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Singularities in global pluripotential theory

– Lectures at Zhejiang University –

Updated on January 7, 2025. The latest version can be found here.
Latest updates: Correct some history, include Theorem 6.1.1

Preface

This book is an expanded version of my lecture notes at the Institute for Advanced Study in Mathematics (IASM) at Zhejiang university. My initial goal was to write a self-contained reference for the participants of the lectures. But I soon realized that many results have never been rigorously proved in any literature. When attempting to resolve these loose ends, the notes grew increasingly lengthy, ultimately resulting in the current book.

In this book, I would like to present my point of view towards the *global* pluripotential theories. There are three different but interrelated theories which deserve this name. They are

(1) the pluripotential theory on compact Kähler manifolds,
(2) the pluripotential theory on the Berkovich analytification of projective varieties, and
(3) the toric pluripotential theory on toric varieties.

We will begin by explaining the picture in the first case. Let us fix a compact Kähler manifold X. The central objects are the *quasi-plurisubharmonic functions* on X.

We are mostly interested in the *singularities* of such functions, that is, the places where a quasi-plurisubharmonic function φ tends to $-\infty$ and how it tends to $-\infty$.

Singularities occur naturally in mathematics. In geometric applications, X should be regarded as the compactified moduli space of certain geometric objects. A Zariski open subset $U \subseteq X$ would parametrize smooth objects. The natural metric on the associated polarizing line bundle is usually smooth only on U, not on X. In case we have suitable positivities, the classical Grauert–Remmert extension theorem (Theorem B.2.2) allows us to extend the metric outside U, but at the cost of introducing singularities.

The classification of singularities is a huge project. Locally near the singularities we know that quasi-plurisubharmonic functions present very complicated behaviours. There are many local invariants associated with the singularities. The most notable ones are the Lelong numbers and the multiplier ideal sheaves. These invariants only reflect the rough behaviour of a quasi-plurisubharmonic function. As an example, a quasi-plurisubharmonic function with log-log singularities have the same local invariants as a bounded one.

The situation changes drastically in the global setting, namely on compact manifolds. In the global setting, there are three different ways to classify quasi-plurisubharmonic functions according to their singularities:

(1) The singularity type characterizing the singularities up to a bounded term.
(2) The P-singularity type associated with global masses.
(3) The $\mathcal{I}$-singularity type associated with all non-Archimedean data.

The classification becomes rougher and rougher as we go downward. In the first case, we say two quasi-plurisubharmonic functions have the same singularity type if their difference lies in L^∞. The corresponding equivalence class gives us essentially the finest information of the singularities we can expect. The other two relations are more delicate, we will study them in detail in Chapter 6.

A natural idea to study the singularities would consist of the following steps:

(1) Classify the $\mathcal{I}$-singularity types.
(2) Classify the P-singularity types within a given $\mathcal{I}$-singularity class.
(3) Classify the singularity types within a given P-equivalence class.

The Step 3 is well-studied in the literature in the last decade under the name of pluripotential theory with prescribed singularities. There are numerous excellent results in this direction. In some sense, this step is already well-understood.

We will give a complete answer to Step 1 in Chapter 7, where we show that $\mathcal{I}$-singularity types can be described very explicitly.

It remains to consider Step 2. This is not an easy task. It is easy to construct examples where a given $\mathcal{I}$-equivalence class consists of a huge amount of P-equivalence classes.

On the other hand, by contrast, in the toric pluripotential theory and non-Archimedean pluripotential theory, Step 2 is essentially trivial: An $\mathcal{I}$-equivalence class consists of a single P-equivalence class. In the toric situation, an $\mathcal{I}$ or P-equivalence class is simply a sub-convex body of the Newton body, while in the non-Archimedean situation, an $\mathcal{I}$ or P-equivalence class is a homogeneous plurisubharmonic metric.

This apparent anomaly and numerous examples show that in the pluripotential theory on compact Kähler manifolds, certain singularities are pathological. Within each $\mathcal{I}$-equivalence class, we could pick up a canonical P-equivalence class, the quasi-plurisubharmonic functions in which are said to be *$\mathcal{I}$-good*. We will study the theory of $\mathcal{I}$-good singularities in Chapter 7. As we will see later on, almost all (if not all) singularities occurring naturally are $\mathcal{I}$-good.

My personal impression is that we are in a situation quite similar to the familiar one in real analysis. There are many non-measurable functions, but in real life, unless you construct a pathological function by force, you only encounter measurable functions. Similarly, although there exist many non-$\mathcal{I}$-good singularities, you would never encounter them in reality!

Having established this general principle, we could content ourselves in the framework of $\mathcal{I}$-good singularities. Then Step 2 is essentially solved, and we have a pretty good understanding of the classification of singularities.

Of course, this classification is a bit abstract. To put it into use, we will introduce two general techniques allowing us to make induction on $\dim X$. For a prime divisor Y in general position, we have the so-called analytic Bertini theorems relating quasi-plurisubharmonic functions on X and on Y. For a non-generic Y, we have the technique of trace operators. These techniques will be explained in Chapter 8.

In the toric situation, these constructions and methods are quite straightforward and are likely known to experts before I entered this field, see Chapter 5 for the toric pluripotential theory on ample line bundles.

The corresponding toric pluripotential theory on big line bundles has never been written down in the literature. We will develop the theory of partial Okounkov bodies in Chapter 10 and the general toric pluripotential theory will be developed as an application in Chapter 12.

Finally, we give applications to non-Archimedean pluripotential theory in Chapter 13 based on the theory of test curves developed in Chapter 9. We also prove the convergence of the partial Bergman kernels in Chapter 14.

The readers are only supposed to be familiar with the basic pluripotential theory. The excellent book [GZ17] is more than enough.

Mingchen Xia
in Hangzhou, March 2024

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Special appreciation goes to Yi Yao and Kewei Zhang for their invaluable contributions to discussions on toric geometry, which ultimately inspired the theory developed in Chapter 12.

Most results in this book are developed in collaboration with Tamás Darvas and Kewei Zhang, whose insights are always crucial in the development of the theories. I would like to thank them for the collaborations over years.

A substantial part of the current book was essentially contained in my PhD thesis. I would like to thank my advisor Robert Berman for his guidance and my colleagues in Göteborg and Paris for constant discussions, especially Bo Berndtsson, David Witt Nyström, Sébastien Boucksom and Elizabeth Wulcan.

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Contents

Conventions

In the whole book, we adopt the following conventions:

- A complex space is always assumed to be *reduced*, *paracompact* and *Hausdorff*.
- A *modification* of a complex space X is proper bimeromorphic morphism $\pi: Y \to X$ that is locally obtained from a finite composition of blow-ups with smooth centers.
- A *subnet* of a net refers to a cofinal subnet.
- A *domain* in $\mathbb{C}^n$ refers to a connected open subset.
- A *complex manifold* is assumed to be paracompact.
- A *submanifold* of a complex manifold means a complex submanifold.
- A *neighborhood* is not necessarily open.
- The set $\mathbb{N}$ of natural numbers includes 0.
- *Increasing functions* and *decreasing functions* are not necessarily strictly monotone.

We will use the following notations throughout the book:

- If I is a non-empty set, then $\mathrm{Fin}(I)$ denote the net of finite non-empty subsets of I, ordered by inclusion.
- $\mathrm{dd^c}$ means $(2\pi)^{-1}\mathrm{i}\partial\overline{\partial}$.

Part I
Preliminaries

In the first two chapters Chapter 1 and Chapter 2 of this part, we recall a few preliminaries about the notion of plurisubharmonic functions and the non-pluripolar products of plurisubharmonic functions.

Most materials in these chapters are standard and are well-documented in other textbooks, so we will be rather sketchy. The readers are encouraged to consult the excellent textbook [GZ17].

In Chapter 3, we develop the techniques of envelope operators. All results in this section are known and are written in various articles.

In Chapter 4, we develop the theory of geodesics in the space of quasi-plurisubharmonic functions. Most results in this chapter are known to different degrees, but not in the fully general form as we present. Most proofs are similar to the known proofs in the literature, but the presence of singularities requires a very careful treatment.

In Chapter 5, we recall the basic results about the toric pluripotential theory on ample line bundles, which will be generalized to big line bundles in Chapter 12.

Experienced readers may safely skip the whole part.

Chapter 1
Plurisubharmonic functions

Once Frigyes Riesz[a] *gave a brilliant explanation of why scientific work is easy. "Everyone has ideas, both right ideas and wrong ideas," he said. "Scientific work consists merely of separating them."*
— Istvan Vincze

[a] Frigyes Riesz (1880–1956), known as Frédéric Riesz in French and Frederic Riesz in English was the first mathematician to define the general notion of subharmonic functions, who also gave these functions a Frenglish name from the very beginning — *fonctions subharmoniques*.

In this chapter, we recall the notion of plurisubharmonic functions and a few basic properties of these functions. The main purpose is to fix the notation for later chapters, so we refer to the literature for most of the proofs.

We give some details about the plurifine topology in Section 1.3, since the related proofs are scattered in a number of articles.

In the literature related to multiplier ideal sheaves and Lelong numbers, there are several different conventions about their normalizations. The readers can find more about the conventions that we adopt throughout the book in Section 1.4.

1.1 The definition of plurisubharmonic functions

In this section, we recall the notion of plurisubharmonic functions. We will also take care of the 0-dimensional case, which makes a number of induction arguments easier to carry out. None of our references treats the 0-dimensional case, but the readers can easily verify that the results in this section hold in this exceptional case.

1.1.1 The 1-dimensional case

Let Ω be a domain (a connected open subset) in $\mathbb{C}$.

Definition 1.1.1 A *subharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) φ satisfies the *sub-mean value inequality*: For any $a \in \Omega$ and $r > 0$ such that $B_1(a, r) \Subset \Omega$, we have

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta.$$

We will denote the set of subharmonic functions on Ω as $\mathrm{SH}(\Omega)$.

Here, $B_1(a, r)$ denotes the open ball with center a and radius r. See (1.1).

In fact, for each $a \in \Omega$, in (3), it suffices to require the sub-mean value inequality for all small enough $r > 0$.

Intuitively, at a specific point $a \in \Omega$, the Condition (2) gives a lower bound of the value of $\varphi(a)$ using the nearby values of φ, while the Condition (3) gives an upper bound. This intuition leads to the following rigidity theorem:

Theorem 1.1.1 *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\Delta\varphi \geq 0$.*
(2) *φ coincides almost everywhere with a subharmonic function ψ on Ω.*

Moreover, the subharmonic function ψ in (2) is unique.

Here in Condition (1), $\Delta\varphi$ is the Laplacian in the sense of currents. This is a special case of Theorem 1.1.2 below.

This theorem gives a very useful way of constructing subharmonic functions.

1.1.2 The higher dimensional case

We will fix $n \in \mathbb{N}$ and a domain Ω (a connected open subset) in $\mathbb{C}^n$.

Definition 1.1.2 When $n \geq 1$, a *plurisubharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) for any complex line $L \subseteq \mathbb{C}^n$ and any connected component U of $L \cap \Omega$, the restriction $\varphi|_U$ is either subharmonic or constantly $-\infty$.[1]

When $n = 0$, the only domain Ω is the singleton. In this case, a *plurisubharmonic function* on Ω is a real-valued function on Ω.

The set of plurisubharmonic functions on Ω is denoted by $\mathrm{PSH}(\Omega)$.

A plurisubharmonic function is also called a psh function for short. The relevant notations are indicated in Fig. 1.1.[2]

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[1] An extremely common mistake in the literature is to replace (3) by the condition that φ is locally integrable and $dd^c\varphi \geq 0$ in the sense of current. For a concrete counterexample, consider a function φ that takes a constant value 0 at all but one single point, at which the value of φ is 1.

[2] We remind the readers that all figures in this book are sometimes misleading: We usually draw a complex dimension as a real dimension. The figures should not be read literally!

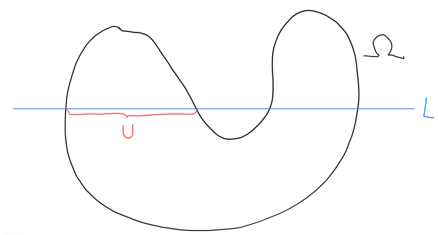

Fig. 1.1 A domain cut by a line

Example 1.1.1 When $n = 0$, we have a canonical bijection $\mathrm{PSH}(\Omega) \cong \mathbb{R}$.

Example 1.1.2 When $n = 1$, we have $\mathrm{PSH}(\Omega) = \mathrm{SH}(\Omega)$.

Similar to Theorem 1.1.1, we have a rigidity theorem for plurisubharmonic functions as well.

Theorem 1.1.2 *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on Ω.*

Moreover, the plurisubharmonic function ψ is unique.

Here, the operator $\mathrm{dd^c}$ is normalized so that

$$\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}.$$

For the proof, we refer to [GZ17, Proposition 1.43].

Plurisubharmonic functions have nice functorialities:

Proposition 1.1.1 *Let $n' \in \mathbb{N}$ and $\Omega' \subseteq \mathbb{C}^{n'}$ be a domain. Given any holomorphic map $f : \Omega \to \Omega'$ and any $\varphi \in \mathrm{PSH}(\Omega')$ exactly one of the following cases occurs:*

(1) *$f^*\varphi \equiv -\infty$;*
(2) *$f^*\varphi \in \mathrm{PSH}(\Omega)$.*

We refer to [GZ17, Proposition 1.44] for the proof[3].

For each $n \in \mathbb{N}$, $a \in \mathbb{C}^n$ and $r > 0$, we write

$$B_n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}\,. \tag{1.1}$$

[3] We remind the readers that the statement of [GZ17, Proposition 1.44] is flawed.

Proposition 1.1.2 *Let $\varphi \in \mathrm{PSH}(B_n(a, r_0))$ for some $r_0 > 0$. Then the function*

$$(-\infty, \log r_0) \to \mathbb{R}, \quad \log r \mapsto \sup_{B_n(a,r)} \varphi$$

is convex and increasing.

See [Bou17, Corollary 2.4].

Proposition 1.1.3 *Let $a < b$ be two real numbers. Let $f : (a, b) \to [-\infty, \infty)$ be a function. Define*

$$g : \{z \in \mathbb{C} : \mathrm{e}^{-b} < |z| < \mathrm{e}^{-a}\} \to [-\infty, \infty), \quad z \mapsto f(-\log |z|).$$

Suppose that g is subharmonic, then f is convex. In particular, f takes real values only.

See [HK76, Theorem 2.12] for a more general result.

1.1.3 The manifold case

Let X be a complex manifold. In the whole book, complex manifolds are assumed to be paracompact, namely, all connected components have countable bases.

Definition 1.1.3 A *plurisubharmonic function* on X is a function $\varphi : X \to [-\infty, \infty)$ such that for any $x \in X$, there exists an open neighborhood U of x in X, an integer $n \in \mathbb{N}$, a domain $\Omega \subseteq \mathbb{C}^n$ and a biholomorphic map $F : \Omega \to U$ such that $F^*(\varphi|_U) \in \mathrm{PSH}(\Omega)$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Example 1.1.3 When X is a domain in $\mathbb{C}^n$, the notions of plurisubharmonic functions in Definition 1.1.3 and in Definition 1.1.2 coincide.

Example 1.1.4 Write $\{X_i\}_{i\in I}$ for the set of connected components of X. Then we have a natural bijection

$$\mathrm{PSH}(X) \cong \prod_{i\in I} \mathrm{PSH}(X_i).$$

Here the product is in the category of sets. In particular, if $X = \varnothing$, then $\mathrm{PSH}(X) = \varnothing$.

This example allows us to reduce to the case of connected manifolds when studying general plurisubharmonic functions.

Proposition 1.1.4 *Let Y be another complex manifold and $f : Y \to X$ be a holomorphic map. Then for any $\varphi \in \mathrm{PSH}(X)$, exactly one of the following cases occurs:*

(1) *$f^*\varphi$ is identically $-\infty$ on some connected component of Y;*
(2) $f^*\varphi \in \mathrm{PSH}(Y)$.

This proposition follows easily from Proposition 1.1.1. We leave the details to the readers.

Theorem 1.1.2 implies immediately the general form of the rigidity theorem:

Theorem 1.1.3 *Let $\varphi\colon X \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on X.*

Moreover, the plurisubharmonic function ψ in (2) is unique.

Definition 1.1.4 A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighborhood U of x in X and a function $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{E\cap U} \equiv -\infty.$$

A subset $E \subseteq X$ is *non-pluripolar* if E is not pluripolar.

A subset $F \subseteq X$ is *co-pluripolar* if $X \setminus F$ is pluripolar.

When X has dimension 1, a pluripolar set is called a *polar set*.

Theorem 1.1.4 (Josefson's theorem) *Let $E \subseteq \mathbb{C}^n$ be a pluripolar set. Then there is $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ such that $\varphi|_E \equiv -\infty$.*

See [GZ17, Corollary 4.41] for the proof of a more general result.

There is also a global version of Josefson's theorem:

Theorem 1.1.5 *Assume that X is a compact complex manifold and $E \subseteq X$ is a pluripolar set. Then there is a quasi-plurisubharmonic function φ on X with $\varphi|_E \equiv -\infty$.*

For a proof, see [Vu19].

1.2 Properties of plurisubharmonic functions

In this section, we explore the basic properties of plurisubharmonic functions.

Let X be a complex manifold.

Proposition 1.2.1

(1) *Assume that $(\varphi_i)_{i\in I}$ is a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. Then ${\sup_{i\in I}}^*\varphi_i \in \mathrm{PSH}(X)$.*
(2) *Assume that $(\varphi_i)_{i\in I}$ is a decreasing net in $\mathrm{PSH}(X)$ such that $\lim_{i\in I}\varphi_i$ is not identically $-\infty$ on each connected component of X, then $\lim_{i\in I}\varphi_i \in \mathrm{PSH}(X)$.*

Here $\sup^*$ denotes the upper semicontinuous regularization of the supremum. When I is a finite family, observe that

$$\sup_{i\in I}{}^*\varphi_i = \sup_{i\in I}\varphi_i.$$

When $I = \{1, \dots, m\}$, we write

$$\varphi_1 \vee \cdots \vee \varphi_m := \sup_{i\in I}\varphi_i.$$

We refer to [GZ17, Proposition 1.28, Proposition 1.40][4].

Proposition 1.2.2 (Choquet's lemma) *Assume that X has countably many connected components. Assume that $(\varphi_i)_{i\in I}$ is a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. There exists a countable subset $J \subseteq I$ such that*

$$\sup_{i\in I}{}^*\varphi_i = \sup_{j\in J}{}^*\varphi_j.$$

Proof We may assume that X is connected. Since by our convention, the complex manifold X is paracompact, it can be covered by countably many open balls, so we can easily reduce to the case where X is an open ball. In this case, the result is proved in [GZ17, Lemma 4.31]. □

Proposition 1.2.3 *Let $\varphi \in \mathrm{PSH}(X)$, then for any $p \geq 1$, $\varphi \in L^p_{\mathrm{loc}}(X)$.*

See [GZ17, Theorem 1.46, Theorem 1.48].

Proposition 1.2.4 *A pluripolar set $E \subseteq X$ is a Lebesgue null set.*

Proof This is a trivial consequence of Proposition 1.2.3. □

Proposition 1.2.5 *Let $(\varphi_i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. Then the set*

$$\left\{x \in X : \sup_{i\in I}\varphi_i < \sup_{i\in I}{}^*\varphi_i\right\}$$

is pluripolar and hence Lebesgue null.

See [GZ17, Corollary 4.28].

Proposition 1.2.6 *Suppose that $\varphi, \psi \in \mathrm{PSH}(X)$. Assume that there is a dense subset $E \subseteq X$ such that $\varphi|_E \leq \psi|_E$, then $\varphi \leq \psi$.*

[4] In [GZ17, Proposition 1.28], the second part is only stated for sequences, the net version is obvious using the sub-mean value inequality.

Proof The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$.

We may assume that $\varphi|_E = \psi|_E$ after replacing φ by $\varphi \vee \psi$. Then we need to show that $\varphi = \psi$.

It follows from [GZ17, Theorem 4.20] that this holds outside a pluripolar set $Y \subseteq X$. In particular, $\varphi = \psi$ almost everywhere. It follows from the uniqueness statement in Theorem 1.1.3 that $\varphi = \psi$. □

Proposition 1.2.7 *Let $(E_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence of pluripolar sets in X. Then*

$$E \coloneqq \bigcup_{i=1}^{\infty} E_i$$

is also pluripolar.

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain. In this case, by Theorem 1.1.4 for each $i \in \mathbb{Z}_{>0}$ we can choose $\psi_i \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\psi_i|_{E_i} \equiv -\infty, \quad \psi_i|_X \leq 0$$

for all $i > 0$. After shrinking X, we may guarantee that $\psi_i|_X \in L^1(X)$ for all $i > 0$. After rescaling, we may also assume that $\|\psi_i\|_{L^1(X)} \leq 1$ for all $i > 0$.

We then define

$$\psi = \sum_{i=1}^{\infty} 2^{-i}\psi_i|_X.$$

Then $\psi \in \mathrm{PSH}(X)$ according to Proposition 1.2.1 and $\psi|_E = -\infty$. □

Corollary 1.2.1 *Let $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in* $\mathrm{PSH}(X)$ *such that $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi \in$* $\mathrm{PSH}(X)$. *Then the set*

$$\left\{x \in X : \varphi(x) \neq \varlimsup_{j\to\infty} \varphi_j(x)\right\}$$

is pluripolar.

Proof We first observe that $(\varphi_j)_j$ is locally uniformly bounded from above. This follows from [GZ17, Exercise 1.20].

For each $j \geq 1$, let

$$\psi_j = \sup_{k\geq j}{}^{*} \varphi_k.$$

Then $\psi_j \in \mathrm{PSH}(X)$ by Proposition 1.2.1. Moreover, $(\psi_j)_j$ is a decreasing sequence and $\psi_j \geq \varphi_j$ for all j. In particular, $\varphi \leq \psi \coloneqq \inf_j$ almost everywhere. By Proposition 1.2.1 again, $\psi \in \mathrm{PSH}(X)$.

On the other hand, by Proposition 1.2.5, there exist pluripolar sets $Z_j \subseteq X$ such that

$$\psi_j = \sup_{k\geq j} \varphi_k$$

on $X \setminus Z_j$. Let

$$Z = \bigcup_{j=1}^{\infty} Z_j.$$

Then Z is a pluripolar set by Proposition 1.2.7, and for any $x \in X \setminus Z$, we have $\psi(x) = \overline{\lim}_j \varphi_j(x)$. Since $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi$, we can find a set $Y \subseteq X$ with zero Lebesgue measure such that $\varphi_j(x) \to \varphi(x)$ for all $x \in X \setminus Y$.

In particular, for any $x \in X \setminus (Y \cup Z)$, we have

$$\psi(x) = \varphi(x).$$

But thanks to Proposition 1.2.6, the equality holds everywhere. Therefore, for all $x \in X \setminus Z$,

$$\varphi(x) = \overline{\lim_{j\to\infty}} \varphi_j(x).$$

Theorem 1.2.1 (Brelot, Grauert–Remmert) *Let Z be an analytic subset in X and $\varphi \in \mathrm{PSH}(X \setminus Z)$. Then the function φ admits an extension to $\mathrm{PSH}(X)$ in the following two cases:*

(1) *The set Z has codimension at least 2 everywhere.*
(2) *The set Z has codimension at least 1 everywhere and is locally bounded from above on an open neighborhood of Z.*

In both cases, the extension is unique and is given by

$$\varphi(x) = \overline{\lim_{X \setminus Z \ni y \to x}} \varphi(y), \quad x \in Z. \tag{1.2}$$

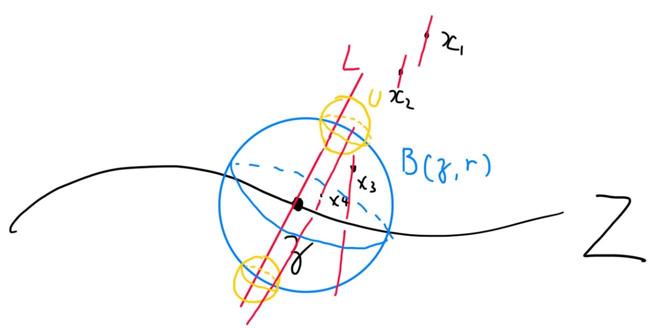

Fig. 1.2 The proof of Grauert–Remmert extension theorem

Proof The extension is unique thanks to Proposition 1.2.6.

(2) Thanks to the uniqueness of the extension, the problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ with $n > 0$ and there is a non-zero holomorphic

function f vanishing identically on Z. For each $\epsilon > 0$, we claim that the function φ_ϵ defined by

$$\varphi_\epsilon(x) := \begin{cases} \varphi(x) + \epsilon \log |f(x)|^2, & x \in X \setminus Z; \\ -\infty, & x \in Z \end{cases}$$

is plurisubharmonic on X. By Definition 1.1.2, it suffices to verify the case $n = 1$. In this case, we may assume that $Z = \{0\}$, It is clear that $\varphi_\epsilon \in \mathrm{SH}(X \setminus Z)$. It suffices to verify the sub-mean value inequality at 0, which is immediate.

Next observe that the sequence φ_ϵ is increasing as $\epsilon \searrow 0$ and φ_ϵ is locally uniformly bounded from above. It follows from Proposition 1.2.1 that $\tilde{\varphi} := {\sup_{\epsilon>0}}^* \varphi_\epsilon \in \mathrm{PSH}(X)$. Moreover, $\tilde{\varphi}$ clearly extends φ. Note that (1.2) follows from the construction.

(1) We invite the readers to have a look at Fig. 1.2 for our notations in the proof.

It suffices to verify that φ is locally bounded from above near each point of Z. The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ with $n \geq 2$.

Assume that our assertion fails. Take $z \in Z$ so that there exists a sequence $(x_j)_j$ in $X \setminus Z$ such that

$$\lim_{j\to\infty} \varphi(x_j) = \infty.$$

Since Z has codimension at least 2[5], we could take a complex line L passing through z and intersects Z only on a discrete set. After shrinking X, we may assume that

$$L \cap Z = \{z\}.$$

Take an open ball $B_n(z,r) \Subset X$. After adding a constant to φ, we may guarantee that $\varphi < 0$ on $L \cap \partial B_n(z,r)$. Since φ is upper semi-continuous, we could find an open neighborhood U of $L \cap \partial B_n(z,r)$ such that

$$\varphi|_U < 0.$$

For each $j \geq 1$, take a complex line L_j passing through x_j and avoiding Z such that $L_j \to L$ as $j \to \infty$. Here we rely on the fact that Z has codimension at least 2. Here the convergence is in the obvious sense. Then for large enough j, we know have

$$L_j \cap \partial B_n(z,r) \subseteq U.$$

It follows from the sub-mean value inequality that $\varphi(x_j) < 0$ for large enough j, which is a contradiction. $\square$

Lemma 1.2.1 *Let $\varphi \in \mathrm{PSH}((\Delta^*)^n)$ be an $(S^1)^n$-invariant plurisubharmonic function. Then φ is finite everywhere.*

Here

$$\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

---

[5] In fact, codimension at least 1 suffices for this step.

Proof It clearly suffices to handle the case $n = 1$. In this case, by [HK76, Theorem 2.12], the map

$$\log r \mapsto \int_0^1 \varphi(r\exp(2\pi \mathrm{i}\theta))\,\mathrm{d}\theta = \varphi(r)$$

is a convex function of $\log r$. So, the set $\{r \in (0,1) : \varphi(r) = -\infty\}$ is convex. But φ is almost everywhere finite by Proposition 1.2.3. Since φ is S^1-invariant, $\varphi|_{(0,1)}$ is almost everywhere finite. It follows from the convexity that it is everywhere finite.□

Proposition 1.2.8 (Kiselman's principle) *Let $\Omega \subseteq \mathbb{C}^m \times \mathbb{C}^n$ be a pseudoconvex domain. Assume that for each $z \in \mathbb{C}^m$, the set*

$$\Omega_z \coloneqq \{w \in \mathbb{C}^n : (z,w) \in \Omega\}$$

has the form $E + \mathrm{i}\mathbb{R}^n$, where $E \subseteq \mathbb{R}^n$ is a subset. Let $\varphi \in \mathrm{PSH}(\Omega)$, assume that φ is independent of the imaginary part of the variable in $\mathbb{C}^n$. Let Ω' be the projection of Ω to $\mathbb{C}^m$. Define $\psi : \Omega' \to [-\infty, \infty)$ as follows:

$$\psi(z) = \inf_{w \in \Omega_z} \varphi(z,w).$$

Then either $\psi \equiv -\infty$ or $\psi \in \mathrm{PSH}(\Omega')$.

See [Dem12b, Theorem 7.5].

Lemma 1.2.2 *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\Omega' \subseteq \Omega$ be a subdomain. Consider $\varphi \in \mathrm{PSH}(\Omega)$ and $\psi \in \mathrm{PSH}(\Omega')$. Assume that*

$$\varliminf_{\substack{\Omega' \ni y \to x,\\ \psi(y) \neq -\infty}} (\varphi(y) - \psi(y)) \geq 0$$

for any $x \in \Omega \cap \partial\Omega'$. Define

$$\eta(z) = \begin{cases} \varphi(z) \vee \psi(z), & \text{if } z \in \Omega', \\ \varphi(z), & \text{if } z \in \Omega \setminus \Omega'. \end{cases}$$

Then $\eta \in \mathrm{PSH}(\Omega)$.

Morally, this is just [GZ17, Proposition 1.30]. But the statement in the reference is slightly misleading, so I reproduced the proof just for clarification.

Proof See Fig. 1.3 for the notations used in the proof.

Take $\epsilon > 0$. We first define

$$\eta_\epsilon(z) = \begin{cases} \varphi(z) \vee (\psi(z) - 2\epsilon), & \text{if } z \in \Omega', \\ \varphi(z), & \text{if } z \in \Omega \setminus \Omega'. \end{cases}$$

We claim that

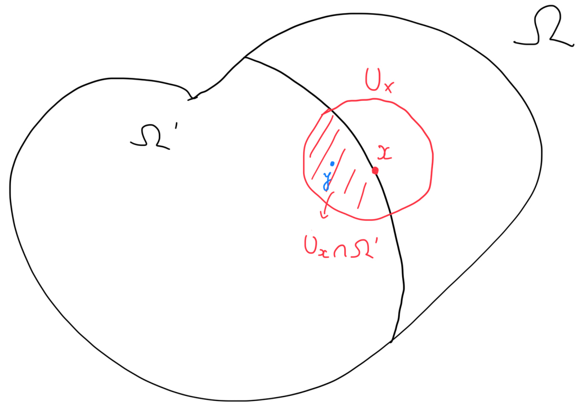

Fig. 1.3 Gluing procedure

$$\eta_\epsilon \in \mathrm{PSH}(\Omega).$$

By our assumption, for each $x \in \Omega \cap \partial\Omega'$, we can find an open neighborhood $U_x \subseteq \Omega$ such that for any $y \in U_x \cap \Omega'$, we have $\varphi(y) \geq \psi(y) - \epsilon$. Therefore, there is an open neighborhood U of $\Omega \cap \partial\Omega'$ such that

$$\varphi(y) \geq \psi(y) - \epsilon, \quad \forall y \in U \cap \Omega'.$$

Therefore, on the open set $(\Omega \setminus \Omega') \cup U$, we have $\eta_\epsilon = \varphi$ and hence η_ϵ is plurisubharmonic there. It is plurisubharmonic on Ω' by Proposition 1.2.1. So our claim follows.

Next we observe that as ϵ decreases to 0, the functions η_ϵ increases to η. Therefore, $\eta^* \in \mathrm{PSH}(\Omega)$ by Proposition 1.2.1. But observe that η is upper semicontinuous. This is only non-trivial on the boundary of Ω': Take $x \in \Omega \cap \partial\Omega'$ and let $(y_i)_{i>0}$ be a sequence in Ω' with limit x. Then we need to show that

$$\overline{\lim_{i\to\infty}} \psi(y_i) \leq \varphi(x). \tag{1.3}$$

We may assume that $\psi(y_i) \neq -\infty$ for all $i > 0$ and the left-hand side of (1.3) is not $-\infty$. Then we can compute

$$\overline{\lim_{i\to\infty}} \psi(y_i) \leq \overline{\lim_{i\to\infty}} \psi(y_i) + \underline{\lim_{i\to\infty}} (\varphi(y_i) - \psi(y_i)) \leq \overline{\lim_{i\to\infty}} \varphi(y_i) \leq \varphi(x).$$

Therefore, $\eta = \eta^* \in \mathrm{PSH}(\Omega)$. □

1.3 Plurifine topology

1.3.1 Plurifine topology on domains

Let $\Omega \subseteq \mathbb{C}^n$ ($n \in \mathbb{N}$) be a domain.

Definition 1.3.1 The *plurifine topology* on Ω is the weakest topology making all $\mathbb{R}$-valued plurisubharmonic functions on Ω continuous.

We want to distinguish the Euclidean topology from the plurifine topology. In the whole book, topological notions without adjectives refer to those with respect to the Euclidean topology. We include the symbol $\mathcal{F}$ in order to denote those with respect to the plurifine topology. For example, we will say $\mathcal{F}$-open subset, $\mathcal{F}$-neighborhood, $\mathcal{F}$-closure, etc. The $\mathcal{F}$-closure of a set $E \subseteq \Omega$ will be denoted by $\bar{E}^{\mathcal{F}}$.

We remind the readers that in the whole book, we follow Bourbaki's convention, a neighborhood is not necessarily open. Similarly, an $\mathcal{F}$-neighborhood is not necessarily $\mathcal{F}$-open.

A priori, we should include Ω into the notations as well, but as we will see shortly in Corollary 1.3.1, this is usually unnecessary.

Proposition 1.3.1 *The plurifine topology on Ω is finer than the Euclidean topology.*

Proof It suffices to show that the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ is $\mathcal{F}$-open. This follows from the observation that this set can be written as

$$\{\psi < 0\} \text{ with } \psi(z) := (\log|z|) \vee (-1).$$

Example 1.3.1 Let $\varphi \in \mathrm{PSH}(\Omega)$ and $C \in \mathbb{R}$. Then the sets $\{\varphi > C\}$ and $\{\varphi < C\}$ are both $\mathcal{F}$-open.

In fact, the later case follows from Proposition 1.3.1. While the former follows from the observation

$$\{\varphi > C\} = \{\varphi \vee (C-1) > C\}.$$

Definition 1.3.2 A subset $E \subseteq \Omega$ is *thin*[6] at $x \in \Omega$ if one of the following conditions holds:

(1) $x \notin \bar{E}$;
(2) $x \in \bar{E}$ and there is an open neighborhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y \to x, y \in E \setminus \{x\}} \varphi(y) < \varphi(x).$$

We say E is *thin* if it is thin at all $x \in \Omega$.

[6] A more proper name would be *plurithin*. But since we will never need the classical notion of thin sets à la Cartan in this book, we prefer omitting the prefix *pluri-*.

Remark 1.3.1 In the second case, we can always arrange that

$$\varphi|_{(E\setminus\{x\})\cap U}$$

is a constant. In fact, we may assume that $\varphi \leq 0$ and $C < 0$ is such that

$$\varlimsup_{y\to x, y\in E\setminus\{x\}} \varphi(y) < C < \varphi(x).$$

We let

$$\psi = (-C)^{-1}(u \vee C) + 1.$$

Then ψ satisfies our requirements for a smaller U.

In the second case, the function φ can be very much improved.

Proposition 1.3.2 (Bedford–Taylor) *Consider a set $E \subseteq \Omega$ and $x \in \bar{E}$. Assume that E is thin at x, then there is $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$:*

(1) *φ is locally bounded outside a neighborhood of x;*
(2) *$\underline{\varphi(x)} > -\infty$;*
(3) $\varlimsup_{y\to x, y\in E\setminus\{x\}} \varphi(y) = -\infty$.

Proof [7] By Remark 1.3.1, there is an open neighborhood $U \subseteq \Omega$ of x and $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{U\cap(E\setminus\{x\})} = -1 < \psi(x) = 0.$$

Without loss of generality, we may assume that $x = 0$, U is the unit ball in $\mathbb{C}^n$.

As ψ is upper semicontinuous, we may choose a decreasing sequence $\delta_j \in (0, 1)$ such that $\psi(y) < 2^{-j-2}$ when $y \in \mathbb{C}^n$ satisfies $|y| < \delta_j$. Set

$$\gamma_j \coloneqq \exp\left(2^{j+1}\log\delta_j\right) \in (0, 1).$$

Observe that γ_j is also decreasing.

We let

$$\varphi_j(z) \coloneqq \begin{cases} \left(\dfrac{2^{-j-1}}{|\log\delta_j|}\log|z|\right) \vee \left(\psi(z) - 2^{-j}\right), & \text{if } |z| < \delta_j, \\ \dfrac{2^{-j-1}}{|\log\delta_j|}\log|z|, & \text{if } |z| \geq \delta_j. \end{cases}$$

Observe that when $|z|$ is sufficiently close to δ_j from below (depending on j), we have

$$\frac{2^{-j-1}}{|\log\delta_j|}\log|z| > 2^{-j-2} - 2^{-j} > \psi(z) - 2^{-j}.$$

[7] The original argument in [BT82, Proposition 10.2] was quite intriguing: Neither the auxiliary functions φ_j's nor the simple computations were correct. However, I believe that Bedford–Taylor had a correct proof in mind. Something more than a typo, but not yet a mistake, could be properly called a *thinkpo*, a terminology invented by R. Berman.

In particular, $\varphi_j \in \mathrm{PSH}(\mathbb{C}^n)$ and $\varphi_j|_U \leq 0$. Moreover, we have

$$\varphi_j(0) = -2^{-j}.$$

Observe that for $z \in U \cap (E \setminus \{0\})$ with $|z| < \gamma_j$, we have $\varphi_j(z) \leq -1$.

We then define

$$\varphi \coloneqq \sum_{j=1}^{\infty} \varphi_j.$$

Since

$$\varphi(0) = -\sum_{j=1}^{\infty} 2^{-j} > -\infty, \quad \sum_{j=1}^{\infty} \frac{2^{-j-1}}{|\log \delta_j|} < \infty,$$

we have $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$. Moreover, fix j, for any $z \in E \setminus \{0\}$ with $|z| < \gamma_j$, we have

$$\varphi(z) \leq \sum_{k=1}^{j} \varphi_k(z) \leq -j.$$

Therefore,

$$\varlimsup_{y \to x, y \in E \setminus \{0\}} \varphi(y) = -\infty.$$

Lemma 1.3.1 *Let $E_1, E_2 \subseteq \Omega$. Assume that E_1, E_2 are both thin at $x \in \Omega$, then so is $E_1 \cup E_2$.*

Proof We may clearly assume that $x \in \overline{E_1} \cap \overline{E_2}$. Take an open neighborhood $U \subseteq \Omega$ of x and $\varphi_1, \varphi_2 \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y \to x, y \in E_i \setminus \{x\}} \varphi_i(y) < \varphi_i(x), \quad i = 1, 2.$$

Then $\varphi_1 + \varphi_2 \in \mathrm{PSH}(U)$ and

$$\varlimsup_{y \to x, y \in (E_1 \cup E_2) \setminus \{x\}} (\varphi_1 + \varphi_2)(y) < \varphi_1(x) + \varphi_2(x).$$

In particular, $E_1 \cup E_2$ is thin at x. □

Theorem 1.3.1 (H. Cartan) *Consider $x \in \Omega$ and a set $E \subseteq \Omega$. Assume that $x \in E$. Then the following are equivalent:*

(1) *E is an $\mathcal{F}$-neighborhood of x;*
(2) *$\Omega \setminus E$ is thin at x.*

Proof (2) $\implies$ (1). We may assume that $x \in \overline{\Omega \setminus E}$. Otherwise, our assertion follows from Proposition 1.3.1.

By Proposition 1.3.2, there is an open neighborhood U of x in Ω and $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\varphi(x) > \sup_{y \in U \cap (\Omega \setminus E)} \varphi(y) =: \lambda.$$

Let $F = \{y \in \Omega : \varphi(y) > \lambda\}$. Then $x \in F$ and F is $\mathcal{F}$-open by Example 1.3.1. Moreover, $U \cap F \subseteq E$. By Proposition 1.3.1, we conclude (1).

(1) $\implies$ (2). We may always replace E by smaller $\mathcal{F}$-neighborhoods of x. In particular, we may assume that E has the following form

$$\{y \in U : \varphi_1(y) > \lambda_1, \dots, \varphi_m(y) > \lambda_m\},$$

where $U \subseteq \Omega$ is an open neighborhood of x, and $\varphi_1, \dots, \varphi_m$ are $\mathbb{R}$-valued psh functions on Ω, and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since a finite union of thin sets is still thin by Lemma 1.3.1, we may assume that $m = 1$. In this case, $\Omega \setminus E$ is clearly thin at x. □

Theorem 1.3.2 *A base of the plurifine topology on Ω is given by sets of the following form:*

$$\{x \in U : \varphi(x) > 0\}\,, \tag{1.4}$$

where $U \subseteq \Omega$ is an open subset and $\varphi \in \mathrm{PSH}(U)$.

Proof Observe that sets of the form (1.4) are $\mathcal{F}$-open.[8] By Theorem 1.3.1, it suffices to show its complement in Ω is thin at each point of (1.4), which is obvious.

Now consider $x \in \Omega$ and an $\mathcal{F}$-open neighborhood $V \subseteq \Omega$ of x. We want to find a set of the form (1.4) contained in V and containing x.

Write $E = \Omega \setminus V$. In case $x \in \operatorname{Int} V$, there is nothing to prove. So we may assume that $x \in \bar{E}$. By Theorem 1.3.1, E is thin at x. By definition, there is an open neighborhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y \to x, y \in U \cap (E \setminus \{x\})} \varphi(y) < \varphi(x).$$

We may assume that $\varphi|_{E \cap U} \leq 0 < \varphi(x)$, Then the set $\{y \in U : \varphi(y) > 0\}$ suffices for our purpose. □

Remark 1.3.2 We remind the readers that in general, an $\mathcal{F}$-open set is *not* a *countable* union of sets of the form (1.4). In fact, an $\mathcal{F}$-open set is not a Borel set in general. See [EK23] for a concrete example.

Corollary 1.3.1 *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n$ be two non-empty open subsets. Then the plurifine topology on Ω_1 is the same as the subspace topology induced from the plurifine topology on Ω_2.*

In particular, when we talk about an $\mathcal{F}$-open set U in $\mathbb{C}^n$, we no longer have to specify the domain $\Omega \supseteq U$.

Corollary 1.3.2 *Let L be an affine subspace of $\mathbb{C}^n$, then the plurifine topology on L is the same as the subspace topology induced from the plurifine topology on $\mathbb{C}^n$.*

Proof We may assume that $L = \mathbb{C}^k \times \{0\}$ for some $k \leq n$. We write the coordinate z on $\mathbb{C}^n$ as (z', z'') with $z \in \mathbb{C}^k$ and $z'' \in \mathbb{C}^{n-k}$.

[8] This is not entirely obvious by definition, as φ is not defined on the whole Ω.

Consider an $\mathcal{F}$-open set $U \subseteq \mathbb{C}^n$ and $x = (x', 0) \in U \cap L$. We want to show that $U \cap L$ (identified with a subset of $\mathbb{C}^k$) is an $\mathcal{F}$-neighborhood of x' in L. By Theorem 1.3.2, we may assume that there are connected open subsets $U' \subseteq \mathbb{C}^k$ containing x' and $U'' \subseteq \mathbb{C}^{n-k}$ containing 0 together with a psh function ψ on $U' \times U''$ such that

$$x \in \{(z', z'') \in U' \times U'' : \psi(z', z'') > 0\} \subseteq \Omega.$$

It follows that

$$x' \in \{z' \in U' : \psi(z', 0) > 0\} \subseteq U \cap L.$$

Thanks to Proposition 1.1.1, $\psi(z', 0)$ is plurisubharmonic in z' because $\psi(x', 0) \neq -\infty$. In particular, $U \cap L$ is an $\mathcal{F}$-neighborhood of x'.

Conversely, if $U \subseteq \mathbb{C}^k$ is an $\mathcal{F}$-open subset, we claim that $U \times \mathbb{C}^{n-k}$ is $\mathcal{F}$-open in $\mathbb{C}^n$. In fact, suppose that $(x', x'') \in U \times \mathbb{C}^{n-k}$. By Theorem 1.3.1, we can find an open neighborhood $V \subseteq \mathbb{C}^k$ of x' and a psh function φ on V such that

$$x' \in \{y \in V : \varphi(y) > 0\} \subseteq U.$$

We define $\psi(z', z'') := \varphi(z')$. Then $\psi \in \mathrm{PSH}(V \times \mathbb{C}^{n-k})$ by Proposition 1.1.1 and

$$(x', x'') \in \{y \in V \times \mathbb{C}^n : \psi(y) > 0\} \subseteq U \times \mathbb{C}^{n-k}.$$

Corollary 1.3.3 *Let $\Omega \subseteq \mathbb{C}^n$ be an $\mathcal{F}$-open subset and $x \in \Omega$. Then x has a compact $\mathcal{F}$-neighborhood contained in Ω.*

Proof By Theorem 1.3.2, we may assume that there is an open set $U \subseteq \mathbb{C}^n$ and a plurisubharmonic function φ on U such that $\Omega = \{y \in U : \varphi(y) > 0\}$.

Take a compact neighborhood K of x in U. Now $\{y \in K : \varphi(y) \geq \varphi(x)/2\}$ is a compact $\mathcal{F}$-neighborhood of x contained in Ω. □

Corollary 1.3.4 *Let $\Omega \Subset \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^{n'}$ be two domains and $F \colon \Omega' \to \Omega$ be a surjective holomorphic map. Then F is $\mathcal{F}$-continuous.*

Proof It suffices to show that the inverse image $F^{-1}(U)$ of each $\mathcal{F}$-open subset $U \subseteq \Omega$ is $\mathcal{F}$-open. By Theorem 1.3.2, after possibly shrinking Ω and Ω', we may assume that U has the form $\{x \in \Omega : \psi(x) > 0\}$, where $\psi \in \mathrm{PSH}(\Omega)$. Since $F^*\psi \in \mathrm{PSH}(\Omega')$ by Proposition 1.1.4, we find that

$$F^{-1}(U) = \{y \in \Omega' : F^*\psi(y) > 0\}$$

is $\mathcal{F}$-open. □

1.3.2 Plurifine topology on manifolds

Let X be a complex manifold.

Definition 1.3.3 The *plurifine topology* on X is the topology with a base consisting of sets of the form $F^{-1}(V)$, where $U \subseteq X$ is an open subset and $F\colon U \to \Omega$ is a biholomorphic morphism with $\Omega \subseteq \mathbb{C}^n$ is a domain for some $n \in \mathbb{N}$ and $V \subseteq \Omega$ is $\mathcal{F}$-open.

Note that these sets form a topological base thanks to Corollary 1.3.4. Moreover, it also follows from Corollary 1.3.4 that the plurifine topologies on domains defined in Definition 1.3.3 and in Definition 1.3.1 coincide.

We refer to Definition 1.5.1 for the notion of quasi-plurisubharmonic functions.

Proposition 1.3.3 *Let $\varphi \in \mathrm{QPSH}(X)$, then $\varphi|_{\{\varphi\neq-\infty\}}$ is $\mathcal{F}$-continuous.*

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain and $\varphi = \psi + g$, where $\psi \in \mathrm{PSH}(X)$ and $g \in C^\infty(X)$ and $|g| \leq C$ for some $C > 0$. Take an open interval $(a, b) \subseteq \mathbb{R}$, it suffices to show that

$$U \coloneqq \{x \in X : a < \varphi(x) < b\} = \{x \in X : a - g(x) < \psi(x) < b - g(x)\}$$

is $\mathcal{F}$-open. Take $x \in U$, we can find an open neighborhood V of x in U such that

$$\sup_{y\in V} (a - g(y)) < \psi(x) < \inf_{y\in V} (b - g(y)) .$$

Therefore,

$$\left\{z \in V : \sup_{y\in V} (a - g(y)) < \psi(z) < \inf_{y\in V} (b - g(y))\right\}$$

is an $\mathcal{F}$-open neighborhood of z in U. We conclude that U is $\mathcal{F}$-open. □

Corollary 1.3.5 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Then the set*

$$\{x \in X : \varphi(x) > \psi(x)\}$$

is $\mathcal{F}$-open.

Proof It suffices to show that for any $x \in X$ such that $\varphi(x) > \psi(x)$, the same holds on an $\mathcal{F}$-neighborhood U of x. Observe that $\varphi(x) \neq -\infty$. If $\psi(x) \neq -\infty$, then it suffices to apply Proposition 1.3.3. Otherwise, take

$$U \coloneqq \{y \in X : \varphi(y) > \varphi(x) - 1\} \cap \{y \in X : \psi(y) < \varphi(x) - 1\}.$$

Lemma 1.3.2 *Let $Z \subseteq X$ be a pluripolar subset. Then*

$$\overline{X \setminus Z}^{\mathcal{F}} = X.$$

Proof The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ and $Z = \{\varphi = -\infty\}$ for some $\varphi \in \mathrm{PSH}(X)$. We need to show that $\{\varphi > -\infty\}$ is $\mathcal{F}$-dense.

Let $x \in X$ be a point with $\varphi(x) = -\infty$ and $U \subseteq X$ be an $\mathcal{F}$-open neighborhood of x in X. We need to show that $U \cap \{\varphi > -\infty\} \neq \varnothing$.

Thanks to Theorem 1.3.2, after shrinking U, we may assume that there is $\psi \in \mathrm{PSH}(X)$ such that $U = \{\psi > 0\}$. Observe that U is not a pluripolar set: Otherwise, $\psi \leq 0$ almost everywhere by Proposition 1.2.4, and hence everywhere by Proposition 1.2.6. So $\varphi|_U \not\equiv -\infty$. We conclude. □

Corollary 1.3.6 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Set*

$$W = \{x \in X : \varphi(x) = -\infty\} \text{ or } W = \{x \in X : \psi(x) = -\infty\}.$$

Then for any pluripolar set $Z \subseteq X$, we have

$$\sup_{X \setminus W} (\varphi - \psi) = \sup_{X \setminus W \cup Z} (\varphi - \psi), \quad \inf_{X \setminus W} (\varphi - \psi) = \inf_{X \setminus W \cup Z} (\varphi - \psi).$$

In particular, taking $\psi = 0$, we find that

$$\sup_{X \setminus Z} \varphi = \sup_X \varphi.$$

Proof This is an immediate consequence of Lemma 1.3.2 and Proposition 1.3.3. □

In the literature about pluripotential theory, one often finds the careless expressions like $\sup_X(\varphi - \psi)$. The issue is that $\varphi - \psi$ is not defined everywhere, and hence this expression does not make sense if you read it literally. Corollary 1.3.6 tells you that you do not have to worry too much about the details on a pluripolar set. In other words, sup and inf could always be understood as a kind of essential supremum and essential infimum modulo pluripolar sets.

1.4 Lelong numbers and multiplier ideal sheaves

Let X be a complex manifold.

Definition 1.4.1 Let $\varphi \in \mathrm{PSH}(X)$ and $x \in X$. The *Lelong number* $\nu(\varphi, x)$ of φ at x is defined as follows: Take an open neighborhood U of x in X and a biholomorphic map $F : U \to \Omega$, where Ω is a domain in $\mathbb{C}^n$. Then we define

$$\nu(\varphi, x) := \sup \left\{ \gamma \in \mathbb{R}_{\geq 0} : \varphi|_U(F^{-1}(y)) \leq \gamma \log |y - F(x)|^2 + O(1) \text{ as } y \to F(x) \right\}. \tag{1.5}$$

Observe that $\nu(\varphi, x)$ does not depend on the choices of U and F. Furthermore, it follows from Proposition 1.4.1 below that the supremum in (1.5) is a maximum.

Remark 1.4.1 Our definition of the Lelong number is not standard. It differs from the standard definition by a factor of 2. As a mnemonic, just remember

$$\nu\left(\log |z|^2, 0\right) = 1 \quad \text{(instead of 2)}.$$

Proposition 1.4.1 *Let $\varphi \in \mathrm{PSH}(B_n(0,1))$. Then*

$$\nu(\varphi, 0) = \lim_{r \searrow 0} \frac{\sup_{B_n(0,r)} \varphi}{\log r^2} \in [0, \infty). \tag{1.6}$$

Proof It follows from Proposition 1.1.2 that the limit in (1.6) exists and is finite. We shall denote the limit by $\nu'(\varphi, 0)$ for the time being.

We first observe that by Proposition 1.1.2,

$$\varphi(x) \leq \nu'(\varphi, 0) \log |x|^2 + \sup_{B_n(0,1)} \varphi \tag{1.7}$$

when $x \in B_n(0,1)$. In particular, $\nu(\varphi, x) \geq \nu'(\varphi, 0)$.

In order to argue the reverse inequality, we may assume that $\nu(\varphi, x) > 0$.

Next observe that by (1.5), for each small enough $\epsilon > 0$, we can find $r_0 \in (0,1)$ and $C > 0$ so that for all $x \in B_n(0, r_0)$, we have

$$\varphi(x) \leq (\nu(\varphi, 0) - \epsilon) \log |x|^2 + C.$$

It follows that $\nu'(\varphi, 0) \geq \nu(\varphi, 0) - \epsilon$. Letting $\epsilon \to 0+$, we conclude. □

We recall Siu's semicontinuity theorem.

Theorem 1.4.1 *Let $\varphi \in \mathrm{PSH}(X)$, then the map $X \ni x \mapsto \nu(\varphi, x)$ is upper semi-continuous with respect to the Zariski topology.*

For an elegant proof we refer to [Dem12a, Theorem 2.10].

Proposition 1.4.2 *Let $\varphi, \psi \in \mathrm{PSH}(X)$, $\lambda \in \mathbb{R}_{>0}$ and $x \in X$, then*

$$\begin{aligned} \nu(\varphi \vee \psi, x) =& \min\{\nu(\varphi, x), \nu(\psi, x)\}, \\ \nu(\varphi + \psi, x) =& \nu(\varphi, x) + \nu(\psi, x), \\ \nu(\lambda\varphi, x) =& \lambda \nu(\varphi, x). \end{aligned}$$

Proof All properties are local, so we may assume that $X = B_n(0,1)$ for some $n \in \mathbb{N}$. All properties follow directly from Proposition 1.4.1. □

Corollary 1.4.1 *Let $(\varphi_i)_{i \in I}$ be a non-empty family in $\mathrm{PSH}(X)$ locally uniformly bounded from above and $x \in X$, then*

$$\nu\left(\sup_{i \in I}{}^* \varphi_i, x\right) = \inf_{i \in I} \nu(\varphi_i, x).$$

Proof We may assume that X is connected. Write $\varphi = \sup_{i \in I}{}^* \varphi_i$. Then $\varphi \in \mathrm{PSH}(X)$ by Proposition 1.2.1.

We observe that the $\leq$ inequality is trivial. It remains to argue the reverse inequality.

It follows from Proposition 1.2.2 that we may assume that I is countable. When I is finite, this is already proved in Proposition 1.4.2. Otherwise, we may further assume that $I = \mathbb{Z}_{>0}$. Thanks to Proposition 1.4.2, we may further assume that $(\varphi_i)_{i \in \mathbb{Z}_{>0}}$ is an increasing sequence. Furthermore, since the problem is local, we may assume that $X = B_n(0,1)$ for some $n \in \mathbb{N}$ and $(\varphi_i)_i$ is uniformly bounded from above. In this case, by (1.7), we have

$$\varphi_i(x) \leq \nu(\varphi_i, 0) \log |x|^2 + C$$

for all $x \in B_n(0,1)$ and all $i \geq 1$ and C is a constant independent of i. In particular, thanks to Proposition 1.2.5, for almost all $x \in B_n(0,1)$, we have

$$\varphi(x) \leq \lim_{i\to\infty} \nu(\varphi_i, 0) \log |x|^2 + C.$$

Thanks of Proposition 1.2.6, the same holds for all x and hence

$$\nu(\varphi, x) \geq \lim_{i\to\infty} \nu(\varphi_i, x).$$

Definition 1.4.2 Let $F \subseteq X$ be a non-empty analytic subset. Then we define the *generic Lelong number* of φ along F as

$$\nu(\varphi, F) \coloneqq \min_{x\in F} \nu(\varphi, x).$$

Note that the minimum is obtained by Theorem 1.4.1.

Definition 1.4.3 Let $\varphi \in \mathrm{PSH}(X)$. Let E be a prime divisor over X (see Definition B.1.1). Take a proper bimeromorphic morphism $\pi\colon Y \to X$ from a complex manifold Y such that E is a prime divisor on Y, then we define the *generic Lelong number* of φ along E as

$$\nu(\varphi, E) \coloneqq \nu(\pi^*\varphi, E).$$

It follows from Theorem 1.4.1 that $\nu(\varphi, E)$ does not depend on the choice of π.

Definition 1.4.4 Let $\varphi \in \mathrm{PSH}(X)$, the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ of φ is by definition the ideal sheaf given by

$$\Gamma(U, \mathcal{I}(\varphi)) = \left\{ f \in \mathcal{O}_X(U) : |f|^2 \exp(-\varphi) \in L^1_{\mathrm{loc}}(U) \right\}$$

for any open subset $U \subseteq X$.

Remark 1.4.2 This definition is different from a few standard references, where instead of $\exp(-\varphi)$, they use $\exp(-2\varphi)$. The conventions adopted in the current book is the most convenient one as far as I know. It simplifies a number of formulae. As a mnemonic, for any real $\lambda > 0$, we have

$$\mathcal{I}(\lambda \log |z|^2) = \mathcal{O}_{\mathbb{C}}(-\lfloor \lambda \rfloor \{0\}),$$

where z is a variable in $\mathbb{C}$ and $\{0\}$ is the divisor defined by $0 \in \mathbb{C}$.

Proposition 1.4.3 (Nadel) *Let $\varphi \in \mathrm{PSH}(X)$. Then $\mathcal{I}(\varphi)$ is coherent.*

See [Dem12a, Proposition 5.7].

Theorem 1.4.2 *Let $\varphi, \psi \in \mathrm{PSH}(X)$, then*

$$\mathcal{I}(\varphi + \psi) \subseteq \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi).$$

See [Dem12a, Theorem 14.2].

The two invariants are related by the following simple result:

Proposition 1.4.4 *Let $\varphi \in \mathrm{PSH}(X)$ and E be a prime divisor over X. Then*

$$\nu(\varphi, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathcal{I}(k\varphi). \tag{1.8}$$

See [DX21, Proposition 2.14].

We remind the readers that this particular form of the formula is compatible with our conventions of ν and $\mathcal{I}$. As a consistency check, consider $\varphi = \log|z|^2$ with $z \in \mathbb{C}$ and E is the divisor defined by $0 \in \mathbb{C}$. Then both sides of (1.8) are equal to 1. See Remark 1.4.1 and Remark 1.4.2.

