

NOTE ON PARTIAL OKOUNKOV BODIES — THE POINT OF VIEW OF B-DIVISORS

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1. INTRODUCTION

In this [note](#), I will explain a more algebraic point view to the partial Okounkov bodies introduced in [\[Xia21\]](#).

The main theorem is the following:

Theorem 1.1. *Let X be an irreducible complex projective manifold of dimension n and (L, ϕ) be a Hermitian pseudo-effective line bundle on X with positive volume. Fix a valuation $\nu: \mathbb{C}(X)^\times \rightarrow \mathbb{Z}^n$ of rank n .*

Then the partial Okounkov body $\Delta_\nu(L, \phi)$ admits the following expression:

$$(1.1) \quad \Delta_\nu(L, \phi) = \nu(\phi) + \lim_{\pi: Z \rightarrow X} \Delta_\nu(c_1(\pi^*L) - \{\text{Sing}_Z(\phi)\}),$$

where π runs over all smooth birational modifications of X .

This theorem needs some explanations. Here $\text{Sing}_Z(\phi)$ denotes the divisorial part of the Siu decomposition of $\text{dd}^c \pi^* \phi$. The notation $\{\bullet\}$ means the associated numerical class. The limit is the Hausdorff limit. The valuation $\nu(\phi)$ is defined in [\[DX24\]](#) using the trace operator.

This theorem shows that the partial Okounkov bodies admit a natural interpretation in terms of the associated b-divisors.

One could easily generalize the argument below to the transcendental case, but I find no applications of such generalizations, so I will content myself to the algebraic setting.

One remark: in [\[DX24\]](#), we only explained how to define the trace operator when the subvariety is smooth. In general, the trace operator gives a P -equivalence class on the normalization of the subvariety. We will use these results freely.

2. SOME PRELIMINARIES

Lemma 2.1. *Let E_j be a countable family of distinct prime divisors on X . Consider $a_{ij} \in \mathbb{R}_{\geq 0}$ for all $i, j > 0$. We assume that the sequence (a_{ij}) for fixed j is increasing in i . Moreover, assume that $a_j := \lim_{i \rightarrow \infty} a_{ij} < \infty$. Assume that the series $\sum_j a_j [E_j]$ converges, then*

$$\lim_{i \rightarrow \infty} \nu \left(\sum_j a_{ij} [E_j] \right) = \nu \left(\sum_j a_j [E_j] \right).$$

*Here implicitly, we assume that ν is surjective.

Proof. We may assume that the valuation ν is induced by a smooth flag Y_\bullet .

We argue by induction on the dimension n . When $n = 1$, there is nothing to argue. Assume that $n > 1$ and the case $n - 1$ is known. We may assume that Y_1 is not among the E_j 's. Write μ for the valuation on Y_1 induced by the truncated flag. Then we need to prove the following:

$$\lim_{i \rightarrow \infty} \mu \left(\sum_j a_{ij} [E_j] |_{Y_1} \right) = \mu \left(\sum_j a_j [E_j] |_{Y_1} \right).$$

Note that $[E_j] |_{Y_1}$ is again the current of integration of an effective divisor on Y_1 (this can be seen using the Lelong–Poincaré formula for example), so the desired convergence follows by induction. \square

Lemma 2.2. *Let T be a closed positive $(1, 1)$ -current on X . Then we have*

$$(2.1) \quad \lim_{\pi: Z \rightarrow X} \nu(\text{Sing}_Z(T)) = \nu(T).$$

Proof. We may assume that ν is induced by a smooth flag Y_\bullet on X .

Given $\pi: Z \rightarrow X$, we let W_1 denote the strict transform of Y_1 in Z . The restriction $\pi_1: W_1 \rightarrow Y_1$ is necessarily birational.

We will argue by induction. The case $n = 1$ is trivial. Assume that $n > 1$ and the case $n - 1$ is known.

We may clearly assume that $\nu(T, Y_1) = 0$. By definition, we have

$$\nu(T) = (0, \mu(\text{Tr}_{Y_1}(T))),$$

where μ denotes the valuation induced by the flag on Y_1 induced by Y_\bullet .

Observe that modifications of the form $\pi_1: W_1 \rightarrow Y_1$ is cofinal in the directed set of modifications of Y_1 . This is obvious since the modifications given by compositions of blow-ups with smooth centers on Y_1 are cofinal.

