Sewing and Propagation of Conformal Blocks

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Contents

Abstract

Propagation is a standard way of producing new conformal blocks from old ones that corresponds to the geometric procedure of adding new distinct points to a pointed compact Riemann surface. On the other hand, sewing conformal blocks corresponds to sewing compact Riemann surfaces.

In this article, we clarify the relations between these two procedures. Most importantly, we show that "sewing and propagation are commuting procedures". More precisely: let ϕ be a conformal block associated to a vertex operator algebra $\mathbb{V}$ and a compact Riemann surface to be sewn, and let $\wr^n\phi$ be its n-times propagation. If the sewing $\widetilde{\mathcal{S}}\phi$ converges, then $\widetilde{\mathcal{S}}\wr^n\phi$ (the sewing of $\wr^n\phi$) converges to $\wr^n\widetilde{\mathcal{S}}\phi$ (the n-times propagation of the sewing $\widetilde{\mathcal{S}}\phi$).

As an application, we can prove the convergence of sewing conformal blocks in certain important cases without assuming $\mathbb{V}$ to be CFT-type, C_2-cofinite, or rational. We also provide a new method of explicitly constructing the permutation-twisted modules associated to the tensor product VOA $\mathbb{V}^{\otimes k}$, originally due to [BDM02]. Our results are crucial for relating the (genus-0) permutation-twisted $\mathbb{V}^{\otimes k}$-conformal blocks and the untwisted $\mathbb{V}$-conformal blocks (of possibly higher genera) [Gui21].

1 Introduction

Propagating conformal blocks

Let $\mathbb{V}$ be a vertex operator algebra (VOA) with vacuum vector $\mathbf{1}$. Let $\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ be an N-pointed compact Riemann surface with local coordinates, namely, each connected component of the compact Riemann surface C contains at least one of the distinct marked points $x_1, \dots, x_N$, and each η_j is an injective holomorphic function on a neighborhood of x_j sending x_j to 0 (i.e., an (analytic) local coordinate at x_j). Associate to each x_j a $\mathbb{V}$-module $\mathbb{W}_j$. Then a conformal block associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ is a linear functional $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ "invariant" under the actions of $\mathbb{V}$ (Cf. [Zhu94, FB04, DGT19a]). When C is the Riemann sphere $\mathbb{P}^1$, the simplest examples of conformal blocks are as follows. (We let ζ be the standard coordinate of $\mathbb{C}$.)

1. $\mathfrak{X} = (\mathbb{P}^1; 0; \zeta)$, $\mathbb{W}$ is associated to the marked point 0. Then each $T \in \mathrm{Hom}_{\mathbb{V}}(\mathbb{W}, \mathbb{V}')$ (where $\mathbb{V}'$ is the contragredient module of the vacuum $\mathbb{V}$) provides a conformal block

$$w \in \mathbb{W} \mapsto \langle Tw, \mathbf{1}\rangle$$

Here $\langle\cdot,\cdot\rangle$ refers to the standard pairing of $\mathbb{V}$ and $\mathbb{V}'$. Of particular interest is the case that an isomorphism of $\mathbb{V}$-modules $T : \mathbb{V} \xrightarrow{\simeq} \mathbb{V}'$ exists and is fixed. Then there is a canonical conformal block associated to $\mathfrak{X}$ and $\mathbb{V}$.

2. $\mathfrak{X} = (\mathbb{P}^1; 0, \infty; \zeta, \zeta^{-1})$, and $\mathbb{W}, \mathbb{W}'$ are associated to $0, \infty$. Then we have a conformal block

$$\tau_{\mathbb{W}} : \mathbb{W} \otimes \mathbb{W}' \to \mathbb{C}, w \otimes w' \mapsto \langle w, w'\rangle. \tag{1.1}$$

3. $\mathfrak{X} = (\mathbb{P}^1; 0, z, \infty; \zeta, \zeta - z, \zeta^{-1})$, and $\mathbb{W}, \mathbb{V}, \mathbb{W}'$ are associated to $0, z, \infty$. The the vertex operation Y for $\mathbb{W}$ defines a conformal block

$$w \otimes v \otimes w' \in \mathbb{W} \otimes \mathbb{V} \otimes \mathbb{W}' \mapsto \langle Y(v, z)w, w' \rangle. \tag{1.2}$$

Now, we add a new point $y \in C \backslash \{x_1, \ldots, x_N\}$ (together with a local coordinate μ) to $\mathfrak{X}$ and call this new data $\wr\mathfrak{X}_y$, and associate the vacuum module $\mathbb{V}$ to y. Then each conformal block $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$ canonically gives rise to one $\wr\phi_y : \mathbb{V} \otimes \mathbb{W} \to \mathbb{C}$ associated to $\wr\mathfrak{X}_y$ and $\mathbb{V} \otimes \mathbb{W}$, called the **propagation** of ϕ at y. The propagation is uniquely determined by the fact that

$$\wr\phi(\mathbf{1} \otimes w_\bullet)_y = \phi(w_\bullet). \tag{1.3}$$

For example, it follows easily from such uniqueness that the third example above is the propagation of the second one at z, i.e.

$$\wr\tau_{\mathbb{W}}(v \otimes w \otimes w)_z = \langle Y(v, z)w, w' \rangle.$$

More generally, when $y \in C$ is close to x_i and the local coordinate μ at y is $\eta_i - \eta_i(y)$,

$$\wr\phi(v \otimes w_\bullet)_y = \phi(w_1 \otimes \cdots \otimes Y(v, \eta_j(y))w_i \otimes \cdots \otimes w_N) \tag{1.4}$$

where the right hand side converges absolutely as a formal Laurent series of $\eta_j(y)$. (Cf. [Zhu94, Thm. 6.2], [FB04, Chapter 10], or Thm. 7.1 of this article.) The uniqueness of $\wr\phi$ satisfying (1.3) is not hard to show; what is more difficult is to prove the existence of propagation (cf. [TUY89, Zhu94, Zhu96, FB04, Cod19, DGT19a]).

Sewing conformal blocks

It is worth noting that the right hand side of (1.4) is the sewing of ϕ and $\wr\tau_{\mathbb{W}}$ (=the conformal block defined in (1.2)) corresponding the geometric sewing of C and $\mathbb{P}^1$ along the points x_i, ∞ with respect to their local coordinates η_i, ζ^{-1}. In general, given an $(N+2)$-pointed compact Riemann surface with local coordinates $\widetilde{\mathfrak{X}} = (\widetilde{C}; x_1, \ldots, x_N, x', x''; \eta_1, \ldots, \eta_N, \xi, \varpi)$ where each connected component of $\widetilde{C}$ intersects $\{x_1, \ldots, x_N\}$, if ξ (resp. ϖ) is defined on a neighborhood W' of x' (resp. W'' of x'') such that $\xi(W')$ is the open disc $\mathcal{D}_r$ with radius r (resp. $\varpi(W'') = \mathcal{D}_\rho$), and that W' (resp. W'') contains only one point among $x_1, \ldots, x_N, x', x''$. Then for each $0 < |q| < r\rho$, we remove

$$F' = \{y \in W' : |\xi(y)| \leqslant |q|/\rho\}, \qquad F'' = \{y \in W'' : |\varpi(y)| \leqslant |q|/r\},$$

from $\widetilde{C}$, and glue the remaining part by identifying all $y' \in W'$ with $y'' \in W''$ if $\xi(y')\varpi(y'') = q$. As a result, we obtain a new compact Riemann surface $\mathcal{C}_q$ with marked points $x_1, \ldots, x_N$ and local coordinates $\eta_1, \ldots, \eta_N$. We denote this data by $\mathfrak{X}_q$. Corresponding to this geometric sewing, we associated $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N, \mathbb{M}, \mathbb{M}'$ to the marked points $x_1, \ldots, x_N, x', x''$ where $\mathbb{M}'$ is contragredient to $\mathbb{M}$, and assume that the modules are $\mathbb{N}$-gradable (i.e., admissible) with grading operator $\widetilde{L}_0$ such that each graded subspace is finite-dimensional. $q^{\widetilde{L}_0} \in \mathrm{End}(\mathbb{M})[[q]]$ can be regarded as an element of $\mathbb{M} \otimes \mathbb{M}'[[q]]$, which we denote by $q^{\widetilde{L}_0} \blacktriangleright \otimes \blacktriangleleft$. If $\psi : \mathbb{W}_\bullet \otimes \mathbb{M} \otimes \mathbb{M}' \to \mathbb{C}$ is a conformal block associated to $\widetilde{\mathfrak{X}}$, we define a linear $\widetilde{\mathcal{S}}\psi : \mathbb{W}_\bullet \to \mathbb{C}[[q]]$ sending each $w \in \mathbb{W}_\bullet$ to

$$\widetilde{\mathcal{S}}\psi(w) = \psi(w \otimes q^{\widetilde{L}_0} \blacktriangleright \otimes \blacktriangleleft). \tag{1.5}$$

It was shown in [DGT19b, Thm. 8.5.1] that the above linear map defines a "formal conformal block" (i.e., a "conformal block" when q is infinitesimal). If this series converges absolutely on $|q| < r\rho$, then it defines an actual conformal block associated to $\mathfrak{X}_q$ [Gui20, Thm. 11.2], called the **sewing** of ψ.

In the above process, if $\widetilde{C}$ is connected, then $\mathcal{C}_q$ is the self-sewing of $\widetilde{C}$. For instance, if we sew the $\mathfrak{X}$ in the above example 3 along 0 and ∞ to get a torus, we accordingly sew the conformal block (1.2) to obtain the (normalized) character of $\mathbb{W}$-module $v \mapsto \mathrm{Tr}(Y(v,z)q^{\widetilde{L}_0})$, which plays an important role in the early development of VOA theory. If $\widetilde{C}$ has two connected component $\widetilde{C}_1, \widetilde{C}_2$, and if we sew $\widetilde{C}$ along $x' \in \widetilde{C}_1, x'' \in \widetilde{C}_2$, we obtain a connected sum of $\widetilde{C}_1$ and $\widetilde{C}_2$. If we choose $\widetilde{C} = C \sqcup \mathbb{P}^1$ and sew $\widetilde{C}$ along $x_i \in C$ and $\infty \in \mathbb{P}^1$, then at $q = 1$, the corresponding sewing of the conformal blocks ϕ and (1.2) is just (1.4), and the new Riemann surface we get is naturally equivalent to C.

Sewing and propagation

Now, (1.4) indicates that propagation and sewing are related: when the inserted point y is close to a marked point x_i, the propagation is defined by sewing. When y is far from the marked points, the propagation is defined by analytic continuation (provided that it exists).

This observation actually gives us a new proof of the existence of propagation: when y is close to a marked point x_i, we simply define the propagation by the series on the right hand side of (1.4). The convergence of that series follows from [FB04, 10.1.1]. Or, if we define conformal blocks to be those vanishing on the actions of certain global meromorphic 1-forms of the sheaf of VOA $\mathscr{V}_C$ on $\mathbb{W}_\bullet$ (which is the definition we take in this article), namely, the "dual definition" in [FB04, 10.1.2], then the convergence of the series, as well as the existence of the analytic continuation, follows from the Strong Residue Theorem ([FB04, 9.2.9], see also Thm. A.1). That the right hand side of (1.4) defines a conformal block follows from the previously mentioned fact that the sewing of a conformal block is again a conformal block. See Sec. 7 for details.

In [Gui20], we have proved a very general result on the convergence of sewing conformal blocks, assuming that $\mathbb{V}$ is CFT-type and C_2-cofinite, and the module $\mathbb{M}$ is semi-simple. To prove the convergence, we have to establish a differential equation satisfied by the formal series (1.5), and the C_2-cofiniteness is crucial for finding that differential equation. Here, we see a different pattern for proving the convergence of sewing: *if this sewing process is related to propagation, then its convergence follows from the Strong Residue Theorem*. In particular, C_2-cofinite (or CFT-type) is not needed for proving the convergence. (Note that in [Gui20], the main reason for assuming CFT-type when proving convergence is to use Buhl's result [Buhl02].)

Main result: sewing and propagation are commuting

A main goal of this article (already mentioned in the abstract) is to take the above idea (i.e. proving the convergence using the Strong Residue Theorem) to the extreme: we prove that if (in the setting of (1.5)) $\widetilde{\mathcal{S}}\psi$ converges absolutely on $|q| < r\rho$, then for distinct $y_1, \dots, y_n \in \widetilde{C}$ away from $\{x_1, \dots, x_N\}$ and from W', W'' (therefor they are also points on $\mathcal{C}_q$), $\widetilde{\mathcal{S}}(\wr^n\psi_{y_1,\dots,y_n})$, the sewing of the n-propagation of ψ at $y_1, \dots, y_n$, converges absolutely to $\wr^n\widetilde{\mathcal{S}}\psi_{y_1,\dots,y_n}$. Moreover, the convergence should be uniform when $y_1, \dots, y_n$ vary on compact sets. More generally, we prove such result for the simultaneous sewing of $\widetilde{C}$ along several pairs of points. See Thm. 9.1 for details. Even more generally, we prove such result for families of compact Riemann surfaces (cf. Rem. 9.2).

The above main result of this article can be summarized by

$$\wr^n\widetilde{\mathcal{S}}\psi = \widetilde{\mathcal{S}}\,\wr^n\,\psi, \tag{1.6}$$

i.e. "the propagation of sewing equals the sewing of propagation". Note that the non-trivial part of this result is that the convergence of $\widetilde{\mathcal{S}}\psi$ guarantees the convergence of $\widetilde{\mathcal{S}}\,\wr^n\,\psi$. For if we assume or already know the convergence, then the equality (1.6) follows directly from the fact that $\wr^n\widetilde{\mathcal{S}}\psi$ is the unique conformal block sending $\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes w \in \mathbb{V}^{\otimes n}\otimes\mathbb{W}_\bullet$ to $\widetilde{\mathcal{S}}\psi(w)$ (cf. (1.3)).

Applications

We give an application of this result. Let $\mathfrak{Y} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$, associate $\mathbb{W}_j$ to x_j for each j, and choose a conformal block $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ associated to $\mathfrak{Y}$. Choose $1 \leqslant i \leqslant N$. Let $\mathfrak{P} = (\mathbb{P}^1; 0, \infty; \zeta, \zeta^{-1})$, and associate $\mathbb{W}_i, \mathbb{W}_i'$ to $0, \infty$. Then $\psi := \phi\otimes\tau_{\mathbb{W}_i} : \mathbb{W}_\bullet\otimes\mathbb{W}_i\otimes\mathbb{W}_i' \to \mathbb{C}$ (recall (1.1)) is a conformal block associate to the disjoint union $\widetilde{\mathfrak{X}} = \mathfrak{Y}\sqcup\mathfrak{P}$. If we sew $\widetilde{\mathfrak{X}}$ along $x_i \in C$ and $\infty \in \mathbb{P}^1$ at q, the new pointed Riemann surface with local coordinates $\mathfrak{X}_q$ is

$$\mathfrak{X}_q = (C; x_1, \dots, x_N; \eta_1, \dots, q^{-1}\eta_i, \dots, \eta_N),$$

and (setting $w_\bullet = w_1\otimes\cdots\otimes w_N$ as usual)

$$\widetilde{\mathcal{S}}\psi(w_\bullet) = \phi(w_1\otimes\cdots\otimes q^{\widetilde{L}_0}w_i\otimes\cdots\otimes w_N), \tag{1.7}$$

which clearly converges absolutely for all q. Assume η_i is defined on an open disc $W_i \ni x_i$ such that $\eta_i(W_i) = \mathcal{D}_{r_i}$ has radius r_i, and that W_i contains only x_i among $x_1, \dots, x_N$. Choose $r > 0$. Then, according to our main result, the sewing of n-propagation

$$\begin{aligned}
&\widetilde{\mathcal{S}}\,\wr^n\,\psi(v_1\otimes\cdots\otimes v_n\otimes w_\bullet)_{\eta_i^{-1}(qz_1),\dots,\eta_i^{-1}(qz_n)}\\
=&\phi(w_1\otimes\cdots\otimes w_{i-1}\otimes q^{\widetilde{L}_0}\blacktriangleright\otimes w_{i+1}\otimes\cdots\otimes w_N)\cdot\wr^n\tau_{\mathbb{W}_i}(v_1\otimes\cdots\otimes v_n\otimes w_i\otimes\blacktriangleleft)_{z_1,\dots,z_n}
\end{aligned}$$

(assuming that the local coordinate at each $z_j \in \mathbb{P}^1$ is $\zeta - z_j$, and the one at $\eta_i^{-1}(qz_j) \in C$ is $q^{-1}\eta_i - z_j$) converges absolutely and uniformly when $z_1, \dots, z_n$ vary on any compact set of the configuration space $\mathrm{Conf}^n(\mathcal{D}_r^\times)$ (where $\mathcal{D}_r^\times = \{z \in \mathbb{C} : 0 < |z| < r\}$) and when $|q| < r_i/r$.

We are especially interested in the case that $q = 1$, which is accessible when $r < r_i$, namely, when $0 < |z_1|, \dots, |z_n| < r_i$. Then $\wr^n \widetilde{\mathcal{S}}\psi = \widetilde{\mathcal{S}} \wr^n \psi$ implies (notice (1.7))

$$\begin{aligned}&\wr^n \phi(v_1 \otimes \cdots \otimes v_n \otimes w_\bullet)_{\eta_i^{-1}(z_1),\dots,\eta_i^{-1}(z_n)}\\ =&\phi(w_1 \otimes \cdots \otimes w_{i-1} \otimes \blacktriangleright \otimes w_{i+1} \otimes \cdots \otimes w_N) \cdot \wr^n \tau_{\mathbb{W}_i}(v_1 \otimes \cdots \otimes v_n \otimes w_i \otimes \blacktriangleleft)_{z_1,\dots,z_n}. \end{aligned} \tag{1.8}$$

In the special case that $0 < |z_1| < \cdots < |z_n| < r_i$, the above relation becomes

$$\begin{aligned}&\wr^n \phi(v_1 \otimes \cdots \otimes v_n \otimes w_\bullet)_{\eta_i^{-1}(z_1),\dots,\eta_i^{-1}(z_n)}\\ =&\phi\big(w_1 \otimes \cdots \otimes Y(v_n, z_n) \cdots Y(v_1, z_1) w_i \otimes \cdots \otimes w_N\big)\end{aligned} \tag{1.9}$$

where the right hand side converges absolutely. Zhu proved relation (1.9) in [Zhu94, Thm. 6.2] when $v_1, \dots, v_n$ are primary, or when the local coordinates are contained in a projective structure (i.e., an atlas whose transition functions are Möbius transforms). But, as explained below, the general case, especially when $0 < |z_1| = \cdots = |z_n| < r_i$, is also important.

Take an automorphism g of $\mathbb{V}^{\otimes k}$ to be the permutation associated to the cycle $(12 \cdots k)$. Starting from a $\mathbb{V}$-module $\mathbb{W}$, Barron-Dong-Mason constructed in [BDM02] a (canonical) g-twisted $\mathbb{V}^{\otimes k}$-module structure on the same vector space $\mathbb{W}$. In particular, they explicitly described the twisted vertex operator Y^g for vectors in $\mathbb{V}^{\otimes k}$ of the form $\mathbf{1} \otimes \cdots \otimes v \otimes \cdots \otimes \mathbf{1}$. For an arbitrary vector of $\mathbb{V}^{\otimes k}$, the Y^g can then be described using normal ordering. Their proof that Y^g satisfies the axioms of a g-twisted module is algebraic, and in particular relies on a previous algebraic result of [Li96]. Recently, another algebraic proof was given by Dong-Xu-Yu in [DXY21] using Zhu's algebras.

Now, our observation in this article is that $Y^g(v_1 \otimes \cdots \otimes v_k, z)$ can be described by $\wr^k \tau_{\mathbb{W}_i}$ at $(z_1, \dots, z_k)$, where $z_1, \dots, z_k$ are the distinct k-th roots of unity of z. Thus, using the consequences of our main result such as relation (1.8), we can give a geometric proof that Y^g satisfies the axioms of a g-twisted module. See (10) for details. Moreover, our point of view will be generalized in [Gui21] to construct permutation twisted conformal blocks from untwisted ones, and vice versa.

Outline

This article is organized as follows. In Section 2, we fix the geometric notations used in later sections, and define the (multi) propagations for an (analytic) family of compact Riemann surfaces. In the case of a single compact Riemann surface C with marked points $S = \{x_1, \dots, x_N\}$, its n-propagation is easy to describe: If we let several distinct points $y_1, \dots, y_n$ move on $C\backslash S$, we obtain a family of compact Riemann surfaces (all isomorphic to C) with N fixed marked points and n varying points over the base manifold $\mathrm{Conf}^n(C\backslash S)$.

We recall the definitions and basic properties of sheaves of VOAs (i.e. VOA bundles) and conformal blocks in Sections 3 and 4. In Section 5, we recall some important facts about the sewing of conformal blocks associated to the sewing of a family of compact Riemann surfaces. In Section 6, we relate sheaves of VOAs and the $\mathscr{W}$-sheaves which were naturally introduced to define (sheaves of) conformal blocks.

In Section 7, we give a new proof of conformal block propagation for (analytic) families of compact Riemann surfaces. In particular, we prove that propagation is compatible (in the complex analytic sense) with the deformation of pointed compact Riemann surfaces. Roughly speaking, this means that if the original conformal blocks are parametrized by $\tau \in \mathcal{B}$ where $\mathcal{B}$ is the base manifold of the family, and if the propagation on each fiber is parametrized by z, then the propagation is a multivariable analytic function of (z, τ). The precise statement is formulated using the language of sheaves; see Thm. 7.1. These results were proved in [Cod19, Thm. 3.6] for CFT type VOAs using a Lie-theoretic method, which relies on the fact that such VOAs have PBW bases. As explained earlier, our proof is based on the idea of sewing, and relies on the Strong Residue Theorem and the fact that the sewing of conformal blocks are conformal blocks [Gui20, Thm. 11.2], whose formal version was proved in [DGT19b].