Also observe the following simple lemma:

Lemma 1.4.1 *Let $x \in X$ and $\varphi \in \mathrm{PSH}(X)$. Let $\pi\colon Y \to X$ be the blow-up of X at x with exceptional divisor E. Then*

$$\nu(\varphi, x) = \nu(\varphi, E),$$

See [Bou02a, Corollaire 1.1.8].

Conversely, the information of the generic Lelong numbers determines the multiplier ideal sheaves:

Theorem 1.4.3 *Let $\varphi \in \mathrm{PSH}(X)$. Let $x \in X$ and $f \in \mathcal{O}_{X,x}$. Then the following are equivalent:*

(1) $f \in \mathcal{I}(\varphi)_x$;
(2) *there exists $\epsilon > 0$ such that for any prime divisor E over X such that x is contained in the center of E on X, we have*

$$\operatorname{ord}_E(f) \geq (1+\epsilon)\nu(\varphi, E) - \frac{1}{2}A_X(E). \tag{1.9}$$

In case φ has analytic singularities and $\pi\colon Y \to X$ is a log resolution with finitely many exceptional divisors $\{E_i\}$ whose centers on X contain x, one may replace (1.9) by

$$\operatorname{ord}_{E_i}(f) > \nu(\varphi, E_i) - \frac{1}{2}A_X(E_i) \quad \forall i. \tag{1.10}$$

Here A_X denotes the log discrepancy. We refer to [Bou17, Corollary 10.18, Proposition 10.12] for the proof and the precise definition of A_X. The formula (1.9) differs from that in Boucksom's notes: The coefficient $\frac{1}{2}$ in front of $A_X(E)$ arises from our convention for ν and $\mathcal{I}$.

The notion of analytic singularities is recalled in Section 1.6.

Theorem 1.4.4 (Guan–Zhou) *Let $\varphi, \psi_j \in \mathrm{PSH}(X)$ ($j \in \mathbb{Z}_{>0}$) such that ψ_j is an increasing sequence converging to φ almost everywhere. Then for any $x \in X$, the germs satisfy*

$$\mathcal{I}(\psi_j)_x = \mathcal{I}(\varphi)_x$$

when j is large enough.

See [GZ15, Hie14] for the proof.

Proposition 1.4.5 *Let $\pi\colon Y \to X$ be a smooth morphism between complex manifolds. Assume that $\varphi \in \mathrm{PSH}(X)$, then*

$$\mathcal{I}(\pi^*\varphi) = \pi^*\mathcal{I}(\varphi).$$

Proof It follows from [Gro60, Théorème 3.10] that locally π can be written as the composition of an étale morphism and a projection. It suffices to handle the two cases separately.

Recall that in the complex analytic setting, an étale morphism is locally biholomorphic, so there is nothing to prove in this case.

Next, assume that $Y = X \times U$, where $U \subseteq \mathbb{C}^n$ is a domain and π is the natural projection. It follows from Fubini's theorem that

$$\mathcal{I}(\pi^*\varphi) \subseteq \pi^*\mathcal{I}(\varphi).$$

The reverse inequality is proved in [Dem12a, Proposition 14.3][9]. □

Definition 1.4.5 Given a coherent ideal sheaf $\mathcal{I}$ on X, the *restriction* $\mathrm{Res}_Y \mathcal{I}$ is the inverse image ideal sheaf given by

$$\mathrm{Res}_Y \mathcal{I} := \mathcal{I}/(\mathcal{I} \cap \mathcal{I}_Y), \tag{1.11}$$

where $\mathcal{I}_Y$ is the ideal sheaf defining Y.

In the literature, it is common to denote this sheaf by the misleading notation $\mathcal{I}|_Y$.

There is a natural morphism

$$i_Y^*\mathcal{I} = \mathcal{I}/(\mathcal{I} \cdot \mathcal{I}_Y) \to \mathrm{Res}_Y \mathcal{I}, \tag{1.12}$$

where $i_Y\colon Y \to X$ is the inclusion.

Theorem 1.4.5 (Ohsawa–Takegoshi) *Let Y be a connected submanifold of X and $\varphi \in \mathrm{PSH}(X)$. Assume that $\varphi|_Y \not\equiv -\infty$, then*

$$\mathcal{I}(\varphi|_Y) \subseteq \mathrm{Res}_Y \mathcal{I}(\varphi).$$

[9] In [Dem12a, Proposition 14.3], Demailly used the highly non-standard notation $f^*\mathcal{I}(\varphi)$ to denote the image of $f^*\mathcal{I}(\varphi) \to \mathcal{O}_X$, even when f is not flat.

See [Dem12a, Theorem 14.1].

1.5 Quasi-plurisubharmonic functions

In practice, it is important to consider a variant of plurisubharmonic functions. We will fix a complex manifold X.

Definition 1.5.1 Let θ be a closed real smooth $(1,1)$-form on X.

A *θ-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that for each $x \in X$ and each open neighborhood U of x in X satisfying the condition that $\theta = \mathrm{dd}^{\mathrm{c}} g$ for some smooth function g on U, we have $g + \varphi|_U \in \mathrm{PSH}(U)$. The set of θ-psh functions on X is denoted by $\mathrm{PSH}(X,\theta)$.

A *quasi-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that there exists a smooth closed real $(1,1)$-form θ' on X such that $\varphi \in \mathrm{PSH}(X,\theta')$. The set of quasi-plurisubharmonic functions on X is denoted by $\mathrm{QPSH}(X)$.

There is a natural non-strict partial order on $\mathrm{QPSH}(X)$ defined as follows:

Definition 1.5.2 Assume that X is compact. Given $\varphi,\psi \in \mathrm{QPSH}(X)$, we say that φ is *more singular* than ψ and write $\varphi \preceq \psi$[10] if there is $C \in \mathbb{R}$ such that $\varphi \leq \psi + C$. We also say ψ is less singular than φ and write $\psi \succeq \varphi$.

In case $\varphi \preceq \psi$ and $\psi \preceq \varphi$, we say φ and ψ have the same *singularity type*. We write $\varphi \sim \psi$ in this case.

When X is not compact, one can still define similar notions, but the generalization is not unique, and we shall not consider them in this book.

Remark 1.5.1 The proceeding results concerning plurisubharmonic functions can be extended *mutatis mutandis* to quasi-plurisubharmonic functions. We will apply these extensions without further explanations.

Proposition 1.5.1 *Assume that X is compact. Let θ be a closed real smooth* $(1,1)$*-form on X. Then for any* $a,b \in \mathbb{R}$*,* $a \leq b$*, the set*

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_X \varphi \in [a,b]\right\}$$

is compact with respect to the L^1*-topology. Moreover,* $\varphi \mapsto \sup_X \varphi$ *is* L^1*-continuous for* $\varphi \in \mathrm{PSH}(X,\theta)$.

This is an immediate consequence of [GZ17, Proposition 8.5, Exercise 1.20].

Remark 1.5.2 More generally, if $K \subseteq X$ is a closed non-polar subset. Then

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_K \varphi \in [a,b]\right\}$$

[10] Some people write $\psi \preceq \varphi$.

is relatively compact with respect to the L^1-topology. See [GZ05, Corollary 4.3].

Proposition 1.5.2 *Assume that X is compact. Let θ be a closed real smooth $(1,1)$-form on X and E be a prime divisor over E. Then*

$$\sup\{\nu(\varphi, E) : \varphi \in \mathrm{PSH}(X,\theta)\} < \infty.$$

Proof It follows from the proof of Corollary 1.4.1 that $\nu(\bullet, E)$ is upper semi-continuous with respect to the L^1-topology on $\mathrm{PSH}(X,\theta)$. Thus, the desired upper bound follows from Proposition 1.5.1. □

Proposition 1.5.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Let θ be a closed real smooth $(1,1)$-form on X. Then the pull-back gives a bijection*

$$\pi^*\colon \mathrm{PSH}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}(Y, \pi^*\theta).$$

This follows from a more general result Theorem B.1.1.

1.6 Analytic singularities

The simplest type of plurisubharmonic singularities is given by the so-called *analytic singularities*. The notion is fairly subtle and there are several mutually *incompatible* definitions in the literature.

Let X be a complex manifold.

Definition 1.6.1 We say $\varphi \in \mathrm{QPSH}(X)$ has *analytic singularities* if for each $x \in X$, we can find an open neighborhood U of x such that $\varphi|_U$ has the form:

$$c\log(|f_1|^2 + \cdots + |f_N|^2) + R, \tag{1.13}$$

where $f_1, \ldots, f_N$ are holomorphic functions on U, $c \in \mathbb{Q}_{>0}$ and R is a bounded function on U.

When R can be taken to be smooth[11], we say φ has *neat analytic singularities*.

Suppose that there is a coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ on X such that we can choose U so that the $f_1, \ldots, f_N$ can be chosen as the generators of $\Gamma(U, \mathcal{I})$ and c is independent of the choice of U, we say φ has analytic singularities of *type* $(c, \mathcal{I})$.

Each potential with analytic singularities has a type. The type is not uniquely determined. We refer to [Bou02a] and [Bou02b] for the details.

Proposition 1.6.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$ be potentials with analytic singularities, then so are $\lambda\varphi$ ($\lambda \in \mathbb{Q}_{>0}$), $\varphi + \psi$ and $\varphi \vee \psi$.*

[11] The decomposition (1.13) is highly non-unique. Here we mean for any x, there is an open neighborhood U and a decomposition of the form (1.13) with R smooth. In the non-trivial cases, R cannot be smooth for all decompositions (1.13).

Proof The $\lambda\varphi$ assertion is trivial. The $\vee$ assertion is proved in [Dem15, Proposition 4.1.8]. The addition assertion is easy and is left to the readers. □

Definition 1.6.2 Let D be an effective $\mathbb{Q}$-divisor[12] on X. We say $\varphi \in \mathrm{QPSH}(X)$ has *log singularities* (along D) on X if for each $x \in X$, there is an open neighborhood U of x such that

(1) $D|_U$ has finitely many irreducible components and can be written as

$$D|_U = \sum_{i=1}^{N} a_i D_i$$

with D_i being prime divisors on U, $a_i \in \mathbb{Q}_{>0}$ and there is a holomorphic function s_i on U defining D_i, and

(2) we have

$$\varphi|_U = a_i \sum_{i=1}^{N} \log |s_i|^2 + R, \tag{1.14}$$

where R is a bounded function on U.

By Proposition 1.6.1, φ has analytic singularities.

Lemma 1.6.1 *Suppose that θ is a closed smooth real $(1,1)$-form on X, a compact Kähler manifold and $\varphi \in \mathrm{PSH}(X,\theta)$. Suppose that φ has log singularities along an effective $\mathbb{Q}$-divisor D on X. Then the cohomology class $[\theta] - [D]$ is nef.*

Moreover, if in addition θ_φ is a Kähler current[13]*, then the cohomology class $[\theta] - [D]$ is ample.*

Proof The first assertion follows immediately from the fact that R in (1.14) has bounded coefficients.

The second assertion follows immediately from the first. □

The following proposition follows immediate from the definitions:

Proposition 1.6.2 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a complex manifold Y. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities (resp. has log singularities along an effective $\mathbb{Q}$-divisor D). Then $\pi^*\varphi$ has analytic singularities (resp. has log singularities along π^*D).*

Definition 1.6.3 Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. A *log resolution* of φ is a modification $\pi\colon Y \to X$ such that $\pi^*\varphi$ has log singularities.

Theorem 1.6.1 *Assume that X is compact. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then there is a log resolution of φ.*

For a proof, we refer to the arguments on [MM07, Page 104].

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[12] Divisors and $\mathbb{Q}$-divisors are implicitly assumed to have locally finite coefficients as usual.

[13] That is, there is a Kähler form ω on X such that $\theta_\varphi \geq \omega$ in the sense of currents.

Definition 1.6.4 Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. A sequence $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathrm{QPSH}(X)$ is *quasi-equisingular approximation* of φ if

(1) φ_j has analytic singularities for each j;
(2) φ_j is decreasing with limit φ;
(3) there is a decreasing sequence $\epsilon_j \geq 0$ with limit 0 and a Kähler form ω on X such that $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$;
(4) for each $\lambda' > \lambda > 0$, there is $j > 0$ such that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

We also say θ_{φ_j} is a quasi-equisingular approximation of θ_φ.

Definition 1.6.5 Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf and $c \in \mathbb{Q}_{>0}$. A function $\varphi \in \mathrm{QPSH}(X)$ is said to have *gentle analytic singularities* (of type $(c,\mathcal{I})$) if

(1) φ has analytic singularities of type $(c,\mathcal{I})$;
(2) $\mathrm{e}^{\varphi/c}\colon X \to \mathbb{R}_{\geq 0}$ is a smooth function;
(3) there is a proper bimeromorphic morphism $\pi\colon \tilde{X} \to X$ from a Kähler manifold $\tilde{X}$ and an effective $\mathbb{Z}$-divisor D on $\tilde{X}$ such that one can write $\pi^*\varphi$ locally as

$$\pi^*\varphi = c\log|g|^2 + h,$$

where g is a local equation of the divisor D and h is smooth.

Theorem 1.6.2 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Then any $\varphi \in \mathrm{PSH}(X,\theta)$ admits a quasi-equisingular approximation $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$.*

Moreover, we can guarantee that for all $j > 0$, φ_j has gentle analytic singularities of type $\left(2^{-j}, \mathcal{I}(2^j\varphi)\right)$.

We refer to [DPS01] for the proof.

Quasi-equisingular approximations are essentially unique in the following sense:

Proposition 1.6.3 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be two quasi-equisingular approximations of φ. Then for any $\epsilon > 0$ and any $j > 0$, we can find $k_0 > 0$ such that for any $k \geq k_0$, we have*

$$\psi_k \preceq (1-\epsilon)\varphi_j.$$

See [Dem15, Corollary 4.1.7].

Definition 1.6.6 Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then we define $\mathcal{I}_\infty(\varphi)$ as the ideal sheaf consisting of germs f of holomorphic functions such that $|f|^2\exp(-\varphi)$ is locally bounded.

Lemma 1.6.2 *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. The sheaf $\mathcal{I}_\infty(\varphi)$ is a coherent sheaf.*

Proof By Theorem 1.6.1, we may find a modification $\pi\colon Y \to X$ such that $\pi^*\varphi$ has log singularities. Observe that

$$\mathcal{I}_\infty(\varphi) = \pi_* \mathcal{I}(\pi^*\varphi),$$

so we may replace X and φ by Y and $\pi^*\varphi$ and assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. We decompose D into its irreducible components:

$$D = \sum_{i=1}^N a_i D_i.$$

In this case, observe that

$$\mathcal{I}_\infty(\varphi) = \mathcal{O}_X\left(-\sum_{i=1}^N (\lceil a_i \rceil D_i)\right)$$

is clearly coherent. □

Lemma 1.6.3 *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then for any $\epsilon > 0$, we can find $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$\mathcal{I}(k(1+\epsilon)\varphi) \subseteq \mathcal{I}_\infty(k\varphi).$$

See [Dem15, Proposition 4.1.6].

Theorem 1.6.3 *Let X be a connected compact Kähler manifold and $Y \subseteq X$ be a connected submanifold. Take a Kähler form ω on X and $\varphi \in \mathrm{PSH}(Y, \omega|_Y)$ such that $\omega|_Y + \mathrm{dd^c}\varphi$ is a Kähler current and that e^φ is a Hölder continuous function on V. Then there exists $\tilde{\varphi} \in \mathrm{PSH}(X, \omega)$ satisfying*

(1) *$\tilde{\varphi}|_Y = \varphi$;*
(2) *$\omega_{\tilde{\varphi}}$ is a Kähler current.*

In addition, if φ has analytic singularities, then so does $\tilde{\varphi}$.

See [DRWN+23, Theorem 6.1].

1.7 The space of currents

Let X be a connected compact Kähler manifold of dimension n and $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$.

Definition 1.7.1 Let Y be a complex manifold and $m \in \mathbb{N}$. We say an (m, m)-current T on Y is *positive*[14] if either $m > n$ or for any smooth $(1, 0)$-forms $\beta_1, \dots, \beta_{n-m}$ on X, the measure

[14] This notion is sometimes known as *weak positivity*.

$$T \wedge \mathrm{i}\beta_1 \wedge \overline{\beta_1} \wedge \cdots \wedge \mathrm{i}\beta_{n-m} \wedge \overline{\beta_{n-m}}$$

is positive.

The basic properties of positive currents can be found in [Dem12b, Section III.1]. We remind the readers that a positive current is necessarily real.

Definition 1.7.2 We say α is *pseudo-effective* if there is a closed positive $(1,1)$-current in α.

We say α is *big* if there is a closed positive $(1,1)$-current T in α dominating a Kähler form. Such currents are called *Kähler currents*.

Definition 1.7.3 We introduce the following notations:

(1) $\mathcal{Z}_+(X)$ denotes the space of closed positive $(1,1)$-currents on X;
(2) given a pseudo-effective $(1,1)$-class α on X, we write $\mathcal{Z}_+(X,\alpha)$ for the set of $T \in \mathcal{Z}_+(X)$ such that $[T] = \alpha$.

Here $[T]$ denotes the cohomology class represented by T.

Definition 1.5.2 has a natural analogue for currents.

Definition 1.7.4 Given $T, T' \in \mathcal{Z}_+(X)$, we write $T \preceq T'$ and say T is *more singular* than T' if when we write $T = \theta + \mathrm{dd}^{\mathrm{c}}\varphi$, $T' = \theta' + \mathrm{dd}^{\mathrm{c}}\varphi'$, we have $\varphi \preceq \varphi'$. We write $T \sim T'$ if $T \preceq T'$ and $T' \preceq T$. In this case, we say T and T' have the same *singularity type*.

Remark 1.7.1 Observe that

$$\mathcal{Z}_+(X)/\sim \cong \mathrm{QPSH}(X)/\sim$$

canonically. The correspondence sends the class of a closed positive current $\theta_\varphi = \theta + \mathrm{dd}^{\mathrm{c}}\varphi$ to the class of φ.

We will adopt the following convention: Whenever we have a notion for quasi-plurisubharmonic functions which depends only on the singularity type, we use the same notation and the same definition for closed positive $(1,1)$-currents.

Example 1.7.1 An an important example of Remark 1.7.1, given $T = \theta + \mathrm{dd}^{\mathrm{c}}\varphi \in \mathcal{Z}_+(X)$ and $x \in X$, we define

$$\nu(T,x) = \nu(\varphi,x). \tag{1.15}$$

Again, as Remark 1.4.1, this differs from the definitions in some literature by a factor of 2. But given our normalization

$$\mathrm{dd}^{\mathrm{c}} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial},$$

(1.15) seems to be the most natural choice.

The key example to keep in mind is the following:

$$\nu([0],0)=1,$$

where $[0]$ is the current of integration at $0\in\mathbb{P}^1$. In fact, as a simple application of the Green's second identity, one can verify that

$$\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log|z|^2=\delta_0,$$

where the right-hand side is the Dirac delta distribution at $0\in\mathbb{C}$.

Definition 1.7.5 Given $T\in\mathcal{Z}_+(X)$. We represent T as $\theta+\mathrm{dd^c}\varphi$ for some closed smooth real $(1,1)$-form θ on X and $\varphi\in\mathrm{PSH}(X,\theta)$, then the *polar locus* of T is defined as the set $\{\varphi=-\infty\}$.

It is clear that the polar locus of T is independent of the choices of θ and φ.

Lemma 1.7.1 (Siu's decomposition) *Let E be a prime divisor on X. Then for any closed positive $(1,1)$-current T on X, the difference $T-\nu(T,E)[E]$ is a closed positive $(1,1)$-current.*

Here $[E]$ is the current of integration associated with E.[15] See [GH94, Page 386, Example 1] for the precise definition. See [Dem12a, Lemma 2.17] for the proof.

It is helpful to check that our conventions are always consistent: There is no extra factor of 2 or $1/2$ anywhere. One could verify this using our favorite example as in Example 1.7.1.

1.8 Plurisubharmonic metrics on line bundles

A natural source of quasi-plurisubharmonic functions is the metrics on line bundles.

Let X be a connected Kähler manifold and L be a holomorphic line bundle on X. Usually, we do not distinguish L from the associated invertible sheaf $\mathcal{O}_X(L)$.

Definition 1.8.1 Let V be a 1-dimensional complex linear space. A *Hermitian form* h on V is a map $h\colon V\times V\to\mathbb{C}$ such that

(1) h is $\mathbb{C}$-linear in the second variable and conjugate linear in the first, and
(2)
$$|v|_h^2:=h(v,v)\in\mathbb{R}_{>0}$$
for each $v\in V\setminus\{0\}$.

We usually identify h with the quadratic form $V\to\mathbb{R}$ sending v to $|v|_h^2$. We write $|v|_h=\sqrt{|v|_h^2}$ for any $v\in V$.

The *singular Hermitian form* on V is the map $V\to\{0,\infty\}$ sending 0 to 0 and other elements to ∞.

[15] We have also used $[E]$ to denote the cohomology class of $[E]$. This should not lead to any confusion.

Definition 1.8.2 Let V_1 and V_2 be 1-dimensional complex linear spaces. Given two maps $h_i\colon V_i \to [0,\infty]$ ($i = 1, 2$) each of which is either a Hermitian form or a singular Hermitian form. Then we define the *tensor product* $h_1 \otimes h_2\colon V_1 \otimes V_2 \to [0,\infty]$ as follows:

(1) If either h_1 or h_2 is singular, we define $h_1 \otimes h_2$ as the singular Hermitian form;
(2) otherwise, define $h_1 \otimes h_2$ as the usual tensor product: For any $v_1 \in V_1$, $v_2 \in V_2$, set

$$h_1 \otimes h_2(v_1 \otimes v_2) = h_1(v_1)h_2(v_2).$$

Definition 1.8.3 A *Hermitian metric* h on L is a family of Hermitian forms $(h_x)_{x\in X}$, such that

(1) for each $x \in X$, h_x is a Hermitian form on L_x, and
(2) for each local section s of $\mathcal{O}_X(L)$, the map $x \mapsto |s(x)|_{h_x}$ is smooth.

The pair (L, h) is called a *Hermitian line bundle*. We shall write $\mathrm{dd^c} h = c_1(L, h)$[16] for the first Chern form of h[17], normalized so that

$$[c_1(L, h)] = c_1(L).$$

The map $x \mapsto |s(x)|_{h_x}$ will be denoted by $|s|_h$.

To be more precise, if $U \subseteq X$ is an open subset on which L admits a nowhere vanishing holomorphic section s, then we define

$$(\mathrm{dd^c} h)|_U = \mathrm{dd^c}\left(-\log |s|_h^2\right).$$

Proposition 1.8.1 (Lelong–Poincaré) *Let $s \in \mathrm{H}^0(X, L)$ be non-zero and h be a Hermitian metric on L. Then*

$$c_1(L, h) + \mathrm{dd^c} \log |s|_h^2 = [Z(s)], \tag{1.16}$$

where $Z(s)$ is the zero divisor defined by s and $[\bullet]$ denote the associated current of integration.

See [Dem12a, (3.11)]. Again, we want to check that our conventions are compatible by investigating the following simple example.

Example 1.8.1 Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. The homogeneous coordinates on $\mathbb{P}^1$ will be denoted by $[X_0 : X_1]$. At a point $x = [X_0 : X_1] \in \mathbb{P}^1$, the fiber L_x is identified with the dual of $[x]$, where $[x] \subseteq \mathbb{C}^2$ is the line represented by x.

In order to introduce the Hermitian metric h on L, we fix the standard Hermitian norm $\|\bullet\|$ on $\mathbb{C}^2$. Then given $\lambda \in L_x = [x]^\vee$, we introduce

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[16] The unusual notation $\mathrm{dd^c} h$ is sometimes referred to as the *Göteborg notation* because it is widely used by the complex geometriers in Göteborg (usually spelled as Gothenburg in English, the second largest (yet very poorly known) city in Sweden). As I identify myself as *Göteborgare*, I do not feel guilty about this notation.

[17] In the literature, people sometimes define the *curvature form* of (L, h) as $\Theta_h = -2\pi i\mathrm{dd^c} h$.

$$|\lambda|_{h_x} = \frac{|\lambda(\tilde{x})|}{\|\tilde{x}\|},$$

where $\tilde{x}$ is an arbitrary non-zero element in $[x]$. The readers can easily verify that h is indeed a Hermitian metric on L. The Hermitian metric h is known as the *Fubini–Study metric*.

A holomorphic section $s \in \mathrm{H}^0(X, L)$ can be formally identified with a linear form $a_0 X_0 + a_1 X_1$: At $x \in X$, the corresponding linear form on $[x]$ is given by sending (X_0, X_1) to $a_0 X_0 + a_1 X_1$.

Next we compute $\mathrm{dd^c} h = c_1(L, h)$. For this purpose, we cover $\mathbb{P}^1$ by $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ and $\mathbb{P}^1 \setminus \{0\}$. Both are holomorphic coordinate charts with coordinate function $z = X_0/X_1$ and $z^{-1} = X_1/X_0$ respectively.

We claim that on $\mathbb{C}$,

$$\mathrm{dd^c} h = \mathrm{dd^c} \log(1 + |z|^2). \tag{1.17}$$

In fact, let t be the nowhere vanishing section of L on $\mathbb{C}$ corresponding to X_1. Then for $z \in \mathbb{C}$, we have an obvious lift $(z, 1) \in [z]$, so

$$|t|_h^2(z) = \frac{1}{|z|^2 + 1}.$$

So (1.17) follows.

In order to obtain a non-trivial case of the Lelong–Poincaré formula, we need to consider a section which vanishes at some points in $\mathbb{C}$. Let s be the holomorphic section of L corresponding to X_0. Then

$$\log |s|_h^2(z) = \log \frac{|z|^2}{|z|^2 + 1}$$

for any $z \in \mathbb{C}$ using the same argument as above. Therefore, we find that restricted to $\mathbb{C}$, we have

$$c_1(L, h) + \mathrm{dd^c} \log |s|_h^2 = \mathrm{dd^c} f = [0],$$

where $f(z) = \log |z|^2$. So the Lelong–Poincaré formula (1.16) is verified in this case.

The Kähler form $\mathrm{dd^c} h$ on $\mathbb{P}^1$ is also known as the *Fubini–Study metric*.

Definition 1.8.4 A (singular) *plurisubharmonic metric* (or *psh metric* for short)[18] h on L is a family $(h_x)_{x \in X}$ such that

(1) for each $x \in X$, h_x is either a Hermitian form on L_x or the singular Hermitian form on L_x, and
(2) there is a Hermitian metric h_0 on L and $\varphi \in \mathrm{PSH}(X, c_1(L, h_0))$ such that for each $x \in X$ and each $v \in L_x$, we have

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[18] In the literature, people usually refer to such metrics as *positively curved singular Hermitian metrics*. I dislike this terminology, as having positive curvature only determines a plurisubharmonic metric almost everywhere, not everywhere.

$$|v|^2_{h_x} = \begin{cases} 0, & \text{if } v = 0; \\ |v|^2_{h_{0,x}} \mathrm{e}^{-\varphi(x)}, & \text{if } v \neq 0. \end{cases} \tag{1.18}$$

The *(first) Chern current* of h is by definition

$$\mathrm{dd^c} h = c_1(L, h) := c_1(L, h_0) + \mathrm{dd^c}\varphi.$$

We shall write the plurisubharmonic metric defined by (1.18) as $h_0 \exp(-\varphi)$[19]. As the readers can easily verify, our conventions guarantee that $c_1(L, h)$ does not depend on the choice of h_0.

Remark 1.8.1 In the literature, some people prefer the convention that in (1.18), neither side has the square. Our choice seems to be the most natural one given our normalization of $\mathrm{dd^c}$.

Observe that once a Hermitian metric h_0 on L is given, the construction in (2) gives a bijection between $\mathrm{PSH}(X, c_1(L, h_0))$ and the set of plurisubharmonic metrics on L.

Definition 1.8.5 Given two holomorphic line bundles L_1, L_2 on X and plurisubharmonic functions h_1 on L_1 and h_2 on L_2, we define the *tensor product* plurisubharmonic metric $h_1 \otimes h_2$ on $L_1 \otimes L_2$ as follows: for each $x \in X$, define

$$(h_1 \otimes h_2)_x = h_{1,x} \otimes h_{2,x}$$

in the sense of Definition 1.8.2.

We can easily verify that $h_1 \otimes h_2$ is indeed a plurisubharmonic metric on $L_1 \otimes L_2$.

Example 1.8.2 We continue with our example Example 1.8.1. Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Let h^0 denote the Fubini–Study metric on L as defined in Example 1.8.1. Note that we have changed the notation from h to h^0. Let $\omega = \mathrm{dd^c} h^0$.

We construct $\varphi \in \mathrm{PSH}(X, \omega)$ as follows: On $\mathbb{C}$, define

$$\varphi(z) = \log \frac{|z|^2}{1 + |z|^2}. \tag{1.19}$$

Then $\varphi \in \mathrm{PSH}(\mathbb{C}, \omega|_{\mathbb{C}})$ by (1.17). Setting $\varphi(\infty) = 0$, we can easily verify that $\varphi \in \mathrm{PSH}(\mathbb{P}^1, \omega)$.[20]

We then get a plurisubharmonic metric $h^0 \exp(-\varphi)$. To be more explicit, h_0 is singular, $h_\infty = h^0_\infty$, while for $z \in \mathbb{C} \setminus \{0\}$ and $\lambda \in [z]^\vee$, we have

$$|\lambda|_{h_z} = \frac{|\lambda(z, 1)|}{|z|}.$$

In the remaining of this section, we assume that X is compact.

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[19] Be careful, this is not $h_0^2 \exp(-\varphi)$, as I prefer to think of h_0 as a quadratic form.

[20] This can also be verified using the Grauert–Remmert extension theorem Theorem 1.2.1.

Definition 1.8.6 Assume that L is a pseudoeffective line bundle on X. A *Fubini–Study metric* on L is a psh metric h on L of the following form: There exists $m \in \mathbb{Z}_{>0}$, finitely many sections $s_1, \ldots, s_N \in \mathrm{H}^0(X, L^m)$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{Q}$ such that for any local nowhere vanishing holomorphic section s of L, we have

$$|s|_h^2 = \min_{i=1,\ldots,N} \left| \frac{s^{\otimes m}}{\mathrm{e}^{\lambda_i/2} s_i} \right|^{2m^{-1}}.$$

We write $\mathrm{FS}(L)$ for the set of Fubini–Study metrics on L.

If we fix a reference smooth Hermitian metric h_0 on L with $\theta = \mathrm{dd^c} h_0$, we can write $h = h_0 \exp(-\varphi)$ with

$$\varphi = \frac{1}{m} \max_{i=1,\ldots,N} \left(\log |s_i|^2_{h_0^m} + \lambda_i \right).$$

Similarly, we write $\mathrm{FS}(X, \theta)$ for the set of such functions.

Definition 1.8.7 Assume that L is a pseudoeffective line bundle on X. The set $\widetilde{\mathrm{FS}}(L)$ of *generalized Fubini metrics* is the smallest subset of $\mathrm{PSH}(L)$ containing $\mathrm{FS}(L)$ which is closed under the following two operations:

(1) $\mathbb{Q}$-convex combinations: if $h_1, h_2 \in \widetilde{\mathrm{FS}}(L)$ and $t \in (0, 1)$, then

$$h_1^t \otimes h_2^{1-t} \in \widetilde{\mathrm{FS}}(L);$$

(2) minima: if $h_1, h_2 \in \widetilde{\mathrm{FS}}(L)$, then

$$\min\{h_1, h_2\} \in \widetilde{\mathrm{FS}}(L).$$

We shall need the following Ohsawa–Takegoshi type extension theorem.

Theorem 1.8.1 *Assume that L is big and T is a holomorphic line bundle on X. Fix a Hermitian metric h_T on T. Take a Kähler form ω on X. Let $Y \subseteq X$ be a connected submanifold of dimension m. Suppose that $\varphi \in \mathrm{PSH}(X, \theta - \delta\omega)$ for some $\delta > 0$ and $\varphi|_Y \not\equiv -\infty$. Then there exists $k_0(\delta, h_T) > 0$ such that for all $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y))$ [21], there exists an extension $\tilde{s} \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ such that*

$$\int_X (h^k \otimes h_T)(\tilde{s}, \tilde{s}) \mathrm{e}^{-k\varphi}\, \omega^n \leq C \int_Y (h^k \otimes h_T)|_Y (s, s) \mathrm{e}^{-k\varphi|_Y}\, \omega|_Y^m,$$

where $C > 0$ is an absolute constant, independent of the data (φ, s, k).

This is a special case of [His12, Theorem 1.4].

Proposition 1.8.2 *Let (L, h) be a Hermitian line bundle on X and set $\theta = c_1(L, h)$. Let (T, h_T) be a Hermitian line bundle on X. Assume that $\varphi \in \mathrm{PSH}(X, \theta)$ is a*

[21] Here and in the sequel, we usually abbreviate $\otimes k$ in the super-index as k to save spaces.

potential with analytic singularities such that θ_φ is a Kähler current. Fix a Kähler form ω on X. For each $k \geq 1$, we let

$$\varphi_k \coloneqq \frac{1}{k} \log \sup_{\substack{s \in \mathrm{H}^0(X, L^k \otimes T) \\ \int_X h^k \otimes h_T(s,s) \mathrm{e}^{-k\varphi} \omega^n \leq 1}} h^k \otimes h_T(s,s). \tag{1.20}$$

Then for any $k \geq 0$,

$$\varphi \preceq \varphi_k \preceq \alpha_k \varphi,$$

where $\alpha_k \in (0,1)$ is an increasing sequence with limit 1.

Note that when k is large enough, $\varphi_k \in \mathrm{PSH}(X, \theta)$. We refer to [DX21, Remark 2.9] for the proof.

Chapter 2
Non-pluripolar products

Pour exprimer d'une manière frappante que le monument que j'élève sera placé sous l'invocation de la Science, j'ai décidé d'inscrire en lettres d'or sur la grande frise du premier étage et à la place d'honneur, les noms des plus grands savants[a] *qui ont honoré la France depuis 1789 jusqu'à nos jours.*
— Gustave Eiffel, 1889

[a] Gaspard Monge, Comte de Péluse (1746—1818), known oddly by his real family name instead of *de Péluse*, is one of the 72 names scribed on the Eiffel tower. He was both a mathematician and a politician, active mainly after the French Revolution.

Let X be a complex manifold and $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$ ($p \in \mathbb{N}$). When the functions $\varphi_1, \ldots, \varphi_p$ are all smooth, there is an obvious definition of a differential form

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p \tag{2.1}$$

by the usual differential calculus. The product is usually known as the *Monge–Ampère product*. It is of interest to extend this construction to the case where the φ_i's have worse regularities.

There are a number of different approaches to this problem. In this book, we will choose the so-called *non-pluripolar theory* due to Bedford, Taylor, Guedj, Zeriahi, Boucksom and Eyssidieux. The reason is that the non-pluripolar theory is the only known theory satisfying the following two features: It is defined for all psh singularities (at least in the global setting) and it satisfies a monotonicity theorem.

We will recall the Bedford–Taylor theory in Section 2.1 and the non-pluripolar theory in Section 2.2.

Some key properties of the non-pluripolar products are recalled in Section 2.3.

The readers who are not familiar with this notion are encouraged to read the original article [BEGZ10].

2.1 Bedford–Taylor theory

Let X be a complex manifold and $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$ ($p \in \mathbb{N}$) be locally bounded plurisubharmonic functions on X[1]. In this case, there is a canonical definition of the Monge–Ampère type product (2.1).

[1] In the literature, some people use $\mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ to denote the set of such functions, which is an abuse of notation. However, this is legitimate thanks to the rigidity Theorem 1.1.3.

Definition 2.1.1 We define the closed positive (p, p)-current (2.1) on X as follows: We make an induction on $p \geq 0$. When $p = 0$, we define (2.1) as the $(0, 0)$-current $[X]$. When $p > 0$, we let

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p := \mathrm{dd^c}\left(\varphi_1 \,\mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p\right).$$

We call this product the *Bedford–Taylor product* .

Remark 2.1.1 There is also a slightly more general version of this construction. Given a closed positive current T, one can also define the product

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p \wedge T$$

in a very similar way.

Proposition 2.1.1 *The product* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is a closed positive* (p, p)*-current on* X*. Moreover, the product is symmetric in the* φ_i*'s.*

See [GZ17, Proposition 3.3, Corollary 3.12]. The proof relies crucially on an important estimate, known as the *Chern–Levine–Nirenberg inequality*. See [GZ17, Theorem 3.9].

The Bedford–Taylor theory has many satisfactory properties.

Theorem 2.1.1 *Let* $(\varphi_i^j)_{j\in\mathbb{Z}_{>0}}$ *be decreasing sequences (resp. increasing sequences) of locally bounded psh functions on* X *converging (resp. converging a.e.) to locally bounded psh function* φ_i*, where* $i = 1, \ldots, p$*. Then*

$$\varphi_0^j \,\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^j \rightharpoonup \varphi_0 \,\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$$

as $j \to \infty$*. In particular, if* φ_0^j *is the constant sequence* 1*, we have*

$$\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^j \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p.$$

Here the notation $\rightharpoonup$ denotes the weak-* convergence of currents.

We refer to [GZ17, Theorem 3.18, Theorem 3.23] for the proofs.

By contrast, we emphasize that the Bedford–Taylor product is not continuous with respect to the L^1_{loc}-convergence in general. A simple example can be found in [GZ17, Example 3.25].

2.2 The non-pluripolar products

The proof of all results in this section can be found in [BEGZ10].

Let X be a complex manifold.

Definition 2.2.1 Let $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$. We set

$$O_k := \bigcap_{j=1}^{p} \{\varphi_j > -k\}, \quad k \in \mathbb{Z}_{>0}.$$

We say that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is *well-defined* if for each connected open subset $U \subseteq X$, any smooth Hermitian form ω on U, for each compact subset $K \subseteq U$, we have

$$\sup_{k\geq 0} \int_{K\cap O_k} \left(\bigwedge_{j=1}^{p} \mathrm{dd^c}(\varphi_j \vee (-k)) \right)\Bigg|_U \wedge \omega^{\dim U - p} < \infty. \tag{2.2}$$

In this case, we define the *non-pluripolar product* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ by

$$\mathbb{1}_{O_k} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c}\left(\varphi_j \vee (-k)\right) \tag{2.3}$$

on $\bigcup_{k\geq 0} O_k$ and make a zero-extension to X.

As recalled in Section 1.3, an $\mathcal{F}$-open subset means an open subset with respect to the plurifine topology.

Proposition 2.2.1 *Let $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$.*

(1) *The product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is local with respect to the plurifine topology in the following sense: Let $O \subseteq X$ be an $\mathcal{F}$-open subset and $\psi_1, \ldots, \psi_p \in \mathrm{PSH}(X)$. Assume that*

$$\varphi_j|_O = \psi_j|_O, \quad j = 1, \ldots, p,$$

and that

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \text{ and } \bigwedge_{j=1}^{p} \mathrm{dd^c}\psi_j$$

are both well-defined, then

$$\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j\right|_O = \left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\psi_j\right|_O. \tag{2.4}$$

If furthermore O is open in the usual topology, then the product

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_O$$

on O is well-defined and

$$\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j\right|_O = \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_O. \tag{2.5}$$

Let $\mathcal{U}$ be an open covering of X. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined if and only if each of the following product is well-defined

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_U, \quad U \in \mathcal{U}.$$

(2) *The current $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ and the fact that it is well-defined depend only on the currents $\mathrm{dd^c}\varphi_j$, not on the choice of the φ_j's nor on the ordering of the φ_j's.*

(3) *When $\varphi_1, \ldots, \varphi_p \in L^\infty_{\mathrm{loc}}(X)$, the product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined and is equal to the Bedford–Taylor product.*

(4) *Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ puts no mass on pluripolar sets.*

(5) *Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j$ is a closed positive (p,p)-current on X.*

(6) *The product is multilinear: Let $\psi_1 \in \mathrm{PSH}(X)$, $a, b > 0$ then*

$$\mathrm{dd^c}(a\varphi_1 + b\psi_1) \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j = a\mathrm{dd^c}\varphi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j + b\mathrm{dd^c}\psi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j \tag{2.6}$$

in the sense that left-hand side is well-defined if and only if both terms on right-hand side are well-defined, and the equality holds in that case.

In view of (3), we do not need to specify whether our product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is the Bedford–Taylor product or the non-pluripolar product when the φ_i's are all locally bounded.

Definition 2.2.2 Let $T_1, \ldots, T_p$ be closed positive $(1,1)$-currents on X. We say that $T_1 \wedge \cdots \wedge T_p$ is *well-defined* if there exists an open covering $\mathcal{U}$ of X, such that on each $U \in \mathcal{U}$, we can find $\varphi_j^U \in \mathrm{PSH}(U)$ ($j = 1, \ldots, p$) such that

$$\mathrm{dd^c}\varphi_j^U = T_j, \quad j = 1, \ldots, p$$

and $\mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U$ is well-defined. In this case, we define the *non-pluripolar product* $T_1 \wedge \cdots \wedge T_p$ as the closed positive (p,p)-current on X defined by

$$\left(T_1 \wedge \cdots \wedge T_p\right)|_U = \mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U, \quad U \in \mathcal{U}. \tag{2.7}$$

The product $T_1 \wedge \cdots \wedge T_p$ is independent of the choices we made thanks to Proposition 2.2.1 (1) and (2).

Proposition 2.2.1 can be formulated in terms of currents without any difficulty.

Remark 2.2.1 Similar to Remark 2.1.1, there is also an extension of the non-pluripolar theory allowing us to define

$$T_1 \wedge \cdots \wedge T_p \cap T$$

for any closed positive current T. This is the *relative non-pluripolar product* introduced by Vu [Vu21]. Unlike the relative Bedford–Taylor products, the relative non-pluripolar products present some pathological behaviors. For example, they are not linear in general.

Remark 2.2.2 Another possible generalization of the non-pluripolar products is motivated by Proposition 2.2.1. One could begin by defining of generalized notion of plurisubharmonic functions on $\mathcal{F}$-open sets, called *$\mathcal{F}$-plurisubharmonic functions* and define their non-pluripolar products. See [EKFW11, EKW14].

Proposition 2.2.2 *Let X be a compact Kähler manifold and $T_1, \dots, T_p$ are closed positive $(1,1)$-currents on X. Then $T_1 \wedge \cdots \wedge T_p$ is well-defined.*

This proposition explains why we usually work in the setting of compact Kähler manifolds.

2.3 Properties of non-pluripolar products

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta_1, \dots, \theta_n$ be closed real smooth $(1,1)$-forms on X.

We write

$$\mathrm{PSH}(X,\theta)_{>0} = \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n > 0 \right\}. \tag{2.8}$$

The non-pluripolar product θ_φ^n is well-defined thanks to Proposition 2.2.2.

Remark 2.3.1 Suppose that X is a connected complex manifold of dimension 0, namely, X is a single point. In this case, by definition, the non-pluripolar product θ_φ^n is given by the current of integration at the unique point. So $\mathrm{PSH}(X,\theta)_{>0} = \mathrm{PSH}(X,\theta) \cong \mathbb{R}$ in this case and $\int_X \theta_\varphi^n = 1$ for all $\varphi \in \mathrm{PSH}(X,\theta)$.

Proposition 2.3.1 *Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y and $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ for $i = 1, \dots, n$. Then*

$$\int_Y \pi^*\theta_{1,\pi^*\varphi_1} \wedge \cdots \wedge \pi^*\theta_{n,\pi^*\varphi_n} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Proof This follows immediately from Proposition 2.2.1 (1) and (4). □

We shall write

$$V_\theta = \sup \{ \varphi \in \mathrm{PSH}(X,\theta) : \varphi \le 0 \}. \tag{2.9}$$

It follows from Proposition 1.2.1 that $V_\theta \in \mathrm{PSH}(X,\theta)$ if $\mathrm{PSH}(X,\theta) \neq \varnothing$.

Theorem 2.3.1 (Semicontinuity theorem) *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X,\theta_j)$ ($k \in \mathbb{Z}_{>0}$, $j = 1, \dots, n$). Let $\chi \ge 0$ be a bounded function such that there are $\eta_1, \eta_2 \in \mathrm{QPSH}(X)$ with $\eta_1 + \chi = \eta_2$.*

Assume that for any $j = 1, \dots, n$, as $k \to \infty$, either φ_j^k decreases to $\varphi_j \in \mathrm{PSH}(X, \theta)$ or increases to $\varphi_j \in \mathrm{PSH}(X, \theta)$ almost everywhere. Then for any open set $U \subseteq X$, we have

$$\varliminf_{k\to\infty} \int_U \chi\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_U \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.10}$$

See [DDNL18b, Theorem 2.3].

Theorem 2.3.2 (Monotonicity theorem) *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X, \theta_j)$ for $j = 1, \dots, n$. Assume that $\varphi_j \succeq \psi_j$ [2] for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n}.$$

In particular, if $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ with $\varphi \succeq \psi$, then

$$\int_X \theta_\varphi^n \geq \int_X \theta_\psi^n.$$

See [DDNL18b, Theorem 1.1]. We will prove a vast extension of this theorem in Proposition 6.1.4.

Thanks to this theorem, the non-pluripolar mass $\int_X \theta_\varphi^n$ could be used as a rough measure of the singularities of $\varphi \in \mathrm{PSH}(X, \theta)$. In Section 3.1, we shall refine this measure by defining the notion of P-envelope.

As a corollary, we obtain that

Corollary 2.3.1 *Fix a directed set I. For each $j = 1, \dots, n$, take an increasing net $(\varphi_j^i)_{i\in I}$ in $\mathrm{PSH}(X, \theta_j)$, uniformly bounded from above. Set*

$$\varphi_j := \sup_{i\in I}{}^* \varphi_j^i.$$

Then

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.11}$$

Proof We may assume that I is infinite as there is nothing to prove otherwise. Thanks to Theorem 2.3.2, we already know the $\leq$ inequality in (2.11). We prove the reverse inequality. When $I \cong \mathbb{Z}_{>0}$ as directed sets, the reverse inequality follows from Theorem 2.3.1. In general, by Choquet's lemma Proposition 1.2.2, we can find a countable infinite subset $R \subseteq I$ such that

$$\sup_{r\in R}{}^* \varphi_j^r = \sup_{i\in I}{}^* \varphi_j^i$$

for all $j = 1, \dots, n$. We fix a bijection $R \cong \mathbb{Z}_{>0}$. For any $j = 1, \dots, n$, we will then denote elements φ_j^r $(r \in R)$ by $\varphi_j^1, \varphi_j^2, \dots$. We shall write

[2] See Definition 1.5.2 for the notation.

$$\psi_j^a = \varphi_j^1 \vee \cdots \vee \varphi_j^a$$

for each $a \in \mathbb{Z}_{>0}$.

It follows from the fact that I is a directed set and Theorem 2.3.2 that

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} \geq \lim_{a\to\infty} \int_X \theta_{1,\psi_1^a} \wedge \cdots \wedge \theta_{n,\psi_n^a}.$$

From the special case mentioned above, we know that the right-hand side is exactly the right-hand side of (2.11), so we conclude. □

The following lemma is striking in that we begin only with an upper bound of φ, but at the end of the day, we get a lower bound almost for free. This powerful method will be employed again and again in the whole book.

Lemma 2.3.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, $\varphi \leq \psi$ and $\int_X \theta_\varphi^n > 0$. Then for any*

$$a \in \left(1, \left(\frac{\int_X \theta_\psi^n}{\int_X \theta_\psi^n - \int_X \theta_\varphi^n}\right)^{1/n}\right), \tag{2.12}$$

there is $\eta \in \mathrm{PSH}(X,\theta)_{>0}$ such that

$$a^{-1}\eta + (1-a^{-1})\psi \leq \varphi.$$

The fraction in (2.12) is understood as ∞ if $\int_X \theta_\psi^n = \int_X \theta_\varphi^n$. Thanks to Theorem 2.3.2, the interval (2.12) is non-empty.

We write

$$\begin{aligned} P_\theta(a\varphi + (1-a)\psi) = &\sup{}^* \left\{\eta \in \mathrm{PSH}(X,\theta) : a^{-1}\eta + (1-a^{-1})\psi \leq \varphi\right\} \\ &\in \mathrm{PSH}(X,\theta). \end{aligned} \tag{2.13}$$

Remark 2.3.2 The notation $P_\theta(a\varphi+(1-a)\psi)$ might lead to some potential confusions since $a\varphi + (1-a)\psi$ is not defined everywhere. But the author cannot come up with a better notation.

Observe that

$$a^{-1}P_\theta(a\varphi + (1-a)\psi) + (1-a^{-1})\psi \leq \varphi. \tag{2.14}$$

In fact, this equation holds outside a pluripolar set by Proposition 1.2.5, hence it holds everywhere by Proposition 1.2.6.

Proof Without loss of generality, we may assume that $\varphi \leq \psi \leq 0$.

We refer to [DDNL21b, Lemma 4.3] for the proof of the existence of $\eta \in \mathrm{PSH}(X,\theta)$ satisfying the given inequality. Next we argue that $P_\theta(a\varphi+(1-a)\psi) \in \mathrm{PSH}(X,\theta)_{>0}$. Choose

$$a' \in \left(a, \left(\frac{\int_X \theta_\psi^n}{\int_X \theta_\psi^n - \int_X \theta_\varphi^n}\right)^{1/n}\right).$$

It follows from (2.13) that

$$P_\theta(a\varphi+(1-a)\psi) \geq \frac{a}{a'}P_\theta(a'\varphi+(1-a')\psi)+\frac{a'-a}{a'}\varphi. \tag{2.15}$$

Therefore, by Theorem 2.3.2, we have

$$\int_X \theta^n_{P_\theta(a\varphi+(1-a)\psi)} \geq \frac{(a'-a)^n}{a'^n}\int_X \theta^n_\varphi > 0. \tag{2.16}$$

Corollary 2.3.2 *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$, $\varphi \leq \psi$. Assume that $\int_X \theta^n_\varphi = \int_X \theta^n_\psi$. Then for any $\epsilon \in (0,1)$, there is $\eta \in \mathrm{PSH}(X,\theta)$ such that*

(1) $\int_X \theta^n_\eta = \int_X \theta^n_\varphi$;
(2) $\epsilon\eta + (1-\epsilon^{-1})\psi \leq \varphi$.

Note that by (2), we trivially have $\eta \leq \psi$.

Proof Fix $\epsilon \in (0,1)$, we define

$$\eta = P_\theta\left(\epsilon^{-1}\varphi+(1-\epsilon^{-1})\psi\right).$$

This is well-defined due to Theorem 2.3.2.

Thanks to (2.16), for each $a' > \epsilon^{-1}$, we have

$$\int_X \theta^n_\eta > \left(\frac{a'-\epsilon^{-1}}{a'}\right)^n \int_X \theta^n_\varphi.$$

Letting $a' \to \infty$, we conclude that

$$\int_X \theta^n_\eta \geq \int_X \theta^n_\varphi.$$

On the other hand, since $\eta \leq \psi$, using Theorem 2.3.2 we find that

$$\int_X \theta^n_\eta \leq \int_X \theta^n_\psi = \int_X \theta^n_\varphi.$$

Hence,

$$\int_X \theta^n_\eta = \int_X \theta^n_\varphi.$$

Proposition 2.3.2 *Assume that $\mathrm{PSH}(X,\theta)_{>0}$ is non-empty, then the cohomology class $[\theta]$ is big.*

See [BEGZ10, Proposition 1.22].

Lemma 2.3.2 *For any $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, there is $\psi \in \mathrm{PSH}(X,\theta)$ such that*

(1) *θ_ψ is a Kähler current, and*
(2) *$\psi \leq \varphi$.*

In particular, there is an increasing sequence $(\varphi_i)_i$ in $\mathrm{PSH}(X,\theta)$ converging almost everywhere to φ such that θ_{φ_i} is a Kähler current for all $i \geq 1$.

Proof Using Lemma 2.3.1, we can find $\epsilon > 0$ and $\gamma \in \mathrm{PSH}(X,\theta)$ such that

$$\frac{\epsilon}{1+\epsilon}V_\theta + \frac{1}{1+\epsilon}\gamma \leq \varphi.$$

We observe that the cohomology class $[\theta]$ is big as a consequence of Proposition 2.3.2. Therefore, we can take $\eta \in \mathrm{PSH}(X,\theta)$ such that θ_η is a Kähler current and $\eta \leq 0$. Then we may take

$$\psi = \frac{\epsilon}{1+\epsilon}\eta + \frac{1}{1+\epsilon}\gamma.$$

Then ψ clearly satisfies (1) and (2).

For the latter claim, it suffices to take

$$\varphi_i = (1-(i+1)^{-1})\varphi + (i+1)^{-1}\psi.$$

Lemma 2.3.3 *Let L be a holomorphic line bundle on X with $\theta \in c_1(L)$. Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then there exists $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) \neq 0.$$

Proof By Lemma 2.3.2, we may further assume that θ_φ is a Kähler current. In this case, the result follows from Hörmander's L^2-estimate, see [Dem12a, Theorem 13.21]. □

Theorem 2.3.3 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. Then the map*

$$[0,1] \ni t \mapsto \log \int_X \theta^n_{t\varphi_1+(1-t)\varphi_0}$$

is concave.

See [DDNL21a] for the proof.

Remark 2.3.3 Here and in the sequel, when we write expressions like $t\varphi + (1-t)\psi$ for $\varphi, \psi \in \mathrm{QPSH}(X)$, we will follow the convention that when $t = 0$, the value is ψ and when $t = 1$, the value is φ.

Chapter 3
The envelope operators

Politiques et scientifiques ont le sens des réalités, mais ce ne sont pas les mêmes. Il en résulte — et ce sera là un principe que le général de Gaulle fera sien que l'activité de recherche ne peut être évaluée, quant à sa qualité propre, que par des hommes qui la pratiquent eux-mêmes.
— Pierre Lelong[a], 1999

[a] Pierre Lelong (1912–2011) was the husband of another famous mathematician Jacqueline Ferrand. During their marriage (1947–1977), the latter published under the name of Jacqueline Lelong-Ferrand.

In this chapter, we study two envelope operators lying at the heart of the whole theory. The first envelope, called the P-envelope, is defined using the non-pluripolar masses, while the second, called the $\mathcal{I}$-envelope, is defined using the multiplier ideal sheaves. The corresponding theories are developed in Section 3.1 and Section 3.2 respectively.

Later on in Chapter 6, we will develop the corresponding P and $\mathcal{I}$-partial orders associated with these envelopes, allowing us to compare the singularities.

3.1 The P-envelope

In this section, X will denote a connected compact Kähler manifold of dimension n.

3.1.1 Rooftop operator and the definition of the P-envelope

We will fix a smooth closed real $(1,1)$-form θ on X.

Definition 3.1.1 Given $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we define their *rooftop operator* as follows:

$$\varphi \wedge \psi = \sup \{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq \varphi, \eta \leq \psi\} . \tag{3.1}$$

For the simplicity of notations, we extend the definition to the case where φ or ψ is constantly $-\infty$, in this case, we simply set

$$\varphi \wedge \psi = -\infty.$$

When we want to be more specific, we could also write $\varphi \wedge_\theta \psi$.

Proposition 3.1.1 *The operator $\wedge$ is a well-defined commutative, associative binary operator*

$$\mathrm{PSH}(X,\theta)\cup\{-\infty\}\times\mathrm{PSH}(X,\theta)\cup\{-\infty\}\to\mathrm{PSH}(X,\theta)\cup\{-\infty\}.$$

Proof We first show that the map is well-defined. For this purpose, take $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. When the set in (3.1) is empty, there is nothing to prove. So let us assume that the set is not empty.

Define

$$\gamma=\sup{}^*\{\eta\in\mathrm{PSH}(X,\theta):\eta\leq\varphi,\eta\leq\psi\}.$$

Then by Proposition 1.2.1, we find that $\gamma\in\mathrm{PSH}(X,\theta)$ and hence γ is a candidate for the supremum in (3.1). Therefore, $\gamma\leq\varphi\wedge\psi$. The reverse inequality is trivial, so

$$\varphi\wedge\psi=\gamma\in\mathrm{PSH}(X,\theta).$$

The commutativity and the associativity of $\wedge$ are both trivial. □

Lemma 3.1.1 *Let $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Assume that $\varphi\wedge\psi\in\mathrm{PSH}(X,\theta)$. Then*

$$\theta^n_{\varphi\wedge\psi}\leq\mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\theta^n_\varphi+\mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\theta^n_\psi.$$

See [DDNL18b, Lemma 3.7] for the proof.

We recall that the relations $\preceq$ and $\sim$ are introduced in Definition 1.5.2.

Definition 3.1.2 Given $\varphi\in\mathrm{PSH}(X,\theta)$, we define its *$P$-envelope* as follows:

$$P_\theta[\varphi]:=\sup{}^*\{\psi\in\mathrm{PSH}(X,\theta):\psi\leq 0,\psi\preceq\varphi\}. \tag{3.2}$$

Observe that by Proposition 1.2.1, we have $P_\theta[\varphi]\in\mathrm{PSH}(X,\theta)$ and $P_\theta[\varphi]\leq 0$. Moreover, the definition can be equivalently described as

$$P_\theta[\varphi]=\sup_{C\in\mathbb{Z}_{>0}}{}^*(\varphi+C)\wedge V_\theta. \tag{3.3}$$

Recall that V_θ is introduced in (2.9). Observe that for any $C\in\mathbb{R}$, we have $(\varphi+C)\wedge V_\theta\in\mathrm{PSH}(X,\theta)$ and

$$(\varphi+C)\wedge V_\theta\sim\varphi.$$

In other words, in (3.2), we may replace the condition $\psi\preceq\varphi$ by $\psi\sim\varphi$.

Morally, the idea lying behind the definition of $P_\theta[\varphi]$ is that we choose the least singular element out of all potentials with the same singularity type as φ. As we shall see in Example 3.1.1 below, $P_\theta[\varphi]$ does not necessarily have the same singularity type as φ. This forces us to define a rougher equivalence relation in Definition 6.1.1.

