \in \mathbb{V}, v' \in \mathbb{V}'$, we have*

$$f_{v,v'}(z_1, z_2) = \sum_{k \in \mathbb{Z}} (z_1 - z_2)^{-k-1} \langle v', (A_k B)(z_2) v \rangle \tag{7.15}$$

where the series on the RHS converges a.l.u. on $\Omega_0 = \{(z_1, z_2) : 0 < |z_1 - z_2| < |z_2|\}$ to the LHS.

Proof. It suffices to prove the claim on $\{(z_1, z_2) : 0 < |z_1 - z_2| < r, r < |z_2|\}$ for all $r > 0$. Then this follows easily from the following basic lemma. □

Lemma 7.13. *Let U be an open subset of $\mathbb{C}^m$ and let $f = f(z_1, \dots, z_m, q_1, \dots, q_n)$ be a holomorphic function on $U \times A_{r_1,R_1} \times \cdots \times A_{r_n,R_n}$ where each $0 \leqslant r_i < R_i \leqslant +\infty$ and $A_{r_i,R_i} = \{q_i \in \mathbb{C} : r_i < |q_i| < R_i\}$. Then f has Laurent series expansion*

$$f(z_\bullet, q_\bullet) = \sum_{k_1, \dots, k_n \in \mathbb{Z}} f_{k_\bullet}(z_\bullet) q_1^{-k_1-1} \cdots q_n^{-k_n-1} \tag{7.16}$$

converging a.l.u. on $U \times A_{r_1,R_1} \times \cdots \times A_{r_n,R_n}$, where each

$$f_{k_\bullet}(z_\bullet) = \oint_{C_n} \cdots \oint_{C_1} f(z_\bullet, q_\bullet) q_1^{k_1} \cdots q_n^{k_n} \frac{dq_1 \cdots dq_n}{(2\mathrm{i}\pi)^n} \tag{7.17}$$

(where C_j is an anticlockwise circle around 0) is clearly holomorphic on U.

Proof. For simplicity, we assume $n = 1$ and write $q_1 = q, r_1 = r, R_1 = R$. We shall prove the a.l.u. convergence on $(z_\bullet, q) \in U \times A_{\widetilde{r},\widetilde{R}}$ for all $\widetilde{r}, \widetilde{R}$ such that $r < \widetilde{r} < \widetilde{R} < R$. Let $C_- = \{q \in \mathbb{C} : |q| = (r + \widetilde{r})/2\}$ and $C_+ = \{q \in \mathbb{C} : |q| = (R + \widetilde{R})/2\}$. Then on $U \times A_{\widetilde{r},\widetilde{R}}$,

$$f(z_\bullet, q) = \mathrm{Res}_{p=q} \frac{f(z_\bullet, p)}{p - q} dp = \Bigg(\oint_{C_+} - \oint_{C_-} \Bigg) \frac{f(z_\bullet, p)}{p - q} \frac{dp}{2\mathrm{i}\pi}.$$

We have $\frac{f(z_\bullet,p)}{p-q} = \sum_{k\leqslant -1} q^{-k-1}p^k f(z_\bullet, p)$ where the RHS converges on $(z_\bullet, q, p) \in (U \times A_{\widetilde{r},\widetilde{R}} \times C_+)$ to the LHS by basic analysis. The same can be said about $\frac{f(z_\bullet,p)}{p-q} = -\sum_{k\geqslant 0} q^{-k-1}p^k f(z_\bullet, p)$ if C_+ is replaced by C_-. So in view of (7.17), and noting that integrals commute with infinite sums due to a.l.u. convergence, the RHS of $\oint_{C_+} \frac{f(z_\bullet,p)}{p-q} \frac{dp}{2\mathrm{i}\pi} = \sum_{k\leqslant -1} f_k(z_\bullet)q^{-k-1}$ (resp. $\oint_{C_-} \frac{f(z_\bullet,p)}{p-q} \frac{dp}{2\mathrm{i}\pi} = -\sum_{k\geqslant 0} f_k(z_\bullet)q^{-k-1}$) converges a.l.u. on $U \times A_{\widetilde{r},\widetilde{R}}$ to the RHS. This completes the proof. $\square$

7.12

We continue our discussion in the previous section. Let $(A_nB)_k : \mathbb{V}' \otimes \mathbb{V} \to \mathbb{C}$ such that

$$\langle v', (A_nB)_k v\rangle = \mathrm{Res}_{z=0}\langle v', (A_nB)(z)v\rangle z^k dz.$$

In other words, $(A_nB)_k$ is a linear map $\mathbb{V} \to (\mathbb{V}')^* = \prod_{n\in\mathbb{N}} \mathbb{V}(n)^{**}$.

Proposition 7.14. *Assume that $A(z), B(z)$ satisfy Def. 7.2. Then the following Jacobi identity holds:*

$$\begin{aligned} &\sum_{l\in\mathbb{N}} \binom{m}{l} (A_{n+l}B)_{m+k-l} \\ =&\sum_{l\in\mathbb{N}} (-1)^l \binom{n}{l} A_{m+n-l}B_{k+l} - \sum_{l\in\mathbb{N}} (-1)^{n+l} \binom{n}{l} B_{n+k-l}A_{m+l}. \end{aligned} \tag{7.18}$$

Remark 7.15. There are two immediate consequences of this proposition. First, by setting $m = 0$, we get a formula to express $(A_nB)_k$ in terms of the modes of $A(z)$ and $B(z)$. From that expression, one easily checks that $(A_nB)_k$ sends each $\mathbb{V}(a)$ to $\mathbb{V}(b)$ where $b = a + \Delta_A + \Delta_B - n - k - 2$. This shows that $(A_nB)_k$ is a linear operator on $\mathbb{V}$, and that $(A_nB)(z)$ is a homogeneous field with weight $\Delta_A + \Delta_B - n - 1$. Second, by setting $n = 0$, we see that $A(z)$ is local to $B(z)$ in the Lie algebraic sense.

Proof of Prop. 7.14. The idea is the same as the proof of VOA Jacobi identity in Subsec. 4.8. (Note that roles of z_1, z_2 in Subsec. 4.8 are switched here.) For each $z_2 \in \mathbb{C}^\times$, we choose a large circle C_+ and a small one C_- centered at 0, and a small one C_0 centered at z_2. Choose $\mu = z_1^m(z_1 - z_2)^n dz_1$. Set $f = f_{v,v'}$. Then

$$\oint_{C_+} \frac{f\mu}{2\mathrm{i}\pi} - \oint_{C_-} \frac{f\mu}{2\mathrm{i}\pi} = \oint_{C_0} \frac{f\mu}{2\mathrm{i}\pi}. \tag{7.19}$$

When z_1 is on C_+, f takes the form (7.2a). Moreover, the RHS of

$$z_1^m(z_1-z_2)^n dz_1 f(z_1,z_2) = \sum_{l,s\in\mathbb{N}} \binom{n}{l}(-z_2)^l z_1^{m+n-l}\langle v', A(z_1)P_sB(z_2)v\rangle$$

converges a.l.u. on $0<|z_2|<|z_1|$ to the LHS. So

$$\begin{aligned}\oint_{C_+}\frac{f\mu}{2\mathbf{i}\pi} &= \oint_{C_+}\sum_{l,s\in\mathbb{N}}\binom{n}{l}(-z_2)^l z_1^{m+n-l}\langle v', A(z_1)P_sB(z_2)v\rangle\frac{dz_1}{2\mathbf{i}\pi}\\ &= \sum_{l,s\in\mathbb{N}}\binom{n}{l}\oint_{C_+}(-z_2)^l z_1^{m+n-l}\langle v', A(z_1)P_sB(z_2)v\rangle\frac{dz_1}{2\mathbf{i}\pi}\\ &= \sum_{l,s\in\mathbb{N}}\binom{n}{l}(-z_2)^l\langle (A_{m+n-l})^{\mathrm{t}}v', P_sB(z_2)v\rangle\frac{dz_1}{2\mathbf{i}\pi}\end{aligned}$$

where the contour integral commutes with the infinite sum due to the a.l.u. convergence; $(A_{m+n-l})^{\mathrm{t}}$ is the transpose of A_{m+n-l}, sending v' to a vector of $\mathbb{V}(s)$ where $s=\mathrm{wt}v'-\Delta_A+m+n-l+1$. So when s is not this weight, the above summand vanishes. We can thus write the above expression as

$$\sum_{l\in\mathbb{N}}\binom{n}{l}(-z_2)^l\langle (A_{m+n-l})^{\mathrm{t}}v', B(z_2)v\rangle\frac{dz_1}{2\mathbf{i}\pi}.$$

The integral on C_- can be treated in a similar way. And by Prop. 7.12,

$$\oint_{C_0}\frac{f\mu}{2\mathbf{i}\pi} = \oint_{C_0}\sum_{l,s\in\mathbb{N}}\binom{m}{l}z_2^{m-l}(z_1-z_2)^{n+l}\cdot(z_1-z_2)^{-s-1}\langle v',(A_sB)(z_2)v\rangle\frac{dz_1}{2\mathbf{i}\pi}$$

where series inside the integrand converge a.l.u. on $0<|z_1-z_2|<|z_2|$. So we can exchange the integral and the sum to compute the result

$$\sum_{l\in\mathbb{N}}\binom{m}{l}z_2^{m-l}\langle v',(A_{n+l}B)(z_2)v\rangle.$$

This computes (7.19). Now all three terms are clearly holomorphic functions of z_2 on $\mathbb{C}^\times$. Multiply them by $z_2^k dz_2$ and evaluate the residue at $z_2=0$, we get (7.18). $\square$

7.13

We are now ready to prove the equivalence of the complex analytic version and the algebraic version of Jacobi identity.

Definition 7.16 (Jacobi identity (complex analytic version)**).** For each $u,v,w\in\mathbb{V}$ and $w'\in\mathbb{V}$, the following series

$$\big\langle w', Y(u,z_1)Y(v,z_2)w\big\rangle := \sum_{n\in\mathbb{N}}\big\langle w', Y(u,z_1)P_nY(v,z_2)w\big\rangle, \tag{7.20a}$$

$$\left\langle w', Y(v, z_2)Y(u, z_1)w\right\rangle := \sum_{n\in\mathbb{N}} \left\langle w', Y(v, z_2)P_nY(u, z_1)w\right\rangle, \tag{7.20b}$$

$$\left\langle w', Y\big(Y(u, z_1 - z_2)v, z_2\big)w\right\rangle := \sum_{n\in\mathbb{N}} \left\langle w', Y\big(P_nY(u, z_1 - z_2)v, z_2\big)\right\rangle \tag{7.20c}$$

converge a.l.u. respectively on

$$\{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < |z_1|\}, \tag{7.21a}$$

$$\{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_2| < |z_2|\}, \tag{7.21b}$$

$$\{(z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1 - z_2| < |z_2|\} \tag{7.21c}$$

and can be extended to the same holomorphic function $f_{w,u,v,w'}$ on $\mathrm{Conf}^2(\mathbb{C}^\times)$.

Theorem 7.17. *The complex analytic and the algebraic versions of Jacobi identity are equivalent.*

Proof. Complex analytic implies algebraic: This follows from the argument in Subsec. 4.8 or the proof of Prop. 7.14.

Algebraic implies complex analytic: Assume that u, v, w, w' are homogeneous. Let $A(z) = Y(u, z)$ and $B(z) = Y(v, z)$. Then A and B are local. Moreover, the VOA Jacobi identity expresses $Y(Y(u)_nv, z)$ in terms of $Y(u, z), Y(v, z)$, and (7.18) expresses $(A_nB)(z)$ in terms of $A(z), B(z)$. From these two expressions, it is clear that

$$Y(Y(u)_nv, z) = (A_nB)(z). \tag{7.22}$$

Thus, the complex analytic locality of A and B proves the complex analytic Jacobi identity. Note that the a.l.u. convergence of (7.20c) $= \sum_{m\in\mathbb{Z}} \left\langle w', Y(Y(u)_mv)w\right\rangle(z_1 - z_2)^{-m-1}$ (note that $P_nY(u, z_1 - z_2)v = Y(u)_m(z_1 - z_2)^{-m-1}v$ where $n = \mathrm{wt}u + \mathrm{wt}v - m - 1$) follows from that of (7.15). □

8 n-point functions for vertex operators; proof of reconstruction theorem

8.1

The goals of this section are twofold. We first prove two analytic properties for n-point functions generalizing Def. 7.2. Then we use these results to prove the reconstruction theorem.

Theorem 8.1. *Assume that the homogeneous fields $A^1(z), \dots, A^M(z) \in (\mathrm{End}\mathbb{V})[[z^{\pm1}]]$ are mutually local. Then for each $v \in \mathbb{V}, v' \in \mathbb{V}'$ and each permutation σ of $\{1, \dots, M\}$, the series*

$$\left\langle v', A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(M)}(z_{\sigma(M)})v\right\rangle \qquad \in (\mathrm{End}\mathbb{V})[[z_1^{\pm1}, \dots, z_M^{\pm1}]] \tag{8.1}$$

converges a.l.u. on

$$\Omega_\sigma = \{z_\bullet \in \mathbb{C}^M : 0 < |z_{\sigma(M)}| < \cdots < |z_{\sigma(1)}|\} \tag{8.2}$$

and can be extended to some $f_{v,v'} \in \mathscr{O}(\mathrm{Conf}^M(\mathbb{C}^\times))$ independent of σ. Moreover, there exists $N \in \mathbb{N}$ for all v, v' such that

$$f_{v,v'}(z_\bullet) \cdot \prod_{1 \leqslant i < j \leqslant M} (z_i - z_j)^N \tag{8.3}$$

is holomorphic on $(\mathbb{C}^\times)^M$. (Indeed, it is an element of $\mathbb{C}[z_1^{\pm 1}, \dots, z_M^{\pm 1}]$.)

$f_{v,v'}$ is called the **$(M+2)$-point (genus 0 correlation) function** associated to the fields $A^\bullet(z)$. In case each $A^i(z)$ is a vertex operator $Y(u_i, z)$, $f_{v,v'}$ is the correlation function associated to (setting ζ to be the standard coordinate of $\mathbb{C}$)

$$(\mathbb{P}^1; 0, z_1, \dots, z_M, \infty; \zeta, \zeta - z_1, \dots, \zeta - z_M, \zeta^{-1}), \tag{8.4}$$

where $v, u_1, \dots, u_M, v'$ are going into the punctures $0, z_1, \dots, z_M, \infty$ respectively.

Remark 8.2. The a.l.u. convergence on Ω_σ of the formal Laurent series (8.1) is equivalent to that of the series of functions

$$\begin{aligned} &\big\langle v', A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(M)}(z_{\sigma(M)}) v \big\rangle \\ := &\sum_{n_2, \dots, n_M \in \mathbb{N}} \big\langle v', A^{\sigma(1)}(z_{\sigma(1)}) P_{n_2} A^{\sigma(2)}(z_{\sigma(2)}) P_{n_3} \cdots P_{n_M} A^{\sigma(M)}(z_{\sigma(M)}) v \big\rangle. \end{aligned} \tag{8.5}$$

Indeed, assume for simplicity that $\sigma = 1$ and v, v' are homogeneous. Then by scale covariance (7.7), the RHS of the above formula equals

$$\begin{aligned} &\sum_{n_2, \dots, n_M \in \mathbb{N}} \big\langle v', A^1(1) P_{n_2} A^2(1) P_{n_3} \cdots P_{n_M} A^M(1) v \big\rangle \\ &\qquad \cdot \Big(\frac{z_2}{z_1}\Big)^{n_2} \Big(\frac{z_3}{z_2}\Big)^{n_3} \cdots \Big(\frac{z_M}{z_{M-1}}\Big)^{n_M} \cdot z_1^{\mathrm{wt} v'} z_M^{-\mathrm{wt} v} \cdot \prod_{i=1}^M z_i^{-\Delta_{A^i}}, \end{aligned} \tag{8.6}$$

which together with Lemma 7.4 proves the claim.

8.2

Proof of Thm. 8.1. The method is the same as in Subsec. 7.9. Choose N such that $(z_i - z_j)^N [A^i(z_i), A^j(z_j)] = 0$ for all i, j. Set

$$f^\sigma(z_\bullet) = \big\langle v', A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(M)}(z_{\sigma(M)}) v \big\rangle \qquad \in \mathbb{C}[[z_\bullet^{\pm 1}]] = \mathbb{C}[[z_1^{\pm 1}, \dots, z_M^{\pm 1}]]. \tag{8.7}$$

Then the formal Laurent series

$$\phi(z_\bullet) = f^\sigma(z_\bullet) \cdot \prod_{1 \leqslant i < j \leqslant M} (z_i - z_j)^N \tag{8.8}$$

is independent of the permutation σ. From

$$f^1(z_\bullet) = \sum_{m,n \in \mathbb{Z}} \langle (A_m^1)^{\mathrm{t}} v', A^2(z_2) \cdots A^{M-1}(z_{m-1}) A_n^M v \rangle \cdot z_1^{-m-1} z_M^{-n-1}$$

and the lower truncation property, we see that the coefficients of $f^1(z_\bullet)$ and hence of $\phi(z_\bullet)$ vanish if the powers of z_M (resp. z_1) is sufficiently negative (resp. positive). Since we can replace M with $\sigma(M)$ and 1 with $\sigma(1)$, we see that the coefficients of ϕ vanish except when the powers $z_1, \ldots, z_M$ are all bounded from below and from above by some fixed constants. Namely, $\phi(z_\bullet) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_M^{\pm 1}]$. In particular, ϕ can be regarded as a holomorphic function on $(\mathbb{C}^\times)^M$.

By expanding $f^1(z_\bullet)$ as a formal Laurent series of $z_1, \ldots, z_M$, it is not hard to see (e.g. by induction on M) that

$$f^1(z_\bullet) \in \mathbb{F} := \mathbb{C}((z_1))((z_2)) \cdots ((z_M)). \tag{8.9}$$

So, for $\sigma = 1$, (8.8) holds in the field $\mathbb{F}$. So

$$f^1(z_\bullet) = \phi(z_\bullet) \cdot \prod_{1 \leqslant i < j \leqslant M} (z_i - z_j)^{-N} \tag{8.10}$$

where $\prod_{1 \leqslant i < j \leqslant M} (z_i - z_j)^{-N} \in \mathbb{F}$ is expanded in the region Ω_1 (defined by (8.2)), cf. Subsec. 7.6. This expansion can be written down explicitly. By basic analysis, one checks that this series, and hence the series on the RHS of (8.10) (regarded as an element of $\mathbb{F}$), converge a.l.u. on Ω_1 to the RHS of (8.10) regarded as a holomorphic function which we denote by $f_{v,v'}$. Since this statement also holds when the permutation 1 is replaced by any σ, our proof is therefore completed. □

8.3

Definition 8.3. Let $A^1, \ldots, A^M$ be mutually local. For each $z_\bullet \in \mathrm{Conf}^M(\mathbb{C}^\times)$,

$$A^1(z_1) \cdots A^M(z_M) : \mathbb{V}' \otimes \mathbb{V} \to \mathbb{C} \tag{8.11}$$

is defined to be the linear map sending $v' \otimes v$ to $f_{v,v'}$ in Thm. 8.1. Equivalently, $A^1(z_1) \cdots A^M(z_M)$ is a linear map from $\mathbb{V}$ to $(\mathbb{V}')^* = \prod_{n \in \mathbb{N}} \mathbb{V}(n)^{**}$.

According to this notation, for local A, B we have

$$A(z_1)B(z_2) = B(z_2)A(z_1), \qquad (A_n B)(z_2) = \mathrm{Res}_{z_1 = z_2} A(z_1)B(z_2)(z_1 - z_2)^n dz_1. \tag{8.12}$$

The following is our second analytic property for n-point functions.

Theorem 8.4. *Assume $A^1, \cdots, A^m, B^1, \ldots, B^n$ are mutually local. Then on*

$$\Omega = \{(z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_n) \in \mathrm{Conf}^{m+n}(\mathbb{C}^\times) : |z_i| > |\zeta_j| \text{ for all } i, j\},$$

for each $v \in \mathbb{V}, v' \in \mathbb{V}'$ the RHS of

$$\begin{aligned} &\langle v', A^1(z_1) \cdots A^m(z_m) B^1(\zeta_1) \cdots B^n(\zeta_n) v \rangle \\ =& \sum_{k \in \mathbb{N}} \langle v', A^1(z_1) \cdots A^m(z_m) P_k B^1(\zeta_1) \cdots B^n(\zeta_n) v \rangle \end{aligned} \tag{8.13}$$

converges a.l.u. to the LHS.

The meaning of the product of $A^1(z_1)\cdots A^m(z_m)$ and $B^1(\zeta_1)\cdots B^n(\zeta_n)$ is clear: it corresponds to the sewing of (setting ζ to be the standard coordinate of $\mathbb{C}$)

$$\mathfrak{X}_1 = (\mathbb{P}^1; 0, z_1, \dots, z_m, \infty; \zeta, \zeta - z_1, \dots, \zeta - z_m, \zeta^{-1}),$$
$$\mathfrak{X}_2 = (\mathbb{P}^1; 0, \zeta_1, \dots, \zeta_n, \infty; \zeta, \zeta - \zeta_1, \dots, \zeta - \zeta_n, \zeta^{-1})$$

along $0 \in \mathfrak{X}_1$ and $\infty \in \mathfrak{X}_2$, in case all these fields are vertex operators. Moreover, this picture, as well as the theorem, can be easily generalized to the products of several strings of mutually local fields.

Remark 8.5. Note that each summand on the RHS of (8.13) is holomorphic on $(z_\bullet, \zeta_\bullet) \in \mathrm{Conf}^m(\mathbb{C}^\times) \times \mathrm{Conf}^n(\mathbb{C}^\times)$. When $\mathbb{V}(k)$ is finite-dimensional, this is due to Thm. 8.1 and that P_k can be written as $\sum_e e\rangle\langle\check{e}$ for a basis $\{e\}$ of $\mathbb{V}(k)$ and dual basis $\{\check{e}\}$.

In the general case that $\mathbb{V}(k)$ is not necessarily finite dimensional, P_k is the projection from $(\mathbb{V}')^*$ onto $\mathbb{V}(k)^{**}$. In each series $A^i(z_i) = \sum_{n\in\mathbb{N}} A^i_n z_i^{-n-1}$ in (8.13), A^i_n is understood as $(A^i_n)^{\mathrm{tt}}$ sending each $\mathbb{V}(a)^{**}$ to $\mathbb{V}(b)^{**}$ where $b = a + \Delta_A - n - 1$. Then, in this sense $A^1, \dots, A^m$ are mutually local. Each summand on the RHS of (8.13) is continuous over $(z_\bullet, \zeta_\bullet) \in \mathrm{Conf}^m(\mathbb{C}^\times) \times \mathrm{Conf}^n(\mathbb{C}^\times)$; for fixed $z_\bullet$, it is holomorphic over $\zeta_\bullet$ by treating $\langle v', A^1(z_1)\cdots A^m(z_m)P_k\cdot\rangle$ as an element of $\mathbb{V}(k)^*$; similarly, it is holomorphic over $z_\bullet$. So, again, it is holomorphic on $\mathrm{Conf}^m(\mathbb{C}^\times) \times \mathrm{Conf}^n(\mathbb{C}^\times)$.

8.4

The idea of the proof of Thm. 8.4 is the following. To show that a series of functions $\sum_n f_n(z_\bullet)$ converges a.l.u. on a domain U: We try to find $r > 1$ and a smaller U' such that $\sum_n f_n(z_\bullet)q^n$ converges a.l.u. on $z_\bullet \in U'$ and $q \in \mathbb{D}_r^\times$ to a function f holomorphic on $U \times \mathbb{D}_r^\times$. Then by Lemma 7.13, $\sum_n f_n(z_\bullet)q^n$ is the series expansion of f, which must converge a.l.u. on $U \times \mathbb{D}_r^\times$.