Note that here we should use the Strong Residue Theorem for analytic families of compact Riemann surfaces. This result is well-known, although we are not able to find a proof in the literature. So we include a proof in the Appendix Section A.

We discuss elementary properties of multi-propagation in Section 8. Most of these important properties were more or less known before (cf. [FB04]) but not explicitly written down. We collect these results under Thm. 8.2 so that they can be directly cited or used in future works on VOA. These results follow rather directly from those in the previous sections.

The main theorem of this article, summarized by the slogan "sewing commutes with propagation", is proved in Section 9. To give an application of this result, we construct in Section 10 permutation-twisted modules for tensor product VOAs.

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2 The geometric setting

We set $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$. Let $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$. For each $r > 0$, we let $\mathcal{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathcal{D}_r^\times = \mathcal{D}_r\backslash\{0\}$. For any topological space X, we define the configuration space $\mathrm{Conf}^n(X) = \{(x_1, \dots, x_N) \in X^n : x_i \neq x_j \ \forall 1 \leqslant i < j \leqslant n\}$.

For each complex manifold X, $\mathscr{O}_X$ is the sheaf of holomorphic functions of X. For each $x \in X$ and any $\mathscr{O}_X$-module $\mathscr{E}$, $\mathscr{E}_x$ is the stalk of $\mathscr{E}$ at x. $\mathfrak{m}_{X,x}$ (or simply $\mathfrak{m}_x$ when no confusion arises) is by definition $\{f \in \mathscr{O}_{X,x} : f(x) = 0\}$. $\mathscr{E}|_x := \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \simeq \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_{X,x}/\mathfrak{m}_x$ is the fiber of $\mathscr{E}$ at x. More generally, if Y is a closed complex submanifold of X with $\mathscr{I}_Y$ being the ideal sheaf (the sheaf of all sections of $\mathscr{O}_X$ vanishing at Y), then the restriction $\mathscr{E}|_Y$ is defined to be $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}_Y$ (restricted to the set Y). We suppress the subscript $\mathscr{O}_X$ under $\otimes$ when taking tensor products of $\mathscr{O}_X$-modules. If s is a section of $\mathscr{E}$, then $s|_Y$ is the corresponding value $s \otimes 1$ in $\mathscr{E}|_Y$.

(For the readers not familiar with the language of sheaf of modules: we only consider the case that $\mathscr{E}$ is locally free (with finite or infinite rank), i.e., a holomorphic vector bundle. Then $\mathscr{E}|_Y$ resp. $s|_Y$ is the usual restriction of the vector bundle resp. vector field to the submanifold Y.)

If $\mathscr{E}$ is locally free, $\mathscr{E}^\vee$ denotes its dual vector bundle.

For a Riemann surface C, its cotangent line bundle is denoted by ω_C.

A **family of compact Riemann surfaces** $\mathfrak{X}$ is by definition a holomorphic proper map of complex manifolds

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$$

that is a submersion and satisfies that each fiber $\mathcal{C}_b := \pi^{-1}(b)$ (where $b \in \mathcal{B}$) is a (non-necessarily connected) compact Riemann surface.

A **family of N-pointed compact Riemann surfaces** is by definition

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \dots, \varsigma_N) \tag{2.1}$$

where $\pi : \mathcal{C} \to \mathcal{B}$ is a family of compact Riemann surfaces, each section $\varsigma_j : \mathcal{B} \to \mathcal{C}$ is holomorphic and satisfies $\pi \circ \varsigma_j = \mathbf{1}_{\mathcal{B}}$, and any two $\varsigma_i(\mathcal{B}), \varsigma_j(\mathcal{B})$ (where $1 \leqslant i < j \leqslant N$) are disjoint. Unless otherwise stated, we also assume that every connected component of each fiber

$$\mathcal{C}_b = \pi^{-1}(b)$$

(where $b \in \mathcal{B}$) contains at least one of $\varsigma_1(b), \dots, \varsigma_N(b)$. We set

$$\mathfrak{X}_b = (\mathcal{C}_b; \varsigma_1(b), \dots, \varsigma_N(b)),$$

which is an N-pointed compact Riemann surface. We define closed submanifold

$$S_{\mathfrak{X}} = \bigcup_{j=1}^{N} \varsigma_j(\mathcal{B}),$$

considered also as a divisor of $\mathcal{C}$. For any sheaf of $\mathscr{O}_{\mathcal{C}}$-module $\mathscr{E}$, and for any $n \in \mathbb{Z}$, we set

$$\mathscr{E}(nS_{\mathfrak{X}}) := \mathscr{E} \otimes \mathscr{O}_X(nS_{\mathfrak{X}}),$$
$$\mathscr{E}(\star S_{\mathfrak{X}}) = \varinjlim_{n \in \mathbb{N}} \mathscr{E}(nS_{\mathfrak{X}}).$$

When $\mathscr{E}$ is a vector bundle, $\mathscr{E}(nS_{\mathfrak{X}})$ is the sheaf of sections of $\mathscr{E}$ which possibly has poles at each $\varsigma_j(\mathcal{B})$ with order at most n.

For each $1 \leqslant j \leqslant N$, a **local coordinate** of $\mathfrak{X}$ at ς_j is defined to be a holomorphic function $\eta_j \in \mathscr{O}(W_i)$ (where W_i is a neighborhood of $\varsigma_i(\mathcal{B})$) which is injective on each fiber $W_i \cap \pi^{-1}(b)$ and has value 0 on $\varsigma_i(\mathcal{B})$. It follows that (π, η_j) is a biholomorphism from W_i to a neighborhood of $\mathcal{B} \times \{0\}$ in $\mathcal{B} \times \mathbb{C}$. $\eta_j|_{\mathcal{C}_b}$ is a local coordinate of the fiber $\mathcal{C}_b$ at the point $\varsigma_j(b)$, which identifies a neighborhood of $\varsigma_j(b)$ (say $W_j \cap \mathcal{C}_b$) with an open subset of $\mathbb{C}$ such that $\varsigma_j(b)$ is identified with the origin. If $\mathfrak{X}$ is equipped with local coordinates $\eta_1, \dots, \eta_N$ at $\varsigma_1(\mathcal{B}), \dots, \varsigma_N(\mathcal{B})$ respectively, we set

$$\mathfrak{X}_b = (\mathcal{C}_b; \varsigma_1(b), \dots, \varsigma_N(b); \eta_1|_{\mathcal{C}_b}, \dots, \eta_N|_{\mathcal{C}_b}).$$

In particular, $S_{\mathfrak{X}_b} = \sum_j \varsigma_j(b)$ is a divisor of $\mathcal{C}_b$.

Now, we let $\mathfrak{X} = $ (2.1) be N-pointed but not necessarily equipped with local coordinates. Define the **propagated family** $\wr\mathfrak{X}$ as follows. Consider the commutative diagram

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}}) \arrow[r] \arrow[d, "\wr\pi"] & \mathcal{C} \arrow[d, "\pi"] \\ \mathcal{C}\backslash S_{\mathfrak{X}} \arrow[r, "\pi"] & \mathcal{B} \end{tikzcd}$$

where $\mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}})$ is the closed submanifold of $\mathcal{C} \times (\mathcal{C}\backslash S_{\mathfrak{X}})$ consisting of all (x, y) satisfying $\pi(x) = \pi(y)$, the first horizontal arrow is the projection onto the first component, and $\wr\pi$ is the projection onto the second component. We set

$$\wr\mathcal{B} = \mathcal{C}\backslash S_{\mathfrak{X}}, \qquad \wr\mathcal{C} = \mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}}).$$

The holomorphic section $\sigma : \mathcal{C}\backslash S_{\mathfrak{X}} \to \mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}})$ is set to be the diagonal map, i.e.,

$$\sigma : x \mapsto (x, x).$$

Define sections

$$\wr\varsigma_j : \mathcal{C}\backslash S_{\mathfrak{X}} \to \mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}}), \qquad x \mapsto (\varsigma_j \circ \pi(x), x).$$

Then we obtain an $(N+1)$-pointed family $\wr\mathfrak{X}$ of compact Riemann surfaces to be

$$\wr\mathfrak{X} = (\wr\pi : \wr\mathcal{C} \to \wr\mathcal{B}; \sigma, \wr\varsigma_1, \dots, \wr\varsigma_N). \tag{2.2}$$

Intuitively, $\wr\mathfrak{X}$ is the result of adding one extra marked point to each fiber $\mathcal{C}_b$ disjoint from $S_{\mathfrak{X}_b}$, letting this marked point vary on $\mathcal{C}_b\backslash S_{\mathfrak{X}_b}$ over all $b \in \mathcal{B}$, and fixing the other marked points.

One can define multi-propagation inductively by $\wr^n\mathfrak{X} = \wr\wr^{n-1}\mathfrak{X}$, which corresponds to varying n extra distinct points of $\mathcal{C}_b\backslash S_{\mathfrak{X}_b}$. Write

$$\wr^n\mathfrak{X} = (\wr^n\pi : \wr^n\mathcal{C} \to \wr^n\mathcal{B}; \sigma_1, \dots, \sigma_n, \wr^n\varsigma_1, \dots, \wr^n\varsigma_N).$$

Then $\wr^n\mathfrak{X}$ can be described in a more explicit way. Let

$$\prod_{\mathcal{B}}^n \mathcal{C}\backslash S_{\mathfrak{X}} = \underbrace{(\mathcal{C}\backslash S_{\mathfrak{X}}) \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}})}_{n}$$

which is the set of all $(x_1, \dots, x_n) \in \prod^n \mathcal{C}\backslash S_{\mathfrak{X}}$ satisfying $\pi(x_1) = \cdots = \pi(x_n)$. Define the relative configuration space

$$\mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) = \Big\{(x_1, \dots, x_N) \in \prod\nolimits^n_{\mathcal{B}} \mathcal{C}\backslash S_{\mathfrak{X}} : x_i \neq x_j \text{ for any } 1 \leqslant i < j \leqslant n\Big\}$$

which clearly admits a submersion $\mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) \to \mathcal{B}$ (sending each $(x_1, \dots, x_n)$ to $\pi(x_1)$). Take

$$\wr^n\pi : \mathcal{C} \times_{\mathcal{B}} \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) \to \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}).$$

to be the pullback of $\pi : \mathcal{C} \to \mathcal{B}$ along $\mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) \to \mathcal{B}$. So we have a commutative diagram

$$\begin{CD}
\mathcal{C} \times_{\mathcal{B}} \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) @>>> \mathcal{C} \\
@VV{\wr^n \pi}V @VV{\pi}V \\
\mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) @>>> \mathcal{B}
\end{CD}$$

Then $\wr^n \mathfrak{X}$ is equivalent to

$$\wr^n \mathfrak{X} \simeq \Big(\wr^n \pi : \mathcal{C} \times_{\mathcal{B}} \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}) \to \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}}); \sigma_1, \dots, \sigma_n, \wr^n \varsigma_1, \dots, \wr^n \varsigma_N \Big)$$

where

$$\begin{gathered}
\sigma_i(x_1, \dots, x_n) = (x_i, x_1, \dots, x_n), \\
\wr^n \varsigma_j(x_1, \dots, x_n) = (\varsigma_j \circ \pi(x_1), x_1, \dots, x_n)
\end{gathered}$$

for each $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant N$, $(x_1, \dots, x_n) \in \mathrm{Conf}^n_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}})$.

3 Sheaves of VOA

For any ($\mathbb{C}$-)vector space W, we define four spaces of formal series

$$\begin{aligned}
W[[z]] &= \Big\{ \sum_{n\in\mathbb{N}} w_n z^n : \text{each } w_n \in W \Big\}, \\
W[[z^{\pm 1}]] &= \Big\{ \sum_{n\in\mathbb{Z}} w_n z^n : \text{each } w_n \in W \Big\}, \\
W((z)) &= \Big\{ f(z) : z^k f(z) \in W[[z]] \text{ for some } k \in \mathbb{Z} \Big\}, \\
W\{z\} &= \Big\{ \sum_{n\in\mathbb{C}} w_n z^n : \text{each } w_n \in W \Big\}.
\end{aligned}$$

Throughout this article, $\mathbb{V}$ is a positive-energy vertex operator algebra (VOA) with vacuum $\mathbf{1}$ and conformal vector $\mathbf{c}$. We write $Y(v,z) = \sum_{z\in\mathbb{Z}} Y(v)_n z^{-n-1}$. Then $\{L_n = Y(\mathbf{c})_{n+1}\}$ are Virasoro algebras, and L_0 gives grading $\mathbb{V} = \bigoplus_{n\in\mathbb{N}} \mathbb{V}(n)$, where each $\mathbb{W}(n)$ is finite-dimensional.

In this article, a $\mathbb{V}$-module $\mathbb{W}$ means a **finitely-admissible $\mathbb{V}$-module**. This means that $\mathbb{W}$ is a weak $\mathbb{V}$-module in the sense of [DLM97] with vertex operators $Y_{\mathbb{W}}(v,z) = \sum_{n\in\mathbb{Z}} Y_{\mathbb{W}}(v)_n z^{-n-1}$, that $\mathbb{W}$ is equipped with a diagonalizable operator $\widetilde{L}_0$ satisfying

$$[\widetilde{L}_0, Y_{\mathbb{W}}(v)_n] = Y_{\mathbb{W}}(L_0 v)_n - (n+1) Y_{\mathbb{W}}(v)_n, \tag{3.1}$$

that the eigenvalues of $\widetilde{L}_0$ are in $\mathbb{N}$, and that each eigenspace $\mathbb{W}(n)$ is finite-dimensional. Let

$$\mathbb{W} = \bigoplus_{n\in\mathbb{N}} \mathbb{W}(n)$$

be the grading given by $\widetilde{L}_0$. Each

$$\mathbb{W}^{\leqslant n} = \bigoplus_{0\leqslant k\leqslant n} \mathbb{W}(k)$$

is finite-dimensional. We choose the $\widetilde{L}_0$ operator on $\mathbb{V}$ to be L_0.

We can define the **contragredient $\mathbb{V}$-module** $\mathbb{W}'$ of $\mathbb{W}$ as in [FHL93]. We choose $\widetilde{L}_0$-grading to be

$$\mathbb{W}' = \bigoplus_{n\in\mathbb{N}} \mathbb{W}'(n), \qquad \mathbb{W}'(n) = \mathbb{W}(n)^*.$$

Therefore, if we let $\langle\cdot,\cdot\rangle$ be the pairing between $\mathbb{W}$ and $\mathbb{W}'$, then $\langle \widetilde{L}_0 w, w'\rangle = \langle w, \widetilde{L}_0 w'\rangle$ for each $w \in \mathbb{W}, w' \in \mathbb{W}'$.

The vertex operator $Y_{\mathbb{W}}$ for $\mathbb{W}$ (abbreviated as Y in the following) gives a linear map $Y : \mathbb{V} \otimes \mathbb{W} \to \mathbb{W}((z))$ sending $v \otimes w$ to $Y(v,z)w$. We will write $Y_{\mathbb{W}}$ as Y when the context is clear. By identifying $\mathbb{V}$ with $\mathbb{V} \otimes 1$ in $\mathbb{V} \otimes \mathbb{C}((z))$ and similarly $\mathbb{W}$ with $\mathbb{W} \otimes 1$ in $\mathbb{W} \otimes \mathbb{C}((z))$, Y can be extended $\mathbb{C}((z))$-bilinearly to

$$\begin{gathered} Y : \Big(\mathbb{V} \otimes \mathbb{C}((z))\Big) \otimes \Big(\mathbb{W} \otimes \mathbb{C}((z))\Big) \to \mathbb{W} \otimes \mathbb{C}((z)), \\ Y(u \otimes f, z)w \otimes g = f(z)g(z)Y(u,z)w \end{gathered} \tag{3.2}$$

(for each $u \in \mathbb{V}, w \in \mathbb{W}, f, g \in \mathbb{C}((z))$). It can furthermore be extended to

$$Y : \Big(\mathbb{V} \otimes \mathbb{C}((z))dz\Big) \otimes \Big(\mathbb{W} \otimes \mathbb{C}((z))\Big) \to \mathbb{W} \otimes \mathbb{C}((z))dz \tag{3.3}$$

in an obvious way. Thus, for each $v \in \mathbb{V} \otimes \mathbb{C}((z))dz$, we can define the residue

$$\mathrm{Res}_{z=0}\, Y(v,z)w, \tag{3.4}$$

which, in case $v = u \otimes f dz, w = m \otimes g$ where $u \in \mathbb{V}$, $m \in \mathbb{W}$, and $f, g \in \mathbb{C}((z))$, is the $\mathbb{W}$-coefficient of $f(z)g(z)Y(v,z)mdz$ before $z^{-1}dz$.

We define a group $\mathbb{G} = \{f \in \mathscr{O}_{\mathbb{C},0} : f(0) = 0, f'(0) \neq 0\}$ where the stalk $\mathscr{O}_{\mathbb{C},0}$ is the set of holomorphic functions defined on a neighborhood of 0. The multiplication rule of $\mathbb{G}$ is the composition $\rho_1 \circ \rho_2$ of any two elements $\rho_1, \rho_2 \in \mathbb{G}$. By [Hua97], for each $\mathbb{V}$-module $\mathbb{W}$, there is a homomorphism $\mathcal{U} : \mathbb{G} \to \mathbb{W}$ defined in the following way: If we choose the unique $c_0, c_1, c_2 \cdots \in \mathbb{C}$ satisfying

$$\rho(z) = c_0 \cdot \exp\Big(\sum_{n>0} c_n z^{n+1}\partial_z\Big) z$$

then we necessarily have $c_0 = \rho'(0)$, and we set

$$\mathcal{U}(\rho) = \rho'(0)^{\widetilde{L}_0} \cdot \exp\Big(\sum_{n>0} c_n L_n\Big).$$

If X is a complex manifold, a (holomorphic) **family of transformations** $\rho : X \to \mathbb{G}$ is by definition an analytic function $\rho = \rho(x,z) = \rho_x(z)$ on a neighborhood of $X \times \{0\} \subset X \times \mathbb{C}$. Then $\mathcal{U}(\rho)$ (on each $\mathbb{W}$) is defined pointwisely, which is an $\mathrm{End}(\mathbb{W})$-valued function on X whose value at each $x \in X$ is $\mathcal{U}(\rho_x)$. $\mathcal{U}(\rho)$ can be regarded as an $\mathscr{O}_X$-module automorphism of $\mathbb{W} \otimes_{\mathbb{C}} \mathscr{O}_X$.

Let $\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B})$ be a family of compact Riemann surfaces. Associated to $\mathfrak{X}$ one can define a sheaf of $\mathscr{O}_X$-modules $\mathscr{V}_{\mathfrak{X}}$ as follows. (Cf. [FB04, Chapter 6, 17]; our presentation follows [Gui20, Sec. 5].) First, suppose $U, V \subset \mathcal{C}$ are open subsets, and we have two holomorphic functions $\eta \in \mathscr{O}(U), \mu \in \mathscr{O}(V)$ locally injective (i.e., étale) on each fiber $U_b := U \cap \pi^{-1}(b), V_b = V \cap \pi^{-1}(b)$ ($b \in \mathcal{B}$) of U and V respectively. We can define a family of transformations $\varrho(\eta|\mu) : U \cap V \to \mathbb{G}$ as follows: for each $p \in \mathcal{C}$, both $\eta - \eta(p)$ and $\mu - \mu(p)$ restricts to an injective holomorphic function on the fiber $(U \cap V)_{\pi(p)} = U \cap V \cap \pi^{-1}(\pi(p))$ vanishing at p. Then $\varrho(\eta|\mu)_p \in \mathbb{G}$ is determined by

$$\boxed{\eta - \eta(p)\big|_{(U\cap V)_{\pi(p)}} = \varrho(\eta|\mu)_p\big(\mu - \mu(p)\big|_{(U\cap V)_{\pi(p)}}\big)} \tag{3.5}$$

on a neighborhood of $0 \in \mathbb{C}$. Then $\mathcal{U}(\varrho(\eta|\mu))$ is an $\mathscr{O}_{U\cap V}$-module automorphism of $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{U\cap V}$ which restricts to an automorphism of $\mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U\cap V}$ for each $n \in \mathbb{N}$. The cocycle condition $\varrho(\eta|\mu)\varrho(\mu|\nu) = \varrho(\eta|\nu)$ holds for any holomorphic function ν on a neighborhood of $\mathcal{C}$ which is injective on each fiber.

Thus, we can define $\mathscr{V}_{\mathfrak{X}}^{\leqslant n}$ to be the holomorphic vector bundle on $\mathcal{C}$ which associates to each open $U \subset \mathcal{C}$ and each $\eta \in \mathscr{O}(U)$ locally injective on fibers a trivialization (i.e., an isomorphism of $\mathscr{O}_U$-modules)

$$\mathcal{U}_\varrho(\eta) : \mathscr{V}_{\mathfrak{X}}^{\leqslant n}|_U \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U \tag{3.6}$$

such that for another similar $V \subset \mathcal{C}, \mu \in \mathscr{O}(V)$, we have the transition function

$$\mathcal{U}_\varrho(\eta)\mathcal{U}_\varrho(\mu)^{-1} = \mathcal{U}(\varrho(\eta|\mu)) : \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U\cap V} \xrightarrow{\simeq} \mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_{U\cap V}. \tag{3.7}$$

If $n' > n$, we have clearly an $\mathscr{O}_{\mathcal{C}}$-module monomorphism $\mathscr{V}_{\mathfrak{X}}^{\leqslant n} \to \mathscr{V}_{\mathfrak{X}}^{\leqslant n'}$ which, for each open $U \subset \mathcal{C}$ and η as above, is transported under the isomorphisms (3.6) to the canonical monomorphism $\mathbb{V}^{\leqslant n} \otimes_{\mathbb{C}} \mathscr{O}_U \to \mathbb{V}^{\leqslant n'} \otimes_{\mathbb{C}} \mathscr{O}_U$ defined by the inclusion $\mathbb{V}^{\leqslant n} \hookrightarrow \mathbb{V}^{\leqslant n'}$. Thus we are allowed to define

$$\mathscr{V}_{\mathfrak{X}} = \varinjlim_{n\in\mathbb{N}} \mathscr{V}_{\mathfrak{X}}^{\leqslant n}.$$

Alternatively, one can directly define $\mathscr{V}_{\mathfrak{X}}$ to be the $\mathscr{O}_{\mathcal{C}}$-module which is locally free (of infinite rank) and isomorphic to $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_U$ via a morphism $\mathcal{U}_\varrho(\eta)$, and whose transition function is given by $\mathcal{U}(\varrho(\eta|\mu))$. We call $\mathscr{V}_{\mathfrak{X}}$ the **sheaf of VOA** associated to $\mathfrak{X}$ and $\mathbb{V}$. If $\mathfrak{X}$ is a single compact Riemann surface C, we write $\mathscr{V}_{\mathfrak{X}}$ as $\mathscr{V}_C$.