Proposition 3.1.2 *Let $\theta'=\theta+\mathrm{dd^c}g$ for some $g\in C^\infty(X)$. Then for any $\varphi\in\mathrm{PSH}(X,\theta)$, we have $\varphi-g\in\mathrm{PSH}(X,\theta')$ and*

$$P_\theta[\varphi] \sim P_{\theta'}[\varphi'].$$

Proof By symmetry, it suffices to show that

$$P_\theta[\varphi] \preceq P_{\theta'}[\varphi'].$$

We may assume that $g \geq 0$. Then for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \preceq \varphi$ and $\psi \leq 0$, we set $\psi' := \psi - g \in \mathrm{PSH}(X,\theta')$. Then $\psi' \preceq \varphi'$ and $\psi' \leq 0$, so $\psi' \leq P_{\theta'}[\varphi']$. Since ψ is arbitrary, it follows that

$$P_\theta[\varphi] - \sup_X g \leq P_\theta[\varphi] - g \leq P_{\theta'}[\varphi'].$$

The P-envelope preserves the non-pluripolar masses:

Proposition 3.1.3 *Suppose that $\theta_1, \ldots, \theta_n$ be smooth closed real $(1,1)$-forms on X. Let $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ for each $i = 1, \ldots, n$. Then*

$$\int_X \theta_{1,P_{\theta_1}[\varphi_1]} \wedge \cdots \wedge \theta_{n,P_{\theta_n}[\varphi_n]} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.4}$$

Proof For each $C \in \mathbb{Z}_{>0}$ and each $i = 1, \ldots, n$, we have

$$(\varphi_i + C) \wedge V_{\theta_i} \sim \varphi_i.$$

It follows from Theorem 2.3.2 that

$$\int_X \theta_{1,(\varphi_1+C)\wedge V_{\theta_1}} \wedge \cdots \wedge \theta_{n,(\varphi_n+C)\wedge V_{\theta_n}} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

So (3.4) follows from (3.3) and Corollary 2.3.1. □

Conversely, Proposition 3.1.3 characterizes the P-envelope:

Theorem 3.1.1 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then*

$$P_\theta[\varphi] = \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \varphi \preceq \psi, \int_X \theta_\varphi^n = \int_X \theta_\psi^n\right\}. \tag{3.5}$$

In particular, in this case,

$$P_\theta\left[P_\theta[\varphi]\right] = P_\theta[\varphi]. \tag{3.6}$$

We refer to [DDNL23, Theorem 3.14] for the proof.

Note that in (3.5) and (3.2), the test function ψ lies on different sides of φ.

In general, we do not know if (3.6) holds when $\int_X \theta_\varphi^n > 0$. We expect it to be wrong. According to our general philosophy, the P-envelope operator is the correct object only when the non-pluripolar mass is positive. We will avoid using the degenerate case in the whole book.

Definition 3.1.3 If $\varphi = P_\theta[\varphi]$ and $\int_X \theta_\varphi^n > 0$, we say φ is a *model potential*.

We remind the readers that the notion of model potentials depends heavily on the choice of θ. When there is a risk of confusion, we also say φ is a model potential in $\mathrm{PSH}(X,\theta)$.

Remark 3.1.1 Definition 3.1.3 is different from the common definition in the literature: We impose the extra condition $\int_X \theta_\varphi^n > 0$. The author believes that this is the only case where this notion is natural. We sometimes emphasize this point by saying $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ is a model potential.

There are plenty of model potentials:

Corollary 3.1.1 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then $P_\theta[\varphi]$ is a model potential in* $\mathrm{PSH}(X,\theta)$. *Moreover,*

$$\int_X \theta^n_{P_\theta[\varphi]} = \int_X \theta^n_\varphi.$$

Proof This follows immediately from Theorem 3.1.1 and Proposition 3.1.3. □

Example 3.1.1 We continue our favorite example Example 1.8.1. Let $X = \mathbb{P}^1$ and ω be the Fubini–Study metric. We define $\varphi \in \mathrm{PSH}(X,\omega)$ as follows: for $z \in \mathbb{C}$, we let

$$\varphi(z) = \begin{cases} -\log(|z|^2+1) + \left(-\log\left(-\log|z|^2\right)\right) \vee \left(2+\log|z|^2\right), & \text{if } |z| < 1/\sqrt{2}, \\ 2+\log\dfrac{|z|^2}{|z|^2+1}, & \text{Otherwise}, \end{cases}$$

while $\varphi(\infty) = 2$. The singularity of φ only occurs at $z = 0$, close to which, $\varphi \sim -\log\left(-\log|z|^2\right)$. This type of singularity is therefore called the *log-log type singularity*.

We claim that

$$P_\omega[\varphi] = 0. \tag{3.7}$$

In particular, we find that φ and $P_\omega[\varphi]$ have different singularity types.

Due to Theorem 3.1.1, in order to verify (3.7), it suffices to verify that

$$\int_X \omega_\varphi = 1. \tag{3.8}$$

Here ω_φ is taken in the non-pluripolar sense. Since $\{0,\infty\} \subseteq \mathbb{P}^1$ is pluripolar, this reduces to show that

$$\int_{\mathbb{C}^*} \mathrm{dd^c}\psi = \frac{1}{4\pi}\int_{\mathbb{C}^*} (\Delta\psi)\,\mathrm{d}\mu = 1,$$

where $\psi(z) = \varphi(z) + \log(|z|^2+1)$ and μ is the standard Lebesgue measure on $\mathbb{C}$.

Note that the Laplacian vanishes outside $\overline{B(0,0.7)}$ since $\psi(z) = 2+\log|z|^2$ there, which is harmonic. Therefore,

$$\int_{\mathbb{C}^*} \mathrm{dd^c}\psi = \frac{1}{4\pi}\int_{|z|<1/\sqrt{2}} (\Delta\psi)(z)\,\mathrm{d}\mu.$$

It is an elementary exercise to see that the right-hand side is exactly equal to 1. If you are familiar with toric geometry, this is more or less trivial since

$$\nabla_r \left((-\log(-r)) \vee (2+r)\right)(-\infty, -\log 2) = [-1, 0).$$

Otherwise, just try to evaluate the integral using Green's identities. Therefore, (3.8) is proved and our assertion (3.7) follows.

Next we give a criterion on when the rooftop operator is not identically $-\infty$.

Proposition 3.1.4 *Assume that* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ *and*

$$\int_X \theta_\varphi^n + \int_X \theta_\psi^n > \int_X \theta_{\varphi \vee \psi}^n .$$

Then $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$.

Proof Without loss of generality, we may assume that $\varphi, \psi \leq 0$. Take

$$\eta \coloneqq P_\theta[(1-\epsilon)(\varphi \vee \psi) + \epsilon V_\theta]$$

for some small enough $\epsilon > 0$, we may guarantee that

$$\int_X \theta_\varphi^n + \int_X \theta_\psi^n > \int_X \theta_\eta^n, \quad \varphi \vee \psi \leq \eta.$$

This is a consequence of Corollary 3.1.1.

Take $C > 0$ large enough, so that

$$\int_{\{\varphi > \eta - C\}} \theta_\varphi^n + \int_{\{\psi > \eta - C\}} \theta_\psi^n > \int_X \theta_\eta^n . \tag{3.9}$$

This is possible thanks to Proposition 2.2.1(4). Fix $C' > C$. Write

$$\gamma_{C'} \coloneqq (\varphi \vee (\eta - C')) \wedge (\psi \vee (\eta - C')) .$$

Then observe that

$$\inf_{C' > C} \gamma_{C'} = \varphi \wedge \psi .$$

Assume by contradiction that $\varphi \wedge \psi \equiv -\infty$, then we have

$$\lim_{C' \to \infty} \sup_X \gamma_{C'} = -\infty .$$

Observe that for each $C' > C$,

$$\sup_X \gamma_{C'} = \sup_{\{\eta \neq -\infty\}} (\gamma_{C'} - \eta)$$

since η is a model potential.[1] It follows that

$$\lim_{C'\to\infty}\sup_{\{\eta\neq-\infty\}}(\gamma_{C'}-\eta)=-\infty. \tag{3.10}$$

For each $C'>C$, we compute

$$\begin{aligned}\int_{\{\gamma_{C'}\leq\eta-C\}}\theta^n_{\gamma_{C'}}\leq&\int_{\{\varphi\vee(\eta-C')\leq\eta-C\}}\theta^n_{\varphi\vee(\eta-C')}+\int_{\{\psi\vee(\eta-C')\leq\eta-C\}}\theta^n_{\psi\vee(\eta-C')}\\=&2\int_X\theta^n_\eta-\int_{\{\varphi>\eta-C\}}\theta^n_\varphi-\int_{\{\psi>\eta-C\}}\theta^n_\psi\\<&\int_X\theta^n_\eta,\end{aligned}$$

where the first line follows from Lemma 3.1.1, the third line follows from (3.9). Using (3.10), we can take C' large enough so that $\gamma_{C'}\leq\eta-C$. Then we find

$$\int_X\theta^n_{\gamma_{C'}}<\int_X\theta^n_\eta,$$

which contradicts Theorem 2.3.2. □

3.1.2 Properties of the P-envelope

Let θ,θ_1,θ_2 be smooth closed real $(1,1)$-forms on X.

Proposition 3.1.5 *Let $\pi\colon Y\to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for any $\varphi\in\mathrm{PSH}(X,\theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi]=\pi^*P_\theta[\varphi].$$

In particular, a potential $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$ is model if and only if $\pi^*\varphi\in\mathrm{PSH}(Y,\pi^*\theta)_{>0}$ is model.

Proof This follows immediately from Proposition 1.5.3. □

We have the following concavity property of the P-envelope.

Proposition 3.1.6

(1) *Suppose that $\varphi\in\mathrm{PSH}(X,\theta)$ and $\lambda\in\mathbb{R}_{>0}$, then*

$$P_{\lambda\theta}[\lambda\varphi]=\lambda P_\theta[\varphi].$$

---

[1] In fact, the $\leq$ direction is trivial, in view of Corollary 1.3.6. As for the reverse inequality, we may assume that the left-hand side is 0, but as η is model and $\gamma_{C'}\leq\eta$, we have $\gamma_{C'}\leq\eta$.

(2) *Suppose that* $\varphi_1 \in \mathrm{PSH}(X, \theta_1)$ *and* $\varphi_2 \in \mathrm{PSH}(X, \theta_2)$, *then*

$$P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2] \geq P_{\theta_1}[\varphi_1] + P_{\theta_2}[\varphi_2].$$

Proof (1) This is obvious by definition.

(2) Suppose that $\psi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\psi_2 \in \mathrm{PSH}(X, \theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \leq \varphi_i$$

for $i = 1, 2$. Then

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \leq \varphi_1 + \varphi_2.$$

It follows from (3.2) that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2].$$

Since ψ_1 and ψ_2 are arbitrary, we conclude. □

Proposition 3.1.7 *Let* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. *Assume that*

$$\varphi = P_\theta[\varphi], \quad \psi = P_\theta[\psi], \quad \varphi \wedge \psi \not\equiv -\infty.$$

Then

$$P_\theta[\varphi \wedge \psi] = \varphi \wedge \psi. \tag{3.11}$$

Proof Observe that we obviously have

$$P_\theta[\varphi \wedge \psi] \leq P_\theta[\varphi] = \varphi, \quad P_\theta[\varphi \wedge \psi] \leq P_\theta[\psi] = \psi.$$

So the $\leq$ direction in (3.11) holds. The reverse direction is trivial. □

Theorem 3.1.2 *Let* $\varphi \in \mathrm{PSH}(X, \theta)$. *Then*

$$\theta^n_{P_\theta[\varphi]} \leq \mathbb{1}_{\{P_\theta[\varphi]=0\}} \theta^n.$$

See [DDNL18b, Theorem 3.8] for the proof.

Theorem 3.1.3 *Assume that* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ *and* $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$. *Then*

$$\int_X \theta^n_\varphi + \int_X \theta^n_\psi \leq \int_X \theta^n_{\varphi \vee \psi} + \int_X \theta^n_{\varphi \wedge \psi}. \tag{3.12}$$

We refer to [DDNL21b, Theorem 5.4] for the proof.

Proposition 3.1.8 *Let* $(\varphi_j)_{j \in I}$ *be a decreasing net of potentials in* $\mathrm{PSH}(X, \theta)$ *satisfying* $P_\theta[\varphi_j] = \varphi_j$ *for each* $j \in I$ *and* $\varphi := \inf_j \varphi_j \not\equiv -\infty$. *Then* $P_\theta[\varphi] = \varphi$.

Proof It follows from Proposition 1.2.1 that $\varphi \in \mathrm{PSH}(X, \theta)$. Therefore, for each $j \in I$,

$$\varphi \leq P_\theta[\varphi] \leq P_\theta[\varphi_j] = \varphi_j.$$

Therefore, $\varphi = P_\theta[\varphi]$. □

Proposition 3.1.9 *Let $(\epsilon_j)_{j\in I}$ be a decreasing net in $\mathbb{R}_{\geq 0}$ with limit 0. Take a Kähler form ω on X. Consider a decreasing net $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$ ($j \in I$) satisfying*

$$P_{\theta+\epsilon_j\omega}[\varphi_j] = \varphi_j \tag{3.13}$$

with pointwise limit $\varphi \not\equiv -\infty$. Then

$$\lim_{j\in I} \int_X (\theta + \epsilon_j\omega)^n_{\varphi_j} = \int_X \theta^n_\varphi. \tag{3.14}$$

Moreover, if $\int_X \theta^n_\varphi > 0$, then for any prime divisor E over X, we have

$$\lim_{j\in I} \nu(\varphi_j, E) = \nu(\varphi, E). \tag{3.15}$$

Proof Observe that $\varphi \in \mathrm{PSH}(X, \theta)$. By Theorem 2.3.2, we have

$$\varliminf_{j\in I} \int_X (\theta + \epsilon_j\omega)^n_{\varphi_j} \geq \varliminf_{j\in I} \int_X (\theta + \epsilon_j\omega)^n_{\varphi} = \int_X \theta^n_\varphi.$$

We now argue the reverse inequality.

Fix $j_0 \in I$, we have

$$\begin{aligned}
\varlimsup_{j\in I} \int_X (\theta + \epsilon_j\omega)^n_{\varphi_j} &= \varlimsup_{j\in I} \int_{\{\varphi_j=0\}} (\theta + \epsilon_j\omega)^n_{\varphi_j} \\
&\leq \varlimsup_{j\in I} \int_{\{\varphi_j=0\}} (\theta + \epsilon_{j_0}\omega)^n_{\varphi_j} \\
&\leq \int_{\{\varphi=0\}} (\theta + \epsilon_{j_0}\omega)^n_{\varphi},
\end{aligned}$$

where in the first line we used (3.13) and Theorem 3.1.2, and in the last line we have used the fact that $\varphi_j \searrow \varphi$ and [DDNL21b, Proposition 4.6] (see also [DDNL23, Lemma 2.11]). Taking limit with respect to j_0, we arrive at the desired conclusion:

$$\varlimsup_{j\in I} \int_X (\theta + \epsilon_j\omega)^n_{\varphi_j} \leq \varliminf_{j_0\in I} \int_{\{\varphi=0\}} (\theta + \epsilon_{j_0}\omega)^n_{\varphi} = \int_{\{\varphi=0\}} \theta^n_\varphi \leq \int_X \theta^n_\varphi.$$

This finishes the proof of (3.14).

It remains to argue (3.15). By Lemma 2.3.1 and (3.14), for any $\epsilon \in (0,1)$ and j big enough there exists $\psi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$ such that $(1-\epsilon)\varphi_j + \epsilon\psi_j \leq \varphi$. This implies that for j big enough we have

$$(1-\epsilon)\nu(\varphi_j, E) + \epsilon\nu(\psi_j, E) \geq \nu(\varphi, E) \geq \nu(\varphi_j, E).$$

On the other hand, the Lelong numbers $\nu(\psi_j, E)$ admit an upper bound for various j by Proposition 1.5.2. So taking limit with respect to j, we conclude (3.15). □

Corollary 3.1.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. Let ω be a Kähler form on X. Then*

$$\varphi = \inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi].$$

Proof Clearly, we have the $\leq$ direction and the right-hand side is non-positive. So by Theorem 3.1.1, it suffices to show that they have the same mass, which follows from Proposition 3.1.9. □

Proposition 3.1.10 *Let $(\varphi_i)_{i\in I}$ be an increasing net of potentials in $\mathrm{PSH}(X,\theta)_{>0}$ uniformly bounded from above. Let $\varphi := {\sup_{i\in I}}^* \varphi_i$. Then*

$$\sup_{i\in I}{}^* P_\theta[\varphi_i] = P_\theta[\varphi].$$

In particular, if φ_i is model for all $i \in I$, then so is φ.

Proof We may assume that I is infinite since otherwise, there is nothing to prove.

We write

$$\eta := \sup_{i\in I}{}^* P_\theta[\varphi_i].$$

Then it is clear that $\eta \leq P_\theta[\varphi]$.

By Corollary 2.3.1, we have

$$\lim_{i\in I} \int_X \theta_{\varphi_i}^n = \int_X \theta_\varphi^n > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0,1)$ and $\psi_i \in \mathrm{PSH}(X,\theta)$ ($i \in I$) such that for all $i \in I$,

$$(1-\epsilon_i)\varphi + \epsilon_i\psi_i \leq \varphi_i.$$

By Proposition 3.1.6, we have

$$P_\theta[\varphi] + \epsilon_i P_\theta[\psi_i] \leq (1-\epsilon_i)P_\theta[\varphi] + \epsilon_i P_\theta[\psi_i] \leq \eta.$$

Taking limit with respect to i, we conclude that $P_\theta[\varphi] \leq \eta$. □

3.1.3 Relative full mass classes

Let θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Definition 3.1.4 We define

$$
\begin{aligned}
\mathrm{PSH}(X,\theta;\phi) &\coloneqq \{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq \phi\}, \\
\mathcal{E}^\infty(X,\theta;\phi) &\coloneqq \{\eta \in \mathrm{PSH}(X,\theta) : \eta \sim \phi\}, \\
\mathcal{E}(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathrm{PSH}(X,\theta;\phi) : \int_X \theta_\varphi^n = \int_X \theta_\phi^n\right\}, \\
\mathcal{E}^1(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathcal{E}(X,\theta;\phi) : \int_X |\phi-\eta|\,\theta_\eta^n < \infty\right\}.
\end{aligned}
$$

Potentials in the last three classes are said to have *relatively minimal singularities*, *full mass* and *finite energy* relative to ϕ respectively.

We have the following inclusions:

$$
\mathcal{E}^\infty(X,\theta;\phi) \subseteq \mathcal{E}^1(X,\theta;\phi) \subseteq \mathcal{E}(X,\theta;\phi) \subseteq \mathrm{PSH}(X,\theta;\phi). \tag{3.16}
$$

The only non-trivial part is the first inclusion, which follows from Theorem 2.3.2.

Remark 3.1.2 Note that this integral

$$
\int_X |\phi-\eta|\,\theta_\eta^n
$$

is defined: The locus where $\phi-\eta$ is undefined is a pluripolar set, while the product θ_η^n puts no mass on pluripolar sets (Proposition 2.2.1).

Similar remarks apply when we talk about similar integrals in the sequel.

When $\phi = V_\theta$, we usually write

$$
\begin{aligned}
\mathcal{E}^\infty(X,\theta;V_\theta) =& \mathcal{E}^\infty(X,\theta), \\
\mathcal{E}(X,\theta;V_\theta) =& \mathcal{E}(X,\theta), \\
\mathcal{E}^1(X,\theta;V_\theta) =& \mathcal{E}^1(X,\theta).
\end{aligned}
$$

Potentials in the three classes are said to have *minimal singularities*, *full mass* and *finite energy* respectively. The relation (3.16) can be written as

$$
\mathcal{E}^\infty(X,\theta) \subseteq \mathcal{E}^1(X,\theta) \subseteq \mathcal{E}(X,\theta)
$$

in this case.

The P-envelope can be used to characterize the full mass classes:

Proposition 3.1.11 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}(X,\theta;\phi)$;
(2) $P_\theta[\varphi] = \phi$.

Proof (2) $\implies$ (1). This follows from Proposition 3.1.3.

(1) $\implies$ (2). Note that ϕ is a candidate of $P_\theta[\varphi]$ as in (3.5). So $P_\theta[\varphi] = \phi$. □

In order to handle the finite energy classes, it is convenient to introduce the following quantity:

Definition 3.1.5 We define the *Monge–Ampère energy* $E_\theta^\phi \colon \mathcal{E}^\infty(X,\theta;\phi) \to \mathbb{R}$ as follows

$$E_\theta^\phi(\varphi) := \frac{1}{n+1}\sum_{j=0}^n \int_X (\varphi-\phi)\,\theta_\varphi^j \wedge \theta_\phi^{n-j}. \tag{3.17}$$

More generally, we extend E_θ^ϕ to a functional $E_\theta^\phi \colon \mathrm{PSH}(X,\theta;\phi) \to [-\infty,\infty)$ as follows

$$E_\theta^\phi(\varphi) := \inf\left\{E_\theta^\phi(\psi) : \psi \in \mathcal{E}^\infty(X,\theta;\phi), \varphi \leq \psi\right\}. \tag{3.18}$$

We write E_θ instead of E_θ^ϕ when $\phi = V_\theta$.

Note that

$$E_\theta^\phi(\varphi + C) = E_\theta^\phi(\varphi) + C\int_X \theta_\varphi^n \tag{3.19}$$

for any $\varphi \in \mathrm{PSH}(X,\theta;\phi)$ and $C \in \mathbb{R}$.

Proposition 3.1.12 *Let $\varphi \in \mathrm{PSH}(X,\theta;\phi)$. The following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X,\theta;\phi)$;
(2) $E_\theta^\phi(\varphi) > -\infty$.

When the conditions are satisfied, (3.17) *holds.*

Given $\varphi, \psi \in \mathcal{E}^1(X,\theta;\phi)$, we have the following cocycle equality

$$E_\theta^\phi(\psi) - E_\theta^\phi(\varphi) = \frac{1}{n+1}\sum_{j=0}^n \int_X (\psi-\varphi)\,\theta_\psi^j \wedge \theta_\varphi^{n-j}. \tag{3.20}$$

See [BEGZ10, Proposition 2.11] and [DDNL18a, Proposition 2.5] for the proofs.[2]

Proposition 3.1.13 *Assume that $\varphi, \psi \in \mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is $\varphi \wedge \psi$.*

Proof The case of $\mathcal{E}^\infty(X,\theta;\phi)$ is trivial.

We consider the case $\mathcal{E}(X,\theta;\phi)$. It follows from Proposition 3.1.4 that $\varphi \wedge \psi \in \mathrm{PSH}(X,\theta)$. By Theorem 3.1.3, we have

$$\int_X \theta_{\varphi\wedge\psi}^n \geq \int_X \theta_\phi^n.$$

By Theorem 2.3.2, equality holds. By Theorem 3.1.1, we conclude that

$$P_\theta[\varphi \wedge \psi] = \phi.$$

Finally, the case $\mathcal{E}^1(X,\theta;\phi)$ is proved in [Xia23a, Theorem 4.13] (the arXiv version). □

Proposition 3.1.14 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ be potentials such that $\psi \leq \phi$ and $\varphi \leq \psi$. Assume that $\varphi \in \mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is ψ.*

[2] In these references, they took $\phi = V_\theta$, but the proof of the general case is almost identical.

Proof The case $\mathcal{E}^\infty(X,\theta;\phi)$ is trivial. The case $\mathcal{E}(X,\theta;\phi)$ follows from Theorem 2.3.2. The case $\mathcal{E}^1(X,\theta;\phi)$ follows from [Xia23a, Proposition 4.5] (arXiv version). □

Proposition 3.1.15 *Let $(\varphi_i)_{i\in I}$ be a uniformly bounded from above non-empty family in $\mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is $\sup^*_{i\in I}\varphi_i$.*

Proof Thanks to Proposition 3.1.14, it suffices to show that

$$\sup_{i\in I}{}^*\varphi_i \leq \phi.$$

Since ϕ is model and $\varphi_i \leq \phi$, we know that

$$\varphi_i - \sup_X \varphi_i \leq \phi$$

for any $i \in I$. By assumption $(\varphi_i)_{i\in I}$ is uniformly bounded from above, our assertion follows. □

Proposition 3.1.16 *Let $\varphi,\psi \in \mathcal{E}(X,\theta;\phi)$. Then*

$$\sup_{C\geq 0}{}^*(\varphi + C)\land\psi = \psi.$$

Proof Since for each $C \geq 0$,

$$(\varphi\land\psi + C)\land\psi \leq (\varphi + C)\land\psi \leq \psi,$$

we may replace φ by $\varphi\land\psi$ (c.f. Proposition 3.1.13) and assume that $\varphi \leq \psi$. In this case, the result is proved in [DDNL18b, Theorem 3.8, Corollary 3.11]. □

Proposition 3.1.17 *Let $\varphi,\psi \in \mathcal{E}^1(X,\theta;\phi)$. Assume that $\varphi \leq \psi$. Then*

$$\int_X (\psi-\varphi)\,\theta_\psi^n \leq E_\theta^\phi(\psi) - E_\theta^\phi(\varphi) \leq \int_X (\psi-\varphi)\,\theta_\varphi^n. \tag{3.21}$$

Proof Thanks to (3.19), we may assume that $\varphi \leq \psi$. Then this result is proved in [Xia23a, Proposition 4.18]. □

3.2 The $\mathcal{I}$-envelope

From the algebraic point of view, a more natural envelope operator is given by the $\mathcal{I}$-envelope.

In this section, X will denote a connected compact Kähler manifold of dimension n.

3.2.1 $\mathcal{I}$-equivalence

Proposition 3.2.1 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *For any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi);$$
(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) = \mathcal{I}(\lambda\psi);$$
(3) *for any modification $\pi\colon Y \to X$ and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y);$$
(4) *for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y);$$
(5) *for any prime divisor E over X, we have*
$$\nu(\varphi, E) = \nu(\psi, E).$$

See Definition B.1.1 for the definition of prime divisors over X. We remind the readers that in the whole book, a *modification* of a compact complex space means a finite composition of blow-ups with smooth centers. This terminology is highly non-standard.

Proof (4) $\iff$ (5). This follows from Lemma 1.4.1.
(3) $\iff$ (5). This follows from Corollary B.1.1.
(1) $\implies$ (5). This follows from Proposition 1.4.4.
(5) $\implies$ (2). This follows from Theorem 1.4.3.
(2) $\implies$ (1). This is trivial. □

Definition 3.2.1 Given $\varphi, \psi \in \mathrm{QPSH}(X)$, we say they are *$\mathcal{I}$-equivalent* and write $\varphi \sim_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 3.2.1 are satisfied.

Clearly, $\sim_{\mathcal{I}}$ is an equivalence relation on $\mathrm{QPSH}(X)$.

Proposition 3.2.2 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for $\varphi, \psi \in \mathrm{QPSH}(X)$, we the following are equivalent:*

(1) *$\varphi \sim_{\mathcal{I}} \psi$;*
(2) *$\pi^*\varphi \sim_{\mathcal{I}} \pi^*\psi$.*

Proof (1) $\implies$ (2). This follows from Proposition 3.2.1(4).
(2) $\implies$ (1). This follows from the simple fact that
$$\mathcal{I}(k\varphi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\varphi)\right), \quad \mathcal{I}(k\psi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\psi)\right)$$
for any $k \in \mathbb{Z}_{>0}$. □

Proposition 3.2.3 *Let $\varphi, \varphi', \psi, \psi' \in \mathrm{QPSH}(X)$ and $\lambda > 0$. Assume that $\varphi \sim_{\mathcal{I}} \psi$ and $\varphi' \sim_{\mathcal{I}} \psi'$, then*

$$\varphi \vee \varphi' \sim_{\mathcal{I}} \psi \vee \psi', \quad \varphi + \varphi' \sim_{\mathcal{I}} \psi + \psi', \quad \lambda\varphi \sim_{\mathcal{I}} \lambda\psi.$$

Similarly, if $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ are two non-empty uniformly bounded from above families in $\mathrm{PSH}(X,\theta)$ for some closed smooth real $(1,1)$-form θ on X such that $\varphi_i \sim_{\mathcal{I}} \psi_i$ for all $i \in I$, then

$$\sup_{i\in I}{}^{*}\varphi_i \sim_{\mathcal{I}} \sup_{i\in I}{}^{*}\psi_i.$$

Proof This follows from Proposition 1.4.2 and Corollary 1.4.1. □

3.2.2 The definition of the $\mathcal{I}$-envelope

We will fix a smooth closed real $(1,1)$-form θ on X.

Definition 3.2.2 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define its *$\mathcal{I}$-envelope* as follows:

$$P_\theta[\varphi]_{\mathcal{I}} := \sup{}^{*}\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim_{\mathcal{I}} \varphi\}. \tag{3.22}$$

If $\varphi = P_\theta[\varphi]_{\mathcal{I}}$, we say φ is an *$\mathcal{I}$-model potential* (in $\mathrm{PSH}(X,\theta)$).

Note that by Proposition 1.2.1, $P_\theta[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$.

Proposition 3.2.4 *Let $\theta' = \theta + \mathrm{dd^c}g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$P_\theta[\varphi]_{\mathcal{I}} \sim P_{\theta'}[\varphi']_{\mathcal{I}}.$$

The proof is similar to that of Proposition 3.1.2, so we omit it.

Proposition 3.2.5 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for $\varphi \in \mathrm{PSH}(X,\theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}} = \pi^* P_\theta[\varphi]_{\mathcal{I}}.$$

Proof The proof is similar to that of Proposition 3.1.5 in view of Proposition 3.2.2. □

Proposition 3.2.6 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\varphi \sim_{\mathcal{I}} P_\theta[\varphi]_{\mathcal{I}}.$$

In particular,

$$P_\theta\left[P_\theta[\varphi]_{\mathcal{I}}\right]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}$$

and the upper semicontinuous regularization in (3.22) is not necessary.

Proof In view of Proposition 3.2.1, it suffices to show that for $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}}). \tag{3.23}$$

By Proposition 1.2.2, we can find $\psi_i \in \mathrm{PSH}(X, \theta)$ $(i \in \mathbb{Z}_{>0})$ such that $\psi_i \leq 0$, $\psi_i \sim_{\mathcal{I}} \varphi$ for all $i \geq 1$ and

$$\sup_{i>0}{}^{*}\psi_i = P_\theta[\varphi]_{\mathcal{I}}.$$

By Proposition 3.2.3, we may replace ψ_i by $\psi_1 \vee \cdots \vee \psi_i$ and assume that the sequence ψ_i is increasing. In this case, it follows from the strong openness theorem Theorem 1.4.4 that for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi_j) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}})$$

for j large enough. □

Definition 3.2.3 Let $\varphi \in \mathrm{PSH}(X, \theta)$, we define the *volume*[3] $\mathrm{vol}(\theta, \varphi)$ as

$$\mathrm{vol}(\theta, \varphi) = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n .$$

Proposition 3.2.7 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X, \theta)$, we have $\varphi' = \varphi - g \in \mathrm{PSH}(X, \theta')$ and*

$$\mathrm{vol}(\theta, \varphi) = \mathrm{vol}(\theta', \varphi').$$

Proof This follows immediately from Proposition 3.2.4 and Theorem 2.3.2. □

In view of Proposition 3.2.7, the volume $\mathrm{vol}(\theta, \varphi)$ depends only on the current θ_φ, and we could write

$$\mathrm{vol}\,\theta_\varphi = \mathrm{vol}(\theta, \varphi). \tag{3.24}$$

The $\mathcal{I}$-envelope and the P-envelope are related in a simple manner.

Proposition 3.2.8 *Let $\varphi \in \mathrm{PSH}(X, \theta)$, then*

$$P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}, \quad \varphi \sim_{\mathcal{I}} P_\theta[\varphi].$$

Proof It suffices to show that $\varphi \sim_{\mathcal{I}} P_\theta[\varphi]$. Namely, for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}\left(kP_\theta[\varphi]\right). \tag{3.25}$$

Fix k for now. It follows from (3.3) and the strong openness theorem Theorem 1.4.4 that

[3] We choose to call this quantity the *volume* instead of the $\mathcal{I}$-*volume* so that the terminology is consistent with the line bundle case.

$$\mathcal{I}(kP_\theta[\varphi]) = \mathcal{I}((k\varphi + C) \wedge kV_\theta)$$

when C is large enough. Since $(k\varphi + C) \wedge kV_\theta \sim k\varphi$, we have

$$\mathcal{I}((k\varphi + C) \wedge kV_\theta) = \mathcal{I}(k\varphi)$$

and (3.25) follows. □

Corollary 3.2.1 *Let $\varphi \in \mathrm{PSH}(X, \theta)$, then*

$$\int_X \theta_\varphi^n \leq \mathrm{vol}\, \theta_\varphi.$$

Proof This follows from Proposition 3.2.8, Theorem 2.3.2 and Proposition 3.1.3. □

The reverse inequality fails in general, see Example 6.1.3.

We note the following special case:

Proposition 3.2.9 *Let $\varphi \in \mathrm{PSH}(X, \theta)$. Assume that φ has analytic singularities, then*

$$\varphi \sim P_\theta[\varphi] \sim P_\theta[\varphi]_{\mathcal{I}}.$$

Proof In view of Proposition 3.2.8, it suffices to show that

$$P_\theta[\varphi]_{\mathcal{I}} \leq \varphi. \tag{3.26}$$

By Proposition 3.2.5, Proposition 3.1.5 and Theorem 1.6.1, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. By rescaling using Proposition 3.2.10, we may assume that D is a divisor. Take quasi-equisingular approximations $(\eta_j)_j$ and $(\varphi_j)_j$ of $P_\theta[\varphi]_{\mathcal{I}}$ and of φ respectively. Recall that by Theorem 1.6.2, we can guarantee that η_j and φ_j both have the singularity type $(2^{-j}, \mathcal{I}(2^j\varphi))$ and hence $\eta_j \sim \varphi_j$ for all large enough j. On the other hand, it is clear that $\varphi_j \sim \varphi$ for all $j \geq 1$. So (3.26) follows. □

3.2.3 Properties of the $\mathcal{I}$-envelope

Let $\theta, \theta_1, \theta_2$ be smooth closed real $(1,1)$-forms on X.

We have the following concavity property of the $\mathcal{I}$-envelope.

Proposition 3.2.10

(1) *Suppose that $\varphi \in \mathrm{PSH}(X, \theta)$ and $\lambda \in \mathbb{R}_{>0}$, then*

$$P_{\lambda\theta}[\lambda\varphi]_{\mathcal{I}} = \lambda P_\theta[\varphi]_{\mathcal{I}}.$$

(2) *Suppose that $\varphi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X, \theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2]_{\mathcal{I}} \geq P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(3) *Suppose that* $\varphi_1 \in \mathrm{PSH}(X, \theta_1)$ *and* $\varphi_2 \in \mathrm{PSH}(X, \theta_2)$*, then*

$$P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2]_{\mathcal{I}} \sim_{\mathcal{I}} P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(4) *Suppose that* $\varphi_1, \varphi_2 \in \mathrm{PSH}(X, \theta)$*, then*

$$P_\theta[\varphi_1 \vee \varphi_2]_{\mathcal{I}} \sim_{\mathcal{I}} P_\theta[\varphi_1]_{\mathcal{I}} \vee P_\theta[\varphi_2]_{\mathcal{I}}.$$

Proof (1) This is obvious by definition.

(2) Suppose that $\psi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\psi_2 \in \mathrm{PSH}(X, \theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \sim_{\mathcal{I}} \varphi_i$$

for $i = 1, 2$. Then thanks to Proposition 3.2.3,

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \sim_{\mathcal{I}} \varphi_1 + \varphi_2.$$

It follows that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2]_{\mathcal{I}}.$$

Since ψ_1 and ψ_2 are arbitrary, we conclude.

(3) and (4) These follow easily from Proposition 3.2.6 and Proposition 3.2.3. □

Lemma 3.2.1 *Let* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$*. Assume that* $\varphi \leq \psi$*, then*

$$P_\theta[\varphi]_{\mathcal{I}} \leq P_\theta[\psi]_{\mathcal{I}}.$$

Proof It suffices to observe that $P_\theta[\varphi]_{\mathcal{I}} \vee \psi \sim_{\mathcal{I}} \psi$ as a consequence of Proposition 1.4.2 and Proposition 3.2.6. □

Proposition 3.2.11 *Consider a decreasing net* $(\varphi_i)_{i\in I}$ *of model potentials in* $\mathrm{PSH}(X, \theta)_{>0}$*. Suppose that* $\varphi := \inf_{i\in I} \varphi_i \not\equiv -\infty$ *and* $\int_X \theta_\varphi^n > 0$*. Then*

$$\inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

Proof Let $\eta = \inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}}$. We clearly have $\eta \geq P_\theta[\varphi]_{\mathcal{I}}$ as a consequence of Lemma 3.2.1.

By Proposition 3.1.9, we have

$$\lim_{i\in I} \int_X \theta_{\varphi_i}^n = \int_X \theta_\varphi^n > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0, 1)$ and $\psi_i \in \mathrm{PSH}(X, \theta)$ such that for all $i \in I$,

$$(1 - \epsilon_i)\varphi_i + \epsilon_i \psi_i \leq \varphi.$$

By Proposition 3.2.10 and Lemma 3.2.1, we have

$$\eta+\epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \le (1-\epsilon_i)\eta+\epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \le (1-\epsilon_i)P_\theta[\varphi_i]_{\mathcal{I}}+\epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \le P_\theta[\varphi]_{\mathcal{I}}.$$

Taking limit with respect to i, we conclude that $\eta \le P_\theta[\varphi]_{\mathcal{I}}$. □

Proposition 3.2.12 *Let $(\varphi_i)_{i\in I}$ be a decreasing net of $\mathcal{I}$-model potentials in* $\mathrm{PSH}(X,\theta)$*. Suppose that $\varphi := \inf_{i\in I}\varphi_i \not\equiv -\infty$, then φ is also $\mathcal{I}$-model in* $\mathrm{PSH}(X,\theta)$.

Proof Observe that $\varphi \le 0$. Let $\eta \in \mathrm{PSH}(X,\theta)$ with $\eta \sim_{\mathcal{I}} \varphi$ and $\eta \le 0$. Then for each $i \in I$, using Proposition 3.2.3, we have $\eta \vee \varphi_i \sim_{\mathcal{I}} \varphi_i$. Therefore,

$$\eta \le \eta \vee \varphi_i \le \varphi_i.$$

It follows that $\eta \le \varphi$. Hence $\varphi = P_\theta[\varphi]_{\mathcal{I}}$. □

Proposition 3.2.13 *Let $(\varphi_i)_{i\in I}$ be an increasing net in* $\mathrm{PSH}(X,\theta)_{>0}$ *uniformly bounded from above. Let $\varphi := \sup_{i\in I}{}^*\varphi_i$. Then*

$$\sup_{i\in I}{}^* P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, if the φ_i's are all $\mathcal{I}$-model, then so is φ.

Proof Let $\eta = \sup_{i\in I}{}^* P_\theta[\varphi_i]_{\mathcal{I}}$. Then $\eta \le P_\theta[\varphi]_{\mathcal{I}}$ as a consequence of Lemma 3.2.1.

By Corollary 2.3.1, we have

$$\lim_{i\in I}\int_X \theta^n_{\varphi_i} = \int_X \theta^n_\varphi > 0.$$

So by Lemma 2.3.1, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0,1)$ and $\psi_i \in \mathrm{PSH}(X,\theta)$ such that for all $i \in I$,

$$(1-\epsilon_i)\varphi+\epsilon_i\psi_i \le \varphi_i.$$

By Proposition 3.2.10 and Lemma 3.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}}+\epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \le (1-\epsilon_i)P_\theta[\varphi]_{\mathcal{I}}+\epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \le P_\theta[\varphi_i]_{\mathcal{I}} \le \eta.$$

Taking limit with respect to i, we conclude that $\eta \ge P_\theta[\varphi]_{\mathcal{I}}$. □

Remark 3.2.1 One could also define the following interpolation between the $\mathcal{I}$-envelope and the P-envelope: Suppose $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, $j \in \{0,\dots,n\}$. Then we let

$$P_{\theta,j}[\varphi] := \sup{}^*\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \le 0, \varphi \le \psi, \int_X \theta^j_\varphi \wedge \theta^{n-j}_{P_\theta[\varphi]_{\mathcal{I}}} = \int_X \theta^j_\psi \wedge \theta^{n-j}_{P_\theta[\psi]_{\mathcal{I}}}\right\}.$$

Based on the techniques developed in Chapter 6, one could show that $P_{\theta,j}[\bullet]$ is a projection operator. When $j=n$, this operator reduces to the P-envelope, while when $j=0$, this operator reduces to the $\mathcal{I}$-envelope.

Chapter 4
Geodesic rays in the space of potentials

In den Dreißiger Jahren besuchte ich regelmäßig die Schweiz, teils um mich auch auf den Viertausendern zu tummeln, zum großen Teil aber auch, um Emigrantenblätter zu lesen und mich mit Kollegen über Naziverbrechen zu unterhalten. Aber auch die Schweizer schauten sich, wenn sie offen reden wollten, ebenso ängstlich um wie das bei uns üblich war.[a]
— Oskar Perron[b]

[a] The recent policy of ETH against Chinese students makes me feel that nothing has changed in Switzerland after the collapsing of Nazi for almost 80 years.

[b] Oskar Perron (1880—1975), after earning himself an *Eisernes Kreuz* during WWI, obtained a position in München in 1922, initiating the glorious period of München. Among his colleagues are Carathéodory, Tietze and Sommerfield.

In this chapter, we study subgeodesics and geodesics in the space of quasi-plurisubharmonic functions. Unlike what one usually finds in the literature, here we are carrying out the constructions in the space of Kähler potentials with prescribed singularities. The usual regularization techniques break down in this setup.

The results in Section 4.2 seem to be new, although they have been applied without proofs in the literature.

4.1 Subgeodesics

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

Definition 4.1.1 Let us fix $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a family $(\varphi_t)_{t\in(0,1)}$ in $\mathrm{PSH}(X,\theta)$ such that

(1) if we define
$$\Phi\colon X\times\{z\in\mathbb{C}: \mathrm{e}^{-1}<|z|<1\}\to[-\infty,\infty),\quad (x,z)\mapsto \varphi_{-\log|z|}(x),$$
then Φ is $p_1^*\theta$-psh, where $p_1\colon X\times\{z\in\mathbb{C}:\mathrm{e}^{-1}<|z|<1\}\to X$ is the natural projection;

(2) when $t\to 0+$ (resp. to $1-$), φ_t converges to φ_0 (resp. φ_1) with respect to the L^1-topology.

We also say $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

We call Φ the *complexification* of the subgeodesic $(\varphi_t)_t$.

When we do not want to specify φ_0 and φ_1, we shall say $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic.

In general, there are no subgeodesics from φ_0 to φ_1. In fact, the existence of a subgeodesic implies that $\varphi_0 \wedge \varphi_1 \not\equiv -\infty$ by Proposition 4.1.2, which does not always hold as we show in Example 5.2.3.

We first note that the subgeodesics are well-behaved under the change of θ:

Proposition 4.1.1 *Let g be a smooth real function on X. Let $\theta' = \theta + \mathrm{dd^c} g$. Suppose that $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$. Then $(\varphi_t - g)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta')$.*

Proof This follows trivially by definition. □

Example 4.1.1 Let $\varphi_0 \in \mathrm{PSH}(X,\theta)$, $C \in \mathbb{R}$. Let

$$\varphi_t = \varphi_0 + tC, \quad t \in (0,1].$$

Then $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

For this purpose, it suffices to observe that $\log|z|$ is a harmonic function in z when $|z| > 0$.

As a consequence, the constant $(\varphi_0)_{t\in[0,1]}$ is a subgeodesic, called the *constant subgeodesic* at φ_0.

A more general version is as follows: Suppose that $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$, $C_1, C_2 \in \mathbb{R}$, then $(\varphi_t + C_1 t + C_2)_{t\in[0,1]}$ is also a subgeodesic.

Proposition 4.1.2 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$ and $(\varphi_t)_{t\in(0,1)}$ be a subgeodesic from φ_0 to φ_1. Then for each $x \in X$, $[0,1] \ni t \mapsto \varphi_t(x)$ is a convex function. In particular,*

$$\inf_{t\in(0,1)} \varphi_t \in \mathrm{PSH}(X,\theta), \quad \inf_{t\in(0,1)} \varphi_t \leq \varphi_0 \wedge \varphi_1.$$

Proof For each $x \in X$, the map

$$\{z \in \mathbb{C} : \mathrm{e}^{-1} < |z| < 1\} \to [-\infty, \infty), \quad z \mapsto \Phi(x,z)$$

is either subharmonic or constantly $-\infty$, as follows from Definition 4.1.1 (1) and Proposition 1.1.4. In the latter case, the convexity of $[0,1] \ni t \mapsto \varphi_t(x)$ is trivial. In the former case, the convexity on the interval $(0,1)$ follows from Proposition 1.1.3.

In order to verify the convexity at the boundary, let us fix $s \in (0,1)$. We need to show that

$$\varphi_s(x) \leq s\varphi_1(x) + (1-s)\varphi_0(x) \tag{4.1}$$

for all $x \in X$. Thanks to Proposition 1.2.6, it suffices to prove this for almost all x.

Take a set $Z \subseteq X$ with zero Lebesgue measure such that for all $x \in X \setminus Z$, we have

(1) $\varphi_t(x) \neq -\infty$ for all $t \in [0,1] \cap \mathbb{Q}$;
(2) $\varphi_t(x) \to \varphi_0(x)$ as $t \to 0+$ and $\varphi_t(x) \to \varphi_1(x)$ as $t \to 1-$.

For all such x, the convexity of $\varphi_t(x)$ for $t \in (0,1)$ guarantees that $\varphi_t(x) \neq -\infty$ for all $t \in [0,1]$ and $t \mapsto \varphi_t(x)$ is convex for $t \in [0,1]$. In particular, (4.1) holds.

Let us prove the last assertion. Let

$$\varphi := \inf_{t\in(0,1)} \varphi_t.$$

By Kiselman's principle Proposition 1.2.8[1], we know that $\varphi \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$. Take $x \in X$ so that

$$\lim_{t\to 0+} \varphi_t(x) = \varphi_0(x) \neq -\infty, \quad \lim_{t\to 1-} \varphi_t(x) = \varphi_1(x) \neq -\infty.$$

Then $\varphi(x) \neq -\infty$. Hence we conclude that $\varphi \in \mathrm{PSH}(X,\theta)$. For any $t \in (0,1)$, using the convexity established above, we have

$$\varphi \le (1-t)\varphi_1 + t\varphi_0.$$

It follows that $\varphi \le \varphi_0$, $\varphi \le \varphi_1$ almost everywhere and hence everywhere by Proposition 1.2.6. Our assertion follows. □

Proposition 4.1.3 *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two non-empty uniformly bounded from above families in* $\mathrm{PSH}(X,\theta)$. *Let $(\varphi_t^i)_{t\in(0,1)}$ be subgeodesics from φ_0^i to φ_1^i for each $i \in I$. Then*

$$\left(\sup_{i\in I}{}^* \varphi_t^i\right)_{t\in(0,1)}$$

is a subgeodesic from $\sup_i{}^*\varphi_0^i$ to $\sup_i{}^*\varphi_1^i$.

Proof We may assume that $\varphi_0^i, \varphi_1^i \le 0$ for all $i \in I$. Then it follows that $\varphi_t^i \le 0$ for all $t \in (0,1)$ and all $i \in I$ by Proposition 4.1.2.

We define

$$\varphi_t := \sup_{i\in I}{}^* \varphi_t^i \in \mathrm{PSH}(X,\theta)$$

for all $t \in [0,1]$. Observe that $[0,1] \ni t \mapsto \varphi_t$ is convex by the same argument leading to (4.1).

Let $(\psi_t)_{t\in(0,1)}$ be the subgeodesic whose complexification Φ_ψ corresponds to $\sup_i{}^*\Phi_{\varphi^i}$, where Φ_{φ^i} is the complexification of $(\varphi_t^i)_{t\in(0,1)}$. Then clearly, $\varphi_t \le \psi_t$ for each $t \in (0,1)$. On the other hand, by Proposition 1.2.5,

$$\psi_t = \sup_{i\in I} \varphi_t^i = \varphi_t \quad \text{almost everywhere}$$

for almost all $t \in (0,1)$. Therefore, using Proposition 1.2.6, we find $\psi_t = \varphi_t$ for almost all $t \in (0,1)$. Since both functions are convex in t, we conclude that $\psi_t = \varphi_t$ for all $t \in (0,1)$.

It remains to argue that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$ and $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. By symmetry, it suffices to argue the former.

Thanks to Proposition 1.2.2, we may further assume that I is a countable set. We know that for any $t \in (0,1)$ and any $j \in I$,

$$\varphi_t^j \le \varphi_t \le t\varphi_1 + (1-t)\varphi_0.$$

---

[1] Applied the the universal cover of the annulus.

Letting $t \to 0+$, we find that

$$\varphi_0^j \leq \varlimsup_{t\to 0+} \varphi_t \leq \varphi_0$$

almost everywhere. Since I is countable, we conclude that

$$\varphi_0 = \varlimsup_{t\to 0+} \varphi_t \tag{4.2}$$

almost everywhere.

Fix $i_0 \in I$. Recall that by Proposition 4.1.2, for each $t \in (0,1)$, we have

$$\inf_{t\in(0,1)} \sup_X \varphi_t \geq \inf_{t\in(0,1)} \sup_X \varphi_t^{i_0} \geq \sup_X \left(\varphi_0^{i_0} \wedge \varphi_1^{i_0}\right) > -\infty,$$

so the set $\{\varphi_t\}_{t\in(0,1)}$ is relatively compact with respect to the L^1-topology by Proposition 1.5.1. Let ψ be a cluster point as $t \to 0+$. It suffices to show that $\psi = \varphi_0$. By Corollary 1.2.1 and (4.2), this holds almost everywhere. Therefore, it holds everywhere by Proposition 1.2.6. □

Proposition 4.1.4 *Let $(\varphi_t)_{t\in[0,1]}$ be a subgeodesic. Then for any $0 \leq a \leq b \leq 1$, the segment $(\varphi_{tb+(1-t)a})_{t\in[0,1]}$ is a subgeodesic.*

Proof It suffices to show that

$$\varphi_{tb+(1-t)a} \xrightarrow{L^1} \varphi_a, \quad \varphi_{tb+(1-t)a} \xrightarrow{L^1} \varphi_b$$

as $t \to 0+$ and $t \to 1-$ respectively. In other words, we need to show that for any $c \in (0,1)$, we have

$$\varphi_t \xrightarrow{L^1} \varphi_c$$

as $t \to c$. For this purpose, observe that by Proposition 4.1.2,

$$\sup_X \inf_{s\in(0,1)} \varphi_t \leq \sup_X \varphi_t \leq \sup_X \varphi_0 + \sup_X \varphi_1$$

for any $t \in (0,1)$. Therefore, $\{\varphi_t\}_{t\in(0,1)}$ is a relatively compact family with respect to the L^1-topology on $\mathrm{PSH}(X,\theta)$ by Proposition 1.5.1. It suffices to show that any cluster point ψ of φ_t as $t \to c$ is equal to φ_c. By Corollary 1.2.1 and the convexity Proposition 4.1.2, we have $\varphi_c = \psi$ almost everywhere and hence everywhere by Proposition 1.2.6. □

Definition 4.1.2 A ray $\ell = (\ell_t)_{t\geq 0}$ is a *subgeodesic ray* in $\mathrm{PSH}(X,\theta)$ if for any $0 \leq a \leq b$, the segment $(\varphi_{tb+(1-t)a})_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$. We say ℓ *emanates* from ℓ_0.

The *complexification* of a subgeodesic ray ℓ is defined as the potential

$$\Phi\colon X \times \{z \in \mathbb{C} : 0 < |z| < 1\} \to [-\infty, \infty), \quad (x,z) \mapsto \ell_{-\log|z|}(x).$$

Note that Φ is $p_1^*\theta$-psh, where $p_1 \colon X \times \{z \in \mathbb{C} : 0 < |z| < 1\} \to X$ is the natural projection.

4.2 Geodesics in the space of potentials

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$. See Definition 3.1.3 for the definition.

Definition 4.2.1 Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. The *geodesic* $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 is the family of potentials $\varphi_t \in \mathrm{PSH}(X,\theta)$ such that

$$\begin{aligned} \varphi_t = \sup{}^* \{\psi_t : (\psi_s)_s \text{ is a subgeodesic from } \psi_0 \text{ to } \psi_1, \\ \psi_0, \psi_1 \in \mathrm{PSH}(X,\theta),\, \psi_0 \leq \varphi_0, \psi_1 \leq \varphi_1\}\,. \end{aligned} \tag{4.3}$$

The envelopes of the form (4.3) are usually referred to as the *Perron envelopes*.

Example 4.2.1 Let $\varphi_0 \in \mathrm{PSH}(X,\theta)$ and $C \in \mathbb{R}$. Then the subgeodesic $(\varphi_0 + tC)_{t\in[0,1]}$ studied in Example 4.1.1 is a geodesic. This follows easily from Proposition 4.1.2.

In particular, when $C = 0$, we find that the constant subgeodesic at φ_0 is indeed a geodesic, which we call the *constant geodesic* at φ.

More generally, suppose that $(\varphi_t)_{t\in[0,1]}$ is a geodesic and $C_1, C_2 \in \mathbb{R}$, then $(\varphi_t + C_1 t + C_2)_{t\in[0,1]}$ is also a geodesic. This follows immediately from Example 4.1.1.

Definition 4.2.2 Let $(\varphi_t)_{t\in[a,b]}$ $(a, b \in \mathbb{R},\, a \leq b)$ be a curve in $\mathcal{E}(X,\theta;\phi)$. We say $(\varphi_t)_{t\in[a,b]}$ is a *geodesic* if the curve $(\varphi_{t(b-a)+a})_{t\in(0,1)}$ is a geodesic from φ_a to φ_b.

We also say $(\varphi_t)_{t\in[a,b]}$ or $(\varphi_t)_{t\in(a,b)}$ is a geodesic in $\mathcal{E}(X,\theta;\phi)$ from φ_a to φ_b.

We refer to Section 3.1.3 for the definition of $\mathcal{E}(X,\theta;\phi)$.

Proposition 4.2.1 *Given $\varphi_0, \varphi_1 \in \mathcal{E}(X,\theta;\phi)$, the geodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 exists and is a subgeodesic from φ_0 to φ_1 and $\varphi_t \in \mathcal{E}(X,\theta;\phi)$ for each $t \in (0,1)$.*

Moreover, for any $0 \leq a \leq b \leq 1$, the restriction $(\varphi_t)_{t\in[a,b]}$ is a geodesic.

If furthermore $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$), then $\varphi_t \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$) for all $t \in (0,1)$.

Proof Without loss of generality, we may assume that $\varphi_0, \varphi_1 \leq \phi$. It follows from Proposition 4.1.2 that $\varphi_t \leq \phi$ for all $t \in (0,1)$. In fact, we have the stronger estimate

$$\varphi_t \leq t\varphi_1 + (1-t)\varphi_0, \quad t \in (0,1). \tag{4.4}$$

We first observe that when $\varphi_0, \varphi_1 \in \mathcal{E}(X,\theta;\phi)$, so is $\varphi_0 \wedge \varphi_1$, see Proposition 3.1.13. In particular, the constant subgeodesic $t \mapsto \varphi_0 \wedge \varphi_1$ is a candidate in (4.3). So

$$\varphi_t \geq \varphi_0 \wedge \varphi_1, \quad t \in (0,1). \tag{4.5}$$

By Proposition 4.1.3, $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic. It follows from Proposition 3.1.14 that $\varphi_t \in \mathcal{E}(X,\theta;\phi)$ for all $t \in (0,1)$.

Next, we show that as $t \to 0+$, we have $\varphi_t \xrightarrow{L^1} \varphi_0$. The corresponding result at $t = 1$ is similar.

We first argue the special case where $\varphi_0 \leq \varphi_1$. Take a constant $C > 0$ such that

$$\varphi_0 - C \leq \varphi_1.$$

Then $(\varphi_0 - Ct)_{t\in(0,1)}$ is clearly a candidate in (4.3), see Example 4.1.1. Therefore, for all $t \in (0,1)$,

$$\varphi_0 - Ct \leq \varphi_t \leq t\varphi_1 + (1-t)\varphi_0. \tag{4.6}$$

It follows that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$.

Let us come back to the general case. By (4.4) and (4.5), we know that for all $t \in (0,1)$,

$$\sup_X \varphi_0 \wedge \varphi_1 \leq \sup_X \varphi_t \leq (\sup_X \varphi_0) \vee (\sup_X \varphi_1).$$

It follows from Proposition 1.5.1 that $\{\varphi_t : t \in (0,1)\}$ is a relatively compact subset of $\mathrm{PSH}(X,\theta)$ with respect to the L^1-topology.

Let ψ be an L^1-cluster point of φ_t as $t \searrow 0$, it suffices to show that $\psi = \varphi_0$.

For each $M \in \mathbb{N}$, we write

$$\varphi_0^M = \varphi_0 \wedge (\varphi_1 + M).$$

Observe that $\varphi_0^M \in \mathcal{E}(X,\theta;\phi)$ by Proposition 3.1.13. Let $(\varphi_t^M)_{t\in(0,1)}$ be the geodesic from φ_0^M to φ_1. Then it is clear that $\varphi_t^M \leq \varphi_t$ for all $t \in (0,1)$. Therefore,

$$\psi \geq \varphi_0 \wedge (\varphi_1 + M)$$

almost everywhere hence everywhere by Proposition 1.2.6. On the other hand, by (4.4), $\psi \leq \varphi_0$. So it suffices to show that

$$\varphi_0 \wedge (\varphi_1 + M) \xrightarrow{L^1} \varphi_0$$

as $M \to \infty$, which is shown in Proposition 3.1.16.

Next, take $0 \leq a \leq b \leq 1$. We want to show that the restriction $(\varphi_t)_{t\in[a,b]}$ is the geodesic from φ_a to φ_b. We may assume that $a < b$. The argument is the standard *balayage* argument.

Let $(\psi_t)_{t\in(a,b)}$ be the (reparameterized) geodesic from φ_a to φ_b. Since $(\varphi_t)_{t\in[a,b]}$ is a (reparameterized) subgeodesic by Proposition 4.1.4. we have $\psi_t \geq \varphi_t$ for all $t \in (a,b)$.

We define

$$\eta_t = \begin{cases} \psi_t, & \text{if } t \in (a,b), \\ \varphi_t, & \text{if } t \in (0,1) \setminus (a,b). \end{cases}$$

We claim that $(\eta_t)_{t\in(0,1)}$ is a subgeodesic from φ_0 to φ_1. This is clear by Lemma 1.2.2 when neither $a=0$ nor $b=1$. Next we handle the case where $a=0$. By the previous part of the proof, we know that $\psi_t \xrightarrow{L^1} \varphi_0$ as $t\to 0+$. But $\psi_t=\eta_t=\eta'_t$ for $t\in(0,b)$. Hence $\eta'_t \xrightarrow{L^1} \varphi_0$ as $t\to 0+$. The case $b=1$ is handled similarly.

Therefore, for all $t\in(0,1)$, we have

$$\varphi_t \geq \eta_t .$$

In particular, for $t\in(a,b)$, we have

$$\varphi_t \geq \eta_t = \psi_t \geq \varphi_t .$$

In other words, $(\varphi_t)_{t\in(a,b)}=(\psi_t)_{t\in(a,b)}$ is the (reparametrized) geodesic from φ_a to φ_b.

