

NOTE ON DUISTERMAAT–HECKMAN MEASURES OF NON-ARCHIMEDEAN METRICS

MINGCHEN XIA

This is an informal note. Please contact me at mingchen@imj-prg.fr for comments.

CONTENTS

1.	Introduction	1
2.	Preliminaries	1
3.	Okounkov test curves	2
4.	The Duistermaat–Heckman measure of a non-Archimedean metric	3
	References	5

1. INTRODUCTION

In this note, we define the Duistermaat–Heckman measure of a non-Archimedean metric using the theory of partial Okounkov bodies developed in [\[Xia21; DX24\]](#). The main result [Theorem 4.3](#) states that the Duistermaat–Heckman measure is canonical (independent of the choice of the flag).

2. PRELIMINARIES

In this section, we recall the theory of Hausdorff metrics on the set of convex bodies following [\[Sch14, Section 1.8\]](#). Fix $n \in \mathbb{N}$. Recall that a convex body in \mathbb{R}^n is a non-empty compact convex subset of \mathbb{R}^n , which may have empty interior. Let \mathcal{K}_n denote the set of convex bodies in \mathbb{R}^n . We will fix the Lebesgue measure $d\lambda$ on \mathbb{R}^n , normalized so that the unit cube has volume 1.

Recall the definition of the Hausdorff metric between $K_1, K_2 \in \mathcal{K}_n$:

$$d_n(K_1, K_2) := \max \left\{ \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

We extend d_n to an extended metric on $\mathcal{K}_n \cup \{\emptyset\}$ by setting

$$d_n(K, \emptyset) = \infty$$

for all $K \in \mathcal{K}_n$.

Theorem 2.1. *The metric space (\mathcal{K}_n, d_n) is complete.*

Theorem 2.2 (Blaschke selection theorem). *Every bounded sequence in \mathcal{K}_n has a convergent subsequence.*

Theorem 2.3. *The Lebesgue volume $\text{vol} : \mathcal{K}_n \rightarrow \mathbb{R}_{\geq 0}$ is continuous.*

Theorem 2.4. *Let $K_i, K \in \mathcal{K}_n$ ($i \in \mathbb{N}$). Then $K_i \xrightarrow{d_n} K$ if and only if the following conditions hold*

- (1) *Each point $x \in K$ is the limit of a sequence $x_i \in K_i$.*
- (2) *The limit of any convergent sequence $(x_{i_j})_{j \in \mathbb{N}}$ with $x_{i_j} \in K_{i_j}$ lies in K , where i_j is a subsequence of $1, 2, \dots$*

Date: February 12, 2024.

The proofs of all these results can be found in [Sch14, Section 1.8].

Lemma 2.5. *Let $K_0, K_1 \in \mathcal{K}_n$. Assume that $K_0 \subseteq K_1$ and*

$$\text{vol } K_0 = \text{vol } K_1 > 0.$$

Then $K_0 = K_1$.

Proof. In fact, if $K_1 \neq K_0$, then $K_1 \setminus K_0$ is a non-empty open subset of K_1 . As $\text{vol } K_1 > 0$, $(K_1 \setminus K_0) \cap \text{Int } K_1 \neq \emptyset$. Thus, $\text{vol } K_1 > \text{vol } K_0$, which is a contradiction. \square

3. OKOUNKOV TEST CURVES

Let $\Delta \in \mathcal{K}^n$. Assume that $V = n! \text{vol } \Delta > 0$.

Definition 3.1. An *Okounkov test curve* relative to Δ is an assignment $(\Delta_\tau)_{\tau < \tau^+}$ ($\tau^+ \in \mathbb{R}$) such that

- (1) Δ_τ is a decreasing assignment of convex bodies in \mathbb{R}^n for $\tau < \tau^+$;
- (2) Δ_τ converges to Δ as $\tau \rightarrow -\infty$ with respect to the Hausdorff metric;
- (3) Δ_τ is concave in the τ variable.

Proposition 3.2. *Any Okounkov test curve $(\Delta_\tau)_{\tau \leq \tau^+}$ relative to Δ is continuous for $\tau < \tau^+$.*

This is proved in [Xia21] for finite energy curves, but the proof works in general as well.

Definition 3.3. A *test function* on Δ is a function $F : \Delta \rightarrow [-\infty, \infty)$ such that

- (1) F is concave;
- (2) F is finite on $\text{Int } \Delta$;
- (3) F is usc.