For each fiber $\mathcal{C}_b$ (where $b \in \mathcal{B}$), we have a canonical equivalence

$$\mathscr{V}_{\mathfrak{X}}|_{\mathcal{C}_b} \simeq \mathscr{V}_{\mathcal{C}_b} \equiv \mathscr{V}_{\mathfrak{X}_b} \tag{3.8}$$

such that if these two $\mathscr{O}_{\mathcal{C}_b}$-modules are identified by this isomorphism, then the restriction of the trivialization (3.6) to $U_b = U \cap \pi^{-1}(b)$ equals

$$\mathcal{U}_\varrho(\eta|_{\mathcal{C}_b}) : \mathscr{V}_{\mathcal{C}_b}|_{U_b} \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{U_b}.$$

Definition 3.1. Since the vacuum vector $\mathbf{1}$ is killed by all L_n (where $n \geqslant 0$), it is fixed by any change of coordinate $\mathcal{U}(\rho)$. It follows that we can define a section $\mathbf{1} \in \mathscr{V}_{\mathfrak{X}}(\mathcal{C})$ which under any trivialization $\mathcal{U}_\varrho(\eta)$ is the constant section $\mathbf{1}$, called the **vacuum section**.

4 Conformal blocks

Let $\mathfrak{X}$ be a family of N-pointed compact Riemann surfaces as in (2.1). We choose $\mathbb{V}$-modules $\mathbb{W}_1, \dots, \mathbb{W}_N$. Set

$$\mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N.$$

$w \in \mathbb{W}_\bullet$ means a vector in $\mathbb{W}_\bullet$, and $w_\bullet \in \mathbb{W}_\bullet$ means a vector of the form $w_1 \otimes \cdots \otimes w_N$ where each $w_i \in \mathbb{W}_i$.

The sheaf of conformal blocks is an $\mathscr{O}_\mathcal{B}$-submodule of an infinite-rank locally free $\mathscr{O}_\mathcal{B}$-module $\mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)$, where the latter is defined as follows. For each open subset $V \subset \mathcal{B}$ such that the restricted family

$$\mathfrak{X}_V := (\pi : \mathcal{C}_V \to V; \varsigma_1|_V, \dots, \varsigma_N|_V)$$

(where $\mathcal{C}_V = \pi^{-1}(V)$) admits local coordinates $\eta_1, \dots, \eta_N$ at $\varsigma_1(V), \dots, \varsigma_N(V)$ respectively, we have a trivialization (i.e., an isomorphism of $\mathscr{O}_V$-modules)

$$\mathcal{U}(\eta_\bullet) \equiv \mathcal{U}(\eta_1) \otimes \cdots \otimes \mathcal{U}(\eta_N) : \mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)|_V \xrightarrow{\simeq} \mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_V.$$

If V is small enough such that we have another set of local coordinates $\mu_1, \dots, \mu_N$ at $\varsigma_1(V), \dots, \varsigma_N(V)$ respectively, for each $1 \leqslant j \leqslant N$ we choose a family of transformations $(\eta_j|\mu_j) : V \to \mathbb{G}$ defined by

$$\boxed{(\eta_j|\mu_j)_b \circ \mu_j|_{\mathcal{C}_b} = \eta_j|_{\mathcal{C}_b}} \tag{4.1}$$

for each $b \in V$. Then each $\mathcal{U}(\eta_j|\mu_j)$ is a holomorphic family of invertible endomorphisms of $\mathbb{W}_j$ associated to $(\eta_j|\mu_j)$ (as defined in Sec. 3). The tensor product of them, as a family of invertible transformations of $\mathbb{W}_\bullet$ (more precisely, an automorphism of the $\mathscr{O}_V$-module $\mathbb{W}_\bullet \otimes_\mathbb{C} \mathscr{O}_V$), is the transition function:

$$\mathcal{U}(\eta_\bullet)\mathcal{U}(\mu_\bullet)^{-1} := \mathcal{U}(\eta_1|\mu_1) \otimes \cdots \otimes \mathcal{U}(\eta_N|\mu_N). \tag{4.2}$$

This gives the definition of $\mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)$.

In particular, $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$ is a vector space equivalent to $\mathbb{W}_\bullet$ through $\mathcal{U}(\eta_\bullet|_{\mathcal{C}_b})$. It is easy to see that for each $b \in \mathcal{B}$, the restriction $\mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)|_b$ (i.e., the fiber of the vector bundle at b) is naturally equivalent to $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$:

$$\mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)|_b \simeq \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet). \tag{4.3}$$

This equivalence is uniquely determined by the fact that if we identify the two spaces, then the restriction of $\mathcal{U}(\eta_\bullet)$ to the map $\mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet) \to \mathbb{W}_\bullet$ equals $\mathcal{U}(\eta_\bullet|_{\mathcal{C}_b})$.

To define conformal blocks, we first consider the case that $\mathcal{B}$ is a single point. Then $C := \mathcal{C}$ is a compact Riemann surface. We can define a linear action of $H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_\mathfrak{X}))$ on $\mathscr{W}_\mathfrak{X}(\mathbb{W}_\bullet)$ as follows. Choose any local coordinate η_i of C at the point $x_j := \varsigma_j(\mathcal{B})$, defined on a neighboorhood W_j of x_j (so, in particular, $\eta_j(x_j) = 0$). Note $S_\mathfrak{X} = \{x_1, \dots, x_N\}$. We assume

$$W_j \cap S_\mathfrak{X} = \{x_j\}.$$

Note that we have a trivialization

$$\mathcal{U}_\varrho(\eta_j) : \mathscr{V}_C|_{W_i} \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{W_i} \simeq \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\eta_i(W_i)}$$

which, tensored by $(\eta_j^{-1})^* : \omega_{W_j} \xrightarrow{\simeq} \omega_{\eta_j(W_j)}$, gives a trivialization

$$\mathcal{V}_\varrho(\eta_j) : \mathscr{V}_C|_{W_i} \otimes \omega_{W_i}(\star S_{\mathfrak{X}}) \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \omega_{\eta_j(W_i)}(\star 0)$$

Then for each $v \in H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}}))$, we have a section $\mathcal{V}_\varrho(\eta_j)v$, which is a $\mathbb{V}$-valued (more precisely, $\mathbb{V}^{\leqslant n}$-valued for some $n \in \mathbb{N}$) holomorphic 1-form on $\eta_j(W_j)$ but possibly has poles at $\eta_j(x_j) = 0$. By taking Laurent series expansions, $\mathcal{V}_\varrho(\eta_j)v$ can be regarded as an element of $\mathbb{V} \otimes \mathbb{C}((z))dz$. We then define, (notice that we have an isomorphism $\mathcal{U}(\eta_\bullet) : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \xrightarrow{\simeq} \mathbb{W}_\bullet$) an action of v on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ by

$$\mathcal{U}(\eta_\bullet) \cdot v \cdot \mathcal{U}(\eta_\bullet)^{-1} w_\bullet = \sum_{j=1}^{N} w_1 \otimes \cdots \otimes \mathrm{Res}_{z=0}\, Y\big(\mathcal{V}_\varrho(\eta_j)v, z\big) w_j \otimes \cdots \otimes w_N$$

for each $w_\bullet \in \mathbb{W}_\bullet$, where the residue is defined as in (3.4). That this definition is independent of the choice of local coordinates $\eta_\bullet$ follows from [FB04, Thm. 6.5.4] (see also [Gui20, Thm. 3.2]), which relies on a crucial change of variable formula (cf. [Gui20, Thm. 3.3]) proved by Huang [Hua97].

Now that we have a linear action of $H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}}))$ on $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$, we say that a linear functional $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathbb{C}$ is a **conformal block** (associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$) exactly when ϕ vanishes on the vector space

$$\mathscr{J} := H^0(C, \mathscr{V}_C \otimes \omega_C(\star S_{\mathfrak{X}})) \cdot \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$$

where $\mathrm{Span}_{\mathbb{C}}$ is suppressed on the right hand side of the equality.

Now we come back to the general setting that $\mathfrak{X}$ is a family of N-pointed compact Riemann surfaces. Let $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ be an $\mathscr{O}_{\mathcal{B}}$-module morphism, which can be understood in the following way: If locally we identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|_V$ (where V is an open subset of $\mathcal{B}$) with $\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_V$, then ϕ associates to each vector $w \in \mathbb{W}_\bullet$ (considered as the constant section $w \otimes 1 \in \mathbb{W}_\bullet \otimes \mathscr{O}(V)$) a holomorphic function $\phi(w)$ on U.

Definition 4.1. Let $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ be an $\mathscr{O}_{\mathcal{B}}$-module morphism. For each $b \in \mathcal{B}$, regard $\phi|_b$ as the restriction of ϕ to the fiber map $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|_b \simeq \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet) \to \mathbb{C}$. Then, we say ϕ is a **conformal block** (over $\mathcal{B}$ associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$) if for each $b \in \mathcal{B}$, $\phi|_b$ is a conformal block associated to $\mathfrak{X}_b$ (i.e., $\phi(b)$ vanishes on $H^0(\mathcal{C}_b, \mathscr{V}_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(\star S_{\mathfrak{X}}|_b)) \cdot \mathscr{W}_{\mathfrak{X}_b}(\mathbb{W}_\bullet)$).

The following proposition is [Gui20, Prop. 6.4].

Proposition 4.2. *Let $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ be an $\mathscr{O}_{\mathcal{B}}$-module morphism. Suppose that each connected component of $\mathcal{B}$ contains a non-empty open subset V such that the restriction of ϕ to $\mathscr{W}_{\mathfrak{X}_V}(\mathbb{W}_\bullet) \to \mathscr{O}_V$ is a conformal block, then the original ϕ is a conformal block associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$.*

5 Sewing conformal blocks

Let $N, M \in \mathbb{Z}_+$. Let

$$\widetilde{\mathfrak{X}} = (\widetilde{\pi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{B}}; \varsigma_1, \dots, \varsigma_N; \varsigma'_1, \dots, \varsigma'_M; \varsigma''_1, \dots, \varsigma''_M;)$$

be a family of $(N + 2M)$-pointed compact Riemann surfaces, assuming that every connected component of each fiber intersects one of $\varsigma_1(\widetilde{\mathcal{B}}), \dots, \varsigma_N(\widetilde{\mathcal{B}})$. For each $1 \leqslant j \leqslant M$, we assume $\widetilde{\mathfrak{X}}$ has local coordinates ξ_j at $\varsigma'_j(\widetilde{\mathcal{B}})$ defined on a neighborhood $W'_j \subset \widetilde{\mathcal{C}}$ of $\varsigma'_j(\widetilde{\mathcal{B}})$ and similarly ϖ_j at $\varsigma''_j(\widetilde{\mathcal{B}})$ defined on a neighborhood W''_j. We assume all W'_j, W''_j $(1 \leqslant j \leqslant M)$ are mutually disjoint and are also disjoint from $\varsigma_1(\widetilde{\mathcal{B}}), \dots, \varsigma_N(\widetilde{\mathcal{B}})$, so that $\varsigma_1(\widetilde{\mathcal{B}}), \dots, \varsigma_N(\widetilde{\mathcal{B}})$ remain after sewing. We also assume that for each $1 \leqslant j \leqslant M$, we can choose $r_j, \rho_j > 0$ such that

$$(\xi_j, \widetilde{\pi}) : W'_j \xrightarrow{\simeq} \mathcal{D}_{r_j} \times \widetilde{\mathcal{B}} \qquad \text{resp.} \qquad (\varpi_j, \widetilde{\pi}) : W''_j \xrightarrow{\simeq} \mathcal{D}_{\rho_j} \times \widetilde{\mathcal{B}} \tag{5.1}$$

is a biholomorphic map. (Recall that $\mathcal{D}_r$ is the open disc at $0 \in \mathbb{C}$ with radius r.)

We do not assume $\widetilde{\mathfrak{X}}$ has local coordinates at $\varsigma_1(\widetilde{\mathcal{B}}), \dots, \varsigma_N(\widetilde{\mathcal{B}})$.

Sewing families of compact Riemann surfaces

We can **sew** $\widetilde{\mathfrak{X}}$ **along all pairs** $\varsigma'_j(\widetilde{\mathcal{B}}), \varsigma''_j(\widetilde{\mathcal{B}})$ to obtain a new family

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{B}; \varsigma_1, \dots, \varsigma_N) \tag{5.2}$$

of compact Riemann surfaces. Here,

$$\mathcal{B} = \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}}, \qquad \mathcal{D}^\times_{r_\bullet \rho_\bullet} = \mathcal{D}^\times_{r_1 \rho_1} \times \cdots \times \mathcal{D}^\times_{r_M \rho_M}.$$

$\mathfrak{X}$ is described as follows.

For each $q_\bullet \in \mathcal{D}^\times_{r_\bullet \rho_\bullet}$ and $b \in \widetilde{\mathcal{B}}$, the fiber $\mathcal{C}_{(q_\bullet, b)}$ is obtained by removing the closed discs

$$F'_{j,b} = \{y \in W'_j \cap \widetilde{\mathcal{C}}_b : |\xi_j(y)| \leqslant |q_j|/\rho_j\}, \qquad F''_{j,b} = \{y \in W''_j \cap \widetilde{\mathcal{C}}_b : |\varpi_j(y)| \leqslant |q_j|/r_j\}$$

(for all j) from $\widetilde{\mathcal{C}}_b$, and gluing the remaining part of the Riemann surface $\widetilde{\mathcal{C}}_b$ by identifying (for all j) $y' \in W'_j \cap \widetilde{\mathcal{C}}_b$ with $y'' \in W''_j \cap \widetilde{\mathcal{C}}_b$ if $\xi_j(y')\varpi_j(y'') = q_j$. This procedure can be performed in a consistent way over all $b \in \widetilde{\mathcal{B}}$, which gives $\pi : \mathcal{C} \to \mathcal{B}$. See for instance [Gui20, Sec. 4] for details.[1]

Since $\Omega = \widetilde{\mathcal{C}} \backslash \bigcup_j (W'_j \cup W''_j)$ is not affected by gluing, $\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \Omega$ can be viewed as a subset of $\mathfrak{X}$, and the restriction of π to this set is $\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \Omega \xrightarrow{1 \otimes \widetilde{\pi}} \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}} = \mathcal{B}$. Thus, for each $1 \leqslant i \leqslant N$ the section ς_i for $\widetilde{\mathfrak{X}}$ defines the corresponding section $1 \times \varsigma_i : \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}} \to \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \Omega$, also denoted by ς_i. A local coordinate η_i of $\widetilde{\mathfrak{X}}$ at $\varsigma_i(\widetilde{\mathcal{B}})$ extends constantly over $\mathcal{D}^\times_{r_\bullet \rho_\bullet}$ to a local coordinate of $\mathfrak{X}$ at $\varsigma_i(\mathcal{B})$, also denoted by η_i.

[1] Indeed, one can extend $\mathfrak{X}$ to a slightly larger flat family of complex curves (with at worst nodal singularities) with base manifold $\mathcal{D}_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}}$ (cf. for instance [Gui20, Sec. 4]).

Sewing conformal blocks

We now define sewing conformal blocks associated to $\widetilde{\mathfrak{X}}$. Associate to $\varsigma_1,\dots,\varsigma_N$ $\mathbb{V}$-modules $\mathbb{W}_1,\dots,\mathbb{W}_N$. Then we have $\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)$ defined by $(\widetilde{\pi}:\widetilde{\mathcal{C}}\to\widetilde{\mathcal{B}};\varsigma_1,\dots,\varsigma_N)$. For each connected open $\widetilde{V}\subset\widetilde{\mathcal{B}}$, $\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)(\widetilde{V})$ can be identified canonically with a subspace of $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V})$ consisting of sections of the latter which are constant with respect to sewing. More precisely, this identification is compatible with restrictions to open subsets of $\widetilde{V}$; moreover, if $\widetilde{V}$ is small enough such that $\widetilde{\mathfrak{X}}|_{\widetilde{V}}$ has local coordinates $\eta_1,\dots,\eta_N$ at $\varsigma_1(\widetilde{V}),\dots,\varsigma_N(\widetilde{V})$ which give rise to $\eta_1,\dots,\eta_N$ of $\eta_1,\dots,\eta_N$ of $\mathfrak{X}|_{\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V}}$ at $\varsigma_1(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V}),\dots,\varsigma_N(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V})$ (which are constant over $\mathcal{D}^\times_{r_\bullet\rho_\bullet}$), then the following diagram commutes:

$$\begin{tikzcd}
\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)(\widetilde{V}) \arrow[r, hook] \arrow[d, "\mathcal{U}(\eta_\bullet)", "\simeq"'] & \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V}) \arrow[d, "\mathcal{U}(\eta_\bullet)"', "\simeq"] \\
\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}(\widetilde{V}) \arrow[r, hook] & \mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V})
\end{tikzcd} \tag{5.3}$$

where the bottom horizontal line is defined by pulling pack the projection $\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\widetilde{V}\to\widetilde{V}$.

Associate to $\varsigma'_1,\dots,\varsigma'_M$ $\mathbb{V}$-modules $\mathbb{M}_1,\dots,\mathbb{M}_M$, whose contragredient modules $\mathbb{M}'_1,\dots,\mathbb{M}'_M$ are associated to $\varsigma''_1,\dots,\varsigma''_M$. We understand $\mathbb{W}_\bullet\otimes\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet$ as

$$\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N\otimes\mathbb{M}_1\otimes\mathbb{M}'_1\otimes\cdots\otimes\mathbb{M}_M\otimes\mathbb{M}'_M,$$

where the order has be changed so that each $\mathbb{M}'_j$ is next to $\mathbb{M}_j$. We can then identify

$$\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet\otimes\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet)=\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)\otimes_{\mathbb{C}}\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet \tag{5.4}$$

such that whenever $\widetilde{V}\subset\widetilde{\mathcal{B}}$ is open such that $\widetilde{\mathfrak{X}}|_{\widetilde{V}}$ has local coordinates $\eta_1,\dots,\eta_N$ at $\varsigma_1(\widetilde{V}),\dots,\varsigma_N(\widetilde{V})$ as before, the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet\otimes\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet)\big|_{\widetilde{V}} \arrow[rr, leftrightarrow, "="] \arrow[dr, "\mathcal{U}(\eta_\bullet,\xi_\bullet,\varpi_\bullet)", "\simeq"'] & & \mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)\big|_{\widetilde{V}}\otimes_{\mathbb{C}}\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet \arrow[dl, "\mathcal{U}(\eta_\bullet)\otimes\mathbf{1}"', "\simeq"] \\
& \mathbb{W}_\bullet\otimes\mathbb{M}_\bullet\otimes\mathbb{M}'_\bullet\otimes_{\mathbb{C}}\mathscr{O}_{\widetilde{V}} &
\end{tikzcd} \tag{5.5}$$

We define

$$q_j^{\widetilde{L}_0}\blacktriangleright\otimes_j\blacktriangleleft=\sum_{n\in\mathbb{N}}q_j^n\sum_{a\in\mathfrak{A}_{j,n}}m(n,a)\otimes\check{m}(n,a)\qquad\in(\mathbb{M}_j\otimes\mathbb{M}'_j)[[q_j]]$$

where for each $n\in\mathbb{N},s\in\mathbb{C}$, $\{m(n,a):a\in\mathfrak{A}_{j,n}\}$ is a basis of $\mathbb{W}(n)$ with dual basis $\{\check{m}(n,a):a\in\mathfrak{A}_{j,n}\}$ in $\mathbb{W}'(n)$.

Now, for any conformal block $\psi : \mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet) \to \mathscr{O}_{\widetilde{\mathcal{B}}}$ associated to the family $\widetilde{\mathfrak{X}}$ and $\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet$, we define an $\mathscr{O}_{\widetilde{\mathcal{B}}}$-module morphism

$$\widetilde{\mathcal{S}}\psi : \mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\widetilde{\mathcal{B}}}[[q_1, \dots, q_M]]$$

by sending each section w over an open $\widetilde{V} \subset \widetilde{\mathcal{B}}$ to

$$\widetilde{\mathcal{S}}\psi(w) = \psi\Big(w \otimes (q_1^{\widetilde{L}_0} \blacktriangleright \otimes_1 \blacktriangleleft) \otimes \cdots \otimes (q_M^{\widetilde{L}_0} \blacktriangleright \otimes_M \blacktriangleleft)\Big) \qquad \in \mathscr{O}(\widetilde{V})[[q_1, \dots, q_M]]. \tag{5.6}$$

The identification (5.4) is used in this definition. $\widetilde{\mathcal{S}}\psi$ is called the (normalized) **sewing of** ψ.

