NOTE ON VANISHING CYCLES AND NEARBY CYCLES

MINGCHEN XIA

Contents

1. Introduction

This note was written when I tried to understand the construction of Beilinson's unipotent nearby cycle. The construction part is just a rewriting of [SGA 7_1, Exposé I], where we use the language of derived categories instead of that of hypercohomology. The constructibility theorem is proved following [SGA $4^1\!/\!_2$].

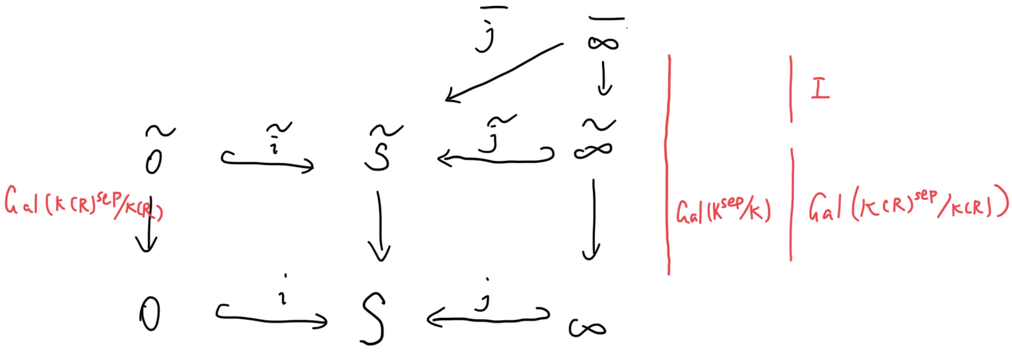
The nearby cycles and the vanishing cycles defined in this note are $[-1]$-shifts of the corresponding definitions in [SGA 7_1; SGA 7_2]. We do so in order to preserve the perversity.

Given a scheme X, $X_{\text{ét}}$ will denote a small étale site associated with X. We refer to [Stacks, Tag 03XB] for the construction.

2. Setup

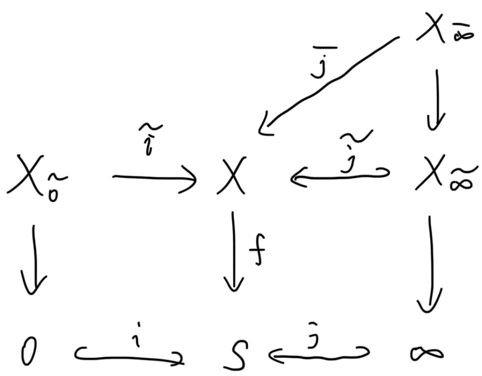
Let R be a Henselian DVR and $S = \operatorname{Spec} R$. We will write 0 and ∞ for the special and the generic point of S. Set $K = \operatorname{Spec} R$ and $\kappa(R)$ as the residue field of R. We write $\tilde{R} = R^{\mathrm{sh}}$, the strict Henselization of R and $\tilde{S} = \operatorname{Spec} \tilde{R}$. The notations $\tilde{0}$ and $\tilde{\infty}$ then denote the special point and the generic point of $\tilde{S}$. Write $K^{\mathrm{sh}} = \operatorname{Spec} \tilde{R}$. Take a separable closure K^{sep}, we can then identify K^{sh} with $K^{\mathrm{sep},I}$ (the upper index I denotes the set of fixed points), where I is the decomposition group of K^{sep}/K. We refer to [Stacks, Tag 0BSD] for the proof. We write $\bar{\infty} = \operatorname{Spec} K^{\mathrm{sep}}$.

Date: January 27, 2023.



The relevant maps are defined in the figure.

Take a separated morphism of finite type $f : X \to S$ of schemes. We can base change the above graph to get



3. The étale nearby cycles

We fix a noetherian torsion ring Λ so that the residue characteristic of R is invertible in Λ. We keep in mind examples like $\Lambda = \mathbb{Z}/(\ell^n)$ with $\ell \in \mathbb{Z}_{>1}$ different from the characteristic of $\kappa(R)$ and $n \in \mathbb{Z}_{>0}$. If one is interested in the case $\Lambda = \mathbb{Z}_\ell$, one could simply replace the étale site below by the pro-étale site.

