NOTES ON THE TRACE OPERATOR

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1. Introduction

In the last year, I have explained to many experts the idea that the trace operator developed in [DX24] could be used to systematically improve inequalities in the literature involving restrictions. In this note, I will explain one example. Probably I will also include other examples later on.

2. Su–Vu's inequality

Let X be a connected compact Kähler manifold of dimension n and α be a nef $(1,1)$-cohomology class on X. Consider a closed positive $(1,1)$-current $T \in \alpha$ such that $\nu(T, W) = 0$ along an irreducible analytic subset $W \subseteq X$. Fix a reference Kähler form ω on X.

In [SV24], Su–Vu proved the following result:

Theorem 2.1. *Let $m \in [1, \dim W]$ be an integer. Then*

$$\frac{m!}{(\dim W - m)!} \sum_{\substack{V \subseteq W \text{ irreducible and analytic} \\ \dim W - \dim V = m \\ \nu(T,V) > 0 \\ V \text{ maximal}}} \nu(T, V)^m \int_V \omega^{\dim V} \leq \int_W \alpha^m \wedge \omega^{\dim V} - \int_W T|_W^m \wedge \omega^{\dim V}. \tag{2.1}$$

We have omitted the obvious restrictions. Here V is called *maximal* if there are no irreducible analytic subsets $V' \subseteq W$ containing V properly such that $\nu(T, V') > 0$.

The restriction $T|_W$ is defined in the usual (naive) way: Write $T = \theta + \mathrm{dd^c}\varphi$ with θ smooth and real and $\varphi \in \mathrm{PSH}(X, \theta)$. We set $T|_W = \theta|_W + \mathrm{dd^c}\varphi|_W$ if $\varphi|_W \not\equiv -\infty$ and $T|_W = 0$ otherwise.

The integral

$$\int_W T|_W^m \wedge \omega^{\dim V}$$

is defined as

$$\int_{\tilde{W}} (\pi^* T|_W)^m \wedge \pi^* \omega^{\dim V},$$

where $\pi \colon \tilde{W} \to W$ is the normalization and the product is the non-pluripolar product.

The formula (2.1) gives very elegant bounds of the the volumes of super level sets of the Lelong numbers. But it is not optimal, as can be seen as follows: The left-hand side of (2.1) is invariant under the $\mathcal{I}$-equivalence relation of T (recall that two closed positive $(1,1)$-currents T, T' in α are $\mathcal{I}$-equivalent if $\mathcal{I}(\lambda T) = \mathcal{I}(\lambda T')$ for all $\lambda > 0$), while the right-hand side is not.

The trace operator provides the natural framework to solve this problem. In fact, we could easily improve (2.1) as follows:

{eq:SV2}

$$\frac{m!}{(\dim W-m)!}\sum_{\substack{V\subseteq W \text{ irreducible and analytic}\\ \dim W-\dim V=m\\ \nu(T,V)>0\\ V \text{ maximal}}}\nu(T,V)^m\int_V\omega^{\dim V}\leq\int_W\alpha^m\wedge\omega^{\dim V}-\int_{\tilde{W}}\mathrm{Tr}_W(T)^m\wedge\pi^*\omega^{\dim V}. \tag{2.2}$$

The right-hand side requires some explanation. The trace operator is defined as in [Xia24], which differs slightly from [DX24]. When $\mathrm{Tr}_W(T)$ admits a representative in $\alpha|_{\tilde{W}}$, it is understood as any such representative. If not, the expression

$$\int_{\tilde{W}}\mathrm{Tr}_W(T)^m\wedge\pi^*\omega^{\dim V}$$

is interpreted as 0.

The argument is quite straightforward. Observe that we only need to handle the case where $\mathrm{Tr}_W(T)$ admits a representative in $\alpha|_{\tilde{W}}$.

We first reduce to the case where T is a Kähler current. This can be done by a simple perturbation on T. Next, take a quasi-equisingular approximation $(T_j)_j$ of T in α. Both sides of (2.2) are continuous along the sequence $T_j\to T$. This is obvious for the left-hand side, while for the right-hand side, it follows from [Xia24, Theorem 6.2.1]. So it suffices to handle the case where T has analytic singularities, in which case we just apply (2.1).

References

DX24 [DX24] T. Darvas and M. Xia. The trace operator of quasi-plurisubharmonic functions on compact Kähler manifolds. 2024. arXiv: 2403.08259 [math.DG].

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