Proof of Thm. 8.4. It suffices to prove the proposition when Ω is replaced by all possible

$$\Omega_r = \{(z_1, \dots, z_m, \zeta_1, \dots, \zeta_n) \in \mathrm{Conf}^{m+n}(\mathbb{C}^\times) : |z_i| > r|\zeta_j| \text{ for all } i, j\}$$

where $r > 1$. To simplify discussions we assume $m = n = 2$. Consider the following element of $\mathscr{O}(\mathrm{Conf}^2(\mathbb{C}^\times)^2)[[q]]$:

$$\sum_{k\in\mathbb{N}} \langle v', A^1(z_1)A^2(z_2)P_k B^1(\zeta_1)B^2(\zeta_2)v\rangle q^k. \tag{8.14}$$

Note that $P_k q^k = P_k q^{L_0}$. By scale covariance, as elements of $\mathrm{Hom}(\mathbb{V}' \otimes \mathbb{V}, \mathbb{C})$,

$$q^{L_0} B^1(\zeta_1)B^2(\zeta_1) = q^{\Delta_{B^1}+\Delta_{B^2}} B^1(q\zeta_1)B^2(q\zeta_2)q^{L_0} \tag{8.15}$$

whenever $0 < |\zeta_2| < |\zeta_1|$. So it holds for all $\zeta_\bullet \in \mathrm{Conf}^2(\mathbb{C}^\times)$ by holomorphicity. Thus, there is $d \in \mathbb{Z}$ such that (8.14), as a series of functions of $(z_\bullet, \zeta_\bullet, q)$, equals

$$\sum_{k\in\mathbb{N}} q^d \langle v', A^1(z_1)A^2(z_2)P_k B^1(q\zeta_1)B^2(q\zeta_2)v\rangle. \tag{8.16}$$

By Thm. 8.1, this series (and hence series (8.14)) converges a.l.u. on $\Omega_r' = \{(z_\bullet,\zeta_\bullet) : 0 < r|\zeta_2| < r|\zeta_1| < |z_2| < |z_1|\}$ and $0 < |q| < r$ to the holomorphic function

$$g(z_\bullet,\zeta_\bullet,q) = q^d\langle v', A^1(z_1)A^2(z_2)B^1(q\zeta_1)B^2(q\zeta_2)v\rangle.$$

Therefore, (8.14) is the Laurent series expansion of g when $(z_\bullet,\zeta_\bullet)\in\Omega_r'$. Namely: the coefficients of (8.14) equal those in the expansion of g when $(z_\bullet,\zeta_\bullet)\in\Omega_r'$. By Lemma 7.13 applied to the holomorphic function $g(z_\bullet,\zeta_\bullet,q)$ on $\Omega_r\times\mathbb{D}_r^\times$, this statement is true when $(z_\bullet,\zeta_\bullet)\in\Omega_r$ (since the coefficients are holomorphic on Ω_r), and the series expansion of g converges a.l.u. on $\Omega_r\times\mathbb{D}_r^\times$ to g. So (8.14) converges a.l.u. on $\Omega_r\times\mathbb{D}_r^\times$ to g. This finishes the proof if we set $q=1$. □

8.5

We now discuss the proof of the reconstruction Thm. 5.12. Assume that the assumptions for graded vertex algebras in Thm. 5.12 hold. We may extend $\mathcal{E}$ to also include the identity field $\mathbf{1}(z)=\mathbf{1}_{\mathbb{V}}$. Namely, $\mathbf{1}_n=\delta_{n,-1}\mathbf{1}_{\mathbb{V}}$. Motivated by (7.22), for each $A^1,\dots,A^k\in\mathcal{E}$ and $n_1,\dots,n_k\in\mathbb{Z}$, we define

$$Y\big(A^1_{n_1}\cdots A^k_{n_k}\mathbf{1},z\big) = \big(A^1_{n_1}\cdots A^k_{n_k}\mathbf{1}\big)(z) \tag{8.17}$$

where the right hand side is defined inductively by

$$\big(A^1_{n_1}\cdots A^k_{n_k}\mathbf{1}\big)(z) = \big(A^1_{n_1}(A^2_{n_2}\cdots A^k_{n_k}\mathbf{1})\big)(z).$$

By the generating property, we can define $Y(u,z)$ for every $u\in\mathbb{V}$ using (8.17) and linearity.

There are two immediate problems with this approach: First, to define the RHS of (8.17) inductively, we need the fact that $A^1_{n_1}(z)$ is local to $(A^2_{n_2}\cdots A^k_{n_k}\mathbf{1})(z)$ ("Dong's lemma"). Second, we need to show that the above definition of $Y(u,z)$ is unique, i.e., independent of how u is written as a linear combination of $A^1_{n_1}\cdots A^k_{n_k}\mathbf{1}$ ("Goddard uniqueness"). Besides these two, we also need to check that such defined $Y(u,z)$ satisfies the translation property. Let us first check the translation property.

8.6

Lemma 8.6. *Assume that homogeneous fields $A(z),B(z)$ are local and satisfy the translation property $[L_{-1},A_k]=-kA_{k-1}$, $[L_{-1},B_k]=-kB_{k-1}$. Then so does each A_nB:*

$$[L_{-1},(A_nB)_k] = -k(A_nB)_{k-1}. \tag{8.18}$$

Proof. By the Jacobi identity (7.18),

$$(A_nB)_k = \sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l}A_{n-l}B_{k+l} - \sum_{l\in\mathbb{N}}(-1)^{n+l}\binom{n}{l}B_{n+k-l}A_l, \tag{8.19}$$

and hence

$$-k(A_nB)_{k-1} = \sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l}(-k)A_{n-l}B_{k+l-1} + \sum_{l\in\mathbb{N}}(-1)^{n+l}\binom{n}{l}kB_{n+k-l-1}A_l.$$

So by the translation property of A, B,

$$[L_{-1}, (A_n B)_k] = \sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l}(-n+l)A_{n-l-1}B_{k+l} + \sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l}(-k-l)A_{n-l}B_{k+l-1}$$
$$+\sum_{l\in\mathbb{N}}(-1)^{n+l}\binom{n}{l}(n+k-l)B_{n+k-l-1}A_l + \sum_{l\geqslant 1}(-1)^{n+l}\binom{n}{l}lB_{n+k-l}A_{l-1}.$$

Look at the RHS. In the first sum, notice $(-1)^l\binom{n}{l}(-n+l) = (-1)^{l+1}\binom{n}{l+1}(l+1)$ and replace l by $l-1$; in the fourth sum, notice $(-1)^{n+l}\binom{n}{l}l = (-1)^{n+l-1}\binom{n}{l-1}(l-1-n)$ and replace l by $l+1$. Then we see why (8.18) is true. $\square$

8.7

Proposition 8.7 (Dong's lemma). *Let $A(z), B(z), C(z)$ be mutually local homogeneous fields. Then for each $n\in\mathbb{Z}$, $C(z)$ is local to $(A_nB)(z)$.*

We prove that $C(z)$ is complex-analytically local to $(A_nB)(z)$.

Proof. Step 1. Choose $v\in\mathbb{V}, v'\in\mathbb{V}'$. Then we have series

$$\sum_{k\in\mathbb{N}}\langle v', (A_nB)(z_2)P_kC(z_3)\rangle = \sum_{k\in\mathbb{N}}\mathrm{Res}_{z_1=z_2}(z_1-z_2)^n\langle v', A(z_1)B(z_2)P_kC(z_3)v\rangle dz_1. \quad (8.20)$$

On the region $\Omega_1 = \mathrm{Conf}^3(\mathbb{C}^\times)\cap\{|z_1|>|z_3|, |z_2|>|z_3|\}$, the RHS of

$$f(z_1,z_2,z_3) := \langle v', A(z_1)B(z_2)C(z_3)v\rangle = \sum_{k\in\mathbb{N}}\langle v', A(z_1)B(z_2)P_kC(z_3)v\rangle$$

converges a.l.u. to the LHS by Thm. 8.4. Therefore, on Ω_1, the sum and the residue (i.e. contour integral) on the RHS of (8.20) commute, and (8.20) converges a.l.u. on $|z_2|>|z_3|>0$ to

$$g(z_2,z_3) = \mathrm{Res}_{z_1=z_2}(z_1-z_2)^nf(z_1,z_2,z_3)dz_1$$

which is holomorphic on $\mathrm{Conf}^2(\mathbb{C}^\times)$ since f is holomorphic on $\mathrm{Conf}^3(\mathbb{C}^\times)$ by Thm. 8.1. Similarly, $\sum_{k\in\mathbb{N}}\langle v', C(z_3)P_k(A_nB)(z_2)\rangle$ converges a.l.u. on $|z_3|>|z_2|>0$ to $g(z_2,z_3)$.

Step 2. To complete the proof, we need to show that $(z_2-z_3)^kg$ is holomorphic near $z_2=z_3$ for some k. By Thm. 8.1, $(z_1-z_2)^nf$ is a linear combination of

$$z_1^az_2^bz_3^c(z_1-z_2)^{n-N}(z_1-z_3)^{-N}(z_2-z_3)^{-N}$$

for some $N\in\mathbb{N}$ and $a,b,c\in\mathbb{Z}$. To prove the claim, we may assume that $(z_1-z_2)^nf$ is just of this form. Then near $z_1=z_2$, $(z_1-z_2)^nf$ has a.l.u. convergent series expansion

$$(z_1-z_2)^nf = \sum_{i,j\geqslant 0}\binom{a}{i}(z_1-z_2)^{n-N+i}z_2^{a-i+b}z_3^c\binom{-N}{j}(z_1-z_2)^j(z_2-z_3)^{-2N-j}.$$

Apply $\mathrm{Rep}_{z_1=z_2}\cdot dz_1$. This means taking the coefficient of $(z_1-z_2)^{n-N+i+j}$ where $n-N+i+j=-1$. So we set $i=N-n-j-1$. Since $i\geqslant 0$, we take $0\leqslant j\leqslant N-n-1$. So

$$g(z_2,z_3) = \sum_{j=0}^{N-n-1}\binom{a}{N-n-j-1}z_2^{a-(N-n-j-1)+b}z_3^c\binom{-N}{j}(z_2-z_3)^{-2N-j},$$

which clearly has finite poles at $z_2=z_3$. $\square$

8.8

Proposition 8.8 (Goddard uniqueness). *Let $\mathcal{E}$ be a set of homogeneous fields satisfying the assumptions for graded vertex algebras in the reconstruction Thm. 5.12. If $A^1(z), A^2(z) \in \mathcal{E}$ satisfy $A^1_{-1}\mathbf{1} = A^2_{-1}\mathbf{1}$, then $A^1(z) = A^2(z)$.*

Proof. Set $A = A^1 - A^2$, and assume without loss of generality that $A \in \mathcal{E}$. Then $A_{-1}\mathbf{1} = 0$. By the generating property, we can show $A(z) = 0$ by show that for any $v' \in \mathbb{V}'$, $B^1, \ldots, B^k \in \mathcal{E}$, and $n, n_1, \ldots, n_k \in \mathbb{Z}$,

$$\langle v', A_n B^1_{n_1} \cdots B^k_{n_k} \mathbf{1} \rangle = 0. \tag{8.21}$$

Suppose we can show that

$$\langle v', A(z) B^1(z_1) \cdots B^k(z_k) \mathbf{1} \rangle = 0 \tag{8.22}$$

as functions on $\mathrm{Conf}^{k+1}(\mathbb{C}^\times)$. Then multiplying it by any Laurent polynomial of $z, z_1, \ldots, z_N$ and taking contour integrals over $|z| = R, |z_1| = r_1, \ldots, |z_k| = r_k$ where $0 < r_k < \cdots < r_1 < R$, we will get (8.21).

Since the LHS of (8.22) is holomorphic, it suffices to prove (8.22) when $0 < |z| < |z_1| < \cdots < |z_k|$, i.e., to prove in this domain that

$$\sum_{s \in \mathbb{N}} \langle v', B^1(z_1) \cdots B^k(z_k) P_s A(z) \mathbf{1} \rangle = 0.$$

Therefore, it suffices to prove $A(z)\mathbf{1} = 0$. Since $A(z)$ satisfies the translation property and the creation property $\lim_{z \to 0} A(z)\mathbf{1} = A_{-1}\mathbf{1}$, similar to the proof of Cor. 3.11 we have $A(z)\mathbf{1} = e^{zL_{-1}} A_{-1}\mathbf{1}$. So $A(z)\mathbf{1}$ must be 0. $\square$

8.9

Proof of the reconstruction Thm. 5.12. Assume that $\mathcal{E}$ contains the identity field $\mathbf{1}(z) = \mathbf{1}_{\mathbb{V}}$. If $A(z), B(z) \in \mathcal{E}$, then using (8.19), one checks easily that $A_n B$ satisfies the creation property with

$$(A_n B)_{-1}\mathbf{1} = A_n B_{-1}\mathbf{1}. \tag{8.23}$$

So by Lemma 8.6 and Dong's lemma, if we include $A_n B$ in $\mathcal{E}$, then the new $\mathcal{E}$ still satisfies the assumptions for graded vertex algebras in Thm. 5.12. By induction, when $A^1, \ldots, A^k \in \mathcal{E}$ we have

$$(A^1_{n_1} \cdots A^k_{n_k} \mathbf{1})_{-1}\mathbf{1} = A^1_{n_1} \cdots A^k_{n_k} \mathbf{1}. \tag{8.24}$$

Therefore, by including any linear combination of vectors of the form $A^1_{n_1} \cdots A^k_{n_k} \mathbf{1}$ in $\mathcal{E}$, we may assume that for each homogeneous $u \in \mathbb{V}$ there exists $A(z) \in \mathcal{E}$ such that $A_{-1}\mathbf{1} = u$. By Goddard uniqueness, such $A(z)$ is unique and hence can be written as $Y(u, z)$.

We now prove the Jacobi identity for Y since the other axioms of graded vertex algebras are obvious. Choose $A(z) = Y(u, z)$ and $B(z) = Y(v, z)$ in $\mathcal{E}$. Note that

$v = B_{-1}\mathbf{1}$. By extending $\mathcal{E}$, we may assume that each A_nB is in $\mathcal{E}$. To show the VOA Jacobi identity (4.12), by (7.18), it suffices to show $Y(Y(u)_nv,z) = (A_nB)(z)$. This follows from Goddard uniqueness and

$$(A_nB)_{-1}\mathbf{1} = A_nB_{-1}\mathbf{1} = A_nv = Y(u)_nv.$$

So $\mathbb{V}$ is a graded vertex algebra. The last paragraph of Thm. 5.12 about VOA is obvious. □

9 VOA modules; contragredient modules

9.1

Let $\mathbb{V}$ be a VOA.

Definition 9.1. A vector space $\mathbb{W}$ equipped with a linear map

$$Y_{\mathbb{W}} : \mathbb{V} \to (\mathrm{End}\mathbb{W})[[z^{\pm 1}]],$$
$$v \mapsto Y_{\mathbb{W}}(v,z) = \sum_{n\in\mathbb{Z}} Y_{\mathbb{W}}(v)_n z^{-n-1}$$

(where each $Y_{\mathbb{W}}(v)_n \in \mathrm{End}\mathbb{W}$) is called a **weak $\mathbb{V}$-module** if the following hold:

- (Lower truncation) $Y_{\mathbb{W}}(v,z)w \in \mathbb{W}((z))$ for each $v \in \mathbb{V}, w \in \mathbb{V}$.
- $Y_{\mathbb{W}}(\mathbf{1},z) = \mathbf{1}_{\mathbb{W}}$.
- (Jacobi identity) For each $u,v \in \mathbb{V}$,

$$\begin{aligned}&\sum_{l\in\mathbb{N}}\binom{m}{l} Y_{\mathbb{W}}\big(Y(u)_{n+l}v\big)_{m+k-l}\\ =&\sum_{l\in\mathbb{N}}(-1)^l\binom{n}{l} Y_{\mathbb{W}}(u)_{m+n-l}Y_{\mathbb{W}}(v)_{k+l} - \sum_{l\in\mathbb{N}}(-1)^{n+l}\binom{n}{l} Y_{\mathbb{W}}(v)_{n+k-l}Y_{\mathbb{W}}(u)_{m+l}.\end{aligned} \tag{9.1}$$

Definition 9.2. An **admissible $\mathbb{V}$-module** $\mathbb{W}$ is a weak $\mathbb{V}$-module such that $\mathbb{W} = \bigoplus_{n\in\mathbb{N}} \mathbb{W}(n)$ is graded by a diagonalizable operator $\widetilde{L}_0$ satisfying the grading property

$$[\widetilde{L}_0, Y_{\mathbb{W}}(v,z)] = Y_{\mathbb{W}}(L_0v,z) + z\partial_z Y_{\mathbb{W}}(v,z) \tag{9.2}$$

for each v. Equivalently, for homogeneous v,

$$[\widetilde{L}_0, Y_{\mathbb{W}}(v)_n] = (\mathrm{wt}v - n - 1)Y_{\mathbb{W}}(v)_n, \tag{9.3}$$

i.e., $Y_{\mathbb{W}}(v,z)$ is $\widetilde{L}_0$-homogeneous with weight $\mathrm{wt}v$. Zero and eigenvectors of $\widetilde{L}_0$ are called **($\widetilde{L}_0$)-homogeneous vectors**. If $w \in \mathbb{W}(n)$, then $\widetilde{\mathrm{wt}}w := n$ is called the **($\widetilde{L}_0$)-weight** of w. If each $\mathbb{W}(n)$ is finite-dimensional, we say $\mathbb{W}$ is **finitely-admissible**.

The lower-truncation property is redundant in the definition of admissible modules since it follows from the grading property.

Convention 9.3. $\mathbb{V}$ itself is an admissible $\mathbb{V}$-module, called the **vacuum module**. (It is analogous to the adjoint representations of Lie algebras.) We always choose the operator $\widetilde{L}_0$ on $\mathbb{V}$ to be L_0.

9.2

Proposition 9.4. *Let $\mathbb{W}$ be a weak $\mathbb{V}$-module. Then for each $u \in \mathbb{V}$, the following translation property holds:*

$$[L_{-1}, Y_{\mathbb{W}}(v,z)] = Y_{\mathbb{W}}(L_{-1}v,z) = \partial_z Y_{\mathbb{W}}(v,z). \tag{9.4}$$

Proof. Applying the Jacobi identity to $[Y_{\mathbb{W}}(\mathbf{c})_0, Y_{\mathbb{W}}(u)_k]$ gives $[L_{-1}, Y(u)_k] = Y(L_{-1}u)_k$. By (3.39), $L_{-1}u = Y(u)_{-2}\mathbf{1}$. Applying the Jacobi identity to $Y_{\mathbb{W}}(Y(u)_{-2}\mathbf{1})_k$ shows that it equals $-kY_{\mathbb{W}}(u)_{k-1}$. □

Proposition 9.5. *Let $\mathbb{W}$ be a weak $\mathbb{V}$-module. Define the action of L_n on $\mathbb{W}$ to be*

$$L_n = Y_{\mathbb{W}}(\mathbf{c})_{n+1} \tag{9.5}$$

Then $(L_n)_{n\in\mathbb{Z}}$ satisfy the Viarsoro relation with the same central charge c as that of $\mathbb{V}$.

Proof. Use the Jacobi identity, the translation property, and Rem. 3.2 to compute $[Y_{\mathbb{W}}(\mathbf{c})_{m+1}, Y_{\mathbb{W}}(\mathbf{c})_{k+1}]$. □

Exercise 9.6. Show that $[L_0, Y_{\mathbb{W}}(v,z)] = Y_{\mathbb{W}}(L_0 v,z) + z\partial_z Y_{\mathbb{W}}(v,z)$.

Remark 9.7. The above exercise shows that if $\mathbb{W}$ is admissible, then $A := \widetilde{L}_0 - L_0$ commutes with the action of $\mathbb{V}$ on $\mathbb{W}$, i.e., $A \in \mathrm{End}_{\mathbb{V}}(\mathbb{W})$. In particular, it commutes with $L_0 = Y_{\mathbb{W}}(\mathbf{c})_1$ and hence with $\widetilde{L}_0$. Therefore, $\widetilde{L}_0 - L_0$ is an endomorphism of the admissible $\mathbb{V}$-module $\mathbb{W}$ commuting with $\widetilde{L}_0$. Note also that by (9.3), L_n lowers the $\widetilde{L}_0$-weights by n:

$$[\widetilde{L}_0, L_n] = -nL_n. \tag{9.6}$$

Convention 9.8. The grading of an admissible $\mathbb{V}$-module always means the $\widetilde{L}_0$-grading, even when L_0 is diagonalizable.

9.3

We discuss some basic properties of irreducible modules.

Convention 9.9. A homomorphism of weak/admissible/finitely admissible modules $A : \mathbb{W}_1 \to \mathbb{W}_2$ always means a linear map intertwining the actions of $\mathbb{V}$.

Definition 9.10. An **irreducible $\mathbb{V}$-module** is a finitely admissible $\mathbb{V}$-module with no proper graded $\mathbb{V}$-invariant subspaces (i.e., no proper $\mathbb{V}$- and $\widetilde{L}_0$-invariant subspaces).

Lemma 9.11 (Schur's lemma)**.** *Let $\mathbb{W}$ be an irreducible $\mathbb{V}$-module. Let $A \in \mathrm{End}_{\mathbb{V}}(\mathbb{W})$ satisfying $[\widetilde{L}_0, A] = 0$. Then $A \in \mathbb{C}\mathbf{1}_{\mathbb{W}}$.*

Proof. By $[\widetilde{L}_0, A] = 0$, A restricts to a linear operator on each $\mathbb{W}(n)$. Choose n such that $\mathbb{W}(n) \neq 0$. Since $\mathbb{W}(n)$ is finite-dimensional, $A|_{\mathbb{W}(n)}$ has an eigenvalue λ. So the (clearly $\mathbb{V}$-invariant) subspace $\mathrm{Ker}(A-\lambda)$ is non-zero. It is also $\widetilde{L}_0$ invariant since $[\widetilde{L}_0, A-\lambda] = 0$. So $\mathrm{Ker}(A-\lambda) = \mathbb{W}$. □

Corollary 9.12. *Let $\mathbb{W}$ be an irreducible $\mathbb{V}$-module. Then $L_0 = \widetilde{L}_0 + \lambda$ for some $\lambda \in \mathbb{C}$. In particular, L_0 is diagonalizable on $\mathbb{W}$.*

Proof. This follows immediately from Rem. 9.7 and Schur's lemma 9.11. □

From this corollary, we see that the $\widetilde{L}_0$-gradings of an irreducible module are unique up to scalar addition.

Corollary 9.13. *Any irreducible $\mathbb{V}$-module $\mathbb{W}$ has no proper $\mathbb{V}$-invariant subspace.*

Proof. Let $\mathbb{M}$ be a $\mathbb{V}$-invariant subspace of $\mathbb{W}$. Then $\mathbb{M}$ is L_0-invariant since $L_0 = Y_{\mathbb{W}}(\mathbf{c})_1$. So $\mathbb{M}$ is $\widetilde{L}_0$-invariant, i.e., a graded subspace. So $\mathbb{M}$ is not proper. □

By the same reasoning, we have:

Corollary 9.14 (Schur's lemma)**.** *Let $\mathbb{W}$ be an irreducible $\mathbb{V}$-module. Then* $\mathrm{End}_{\mathbb{V}}(\mathbb{W}) = \mathbb{C}\mathbf{1}_{\mathbb{W}}$.

Definition 9.15. We say that $\mathbb{V}$ is **rational** if any admissible $\mathbb{V}$-module $\mathbb{W}$ is completely reducible, i.e., $\mathbb{W}$ is a direct sum of irreducible $\mathbb{V}$-modules.

9.4

By replacing L_0 with $\widetilde{L}_0$, all the results in Sec. 7 and Subsec. 8.1-8.7 hold for admissible modules.

Exercise 9.16. Give a complex analytic definition of Jacobi identity for admissible $\mathbb{V}$-modules that is equivalent to the algebraic Jacobi identity (9.1).

However, due to the lack of vacuum vector $\mathbf{1}$, the Goddard uniqueness and hence the reconstruction theorem do not hold for modules. Therefore, checking locality is not enough to prove the existence of $\mathbb{V}$-module structures. To construct examples of modules, new methods are needed.

Here is one easy method to construct VOA modules. Suppose $\mathbb{V}$ is a subalgebra of a VOA $\mathbb{U}$ such that the L_0 on $\mathbb{U}$ restricts to that of $\mathbb{V}$. (We do not assume $\mathbb{V}$ and $\mathbb{U}$ have the same conformal vector.) If we have constructed an admissible $\mathbb{U}$-module $\mathbb{M}$ (for instance, $\mathbb{M} = \mathbb{U}$), then by regarding $\mathbb{M}$ as a $\mathbb{V}$-module, any $\mathbb{V}$-invariant graded subspace of $\mathbb{M}$ is clearly a $\mathbb{V}$-module. In particular, if we already know that $\mathbb{M}$ is a unitary $\mathbb{U}$-module, then such constructed $\mathbb{V}$-modules are unitary.

9.5

Here we state some results on the irreducible modules associated to affine and Virasoro VOAs without providing proofs. The readers are referred to [LL, Chapter 6], [Was10], and [FZ92, Wang93] for details.