Given $L \in \mathbf{D}^+(X_{\bar{\infty},\text{ét}}; \Lambda)$, we let

{eq:Psif_def} (3.1) $$\Psi_f(L) := \tilde{i}^* \mathbf{R}\bar{j}_* L[-1] \in \mathbf{D}^+(X_{\tilde{0},\text{ét}}; \Lambda).$$

When $L \in \mathbf{D}^+(X_{\infty,\text{ét}}; \Lambda)$, we can view it as an object in $\mathbf{D}^+(X_{\bar{\infty},\text{ét}}; \Lambda)$ endowed with a compatible $\mathrm{Gal}(K^{\text{sep}}/K)$-action (here compatible means compatible with the $\mathrm{Gal}(K^{\text{sep}}/K)$-action on $X_{\bar{\infty}}$). It follows that $\Psi_f(L)$ is endowed with a $\mathrm{Gal}(K^{\text{sep}}/K)$-action compatible with the $\mathrm{Gal}(K^{\text{sep}}/K)/I$-action on $X_{\tilde{0}}$. In particular, we can view Ψ_f as a functor

(3.2) $$\Psi_f : \mathbf{D}^+(X_{\infty,\text{ét}}; \Lambda) \to \mathbf{D}^+(X_{\tilde{0},\text{ét}}; \Lambda).$$

The object $\Psi_f(L)$ is called the *nearby cycle* of L. Observe that if $\bar{x}$ is a geometric point of X_0, we have

{eq:stalk_Psif} (3.3) $$\Psi_f(L)_{\bar{x}} \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{ét}}(X_{\bar{x},\bar{\infty}}, L[-1]),$$

where $X_{\bar{x}}$ is the strict Henselization of X at $\bar{x}$ and $X_{\bar{x},\bar{\infty}}$ is the geometric fiber of $X_{\bar{x}} \to S$ at $\bar{\infty}$. In fact, we have the following Cartesian square:

{eq:Xxinfty_square}

$$
\begin{tikzcd}
X_{(\bar{x}),\bar{\infty}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_{(\bar{x})} \arrow[d] \\
X_{\bar{\infty}} \arrow[r, "\bar{j}"] & X
\end{tikzcd} \tag{3.4}
$$

The first property is

p:nearbycycle_proper

Proposition 3.1. *Assume that f is proper. Then for any $L \in \mathbf{D}^+(X_{\infty,\acute{e}t};\Lambda)$, we have canonical isomorphisms*

$$\mathbf{R}\Gamma_{\acute{e}t}(X_{\bar{\infty}}, L) \xrightarrow{\sim} \mathbf{R}\Gamma_{\acute{e}t}(X, \mathbf{R}\bar{j}_* L) \xrightarrow{\sim} \mathbf{R}\Gamma_{\acute{e}t}(X_{\bar{0}}, \Psi_f(L)[1]).$$

The isomorphisms are compatible with the action of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$.

The importance of this proposition is that it transfers the cohomological information of L on the fiber at infinity to the special fiber.

For the proof, we need the following elegant consequence of the proper base change theorem:

p:proper_base_change

Proposition 3.2. *Assume that f is proper, then for any $L \in \mathbf{D}^+(X_{\acute{e}t};\Lambda)$, we have a canonical isomorphism*

$$\mathbf{R}\Gamma_{\acute{e}t}(X, L) \xrightarrow{\sim} \mathbf{R}\Gamma_{\acute{e}t}(X_0, L_0),$$

where L_0 is the pull-back of L of X_0.

Proof. When R is strictly Henselian, 0 is a geometric point. It suffices to apply the proper base change theorem in this case, see [Stacks, Tag 095T] *.

In general, we make a base change to $\tilde{S}$, the strict Henselization of $\tilde{S}$. Then $\tilde{S} \to S$ is a Galois covering with Galois group $G = \mathrm{Gal}(K^{\mathrm{sep}}/K) = \mathrm{Gal}(\kappa(R)^{\mathrm{sep}}/\kappa(R))$. See [BLR90, Page 139] for the proof. By the way-out argument, we may assume that L is of the form $\mathcal{F}[0]$, where $\mathcal{F}$ is a sheaf of Λ-module on $X_{\text{ét}}$. Then we can apply the functoriality of the Hochschild–Serre spectral sequence (see [Mil80, Theorem 14.9] and its proof, the functoriality is essentially the functoriality of the Grothendieck spectral sequence) to get a morphism of spectral sequences

$$
\begin{tikzcd}
H^p_{\text{ét}}(G, H^q_{\text{ét}}(X \times_S \tilde{S}, \mathcal{F}_{X\times_S\tilde{S}})) \arrow[r, Rightarrow] \arrow[d] & H^{p+q}_{\text{ét}}(X, \mathcal{F}) \arrow[d] \\
H^p_{\text{ét}}(G, H^q_{\text{ét}}(X_0 \times_0 \tilde{0}, \mathcal{F}_{X_0\times_0\tilde{0}})) \arrow[r, Rightarrow] & H^{p+q}_{\text{ét}}(X, \mathcal{F}|_{X_0})
\end{tikzcd}.
$$

Now we know that the left vertical maps are isomorphisms, so we conclude that the right vertical map is also an isomorphism. ☐

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Only the second isomorphism needs argument, but this follows immediately from Proposition 3.2. ☐

*Strictly speaking, the cited theorem only considers the case of sheaves, not complexes. One argues the complex case using the usual way-out argument. One can also apply [Stacks, Tag 09C9] directly.