Therefore, by induction, it suffices to argue that for any $\pi: Z \rightarrow X$, we have

$$(2.2) \quad \nu(\text{Sing}_Z(T)) = (0, \mu(\text{Sing}_{\widetilde{W}_1}(\text{Tr}_{Y_1}(T)))) ,$$

where \widetilde{W}_1 is the normalization of W_1 . Let $\widetilde{\pi}_1$ denote the normalization of π_1 so that we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{W}_1 & \longrightarrow & W_1 & \hookrightarrow & Z \\ \downarrow \widetilde{\pi}_1 & & \downarrow \pi_1 & & \downarrow \pi \\ Y_1 & \xrightarrow{\quad} & Y_1 & \hookrightarrow & X. \end{array}$$

From the birational behaviour of the trace operator proved in [Xia23, DX24], we know that

$$\widetilde{\pi}_1^* \text{Tr}_{Y_1}(T) \sim_P \text{Tr}_{W_1}(\pi^* T).$$

So we only need to prove

$$\nu(\text{Sing}_Z(\pi^* T)) = (0, \mu(\text{Sing}_{\widetilde{W}_1}(\text{Tr}_{W_1}(\pi^* T)))) ,$$

In order to prove this, we may add a Kähler form to T and assume that T is a Kähler current. Take a quasi-equisingular approximation T_j of T . Then $\pi^* T_j$ is a quasi-equisingular approximation of $\pi^* T$. By the decreasing continuity of the trace operator proved in [Xia22, DX24], the d_S -continuity of Lelong numbers proved in [Xia22] and Lemma 2.1, both sides are continuous along quasi-equisingular approximations, we reduce to the case where $\pi^* T$ has analytic singularities. In this case, take a suitable resolution and argue as before, we may assume that $\pi^* T = [D]$ for a snc

\mathbb{Q} -divisor D . By additivity, we finally reduce to the case where D is a prime divisor on X different from Y_1 . The problem is reduced to

$$\nu([D]) = (0, \mu([D]|_{Y_1})),$$

which is clear by definition. \square

3. THE PROOF

Now let us begin the argument of [Theorem 1.1](#). We argue by induction on n . The case $n = 1$ is of course trivial. Let us assume that $n > 1$ and the result is known in dimension $n - 1$.

We first make a few simplifications. Observe that [\(1.1\)](#) is birationally invariant, so we may assume that ν is equivalent to the valuation induced by a smooth flag. Furthermore, we reduce to the case that ν is the valuation induced by a smooth flag Y_\bullet .

It would be more convenient to use the language of currents. We shall write $T = dd^c \phi$. Then one needs to prove two things: first of all, the limit in [\(1.1\)](#) exists; secondly,

$$(3.1) \quad \Delta_\nu(T) = \nu(T) + \lim_{\pi: Z \rightarrow X} \Delta_\nu(c_1(\pi^*L) - \{\text{Sing}_Z(T)\}).$$

We may replace T by $T - \nu(T, Y_1)[Y_1]$ and L by the numerical class $L - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$. But now L is replaced by a big numerical class α on X in the real Néron–Severi group of X . By perturbation, we may assume α lies in the rational Néron–Severi group. After a rescaling, we reduce back to the case where α is represented by a line bundle L .

Eventually we want to show [\(3.1\)](#) assuming that $\nu(T, Y_1) = 0$.

Let us prove [\(3.1\)](#). As shown in [\[Xia23, DX24\]](#), we have

$$\Delta_\nu(c_1(\pi^*L) - \{\text{Sing}_Z(T)\}) = \overline{\{\nu(S) : S \in c_1(\pi^*L) - \{\text{Sing}_Z(T)\}\}}.$$

Therefore,

$$\Delta_\nu(c_1(\pi^*L) - \{\text{Sing}_Z(T)\}) + \nu(\text{Sing}_Z(T)) \subseteq \overline{\{\nu(S) : S \in c_1(L), \pi^*S \geq \text{Sing}_Z(T)\}}.$$

We observe that the right-hand side is decreasing with respect to π , which together with [Lemma 2.2](#) implies that the net of convex bodies $\Delta_\nu(c_1(\pi^*L) - \{\text{Sing}_Z(T)\})$ for various Z is uniformly bounded. Suppose that Δ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in c_1(L), S \preceq_T T\}}.$$

As shown in [\[Xia23, DX24\]](#), the right-hand side is exactly $\Delta_\nu(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_\nu(T).$$

But observe that both sides have the same volume, as computed in [\[Xia23, DX24\]](#) and [\[Xia22\]](#). So equality holds.

It follows from the Blaschke selection theorem that the limit in [\(3.1\)](#) exists and [\(3.1\)](#) holds.

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