Let $\mathfrak{g}$ be either abelian or a simple Lie algebras. Let W be a finite dimensional irreducible representation of $\mathfrak{g}$. Fix a level $l \in \mathbb{C}$ such that $l + h^\vee \neq 0$. Recall the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ into Lie subalgebras defined in Subsec. 6.8:

$$\widetilde{\mathfrak{g}}_- = \mathrm{Span}\{X_n : X \in \mathfrak{g}, n < 0\}, \qquad \widetilde{\mathfrak{g}}_+ = \mathrm{Span}\{X_n, K, D : X \in \mathfrak{g}, n \geqslant 0\}. \tag{9.7}$$

Then the $\mathfrak{g}$-module W extends to a $\widetilde{\mathfrak{g}}_+$-module structure such that X_n acts trivially on W if $n > 0$, X_0 acts as X, and $D = 0, K = l$ on W. Let

$$V_{\mathfrak{g}}(l, W) = \mathrm{Ind}_{\widetilde{\mathfrak{g}}_+}^{\widetilde{\mathfrak{g}}}(W) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{g}}_+)} W, \qquad L_{\mathfrak{g}}(l, W) = V_{\mathfrak{g}}(l, W)/I$$

where I is the largest proper D- and $\widetilde{\mathfrak{g}}$-invariant subspace. Then $V_{\mathfrak{g}}(l, W)$ and $L_{\mathfrak{g}}(l, W)$ have unique finitely admissible $V_{\mathfrak{g}}(l, 0)$-module structures such that $D = \widetilde{L}_0$ and that, letting $\mathbb{W}$ be either of them, $Y_{\mathbb{W}}(X_{-1}\mathbf{1})_n$ equals the action of X_n on $\mathbb{W}$ for each $X \in \mathfrak{g}, n \in \mathbb{Z}$. $L_{\mathfrak{g}}(l, W)$ is irreducible. When $\mathfrak{g}$ is abelian, W is unitary, and $l > 0$, then $V_{\mathfrak{g}}(l, W) = L_{\mathfrak{g}}(l, W)$ are unitary modules.

Assume that $\mathfrak{g}$ is simple and $l \in \mathbb{N}$. Then W is naturally a unitary $\mathfrak{g}$-module. Then all irreducible modules of the WZW model $L_{\mathfrak{g}}(l, 0)$ are unitary, and are given by all $L_{\mathfrak{g}}(l, W)$ where W is an irreducible $\mathfrak{g}$-module satisfying the following property: (Skip this part if you are not familiar with Lie algebra representations.) Let λ be the highest weight of W. Let θ be the highest root (which is also a longest root) of $\mathfrak{g}$, namely, the highest weight of the adjoint representation of $\mathfrak{g}$. Recall the inner product $(\cdot|\cdot)$ on $\mathfrak{g}$ satisfying $(\theta|\theta) = 2$, which restricts an inner product on the Cartan subalgebra $\mathfrak{h}$. It gives canonically an inner product on the dual space $\mathfrak{h}^*$ (i.e. the weight space). Then $(\theta|\lambda)$ (which is always $\geqslant 0$) should be $\leqslant l$. There are only finitely many equivalence classes of such W.

Similarly, $\mathrm{Vir} = \mathrm{Vir}^+ \oplus \mathrm{Vir}^-$ where

$$\mathrm{Vir}^- = \mathrm{Span}\{L_n : n \leqslant -1\}, \qquad \mathrm{Vir}^+ = \mathrm{Span}\{K, L_n : n \geqslant 0\}.$$

For each $c, h \in \mathbb{C}$, let $\mathbb{C}_{c,h}$ be the one dimensional Vir^+-module such on which $K = c, L_0 = h$ and $L_n = 0$ for all $n > 0$. Let

$$M_{\mathrm{Vir}}(c, h) = \mathrm{Ind}_{\mathrm{Vir}^+}^{\mathrm{Vir}} \mathbb{C}_{c,h} = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}^+)} \mathbb{C}_{c,h}, \qquad L_{\mathrm{Vir}}(c, h) = M_{\mathrm{Vir}}(c, h)/I$$

where I is again the largest proper submodule. Then there exist unique finitely admissible $V_{\mathrm{Vir}}(l, 0)$-module structure on $\mathbb{W} = M_{\mathrm{Vir}}(l, h)$ or $\mathbb{W} = L_{\mathrm{Vir}}(l, h)$ with $\widetilde{L} = L_0 - h$ such that $Y_{\mathbb{W}}(\mathbf{c})_n$ is the action of L_{n-1} on $\mathbb{W}$.

When c satisfies (6.4), the irreducible modules of the minimal model $L_{\mathrm{Vir}}(c, 0)$ are classified by all $L_{\mathrm{Vir}}(c, h_{m,n})$ where m, n are integers with $0 < m < p, 0 < n < q$ and

$$h_{m,n} = \frac{(np - mq)^2 - (p - q)^2}{4pq}. \tag{9.8}$$

When c satisfies (6.5), $L_{\mathrm{Vir}}(c, 0)$ and all its irreducible modules are unitary.

9.6

The remaining part of this section is devoted to the study of contragredient modules (i.e., dual modules). Let $\mathbb{W} = \bigoplus_{n \in \mathbb{N}} \mathbb{W}(n)$ be an admissible $\mathbb{V}$-module. As usual, for each n we define the projection of algebraic completion to $\mathbb{W}(n)$ in the canonical way:

$$P_n : \mathbb{W}^{\mathrm{cl}} = \prod_{n \in \mathbb{N}} \mathbb{W}(n) \to \mathbb{W}(n). \tag{9.9}$$

Define the graded dual space

$$\mathbb{W}' = \bigoplus_{n\in\mathbb{N}} \mathbb{W}'(n) := \bigoplus_{n\in\mathbb{N}} \mathbb{W}(n)^*$$

as usual. Then $P_n : \mathbb{W}' \to \mathbb{W}(n)^*$ is defined in an obvious way.

9.7

Our goal is to define an admissible $\mathbb{V}$-module structure $Y_{\mathbb{W}'}$ on $\mathbb{W}'$. To find the formula of $Y_{\mathbb{W}'}$, consider the data

$$\mathfrak{X} = (\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1})$$

where ζ is the standard coordinate of $\mathbb{C}$. If $\mathcal{H}^{\mathrm{fin}}$ contains $\mathbb{W}\otimes\widehat{\mathbb{W}}$ where $\widehat{\mathbb{W}}$ is a $\widehat{\mathbb{V}}$-module, and if $w \otimes \widehat{w} \in \mathbb{W} \otimes \widehat{\mathbb{W}}$, $v \otimes \widehat{v} \in \mathbb{V} \otimes \widehat{\mathbb{V}}$, $w' \otimes \widehat{w}' \in \mathbb{W}' \otimes \widehat{\mathbb{W}}'$ are going into the punctures $0, z, \infty$ respectively, then the correlation function is given by

$$\langle w' \otimes \widehat{w}', Y_{\mathbb{W}}(v, z)w \otimes W_{\widehat{\mathbb{W}}}(\widehat{v}, \overline{z})\widehat{w}\rangle = \langle w', Y_{\mathbb{W}}(v, z)w\rangle\langle \widehat{w}', Y_{\widehat{\mathbb{W}}}(\widehat{v}, \overline{z})\widehat{w}\rangle. \tag{9.10}$$

To simplify discussions, we focus on the chiral halves. The standard conformal block for $\mathbb{W}, \mathbb{V}, \mathbb{W}'$ associated to $0, z, \infty$ is given by $\langle w', Y_{\mathbb{W}}(v, z)w\rangle$. Indeed, if we choose $\widehat{v} = \mathbf{1}$, choose $\widehat{w}, \widehat{w}'$ such that $\langle \widehat{w}', \widehat{w}\rangle = 1$, and identify $\mathbb{W}$ with $\mathbb{W} \otimes \widehat{w}$ by identifying w with $w \otimes \widehat{w}$, and similarly identify $\mathbb{W}'$ with $\mathbb{W}' \otimes \widehat{w}'$, then the correlation function (9.10) becomes exactly the conformal block $\langle w', Y_{\mathbb{W}}(v, z)w\rangle$. So we can also view $\langle w', Y_{\mathbb{W}}(v, z)w\rangle$ as a (restricted) correlation function.

We wish that the correlation function/standard conformal block associated to

$$\mathfrak{Z} = (\mathbb{P}^1; 0, z^{-1}, \infty; \zeta, \zeta - z^{-1}, \zeta^{-1})$$

is

$$\psi(w' \otimes v \otimes w) = \langle Y_{\mathbb{W}'}(v, z^{-1})w', w\rangle$$

where $\mathbb{W}', \mathbb{V}, \mathbb{W}$ are associated to $0, z^{-1}, \infty$. Now, the biholomorphism $\gamma \in \mathbb{P}^1 \mapsto \gamma^{-1} \in \mathbb{P}^1$ gives almost an equivalence of $\mathfrak{X}$ and $\mathfrak{Y}$: the only exception is that the local coordinate $\zeta - z$, pulled back along this map, is $\zeta^{-1} - z$ but not $\zeta - z^{-1}$. So let us consider

$$\mathfrak{Y} = (\mathbb{P}^1; 0, z^{-1}, \infty; \zeta, \zeta^{-1} - z, \zeta^{-1})$$

equivalent to $\mathfrak{X}$ via $\gamma \mapsto \gamma^{-1}$. Again, we associate $\mathbb{W}', \mathbb{V}, \mathbb{W}$ to $0, z^{-1}, \infty$ as for $\mathfrak{Z}$. Then the standard conformal block for $\mathfrak{Y}$ is still

$$\phi(w' \otimes v \otimes w) = \langle w', Y_{\mathbb{W}}(v, z)w\rangle.$$

Now we relate ϕ and ψ using the change of coordinate formula, noting that $\zeta - z^{-1} = \vartheta_z \circ (\zeta^{-1} - z)$ where (for each $t \in \mathbb{P}^1$)

$$\vartheta_z(t) = \frac{1}{z + t} - \frac{1}{z}. \tag{9.11}$$

Therefore

$$\phi(w' \otimes v \otimes w) = \psi(w' \otimes \mathcal{U}(\vartheta_z)v \otimes w) \tag{9.12}$$

where $\mathcal{U}(\vartheta_z)$ is the operator on (the Hilbert space completion of) $\mathbb{W}$ associated to $\mathbb{V}$.

9.8

It remains to find $\mathcal{U}(\vartheta_z)$. To avoid conflict of notations, we write $z^{n+1}\partial_z$ in Sec. 2 as $\zeta^{n+1}\partial_\zeta$. Then by (2.15), $\exp(z\zeta^2\partial_\zeta)$ sends γ to $(1/\gamma - z)^{-1}$ and hence $-z^{-2}\gamma$ to $\vartheta_z(\gamma)$. This means

$$\vartheta_z = \exp(z\zeta^2\partial_\zeta) \circ \exp(\log(-z^{-2})\zeta\partial_\zeta) = \exp(z\zeta^2\partial_\zeta) \circ (-z^{-2})^{\zeta\partial_\zeta}. \tag{9.13}$$

Thus, on $\mathbb{V}$,

$$\mathcal{U}(\vartheta_z) = e^{zL_1}(-z^{-2})^{L_0}. \tag{9.14}$$

Expanding (9.12), we get

$$\big\langle w', Y_{\mathbb{W}}(v,z)w\big\rangle = \big\langle Y_{\mathbb{W}'}\big(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}\big)w', w\big\rangle$$

Exchange the role of $\mathbb{W}$ and $\mathbb{W}'$, and we get our definition:

Definition 9.17. Let $\mathbb{W}$ be an admissible $\mathbb{V}$-module. Then $Y_{\mathbb{W}'}: \mathbb{V} \to (\mathrm{End}\mathbb{W}')[[z^{\pm 1}]]$ is defined by

$$\big\langle Y_{\mathbb{W}'}(v,z)w', w\big\rangle = \big\langle w', Y_{\mathbb{W}}\big(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}\big)w\big\rangle \tag{9.15}$$

for each $v \in \mathbb{V}, w \in \mathbb{W}, w' \in \mathbb{W}'$. Assuming v to be homogeneous, this means

$$Y_{\mathbb{W}'}(v,z) = \sum_{k\in\mathbb{N}} \frac{z^k}{k!} \cdot (-z^{-2})^{\mathrm{wt}v} \cdot Y_{\mathbb{W}}\big(L_1^k v, z^{-1}\big)^{\mathrm{t}}. \tag{9.16}$$

Expanding both sides, we see that for each $n \in \mathbb{Z}$,

$$Y_{\mathbb{W}'}(v)_n = \sum_{k\in\mathbb{N}} \frac{(-1)^{\mathrm{wt}v}}{k!} \big(Y_{\mathbb{W}}(L_1^k v)_{-n-k-2+2\mathrm{wt}v}\big)^{\mathrm{t}}. \tag{9.17}$$

Exercise 9.18. Let L_n be $Y_{\mathbb{W}'}(\mathbf{c})_{n+1}$ on $\mathbb{W}'$. Use (9.17) to show that for each $w \in \mathbb{W}, w' \in \mathbb{W}'$,

$$\langle L_n w', w\rangle = \langle w', L_{-n}w\rangle. \tag{9.18}$$

9.9

The purpose of this subsection is to prove Cor. 9.20.

Exercise 9.19. Use $[\widetilde{L}_0, L_1] = -L_1$ to show that when acting $w \in \mathbb{W}$,

$$L_1\lambda^{\widetilde{L}_0} = \lambda^{\widetilde{L}_0+1}L_1, \tag{9.19a}$$

$$e^{\tau L_1}\lambda^{\widetilde{L}_0} = \lambda^{\widetilde{L}_0}e^{\tau\lambda L_1} \tag{9.19b}$$

in $\mathbb{W}[\lambda]$ and $\mathbb{W}[\lambda,\tau]$ respectively.

(Hint. Method 1: Compute ∂_λ for the first equation, ∂_τ for the second one, and apply Lemma 3.7. Method 2: Use the fact that L_1 lowers the weights by 1 to verify the equations when v is homogeneous.)

By taking ∂_λ of (9.19b) at $\lambda = 1$, we get

$$e^{\tau L_1}\widetilde{L}_0 = \widetilde{L}_0 e^{\tau L_1} + \tau L_1 e^{\tau L_1}. \tag{9.20}$$

Corollary 9.20. *we have*

$$Y_{\mathbb{W}}(v,z) = Y_{\mathbb{W}'}\big(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}\big)^{\mathrm{t}}. \tag{9.21}$$

Thus, if $\mathbb{W}$ is finitely admissible, then $\mathbb{W}'' = \mathbb{W}$ and $Y_{\mathbb{W}''} = Y_{\mathbb{W}}$.

Proof. By (9.15),

$$Y_{\mathbb{W}'}\big(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}\big)^{\mathrm{t}} = Y_{\mathbb{W}}\big(e^{z^{-1}L_1}(-z^2)^{L_0}e^{zL_1}(-z^{-2})^{L_0}v, z^{-1}\big)^{\mathrm{t}},$$

which equals $Y_{\mathbb{W}}(v,z)$ since $(-z^2)^{L_0}e^{zL_1} = e^{-z^{-1}L_1}(-z^2)^{L_0}$ due to (9.19b). $\square$

9.10

In the rest of this section, we prove the following main result of this section.

Theorem 9.21. *Let $(\mathbb{W}, Y_{\mathbb{W}})$ be an admissible $\mathbb{V}$-module. Then $(\mathbb{W}', Y_{\mathbb{W}'})$ is an admissible $\mathbb{V}$-module. If $\mathbb{W}$ is finitely-admissible, then so is $\mathbb{W}'$, and under the canonical identification $\mathbb{W} = \mathbb{W}''$ we have $Y_{\mathbb{W}} = Y_{\mathbb{W}''}$.*

The very last sentence of this theorem is proved. To verify that $\mathbb{W}'$ is an admissible module, we begin with the following simple observation.

Lemma 9.22. *If $v \in \mathbb{V}$ is homogeneous, then $Y_{\mathbb{W}'}(v,z)$ is $\widetilde{L}_0$-homogeneous with weight $\mathrm{wt}v$.*

Proof. Using (9.6) and (9.17), one easily computes $[\widetilde{L}_0, Y_{\mathbb{W}'}(v)_n] = (\mathrm{wt}v - n - 1)Y_{\mathbb{W}'}(v)_n$. $\square$

It is clear that $Y_{\mathbb{W}'}(\mathbf{1}, z) = \mathbf{1}_{\mathbb{W}'}$. To prove that $Y_{\mathbb{W}'}$ satisfies the axioms of an admissible module, it remains to check the Jacobi identity. We first prove the locality:

Lemma 9.23. *Let $u, v \in \mathbb{V}$ be homogeneous. Then $Y_{\mathbb{W}'}(u,z)$ and $Y_{\mathbb{W}'}(v,z)$ are local.*

Proof. We prove the complex analytic locality. For each $w \in \mathbb{W}, w' \in \mathbb{W}'$,

$$\sum_{n\in\mathbb{N}}\big\langle Y_{\mathbb{W}'}(u,z_1)P_nY_{\mathbb{W}'}(v,z_2)w', w\big\rangle \tag{9.22}$$

$$= \sum_{n\in\mathbb{N}}\big\langle w', Y_{\mathbb{W}}\big(e^{z_2L_1}(-z_2^{-2})^{L_0}v, z_2^{-1}\big)P_nY_{\mathbb{W}}\big(e^{z_1L_1}(-z_1^{-2})^{L_0}u, z_1^{-1}\big)w\big\rangle \tag{9.23}$$

which converges a.l.u. on $0 < |z_1^{-1}| < |z_2^{-1}|$ by the locality of $Y_{\mathbb{W}}(u,z)$ and $Y_{\mathbb{W}}(v,z)$. Moreover, this locality shows that the above expression and

$$\sum_{n\in\mathbb{N}}\big\langle Y_{\mathbb{W}'}(v,z_2)P_nY_{\mathbb{W}'}(u,z_1)w', w\big\rangle \tag{9.24}$$

$$= \sum_{n\in\mathbb{N}}\big\langle w', Y_{\mathbb{W}}\big(e^{z_1L_1}(-z_1^{-2})^{L_0}u, z_1^{-1}\big)P_nY_{\mathbb{W}}\big(e^{z_2L_1}(-z_2^{-2})^{L_0}v, z_2^{-1}\big)w\big\rangle \tag{9.25}$$

(which converges a.l.u. on $0 < |z_2^{-1}| < |z_1^{-1}|$) can be extended to the same holomorphic function f on $\mathrm{Conf}^2(\mathbb{C}^\times)$. This function is a $\mathbb{C}[z_1^{\pm1}, z_2^{\pm1}]$-linear combination of 4-point correlations functions of the form $\langle w', Y_{\mathbb{W}}(\cdot, z_1)Y_{\mathbb{W}}(\cdot, z_2)w\rangle$ which is holomorphic on $\mathbb{C}^\times \times \mathbb{C}^\times$ when multiplied by $(z_1 - z_2)^N$ for some N. So f shares the same property. $\square$

9.11

Write $A(z) = Y_{\mathbb{W}'}(u, z)$ and $B(z) = Y_{\mathbb{W}'}(v, z)$. Since A and B are local, we have the Jacobi identity (7.18) for $A(z), B(z), (A_\bullet B)(z)$, which implies the Jacobi identity for $Y_{\mathbb{W}'}$ if we can show that for all $k \in \mathbb{Z}$ and homogeneous $w \in \mathbb{W}, w' \in \mathbb{W}'$, as elements of $\mathbb{C}[z_2^{\pm 1}]$ we have

$$\big\langle (A_k B)(z_2)w', w\big\rangle = \big\langle Y_{\mathbb{W}'}(Y(u)_k v, z_2)w', w\big\rangle. \tag{9.26}$$

By (7.14), the LHS of (9.26) is $\mathrm{Res}_{z_1=z_2}(z_1 - z_2)^k f(z_1, z_2)dz_1$ where f was defined in the proof of Lemma 9.23. By (9.25) and the complex analytic Jacobi identity for $Y_{\mathbb{W}}$, the RHS of

$$f(z_1, z_2) = \sum_{n\in\mathbb{N}} \big\langle w', Y_{\mathbb{W}}\big(P_n Y\big(e^{z_1L_1}(-z_1^{-2})^{L_0}u, z_1^{-1} - z_2^{-1}\big) \cdot e^{z_2L_1}(-z_2^{-2})^{L_0}v, z_2^{-1}\big)w\big\rangle \tag{9.27}$$

converges a.l.u. on $0 < |z_1^{-1} - z_2^{-1}| < |z_2^{-1}|$ to the LHS.

The RHS of (9.26) is the application of $\mathrm{Res}_{z_1-z_2=0}\cdot(z_1 - z_2)^k d(z_1 - z_2)$ to the following elements of $\mathbb{C}[z_2^{\pm 1}][[(z_1 - z_2)^{\pm 1}]]$:

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}(z_1 - z_2)^{-n-1}\big\langle Y_{\mathbb{W}'}\big(Y(u)_n v, z_2\big)w', w\big\rangle\\
=&\big\langle Y_{\mathbb{W}'}\big(Y(u, z_1 - z_2)v, z_2\big)w', w\big\rangle\\
=&\big\langle w', Y_{\mathbb{W}}\big(e^{z_2L_1}(-z_2^{-2})^{L_0}Y(u, z_1 - z_2)v, z_2\big)w\big\rangle\\
=&\big\langle w', Y_{\mathbb{W}}\big(e^{z_2L_1}Y\big((-z_2^{-2})^{L_0})u, z_2^{-2}(z_2 - z_1)\big)(-z_2^{-2})^{L_0}v, z_2\big)w\big\rangle.
\end{aligned} \tag{9.28}$$

where the scale covariance is used in the last equality.

Exercise 9.24. Set the following element of $\mathbb{C}[z_2^{\pm 1}][[(z_1 - z_2)^{\pm 1}]]$:

$$g_n(z_2, z_1 - z_2) = \big\langle w', Y_{\mathbb{W}}\big(P_n e^{z_2L_1}Y\big((-z_2^{-2})^{L_0})u, z_2^{-2}(z_2 - z_1)\big)(-z_2^{-2})^{L_0}v, z_2\big)w\big\rangle.$$

Show that for each $k \in \mathbb{Z}$,

$$\mathrm{Res}_{z_1-z_2=0}\,(z_1 - z_2)^k g_n(z_2, z_1 - z_2)d(z_1 - z_2) \tag{9.29}$$

is a monomial of $z_2^{\pm 1}$ that vanishes when $n > \mathrm{wt}u + \mathrm{wt}v - k - 1$. Conclude that the application of $\mathrm{Res}_{z_1-z_2=0}\cdot(z_1 - z_2)^k d(z_1 - z_2)$ to (9.28) equals the (automtically finite) sum over all n of (9.29).

It follows that (9.26) holds if we can show: For any $v' \in \mathbb{V}'(n) = \mathbb{V}(n)^*$ (e.g., $\langle v', \cdot\rangle = \langle w', Y_{\mathbb{W}}(P_n\cdot, z_2)w\rangle$), as holomorphic functions of z_2 on $\mathbb{C}^\times$,

$$\begin{aligned}
&\mathrm{Res}_{z_1=z_2}(z_1 - z_2)^k\big\langle v', Y\big(e^{z_1L_1}(-z_1^{-2})^{L_0}u, z_1^{-1} - z_2^{-1}\big) \cdot e^{z_2L_1}(-z_2^{-2})^{L_0}v\big\rangle dz_1\\
=&\mathrm{Res}_{z_1-z_2=0}(z_1 - z_2)^k\big\langle v', e^{z_2L_1}Y\big((-z_2^{-2})^{L_0}u, z_2^{-2}(z_2 - z_1)\big)(-z_2^{-2})^{L_0}v\big\rangle d(z_1 - z_2)
\end{aligned} \tag{9.30}$$

where the LHS is the residue of a holomorphic function and the RHS is that of a formal Laurent series. This follows if we can show that

$$\big\langle v', Y\big(e^{z_1L_1}(-z_1^{-2})^{L_0}u, z_1^{-1} - z_2^{-1}\big) \cdot e^{z_2L_1}(-z_2^{-2})^{L_0}v\big\rangle$$

$$=\left\langle v', e^{z_2L_1}Y\big((-z_2^{-2})^{L_0}u, z_2^{-2}(z_2-z_1)\big)(-z_2^{-2})^{L_0}v\right\rangle \tag{9.31}$$

where the RHS as a formal Laurent series of z_2, z_1-z_2 converges a.l.u. on $0<|z_1-z_2|<|z_2|$ to the LHS as a holomorphic function of z_1, z_2.

Clearly, as elements of $\mathbb{C}[[z_2^{\pm 1}, (z_1-z_2)^{\pm 1}]]$ the sum

$$\begin{aligned}&\left\langle v', e^{z_2L_1}Y\big((-z_2^{-2})^{L_0}u, z_2^{-2}(z_2-z_1)\big)(-z_2^{-2})^{L_0}v\right\rangle\\ =&\sum_{n\in\mathbb{N}}\left\langle v', e^{z_2L_1}P_nY\big((-z_2^{-2})^{L_0}u, z_2^{-2}(z_2-z_1)\big)(-z_2^{-2})^{L_0}v\right\rangle\end{aligned} \tag{9.32}$$

satisfies the conditions in Lemma 7.4. If we can prove the claim that the RHS converges a.l.u. on $0<|z_1-z_2|<|z_2|$ to the LHS of (9.31), then by Lemma 7.4, we are done with the proof. The claim follows from the following "$e^{\tau L_1}$-covariance" (where $\tau=z_2, z=z_2^{-2}(z_2-z_1)$), which we prove for $Y_{\mathbb{W}}$ though we actually just need it for $Y=Y_{\mathbb{V}}$.