Finally, assume furthermore that $\varphi_0,\varphi_1\in\mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$). Thanks to (4.5), Proposition 3.1.13 and Proposition 3.1.14, we find $\varphi_t\in\mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$) for all $t\in(0,1)$. □

Proposition 4.2.2 *Let $\varphi_1,\varphi_0\in\mathcal{E}(X,\theta;\phi)$ with $\varphi_1\leq\varphi_0$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$s \sup_{\{\varphi_0\neq-\infty\}}(\varphi_1-\varphi_0) = \sup_{\{\varphi_0\neq-\infty\}}(\varphi_s-\varphi_0) \tag{4.7}$$

for all $s\in[0,1]$.

Proof The notations in the proof are indicated in Fig. 4.1.[2]

We may assume that $s\in[0,1)$ since there is nothing to prove when $s=1$.

After replacing φ_t by $\varphi_t - C't$ for some large enough $C'>0$, we may assume that $\varphi_1\leq\varphi_0$. This procedure preserves the geodesic property by Example 4.2.1.

Since the constant geodesic at φ_1 is a candidate in (4.3), it follows that $\varphi_1\leq\varphi_t$ for all $t\in[0,1]$. Similarly, $[0,1]\ni t\mapsto\varphi_t$ is decreasing.

Let

$$C = \sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_0) \leq 0. \tag{4.8}$$

Then by Proposition 1.2.6, we have

$$\varphi_1 \leq \varphi_0 + C.$$

So $(\varphi_1 - C(1-t))_{t\in(0,1)}$ is a candidate in (4.3) and hence

$$\varphi_1 - C(1-t) \leq \varphi_t, \quad t\in(0,1). \tag{4.9}$$

[2] When dealing with convex functions, drawing a picture is the easiest way to keep track of the directions of inequalities.

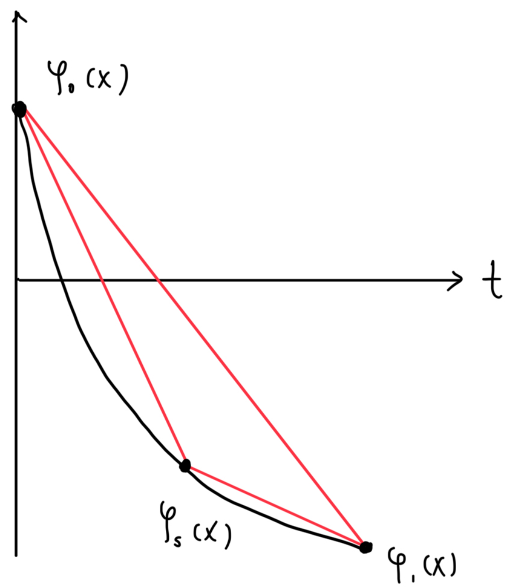

Fig. 4.1 The typical behavior of $\varphi_t(x)$

By Proposition 4.2.1, we have $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. Since φ_t is decreasing in $t \in (0,1)$. It follows that $\varphi_1 = \inf_{t\in(0,1)} \varphi_t$. Therefore, we can find a pluripolar set $Z \subseteq X$ such that $\varphi_t(x) \to \varphi_1(x) > -\infty$ as $t \to 1-$ for all $x \in X \setminus Z$.

Similarly, since $\varphi_0 = \sup_{t\in(0,1)}{}^*\varphi_t$, after enlarging Z, we may also guarantee that $\varphi_t(x) \to \varphi_0(x) > -\infty$ as $t \to 0+$ for all $x \in X \setminus Z$ by Proposition 1.2.5.

For any such $x \in X \setminus Z$, the function $t \mapsto \varphi_t(x)$ is a real-valued continuous convex function on $[0,1]$. In particular, $t \mapsto \varphi_t(x)$ is absolutely continuous on $[0,1]$. Hence, for any $s \in [0,1)$, we have

$$\varphi_1(x) - \varphi_s(x) = \int_s^1 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\,\mathrm{d}t \leq (1-s)\lim_{t\to 1-} \frac{\varphi_1(x)-\varphi_t(x)}{1-t} \leq (1-s)C, \tag{4.10}$$

where the second inequality follows from (4.9).

Taking supremum in (4.10), we find that

$$\sup_{X\setminus Z}(\varphi_1 - \varphi_s) \leq (1-s)\sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t} \leq (1-s)C. \tag{4.11}$$

When $s = 0$, we deduce from Corollary 1.3.6 and (4.8) that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0) = \sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t}.$$

But this equality works equally well for the geodesic $(\varphi_{(1-s)t+s})_{t\in[0,1]}$. It follows that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_s)=(1-s)\sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t}=(1-s)C.$$

Therefore, invoking Corollary 1.3.6 again, we deduce that all inequalities in (4.11) are in fact equalities. In other words,

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_0)=\sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t}=\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_1-\varphi_s}{1-s}. \tag{4.12}$$

On the other hand, we have the trivial inequality

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_0)\leq s\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s}+(1-s)\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_1-\varphi_s}{1-s}.$$

Together with (4.12), we find that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_0)\leq\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s}.$$

The reverse inequality follows from the convexity,

$$\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s}=\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1-\varphi_0).$$

Using Corollary 1.3.6, we conclude (4.7). □

With an almost identical proof, we find

Proposition 4.2.3 *Let $\varphi_1,\varphi_0\in\mathcal{E}^\infty(X,\theta;\phi)$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$t\inf_{\{\phi\neq-\infty\}}(\varphi_1-\varphi_0)=\inf_{\{\phi\neq-\infty\}}(\varphi_t-\varphi_0)$$

for all $t\in(0,1]$.

Definition 4.2.3 Let $\ell=(\ell_t)_{t\geq 0}$ be a curve in $\mathcal{E}(X,\theta;\phi)$. We say ℓ is a *geodesic ray* in $\mathcal{E}(X,\theta;\phi)$ emanating from ℓ_0 if for each $0\leq a\leq b$, the restriction $(\ell_t)_{t\in[a,b]}$ is a geodesic.

The set of geodesic rays in $\mathcal{E}(X,\theta;\phi)$ emanating from ϕ is denoted by $\mathcal{R}(X,\theta;\phi)$.

We say a geodesic ray $\ell\in\mathcal{R}(X,\theta;\phi)$ has *finite energy* if $\ell_t\in\mathcal{E}^1(X,\theta;\phi)$ for all $t>0$. The set of geodesic rays with finite energy is denoted by $\mathcal{R}^1(X,\theta;\phi)$.

We say a geodesic ray $\ell\in\mathcal{R}(X,\theta;\phi)$ is *bounded* if $\ell_t\in\mathcal{E}^\infty(X,\theta;\phi)$ for all $t\geq 0$. The set of bounded geodesic rays is denoted by $\mathcal{R}^\infty(X,\theta;\phi)$.

Given $\ell,\ell'\in\mathcal{R}(X,\theta;\phi)$, we write $\ell\leq\ell'$ if $\ell_t\leq\ell'_t$ for each $t\geq 0$.

When $\phi=V_\theta$, we usually omit it from the notations and write $\mathcal{R}(X,\theta)$, $\mathcal{R}^1(X,\theta)$ and $\mathcal{R}^\infty(X,\theta)$ respectively.

Proposition 4.2.4 *Let $\ell\in\mathcal{R}(X,\theta;\phi)$. Then there is a constant $C\in\mathbb{R}$ such that*

$$\sup_X\ell_t=Ct,\quad t\geq 0.$$

Proof It follows from Proposition 4.2.2 that

$$\sup_{\{\phi\neq-\infty\}}(\ell_t-\phi)=t\sup_X(\ell_1-\phi)$$

for all $t\geq 0$.

It suffices to show that for any $t\geq 0$,

$$\sup_{\{\phi\neq-\infty\}}(\ell_t-\phi)=\sup_X \ell_t.$$

The $\geq$ direction follows easily from Corollary 1.3.6. In order to argue the reverse inequality, let us observe that for any $t\geq 0$,

$$\ell_t-\sup_X\ell_t\leq 0,\quad \ell_t-\sup_X\ell_t\leq\phi.$$

Since ϕ is a model potential, it follows that

$$\ell_t-\sup_X\ell_t\leq\phi.$$

Our assertion follows. □

Definition 4.2.4 We define the *radial Monge–Ampère energy* $\mathbf{E}^{\phi}\colon\mathcal{R}(X,\theta;\phi)\to\mathbb{R}\cup\{\infty\}$ as follows:

$$\mathbf{E}^{\phi}(\ell)\coloneqq\varlimsup_{t\to\infty}\frac{E^{\phi}_{\theta}(\ell_t)}{t}.$$

When $\phi=V_\theta$, we write $\mathbf{E}$ instead of $\mathbf{E}^{V_\theta}$.

Thanks to Proposition 4.2.2, $\mathbf{E}^{\phi}(\ell)<\infty$ for any $\ell\in\mathcal{R}^1(X,\theta;\phi)$.

Definition 4.2.5 Let $\varphi,\psi\in\mathcal{E}^1(X,\theta;\phi)$, we define

$$d_1(\varphi,\psi)=E^{\phi}_{\theta}(\varphi)+E^{\phi}_{\theta}(\psi)-2E^{\phi}_{\theta}(\varphi\wedge\psi).$$

Note that by Proposition 3.1.13, $\varphi\in\psi\in\mathcal{E}^1(X,\theta;\phi)$.

In particular, if $\varphi\leq\psi$, we have

$$d_1(\varphi,\psi)=E^{\phi}_{\theta}(\psi)-E^{\phi}_{\theta}(\varphi). \tag{4.13}$$

Theorem 4.2.1 *The function d_1 defined in Definition 4.2.5 is a complete metric on $\mathcal{E}^1(X,\theta;\phi)$.*

The function $E^{\phi}_{\theta}\colon\mathcal{E}^1(X,\theta;\phi)\to\mathbb{R}$ is continuous with respect to d_1.

Moreover, given a decreasing (resp. increasing) sequence $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathcal{E}^1(X,\theta;\phi)$ converging (resp. converging almost everywhere) to $\varphi\in\mathcal{E}^1(X,\theta;\phi)$, then $\varphi_j\xrightarrow{d_1}\varphi$. Conversely, if a monotone sequence $(\varphi_j)_j$ converges in $\mathcal{E}^1(X,\theta;\phi)$, then the limit is almost everywhere equal to the pointwise limit of the sequence.

See [DDNL18a, Theorem 1.1, Proposition 2.9, Proposition 2.7]. The readers should have no difficulty in generalizing all arguments to the current setting.

Next we recall a few particular properties when $\phi = V_\theta$.[3]

Proposition 4.2.5 *Let $(\varphi_t)_{t\in[a,b]}$ be a geodesic in $\mathcal{E}^1(X,\theta)$, then $t \mapsto E_\theta(\varphi_t)$ is a linear function of $t \in [a,b]$.*

See [DDNL18c, Theorem 3.12].

Proposition 4.2.6 *Let $\ell, \ell' \in \mathcal{R}^1(X,\theta)$ and $\ell \leq \ell'$. Then*

$$d_1(\ell,\ell') = \mathbf{E}(\ell') - \mathbf{E}(\ell). \tag{4.14}$$

Proof This is a direct consequence of (4.13). □

Proposition 4.2.7 *Let $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. Then the map*

$$t \mapsto d_1(\ell_t, \ell'_t)$$

is convex.

See [DDNL21b, Proposition 2.10] for the proof. In particular, we can introduce:

Definition 4.2.6 Let $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. We define

$$d_1(\ell,\ell') := \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t, \ell'_t).$$

Theorem 4.2.2 *The function d_1 defined in Definition 4.2.6 is a metric and $(\mathcal{R}^1(X,\theta), d_1)$ is a complete metric space.*

See [DDNL21b, Theorem 2.14] for the proof.

Proposition 4.2.8 *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two uniformly bounded from above increasing nets in $\mathcal{E}^\infty(X,\theta)$. Let $(\varphi_t^i)_{t\in(0,1)}$ be the geodesic from φ_0^i to φ_1^i for each $i \in I$. Then*

$$\left(\sup_{i\in I}{}^* \varphi_t^i\right)_{t\in(0,1)}$$

is the geodesic from $\sup_i{}^*\varphi_0^i$ to $\sup_i{}^*\varphi_0^i$.

Proof By Proposition 1.2.2 and Proposition 4.1.3, we may assume that I is countable. In this case, the assertion follows from [DDNL18c, Proposition 3.3] and Theorem 2.1.1. □

Next we recall that $\vee$ operator at the level of geodesic rays.

Definition 4.2.7 Let $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$. We define $\ell \vee \ell'$ as the minimal ray in $\mathcal{R}^\infty(X,\theta)$ lying above both ℓ and ℓ'.

[3] I expect that these assertions hold even when $\phi \neq V_\theta$. But I am unable to prove them.

Proposition 4.2.9 *Given $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$. Then $\ell \vee \ell' \in \mathcal{R}^\infty(X,\theta)$ exists, and*

$$\mathbf{E}(\ell \vee \ell') = \lim_{t\to\infty} \frac{1}{t} E_\theta(\ell_t \vee \ell'_t). \tag{4.15}$$

Proof For each $t > 0$, let $(\ell''^t_s)_{s\in[0,t]}$ be the geodesic from V_θ to $\ell_t \vee \ell'_t$.

Step 1. We first show that for each fixed $s \geq 0$, ℓ''^t_s is increasing in $t \in [s,\infty)$.

To see this, fix $s \geq 0$ and choose $t' > t \geq s$. We need to show that

$$\ell''^{t'}_s \geq \ell''^t_s. \tag{4.16}$$

Since $(\ell''^{t'}_a)_{a\in[0,t]}$ is a geodesic. It suffices to show that $(\ell''^t_a)_{a\in[0,t]}$ is a candidate in the Perron envelope defining the former geodesic. In other words, in verifying (4.16), we may assume that either $s = 0$ or $s = t$. The case $s = 0$ is of course trivial. So it remains to prove the following:

$$\ell''^{t'}_t \geq \ell_t \vee \ell'_t.$$

By symmetry, it suffices to prove

$$\ell''^{t'}_t \geq \ell_t.$$

But since $(\ell_a)_{a\in[0,t']}$ is a candidate in the Perron envelope defining $\ell''^{t'}$, this inequality follows.

Step 2. Next, observe that for a fixed $s \geq 0$, we have

$$\sup_X \ell''^t_s \leq \frac{s}{t}\sup_X \ell''^t_t + \frac{t-s}{t}\sup_X \ell''^t_0 = \frac{s}{t}\left(\sup_X \ell_t\right) \vee \left(\sup_X \ell'_t\right)$$

for all $t \geq s$. The right-hand side is bounded from above by a constant independent of $t \geq s$ by Proposition 4.2.4. Let

$$(\ell \vee \ell')_s = \sup_{t\geq s}{}^* \ell''^t_s.$$

Then Proposition 4.2.8 guarantees that $\ell \vee \ell' \in \mathcal{R}^\infty(X,\theta)$.

Step 3. We need to show that $\ell \vee \ell'$ defined in this way is indeed the minimal ray lying above ℓ and ℓ'.

First, by Step 1, we have

$$\ell''^t_s \geq \ell''^s_s \geq \ell_s$$

for any $t \geq s \geq 0$. Therefore,

$$(\ell \vee \ell')_s \geq \ell_s$$

for all $s \geq 0$. In other words, $\ell \vee \ell' \geq \ell$. Similarly, $\ell \vee \ell' \geq \ell'$.

Next, let $L \in \mathcal{R}^\infty(X,\theta)$ be a ray lying above both ℓ and ℓ'. Then we have

$$L_t \geq \ell_t \wedge \ell'_t$$

for all $t \geq 0$. In particular,

$$L_s \geq \ell''^t_s$$

for all $t \geq s \geq 0$. It follows that

$$L_s \geq (\ell \vee \ell')_s$$

for all $s \geq 0$.

Step 4. It remains to argue (4.15):

$$\mathbf{E}(\ell \vee \ell') = E_\theta(\ell \vee \ell')_1 = \lim_{t\to\infty} E_\theta(\ell''^t_1) = \lim_{t\to\infty} \frac{1}{t} E_\theta(\ell_t \vee \ell'_t),$$

where we applied Proposition 4.2.5 and Theorem 4.2.1. □

Lemma 4.2.1 *For any $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$, we have*

$$d_1(\ell,\ell') \leq d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq C_n d_1(\ell,\ell'), \tag{4.17}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof The first inequality is trivial. As for the second, we estimate

$$\begin{aligned} d_1(\ell,\ell\vee\ell') =& \mathbf{E}(\ell\vee\ell') - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty} \frac{1}{t}\mathbf{E}(\ell_t\vee\ell'_t) - \mathbf{E}(\ell) \\ =& \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t\vee\ell'_t,\ell_t), \end{aligned}$$

where one the first line and the third, we applied Proposition 4.2.6, on the second line, we used (4.15). In all, we find

$$d_1(\ell,\ell\vee\ell') + d_1(\ell',\ell\vee\ell') \leq \lim_{t\to\infty} \frac{1}{t}\left(d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t)\right).$$

By [DDNL18a, Theorem 3.7],

$$d_1(\ell_t\vee\ell'_t,\ell_t) + d_1(\ell_t\vee\ell'_t,\ell'_t) \leq 3(n+1)2^{n+2} d_1(\ell_t,\ell'_t).$$

Now (4.17) follows. □

Example 4.2.2 Let $\varphi \in \mathrm{PSH}(X,\theta)$. For each $C > 0$, let $(\ell^{\varphi,C}_t)_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta - C)\vee\varphi$. For each $t \geq 0$, there is $\ell^\varphi_t \in \mathcal{E}^\infty(X,\theta)$ such that

$$\ell^{\varphi,C}_t \xrightarrow{d_1} \ell^\varphi_t \tag{4.18}$$

as $C \to \infty$. Then $\ell^\varphi \in \mathcal{R}^\infty(X,\theta)$ and

$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1}\sum_{j=0}^n \left(\int_X \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^n_{V_\theta}\right). \tag{4.19}$$

From the proof below, we see that $\ell^{\varphi+C} = \ell^\varphi$ for any $C \in \mathbb{R}$.

Proof **Step 1**. We first assume that $\varphi \leq 0$.

We first show that for each fixed $t \geq 0$, $\ell_t^{\varphi,C}$ is increasing in $C \geq t$.

To see this, choose $t \leq C_1 < C_2$. We need to show that

$$\ell_t^{\varphi,C_1} \leq \ell_t^{\varphi,C_2}.$$

Since both sides are geodesics for $t \in [0, C_1]$, it suffices to show that

$$(V_\theta - C_1) \vee \varphi \leq \ell_{C_1}^{\varphi,C_2}. \tag{4.20}$$

Now $((V_\theta - t) \vee \varphi)_{t\in[0,C_2]}$ is a subgeodesic from V_θ to $(V_\theta - C_2) \vee \varphi$ by Proposition 4.1.3.[4] At $t = 0$ and $t = C_1$, it is dominated by the geodesic ℓ_t^{φ,C_2}, hence we conclude that the same holds at $t = C_1$, which is exactly (4.20).

From Proposition 4.1.2, we know that for any $C > t > 0$, we have

$$\ell_t^{\varphi,C} \leq \frac{t}{C}\left((V_\theta - C) \vee \varphi\right) + \frac{C-t}{C} \cdot V_\theta \leq 0,$$

so by Proposition 1.2.1,

$$\ell_t^\varphi := \sup_{C>t}{}^* \ell_t^{\varphi,C} \in \mathcal{E}^\infty(X,\theta) \tag{4.21}$$

for all $t \geq 0$. Thanks to Theorem 4.2.1, we have

$$\ell_t^{\varphi,C} \xrightarrow{d_1} \ell_t^\varphi$$

as $C \to \infty$ for all $t \geq 0$. It follows from Proposition 4.2.8 that $\ell^\varphi \in \mathcal{R}^\infty(X,\theta)$.

It remains to compute the energy of ℓ^φ. We first fix $C \geq t > 0$ and compute using Proposition 4.2.5:

$$E_\theta\left(\ell_t^{\varphi,C}\right) = \frac{t}{C} E_\theta\left((V_\theta - C) \vee \varphi\right).$$

Letting $C \to \infty$ and applying Theorem 4.2.1, we find that

$$E_\theta(\ell_t^\varphi) = \lim_{C\to\infty} \frac{t}{C} E_\theta\left((V_\theta - C) \vee \varphi\right)$$

for any $t \geq 0$. It follows that

$$\mathbf{E}(\ell^\varphi) = \lim_{C\to\infty} \frac{1}{C} E_\theta\left((V_\theta - C) \vee \varphi\right).$$

Using the definition of E_θ, in order to obtain (4.19), it suffices to show that for each $j = 0, \ldots, n$, we have

[4] Here we need $\varphi \leq 0$.

$$\lim_{C\to\infty}\int_X \frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta} = \int_X \theta^j_\varphi\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^n_{V_\theta}. \quad (4.22)$$

For this purpose, for each $C > 0$, we decompose X as $\{\varphi > V_\theta - C\}$ and $\{\varphi \leq V_\theta - C\}$. We have

$$\begin{aligned}
&\int_{\{\varphi > V_\theta - C\}} \frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\
=&\int_{\{\varphi > V_\theta - C\}} \frac{\varphi - V_\theta}{C}\theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\{\varphi \leq V_\theta - C\}} \frac{(V_\theta - C)\vee\varphi - V_\theta}{C}\theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\
=&-\int_{\{\varphi \leq V_\theta - C\}} \theta^j_{(V_\theta-C)\vee\varphi}\wedge\theta^{n-j}_{V_\theta}\\
=&-\int_X \theta^n_{V_\theta} + \int_{\{\varphi > V_\theta - C\}} \theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}.
\end{aligned}$$

Observe that for $C > 0$, the functions $\mathbb{1}_{\{\varphi > V_\theta - C\}}C^{-1}(\varphi - V_\theta)$ is defined almost everywhere and is bounded. When $C \to \infty$, these functions converge to 0 almost everywhere. Therefore, (4.22) follows.

Finally, let us observe that ℓ_t is decreasing in $t \geq 0$ by the argument in the proof of Proposition 4.2.2.

Step 2. We assume that $D = \sup_X \varphi > 0$.

Then

$$(V_\theta - C)\vee\varphi = (V_\theta - C - D)\vee(\varphi - D) + D.$$

Therefore,

$$\ell^{\varphi,C}_t = \ell^{\varphi-D,C+D}_{\frac{C+D}{C}\cdot t} + \frac{D}{C}\cdot t \quad (4.23)$$

for all $C > 0$ and $t \in [0, C]$, since both sides are geodesics with the same endpoints.

Next, observe that for any fixed $t \geq 0$, as $C \to \infty$, we have

$$\ell^{\varphi-D,C+D}_{\frac{C+D}{C}\cdot t} \xrightarrow{d_1} \ell^{\varphi-D}_t. \quad (4.24)$$

In fact, we may assume that $t > 0$, then for any $\delta \in (0, t)$, we have

$$
\begin{aligned}
&\varlimsup_{C\to\infty} d_1\left(\ell^{\varphi-D,C+D}_{\frac{C+D}{C}\cdot t},\ell^{\varphi-D}_t\right)\\
\leq &\varlimsup_{C\to\infty} d_1\left(\ell^{\varphi-D,C+D}_{\frac{C+D}{C}\cdot t},\ell^{\varphi-D,C+D}_t\right)+\varlimsup_{C\to\infty} d_1\left(\ell^{\varphi-D}_t,\ell^{\varphi-D,C+D}_t\right)\\
= &\varlimsup_{C\to\infty} d_1\left(\ell^{\varphi-D,C+D}_{\frac{C+D}{C}\cdot t},\ell^{\varphi-D,C+D}_t\right)\\
\leq &\varlimsup_{C\to\infty} d_1\left(\ell^{\varphi-D,C+D}_{t-\delta},\ell^{\varphi-D,C+D}_t\right)\\
= &d_1(\ell^{\varphi-D}_{t-\delta},\ell^{\varphi-D}_t),
\end{aligned}
$$

where on the third line, we applied Step 1. Let $\delta\to 0+$, using Theorem 4.2.1, we find that

$$\lim_{\delta\to 0+} d_1(\ell^{\varphi-D}_{t-\delta},\ell^{\varphi-D}_t)=0.$$

Therefore, (4.24) follows.

Taking (4.23) into account, we conclude that

$$\ell^{\varphi,C}_t \xrightarrow{d_1} \ell^{\varphi-D}_t$$

as $C\to\infty$ for any $t\geq 0$. Namely,

$$\ell^{\varphi}=\ell^{\varphi-D}.$$

In particular,

$$\mathbf{E}(\ell^{\varphi})=\mathbf{E}(\ell^{\varphi-D})$$

and (4.19) follows.

Chapter 5
Toric pluripotential theory on ample line bundles

There are two principal ways to formulate mathematical assertions (problems, conjectures, theorems, . . .): Russian and French. The Russian way is to choose the most simple and specific case (so that nobody could simplify the formulation preserving the main point). The French way is to generalize the statement as far as nobody could generalize it further.
— Vladimir Arnold[a]

[a] Vladimir Igorevich Arnold (1937–2010), who became a professor at l'Université Paris IX after the dissolution of USSR, was always sick of France (so am I!). In the public lecture entitled "Sur l'éducation mathématique" in 1997, he invented the famous joke "Combien font 2 + 3?" to question the french education system.

In this chapter, we briefly recall the toric pluripotential theory relative to an ample line bundle. The general case of big line bundles will be handled in Chapter 12 after developing the powerful machinery of partial Okounkov bodies in Chapter 10. The main new result is Theorem 5.2.2 computing the L^2-sections of a Hermitian big line bundle in the toric setting.

We assume that the readers are familiar with basic toric geometry, such as the materials in [CLS11]. If not, this section can be safely skipped.

Some basic facts about convex functions and convex bodies are recalled in Appendix A.

5.1 Toric setup

Let T be a complex torus of dimension n[1] and $T_c \subset T(\mathbb{C})$ denotes the corresponding compact torus. Write M for the character lattice of T, which is a free Abelian group of rank n. Similarly, let N be cocharacter lattice of T, which is the dual lattice of M. Given $m \in M$, the corresponding character of M is denoted by χ^m. Write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is denoted by $\langle \bullet, \bullet \rangle$.

Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional *smooth*[2] lattice polytope[3].

Given any (closed) facet F of P, let $u_F \in N$ denote the unique ray generator (the first non-zero integral element) of the inward normal ray of F. Then P can be

[1] Namely, an algebraic group defined over $\mathbb{C}$, which is isomorphic to $\mathbb{G}_{\mathrm{m}}^n$.

[2] Recall that *smooth* means that for every vertex $v \in P$, if we take the first lattice point w_E apart from v as one transverses each edge E of P containing v from v, then $\{w_E - v\}_E$ forms a basis of M. See [CLS11, Definition 2.4.2]. We also say P is a *Delzant polytope* in this case.

[3] A *lattice polytope* in $M_{\mathbb{R}}$ is the convex hull of finitely many points in M.

represented as

$$P = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \text{ of } P\} \tag{5.1}$$

for some uniquely determined integers a_F. The presentation is called the *facet presentation* of P.

Given any (closed) face Q of P, we let $\sigma_Q \subseteq N_{\mathbb{R}}$ be the closed convex cone generated by the u_F's, where F runs over all facets of P containing Q. When $Q = P$, σ_P is understood as $\{0\}$.

Let Σ be the *(inner) normal fan* of P. Namely,

$$\Sigma = \left\{\sigma_Q : Q \text{ is a face of } P\right\}.$$

The notation $\Sigma(1)$ denotes the set of rays in Σ. Note that $\Sigma(1)$ is in bijective correspondence with the set of facets of P. In fact, given any facet F of P, the cone σ_F is just the ray generated by u_F, namely, the inward normal ray of F.

For any $\rho \in \Sigma(1)$, let $u_\rho \in N$ denote the ray generator of ρ, namely the first non-zero element in $N \cap \rho$. If $\rho = \sigma_F$ for some facet F of P, then $u_\rho = u_F$.

Now the facet presentation (5.1) can be equivalently rewritten as

$$P = \left\{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\right\}.$$

Let $\mathrm{Supp}_P : N_{\mathbb{R}} \to \mathbb{R}$ denote the *support function* of P. Recall that the support function (Example A.1.2) of P is defined as

$$\mathrm{Supp}_P(n) = \max\left\{\langle m, n \rangle : m \in P\right\}.$$

Note that our support function differs from [CLS11, Proposition 4.2.14], where instead of a maximum, they took the minimum.

Recall that the *characteristic function* $\chi_P : N_{\mathbb{R}} \to \{0, \infty\}$ of P is defined as in Example A.1.1:

$$\chi_P(n) := \begin{cases} 0, & n \in P; \\ \infty, & n \notin A. \end{cases}$$

Let $X = X_\Sigma$ be the smooth projective toric variety corresponding to Σ. See [CLS11, Theorem 3.1.5] for the construction of X and [CLS11, Theorem 3.1.19] for the smoothness of X. There is a canonical embedding $T \subseteq X$ as a dense Zariski open subset.

Let D be the Cartier divisor on X defined by P:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where D_ρ is the toric prime divisor defined by ρ under the orbit-cone correspondence [CLS11, Theorem 3.2.6].

Let L be the toric line bundle induced by P, namely $L = \mathcal{O}_X(D)$. Since P has full dimension, L^k is very ample for each $k \geq n-1$ by [CLS11, Corollary 2.2.19], we actually know that L is ample.

We will choose the base e for the logarithm map

$$\mathbb{C}^* \to \mathbb{R}, \quad z \mapsto \log |z|^2. \tag{5.2}$$

This choice will be fixed throughout the whole book. Since we have a canonical identification $T(\mathbb{C}) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$, the logarithm map then induces a tropicalization map after tensoring with N:

$$\mathrm{Trop}\colon T(\mathbb{C}) \to N_{\mathbb{R}}. \tag{5.3}$$

Before proceeding, it is always helpful to understand everything in our favorite example.

Example 5.1.1 We take $n = 1$ and $P = [0,1] \subseteq M_{\mathbb{R}} = \mathbb{R}$. In this case, the facet representation (5.1) becomes

$$P = \{m \in \mathbb{R} : \langle m, 1\rangle \geq 0, \langle m, -1\rangle \geq -1\},$$

with $u_{\{0\}} = 1$, $u_{\{1\}} = -1$, $a_{\{0\}} = 0$ and $a_{\{1\}} = 1$. The normal fan Σ is

$$\Sigma = \{(-\infty, 0], \{0\}, [0, \infty)\}.$$

The corresponding toric variety is just $X = \mathbb{P}^1$. Under the orbit-cone correspondence, we have

$$D_{\{[0,\infty)\}} = [0], \quad D_{\{(-\infty,0]\}} = [\infty].$$

The canonical divisor $D = [\infty]$ and therefore,

$$L = \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}(1).$$

5.2 Toric plurisubharmonic functions

We continue to use the notations of Section 5.1.

Lemma 5.2.1 *Let $F\colon N_{\mathbb{R}} \to [-\infty, \infty]$ be a function. Then the following are equivalent:*

(1) *F is convex and takes values in $\mathbb{R}$, and*
(2) *$\mathrm{Trop}^* F$ is plurisubharmonic on $T(\mathbb{C})$.*

Proof We may choose an identification $N \cong \mathbb{Z}^n$ so that we have an identification $T(\mathbb{C}) \cong \mathbb{C}^{*n}$. Then Trop is identified with the map

$$\mathrm{Trop}\colon \mathbb{C}^{*n} \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto \left(\log |z_1|^2, \dots, \log |z_n|^2\right).$$

(1) $\implies$ (2). Let $F_k \in C^\infty(\mathbb{R}^n) \cap \mathrm{Conv}(\mathbb{R}^n)$ be a decreasing sequence with limit F (see Proposition A.3.3). It follows from a straightforward computation that

$$\mathrm{dd^c}\,\mathrm{Trop}^* F_k(z_1,\dots,z_n) = \frac{\mathrm{i}}{2\pi}\sum_{i,j=1}^n \partial_{ij}F_k\left(\log|z_1|^2,\dots,\log|z_n|^2\right) z_i^{-1}\overline{z_j}^{-1}\mathrm{d}z_i\wedge\mathrm{d}\overline{z_j}. \tag{5.4}$$

So $\mathrm{Trop}^* F_k$ is plurisubharmonic. It follows from Proposition 1.2.1 that $\mathrm{Trop}^* F$ is plurisubharmonic.

(2) $\implies$ (1). It follows from Lemma 1.2.1 that F is finite. Moreover, take a radial mollifier, we may find a decreasing sequence φ_k of $(S^1)^n$-invariant smooth psh functions on $\mathbb{C}^{*n}$ with limit $\mathrm{Trop}^* F$. Write $\varphi_k = \mathrm{Trop}^* F_k$ for some function $F_k\colon \mathbb{R}^n \to \mathbb{R}$, it follows from (5.4) that F_k is convex for all k. Therefore, F is convex by Lemma A.1.2. □

Next we define a canonical Kähler form in $c_1(L)$.

Let $G_0\colon M_{\mathbb{R}} \to (-\infty,\infty]$ be defined as

$$G_0(m) := \begin{cases} \sum_{\rho\in\Sigma(1)} \left(\langle m,u_\rho\rangle + a_\rho\right)\log\left(\langle m,u_\rho\rangle + a_\rho\right)^{4}, & \text{if } m\in P,\\ \infty, & \text{otherwise.}\end{cases} \tag{5.5}$$

This is a closed proper convex function and $G_0 \sim \chi_P$, where $\sim$ is the relation defined in Definition A.1.8.

Let

$$F_0 = G_0^* \in \mathcal{E}^\infty(N_{\mathbb{R}}, P). \tag{5.6}$$

Recall that G_0^* is the Legendre transform of G_0, as recalled in Definition A.2.1. The set $\mathcal{E}^\infty(N_{\mathbb{R}}, P)$ is defined in Definition A.3.1.

By Guillemin's theorem [Gui94, CDG03], $\mathrm{dd^c}\,\mathrm{Trop}^* F_0$ can be extended to a unique Kähler form ω in $c_1(L)$. The Kähler form ω is clearly T_c-invariant.

For each $\rho \in \Sigma(1)$, we write

$$r_\rho(m) = \log\left(\langle m,u_\rho\rangle + a_\rho\right) + 1, \quad m\in P.$$

It follows from (5.5) that

$$\nabla G_0(m) = \sum_{\rho\in\Sigma(1)} r_\rho(m)u_\rho. \tag{5.7}$$

Example 5.2.1 Let us move on with our favorite example Example 5.1.1. We continue to use the same notations. In this case,

$$G_0(m) = \begin{cases} m\log m + (1-m)\log(1-m), & \text{if } m\in[0,1],\\ \infty, & \text{otherwise.}\end{cases}$$

[4] We understand that $0\log 0 = 0$ in this expression.

The Legendre transform is given[5] by

$$F_0(n) = \log\left(1 + \mathrm{e}^n\right).$$

Composing with the tropicalization map, we find that

$$\omega|_{\mathbb{C}^*}(z) = \log\left(1 + |z|^2\right).$$

This is exactly the Fubini–Study metric as we have seen in Example 1.8.1.

Now we could explain one subtlety: In our expression (5.5), there is no factor $1/2$ before the sum, this is due to the presence of the square in our choice of the tropicalization map (5.2).

Let $\mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ denote the set of T_c-invariant ω-psh functions.

Theorem 5.2.1 *There are canonical bijections between the following three sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$*,*
(2) *the set* $\mathcal{P}(N_{\mathbb{R}}, P)$ *in Definition A.3.1, namely, the set of convex functions* $F\colon N_{\mathbb{R}} \to \mathbb{R}$ *satisfying* $F \leq \mathrm{Supp}_P$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G|_{M_{\mathbb{R}} \setminus P} \equiv \infty.$$

For the notion of closeness and properness, we refer to Definition A.1.2 and Definition A.1.7.

Proof The bijection between (2) and (3) is the classical Legendre duality. Given F as in (2), we construct $G = F^*$ and *vice versa*, see Proposition A.2.5.

The map from (1) to (2) is given as follows: Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$, since φ is T_c-invariant, we can find $f\colon N_{\mathbb{R}} \to [-\infty, \infty)$ such that

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^* f. \tag{5.8}$$

We then define $F = f + F_0$. Then $\mathrm{Trop}^* F \in \mathrm{PSH}(T(\mathbb{C}))$. By Lemma 5.2.1, $F(n)$ is finite for any $n \in N_{\mathbb{R}}$ and F is convex. Moreover, $F \leq \mathrm{Supp}_P$ since this holds for F_0.

Conversely, given a map $F \in \mathcal{P}(N_{\mathbb{R}}, P)$, then

$$\mathrm{Trop}^*(F - F_0) \in \mathrm{PSH}(T(\mathbb{C}), \omega|_{T(\mathbb{C})}).$$

It follows from Theorem 1.2.1 that this function can be extended uniquely to an ω-psh function on X. The uniqueness of the extension guarantees its T_c-invariance.

The two maps are clearly inverse to each other. □

[5] While reading an advanced mathematical textbook/paper, I usually tend to trust the author for their elementary computations. A few years ago, I was asked to present the result of a landmark paper written by two respected mathematicians on a conference. After spending a few days on the elementary integrals, I found out that all non-trivial constants in that paper were wrong. So I ask the readers to really verify this expression, if it is not obvious to you.

Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$, we will write F_φ and G_φ for the convex functions given by Theorem 5.2.1. From the proof, we have the following relations:

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^*(F_\varphi - F_0), \quad G_\varphi = F_\varphi^*. \tag{5.9}$$

Example 5.2.2 Let us take our favorite example Example 5.2.1 again. We will continue to use the same notations.

Recall that in Example 1.8.2 and Example 3.1.1, we constructed two S^1-invariant functions in $\mathrm{PSH}(X, \omega)$.

We begin with the function φ in Example 1.8.2. Recall that

$$\varphi(z) = \log \frac{|z|^2}{|z|^2 + 1}$$

for $z \in \mathbb{C}$. The function $f : \mathbb{R} \to \mathbb{R}$ in (5.8) is therefore

$$f(n) = \log \frac{\mathrm{e}^n}{1 + \mathrm{e}^n}.$$

Therefore, $F_\varphi : \mathbb{R} \to \mathbb{R}$ is

$$F_\varphi(n) = n.$$

Correspondingly, $G_\varphi : \mathbb{R} \to \mathbb{R}$ is

$$G_\varphi(m) = \begin{cases} 0, \text{ if } m = 1, \\ \infty, \text{ otherwise.} \end{cases}$$

Similarly, if ψ denote the function in Example 3.1.1, then the function f in (5.8) is

$$f(n) = \begin{cases} -\log(\mathrm{e}^n + 1) + (-\log(-n)) \vee (n+2), \text{ if } n < -\log 2, \\ 2 + \log \dfrac{\mathrm{e}^n}{1 + \mathrm{e}^n}, \text{ otherwise.} \end{cases}$$

Therefore,

$$F_\psi(n) = \begin{cases} (-\log(-n)) \vee (n+2), \text{ if } n < -\log 2, \\ 2 + n, \text{ otherwise.} \end{cases}$$

The Legendre transform is tricky to compute. Let λ be the large solution of $\log x = x - 2$. So $\lambda \approx 3.146$. The smaller solution is around $0.159 < \log 2 \approx 0.693$. It might be helpful to have a look at the poorly drawn picture Fig. 5.1.

It is immediate that $G_\psi(m) = -\infty$ unless $m \in [0, 1]$. Let us assume that $m \in [0, 1]$. Then

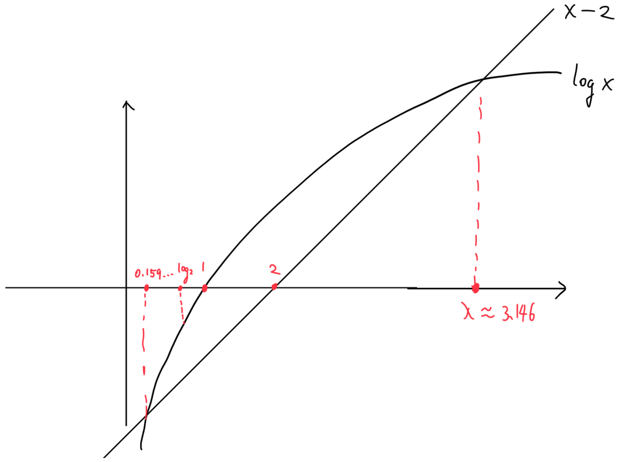

Fig. 5.1 The graphs of $\log x$ and $x - 2$.

$$
\begin{aligned}
G_\psi(m) &= \sup_{n\in\mathbb{R}} \big(mn - F_\psi(n)\big) \\
&= \sup_{n<-\log 2} \big(mn - (-\log(-n)) \vee (n+2)\big) \vee \sup_{n\geq -\log 2} (mn - n - 2) \\
&= \sup_{n>\log 2} \big(-mn + (\log n) \wedge (n-2)\big) \vee \big((1-m)\log 2 - 2\big)\,.
\end{aligned}
$$

Let us focus on the first part, which can be decomposed further into

$$
\begin{aligned}
&\sup_{n>\log 2} \big(-mn + (\log n) \wedge (n-2)\big) \\
={}& \sup_{n\in(\log 2,\lambda]} (n - 2 - mn) \vee \sup_{n>\lambda}(\log n - mn) \\
={}& ((1-m)\lambda - 2) \vee \sup_{n>\lambda}(\log n - mn).
\end{aligned}
$$

The latter part can be computed easily:

$$
\sup_{n>\lambda}(\log n - mn) = \begin{cases} -\log m - 1, \text{ if } m \in [0, \lambda^{-1}], \\ \log\lambda - m\lambda, \text{ if } m \in (\lambda^{-1}, 1]. \end{cases}
$$

Putting everything together, we find

$$
G_\psi(m) = \begin{cases} (-\log m - 1) \vee ((1-m)\lambda - 2)\,, \text{ if } m \in [0, \lambda^{-1}], \\ (\log\lambda - m\lambda) \vee ((1-m)\lambda - 2)\,, \text{ if } m \in (\lambda^{-1}, 1]. \end{cases}
$$

This can be further simplified, the final result is

$$G_\psi(m) = \begin{cases} -\log m - 1, \text{ if } m \in [0, \lambda^{-1}], \\ (1-m)\lambda - 2, \text{ if } m \in (\lambda^{-1}, 1], \\ \infty, \text{ otherwise.} \end{cases}$$

The graph of G_ψ on $(0, 1]$ is sketched in Fig. 5.2.

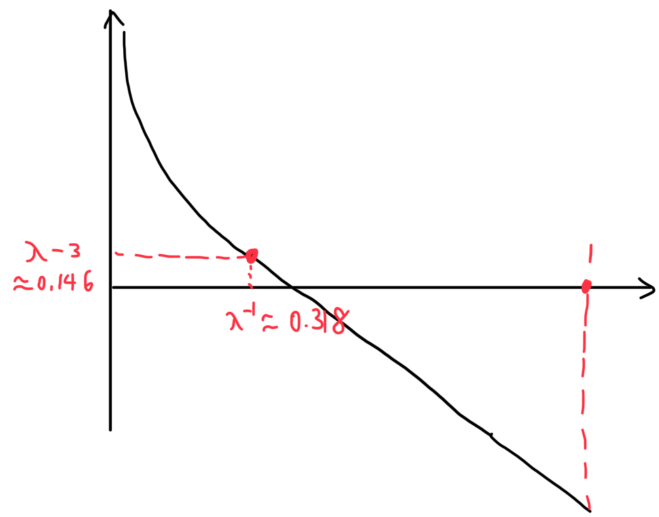

Fig. 5.2 The graph of G_ψ.

We observe a few elementary facts.

Proposition 5.2.1 *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. The following are equivalent:*

(1) $\varphi \le \psi$,
(2) $F_\varphi \le F_\psi$, *and*
(3) $G_\psi \le G_\varphi$.

The same holds if we replace all $\le$'s by $\preceq$.

In particular, $\varphi \in \mathcal{E}^\infty(X, \omega)$ if and only if $F_\varphi \in \mathcal{E}^\infty(N_\mathbb{R}, P)$.

Proof The equivalence between (1) and (2) follows from the definition (5.9). The equivalence between (2) and (3) follows from the definition of the Legendre transform. □

Similarly, we have

Proposition 5.2.2 *Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ and $C \in \mathbb{R}$. We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

Proposition 5.2.3 *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ with $\varphi \wedge \psi \not\equiv -\infty$, then $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and*

$$F_{\varphi\wedge\psi} = F_\varphi \wedge F_\psi, \quad G_{\varphi\wedge\psi} = G_\varphi \vee G_\psi.$$

The operators $\wedge$ and $\vee$ are defined in Definition A.1.5 and Definition A.1.6.

Proof It is clear that $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. So $\varphi \wedge \psi$ is the biggest element in $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ which id dominated by both φ and ψ. In view of Theorem 5.2.1 and Proposition 5.2.1, $G_{\varphi\wedge\psi}$ is the smallest closed proper convex function G on $M_{\mathbb{R}}$ dominating both G_φ and G_ψ, which is just $G_\varphi \vee G_\psi$.

The claim for F follows from Proposition A.2.3. □

Example 5.2.3 Now we can give an example of $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ with $\varphi\wedge\psi \equiv -\infty$.

We take $P = [0,1]$ so that $X = \mathbb{P}^1$ and ω is the Fubini–Study metric. Let $\varphi \in \mathrm{PSH}(X,\omega)$ be such that

$$\varphi(z) = \log \frac{|z|^2}{|z|^2+1}$$

for $z \in \mathbb{C}$. We have computed that G_φ in Example 5.2.2:

$$G_\varphi(m) = \begin{cases} 0, \text{ if } m = 1, \\ \infty, \text{ otherwise.} \end{cases}$$

Now we define $\psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ as the unique function such that

$$\psi(z) = \log \frac{1}{|z|^2+1}$$

for $z \in \mathbb{C}$. Then a similar computation shows that

$$G_\psi(m) = \begin{cases} 0, \text{ if } m = 0, \\ \infty, \text{ otherwise.} \end{cases}$$

Now we claim that $\varphi \wedge \psi \equiv -\infty$. Otherwise, we would have

$$G_{\varphi\vee\psi} = G_\varphi \vee G_\psi \equiv \infty,$$

which is not proper.

Proposition 5.2.4 *Let $\{\varphi_i\}_{i\in I}$ be a non-empty family in $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ uniformly bounded from above. Then ${\sup_{i\in I}}^* \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and*

$$F_{{\sup_{i\in I}}^*\varphi_i} = \bigvee_{i\in I} F_{\varphi_i}, \quad G_{{\sup_{i\in I}}^*\varphi_i} = \mathrm{cl} \bigwedge_{i\in I} G_{\varphi_i}.$$

Moreover, if I is finite, then

$$G_{\max_{i\in I}\varphi_i} = \bigwedge_{i\in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i\in I}$ *is a decreasing net in* $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *such that* $\inf_{i\in I}\varphi_i \not\equiv -\infty$, *then* $\inf_{i\in I}\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and*

$$F_{\inf_{i\in I}\varphi_i} = \inf_{i\in I} F_{\varphi_i}, \quad G_{\inf_{i\in I}\varphi_i} = \bigvee_{i\in I} G_{\varphi_i}.$$

Recall that the closure cl is defined in Definition A.1.7.

Proof Thanks to Lemma A.1.2 and Proposition A.1.1, in both cases, the statement for F is clear. The corresponding statement for G is obtained via Proposition A.2.3. □

The complex Monge–Ampère operator is closely related to the real one:

Proposition 5.2.5 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, *then*

$$\mathrm{Trop}_* \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi). \tag{5.10}$$

In particular,

$$\int_X \omega_\varphi^n = \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F_\varphi) = n!\,\mathrm{vol}\,\overline{\{G_\varphi < \infty\}}$$

and

$$\int_X \omega^n = n!\,\mathrm{vol}\,P.$$

Here the real Monge–Ampère operator is defined in Definition A.4.1. The normalization of the Lebesgue measure vol on $M_{\mathbb{R}}$ is such that the fundamental lattice cube as measure 1.

Proof We only need to prove (5.10). By Proposition A.3.3, we can find a decreasing sequence of smooth convex functions F_j on $N_{\mathbb{R}}$ with limit F_φ. We write $F_j = F_{\varphi_j}$ for some $\varphi_j \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. By Theorem 2.1.1 and Theorem A.4.1, it suffices to establish (5.10) for the φ_j's. We may therefore reduce to the case where F_φ is smooth. We write $F = F_\varphi$ to simplify the notations. The notations $a_i = \log|z_i|^2$ will be used, where $i = 1,\dots,n$.

Next we fix an identification $N = \mathbb{Z}^n$. Fix a test function $f \in C_c^0(N_{\mathbb{R}})$, we need to show that

$$\int_{\mathbb{C}^{*n}} f(a_1,\dots,a_n)\,(\mathrm{dd^c}\,\mathrm{Trop}^*\,F(z_1,\dots,z_n))^n = \int_{\mathbb{R}^n} f\ \mathrm{MA}_{\mathbb{R}}(F).$$

Using Proposition A.4.1 and (5.4), this reduces to

$$\begin{split}\left(\frac{\mathrm{i}}{2\pi}\right)^n \int_{\mathbb{C}^{*n}} f(a_1,\dots,a_n)\left(\sum_{i,j=1}^n \partial_{i,j}F(a_1,\dots,a_n) z_i^{-1}\overline{z_j}^{-1}\mathrm{d}z_i\wedge\mathrm{d}\overline{z_j}\right)^n = \\ n!\int_{\mathbb{R}^n} f\det\nabla^2 F\,\mathrm{d}\,\mathrm{vol}\,.\end{split} \tag{5.11}$$

Expanding the bracket, we get

$$\left(\sum_{i,j=1}^{n} \partial_{i,j} F z_i^{-1} \overline{z_j}^{-1} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j}\right)^n = \sum_{i_1,\ldots,i_n=1}^{n} \sum_{j_1,\ldots,j_n=1}^{n} \partial_{i_1 j_1} F \cdots \partial_{i_n j_n} F \cdot \\ \mathrm{d}\log z_{i_1} \wedge \mathrm{d}\log \overline{z_{j_1}} \wedge \cdots \wedge \mathrm{d}\log z_{i_n} \wedge \mathrm{d}\log \overline{z_{j_n}},$$

where $\mathrm{d}\log z_i = z_i^{-1}\mathrm{d}z_i$ and $\mathrm{d}\log \overline{z_i} = \overline{z_i}^{-1}\mathrm{d}\overline{z_i}$ are understood.

Using the apparent symmetry, the expression on the right-hand side becomes

$$\begin{aligned}
&\sum_{\sigma,\tau\in\mathfrak{S}_n} \prod_{k=1}^{n} \partial_{\sigma(k)\tau(k)} F \,\mathrm{d}\log z_{\sigma(1)} \wedge \mathrm{d}\log \overline{z_{\tau(1)}} \wedge \cdots \wedge \mathrm{d}\log z_{\sigma(n)} \wedge \mathrm{d}\log \overline{z_{\tau(n)}},\\
=&n! \sum_{\tau\in\mathfrak{S}_n} \prod_{k=1}^{n} \partial_{k\tau(k)} F \,\mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_{\tau(1)}} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_{\tau(n)}}\\
=&n! \sum_{\tau\in\mathfrak{S}_n} (-1)^{\mathrm{Sign}\,\tau} \prod_{k=1}^{n} \partial_{k\tau(k)} F \,\mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_1} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_n}\\
=&n! \det \nabla^2 F \mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_1} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_n},
\end{aligned}$$

where $\mathfrak{S}_n$ is the permutation group on $\{1,\ldots,n\}$ and $\mathrm{Sign}(\tau)$ is the sign of τ.

Next, switch to polar coordinates for each z_i: Let $z_i = r_i \exp(\mathrm{i}\theta_i)$ and recall that $r_i = \exp(a_i/2)$, then the left-hand side of (5.11) becomes

$$\begin{aligned}
&\frac{n!}{(2\pi)^n} \int_{\mathbb{R}^n\times[0,2\pi)^n} f \det \nabla^2 F \,\mathrm{d}a_1 \wedge \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}a_n \wedge \mathrm{d}\theta_n\\
=&n! \int_{\mathbb{R}^n} f \det \nabla^2 F \,\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n,
\end{aligned}$$

which is exactly what we have expected. □

Next we study the envelope operators developed in Chapter 3 in the toric setting.

Definition 5.2.1 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. We define its *Newton body* as

$$\Delta(\omega,\varphi) \coloneqq \overline{\{G_\varphi < \infty\}} \subseteq P.$$

Note that $\Delta(\omega,\varphi)$ is a convex body.

By Proposition A.2.2, we have

$$\Delta(\omega,\varphi) = \overline{\nabla F_\varphi(N_{\mathbb{R}})}.$$

Example 5.2.4 By (5.5), we have

$$\Delta(\omega,0) = P.$$

In the case of Example 5.2.2, we have

$$\Delta(\omega, \varphi) = \{1\}, \quad \Delta(\omega, \psi) = [0, 1].$$

Observe that in the latter case,

$$\{G_\varphi < \infty\} \subsetneq P.$$

Proposition 5.2.6 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Then $P_\omega[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$ and*

$$G_{P_\omega[\varphi]}(x) = \begin{cases} G_0(x), & \text{if } x \in \Delta(\omega, \varphi); \\ \infty, & \text{otherwise.} \end{cases} \tag{5.12}$$

Proof By (3.3), we have

$$P_\omega[\varphi] = \sup_{C \in \mathbb{R}}{}^* \left((\varphi + C) \wedge 0\right).$$

It follows from Proposition 5.2.2, Proposition 5.2.3 and Proposition 5.2.4 that $P_\omega[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Moreover, by the same propositions, we have

$$G_{P_\omega[\varphi]} = \mathrm{cl} \inf_{C \in \mathbb{R}} \left(G_0 \vee (G_\varphi - C)\right),$$

which is clearly equal to the right-hand side of (5.12).

Recall that $\mathrm{H}^0(X, L)$ can be identified with the vector space generated by χ^m for all $m \in P \cap M$, see [CLS11, Proposition 4.3.3]. In other words, a character χ^m of T can be extended to a regular function on X if and only if $m \in P$. This gives a beautiful characterization of the lattice points in P. The following theorem of Yi Yao gives an analogous characterization of the lattice points in the Newton body.

Theorem 5.2.2 (Yao) *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Given $m \in M$, the corresponding character χ^m can be extended to a section in $\mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$ if $m \in \Delta(\omega, \varphi) \cap M$.*

Fix a norm on $N_{\mathbb{R}}$. There is a constant $C_0 > 0$ depending only on n and the norm such that for any $m \in M \cap (P \setminus \Delta(\omega, \varphi))$, if there is $n_0 \in N_{\mathbb{R}}$ such that

$$\langle m, n_0 \rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n_0) > C_0 |n_0|,$$

then $\chi^m \notin \mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$.

Proof It is convenient to use explicit coordinates. We will identify N with $\mathbb{Z}^n$ after choosing a basis. In this way, we get an identification $M = \mathbb{Z}^n$ and $T(\mathbb{C}) = \mathbb{C}^{*n}$. In this case, we have

$$\chi^m(z) = z^m$$

with the multi-index notation.

Observe that $\mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$ is a $\mathbb{C}^{*n}$-invariant subspace of $\mathrm{H}^0(X, L)$, it follows that $\mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$ is the direct sum of suitable χ^m's. Due to Proposition 3.2.8, we may replace φ by $P_\omega[\varphi]$ and thanks to Proposition 5.2.6, we may assume that G_φ has the following form:

$$G_\varphi(x) = \begin{cases} G_0(x), & \text{if } x \in \Delta(\omega,\varphi); \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, $F_\varphi \sim \mathrm{Supp}_{\Delta(\omega,\varphi)}$.

Now given $m \in M \cap P$, we need to know whether the following expression is finite or not:

$$\int_{\mathbb{C}^{*n}} |\chi^m|^2 \exp(-\operatorname{Trop}^* F_0 - \varphi)\, \omega^n. \tag{5.13}$$

By Proposition 5.2.5, (5.13) is finite if and only if the following integral is finite:

$$\int_{\mathbb{R}^n} \exp\left(\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n)\right) \mathrm{MA}_{\mathbb{R}}(F_0)(n).$$

By a change of variable, this integral is finite if and only if the following integral is:

$$\int_P \exp\left(\langle m, \nabla G_0(m')\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(\nabla G_0(m'))\right) \mathrm{d}m'. \tag{5.14}$$

Suppose that $m \in \Delta(\omega,\varphi)$, then the integrand in (5.14) is bounded from above by e, so we are done.

Next suppose that $m \notin \Delta(\omega,\varphi)$. Suppose that we can find $n_0 \in \mathbb{R}^n$ such that

$$\langle m, n_0\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n_0) > C_0|n_0|.$$

In particular, there is a closed convex cones C containing n_0 in their interiors such that there exists $\epsilon > 0$ such that

$$\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n) \geq C_0|n|$$

for all $n \in C$.

Thus, it would suffice to prove

$$\int_{P\cap\{\nabla G_0 \subseteq C\}} \exp\left(C_0|\nabla G_0(m')|\right) \mathrm{d}m' = \infty. \tag{5.15}$$

Take a cone σ in Σ such that $n_0 \in -\operatorname{RelInt}\sigma$. Let $\rho_1,\ldots,\rho_a$ be the minimal number of rays of σ such that n_0 lies in the closed convex cone they generated. Then $u_{\rho_1},\ldots,u_{\rho_a}$ are linearly independent. We may find rays $\rho_{a+1},\ldots,\rho_n \in \Sigma(1)$ such that $u_{\rho_1},\ldots,u_{\rho_n}$ form a basis of $\mathbb{R}^n$.

Taking the form (5.7) of ∇G_0 into account, we find that there is a subset of $P\cap\{\nabla G_0 \subseteq C\}$ given by those $m' \in P$ such that for all $\rho \in \Sigma(1)$ different from $\rho_1,\ldots,\rho_a$, the function $r_\rho(m')$ is uniformly bounded, while m' is close enough to the faces corresponding to the rays $\rho_1,\ldots,\rho_n$ and $\sum_{i=1}^a r_{\rho_i}(m')u_{\rho_i} \in C$. Replacing the domain of integration in (5.15) by this region, we conclude that the integral (5.15) diverges when C_0 is large enough. □

Corollary 5.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and $\int_X \omega_\varphi^n > 0$, then*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) = n! \operatorname{vol} \Delta(\omega, \varphi).$$

Example 5.2.5 In general, in the setup of Theorem 5.2.2, there exists $m \in M \cap (P \setminus \Delta(\omega, \varphi))$ such that $\chi^m \in \mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$.

As a concrete example, let us take $P = [0, 1]$. Take φ so that $\Delta(\omega, \varphi) = [0, 1/2]$. We claim that χ^1 is L^2-integrable.