Definition 5.1. Let X be a complex manifold. Consider an element

$$f = \sum_{n_1, \dots, n_M \in \mathbb{C}} f_{n_1, \dots, n_M} q_1^{n_1} \cdots q_M^{n_M} \qquad \in \mathscr{O}(X)\{q_1, \dots, q_M\}$$

where each $f_{n_1,\dots,n_M} \in \mathscr{O}(X)$. Let $R_1, \dots, R_M \in [0, +\infty]$ and $\mathcal{D}^\times_{R_\bullet} = \mathcal{D}^\times_{R_1} \times \cdots \times \mathcal{D}^\times_{R_M}$. For any locally compact subset Ω of $\mathcal{D}^\times_{R_\bullet} \times X$, we say that formal series f **converges absolutely and locally uniformly (a.l.u.) on** Ω, if for any compact subsets $K \subset \Omega$, we have

$$\sup_{(q_\bullet, x) \in K} \sum_{n_1, \dots, n_M \in \mathbb{C}} \left| f_{n_1, \dots, n_M}(x) q_1^{n_1} \cdots q_M^{n_M} \right| < +\infty.$$

In the case that $f \in \mathscr{O}(X)[[q_1^{\pm 1}, \dots, q_M^{\pm 1}]]$, it is clear from complex analysis that f converges a.l.u. on $\mathcal{D}^\times_{R_\bullet} \times X$ if and only if f is the Laurent series expansion of an element (also denoted by f) of $\mathscr{O}(\mathcal{D}^\times_{R_\bullet} \times X)$.

Definition 5.2. We say that $\widetilde{\mathcal{S}}\psi$ **converges a.l.u.** (on $\mathcal{B} = \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}}$), if for any open subset $\widetilde{V} \subset \widetilde{\mathcal{B}}$ and any section w of $\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet)(\widetilde{V})$, $\widetilde{\mathcal{S}}\psi(w)$ converges a.l.u. on $\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{V}$.

Theorem 5.3 ([Gui20], Thm. 11.2). *If $\widetilde{\mathcal{S}}\psi$ converges a.l.u. on $\mathcal{B} = \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}}$, then $\widetilde{\mathcal{S}}\psi$ (resp. $\mathcal{S}\psi$), when extended $\mathscr{O}_{\mathcal{B}}$-linearly to an $\mathscr{O}_{\mathcal{B}}$-module homomorphism $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ using the inclusion $\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet) \subset \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ defined by (5.3), is a conformal block associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$.*

Example 5.4. Let $\mathfrak{Y} = (C; x_1, \dots, x_N)$ be an N-pointed compact Riemann surface with local coordinates $\eta_1, \dots, \eta_N$ at $x_1, \dots, x_N$, defined on neighborhoods $W_1, \dots, W_N$ satisfying $W_j \cap \{x_1, \dots, x_N\} = x_j$ for each $1 \leqslant j \leqslant N$. Assume $\eta_1(W_1) = \mathcal{D}_r$ for some $r > 0$. Let ζ be the standard coordinate of $\mathbb{C}$. Let $\widetilde{\mathfrak{X}}$ be the disjoint union of $\mathfrak{Y}$ and $(\mathbb{P}^1; 0, 1, \infty)$, namely, we have an $(N+3)$-pointed compact Riemann surface

$$\widetilde{\mathfrak{X}} = (C \sqcup \mathbb{P}^1; x_1, \dots, x_N, 0, 1, \infty).$$

We equip $\widetilde{\mathfrak{X}}$ with local coordinates $\eta_1, \dots, \eta_N, \zeta, (\zeta - 1), \zeta^{-1}$. The local coordinate ζ at 0 should be defined at $|z| < 1$ so that no marked points other than 0 is inside this region.

We sew $\widetilde{\mathfrak{X}}$ along $x_1 \in C$ and $\infty \in \mathbb{P}^1$ using the chosen local coordinates η_1 and $1/\zeta$ to obtain a family $\mathfrak{X}$. Then

$$\mathfrak{X} = (\pi : C \times \mathcal{D}^\times_r \to \mathcal{D}^\times_r; x_1, x_2, \dots, x_N, \varsigma)$$

where π is the projection onto the $\mathcal{D}_r^\times$-component, the sections $x_1,\dots,x_N$ are (rigorously speaking) sections sending q to $(x_1,q),\dots,(x_N,q)$. The section ς is defined by $\varsigma(q)=(\eta_1^{-1}(q),q)$, where η_1^{-1} sends $\mathcal{D}_r$ biholomorphically to W_1. Moreover, the local coordinates of $\mathfrak{X}$ defined naturally by those of $\widetilde{\mathfrak{X}}$ are described as follows: For each $|q|<r$, their restrictions to

$$\mathfrak{X}_q=(C;x_1,x_2,\dots,x_N,\eta_1^{-1}(q)) \tag{5.7}$$

are $q^{-1}\eta_1,\eta_2,\dots,\eta_N,q^{-1}(\eta_1-q)$.

Attach $\mathbb{V}$-modules $\mathbb{W}_1,\dots,\mathbb{W}_N,\mathbb{W}_1,\mathbb{V},\mathbb{W}_1'$ with simple L_0-grading to the marked points $x_1,\dots,x_N,0,1,\infty$ respectively of $\widetilde{\mathfrak{X}}$. Fix the trivializations of $\mathscr{W}$-sheaves using the chosen local coordinates. Let $\phi:\mathbb{W}_1\otimes\cdots\otimes\mathbb{W}_N\to\mathbb{C}$ be a conformal block associated to $(C;x_1,\dots,x_N)$ and $\mathbb{W}_1,\dots,\mathbb{W}_N$. Let

$$\begin{gathered}\omega:\mathbb{W}_1\otimes\mathbb{V}\otimes\mathbb{W}_1'\to\mathbb{C},\\ w\otimes u\otimes w'\mapsto\langle Y(u,1)w,w'\rangle=\sum_{n\in\mathbb{Z}}\langle Y(u)_nw,w'\rangle,\end{gathered}$$

which is a conformal block associated to $(\mathbb{P}^1;0,1,\infty)$ and $\mathbb{W}_1,\mathbb{V},\mathbb{W}_1'$. Then $\psi:=\phi\otimes\omega$ is a conformal block for $\widetilde{\mathfrak{X}}$. Note that when u,w_1 are $\widetilde{L}_0$-homogeneous (i.e. eigenvectors of $\widetilde{L}_0$) with eigenvalues (weights) $\widetilde{\mathrm{wt}}(u),\widetilde{\mathrm{wt}}(w_1)\in\mathbb{N}$ respectively, by (3.1), $Y(u)_nw_1$ is $\widetilde{L}_0$-homogeneous with weight $\widetilde{\mathrm{wt}}(u)+\widetilde{\mathrm{wt}}({}_1)-n-1$. Then

$$\begin{gathered}\widetilde{\mathcal{S}}\psi:\mathbb{W}_1\otimes\cdots\mathbb{W}_N\otimes\mathbb{V}\to\mathbb{C}[[q]]\\ \widetilde{\mathcal{S}}\psi(w_1\otimes\cdots\otimes w_N\otimes u)=\sum_{n\in\mathbb{Z}}q^{\widetilde{\mathrm{wt}}(u)+\widetilde{\mathrm{wt}}(w_1)-n-1}\cdot\psi\big(Y(u)_nw_1\otimes w_2\otimes\cdots\otimes w_N\big)\end{gathered} \tag{5.8}$$

when u,w_1 are $\widetilde{L}_0$-homogeneous.

From [FB04, Sec. 10.1], this series converges a.l.u. on $\mathcal{D}_r^\times$ (i.e. when $0<|q|<r$). (See the proof of Thm. 7.1 for the detailed explanation.) Then, by Theorem 5.3, for each $0<|q|<r$, (5.8) converges to a conformal block associated to $\mathfrak{X}_q$ and the local coordinates mentioned after (5.7). If we change the coordinates at x_1 and $\eta_1^{-1}(q)$ to η_1 and η_1-q respectively, then in the formula (5.8), u and w_1 should be multiplied both by $q^{-\widetilde{L}_0}$. Under the trivialization given by the new coordinates, $\widetilde{\mathcal{S}}\psi(w_1\otimes\cdots\otimes w_N\otimes u)$ equals

$$\psi\big(Y(u,q)w_1\otimes w_2\otimes\cdots\otimes w_N\big):=\sum_{n\in\mathbb{Z}}q^{-n-1}\cdot\psi\big(Y(u)_nw_1\otimes w_2\otimes\cdots\otimes w_N\big). \tag{5.9}$$

We conclude that (once the a.l.u. convergence is established) for all $0<|q|<r$, (5.9) is a conformal block associated to $\mathfrak{X}_q$, local coordinates $\eta_1,\eta_2\dots,\eta_N,\eta_1-q$, and modules $\mathbb{W}_1,\dots,\mathbb{W}_N,\mathbb{V}$.

6 An equivalence of sheaves

Recall $\wr\mathfrak{X}=(\wr\pi:\wr\mathcal{C}\to\wr\mathcal{B};\sigma,\wr\varsigma_1,\dots,\wr\varsigma_N)$ in (2.2). In particular, $\wr\mathcal{C}=\mathcal{C}\times_{\mathcal{B}}(\mathcal{C}\backslash S_{\mathfrak{X}})$, $\wr\mathcal{B}=\mathcal{C}\backslash S_{\mathfrak{X}}$. The goal of this section is to establish a canonical isomorphism

$$\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)\simeq\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)|_{\mathcal{C}\backslash S_{\mathfrak{X}}},$$

which relates the sheaves of VOAs and the $\mathscr{W}$-sheaves.

Note that $\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ is the pullback sheaf $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\otimes_{\mathscr{O}_{\mathcal{B}}}\mathscr{O}_{\mathcal{C}}$. This is the sheaf for the presheaf associating to each open $U\subset\mathcal{C}$ the $\mathscr{O}(U)$-module $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big(\pi(U)\big)\otimes_{\mathscr{O}(\pi(U))}\mathscr{O}(U)$. (Note that π is an open map.) Assume the restriction $\mathfrak{X}_{\pi(U)}$ has local coordinates $\eta_1,\dots,\eta_N$ at $\varsigma_1(\pi(U)),\dots,\varsigma_N(\pi(U))$. We write

$$\pi^*w:=w\otimes 1\in\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\otimes_{\mathcal{B}}\mathscr{O}_{\mathcal{C}}=\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$$

for any section $w\in\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$. Sheafifying the tensor product $\mathcal{U}(\eta_\bullet)\otimes 1$ on the presheaf provides an isomorphism of $\mathscr{O}_{\mathcal{C}}$-modules

$$\pi^*\mathcal{U}(\eta_\bullet)\equiv\mathcal{U}(\eta_\bullet)\otimes 1:\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_U\otimes_{\mathscr{O}_{\pi(U)}}\mathscr{O}_U\xrightarrow{\simeq}\Big(\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_{\pi(U)}\Big)\otimes_{\mathscr{O}_{\pi(U)}}\mathscr{O}_U \tag{6.1}$$

or simply a trivialization

$$\pi^*\mathcal{U}(\eta_\bullet):\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_U\xrightarrow{\simeq}\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U. \tag{6.2}$$

Choose $\mu\in\mathscr{O}(U)$ injective on each fiber of U. Then we have a trivialization

$$\boxed{\mathcal{U}_\varrho(\mu)\otimes\pi^*\mathcal{U}(\eta_\bullet):\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_U\xrightarrow{\simeq}\mathbb{V}\otimes\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U} \tag{6.3}$$

Now assume $U\subset\mathcal{C}\backslash S_{\mathfrak{X}}=\wr\mathcal{B}$. Then we can equip the family $\wr\mathfrak{X}_U$ with local coordinates as follows. For the local coordinate at each submanifold $\wr\varsigma_j(U)$ of $\wr\mathcal{C}_U=\wr\mathcal{C}\cap\wr\pi^{-1}(U)$, we choose $\wr\eta_j$ defined by

$$\wr\eta_j(x,y)=\eta_j(x) \tag{6.4}$$

whenever $(x,y)\in\mathcal{C}\times_{\mathcal{B}}\mathcal{C}\backslash S_{\mathfrak{X}}$ makes the above definable. The local coordinate at $\sigma(U)$ is $\triangle\mu$ given by

$$\triangle\mu(x,y)=\mu(x)-\mu(y) \tag{6.5}$$

when $(x,y)\in U\times_{\mathcal{B}}U$. (Recall that σ is the diagonal map.) We can then use $\triangle\mu,\wr\eta_\bullet=(\wr\eta_1,\dots,\wr\eta_N)$ to obtain a trivialization

$$\boxed{\mathcal{U}(\triangle\mu,\wr\eta_\bullet):\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)|_U\xrightarrow{\simeq}\mathbb{V}\otimes\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U} \tag{6.6}$$

We shall relate the two trivializations. First, we need a lemma. Recall $U\subset\mathcal{C}\backslash S_{\mathfrak{X}}$. Recall (3.5) and (4.1).

Lemma 6.1. *Suppose $\eta'_1,\dots,\eta'_N$ are local coordinates of $\mathfrak{X}_{\pi(U)}$ at $\varsigma_1(\pi(U)),\dots,\varsigma_N(\pi(U))$ respectively, and $\mu'\in\mathscr{O}(U)$ is injective on each fiber of U. Then, for each $x\in U$, we have*

$$(\wr\eta_j|\wr\eta'_j)_x=(\eta_j|\eta'_j)_{\pi(x)},\qquad(\triangle\mu|\triangle\mu')_x=\varrho(\mu|\mu')_x.$$

Note that $(\wr\eta_j|\wr\eta'_j)$ is a family of transformations over $U\subset\wr\mathcal{B}=\mathcal{C}\backslash S_{\mathfrak{X}}$, and the transformation over the point x is $(\wr\eta_j|\wr\eta'_j)_x$. $(\triangle\mu|\triangle\mu')_x$ is understood in a similar way.

Proof. We identify $\wr\mathcal{C}_x$ with $\mathcal{C}_{\pi(x)}$ by identifying $(y,x)\in\mathcal{C}\times_{\mathcal{B}}\mathcal{C}\backslash S_{\mathfrak{X}}$ with $y\in\mathcal{C}_{\pi(x)}$. Then, from the definition of $\wr\eta_j,\wr\eta_j'$, we clearly have $\wr\eta_j|_{\wr\mathcal{C}_x}=\eta_j|_{\mathcal{C}_{\pi(x)}}$ and $\wr\eta_j'|_{\wr\mathcal{C}_x}=\eta_j'|_{\mathcal{C}_{\pi(x)}}$. By (4.1), we have

$$(\wr\eta_j|\wr\eta_j')_x\circ\wr\eta_j'|_{\wr\mathcal{C}_x}=\wr\eta_j|_{\wr\mathcal{C}_x},$$
$$(\eta_j|\eta_j')_{\pi(x)}\circ\eta_j'|_{\mathcal{C}_{\pi(x)}}=\eta_j|_{\mathcal{C}_{\pi(x)}}.$$

This proves $(\wr\eta_j|\wr\eta_j')_x=(\eta_j|\eta_j')_{\pi(x)}$.

Similarly,

$$(\triangle\mu|\triangle\mu')_x\circ\triangle\mu'|_{\wr\mathcal{C}_x}=\triangle\mu|_{\wr\mathcal{C}_x}.$$

By (6.5), we have $\triangle\mu|_{\wr\mathcal{C}_x}=(\mu-\mu(x))|_{\mathcal{C}_{\pi(x)}}$ and $\triangle\mu'|_{\wr\mathcal{C}_x}=(\mu'-\mu'(x))|_{\mathcal{C}_{\pi(x)}}$. These imply

$$(\triangle\mu|\triangle\mu')_x\circ(\mu'-\mu'(x))|_{\mathcal{C}_{\pi(x)}}=(\mu-\mu(x))|_{\mathcal{C}_{\pi(x)}}.$$

Comparing this relation with (3.5) shows that $(\triangle\mu|\triangle\mu')_x=\varrho(\mu|\mu')_x$. □

Proposition 6.2. *We have a unique isomorphism*

$$\Psi_{\mathfrak{X}}:\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)\xrightarrow{\simeq}\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_{\mathcal{C}\backslash S_{\mathfrak{X}}}\tag{6.7}$$

such that for any open $U\subset\mathcal{C}\backslash S_{\mathfrak{X}}$ and $\mu,\triangle\mu,\wr\eta_\bullet$ as above, the restriction of this isomorphism to U makes the following diagram commutes.

$$\begin{tikzcd}[column sep=small]
\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)\big|_U \arrow[rr, "\Psi_{\mathfrak{X}}", "\simeq"'] \arrow[dr, "\mathcal{U}(\triangle\mu,\wr\eta_\bullet)", "\simeq"'] & & \mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_U \arrow[dl, "\mathcal{U}_\varrho(\mu)\otimes\pi^*\mathcal{U}(\eta_\bullet)"', "\simeq"] \\
& \mathbb{V}\otimes\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U &
\end{tikzcd}\tag{6.8}$$

Proof. One can define an isomorphism $\Psi_{\mathfrak{X}}$ such that the above diagram commutes. Such isomorphism is clearly unique. Thus, it remains to check that $\Psi_{\mathfrak{X}}$ is well-defined. We will do so by checking that the transition functions of the two sheaves agree.

Assume U is small enough such that we can have another set of $\mu',\eta_\bullet'$ similar to $\mu,\eta_\bullet$. Then by (4.2) and Lemma 6.1, for each $x\in U$, we have equalities

$$\begin{aligned}&\mathcal{U}(\triangle\mu,\wr\eta_\bullet)_x\cdot\mathcal{U}(\triangle\mu',\wr\eta_\bullet')_x^{-1}=\mathcal{U}(\triangle\mu|\triangle\mu')_x\otimes\mathcal{U}(\wr\eta_1|\wr\eta_1')_x\otimes\cdots\otimes\mathcal{U}(\wr\eta_N|\wr\eta_N')_x\\ =&\mathcal{U}(\varrho(\mu|\mu')_x)\otimes\mathcal{U}(\eta_1|\eta_1')_{\pi(x)}\otimes\cdots\otimes\mathcal{U}(\eta_N|\eta_N')_{\pi(x)}\end{aligned}\tag{6.9}$$

for transformations on $\mathbb{V}\otimes\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U|_x\simeq\mathbb{V}\otimes\mathbb{W}_\bullet$.

By (4.2) and (6.1), we have

$$\big(\pi^*\mathcal{U}(\eta_\bullet)\big)_x\cdot\big(\pi^*\mathcal{U}(\eta_\bullet')\big)_x^{-1}=\mathcal{U}(\eta_1|\eta_1')_{\pi(x)}\otimes\cdots\otimes\mathcal{U}(\eta_N|\eta_N')_{\pi(x)}\tag{6.10}$$

for automorphisms of $\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_U|_x\simeq\mathbb{W}_\bullet$. Thus, by (4.2) and (3.7),

$$\begin{aligned}&\big(\mathcal{U}_\varrho(\mu)\otimes\pi^*\mathcal{U}(\eta_\bullet)\big)_x\cdot\big(\mathcal{U}_\varrho(\mu')\otimes\pi^*\mathcal{U}(\eta_\bullet')\big)_x^{-1}\\ =&\mathcal{U}(\varrho(\mu|\mu'))_x\otimes\mathcal{U}(\eta_1|\eta_1')_{\pi(x)}\otimes\cdots\otimes\mathcal{U}(\eta_N|\eta_N')_{\pi(x)},\end{aligned}$$

which equals (6.9). □

7 Propagation of conformal blocks

Let $\phi : \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{B}}$ be a conformal block associated to $\mathbb{W}_\bullet = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ and a family $\mathfrak{X}$ of N-pointed compact Riemann surfaces. Recall $\wr\mathcal{C} = \mathcal{C} \times_{\mathcal{B}} (\mathcal{C}\backslash S_{\mathfrak{X}})$, $\wr\mathcal{B} = \mathcal{C}\backslash S_{\mathfrak{X}}$. The goal of this section is to prove the following theorem. (Cf. [Zhu94, Sec. 6], [FB04, Thm. 10.3.1], [Cod19, Thm. 3.6].)

Theorem 7.1. *There is a unique $\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$-module morphism $\wr\phi : \mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V} \otimes \mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$ satisfying the following property:*

"Choose any open subset $V \subset \mathcal{B}$ such that the restricted family $\mathfrak{X}_V$ has local coordinates $\eta_1, \dots, \eta_N$ at $\varsigma_j(V)$. For each j, we choose a neighborhood $W_j \subset \mathcal{C}_V$ of $\varsigma_j(V)$ on which η_j is defined, such that W_j intersects only $\varsigma_j(V)$ among $\varsigma_1(V), \dots, \varsigma_N(V)$. Identify

$$W_j = (\eta_j, \pi)(W_j) \qquad \text{via } (\eta_j, \pi)$$

so that W_j is a neighborhood of $\{0\} \times V$ in $\mathbb{C} \times V$. Let

$$U_j := W_j \backslash S_{\mathfrak{X}} = W_j \backslash (\{0\} \times V)$$

which is inside $\mathbb{C}^\times \times V$. Let z be the standard coordinate of $\mathbb{C}$. Identify

$$\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_V = \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_V \qquad \text{via } \mathcal{U}(\eta_\bullet).$$

Identify

$$\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V} \otimes \mathbb{W}_\bullet)\big|_{U_j} = \mathbb{V} \otimes \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_{U_j} \qquad \text{via } \mathcal{U}(\triangle\eta_j, \wr\eta_\bullet) \tag{7.1}$$

(cf. (6.6)). For each $u \in \mathbb{V}, w_\bullet \in \mathbb{W}_\bullet$, consider each vector of $\mathbb{W}_\bullet$ as a constant section of $\mathbb{W}_\bullet \otimes \mathscr{O}(U_j)$ and $u \otimes w_\bullet$ as a constant section of $\mathbb{V} \otimes \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}(U_j)$. Then the following equation holds on the level of $\mathscr{O}(V)[[z^{\pm 1}]]$:

$$\boxed{\phi\big(w_1 \otimes \cdots \otimes Y(u,z)w_j \otimes \cdots \otimes w_N\big) = \wr\phi(u \otimes w_\bullet)} \tag{7.2}$$

where $Y(u,z)w := \sum_{n\in\mathbb{Z}} Y(u)_n w \cdot z^{-n-1}$ is an element of $\mathbb{W}_j((z))$, and $\wr\phi(u \otimes w_\bullet) \in \mathscr{O}(U_j)$ is regarded as an element of $\mathscr{O}(V)[[z^{\pm 1}]]$ by taking Laurent series expansion."