Let Δ_\bullet be an Okounkov test curve relative to Δ . We define the *Legendre transform* of Δ_\bullet as

$$G[\Delta_\bullet] : \Delta \rightarrow [-\infty, \infty), \quad a \mapsto \sup \left\{ \tau < \tau^+ : a \in \Delta_\tau \right\}.$$

Conversely, a test function F on Δ , set $\tau^+ = \sup_\Delta F$. We define the *inverse Legendre transform* of F as

$$\Delta[F] : (-\infty, \tau^+] \rightarrow \mathcal{K}_n, \quad \Delta[F]_\tau = \{F \geq \tau\}.$$

Theorem 3.4. *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between the set of Okounkov test curves relative to Δ and test functions on Δ .*

The proof is essentially contained in [Xia21].

Definition 3.5. Let Δ_\bullet be an Okounkov test curve relative to Δ . We define the *Duistermaat–Heckman measure* $\text{DH}(\Delta_\bullet)$ as

$$\text{DH}(\Delta_\bullet) := G[\Delta_\bullet]_*(d\lambda).$$

It is a Radon measure on \mathbb{R} .

In other words, $\text{DH}(\Delta_\bullet)$ is the probability distribution of the random variable $G[\Delta_\bullet]$ on the measure space $(\Delta, d\lambda)$.

Lemma 3.6. *Suppose that Δ_\bullet^k is a decreasing sequence of Okounkov test curves relative to Δ with the same τ^+ . Assume that the pointwise Hausdorff limit Δ_\bullet is still a Okounkov test curve relative to Δ . Then $\text{DH}(\Delta_\bullet^k) \rightarrow \text{DH}(\Delta_\bullet)$ as $k \rightarrow \infty$.*

Proof. Observe that

$$G[\Delta_\bullet^k] \rightarrow G[\Delta_\bullet]$$

pointwisely as $k \rightarrow \infty$. It follows from the dominated convergence theorem that $\text{DH}(\Delta_\bullet^k) \rightarrow \text{DH}(\Delta_\bullet)$ as $k \rightarrow \infty$. \square

Observe that

$$(3.1) \quad \int_{\mathbb{R}} \mathrm{DH}(\Delta_{\bullet}) = \mathrm{vol} \Delta.$$

More generally, we compute the characteristic function of $G[\Delta_{\bullet}]$ as follows: for any $t \in \mathbb{C}$,

$$(3.2) \quad \int_{\Delta} e^{itG[\Delta_{\bullet}]} d\lambda = e^{it\tau^+} \mathrm{vol} \Delta - it \int_{-\infty}^{\tau^+} (\mathrm{vol} \Delta - \mathrm{vol} \Delta_{\tau}) e^{it\tau} d\tau.$$

In particular, the moments are given by

$$\int_{\mathbb{R}} x^m \mathrm{DH}(\Delta_{\bullet})(x) = \int_{\Delta} G[\Delta_{\bullet}]^m d\lambda = (\tau^+)^m \mathrm{vol} \Delta - \int_{-\infty}^{\tau^+} m\tau^{m-1} (\mathrm{vol} \Delta - \mathrm{vol} \Delta_{\tau}) d\tau.$$

4. THE DUISTERMAAT–HECKMAN MEASURE OF A NON-ARCHIMEDEAN METRIC

Let X be an connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$ -form on X such that $\mathrm{PSH}(X, \theta) \neq \emptyset$. We will define the Duistermaat–Heckman measure of elements in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as studied in [DXZ23; Xia23]. We will follow the notations in [Xia23].

4.1. Non-Archimedean metrics. Consider an element $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, recall that by definition, Γ is an inverse system $(\Gamma^{\theta+\omega})_{\omega}$ indexed by the directed set of Kähler forms on X ordered by reverse of the usual comparison. For each ω ,

$$\Gamma^{\theta+\omega} : (-\infty, \Gamma_{\max}) \rightarrow \mathrm{PSH}(X, \theta + \omega)$$

is a decreasing concave curve of \mathcal{I} -model potentials. The number $\Gamma_{\max} \in \mathbb{R}$ is independent of the choice of ω . The transition map from the index ω to $\omega + \omega'$ sends $\Gamma^{\theta+\omega}$ to the following map

$$(-\infty, \Gamma_{\max}) \rightarrow \mathrm{PSH}(X, \theta + \omega + \omega'), \quad \tau \mapsto P_{\theta+\omega+\omega'} \left[\Gamma_{\tau}^{\theta+\omega} \right]_{\mathcal{I}}.$$

The volume of Γ is defined as the limit

$$\lim_{\omega} \left(\theta + \omega + \mathrm{dd}^c \Gamma_{-\infty}^{\theta+\omega} \right)^n.$$

Here $\Gamma_{-\infty}^{\theta+\omega} = \sup_{\tau < \Gamma_{\max}}^* \Gamma_{\tau}^{\theta+\omega}$.