It suffices to verify the convergence of (5.14). Recall that

$$\nabla G_0(m') = \log \frac{m'}{1 - m'}, \quad m' \in [0, 1],$$

while

$$\mathrm{Supp}_{[0,1/2]}(a) = \begin{cases} a/2, \text{ if } a > 0; \\ \quad 0, \text{ otherwise.} \end{cases}$$

Therefore, (5.14) becomes

$$\int_0^{1/2} \frac{m'}{1 - m'} \, \mathrm{d}m' + \int_{1/2}^1 \left(\frac{m'}{1 - m'} \right)^{1/2} \mathrm{d}m' < \infty.$$

We interpret various classes of potentials studied in Section 3.1.3 in the toric setting.

Proposition 5.2.7 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^\infty(X, \omega)$;
(2) $F_\varphi \in \mathcal{E}^\infty(N_\mathbb{R}, P)$;
(3) $G_\varphi \sim G_0$.

The notation $\mathcal{E}^\infty(N_\mathbb{R}, P)$ is defined in Definition A.3.1.

Proof This follows immediately from Proposition 5.2.1. □

Proposition 5.2.8 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}(X, \omega)$;
(2) $F_\varphi \in \mathcal{E}(N_\mathbb{R}, P)$;
(3) $\overline{\mathrm{Dom}\, G_\varphi} = P$.

The notation $\mathcal{E}(N_\mathbb{R}, P)$ is defined in Definition A.3.1.

Proof (1) $\iff$ (3). By Proposition 5.2.5

$$\int_X \omega_\varphi^n = \int_{T(\mathbb{C})} \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c} \varphi|_{T(\mathbb{C})} \right)^n = n! \operatorname{vol} \overline{\mathrm{Dom}\, G_\varphi}, \quad \int_X \omega^n = n! \operatorname{vol} P.$$

Therefore, (1) and (3) are equivalent.

(2) $\iff$ (3). This follows from Proposition A.2.2. □

Proposition 5.2.9 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$*, then*

$$E_\omega(\varphi) = n! \int_P \left(G_0 - G_\varphi\right) \,\mathrm{d}\,\mathrm{vol}\,.$$

Proof It suffices to consider the case where φ is bounded. In this case, one could apply [BB13, Proposition 2.9]. □

Corollary 5.2.2 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$*. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X, \omega)$*;*
(2) $F_\varphi \in \mathcal{E}^1(N_{\mathbb{R}}, P)$*;*
(3) $G_\varphi \in L^1(P)$.

The notation $\mathcal{E}^1(N_{\mathbb{R}}, P)$ is defined in Definition A.3.1.

Definition 5.2.2 We define

$$\begin{aligned}
\mathcal{E}^\infty_{\mathrm{tor}}(X, \omega) =& \mathcal{E}^\infty(X, \omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X, \omega),\\
\mathcal{E}^1_{\mathrm{tor}}(X, \omega) =& \mathcal{E}^1(X, \omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X, \omega),\\
\mathcal{E}_{\mathrm{tor}}(X, \omega) =& \mathcal{E}(X, \omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X, \omega).
\end{aligned}$$

Corollary 5.2.3 *Let* $\varphi, \psi \in \mathcal{E}^1_{\mathrm{tor}}(X, \omega)$*, then*

$$d_1(\varphi, \psi) = -n! \int_P \left(G_\varphi + G_\psi - 2G_{\varphi \vee \psi}\right) \,\mathrm{d}\,\mathrm{vol}\,.$$

Proof This follows from (5.2.9), Proposition 5.2.3 and Definition 4.2.5. □

Proposition 5.2.10 *Let* $\varphi_0, \varphi_1 \in \mathcal{E}^1_{\mathrm{tor}}(X, \omega)$*. The geodesic* $(\varphi_t)_{t \in (0,1)}$ *from* φ_0 *to* φ_1 *satisfies the following: For each* $t \in (0, 1)$*,* $\varphi_t \in \mathcal{E}^1_{\mathrm{tor}}(X, \omega)$ *and*

$$G_{\varphi_t} = (1 - t)G_{\varphi_0} + tG_{\varphi_1}.$$

This will be proved more generally in Corollary 12.3.4.

Definition 5.2.3 We define

$$\mathcal{R}^1_{\mathrm{tor}}(X, \omega) := \left\{\ell \in \mathcal{R}^1(X, \omega) : \ell_t \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega) \text{ for all } t \geq 0\right\}.$$

Part II
The theory of $\mathcal{I}$-good singularities

This part is the technical core of the whole book. We will develop the theory of $\mathcal{I}$-good singularities.

We first develop some general techniques to compare the singularities in Chapter 6: The P-partial order, the $\mathcal{I}$-partial order and the d_S-pseudometric.

The P-partial order seems to be new. Some basic properties of the d_S-pseudometric have never appeared in the literature either.

Then in Chapter 7, we introduce the notion of $\mathcal{I}$-good singularities and characterize $\mathcal{I}$-good singularities in different ways. Then we establish the asymptotic Riemann–Roch formula for Hermitian pseudoeffective line bundles.

In Chapter 8, we develop two key techniques in the inductive study of singularities: The trace operator and the analytic Bertini theorem. Roughly speaking, the latter tells us the behavior of a quasi-plurisubharmonic function along a general divisor, while the former handles the case of special divisors. We will establish a relative version of the asymptotic Riemann–Roch formula as well.

In Chapter 9, we develop the theory of test curves. These are curves of model potentials. The key technique is the Ross–Witt Nyström correspondence, which relates test curves with geodesic rays. The complete proof of the most general form of this correspondence has never appeared in the literature, so we will give the full details.

In Chapter 10, we develop the theory of partial Okounkov bodies, in both algebraic and transcendental setting. The partial Okounkov bodies can be regarded as non-toric extensions of the Newton bodies. It turns out that even in the toric setting, our techniques give non-trivial new results.

In Chapter 11, we develop the theory of b-divisors in the algebraic setting. We formulate the general form of the Chern–Weil formula in terms of b-divisors. We also relate the theory of partial Okounkov bodies to b-divisors.

Chapter 6
Comparison of singularities

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.
— Michael Atiyah[a]

[a] Sir Michael Francis Atiyah (1929–2019) wrote the influential *Introduction to commutative algebra* together with I. G. MacDonald, a poor guy whose name is often omitted or misspelled.

In this chapter, we study several ways of comparing the singularities of quasi-plurisubharmonic functions. In Section 6.1, we will introduce the P and $\mathcal{I}$-partial orders, closely related to the P and $\mathcal{I}$-equivalence relations introduced in Chapter 3.

In Section 6.2, we introduce and study the d_S-pseudometric characterizing the differences between singularities. We will prove that a number of continuity results with respect to d_S.

6.1 The P and $\mathcal{I}$-partial orders

Let X be a connected compact Kähler manifold of dimension n.

Recall that we have defined a (non-strict) partial order on $\mathrm{QPSH}(X)$ in Definition 1.5.2 to compare the singularity types of quasi-plurisubharmonic functions. The problem with this partial order is that it is too fine. In general, for our interest, it is helpful to consider rougher relations.

6.1.1 The definitions of the partial orders

Recall that the P-envelope is defined in Definition 3.1.2.

Definition 6.1.1 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is *P-more singular than* ψ and write $\varphi \preceq_P \psi$ if for some closed smooth real $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, we have

$$P_\theta[\varphi] \leq P_\theta[\psi]. \tag{6.1}$$

Suppose that $\varphi \preceq_P \psi$ and $\psi \preceq_P \varphi$, we shall write $\varphi \sim_P \psi$ and say φ and ψ have the same *P-singularity type*.

The condition (6.1) is independent of the choice of θ:

Lemma 6.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. For any Kähler form ω on X, the following are equivalent:*

(1) $P_\theta[\varphi] \leq P_\theta[\psi]$;
(2) $P_{\theta+\omega}[\varphi] \leq P_{\theta+\omega}[\psi]$.

In particular, $\preceq_P$ defines a non-strict partial order on $\mathrm{QPSH}(X)$.

Proof (1) $\implies$ (2). Observe that

$$\varphi \leq P_\theta[\varphi] \leq P_{\theta+\omega}[\varphi].$$

It follows from Theorem 3.1.1 that

$$P_{\theta+\omega}[\varphi] = P_{\theta+\omega}[P_\theta[\varphi]]. \tag{6.2}$$

A similar formula holds for ψ. So we see that (2) holds.

(2) $\implies$ (1). By (6.2), we may assume that φ and ψ are both model potentials in $\mathrm{PSH}(X,\theta)_{>0}$.

Observe that $\varphi \vee \psi \leq P_{\theta+\omega}[\psi]$. It follows that $P_{\theta+\omega}[\varphi \vee \psi] \leq P_{\theta+\omega}[\psi]$. The reverse inequality is trivial, so

$$P_{\theta+\omega}[\varphi \vee \psi] = P_{\theta+\omega}[\psi].$$

From the direction we have proved, for any $C \geq 1$,

$$P_{\theta+C\omega}[\varphi \vee \psi] = P_{\theta+C\omega}[\psi].$$

So by Proposition 3.1.3,

$$\int_X \left(\theta + C\omega + \mathrm{dd^c}(\varphi \vee \psi)\right)^n = \int_X \left(\theta + C\omega + \mathrm{dd^c}\psi\right)^n .$$

Since both sides are polynomials in C, the equality extends to $C = 0$, namely,

$$\int_X \theta_{\varphi\vee\psi}^n = \int_X \theta_\psi^n .$$

In particular, $\varphi \vee \psi \leq P_\theta[\psi] = \psi$ by (3.5). So (1) follows. □

As a first example of P-equivalence, we have:

Example 6.1.1 Let θ be a closed smooth real $(1,1)$-form on X and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then

$$\varphi \sim_P P_\theta[\varphi].$$

This follows immediately from Theorem 3.1.1.

We give a very useful criterion of the P-equivalence in terms of the non-pluripolar masses.

Proposition 6.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ and $\varphi \leq \psi$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) *for each $j = 0, \ldots, n$, we have*

$$\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}. \tag{6.3}$$

Assume furthermore that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$, then these conditions are equivalent to the following:

(3) *We have*

$$\int_X \theta_\varphi^n = \int_X \theta_\psi^n.$$

Recall that V_θ is introduced in (2.9).

Proof We first prove the equivalence between (1) and (3) when $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$.

(1) $\implies$ (3). Assume that $\varphi \sim_P \psi$. By Lemma 6.1.1, we have

$$P_\theta[\varphi] = P_\theta[\psi].$$

So (3) follows from Proposition 3.1.3.

(3) $\implies$ (1). It follows from Theorem 3.1.1 that $P_\theta[\varphi] = P_\theta[\psi]$, so (1) follows.

Let us come back to the general case.

(1) $\implies$ (2). Fix $j \in \{0, \ldots, n\}$, we argue (6.3).

Take a Kähler form ω on X. By Lemma 6.1.1, for each $\epsilon > 0$, we have

$$P_{\theta+\epsilon\omega}[\varphi] = P_{\theta+\epsilon\omega}[\psi].$$

It follows from Proposition 3.1.3 that

$$\begin{aligned}
\int_X (\theta + \epsilon\omega + \mathrm{dd^c}\psi)^j \wedge \theta_{V_\theta}^{n-j} &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\psi])^j \wedge \theta_{V_\theta}^{n-j} \\
&= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi])^j \wedge \theta_{V_\theta}^{n-j} \\
&= \int_X (\theta + \epsilon\omega + \mathrm{dd^c}\varphi)^j \wedge \theta_{V_\theta}^{n-j}.
\end{aligned}$$

Since the two extremes are both polynomials in ϵ, we conclude that the same holds when $\epsilon = 0$, that is, (6.3) holds.

(2) $\implies$ (1). Assume (6.3) holds for all $j = 0, \ldots, n$. For each $t \in (0, 1)$, we have

$$\int_X \theta_{t\varphi+(1-t)V_\theta}^n = \int_X \theta_{t\psi+(1-t)V_\theta}^n$$

by the binomial expansion. By the implication (3) $\implies$ (1), we have

$$t\varphi + (1-t)V_\theta \sim_P t\psi + (1-t)V_\theta$$

for each $t \in (0,1)$.

Fix a Kähler form ω on X. From the implication (1) $\implies$ (3), we have

$$\int_X (\theta+\omega)^n_{t\varphi+(1-t)V_\theta} = \int_X (\theta+\omega)^n_{t\psi+(1-t)V_\theta}.$$

Since both sides are polynomials in t, the same holds when $t=1$. From the implication (3) $\implies$ (1) again, we have $\varphi \sim_P \psi$. □

Proposition 6.1.2 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *For any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) \subseteq \mathcal{I}(k\psi);$$

(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) \subseteq \mathcal{I}(\lambda\psi);$$

(3) *for any modification $\pi\colon Y \to X$ and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y);$$

(4) *for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y);$$

(5) *for any prime divisor E over X, we have*
$$\nu(\varphi, E) \geq \nu(\psi, E).$$

Proof The proof is almost identical to that of Proposition 3.2.1. □

Definition 6.1.2 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is $\mathcal{I}$*-more singular than* ψ and write $\varphi \preceq_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 6.1.2 are satisfied.

It is clear that $\preceq_{\mathcal{I}}$ is a non-strict partial order on $\mathrm{QPSH}(X)$.

Note that $\varphi \preceq_{\mathcal{I}} \psi$ and $\psi \preceq_{\mathcal{I}} \varphi$ both hold if and only if $\varphi \sim_{\mathcal{I}} \psi$ in the sense of Definition 3.2.1.

Lemma 6.1.2 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, then*

$$P_\theta[\varphi \vee \psi] = P_\theta\left[P_\theta[\varphi] \vee P_\theta[\psi]\right]. \tag{6.4}$$

Proof Since $\varphi \vee \psi \leq P_\theta[\varphi] \vee P_\theta[\psi]$, the $\leq$ direction of (6.4) follows. Conversely, it suffices to show that

$$P_\theta[\varphi \vee \psi] \geq P_\theta[\varphi] \vee P_\theta[\psi],$$

which is obvious. □

Lemma 6.1.3 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Then the following are equivalent:*

(1) $\varphi \preceq_P \psi$ (resp. $\varphi \preceq_{\mathcal{I}} \psi$);
(2) $\varphi \vee \psi \sim_P \psi$ (resp. $\varphi \vee \psi \sim_{\mathcal{I}} \psi$).

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. We only prove the P case, the $\mathcal{I}$ case is similar.

(2) $\implies$ (1). By (2) and Example 6.1.1, $P_\theta[\varphi \vee \psi] = P_\theta[\psi] \sim_P \psi$. But $\varphi \leq P_\theta[\varphi \vee \psi]$, so (1) follows.

(1) $\implies$ (2). We may assume that φ, ψ are both model by Lemma 6.1.2. Then $\varphi \leq \psi$ and (2) follows. □

Corollary 6.1.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Assume that $\varphi \preceq_P \psi$, then $\varphi \preceq_{\mathcal{I}} \psi$.*

Proof This follows from Lemma 6.1.3 and Proposition 3.2.8. □

Next we give a few extra characterizations of the P-envelope.

Corollary 6.1.2 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then*

$$\begin{aligned} P_\theta[\varphi] &= \sup \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim_P \varphi\} \\ &= \sup \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq_P \varphi\}. \end{aligned}$$

Just for comparison, let us recall a few other characterizations of the P-envelope for $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$:

$$\begin{aligned} P_\theta[\varphi] &= \sup{}^* \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq \varphi\} \\ &= \sup{}^* \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim \varphi\} \\ &= \sup_{C \in \mathbb{Z}_{>0}}{}^* (\varphi + C) \wedge V_\theta \\ &= \sup \left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \varphi \preceq \psi, \int_X \theta_\varphi^n = \int_X \theta_\psi^n \right\}. \end{aligned}$$

Proof Note that $\psi \sim_P \varphi$ implies that $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ by Proposition 6.1.4. We observe that

$$\begin{aligned} &\sup \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim_P \varphi\} \\ =&\sup \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \varphi \leq \psi, \psi \sim_P \varphi\} \end{aligned}$$

by Lemma 6.1.3. So the first equality is a direct consequence of Proposition 6.1.1 and Theorem 3.1.1.

Next we prove the second equality. We only need to show that for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \leq 0$ and $\psi \preceq_P \varphi$, we have $\psi \leq P_\theta[\varphi]$.

By Lemma 6.1.3 and Example 6.1.1, we know that $P_\theta[\varphi] \vee \psi \sim_P \varphi$ and $P_\theta[\varphi] \vee \psi \leq 0$. It follows from the first equality that $\psi \leq P_\theta[\varphi]$. □

Similarly, we have a new characterization of the $\mathcal{I}$-envelope.

Corollary 6.1.3 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi]_{\mathcal{I}} = \sup\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq_{\mathcal{I}} \varphi\}.$$

Proof It suffices to show that for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \leq 0$ and $\psi \preceq_{\mathcal{I}} \varphi$, we have $\psi \leq P_\theta[\varphi]_{\mathcal{I}}$. By Lemma 6.1.3 and Proposition 3.2.6, we know that $P_\theta[\varphi]_{\mathcal{I}} \vee \psi \sim_{\mathcal{I}} \varphi$. Therefore,

$$\psi \leq P_\theta[\varphi]_{\mathcal{I}} \vee \psi \leq P_\theta[\varphi]_{\mathcal{I}}.$$

Proposition 6.1.3 *Suppose that $\varphi, \psi \in \mathrm{QPSH}(X)$ and θ is a closed real smooth $(1,1)$-form on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \preceq_{\mathcal{I}} \psi$;
(2) $P_\theta[\varphi]_{\mathcal{I}} \leq P_\theta[\psi]_{\mathcal{I}}$.

Proof (1) $\implies$ (2). This follows immediately from Corollary 6.1.3.
(2) $\implies$ (1). This follows from Proposition 3.2.6. □

Example 6.1.2 Let us continue our example Example 3.1.1, where $X = \mathbb{P}^1$, ω is the Fubini–Study metric and $\varphi \in \mathrm{PSH}(X,\omega)$ has log-log singularity at 0. We have shown that $P_\omega[\varphi] = 0$ in (3.7), so $\varphi \sim_P 0$ and hence $\varphi \sim_{\mathcal{I}} 0$. In particular, P-equivalence is not equivalent to the equivalence of singularity types.

On the other hand, consider a potential $\psi \in \mathrm{PSH}(X,\omega)$ with log singularity at 0, as in Example 1.8.2. We know that $\nu(\psi, 0) = 1$ from the explicit expression (1.19). So $\psi \nsim_{\mathcal{I}} 0$ and hence $\psi \nsim_P 0$.

Moreover, $\psi \preceq_P \varphi$ and hence $\psi \preceq_{\mathcal{I}} \varphi$.

We give an example showing that P-equivalence is not equivalent to $\mathcal{I}$-equivalence.

Example 6.1.3 Let $X = \mathbb{P}^1$ and ω be the Fubini–Study metric. Let $K \subseteq \mathbb{P}^1$ be a polar Cantor sets carrying an atom free probability measure μ supported on K (see [Car83, Page 31]). Write $\mu = \omega + \mathrm{dd}^{\mathrm{c}}\varphi$ for some ω-subharmonic function φ. Since μ is atom free, we know that all Lelong numbers of φ are 0. On the other hand, φ has 0 non-pluripolar mass since K is pluripolar.

Then observe that $\varphi \sim_{\mathcal{I}} 0$ while $\varphi \nsim_P 0$.

For later use, we give the following definition.

Definition 6.1.3 Let L be a pseudoeffective line bundle on X. An *elementary metric* on L is a psh metric h on L such that there is a generalized Fubini–Study metric h' on L such that

$$\mathrm{dd}^{\mathrm{c}}h \sim_P \mathrm{dd}^{\mathrm{c}}h'.$$

The set of elementary metrics on L is denoted by $\mathrm{Ele}(L)$.

We also say $\mathrm{dd}^{\mathrm{c}}h$ is elementary. If we have fixed a Hermitian metric h_0 on L, and if we represent h as $h_0\exp(-\varphi)$, we also say the quasi-psh function φ is elementary.

Recall that the generalized Fubini–Study metrics are defined in Definition 1.8.7.

6.1.2 Properties of the partial orders

Now we state a more natural version of the monotonicity theorem Theorem 2.3.2.

Proposition 6.1.4 *Let $\theta_1, \ldots, \theta_n$ be closed real smooth $(1,1)$-forms on X. Let $\varphi_i, \psi_i \in \mathrm{PSH}(X, \theta_i)$ for $i = 1, \ldots, n$. Assume that $\varphi_i \preceq_P \psi_i$ for each i. Then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} \leq \int_X \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{n,\psi_n}.$$

Proof Fix a Kähler form ω on X. For each $i = 1, \ldots, n$, since $\varphi_i \preceq_P \psi_i$, we have

$$P_{\theta_i+\epsilon\omega}[\varphi_i] \leq P_{\theta_i+\epsilon\omega}[\psi_i]$$

for all $\epsilon > 0$. Therefore, by Proposition 3.1.3 and Theorem 2.3.2, we have

$$\int_X (\theta_1 + \epsilon\omega)_{\varphi_1} \wedge \cdots \wedge (\theta_n + \epsilon\omega)_{\varphi_n} \leq \int_X (\theta_1 + \epsilon\omega)_{\psi_1} \wedge \cdots \wedge (\theta_n + \epsilon\omega)_{\psi_n}.$$

Letting $\epsilon \to 0+$, we find the desired inequality. □

Next we show that the P and $\mathcal{I}$-partial orders are preserved by some natural operations.

Proposition 6.1.5 *Let $\varphi, \psi, \varphi', \psi' \in \mathrm{QPSH}(X)$. Assume that*

$$\varphi \preceq_P \psi, \quad \varphi' \preceq_P \psi'.$$

Then

$$\varphi + \varphi' \preceq_P \psi + \psi'.$$

The same holds with $\preceq_{\mathcal{I}}$ in place of $\preceq_P$.

Proof Take a Kähler form ω on X such that $\varphi, \psi, \varphi', \psi' \in \mathrm{PSH}(X, \omega)_{>0}$. The statement for $\preceq_{\mathcal{I}}$ is a simple consequence of Proposition 1.4.2. We only need to handle the case of $\preceq_P$.

Step 1. We first show that

$$P_\omega[\varphi] + P_\omega[\varphi'] \sim_P \varphi + \varphi'.$$

In fact, we clearly have

$$P_\omega[\varphi] + P_\omega[\varphi'] \geq \varphi + \varphi'.$$

So by Proposition 6.1.1, it suffices to show that they have the same mass. We compute

$$
\begin{aligned}
&\int_X (2\omega + \mathrm{dd^c} P_\omega[\varphi] + \mathrm{dd^c} P_\omega[\varphi'])^n \\
&= \sum_{j=0}^{n} \binom{n}{j} \int_X (\omega + \mathrm{dd^c} P_\omega[\varphi])^j \wedge (\omega + \mathrm{dd^c} P_\omega[\varphi'])^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} \int_X \omega_\varphi^j \wedge \omega_{\varphi'}^{n-j} \\
&= \int_X (2\omega + \varphi + \varphi')^n ,
\end{aligned}
$$

where we applied Proposition 3.1.3 on the third line.

Step 2. By Step 1, we may assume that $\varphi, \psi, \varphi', \psi'$ are all model potentials. So $\varphi \leq \psi$ and $\varphi' \leq \psi'$. Our assertion follows. □

Proposition 6.1.6 *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be uniformly bounded from above non-empty families in* $\mathrm{QPSH}(X)$. *Assume that there exists a closed smooth real $(1,1)$-form θ such that $\varphi_i, \psi_i \in \mathrm{PSH}(X,\theta)$ and $\varphi_i \preceq_P \psi_i$ for all $i \in I$. Then*

$$
\sup_{i\in I}{}^*\varphi_i \preceq_P \sup_{i\in I}{}^*\psi_i .
$$

The same holds with $\preceq_{\mathcal{I}}$ in place of $\preceq_P$.

Proof By increasing θ, we may assume that $\varphi_i, \psi_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$. The statement for $\preceq_{\mathcal{I}}$ is a simple consequence of Corollary 1.4.1, we only have to consider the statement for $\preceq_P$.

Step 1. We first handle the case where I is a directed set and $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are increasing nets.

In this case, our assertion follows simply from Proposition 3.1.10.

Step 2. We handle the case where I is finite. We may assume that $I = \{0,1\}$. It suffices to show that

$$
P_\theta[\varphi_0] \vee P_\theta[\varphi_1] \sim_P \varphi_0 \vee \varphi_1,
$$

which follows from Lemma 6.1.2.

Step 3. The general case can be reduced to the two cases handled in Step 1 and Step 2. More precisely, by Proposition 1.2.2, we could find a countable subset $J \subseteq I$ such that

$$
\sup_{j\in J}{}^*\varphi_j = \sup_{i\in I}{}^*\varphi_i, \quad \sup_{i\in I}{}^*\psi_j = \sup_{i\in I}{}^*\psi_i .
$$

We may replace I by J and assume that I is countable. We may assume that I is infinite, as otherwise, we could apply Step 2 directly. So let us assume that $J = \mathbb{Z}_{>0}$. In this case, by Step 2 again, we may assume that both $(\varphi_i)_i$ and $(\psi_i)_i$ are increasing, which is the situation of Step 1.

Proposition 6.1.7 *Let $\varphi, \psi, \varphi', \psi' \in \mathrm{PSH}(X,\theta)_{>0}$ for some closed smooth real $(1,1)$-form θ on X. Assume that*

$$\varphi \sim_P \varphi', \quad \psi \sim_P \psi', \quad \varphi' \wedge \psi' \in \mathrm{PSH}(X,\theta)_{>0}.$$

Then

$$\varphi \wedge \psi \in \mathrm{PSH}(X,\theta)_{>0}, \quad \varphi \wedge \psi \sim_P \varphi' \wedge \psi'.$$

Proof Without loss of generality, we may assume that $\psi = \psi'$. Replacing φ' by $P_\theta[\varphi'] + C$ for some constant C, we may also assume that $\varphi \leq \varphi'$.

Using Corollary 2.3.2, for each $\epsilon \in (0,1)$, we can find $\eta \in \mathrm{PSH}(X,\theta)$ such that

$$\int_X \theta_\eta^n = \int_X \theta_\varphi^n, \quad \epsilon\eta + (1-\epsilon)\varphi' \leq \varphi, \quad \eta \leq \varphi'.$$

Since

$$\int_X \theta_\eta^n + \int_X \theta_{\varphi'\wedge\psi}^n > \int_X \theta_\varphi^n = \int_X \theta_{\varphi'}^n \geq \int_X \theta_{\eta\vee(\varphi'\wedge\psi)},$$

by Proposition 3.1.4, we find $\eta \wedge \psi \in \mathrm{PSH}(X,\theta)$. Now observe that

$$\epsilon(\eta\wedge\psi) + (1-\epsilon)(\varphi'\wedge\psi) \leq \varphi\wedge\psi.$$

Hence $\varphi\wedge\psi \in \mathrm{PSH}(X,\theta)$. By Theorem 2.3.2, we find that

$$(1-\epsilon)^n \int_X \theta_{\varphi'\wedge\psi}^n \leq \int_X \theta_{\varphi\wedge\psi}^n.$$

Letting $\epsilon \to 0+$ and applying Theorem 2.3.2, we find that

$$\int_X \theta_{\varphi'\wedge\psi}^n = \int_X \theta_{\varphi\wedge\psi}^n.$$

We conclude by Proposition 6.1.1.

Theorem 6.1.1 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) *There is a subgeodesic from φ_0 to φ_1;*
(2) $\varphi_0 \sim_P \varphi_1$.

Proof (2) $\implies$ (1). This follows from Proposition 4.2.1.

(1) $\implies$ (2). Let $(\varphi_t)_{t\in(0,1)}$ be a subgeodesic from φ_0 to φ_1.

We first reduce to the case where $\varphi_0 \geq \varphi_1$. Observe that $(\varphi_t \vee \varphi_1)_{t\in(0,1)}$ is a subgeodesic from $\varphi_0 \vee \varphi_1$ to φ_1. Assume that the special case has been proved, then we know that

$$\varphi_0 \vee \varphi_1 \sim_P \varphi_1.$$

Hence $\varphi_0 \leq_P \varphi_1$ by Proposition 6.1.3. The converse is proved similarly. Hence (2) follows.

Now we assume that $\varphi_0 \geq \varphi_1$. Next we reduce to the case where $t \mapsto \varphi_t$ is decreasing.

We replace $(\varphi_t)_t$ by the geodesic, which exists since a subgeodesic exists. Fix $t_0 \in (0,1)$, it suffices to argue that

$$\varphi_0 \geq \varphi_t \geq \varphi_1, \tag{6.5}$$

since $(\varphi_t)_{t\in[0,t_0]}$ and $(\varphi_t)_{t\in[t_0,1]}$ are both geodesics (proved as in Proposition 4.2.1).

The first part of (6.5) is obvious, since $(\varphi_t)_{t\in(0,1)}$ is a candidate in the Perron envelope defining the constant geodesic at φ_0. The latter is also obvious since $(\varphi_1)_{t\in[0,1]}$ is a subgeodesic.

Let $\varphi_t = \varphi_1$ for all $t > 1$. Then by the gluing lemma Lemma 1.2.2, we find that $(\varphi_t)_{t\geq 0}$ is a subgeodesic ray.

Next, we consider the Legendre transform

$$\Gamma_\tau \coloneqq \inf_{t\geq 0}(\varphi_t - t\tau), \quad \tau \in \mathbb{R}.$$

It follows from Kiselman's principle that $\Gamma_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$. Note that for $\tau > 0$, we clearly have $\Gamma_\tau \equiv -\infty$. On the other hand, for $\tau \leq 0$,

$$\Gamma_\tau = \inf_{t\in[0,1]}(\varphi_t - t\tau) \in \mathrm{PSH}(X,\theta).$$

See Proposition 4.1.2.

By Legendre inversion, for $t > 0$,

$$\varphi_t = \sup_{\tau\in\mathbb{R}}(\Gamma_\tau + t\tau).$$

Fix a Kähler form ω on X. It follows from Proposition 6.1.6 that for each $t > 0$,

$$\varphi_t \sim_P \sup_{\tau<0}{}^* P_{\theta+\omega}[\Gamma_\tau].$$

The right-hand side is independent of t. Here by adding ω, we no longer have to worry about the possibility where Γ_τ has vanishing mass.

By Proposition 6.1.6 again, the same holds for $t = 0$ as well. Our assertion follows. □

Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. We write $p_1 \colon X \times S \to X$ for the natural projection.

Corollary 6.1.4 *Let $\Phi \in \mathrm{PSH}(X \times S, p_1^*\theta)$. Assume that for any $c \in \mathbb{R}$, $x \in X$ and $z \in S$, we have*

$$\Phi(x,z) = \Phi(x, z + \mathrm{i}c).$$

Then $\int_X (\theta + \mathrm{dd^c}\Phi_z)^n$ is independent of $z \in S$, where $\Phi_z \in \mathrm{PSH}(X,\theta)$ is given by $\Phi_z(x) = \Phi(x,z)$.

This seems to be the first non-trivial result concerning the variation of non-pluripolar masses.

6.2 The d_S-pseudometric

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. The goal of this section is to study a pseudometric on the space $\mathrm{PSH}(X,\theta)$.

6.2.1 The definition of the d_S-pseudometric

Recall that for any $\varphi \in \mathrm{PSH}(X,\theta)$, the geodesic ray $\ell^\varphi \in \mathcal{R}^1(X,\theta)$ is defined in Example 4.2.2.

Definition 6.2.1 For $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we define

$$d_S(\varphi,\psi) \coloneqq d_1(\ell^\varphi, \ell^\psi).$$

When we want to be more specific, we write $d_{S,\theta}$ instead of d_S.

The d_1 distance of geodesic rays is defined in Definition 4.2.6.

Proposition 6.2.1 *The function d_S defined in Definition 6.2.1 is a pseudometric on* $\mathrm{PSH}(X,\theta)$.

Proof This follows immediately from Theorem 4.2.2. □

When studying a pseudometric, the first thing is to understand when the distance between two elements vanishes.

We first prove a preparation:

Lemma 6.2.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then*

$$d_S(\varphi,\psi) \leq d_S(\varphi, \varphi \vee \psi) + d_S(\psi, \varphi \vee \psi) \leq C_n d_S(\varphi,\psi),$$

where $C_n = 3(n+1)2^{n+2}$.

We shall use the notations introduced in Example 4.2.2.

Proof Observe that

$$\ell^\varphi \vee \ell^\psi = \ell^{\varphi \vee \psi}. \tag{6.6}$$

Recall that $\vee$ is defined in Definition 4.2.7. Note that this assertion implies our desired inequality by Lemma 4.2.1.

In proving this assertion, we may assume that $\varphi, \psi \leq 0$ since

$$\ell^{\varphi+C} = \ell^\varphi, \quad \ell^{\psi+C} = \ell^\psi, \quad \ell^{(\varphi+C)\vee(\psi+C)} = \ell^{\varphi\vee\psi}$$

for any $C \in \mathbb{R}$.

In fact, it is clear that

$$\ell^{\varphi} \leq \ell^{\varphi\vee\psi}, \quad \ell^{\psi} \leq \ell^{\varphi\vee\psi},$$

so the $\leq$ direction in (6.6) holds.

Conversely, if $\ell' \in \mathcal{R}^1(X,\theta)$ and $\ell' \geq \ell^{\varphi} \vee \ell^{\psi}$, then for each $t \geq 0$,

$$\ell'_t \geq ((V_\theta - t) \vee \varphi) \vee ((V_\theta - t) \vee \psi) = (V_\theta - t) \vee (\varphi \vee \psi).$$

Therefore,

$$\ell'_s \geq \ell_s^{\varphi\vee\psi,t}$$

for any $0 \leq s \leq t$. It follows from (4.21) that $\ell'_s \geq \ell_s^{\varphi\vee\psi}$ for any $s \geq 0$. □

Proposition 6.2.2 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) $d_S(\varphi,\psi) = 0$.

In particular, $d_S(\varphi, P_\theta[\varphi]) = 0$ for all $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proof By Lemma 6.1.3, we have $\varphi \sim_P \psi$ if and only if $\varphi \sim_P \varphi \vee \psi$ and $\psi \sim_P \varphi \vee \psi$. By Lemma 6.2.1, $d_S(\varphi,\psi) = 0$ if and only if $d_S(\varphi, \varphi\vee\psi) = 0$ and $d_S(\psi, \varphi\vee\psi) = 0$. So it suffices to prove the assertion when $\varphi \leq \psi$. Assuming this, by Proposition 4.2.6 we have that (2) holds if and only if

$$\mathbf{E}(\ell^{\varphi}) = \mathbf{E}(\ell^{\psi}),$$

where $\mathbf{E}$ is introduced in Definition 4.2.4. But by (4.19), this holds if and only if

$$\sum_{j=0}^{n} \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \sum_{j=0}^{n} \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}.$$

Thanks to Theorem 2.3.2, this holds if and only if for all $j = 0, \ldots, n$,

$$\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j},$$

which is equivalent to (1) by Proposition 6.1.1. □

Lemma 6.2.2 *Suppose that $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi$, then*

$$d_S(\varphi,\psi) = \frac{1}{n+1} \sum_{j=0}^{n} \left(\int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} \right).$$

Proof This follows trivially from (4.19). □

Corollary 6.2.1 *Suppose that $\varphi, \psi, \eta \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi \preceq_P \eta$. Then*

$$d_S(\varphi,\eta) \geq d_S(\varphi,\psi), \quad d_S(\varphi,\eta) \geq d_S(\psi,\eta).$$

Proof This is an immediate consequence of Lemma 6.2.2 and Proposition 6.1.4. □

Corollary 6.2.2 *For any $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we have*

$$
\begin{aligned}
d_S(\varphi,\psi) \leq & \frac{1}{n+1}\sum_{j=0}^{n}\left(2\int_X \theta^j_{\varphi\vee\psi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\psi}\wedge\theta^{n-j}_{V_\theta}\right) \\
\leq & C_n d_S(\varphi,\psi),
\end{aligned}
\tag{6.7}
$$

where $C_n = 3(n+1)2^{n+2}$.

In particular, if $(\varphi_i)_{i\in I}$ is a net in $\mathrm{PSH}(X,\theta)$ with d_S-limit φ, then for each $j = 0,\ldots,n$,

$$
\lim_{i\in I}\int_X \theta^j_{\varphi_i}\wedge\theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta} = \lim_{i\in I}\int_X \theta^j_{\varphi_i\vee\varphi}\wedge\theta^{n-j}_{V_\theta}.
$$

Proof The estimates (6.7) follows from the combination of Lemma 6.2.2 and Lemma 6.2.1.

Suppose that $\varphi_i \xrightarrow{d_S} \varphi$, then $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ by Lemma 6.2.1. Therefore, Theorem 2.3.2 and Lemma 6.2.2 imply that

$$
\lim_{i\in I}\int_X \theta^j_{\varphi_i\vee\varphi}\wedge\theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}
$$

for any $j = 0,\ldots,n$. The last assertion now follows from (6.7) and Theorem 2.3.2. □

Corollary 6.2.3 *Suppose that $\varphi_i \in \mathrm{PSH}(X,\theta)$ ($i \in I$) be an increasing net, uniformly bounded from above. Then*

$$
\varphi_i \xrightarrow{d_S} \sup_{j\in I}{}^*\varphi_j.
$$

If the φ_i's are all model potentials in $\mathrm{PSH}(X,\theta)_{>0}$, then so is $\sup_{j\in I}{}^*\varphi_j$, as we have seen in Proposition 3.1.10.

Proof Write $\varphi = \sup_{j\in I}{}^*\varphi_j$. Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X,\theta)$. By Lemma 6.2.2, it suffices to show that for each $k = 0,\ldots,n$, we have

$$
\lim_{j\in I}\int_X \theta^k_{\varphi_j}\wedge\theta^{n-k}_{V_\theta} = \int_X \theta^k_{\varphi}\wedge\theta^{n-k}_{V_\theta}.
$$

The latter follows from Corollary 2.3.1. □

Corollary 6.2.4 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Then*

$$
\left|\int_X \theta^n_\varphi - \int_X \theta^n_\psi\right| \leq D_n d_S(\varphi,\psi),
$$

where $D_n = 3(n+1)C_n$ with C_n being the same constant as in Lemma 6.2.1.

Proof We compute

$$\begin{aligned}\left|\int_X \theta_\varphi^n - \int_X \theta_\psi^n\right| \leq & \left|2\int_X \theta_{\varphi\vee\psi}^n - \int_X \theta_\varphi^n - \int_X \theta_\psi^n\right| + 2\left|\int_X \theta_{\varphi\vee\psi}^n - \int_X \theta_\varphi^n\right| \\ \leq & (n+1)C_n d_S(\varphi,\psi) + 2(n+1)d_S(\varphi,\varphi\vee\psi) \\ \leq & (n+1)C_n d_S(\varphi,\psi) + 2(n+1)C_n d_S(\varphi,\psi),\end{aligned}$$

where the first line is just the triangle inequality, the second line follows from Corollary 6.2.2 and the third line follows from Lemma 6.2.1. □

By contrast, for decreasing nets, the situation is different:

Corollary 6.2.5 *Suppose that $(\varphi_i)_{i\in I}$ is a decreasing net in* $\mathrm{PSH}(X,\theta)$ *such that $\varphi := \inf_{i\in I}\varphi_i \not\equiv -\infty$. Then the following are equivalent:*

(1) *We have*

$$\varphi_i \xrightarrow{d_S} \varphi;$$

(2) *for each $k = 0,\ldots,n$, we have*

$$\lim_{j\in I}\int_X \theta_{\varphi_j}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}. \tag{6.8}$$

If we assume furthermore that $\int_X \theta_\varphi^n > 0$, then the above conditions are equivalent to the following:

(3) *We have*

$$\lim_{j\in I}\int_X \theta_{\varphi_j}^n = \int_X \theta_\varphi^n.$$

In the latter case, we also have

$$P_\theta[\varphi] = \inf_{j\in I} P_\theta[\varphi_j]. \tag{6.9}$$

Proof Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X,\theta)$.

(1) $\iff$ (2). This follows immediately from Lemma 6.2.2.

Assume that $\int_X \theta_\varphi^n > 0$.

(2) $\implies$ (3). This is trivial.

(3) $\implies$ (2). Let $(b_j)_{j\in I}$ be a net converging to ∞ such that

$$b_j \in \left(1, \left(\frac{\int_X \theta_{\varphi_j}^n}{\int_X \theta_{\varphi_j}^n - \int_X \theta_\varphi^n}\right)^{1/n}\right).$$

By Lemma 2.3.1, for each $j \in I$, we can find $\eta_j \in \mathrm{PSH}(X,\theta)$ such that

$$b_j^{-1}\eta_j + (1-b_j^{-1})\varphi_j \leq \varphi.$$

It follows from Theorem 2.3.2 that for any $k = 0,\ldots,n$,

$$\int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k} \geq \left(1-b_j^{-1}\right)^k \int_X \theta_{\varphi_j}^k \wedge \theta_{V_\theta}^{n-k}.$$

Taking the limit, we conclude the $\leq$ direction in (6.8). The $\geq$ direction follows from Theorem 2.3.2.

Finally, we argue (6.9). We may assume that $\varphi_j \leq 0$ for all $j \in I$. Let $\psi_j = P_\theta[\varphi_j] \geq \varphi_j$. It follows from Corollary 3.1.1 that ψ_j is a model potential. Let

$$\psi = \inf_{j\in I} \psi_j \geq \varphi.$$

It follows from Proposition 3.1.3 and Proposition 3.1.9 that

$$\int_X \theta_\psi^n = \lim_{j\in I} \int_X \theta_{\psi_j}^n = \lim_{j\in I} \int_X \theta_{\varphi_j}^n = \int_X \theta_\varphi^n.$$

By Proposition 3.1.8, ψ is a model potential. Hence $\psi = P_\theta[\varphi]$ by Theorem 3.1.1. □

Having understood the increasing and decreasing cases, we shall handle more general convergent sequences. In fact, since d_S is a pseudometric, the topology is completely determined by convergent sequences, so we do not need to consider nets in general.

Proposition 6.2.3 *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ $(j \geq 1)$, $\varphi_j \xrightarrow{d_S} \varphi$. Assume that there is $\delta > 0$ such that*

$$\int_X \theta_{\varphi_j}^n \geq \delta$$

for all j and the φ_j's and φ are all model potentials. Then up to replacing $(\varphi_j)_j$ by a subsequence, there is a decreasing sequence $(\psi_j)_j$ and an increasing sequence $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)$ such that

(1) $\psi_j \xrightarrow{d_S} \varphi$, $\eta_j \xrightarrow{d_S} \varphi$;
(2) $\psi_j \geq \varphi_j \geq \eta_j$ for all j.

In fact, for any $j \geq 1$, we will take

$$\eta_j = \inf_{k\in\mathbb{N}} \varphi_j \wedge \varphi_{j+1} \wedge \cdots \wedge \varphi_{j+k}, \quad \psi_j = \sup_{k\geq j}{}^* \varphi_k.$$

Proof We are free to replace $(\varphi_j)_j$ by a subsequence. So we may assume that

$$d_S(\varphi_j, \varphi_{j+1}) \leq C_n^{-2j}, \quad d_S(\varphi, \varphi_j) \leq \frac{2^{-j}}{D_n}, \tag{6.10}$$

where C_n is the constant in Corollary 6.2.2, D_n is the constant in Corollary 6.2.4.

In particular, by Corollary 6.2.4,

$$\left| \int_X \theta_{\varphi_j}^n - \int_X \theta_\varphi^n \right| \leq 2^{-j}. \tag{6.11}$$

Step 1. We handle the ψ_j's. For each $j \geq 1$ and $k \geq 1$, by Lemma 6.2.1 we have

$$
\begin{aligned}
d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \leq & C_n d_S(\varphi_j, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \\
\leq & C_n d_S(\varphi_j, \varphi_{j+1}) + C_n d_S(\varphi_{j+1}, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}).
\end{aligned}
$$

By iteration, we find

$$
\begin{aligned}
d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \leq & \sum_{a=j}^{j+k-1} C_n^{a+1-j} d_S(\varphi_a, \varphi_{a+1}) \\
\leq & \sum_{a=j}^{j+k-1} C_n^{a+1-j} C_n^{-2a} \leq \frac{C_n^{1-2j}}{1 - C_n^{-1}}.
\end{aligned}
$$

Using Corollary 6.2.3, we have

$$
\varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k} \xrightarrow{d_S} \psi_j
$$

as $k \to \infty$. Hence

$$
d_S(\varphi_j, \psi_j) \leq \frac{C_n^{1-2j}}{1 - C_n^{-1}}. \tag{6.12}
$$

We conclude that $\psi_j \xrightarrow{d_S} \varphi$.

Moreover, we observe that

$$
\varphi = \inf_{j \geq 1} P_\theta[\psi_j] \tag{6.13}
$$

by Corollary 6.2.5.

Step 2. We consider the η_j's.

For each $j \geq 1$ and $k \geq 0$, we let

$$
\eta_j^k := \varphi_j \wedge \cdots \wedge \varphi_{j+k}.
$$

Using (6.12) and Corollary 6.2.4, we have

$$
\left| \int_X \theta_{\psi_j}^n - \int_X \theta_\varphi^n \right| \leq 2^{-j-1}
$$

when $j \geq j_0$ for some large j_0. Taking (6.11), we have

$$
\left| \int_X \theta_{\varphi_j}^n - \int_X \theta_{\psi_{j-1}}^n \right| \leq 2^{1-j} \tag{6.14}
$$

for $j > j_0$. Take $j_1 > j_0$ so that for $j \geq j_1$, $2^{1-j} < \delta$.

Step 2.1. We claim that for a fixed $j \geq j_1$, for any $k \in \mathbb{N}$, we have $\eta_j^k \in \mathrm{PSH}(X, \theta)$ and

$$
\int_X \theta_{\eta_j^k}^n \geq \int_X \theta_{\varphi_j}^n - \sum_{a=1}^{k} 2^{1-j-a}. \tag{6.15}
$$

We argue by induction on $k \geq 0$. The case $k = 0$ is trivial. When $k > 0$, assume that the case $k - 1$ is known. Then

$$\begin{aligned}\int_X \theta^n_{\eta_j^{k-1}} + \int_X \theta^n_{\varphi_{j+k}} &\geq \int_X \theta^n_{\varphi_j} - \sum_{a=1}^{k-1} 2^{1-j-a} + \int_X \theta^n_{\psi_{j+k-1}} - 2^{1-j-k} \\ &> \int_X \theta^n_{\varphi_j} - 2^{1-j} + \int_X \theta^n_{\psi_{j+k-1}} > \int_X \theta^n_{\psi_{j+k-1}},\end{aligned}$$

where the first inequality follows from the inductive hypothesis and (6.14).

Observe that

$$\eta_j^{k-1} \vee \varphi_{j+k} \leq \psi_{j+k-1},$$

it follows from Proposition 3.1.4 that $\eta_j^k \in \mathrm{PSH}(X, \theta)$. By Theorem 3.1.3, we deduce that

$$\begin{aligned}\int_X \theta^n_{\eta_j^k} &\geq \int_X \theta^n_{\varphi_{j+k}} + \int_X \theta^n_{\eta_j^{k-1}} - \int_X \theta^n_{\psi_{j+k-1}} \\ &\geq \int_X \theta^n_{\varphi_j} - \sum_{a=1}^{k} 2^{1-j-a},\end{aligned}$$

where the second inequality follows from the inductive hypothesis and (6.14). Therefore, (6.15) follows.

Step 2.2. It follows from Proposition 3.1.7 that for any $j \geq j_1$, $k \geq 0$,

$$P_\theta\left[\eta_j^k\right] = \eta_k^j.$$

By Proposition 3.1.9, we have

$$\lim_{k\to\infty} \int_X \theta^n_{\eta_j^k} = \int_X \theta^n_{\eta_j}$$

for any $j \geq j_1$. Letting $k \to \infty$ in (6.15), we find that

$$\int_X \theta^n_{\eta_j} \geq \int_X \theta^n_{\varphi_j} - 2^{1-j} > 0$$

for $j \geq j_1$. Observe that we also have

$$\int_X \theta^n_{\eta_j} \leq \int_X \theta^n_{\varphi_j} \leq \int_X \theta^n_{\psi_j}$$

for $j \geq j_1$ by Theorem 2.3.2. It follows from Corollary 2.3.1 that

$$\int_X \theta^n_{\eta} = \lim_{j\to\infty} \int_X \theta^n_{\varphi_j} = \lim_{j\to\infty} \int_X \theta^n_{\psi_j} = \int_X \theta^n_{\varphi},$$

where $\eta = \sup_{j\geq j_1}{}^*\eta_j$. Since $\eta_j \leq \varphi_j \leq \psi_j \leq 0$, we also have that $\eta_j \leq P_\theta[\psi_j]$. Therefore, by (6.13), we also have $\eta \leq \varphi$. It follows from Proposition 6.1.1 that $\eta \sim_P \varphi$. By Corollary 6.2.3 and Proposition 6.2.2, we have $\eta_j \xrightarrow{d_S} \varphi$. □

Corollary 6.2.6 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$. Assume that there is $\delta > 0$ such that $\int_X \theta^n_{\varphi_j} \geq \delta$ for all $j \in I$. Then $(\varphi_j)_{j\in I}$ has a d_S-convergent subnet.*

If moreover $(\varphi_j)_{j\in I}$ is decreasing, then $(\varphi_j)_{j\in I}$ itself is convergent.

Proof If the net $(\varphi_j)_{j\in I}$ is decreasing, then it is convergent by Corollary 6.2.5 and Proposition 3.1.9.

It remains to prove the first assertion. Since the space of $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta^n_\varphi \geq \delta$ is a pseudometric space, its completeness can be characterized using sequences instead of nets. So we may assume that $(\varphi_j)_{j\in I}$ is a sequence and $I = \mathbb{Z}_{>0}$.

Replacing $(\varphi_j)_{j>0}$ by a subsequence, we may assume that (6.10) holds. Define

$$\psi_j = \sup_{k\geq j}{}^* \varphi_k$$

for each $j > 0$. As in the proof of Proposition 6.2.3 Step 1, especially (6.12), we know that

$$\lim_{j\to\infty} d_S(\varphi_j, \psi_j) = 0.$$

It suffices to prove our assertion for $(\psi_j)_j$ in place of $(\varphi_j)_j$. But since $(\psi_j)_j$ is decreasing, this case has already been handled at the beginning of the proof. □

Lemma 6.2.3 *There is a constant $C > 0$ depending only on X and θ such that for any $\varphi \in \mathrm{PSH}(X,\theta)$ satisfying that θ_φ is a Kähler current, we have*

$$d_{S,\theta}((1-\epsilon)\varphi, \varphi) \leq C\epsilon$$

for $\epsilon > 0$ such that $(1-\epsilon)\varphi \in \mathrm{PSH}(X,\theta)$.

Proof By Lemma 6.2.2, we can compute

$$\begin{aligned} d_{S,\theta}((1-\epsilon)\varphi, \varphi) =& \frac{1}{n+1}\sum_{j=0}^n \left(\int_X \theta^j_{(1-\epsilon)\varphi} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} \right) \\ =& \frac{1}{n+1}\sum_{j=0}^n \left(\int_X (1-\epsilon)^j \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} \right) \\ &+ \frac{1}{n+1}\sum_{j=0}^n \sum_{k=0}^{j-1} \binom{j}{k} (1-\epsilon)^k \epsilon^{j-k} \int_X \theta^{j-k} \wedge \theta^k_\varphi \wedge \theta^{n-j}_{V_\theta}. \end{aligned}$$

Both terms are of the order of $O(\epsilon)$. □

6.2.2 Convergence theorems

Next we establish some important convergence theorems, allowing us to effectively manipulate the d_S-convergence.

Lemma 6.2.4 *Let $(\varphi_i)_{i\in I}$ be a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi \in \mathrm{PSH}(X,\theta)$. *Assume that* $\varphi_i \xrightarrow{d_S} \varphi$. *Then for any* $t \in (0,1]$,

$$(1-t)\varphi_i + tV_\theta \xrightarrow{d_S} (1-t)\varphi + tV_\theta .$$

When $t = 1$, the sum is understood as in Remark 2.3.3.

Proof Fix $t \in (0,1]$, we write

$$\varphi_{i,t} = (1-t)\varphi_i + tV_\theta , \quad \varphi_t = (1-t)\varphi + tV_\theta$$

for any $i \in I$.

By Corollary 6.2.2, it suffices to show that for each $j = 0, \ldots, n$,

$$2\int_X \theta^j_{\varphi_{i,t}\vee\varphi_t} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_{i,t}} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} \to 0. \tag{6.16}$$

Observe that

$$\varphi_{i,t} \vee \varphi_t = (1-t)(\varphi \vee \varphi_i) + tV_\theta .$$

So after binomial expansion, (6.16) follows from Corollary 6.2.2. □

Lemma 6.2.5 *Let* $\varphi \in \mathrm{PSH}(X,\theta)$. *For each* $t \in (0,1)$, *let* $\varphi_t = (1-t)\varphi + tV_\theta$. *Then*

$$\varphi_t \xrightarrow{d_S} \varphi$$

as $t \to 0+$.

Proof By Lemma 6.2.2, we need to show that for each $j = 1, \ldots, n$, we have

$$\lim_{t\to 0+} \int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta} .$$

For this purpose, we compute

$$\begin{aligned}&\int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta} \\ =& \sum_{i=0}^{j-1} \binom{j}{i} (1-t)^i t^{j-i} \int_X \theta^i_{\varphi} \wedge \theta^{n-i}_{V_\theta} .\end{aligned}$$

As $t \to 0+$, the right-hand side clearly tends to 0. □

The following convergent theorem lies at the heart of the whole theory.

Theorem 6.2.1 *Let $\theta_1, \ldots, \theta_n$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi_j^k)_{k\in I}$ are nets in $\mathrm{PSH}(X,\theta_j)$ and $\varphi_j \in \mathrm{PSH}(X,\theta_j)$ for $j = 1, \ldots, n$. We assume that $\varphi_j^k \xrightarrow{d_S} \varphi_j$ for each $j = 1, \ldots, n$. Then*

$$\lim_{k\in I} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{6.17}$$

Proof Since d_S is a pseudometric, in order to establish the continuity of mixed masses, it suffices to consider sequences instead of nets. So we may assume that $I = \mathbb{Z}_{>0}$ as ordered sets.

Step 1. We reduce to the case where φ_j^k, φ_j all have positive masses and there is a constant $\delta > 0$, such that for all j and k,

$$\int_X \theta^n_{j,\varphi_j^k} > \delta.$$

Take $t \in (0,1)$. By Lemma 6.2.4, we have

$$(1-t)\varphi_j^k + tV_{\theta_j} \xrightarrow{d_S} (1-t)\varphi_j + tV_{\theta_j}$$

as $k \to \infty$ for each j. Assume that we have proved the special case of the theorem, we have

$$\begin{aligned}&\lim_{k\to\infty} \int_X \theta_{1,(1-t)\varphi_1^k + tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n^k + tV_{\theta_n}} \\ =& \int_X \theta_{1,(1-t)\varphi_1 + tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n + tV_{\theta_n}}.\end{aligned}$$

Since both sides are polynomials in t, by Lagrange interpolation formula, the limit exists at $t = 0$ as well and the same formula holds at $t = 0$. From this, (6.17) follows.

Step 2. Next we may assume that φ_j^k, φ_j are model potentials for all $j = 1, \ldots, n$, $k > 0$ by Proposition 6.2.2 and Corollary 3.1.1.

It suffices to prove that any subsequence of $\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}$. Thus, by Proposition 6.2.3 and Theorem 2.3.2, we may assume that for each fixed i, $(\varphi_i^k)_k$ is either increasing or decreasing. We may assume that there is $i_0 \in \{0, \ldots, n\}$ such that for $i \leq i_0$, the sequence is decreasing and for $i > i_0$, the sequence is increasing.

Thanks to Corollary 6.2.5, Corollary 6.2.3 and Proposition 3.1.10, we have

$$\varphi_i = \inf_{k>0} \varphi_i^k, \quad i \leq i_0$$

and

$$\varphi_i = \sup_{k>0}{}^* \varphi_j^k, \quad i > i_0.$$

Therefore, for each $k > 0$, using Theorem 2.3.2, we have

$$\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{i_0,\varphi_{i_0}} \wedge \theta_{i_0+1,\varphi_n^{i_0+1}} \wedge \cdots \wedge \theta_{n,\varphi_n^k}.$$

Using Corollary 2.3.1, we therefore conclude that

$$\varliminf_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

It remains to prove

$$\varlimsup_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{6.18}$$

By Theorem 2.3.2, for each $k > 0$, we have

$$\int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{i_0,\varphi_{i_0}^k} \wedge \theta_{i_0+1,\varphi_{i_0+1}} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

When proving (6.18), we may replace φ_j^k by φ_j whenever $j > i_0$, $k > 0$. Thus, we are reduced to the case where for all i, $(\varphi_i^k)_k$ is decreasing.

Thanks to Lemma 2.3.1, for each $i = 1, \ldots, n$, we may take an increasing sequence $(b_i^k)_k$ tending to ∞ satisfying

$$b_i^k \in \left(1, \left(\frac{\int_X \theta^n_{i,\varphi_i^k}}{\int_X \theta^n_{i,\varphi_i^k} - \int_X \theta^n_{i,\varphi_i}}\right)^{1/n}\right)$$

and a sequence $(\psi_i^k)_k$ in $\mathrm{PSH}(X, \theta_i)$ such that

$$(b_i^k)^{-1}\psi_i^k + \left(1 - (b_i^k)^{-1}\right)\varphi_i^k \leq \varphi_i.$$

Then by Theorem 2.3.2 again,

$$\prod_{i=1}^n \left(1 - (b_i^k)^{-1}\right) \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \leq \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Letting $k \to \infty$, we conclude (6.18). □

Corollary 6.2.7 *Suppose that $(\varphi_i)_{i\in I}$ is a net in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;

(2) $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ *and*

$$\lim_{i\in I} \int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_\varphi \wedge \theta^{n-j}_{V_\theta} \tag{6.19}$$

for each $j = 0, \dots, n$*;*

(3) *for each* $j = 0, \dots, n$*, (6.19) holds and*

$$\lim_{i\in I}\int_X \theta^j_{\varphi_i\vee\varphi}\wedge\theta^{n-j}_{V_\theta} = \int_X \theta^j_\varphi\wedge\theta^{n-j}_{V_\theta}. \tag{6.20}$$

The corollary allows us to reduce a number of convergence problems related to d_S to the case $\varphi_i \geq \varphi$. This is the most handy way of establishing d_S-convergence in practice.

Proof The equivalence between (2) and (3) follows directly from Lemma 6.2.2.

(1) $\implies$ (2). That $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ follows from Corollary 6.2.2. While (6.19) follows from Theorem 6.2.1.