Moreover, $\wr\phi$ is a conformal block associated to $\wr\mathfrak{X}$ and $\mathbb{V} \otimes \mathbb{W}_\bullet$.

Note that the left hand side of (7.2) is understood as

$$\sum_{n\in\mathbb{Z}} \phi\big(w_1 \otimes \cdots \otimes Y(u)_n w_j \otimes \cdots \otimes w_N\big) z^{-n-1},$$

which is in $\mathscr{O}(U_j)((z))$.

Proof of the uniqueness of $\wr\phi$. It suffices to restrict to the propagation of each fiber $\mathfrak{X}_b$, i.e., restrict $\wr\phi$ to a morphism $\phi|_{\wr(\mathfrak{X}_b)} : \mathscr{W}_{\wr(\mathfrak{X}_b)}(\mathbb{V} \otimes \mathbb{W}_\bullet) \to \mathscr{O}_{\mathcal{C}_b\backslash S_{\mathfrak{X}_b}}$. (Note that $\wr(\mathfrak{X}_b)$ is $\mathcal{C}_b \times (\mathcal{C}_b\backslash S_{\mathfrak{X}_b}) \to \mathcal{C}_b\backslash S_{\mathfrak{X}_b}$ with marked points.) By (7.2), we know $\wr\phi|_{\wr(\mathfrak{X}_b)}$ is uniquely determined on $(W_1 \cup \cdots \cup W_N) \cap \mathcal{C}_b$. For two possible propagations $\wr_1\phi, \wr_2\phi$, let Ω be the set of all $x \in \mathcal{C}_b\backslash S_{\mathfrak{X}_b}$ on a neighborhood of which $\wr_1\phi|_{\wr(\mathfrak{X}_b)}$ agrees with $\wr_2\phi|_{\wr(\mathfrak{X}_b)}$. Then Ω is open and intersect any connected component of $\mathcal{C}_b$. By complex analysis, it is clear that if U is a connected open subset of $\mathcal{C}_b\backslash S_{\mathfrak{X}_b}$ intersecting Ω such that the restriction $\mathscr{W}_{\wr(\mathfrak{X}_b)}(\mathbb{V} \otimes \mathbb{W}_\bullet)|_U$ is equivalent to $\mathbb{V} \otimes \mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_U$, then $U \subset \Omega$. So Ω is closed, and hence must be $\mathcal{C}\backslash S_{\mathfrak{X}_b}$. This proves the uniqueness. $\square$

Proof of the existence of $\wr\phi$. We identify $\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)$ with $\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_{\mathcal{C}\backslash S_{\mathfrak{X}}}$ as in Prop. 6.2, and construct an $\mathscr{O}_{\mathcal{C}_{S_{\mathfrak{X}}}}$-module morphism $\wr\phi:\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_{\mathcal{C}\backslash S_{\mathfrak{X}}}\to\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$ satisfying (7.2). By the uniqueness proved above, we can safely restrict the base manifold $\mathcal{B}$ to V. So we assume in the following that $\mathcal{B}=V$ and hence $\mathfrak{X}$ has local coordinates $\eta_\bullet$ at marked points. So we identify $\mathscr{W}_{\mathfrak{X}(\mathbb{W}_\bullet)}$ with $\mathbb{W}_\bullet\otimes_{\mathbb{C}}\mathscr{O}_{\mathcal{B}}$ through $\mathcal{U}(\eta_\bullet)$, which yields

$$\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)=\mathscr{V}_{\mathfrak{X}}\otimes_{\mathbb{C}}\mathbb{W}_\bullet \tag{7.3}$$

For each $k\in\mathbb{N}$, we let

$$\mathscr{E}=(\mathscr{V}_{\mathfrak{X}}^{\leqslant k})^\vee$$

be the dual bundle of $\mathscr{V}_{\mathfrak{X}}^{\leqslant k}$. Then the identifications $W_j=(\eta_j,\pi)(W_j)$ and

$$\mathscr{V}_{\mathfrak{X}}^{\leqslant k}|_{W_j}=\mathbb{V}^{\leqslant k}\otimes_{\mathbb{C}}\mathscr{O}_{W_j}\qquad\text{via }\mathcal{U}_\varrho(\eta_j) \tag{7.4}$$

are compatible with the identifications in Sec. A if we set the E_i in that section to be $(\mathbb{V}^{\leqslant k})^\vee$. Choose any $w_\bullet\in\mathbb{W}_\bullet$. Let $s_j=\sum_{n\in\mathbb{Z}}e_{j,n}\cdot z^n$ as in Sec. A where each $e_{j,n}\in(\mathbb{V}^{\leqslant k})^\vee\otimes_{\mathbb{C}}\mathscr{O}(\mathcal{B})$ is defined by

$$u\in\mathbb{V}^{\leqslant k}\mapsto\phi\big(w_1\otimes\cdots\otimes Y(u)_{-n-1}w_j\otimes\cdots\otimes w_N\big)\in\mathscr{O}(\mathcal{B}).$$

For each $b\in\mathcal{B}$, since $\phi|_b$ is a conformal block, it vanishes on $H^0(\mathcal{C}_b,\mathscr{V}_{\mathcal{C}_b}^{\leqslant k}\otimes\omega_{\mathcal{C}_b}(\star S_{\mathfrak{X}_b}))\cdot w_\bullet$. This means that $s_1,\dots,s_N$ satisfy condition (c) of Theorem A.1. Hence, by that theorem, $s_1,\dots,s_N$ are series expansions of a unique element $s\in H^0(\mathcal{C},(\mathscr{V}_{\mathfrak{X}}^{\leqslant k})^\vee(\star S_{\mathfrak{X}}))$, which restricts to $s\in H^0(\mathcal{C}\backslash S_{\mathfrak{X}},(\mathscr{V}_{\mathfrak{X}}^{\leqslant k})^\vee)$ and hence defines an $\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$-module morphism $\mathscr{V}_{\mathfrak{X}}^{\leqslant k}|_{\mathcal{C}\backslash S_{\mathfrak{X}}}\otimes_{\mathbb{C}}w_\bullet\to\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$. These morphisms are compatible for different k, and is extended $\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$-linearly to a morphism $\wr\phi:\mathscr{V}_{\mathfrak{X}}\otimes\pi^*\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)\big|_{\mathcal{C}\backslash S_{\mathfrak{X}}}\to\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$ (recall (7.3)).

By Prop. 6.2, we can regard $\wr\phi$ as a morphism $\wr\phi:\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)\to\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$. Note that the identifications (7.3) and (7.4) are compatible with (7.1), thanks to the commutative diagram (6.8). Thus, $\wr\phi$ satisfies (7.2) under the required identifications. □

Proof that $\wr\phi$ *is a conformal block.* Since being a conformal block is a fiberwise condition, we may prove $\wr\phi$ is a conformal block by restricting it to each fiber $\mathfrak{X}_b$ and its propagation $\wr(\mathfrak{X}_b)$. Therefore, we may assume that $\mathcal{B}$ is a single point. So $C:=\mathcal{C}$ is a compact Riemann surface. We trim each W_j so that $\eta_j(W_j)=\mathcal{D}_{r_j}$ for some $r_j>0$.

From the previous proof, we have a morphism $\wr\phi:\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V}\otimes\mathbb{W}_\bullet)\to\mathscr{O}_{\mathcal{C}\backslash S_{\mathfrak{X}}}$ which, given the trivializations in the statement of Theorem 7.1, is equal to (7.2) when restricted to $W_j\backslash S_{\mathfrak{X}}=W_j\backslash\{\varsigma_j\}$. This shows that the series (7.2) converges a.l.u. on $0<|z|<r_j$. Therefore, as explained in Example 5.4, we can use Thm. 5.3 to conclude that $\wr\phi$ is a conformal block when restricted to each W_j. By Prop. 4.2, $\wr\phi$ is globally a conformal block. □

The proof of Thm. 7.1 is completed.

We now give an application of this theorem. Suppose $\mathbb{E}$ is a set of vectors in a $\mathbb{V}$-module $\mathbb{W}$. We say $\mathbb{E}$ **generates** $\mathbb{W}$ if $\mathbb{W}$ is spanned by vectors of the form $Y(u_1)_{n_1}\cdots Y(u_k)_{n_k}w$ where $k\in\mathbb{Z}_+$, $u_1,\dots,u_k\in\mathbb{V}$, $n_1,\dots,n_k\in\mathbb{Z}$, $w\in\mathbb{E}$.

Proposition 7.2. *Let $\mathfrak{X} = (C; x_1, \ldots, x_N)$ be an N-pointed connected compact Riemann surface, where $N \geqslant 2$. Choose local coordinate η_j at x_j. Associate $\mathbb{V}$-modules $\mathbb{W}_1, \ldots, \mathbb{W}_N$ to $x_1, \ldots, x_N$. Identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) = \mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_N$ via $\mathcal{U}(\eta_\bullet)$. Suppose that for each $2 \leqslant i \leqslant N$, $\mathbb{E}_i$ is a generating subset of $\mathbb{W}_i$. Then any conformal block $\phi : \mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \cdots \mathbb{W}_N \to \mathbb{C}$ is determined by its values on $\mathbb{W}_1 \otimes \mathbb{E}_2 \otimes \cdots \otimes \mathbb{E}_N$.*

Proof. Assume ϕ vanishes on $\mathbb{W}_1 \otimes \mathbb{E}_2 \otimes \cdots \otimes \mathbb{E}_N$. We shall show that ϕ vanishes on $\mathbb{W}_1 \otimes Y(u)_n \mathbb{E}_2 \otimes \cdots \otimes \mathbb{E}_N$ for each $u \in \mathbb{V}, n \in \mathbb{Z}$. Then, by successively applying this result, we see that ϕ vanishes on $\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \mathbb{E}_3 \otimes \cdots \otimes \mathbb{E}_N$, and hence (by repeating again this procedure several times) vanishes on $\mathbb{W}_1 \otimes \mathbb{W}_2 \otimes \cdots \otimes \mathbb{W}_N$.

Identify $\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V} \otimes \mathbb{W}_\bullet) = \mathscr{V}_{\mathfrak{X}}\big|_{C\backslash S_{\mathfrak{X}}} \otimes_{\mathbb{C}} \mathbb{W}_\bullet$ using (6.7). Then we can consider $\wr\phi$ as a morphism $\wr\phi : \mathscr{V}_{\mathfrak{X}}\big|_{C\backslash S_{\mathfrak{X}}} \otimes_{\mathbb{C}} \mathbb{W}_\bullet \to \mathscr{O}_{C\backslash S_{\mathfrak{X}}}$. Let Ω be the open set of all $x \in C\backslash S_{\mathfrak{X}}$ such that x has a neighborhood $U \subset C\backslash S_{\mathfrak{X}}$ such that the restriction

$$\wr\phi|_U : \mathscr{V}_{\mathfrak{X}}|_U \otimes_{\mathbb{C}} \mathbb{W}_1 \otimes \mathbb{E}_2 \otimes \cdots \otimes \mathbb{E}_N \to \mathscr{O}_U$$

vanishes. We note that if U is connected, and if we can find an injective $\eta \in \mathscr{O}(U)$ (so that $\mathscr{V}_{\mathfrak{X}}|_U$ is trivialized to $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_U$), then by complex analysis, $\wr\phi|_U$ vanishes whenever $\wr\phi|_V$ vanishes for some non-empty open $V \subset U$. We conclude that if such U intersects Ω, then U must be inside Ω. So Ω is closed. It is clear that for each $w_1 \in \mathbb{W}_1, w_2 \in \mathbb{E}_2, \ldots, w_N \in \mathbb{E}_N$, the following formal series of z

$$\phi(Y(u, z)w_1 \otimes w_2 \otimes \cdots \otimes w_N)$$

vanishes. Thus, by Thm. 7.1, Ω contains $W_0 \backslash \{x_0\}$ for some neighborhood W_0 of x_0. Therefore $\Omega = C\backslash S_{\mathfrak{X}}$. By Thm. 7.1 again, we see

$$\phi(w_1 \otimes Y(u, z)w_2 \otimes \cdots \otimes w_N)$$

also vanishes. This finishes the proof. □

Remark 7.3. Since $\mathbf{1}$ generates $\mathbb{V}$, we see that if $\mathbb{V}, \mathbb{W}_2, \ldots, \mathbb{W}_N$ (where $N \geqslant 2$) are associated to a connected $\mathfrak{X} = (C; x_1, \ldots, x_N)$, then any conformal block $\phi : \mathbb{V} \otimes \mathbb{W}_2 \otimes \cdots \otimes \mathbb{W}_N \to \mathbb{C}$ is determined by its values on $\mathbf{1} \otimes \mathbb{W}_2 \otimes \cdots \otimes \mathbb{W}_N$. This proves the following two well-known results. In fact, in the literature, the propagation of conformal blocks is best known in the form of the following two corollaries.

Corollary 7.4. *Let $\mathfrak{X} = (C; x_1, \ldots, x_N)$ be an N-pointed compact Riemann surface associated with $\mathbb{V}$-module $\mathbb{W}_1, \ldots, \mathbb{W}_N$. Identify $\mathscr{W}_{\wr\mathfrak{X}}(\mathbb{V} \otimes \mathbb{W}_\bullet) = \mathscr{V}_{\mathfrak{X}}\big|_{C\backslash S_{\mathfrak{X}}} \otimes_{\mathbb{C}} \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$ via (6.7). Then for each $x \in C\backslash S_{\mathfrak{X}}$, $\wr\phi|_x$ is the unique linear map $\mathscr{V}_{\mathfrak{X}}\big|_x \otimes_{\mathbb{C}} \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathbb{C}$ which is a conformal block and satisfies*

$$\wr\phi|_x(\mathbf{1} \otimes w) = \phi(w)$$

for each vector $w \in \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$.

Proof. The uniqueness follows from the previous remark. We shall show that $\wr\phi(\mathbf{1} \otimes w)$, which is an element of $\mathscr{O}(C\backslash S_{\mathfrak{X}})$, equals the constant function $\phi(w)$. By complex analysis, it suffices to prove $\wr\phi(\mathbf{1} \otimes w) = \phi(w)$ when restricted to each $W_j \backslash \{x_j\}$ (where W_j is a small disc containing x_j on which a local coordinate is defined). This is true by (7.2). □

Corollary 7.5. *Let $\mathfrak{X} = (C; x_1, \dots, x_N)$ be an N-pointed connected compact Riemann surface associated with $\mathbb{V}$-module $\mathbb{W}_1, \dots, \mathbb{W}_N$. Choose $x \in C \backslash \{x_1, \dots, x_N\}$. Then the space of conformal blocks associated to $\mathfrak{X}$ and $\mathbb{W}_\bullet$ is isomorphic to the space of conformal blocks associated to $(\wr\mathfrak{X})_x = (C; x, x_1, \dots, x_N)$ and $\mathbb{V}, \mathbb{W}_1, \dots, \mathbb{W}_N$.*

Proof. We assume the identifications in Cor. 7.4. The linear map F from the first space to the second one is defined by $\upphi \mapsto \wr\upphi|_x$. The linear map G from the second one to the first one is defined by $\uppsi \mapsto \uppsi(\mathbf{1} \otimes \cdot)$. By Cor. 7.4, we have $G \circ F = 1$. By Remark 7.3, G is injective. So G is bijective. □

8 Multi-propagation

Let $\mathfrak{X} = (C; x_1, \dots, x_N)$ be an N-pointed compact Riemann surface. Recall $S_{\mathfrak{X}} = \{x_1, \dots, x_N\}$. We choose local coordinates $\eta_1 \in \mathscr{O}(W_1), \dots, \eta_N \in \mathscr{O}(W_N)$ of $\mathfrak{X}$ at $x_1, \dots, x_N$, where each W_j is a neighborhood of x_j satisfying $W_j \cap S_{\mathfrak{X}} = \{x_j\}$.

Let $n \in \mathbb{Z}_+$. By Section 2, $\wr^n\mathfrak{X}$ is

$$\wr^n\mathfrak{X} = (\wr^n\pi : C \times \mathrm{Conf}^n(C\backslash S_{\mathfrak{X}}) \to \mathrm{Conf}^n(C\backslash S_{\mathfrak{X}}); \sigma_1, \dots, \sigma_n, \wr^n x_1, \dots, \wr^n x_N)$$

where $\wr^n\pi$ is the projection onto the second component, and the sections are given by

$$\wr^n x_j(y_1, \dots, y_n) = (x_j, y_1, \dots, y_n),$$
$$\sigma_i(y_1, \dots, y_n) = (y_i, y_1, \dots, y_n).$$

We define local coordinate

$$\wr^n\eta_j(x, y_1, \dots, y_n) = \eta_j(x) \tag{8.1}$$

of $\wr^n\mathfrak{X}$ at $x_j \times \mathrm{Conf}^n(C\backslash S_{\mathfrak{X}})$, defined on $W_j \times \mathrm{Conf}^n(C\backslash S_{\mathfrak{X}})$. Suppose U is an open subset of $C\backslash S_{\mathfrak{X}}$ which admits an injective $\mu \in \mathscr{O}(U)$. Then a local coordinate $\triangle_i\mu$ of $(\wr^n\mathfrak{X})_U$ at $\sigma_i(U)$ is defined by

$$\triangle_i\mu(x, y_1, \dots, y_n) = \mu(x) - \mu(y_i) \tag{8.2}$$

whenever this expression is definable.

We shall relate the $\mathscr{W}$-sheaves with the exterior product $\mathscr{V}_C^{\boxtimes n}$, which is an $\mathscr{O}_{C^n}$-module defined by

$$\mathscr{V}_C^{\boxtimes n} := \mathrm{pr}_1^*\mathscr{V}_C \otimes \mathrm{pr}_2^*\mathscr{V}_C \otimes \cdots \otimes \mathrm{pr}_n^*\mathscr{V}_C. \tag{8.3}$$

Here, each $\mathrm{pr}_i : C^n = \underbrace{C \times \cdots \times C}_{n} \to C$ is the projection onto the i-th component. The tensor products are over $\mathscr{O}_{C^n}$ as usual. Similar to the description in Section 6, the $\mathscr{O}_{C^n}$-module $\mathrm{pr}_i^*\mathscr{V}_C$ is the pullback of the (infinite-rank) vector bundle $\mathscr{V}_C$ along pr_i to C^n, i.e., $\mathscr{V}_C \otimes_{\mathscr{O}_C} \mathscr{O}_{C^n}$ where the action of $f \in \mathscr{O}_C$ on $\mathscr{O}_{C^n}$ is defined by the multiplication of $f \circ \mathrm{pr}_i$. If $U \subset C$ is open and $\mu \in \mathscr{O}(U)$ is injective, we then have a trivilization

$$\mathrm{pr}_i^*\mathcal{U}_\varrho(\mu) : \mathrm{pr}_i^*\mathscr{V}_C\big|_{\mathrm{pr}_i^{-1}(U)} \xrightarrow{\simeq} \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\mathrm{pr}_i^{-1}(U)}.$$

Proposition 8.1. *We have a unique isomorphism*

$$\mathscr{W}_{\wr^n\mathfrak{X}}(\mathbb{V}^{\otimes n} \otimes \mathbb{W}_\bullet) \xrightarrow{\simeq} \mathscr{V}_C^{\boxtimes n}\big|_{\mathrm{Conf}^n(C\backslash S_{\mathfrak{X}})} \otimes_{\mathbb{C}} \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \tag{8.4}$$

such that for any n mutually disjoint open subsets $U_1, \dots, U_n \subset C\backslash S_{\mathfrak{X}}$ and any injective $\mu_1 \in \mathscr{O}(U_1), \dots, \mu_n \in \mathscr{O}(U_n)$, the restriction of this isomorphism to U makes the following diagram commutes.

$$\begin{tikzcd}[column sep=small]
\mathscr{W}_{\wr^n\mathfrak{X}}(\mathbb{V}^{\otimes n} \otimes \mathbb{W}_\bullet)\big|_{U_1\times\cdots\times U_n} \arrow[rr, "\simeq"] \arrow[dr, "\simeq", "\mathcal{U}(\triangle_\bullet\mu_\bullet, \wr^n\eta_\bullet)"'] & & \mathscr{V}_C^{\boxtimes n}\big|_{U_1\times\cdots\times U_n} \otimes_{\mathbb{C}} \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \arrow[dl, "\simeq"', "\mathrm{pr}_1^*\mathcal{U}_\varrho(\mu_1)\otimes\cdots\otimes\mathrm{pr}_n^*\mathcal{U}_\varrho(\mu_n)\otimes\mathcal{U}(\eta_\bullet)"] \\
& \mathbb{V}\otimes\mathbb{W}_\bullet \otimes_{\mathbb{C}} \mathscr{O}_{U_1\times\cdots\times U_n} &
\end{tikzcd} \tag{8.5}$$

Here,

$$(\triangle_\bullet\mu_\bullet, \wr^n\eta_\bullet) := (\triangle_1\mu_1, \dots, \triangle_n\mu_n, \wr^n\eta_1, \dots, \wr^n\eta_n).$$

Proof. Suppose we have another injective $\mu_i' \in \mathscr{O}(U_i)$. Similar to the proof of Lemma 6.1, we see that for each $y_i \in U_i$,

$$(\triangle_i\mu_i|\triangle_i\mu_i')_{(y_1,\dots,y_n)} = \varrho(\mu_i|\mu_i')_{y_i}.$$

(See (3.5) and (4.1) for the meaning of notations.) Using this relation, one shows, as in the proof of Prop. 6.2, that the transition functions for the two trivializations in (8.5) are equal. This finishes the proof. □

Choose a conformal block $\phi : \mathbb{W}_\bullet \to \mathbb{C}$ associated to $\mathfrak{X}$ and $\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$. By Theorem 7.1, we have n-propagation $\wr^n\phi$, which is a conformal block associated to $\wr^n\mathfrak{X}$ and $\mathbb{V}^{\otimes n}\otimes\mathbb{W}_\bullet$. By Prop. 8.1, we can regard $\wr^n\phi$ as a morphism

$$\wr^n\phi : \mathscr{V}_C^{\boxtimes n}\big|_{\mathrm{Conf}^n(C\backslash S_{\mathfrak{X}})} \otimes_{\mathbb{C}} \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) \to \mathscr{O}_{\mathrm{Conf}^n(C\backslash S_{\mathfrak{X}})}.$$

Important facts about $\wr^n\phi$

Choose (non-necessarily disjoint) open $U_1, \dots, U_n \subset C$ and write

$$\mathrm{Conf}(U_\bullet\backslash S_{\mathfrak{X}}) = (U_1 \times \cdots \times U_n) \cap \mathrm{Conf}^n(C\backslash S_{\mathfrak{X}}).$$

For any sections $v_i \in \mathscr{V}_C(U_i)$ and any $w \in \mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet)$, we write

$$\wr^n\phi(v_1, \dots, v_n, w) := \wr^n\phi\Big(\mathrm{pr}_1^*v_1 \otimes \cdots \otimes \mathrm{pr}_n^*v_n \otimes w\big|_{\mathrm{Conf}(U_\bullet\backslash S_{\mathfrak{X}})}\Big)$$
$$\in \mathscr{O}\big(\mathrm{Conf}(U_\bullet\backslash S_{\mathfrak{X}})\big). \tag{8.6}$$

We now summarize some important properties of $\wr^n\phi$ in this setting.