Proposition 9.25. *Let $\mathbb{W}$ be admissible. Then for each $v\in\mathbb{V}, w\in\mathbb{W}, w'\in\mathbb{W}'$, the LHS of*

$$\sum_{n\in\mathbb{N}}\left\langle w', e^{\tau L_1}P_nY_{\mathbb{W}}(v,z)w\right\rangle=\left\langle w', Y_{\mathbb{W}}\big(e^{\tau(1-\tau z)L_1}(1-\tau z)^{-2L_0}v, z/(1-\tau z)\big)e^{\tau L_1}w\right\rangle \tag{9.33}$$

converges a.l.u. on $\{(z,\tau)\in\mathbb{C}^\times\times\mathbb{C}: |\tau|<|z^{-1}|\}$ to the RHS.

This theorem is a special case of the conformal covariance Thm. 10.7 which will be explained later. However, the proof of Thm. 10.7 is quite involved. So in the following we give an elementary proof of Prop. 9.25.

9.12

We view the $e^{\tau L_1}$-covariance of $Y_{\mathbb{W}}$ as the transpose of the translation covariance of $Y_{\mathbb{W}'}$. So we first need to prove the latter.

Lemma 9.26. *We have $[L_{-1}, Y_{\mathbb{W}'}(v,z)]=\partial_z Y_{\mathbb{W}'}(v,z)$.*

Proof. Assume v is homogeneous. It suffices prove

$$\left[L_1, Y_{\mathbb{W}}\big(e^{zL_1}z^{-2L_0}v, z^{-1}\big)\right]=-\partial_z Y_{\mathbb{W}}\big(e^{zL_1}z^{-2L_0}v, z^{-1}\big). \tag{9.34}$$

which is the transpose of the formula we want to prove multiplied by $(-1)^{\mathrm{wt}v}$. The Jacobi identity for $Y_{\mathbb{W}}$ implies (5.3) where Y is replaced by $Y_{\mathbb{W}}$. By (5.3),

$$[L_1, Y_{\mathbb{W}}(v,z)]=z^2Y_{\mathbb{W}}(L_{-1}v,z)+2zY_{\mathbb{W}}(L_0v,z)+Y_{\mathbb{W}}(L_1v,z). \tag{9.35}$$

Using this relation, one checks that the LHS of (9.34) equals

$$Y_{\mathbb{W}}(L_1e^{zL_1}z^{-2L_0}v, z^{-1})+2z^{-1}Y_{\mathbb{W}}(L_0e^{zL_1}z^{-2L_0}v, z^{-1})+z^{-2}Y_{\mathbb{W}}(L_{-1}e^{zL_1}z^{-2L_0}v, z^{-1}). \tag{9.36}$$

It is easy to guess by chain rule and verify rigorously by series expansions that

$$-\partial_z Y_{\mathbb{W}}(v, z^{-1})=z^{-2}Y_{\mathbb{W}}(L_{-1}v, z^{-1}).$$

Thus, the RHS of (9.34) is

$$-Y_{\mathbb{W}}(L_1e^{zL_1}z^{-2L_0}v, z^{-1})+2z^{-1}Y_{\mathbb{W}}(e^{zL_1}L_0z^{-2L_0}v, z^{-1})+z^{-2}Y_{\mathbb{W}}(L_{-1}e^{zL_1}z^{-2L_0}v, z^{-1})$$

which equals (9.36) due to (9.20). □

9.13

Now that we have the translation property for $Y_{\mathbb{W}'}$, we have the translation covariance in the form of (3.36) or (equivalently) Exercise 7.5. We need the latter form: the LHS of

$$\sum_{n\in\mathbb{N}}\big\langle Y_{\mathbb{W}'}(u,z)P_n e^{\tau L_{-1}}w',w\big\rangle=\big\langle Y_{\mathbb{W}'}(u,z-\tau)w',e^{\tau L_1}w\big\rangle,\tag{9.37}$$

converges a.l.u. on $|\tau|<|z|$ to the RHS.

Proof of Prop. 9.25. By Cor. 9.20, as sums of holomorphic functions we have

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\big\langle w',e^{\tau L_1}P_nY_{\mathbb{W}}(v,z)w\big\rangle=\sum_{n\in\mathbb{N}}\big\langle P_ne^{\tau L_{-1}}w',Y_{\mathbb{W}}(v,z)w\big\rangle\\ =&\sum_{n\in\mathbb{N}}\big\langle Y_{\mathbb{W}'}(e^{zL_1}(-z^{-2})^{L_0}v,z^{-1})P_ne^{\tau L_{-1}}w',w\big\rangle,\end{aligned}$$

which by (9.37) converges a.l.u. on $|\tau|<|z^{-1}|$ to

$$\big\langle Y_{\mathbb{W}'}\big(e^{zL_1}(-z^{-2})^{L_0}v,z^{-1}-\tau\big)w',e^{\tau L_1}w\big\rangle.$$

We move $Y_{\mathbb{W}'}$ to the right using (9.15). Then the above becomes

$$\big\langle w',Y_{\mathbb{W}}\big(e^{(z^{-1}-\tau)L_1}(-(z^{-1}-\tau)^{-2})^{L_0}e^{zL_1}(-z^{-2})^{L_0}v,(z^{-1}-\tau)^{-1}\big)e^{\tau L_1}w\big\rangle.$$

This equals the RHS of (9.33) since, by (9.19b), when acting on $\mathbb{V}$,

$$(-(z^{-1}-\tau)^{-2})^{L_0}e^{zL_1}=e^{-z(z^{-1}-\tau)^2L_1}(-(z^{-1}-\tau)^{-2})^{L_0}.$$

$\square$

The proof of Thm. 9.21 is complete.

9.14

Definition 9.27. We say that $\mathbb{V}$ is **self-dual** if the vacuum module $\mathbb{V}$ (with grading $\widetilde{L}_0=L_0$) is isomorphic to its contragredient module $\mathbb{V}'$.

The construction of tensor product modules is much easier:

Proposition 9.28. *Let $\mathbb{V}_1,\mathbb{V}_2$ be VOAs and $\mathbb{W}_i$ be an admissible $\mathbb{V}_i$-module. Then the vector space $\mathbb{W}_1\otimes\mathbb{W}_2$ has a unique admissible $\mathbb{V}_1\otimes\mathbb{V}_2$-module structure with grading $\widetilde{L}_0\otimes\mathbf{1}_{\mathbb{W}_2}+\mathbf{1}_{\mathbb{W}_1}\otimes\widetilde{L}_0$ such that for each $v_i\in\mathbb{V}_i$,*

$$Y_{\mathbb{W}_1\otimes\mathbb{W}_2}(v_1\otimes v_2,z)=Y_{\mathbb{W}_1}(v_1,z)\otimes Y_{\mathbb{W}_2}(v_2,z).\tag{9.38}$$

Equivalently, for each $w_i\in\mathbb{W}_i,w_i'\in\mathbb{W}_i'$,

$$\big\langle w_1'\otimes w_2',Y_{\mathbb{W}_1\otimes\mathbb{W}_2}(v_1\otimes v_2,z)(w_1\otimes w_2)\big\rangle=\langle w_1',Y_{\mathbb{W}_1}(v_1,z)w_1\rangle\cdot\langle w_2',Y_{\mathbb{W}_2}(v_2,z)w_2\rangle.\tag{9.39}$$

Proof. Using (9.39), it is easy to verify that $Y_{\mathbb{W}_1\otimes\mathbb{W}_2}$ satisfies the complex analytic Jacobi identity. $\square$

10 Change of coordinate theorems

10.1

The goal of this section is to study the change of local coordinates in a rigorous way. Due to some convergence issues, it is very difficult to show that a given local coordinate of $\mathbb{C}$ at 0 can be written as $\exp(f\partial_z)$ for a holomorphic vector field $f\partial_z$. So we first discuss formal coordinates and find the formal vector fields generating them.

Define the following two subspaces of $z\cdot\mathbb{C}[[z]]$

$$\mathcal{G}=\Big\{\sum_{n\in\mathbb{Z}_+}a_nz^n:a_1\neq 0\Big\},\qquad \mathcal{G}_+=\Big\{z+\sum_{n\geqslant 2}a_nz^n\in\mathcal{G}\Big\}. \tag{10.1}$$

Elements of $\mathcal{G}$ are viewed as formal local coordinates of $\mathbb{C}$ at 0. Likewise, set

$$\mathbb{G}=\Big\{\alpha(z)\in\mathcal{G}:\sum_n|a_n|r^n<+\infty\text{ for some }r>0\Big\},\qquad \mathbb{G}_+=\mathbb{G}\cap\mathcal{G}_+. \tag{10.2}$$

Then elements of $\mathbb{G}$ are local coordinates of $\mathbb{C}$ at 0, or equivalently, transformations of local coordinates.

There is an obvious right action of $\mathcal{G}$ on $\mathbb{C}((z))$ defined by composition $f\mapsto f\circ\alpha$ if $f\in\mathbb{C}((z))$ and $\alpha\in\mathcal{G}$. We leave it to the readers to check that it is well-defined. So $\mathcal{G}$ is a group whose product is the composition and whose identity is z.

10.2

According to Sec. 2, to find the change of coordinate operator $\mathcal{U}(\alpha)$ for each $\alpha\in\mathcal{G}$, we need to write it as $\alpha=\exp(\sum_{n\geqslant 0}c_nz^{n+1}\partial_z)$. This task is easy if $\alpha\in\mathcal{G}_+$. Indeed, write

$$\alpha(z)=z+\sum_{n\geqslant 2}a_nz^n. \tag{10.3}$$

Then we can indeed choose $c_0=0$, and

$$\begin{aligned}\alpha(z)=&\sum_{k\in\mathbb{N}}\frac{1}{k!}\Big(\sum_{n\geqslant 1}c_nz^{n+1}\partial_z\Big)^k(z)\\ =&z+\sum_{n_1\geqslant 1}c_{n_1}z^{n_1+1}+\frac{1}{2!}\sum_{n_1,n_2\geqslant 1}(n_1+1)c_{n_1}c_{n_2}z^{n_1+n_2+1}\\ &+\frac{1}{3!}\sum_{n_1,n_2,n_3\geqslant 1}(n_1+1)(n_1+n_2+1)c_{n_1}c_{n_2}c_{n_3}z^{n_1+n_2+n_3+1}+\cdots.\end{aligned} \tag{10.4}$$

This means that for each $m\geqslant 1$,

$$a_{m+1}=c_m+\sum_{\substack{2\leqslant l\leqslant m\\ n_1,\dots,n_l\in\mathbb{Z}_+\\ n_1+\cdots+n_l=m}}\frac{1}{l!}(n_1+1)\cdots(n_1+n_2+\cdots+n_{l-1}+1)c_{n_1}\cdots c_{n_l}. \tag{10.5}$$

This shows that one can solve $c_1,c_2,\dots$ given the coefficients $a_2,a_3,\dots$.

For a general $\alpha \in \mathcal{G}$, instead of solving $\alpha = \exp(\sum_{n\geqslant 0} c_n z^{n+1}\partial_z)$, it is easier to solve

$$\alpha(z) = \alpha'(0)\cdot \exp\Big(\sum_{n\geqslant 1} c_n z^{n+1}\partial_z\Big)(z). \tag{10.6}$$

since $\alpha(z)/\alpha'(0) \in \mathcal{G}_+$. The first several terms are

$$c_1 = \frac{1}{2}\frac{\alpha''(0)}{\alpha'(0)}, \tag{10.7a}$$

$$c_2 = \frac{1}{6}\frac{\alpha'''(0)}{\alpha'(0)} - \frac{1}{4}\Big(\frac{\alpha''(0)}{\alpha'(0)}\Big)^2. \tag{10.7b}$$

The corresponding linear operator on an admissible $\mathbb{V}$-module $\mathbb{W}$ is given by

$$\mathcal{U}(\alpha) = \alpha'(0)^{\widetilde{L}_0}\exp\Big(\sum_{n\geqslant 1} c_n L_n\Big) = \alpha'(0)^{\widetilde{L}_0}\sum_{k\in\mathbb{N}}\frac{1}{k!}\Big(\sum_{n\geqslant 1} c_n L_n\Big)^k. \tag{10.8}$$

Its inverse is $\mathcal{U}(\alpha)^{-1} = \exp(-\sum_{n\geqslant 1} c_n L_n)\alpha'(0)^{-\widetilde{L}_0}$.

The point of replacing L_0 with $\widetilde{L}_0$ is to avoid the ambiguity caused by the non-integral eigenvalues of L_0. Since (by Cor. 9.12) $\widetilde{L}_0 - L_0$ is a constant if $\mathbb{W}$ is irreducible, it is not a big deal to make such a replacement.

Remark 10.1. By the fact that L_n lowers the weights by n, the above double sum is finite when $\mathcal{U}(\alpha)$ (and similarly $\mathcal{U}(\alpha)^{-1}$) is acting on any vector. Moreover, they preserve $\mathbb{W}^{\leqslant n}$ for each $n \in \mathbb{N}$ where

$$\mathbb{W}^{\leqslant n} = \bigoplus_{0\leqslant j\leqslant n}\mathbb{W}(j). \tag{10.9}$$

So $\mathcal{U}(\alpha)$ restricts to a linear isomorphism on each $\mathbb{W}^{\leqslant n}$. Note that $\mathcal{U}(\alpha)$ does not preserve $\mathbb{W}(n)$.

10.3

In applications, we need to consider a **holomorphic family of (analytic) transformations** $\rho : X \to \mathbb{G}$, which means that $\rho = \rho_x(z)$ is a holomorphic function on a neighborhood of $X \times \{0\}$ in $X \times \mathbb{C}$ where X is a complex manifold (here $(x,z) \in X \times \mathbb{C}$), and $\rho_x(0) = 0$ and $\rho_x'(0) \equiv \partial_z\rho_x(0) \neq 0$ for all $x \in X$.

We now restrict to the case that X is an open subset U of $\mathbb{C}$ and let ζ be the standard variable of U, but consider a slightly more general situation that $\rho = \mathscr{O}(U)[[z]]$ with $\rho_\zeta(0) = 0$ and $\rho_\zeta'(0) \neq 0$ for all $\zeta \in U$. Equivalently,

$$\rho_\zeta(z) = \sum_{n\geqslant 1}\frac{1}{n!}\rho_\zeta^{(n)}(0)z^n \tag{10.10}$$

where each $\zeta \mapsto \rho_\zeta^{(n)}(0)$ is an element of $\mathscr{O}(U)$ and $\rho_\zeta'(0) \neq 0$. Note that when $z \neq 0$, $\rho_\zeta(z)$ does not make sense as a value. We call $\rho : U \to \mathcal{G}$ a **family of formal coordinates**.

Remark 10.2. We can take limits and derivatives for elements of $\mathscr{O}(U)[[z^{\pm 1}]]$ by treating each $\mathscr{O}(U)$-coefficient. So, for instance, the derivative $\partial_z\rho_\zeta(z)$ at $\zeta_0\in U$ makes sense analytically as the value of the limit $\lim_{\zeta\to\zeta_0}\frac{\rho_\zeta(z)-\rho_{\zeta_0}(z)}{\zeta-\zeta_0}$.

By (10.5),

$$\rho_\zeta=\rho_\zeta'(0)\exp\Big(\sum_{n\geqslant 1}c_n(\zeta)z^{n+1}\partial_z\Big) \tag{10.11}$$

where $c_1,c_2,\dots\in\mathscr{O}(U)$. So

$$\mathcal{U}(\rho_\zeta)=\rho_\zeta'(0)^{\widetilde{L}_0}\exp\Big(\sum_{n\geqslant 1}c_n(\zeta)L_n\Big), \tag{10.12}$$

which shows that

$$\mathcal{U}(\rho_\zeta)\big|_{\mathbb{W}^{\leqslant k}}\in\mathrm{End}(\mathbb{W}^{\leqslant k})\otimes\mathscr{O}(U) \tag{10.13}$$

for each $k\in\mathbb{N}$.

10.4

Let $\rho:U\to\mathcal{G}$ be a family of formal coordinates.

Proposition 10.3. *Suppose $0\in U$ and $\rho_0(z)=z$. Then, when acting on each vector of $\mathbb{W}$, or equivalently, when restricted to each $\mathbb{W}^{\leqslant k}$,*

$$\partial_\zeta\mathcal{U}(\rho_\zeta)\Big|_{\zeta=0}=\sum_{n\geqslant 1}\frac{1}{n!}\Big(\partial_\zeta\rho_\zeta^{(n)}(0)\Big|_{\zeta=0}\Big)\widetilde{L}_{n-1} \tag{10.14}$$

where $\widetilde{L}_n=L_n$ when $n\geqslant 1$, and $\partial_\zeta\rho_\zeta^{(n)}(z)=\partial_\zeta\partial_z^n\rho_\zeta(z)$.

Remark 10.4. The geometric meaning of Prop. 10.3 is the following. Assume that $\rho:U\to\mathbb{G}$ is a holomorphic family with $\rho_0(z)=z$. Then for each z_0 near 0, $\zeta\mapsto\rho_\zeta(z_0)$ is a path in $\mathbb{C}$ whose initial value is z_0. So $\partial_\zeta\rho_\zeta(z_0)\partial_z\big|_{\zeta=0}$ is the tangent vector at z_0 describing the velocity of the path. By assembling these tangent vectors, we get a holomorphic tangent vector field $\partial_\zeta\rho_\zeta(z)\partial_z\big|_{\zeta=0}$, which equals

$$\partial_\zeta\rho_\zeta(z)\partial_z\Big|_{\zeta=0}=\sum_{n\geqslant 1}\frac{1}{n!}\partial_\zeta\rho_\zeta^{(n)}(0)z^n\partial_z\Big|_{\zeta=0}. \tag{10.15}$$

In view of the correspondence $z^n\partial_z\leftrightarrow L_{n-1}$, Prop. 10.3 says that $\partial_\zeta\mathcal{U}(\rho_\zeta)\big|_{\zeta=0}$ is exactly the linear operator corresponding to the tangent vector field.

Proof of Prop. 10.3. From (10.12), $\partial_\zeta\mathcal{U}(\rho_\zeta)$ is expressed in terms of c_n. So we need to express c_n in terms of $\partial_\zeta\rho_\zeta^{(n)}(0)$. By (10.11) and (10.4),

$$\rho_\zeta(z)=\rho_\zeta'(0)\Big(z+\sum_{n\geqslant 1}c_n(\zeta)z^{n+1}\Big)+R_\zeta(z)$$

where $R_\zeta(z)$ is a sum of polynomials of z multiplied by at least two terms of $c_1(\zeta), c_2(\zeta), \ldots$. Since $\rho_0(z) = z$, equivalently, $\rho_0'(0) = 1$ and $c_1(0) = c_2(0) = \cdots = 0$, we have $\partial_\zeta R(z)|_{\zeta=0} = 0$. So

$$\partial_\zeta \rho_\zeta(z)\Big|_{\zeta=0} = \partial_\zeta \rho_\zeta'(0) z + \sum_{n\geqslant 1} \partial_\zeta c_n(0) z^{n+1}. \tag{10.16}$$

A similar argument applied to the derivative of (10.12) shows

$$\partial_\zeta \mathcal{U}(\rho_\zeta)\Big|_{\zeta=0} = \partial_\zeta \rho_\zeta'(0)\widetilde{L}_0 + \sum_{n\geqslant 1} \partial_\zeta c_n(0) L_n. \tag{10.17}$$

By (10.16), for all $n \geqslant 2$,

$$\frac{1}{n!}\partial_\zeta \rho_\zeta^{(n)}(0)\Big|_{\zeta=0} = \partial_\zeta c_{n-1}(0).$$

Substituting this relation into (10.17) proves (10.14). $\square$

10.5

Theorem 10.5 ([Hua97, Sec. 4.2]). $\mathcal{U} : \mathcal{G} \to \mathrm{End}(\mathbb{W})$ *is a group representation. Namely,* $\mathcal{U}(\alpha \circ \beta) = \mathcal{U}(\alpha)\mathcal{U}(\beta)$ *for all* $\alpha, \beta \in \mathcal{G}$.

With the help of this theorem, we can calculate $\partial_\zeta \mathcal{U}(\rho_\zeta)$ at $\zeta = 0$ without assuming $\rho_0(z) = z$ by computing $\partial_\zeta \mathcal{U}(\rho_\zeta \circ \rho_0^{-1})$ using Prop. 10.3.

Proof. It suffices to consider the following two cases: (a) $\alpha, \beta \in \mathcal{G}_+$ (b) $\alpha \in \mathcal{G}_+$ and β is a scaling. Let $l_n = z^{n+1}\partial_z$.

Case (a). We write $\alpha(z) = \exp(\sum_{n\geqslant 1} a_n l_n)(z) = \exp(X)(z)$ and $\beta(z) = \exp(\sum_{n\geqslant 1} b_n l_n)(z) = \exp(Y)(z)$. By the Campbell-Hausdorff theorem [Jac, Sec. V.5], $\alpha \circ \beta = \exp(Z)$ where

$$\begin{aligned} Z =& X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) - \frac{1}{24}[X,[Y,[X,Y]]] \\ &+ H_5 + H_6 + \cdots \end{aligned}$$

where each H_n is a finite sum of $n-1$ iterated brackets of X and Y, and hence an infinite linear combination of $l_n, l_{n+1}, \ldots$. So H_n increases the powers of z by at least n. From this we see that Z is also of the form $\sum_{n\geqslant 1} c_n l_n$ for some $c_1, c_2, \cdots \in \mathbb{C}$.

The representation $l_n \mapsto \pi(l_n) = L_n$ is a representation of the Lie subalgebra $\mathrm{Span}_{\mathbb{C}}\{l_1, l_2, \ldots\}$ of the Witt algebra. (There is no central term!) Write $\pi(X) = \sum_{n\geqslant 1} a_n L_n$ and $\pi(Y), \pi(Z), \pi(H_n)$ in a similar way. Note that each $\pi(H_n) = \bullet L_n + \bullet L_{n+1} + \cdots$ lowers the $\widetilde{L}_0$-weights by at least n. So $\sum_{n\geqslant 1} \pi(H_n)$ is well defined. By Campbell-Hausdorff theorem (applied to $\pi(X)$ and $\pi(Y)$), we have

$$\mathcal{U}(\alpha)\mathcal{U}(\beta) = \exp(\pi(X))\exp(\pi(Y)) = \exp\Big(\sum_{n\geqslant 1}\pi(H_n)\Big) = \exp(\pi(Z)) = \mathcal{U}(\alpha \circ \beta).$$

Case (b). Write $\alpha(z) = \exp(\sum_{n\geqslant 1} a_n l_n)(z)$ and $\beta(z) = \lambda z$ where $\lambda \neq 0$. One checks easily that

$$\alpha \circ \beta(z) = \lambda \cdot \exp\Big(\sum_{n\geqslant 1} a_n \lambda^n l_n\Big)(z).$$

Similar to the argument in Exercise 9.19, $[\widetilde{L}_0, L_n] = -L_n$ implies

$$\exp\Big(\sum_{n\geqslant 1} a_n L_n\Big)\lambda^{\widetilde{L}_0} = \lambda^{\widetilde{L}_0} \exp\Big(\sum_{n\geqslant 1} a_n \lambda^n L_n\Big)$$

This finishes the proof. □

10.6

Our goal is to find the covariance formula for $Y_{\mathbb{W}}$ under the change of local coordinate of $0 \in \mathbb{C}$ from the standard one ζ to any $\alpha \in \mathbb{G}$ defined on $\mathbb{D}_r$. Choose $z \in \mathbb{D}_r^\times$, and consider

$$\mathfrak{A} = (\mathbb{P}^1; 0, \infty; \alpha^{-1}, 1/\zeta), \qquad \mathfrak{P} = (\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, 1/\zeta). \tag{10.18}$$

where α^{-1} is the inverse function of α, not to be confused with $1/\alpha$.