(2) $\implies$ (1). By (6.7), we need to show that for each $j = 0, \dots, n$, we have

$$2\int_X \theta^j_{\varphi_i\vee\varphi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_\varphi\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_i}\wedge\theta^{n-j}_{V_\theta} \to 0.$$

This follows from Theorem 6.2.1 and (6.19). □

Corollary 6.2.8 *Let* $(\varphi_i)_{i\in I}$ *be a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi \in \mathrm{PSH}(X,\theta)$*. Let* ω *be a Kähler form on* X*. Then the following are equivalent:*

(1) $\varphi_i \xrightarrow{d_{S,\theta}} \varphi$*;*

(2) $\varphi_i \xrightarrow{d_{S,\theta+\omega}} \varphi$.

In particular, there is no risk when we simply write $\varphi_i \xrightarrow{d_S} \varphi$.

Proof (1) $\implies$ (2). It suffices to show that for each $j = 0, \dots, n$, we have

$$\begin{aligned}2\int_X (\theta+\omega)^j_{\varphi_i\vee\varphi}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}} &- \int_X (\theta+\omega)^j_{\varphi_i}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\ &- \int_X (\theta+\omega)^j_{\varphi}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}} \to 0.\end{aligned}$$

Note that this quantity is a linear combination of terms of the following form:

$$\begin{aligned}2\int_X \theta^r_{\varphi_i\vee\varphi}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}} &- \int_X \theta^r_{\varphi_i}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\ &- \int_X \theta^r_{\varphi}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}},\end{aligned}$$

where $r = 0, \dots, j$. By Theorem 6.2.1, it suffices to show that $\varphi \vee \varphi_i \xrightarrow{d_S} \varphi$. But this follows from Corollary 6.2.7.

(2) $\implies$ (1). From the direction we already proved, for each $C \geq 1$, we have that

$$\varphi_i \xrightarrow{d_{S,\theta+C\omega}} \varphi.$$

By Theorem 6.2.1, it follows that

$$\lim_{i\in I}\int_X (\theta + C\omega)^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X (\theta + C\omega)^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}$$

for all $j = 0, \ldots, n$. It follows that

$$\lim_{i\in I}\int_X \theta^j_{\varphi_i} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}. \tag{6.21}$$

By Corollary 6.2.7, it remains to show that $\varphi_i \vee \varphi \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 6.2.7 again, we know that $\varphi_i \vee \varphi \xrightarrow{d_{S,\theta+\omega}} \varphi$. So it suffices to apply (6.21) to $\varphi_i \vee \varphi$ instead of φ_i, and we conclude by Lemma 6.2.2. □

We sometimes need a slightly more general form.

Corollary 6.2.9 *Let $(\varphi_j)_{j\in I}$, $(\psi_j)_{j\in I}$ be nets in* $\mathrm{PSH}(X,\theta)$. *Consider a Kähler form ω on X. Then the following are equivalent:*

(1) $d_{S,\theta}(\varphi_i,\psi_i) \to 0$;
(2) $d_{S,\theta+\omega}(\varphi_i,\psi_i) \to 0$.

In particular, we can write $d_S(\varphi_i,\psi_i) \to 0$ without ambiguity.

Proof The proof is similar to that of Corollary 6.2.8, which is therefore left to the readers. □

Corollary 6.2.10 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. Define $\varphi_t = t\varphi_1 + (1-t)\varphi_0$ for $t \in (0,1)$. Then*

$$\varphi_t \xrightarrow{d_S} \varphi_0$$

as $t \to 0+$.

Proof First note that for each $j = 0, \ldots, n$,

$$\lim_{t\to 0+}\int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi_0} \wedge \theta^{n-j}_{V_\theta}.$$

So thanks to Corollary 6.2.7, it remains to argue that for all $j = 0, \ldots, n$,

$$\lim_{t\to 0+}\int_X \theta^j_{\varphi_t\vee\varphi_0} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi_0} \wedge \theta^{n-j}_{V_\theta}.$$

Observe that for $t \in (0,1)$, we have

$$\varphi_t \vee \varphi_0 = t(\varphi_1 \vee \varphi_0) + (1-t)\varphi_0,$$

so the desired inequality follows. □

We have the following sandwich criterion:

Corollary 6.2.11 *Let $(\varphi_i)_{i\in I}, (\psi_i)_{i\in I}, (\eta_i)_{i\in I}$ be three nets in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi \in \mathrm{PSH}(X,\theta)$. *Assume that*

(1) $\psi_i \preceq_P \varphi_i \preceq_P \eta_i$ *for each* $i \in I$;

(2) $\eta_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \varphi$.

Then $\varphi_i \xrightarrow{d_S} \varphi$.

Proof By Corollary 6.2.8, we may replace θ by $\theta+\omega$, where ω is a Kähler form on X. In particular, we may assume that $\varphi_i, \psi_i, \eta_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$. By Proposition 6.2.2, we may assume that $\varphi_i, \psi_i, \eta_i$ are model potentials for all $i \in I$ and hence $\varphi_i \le \psi_i \le \eta_i$ for all $i \in I$.

It follows from Theorem 2.3.2 that for each $k = 0, \ldots, n$, we have

$$\int_X \theta_{\psi_i}^k \wedge \theta_{V_\theta}^{n-k} \le \int_X \theta_{\varphi_i}^k \wedge \theta_{V_\theta}^{n-k} \le \int_X \theta_{\eta_i}^k \wedge \theta_{V_\theta}^{n-k}$$

for all $i \in I$. By Theorem 6.2.1, the limits with respect to $i \in I$ of the both ends are $\int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}$. It follows that

$$\lim_{i\in I} \int_X \theta_{\varphi_i}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}. \tag{6.22}$$

By Corollary 6.2.7, it remains to prove that $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$. By Corollary 6.2.7 and Proposition 6.1.6, up to replacing ψ_i (resp. φ_i, η_i) by $\psi_i \vee \varphi$ (resp. $\varphi_i \vee \varphi$, $\eta_i \vee \varphi$), we may assume from the beginning that $\psi_i, \varphi_i, \eta_i \ge \varphi$. Now $\varphi_i \xrightarrow{d_S} \varphi$ by (6.22) and Lemma 6.2.2. □

Proposition 6.2.4 *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be nets in* $\mathrm{PSH}(X,\theta)$ *such that* $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$ *and* $\psi_i \xrightarrow{d_S} \psi \in \mathrm{PSH}(X,\theta)$. *Assume that* $\varphi_i \preceq_P \psi_i$ *for all* $i \in I$. *Then* $\varphi \preceq_P \psi$.

Proof It follows from Proposition 6.2.5 that

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

By Lemma 6.1.3, we have $\varphi_i \vee \psi_i \sim_P \psi_i$ for all $i \in I$. In particular, by Proposition 6.2.2,

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \psi.$$

By Proposition 6.2.2 again, $\varphi \vee \psi \sim_P \psi$ and hence $\varphi \preceq_P \psi$ by Lemma 6.1.3. □

Proposition 6.2.5 *Let $(\varphi_i)_{i\in I}$ (resp. $(\psi_i)_{i\in I}$) be a net in* $\mathrm{PSH}(X,\theta)$ *such that* $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$ *(resp.* $\varphi_i \xrightarrow{d_S} \psi \in \mathrm{PSH}(X,\theta)$*). Then*

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

Proof Since d_S is a pseudometric, we may assume that both nets are actually sequences and $I = \mathbb{Z}_{>0}$. By Corollary 6.2.8, we may assume that the masses $\int_X \theta_\varphi^n > 0$, $\int_X \theta_\psi^n > 0$.

Using Proposition 6.2.3, we may assume that both sequences are monotone and lie in $\mathrm{PSH}(X,\theta)_{>0}$.

Thanks to Proposition 6.1.6, we may assume that the φ_j's, the ψ_j's, φ and ψ are all model. In particular, $(\varphi_j)_j$ (resp. $(\psi_j)_j$) converges to φ (rest. ψ) almost everywhere.

We handle three cases separately.

Step 1. Assume that both sequences are increasing.

In this case, we have $\varphi_j \vee \psi_j \nearrow \varphi \vee \psi$ almost everywhere. Therefore, $\varphi_j \vee \psi_j \xrightarrow{d_S} \varphi \vee \psi$ by Corollary 6.2.3.

Step 2. Assume that one sequence, say $(\varphi_j)_j$ is increasing while the other is decreasing. Then we have

$$\varphi_j \vee \psi \leq \varphi_j \vee \psi_j \leq \varphi \vee \psi_j.$$

Thanks to Corollary 6.2.11, it suffices to show that both sides converge to $\varphi \vee \psi$ with respect to d_S. So we reduce to the case where both sequences are decreasing.

Step 3. Assume that both sequences are decreasing.

In this case, due to Corollary 6.2.5, it suffices to show that

$$\lim_{j\to\infty} \int_X \theta^n_{\varphi_j \vee \psi_j} = \int_X \theta^n_{\varphi \vee \psi}. \tag{6.23}$$

The $\geq$ direction follows from Theorem 2.3.2, it remains to argue the $\leq$ direction.

Thanks to Lemma 2.3.1, we may find a sequence $(\epsilon_j)_j$ in $(0,1)$ with limit 0 and a sequences $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ such that

$$(1-\epsilon_j)\varphi_j + \epsilon_j \eta_j \leq \varphi, \quad \eta_j \leq \varphi_j.$$

It follows that for each $j \geq 1$, we have

$$(1-\epsilon_j)(\varphi_j \vee \psi_j) + \epsilon_j \eta_j \leq \varphi \vee \psi_j.$$

Therefore by Theorem 2.3.2,

$$(1-\epsilon_j)^n \int_X \theta^n_{\varphi_j \vee \psi_j} \leq \int_X \theta^n_{\varphi \vee \psi_j}.$$

Letting $j \to \infty$, we find that

$$\lim_{j\to\infty} \int_X \theta^n_{\varphi_j \vee \psi_j} \leq \lim_{j\to\infty} \int_X \theta^n_{\varphi \vee \psi_j}.$$

Therefore, in order to prove (6.23), we may assume that one of the sequences is constant, let us say $\psi_j = \psi$ for all j. Repeating the same argument as before and constructing $(\epsilon_j)_j$, $(\eta_j)_j$ as above, we get

$$(1-\epsilon_j)^n \int_X \theta^n_{\varphi_j\vee\psi} \leq \int_X \theta^n_{\varphi\vee\psi}.$$

Letting $j \to \infty$, we conclude (6.23). □

Theorem 6.2.2 *Let θ_1, θ_2 be smooth real closed $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi_i)_{i\in I}$ (resp. $(\psi_i)_{i\in I}$) be a net in $\mathrm{PSH}(X,\theta_1)$ (resp. $\mathrm{PSH}(X,\theta_2)$) and $\varphi \in \mathrm{PSH}(X,\theta_1)$ (resp. $\psi \in \mathrm{PSH}(X,\theta_2)$). Consider the following three conditions:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;

(2) $\psi_i \xrightarrow{d_S} \psi$;

(3) $\varphi_i + \psi_i \xrightarrow{d_S} \varphi + \psi$.

Then any two of these conditions imply the third.

Proof By Corollary 6.2.8, we may assume that θ_1, θ_2 are both Kähler forms. We denote them by ω_1, ω_2 instead. Let $\omega = \omega_1 + \omega_2$.

(1)+(2) $\implies$ (3). It suffices to show that for each $r = 0, \ldots, n$,

$$2\int_X \omega^r_{(\varphi_j+\psi_j)\vee(\varphi+\psi)} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi_j+\psi_j} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi+\psi} \wedge \omega^{n-r} \to 0.$$

Observe that for each $j \in I$,

$$(\varphi_j + \psi_j) \vee (\varphi + \psi) \leq \varphi_j \vee \varphi + \psi_j \vee \psi.$$

Thus, it suffices to show that

$$2\int_X \omega^r_{\varphi_j\vee\varphi+\psi_j\vee\psi} \wedge \omega - \int_X \omega^r_{\varphi_j+\psi_j} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi+\psi} \wedge \omega^{n-r} \to 0.$$

The left-hand side is a linear combination of

$$2\int_X \omega^a_{1,\varphi_j\vee\varphi} \wedge \omega^{r-a}_{2,\psi_j\vee\psi} \wedge \omega^{n-r} - \int_X \omega^a_{1,\varphi_j} \wedge \omega^{r-a}_{2,\psi_j} \wedge \omega^{n-r} - \int_X \omega^a_{1,\varphi} \wedge \omega^{r-a}_{2,\psi} \wedge \omega^{n-r}$$

with $a = 0, \ldots, r$. Observe that $\varphi_j \vee \varphi \xrightarrow{d_S} \varphi$ and $\psi_j \vee \psi \xrightarrow{d_S} \psi$ by Corollary 6.2.2, each term tends to 0 by Theorem 6.2.1.

(1)+(3) $\implies$ (2). For each $C \geq 1$, from the direction we already proved,

$$C\varphi_i + \psi_i \xrightarrow{d_S} C\varphi + \psi.$$

By Theorem 6.2.1, for each $j = 0, \ldots, n$,

$$\begin{aligned}&\lim_{i\in I} \int_X (C\omega_1 + \omega_2 + \mathrm{dd^c}(C\varphi_i + \psi_i))^j \wedge \omega_2^{n-j}\\ =&\int_X (C\omega_1 + \omega_2 + \mathrm{dd^c}(C\varphi + \psi))^j \wedge \omega_2^{n-j}.\end{aligned}$$

It follows that

$$\lim_{i\in I}\int_X \omega_{2,\psi_i}^j \wedge \omega_2^{n-j} = \int_X \omega_{2,\psi}^j \wedge \omega_2^{n-j}. \tag{6.24}$$

Therefore, (2) follows if $\psi_i \geq \psi$ for each i by Lemma 6.2.2.

Next we prove the general case. By the direction that we already proved, we know that $\varphi_i + \psi \xrightarrow{d_S} \varphi + \psi$. By Proposition 6.2.5, we have that

$$\varphi_i + \psi_i \vee \psi \xrightarrow{d_S} \varphi + \psi.$$

It follows from the special case above that $\psi_i \vee \psi \xrightarrow{d_S} \psi$. It follows from (6.24) and Corollary 6.2.7 that (2) holds.

(2)+(3) $\implies$ (1). This is similar.

Theorem 6.2.3 *The map*

$$P_\theta[\bullet]_{\mathcal{I}}\colon \mathrm{PSH}(X,\theta)_{>0} \to \mathrm{PSH}(X,\theta)_{>0}$$

is continuous with respect to d_S.

Proof Let $(\varphi_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{PSH}(X,\theta)_{>0}$ such that $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)_{>0}$. We want to show that

$$P_\theta[\varphi_i]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}. \tag{6.25}$$

We may assume that the φ_i's and φ are all model potentials by Proposition 6.2.2.

By Proposition 6.2.3 and Corollary 6.2.11, we may assume that $(\varphi_i)_i$ is either increasing or decreasing. In the increasing case, we apply Proposition 3.2.13 and Corollary 6.2.3, while in the decreasing case, we apply Proposition 3.2.11, Proposition 3.1.9 and Corollary 6.2.5. ◻

6.2.3 Continuity of invariants

Theorem 6.2.4 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. Then for any prime divisor E over X, we have*

$$\lim_{j\in I} \nu(\varphi_j, E) = \nu(\varphi, E). \tag{6.26}$$

Proof First observe that since d_S is a pseudometric, it suffices to prove (6.26) when $I = \mathbb{Z}_{>0}$ as partially ordered sets.

By Corollary 6.2.8, we may assume that the masses of φ_j and of φ are bounded from below by a positive constant.

By Theorem 6.2.3, we may assume that φ_i and φ are both $\mathcal{I}$-model and hence model. When proving (6.26), we are free to pass to subsequences.

By Proposition 6.2.3, we may assume that the sequence (φ_i) is either increasing or decreasing. In the increasing case, there is nothing to prove. In the decreasing case, (6.26) follows from Proposition 3.1.9. □

Theorem 6.2.5 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Assume that $\varphi_j \xrightarrow{d_S} \varphi$, then*

$$\operatorname{vol}\theta_{\varphi_j} \to \operatorname{vol}\theta_{\varphi}, \quad \int_X \theta_{\varphi_j}^n \to \int_X \theta_{\varphi}^n. \tag{6.27}$$

Recall the volume is defined in Definition 3.2.3.

Proof The latter part of (6.27) is just a special case of Theorem 6.2.1.

We may therefore assume that $\int_X \theta_{\varphi_j}^n > 0$ for all $j \in I$. Then by Theorem 6.2.3, we have

$$P_\theta[\varphi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}.$$

Therefore, the first part of (6.27) follows again from Theorem 6.2.1. □

Theorem 6.2.6 *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \xrightarrow{d_S} \varphi$. Then for each $\lambda' > \lambda > 0$, there is $j_0 > 0$ so that for $j \geq j_0$,*

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi). \tag{6.28}$$

Proof Fix $\lambda' > \lambda > 0$, we want to find $j_0 > 0$ so that for $j \geq j_0$, (6.28) holds.

Step 1. We first assume that φ has analytic singularities.

Let $\pi\colon Y \to X$ be a log resolution of φ and let $E_1,\dots,E_N$ be all prime divisors in the polar locus of φ on Y. Recall that by Theorem 1.4.3, a local holomorphic function f lies in the right-hand side of (6.28) if and only if

$$\operatorname{ord}_{E_i}(f) > \lambda\nu(\varphi, E_i) - \frac{1}{2}A_X(E_i) \tag{6.29}$$

whenever they make sense. Here A_X denotes the log discrepancy. Similarly, f lies in the left-hand side of (6.28) implies that there is $\epsilon > 0$ so that

$$\operatorname{ord}_{E_i}(f) \geq (1+\epsilon)\lambda'\nu(\varphi_j, E_i) - \frac{1}{2}A_X(E_i).$$

As Lelong numbers are continuous with respect to d_S by Theorem 6.2.4, we can find $j_0 > 0$ so that when $j \geq j_0$, $\lambda'\nu(\varphi_j, E_i) \geq \lambda\nu(\varphi, E_i)$ for all i. In particular, (6.29) follows.

Step 2. We handle the general case.

By Corollary 6.2.8, we are free to increase θ and assume that θ_φ is a Kähler current.

Take a quasi-equisingular approximation $(\psi_k)_k$ of φ in $\mathrm{PSH}(X,\theta)$. The existence is guaranteed by Theorem 1.6.2. Take $\lambda'' \in (\lambda, \lambda')$, then by definition, we can find $k > 0$ so that

$$\mathcal{I}(\lambda''\psi_k) \subseteq \mathcal{I}(\lambda\varphi).$$

Observe that $\varphi_j \vee \psi_k \xrightarrow{d_S} \psi_k$ as $j \to \infty$ by Proposition 6.2.5. By Step 1, we can find $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}(\lambda'(\varphi_j \vee \psi_k)) \subseteq \mathcal{I}(\lambda''\psi_k).$$

It follows that for $j \geq j_0$,

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

Chapter 7
$\mathcal{I}$-good singularities

Le but de cette thèse est de munir son auteur du titre de Docteur.[a]
— Adrien Douady, at the beginning of his thesis

[a] Similarly, the purpose of the current book is to make my complaints about France in the acknowledgments published.

In this chapter, we study the key notion in the whole theory: The $\mathcal{I}$-good singularities. We will give several useful characterizations of $\mathcal{I}$-good singularities. The key result is the asymptotic Riemann–Roch formula for Hermitian pseudoeffective line bundles Theorem 7.3.1.

7.1 The notion of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold of dimension n.

Theorem 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) *There exists a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$;*

(2) *we have*

$$\int_X \theta_\varphi^n = \operatorname{vol}\theta_\varphi; \tag{7.1}$$

(3) *we have*

$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}. \tag{7.2}$$

In (1), we could in addition require that each θ_{φ_j} is a Kähler current.

Moreover, if θ_φ is a Kähler current, the sequence in (1) can be taken as any quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

Recall that according to Corollary 3.2.1, we always have

$$\int_X \theta_\varphi^n \leq \operatorname{vol}\theta_\varphi.$$

Proof (1) $\implies$ (2). By Theorem 6.2.1, we may assume that $\int_X \theta_{\varphi_j}^n > 0$ for all $j \geq 1$. It follows from Proposition 3.2.9 that

$$\int_X \theta_{\varphi_j}^n = \operatorname{vol}\theta_{\varphi_j}$$

for any $j \geq 1$. Using Theorem 6.2.5, we conclude (7.1).

(2) $\iff$ (3). This follows from Theorem 3.1.1.

(3) $\implies$ (1). Note that the condition in (1) characterizes the closure of analytic singularities in $\mathrm{PSH}(X, \theta)$.

Step 1. We first assume that θ_φ is a Kähler current, but φ does not necessarily satisfy (3). We show that $P_\theta[\varphi]_{\mathcal{I}}$ lies in the closure of analytic singularities.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X, \theta)$. We will show that $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$. Let

$$\psi = \inf_{j \in \mathbb{Z}_{>0}} P_\theta[\varphi_j].$$

We know that $\varphi_j \xrightarrow{d_S} \psi$ by Proposition 6.2.2, Proposition 3.1.9 and Corollary 6.2.5.

Moreover, observe that ψ is $\mathcal{I}$-model by Proposition 3.2.11 and Proposition 3.2.9. So it suffices to show that $\varphi \sim_{\mathcal{I}} \psi$.

First observe that since for all $j > 0$, $\varphi \leq \varphi_j$, we have

$$\varphi - \sup_X \varphi \leq P_\theta[\varphi_j].$$

Therefore,

$$\varphi - \sup_X \varphi \leq \psi.$$

Conversely, it remains to argue that $\psi \preceq_{\mathcal{I}} \varphi$. For this purpose, take $\lambda > 0$, we need to show that

$$\mathcal{I}(\lambda\psi) \subseteq \mathcal{I}(\lambda\varphi).$$

By the strong openness Theorem 1.4.4, we may take $\lambda' > \lambda$ such that $\mathcal{I}(\lambda\psi) = \mathcal{I}(\lambda'\psi)$, then it follows from the definition of the quasi-equisingular approximation that

$$\mathcal{I}(\lambda'\psi) \subseteq \mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi)$$

for large enough j. Our assertion follows.

It follows from the proof that we may take φ_j so that θ_{φ_j} is a Kähler current for all $j \geq 1$.

Step 2. We handle the general case.

By Lemma 2.3.2, we can find $\psi \in \mathrm{PSH}(X, \theta)$ so that θ_ψ is a Kähler current and $\psi \leq \varphi$. We let

$$\psi_j = (1 - j^{-1})\varphi + j^{-1}\psi$$

for each $j \in \mathbb{Z}_{>1}$. Then $(\psi_j)_j$ is an increasing sequence converging almost everywhere to φ. Then

$$P_\theta[\psi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} = P_\theta[\varphi]$$

by Proposition 3.2.13, Corollary 6.2.3. From Step 1, we know that each $P_\theta[\psi_j]_\mathcal{I}$ lies in the closure of analytic singularities, hence so is $P_\theta[\varphi] \sim_P \varphi$. □

Definition 7.1.1 We say a potential $\varphi \in \mathrm{QPSH}(X)$ is $\mathcal{I}$*-good* if for some smooth closed real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, we have

$$P_\theta[\varphi] = P_\theta[\varphi]_\mathcal{I}. \tag{7.3}$$

Remark 7.1.1 In view of Theorem 7.1.1 and Corollary 3.2.1, the failure of $\mathcal{I}$-goodness of a given $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ can be characterized using the difference between the volume and the mass. We therefore introduce

$$\mathrm{Macron}(\theta_\varphi) \coloneqq \operatorname{vol}\theta_\varphi - \int_X \theta_\varphi^n.$$

As we mentioned in the introduction, all potentials in practice are expected to be $\mathcal{I}$-good. The evil guy Macron is bound to be eliminated.

An immediate question is to verify that this definition is in dependent of the choice of θ.

Lemma 7.1.1 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ for some smooth closed real $(1,1)$-form θ on X. Take a Kähler form ω on X. Then the following are equivalent:*

(1) $P_\theta[\varphi] = P_\theta[\varphi]_\mathcal{I}$;
(2) $P_{\theta+\omega}[\varphi] = P_{\theta+\omega}[\varphi]_\mathcal{I}$.

Proof (1) $\implies$ (2). By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 6.2.8, we have $\varphi_j \xrightarrow{d_{S,\theta+\omega}} \varphi$. Therefore, by Theorem 7.1.1 again, (2) holds.

(2) $\implies$ (1). Suppose that (1) fails, so that

$$\int_X (\theta + \mathrm{dd^c}\varphi)^n < \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_\mathcal{I})^n.$$

It follows that

$$\begin{aligned}
\int_X (\theta+\omega+\mathrm{dd^c}\varphi)^n &= \sum_{i=0}^n \binom{n}{i} \int_X \theta_\varphi^i \wedge \omega^{n-i} \\
&< \sum_{i=0}^n \binom{n}{i} \int_X \theta_{P_\theta[\varphi]_\mathcal{I}}^i \wedge \omega^{n-i} \\
&= \int_X (\theta+\omega+\mathrm{dd^c} P_\theta[\varphi]_\mathcal{I})^n \\
&\leq \int_X (\theta+\omega+\mathrm{dd^c} P_{\theta+\omega}[\varphi]_\mathcal{I})^n.
\end{aligned}$$

So (2) fails as well. □

Corollary 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. Let $(\varphi_j)_{j\in I}$ be a net of $\mathcal{I}$-good potentials in $\mathrm{PSH}(X,\theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi$. Then φ is $\mathcal{I}$-good.*

Note that we do not need to assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proof By Corollary 6.2.8, we may assume that $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)_{>0}$ for all $j \in I$. It follows from Theorem 7.1.1 that

$$\int_X \theta_{\varphi_j}^n = \operatorname{vol}\theta_{\varphi_j}$$

for all $j \in I$. Taking limit with respect to j with the help of Theorem 6.2.5, we conclude that

$$\int_X \theta_{\varphi}^n = \operatorname{vol}\theta_{\varphi}.$$

Therefore, by Theorem 7.1.1 again, we find that φ is $\mathcal{I}$-good. □

Example 7.1.1 Assume that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then φ is $\mathcal{I}$-good. This is proved in Proposition 3.2.9.

In particular, the potential in Example 1.8.2 is $\mathcal{I}$-good.

Example 7.1.2 Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ is an $\mathcal{I}$-model potential for some closed real smooth $(1,1)$-form θ on X. Then φ is $\mathcal{I}$-good.

Example 7.1.3 Let $\varphi \in \mathcal{E}(X,\theta)$. Then φ is $\mathcal{I}$-good. In fact, since $P_\theta[\varphi] = V_\theta$, we deduce that $P_\theta[\varphi]_{\mathcal{I}} = V_\theta$ as well.

In particular, the potential in Example 3.1.1 is $\mathcal{I}$-good.

A further class of examples of $\mathcal{I}$-good singularities will be given in Example 7.3.1 below.

Unfortunately, there exists non-$\mathcal{I}$-good potentials.

Example 7.1.4 The potential in Example 6.1.3 is not $\mathcal{I}$-good. In fact, since φ has no non-vanishing Lelong numbers, we know that $\varphi \sim_{\mathcal{I}} 0$, hence

$$P_{2\omega}[\varphi] = 0.$$

On the other hand,

$$\int_X (2\omega + \mathrm{dd^c}\varphi) = \int_X \omega < \int_X (2\omega),$$

where $2\omega + \mathrm{dd^c}\varphi$ is understood in the non-pluripolar sense.

Corollary 7.1.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ and $(\epsilon_j)_j$ be a decreasing sequence in $\mathbb{R}_{\geq 0}$ with limit 0. Fix a Kähler form ω on X. Consider a decreasing sequence $\varphi_j \in \mathrm{PSH}(X,\theta+\epsilon_j\omega)$ of potentials with analytic singularities for each $j \geq 1$. Assume that $\varphi = \inf_j \varphi_j$. Then the following are equivalent:*

(1) $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$ [1], *and*
(2) $(\varphi_j)_j$ *is a quasi-equisingular approximation of* φ.

Proof By Corollary 6.2.8 and Example 7.1.2, we may replace θ by $\theta + C\omega$ for some large constant $C > 0$ and assume that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta - \omega)$ for all $j \geq 1$.

(2) $\implies$ (1). This is already proved in the proof of Theorem 7.1.1.

(1) $\implies$ (2). This follows from Theorem 6.2.6. □

7.2 Properties of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold of dimension n.

Proposition 7.2.1 *Let* $\varphi, \psi \in \mathrm{QPSH}(X)$ *be* $\mathcal{I}$*-good and* $\lambda > 0$. *Then the following potentials are all* $\mathcal{I}$*-good:*

(1) $\varphi + \psi$;
(2) $\varphi \vee \psi$;
(3) $\lambda\varphi$.

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. It follows from Theorem 7.1.1 that there are sequences $(\varphi_j)_j, (\psi_j)_j$ in $\mathrm{PSH}(X, \theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$ and $\psi_j \xrightarrow{d_S} \psi$.

By Theorem 6.2.2, Proposition 6.2.5, we have

$$\varphi_j + \psi_j \xrightarrow{d_S} \varphi + \psi, \quad \varphi_j \vee \psi_j \xrightarrow{d_S} \varphi \vee \psi.$$

On the other hand, it is clear that

$$\lambda\varphi_j \xrightarrow{d_S} \lambda\varphi.$$

Therefore, our assertions follow from Theorem 7.1.1. □

Example 7.2.1 Let L be a pseudoeffective line bundle on X. Elementary metrics on L are defined in Definition 6.1.3. Let h be an elementary metric on L, then $\mathrm{dd^c} h$ is $\mathcal{I}$-good.

This is a direct consequence of Proposition 7.2.1 and Example 7.1.1.

Proposition 7.2.2 *Let* $(\varphi_j)_{j\in I}$ *be a non-empty family of* $\mathcal{I}$*-good potentials in* $\mathrm{PSH}(X, \theta)$ *for some closed real smooth* $(1,1)$*-form* θ *on* X. *Then* $\sup^*_{j\in I}\varphi_j$ *is* $\mathcal{I}$*-good.*

[1] Just to be sure, this means $\varphi_j \xrightarrow{d_{S,\theta+\epsilon\omega}} P_\theta[\varphi]_{\mathcal{I}}$ for any $\epsilon > 0$. The choice of ϵ is irrelevant due to Corollary 6.2.8.

Proof After adding a Kähler form to θ, we may assume that $\varphi_j \in \mathrm{PSH}(X,\theta)_{>0}$ for all $j \in I$.

When I is finite, this result follows from Proposition 7.2.1. When I is infinite, we may assume that $I = \mathbb{Z}_{>0}$ by Proposition 1.2.2. By Proposition 7.2.1, we may assume that the sequence $(\varphi_j)_j$ is increasing. In this case, as shown in Corollary 6.2.3,

$$\varphi_j \xrightarrow{d_S} \sup_{i\in\mathbb{Z}_{>0}}{}^*\varphi_i.$$

Therefore, $\sup_{i\in\mathbb{Z}_{>0}}{}^*\varphi_i$ is $\mathcal{I}$-good by Corollary 7.1.1. □

Theorem 7.2.1 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. Assume that φ is $\mathcal{I}$-good, then we have*

$$\operatorname{vol}\theta_{\varphi_j} \to \operatorname{vol}\theta_\varphi. \tag{7.4}$$

This partially extends Theorem 6.2.5 to the case where φ has vanishing mass.

Proof Fix a Kähler form ω on X. Then for any $\epsilon > 0$, we have

$$\begin{aligned}\operatorname{vol}(\theta+\epsilon\omega)_\varphi &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}\right)^n\\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}\varphi\right)^n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}\right)^n &\geq \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta}[\varphi]_{\mathcal{I}}\right)^n\\ &\geq \int_X \left(\theta+\mathrm{dd^c}P_{\theta}[\varphi]_{\mathcal{I}}\right)^n\\ &\geq \int_X \theta_\varphi^n.\end{aligned}$$

Therefore,

$$0 \leq \operatorname{vol}(\theta+\epsilon\omega)_\varphi - \operatorname{vol}\theta_\varphi \leq \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}\varphi\right)^n - \int_X \theta_\varphi^n.$$

The difference can be controlled by a polynomial in ϵ without constant term independent of the choice of φ. We have a similar estimate for φ_j as well. So our assertion follows from Theorem 6.2.5. □

Proposition 7.2.3 *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$. Then*

(1) *we have*

$$\lim_{\epsilon\to 0+} \operatorname{vol}(\theta,(1-\epsilon)\varphi+\epsilon\psi) = \operatorname{vol}(\theta,\varphi).$$

(2) *Let ω be a Kähler form on X, then*

$$\operatorname{vol}\theta_\varphi = \lim_{\epsilon\to 0+} \operatorname{vol}(\theta+\epsilon\omega)_\varphi .$$

(3) *Consider a prime divisor E on X. Then*

$$\operatorname{vol}\theta_\varphi = \operatorname{vol}(\theta_\varphi - \nu(\varphi,E)[E]).$$

In the proof below, we shall freely use the d_S-convergence and $\mathcal{I}$-goodness to currents. Should the readers have any doubt, please refer to Remark 1.7.1.

Proof (1) This follows after combining Corollary 6.2.10 with Theorem 6.2.5.

(2) For each $\epsilon > 0$,

$$\begin{aligned}\operatorname{vol}(\theta+\epsilon\omega)_\varphi &= \int_X (\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}})^n\\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_{\theta+\epsilon\omega}\left[P_\theta[\varphi]_{\mathcal{I}}\right]\right)^n\\ &= \int_X \left(\theta+\epsilon\omega+\mathrm{dd^c}P_\theta[\varphi]_{\mathcal{I}}\right)^n,\end{aligned}$$

where the second equality follows from Example 7.1.2. Letting $\epsilon\to 0+$, we conclude.

(3) By (2), we may assume that θ_φ is a Kähler current. Take a quasi-equisingular approximation $(S_j)_j$ of $\theta_\varphi - \nu(\varphi,E)[E]$. By Theorem 6.2.2,

$$S_j + \nu(\varphi,E)[E] \xrightarrow{d_S} \theta_\varphi .$$

For each $j \geq 1$, the currents $S_j + \nu(\varphi,E)[E]$ and S_j are $\mathcal{I}$-good as follows from Proposition 7.2.1, we have

$$\operatorname{vol}(S_j+\nu(\varphi,E)[E]) = \int_X (S_j+\nu(\varphi,E)[E])^n = \int_X S_j^n = \operatorname{vol} S_j .$$

Letting $j\to\infty$, we conclude by Theorem 6.2.6. □

7.3 The volume of Hermitian pseudoeffective line bundles

Let X be a connected compact Kähler manifold of dimension n.

Definition 7.3.1 A *Hermitian pseudoeffective line bundle* (L,h) on a complex manifold Y consists of a holomorphic line bundle L on Y together with a plurisubharmonic metric h on L.

Theorem 7.3.1 *Let (L,h) be a Hermitian pseudoeffective line bundle and T be a holomorphic line bundle on X. We have*

$$\lim_{k\to\infty}\frac{n!}{k^n}h^0(X,T\otimes L^k\otimes\mathcal{I}(h^k))=\operatorname{vol}(\mathrm{dd^c}h). \tag{7.5}$$

In particular, the limit exists.

For the proof, let us fix a smooth Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$. We identify h with $h_0\exp(-\varphi)$ for some $\varphi \in \operatorname{PSH}(X,\theta)$. See Section 1.8 for the relevant notations.

Recall that when X admits a big line bundle, it is necessarily projective. See [MM07, Theorem 2.2.26].

We first handle the case where φ has analytic singularities.

Proposition 7.3.1 *Under the assumptions of Theorem 7.3.1, assume furthermore that φ has analytic singularities, then (7.5) holds.*

Proof **Step 1**. Reduce to the case of log singularities.

Let $\pi\colon Y\longrightarrow X$ be a log resolution of φ. In this case, for each $k\in\mathbb{Z}_{>0}$, we have

$$h^0(X,T\otimes L^k\otimes\mathcal{I}(kh))=h^0(Y,K_{Y/X}\otimes\pi^*T\otimes\pi^*L^k\otimes\mathcal{I}(k\pi^*h)).$$

By Proposition 3.2.5, we have

$$\operatorname{vol}(\mathrm{dd^c}h)=\operatorname{vol}(\mathrm{dd^c}\pi^*h).$$

Therefore, it suffices to argue (7.5) with $K_{Y/X}\otimes\pi^*T$, π^*L and π^*h in place of T, L and h.

Step 2. Assume that D has log singularities along an effective $\mathbb{Q}$-divisor D, we decompose D into irreducible components, say

$$D=\sum_{i=1}^N a_iD_i.$$

In this case, we can easily compute

$$\mathcal{I}(k\varphi)=\mathcal{O}_X\left(-\sum_{i=1}^N\lfloor ka_i\rfloor D_i\right)$$

for each $k\in\mathbb{Z}_{>0}$. Observe that $L-D$ is nef (see Lemma 1.6.1), so we could apply the asymptotic Riemann–Roch theorem to conclude that

$$\lim_{k\to\infty}\frac{n!}{k^n}h^0\left(X,T\otimes L^k\otimes\mathcal{O}_X\left(-\sum_{i=1}^N\lfloor ka_i\rfloor D_i\right)\right)=(L-D)^n.$$

Observe that by Proposition 1.8.1,

$$\theta_\varphi=[D]+T,$$

where T is a closed positive $(1,1)$-current with bounded potential. Therefore,

$$(L-D)^n = \int_X T^n = \int_X \theta_\varphi^n.$$

By Example 7.1.1, we know that the right-hand side is exactly $\operatorname{vol}\theta_\varphi$. □

Proof (Proof of Theorem 7.3.1) **Step 1**. We first handle the case where θ_φ is a Kähler current. Fix a Kähler form $\omega \geq \theta$ on X such that $\theta_\varphi \geq 2\delta\omega$ for some $\delta \in (0,1)$.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j} \geq \delta\omega$ for all j. From Proposition 7.3.1, we know that for each $j \geq 1$,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_j)) = \operatorname{vol}\theta_{\varphi_j}.$$

It follows from Theorem 7.1.1 and Theorem 6.2.5 that the right-hand side converges to $\operatorname{vol}\theta_\varphi$ as $j \to \infty$. Therefore,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \operatorname{vol}\theta_\varphi.$$

Conversely, fix an integer $N > \delta^{-1}$. From Theorem 7.1.1 and Theorem 6.2.1, we know that

$$\lim_{j\to\infty} \int_X \theta_{\varphi_j}^n = \int_X \theta_{P_\theta[\varphi]_{\mathcal{I}}}^n > 0. \tag{7.6}$$

Therefore, by Lemma 2.3.1, we can find $j_0 > 0$ such that for $j \geq j_0$, there is $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ (depending on j) with

$$(1-N^{-1})\varphi_j + N^{-1}\psi \leq P_\theta[\varphi]_{\mathcal{I}}. \tag{7.7}$$

For each $k > 0$, we write $k = k'N - r$, where $k' \in \mathbb{N}$ and $r \in \{0,1,\ldots,N-1\}$. Then we compute for $j > j_0$ and large enough k that

$$\begin{aligned}
&h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))\\
\geq & h^0(X, T\otimes L^{-r} \otimes L^{k'N} \otimes \mathcal{I}(k'N\varphi))\\
\geq & h^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}\left(k'(\psi+(N-1)\varphi_j)\right)\right)\\
\geq & h^0\left(X, T\otimes L^{-r}\otimes L^{k'(N-1)}\otimes \mathcal{I}\left(k'N\varphi_j\right)\right),
\end{aligned}$$

where the third line follows from (7.7), the fourth line can be argued as follows: For large enough k, there is a non-zero section $s \in H^0(X, L^{k'} \otimes \mathcal{I}(k'\psi))$ by Lemma 2.3.3. It follows from Lemma 1.6.3 that for large enough k,

$$\mathcal{I}\left(k'N\varphi_j\right) \subseteq \mathcal{I}_\infty\left(k'(N-1)\varphi_j\right).$$

It follows that multiplication by s gives an injective map

$$\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'(N-1)}\otimes \mathcal{I}\left(k'N\varphi_j\right)\right)\hookrightarrow$$
$$\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}\left(k'\psi+k'(N-1)\varphi_j\right)\right).$$

Next observe that

$$(N-1)\theta+N\mathrm{dd^c}\varphi_j\geq 0.$$

So Proposition 7.3.1 is applicable. We let $k\to\infty$ to conclude that

$$\begin{aligned}\varliminf_{k\to\infty}\frac{n!}{k^n}h^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\geq & N^{-n}\int_X\left((N-1)\theta+N\mathrm{dd^c}\varphi_j\right)^n\\ =&\int_X\left((1-N^{-1})\theta+\mathrm{dd^c}\varphi_j\right)^n\\ \geq&\int_X\left(\theta+\mathrm{dd^c}\varphi_j\right)^n-CN^{-1},\end{aligned}$$

where C is a constant independent of N and j. Letting $j\to\infty$ and then $N\to\infty$ and using (7.6), we find that

$$\varliminf_{k\to\infty}h^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\geq\int_X\theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

Therefore, (7.5) follows.

Step 2. We handle the case where $\operatorname{vol}\theta_\varphi>0$. We may assume that φ is $\mathcal{I}$-model.

Fix observe that L is big by Proposition 2.3.2. Hence X is projective. Take a very ample line bundle A on X and a Kähler form ω in $c_1(A)$.

Fix $N\in\mathbb{Z}_{>0}$, we decompose any $k>0$ as $k=k'N+r$ with $k'\in\mathbb{N}$ and $r\in\{0,1,\ldots,N-1\}$. Then

$$h^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\leq h^0(X,T\otimes L^r\otimes L^{k'N}\otimes\mathcal{I}(k'N\varphi)).$$

Therefore,

$$\begin{aligned}&\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))\\ \leq&\max_{r=0,\ldots,N-1}\varlimsup_{k'\to\infty}\frac{1}{k'^nN^n}h^0(X,T\otimes L^r\otimes L^{k'N}\otimes\mathcal{I}(k'N\varphi))\\ \leq&\max_{r=0,\ldots,N-1}\varlimsup_{k'\to\infty}\frac{1}{k'^nN^n}h^0(X,T\otimes L^r\otimes L^{k'N}\otimes A^{k'}\otimes\mathcal{I}(k'N\varphi))\\ =&\int_X\left(N^{-1}\omega+\theta+\mathrm{dd^c}P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}}\right)^n,\end{aligned}$$

where we have applied Step 1 on the fourth line. On the other hand, since φ is $\mathcal{I}$-good by Example 7.1.2, we have

$$P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}}=P_{\theta+N^{-1}\omega}[\varphi].$$

It follows from Proposition 3.1.3 that

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \int_X \left(\theta + N^{-1}\omega + \mathrm{dd^c}\varphi\right)^n .$$

Letting $N \to \infty$, we conclude

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \leq \int_X \theta_\varphi^n .$$

It remains to argue the reverse inequality.

Choose $\psi \in \mathrm{PSH}(X,\theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. The existence of ψ is guaranteed by Lemma 2.3.2. Then for any $t \in (0,1)$, we set

$$\varphi_t = (1-t)\varphi + t\psi .$$

It follows again from Step 1 that

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \geq \lim_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_t)) = \mathrm{vol}\,\theta_{\varphi_t} .$$

On the other hand, by Corollary 6.2.3, we have $\varphi_t \xrightarrow{d_S} \varphi$ as $t \to 0+$. It follows from Theorem 6.2.5 that

$$\lim_{t\to 0+} \mathrm{vol}\,\theta_{\varphi_t} = \mathrm{vol}\,\theta_\varphi .$$

So we find

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) \geq \mathrm{vol}\,\theta_\varphi .$$

We conclude (7.5) in this case.

Step 3. We finally handle the case where $\mathrm{vol}\,\theta_\varphi = 0$. Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-model.

Assume that (7.5) fails. That is,

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k) \geq \varlimsup_{k\to\infty} \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)) > 0,$$

then L is a big line bundle and hence X is projective.

Fix a very ample line bundle A on X and a Kähler form $\omega \in c_1(A)$. Take a decreasing sequence $(\epsilon_j)_j$ of rational numbers with limit 0 and a quasi-equisingular approximation $(\varphi_j)_j$ of φ with $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)_{>0}$.

We claim that as $j \to \infty$, the sequence $P_{\theta+\epsilon_j\omega}[\varphi_j]$ is decreasing with limit φ.

It is clear that this sequence is decreasing. Let ψ denote its limit for the moment. It is also clear that $\psi \geq \varphi$. Since φ is $\mathcal{I}$-model, it remains to show that $\psi \leq_{\mathcal{I}} \varphi$. But the argument is exactly as in the proof of Theorem 7.1.1. So we conclude.

By our claim and Proposition 3.1.9, we find that

$$\lim_{j\to\infty}\int_X(\theta+\epsilon_j\omega+\mathrm{dd}^{\mathrm{c}}\varphi_j)^n=\int_X\theta_\varphi^n=0. \tag{7.8}$$

Fix $j>0$, take an integer $N>0$ so that $N\epsilon_j$ is an integer. Then we compute

$$\begin{aligned}
&\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0\left(X,T\otimes L^k\otimes\mathcal{I}(k\varphi)\right)\\
\le&\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0\left(X,T\otimes L^k\otimes\mathcal{I}(k\varphi_j)\right)\\
\le&\max_{a=0,\dots,N-1}\varlimsup_{k'\to\infty}\frac{n!}{(k'N)^n}h^0\left(X,T\otimes L^a\otimes L^{Nk'}\otimes\mathcal{I}(Nk'\varphi_j)\right)\\
\le&\max_{a=0,\dots,N-1}\varlimsup_{k'\to\infty}\frac{n!}{(k'N)^n}h^0\left(X,T\otimes L^a\otimes L^{Nk'}\otimes A^{k'N\epsilon_j}\otimes\mathcal{I}(Nk'\varphi_j)\right)\\
=&\frac{1}{N^n}\int_X\left(N\theta+\epsilon_jN\omega+N\mathrm{dd}^{\mathrm{c}}\varphi_j\right)^n,
\end{aligned}$$

where the third line follows by writing $k=Nk'+a$ as before, we applied Step 2 on the last line. Letting $N\to\infty$, we find that

$$\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0\left(X,T\otimes L^k\otimes\mathcal{I}(k\varphi)\right)\le\int_X(\theta+\epsilon_j\omega+\mathrm{dd}^{\mathrm{c}}\varphi_j)^n.$$

Since we know (7.8), letting $j\to\infty$, we conclude that

$$\varlimsup_{k\to\infty}\frac{n!}{k^n}h^0\left(X,T\otimes L^k\otimes\mathcal{I}(k\varphi)\right)=0,$$

which is a contradiction. Hence (7.5) is established in full generality. □

Corollary 7.3.1 *Let L be a pseudoeffective line bundle on X, h be a Hermitian metric on L with $\theta=c_1(L,h)$. Then we have*

$$\lim_{k\to\infty}\frac{n!}{k^n}h^0(X,L^k)=\int_X\theta_{V_\theta}^n. \tag{7.9}$$

This common quantity is the *volume* of L, usually denoted by $\operatorname{vol}L$.

Example 7.3.1 If X is a toric smooth projective variety and θ is invariant under the action of the compact torus. Then any $\varphi\in\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ is $\mathcal{I}$-good.

Proof Thanks to Lemma 7.1.1, we may assume that $\theta\in c_1(L)$ for some toric invariant ample line bundle L. In this case, the result follows from Theorem 7.1.1, Theorem 7.3.1 and Theorem 5.2.2. □

Chapter 8
The trace operator

The difference between mathematicians and physicists is that after physicists prove a big result they think it is fantastic but after mathematicians prove a big result they think it is trivial.
— Lucien Szpiro

In this chapter, we develop the theory of trace operators and prove the analytic Bertini theorem. These techniques allow us to make induction on the dimension while studying the singularities. Roughly speaking, the analytic Bertini theorem allows us to study generic restrictions, while the trace operator handles the remaining cases.

In Section 8.3, we establish a relative version of the Theorem 7.3.1.

8.1 The definition of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset.

The trace operator gives a way to restrict a quasi-plurisubharmonic function on X to $\tilde{Y}$, the normalization of Y. It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space. We refer to Appendix B for the pluripotential theory on unibranch Kähler spaces.

For later applications, we need this generality even if initially we are only interested in the smooth case.

We first observe that given $\varphi \in \mathrm{QPSH}(X)$ with analytic singularities such that $\nu(\varphi, Y) = 0$, then $\varphi|_Y \not\equiv -\infty$. This observation will be crucial in the sequel.

Proposition 8.1.1 *Let $\varphi \in \mathrm{QPSH}(X)$ be a function such that $\nu(\varphi, Y) = 0$. Let $(\varphi_i)_i$, $(\psi_i)_i$ be quasi-equisingular approximations of φ. Then*

$$\lim_{i\to\infty} d_S\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) = 0. \tag{8.1}$$

The meaning of (8.1) is explained in Corollary 6.2.9.

Proof Take a Kähler form ω on X such that $\varphi_i, \psi_i \in \mathrm{PSH}(X, \omega/2)$ for all $i \geq 1$. By Corollary 6.2.9, we need to show that

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) = 0.$$

Assume that this fails, then up to replacing the sequences by subsequences, we may assume that the following limit exists and

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}\left(\varphi_i|_{\tilde{Y}},\psi_i|_{\tilde{Y}}\right) > 0.$$

Take a Kähler form $\tilde{\omega}$ on $\tilde{Y}$, then

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\varphi_i|_{\tilde{Y}},\psi_i|_{\tilde{Y}}\right) > 0$$

by Corollary 6.2.9.

Replacing φ by $P_\omega[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good. In particular, $\varphi_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \varphi$. Therefore,

$$\varphi_i \vee \psi \xrightarrow{d_S} \varphi$$

due to Proposition 6.2.5. We may replace $(\psi_i)_i$ with $(\varphi_i \vee \psi_i)_i$ and assume that $\varphi_i \leq \psi_i$ for all $i \geq 1$.

Take a decreasing sequence $(\epsilon_j)_j$ in $\mathbb{R}_{>0}$ with limit 0 such that $(1-\epsilon_j)\varphi_j \in \mathrm{PSH}(X,\omega)$. We first observe that

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}(\varphi_i|_{\tilde{Y}},(1-\epsilon_i)\varphi_i|_{\tilde{Y}}) = 0.$$

This is a consequence of Lemma 6.2.3. Hence, by Corollary 6.2.9, we find

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}(\varphi_i|_{\tilde{Y}},(1-\epsilon_i)\varphi_i|_{\tilde{Y}}) = 0.$$

But thanks to Corollary 6.2.6, there is $\psi \in \mathrm{PSH}(\tilde{Y},\omega|_{\tilde{Y}}+\tilde{\omega})$ such that

$$\varphi_i|_{\tilde{Y}} \xrightarrow{d_S} \psi.$$

Hence,

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}(\epsilon,(1-\epsilon_i)\varphi_i|_{\tilde{Y}}) = 0.$$

Next by Proposition 1.6.3, we could find a subsequence $(\psi_{j_i})_{i\in\mathbb{Z}_{>0}}$ of $(\psi_j)_j$ such that for each $i \geq 1$,

$$\varphi_{j_i} \leq \psi_{j_i} \preceq (1-\epsilon_i)\varphi_i.$$

Hence,

$$\varphi_{j_i}|_{\tilde{Y}} \leq \psi_{j_i}|_{\tilde{Y}} \preceq (1-\epsilon_i)\varphi_i|_{\tilde{Y}}.$$

Therefore, by Corollary 6.2.1,

$$\begin{aligned}
\varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}(\varphi_{j_i}|_{\tilde{Y}},\psi_{j_i}|_{\tilde{Y}}) \leq& \varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}(\varphi_{j_i}|_{\tilde{Y}},(1-\epsilon_i)\varphi_i|_{\tilde{Y}})\\
=& \varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}(\psi,(1-\epsilon_i)\varphi_i|_{\tilde{Y}})\\
=&0,
\end{aligned}$$

which is a contradiction. □

Definition 8.1.1 Let $\varphi \in \mathrm{QPSH}(X)$ be a function such that $\nu(\varphi, Y) = 0$. We say a potential $\psi \in \mathrm{QPSH}(\tilde{Y})$ is a *trace operator* of φ along Y if there is a quasi-equisingular approximation $(\varphi_j)_j$ of φ such that

$$\varphi_j|_{\tilde{Y}} \xrightarrow{d_S} \psi\text{[1]}. \tag{8.2}$$

By Corollary 6.2.6, the trace operator is always defined. Observe that by Proposition 8.1.1, the condition (8.2) is independent of the choice of $(\varphi_j)_j$.

Proposition 8.1.2 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Suppose that ψ and ψ' are trace operators of φ along Y. Then ψ and ψ' are $\mathcal{I}$-good and $\psi \sim_P \psi'$.*

Proof That ψ and ψ' are $\mathcal{I}$-good follows from Theorem 7.1.1. The fact that $\psi \sim_P \psi'$ follows from Proposition 8.1.1 and Proposition 6.2.2. □

Definition 8.1.2 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. We write $\mathrm{Tr}_Y(\varphi)$ for any trace operator of φ along Y.

Given a closed smooth real $(1,1)$-form θ on X. When $\mathrm{Tr}_Y(\varphi)$ can be chosen to lie in $\mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})_{>0}$, we write

$$\mathrm{Tr}_Y^\theta(\varphi) \coloneqq P_{\theta|_{\tilde{Y}}}[\mathrm{Tr}_Y(\varphi)] = P_{\theta|_{\tilde{Y}}}[\mathrm{Tr}_Y(\varphi)]_{\mathcal{I}}\,.$$

The trace operator $\mathrm{Tr}_Y(\varphi)$ is therefore well-defined only up to P-equivalence by Proposition 8.1.2. Also observe that if $\varphi \in \mathrm{PSH}(X, \theta)$ for some smooth closed real $(1,1)$-form θ on X, then for any Kähler form ω on X, the trace operator $\mathrm{Tr}_Y^{\theta+\omega}(\varphi)$ is always defined. In particular, if θ_φ is a Kähler current, $\mathrm{Tr}_Y^\theta(\varphi)$ is always defined.

Remark 8.1.1 As in Remark 1.7.1, the trace operator could also be applied to closed positive $(1,1)$-currents on X. If $T \in \mathcal{Z}_+(X, \alpha)$ for some pseudoeffective class α on X (see Definition 1.7.3) and $\beta \in \mathrm{H}^{1,1}(\tilde{Y}, \mathbb{R})$, then we write

$$\mathrm{Tr}_Y^\beta(T)$$

for any (if exists) closed positive $(1,1)$-current in β representing $\mathrm{Tr}_Y(T)$ when $\nu(T, Y) = 0$.

Proposition 8.1.3 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that $\varphi|_Y \not\equiv -\infty$. Then*

$$\varphi|_{\tilde{Y}} \preceq_P \mathrm{Tr}_Y(\varphi).$$

[1] To be more precise, what we mean is the following: We can find a closed smooth real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X, \theta)$. Then there is a Kähler form such that $\omega + \theta + \mathrm{dd^c}\varphi_j \geq 0$ for all $j \geq 1$. Take a Kähler form $\tilde{\omega}$ on $\tilde{Y}$ so that $\tilde{\omega} \geq (\theta + \omega)|_{\tilde{Y}}$ and that $\psi \in \mathrm{PSH}(\tilde{Y}, \tilde{\omega})$. Then our condition means that $\varphi_j|_{\tilde{Y}} \xrightarrow{d_{S,\tilde{\omega}}} \psi$. This condition is independent of the choices of θ, ω and $\tilde{\omega}$ by Corollary 6.2.8.

Proof Take a Kähler form ω such that ω_φ is a Kähler current. Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\omega)_{>0}$. We may assume that $\varphi_j \leq 0$ for all $j \geq 1$.

Then

$$\varphi_j|_{\tilde{Y}} \leq P_{\omega|_{\tilde{Y}}}\left[\varphi_j|_{\tilde{Y}}\right] \tag{8.3}$$

for all $j \geq 1$. In particular,

$$\varphi|_{\tilde{Y}} \leq \inf_{j\geq 1} P_{\omega|_{\tilde{Y}}}\left[\varphi_j|_{\tilde{Y}}\right].$$

Thanks to Corollary 6.2.5,

$$\mathrm{Tr}_Y(\varphi) \sim_P \inf_{j\geq 1} P_{\omega|_{\tilde{Y}}}[\varphi_j|_{\tilde{Y}}]. \tag{8.4}$$

We conclude our assertion. □

Example 8.1.1 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that φ has analytic singularities, then

$$\mathrm{Tr}_Y(\varphi) \sim_P \varphi|_{\tilde{Y}}.$$

Example 8.1.2 Let $\varphi \in \mathrm{QPSH}(X)$. Take a closed real smooth $(1,1)$-form θ on X such that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then

$$\mathrm{Tr}_X(\varphi) \sim_P P_\theta[\varphi]_{\mathcal{I}}, \quad \mathrm{Tr}_X^\theta(\varphi) = P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, the trace operator can be regarded as a generalization of the $\mathcal{I}$-envelope.

Example 8.1.3 Assume that $\varphi \in \mathrm{PSH}(X,\theta)$ for some closed smooth real $(1,1)$-form θ on X and

$$\lim_{\epsilon\searrow 0} \int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right)^{\dim Y} > 0 \tag{8.5}$$

for any arbitrary choice of a Kähler form ω on X. Then it follows from Proposition 3.1.9 that $\mathrm{Tr}_Y^\theta(\varphi)$ is defined, and its mass is exact the above limit.

Remark 8.1.2 The trace operator allows us to introduce the following extension of the moving Seshadri constant: Let $T \in \mathcal{Z}_+(X,\alpha)$ and $x \in X$, we define

$$\epsilon(T,x) := \inf_{V\ni x} \left(\frac{\mathrm{vol}\,\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T}{\mathrm{mult}_x V}\right)^{\frac{1}{\dim V}},$$

where $\mathrm{vol}\,\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T = 0$ if $\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T$ is not defined. Here V runs over all positive-dimensional closed irreducible analytic subsets of X containing x.

These moving Seshadri constants seem to be new. But since I do not have particularly good applications in mind, I will not study these objects in this book.

8.2 Properties of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset.

Proposition 8.2.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$, $\lambda > 0$. Assume that $\nu(\varphi, Y) = \nu(\psi, Y) = 0$. Then we have the following:*

(1) *Suppose that $\varphi \preceq_{\mathcal{I}} \psi$, then $\mathrm{Tr}_Y(\varphi) \preceq_P \mathrm{Tr}_Y(\psi)$.*
(2) *We have*
$$\mathrm{Tr}_Y(\varphi + \psi) \sim_P \mathrm{Tr}_Y(\varphi) + \mathrm{Tr}_Y(\psi).$$
(3) *We have*
$$\mathrm{Tr}_Y(\lambda\varphi) \sim_P \lambda\,\mathrm{Tr}_Y(\varphi).$$
(4) *We have*
$$\mathrm{Tr}_Y(\varphi \vee \psi) \sim_P \mathrm{Tr}_Y(\varphi) \vee \mathrm{Tr}_Y(\psi).$$

Proof Take a closed smooth real $(1,1)$-form θ on X such that $\theta_\varphi, \theta_\psi$ are both Kähler currents. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be quasi-equisingular approximations of φ and ψ in $\mathrm{PSH}(X, \theta)$ respectively. We may assume that $\varphi_j \leq 0$ and $\psi_j \leq 0$ for all $j \geq 1$.