As an elementary fact, the map $(v_1, \dots, v_n) \mapsto \wr^n\phi(v_1, \dots, v_n, w)$ intertwines the action of each $\mathscr{O}(U_i)$ on the i-th component. (Here, each $f \in \mathscr{O}(U_i)$ acts on $\mathscr{O}(\mathrm{Conf}(U_\bullet\backslash S_{\mathfrak{X}}))$ by the multiplication of $(f\circ\mathrm{pr}_i)|_{\mathrm{Conf}(U_\bullet\backslash S_{\mathfrak{X}})}$). Moreover, it is compatible with restricting to open subsets of U_i.

We set $\wr^0\phi = \phi$.

Theorem 8.2. *Identify*

$$\mathscr{W}_{\mathfrak{X}}(\mathbb{W}_\bullet) = \mathbb{W}_\bullet \qquad \text{via } \mathcal{U}(\eta_\bullet).$$

Choose any $w_\bullet \in \mathbb{W}_\bullet$. *For each* $1 \leqslant i \leqslant n$, *choose an open subset* U_i *of* C *equipped with an injective* $\mu_i \in \mathscr{O}(U_i)$. *Identify*

$$\mathscr{V}_C\big|_{U_i} = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{U_i} \qquad \text{via } \mathcal{U}_\varrho(\mu_i).$$

Choose $v_i \in \mathscr{V}_C(U_i) = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(U_i)$. *Choose* $(y_1, \dots, y_n) \in \mathrm{Conf}(U_\bullet \backslash S_{\mathfrak{X}})$. *Then the following are true.*

(1) *If* $U_1 = W_j$ *(where* $1 \leqslant j \leqslant N$*) and contains only* y_1, x_j *among all* $x_\bullet, y_\bullet$, *if* $\mu_1 = \eta_j$, *and if* U_1 *contains the closed disc with center* x_j *and radius* $|\eta_j(y_1)|$ *(under the coordinate* η_j*), then*

$$\begin{aligned} &\wr^n \, \phi(v_1, v_2, \dots, v_n, w_\bullet)\big|_{y_1, y_2, \dots, y_n} \\ &= \wr^{n-1} \, \phi\big(v_2, \dots, v_n, w_1 \otimes \cdots \otimes Y(v_1, z) w_j \otimes \cdots \otimes w_N\big)\big|_{y_2, \dots, y_n} \big|_{z = \eta_j(y_1)} \end{aligned} \tag{8.7}$$

where the series of z *on the right hand side converges absolutely, and* v_1 *is considered as an element of* $\mathbb{V} \otimes \mathbb{C}((z))$ *by taking Taylor series expansion with respect to the variable* η_j *at* x_j.

(2) *If* $U_1 = U_2$ *and contains only* y_1, y_2 *among all* $x_\bullet, y_\bullet$, *if* $\mu_1 = \mu_2$, *and if* U_2 *contains the closed disc with center* y_2 *and radius* $|\mu_2(y_1) - \mu_2(y_2)|$ *(under the coordinate* μ_2*), then*

$$\begin{aligned} &\wr^n \, \phi(v_1, v_2, v_3, \dots, v_n, w_\bullet)\big|_{y_1, y_2, \dots, y_n} \\ &= \wr^{n-1} \, \phi\big(Y(v_1, z) v_2, v_3, \dots, v_n, w_\bullet\big)\big|_{y_2, \dots, y_n} \big|_{z = \mu_2(y_1) - \mu_2(y_2)} \end{aligned} \tag{8.8}$$

where the series of z *on the right hand side converges absolutely, and* v_1 *is considered as an element of* $\mathbb{V} \otimes \mathbb{C}((z))$ *by taking Taylor series expansion with respect to the variable* $\mu_2 - \mu_2(y_2)$ *at* y_2.

(3) *We have*

$$\wr^n \phi(\mathbf{1}, v_2, v_3, \dots, v_n, w_\bullet) = \wr^{n-1} \phi(v_2, \dots, v_n, w_\bullet). \tag{8.9}$$

(4) *For any permutation* π *of the set* $\{1, 2, \dots, n\}$, *we have*

$$\wr^n \phi(v_{\pi(1)}, \dots, v_{\pi(n)}, w_\bullet)\big|_{y_{\pi(1)}, \dots, y_{\pi(n)}} = \wr^n \phi(v_1, \dots, v_n, w_\bullet)\big|_{y_1, \dots, y_n}. \tag{8.10}$$

Proof. When v_1, v_2 are constant sections (i.e. in $\mathbb{V}$), (1) and (2) follow from Thm. 7.1 and especially formula (7.2). The general case follows immediately. (3) follows from Cor. 7.4. By (3), part (4) holds when $v_1, \dots, v_n$ are all the vacuum section $\mathbf{1}$. Thus, it hols for all $v_1, \dots, v_n$ due to Prop. 7.2. $\square$

9 Sewing and multi-propagation

We assume, in addition to the setting of Section 5, that $\widetilde{\mathcal{B}}$ is a single point. Namely, we have an $(N+2M)$-pointed compact Riemann surface

$$\widetilde{\mathfrak{X}} = (\widetilde{C}; x_1, \dots, x_N; x'_1, \dots, x'_M; x''_1, \dots, x''_M),$$

where each connected component of $\widetilde{C}$ contains one of $x_1, \dots, x_N$. For each $1 \leqslant j \leqslant M$, $\widetilde{\mathfrak{X}}$ has local coordinates ξ_j at x'_j and ϖ_j at x''_j defined respectively on neighborhoods $W'_j \ni x'_j, W''_j \ni x''_j$. All W'_j, W''_j (where $1 \leqslant j \leqslant M$) are mutually disjoint and do not contain $x_1, \dots, x_N$. $\xi_j(W'_j) = \mathcal{D}_{r_j}$, and $\varpi_j(W''_j) = \mathcal{D}_{\rho_j}$. For each marked point x_i we associate a $\mathbb{V}$-module $\mathbb{W}_i$. To x'_j and x''_j to we associate respectively a $\mathbb{V}$-module $\mathbb{M}_j$ and its contragredient $\mathbb{M}'_j$. We set

$$S_{\widetilde{\mathfrak{X}}} = \{x_1, \dots, x_N\}.$$

Also, for each $1 \leqslant i \leqslant N$, choose a local coordinate η_i at x_i. Identify

$$\mathscr{W}_{\widetilde{\mathfrak{X}}}(\mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet) = \mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet \qquad \text{via } \mathcal{U}(\eta_\bullet, \xi_\bullet, \varpi_\bullet).$$

We sew $\widetilde{\mathfrak{X}}$ along each x'_j, x''_j to obtain a family

$$\mathfrak{X} = (\pi : \mathcal{C} \to \mathcal{D}^\times_{r_\bullet \rho_\bullet}; x_1, \dots, x_N),$$

where the points $x_1, \dots, x_N$ on $\widetilde{C}$ and the local coordinates $\eta_1, \dots, \eta_N$ at these points extend constantly (over $\mathcal{D}^\times_{r_\bullet \rho_\bullet}$) to sections and local coordinates of $\mathfrak{X}$, denoted by the same symbols. (Cf. Sec. 5.) For each $q_\bullet \in \mathcal{D}^\times_{r_\bullet \rho_\bullet}$, we identify

$$\mathscr{W}_{\mathfrak{X}_{q_\bullet}}(\mathbb{W}_\bullet) = \mathbb{W}_\bullet \qquad \text{via } \mathcal{U}(\eta_\bullet).$$

Let $\phi : \mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet \to \mathbb{C}$ be a conformal block associated to $\widetilde{\mathfrak{X}}$ that converges a.l.u. on $\mathcal{D}^\times_{r_\bullet \rho_\bullet}$. Let $U_1, \dots, U_n \subset \widetilde{C}$ be open and disjoint from each W'_j, W''_j. For each $q_\bullet \in \mathcal{D}^\times_{r_\bullet \rho_\bullet}$, since the fiber $\mathcal{C}_{q_\bullet}$ is obtained by removing a small part of each $W'_j, W''_j \subset \widetilde{C}$ and gluing the remaining part of $\widetilde{C}$, we see that each U_i can be regarded as an open subset of the fiber $\mathcal{C}_{q_\bullet}$. By Thm. 5.3,

$$\widetilde{\mathcal{S}}_{q_\bullet} \phi := \widetilde{\mathcal{S}} \phi|_{q_\bullet}$$

is a conformal block associated to $\mathfrak{X}_{q_\bullet}$. Thus, we can consider its n-propagation $\wr^n \widetilde{\mathcal{S}}_{q_\bullet} \phi$. In the setting of Thm. 8.2, and setting

$$\mathrm{Conf}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}}) = (U_1 \times \cdots \times U_n) \cap \mathrm{Conf}^n(\widetilde{C} \backslash S_{\widetilde{\mathfrak{X}}}),$$

for each $v_i \in \mathscr{V}_{\widetilde{C}}(U_i) = \mathscr{V}_{\mathcal{C}_{q_\bullet}}(U_i)$ and $w_\bullet \in \mathbb{W}_\bullet$,

$$\wr^n \widetilde{\mathcal{S}}_{q_\bullet} \phi(v_1, \dots, v_n, w_\bullet) \qquad \in \mathscr{O}(\mathrm{Conf}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})).$$

This expression relies holomorphically on $q_\bullet$ due to Thm. 7.1 (applied n times). Thus, by varying $q_\bullet$, we obtain

$$\wr^n \widetilde{\mathcal{S}}\phi(v_1,\dots,v_n,w_\bullet) \qquad \in \mathscr{O}\big(\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times \mathrm{Conf}(U_\bullet\backslash S_{\widetilde{\mathfrak{X}}})\big). \tag{9.1}$$

Since $\wr^n\phi$ is a conformal block associated to $\wr^n\widetilde{\mathfrak{X}}$, we can talk about the a.l.u. convergence of its sewing $\widetilde{\mathcal{S}}\wr^n\phi$, which is a conformal block by Thm. 5.3 again. In the setting of Thm. 8.2, this means for each $v_i\in\mathscr{V}_{\widetilde{C}}(U_i)$ and $w_\bullet\in\mathbb{W}_\bullet$ the a.l.u. convergence of

$$\begin{aligned}\widetilde{\mathcal{S}}\wr^n\phi(v_1,\dots,v_n,w_\bullet) &:= \wr^n\phi\Big(v_1,\dots,v_n,w_\bullet\otimes(q_1^{\widetilde{L}_0}\blacktriangleright\otimes_1\blacktriangleleft)\otimes\cdots\otimes(q_M^{\widetilde{L}_0}\blacktriangleright\otimes_M\blacktriangleleft)\Big) \\ &\in\mathscr{O}(\mathrm{Conf}(U_\bullet\backslash S_{\widetilde{\mathfrak{X}}}))[[q_1,\dots,q_M]]\end{aligned} \tag{9.2}$$

on $\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\mathrm{Conf}(U_\bullet\backslash S_{\widetilde{\mathfrak{X}}})$ in the sense of Def. 5.1. We may ask whether this convergence is true, and if it is true, whether the value of this expression at $q_\bullet$ equals (9.1). The answer is Yes.

Theorem 9.1. *Assume $\widetilde{\mathcal{S}}\phi$ converges a.l.u. on $\mathcal{D}^\times_{r_\bullet\rho_\bullet}$. Then for each open $U_1,\dots,U_n\subset\widetilde{C}$ disjoint from W'_j,W''_j ($1\leqslant j\leqslant N$), each $v_i\in\mathscr{V}_{\widetilde{C}}(U_i)$ and $w_\bullet\in\mathbb{W}_\bullet$, the relation*

$$\widetilde{\mathcal{S}}\wr^n\phi(v_1,\dots,v_n,w_\bullet)=\wr^n\widetilde{\mathcal{S}}\phi(v_1,\dots,v_n,w_\bullet) \tag{9.3}$$

holds on the level of $\in\mathscr{O}(\mathrm{Conf}(U_\bullet\backslash S_{\widetilde{\mathfrak{X}}}))[[q_1^{\pm1},\dots,q_M^{\pm1}]]$. In particular, the left hand side converges a.l.u. on $\mathcal{D}^\times_{r_\bullet\rho_\bullet}\times\mathrm{Conf}(U_\bullet\backslash S_{\widetilde{\mathfrak{X}}})$.

We note that the right hand side of (9.3) is considered as a series of $q_1,\dots,q_M$ by taking Laurent series expansion.

Proof. We prove this theorem by induction on n. Let us assume the case for $n-1$ is proved. For each $1\leqslant i\leqslant N$ we choose a neighborhood $W_i\subset\widetilde{C}$ of x_i on which η_i is defined. We assume W_i is small enough such that it does not intersect any W'_j,W''_j ($1\leqslant j\leqslant N$) and contains only x_1 of $x_1,\dots x_N$.

Step 1. Note that we can clearly shrink $\mathcal{D}^\times_{r_\bullet\rho_\bullet}$ since the formal series in (9.3) are independent of the size of this punctured polydisc. Therefore, we can also shrink each W'_j,W''_j to smaller discs, so that the interior of $\widetilde{C}\backslash\bigcup_{1\leqslant j\leqslant M}(W'_j\cup W''_j)$ (denoted by $\mathbf{H}$) is homotopic to $\mathbf{H}_0=\widetilde{C}\backslash\{x'_1,\dots,x'_M,x''_1,\dots,x''_M\}$. Therefore, since each connected component of $\widetilde{C}$ (and hence each one of $\mathbf{H}_0$) intersects $x_1,\dots,x_N$, each one of $\mathbf{H}_0$ contains at least one of $W_1,\dots,W_N$. The same is true for $\mathbf{H}$. So each connected component of $\mathbf{H}\backslash S_{\widetilde{\mathfrak{X}}}$ contains at lease one $W_j\backslash\{x_j\}$.

Fix $U_2,\dots,U_n$ and $v_2,\dots,v_n,w_\bullet$ as in the statement of this theorem. Let Ω be the open set of all $y_1\in\mathbf{H}\backslash S_{\widetilde{\mathfrak{X}}}$ contained in an open $U_1\subset\mathbf{H}\backslash S_{\widetilde{\mathfrak{X}}}$ such that (9.3) holds for all $v_1\in\mathscr{V}_{\widetilde{C}}(U_1)$. By complex analysis, if $V_1\subset\mathbf{H}\backslash S_{\widetilde{\mathfrak{X}}}$ is open such that $\mathscr{V}_{\widetilde{C}}|_{V_1}$ is trivializable (e.g., when there is an injective element of $\mathscr{O}(V_1)$), then $V_1\subset\Omega$ whenever $V_1\cap\Omega\neq\varnothing$. So Ω is closed. Thus, if Ω intersects $W_1\backslash\{x_1\},\dots,W_N\backslash\{x_N\}$, then $\Omega=\mathbf{H}\backslash S_{\widetilde{\mathfrak{X}}}$, which finishes the proof.

Step 2. We show Ω intersects $W_1\backslash\{x_1\}$, and hence intersects the other $W_i\backslash\{x_i\}$ by a similar argument. Indeed, we shall show that (9.3) holds whenever $U_1=W_1$.

Note $w_\bullet = w_1 \otimes w_2 \otimes \cdots \otimes w_N$ by convention. We let $w_\circ = w_2 \otimes \cdots \otimes w_N$. Identify W_1 with $\eta_1(W_1)$ via η_1 so that η_1 is identified with the standard coordinate z. Let $\mathrm{Conf}(U_\circ \backslash S_{\widetilde{\mathfrak{X}}}) = (U_2 \times \cdots \times U_n) \cap \mathrm{Conf}^{n-1}(\widetilde{C} \backslash S_{\widetilde{\mathfrak{X}}})$. Identify $\mathscr{V}_{\widetilde{C}}|_{W_1}$ with $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{W_1}$ using $\mathcal{U}_\varrho(\eta_1)$. Choose any $v_1 \in \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(W_1)$. Then by Thm. 8.2,

$$\widetilde{\mathcal{S}}\, \wr^n\, \phi(v_1, v_2, \dots, v_n, w_\bullet) = \widetilde{\mathcal{S}}\, \wr^{n-1}\, \phi(v_2, \dots, v_n, Y(v_1, z) w_1 \otimes w_\circ)$$

on the level of $\mathscr{O}(\mathrm{Conf}(U_\circ \backslash S_{\widetilde{\mathfrak{X}}}))[[z^{\pm 1}, q_1^{\pm 1}, \dots, q_M^{\pm 1}]]$. By our assumption on the $(n-1)$-case, this expression can be regarded as an element of (and hence this equation holds on the level of) $\mathscr{O}(\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \mathrm{Conf}(U_\circ \backslash S_{\widetilde{\mathfrak{X}}}))[[z^{\pm 1}]]$, and we have

$$\widetilde{\mathcal{S}}\, \wr^n\, \phi(v_1, v_2, \dots, v_n, w_\bullet) = \wr^{n-1} \widetilde{\mathcal{S}} \phi(v_2, \dots, v_n, Y(v_1, z) w_1 \otimes w_\circ)$$

also on this level. By Thm. 8.2 again, this expression equals

$$\wr^n \widetilde{\mathcal{S}} \phi(v_1, v_2, \dots, v_n, w_1 \otimes w_\circ)$$

on this level. Since the above is an element of $\mathscr{O}(\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \mathrm{Conf}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}}))$, by the uniqueness of Laurent series expansion, we see the left hand side of (9.3) is also an element of this ring, and (9.3) holds on this level. □

Remark 9.2. We discuss how to generalize Thm. 9.1 to the case that $\widetilde{\mathfrak{X}}$ is a family of compact Riemann surfaces as in Sec. 5. We assume the setting of that section, together with one more assumption that $\widetilde{\mathfrak{X}}$ has local coordinates $\eta_1, \dots, \eta_N$ at $\varsigma_1(\widetilde{\mathcal{B}}), \dots, \varsigma_N(\widetilde{\mathcal{B}})$ so that we can identify the $\mathscr{W}$-sheaves with the free ones using the trivialization $\mathcal{U}(\eta_\bullet)$ or $\mathcal{U}(\eta_\bullet, \xi_\bullet, \varpi_\bullet)$.

We use freely the notations in Sec. 5. Let $S_{\widetilde{\mathfrak{X}}} = \bigcup_{1 \leqslant i \leqslant M} \varsigma_i(\widetilde{\mathcal{B}})$. Let

$$\phi : \mathbb{W}_\bullet \otimes \mathbb{M}_\bullet \otimes \mathbb{M}'_\bullet \otimes_{\mathbb{C}} \mathscr{O}_{\widetilde{\mathcal{B}}} \to \mathscr{O}_{\widetilde{\mathcal{B}}}$$

be a conformal block associated to $\widetilde{\mathfrak{X}}$ converging a.l.u. on $\mathcal{B} = \mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{B}}$. Choose any open $U_1, \dots, U_n \subset \widetilde{\mathcal{C}}$ disjoint from all W'_j, W''_j. Choose $v_i \in \mathscr{V}_{\widetilde{\mathfrak{X}}}(U_i)$ and $w_\bullet \in \mathbb{W}_\bullet$. Let $\mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})$ be the set of all $(y_1, \dots, y_n) \in \mathrm{Conf}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})$ satisfying $\widetilde{\pi}(y_1) = \cdots = \widetilde{\pi}(y_n)$. For each $m_j \in \mathbb{M}_j, m'_j \in \mathbb{M}'_j$, we have

$$\wr^n \phi(v_1, \dots, v_n, w_\bullet \otimes m_\bullet \otimes m'_\bullet) \qquad \in \mathscr{O}(\mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}}))$$

whose restriction to each $\widetilde{\mathcal{C}}_b^{\times n}$ (where $b \in \widetilde{\mathcal{B}}$ is such that $\widetilde{\mathcal{C}}_b$ intersects $U_1, \dots, U_n$) is $\wr^n(\phi|_b)(v_1, \dots, v_n, w_\bullet \otimes m_\bullet \otimes m'_\bullet)$. (Indeed, this expression is a priori only a function holomorphic when restricted to each $\widetilde{\mathcal{C}}_b^{\times n}$; that it is holomorphic on $\mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})$ (i.e., holomorphic when b also varies) is due to Thm. 7.1.) Thus, we can define

$$\widetilde{\mathcal{S}}\, \wr^n\, \phi(v_1, \dots, v_n, w_\bullet) \qquad \in \mathscr{O}(\mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}}))[[q_1^{\pm 1}, \dots, q_M^{\pm 1}]] \tag{9.4}$$

using (9.2). Similarly, with the aid of Thm. 7.1 we can define

$$\wr^n \widetilde{\mathcal{S}} \phi(v_1, \dots, v_n, w_\bullet) \qquad \in \mathscr{O}\big(\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})\big) \tag{9.5}$$

whose restriction to each $\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \widetilde{\mathcal{C}}_b^{\times n}$ is $\wr^n \widetilde{\mathcal{S}}(\phi|_b)(v_1, \dots, v_n, w_\bullet)$.