The subset $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$ of $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ consisting of elements with positive volume can be identified with the set of concave curves of \mathcal{I} -model potentials $(\Gamma_{\tau})_{\tau < \Gamma_{\max}}$ in $\mathrm{PSH}(X, \theta)$ for some $\Gamma_{\max} \in \mathbb{R}$ such that the volume $\int_X (\theta + \mathrm{dd}^c \Gamma_{-\infty})^n > 0$.

4.2. The Duistermaat–Heckman measure. We fix a smooth flag Y_{\bullet} on X .

Now suppose that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$. We define the Okounkov test curve $(\Delta_{Y_{\bullet}}(\Gamma)_{\tau})_{\tau < \Gamma_{\max}}$ associated with Γ as follows: given $\tau < \Gamma_{\max}$, we set

$$\Delta_{Y_{\bullet}}(\Gamma)_{\tau} := \Delta_{Y_{\bullet}}(\theta + \mathrm{dd}^c \Gamma_{\tau}).$$

The right-hand side is the partial Okounkov body studied in [Xia23; DX24].

Proposition 4.1. *Given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$, the curve $(\Delta_{Y_{\bullet}}(\Gamma)_{\tau})_{\tau < \Gamma_{\max}}$ is an Okounkov test curve relative to $\Delta_{Y_{\bullet}}(\theta + \mathrm{dd}^c \Gamma_{-\infty})$.*

Proof. This is a simple consequence of the properties proved in [Xia23; DX24]. \square

Definition 4.2. The *Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$ is defined as the Duistermaat–Heckman measure of the Okounkov test curve $\Delta_{Y_{\bullet}}(\Gamma)$.

Theorem 4.3. *The Duistermaat–Heckman measure $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$ is independent of the choice of the flag Y_{\bullet} .*

Proof. Assume further more that Γ is bounded ($\Gamma_\tau = \Gamma_{-\infty}$ for small enough τ). we observe that the characteristic function of the random variable $G[\Delta_{Y_\bullet}(\Gamma)]$ as computed in [\(B.2\)](#) [eq:char](#) is independent of the choice of the flag and is entire. It is a classical result that in this case, the corresponding probability distribution is determined by the moments.

In general, Γ is the decreasing limit of the sequence $\Gamma \vee \Gamma^k$ as $k \rightarrow \infty$, where $\Gamma^k: (-\infty, -k) \rightarrow \text{PSH}(X, \theta)$ takes the constant value $\Gamma_{-\infty}$. It follows from the general continuity result proved in [\[DX24\]](#) [Xia23](#) that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of $\Delta_{Y_\bullet}(\Gamma \vee \Gamma^k)_\tau$ for any $\tau < \Gamma_{\max}$. So $\text{DH}(\Gamma \vee \Gamma^k) \rightarrow \text{DH}(\Gamma)$ by [Lemma 3.6](#). It follows that $\text{DH}(\Gamma)$ is independent of the choice of the flag. \square

REFERENCES

- [Xia23] [DX24] Tamás Darvas and Mingchen Xia. The trace operator of quasi-plurisubharmonic functions on compact Kähler manifolds (to appear). 2024.
- [DXZ23] [DXZ23] Tamás Darvas, Mingchen Xia, and Kewei Zhang. A transcendental approach to non-Archimedean metrics of pseudoeffective classes. 2023. arXiv: [2302.02541 \[math.AG\]](#).
- [Sch14] [Sch14] R. Schneider. Convex bodies: the Brunn–Minkowski theory. 151. Cambridge university press, 2014.
- [Xia21] [Xia21] M. Xia. Partial Okounkov bodies and Duistermaat–Heckman measures of non-Archimedean metrics. 2021. arXiv: [2112.04290 \[math.AG\]](#).
- [XiaNA] [Xia23] Mingchen Xia. Operations on transcendental non-Archimedean metrics. 2023. eprint: <https://mingchenxia.github.io/home/Notes/OTNA.pdf>.

Mingchen Xia, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, PARIS

Email address, mingchen@imj-prg.fr

Homepage, <http://mingchenxia.github.io/home/>.