(1) By Corollary 7.1.2 and Proposition 6.2.5, we may assume that $\varphi_j \leq \psi_j$ for all j. Then our assertion follows from Proposition 6.2.4.

(2) It follows from Theorem 6.2.2 that $\varphi_j + \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}}$. However, by Proposition 3.2.10 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi + \psi]_{\mathcal{I}}.$$

Therefore, by Proposition 6.2.2, Corollary 7.1.2 and Proposition 1.6.1, $(\varphi_j + \psi_j)_j$ is a quasi-equisingular approximation of $\varphi + \psi$. We conclude using Theorem 6.2.2.

(3) Let $(\lambda_j)_j$ be an increasing sequence of positive rational numbers with limit λ. Then $(\lambda_j\varphi_j)_j$ is a quasi-equisingular approximation of φ. Our assertion follows Lemma 6.2.3.

(4) By Proposition 6.2.5, we have

$$\varphi_j \vee \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}}.$$

By Proposition 3.2.10 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi \vee \psi]_{\mathcal{I}}.$$

Therefore, our assertion follows exactly as in the proof of (2). □

Proposition 8.2.2 *Let $(\varphi_j)_{j \in I}$ be a decreasing net in $\mathrm{QPSH}(X)$. Assume that there exists a closed real smooth $(1,1)$-form θ such that $\varphi_j \in \mathrm{PSH}(X, \theta)$ for each $j \in I$. Assume that $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{QPSH}(X)$ and $\nu(\varphi, Y) = 0$. Then*

$$\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \mathrm{Tr}_Y(\varphi).$$

Proof By Corollary 6.2.8, we may assume that there is a Kähler form ω on X such that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta - \omega)$ for all $j \in I$. Thanks to Proposition 8.2.1, for each $j \geq 1$,

$$\mathrm{Tr}_Y(\varphi_{j+1}) \preceq_P \mathrm{Tr}_Y(\varphi_j).$$

It follows from Proposition 8.2.1 and Corollary 6.2.6 that there exists $\psi \in \mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})$ such that $\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \psi$.

For each $j \geq 1$, we take a quasi-equisingular approximation $(\varphi_j^k)_k$ in $\mathrm{PSH}(X, \theta)$ of φ_j. Using Theorem 1.6.2, we may guarantee that

$$\varphi_{j+1}^k \leq \varphi_j^k$$

for each $j, k \geq 1$. In particular, $(\varphi_j^j)_j$ is a quasi-equisingular approximation of φ. By Proposition 6.2.4, we have $\psi \preceq_P \mathrm{Tr}_Y(\varphi)$.

Conversely, by Proposition 8.2.1, $\mathrm{Tr}_Y(\varphi_j) \succeq_P \mathrm{Tr}_Y(\varphi)$. It follows again from Proposition 6.2.4 that $\mathrm{Tr}_Y(\varphi) \preceq_P \psi$. □

Example 8.2.1 The trace operator is not continuous along increasing sequences. Let us consider the case $X = \mathbb{P}^2$ with coordinates (z_1, z_2) on $\mathbb{C}^2 \subseteq X$. Let ω_{FS} denote the Fubini–Study metric. The subvariety $Y \cong \mathbb{P}^1$ is defined by $z_2 = 0$. Consider an increasing sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X, \omega_{\mathrm{FS}})$, whose potentials near $(0,0)$ are given by

$$\log |z_1|^2 \vee \left(k^{-1} \log |z_2|^2\right) + O(1).$$

The pointwise restriction of these potentials to Y are given locally by

$$\log |z_1|^2 + O(1).$$

On the other hand, locally

$$\log |z_1|^2 \vee \left(k^{-1} \log |z_2|^2\right) \to 0$$

almost everywhere as $k \to \infty$. So the trace operator is not continuous along the sequence $(\varphi_j)_j$.

Lemma 8.2.1 *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a connected Kähler manifold. Assume that W (resp. Y) be analytic subsets in Z (resp. X) of codimension 1 such that the restriction $\Pi\colon W \to Y$ of π is defined and is bimeromorphic, so that we have the following commutative diagram*

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d, "\tilde{\Pi}"] & W \arrow[r, hook] \arrow[d, "\Pi"] & Z \arrow[d, "\pi"] \\
\tilde{Y} \arrow[r] & Y \arrow[r, hook] & X.
\end{tikzcd}$$

Then for any $\varphi \in \mathrm{QPSH}(X)$ with $\nu(\varphi, Y) = 0$, we have

$$\tilde{\Pi}^* \operatorname{Tr}_Y(\varphi) \sim_P \operatorname{Tr}_W(\pi^*\varphi). \tag{8.6}$$

Proof We first observe that by Zariski's main theorem, $\nu(\pi^*\varphi, W) = 0$. So the right-hand side of (8.6) makes sense.

Step 1. Assume that φ has analytic singularities. It suffices to apply Example 8.1.1 to reformulate (8.6) as

$$\tilde{\Pi}^*(\varphi|_{\tilde{Y}}) \sim_P (\pi^*\varphi)|_{\tilde{W}}.$$

In fact, the strict equality holds, which is nothing but the functoriality of pullbacks.

Step 2. Next we handle the general case. Choose a smooth closed real $(1,1)$-form θ such that θ_φ is a Kähler current. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\operatorname{PSH}(X,\theta)$. By Corollary 7.1.2, $(\pi^*\varphi_j)_j$ is a quasi-equisingular approximation of $\pi^*\varphi$. From Step 1, we know that for each j,

$$\tilde{\Pi}^* \operatorname{Tr}_Y(\varphi_j) \sim_P \operatorname{Tr}_W(\pi^*\varphi_j).$$

Letting $j \to \infty$, we conclude (8.6) using Proposition 8.2.2. □

Proposition 8.2.3 *Let $\varphi \in \operatorname{QPSH}(X)$ with $\nu(\varphi, Y) = 0$. Assume that Y is smooth. Then for any $\lambda > 0$, we have*

$$\mathcal{I}(\lambda \operatorname{Tr}_Y(\varphi)) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi). \tag{8.7}$$

Proof Take a Kähler form ω on X such that ω_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\operatorname{PSH}(X,\omega)$.

By definition, for each $j \geq 1$, we get that

$$\operatorname{Tr}_Y(\varphi) \preceq_P \varphi_j|_Y.$$

For any $\lambda' > \lambda > 0$, we can find $j > 0$ so that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

By Theorem 1.4.5, we have

$$\mathcal{I}(\lambda' \operatorname{Tr}_Y(\varphi)) \subseteq \mathcal{I}(\lambda'\varphi_j|_Y) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda'\varphi_j) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi).$$

Thanks to Theorem 1.4.4, we conclude (8.7). □

Lastly, we turn our attention to global sections. For this we will need the following global Ohsawa–Takegoshi extension theorem for the trace operator:

Theorem 8.2.1 *Let L be a big line bundle on X and θ is a closed real smooth $(1,1)$-form on X representing $c_1(L)$. Suppose that $\varphi \in \operatorname{PSH}(X,\theta)$ and θ_φ is a Kähler current. Assume that $\nu(\varphi, Y) = 0$. Let T be a holomorphic line bundle on X. Then there exists k_0 such that for all $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k \operatorname{Tr}_Y^\theta(\varphi)))$, there exists an extension $\tilde{s} \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$.*

It is of interest to know if one could control the L^2-norm of $\tilde{s}$ in the above result.

Proof Fix a Kähler form ω on X. We may assume that $Y \neq X$ and that $\theta_\varphi \geq 3\delta\omega$ for some $\delta > 0$. Let $(\varphi_j)_j$ be the decreasing quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We can assume that $\theta_{\varphi_j} \geq 2\delta\omega$ for all $j \geq 1$. Also, there exists $\epsilon_0 > 0$ such that $\theta_{(1+\epsilon)\varphi_j} \geq \delta\omega$ for any $\epsilon \in (0,\epsilon_0)$. Take $k_0 = k_0(\delta)$ as in Theorem 1.8.1.

We fix $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\,\mathrm{Tr}_Y^\theta(\varphi)))$. By Theorem 1.4.4, there exists $\epsilon \in (0,\epsilon_0)$ such that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon)\,\mathrm{Tr}_Y^\theta(\varphi)))$.

Since $\mathrm{Tr}_Y^\theta(\varphi) \leq \varphi_j|_Y$, we obtain that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j|_Y))$. Due to Theorem 1.8.1 there exists $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j))$ such that $\tilde{s}_j|_Y = s$, for all j.

But by definition of quasi-equisingular approximation, we obtain that for high enough j the inclusion $\mathcal{I}(k(1+\epsilon)\varphi_j) \subseteq \mathcal{I}(k\varphi)$ holds. As a result, $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ for high enough j, finishing the argument. □

8.3 Restricted volumes

Let X be a connected projective manifold of dimension n and $Y \subseteq X$ be a connected submanifold of dimension m. Consider a big line bundle L on X, a Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$. Let A be a very ample line bundle on X. Take a Hermitian metric h_A on A such that $\omega = \mathrm{dd^c} h_A$ is a Kähler form.

Using the trace operator, one could prove the following generalization of Theorem 7.3.1.

Theorem 8.3.1 *Let h be a singular plurisubharmonic metric on L with $\nu(\mathrm{dd^c} h, Y) = 0$. Assume that*

$$\lim_{\epsilon \searrow 0} \left(\mathrm{Tr}_Y^{c_1(L|_Y)+\epsilon\omega}(c_1(L,h))\right)^m > 0. \tag{8.8}$$

Then for any holomorphic line bundle T on X we have that

$$\int_Y \left(\mathrm{Tr}_Y^{c_1(L|_Y)}(c_1(L,h))\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathrm{Res}_Y(\mathcal{I}(h^k))\right). \tag{8.9}$$

Recall that Res_Y is defined in Definition 1.4.5. Observe that by Example 8.1.3, (8.8) implies that $\mathrm{Tr}_Y^{c_1(L|_Y)}(c_1(L,h))$ is defined. So (8.9) is defined.

We will identify h with $\varphi \in \mathrm{PSH}(X,\theta)$ as in (1.18).

We only need to consider the case $Y \neq X$, since otherwise, the result is proved in Theorem 7.3.1. We will always assume $Y \neq X$ in the sequel.

Lemma 8.3.1 *There is $\psi_Y \in \mathrm{QPSH}(X)$ with neat analytic singularities such that $\{\psi_Y = -\infty\} = Y$ and in an open neighborhood of Y, we have*

$$\psi_Y(x) = 2(n-m)\log \mathrm{dist}(x, Y) \tag{8.10}$$

for some Riemannian distance function $\mathrm{dist}(\cdot, Y)$.

See Definition 1.6.1 for the definition of neat analytic singularities.

See [Fin22, Lemma 2.3] for the proof.

Lemma 8.3.2 *The multiplier ideal sheaf of ψ_Y can be calculated as*

$$\mathcal{I}(\psi_Y) = \mathcal{I}_Y. \tag{8.11}$$

Moreover, given $y \in Y$ and $\epsilon > 0$, for any germ $f \in \mathcal{I}_{Y,y}$ we have

$$\int_U |f|^\epsilon \mathrm{e}^{-\psi_Y} \omega^n < \infty, \tag{8.12}$$

where U is an open neighborhood of y in X.

In other words, ψ_Y has *log canonical singularities*.

Proof Since ψ_Y is locally bounded away from Y, it suffices to prove (8.11) along Y. Fix $y \in Y$, and we will verify (8.11) germ-wise at y.

Take an open neighbourhood $U \subset X$ of y and a biholomorphic map $F \colon U \to V \times W$, where V is an open neighbourhood of y in Y and W is a connected open subset in $\mathbb{C}^{n-m}$ containing 0, such that $F(Y \cap U) = V \times \{0\}$. For any $x \in U$, write x_V, x_W for the two components of $F(x)$ in V and W respectively. We denote the coordinates in $\mathbb{C}^{n-m}$ as $w_1, \dots, w_{n-m}$.

Due to (8.10), after possibly shrinking U, we may assume that

$$\exp(-\psi_Y(x)) = |x_W|^{2m-2n} + O(1)$$

for any $x \in U \setminus Y$.

Given $f \in \mathcal{I}_{Y,y}$, after shrinking U, we may assume that there exists $g_1, \dots, g_{n-m} \in \mathrm{H}^0(V \times W, \mathcal{O}_{V \times W})$ such that

$$f = \sum_{i=1}^{n-m} w_i g_i.$$

In order to verify $f \in \mathcal{I}(\psi_Y)_y$, it suffices to show $w_i g_i \in \mathcal{I}\left(\left(\sum_{i=1}^{n-m} |w_i|^2\right)^{m-n}\right)_{F(y)}$, which follows from Fubini's theorem. The proof of (8.12) is similar.

Conversely, take $f \in \mathcal{I}(\psi_Y)$, the similar application of Fubini's theorem shows that after possible shrinking U, we have $f|_Y = 0$. By Rückert's Nullstellensatz [GR84, Page 67], it follows that $f \in \mathcal{I}_Y$. □

Lemma 8.3.3 *Assume that φ has analytic singularity type and θ_u is a Kähler current. Suppose that $\varphi|_Y \not\equiv -\infty$. Then*

$$\int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m = \lim_{k \to \infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\}. \tag{8.13}$$

Recall that $\mathcal{I}_\infty$ is defined in Definition 1.6.6.

Proof Suppose that $\epsilon \in (0,1)$ is small enough so that $(1-\epsilon)u \in \mathrm{PSH}(X, \theta)$.

Using Theorem 7.3.1 we can start to write the following sequence of inequalities:

$$
\begin{aligned}
&\frac{1}{m!}\int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m\\
=&\lim_{k\to\infty}\frac{1}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y))\\
\leq&\varliminf_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\right\} \quad \text{by Theorem 1.8.1}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}(k\varphi))\right\}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim\left\{s|_Y : s\in \mathrm{H}^0(X, T\otimes L^k\otimes \mathcal{I}_\infty((1-\epsilon)k\varphi))\right\} \quad \text{by Lemma 1.6.3}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim_{\mathbb{C}}\left\{s\in \mathrm{H}^0(Y, T|_Y\otimes L|_Y^k) : \log h^k(s,s)\leq (1-\epsilon)k\varphi|_Y\right\}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m} h^0\left(Y, T|_Y\otimes L|_Y^k\otimes \mathcal{I}((1-\epsilon)k\varphi|_Y)\right)\\
=&\frac{1}{m!}\int_Y (\theta|_Y + (1-\epsilon)\mathrm{dd^c}\varphi|_Y)^m \quad \text{by Theorem 7.3.1.}
\end{aligned}
$$

Letting $\epsilon \to 0$, (8.13) follows from multi-linearity of the non-pluripolar product. □

Proposition 8.3.1 *In the setting of Theorem 8.3.1, assume that* $\mathrm{dd^c}h$ *is a Kähler current. Then* (8.9) *holds.*

Proof Let $(\varphi_j)_j$ a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. After possibly replacing $(\varphi_j)_j$ by a subsequence, there exists $\epsilon_0 \in (0,1)\cap\mathbb{Q}$ such that $\theta_{(1-\epsilon)^2\varphi_j}$ and $\theta_{(1-\epsilon)\varphi_j}$ are also Kähler currents for any $\epsilon\in(0,\epsilon_0)$.

We claim that for any $j\geq 1$ and $k\in\mathbb{N}$, we have

$$
\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\cap\mathcal{I}(\psi_Y)\subseteq\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y). \tag{8.14}
$$

Take $x\in X$, and it suffices to argue (8.14) along the germ of x. Since ψ_Y is locally bounded outside Y, we may assume that $x\in Y$. Recall that by Lemma 8.3.2, $\mathcal{I}(\psi_Y)=\mathcal{I}_Y$.

Let $f\in\mathcal{I}_\infty((1-\epsilon)k\varphi_j)_x\cap\mathcal{I}(\psi_Y)_x$. Then there is an open neighbourhood U of x in X such that $|f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j}\leq C$ holds on $U\setminus\{\varphi_j=-\infty\}$ for some $C>0$, hence

$$
\begin{aligned}
\int_U |f|^2\mathrm{e}^{-k(1-\epsilon)^2\varphi_j-\psi_Y}\,\omega^n =& \int_U |f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j}|f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n\\
\leq& C\int_U |f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n<\infty,
\end{aligned}
$$

where the last inequality follows from Lemma 8.3.2. We have proved the claim (8.14).

Next we consider the following composition morphism of coherent sheaves on Y:

$$
\mathrm{Res}_Y\,\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\hookrightarrow\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\cap\mathcal{I}_Y}\to\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}. \tag{8.15}
$$

Here we have identified the coherent $\mathcal{O}_X$-modules supported on Y with coherent $\mathcal{O}_Y$-modules. Note that the target of (8.15) is also supported on Y as ψ_Y is locally bounded outside Y. We denote the coherent $\mathcal{O}_Y$-module whose pushforward to X gives $\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}$ by $\mathcal{I}_{k,j}$.

In (8.15), the first map is the inclusion and the second one is the obvious projection induced by (8.14). Although in general the second map fails to be injective, we observe that the composition is still injective as $\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)\subseteq\mathcal{I}(\psi_Y)=\mathcal{I}_Y$. Therefore, for any $k\in\mathbb{N}$, we have an injective morphism of coherent $\mathcal{O}_Y$-modules:

$$L|_Y^k\otimes T|_Y\otimes \operatorname{Res}_Y\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\hookrightarrow L|_Y^k\otimes T|_Y\otimes\mathcal{I}_{k,j}. \tag{8.16}$$

Using Theorem 7.3.1 we can start the following inequalities:

$$\begin{aligned}
&\frac{1}{m!}\int_Y\left(\theta|_Y+\mathrm{dd^c}\operatorname{Tr}_Y^\theta(\varphi)\right)^m\\
=&\lim_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\mathcal{I}(k\operatorname{Tr}_Y^\theta(\varphi)))\quad\text{by Theorem 7.3.1}\\
\leq&\varliminf_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\operatorname{Res}_Y(\mathcal{I}(k\varphi)))\quad\text{by Theorem 1.4.5}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\operatorname{Res}_Y(\mathcal{I}(k\varphi)))\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\mathcal{I}(k\varphi_j)|_Y)\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\mathcal{I}_\infty((1-\epsilon)k\varphi_j)|_Y)\quad\text{by Lemma 1.6.3}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}h^0(Y,T|_Y\otimes L|_Y^k\otimes\mathcal{I}_{k,j})\quad\text{by (8.16)}\\
\leq&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim_{\mathbb{C}}\left\{s|_Y: s\in\mathrm{H}^0\left(X,T\otimes L^k\otimes\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}\right)\right\}\\
=&\varlimsup_{k\to\infty}\frac{1}{k^m}\dim_{\mathbb{C}}\left\{s|_Y: s\in\mathrm{H}^0(X,T\otimes L^k\otimes\mathcal{I}((1-\epsilon)^2k\varphi_j))\right\}\quad\text{(see below)}\\
=&\frac{1}{m!}\int_Y\left(\theta|_Y+(1-\epsilon)^2\mathrm{dd^c}\varphi_j|_Y\right)^m\quad\text{by Lemma 8.3.3,}
\end{aligned}$$

where in the penultimate line we used [CDM17, Theorem 1.1(6)] for $q=0$. Letting $\epsilon\to\infty$ and then $j\to\infty$ the result follows. □

Proof (Proof of Theorem 8.3.1) Using Proposition 8.2.3 and Theorem 7.3.1 we obtain that

$$\begin{aligned}\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m &= \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\,\mathrm{Tr}_Y^\theta(\varphi)))\\ &\leq \varliminf_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathrm{Res}_Y(\mathcal{I}(k\varphi))).\end{aligned}$$

Now we address the other direction in (8.9). Let $\phi \in \mathrm{H}^0(X, A)$ be a section that does not vanish identically on Y. Such ϕ exists since A is very ample.

We fix $k_0 \in \mathbb{N}$. For any $k \geq 0$, we have that $k = qk_0 + r$ with $q, r \in \mathbb{N}$ and $r \in \{0, \dots, k_0 - 1\}$. Also, we have an injective linear map

$$\mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)) \xrightarrow{\cdot\phi^{\otimes q}} \mathrm{H}^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}(k\varphi|_Y)\right).$$

Therefore,

$$\begin{aligned}&\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)\right)\\ \leq &\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}(k\varphi|_Y)\right)\\ = &\frac{1}{k_0^m} \varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}(k\varphi|_Y)\right)\\ \leq &\frac{1}{k_0^m} \varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}(k_0 q\varphi|_Y)\right)\\ = &\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta + k_0^{-1}\omega}(\varphi)\right)^m\\ = &\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m,\end{aligned}$$

where in the fourth line we have used that $k_0 q \leq k$ and in the last line we have used Proposition 8.3.1 for the big line bundle $L^{k_0} \otimes A$, the Kähler current $k_0\theta_u - \mathrm{dd^c}\log g = k_0\theta_u + \omega$, and twisting bundle $T \otimes L^r$. Letting $k_0 \to \infty$, we conclude that

$$\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)\right) \leq \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m.$$

Theorem 8.3.2 *Let $\varphi \in \mathrm{PSH}(X, \theta)$ such that $\nu(\varphi, Y) = 0$. Assume that θ_φ is a Kähler current. Then*

$$\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\}.$$

Proof This is a consequence of Theorem 7.3.1, Theorem 8.2.1 and Theorem 8.3.1:

$$\begin{aligned}\int_Y \left(\theta|_Y + \mathrm{dd^c}\, \mathrm{Tr}_Y^\theta(\varphi)\right)^m &= \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\, \mathrm{Tr}_Y^\theta(\varphi))) \\ &\leq \varliminf_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\} \\ &\leq \varlimsup_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)) \right\} \\ &\leq \lim_{k\to\infty} \frac{m!}{k^m} h^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi)|_Y) \\ &= \int_Y \left(\theta|_Y + \mathrm{dd^c}\, \mathrm{Tr}_Y^\theta(\varphi)\right)^m .\end{aligned}$$

Remark 8.3.1 One could also show that when (8.8) fails, the right-hand side of (8.9) is 0. See [DX24].

8.4 Analytic Bertini theorems

Let X be a connected projective manifold of dimension $n \geq 1$.

The analytic Bertini theorem handles the restriction along a generic subvariety.

Theorem 8.4.1 *Let $\varphi \in \mathrm{QPSH}(X)$. Let $p\colon X \to \mathbb{P}^N$ be a morphism ($N \geq 1$). Define*

$$\mathcal{G} := \{H \in |\mathcal{O}_{\mathbb{P}^N}(1)| : H' := H \cap X \text{ is smooth and } \mathcal{I}(\varphi|_{H'}) = \mathrm{Res}_{H'}(\mathcal{I}(\varphi))\} .$$

Then $\mathcal{G} \subseteq |\mathcal{O}_{\mathbb{P}^N}(1)|$ is co-pluripolar.

Recall that co-pluripolar sets are defined in Definition 1.1.4. We adopt the convention that $\mathcal{I}(-\infty) = 0$.

Remark 8.4.1 Here and in the sequel, we slightly abuse the notation by writing $H \cap X$ for $p^{-1}H$, the scheme-theoretic inverse image of H. In other words, $H \cap X := H \times_{\mathbb{P}^N} X$.

By definition, any $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ such that $p^{-1}H = \emptyset$ lies in $\mathcal{G}$.

Proof Take an ample line bundle L with a smooth Hermitian metric h such that $c_1(L, h) + \mathrm{dd^c}\varphi \geq 0$, where $c_1(L, h)$ is the first Chern form of (L, h), namely the curvature form of h. We introduce $\Lambda := |\mathcal{O}_{\mathbb{P}^N}(1)|$ to simplify our notations.

Step 1. We prove that the following set is co-pluripolar:

$$\begin{aligned}\mathcal{G}_L := \big\{H \in \Lambda : H \cap X \text{ is smooth and } &\mathrm{H}^0\left(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})\right) = \\ &\mathrm{H}^0\left(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))\right)\big\} .\end{aligned}$$

Here $\omega_{H\cap X}$ denotes the dualizing sheaf of $H \cap X$.

Let $U \subseteq \Lambda \times X$ be the closed subvariety whose $\mathbb{C}$-points correspond to pairs $(H, x) \in \Lambda \times X$ with $p(x) \in H$. Let $\pi_1 : U \to \Lambda$ be the natural projection. We may assume that π_1 is surjective, as otherwise there is nothing to prove.

Observe that U is a local complete intersection scheme by *Krulls Hauptidealsatz* and *a fortiori* a Cohen–Macaulay scheme. It follows from miracle flatness [Mat89, Theorem 23.1] that the natural projection $\pi_2 \colon U \to X$ is flat. As the fibers of π_2 over closed points of X are isomorphic to $\mathbb{P}^{N-1}$, it follows that π_2 is smooth. Thus, U is smooth as well. Moreover, observe that

$$\mathcal{I}(\pi_2^*\varphi) = \pi_2^*\mathcal{I}(\varphi) \tag{8.17}$$

by Proposition 1.4.5.

In the following, we will construct pluripolar sets $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subsetneq \Lambda$ such that the behaviour of π_1 is improved successively on the complement of Σ_i.

Step 1.1. The usual Bertini theorem shows that there is a proper Zariski closed set $\Sigma_1 \subsetneq \Lambda$ such that π_1 has smooth fibres outside Σ_1. Enlarging Σ_1, we could guarantee that π_1 and $\mathcal{I}(\pi_2^*\varphi)$ are both flat outside Σ_1. See [DG65, Théorème 6.9.1]. Then after further enlarging Σ_1 so that H avoids all associated points of $\mathcal{O}_X/\mathcal{I}(\varphi)$, for all $H \in \Lambda \setminus \Sigma_1$. Let $\pi_{1,H}$ denote the fibre of π_1 at H and write $i_H \colon \pi_{1,H} \to U$ for the inclusion morphism. We arrive at

$$\mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi)) = i_H^*\mathcal{I}(\pi_2^*\varphi)$$

for all $H \in \Lambda \setminus \Sigma_1$.[2]

Step 1.2. By Grauert's coherence theorem,

$$\mathcal{F}^i := R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L \otimes \mathcal{I}(\pi_2^*\varphi)\right)$$

is coherent for all i. Here $\omega_{U/\Lambda}$ denotes the relative dualizing sheaf of the morphism $U \to \Lambda$. Thus, there is a proper Zariski closed set $\Sigma_2 \subsetneq \Lambda$ such that

(1) $\Sigma_2 \supseteq \Sigma_1$.
(2) The $\mathcal{F}^i$'s are locally free outside Σ_2.

We write $\mathcal{F} = \mathcal{F}^0$. By cohomology and base change [Har77, Theorem III.12.11], for any $H \in \Lambda \setminus \Sigma_2$, the fibre $\mathcal{F}|_H$ of $\mathcal{F}$ is given by

$$\mathcal{F}|_H = \mathrm{H}^0\left(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \pi_2^*L|_{\pi_{1,H}} \otimes \mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi))\right).$$

Step 1.3. In order to proceed, we need to make use of the Hodge metric $h_{\mathcal{H}}$ on $\mathcal{F}$ defined in [HPS18]. We briefly recall its definition in our setting. By [HPS18, Section 22], we can find a proper Zariski closed set $\Sigma_3 \subsetneq \Lambda$ such that

(1) $\Sigma_3 \supseteq \Sigma_2$,
(2) π_1 is smooth outside Σ_3,
(3) both $\mathcal{F}$ and $\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)/\mathcal{F}$ are locally free outside Σ_3, and
(4) for each i,

$$R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)$$

is locally free outside Σ_3.

[2] This subtle point was overlooked in the proof of [Xia22a].

Then for any $H \in \Lambda \setminus \Sigma_3$,

$$\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \subseteq \mathcal{F}|_H \subseteq \mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X}).$$

See [HPS18, Lemma 22.1].

Now we can give the definition of the Hodge metric on $\Lambda\setminus\Sigma_3$. Given any $H \in \Lambda\setminus\Sigma_3$, any $\alpha \in \mathcal{F}|_H$, the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha, \alpha) := \int_{X\cap H} |\alpha|_h^2 \mathrm{e}^{-\varphi} \in [0, \infty].$$

Observe that $h_{\mathcal{H}}(\alpha, \alpha) < \infty$ if and only if $\alpha \in \mathrm{H}^0(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))$. Moreover, $h_{\mathcal{H}}(\alpha, \alpha) > 0$ if $\alpha \neq 0$. It is shown in [HPS18] (c.f. [PT18, Theorem 3.3.5]) that $h_{\mathcal{H}}$ is indeed a singular Hermitian metric, and it extends to a positive metric on $\mathcal{F}$.

Step 1.4. The determinant $\det h_{\mathcal{H}}$ is singular at all $H \in \Lambda \setminus \Sigma_3$ such that

$$\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \neq \mathcal{F}|_H.$$

As the map π_2 is smooth, we have $\pi_2^*\mathcal{I}(\varphi) = \mathcal{I}(\pi_2^*\varphi)$ by Proposition 1.4.5. Under the identification $\pi_{1,H} \cong H \cap X$, we have

$$\mathrm{Res}_{\pi_{1,H}}\left(\pi_2^*\mathcal{I}(\varphi)\right) \cong \mathrm{Res}_{H\cap X}\left(\mathcal{I}(\varphi)\right).$$

Thus, we have the following inclusions:

$$\begin{aligned} &\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \\ \subseteq &\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))), \end{aligned}$$

the right-hand side being $\mathcal{F}|_H$.

Recall that the first inclusion follows from Theorem 1.4.5. Hence, $\det h_{\mathcal{H}}$ is singular at all $H \in |\mathcal{O}_{\mathbb{P}^N}(1)| \setminus \Sigma_3$ such that

$$\begin{aligned} &\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \\ \neq &\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))). \end{aligned}$$

Let Σ_4 be the union of Σ_3 and the set of all such H. Since the Hodge metric $h_{\mathcal{H}}$ is positive ([PT18, Theorem 3.3.5] and [HPS18, Theorem 21.1]), its determinant $\det h_{\mathcal{H}}$ is also positive ([Rau15, Proposition 1.3] and [HPS18, Proposition 25.1]), it follows that Σ_4 is pluripolar. As a consequence, $\mathcal{G}_L$ is co-pluripolar.

Step 2.

Fix an ample invertible sheaf S on X. The same result holds with $L \otimes S^{\otimes a}$ in place of L. Thus, the set

$$A := \bigcap_{a=0}^{\infty} \mathcal{G}_{L\otimes S^{\otimes a}}$$

is co-pluripolar. For each $H \in W$ such that $X \cap H$ is smooth and $\mathcal{I}(\varphi|_{X\cap H}) \neq \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))$, let $\mathcal{K}$ be the following cokernel:

$$0 \to \mathcal{I}(\varphi|_{X\cap H}) \to \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi)) \to \mathcal{K} \to 0.$$

By Serre vanishing theorem, taking a large enough, we may guarantee that

$$\mathrm{H}^1(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{I}(\varphi|_{X\cap H})) = 0$$

and

$$\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{K}) \neq 0.$$

Then

$$\begin{aligned} &\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathcal{I}(\varphi|_{X\cap H})) \neq \\ &\mathrm{H}^0(X \cap H, \omega_{X\cap H} \otimes (L \otimes S^{\otimes a})|_{X\cap H} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))). \end{aligned}$$

Thus, $H \notin A$. We conclude that $\mathcal{G}$ is co-pluripolar. □

Remark 8.4.2 More generally, the same technique implies the following general result: Let $f : X \to Y$ be a projective morphism between complex manifolds and (L, h) be a Hermitian pseudoeffective line bundle on X. Then for quasi-every[3] $y \in Y$, the fiber X_y is smooth and

$$\mathcal{I}(\lambda h|_{X_y}) = \mathrm{Res}_{X_y}(\mathcal{I}(\lambda h)).$$

In the sequel of this section, we fix a base-point free linear system Λ on X.

Corollary 8.4.1 *Let $\varphi \in \mathrm{QPSH}(X)$. Then for quasi-every $H \in \Lambda$, we have $\varphi|_H \not\equiv -\infty$.*

Proof This follows immediately from Theorem 8.4.1. □

Corollary 8.4.2 *Assume that $n \geq 2$. Let $\varphi \in \mathrm{QPSH}(X)$. Then quasi-every $H \in \Lambda$ is connected and smooth, satisfies $\nu(\varphi, H) = 0$ and we have*

$$\mathrm{Tr}_H(\varphi) \sim_{\mathcal{I}} \varphi|_H.$$

The assumption $n \geq 2$ is only to guarantee that a general element $H \in \Lambda$ is connected, since we developed most of our theories only in this case.

Proof First observe that the set $\{x \in X : \nu(\varphi, x) > 0\}$ is a countable union of proper analytic subsets by Theorem 1.4.1. It follows that a very general element in Λ is not contained in this set.

Fix an ample line bundle L so that there is a smooth psh metric h_L such that $c_1(L, h_L) + \mathrm{dd^c}\varphi$ is a Kähler current. Thanks to Theorem 8.4.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that each $H \in \Lambda'$ satisfies the following:

(1) H is smooth;
(2) $\nu(\varphi, H) = 0$;

[3] That is, for all y outside a pluripolar subset of Y.

(3) $\mathcal{I}(k\varphi|_H) = \mathrm{Res}_H(\mathcal{I}(k\varphi))$ for all $k > 0$.

It follows from Theorem 8.3.1 and Theorem 7.3.1 that

$$\int_H \left(c_1(L,h_L)|_H + \mathrm{dd^c}\,\mathrm{Tr}_Y^{c_1(L,h_L)}(\varphi)\right)^{n-1} = \int_H (c_1(L,h_L)|_H + \mathrm{dd^c}\varphi|_H)^{n-1}.$$

Since $\varphi|_H \preceq_P \mathrm{Tr}_Y(\varphi)$ by Proposition 8.1.3, our assertion follows. □

Lemma 8.4.1 *Assume that $n \geq 2$. Let T be a closed positive $(1,1)$-current on X with $\int_X T^n > 0$. Then quasi-every $H \in \Lambda$ is connected and smooth, $T|_H$ is well-defined and satisfies*

$$\int_H T|_H^{n-1} > 0.$$

Proof Write $T = \theta_\varphi$ for some smooth closed real $(1,1)$-form θ on X and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Thanks to Lemma 2.3.2, we can find $\psi \in \mathrm{PSH}(X,\theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. By Corollary 8.4.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that each $H \in \Lambda'$ satisfies:

(1) H is smooth and connected;
(2) the restriction $\psi|_H$ is not identically $-\infty$.

Therefore, $\psi|_H \leq \varphi|_H$ are two potentials in $\mathrm{PSH}(H,\theta|_H)$ for any $H \in \Lambda'$. Our assertion follows from Theorem 2.3.2. □

Corollary 8.4.3 *Assume that $n \geq 2$. Let T be a closed positive $(1,1)$-current on X with $\mathrm{vol}\,T > 0$. Then quasi-every $H \in \Lambda$ is connected and smooth, and $\mathrm{Tr}_H^{[T]|_H}(T)$ is well-defined.*

Proof This follows from Example 8.1.3, Corollary 8.4.2 and Lemma 8.4.1. □

Proposition 8.4.1 *Assume that $n \geq 2$. Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Assume that $\varphi \preceq_P \psi$. Then quasi-every $H \in \Lambda$ is connected and smooth, and $\varphi|_H \preceq_P \psi|_H$.*

Proof Thanks to Lemma 6.1.3, we may replace φ by $\varphi \vee \psi$ and assume that $\varphi \sim_P \psi$. It suffices to show that $\varphi|_H \sim_P \psi|_H$ for quasi-every $H \in \Lambda$.

Take a smooth closed real $(1,1)$-form θ on X so that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. It suffices to compare φ and ψ with $P_\theta[\varphi]$, so without loss of generality, we may assume that ψ is a model potential in $\mathrm{PSH}(X,\theta)_{>0}$. Up to adding a constant to φ, we may then assume that $\varphi \leq \psi$. It follows from Lemma 2.3.1 that we can find a sequence $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ such that

$$j^{-1}\eta_j + \left(1 - j^{-1}\right)\psi \leq \varphi$$

for all $j \geq 2$. By Corollary 8.4.1, Lemma 8.4.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that any $H \in \Lambda'$ satisfies:

(1) H is smooth and connected;
(2) $\eta_j|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$ for all $j \geq 2$ and $\psi|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$.

Therefore, taking Proposition 3.1.6 into account, we arrive at

$$j^{-1}P_{\theta|_H}[\eta_j|_H] + \left(1 - j^{-1}\right) P_{\theta|_H}[\psi|_H] \leq P_{\theta|_H}[\varphi|_H]$$

for all $j \geq 2$. Letting $j \to \infty$, we conclude that

$$P_{\theta|_H}[\psi|_H] \leq P_{\theta|_H}[\varphi|_H]$$

and hence $\psi|_H \preceq_P \varphi|_H$. □

Lemma 8.4.2 *Assume that $n \geq 2$. Let θ be a closed smooth $(1,1)$-form on X representing a big cohomology class and $(\varphi_j)_j$ be a decreasing sequence in* $\mathrm{PSH}(X,\theta)$. *Assume that $\varphi \in \mathrm{PSH}(X,\theta)$ and $\varphi_j \xrightarrow{d_S} \varphi$. Then quasi-every $H \in \Lambda$ is connected and smooth, $\varphi_j|_H \not\equiv -\infty$ for all $j \geq 1$, $\varphi|_H \not\equiv -\infty$, and*

$$\varphi_j|_H \xrightarrow{d_S} \varphi|_H.$$

Proof By Corollary 6.2.8, we may assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Using Lemma 2.3.1, we could find a decreasing sequence $(\epsilon_j)_j$ in $(0,1)$ with limit 0 and $\eta_j \in \mathrm{PSH}(X,\theta)_{>0}$ such that $\eta_j \leq \varphi_j$ and

$$\epsilon_j\eta_j + (1-\epsilon_j)\varphi_j \leq \varphi.$$

By Corollary 8.4.1, Lemma 8.4.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that any $H \in \Lambda'$ satisfies:

(1) H is smooth and connected;
(2) $\eta_j|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$ for all $j \geq 1$ and $\varphi|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$.

Therefore, taking Proposition 3.1.6 into account, we arrive at

$$\epsilon_j P_{\theta|_H}[\eta_j|_H] + \left(1-\epsilon_j\right) P_{\theta|_H}[\varphi_j|_H] \leq P_{\theta|_H}[\varphi|_H].$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} P_{\theta|_H}[\varphi_j|_H] \leq P_{\theta|_H}[\varphi|_H].$$

By Theorem 2.3.2 and Proposition 3.1.9, we conclude that

$$\lim_{j\to\infty}\int_H (\theta|_H + \mathrm{dd^c}\varphi_j|_H)^{n-1} = \int_H (\theta|_H + \mathrm{dd^c}\varphi|_H)^{n-1}.$$

Therefore, using Corollary 6.2.5, we conclude that $\varphi_j|_H \xrightarrow{d_S} \varphi|_H$. □

Corollary 8.4.4 *Assume that $n \geq 2$. Let $\varphi \in$* $\mathrm{QPSH}(X)$ *be an $\mathcal{I}$-good potential. Then quasi-every $H \in \Lambda$ satisfies:*

(1) H is connected and smooth;
(2) $\varphi|_H \in$ $\mathrm{PSH}(X,\theta|_H)$ *is $\mathcal{I}$-good;*

(3) $\nu(\varphi, H) = 0$;
(4) $\mathrm{Tr}_H\, \varphi \sim_P \varphi|_H$.

Furthermore, if θ *is a closed smooth real* $(1,1)$*-form on* X *such that* $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$*, then we could further guarantee that* $\mathrm{Tr}_H(\varphi)$ *has a representative* $\mathrm{Tr}_H(\varphi) \in \mathrm{PSH}(H, \theta|_H)_{>0}$ *for all* $H \in \Lambda'$.

Proof This is a consequence of Lemma 8.4.2, Theorem 7.1.1, Corollary 8.4.2 and Corollary 8.4.3. □

For later use, let us also prove a reverse Bertini theorem herem.

Lemma 8.4.3 (Reverse Bertini theorem) *Let* X *be a complex manifold,* $f : X \to \Delta^*$ *be a projective surjective morphism to the punctured unit disk* Δ^**. Let* (L,h)*,* (L,h') *be Hermitian pseudo-effective line bundles on* X *with the same underlying line bundle. Assume that there is a biholomorphic* S^1*-action on* (X,L) *making* f *equivariant and such that* h *and* h' *are invariant under this action. Assume that for quasi-every* $z \in \Delta^*$*,* X_z *is smooth and* $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$*, then* $h \sim_{\mathcal{I}} h'$.

Proof We need to show that $\mathcal{I}(kh) = \mathcal{I}(kh')$ for all positive integer k. Clearly, it suffices to prove the case $k = 1$. We will therefore prove $\mathcal{I}(h) = \mathcal{I}(h')$. First observe that it suffices to prove that

$$f_*(K_X \otimes L \otimes \mathcal{I}(h)) = f_*(K_X \otimes L \otimes \mathcal{I}(h')) \tag{8.18}$$

as subsheaves of $f_*(K_X \otimes L)$. In fact, suppose that (8.18) holds. Let H be a f-ample invertible sheaf on X, then (8.18) also holds with $L \otimes H^m$ in place of L. It follows from Grauert–Remmert's version of Serre vanishing theorem [BS76, Theorem 2.1(A)] that $\mathcal{I}(h) = \mathcal{I}(h')$.

It remains to prove (8.18). Observe that both sides of (8.18) are locally free by [Mat18, Corollary 1.5]. We claim that it suffices to show that

$$f_*(K_X \otimes L \otimes \mathcal{I}(h))_z = f_*(K_X \otimes L \otimes \mathcal{I}(h'))_z \tag{8.19}$$

for one $z \in \Delta^*$. In fact, this implies that the same holds outside a countable subset of Δ^*. But the set where (8.19) fails has to be S^1-invariant, it has to be empty.

In fact, we will prove (8.19) for quasi-every $z \in \Delta^*$. By cohomology and base change together with the analytic Bertini theorem Remark 8.4.2, for quasi-every $z \in \Delta^*$, we have

$$\begin{aligned} f_*(K_X \otimes L \otimes \mathcal{I}(h))_z &= \mathrm{H}^0(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h|_{X_z})), \\ f_*(K_X \otimes L \otimes \mathcal{I}(h'))_z &= \mathrm{H}^0(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h'|_{X_z})). \end{aligned}$$

But we assumed that for quasi-every z, $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$, it follows that for quasi-every $z \in \Delta^*$, (8.19) holds. The proof is complete. □

Chapter 9
Test curves

Comment se fait-il que M. Gauss ait osé vous faire dire que la plupart de vos théorèmes lui étaient connus et qu'il en avait fait la découverte dès 1808. Cet excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertss des autres.
— Adrien-Marie Legendre, in a letter to Jacobi in 1827

In this chapter, we develop the theory of test curves. Roughly speaking, a test curve is a concave curve of model potentials. In Section 9.2, we will prove the Ross–Witt Nyström correspondence, through which the test curves are related to geodesic rays in the space of quasi-plurisubharmonic functions. In Section 9.4, we define operations on test curves, anticipating applications in non-Archimedean pluripotential theory in Chapter 13.

9.1 The notion of test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

Definition 9.1.1 A *test curve* Γ in $\mathrm{PSH}(X,\theta)$ consists of a real number $\Gamma_{\max}$ together with a map $(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta)$ denoted by $\tau \mapsto \Gamma_\tau$ satisfying the following conditions:

(1) The map $\tau \mapsto \Gamma_\tau$ is concave and decreasing;
(2) each Γ_τ is a model potential;
(3) the potential
$$\Gamma_{-\infty} \coloneqq \sup_{\tau<\Gamma_{\max}}{}^*\Gamma_\tau \tag{9.1}$$
satisfies
$$\int_X (\theta + \mathrm{dd^c}\Gamma_{-\infty})^n > 0.$$

Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. The set of test curves Γ with $\Gamma_{-\infty} = \phi$ is denoted by $\mathrm{TC}(X,\theta;\phi)$.

The union of all $\mathrm{TC}(X,\theta;\phi)$'s for various model potentials $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ is denoted by $\mathrm{TC}(X,\theta)_{>0}$[1].

[1] This is a poorly chosen notation. Considering the analogy with $\mathcal{E}(X,\theta)$ and $\mathcal{E}(X,\theta;\phi)$, we should have reserved $\mathrm{TC}(X,\theta)$ to $\mathrm{TC}(X,\theta;V_\theta)$.

By (2), $\sup_X \Gamma_\tau = 0$ for each $\tau < \Gamma_{\max}$. So $\Gamma_{-\infty} \in \mathrm{PSH}(X, \theta)$ by Proposition 1.2.1. Moreover, $\Gamma_{-\infty}$ is a model potential by Proposition 3.1.10.

Remark 9.1.1 Sometimes it is convenient to extend Γ_τ to $\tau \geq \Gamma_{\max}$ as well. This can be done as follows: For $\tau > \Gamma_{\max}$, we set $\Gamma_\tau \equiv -\infty$. For $\tau = \Gamma_{\max}$, we set

$$\Gamma_\tau \coloneqq \inf_{\tau' < \Gamma_{\max}} \Gamma_{\tau'} \in \mathrm{PSH}(X, \theta).$$

We will always make this extension in the sequel.

Recall that according to our general principle, we only talk about model potentials when a potential has positive mass. Fortunately, this principle is not violated in the above definition, as shown below:

Lemma 9.1.1 *Assume that $\Gamma \in \mathrm{TC}(X, \theta)_{>0}$. Then for each $\tau < \Gamma_{\max}$, we have*

$$\int_X (\theta + \mathrm{dd^c}\Gamma_\tau)^n > 0. \tag{9.2}$$

Proof Fix $\tau \in (-\infty, \Gamma_{\max})$.

By assumption, $\Gamma_{-\infty}$ has positive mass. By Corollary 2.3.1, we have

$$\int_X \theta^n_{\Gamma_{-\infty}} = \lim_{\tau \to -\infty} \int_X \theta^n_{\Gamma_\tau}.$$

In particular, for a sufficiently small $\tau_0 < \tau$, we have

$$\int_X \theta^n_{\Gamma_{\tau_0}} > 0.$$

Now take $\tau' \in (\tau, \Gamma_{\max})$ and $t \in (0, 1)$ so that

$$\tau = (1 - t)\tau' + t\tau_0.$$

From the concavity of Γ, we find that

$$\Gamma_\tau \geq (1 - t)\Gamma_{\tau'} + t\Gamma_{\tau_0}.$$

By Theorem 2.3.2,

$$\int_X \theta^n_{\Gamma_\tau} \geq \int_X \theta^n_{(1-t)\Gamma_{\tau'} + t\Gamma_{\tau_0}} \geq t^n \int_X \theta^n_{\Gamma_{\tau_0}} > 0$$

and (9.2) follows. □

Proposition 9.1.1 *Let $\Gamma \in \mathrm{TC}(X, \theta)_{>0}$. Then the map*

$$[-\infty, \Gamma_{\max}) \to \mathbb{R}, \quad \tau \mapsto \log \int_X \theta^n_{\Gamma_\tau}$$

is concave and continuous.

Proof The concavity of this function follows from Theorem 2.3.3 and Theorem 2.3.2. The continuity at $-\infty$ is a consequence of Corollary 2.3.1. □

Definition 9.1.2 Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential.

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to be *bounded* if for τ small enough, $\Gamma_\tau = \phi$. The subset of bounded test curves in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathrm{TC}^\infty(X,\theta;\phi)$. In this case, we write

$$\Gamma_{\min} := \max\{\tau \in \mathbb{R} : \Gamma_\tau = \phi\}.$$

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to have *finite energy* if

$$\mathbf{E}^\phi(\Gamma) := \Gamma_{\max}\int_X \theta_\phi^n + \int_{-\infty}^{\Gamma_{\max}}\left(\int_X \theta_{\Gamma_\tau}^n - \int_X \theta_\phi^n\right)\mathrm{d}\tau > -\infty. \tag{9.3}$$

When $\phi = V_\theta$, we write $\mathbf{E}$ instead of $\mathbf{E}^\phi$.

The subset of test curves with finite energy is denoted by $\mathrm{TC}^1(X,\theta;\phi)$.

Example 9.1.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, there is a canonically associated test curve $\Gamma^\varphi \in \mathrm{TC}^\infty(X,\theta;V_\theta)$: Set $\Gamma^\varphi_{\max} = 0$ and

$$\Gamma^\varphi_\tau = \begin{cases} V_\theta, & \text{if } \tau \leq -1;\\ P_\theta\left[(1+\tau)\varphi - \tau V_\theta\right], & \text{if } -1 < \tau < 0.\end{cases}$$

Note that Γ^φ is indeed a test curve, as follows from Proposition 3.1.6.

We first observe that the notion of test curves does not really depend on the choice of θ within its cohomology class.

Proposition 9.1.2 *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. Let $\phi' \in \mathrm{PSH}(X,\theta')_{>0}$ be the unique model potential satisfying $\phi \sim \phi'$.*

Then there is a canonical bijection

$$\mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(X,\theta';\phi').$$

This bijection induces the following bijections:

$$\mathrm{TC}^1(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^1(X,\theta';\phi'),\quad \mathrm{TC}^\infty(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^\infty(X,\theta';\phi').$$

These bijections satisfy the obvious cocycle conditions.

Proof Choose $g \in C^\infty(X)$ such that $\theta' = \theta + \mathrm{dd^c} g$. Given any $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we observe that $\Gamma' : (-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X,\theta')$ defined as

$$\tau \mapsto P_{\theta'}[\Gamma_\tau - g]$$

lies in $\mathrm{TC}(X,\theta';\phi')$. Moreover, the choice of g is irrelevant since for any other choice of g, say g', we have

$$\Gamma_\tau - g \sim \Gamma_\tau - g'$$

for all $\tau < \Gamma_{\max}$. All assertions follow directly from the definition. □

Proposition 9.1.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold. Then the pointwise pull-back induces a bijection*

$$\pi^*\colon \mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This follows immediately from Proposition 3.1.5. □

Proposition 9.1.4 *Let Γ be a test curve in $\mathrm{PSH}(X,\theta)$. For each $x \in X$, the map $\mathbb{R} \ni \tau \mapsto \Gamma_\tau(x)$ is a closed concave function. Moreover, the map is proper as long as $\Gamma_{\Gamma_{\max}}(x) \neq -\infty$.*

The notion of closeness is recalled in Definition A.1.7.

Proof We argue the closeness. Fix $x \in X$. Assume that $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. We only need to argue the upper-semicontinuity of $\tau \mapsto \Gamma_\tau(x)$. The upper semi-continuity is clear at $\tau \geq \Gamma_{\max}$, so we are reduced to prove the following:

$$\Gamma_\tau = \inf_{\tau'<\tau} \Gamma_{\tau'} \tag{9.4}$$

for any $\tau < \Gamma_{\max}$. Take $\tau'' \in (\tau, \Gamma_{\max})$. Outside the polar locus of $\Gamma_{\tau''}$, we know that (9.4) holds by continuity. So (9.4) holds everywhere by Proposition 1.2.6.

The final assertion is trivial. □

Definition 9.1.3 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a smooth closed real positive $(1,1)$-form. Then we define $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$ as follows:

(1) Define

$$P_{\theta+\omega}[\Gamma]_{\max} = \Gamma_{\max};$$

(2) for each $\tau < \Gamma_{\max}$, define

$$P_{\theta+\omega}[\Gamma]_\tau = P_{\theta+\omega}[\Gamma_\tau].$$

It follows form Proposition 3.1.6 that $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$.

9.2 Ross–Witt Nyström correspondence

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proposition 9.1.4 allows us to talk about the Legendre transforms in the expected way.

The general definition of the Legendre transform Definition A.2.1 can be translated as follows:

Definition 9.2.1 Let $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. We define its *Legendre transform* as $\Gamma^* : (0, \infty) \to \mathrm{PSH}(X, \theta)$ given by

$$\Gamma_t^* = \sup_{\tau \in \mathbb{R}} (t\tau + \Gamma_\tau) . \tag{9.5}$$

Thanks to Remark 9.1.1, (9.5) can be equivalently written as

$$\Gamma_t^* = \sup_{\tau < \Gamma_{\max}} (t\tau + \Gamma_\tau) = \sup_{\tau \le \Gamma_{\max}} (t\tau + \Gamma_\tau) .$$

It is sometimes handy to *define*

$$\Gamma_0^* \coloneqq \phi \tag{9.6}$$

at $t = 0$. But it is important to remember by doing so, (9.5) is not true at $t = 0$.

Remark 9.2.1 Here we do not talk about the case $t < 0$ because its behavior is pretty trivial: Take $x \in X$, if $\Gamma_\tau(x) = -\infty$ for all $\tau < \Gamma_{\max}$, then $\Gamma_t^* = -\infty$; otherwise, $\Gamma_t^* = \infty$.

The information about $t > 0$ suffices to characterize Γ.

Proposition 9.2.1 *Let $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. Then*

$$\Gamma_\tau = \inf_{t > 0} \left(\Gamma_t^* - t\tau \right) \tag{9.7}$$

for all $\tau \in \mathbb{R}$.

Due to our convention (9.6), in (9.7) we could as well take $t \ge 0$.

Proof Fix $x \in X$. We want to establish (9.7) at x. We distinguish two cases. First suppose that $\Gamma_\tau(x) = -\infty$ for all $\tau < \Gamma_{\max}$ and hence all $\tau \in \mathbb{R}$. In this case, we have $\Gamma_t^*(x) = -\infty$ for all $t > 0$. Therefore, (9.7) follows trivially.

Otherwise, by Remark 9.2.1, we know that $\Gamma_t^*(x) = \infty$ for all $t < 0$. The relative interior of the domain of $t \mapsto \Gamma_t^*(x)$ is contained in $(0, \infty)$. Therefore, (9.7) follows from Theorem A.2.1, Proposition 9.1.4. □

In Definition 9.2.1, we have made a non-trivial claim that $\Gamma_t^* \in \mathrm{PSH}(X, \theta)$ for all $t > 0$. Let us prove this.

Lemma 9.2.1 *Let $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. Then $\Gamma_t^* \in \mathrm{PSH}(X, \theta)$ for all $t > 0$. In fact, Γ is upper semicontinuous as a function of $X \times (0, \infty)$.*

Proof We first observe that for each $x \in X$, we have

$$\Gamma_t^*(x) \le t\Gamma_{\max} < \infty.$$

Let $R = \{a + \mathrm{i}b \in \mathbb{C} : a > 0, b \in \mathbb{R}\}$. We consider

$$F : X \times R \to [-\infty, \infty), \quad (x, a + \mathrm{i}b) \mapsto \Gamma_a^*(x).$$

Let $\pi\colon X\times R\to X$ be the natural projection. Observe that the upper semicontinuous envelope G of F is $\pi^*\theta$-psh by Proposition 1.2.1. It suffices to show that $F=G$. We let

$$E\coloneqq\{(x,z)\in X\times R : F(x,z)<G(x,z)\}\,.$$

We want to argue that $E=\varnothing$. Clearly, E can be written as $B\times \mathrm{i}\mathbb{R}$ for some set $B\subseteq X\times(0,\infty)$. Since E is a pluripolar set by Proposition 1.2.5, it has zero Lebesgue measure. Hence, B has zero Lebesgue measure. For each $x\in X$, write

$$B_x=\{t\in(0,\infty):(t,x)\in B\}\,.$$

By Fubini's theorem, B_x has vanishing 1-dimensional Lebesgue measure for all $x\in X\setminus Z$, where $Z\subseteq X$ is a subset of measure 0. We may assume that $Z\supseteq\{\Gamma_{\Gamma_{\max}}=-\infty\}$ so that for $x\in X\setminus Z$, $\Gamma_t(x)\neq-\infty$ for all $t>0$.

For any $x\in X\setminus Z$, both $t\mapsto F(x,t)$ and $G(x,t)$ are convex functions with values in $\mathbb{R}$ on $(0,\infty)$. They agree almost everywhere, hence everywhere by their continuity. It follows that for $x\in X\setminus Z$, we have $B_x=\varnothing$.

By Proposition 9.2.1, for any $x\in X$, we have

$$\Gamma_\tau(x)=\inf_{t>0}(F(x,t)-t\tau),\quad \tau<\Gamma_{\max}.$$

On the other hand, let

$$\chi_\tau(x)=\inf_{t>0}(G(x,t)-t\tau),\quad \tau<\Gamma_{\max},\quad x\in X. \tag{9.8}$$

By Kiselman's principle Proposition 1.2.8, $\chi_\tau\in\mathrm{PSH}(X,\theta)$. But on $X\setminus Z$, we already know that $\Gamma_\tau=\chi_\tau$ for all $\tau<\Gamma_{\max}$. By Proposition 1.2.6,

$$\Gamma_\tau=\chi_\tau,\quad \tau<\Gamma_{\max}.$$

Now we conclude that $F(x,t)=G(x,t)$ by Corollary A.2.1. □

Corollary 9.2.1 *Let $\Gamma\in\mathrm{TC}(X,\theta;\phi)$. Then $\Gamma^*_t\in\mathcal{E}(X,\theta;\phi)$ for all $t>0$.*

Proof Fix $t>0$. We already know that $\Gamma^*_t\in\mathrm{PSH}(X,\theta)$ by Lemma 9.2.1. It suffices to show that

$$\Gamma^*_t\sim_P\phi.$$

From (9.5) and Proposition 6.1.6, we know that

$$\Gamma^*_t\sim_P\sup_{\tau<\Gamma_{\max}}{}^*\Gamma_\tau=\phi.$$

Lemma 9.2.2 *Let $\Gamma\in\mathrm{TC}(X,\theta;\phi)$, then*

$$\sup_X\Gamma^*_t=t\Gamma_{\max}$$

for all $t>0$.

In particular, $t \mapsto \Gamma^*_t - t\Gamma_{\max}$ is a decreasing function in $t > 0$.

Proof Choose $x \in X$ such that $\Gamma_{\Gamma_{\max}}(x) = 0$. Then $\Gamma_\tau(x) = 0$ for all $\tau < \Gamma_{\max}$, and hence for all $t > 0$,

$$\Gamma^*_t(x) = t\Gamma_{\max}$$

by definition. On the other hand, since $\Gamma_\tau \leq 0$ for all $\tau < \Gamma_{\max}$, we have

$$\sup_X \Gamma^*_t \leq t\Gamma_{\max}.$$

Lemma 9.2.3 *Given $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we have $\Gamma^* \in \mathcal{R}(X,\theta;\phi)$.*

See Definition 4.2.3 for the notation $\mathcal{R}(X,\theta;\phi)$.

Proof It follows from Lemma 9.2.1, (9.5) and Proposition 1.2.1 that Γ^* is a subgeodesic ray. By Corollary 9.2.1, for any $t > 0$, $\Gamma^*_t \in \mathcal{E}(X,\theta;\phi)$.