Consider (9.5) on the level of $\mathscr{O}(\mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}}))[[q_1^{\pm 1}, \dots, q_M^{\pm 1}]]$. By applying Thm. 9.1 to $\phi|_b$, we see that the coefficients before $q_1, \dots, q_N$ of (9.4) and (9.5) agree when restricted to each $\widetilde{\mathcal{C}}_b^{\times n}$. So (9.4) = (9.5). In particular, (9.4) converges a.l.u. on $\mathcal{D}^\times_{r_\bullet \rho_\bullet} \times \mathrm{Conf}_{\widetilde{\mathcal{B}}}(U_\bullet \backslash S_{\widetilde{\mathfrak{X}}})$.

10 A geometric construction of permutation-twisted $\mathbb{V}^{\otimes k}$-modules

Let $\mathbb{U}$ be a (positive energy) VOA, and let g be an automorphism of $\mathbb{U}$ fixing the vacuum and the conformal vector of $\mathbb{U}$. In particular, g preserves the L_0-grading of $\mathbb{U}$. We assume g has finite order k.

A (finitely-admissible) **g-twisted $\mathbb{U}$-module** is a vector space $\mathcal{W}$ together with a diagonalizable operator $\widetilde{L}_0^g$, and an operation

$$\begin{aligned} Y^g : \mathbb{U} \otimes \mathcal{W} &\to \mathcal{W}[[z^{\pm 1/k}]] \\ u \otimes w &\mapsto Y^g(u,z)w = \sum_{n \in \frac{1}{k}\mathbb{Z}} Y^g(u)_n w \cdot z^{-n-1} \end{aligned}$$

satisfying the following conditions:

1. $\mathcal{W}$ has $\widetilde{L}_0^g$-grading $\mathcal{W} = \bigoplus_{n \in \frac{1}{k}\mathbb{N}} \mathcal{W}(n)$, each eigenspace $\mathcal{W}(n)$ is finite-dimensional, and for any $u \in \mathbb{U}$ we have
$$[\widetilde{L}_0^g, Y^g(u)_n] = Y^g(L_0 u)_n - (n+1) Y^g(u)_n. \tag{10.1}$$
In particular, for each $w \in \mathcal{W}$ the lower truncation condition follows: $Y^g(u)_n w = 0$ when n is sufficiently small.

2. $Y^g(\mathbf{1}, z) = \mathbf{1}_{\mathcal{W}}$.

3. (g-equivariance) For each $u \in \mathbb{U}$,
$$Y^g(gu, z) = Y^g(u, e^{-2\mathbf{i}\pi} z) := \sum_{n \in \frac{1}{k}\mathbb{Z}} Y^g(u)_n w \cdot e^{2(n+1)\mathbf{i}\pi} z^{-n-1}. \tag{10.2}$$

4. (Jacobi identity-analytic version) Let $\mathcal{W}' = \bigoplus_{n \in \frac{1}{k}\mathbb{N}} \mathcal{W}(n)^*$. Let P_n be the projection of $\overline{\mathcal{W}} = \prod_{n \in \frac{1}{k}\mathbb{N}} \mathcal{W}(n)^*$ (the dual space of $\mathcal{W}'$) onto $\mathcal{W}(n)$ and similarly $\overline{\mathbb{U}}$ (the dual space of $\mathbb{U}'$) onto $\mathbb{U}(n)$. Then for each $u, v \in \mathbb{U}$, $w \in \mathcal{W}$, $w' \in \mathcal{W}'$, and for each $z \neq \xi$ in $\mathbb{C}^\times$ with chosen $\arg \xi$, the following series of n
$$\langle Y^g(u,z) Y^g(v,\xi) w, w' \rangle := \sum_{n \in \frac{1}{k}\mathbb{N}} \langle Y^g(u,z) P_n Y^g(v,\xi) w, w' \rangle \tag{10.3}$$
$$\langle Y^g(v,\xi) Y^g(u,z) w, w' \rangle := \sum_{n \in \frac{1}{k}\mathbb{N}} \langle Y^g(v,\xi) P_n Y^g(u,z) w, w' \rangle \tag{10.4}$$
$$\langle Y^g(Y(u,z-\xi)v,\xi) w, w' \rangle := \sum_{n \in \mathbb{N}} \langle Y^g(P_n Y(u,z-\xi)v,\xi) w, w' \rangle \tag{10.5}$$
(where ξ is fixed) converge a.l.u. for z in $|z| > |\xi|$, $|z| < |\xi|$, $|z - \xi| < |\xi|$ respectively. Moreover, for any fixed $\xi \in \mathbb{C}^\times$ with chosen argument $\arg \xi$, let R_ξ be the ray with argument $\arg \xi$ from 0 to ∞, but with $0, \xi, \infty$ removed. Any point on R_ξ

is assumed to have argument $\arg \xi$. Then the above three expressions, considered as functions of z defined on R_ξ satisfying the three mentioned inequalities respectively, can be analytically continued to the same holomorphic function on the open set

$$\Delta_\xi = \mathbb{C}\backslash\{\xi, -t\xi : t \geqslant 0\},$$

which can furthermore be extended to a multivalued holomorphic function $f_\xi(z)$ on $\mathbb{C}^\times\backslash\{\xi\}$ (i.e., a holomorphic function on the universal cover of $\mathbb{C}^\times\backslash\{\xi\}$).

In the above Jacobi identity, if we let the series $\sum_n h_n(z)$ be any of (10.3), (10.4), (10.5), then by saying that this series converges a.l.u. for z in an open set Ω, we mean $\sup_{z\in K}\sum_n |f_n(z)| < +\infty$ for each compact $K \subset \Omega$; the sup is over all $z \in K$ with all possible $\arg z$.

Remark 10.1. The above analytic version of Jacobi identity is equivalent to the usual algebraic one. Indeed, assume without loss of generality that $gu = e^{2\mathbf{i}j\pi/k}u$. Then the g-equivariance condition shows that $z^{\frac{j}{k}}Y^g(u,z)$ is single-valued over z. Thus, $z^{\frac{j}{k}}$ times (10.3), (10.4), (10.5) are series expansions on $|z| > |\xi|$, $|z| < |\xi|$, $|z-\xi| < |\xi|$ respectively (not necessarily restricting to R_ξ) of the same single-valued holomorphic function $z^{\frac{j}{k}}f_\xi$ on $\mathbb{C}^\times\backslash\{\xi\}$. By Strong Residue Theorem, this is equivalent to that for each $m, n \in \mathbb{Z}$,

$$\Big(\oint_{|z|=2|\xi|} - \oint_{|z|=|\xi|/3} - \oint_{|z-\xi|=|\xi|/3}\Big) z^{\frac{j}{k}+m}(z-\xi)^n f_\xi(z)dz = 0,$$

where in these integrals, $f_\xi(z)$ is replaced by (10.3), (10.4), (10.5) respectively. Equivalently,

$$\begin{aligned}
&\sum_{l\in\mathbb{N}}\binom{\frac{j}{k}+m}{l}\big\langle Y^g\big(Y(u)_{n+l}v,\xi\big)w,w'\big\rangle\xi^{\frac{j}{k}+m-l}\\
=&\sum_{l\in\mathbb{N}}\binom{n}{l}(-1)^l\big\langle Y^g(u)_{\frac{j}{k}+m+n-l}Y^g(v,\xi)w,w'\big\rangle\xi^l\\
&-\sum_{l\in\mathbb{N}}\binom{n}{l}(-1)^{n-l}\big\langle Y^g(v,\xi)Y^g(u)_{\frac{j}{k}+m+l}w,w'\big\rangle\xi^{n-l}.
\end{aligned}\tag{10.6}$$

By comparing the coefficients before ξ^{-h-1}, the above is equivalent to that for each $m, n \in \mathbb{Z}, h \in \frac{1}{k}\mathbb{Z}$, (suppressing w, w')

$$\begin{aligned}
&\sum_{l\in\mathbb{N}}\binom{\frac{j}{k}+m}{l}Y^g\big(Y(u)_{n+l}\,v\big)_{\frac{j}{k}+m+h-l}\\
=&\sum_{l\in\mathbb{N}}\binom{n}{l}(-1)^lY^g(u)_{\frac{j}{k}+m+n-l}Y^g(v)_{h+l}-\sum_{l\in\mathbb{N}}\binom{n}{l}(-1)^{n-l}Y^g(v)_{n+h-l}Y^g(u)_{\frac{j}{k}+m+l}
\end{aligned}\tag{10.7}$$

which is the algebraic Jacobi identity.

Construction of twisted representations associated to cyclic permutation actions of $\mathbb{V}^{\otimes k}$

We let $\mathbb{U} = \mathbb{V}^{\otimes k}$ with conformal vector $\mathbf{c} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{c}$, and g an automorphism defined by

$$g : (v_1, v_2, \ldots, v_k) \in \mathbb{V}^{\otimes k} \mapsto (v_k, v_1, \ldots, v_{k-1}).$$

For each $\mathbb{V}$-module with $\widetilde{L}_0$-operator, we define an associated g-twisted $\mathbb{U}$-module $\mathcal{W}$ as follows.

As a vector space, $\mathcal{W} = \mathbb{W}$. We define $\widetilde{L}_0^g = \frac{1}{k}\widetilde{L}_0$.

Let ζ be the standard coordinate of $\mathbb{C}$. Let $\mathfrak{X} = (\mathbb{P}^1; 0, \infty)$. We associate to $0, \infty$ local coordinates local coordinates ζ, ζ^{-1} and $\mathbb{V}$-modules $\mathbb{W}, \mathbb{W}'$. Note

$$\mathcal{U}(\zeta, \zeta^{-1}) : \mathscr{W}_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{W}') \xrightarrow{\simeq} \mathbb{W} \otimes \mathbb{W}'$$

Let $\langle \cdot, \cdot \rangle$ be the pairing for $\mathbb{W}$ and $\mathbb{W}'$. We define a conformal block

$$\begin{gathered} \uptau_{\mathbb{W}} : \mathscr{W}_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{W}') \to \mathbb{C}, \\ \mathcal{U}(\eta_0, \eta_\infty)^{-1}(w \otimes w') \mapsto \langle w, w' \rangle \end{gathered}$$

whenever the local coordinates η_0, η_∞ at $0, \infty$ are such that $(\mathbb{P}^1; 0, \infty; \eta_0, \eta_\infty) \simeq (\mathbb{P}^1; 0, \infty; \zeta, \zeta^{-1})$. It is easy to see that this definition is independent of the choice of such η_0, η_∞.

In the setting of Thm. 8.2, we have

$$\wr^k \uptau_{\mathbb{W}} : \underbrace{\mathscr{V}_{\mathfrak{X}}(\mathbb{C}^\times) \otimes \cdots \otimes \mathscr{V}_{\mathfrak{X}}(\mathbb{C}^\times)}_{k} \otimes \mathscr{W}_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{W}') \to \mathscr{O}(\mathrm{Conf}^k(\mathbb{C}^\times))$$

where all the $\otimes$ are over $\mathbb{C}$. Let

$$\omega_k = e^{-2\mathbf{i}\pi/k}.$$

Since $\zeta^k : z \mapsto z^k$ is locally injective holomorphic on $\mathbb{C}^\times$, we have a trivilization

$$\mathcal{U}_\varrho(\zeta^k) : \mathscr{V}_{\mathfrak{X}}|_{\mathbb{C}^\times} \xrightarrow{\quad\simeq\quad} \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^\times}.$$

Then, for each $w \in \mathbb{W}, w' \in \mathbb{W}'$, and for each $v_1, \ldots, v_n \in \mathbb{V}$ (considered as a constant section of $\mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}(\mathbb{C}^\times)$) we define, for $v_\bullet = v_1 \otimes \cdots \otimes v_k \in \mathbb{V}^{\otimes k}$,

$$\boxed{\begin{aligned} &\langle Y^g(v_\bullet, z)w, w' \rangle \\ &= \wr^k \uptau_{\mathbb{W}} \big(\mathcal{U}_\varrho(\zeta^k)^{-1} v_1, \ldots, \mathcal{U}_\varrho(\zeta^k)^{-1} v_k, \mathcal{U}(\zeta, \zeta^{-1})^{-1}(w \otimes w')\big)\Big|_{\omega_k^{\bullet-1}\sqrt[k]{z}} \end{aligned}} \tag{10.8}$$

where, for each $z \in \mathbb{C}^\times$ with argument $\arg z$,

$$\omega_k^{\bullet-1}\sqrt[k]{z} := (\sqrt[k]{z}, \omega_k \sqrt[k]{z}, \omega_k^2 \sqrt[k]{z}, \ldots, \omega_k^{k-1}\sqrt[k]{z}) \qquad \in \mathrm{Conf}^k(\mathbb{C}^\times), \tag{10.9}$$

and $\sqrt[k]{z}$ is assumed to have argument $\frac{1}{k}\arg z$.

(10.8) is a multi-valued function of z, single-valued of $\sqrt[k]{z} \in \mathbb{C}^\times$. So we have Laurent series expansion

$$\langle Y^g(v_\bullet, z)w, w'\rangle = \sum_{n \in \frac{1}{k}\mathbb{Z}} \langle Y^g(v_\bullet)_n w, w'\rangle z^{-n-1}$$

which defines $Y^g(v_\bullet)_n$ as a linear map $\mathbb{W} \otimes \mathbb{W}' \to \mathbb{C}$.

Lemma 10.2. *Each $Y^g(v_\bullet)_n$ is a linear operator on $\mathbb{W}$. Moreover, (10.1) is satisfied.*

Proof. For each $q \in \mathbb{C}^\times$ with chosen $\arg q$, by (4.2) we have

$$\mathcal{U}(q^{\frac{1}{k}}\zeta, q^{-\frac{1}{k}}\zeta^{-1})\mathcal{U}(\zeta, \zeta^{-1})^{-1} = q^{\frac{1}{k}\widetilde{L}_0} \otimes q^{-\frac{1}{k}\widetilde{L}_0} = q^{\widetilde{L}_0^g} \otimes q^{-\widetilde{L}_0^g}.$$

Thus

$$\begin{aligned}
&\langle Y^g(v_\bullet, z)q^{-\widetilde{L}_0^g}w, q^{\widetilde{L}_0^g}w'\rangle \\
&= \wr^k\, \tau_{\mathbb{W}}\big(\mathcal{U}_\varrho(\zeta^k)^{-1}v_1, \dots, \mathcal{U}_\varrho(\zeta^k)^{-1}v_k, \mathcal{U}(q^{\frac{1}{k}}\zeta, q^{-\frac{1}{k}}\zeta^{-1})^{-1}(w \otimes w')\big)\Big|_{\omega_k^{\bullet -1}\sqrt[k]{z}}.
\end{aligned} \tag{10.10}$$

We have an equivalence of pointed Riemann spheres with locally injective functions and local coordinates (at the last two marked points)

$$\begin{aligned}
&(\mathbb{P}^1; \omega_k^{\bullet -1}\sqrt[k]{z}, 0, \infty; \zeta^k, q^{\frac{1}{k}}\zeta, q^{-\frac{1}{k}}\zeta^{-1}) \\
\simeq &(\mathbb{P}^1; \omega_k^{\bullet -1}\sqrt[k]{qz}, 0, \infty; q^{-1}\zeta^k, \zeta, \zeta^{-1})
\end{aligned}$$

defined by $z \in \mathbb{P}^1 \mapsto \sqrt[k]{q}z \in \mathbb{P}^1$, where $\sqrt[k]{q}$ has argument $\frac{1}{k}\arg q$. By (3.7) and (3.5), on $\mathbb{V}$ we have

$$\mathcal{U}_\varrho(\zeta^k)\mathcal{U}_\varrho(q^{-1}\zeta^k)^{-1} = \mathcal{U}(\varrho(\zeta^k|q^{-1}\zeta^k)) = q^{L_0}.$$

So (10.10) equals

$$\begin{aligned}
&\wr^k\, \tau_{\mathbb{W}}\big(\mathcal{U}_\varrho(q^{-1}\zeta^k)^{-1}v_1, \dots, \mathcal{U}_\varrho(q^{-1}\zeta^k)^{-1}v_k, \mathcal{U}(\zeta, \zeta^{-1})^{-1}(w \otimes w')\big)\Big|_{\omega_k^{\bullet -1}\sqrt[k]{qz}} \\
&= \wr^k\, \tau_{\mathbb{W}}\big(\mathcal{U}_\varrho(\zeta^k)^{-1}q^{L_0}v_1, \dots, \mathcal{U}_\varrho(\zeta^k)^{-1}q^{L_0}v_k, \mathcal{U}(\zeta, \zeta^{-1})^{-1}(w \otimes w')\big)\Big|_{\omega_k^{\bullet -1}\sqrt[k]{qz}}
\end{aligned}$$

We conclude

$$\langle Y^g(v_\bullet, z)q^{-\widetilde{L}_0^g}w, q^{\widetilde{L}_0^g}w'\rangle = \langle Y^g(q^{L_0}v_\bullet, qz)w, w'\rangle.$$

So, if $L_0 v_\bullet = \alpha v_\bullet$, $\widetilde{L}_0^g w = \beta w$, $\widetilde{L}_0^g w' = \gamma w'$, then

$$\langle Y^g(v_\bullet, z)w, w'\rangle = q^{\alpha+\beta-\gamma}\langle Y^g(v_\bullet, qz)w, w'\rangle,$$

which shows, by looking at the coefficients before z^{-n-1}, that $\langle Y^g(v_\bullet)_n w, w'\rangle$ equals 0 unless $\alpha+\beta-\gamma-n-1 = 0$. This proves $Y^g(v_\bullet)_n\mathcal{W}(\beta) \subset \mathcal{W}(\alpha+\beta-n-1)$. In particular, $Y^g(v_\bullet)_n$ can be regarded as a linear operator on $\mathcal{W}$. $\square$

Using part (3) and (4) of Thm. 8.2, it is easy to show $Y^g(\mathbf{1}, z) = \mathbf{1}_{\mathcal{W}}$ and show (10.2). Moreover:

Theorem 10.3. *Y^g satisfies the Jacobi identity. Therefore, $(\mathcal{W}, Y^g)$ is a g-twisted $\mathbb{V}^{\otimes k}$-module.*

Proof. Choose the two vectors of $\mathbb{U}$ to be $u_\bullet = u_1 \otimes \cdots \otimes u_k, v_\bullet = v_1 \otimes \cdots \otimes v_k \in \mathbb{V}^{\otimes k}$. Identify $\mathscr{W}_{\mathfrak{X}}(\mathbb{W} \otimes \mathbb{W}') = \mathbb{W} \otimes \mathbb{W}'$ via $\mathcal{U}(\zeta, \zeta^{-1})$. Identify $\mathscr{V}_{\mathfrak{X}}|_{\mathbb{C}^\times} = \mathbb{V} \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{C}^\times}$ via $\mathcal{U}_\varrho(\zeta^k)$. For each $\xi \in \mathbb{C}^\times$ with chosen $\arg \xi$, we define

$$f_\xi(z) = \wr^{2k} \uptau_{\mathbb{W}}(u_1, \ldots, u_k, v_1, \ldots, v_k, w \otimes w')\Big|_{\omega_k^{\bullet-1} \sqrt[k]{z},\, \omega_k^{\bullet-1} \sqrt[k]{\xi}} \tag{10.11}$$

where $\omega_k^{\bullet-1} \sqrt[k]{\xi}$ is a k-tuple understood in a similar way as (10.9). Then f_ξ is a multi-valued holomorphic function which lifts to a single-valued one on the k-fold covering space $\mathbb{C}^\times \backslash (\omega_k^{\bullet-1} \sqrt[k]{\xi})$ of $\mathbb{C}^\times \backslash \{\xi\}$.