We associate $\mathbb{W}, \mathbb{W}', \mathbb{W}, \mathbb{V}, \mathbb{W}'$ to the five marked points in the order listed above. By the change of coordinates formula in Sec. 2, the standard conformal blocks associated to these two are

$$\langle w', \mathcal{U}(\alpha)w\rangle, \qquad \langle w', Y_{\mathbb{W}}(v, z)w\rangle. \tag{10.19}$$

We sew $\mathfrak{A}$ and $\mathfrak{P}$ along $0 \in \mathfrak{A}$ and $\infty \in \mathfrak{P}$. We follow Rem. 4.4 to change the α^{-1} of $\mathfrak{A}$ to α^{-1}/r and the $1/\zeta$ of $\mathfrak{P}$ to r/ζ. Replace r by a slightly smaller number $> |z|$. Then the range of α^{-1}/r contains $\mathbb{D}_1^{\mathrm{cl}}$ (which is pulled back to $\alpha(\mathbb{D}_r^{\mathrm{cl}})$ in $\mathfrak{A}$), and the pullback of the unit disc under r/ζ is $\mathbb{P}^1 \backslash \mathbb{D}_r$, which is disjoint from z and 0. So Assumption 4.3 is satisfied.

This sewing identifies the following parts of $\mathfrak{A}, \mathfrak{P}$ respectively

$$A_1 = \{\gamma : 0 < |\alpha^{-1}(\gamma)| < r\}$$
$$A_2 = \{\gamma : 1/r < |1/\gamma| < +\infty\} = \{\gamma : 0 < |\gamma| < r\}$$

(cf. (4.2)) via the rule $\alpha^{-1}(\gamma_1) \cdot 1/\gamma_2 = 1$, or more precisely,

$$\gamma_1 \in A_1 \text{ is glued to } \gamma_2 \in A_2 \qquad \Longleftrightarrow \qquad \gamma_1 = \alpha(\gamma_2). \tag{10.20}$$

The point 0 of $\mathfrak{A}$ and the part $\{\gamma : |1/\gamma| \leqslant 1/r\} = \{\gamma : r \leqslant |\gamma| \leqslant +\infty\}$ of $\mathfrak{P}$ are discarded. We thus have an isomorphism

$$\mathfrak{A}\#\mathfrak{P} \qquad \xrightarrow{\simeq} \qquad \mathfrak{X} = \big(\mathbb{P}^1; 0, \alpha(z), \infty; \alpha^{-1}, \alpha^{-1} - z, 1/\zeta\big) \tag{10.21}$$

where any $\gamma_1 \in \mathbb{P}^1\backslash\{0\}$ of $\mathfrak{A}$ is identified with $\gamma_1 \in \mathfrak{X}$, and any $\gamma_2 \in \mathbb{D}_r$ of $\mathfrak{P}$ is identified with $\alpha(\gamma_2)$ of $\mathfrak{P}$.

10.7

On the one hand, the standard conformal block for $\mathfrak{A}\#\mathfrak{P}$ is the contraction of the two in (4.2), which is

$$\langle w', \mathcal{U}(\alpha)Y_{\mathbb{W}}(v,z)w\rangle. \tag{10.22}$$

On the other hand, since $\langle w', Y_{\mathbb{W}}(v,\alpha(z))w\rangle$ is the standard conformal block for $(\mathbb{P}^1; 0, \alpha(z), \infty; \zeta, \zeta-\alpha(z), 1/\zeta)$, by the change or coordinate formula in Sec. 2, the standard conformal block of $\mathfrak{P}$ should be

$$\big\langle w', Y_{\mathbb{W}}\big(\mathcal{U}(\varrho(\alpha|\mathbf{1})_z)v, \alpha(z)\big)\mathcal{U}(\alpha)w\big\rangle \tag{10.23}$$

where $\varrho(\alpha|\mathbf{1})_z \in \mathbb{G}$ is the change from $\alpha^{-1}-z$ to $\zeta-\alpha(z)$, namely,

$$\varrho(\alpha|\mathbf{1})_z(t) = \alpha(z+t)-\alpha(z). \tag{10.24}$$

(The meaning of the notation $\varrho(\alpha|\mathbf{1})$ will be explained in (11.9).) So (10.22) and (10.23) should be equal. That this result is a rigorous mathematical theorem is due to Huang.

Theorem 10.6 ([Hua97]). *Let $\mathbb{W}$ be an admissible $\mathbb{V}$-module. Then for each $w\in\mathbb{W}, w'\in\mathbb{W}', v\in\mathbb{V}$ and $\alpha\in\mathbb{G}$, the following equation holds in $\mathbb{C}((z))$*

$$\langle w', \mathcal{U}(\alpha)Y_{\mathbb{W}}(v,z)w\rangle = \big\langle w', Y_{\mathbb{W}}\big(\mathcal{U}(\varrho(\alpha|\mathbf{1})_z)v, \alpha(z)\big)\mathcal{U}(\alpha)w\big\rangle. \tag{10.25}$$

Equivalently, in $\mathbb{C}((z))$,

$$\langle w', \mathcal{U}(\alpha)Y_{\mathbb{W}}(v,z)\mathcal{U}(\alpha)^{-1}w\rangle = \big\langle w', Y_{\mathbb{W}}\big(\mathcal{U}(\varrho(\alpha|\mathbf{1})_z)v, \alpha(z)\big)w\big\rangle. \tag{10.26}$$

10.8

We explain the meanings of both sides of (10.25); (10.26) is understood in the similar way.

The meaning of the LHS of (10.25) is clear. Suppose $\alpha\in\mathscr{O}(\mathbb{D}_r)$. Then $\langle w', Y(v,\alpha(z))w\rangle$ is a Laurent polynomial of $\alpha(z)$, which is clearly holomorphic on $\mathbb{D}_r^\times$ with finite poles at 0. $z\mapsto\varrho(\alpha|\mathbf{1})_z$ is a holomorphic family of transformations. So $\mathcal{U}(\varrho(\alpha|\mathbf{1}))v$ is in $\mathbb{V}\otimes\mathscr{O}(\mathbb{D}_r)$ by (10.13). By linearity, the holomorphicity of $\langle w', Y(v,\alpha(z))w\rangle\in\mathbb{C}[z^{\pm1}]$ implies that the RHS of (10.25) is also holomrophic on $\mathbb{D}_r^\times$ with finite poles at 0. So, *the RHS of* (10.25) *is understood as an element of* $\mathbb{C}((z))$ *by taking Laurent series expansion of the holomorphic function*.

More generally, let $\alpha: X\to\mathbb{G}$ be a holomorphic family of transformations over a Riemann surface X. If α is holomorphic on $X\times\mathbb{D}_r$, then the RHS of (10.25) is naturally a holomorphic function on $X\times\mathbb{D}_r^\times$ with finite poles at $z=0$. Thus, as an element of $\mathscr{O}(X)((z))$ obtained by taking Laurent series expansion, it converges a.l.u. on $X\times\mathbb{D}_r^\times$ by Lemma 7.13. So is the LHS. We conclude:

Theorem 10.7. *Suppose $\alpha: X\to\mathbb{G}$ is a holomorphic family of transformations that is holomorphic on $X\times\mathbb{D}_r$. Then both sides of* (10.25) *and* (10.26) *are elements of $\mathscr{O}(X)((z))$ and converge a.l.u. on $X\times\mathbb{D}_r^\times$ to the same function. Moreover, the following series*

$$\sum_{n\in\mathbb{N}}\langle w', \mathcal{U}(\alpha)P_nY_{\mathbb{W}}(v,z)w\rangle \tag{10.27}$$

of elements of $\mathscr{O}(X\times\mathbb{C}^\times)$ converges a.l.u. on $X\times\mathbb{D}_r^\times$ to (10.25).

Proof. The last statement is due to Lemma 7.4 when v, w, w' are homogeneous. □

10.9 ⋆

We present the proof of (10.26) below. The idea is the same as in the proofs of scale and translation covariance. Also, it is not hard to see that the following proof works for all α in $\mathcal{G}$.

Proof of Thm. 10.6. Step 1. Let us first assume $\alpha \in \mathbb{G}_+$ so that $\alpha'(0) = 1$. Choose $c_1, c_2, \dots \in \mathbb{C}$ such that

$$\alpha(z) = \exp\Big(\sum_{n\geqslant 1} c_n z^{n+1}\partial_z\Big)(z), \tag{10.28}$$

and set

$$\alpha_\tau(z) = \exp\Big(\sum_{n\geqslant 1} \tau c_n z^{n+1}\partial_z\Big)(z) \qquad \in \mathbb{C}[\tau][[z]]$$

so that $\alpha_1(z) = \alpha(z)$. Note that we can write

$$\alpha_\tau(z) = z + \sum_{n\geqslant 2} p_n(\tau) z^n \tag{10.29}$$

where $p_n(\tau) \in \mathbb{C}[\tau]$. So we can view $\alpha_\tau(z)$ as a $\mathbb{C}[[z]]$-valued holomorphic function. The limit $\partial_\tau \alpha_\tau(z) = \lim_{\gamma\to\tau} \frac{\alpha_\gamma(z)-\alpha_\tau(z)}{\gamma-\tau}$ makes sense analytically as in Rem. 10.2.

(10.29) shows that

$$1/\alpha_\tau(z) \in z^{-1}\mathbb{C}[\tau][[z]].$$

Therefore, $\langle w', Y_{\mathbb{W}}(v, \alpha_\tau(z))w\rangle$, which is a Laurent polynomial of $\alpha_\tau(z)$, must also be in $\mathbb{C}[\tau]((z))$. It is not hard to verify that $\partial_\tau \alpha_\tau(z)|_{\tau=0} = \sum c_n z^{n+1}$ and that $\alpha_\gamma \circ \alpha_\tau(z) = \alpha_{\gamma+\tau}(z)$ for each $\gamma, \tau \in \mathbb{C}$. By taking derivative in the sense of Rem. 10.2, we obtain

$$\partial_\tau \alpha_\tau(z) = \sum_{n\geqslant 1} c_n \alpha_\tau(z)^{n+1}.$$

From this and the translation property, we obtain in $\mathbb{C}[\tau]((z))$ that

$$\partial_\tau \langle w', Y_{\mathbb{W}}(v, \alpha_\tau(z))w\rangle = \sum_{n\geqslant 1} c_n \alpha_\tau(z)^{n+1} \cdot \langle w', Y_{\mathbb{W}}(L_{-1}v, \alpha_\tau(z))w\rangle \tag{10.30}$$

as $\mathbb{C}((z))$-valued holomorphic functions of $\tau \in \mathbb{C}$.

Step 2. Let us calculate $\partial_\tau \mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v$. Note that any formal power series composed with $z+t$ is an element of $\mathbb{C}[[z,t]]$. So, even though α_τ is a formal coordinate, $\alpha_\tau(z+t)$ still makes sense, and we can use (10.24) again to define $\varrho(\alpha_\tau|\mathbf{1})_z$. Namely, in view of (10.29),

$$\varrho(\alpha_\tau|\mathbf{1})_z(t) = t + \sum_{n\geqslant 2} p_n(\tau) \sum_{j=1}^{n} \binom{n}{j} z^{n-j} t^j \qquad \in \mathbb{C}[\tau][[z]][[t]].$$

Similarly,

$$\begin{aligned}\varrho(\alpha_\zeta|\mathbf{1})_{\alpha_\tau(z)}(t) :=& \alpha_\zeta(\alpha_\tau(z)+t)-\alpha_\zeta(\alpha_\tau(z))\\ =&t+\sum_{n\geqslant 2}p_n(\zeta)\sum_{j=1}^{n}\binom{n}{j}\alpha_\tau(z)^{n-j}t^j\end{aligned} \tag{10.31}$$

makes sense as an element of $\mathbb{C}[\zeta,\tau][[z]][[t]]$. Using $\alpha_\zeta(\alpha_\tau(z))=\alpha_{\zeta+\tau}(z)$, one checks easily that

$$\varrho(\alpha_\zeta|\mathbf{1})_{\alpha_\tau(z)}\circ\varrho(\alpha_\tau|\mathbf{1})_z(t)=\varrho(\alpha_{\zeta+\tau}|\mathbf{1})_z(t).$$

Apply Thm. 10.5 to the above relation and take ∂_ζ at $\zeta=0$, we obtain

$$\partial_\tau\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v=\partial_\zeta\mathcal{U}(\varrho(\alpha_\zeta|\mathbf{1})_{\alpha_\tau(z)})\big|_{\zeta=0}\cdot\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1}_z))v. \tag{10.32}$$

Clearly $\varrho(\alpha_0|\mathbf{1})_{\alpha_\tau(z)}(t)=t$. By going through the proof of Prop. 10.3, we see that Prop. 10.3 also applies to the present situation: acting on $\mathbb{V}$ we have

$$\partial_\zeta\mathcal{U}(\varrho(\alpha_\zeta|\mathbf{1})_{\alpha_\tau(z)})\Big|_{\zeta=0}=\sum_{k\geqslant 1}\frac{1}{k!}\Big(\partial_\zeta\varrho(\alpha_\zeta|\mathbf{1})^{(k)}_{\alpha_\tau(z)}(0)\Big|_{\zeta=0}\Big)L_{k-1}. \tag{10.33}$$

By (10.31), it is clear that

$$\partial_\zeta\varrho(\alpha_\zeta|\mathbf{1})^{(k)}_{\alpha_\tau(z)}(0)=\partial_\zeta\alpha_\zeta^{(k)}(\alpha_\tau(z)).$$

Since, by (10.28), we have $\partial_\zeta\alpha_\zeta(z)\big|_{\zeta=0}=\sum_{n\geqslant 1}c_nz^{n+1}$ and hence

$$\frac{1}{k!}\partial_\zeta\alpha_\zeta^{(k)}(z)\big|_{\zeta=0}=\sum_{n\geqslant 1}\binom{n+1}{k}c_nz^{n-k+1},$$

we obtain

$$\begin{aligned}&\partial_\zeta\mathcal{U}(\varrho(\alpha_\zeta|\mathbf{1})_{\alpha_\tau(z)})\Big|_{\zeta=0}=\sum_{k,n\geqslant 1}\binom{n+1}{k}c_n\alpha_\tau(z)^{n-k+1}L_{k-1}\\ =&\sum_{n\geqslant 1}c_n\sum_{l\geqslant 0}\binom{n+1}{l+1}\alpha_\tau(z)^{n-l}L_l.\end{aligned} \tag{10.34}$$

To sum up, we get

$$\begin{aligned}&\partial_\tau\big\langle w',Y_{\mathbb{W}}(\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v,z)w\big\rangle\\ =&\sum_{n\geqslant 1}c_n\sum_{l\geqslant 0}\binom{n+1}{l+1}\alpha_\tau(z)^{n-l}\big\langle w',Y_{\mathbb{W}}(L_l\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v,z)w\big\rangle.\end{aligned} \tag{10.35}$$

Combining this relation with (10.30) and (5.3) yields

$$\partial_\tau\big\langle w',Y_{\mathbb{W}}(\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v,\alpha_\tau(z))w\big\rangle=\sum_{n\geqslant 1}c_n\big\langle w',[L_n,Y_{\mathbb{W}}(\mathcal{U}(\varrho(\alpha_\tau|\mathbf{1})_z)v,\alpha_\tau(z))]w\big\rangle. \tag{10.36}$$

(We leave it to the readers to check that this infinite sum is well-defined.) A similar calculation shows

$$\partial_\tau\big\langle w',\mathcal{U}(\alpha_\tau)Y_{\mathbb{W}}(v,z)\mathcal{U}(\alpha_\tau)^{-1}w\big\rangle=\sum_{n\geqslant 1}c_n\big\langle w',[L_n,\mathcal{U}(\alpha_\tau)Y_{\mathbb{W}}(v,z)\mathcal{U}(\alpha_\tau)^{-1}]w\big\rangle. \tag{10.37}$$

Thus, by Lemma 3.7, we get (10.26) for all $\alpha\in\mathbb{G}_+$. We have also proved (10.26) when α is a scaling. The general case follows from the combination of these two cases. We leave the details the readers. □

11 Definitions of conformal blocks and sheaves of VOAs

11.1

The goal of this section is to give two equivalent definitions of conformal blocks, both due to [FB04].

Assumption 11.1. Starting from this section, we assume $\dim\mathbb{V}(n)<+\infty$ for each n, and write $Y_{\mathbb{W}}$ as Y when possible.

Let

$$\mathfrak{X}=(C;x_1,\dots,x_N;\eta_1,\dots,\eta_N) \tag{11.1}$$

be an N-pointed compact Riemann surface with local coordinates. Assume that η_j is holomorphic (and injective) on an neighborhood U_j of x_j. Assume that $x_j\notin U_i$ if $i\neq j$.

Assumption 11.2. Unless otherwise stated, by an N-pointed compact Riemann surface, we assume that each connected component contains at least one marked point.

Recall that in Segal's picture, we have decomposition $\mathcal{H}^{\mathrm{fin}}=\bigoplus\mathbb{W}_i\otimes\widehat{\mathbb{W}}_i$, and the correlation function decomposes to $\mathbb{V}$- and $\widehat{\mathbb{V}}$-conformal blocks $T_{\mathfrak{X}}=\sum_{i_1,\dots,i_N\in\mathfrak{I}}\phi_{\mathfrak{X},i_\bullet}\otimes\psi_{\overline{\mathfrak{X}},i_\bullet}$ as in (1.14), where each $\phi_{\mathfrak{X},i_\bullet}$ is a linear functional on $\mathbb{W}_{i_\bullet}:=\mathbb{W}_{i_1}\otimes\cdots\otimes\mathbb{W}_{i_N}$.

In the following discussions, we fix a vector $\widehat{w}_i$ in each $\widehat{\mathbb{W}}_i$, and identify each $\mathbb{W}_i$ with $\mathbb{W}_i\otimes\widehat{w}_i$ so that we can restrict the correlation function $T_{\mathfrak{X}}$ onto $\mathbb{W}_{i_\bullet}$ to get a conformal block. Thus, we shall not distinguish between conformal blocks and (restrictions of) correlation functions.

11.2

We write $\mathbb{W}_{i_k}=\mathbb{W}_k$ and $\phi_{\mathfrak{X}_{i_\bullet}}=\phi$ for simplicity. So the $\mathbb{V}$-modules $\mathbb{W}_1,\dots,\mathbb{W}_N$ are associated to $x_1,\dots,x_N$. Recall the notation $\mathbb{W}_\bullet=\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N$.

We add a point x to $\mathfrak{X}$ different from $x_1,\dots,x_N$. Then we get a new $(N+1)$-pointed compact Riemann surface $\wr\mathfrak{X}_x$. We insert vectors of $\mathbb{V}\simeq\mathbb{V}\otimes\mathbf{1}$ to x. Then we get a new conformal block $\wr\phi_x:\mathbb{V}\otimes\mathbb{W}_\bullet\to\mathbb{C}$, which is the restriction of the correlation function $T_{\wr\mathfrak{X}_x}$ to $\mathbb{V}\otimes\mathbb{W}_\bullet$. $\wr\phi_x$ has the following two features. (Let ζ be the standard coordinate of $\mathbb{C}$.)

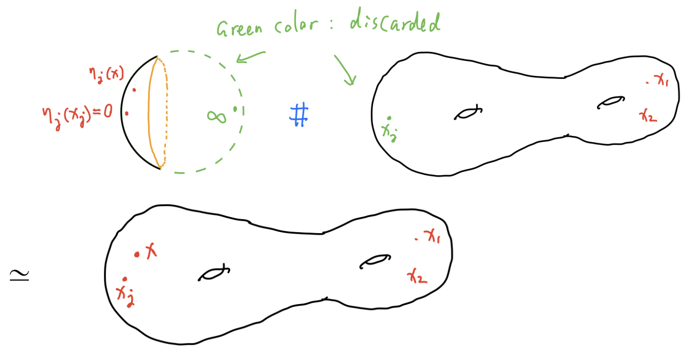
First, assume $\eta_j(U_j) \supset \mathbb{D}_{r_j}$. Let $x \in \eta_j^{-1}(\mathbb{D}_{r_j})$. We assign local coordinate $\eta_j - \eta_j(x)$ to x so that every marked point of $\wr\mathfrak{X}_x$ has an associated local coordinate. Let

$$\mathfrak{P}_{\eta_j(x)} = (\mathbb{P}^1; 0, \eta_j(x), \infty; \zeta, \zeta - \eta_j(x), 1/\zeta). \tag{11.2}$$

Consider the sewing $\mathfrak{P}_{\eta_j(x)}\#\mathfrak{X}$ along $\infty \in \mathfrak{P}_{\eta_j(x)}$ and $x_j \in \mathfrak{X}$. We have an equivalence

$$\mathfrak{P}_{\eta_j(x)}\#\mathfrak{X} \simeq \wr\mathfrak{X}_x \tag{11.3}$$

where the parts $\mathbb{P}^1\backslash\mathbb{D}_{r_j}$ and x_j of $\mathfrak{P}_{\eta_j(x)}$ and $\mathfrak{X}$ are discarded; any $\gamma \in \mathbb{D}_{r_j}$ is equivalent to $\eta_j^{-1}(\gamma)$ of $\wr\mathfrak{X}_x$, and is glued with $\eta_j^{-1}(\gamma)$ of $\mathfrak{X}$ when $\gamma \in \mathbb{D}_{r_j}^\times$; in particular, the marked points $0, \eta_j(x)$ of $\mathfrak{P}_{\eta_j(x)}$ (which are not discarded) are identified respectively with x_j, x of $\wr\mathfrak{X}_x$.



(11.4)

Therefore, by the sewing-contraction correspondence, the conformal block $\wr\upphi_x$ associated to $\wr\mathfrak{X}_x$ (where the local coordinate at x is $\eta_j - \eta_j(x)$) is

$$\wr\upphi_x(v \otimes w_\bullet) = \upphi\big(w_1 \otimes \cdots \otimes Y(v, \eta_j(x))w_j \otimes \cdots \otimes w_N\big) \tag{11.5}$$

where the RHS is short for the following two equivalent series (cf. Lemma 7.4) and is converging a.l.u. to the LHS of (11.5):

$$\begin{aligned}\text{RHS of (11.5)} &= \sum_{n\in\mathbb{Z}} \upphi\big(w_1 \otimes \cdots \otimes Y(v)_n w_j \otimes \cdots \otimes w_N\big) z^{-n-1}\big|_{z=\eta_j(x)}\\ &= \sum_{n\in\mathbb{N}} \upphi\big(w_1 \otimes \cdots \otimes P_n Y(v, \eta_j(x))w_j \otimes \cdots \otimes w_N\big).\end{aligned} \tag{11.6}$$

11.3

The second feature is: according to (1.12), for any x on C not necessarily close to any of $x_\bullet$, $\wr\upphi_x(v \otimes w_\bullet)$ is holomorphic with respect to the motion of x. A downside of this description is that it depends on a particular choice of local coordinates at x: if in one local coordinate v is a constant, then in another one v will vary. So let us give an coordinate-independent descrption:

Besides the translation of x, we also allow v to vary holomorphically with respect to x. Namely, let $U \subset C$ be open, choose a sufficiently large $n \in \mathbb{N}$, and assume v is a $\mathbb{V}^{\leqslant n}$-valued holomorphic function on U. (Recall that $\mathbb{V}^{\leqslant n}$ is finite dimensional by Convention 11.1.) Namely,

$$v \in \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}(U). \tag{11.7}$$

Assume that there is a **univalent** (i.e., holomorphic+injective) function $\mu : U \to \mathbb{C}$.[4] (It is helpful to think of μ vanishing at some point $y \in U$, i.e., μ is a local coordinate at y. But technically this is not necessary.) Then at each $x \in U$ there is a natural local coordinate $\mu - \mu(x)$. If we let $\wr\phi_x$ act on abstract vectors instead of concrete ones, then for each v as above (so that each $\mathcal{U}(\mu - \mu(x))^{-1}v(x)$ is a concrete vector)

$$x \in U \mapsto \wr\phi_x\big(\mathcal{U}(\mu - \mu(x))^{-1}v(x) \otimes w_\bullet\big) \tag{11.8}$$

is a holomorphic function. The choice of local coordinate $\mu - \mu(x)$ is in accordance with $(\zeta - z)/r$ in (1.10) if we assume $r = 1$ and identify μ with the standard coordinate ζ of $\mathbb{C}$.

11.4

We explain why this description is independent of the choice of μ. Let $\eta \in \mathscr{O}(U)$ be also univalent. Let $\varrho(\eta|\mu)_x \in \mathbb{G}$ be the change of coordinate from $\mu - \mu(x)$ to $\eta - \eta(x)$. Namely

$$\varrho(\eta|\mu)_x\big(\mu(y) - \mu(x)\big) = \eta(y) - \eta(x) \tag{11.9}$$

for any $y \in C$ close to x. Equivalently,

$$\varrho(\eta|\mu)_x(z) = \eta \circ \mu^{-1}(z + \mu(x)) - \eta(x), \tag{11.10}$$

from which we see that $\varrho(\eta|\mu) : X \to \mathbb{G}, x \mapsto \varrho(\eta|\mu)_x$ is a holomorphic family of transformations. Thus, by (10.13), $\mathcal{U}(\varrho(\eta|\mu))\big|_{\mathbb{V}^{\leqslant n}}$ is in $\mathrm{End}\mathbb{V}^{\leqslant n} \otimes \mathscr{O}(U)$. Thus, by $\mathscr{O}(U)$-linearity, $\mathcal{U}(\varrho(\eta|\mu))$ sends each section of $\mathbb{V}^{\leqslant n} \otimes \mathscr{O}(U)$ to $\mathbb{V}^{\leqslant n} \otimes \mathscr{O}(U)$ such that its valued at each x is an automorphism of $\mathbb{V}^{\leqslant n}$.