Next we define the *vanishing cycle* functor

$$\Phi_f : \mathbf{D}^+(X_{\text{ét}};\Lambda) \to \mathbf{D}^+(X_{\tilde{0},\text{ét}};\Lambda). \tag{3.5}$$

Let $L \in \mathbf{D}^+(X_{\text{ét}};\Lambda)$, we define

$$\Phi_f(L) \coloneqq \tilde{i}^!(L).$$

From the well-known exact triangle

$$\tilde{i}^! \to \tilde{i}^* \to \tilde{i}^*\mathbf{R}\tilde{j}_*\tilde{j}^* \xrightarrow{+1},$$

we find an exact triangle

{eq:fundam_triang}

$$\Psi_f(L|_{X_\infty}) \to \Phi_f(L) \to L|_{X_{\tilde{0}}} \xrightarrow{+1} . \tag{3.6}$$

We remind the readers that our vanishing cycle and nearby cycle are both shifted from the SGA 7 definition by 1. In terms of cohomology, (3.6) induces an exact triangle

$$\mathbf{R}\Gamma_{\text{ét}}(X_{\tilde{0}},\Phi_f(L)) \to \mathbf{R}\Gamma_{\text{ét}}(X_{\tilde{0}},L|_{X_{\tilde{0}}}) \to \mathbf{R}\Gamma_{\text{ét}}(X_{\bar{\infty}},L|_{X_\infty}) \xrightarrow{+1} \tag{3.7}$$

if f is proper. Here we used Proposition 3.1. This means that the vanishing cycle characterizes the difference between the cohomology of the general fiber and the cohomology of the special fiber.

Our next goal is to extend Ψ_f and Φ_f to the full derived category. This is easy for Ψ_f as $\mathbf{R}\bar{j}_*$ has finite cohomological dimension:

Lemma 3.3. *The morphism $\bar{j}$ has cohomological dimension $\leq \dim X_{\bar{\infty}} + 1$. In particular, Ψ_f has cohomological dimension $\leq \dim X_{\bar{\infty}} + 1$.*

Proof. By (3.3), it suffices to show that for any sheaf $\mathcal{F}$ of Λ-modules on $X_{\infty,\text{ét}}$ and any geometric point $\bar{x}$ of X_0, we have

$$H^q_{\text{ét}}(X_{\bar{x},\bar{\infty}},\mathcal{F}) = 0$$

whenever $q > \dim X_{\bar{\infty}}$. We can write $X_{\bar{x},\bar{\infty}}$ as the projective limit of affine schemes of dimension at most $\dim X_{\bar{\infty}}$ (as can be seen from (3.4) by writing $X_{(\bar{x})}$ as a projective limit of affine étale schemes over X), so the vanishing is a consequence of the affine Lefschetz theorem, see [Stacks, Tag 0F0V] for the argument. □

The well-known argument allows us to extend Ψ_f to

$$\Psi_f : \mathbf{D}(X_{\infty,\text{ét}};\Lambda) \to \mathbf{D}(X_{\tilde{0},\text{ét}};\Lambda). \tag{3.8}$$

Next we extend the vanishing cycle. As per our definition, the vanishing cycle is just given by an upper shriek, it is well-known that the definition makes sense, see [Stacks, Tag 0G2B]. So we get

$$\Phi_f : \mathbf{D}(X_{\text{ét}};\Lambda) \to \mathbf{D}(X_{\tilde{0},\text{ét}};\Lambda). \tag{3.9}$$

The usual truncation argument shows that the exact triangle (3.6) extends to the unbounded case.

4. Constructibility

thm:constru **Theorem 4.1.** *Both Ψ_f and Φ_f preserve constructibility. In other words, we have functors*

$$\begin{aligned} \Psi_f &: \mathbf{D}_c(X_{\infty,\acute{e}t};\Lambda) \to \mathbf{D}_c(X_{\tilde{0},\acute{e}t};\Lambda), \\ \Phi_f &: \mathbf{D}_c(X_{\acute{e}t};\Lambda) \to \mathbf{D}_c(X_{\tilde{0},\acute{e}t};\Lambda). \end{aligned} \tag{4.1}$$

We reproduce the proof following [SGA4.5 SGA 4½].