Let $(m_{n,\alpha})_{\alpha \in \mathfrak{A}}$ be a set of basis of $\mathbb{W}(n)$ with dual basis $(\check{m}_{n,\alpha})_{\alpha \in \mathfrak{A}}$. Assume $0 < |z| < |\xi|$. We shall show that the following infinite sum over n

$$\begin{aligned}
&\langle Y^g(v_\bullet, \xi) Y^g(u_\bullet, z) w, w' \rangle \\
=& \sum_{n \in \mathbb{N}} \sum_{\alpha \in \mathfrak{A}} \wr^k \uptau_{\mathbb{W}}(u_1, \ldots, u_k, w \otimes \check{m}_{n,\alpha})_{\omega_k^{\bullet-1} \sqrt[k]{z}} \cdot \wr^k \uptau_{\mathbb{W}}(v_1, \ldots, v_k, m_{n,\alpha} \otimes w')_{\omega_k^{\bullet-1} \sqrt[k]{\xi}}
\end{aligned} \tag{10.12}$$

converges a.l.u. to $f_\xi(z)$. Indeed, this expression is the sewing at $q = 1$ of the $2k$-propagation of the conformal block

$$\begin{gathered}
\upphi : \mathbb{W} \otimes \mathbb{W}' \otimes \mathbb{W} \otimes \mathbb{W}' \to \mathbb{C}, \\
w_1 \otimes w_1' \otimes w_2 \otimes w_2' \mapsto \langle w_1, w_1' \rangle \cdot \langle w_2, w_2' \rangle
\end{gathered}$$

associated to $(\mathbb{P}_a^1 \sqcup \mathbb{P}_b^1; 0_a, \infty_a, 0_b, \infty_b)$. Here, $\mathbb{P}_a^1, \mathbb{P}_b^1$ are two identical Riemann spheres. The sewing is along ∞_a and 0_b using local coordinates ζ, ζ^{-1}, and by choosing suitable open discs $W' \ni \infty_a, W'' \ni 0_b$ with radius r, ρ satisfying $r\rho > 1$ such that W', W'' do not intersect $\omega_k^{\bullet-1} \sqrt[k]{z}$ and $\omega_k^{\bullet-1} \sqrt[k]{\xi}$. (Note that $|z| < |\xi|$ guarantees the existence of such W', W''.) Since the sewing of $\upphi$ clearly converges a.l.u. on $\mathcal{D}_{r\rho}^\times$, by Thm. 9.1, the sewing at $q = 1$ of $\wr^{2k}\upphi$ (which is (10.12)) converges a.l.u. (for varying z) to the $2k$-propagation of the sewing, which is just $f_\xi(z)$. A similar argument shows that when $0 < |\xi| < |z|$, $\langle Y^g(u_\bullet, z) Y^g(v_\bullet, \xi) w, w' \rangle$ converges a.l.u. (for varying z) to $f_\xi(z)$.

Consider $g_\xi \in \mathrm{Conf}^k(\mathbb{C} \backslash \omega_k^{\bullet-1} \sqrt[k]{\xi})$ defined by

$$g_\xi(z_1, \ldots, z_k) = \wr^{2k} \uptau_{\mathbb{W}}(u_1, \ldots, u_k, v_1, \ldots, v_k, w \otimes w')\Big|_{z_1, \ldots, z_k,\, \omega_k^{\bullet-1} \sqrt[k]{\xi}}.$$

The region $\Omega = \{z \in \mathbb{C}^\times : |z^k - \xi| < |\xi|\}$ has k connected components $\Omega_1, \ldots, \Omega_k$, each one Ω_i contains exactly one element $\omega_k^{i-1} \sqrt[k]{\xi}$ of $\omega_k^{\bullet-1} \sqrt[k]{\xi}$, and $\Omega_i \simeq \zeta^k(\Omega_i)$ where $\zeta^k(\Omega_i)$ is the open disc with center ξ and radius $|\xi|$. By Thm. 8.2 and the definition (10.8), whenever $z_i \in \Omega_i$ for each i, we have (letting $x_1, \ldots, x_k$ be formal variables)

$$\begin{aligned}
&g_\xi(z_1, \ldots, z_k) \\
=& \wr^k \, \uptau_{\mathbb{W}}(Y(u_1, x_1) v_1, \ldots, Y(u_k, x_k) v_k, w \otimes w')\Big|_{\omega_k^{\bullet-1} \sqrt[k]{\xi}}\Big|_{x_k = z_k^k - \xi} \cdots \Big|_{x_1 = z_1^k - \xi}
\end{aligned}$$

$$=\langle Y^g(Y(u_1,x_1)v_1\otimes\cdots\otimes Y(u_k,x_k)v_k,\xi)w,w'\rangle\Big|_{x_k=z_k^k-\xi}\cdots\Big|_{x_1=z_1^k-\xi}. \tag{10.13}$$

where the right hand side converges absolutely and successively for $x_k,x_{k-1},\dots,x_1$. Since the simultaneous Laurent series expansion of the holomorphic function $h(\varkappa_1,\dots\varkappa_k)=g_\xi(\sqrt[k]{\xi+\varkappa_1},\omega_k\sqrt[k]{\xi+\varkappa_2},\dots,\omega_k^{k-1}\sqrt[k]{\xi+\varkappa_k})$ in the region $0<|\varkappa_i|<|\xi|$ (for all i) clearly converges a.l.u., and since the coefficients of these series agree with those before the powers of $x_1,\dots,x_k$ on the right hand side of (10.13) (by taking Laurent series expansion through contour integrals), we see that (10.13) converges absolutely (as a multi-variable series) to $g_\xi(z_1,\dots,z_k)$ at the desired points.

Now we assume $0<|z-\xi|<|\xi|$, assume $\arg z$ is such that $\sqrt[k]{z}\in\Omega_1\ni\sqrt[k]{\xi}$ (which is true when $\arg z=\arg\xi$), and set $(z_1,\dots,z_k)=\omega_k^{\bullet-1}\sqrt[k]{z}$. Then we see that $\langle Y^g(Y(u_\bullet,z-\xi)v_\bullet,\xi)w,w'\rangle$ converges a.l.u. to $g_\xi(\omega_k^{\bullet-s}\sqrt[k]{z})=f_\xi(z)$. This finishes the verification of the Jacobi identity. $\square$

Remark 10.4. Using Thm. 8.2, it is easy to see that

$$\wr^k\tau_{\mathbb{W}}(\mathbf{1},\cdots,\mathcal{U}_\varrho(\zeta)^{-1}v_i,\cdots,\mathbf{1},w\otimes w')|_z=\langle Y(v,z)w,w'\rangle.$$

By (3.7), $\mathcal{U}_\varrho(\zeta)\mathcal{U}_\varrho(\zeta^k)^{-1}=\mathcal{U}(\varrho(\zeta|\zeta^k))$. Thus, when $v_\bullet=v_1\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}$, (10.8) becomes

$$\langle Y(\mathcal{U}(\varrho(\zeta|\zeta^k)_{\sqrt[k]{z}})v_1,\sqrt[k]{z})w,w'\rangle.$$

By (3.5), $\varrho(\zeta|\zeta^k)_{\sqrt[k]{z}}$ sends z_1^k-z to $z_1-\sqrt[k]{z}$ when z_1 is close to $\sqrt[k]{z}$. Hence this transformation equals $\delta_{k,z}$ where

$$\delta_{k,z}(t)=(z+t)^{\frac{1}{k}}-z^{\frac{1}{k}}.$$

We conclude

$$Y^g(v_1\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},z)=Y(\mathcal{U}(\delta_{k,z})v_1,\sqrt[k]{z}). \tag{10.14}$$

This equation uniquely determines the g-twisted module structure of $\mathcal{W}$, since $\mathbb{V}^{\otimes k}$ is g-generated by vectors of the form $v_1\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}$.

It is not hard to check that $\mathcal{U}(\delta_{k,z})$ agrees with the operator $\Delta_k(z)$ in [BDM02]. Thus, our g-twisted module $(\mathcal{W},Y^g)$ agrees with the one $(T_g^k(\mathbb{W}),Y_g)$ in [BDM02, Thm. 3.9].

A Strong residue theorem for analytic families of curves

Let $\mathfrak{X}=(\pi:\mathcal{C}\to\mathcal{B};\varsigma_1,\dots,\varsigma_N)$ be a (holomorphic) family of N-pointed compact Riemann surfaces. Recall the definition in Sec. 2. In particular, we assume each connected component of each fiber $\mathcal{C}_b=\pi^{-1}(b)$ contains at least one of $\varsigma_1(b),\dots,\varsigma_N(b)$. We let $\mathscr{E}$ be a holomorphic vector bundle on $\mathcal{C}$ with finite rank, and let $\mathscr{E}^\vee$ be its dual bundle.

We assume that $\mathfrak{X}$ is equipped with local coordinates $\eta_1,\dots,\eta_N$ at $\varsigma_1(\mathcal{B}),\dots,\varsigma_N(\mathcal{B})$ respectively. Assume for each j that η_j is defined on a neighborhood $W_j\subset\mathcal{C}$ of $\varsigma_j(\mathcal{B})$ which intersects only $\varsigma_j(\mathcal{B})$ among $\varsigma_1(\mathcal{B}),\dots,\varsigma_N(\mathcal{B})$, and that there is a trivialization

$$\mathscr{E}_j|_{W_j}\simeq E_j\otimes_{\mathbb{C}}\mathscr{O}_{W_j}$$

with dual trivialization

$$\mathscr{E}_j^\vee|_{W_j} \simeq E_j^\vee \otimes_{\mathbb{C}} \mathscr{O}_{W_j},$$

where E_j is a finite-dimensional vector space and $E_j^\vee$ is its dual space. We identify $\mathscr{E}_j|_{W_j}$ and $\mathscr{E}^\vee|_{W_j}$ with their trivializations.

For each j, we identify

$$W_j = (\pi, \eta_j)(W_j) \qquad \text{via } (\pi, \eta_j).$$

Then W_j is a neighborhood of $\mathcal{B} \times \{0\}$ in $\mathcal{B} \times \mathbb{C}$. We let z be the standard coordinate of $\mathbb{C}$. Consider

$$s_j = \sum_{n\in\mathbb{Z}} e_{j,n} \cdot z^n \qquad \in \big(E_j \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})\big)((z)), \tag{A.1}$$

where each $e_{j,n} \in E_j \otimes_{\mathbb{C}} \mathscr{O}(\mathcal{B})$ is 0 when n is sufficiently small. Considering $e_{j,n}$ as an E_j-valued holomorphic on $\mathscr{O}(\mathcal{B})$, we let $e_{j,n}(b) \in E_j$ be its value at $b \in \mathcal{B}$. Then $s_j(b)$, the restriction of s_j to $\mathcal{C}_b$, is represented by

$$s_j(b) = \sum_n e_{j,n}(b) z^n \qquad \in E_j((z)).$$

Suppose that s is a section of $\mathscr{E}(\star S_{\mathfrak{X}})$ defined on W_j. Then $s|_{W_j} = s|_{W_j}(b,z)$ is an E_j-valued meromorphic function on W_j with poles at $z = 0$. We say that s **has series expansion** s_j **at** $\varsigma_j(\mathcal{B})$ if for each $b \in \mathcal{B}$, the meromorphic function $s|_{W_j}(b,z)$ of z has Laurent series expansion (A.1) at $z = 0$.

For each $b \in \mathcal{B}$, choose $\sigma_b \in H^0(\mathcal{C}_b, \mathscr{E}^\vee|_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(\star S_{\mathfrak{X}_b}))$. Then in $W_{j,b} = W_j \cap \pi^{-1}(B)$, σ_b can be regarded as an $E_j^\vee \otimes dz$-valued holomorphic function but with possibly finite poles at $z = 0$. So it has series expansion at $z = 0$:

$$\sigma_b|_{W_{j,b}}(z) = \sum_n \phi_{j,n} z^n dz \qquad \in E_j^\vee((z))dz$$

where $\phi_{j,n} \in E_j^\vee$. We define the residue pairing

$$\begin{aligned}\mathrm{Res}_j\langle s_j, \sigma_b\rangle =& \mathrm{Res}_{z=0}\langle s_j(b), \sigma_b|_{U_j,b}(z)\rangle\\ =& \mathrm{Res}_{z=0}\bigg(\Big\langle \sum_n e_{j,n}(b)z^n, \sum_n \phi_{j,n}z^n\Big\rangle dz\bigg).\end{aligned} \tag{A.2}$$

in which the pairing between E_j and $E_j^\vee$ is denoted by $\langle\cdot,\cdot\rangle$.

We now prove the Strong Residue Theorem for $\mathscr{E}$. Our proof is inspired by that of [Ueno08, Thm. 1.22].

Theorem A.1. *For each $1 \leqslant j \leqslant N$, choose s_j as in (A.1). Then the following statements are equivalent.*

(a) There exists $s \in H^0(\mathcal{C}, \mathscr{E}(\star S_{\mathfrak{X}}))$ whose series expansion at $\varsigma_j(\mathcal{B})$ (for each $1 \leqslant j \leqslant N$) is s_j.

(b) For each $b \in \mathcal{B}$, there exists $s_b \in H^0(\mathcal{C}_b, \mathscr{E}|_{\mathcal{C}_b}(\star S_{\mathfrak{X}_b}))$ whose series expansion at $\varsigma_j(b)$ (for each $1 \leqslant j \leqslant N$) is $s_j(b)$.

(c) For any $b \in \mathcal{B}$ and any $\sigma_b \in H^0\big(\mathcal{C}_b, \mathscr{E}^\vee|_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(\star S_{\mathfrak{X}_b})\big)$,

$$\sum_{j=1}^{N} \mathrm{Res}_j \langle s_j, \sigma_b \rangle = 0. \tag{A.3}$$

Moreover, when these statements hold, there is only one $s \in H^0(\mathcal{C}, \mathscr{E}(\star S_{\mathfrak{X}}))$ satisfying (a).

Proof. (a) trivially implies (b). That (b) implies (c) follows from Residue theorem (i.e., Stokes theorem): The evaluation between s_b and σ_b is an element of $H^0(\mathcal{C}_b, \omega_{\mathcal{C}_b}(\star S_{\mathfrak{X}_b}))$ whose total residue over all poles is 0.

If s satisfies (a), then for each $b \in \mathcal{B}$, $s|_{\mathcal{C}_b}$ is uniquely determined by its series expansions near $\varsigma_1(b), \dots, \varsigma_N(b)$ (since each component of $\mathcal{C}_b$ contains some $\varsigma_j(b)$). Therefore the sections satisfying (a) is unique.

Now assume (c) is true. We shall prove (a). Suppose that for each $b \in \mathcal{B}$ we can find a neighborhood $V \subset \mathcal{B}$ such that an s satisfying (a) exists for the family $\mathfrak{X}_V$. Then, by the uniqueness proved above, we can glue all these locally defined s to a global one. Thus, we may shrink $\mathcal{B}$ to a small neighborhood of a given $b_0 \in \mathcal{B}$ when necessary.

We first note that, by replacing $\mathcal{B}$ with a neighborhood of a given $b_0 \in \mathcal{B}$, we may assume $\pi_* \mathscr{E}(kS_{\mathfrak{X}}) = 0$ for sufficiently large k. Indeed, choose any $b_0 \in \mathcal{B}$. Then by Serre duality,

$$H^0\big(\mathcal{C}_b, \mathscr{E}|_{\mathcal{C}_b}(-kS_{\mathfrak{X}_b})\big) \simeq H^1\big(\mathcal{C}_b, \mathscr{E}^\vee|_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}_b})\big), \tag{A.4}$$

which, by Serre vanishing theorem, equals 0 for some $k = k_0$ when $b = b_0$. Since π is open, $\mathfrak{X}$ is a flat family ([GPR, Thm. II.2.13] or [Fis76, Sec. 3.20]). Thus, we can apply the upper-semicontinuity theorem ([GPR, Thm. III.4.7] or [BS76, Thm. III.4.12]) to see that (A.4) vanishes for $k = k_0$ and (by shrinking $\mathcal{B}$ to a neighborhood of b_0) any $b \in \mathcal{B}$. Since the vector space $H^0\big(\mathcal{C}_b, \mathscr{E}|_{\mathcal{C}_b}(-kS_{\mathfrak{X}_b})\big)$ shrinks as k increases, (A.4) is constantly zero for all $b \in \mathcal{B}$ and $k \geqslant k_0$. This implies $\pi_* \mathscr{E}(-kS_{\mathfrak{X}}) = 0$ for all $k \geqslant k_0$ ([GPR, Thm. III.4.7-(d)] or [BS76, Cor. III.3.5]).

Choose $p \in \mathbb{N}$ such that for each $1 \leqslant j \leqslant N$, the $e_{j,n}$ in (A.1) equals 0 when $n < -p$. For any $k \geqslant k_0$, as $\pi_* \mathscr{E}(-kS_{\mathfrak{X}}) = 0$, the short exact sequence

$$0 \to \mathscr{E}(-kS_{\mathfrak{X}}) \to \mathscr{E}(pS_{\mathfrak{X}}) \to \mathscr{E}(pS_{\mathfrak{X}})/\mathscr{E}(-kS_{\mathfrak{X}}) \to 0$$

induces a long one

$$0 \to \pi_* \mathscr{E}(pS_{\mathfrak{X}}) \to \pi_* \big(\mathscr{E}(pS_{\mathfrak{X}})/\mathscr{E}(-kS_{\mathfrak{X}})\big) \xrightarrow{\delta} R^1\pi_* \mathscr{E}(-kS_{\mathfrak{X}}). \tag{A.5}$$

For each $1 \leqslant j \leqslant N$, set $s_j|_k = \sum_{n<k} e_{j,n} \cdot z^n$, which can be regarded as a section in $\mathscr{E}(pS_{\mathfrak{X}})(W_j)$. Let $W_0 = \mathcal{C} \backslash S_{\mathfrak{X}}$. Then $\mathfrak{U} = \{W_0, W_1, \dots, W_N\}$ is an open cover of $\mathcal{C}$. Define Čech 0-cocycle $\psi = (\psi_j)_{0 \leqslant j \leqslant N} \in Z^0(\mathfrak{U}, \mathscr{E}(pS_{\mathfrak{X}})/\mathscr{E}(-kS_{\mathfrak{X}}))$ by setting

$$\psi_0 = 0, \qquad \psi_j = s_j|_k \quad (1 \leqslant j \leqslant N).$$

Then $\delta\psi = \big((\delta\psi)_{i,j}\big)_{0 \leqslant i,j \leqslant N} \in Z^1(\mathfrak{U}, \mathscr{E}(-kS_{\mathfrak{X}}))$ is described as follows: $(\delta\psi)_{0,0} = 0$; if $i, j > 0$ then $(\delta\psi)_{i,j}$ is not defined since $W_i \cap W_j = \varnothing$; if $1 \leqslant j \leqslant N$ then $(\delta\psi)_{j,0} = -(\delta\psi)_{0,j}$ equals $s_j|_k$ (considered as a section in $\mathscr{E}(-kS_{\mathfrak{X}})(W_j \cap W_0)$).

Consider $\delta\psi$ as a section of $R^1\pi_*\mathscr{E}(-kS_{\mathfrak{X}})$. We shall show that $\delta\psi = 0$. By the fact that (A.4) vanishes and the invariance of Euler characteristic, $\dim H^1\big(\mathcal{C}_b, (\mathscr{E}|_{\mathcal{C}_b})(-kS_{\mathfrak{X}_b})\big)$ is locally constant over $b \in \mathcal{B}$, which shows that $R^1\pi_*(\mathcal{C}, \mathscr{E}(-kS_{\mathfrak{X}}))$ is locally free and its fiber at b is naturally equivalent to $H^1\big(\mathcal{C}_b, (\mathscr{E}|_{\mathcal{C}_b})(-kS_{\mathfrak{X}_b})\big)$. (Cf. [GPR, Thm. III.4.7] or [BS76, Thm. III.4.12].) Thus, it suffices to show that for each fiber $\mathcal{C}_b$, the restriction $\delta\psi|_{\mathcal{C}_b} \in H^1(\mathcal{C}_b, \mathscr{E}|_{\mathcal{C}_b}(-kS_{\mathfrak{X}_b}))$ is zero.

The residue pairing for the Serre duality

$$H^1(\mathcal{C}_b, \mathscr{E}|_{\mathcal{C}_b}(-kS_{\mathfrak{X}})) \simeq H^0\big(\mathcal{C}_b, \mathscr{E}^\vee|_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}_b})\big)^*$$

applied to $\delta\psi|_{\mathcal{C}_b}$ and any $\sigma_b \in H^0\big(\mathcal{C}_b, \mathscr{E}^\vee|_{\mathcal{C}_b} \otimes \omega_{\mathcal{C}_b}(kS_{\mathfrak{X}_b})\big)$, is given by

$$\langle \delta\psi|_{\mathcal{C}_b}, \sigma_b\rangle = \sum_{j=1}^N \mathrm{Res}_j \langle s_j|_k, \sigma_b\rangle.$$

Since for each $1 \leqslant j \leqslant N$, $\langle s_j - s_j|_k, \sigma_b\rangle$ has removable singularity at $z = 0$, we have $\mathrm{Res}_j\langle s_j - s_j|_k, \sigma_b\rangle = 0$. Therefore,

$$\langle \delta\psi|_{\mathcal{C}_b}, \sigma_b\rangle = \sum_{j=1}^N \mathrm{Res}_j \langle s_j, \sigma_b\rangle = 0.$$

Thus $\delta\psi|_{\mathcal{C}_b} = 0$ for any b. This proves that $\delta\psi = 0$.

By (A.5), for each $k \geqslant k_0$, there is a unique $s|_k \in \big(\pi_*\mathscr{E}(pS_{\mathfrak{X}})\big)(\mathcal{B}) = H^0(\mathcal{C}, \mathscr{E}(pS_{\mathfrak{X}}))$ which is sent to $\psi \in \pi_*\big(\mathscr{E}(pS_{\mathfrak{X}})/\mathscr{E}(-kS_{\mathfrak{X}})\big)(\mathcal{B})$. So near $\varsigma_j(\mathcal{B})$, $s|_k$ has series expansion

$$s|_k = s_j|_k + \bullet z^k + \bullet z^{k+1} + \cdots . \tag{A.6}$$

By this uniqueness, we must have $s|_{k_0} = s|_{k_0+1} = s|_{k_0+2} = \cdots$. Let $s = s|_{k_0}$. Then s has series expansion s_j at $\varsigma_j(\mathcal{B})$ for each j. □

We remark that the above proof also applies to locally free sheaves over a proper flat family of pointed complex curves (with at worst nodal singularities) such that each $S_{\mathfrak{X}_b}$ does not intersect the node of $\mathcal{C}_b$, and that $S_{\mathfrak{X}_b}$ intersects each irreducible component of $\mathcal{C}_b$. This is because the residue pairing for Serre duality is described in the same way as in the smooth case.

Index

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