This property can be summarized in the following way: Let $\mathscr{O}_U$ be the trivial holomorphic line (i.e. 1-dimensional vector bundle) over U. So $\mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U$ is the trivial (holomorphic) vector bundle[5] with fiber $\mathbb{V}^{\leqslant n}$. Then we have an automorphism of vector bundle (equivalently, an automorphism of $\mathscr{O}_U$-module)

$$\mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U.$$

By Subsec. 2.11,

$$\wr\phi_x\big(\mathcal{U}(\mu - \mu(x))^{-1}v(x) \otimes w_\bullet\big) = \wr\phi_x\big(\mathcal{U}(\eta - \eta(x))^{-1}u(x) \otimes w_\bullet\big)$$

where $u(x) = \mathcal{U}(\varrho(\eta|\mu)_x)v(x)$. Thus, the function v on U is holomorphic iff u is so. This implies that the holomorphicity of (11.8) is independent of the choice of μ.

Example 11.3. Let ζ be the standard coordinate of $\mathbb{C}^\times$. Then for each $\gamma \in \mathbb{C}^\times$,

$$\varrho\big(1/\zeta\big|\zeta\big)_\gamma = \varrho\big(\zeta\big|1/\zeta\big)_{1/\gamma} = \vartheta_\gamma \tag{11.11}$$

where $\vartheta_\gamma(z) = \frac{1}{\gamma+z} - \frac{1}{\gamma}$ (cf. (9.11)). Therefore, by (9.14),

$$\mathcal{U}\big(\varrho\big(1/\zeta\big|\zeta\big)_\gamma\big) = \mathcal{U}\big(\varrho\big(\zeta\big|1/\zeta\big)_{1/\gamma}\big) = e^{\gamma L_1}(-\gamma^{-2})^{L_0}. \tag{11.12}$$

[4]Indeed, one only needs to assume that $d\mu$ is nowhere zero on U. Then μ must be locally univalent, which is sufficient for applications.

[5]In our notes, all vector bundles are holomorphic with finite ranks unless otherwise stated.

11.5

The combination of these two features gives the definition of conformal blocks. To simplify the definition and make it more precise, let us introduce some new notions.

We define a vector bundle $\mathscr{V}_C^{\leqslant n}$ over C whose fibers are equivalent to $\mathbb{V}^{\leqslant n}$ as follows. Recall that holomorphic vector bundles can be constructed once we have holomorphic transation functions. By (7.7), for univalent $\eta_i \in \mathscr{O}(U)$, $i = 1, 2, 3$, we have

$$\varrho(\eta_1|\eta_2)_x \circ \varrho(\eta_2|\eta_3)_x = \varrho(\eta_1|\eta_3)_x \tag{11.13}$$

and hence the cocycle condition

$$\mathcal{U}(\varrho(\eta_1|\eta_2))\mathcal{U}(\varrho(\eta_2|\eta_3)) = \mathcal{U}(\varrho(\eta_1|\eta_3)) \tag{11.14}$$

due to Thm. 10.5. Thus, we have a unique (up to equivalence) vector bundle $\mathscr{V}_C^{\leqslant n}$ whose transition functions are of the form $\mathcal{U}(\varrho(\eta|\mu))$. More precisely, for any open $U \subset C$ with a univalent $\eta \in \mathscr{O}(U)$ is associated with a trivialization (i.e., an equivalence of vector bundles/$\mathscr{O}_U$-modules)

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_C^{\leqslant n}\big|_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U \tag{11.15}$$

compatible with the restriction of η to open subsets (i.e., if $V \subset U$ is open then $\mathcal{U}_\varrho(\eta|_V) = \mathcal{U}_\varrho(\eta)|_V$) such that if $\mu \in \mathscr{O}(U)$ is also univalent, then

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} = \mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U. \tag{11.16}$$

Remark 11.4. Intuitively, the fiber of $\mathscr{V}_C^{\leqslant n}$ at each $x \in C$ is the vector space $\mathscr{W}(\mathbb{V}^{\leqslant n})$ of abstract VOA vectors whose energies are $\leqslant n$. The trivialization $\mathcal{U}_\varrho(\eta)$ sends each fiber $\mathscr{V}_C^{\leqslant n}|_x$ at x to $\mathbb{V}^{\leqslant n}$ via the isomorphism $\mathcal{U}(\eta - \eta(x))$, and sends each abstract VOA vector to its $(\eta - \eta(x))$-coordinate representation. If $v \in \mathbb{V}^{\leqslant n} \otimes \mathscr{O}(U)$, then the map $x \mapsto \mathcal{U}(\eta - \eta(x))^{-1}v(x)$ is just the section $\mathcal{U}_\varrho(\eta)^{-1}v$ of $\mathscr{V}_C^{\leqslant n}$ on U, and any section on U is of this form. $\mathscr{V}_C^{\leqslant n}(U)$, the space of all sections of $\mathscr{V}_C^{\leqslant n}$ on U, *is the space of all VOA vectors with energies $\leqslant n$ varying and moving holomorphically on U.*

Remark 11.5. The vacuum vector $\mathbf{1}$ is fixed by any change of coordinate operator $\mathcal{U}(\varrho(\eta|\mu))$ since it is killed by $L_{\geqslant 0}$. So we let $\mathbf{1}$ denote also the element of $\mathscr{V}_C^{\leqslant n}(C)$ whose trivialization under any local univalent map η is the vacuum vector $\mathbf{1}$. We call $\mathbf{1}$ the **vacuum section**.

11.6

Now, the property that $\wr\phi$ is holomorphic with respect to the motion and variation of the inserted VOA vectors can be expressed in the following form:

1. For each open subset U of $C \backslash \{x_\bullet\}$,

$$\wr\phi(\cdot \otimes w_\bullet) : \mathscr{V}_C^{\leqslant n}(U) \to \mathscr{O}(U), \qquad \mathbf{v} \mapsto \wr\phi(\mathbf{v} \otimes w_\bullet) \tag{11.17}$$

   is an $\mathscr{O}(U)$-module (homo)morphism. (The reason that it intertwines the actions of $\mathscr{O}(U)$ is clear.)

2. $\wr\phi(\cdot\otimes w_\bullet)$ is compatible with the restriction to open subsets. Namely, if $V\subset U$ is open, then $\wr\phi(\mathbf{v}|_V\otimes w_\bullet)=\wr\phi(\mathbf{v}\otimes w_\bullet)|_V$.

The above two points can be summarized using the sheaf theoretic language: $\wr\phi(\cdot\otimes w_\bullet)$ is a morphism of $\mathscr{O}_{C\backslash\{x_\bullet\}}$-modules $\mathscr{V}^{\leqslant n}_{C\backslash\{x_\bullet\}}\to\mathscr{O}_{C\backslash\{x_\bullet\}}$. Equivalently,

$$\wr\phi(\cdot\otimes w_\bullet)\in H^0\big(C\backslash\{x_\bullet\},(\mathscr{V}_C^{\leqslant n})^\vee\big).$$

11.7

To simplify the formulation of definitions and theorems, we consider the direct limit sheaf

$$\mathscr{V}_C=\varinjlim_{n\in\mathbb{N}}\mathscr{V}_C^{\leqslant n}$$

whose space of sections on any connected open $U\subset C$ (or more generally, any open U with finitely many connected component) is

$$\mathscr{V}_C(U)=\varinjlim_{n\in\mathbb{N}}\mathscr{V}_C^{\leqslant n}(U).$$

This is possible since for each $n_1\leqslant n_2$ we have an obvious injective $\mathscr{O}_C$-module morphism (i.e., morphism of vector bundles) $\mathscr{V}_C^{\leqslant n_1}\to\mathscr{V}_C^{\leqslant n_2}$ which under any trivialization as in (11.15) is the obvious inclusion $\mathbb{V}^{\leqslant n_1}\otimes\mathscr{O}_U\hookrightarrow\mathbb{V}^{\leqslant n_2}\otimes\mathscr{O}_U$. Both $\mathscr{V}_C$ and $\mathscr{V}_C^{\leqslant n}$ are called **sheaves of VOAs** associated to C and $\mathbb{V}$.

Equivalently, $\mathscr{V}_C$ is an infinite-rank vector bundle such that for each connected open $U\subset C$ with a univalent U, we have a trivialization

$$\mathcal{U}_\varrho(\eta):\mathscr{V}_C|_U\xrightarrow{\simeq}\mathbb{V}\otimes\mathscr{O}_U$$

compatible with the restriction of η to connected open subsets, such that for any another univalent $\mu\in\mathscr{O}(U)$ we also have $\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1}=\mathcal{U}(\varrho(\eta|\mu))$ as an automorphism of the $\mathscr{O}_U$-module $\mathbb{V}\otimes\mathscr{O}_U$.

Thus, roughly speaking, $\mathscr{V}_C(U)$ is the set of all sections v belonging to $\mathscr{V}_C^{\leqslant n}(U)$ for some $n\in\mathbb{N}$.

In the rest of these notes, the readers may replace $\mathscr{V}_C$ by $\mathscr{V}_C^{\leqslant n}$ for all possible n if they are not comfortable with locally free sheaves of infinite ranks.

11.8

We are now ready to state the definition of conformal blocks. Recall the data $\mathfrak{X}$ in (11.1) and that each η_i is defined on $U_i\ni x_i$. Let $\mathbb{V}$ be a VOA, and let $\mathbb{W}_1,\dots,\mathbb{W}_N$ be admissible $\mathbb{V}$-modules associated respectively to the marked points $x_1,\dots,x_N$.

Definition 11.6 (Complex analytic version)**.** A linear functional $\phi:\mathbb{W}_\bullet=\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N\to\mathbb{C}$ is called a **conformal block** associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$ if the following holds: For each $w_\bullet\in\mathbb{W}_\bullet$, there exists a (necessarily unique) $\mathscr{O}_{C\backslash\{x_\bullet\}}$-module morphism

$$\wr\phi(\cdot,w_\bullet):\mathscr{V}_{C\backslash\{x_\bullet\}}\to\mathscr{O}_{C\backslash\{x_\bullet\}}$$

(equivalently, $\wr\phi(\cdot, w_\bullet) \in H^0(C\backslash\{x_\bullet\}, \mathscr{V}_C^\vee)$) such that for each $1 \leqslant i \leqslant N$, by identifying

$$\mathscr{V}_C|_{U_i} = \mathbb{V} \otimes \mathscr{O}_{U_i} \qquad \text{via } \mathcal{U}_\varrho(\eta_i) \tag{11.18}$$

and identifying

$$U_i = \eta_i(U_i) \qquad \text{via } \eta_i \tag{11.19}$$

so that η_i becomes the standard coordinate z, for each $v \in \mathscr{V}_C(U_i) = \mathbb{V} \otimes \mathscr{O}(U_i)$ (restricted to $U_i\backslash\{x_i\} = \eta_i(U_i)\backslash\{0\}$), the equality

$$\wr\phi(v, w_\bullet)_z = \phi\big(w_1 \otimes \cdots \otimes Y(v(z), z)w_i \otimes \cdots \otimes w_N\big) \tag{11.20}$$

holds in $\mathbb{C}[[z^{\pm 1}]]$. $\square$

Note that the LHS of (11.20) is an element of $\mathscr{O}(\eta_i(U_i)\backslash\{0\})$, regarded as one in $\mathbb{C}[[z^{\pm 1}]]$ by taking Laurent series expansions. The RHS is understood as

$$\sum_{m\in\mathbb{N}, n\in\mathbb{Z}} \phi(\cdots \otimes Y(v_m)_n w_i \otimes \cdots) z^{m-n-1}$$

if v has expansion $v(z) = \sum_{m\geqslant 0} v_m z^m$ where each $v_m \in \mathbb{V}$. In particular, (11.20) is in $\mathbb{C}((z))$.

11.9

Let us make some comments on this definition.

Remark 11.7. By Lemma 7.13, we see that if $\eta_i(U_i) \supset \mathbb{D}_{r_i}$, then the formal Laurent series of z on the RHS of (11.20), and equivalently (cf. (11.6)), the series of functions

$$\sum_{n\in\mathbb{N}} \phi\big(w_1 \cdots \otimes P_n Y(v(z), z)w_i \otimes \cdots \otimes w_N\big)$$

converge a.l.u. on $z \in \mathbb{D}_{r_i}^\times$ to the LHS of (11.20). This explains why Def. 11.6 is viewed as a complex analytic definition.

Remark 11.8. The uniqueness of $\wr\phi(\cdot, w_\bullet)$ is due to the following reason. It suffices to restrict $\omega = \wr\phi(\cdot, w_\bullet)$ to $\mathscr{V}_C^{\leqslant n}$ for each $n \geqslant 0$. Suppose $\omega' = \wr'\phi(\cdot, w_\bullet)$ is another morphism satisfying the descriptions in Def. 11.6. Then ω and ω' are sections of the vector bundle $(\mathscr{V}_C^{\leqslant n})^\vee$ over $C\backslash\{x_\bullet\}$. Moreover, by (11.20), $\omega - \omega'$ vanishes on each $U_i\backslash\{x_i\}$. Thus, if we let $\Omega \subset C\backslash\{x_\bullet\}$ be the set of all points y such that $\omega - \omega'$ vanishes on a neighborhood of y, then by Assumption 11.2, Ω intersects each connected component of C. By complex analysis, Ω is both open and closed. So $\Omega = C\backslash\{x_\bullet\}$.

Remark 11.9. By this uniqueness, we may define $\wr\phi(\cdot, w)$ for all $w \in \mathbb{W}_\bullet$ such that $\wr\phi(\cdot, w)$ is linear over w.

Remark 11.10. By complex analysis, it is clear that the definition of conformal blocks is independent of the sizes and shapes of the neighborhoods $U_1, U_2, \ldots$ of $x_\bullet$.

Remark 11.11. By $\mathscr{O}(U_i)$-linearity, to verify (11.20) for all $v \in \mathbb{V} \otimes \mathscr{O}(U_i)$, it suffices to verify it for all constant $v \in \mathbb{V} \simeq \mathbb{V} \otimes \mathbf{1}$.

11.10

Example 11.12. Fix $\gamma \in \mathbb{C}^\times$, and let $\mathfrak{P} = (\mathbb{P}^1; 0, \gamma, \infty; \zeta, \zeta - \gamma, 1/\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$. Let $\mathbb{W}$ be an admissible $\mathbb{V}$-module, and associate $\mathbb{W}, \mathbb{V}, \mathbb{W}'$ to $0, \gamma, \infty$. Then the following linear functional is a conformal block, called the **conformal block associated to the vertex operation** $Y_{\mathbb{W}}$.

$$\omega : \mathbb{W} \otimes \mathbb{V} \otimes \mathbb{W}' \to \mathbb{C}, \qquad w_\bullet = w \otimes v \otimes w' \mapsto \langle w', Y(v, \gamma) w \rangle \tag{11.21}$$

Proof. We construct the $\mathscr{O}_{\mathbb{C}^\times \backslash \{\gamma\}}$-module morphism $\wr\omega(\cdot, w_\bullet) : \mathscr{V}_{\mathbb{C}^\times \backslash \{\gamma\}} \to \mathscr{O}_{\mathbb{C}^\times \backslash \{\gamma\}}$ as follows. For every open $U \subset \mathbb{C}^\times \backslash \{\gamma\}$, set

$$\begin{gathered}\wr\omega(\cdot, w_\bullet) : \mathscr{V}_{\mathbb{C}^\times \backslash \{\gamma\}}(U) \to \mathscr{O}(U), \\ \mathcal{U}_\varrho(\zeta)^{-1} u \mapsto \langle w', Y(u(z), z) Y(v, \gamma) w \rangle\end{gathered}$$

where $u \in \mathbb{V} \otimes \mathscr{O}(U)$, and we have used the convention in Def. 8.3 so that the above termed is defined and holomorphic when $z \neq 0, \gamma, \infty$ and u is holomorphic.

Assume without loss of generality that u is a constant section, i.e. $u \in \mathbb{V}$. By the complex analytic Jacobi identity for $Y_{\mathbb{W}}$, (11.20) holds for ω when γ is close to 0 or γ. When z is close to ∞, $\wr\omega(\cdot, w_\bullet)$ sends $\mathcal{U}_\varrho(\zeta)^{-1} u$ to $\langle w', Y(u, z) Y(v, \gamma) w \rangle$. Thus, it sends

$$\mathcal{U}_\varrho(1/\zeta)^{-1} u = \mathcal{U}_\varrho(\zeta)^{-1} \mathcal{U}(\varrho(\zeta|1/\zeta)) u$$

to

$$\big\langle w', Y\big(\mathcal{U}(\varrho(\zeta|1/\zeta)_z) u, z\big) Y(v, \gamma) w \big\rangle \xlongequal{(11.12)} \big\langle w', Y\big(e^{z^{-1} L_1} (-z^2)^{L_0} u, z\big) Y(v, \gamma) w \big\rangle,$$

which by (9.15) equals

$$\big\langle Y\big(u, z^{-1}\big) w', Y(v, \gamma) w \big\rangle = \big\langle Y\big(u, \eta_\infty(z)\big) w', Y(v, \gamma) w \big\rangle$$

where $\eta_\infty = 1/\zeta$ is the local coordinate at ∞. This proves (11.20) when z is near ∞. □

Exercise 11.13. Let $\mathbb{W}_1, \mathbb{W}_2$ be admissible $\mathbb{V}$-modules, and let $T : \mathbb{W}_1 \to \mathbb{W}_2$ be a $\mathbb{V}$-module homomorphism, i.e., a linear map intertwines the $\mathbb{V}$-actions. Let $\mathfrak{P} = (\mathbb{P}^1; 0, \infty; \zeta, 1/\zeta)$, and associate $\mathbb{W}_1, \mathbb{W}_2'$ to $0, \infty$ respectively. Show that the following linear functional is a conformal block associated to $\mathfrak{P}$ and $\mathbb{W}_1, \mathbb{W}_2'$.

$$\mathbb{W}_1 \otimes \mathbb{W}_2' \to \mathbb{C} \qquad w_1 \otimes w_2' \mapsto \langle T w_1, w_2' \rangle \tag{11.22}$$

11.11

Due to the fact that (11.20) belongs to $\mathbb{C}((z))$, we may regard $\wr\phi(\cdot, w_\bullet)$ as a section of $(\mathscr{V}_C^{\leqslant n})^\vee$ that has finite poles at $x_\bullet$:

$$\wr\phi(\cdot, w_\bullet) \in H^0(C, (\mathscr{V}_C^{\leqslant n})^\vee(\star x_\bullet)). \tag{11.23}$$

The meaning of the notation is the following. Let $\mathscr{E}$ be a vector bundle over C. Then for each $k_1, \dots, k_N \in \mathbb{Z}$,

$$\mathscr{E}(k_1 x_1 + \cdots + k_N x_N)$$

denotes the $\mathscr{O}_C$-module whose space of sections on each open $U \subset C$ are all $s \in \mathscr{E}(U\backslash\{x_\bullet\})$ such that for each $1 \leqslant i \leqslant N$ the function $\eta_i^{k_i} \cdot s : x \mapsto \eta_i(x)^{k_i} s(x)$ is holomorphic on a neighborhood of x_i (equivalently, on U_i). Thus, when $k_1, \dots, k_N \geqslant 0$, it is the sheaf of sections of $\mathscr{E}_{C\backslash\{x_\bullet\}}$ that have poles of order at most k_i at x_i. Then

$$\mathscr{E}(\star x_\bullet) = \varinjlim_{k_1,\dots,k_N \in \mathbb{N}} \mathscr{E}(k_1 x_1 + \dots + k_N x_N) \tag{11.24}$$

is the sheaf of sections of $\mathscr{E}_{C\backslash\{x_\bullet\}}$ that have finite poles at $x_1, \dots, x_N$.

This viewpoint allows us to use the strong residue theorem to obtain the algebraic definition of conformal blocks. Let ω_C be holomorphic cotangent line bundle of C, i.e., the sheaf of holomorphic 1-forms on the open subsets of C. The by residue theorem/Stokes' theorem,

$$\sum_{i=1}^N \mathrm{Res}_{x_i} \lambda = 0 \tag{11.25}$$

for all $\lambda \in H^0(C\backslash\{x_\bullet\}, \omega_C)$, and hence for all $\lambda \in H^0(C, \omega_C(\star x_\bullet))$.

Theorem 11.14 (**Strong residue theorem**). *Let $\mathscr{E}$ be a vector bundle on C. For each $1 \leqslant i \leqslant N$, use a trivialization of $\mathscr{E}$ and the corresponding dual trivialization for the dual vector bundle $\mathscr{E}^\vee$ to fix an identification*

$$\mathscr{E}|_{U_i} = E_i \otimes \mathscr{O}_{U_i}, \qquad \mathscr{E}^\vee|_{U_i} = E_i^* \otimes \mathscr{O}_{U_i} \tag{11.26}$$

where E_i is a finite dimensional vector space and E_i^* is its dual space. Choose

$$s_i = \sum_{n \in \mathbb{Z}} e_{i,n} \eta_i^n \qquad \in E_i((\eta_i)). \tag{11.27}$$

Then the following are equivalent.

(a) There exists $s \in H^0(C, \mathscr{E}(\star x_\bullet))$ whose Laurent series expansion at each x_i is s_i.

(b) For each $\sigma \in H^0(C, \mathscr{E}^\vee \otimes \omega_C(\star x_\bullet))$,

$$\sum_{i=1}^N \mathrm{Res}_{x_i} \langle s_i, \sigma \rangle = 0. \tag{11.28}$$

Here, $\mathscr{E}^\vee \otimes \omega_C$ is the tensor product of the two vector bundles. Recall that in general, if $\mathscr{E}$ and $\mathscr{F}$ are vector bundles over a complex manifold X, then $\mathscr{E} \otimes \mathscr{F}$ (or more precisely, $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$) is the one whose transition functions are given by the tensor products of those of $\mathscr{E}$ and $\mathscr{F}$. Equivalently, $\mathscr{E} \otimes \mathscr{F}$ is the sheafification of the presheaf whose space of sections over any open $U \subset X$ is $\mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$. $(\mathscr{E} \otimes \mathscr{F})(U)$ equals $\mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$ when $\mathscr{E}_U$ and $\mathscr{F}_U$ are trivializable (i.e. equivalent to free $\mathscr{O}_U$-modules). (To see this, simply assume $\mathscr{E}_U = \mathscr{O}_U^{\oplus m}$ and $\mathscr{E}_U = \mathscr{O}_U^{\oplus n}$.)

11.12

The LHS of (11.28) is understood in the following way. In view of (11.26), at each x_i, σ has expansion $\sigma = \sum_{n\in\mathbb{Z}} \varepsilon_{i,n}\eta_i^n d\eta_i$ where $\varepsilon_{i,n} \in E_i^*$. Then $\langle s_i, \sigma\rangle = \sum_{m,n\in\mathbb{Z}}\langle e_{i,m}, \varepsilon_{i,n}\rangle\eta_i^{m+n}d\eta_i$. So (11.28) reads

$$\sum_{i=1}^N \sum_{m+n=-1} \langle e_{i,m}, \varepsilon_{i,n}\rangle = 0$$

where the sum over all $m, n \in \mathbb{Z}$ satisfying $m + n = -1$ is finite.

Remark 11.15. Suppose $U_i \supset \eta_i(\mathbb{D}_{r_i})$. Then it is clear that if (a) or (b) holds, then the series $s_i = \sum_{n\in\mathbb{Z}} e_{i,n}\eta_i^n$ converges a.l.u. on $\eta_i \in \mathbb{D}_{r_i}^\times$. It is remarkable that this analytic property follows from the algebraic condition (11.28). This is analogous to that the formal variable version of local fields implies the complex analytic one, and that the algebraic Jacobi identity for VOAs implies the complex analytic one.

That $(a) \Rightarrow (b)$ follows from the residue theorem, since $\langle s, \sigma\rangle$ is an element of $H^0(C, \omega_C(\star x_\bullet))$. The other direction is more difficult. To prove it one needs more advance tools such as sheaf cohomology and Serre duality, which we are not able to present here due to page limitations. We refer the readers to [Muk10, Sec. 1.2.2][6] or [Gui, Sec. 1.4] for details.