We make some preliminary reductions. First of all, by the fundamental triangle (3.6), it suffices to consider Ψ_f. By the usual way-out argument, it suffices to consider the case of a single sheaf $\mathcal{F}$ of Λ-modules on $X_{\infty,\text{ét}}$ and show that $\Psi_f(\mathcal{F}[0])$ is constructible. By definition (3.1), we may assume that R is strictly Henselian. Finally, we may assume that X_∞ is dense in X as otherwise, we may replace X by the closure of X_∞.

For simplicity, we will write

$$\Psi_f^i(\mathcal{F}) := \mathcal{H}^i\left(\Psi_f(\mathcal{F}[0])\right) \in \mathrm{Mod}_\Lambda(X_{0,\text{ét}}), \quad i \in \mathbb{N}.$$

The argument involves an induction on $n := \dim X_\infty \geq 0$. Assume that the theorem is prove until dimension $n-1$. We first prove a weaker claim.

lma:weakconstr **Lemma 4.2.** *Assume that the theorem is prove until dimension $n-1$. For each $i \in \mathbb{N}$, there is a constructible sub-Λ-module $\mathcal{G}_i$ of $\Psi_f^i(\mathcal{F})$ such that the local sections of $\Psi_f^i(\mathcal{F})/\mathcal{G}_i$ have finite supports.*

Let us first see how Lemma 4.2 implies Theorem 4.1.

Proof of Theorem 4.1. The problem is local, so we may assume that X is affine. Then we may assume that X is projective over S. By Proposition 3.1, we then have a spectral sequence

{eq:nearby_ss}
$$H^p_{\text{ét}}(X_0, \Psi_f^q(\mathcal{F})) \implies H^{p+q-1}_{\text{ét}}(X_{\bar{\infty}}, \mathcal{F}). \tag{4.2}$$

Take $\mathcal{G}_q$ as in Lemma 4.2 and write

$$\mathcal{H}^q := \Psi_f^q(\mathcal{F})/\mathcal{G}_q.$$

It suffices to prove the constructiblity of $\mathcal{H}^q$. But the local sections of $\mathcal{H}^q$ have finite supports, so it suffices to show that $H^0(X_0, \mathcal{H}^q)$ is of finite type as an Λ-module.

We regard (4.2) as a spectral sequence E_2^{pq} in the quotient $\mathrm{Mod}_\Lambda / \mathrm{Mod}_\Lambda^f$, where Mod_Λ^f is the thick subcategory of Mod_Λ consisting of finite Λ-modules. We see immediately that $E_2^{pq} = 0$ as long as $p \neq 0$, so the spectral sequence degenerates at E_2. It follows that

$$E_2^{0q} \cong H^q(X_{\bar{\infty}}, \mathcal{F})$$

in $\mathrm{Mod}_\Lambda / \mathrm{Mod}_\Lambda^f$. The latter space is a finite Λ-module as X is projective. We conclude. □

It remains to argue Lemma 4.2. We will need a well-known trick.

The problem is local on X, so we may assume that X is a closed subscheme of $\mathbb{A}^n_S$. Let $\pi : X \to \mathbb{A}^1_S$ be the projection onto any coordinate.

Let $0'$ be a geometric point over the generic point of $\mathbb{A}^1_0$, where $0 = \operatorname{Spec} \kappa(R)$. Let

$$S' := \mathbb{A}^1_{S,(0')}.$$

Write $S' = \operatorname{Spec} R'$ and $K' = \operatorname{Spec} R'$. Now R' is an unramified extension of the DVR R. We have a commutative diagram:

$$\begin{tikzcd}
X' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & S' \arrow[d] \\
X \arrow[r, hook] \arrow[dr] & \mathbb{A}^n_S \arrow[r, "\pi"] \arrow[d] & \mathbb{A}^1_S \arrow[dl] \\
& S &
\end{tikzcd} .$$