11.13

We now apply the strong residue theorem to the case that $\mathscr{E} = (\mathscr{V}_C^{\leqslant n})^\vee$. The trivialization (11.26) is given by $\mathcal{U}_\varrho(\eta_i)$ (cf. (11.27)) and its dual. In particular, $E_i = (\mathbb{V}^{\leqslant n})^*$. The series s_i we choose is the RHS of (11.20), namely,

$$s_i = \sum_{n\in\mathbb{Z}} s_{i,n}\eta_i^n \qquad \in (\mathbb{V}^{\leqslant n})^*((\eta_i))$$

where $s_{i,n} \in (\mathbb{V}^{\leqslant n})^*$ sends each $v \in \mathbb{V}^{\leqslant n}$ to

$$s_{i,n}(v) = \phi\big(w_1 \otimes \cdots \otimes Y(v)_{-n-1}w_i \otimes \cdots \otimes w_N\big).$$

Now, Def. 11.6 says simply that (for all n) all $s_1, \ldots, s_N$ are series expansions at $x_1, \ldots, x_N$ of the same section of $H^0(C, \mathscr{E}(\star x_\bullet))$, namely $\wr\phi(\cdot, w_\bullet)$. Thus, by the strong residue Thm. 11.14, the statements in Def. 11.6 (when restricted to $\mathscr{V}_C^{\leqslant n}$) are equivalent to $\sum_{i=1}^N \mathrm{Res}_{x_i}\langle s_i, \sigma\rangle = 0$ for all $\sigma \in H^0(C, \mathscr{V}_C^{\leqslant n} \otimes \omega_C(\star x_\bullet))$. Namely, ϕ vanishes on

$$\sigma \cdot w_\bullet = \sum_{i=1}^N w_1 \otimes \cdots \otimes \sigma \cdot w_i \otimes \cdots \otimes w_N \tag{11.29}$$

where for each i,

$$\sigma \cdot w_i = \mathrm{Res}_{z=0}\, Y(v_i(z), z)w_i dz \qquad \in \mathbb{W}_i \tag{11.30}$$

[6] Though [Muk10] only discusses the case that $\mathscr{E} = \mathscr{O}_C$, its proof applies to all vector bundles.

and $\sigma|_{U_i} = v_i(z)dz$ under the identifications (11.18) and (11.19).

For instance, if $\sigma|_{U_i} = uz^k dz$ where $u \in \mathbb{V}$, then

$$(uz^k dz) \cdot w_i = Y(u)_k w_i. \tag{11.31}$$

Definition 11.16. We define a linear action of $H^0\big(C, \mathscr{V}_C \otimes \omega_C(\star x_\bullet)\big)$ on $\mathbb{W}_\bullet$ such that for each $\sigma, w_\bullet$ in the two vector spaces respectively, $\sigma \cdot w_\bullet$ is defined by (11.29) and (11.30). We call it the **residue action**.

Thus, taking all $n \in \mathbb{N}$ into account, we see that the complex analytic Def. 11.6 of conformal blocks is equivalent to the following algebraic one:

Definition 11.17 (Algebraic version)**.** A linear functional $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ is called a **conformal block** associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$ if it vanishes on the following subspace

$$H^0\big(C, \mathscr{V}_C \otimes \omega_C(\star x_\bullet)\big) \cdot \mathbb{W}_\bullet \tag{11.32}$$

of $\mathbb{W}_\bullet$, where we have suppressed $\mathrm{Span}_{\mathbb{C}}$ in (11.32).

Definition 11.18. The vector space

$$\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet) = \frac{\mathbb{W}_\bullet}{H^0\big(C, \mathscr{V}_C \otimes \omega_C(\star x_\bullet)\big) \cdot \mathbb{W}_\bullet} \tag{11.33}$$

is called the **space of coinvariants** (also called space of covacua) associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$. Its dual space is denoted by $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)$ and called the **space of conformal blocks** (or space of vacua, space of invariants).

12 Pushforward and Lie derivatives in sheaves of VOAs

12.1

We continue our discussions in the previous section. The residue action of σ on w_i is crucial in the theory conformal blocks. Let us present its definition in a form that indicates the choice of local coordinate η_i.

We now only assume that σ is a section of $\mathscr{V}_C \otimes \omega_C(\star x_\bullet)$ defined on a neighborhood of x_i, say on U_i. (Namely, σ is a section of $\mathscr{V}_C \otimes \omega_C$ on $U_i \backslash \{x_i\}$ with finite poles at x_i.) Let $\mathcal{V}_\varrho(\eta_i)\sigma$ be $v_i(z)dz$ in (11.30). Then (11.30) reads

$$\sigma \cdot w_i = \mathrm{Res}_{z=0}\, Y(\mathcal{V}_\varrho(\eta_i)\sigma, z)w_i. \tag{12.1}$$

Let us describe $\mathcal{V}_\varrho(\eta_i)$ in a more geometric way. Notice that we have an obvious equivalence

$$(\eta_i)_* : \mathscr{O}_{U_i} \xrightarrow{\simeq} \mathscr{O}_{\eta_i(U_i)}$$

sending f to $f \circ \eta_i^{-1}$. Then $\mathbf{1}_{\mathbb{V}} \otimes (\eta_i)_* : \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{U_i} \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\eta_i(U_i)}$. We define the **pushforward**

$$\begin{gathered} \mathcal{V}_\varrho(\eta_i) : \mathscr{V}_{U_i} \xrightarrow{\simeq} \mathbb{V} \otimes \mathscr{O}_{\eta_i(U_i)} \\ \mathcal{V}_\varrho(\eta_i) = (\mathbf{1}_{\mathbb{V}} \otimes (\eta_i)_*)\mathcal{U}_\varrho(\eta_i) \end{gathered} \tag{12.2}$$

Its tensor product with $(\eta_i)_* = (\eta_i^{-1})^* : \omega_{U_i} \xrightarrow{\simeq} \omega_{\eta_i(U_i)}$ is also denoted by $\mathcal{V}_\varrho(\eta_i)$:

$$\mathcal{V}_\varrho(\eta_i) \equiv \mathcal{V}_\varrho(\eta_i) \otimes (\eta_i)_* : \mathscr{V}_{U_i} \otimes \omega_{U_i}(\star x_i) \xrightarrow{\simeq} \mathbb{V} \otimes \omega_{\eta_i(U_i)}(\star 0). \tag{12.3}$$

12.2

The above geometric description is convenient when treating simultaneously more than one local coordinate at x_i and the corresponding trivializations. As an application, let us show that the action of σ on $\mathbb{W}_i$ can be formulated in a coordinate-independent way.

From now on, we do not fix the local coordinates of $\mathfrak{X} = (C; x_1, \ldots, x_N)$. Let $\mathscr{W}(\mathbb{W}_i)$ be an abstract vector space isomorphic to $\mathbb{W}_i$. To be more precise, we consider $\mathscr{W}(\mathbb{W}_i)$ as a (infinite rank) vector bundle over a single point with trivialization

$$\mathcal{U}(\eta_i) : \mathscr{W}(\mathbb{W}_i) \xrightarrow{\simeq} \mathbb{W}_i \tag{12.4}$$

for any local coordinate η_i of C at x_i, such that if μ_i is also a local coodinate at x_i, then the transition function is

$$\mathcal{U}(\eta_i)\mathcal{U}(\mu_i)^{-1} = \mathcal{U}(\eta_i \circ \mu_i^{-1}) : \mathbb{W}_i \xrightarrow{\simeq} \mathbb{W}_i. \tag{12.5}$$

Note that $\eta_i \circ \mu_i^{-1} \in \mathbb{G}$ is the change of coordinate from μ_i to η_i, and $\mathcal{U}(\eta_i \circ \mu_i^{-1})$ is the corresponding invertible operator on $\mathbb{W}_i$ defined by (10.8).

For each $\sigma \in H^0(U_i, \mathscr{V}_{U_i} \otimes \omega_{U_i}(\star x_i))$ and $\mathbf{w} \in \mathscr{W}(\mathbb{W}_i)$, define

$$\sigma \cdot \mathbf{w} = \mathcal{U}(\eta_i)^{-1} \cdot \sigma \cdot \mathcal{U}(\eta_i)\mathbf{w} \tag{12.6}$$

where the action of σ on $\mathcal{U}(\eta_i)\mathbf{w}$ is defined by (12.1).

12.3

Proposition 12.1. *The definition of residue action $\sigma \cdot \mathbf{w}$ is independent of the choice of local coordinates η_i at x_i.*

The proof of this proposition is a good exercise of computing $\mathcal{V}_\varrho(\eta)\sigma$ when $\eta, \mathscr{V}_{U_i}, \mathscr{W}(\mathbb{W}_i)$ are not identified with the standard ones using the trivializations.

Proof. Write $x_i = x, U_i = U, \mathbb{W}_i = \mathbb{W}$ for simplicity. Let $\mu, \eta \in \mathscr{O}(U)$ be coordinates of U at x. (So $\eta(x) = \mu(x) = 0$.) Identify U with $\mu(U)$ via μ so that μ is identified with the standard coordinate $\mathbf{1}_{\mathbb{C}}$ of $\mathbb{C}$. We have $\eta \in \mathbb{G}$. Identify $\mathscr{W}(\mathbb{W})$ with $\mathbb{W}$ via $\mathcal{U}(\mu)$. So $\mathcal{U}(\mu) = \mathbf{1}$, and $\mathcal{U}(\eta) : \mathscr{W}(\mathbb{W}) \to \mathbb{W}$ agrees with the operator associated with the transformation η. We write $\mathbf{w} \in \mathscr{W}(\mathbb{W})$ as $w \in \mathbb{W}$.

Due to the above identifications, we have $\mu_* = \mathbf{1}$ and hence $\mathcal{V}_\varrho(\mu) = \mathcal{U}_\varrho(\mu)$. Write

$$\mathcal{V}_\varrho(\mu)\sigma = \mathcal{U}_\varrho(\mu)\sigma = u(z)dz$$

where $u = u(z)$ belongs to $H^0(U, \mathbb{V} \otimes \mathscr{O}_U(\star 0))$. So the action $\sigma \cdot w$ defined by μ is simply $\mathrm{Res}_{z=0} Y(u(z), z) w dz$.

Let us compute $\sigma \cdot w$ using η. In view of (12.1) and (12.6), we compute $\mathcal{V}_\varrho(\eta)\sigma$. First,

$$\mathcal{U}_\varrho(\eta)\sigma = \mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1}u(z)dz = \mathcal{U}(\varrho(\eta|\mu))u(z)dz = \mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}})_z)u(z)dz.$$

Here z is the standard variable of $\mathbb{C}$. Applying $\eta_* = (\eta^{-1})^*$, we get

$$\mathcal{V}_\varrho(\eta)\sigma = \mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}})_{\eta^{-1}(z)})u(\eta^{-1}(z))d\eta^{-1}(z)$$

defined on $\eta(U) \subset \mathbb{C}$. Thus, when evaluated with any vector $w' \in \mathbb{W}'$, we have

$$\mathcal{U}(\eta)^{-1} \cdot \sigma \cdot \mathcal{U}(\eta)w = \sum_{n\in\mathbb{N}} \mathcal{U}(\eta)^{-1} P_n \cdot \sigma \cdot \mathcal{U}(\eta)w$$
$$= \sum_{n\in\mathbb{N}} \mathrm{Res}_{z=0}\, \mathcal{U}(\eta)^{-1} P_n Y\big(\mathcal{V}_\varrho(\eta)\sigma, z\big)\mathcal{U}(\eta)w$$
$$= \sum_{n\in\mathbb{N}} \mathrm{Res}_{z=0}\, \underbrace{\mathcal{U}(\eta)^{-1} P_n Y\big(\mathcal{U}(\varrho(\eta|\mathbf{1}_{\mathbb{C}})_{\eta^{-1}(z)})u(\eta^{-1}(z)), z\big)\mathcal{U}(\eta)w}_{A_n} \cdot d\eta^{-1}(z).$$

By the change of coordiante Thm. 10.7, $\sum_n \langle w', A_n\rangle$ converges a.l.u. when $z \neq 0$ is small. Thus we can move the infinite sum into the residue, and by Thm. 10.7 again, the above equals

$$\mathrm{Res}_{z=0}\, Y(u(\eta^{-1}(z)), \eta^{-1}(z))w \cdot d\eta^{-1}(z) \xlongequal{\zeta=\eta^{-1}(z)} \mathrm{Res}_{\zeta=0}\, Y(u(\zeta), \zeta)w \cdot d\zeta.$$

This finishes the proof. □

12.4

We are now ready to give a coordinate independent definition of conformal blocks. Let $\mathfrak{X} = (C; x_\bullet)$ be an N-pointed compact Riemann surface, for which we do not fix local coordinates. Again, we associate admissible $\mathbb{V}$-modules $\mathbb{W}_\bullet$ to the markd points $x_\bullet$. Let

$$\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) = \mathscr{W}(\mathbb{W}_1) \otimes \cdots \otimes \mathscr{W}(\mathbb{W}_N). \tag{12.7}$$

Then for each choice of local coordinates $\eta_\bullet$, we have trivialization

$$\mathcal{U}(\eta_\bullet) := \mathcal{U}(\eta_1) \otimes \cdots \otimes \mathcal{U}(\eta_N) : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \xrightarrow{\simeq} \mathbb{W}_\bullet. \tag{12.8}$$

If $\mu_\bullet$ is another set of local coordinates, then we have transition function

$$\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} = \mathcal{U}(\eta_\bullet \circ \mu_\bullet^{-1}) := \mathcal{U}(\eta_1 \circ \mu_1^{-1}) \otimes \cdots \otimes \mathcal{U}(\eta_N \circ \mu_N^{-1}). \tag{12.9}$$

For each $\mathbf{w}_\bullet = \mathbf{w}_1 \otimes \cdots \mathbf{w}_N \in \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ and $\sigma \in H^0(C; \mathscr{V}_C \otimes \omega_C(\star x_\bullet))$, define the residue action

$$\sigma \cdot \mathbf{w}_\bullet = \sum_{i=1}^N \mathbf{w}_1 \otimes \cdots \otimes \sigma \cdot \mathbf{w}_i \otimes \cdots \otimes \mathbf{w}_N \tag{12.10}$$

(where each $\sigma \cdot \mathbf{w}_i$ is defined by (12.6)). This gives a linear action of $H^0(C; \mathscr{V}_C \otimes \omega_C(\star x_\bullet))$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$.

Definition 12.2. The vector space

$$\mathscr{T}_{\mathfrak{X}}(\mathbb{W}_\bullet) = \frac{\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)}{H^0\big(C, \mathscr{V}_C \otimes \omega_C(\star x_\bullet)\big) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)} \tag{12.11}$$

and its dual space $\mathscr{T}^*_{\mathfrak{X}}(\mathbb{W}_\bullet)$ are called respectively the **space of coinvariants** and the **space of conformal blocks** associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$.

12.5

Let us generalize the pushforward in Subsec. 12.1 to a more general geometric setting. Let X and Y be (non-necessarily compact) Riemann surfaces, and let $\varphi : X \xrightarrow{\simeq} Y$ be a bi-holomorphism. Let

$$\varphi_* : \mathscr{O}_X \to \mathscr{O}_Y, \qquad f \mapsto f \circ \varphi^{-1} \tag{12.12}$$

be the pushforward of the structure sheaves. We let φ_* also denote

$$\varphi_* \equiv \mathbf{1}_{\mathbb{V}} \otimes \varphi_* : \mathbb{V} \otimes \mathscr{O}_X \xrightarrow{\simeq} \mathbb{V} \otimes \mathscr{O}_Y. \tag{12.13}$$

Let $U \subset X$ and $V \subset Y$ be open and connected such that $V = \varphi(U)$. Suppose there is a univalent $\eta \in \mathscr{O}(Y)$. Recall that we have an equivalence

$$\mathcal{V}_\varrho(\eta) = \eta_* \cdot \mathcal{U}_\varrho(\eta) : \mathscr{V}_V \xrightarrow{\simeq} \mathbb{V} \otimes \mathscr{O}_{\eta(V)} \tag{12.14}$$

where the pushforward $\eta_* : \mathbb{V} \otimes \mathscr{O}_V \to \mathbb{V} \otimes \mathscr{O}_{\eta(V)}$ is similar to (12.13). We define

$$\mathcal{V}_\varrho(\varphi) : \mathscr{V}_U \xrightarrow{\simeq} \mathscr{V}_V, \qquad \mathcal{V}_\varrho(\eta)\mathcal{V}_\varrho(\varphi) = \mathcal{V}_\varrho(\eta \circ \varphi). \tag{12.15}$$

Equivalently,

$$\mathcal{U}_\varrho(\eta)\mathcal{V}_\varrho(\varphi) = \varphi_* \cdot \mathcal{U}_\varrho(\eta \circ \varphi). \tag{12.16}$$

Proof. Note that $\mathcal{V}_\varrho(\eta) = \eta_* \cdot \mathcal{U}_\varrho(\eta)$, $\mathcal{V}_\varrho(\eta\circ\varphi) = (\eta\circ\varphi)_* \cdot \mathcal{U}_\varrho(\eta\circ\varphi)$, and $(\eta\circ\varphi)_* = \eta_* \cdot \varphi_*$. $\square$

Lemma 12.3. *The definition of $\mathcal{V}_\varrho(\varphi)$ is independent of the choice of univalent map η.*

Proof. Let $\mu \in \mathscr{O}(V)$ be univalent. Using (11.9), one checks easily that

$$\mathcal{U}(\varrho(\eta \circ \varphi | \mu \circ \varphi)) = \varphi_*^{-1} \cdot \mathcal{U}(\varrho(\eta|\mu)) \cdot \varphi_*$$

as morphisms $\mathbb{V} \otimes \mathscr{O}_U \to \mathbb{V} \otimes \mathscr{O}_U$. This means

$$\mathcal{U}_\varrho(\eta \circ \varphi)\mathcal{U}_\varrho(\mu \circ \varphi)^{-1} = \varphi_*^{-1} \cdot \mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} \cdot \varphi_*. \tag{12.17}$$

The independence follows immediately from the above formula and (12.16). $\square$

By this lemma, we have a global equivalence

$$\mathcal{V}_\varrho(\varphi) : \mathscr{V}_X \xrightarrow{\simeq} \mathscr{V}_Y \tag{12.18}$$

defined locally by (12.15) or (12.16). We call $\mathcal{V}_\varrho(\varphi)$ the **pushforward** associated to φ. We also use $\mathcal{V}_\varrho(\varphi)$ to denote

$$\mathcal{V}_\varrho(\varphi) \equiv \mathcal{V}_\varrho(\varphi) \otimes \varphi_* : \mathscr{V}_X \otimes \omega_X \xrightarrow{\simeq} \mathscr{V}_Y \otimes \omega_Y \tag{12.19}$$

where φ_* is $(\varphi^*)^{-1} = (\varphi^{-1})^* : \omega_X \to \omega_Y$.

12.6

Remark 12.4. From (12.15), it is clear that $\mathcal{V}_\varrho(\psi \circ \varphi) = \mathcal{V}_\varrho(\psi)\mathcal{V}_\varrho(\varphi)$ if $\psi : Y \to Z$ is a bi-holomorphism of complex manifolds.

Remark 12.5. The geometric meanings of $\mathcal{V}_\varrho(\varphi) : \mathscr{V}_X \to \mathscr{V}_Y$ and the formula (12.16) are as follows. Let $x \in X$. Choose a vector $\mathbf{u}$ in the fiber $\mathscr{V}_X|x$, considered an abstract VOA vector. Let $\mathbf{v} = \mathcal{V}_\varrho(\varphi)\mathbf{u}$. Then by (12.16) and the geometric meanings of $\mathcal{U}_\varrho(\eta)$ and $\mathcal{U}_\varrho(\mu)$ (cf. Rem. 11.4), $\mathbf{u}$ and $\mathbf{v}$ are related by the property that for any univalent η holomorphic on a neighborhood y, if we set $\mu = \eta \circ \varphi$, then the coordinate representation of $\mathbf{u}$ under $\mu - \mu(x)$ is the same as that of $\mathbf{v}$ under $\eta - \eta(y)$.

We will simply say that the μ-trivialization of $\mathbf{u}$ and the η-trvialization of $\mathbf{v}$ are equal.

Remark 12.6. Now $\mathcal{V}_\varrho(\eta)$ has two meanings: as an equivalence $\mathscr{V}_V \to \mathbb{V}\otimes\mathscr{O}_{\eta(V)}$ defined by (12.14), and as an equivalence $\mathscr{V}_V \to \mathscr{V}_{\eta(V)}$ defined similar to $\mathcal{V}_\varrho(\varphi)$. These two meanings agree if we identify $\mathscr{V}_{\eta(V)}$ with $\mathbb{V} \otimes \mathscr{O}_{\eta(V)}$ via the trivialization $\mathcal{U}_\varrho(\zeta)$ where ζ is the standard coordinate of $\mathbb{C}$.

12.7

That one can define pushforward for (co)tangent bundles as well as for sheaves of VOAs implies that these two classes of objects are closely related. Indeed, one can view $\mathscr{V}_C$ as a twisted direct sum of tensor products of the holomorphic tangent line bundle Θ_C of C. (Note that ω_C is the dual of Θ_C.)

To see this, let us look at the transition function $\mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes \mathscr{O}_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes \mathscr{O}_U$ where $\mu, \eta \in \mathscr{O}(U)$ are univalent. By (10.8), $\mathcal{U}(\varrho(\eta|\mu))_x = \varrho(\eta|\mu)'_x(0)^{L_0}(1 +$ products of $L_{>0})$ on $\mathbb{V}$. From (11.9), $\varrho(\eta|\mu)'_x(0) = \frac{\partial\eta}{\partial\mu}(x)$. Thus, as $L_{>0}$ lowers weights, we conclude that for each $v = v(x) \in \mathbb{V}^{\leqslant n} \otimes \mathscr{O}_U$,

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1}v = \mathcal{U}(\varrho(\eta|\mu))v = (\partial\eta/\partial\mu)^n v \qquad \mod \mathbb{V}^{\leqslant n-1} \otimes \mathscr{O}_U. \tag{12.20}$$

Thus, the transition function $\mathcal{U}(\varrho(\eta|\mu))$ from the μ-coordinate to the η-coordinate for the quotient bundle $\mathscr{V}_C^{\leqslant n}/\mathscr{V}_C^{\leqslant n-1}$ is $(\partial\eta/\partial\mu)^n$, which agrees that of $\mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes n}$. We conclude:

Proposition 12.7. *There is an equivalence of $\mathscr{O}_C$-modules*

$$\mathscr{V}_C^{\leqslant n}/\mathscr{V}_C^{\leqslant n-1} \simeq \mathbb{V}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes n} \tag{12.21}$$

such that if $U \subset C$ is open and $\eta \in \mathscr{O}(U)$ is univalent, then for each $v \in \mathbb{V}(n)$, $v \otimes \partial_\eta^n$ (which is an element in the RHS of (12.21)) is equivalent to the equivalence class of $\mathcal{U}_\varrho(\eta)^{-1}v$ in the LHS of (12.21).

Thus, in general, an element of $\mathbb{V}(n) \otimes \Theta_C^{\otimes n}(U)$ is a sum of those of the form $v \otimes f\partial_\eta^n$ where $v \in \mathbb{V}(n)$ and $f \in \mathscr{O}(U)$. It is identified with $f \cdot \mathcal{U}_\varrho(\eta)^{-1}v$ in the LHS of (12.21).

12.8

If we focus on only primary vectors, we can get subbundles of $\mathscr{V}_C$ naturally equivalent to direct sums of tensor products of Θ_C without taking quotient. Recall that a primary vector v in $\mathbb{V}(n)$ is one killed by $L_{>0}$. So the change of coordinate formula for v is $\mathcal{U}(\varrho(\eta|\mu))v = (\partial\eta/\partial\mu)^n v$. Thus, if we let $\mathbf{P}(n)$ be the subspace of weight n primary vectors of $\mathbb{V}$, then $\mathscr{V}_C$ has a vector subbundle $\mathscr{P}_C^n$ with local trivialization $\mathcal{U}_\varrho(\eta) : \mathscr{P}_C^n|_U \xrightarrow{\simeq} \mathbf{P}(n) \otimes_{\mathbb{C}} \mathscr{O}_U$ for any univalent $\eta \in \mathscr{O}(U)$. Moreover, $\mathscr{P}_C^n$ has the same transition functions as $\Theta_C^{\otimes n}$. So $\mathscr{P}_C^n \simeq \mathbf{P}(n) \otimes_{\mathbb{C}} \Theta_C^{\otimes n}$.