Write $\mathcal{F}'$ for the pull-back of $\mathcal{F}$ to $X'_{\text{ét}}$. We also write

$$k' = K^{\text{sep}} \otimes_K K'.$$

Observe that k' is indeed a field as it is the fraction field of $(\mathbb{A}^1_{\bar{S}})_{(0')}$ with $\bar{S}$ being the normalization of S in K^{sep}. The group

$$P = \operatorname{Gal}(k'^{\text{sep}}/k')$$

is a pro-p-group, where p is the exponent characteristic of $\kappa(R)$. See [stacks-project Stacks, Tag 0BSD] for the proofs. We observe that if X' is a separated scheme of finite type over S' and $\mathcal{F}$ is a sheaf of Λ-modules on $X'_{\infty,\text{ét}}$, then we have a canonical identification

{eq:wild_Psi} (4.3) $$\Psi^q_{X \to S}(\mathcal{F}) \xrightarrow{\sim} \Psi^q_{X' \to S}(\mathcal{F})^P, \quad q \in \mathbb{N}.$$

This is a simple consequence of the Hochschild–Serre spectral sequence.

Observe that on X'_0, the pull-back of $\mathbf{R}^i\Psi_{X \to S}(\mathcal{F})$ is just

$$\mathbf{R}^i\Psi_{X \to S}(\mathcal{F}') = \mathbf{R}^i\Psi_{X' \to S}(\mathcal{F}')^P$$

by (4.3). This sheaf is constructible by inductive hypothesis.

Now the proof of Lemma 4.2 is reduced to the following lemma:

Lemma 4.3. *Let k be a field and $n \in \mathbb{N}$. Suppose that X is a closed subscheme of $\mathbb{A}^n$. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{\text{ét}}$ and $\bar{\eta}$ be a geometric point over the generic point of $\mathbb{A}^1_k$. For each $i = 1, \ldots, n$, let $\pi_i : X \to \mathbb{A}^1_k$ be the projection onto the i-th coordinate and $X_{\bar{\eta},i}$ be the geometric fiber of π_i over $\bar{\eta}$. Write $\mathcal{F}_{\bar{\eta},i}$ for the restriction of $\mathcal{F}$ to $X_{\bar{\eta},i}$. Suppose that the $\mathcal{F}_{\bar{\eta},i}$'s are all constructible, then there is a constructible sub-Λ-module $\mathcal{F}' \subseteq \mathcal{F}$ such that the local sections of $\mathcal{F}/\mathcal{F}'$ have finite supports.*

Proof. By spreading out, for each $i = 1, \ldots, n$, we can find an étale neighbourhood U of $\bar{\eta}$ and a constructible sheaf $\mathcal{H}$ on $X_{U,i} := X \times_{\mathbb{A}^1_k, \pi_i} U$ extending $\mathcal{F}_{\bar{\eta},i}$ on $X_{\bar{\eta},i}$. By further shrinking U, we may assume that the isomorphism $\mathcal{H}_{\bar{\eta},i} \to \mathcal{F}_{\bar{\eta},i}$ extends to a morphism

$$\mathcal{H} \to \mathcal{F}_{U,i} := \mathcal{F}|_{X_{U,i}}.$$

Write $\varphi : X_{U,i} \to X$ for the natural inclusion. Then by adjunction, we have a morphism

$$\varphi_! \mathcal{H} \to \mathcal{F}.$$

Write $\mathcal{F}'_i$ for its image. Then

$$(\mathcal{F}/\mathcal{F}'_i)_{\bar{\eta},i} = 0.$$

Take $\mathcal{F}'$ as the sum of the $\mathcal{F}'_1, \ldots, \mathcal{F}'_n$. $\square$

References

BLR90 [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. Néron models. Vol. 21. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990, pp. x+325. URL: https://doi.org/10.1007/978-3-642-51438-8.

Mil80 [Mil80] J. S. Milne. Étale cohomology. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980, pp. xiii+323.

SGA4.5 [SGA 4½] P. Deligne. Cohomologie étale. Vol. 569. Lecture Notes in Mathematics. Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$. Springer-Verlag, Berlin, 1977, pp. iv+312. URL: https://doi.org/10.1007/BFb0091526.

SGA7-1 [SGA 7_1] Groupes de monodromie en géométrie algébrique. I. Lecture Notes in Mathematics, Vol. 288. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. Springer-Verlag, Berlin-New York, 1972, pp. viii+523.

SGA7-2 [SGA 7_2] Groupes de monodromie en géométrie algébrique. II. Lecture Notes in Mathematics, Vol. 340. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz. Springer-Verlag, Berlin-New York, 1973, pp. x+438.

stacks-project [Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.

Mingchen Xia, Department of Mathematics, Institut de Mathématiques de Jussieu-Paris Rive Gauche

Email address, mingchen@imj-prg.fr

Homepage, https://mingchenxia.github.io/home/.