Since the basic properties of line bundles $\Theta_C^{\otimes n}$ are well known, in the early development of the mathematical theory of conformal blocks, sheaves of VOAs were not yet defined, and the sheaves $\mathscr{P}_C^n$ were sometimes used instead to define and study conformal blocks. Specifically, in the landmark paper [TUY89], conformal blocks for a WZW model $\mathbb{V} = L_l(\mathfrak{g}, 0)$ (where $\mathfrak{g}$ is simple and $l \in \mathbb{N}$) was defined using

$$\mathscr{P}_C^1 \otimes \omega_C(\star x_\bullet) \simeq \mathbf{P}(1) \otimes_{\mathbb{C}} \Theta_C \otimes \omega_C(\star x_\bullet) = \mathfrak{g} \otimes_{\mathbb{C}} \mathscr{O}_C(\star x_\bullet).$$

(Note that $\Theta_C \otimes \omega_C \simeq \mathscr{O}_C$ since ω is dual to Θ_C.) Thus, for WZW models, the space of vacuua was defined (for $\mathfrak{X}$ with local coordinates) in [TUY89] to be

$$\frac{\mathbb{W}_\bullet}{H^0\big(C, \mathfrak{g} \otimes_{\mathbb{C}} \mathscr{O}_C(\star x_\bullet)\big) \cdot \mathbb{W}_\bullet}$$

Fortunately, this definition agrees with the one defined using $H^0(C, \mathscr{V}_C \otimes \omega_C(\star x_\bullet))$. See [FB04, Sec. 9.3].

12.9

In differential geometry, the Lie derivatives of sections of (tensor products of) tangent and cotangent bundles are defined using the pushforward or the pullback maps associated to flows. Likewise, we can define Lie derivatives for sections of $\mathscr{V}_C$.

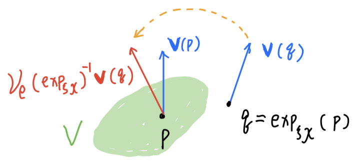
Let $W \subset C$ be an open subset, and choose $\mathfrak{x} \in \Theta_C(W)$, namely, $\mathfrak{X}$ is a holomorphic tangent field on W. Note that for any precompact open subset $V \subset W$ (i.e., the closure of V in W is compact), there is a neighborhood $T \subset \mathbb{C}$ of 0 (with variable ζ) such that the holomorphic flow $\exp(\zeta\mathfrak{x})$ is holomorphic on $T \times V$ and is injective as a function on V for each $\zeta \in T$. (Cf. Subsec. 2.6.)

In the following, we write $\exp(\zeta\mathfrak{x})(x)$ as $\exp_{\zeta\mathfrak{x}}(x)$.

Definition 12.8. For any $\mathbf{v} \in \mathscr{V}_C^{\leq n}(W)$ and $\mathfrak{x} \in \Theta_C(W)$, define the **Lie derivative** $\mathcal{L}_{\mathfrak{x}}\mathbf{v}$ to be an element of $\mathscr{V}_C^{\leq n}(W)$ (if the limit exists) as follows . Choose any precompact open subset V in W. Then

$$\mathcal{L}_{\mathfrak{x}}\mathbf{v}\big|_V = \lim_{\zeta\to 0} \frac{\mathcal{V}_\varrho(\exp_{\zeta\mathfrak{x}})^{-1}\big(\mathbf{v}\big|_{\exp_{\zeta\mathfrak{x}}(V)}\big) - \mathbf{v}\big|_V}{\zeta} \tag{12.22}$$

Intuition: For each $p \in V$, $\mathbf{v}(p) \in \mathscr{V}_C^{\leq n}|p$ is an abstract VOA vector at p. Let $q = \exp_{\zeta\mathfrak{x}}(p)$. Then $\mathbf{v}(q) \in \mathscr{V}_C^{\leq n}|q$ is an abstract VOA vector at $\mathbf{v}(q)$, which is pulled back to the vector $\mathcal{V}_\varrho(\exp_{\zeta\mathfrak{x}})^{-1}\mathbf{v}(q) \in \mathscr{V}_C^{\leqslant n}|p$ via the map $\exp_{\zeta\mathfrak{x}}$.



(12.23)

Then for small ζ,

$$(\mathcal{L}_\mathfrak{x}\mathbf{v})(p) \approx \frac{\mathcal{V}_\varrho(\exp_{\zeta\mathfrak{x}})^{-1}\mathbf{v}(q) - \mathbf{v}(p)}{\zeta} \tag{12.24}$$

12.10

Proposition 12.9. *Assume that $\eta \in \mathscr{O}(W)$ is univalent, and set*

$$u = \mathcal{U}_\varrho(\eta)\mathbf{v} \qquad \in \mathbb{V}^{\leqslant n} \otimes_\mathbb{C} \mathscr{O}(W).$$

Write $\mathfrak{x} = h\partial_\eta$ where $h \in \mathscr{O}(W)$. Then $\mathcal{L}_\mathfrak{x}\mathbf{v}$ exists (i.e. the limit on the RHS of (12.22) *exists) as an element of $\mathscr{V}^{\leqslant n}(W)$, and its η-trivialization is*

$$\mathcal{U}_\varrho(\eta)\mathcal{L}_\mathfrak{x}\mathbf{v} = h\partial_\eta u - \sum_{k\geqslant 1}\frac{1}{k!}\partial_\eta^k h \cdot L_{k-1}u. \tag{12.25}$$

Proof. We need to find the η-trivialization of $\mathcal{V}_\varrho(\exp_{\zeta\mathfrak{x}})^{-1}\big(\mathbf{v}|_{\exp_{\zeta\mathfrak{x}}(V)}\big)$ at any $p \in V$, namely, the η-trivialization of the red vector in (12.23). Since the η-trivialization of $\mathbf{v}(q)$ is $u(q)$, by (12.16) or Rem. 12.5, the $\eta \circ \exp_{\zeta\mathfrak{x}}$-trivialization of the red vector is also $u(q)$. So the η-trivialization of the red vector (which is at p) is

$$\mathcal{U}(\varrho(\eta|\eta \circ \exp_{\zeta\mathfrak{x}})_p)u(\exp_{\zeta\mathfrak{x}}(p)). \tag{12.26}$$

Its derivative over ζ at $\zeta = 0$ gives $\mathcal{L}_\mathfrak{x}\mathbf{v}(p)$ under the η-trivialization. (The readers can check [Gui, Sec. 2.6] if they are not satisfied with the rigorousness of the proof here.)

The derivative at $\zeta = 0$ of $u(\exp_{\zeta\mathfrak{x}}(p))$ is just the action of the vector field $\mathfrak{x}$ on u, namely $h\partial_\eta u$ at p. (Notice (2.9).) The derivative of $\mathcal{U}(\varrho(\eta|\eta \circ \exp_{\zeta\mathfrak{x}})_p)$ at 0 can be calculated using Prop. 10.3: if we identify η with the standard coordinate of $\mathbb{C}$, then

$$\begin{aligned}&\partial_\zeta\mathcal{U}(\varrho(\eta|\eta \circ \exp_{\zeta\mathfrak{x}})_p)(t)\big|_{\zeta=0} \xlongequal{(11.10)} \partial_\zeta\big(\exp_{-\zeta\mathfrak{x}}(t + \exp_{\zeta\mathfrak{x}}(p)) - p\big)\big|_{\zeta=0}\\ =&-h(t+p) + h(p).\end{aligned}$$

Its k-th derivative over t at $t = 0$ is then $-\partial_\eta^k h(p)$. Thus, by Prop. 10.3,

$$\partial_\zeta\mathcal{U}(\varrho(\eta|\eta \circ \exp_{\zeta\mathfrak{x}})_p)\big|_{\zeta=0} = -\sum_{k\geqslant 1}\frac{1}{k!}\partial_\eta^k h \cdot L_{k-1}.$$

□

12.11

In Prop. 12.9, if we assume that $u \in \mathbf{P}(n) \otimes_{\mathbb{C}} \mathscr{O}(W)$, i.e., the values of u are primary with weights n, then the Lie derivative formula is $h\partial_\eta u - n\partial_\eta h \cdot u$. Not surprisingly, this result agrees with the formula of Lie derivatives in $\Theta_C^{\otimes n}$, including the case $n = -m < 0$ where we understand $\Theta_C^{\otimes(-m)} = \omega_C^{\otimes m}$.

Since we have pushforward for sections of $\mathscr{V}_C^{\leqslant n} \otimes \omega_C$ (cf. (12.19)), we can also define Lie derivatives in this bundle using the same formula (12.22). The result is easy to guess by Leibniz rule and prove rigorously:

Corollary 12.10. *Let $\sigma \in \mathscr{V}_C^{\leqslant n} \otimes \omega_C(W)$, and set*

$$u \cdot d\eta = \mathcal{U}_\varrho(\eta)\sigma \qquad \in \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \omega_C(W)$$

where $u \in \mathbb{V}^{\leqslant n} \in \mathscr{O}(W)$. Write $\mathfrak{x} = h\partial_\eta$ where $h \in \mathscr{O}(W)$. Then

$$\mathcal{U}_\varrho(\eta)\mathcal{L}_{\mathfrak{x}}\sigma = h\partial_\eta u \cdot d\eta - \sum_{k\geqslant 1} \frac{1}{k!}\partial_\eta^k h \cdot L_{k-1}u \cdot d\eta + \partial_\eta h \cdot u \cdot d\eta. \tag{12.27}$$

13 Families of compact Riemann surfaces and parallel sections of conformal blocks

13.1

Definition 13.1. A **family of compact Riemann surfaces** is the data $\pi : \mathcal{C} \to \mathcal{B}$ where $\mathcal{B}, \mathcal{C}$ are Riemann surfaces, the surjective holomorphic map π is proper (i.e. π^{-1}(compact) is compact) and a submersion (i.e. the linear map $d\pi$ between holomorphic tangent spaces is everywhere surjective), and for each $b \in \mathcal{B}$ the fiber $\mathcal{C}_b = \pi^{-1}(b)$ is a compact Riemann surface. Clearly, π is an open map.

By Ehresmann's fibration theorem, if $\mathcal{B}$ is connected, then all fibers of the family are diffeomorphic; moreover, as a family of differential manifolds, $\pi : \mathcal{C} \to \mathcal{B}$ is locally trivial, i.e. as a projection of $C \times V \to V$ when $V \subset \mathcal{B}$ is open and C is a surface. However, it is not locally trivial as a family of complex manifolds.

Definition 13.2. A **family of N-pointed compact Riemann surfaces** is the data $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \dots, \varsigma_N)$ where $\pi : \mathcal{C} \to \mathcal{B}$ is a family of compact Riemann surfaces, and the following conditions hold:

(a) Each $\varsigma_i : \mathcal{B} \to \mathcal{C}$ is a section, i.e., a holomorphic map such that $\pi \circ \varsigma_i = \mathbf{1}_{\mathcal{B}}$. (So $\varsigma_i(b)$ is are points on the fiber $\mathcal{C}_b$.)

(b) $\varsigma_1(b), \dots, \varsigma_N(b)$ are distinct, considered as marked points of each fiber $\mathcal{C}_b$.

(c) Each connected component of each fiber $\mathcal{C}_b$ contains at least one of $\varsigma_1(b), \dots, \varsigma_N(b)$.

The following is a hypersurface in $\mathcal{C}$.

$$S_{\mathfrak{X}} = \bigcup_{j=1}^{N} \varsigma_j(\mathcal{B}) \tag{13.1}$$

A **local coordinate** η_i of the family at ς_i is a holomorphic function on a neighborhood U_i of $\varsigma_i(\mathcal{B})$ that restricts to a local coordinate $\eta_i|_{\mathcal{C}_b \cap U_i}$ of $\mathcal{C}_b$ at $\varsigma_i(b)$ for each $b \in \mathcal{B}$, i.e., $\eta_i(\varsigma_i(b)) = 0$ and η_i is injective on the fiber

$$U_{i,b} = \mathcal{C}_b \cap U_i.$$

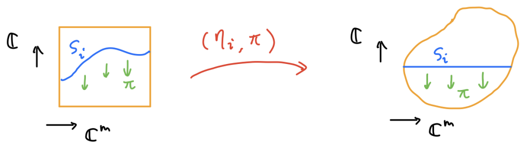
We call the data $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \ldots, \varsigma_N; \eta_1, \ldots, \eta_N)$ a **family of N-pointed compact Riemann surfaces with local coordinates.** We define the fiber

$$\mathfrak{X}_b = (\mathcal{C}_b; \varsigma_i(b), \ldots, \varsigma_N(b); \eta_1|_{\mathcal{C}_b}, \ldots, \eta_N|_{\mathcal{C}_b}) \tag{13.2}$$

which is an N-pointed compact Riemann surface with local coordinates. □

13.2

Since π is a submersion, on a neighborhood of $p \in \varsigma_i(\mathcal{B})$, π is equivalent to the projection $D \times V \to V$ where $D \subset \mathbb{C}, V \in \mathbb{C}^m$ are open. So ς_i restricted to $V \subset \mathcal{B}$ is written as $\varsigma_i(b) = (\sigma_i(b), b)$ where $\sigma_i : V \to D$ is holomorphic. Namely, $\varsigma_i|_V$ is the graph of σ_i.



(13.3)

By the fact that η_i is injective on each fiber, $\partial_{z_1}\eta_i$ is nowhere zero where z_1 is the coordinate for D. So the Jacobian of (η_i, π) is nowhere zero. Thus, by the inverse mapping theorem, together with the easy fact that (η_i, π) is injective on U_i, we see that *(η_i, π) is a biholomorphism from U_i to a neighborhood of $\{0\} \times \mathcal{B}$ in $\mathbb{C} \times \mathcal{B}$.*

Thus, by identifying U_i with its image W (which is a neighborhood of $\{0\} \times \mathcal{B}$) under (η_i, π), we may assume that π is the projection of W onto $\mathcal{B}$, ς_i is the canonical map $\mathcal{B} \to \{0\} \times \mathcal{B}$, and η_i is the projection of $W \subset \mathbb{C} \times \mathcal{B}$ onto the $\mathbb{C}$-axis.

13.3

Example 13.3. Let C be a connected compact Riemann surface. Then

$$\mathfrak{X} = (\pi : C \times \mathrm{Conf}^N(C) \to \mathrm{Conf}^N(C); \varsigma_1, \ldots, \varsigma_N)$$

is a family of N-pointed compact Riemann surface, where π is the projection onto the second component, and $\varsigma_i : \mathrm{Conf}^N(C) \to C \times \mathrm{Conf}^N(C)$ sends each $(x_1, \ldots, x_N)$ to $(x_i, x_1, \ldots, x_N)$. The fibers are $\mathfrak{X}_{x_\bullet} = (C; x_1, \ldots, x_N)$.

Example 13.4. Let $\mathfrak{X} = (\pi : \mathbb{P}^1 \times \mathrm{Conf}^N(\mathbb{C}^\times) \to \mathrm{Conf}^N(\mathbb{C}^\times); 0, \varsigma_1, \ldots, \varsigma_N, \infty)$ where $0, \infty$ as sections sending $x_\bullet$ to $(0, x_\bullet)$ and $(\infty, x_\bullet)$ respectively, and ς_i is as in the previous example. Then $\mathfrak{X}$ is $(N+2)$-pointed. Moreover, $\mathfrak{X}$ can be equipped with local coordinates $\zeta, \eta_1, \ldots, \eta_N, 1/\zeta$ at $0, \varsigma_1, \ldots, \varsigma_N, \infty$ respectively, where ζ sends $(z, z_\bullet)$ to z, $1/\zeta$ sends $(z, z_\bullet)$ to $1/z$, and each η_i sends $(z, z_\bullet)$ to $z - z_i$. The fibers are

$$\mathfrak{X}_{z_\bullet} = (\mathbb{P}^1; 0, z_1, \ldots, z_N, \infty; \zeta, \zeta - z_1, \ldots, \zeta - z_N, 1/\zeta)$$

where ζ is now the standard coordinate of $\mathbb{C}$.

13.4

Example 13.5. Let

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; x_1, \ldots, x_N, x', x''; \eta_1, \ldots, \eta_N; \xi, \varpi)$$

be an $(N+2)$-pointed compact Riemann surface with local coordinates such that each connected component contains one of $x_1, \ldots, x_N$. Let U', U'' be respectively open discs centered at x', x'' with radii r, ρ. More precisely, we assume ξ, ϖ are defined on U', U'', and we have biholomorphisms

$$\xi : U' \xrightarrow{\simeq} \mathbb{D}_r, \qquad \varpi : U'' \xrightarrow{\simeq} \mathbb{D}_\rho.$$

We assume moreover that $U', U'', x_1, \ldots, x_N$ are mutually disjoint.

For each $q \in \mathbb{D}^\times_{r\rho}$ we can define an N-pointed $\mathfrak{X}_q$ by the sewing operation as follows. We glue the following annuli

$$\begin{gathered} \xi^{-1}(A_{|q|/\rho, r}) = \{x \in U' : |q|/\rho < |\xi(x)| < r\} \\ \Big\updownarrow \text{identify} \\ \varpi^{-1}(A_{|q|/r, \rho}) = \{y \in U'' : |q|/r < |\varpi(y)| < \rho\} \end{gathered} \tag{13.4}$$

where the rule for identification is

$$x = y \qquad \text{iff} \qquad \xi(x)\varpi(y) = q. \tag{13.5}$$

The parts $Z'_q = \{x \in U' : |\xi(x)| \leqslant |q|/\rho\}$ and $Z''_q = \{y \in U'' : |\varpi(y)| \leqslant |q|/r\}$ are discarded. By this gluing procedure we obtain the sewn Riemann surface $\mathcal{C}_q$ with marked points $x_1, \ldots, x_N$ (the same as the first N marked points of $\widetilde{\mathfrak{X}}$). The local coordinate at x_i is also chosen to be η_i. This defines $\mathfrak{X}_q = (\mathcal{C}_q; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$.

One can assemble all $\mathfrak{X}_q$ to form a family

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathbb{D}^\times_{r\rho}; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$$

whose fiber at each $q \in \mathbb{D}^\times_{r\rho}$ is $\mathfrak{X}_q$. (We have abused notations here to let x_i denote a section and η_i a local coordinate at the section x_i.) It could be obtained in the following way:

- We have closed subsets $E' = \bigcup_{q \in \mathbb{D}^\times_{r\rho}} Z'_q \times \{q\}$ and $E'' = \bigcup_{q \in \mathbb{D}^\times_{r\rho}} Z''_q \times \{q\}$ of $\widetilde{C} \times \mathbb{D}^\times_{r\rho}$. Consider the projection

  $$\pi : (\widetilde{C} \times \mathbb{D}^\times_{r\rho}) \backslash (E' \cup E'') \to \mathbb{D}^\times_{r\rho}.$$

  Each x_i is the section sending $q \in \mathcal{D}^\times_\rho$ to (x_i, q), and η_i sends (x, q) to $\eta_i(x)$ when x is close to x_i. Modding this data by a suitable holomorphic relation gives the family $\mathfrak{X}$.

$\square$

In the above example, we can in fact extend $\mathfrak{X}$ to a family over $\mathcal{D}_{r\rho}$ where $\mathfrak{X}_0 = (\mathcal{C}_0; x_\bullet; \eta_\bullet)$ is the "limit" of $\mathfrak{X}_q$ as $q \to 0$. As a topological space, $\mathcal{C}_0$ is obtained by gluing x' and x'' of $\widetilde{C}$. $\mathcal{C}_0$ is not a smooth manifold, and hence cannot be a Riemann surface. However, one can make $\mathcal{C}_0$ a singular complex manifold (more precisely: a complex space) by defining a suitable structure sheaf $\mathscr{O}_{\mathcal{C}_0}$. $\mathcal{C}_0$ is called a **nodal curve**. Nodal curves are crucial to the proof of sewing and factorization of conformal blocks. However, this topic is out of the scope of our notes. We refer the readers to [Gui] for a detailed discussion of this topic.

13.5

Example 13.6. Let $\mathfrak{X}_0 = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ be an N-pointed compact Riemann surface with local coordinates. Write $x_1 = x$ and $\eta_1 = \eta$ for simplicity. Let η be defined on a neighborhood $U = U_1 \ni x_1$ disjoint from $x_2, \dots, x_N$. Assume that $\eta(U)$ is an open disc centered at 0 with radius > 1.

Let h be a holomorphic function on a neighborhood of $\mathbb{S}^1$. Then $\mathfrak{x} = h\partial_z$ is a holomorphic tangent field near $\mathbb{S}^1$. We choose $0 < r < 1 < R$ such that h is defined on an open set containing the closure of $A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$. Moreover, we choose a connected neighborhood $\Delta \subset \mathbb{C}$ of 0 such that the following hold.

1. There is a neighborhood $\Delta \subset \mathbb{C}$ of 0 such that the holomorphic flow $\tau \in \Delta \mapsto \exp(\tau\mathfrak{x}) = \exp_{\tau\mathfrak{x}}$ is defined on $(z, \tau) \in A_{r,R} \times \Delta$ and is injective on $z \in A_{r,R}$ for any fixed τ. (Cf. Subsec. 2.6.)

2. For each $\tau \in \Delta$, we have $0 \notin \exp_{\tau\mathfrak{x}}(\mathbb{S}^1)$.

Let Γ_τ be the simple closed curve $\exp_{\tau\mathfrak{x}} : \mathbb{S}^1 \to \mathbb{C}$. Then by the Jordan curve theorem, for each $\tau \in \Delta$, $\mathbb{P}^1 \backslash \Gamma_\tau$ has two connected components

$$\mathbb{P}^1 \backslash \Gamma_\tau = \Omega_\tau \sqcup \Omega'_\tau$$

where Ω'_τ is the one containing ∞. In the following, we give some technical remarks which can be skipped on first reading:

- By Stokes' theorem, for each $z \in \mathbb{P}^1$, $z \in \Omega_\tau$ (resp. $z \in \Omega'_\tau$) iff $\oint_{\Gamma_\tau} \frac{d\zeta}{\zeta - z}$ equals $2\mathbf{i}\pi$ (resp. 0). This implies that

$$O = \{(z, \tau) \in \mathbb{P}^1 \times \Delta : \exp_{\tau\mathfrak{x}}(z) \in \Omega_\tau\} \qquad O' = \{(z, \tau) \in \mathbb{P}^1 \times \Delta : \exp_{\tau\mathfrak{x}}(z) \in \Omega'_\tau\}$$

  are both closed and open inside $\mathbb{P}^1 \times \Delta$. In summary: the property that z is inside (resp. outside) Γ_τ is continuous with respect to the variation of τ and z.

- Consequently, for each $z \in A_{r,R} \backslash \mathbb{S}^1$, the subset of all $\tau \in \Delta$ such that $\exp_{\tau\mathfrak{x}}(z)$ belongs to Ω_τ (resp. Ω'_τ) is an open subset of Δ, and hence also closed, and hence must be $\varnothing$ or Δ. This shows that for each $z \in A_{r,R}$,

$$\begin{aligned} |z| < 1 \qquad &\Longleftrightarrow \qquad \exp_{\tau\mathfrak{x}}(z) \in \Omega_\tau \text{ for all } \tau \in \Delta \\ |z| > 1 \qquad &\Longleftrightarrow \qquad \exp_{\tau\mathfrak{x}}(z) \in \Omega'_\tau \text{ for all } \tau \in \Delta \end{aligned} \tag{13.6}$$

A similar argument shows that if $z \in \mathbb{P}^1$ and $z \notin \exp_{\tau\mathfrak{r}}(\mathbb{S}^1)$ for all $\tau \in \Delta$, then

$$\begin{aligned} |z| < 1 \qquad &\Longleftrightarrow \qquad z \in \Omega_\tau \text{ for all } \tau \in \Delta \\ |z| > 1 \qquad &\Longleftrightarrow \qquad z \in \Omega'_\tau \text{ for all } \tau \in \Delta \end{aligned} \tag{13.7}$$

In particular, $0 \in \Omega_\tau$ for all $\tau \in \Delta$.

The family $\mathfrak{X}$ we shall construct has base manifold Δ. For each $\tau \in \Delta$, let

$$\mathcal{R}_\tau = \exp_{\tau\mathfrak{r}}(A_{r,R}) \cup \Omega_\tau.$$

Then the fiber $\mathcal{C}_\tau$ is obtained by gluing $C \backslash \eta^{-1}(\mathbb{D}_r^{\mathrm{cl}})$ with $\mathcal{R}_\tau$ by identifying there subsets $\eta^{-1}(A_{r,R})$ and $\exp_{\tau\mathfrak{r}}(A_{r,R})$ via the biholomorphism $\exp_{\tau\mathfrak{r}} \circ \eta$. (We leave it to the readers to check that $\mathcal{C}_\tau$ is a compact Riemann surface. (13.6) is needed when checking the sequential compactness.)

The marked points of $\mathcal{C}_\tau$, together with local coordinates, are chosen to be $0 \in \mathcal{R}_\tau$ with the standard coordinate ζ of $\mathcal{R}_\tau \subset \mathbb{C}$, and $x_2, \dots, x_N \in C \backslash \eta^{-1}(\mathbb{D}_r^{\mathrm{cl}})$ together with $\eta_2, \dots, \eta_N$. This give an N-pointed compact Riemann surface with local coordinates $\mathfrak{X}_\tau$. We leave it to the readers to construct a family $\mathfrak{X}$ over Δ whose fibers are $\mathfrak{X}_\tau$. $\square$

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