First observe that as $t \to 0+$, we have

$$\Gamma^*_t \xrightarrow{L^1} \phi. \tag{9.9}$$

By Lemma 9.2.2 and Proposition 1.5.1, it suffices to show each L^1-cluster point $\psi \in \mathrm{PSH}(X,\theta)$ as Γ^*_t as $t \to 0$ is equal to ϕ.

To see this, first observe that by (9.5), for any fixed $t > 0$,

$$\Gamma^*_t(x) \leq t\Gamma_{\max} + \phi(x).$$

Therefore, $\psi \leq \phi$. On the other hand, for any fixed $\tau < \Gamma_{\max}$, by (9.5), we have

$$\Gamma^*_t \geq \Gamma_\tau + t\tau$$

for any $t > 0$. So $\psi \geq \Gamma_\tau$ almost everywhere and hence everywhere by Proposition 1.2.6. It follows that $\psi \geq \phi$. Therefore, $\psi = \phi$.

Assume that Γ^* is not a geodesic ray. Then we can find $0 \leq a < b$ such that $(\Gamma^*_t)_{t\in(a,b)}$ differs from the geodesic $(\eta_t)_{t\in(a,b)}$ from Γ^*_a to Γ^*_b. The existence of $(\eta_t)_t$ is guaranteed by Proposition 4.2.1. We consider the subgeodesic $(\ell_t)_{t>0}$ given by $\ell_t = \eta_t$ for $t \in (a,b)$ and $\ell_t = \Gamma^*_t$ otherwise. Note that ℓ is a subgeodesic due to Lemma 1.2.2.

Consider the Legendre transform

$$\Gamma'_\tau = \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}.$$

Then $\Gamma'_\tau \geq \Gamma_\tau$ and $\Gamma'_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$ by Proposition 1.2.8 for all $\tau \in \mathbb{R}$.

We claim that

$$\Gamma'_\tau \leq \Gamma_\tau + (b-a)(\Gamma_{\max} - \tau), \quad \tau \in \mathbb{R}. \tag{9.10}$$

Observe that $\Gamma'_\tau \equiv -\infty$ when $\tau > \Gamma_{\max}$ by Lemma 9.2.2. So it suffices to consider $\tau \leq \Gamma_{\max}$. In this case, we compute

$$\inf_{t\in[a,b]}(\ell_t - t\tau) \le \Gamma_b^* - b\tau \le (b-a)(\Gamma_{\max} - \tau) + \inf_{t\in[a,b]}(\Gamma_t^* - t\tau),$$

where we applied Lemma 9.2.2. Therefore, (9.10) follows. In particular, for any $\tau < \Gamma_{\max}$, we have $\Gamma'_\tau \sim \Gamma_\tau$. On the other hand, by definition of Γ'_τ, we clearly have $\Gamma'_\tau \le 0$ for all $\tau < \Gamma_{\max}$. It follows from the fact that Γ_τ is a model potential that $\Gamma_\tau = \Gamma'_\tau$ for all $\tau < \Gamma_{\max}$. Therefore, by Theorem A.2.1, we have $\Gamma_t^* = \ell'_t$ for all $t > 0$, which is a contradiction. □

Given $\ell \in \mathcal{R}(X, \theta; \phi)$, define its Legendre transform

$$\ell_\tau^* := \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}. \tag{9.11}$$

Lemma 9.2.4 *Given $\ell \in \mathcal{R}(X, \theta; \phi)$, then $\ell^* = (\ell^*_\tau)_{\tau < \sup_X \ell_1} \in \mathrm{TC}(X, \theta)$.*

Proof Note that it follows from Proposition 1.2.8 that $\ell^*_\tau \in \mathrm{PSH}(X, \theta) \cup \{-\infty\}$ for all $\tau \in \mathbb{R}$. It is clear that $\mathbb{R} \ni \tau \mapsto \ell^*_\tau$ is a decreasing and concave function.

By Proposition 4.2.4,

$$\sup_X \ell_t = t \sup_X \ell_1 \quad \forall t \ge 0.$$

Observe that $(0, \infty) \ni t \mapsto \ell_t - t \sup_X \ell_1$ is a decreasing sequence in $\mathrm{PSH}(X, \theta)$ with $\sup_X(\ell_t - t \sup_X \ell_1) = 0$. It follows that

$$\ell^*_{\sup_X \ell_1} = \inf_{t>0}\left(\ell_t - t\sup_X \ell_1\right) \in \mathrm{PSH}(X, \theta).$$

On the other hand, for $\tau > \sup_X \ell_1$, the same argument shows that

$$\ell^*_\tau \equiv -\infty.$$

Therefore, $\ell^*_\tau \in \mathrm{PSH}(X, \theta)$ if and only if $\tau \le \ell^*_{\max} := \sup_X \ell_1$.

We claim that $(\ell^*_\tau)_{\tau < \ell^*_{\max}}$ is a test curve. We first observe that for $\tau < \ell^*_{\max}$, we have

$$\ell_\tau \le \ell_1 - \tau \sim_P \phi.$$

Therefore,

$$\ell_\tau \preceq_P \phi, \quad \forall \tau < \ell^*_{\max}. \tag{9.12}$$

Also observe that for any $\tau \le \ell^*_{\max}$ and any $t > 0$, we have

$$\sup_X \ell^*_\tau \le \sup_X \ell_t - t\tau = \ell^*_{\max} t - t\tau.$$

Letting $t \to 0+$, we find that for any $\tau \le \ell^*_{\max}$, we have

$$\sup_X \ell^*_\tau \le 0. \tag{9.13}$$

Fix $\tau < \ell^*_{\max}$, we want to argue that

$$P_\theta[\ell^*_\tau] = \ell^*_\tau. \tag{9.14}$$

First we claim that for any $C > 0$, we have

$$(\ell^*_\tau + C) \wedge \phi = (\ell^*_\tau + C) \wedge V_\theta. \tag{9.15}$$

The $\leq$ direction is trivial. We argue the reverse inequality, which reduces to

$$\phi \geq (\ell^*_\tau + C) \wedge V_\theta.$$

Since ϕ is model and $(\ell^*_\tau + C) \wedge V_\theta \leq 0$, it suffices to show that

$$\phi \geq_P (\ell^*_\tau + C) \wedge V_\theta,$$

which follows from (9.12). Therefore, (9.15) is established. Thanks to (9.13), we have the obvious inequality

$$(\ell^*_\tau + C) \wedge V_\theta \geq \ell^*_\tau$$

for any $C > 0$. Therefore, in order to prove (9.14), it remains to argue that for any $C > 0$,

$$(\ell^*_\tau + C) \wedge \phi \leq \ell^*_\tau. \tag{9.16}$$

For this purpose, let us consider the following geodesics: For any $M > 0$ and $t \in [0, 1]$, let

$$\ell^{1,M}_t = \ell_{tM} - tM\tau, \quad \ell^{2,M}_t = (\ell^*_\tau + C) \wedge \phi - Ct.$$

It is clear that at $t = 0, 1$, we have $\ell^{2,M}_t \leq \ell^{1,M}_t$. Hence, the same holds for all $t \in [0, 1]$. In particular, for any fixed $s \in (0, 1]$, we have

$$(\ell^*_\tau + C) \wedge \phi - Cs \leq \ell_{sM} - sM\tau$$

for all $M > 0$. Taking infimum with respect to $M > 0$, we find

$$(\ell^*_\tau + C) \wedge \phi - Cs \leq \ell^*_\tau.$$

Since $s \in (0, 1]$ is arbitrary, we conclude (9.16). □

Theorem 9.2.1 *The Legendre transform in Definition 9.2.1 is a bijection*

$$\mathrm{TC}(X, \theta; \phi) \xrightarrow{\sim} \mathcal{R}(X, \theta; \phi). \tag{9.17}$$

Moreover, this bijection restricts to the following bijections:

$$\mathrm{TC}^1(X, \theta; \phi) \xrightarrow{\sim} \mathcal{R}^1(X, \theta; \phi), \quad \mathrm{TC}^\infty(X, \theta; \phi) \xrightarrow{\sim} \mathcal{R}^\infty(X, \theta; \phi). \tag{9.18}$$

For any $\Gamma \in \mathrm{TC}^1(X, \theta; \phi)$, we have

$$\mathbf{E}^\phi(\Gamma) = \mathbf{E}^\phi(\Gamma^*). \tag{9.19}$$

The correspondence (9.17) will be referred to as the Ross–Nyström correspondence.

Proof **Step 1**. We first establish (9.17).

It follows from Lemma 9.2.3 that the forward map is well-defined. The inverse map is given by (9.11). We show that the inverse map is also well-defined. Given $\ell \in \mathcal{R}(X,\theta;\phi)$, we know from Lemma 9.2.4 that $\ell^* \in \mathrm{TC}(X,\theta)$. We need to show that $\ell^* \in \mathrm{TC}(X,\theta;\phi)$.

By Corollary A.2.1 and Lemma 9.2.3, we know that

$$\ell = (\ell^*)^* \in \mathcal{R}(X,\theta;\ell^*_{-\infty}).$$

So it follows that $\ell^*_{-\infty} = \phi$. Therefore, $\ell^* \in \mathrm{TC}(X,\theta;\phi)$ as expected.

The two operations are inverse to each other thanks to Corollary A.2.1. Hence, (9.17) is established.

Step 2. Next we consider the bounded situation. Namely, we want to establish the second half of (9.18).

Suppose that $\Gamma \in \mathrm{TC}^\infty(X,\theta;\phi)$. Take $\tau_0 \in \mathbb{R}$ so that $\Gamma_\tau = \phi$ for all $\tau \leq \tau_0$. It follows from (9.5) that

$$\Gamma^*_t \geq \phi + t\tau_0$$

for all $t > 0$. Therefore, $\Gamma^*_t \sim \phi$ for all $t > 0$ and hence $\Gamma^* \in \mathcal{R}^\infty(X,\theta;\phi)$.

Conversely, suppose that $\ell \in \mathcal{R}^\infty(X,\theta;\phi)$. Thanks to Proposition 4.2.3, there is a constant $C > 0$ such that

$$\ell_t \geq \phi - Ct.$$

Therefore, according to (9.11), we have

$$\ell^*_\tau \geq \inf_{t>0} (\phi - (C+\tau)t) = \phi$$

if $\tau \leq -C$. Therefore, $\ell^*_\tau = \phi$ for all $\tau \leq -C$.

Step 3. We establish (9.19) and the first half of (9.18).

Step 3.1. We reduce to the case where $\Gamma_{\max} = 0$.

Suppose that we define

$$\Gamma'_\tau = \Gamma_{\tau+\Gamma_{\max}}, \quad \forall \tau < 0.$$

Then $\Gamma' \in \mathrm{TC}(X,\theta;\phi)$ as well and for all $t \geq 0$,

$$\Gamma'^*_t = \sup_{\tau<0} \left(t\tau + \Gamma'_\tau\right) = \sup_{\tau<\Gamma_{\max}} (t\tau + \Gamma_\tau) - t\Gamma_{\max} = \Gamma^*_t - t\Gamma_{\max}.$$

Therefore,

$$\mathbf{E}^\phi\left(\Gamma'^*\right) = \mathbf{E}^\phi\left(\Gamma^*\right) - \Gamma_{\max}\int_X \theta^n_\phi.$$

by (3.19). Using (9.3), we also have

$$\begin{aligned}\mathbf{E}^\phi(\Gamma') &= \int_{-\infty}^{0}\left(\int_X \theta^n_{\Gamma'_\tau} - \int_X \theta^n_\phi\right)\mathrm{d}\tau\\ &= \int_{-\infty}^{\Gamma_{\max}}\left(\int_X \theta^n_{\Gamma_\tau} - \int_X \theta^n_\phi\right)\mathrm{d}\tau\\ &= \mathbf{E}^\phi(\Gamma) - \Gamma_{\max}\int_X \theta^n_\phi.\end{aligned}$$

Therefore, it suffices to establish (9.19) for Γ' in place of Γ.

Step 3.2. We assume that $\Gamma_{\max} = 0$ and $\Gamma \in \mathrm{TC}^\infty(X,\theta;\phi)$. We prove (9.19).

For $N \in \mathbb{Z}_{>0}$, $M \in \mathbb{Z}$, we introduce the following:

$$\Gamma_t^{*,N,M} := \max_{\substack{k\in\mathbb{Z}\\ k\leq M}}\left(\Gamma_{k/2^N} + tk/2^N\right) \in \mathcal{E}^\infty(X,\theta;\phi),\qquad t>0.$$

We first claim that for all $t > 0$, $N \in \mathbb{Z}_{>0}$ and $M \in \mathbb{Z}$,

$$\frac{t}{2^N}\int_X \theta^n_{\Gamma_{(M+1)/2^N}} \leq E^\phi_\theta\left(\Gamma_t^{*,N,M+1}\right) - E^\phi_\theta\left(\Gamma_t^{*,N,M}\right) \leq \frac{t}{2^N}\int_X \theta^n_{\Gamma_{M/2^N}}. \tag{9.20}$$

Assuming this, let us prove (9.19).

Fixing N, let $M = \lfloor 2^N\Gamma_{\min}\rfloor$. Then repeated application of (9.20) yields

$$\sum_{j=M+1}^{0}\frac{t}{2^N}\int_X \theta^n_{\Gamma_{j/2^N}} \leq E^\phi_\theta\left(\Gamma_t^{*,N,0}\right) - E^\phi_\theta\left(E_t^{*,N,M}\right) \leq \sum_{j=M}^{-1}\frac{t}{2^N}\int_X \theta^n_{\Gamma_{j/2^N}}.$$

Since $M \leq 2^N\Gamma_{\min}$, we have that

$$\Gamma_t^{*,N,M} = \phi + tM/2^N,$$

using (3.19), we can continue to write

$$\sum_{j=M+1}^{0}\frac{t}{2^N}\left(\int_X \theta^n_{\Gamma_{j/2^N}} - \int_X \theta^n_\phi\right) \leq E^\phi_\theta\left(\Gamma_t^{*,N,0}\right) \leq \sum_{j=M}^{-1}\frac{t}{2^N}\left(\int_X \theta^n_{\Gamma_{j/2^N}} - \int_X \theta^n_\phi\right).$$

We now notice that we have Riemann sums on both the left and right of the above inequality. Using Proposition 9.1.1, it is possible to let $N \to \infty$ and obtain

$$E^\phi_\theta(\Gamma^*_t) = t\mathbf{E}^\phi(\Gamma) \tag{9.21}$$

So (9.19) follows as desired.

It remains to argue (9.20). Fix $t > 0$, $N \in \mathbb{Z}_{>0}$ and $M \in \mathbb{Z}$. By Proposition 3.1.17,

$$\begin{aligned}\int_X\left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right)\theta^n_{\Gamma_t^{*,N,M+1}} \leq& E^\phi_\theta(\Gamma_t^{*,N,M+1}) - E^\phi_\theta(\Gamma_t^{*,N,M})\\ \leq& \int_X\left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right)\theta^n_{\Gamma_t^{*,N,M}}.\end{aligned} \tag{9.22}$$

Clearly $\Gamma_t^{*,N,M+1} \geq \Gamma_t^{*,N,M}$. Moreover, since $\mathbb{R} \ni \tau \mapsto \Gamma_\tau + t\tau$ is concave, we notice that

$$U_t := \left\{\Gamma_t^{*,N,M+1} > \Gamma_t^{*,N,M}\right\} = \left\{\Gamma_{(M+1)/2^N} + 2^{-N}t > \Gamma_{M/2^N}\right\},$$

and on U_t we have

$$\Gamma_t^{*,N,M+1} = \Gamma_{(M+1)/2^N} + t(M+1)/2^N, \quad \Gamma_t^{*,N,M} = \Gamma_{M/2^N} + tM/2^N. \tag{9.23}$$

We also note that U_t is $\mathcal{F}$-open by Corollary 1.3.5. So from the lower bound in (9.22), we have

$$\begin{aligned}
E_\theta^\phi(\Gamma_t^{*,N,M+1}) - E_\theta^\phi(\Gamma_t^{*,N,M}) &\geq \int_{U_t} \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M+1}} \\
&= \int_{U_t} \left(\Gamma_{(M+1)/2^N} - \Gamma_{M/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{(M+1)/2^N}} \\
&\geq \int_{\{\Gamma_{(M+1)/2^N}=0\}} t2^{-N}\, \theta^n_{\Gamma_{(M+1)/2^N}},
\end{aligned}$$

where on the second line, we applied (9.23) and Proposition 2.2.1, on the third line, we applied the fact that $\theta^n_{\Gamma_{(M+1)/2^N}}$ is supported on the set

$$\left\{\Gamma_{(M+1)/2^N} = 0\right\} \subseteq U_t \cap \left\{\Gamma_{M/2^N} = 0\right\},$$

see Theorem 3.1.2. We have deduced the first inequality in (9.20). Next, we apply the upper bound part in (9.22) and compute similarly

$$\begin{aligned}
E_\theta^\phi(\Gamma_t^{*,N,M+1}) - E_\theta^\phi(\Gamma_t^{*,N,M}) &\leq \int_X \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M}} \\
&= \int_{U_t} \left(\Gamma_{(M+1)/2^N} - \Gamma_{M/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{M/2^N}} \\
&\leq \int_{\{\Gamma_{M/2^N}=0\}\cap U_t} \left(\Gamma_{(M+1)/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{M/2^N}} \\
&\leq \int_{\{\Gamma_{M/2^N}=0\}\cap U_t} t2^{-N}\, \theta^n_{\Gamma_{M/2^N}}.
\end{aligned}$$

We conclude the latter half of (9.20).

Step 3.3. We assume that $\Gamma_{\max} = 0$. Now $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ only.

For each $\epsilon > 0$, we introduce $\Gamma^\epsilon \in \mathrm{TC}^\infty(X,\theta;\phi)$ as follows:

(1) Let $\Gamma^\epsilon_{\max} = 0$, and
(2) we set

$$\Gamma^\epsilon_\tau = \begin{cases} \phi, \text{ if } \tau \leq -\epsilon^{-1}, \\ P_\theta\left[(1+\epsilon\tau)\Gamma_\tau - \epsilon\tau\phi\right], \text{ if } \tau \in \left(-\epsilon^{-1}, 0\right). \end{cases}$$

It follows from Corollary 6.2.10 and Corollary 6.2.5 that for each $\tau < 0$, the sequence Γ^ϵ_τ is a decreasing sequence with limit Γ_τ as $\epsilon \searrow 0$. Therefore, by Proposition 3.1.9, we have

$$\lim_{\epsilon\to 0+} \int_X \left(\theta + \mathrm{dd^c}\Gamma^\epsilon_\tau\right)^n = \int_X (\theta + \mathrm{dd^c}\Gamma_\tau)^n$$

for all $\tau < 0$. Hence, by the monotone convergence theorem and Step 3.2, we find

$$\mathbf{E}^\phi(\Gamma) = \lim_{\epsilon\to 0+} \mathbf{E}^\phi(\Gamma^\epsilon) = \lim_{\epsilon\to 0+} \mathbf{E}^\phi(\Gamma^{\epsilon *}) = \lim_{\epsilon\to 0+} E^\phi_\theta(\Gamma^{\epsilon *}_1), \tag{9.24}$$

where the last equality follows from (9.21). Furthermore, according to Proposition A.2.3, we have

$$\Gamma^*_t = \inf_{\epsilon>0} \Gamma^{\epsilon *}_t$$

for all $t > 0$. Note that we do not have to take the closure of the right-hand side since it is automatically upper semicontinuous in t.

Now suppose that $\Gamma \in \mathrm{TC}^1(X,\theta;\phi)$. Then by (9.24), as $\epsilon \to 0+$, $(\Gamma^{\epsilon *}_t)_\epsilon$ is a decreasing Cauchy net in $\mathcal{E}^1(X,\theta;\phi)$ and hence by Theorem 4.2.1 for each $t > 0$,

$$E^\phi_\theta(\Gamma^*_t) = \lim_{\epsilon\to 0+} E^\phi_\theta(\Gamma^{\epsilon,*}_t) = t\mathbf{E}^\phi(\Gamma) > -\infty,$$

where we have applied (9.21) and (9.24). Hence, $\Gamma^* \in \mathcal{E}^1(X,\theta;\phi)$. Moreover, (9.19) follows.

Conversely, suppose that $\Gamma^* \in \mathcal{R}^1(X,\theta;\phi)$. Then (9.24) implies that

$$\mathbf{E}^\phi(\Gamma) = \lim_{\epsilon\to 0+} E^\phi_\theta(\Gamma^{\epsilon *}_1) \geq E^\phi_\theta(\Gamma^*_1) > -\infty.$$

Hence, $\Gamma \in \mathrm{TC}^1(X,\theta;\phi)$. □

As an immediate consequence of the proof, we have

Corollary 9.2.2 *Let $\ell \in \mathcal{R}^1(X,\theta;\phi)$, then $[0,\infty) \ni t \mapsto E^\phi_\theta(\ell_t)$ is linear.*

Proof This follows from the same argument as that of (9.24). □

Corollary 9.2.3 *Let $\ell \in \mathcal{R}(X,\theta;\phi)$. Then $\sup_X \ell_t = \ell^*_{\max} t$.*

Proof This follows from Lemma 9.2.2 and Theorem 9.2.1. □

Example 9.2.1 Let us see what the test curve in Example 9.1.1 correspond to under the Ross–Nyström correspondence. Fix $\varphi \in \mathrm{PSH}(X,\theta)$. We claim that

$$\ell^\varphi = \Gamma^{\varphi *}, \tag{9.25}$$

where ℓ^φ is as in Example 4.2.2. We may assume that $\varphi \leq 0$ since both sides are invariant after adding a constant to φ.

We first prove the easy direction $\ell^\varphi \geq \Gamma^{\varphi *}$, which is equivalent to $\ell^{\varphi *} \geq \Gamma^\varphi$. Since $\ell^{\varphi *}$ is a test curve, the latter is equivalent to

$$\ell_\tau^{\varphi *} \geq (1+\tau)\varphi - \tau V_\theta$$

for all $\tau \in (-1,0)$. By Legendre duality, this is equivalent to

$$\ell_t^{\varphi} \geq \sup_{\tau\in(-1,0)} ((1+\tau)\varphi - \tau V_\theta + t\tau) = \varphi \vee (V_\theta - t)$$

for all $t \geq 0$.

Using the notations of Example 4.2.2, we find easily that

$$\ell_t^{\varphi,C} \geq \varphi \vee (V_\theta - t)$$

for any $C > 0$ and $t \in [0,C]$, since it holds at $t=0$ and $t=C$. Letting $C \to \infty$, we find that

$$\ell_t^{\varphi} \geq \varphi \vee (V_\theta - t).$$

Therefore, $\ell^\varphi \geq \Gamma^{\varphi *}$ follows.

In order to prove the equality in (9.25), it suffices to show that the two sides have the same energy, as a consequence of (4.14). So we compute

$$\begin{aligned}
\mathbf{E}\left(\Gamma^{\varphi *}\right) =& \mathbf{E}\left(\Gamma^{\varphi}\right)\\
=& \int_{-1}^{0}\left(\int_X \theta^n_{(1+\tau)V_\theta - \tau\varphi} - \int_X \theta^n_{V_\theta}\right) \mathrm{d}\tau\\
=& \sum_{j=0}^{n}\binom{n}{j}\int_X \theta^j_{V_\theta}\wedge\theta_\varphi^{n-j}\int_0^1 \tau^j(1-\tau)^{n-j}\,\mathrm{d}\tau - \int_X \theta^n_{V_\theta}\\
=& \sum_{j=0}^{n}\binom{n}{j}\frac{j!(n-j)!}{(n+1)!}\int_X \theta^j_{V_\theta}\wedge\theta_\varphi^{n-j} - \int_X \theta^n_{V_\theta}\\
=& \mathbf{E}(\ell^\varphi),
\end{aligned}$$

where we used the value of the β-function[2] on the fourth line, and on the last line is just (4.19).

Proposition 9.2.2 (He–Testorf–Wang) *Let $\ell \in \mathcal{R}(X,\theta;\phi)$. Given any $\tau < \ell^*_{\max}$ and $x \in X$, we have*

$$\mathcal{I}(\ell_\tau^*)_x = \left\{ f \in \mathcal{O}_{X,x} : |f|^2 \int_0^\infty \exp(-\ell_t + t\tau)\,\mathrm{d}t \text{ is locally integrable near } x \right\}. \tag{9.26}$$

Proof Fix $x \in X$, $\tau < \ell^*_{\max}$ and $f \in \mathcal{O}_{X,x}$. Fix a Kähler form ω on X.

Step 1. We first assume that f lies in the right-hand side of (9.26).

Given any $y \in X$, it follows from (9.11) that there is $t_0 > 0$ with

$$\ell^*_\tau(y) + 1 \geq \ell_{t_0}(y) - t_0\tau.$$

[2] Also known as Euler integral of the first kind.

Observe that $t \mapsto \ell_t - t\ell^*_{\max}$ is decreasing in t, it follows that for $t \in [t_0, t_0+1]$, we have

$$\ell^*_\tau(y) + 1 - t_0(\ell^*_{\max} - \tau) \geq \ell_{t_0}(y) - t_0\ell^*_{\max} \geq \ell_t(y) - t\ell^*_{\max}.$$

Since $\tau < \ell^*_{\max}$, we deduce that

$$\ell^*_\tau(y) + 1 + \ell^*_{\max} - \tau \geq \ell_t(y) - t\tau, \quad t \in [t_0, t_0+1]. \tag{9.27}$$

Take a sufficiently small open neighborhood U of x such that

$$\int_U |f|^2 \int_0^\infty \exp(-\ell_t + t\tau)\,\mathrm{d}t\,\omega^n < \infty.$$

Applying (9.27), we deduce that

$$\int_U |f|^2 \exp(-\ell^*_\tau)\,\omega^n < \infty.$$

Therefore, $f \in \mathcal{I}(\ell^*_\tau)_x$.

Step 2. Assume that $f \in \mathcal{I}(\ell^*_\tau)_x$.

It follows from Theorem 1.4.4 that $f \in \mathcal{I}(\ell^*_{\tau+\epsilon})_x$ for some small enough $\epsilon > 0$ with $\tau + \epsilon < \ell^*_{\max}$. Take a sufficiently small open neighborhood U of x such that

$$\int_U |f|^2 \exp(-\ell^*_{\tau+\epsilon})\,\omega^n < \infty.$$

We compute

$$\begin{aligned}\int_U |f|^2 \int_0^\infty \exp(-\ell_t + t\tau)\,\mathrm{d}t\,\omega^n \leq &\int_U |f|^2 \int_0^\infty \exp(-\ell^*_{\tau+\epsilon} - t\epsilon)\,\mathrm{d}t\,\omega^n\\ =&\frac{1}{\epsilon}\int_U |f|^2 \exp(-\ell^*_{\tau+\epsilon})\,\omega^n\\ <&\infty.\end{aligned}$$

Therefore, f lies in the right-hand side of (9.26). □

9.3 $\mathcal{I}$-model test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Definition 9.3.1 A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is *$\mathcal{I}$-model* if for any $\tau < \Gamma_{\max}$, the potential Γ_τ is $\mathcal{I}$-model.

The subset of $\mathcal{I}$-model test curves in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathcal{E}^{\mathrm{NA}}(X,\theta;\phi)$. When $\phi = V_\theta$, we omit ϕ and write $\mathcal{E}^{\mathrm{NA}}(X,\theta)$ instead.

The union of the sets of $\mathcal{I}$-model test curves in $\mathrm{PSH}(X,\theta)$ for all model potentials $\phi\in\mathrm{PSH}(X,\theta)_{>0}$ is denoted by $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

Note that $\Gamma_{\Gamma_{\max}}$ is automatically $\mathcal{I}$-model by Proposition 3.2.12.

Proposition 9.3.1 *Let $\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Then $\Gamma_{-\infty}$ is an $\mathcal{I}$-model potential.*

Proof This follows from Proposition 3.2.13. □

Proposition 9.3.2 *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Then there is a canonical bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}\xrightarrow{\sim}\mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}.$$

This bijection satisfies the obvious cocycle condition.

Proof This is an immediate consequence of Proposition 9.1.2 and Example 7.1.2. □

Proposition 9.3.3 *Let $\pi\colon Y\to X$ be a proper bimeromorphic morphism from a Kähler manifold. Then the pointwise pull-back induces a bijection*

$$\pi^*\colon\mathrm{PSH}^{\mathrm{NA}}(X,\theta;\phi)\xrightarrow{\sim}\mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This is an immediate consequence of Proposition 9.1.3 and Proposition 3.2.5. □

Definition 9.3.2 Given $\Gamma\in\mathrm{TC}(X,\theta;\phi)$, we define its *$\mathcal{I}$-envelope* $P_\theta[\Gamma]_{\mathcal{I}}$ as the map

$$(-\infty,\Gamma_{\max})\to\mathrm{PSH}(X,\theta),\quad \tau\mapsto P_\theta\left[\Gamma_\tau\right]_{\mathcal{I}}.$$

More generally, for any closed real smooth positive $(1,1)$-form ω on X, we define $P_{\theta+\omega}[\Gamma]_{\mathcal{I}}$ as the map

$$(-\infty,\Gamma_{\max})\to\mathrm{PSH}(X,\theta),\quad \tau\mapsto P_{\theta+\omega}\left[\Gamma_\tau\right]_{\mathcal{I}}.$$

Proposition 9.3.4 *Let $\Gamma\in\mathrm{TC}(X,\theta;\phi)$, then*

$$P_\theta[\Gamma]_{\mathcal{I}}\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta;P_\theta[\phi]_{\mathcal{I}}).$$

More generally, for any closed real smooth positive $(1,1)$-form ω on X, we have

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}}\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega;P_{\theta+\omega}[\phi]_{\mathcal{I}}).$$

Proof The only non-trivial point is to show that

$$\sup_{\tau<\Gamma_{\max}}{}^*P_\theta[\Gamma_\tau]_{\mathcal{I}}=P_\theta[\phi]_{\mathcal{I}},\qquad \sup_{\tau<\Gamma_{\max}}{}^*P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}}=P_{\theta+\omega}[\phi]_{\mathcal{I}}.$$

These follow from Proposition 3.2.13. □

Definition 9.3.3 Let $\phi\in\mathrm{PSH}(X,\theta)_{>0}$ be a model potential. A geodesic ray $\ell\in\mathcal{R}(X,\theta;\phi)$ is *maximal* if ℓ^* is $\mathcal{I}$-model.

An important class of $\mathcal{I}$-model test curves is given by filtrations. We briefly recall the corresponding terminology.

Definition 9.3.4 Let L be a big line bundle. We write

$$R(X,L) = \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X,L^k)$$

for the section ring of L.

A *filtration* on $R(X,L)$ is a decreasing family of graded linear subspaces $(\mathcal{F}^\lambda)_{\lambda\in\mathbb{R}}$ of $R(X,L)$ with graded pieces

$$\mathcal{F}^\lambda = \bigoplus_{k=0}^{\infty} \mathcal{F}_k^\lambda,$$

such that the following conditions are satisfied:

- The filtration is left-continuous: For any $\lambda \in \mathbb{R}$, we have
$$\mathcal{F}^\lambda = \bigcap_{\lambda'<\lambda} \mathcal{F}^{\lambda'};$$
- the filtration is multiplicative: For any $\lambda, \lambda' \in \mathbb{R}$ and any $k, k' \in \mathbb{N}$, we have
$$\mathcal{F}_k^\lambda \cdot \mathcal{F}_{k'}^{\lambda'} \subseteq \mathcal{F}_{k+k'}^{\lambda+\lambda'};$$
- There is an integer $C > 0$ such that
$$\mathcal{F}_m^{Cm} = 0, \quad \mathcal{F}_m^{-Cm} = \mathrm{H}^0(X,L^k) \tag{9.28}$$
for all $m \in \mathbb{N}$.

Given a filtration $\mathcal{F}$ on $R(X,L)$, we define

$$\tau_k(\mathcal{F}) = \max\left\{\lambda \in \mathbb{R} : \mathcal{F}_k^\lambda \neq 0\right\}.$$

By Fekete's lemma, we can introduce

$$\tau(\mathcal{F}) = \lim_{k\to\infty} \frac{1}{k}\tau_k(\mathcal{F}) = \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\tau_k(\mathcal{F}).$$

Note that $\tau(\mathcal{F})$ is bounded from above by the constant C in (9.28), hence finite.

Example 9.3.1 Let L be a big line bundle on X and $\mathcal{F}$ be a filtration on $R(X,L)$. Fix a smooth Hermitian metric h on L and write $\theta = c_1(L,h)$.

We introduce a few auxiliary functions. For each $k \in \mathbb{Z}_{>0}$, we introduce

$$\Gamma_\tau^{\mathcal{F},k} := \sup{}^* \left\{\log|s|_{h^k}^2 : s \in \mathcal{F}_k^{k\tau}, |s|_{h^k}^2 \le 1\right\}.$$

When $k\tau \leq \tau_k(\mathcal{F})$, we know that $\mathcal{F}_k^{k\tau} \neq 0$. Moreover, Proposition 1.8.1 and Proposition 1.2.1 imply that

$$\Gamma_\tau^{\mathcal{F},k} \in \mathrm{PSH}(X, k\theta), \quad \tau \leq k^{-1}\tau_k(\mathcal{F}).$$

Observe that for $k, k' \in \mathbb{Z}_{>0}$, we have

$$\Gamma_\tau^{\mathcal{F},k+k'} \geq \Gamma_\tau^{\mathcal{F},k} + \Gamma_\tau^{\mathcal{F},k'}.$$

In particular, by Fekete's lemma,

$$\lim_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_\tau^{\mathcal{F},k} = \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_\tau^{\mathcal{F},k} \tag{9.29}$$

exists for any $\tau < \tau(\mathcal{F})$.

We define $(\Gamma_\tau^{\mathcal{F}})_{\tau<\tau(\mathcal{F})}$ as follows:

$$\Gamma_\tau^{\mathcal{F}} := P_\theta\left[\sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_\tau^{\mathcal{F},k}\right]^{3}.$$

We claim that $\Gamma^{\mathcal{F}} \in \mathcal{E}^{\mathrm{NA}}(X,\theta)$.

It is clear that $(-\infty, \tau(\mathcal{F})) \ni \tau \mapsto \Gamma_\tau^{\mathcal{F}}$ is decreasing. We prove its concavity. By Proposition 3.1.6, it suffices to show that

$$(-\infty, \tau(\mathcal{F})) \ni \tau \mapsto \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_\tau^{\mathcal{F},k}$$

is concave. In other words, we need to prove the following: Given $\tau_0 < \tau_1 < \tau(\mathcal{F})$ and $t \in (0,1)$, we have

$$\sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_{t\tau_1+(1-t)\tau_0}^{\mathcal{F},k} \geq t \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_{\tau_1}^{\mathcal{F},k} + (1-t)\sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_{\tau_0}^{\mathcal{F},k}.$$

But thanks to Proposition 1.2.6 and Proposition 1.2.5, it suffices to show that

$$\sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_{t\tau_1+(1-t)\tau_0}^{\mathcal{F},k} \geq t \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_{\tau_1}^{\mathcal{F},k} + (1-t)\sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_{\tau_0}^{\mathcal{F},k}$$

for all $t \in (0,1)$. Take $s_i \in \mathcal{F}_{k_i}^{k_i\tau_i}$ for $i = 0,1$ with $|s|_{h^{k_i}}^2 \leq 1$, where $k_0, k_1 \in \mathbb{Z}_{>0}$. We need to prove that

$$\sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_{t\tau_1+(1-t)\tau_0}^{\mathcal{F},k} \geq \frac{1-t}{k_0}\log|s_0|_{h^{k_0}}^2 + \frac{t}{k_1}\log|s_1|_{h^{k_1}}^2. \tag{9.30}$$

---

[3] It is not clear if $P_\theta[\bullet]$ is necessary here. When L is ample, it is shown in [RWN14, Proposition 7.11] that it is not necessary. The proof in the reference relies on a Skoda division theorem [RWN14, Theorem 7.10], which is not known in the case of big line bundles.

Approximate t by rational number from above, we may reduce to the case where $t \in \mathbb{Q}$. Write $t = p/q$ with $p, q \in \mathbb{Z}_{>0}$. Then

$$s := s_0^{k_1(q-p)} \otimes s_1^{k_0 p} \in \mathcal{F}_{k_0 k_1 q}^{k_0 k_1 \tau_0 (q-p) + k_0 k_1 \tau_1 p},$$

and

$$\begin{aligned}
&\frac{1}{k_0 k_1 q} \log |s|^2_{h^{k_0 k_1 q}} \\
=&\frac{1}{k_0 k_1 q} \left(k_1 (q-p) \log |s_0|^2 + k_0 p \log |s_1|^2 \right) \\
=&\frac{1-t}{k_0} \log |s_0|^2_{h^{k_0}} + \frac{t}{k_1} \log |s_1|^2_{h^{k_1}}.
\end{aligned}$$

So (9.30) follows.

Note that for each $k \in \mathbb{Z}_{>0}$, $\tau \leq k^{-1} \tau_k(\mathcal{F})$, we know that $\Gamma_\tau^{\mathcal{F},k}$ is $\mathcal{I}$-good by Proposition 7.2.2. It follows that for each $\tau < \tau(\mathcal{F})$, the potential $\Gamma_\tau^{\mathcal{F}}$ is also $\mathcal{I}$-good.

It remains to show that the test curve $\Gamma^{\mathcal{F}}$ is bounded. Fix $\tau \leq -C$, where C is as in (9.28), we will show that

$$\Gamma_\tau^{\mathcal{F}} = V_\theta. \tag{9.31}$$

Of course, this follows from the Bergman kernel technique. But based on the theory we have developed so far, we could give an elegant and elementary argument.

Fix $k > 0$. Observe that for any $s \in \mathrm{H}^0(X, L^k)$, we have

$$s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\Gamma_\tau^{\mathcal{F}})).$$

In fact, by definition of $\Gamma_\tau^{\mathcal{F}}$, it suffices to show that

$$s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(\Gamma_\tau^{\mathcal{F},k})),$$

which is clear by definition. Therefore, by Theorem 7.3.1,

$$\operatorname{vol}(\theta + \mathrm{dd^c} \Gamma_\tau^{\mathcal{F}}) = \operatorname{vol} L.$$

But since $\Gamma_\tau^{\mathcal{F}}$ is $\mathcal{I}$-model, this implies (9.31).

Remark 9.3.1 There is an important special case of Example 9.3.1: Suppose that L is ample and $\mathcal{F}$ is the filtration induced by a smooth test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L). Then the geodesic ray $\Gamma^{\mathcal{F}*}$ is exactly the Phong–Sturm geodesic ray associated with $(\mathcal{X}, \mathcal{L})$. See [RWN14, Section 9].

Remark 9.3.2 We deduce from Example 9.3.1 that the ray $\Gamma^{\mathcal{F}*}$ induced by a filtration $\mathcal{F}$ is maximal.

9.4 Operations on test curves

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta', \theta''$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes.

Definition 9.4.1 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. For any closed real smooth positive $(1,1)$-form ω on X, we define $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$ as the following map:

$$(-\infty, \Gamma_{\max}) \to \mathrm{PSH}(X,\theta), \quad \tau \mapsto P_{\theta+\omega}\left[\Gamma_\tau\right].$$

Proposition 9.4.1 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a closed real smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma]_{-\infty} = P_{\theta+\omega}[\Gamma_{-\infty}].$$

Proof This follows from Proposition 3.1.10. □

Definition 9.4.2 Given $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, we say $\Gamma \leq \Gamma'$ if for all $\Gamma_{\max} \leq \Gamma'_{\max}$ and for all $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \leq_P \Gamma'_\tau. \tag{9.32}$$

Observe that (9.32) actually holds for all $\tau \in \mathbb{R}$ if $\theta = \theta'$. It is easy to verify that for all $\leq$ defines a partial order on $\mathrm{TC}(X,\theta)_{>0}$.

Lemma 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a closed real smooth positive $(1,1)$-form on X. Then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma] \leq P_{\theta+\omega}[\Gamma']$.

Proof This follows from Example 6.1.1. □

Definition 9.4.3 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then we define $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$ as follows:

(1) We set
$$(\Gamma+\Gamma')_{\max} := \Gamma_{\max} + \Gamma'_{\max};$$
(2) for any $\tau < (\Gamma+\Gamma')_{\max}$, we define
$$(\Gamma+\Gamma')_\tau := P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}}\left(\Gamma_t + \Gamma'_{\tau-t}\right)\right]. \tag{9.33}$$

Lemma 9.4.2 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then for any $\tau < (\Gamma + \Gamma')_{\max}$, we have*
$$\sup_{t\in\mathbb{R}}\left(\Gamma_t + \Gamma'_{\tau-t}\right) \in \mathrm{PSH}(X,\theta).$$
This potential is $\mathcal{I}$-good if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

In particular, (9.33) in Definition 9.4.3 makes sense.

Proof Let

$$\eta_\tau = \sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-t}\right) = \sup_{t<\Gamma_{\max},\tau-t<\Gamma'_{\max}} \left(\Gamma_t + \Gamma'_{\tau-t}\right)$$

for all $\tau \in \mathbb{R}$. Set

$$Z = \left\{x \in X : \Gamma_T(x) = -\infty \text{ or } \Gamma'_T(x) = -\infty \text{ for small enough } T\right\}.$$

It follows from Proposition A.2.4 that for any $x \in X \setminus Z$, we have

$$\eta_t^*(x) = \Gamma_t^*(x) + \Gamma_t'^*(x)$$

for all $t > 0$. The same trivially holds when $x \in Z$, so the equation holds everywhere. In particular, by Corollary A.2.1 and Proposition 1.2.8, we have

$$\eta_\tau = (\Gamma^* + \Gamma'^*)^*_\tau \in \mathrm{PSH}(X, \theta + \theta')$$

when $\tau < \Gamma_{\max} + \Gamma'_{\max}$.

Next, assume that Γ and Γ' are $\mathcal{I}$-model. We need to argue that so is $\Gamma + \Gamma'$. Fix $\tau < \Gamma_{\max} + \Gamma'_{\max}$. Then for each $t \in \mathbb{R}$ such that $t < \Gamma_{\max}$ and $\tau - t < \Gamma'_{\max}$, we know that $\Gamma_t \in \mathrm{PSH}(X,\theta)_{>0}$ and $\Gamma'_{\tau-t} \in \mathrm{PSH}(X,\theta')_{>0}$ by Lemma 9.1.1. It follows from Example 7.1.2 that Γ_t and $\Gamma'_{\tau-t}$ are both $\mathcal{I}$-good, hence so is $\Gamma_t + \Gamma'_{\tau-t} \in \mathrm{PSH}(X,\theta+\theta')_{>0}$ by Proposition 7.2.1. Therefore, η_τ is $\mathcal{I}$-good by Proposition 7.2.2. Therefore, $\Gamma + \Gamma'$ is $\mathcal{I}$-model. □

Proposition 9.4.2 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$. Moreover,*

$$(\Gamma + \Gamma')_{-\infty} = P_{\theta+\theta'}[\Gamma_{-\infty} + \Gamma'_{-\infty}]. \tag{9.34}$$

When $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$, we have $\Gamma + \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$.

The operation + is commutative and associative.

Proof It follows immediately from Lemma 9.4.2 that $\Gamma + \Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$ if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

We argue (9.34). By definition, for any small enough τ, we have

$$(\Gamma+\Gamma')_{-\infty} \geq (\Gamma+\Gamma')_{2\tau} \geq_P \Gamma_\tau + \Gamma'_\tau.$$

Letting $\tau \to -\infty$ and applying Proposition 6.2.4 and Theorem 6.2.2, we find that

$$(\Gamma+\Gamma')_{-\infty} \geq_P \Gamma_{-\infty} + \Gamma'_{-\infty}.$$

On the other hand, for each small enough τ, we have

$$(\Gamma+\Gamma')_\tau \sim_P \sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau-t}\right) \leq_P \Gamma_{-\infty} + \Gamma'_{-\infty}$$

by Proposition 6.1.5 and Proposition 6.2.4. We apply Proposition 6.2.4 again, we conclude that

$$(\Gamma+\Gamma')_{-\infty} \preceq_P \Gamma_{-\infty}+\Gamma'_{-\infty}.$$

So (9.34) follows.

Finally, let us show that + is commutative and associative. Commutativity is obvious. Let $\Gamma'' \in \mathrm{TC}(X,\theta'')_{>0}$. Then we want to show that

$$(\Gamma+\Gamma')+\Gamma'' = \Gamma+(\Gamma'+\Gamma'').$$

First observe that

$$((\Gamma+\Gamma')+\Gamma'')_{\max} = (\Gamma+(\Gamma'+\Gamma''))_{\max}\,.$$

Fix τ less than this common value. We observe that

$$\begin{aligned}
&((\Gamma+\Gamma')+\Gamma'')_\tau\\
=&P_\theta\left[\sup_{t_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{t_1}+\Gamma''_{\tau-t_1}\right)\right]\\
\sim_P&\sup_{t_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{t_1}+\Gamma''_{\tau-t_1}\right)\\
\sim_P&\sup_{t_1,t_2\in\mathbb{R}}\left(\Gamma_{t_2}+\Gamma'_{t_1-t_2}+\Gamma''_{\tau-t_1}\right),
\end{aligned}$$

where in the last line, we applied Proposition 6.2.4 and Proposition 6.1.5. Similarly, for $(\Gamma+(\Gamma'+\Gamma''))_\tau$, we get the same expression. The associativity follows. □

Lemma 9.4.3 *Let $\Gamma\in\mathrm{TC}(X,\theta)_{>0}$ and $\Gamma'\in\mathrm{TC}(X,\theta')_{>0}$, then for any closed smooth positive $(1,1)$-forms ω and ω' on X, we have*

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma'] = P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma].$$

Proof Observe that

$$\begin{aligned}
P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma']_{\max} &= (P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma])_{\max}\\
&=P_{\theta+\omega}[\Gamma]_{\max}+P_{\theta'+\omega'}[\Gamma]_{\max}.
\end{aligned}$$

Take $\tau\in\mathbb{R}$ less than this common value, we need to verify that

$$(\Gamma+\Gamma')_\tau \sim_P (P_{\theta+\omega}[\Gamma]+P_{\theta'+\omega'}[\Gamma])_\tau\,.$$

By definition, this means that

$$\sup_{t<\Gamma_{\max},t-\tau<\Gamma'_{\max}}\left(\Gamma_t+\Gamma'_{\tau-t}\right)\sim_P \sup_{t<\Gamma_{\max},t-\tau<\Gamma'_{\max}}\left(P_{\theta+\omega}[\Gamma_t]+P_{\theta'+\omega'}[\Gamma'_{\tau-t}]\right).$$

This is a consequence of Proposition 6.1.5 and Proposition 6.1.6. □

Definition 9.4.4 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $C \in \mathbb{R}$, we define $\Gamma + C \in \mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set
$$(\Gamma + C)_{\max} \coloneqq \Gamma_{\max} + C;$$

(2) for any $\tau < (\Gamma + C)_{\max}$, we set
$$\Gamma_\tau \coloneqq \Gamma_{\tau - C}.$$

It is obvious that if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then so is $\Gamma + C$.

Proposition 9.4.3 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma \in \mathrm{TC}(X,\theta')_{>0}$ *and* $C, C' \in \mathbb{R}$*, then*

(1) $(\Gamma + \Gamma') + C = \Gamma + (\Gamma' + C) = (\Gamma + C) + \Gamma'$*;*
(2) $\Gamma + (C + C') = (\Gamma + C) + C'$.

Proof (1) We first observe that
$$((\Gamma + \Gamma') + C)_{\max} = (\Gamma + (\Gamma' + C))_{\max} = ((\Gamma + C) + \Gamma')_{\max} = \Gamma_{\max} + \Gamma'_{\max} + C.$$

Take any $\tau \in \mathbb{R}$ less than this common value. We compute
$$\begin{aligned}
((\Gamma + \Gamma') + C)_\tau =& (\Gamma + \Gamma')_{\tau - C} = P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau - C - t}\right)\right],\\
(\Gamma + (\Gamma' + C))_\tau =& P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + (\Gamma' + C)_{\tau - t}\right)\right] = P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau - C - t}\right)\right],\\
((\Gamma + C) + \Gamma')_\tau =& P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left((\Gamma + C)_{C+t} + \Gamma'_{\tau - C - t}\right)\right]\\
=& P_{\theta+\theta'}\left[\sup_{t\in\mathbb{R}} \left(\Gamma_t + \Gamma'_{\tau - C - t}\right)\right].
\end{aligned}$$

(2) Observe that
$$(\Gamma + (C + C'))_{\max} = ((\Gamma + C) + C')_{\max} = \Gamma_{\max} + C + C'.$$

For any $\tau \in \mathbb{R}$ less than this value, we have
$$(\Gamma + (C + C'))_\tau = \Gamma_{\tau - C - C'} = ((\Gamma + C) + C')_\tau\,.$$

Definition 9.4.5 Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$. We define $\Gamma \vee \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set
$$(\Gamma \vee \Gamma')_{\max} \coloneqq \Gamma_{\max} \vee \Gamma'_{\max},$$
and

(2) for any $\tau < (\Gamma \vee \Gamma')_{\max}$, we define
$$(\Gamma \vee \Gamma')_\tau \coloneqq P_\theta\left[\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)\right]. \tag{9.35}$$

Recall that the upper convex envelope CE is defined in Definition A.1.4. Trivially, we have $\Gamma \vee \Gamma' \geq \Gamma$ and $\Gamma \vee \Gamma' \geq \Gamma'$.

Lemma 9.4.4 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$. Then for any $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$, we have*

$$\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau \in \mathrm{PSH}(X, \theta).$$

This potential is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$.

In particular, (9.35) *in Definition 9.4.5 makes sense.*

Proof To simply the notations, we write

$$\psi_\tau = \mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau$$

for all $\tau \in \mathbb{R}$. Thanks to Proposition A.2.3, we have

$$\psi^*_t(x) = \Gamma^*_t(x) \vee \Gamma'^*_t(x) \tag{9.36}$$

for all $t > 0$ as long as $\Gamma_\tau(x) \neq -\infty$ and $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. Otherwise, assume that $x \in X$ is such that $\Gamma_\tau = -\infty$ for all $\tau \in \mathbb{R}$, then by definition, $\psi_\tau(x) = \Gamma'_\tau(x)$ for all $\tau \in \mathbb{R}$. Therefore, $\Gamma^*_t(x) = -\infty$ for all $t > 0$ and hence (9.36) continues to hold. Therefore, we have shown that

$$\psi^*_t = \Gamma^*_t \vee \Gamma'^*_t \in \mathrm{PSH}(X, \theta).$$

It follows from Proposition 4.1.3 that $(\psi^*_t)_{t\in[a,b]}$ is a subgeodesic for any $0 < a < b$.

Next we observe that $\psi_\bullet$ is closed by definition. So it follows from Proposition A.2.3 and Proposition 1.2.8 that

$$\psi_\tau = (\psi^*_\bullet)^*_\tau \in \mathrm{PSH}(X, \theta) \cup \{-\infty\}.$$

Due to Proposition 9.1.4 and Proposition A.1.2, there is a pluripolar set $Z \subseteq X$ such that for $x \in X \setminus Z$, we have

$$\psi_\tau(x) = \sup\left\{\lambda\Gamma_\rho(x) + (1-\lambda)\Gamma'_{\rho'}(x) : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\}$$

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$. It follows from Proposition 1.2.6 that

$$\psi_\tau = \sup{}^*\left\{\lambda\Gamma_\rho + (1-\lambda)\Gamma'_{\rho'} : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\} \tag{9.37}$$

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$.

It follows from (9.37) that ψ_τ is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$, thanks to Proposition 7.2.1 and Proposition 7.2.2. □

Corollary 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$. Then $\Gamma \vee \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$ and*

$$(\Gamma \vee \Gamma')_{-\infty} = P_\theta\left[\Gamma_{-\infty} \vee \Gamma'_{-\infty}\right]. \tag{9.38}$$

If $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$*, then* $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

For each $\Gamma'' \in \mathrm{TC}(X,\theta)_{>0}$ *and each* $\Gamma'' \geq \Gamma$ *and* $\Gamma'' \geq \Gamma'$*, we have* $\Gamma'' \geq \Gamma \vee \Gamma'$.

Moreover, the operation $\vee$ *is associative and commutative.*

In particular, given a finite family $\{\Gamma_i\}_{i\in I}$ in $\mathrm{TC}(X,\theta)_{>0}$, we can define

$$\bigvee_{i\in I} \Gamma_i$$

without ambiguity.

Proof It follows immediately from Lemma 9.4.4 that $\Gamma \vee \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

The argument of (9.38) is very similar to that of (9.34), which we leave to the readers.

Take Γ'' as in the statement of the proposition. First observe that

$$\Gamma''_{\max} \geq \Gamma_{\max} \vee \Gamma'_{\max} = (\Gamma \vee \Gamma')_{\max}.$$

Take $\tau < (\Gamma \vee \Gamma')_{\max}$, we argue that

$$\Gamma''_\tau \geq (\Gamma \vee \Gamma')_\tau.$$

By the concavity of Γ'', this is equivalent to

$$\Gamma''_\tau \geq \Gamma_\tau \vee \Gamma'_\tau.$$

Therefore,

$$\Gamma'' \geq \Gamma \vee \Gamma'.$$

The commutativity and associativity of $\vee$ are trivial. □

Lemma 9.4.5 *Let* $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ *and* ω *be a closed smooth positive* $(1,1)$*-form on* X*. Then*

$$P_{\theta+\omega}[\Gamma \vee \Gamma'] = P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'].$$

Proof We first observe that

$$(P_{\theta+\omega}[\Gamma \vee \Gamma'])_{\max} = (P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'])_{\max} = \Gamma_{\max} \vee \Gamma'_{\max}.$$

Let $\tau \in \mathbb{R}$ be less than this common value. We need to show that

$$(\Gamma \vee \Gamma')_\tau \sim_P (P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'])_\tau\,.$$

We need the formula (9.37) proved in the proof of Lemma 9.4.4:

$$(\Gamma \vee \Gamma')_\tau = \sup{}^* \left\{ \lambda\Gamma_\rho + (1-\lambda)\Gamma'_{\rho'} : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau \right\}.$$

A similar result holds with $P_{\theta+\omega}[\Gamma]$ and $P_{\theta+\omega}[\Gamma']$ in place of Γ and Γ'. So our assertion is a direct consequence of Proposition 6.1.5 and Proposition 6.1.6. □

Definition 9.4.6 Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$. Assume that

$$\sup_{i\in I}\Gamma^i_{\max}<\infty. \tag{9.39}$$

Then we define $\sup^*_{i\in I}\Gamma^i\in\mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set

$$\left(\sup^*_{i\in I}\Gamma^i\right)_{\max}=\sup_{i\in I}\Gamma^i_{\max};$$

(2) for any $\tau<\sup_{i\in I}\Gamma^i_{\max}$, we let

$$\left(\sup^*_{i\in I}\Gamma^i\right)_{\tau}\coloneqq\sup^*_{i\in I}\Gamma^i_{\tau}.$$

Proposition 9.4.4 *Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.39). Then $\sup^*_{i\in I}\Gamma^i$ as defined in Definition 9.4.6 lies in $\mathrm{TC}(X,\theta)_{>0}$. Moreover, if $\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ for all $i\in I$, then $\sup^*_{i\in I}\Gamma^i$ lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ as well.*

Moreover, we have

$$\left(\sup^*_{i\in I}\Gamma^i\right)_{-\infty}=\sup^*_{i\in I}\Gamma^i_{-\infty}. \tag{9.40}$$

Proof The first assertion follows easily from Proposition 3.1.10, while the second follows from Proposition 3.2.13.

It remains to argue (9.40). Without loss of generality, we may assume that I contains a minimal element i_0.

By Proposition 1.2.5, there is a pluripolar set $Z\subseteq X$ such that for any $x\in X\setminus Z$,

$$\left(\sup^*_{i\in I}\Gamma^i\right)_{-\infty}(x)=\sup_{\mathbb{Q}\ni\tau<\Gamma^{i_0}_{\max}}\left(\sup^*_{i\in I}\Gamma^i_{\tau}\right)(x)=\sup_{\mathbb{Q}\ni\tau\tau<\Gamma^{i_0}_{\max},i\in I}\Gamma^i_{\tau}(x)=\sup_{i\in I}\Gamma^i_{-\infty}(x).$$

So they are equal everywhere by Proposition 1.2.6. □

Lemma 9.4.6 *Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.39). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup^*_{i\in I}\Gamma^i\right]=\sup^*_{i\in I}P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof Observe that

$$\left(P_{\theta+\omega}\left[\sup^*_{i\in I}\Gamma^i\right]\right)_{\max}=\left(\sup^*_{i\in I}P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\max}=\sup_{i\in I}\Gamma^i_{\max}.$$

Fix $\tau\in\mathbb{R}$ less than this common value.

It suffices to show that

$$\left(\sup^*_{i\in I}\Gamma^i\right)_{\tau}\sim_P\left(\sup^*_{i\in I}P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\tau}.$$

This is an immediate consequence of Proposition 6.1.6. □

Definition 9.4.7 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.39). Then we define

$$\sup_{i\in I}{}^*\Gamma^i := \sup_{J\in \mathrm{Fin}(I)}{}^* \left(\bigvee_{j\in J}\Gamma^j\right). \tag{9.41}$$

Recall that $\mathrm{Fin}(I)$ is the net of non-empty finite subsets of I, ordered by inclusion.

Observe that by Definition 9.4.5, we have

$$\sup_{J\in \mathrm{Fin}(I)} \left(\bigvee_{j\in J}\Gamma^j\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max} < \infty.$$

So (9.41) makes sense. In particular,

$$\left(\sup_{i\in I}\Gamma^i\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max}. \tag{9.42}$$

It is clear that Definition 9.4.7 extends both Definition 9.4.6 and Definition 9.4.5.

Proposition 9.4.5 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.39). Then $\sup_{i\in I}{}^*\Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$. Moreover, if $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then so is $\sup_{i\in I}{}^*\Gamma^i$.*

Finally, we have

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{-\infty} = \sup_{i\in I}{}^*\Gamma^i_{-\infty}. \tag{9.43}$$

Proof The first assertion and the second follow from Proposition 9.4.4 and Corollary 9.4.1.

It remains to argue (9.43). For this purpose, it suffices to show that

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{-\infty} \sim_P \sup_{i\in I}{}^*\Gamma^i_{-\infty}.$$

For any $J\in \mathrm{Fin}(I)$, it follows from Corollary 9.4.1 and Proposition 6.1.6 that

$$\left(\bigvee_{j\in J}\Gamma^j\right)_{-\infty} \sim_P \bigvee_{j\in J}\Gamma^j_{-\infty}.$$

From this, applying Proposition 3.1.10, Proposition 6.1.6 and Proposition 9.4.4, we conclude our assertion. □

Lemma 9.4.7 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.39). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right] = \sup_{i\in I}{}^* P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof This is a direct consequence of Lemma 9.4.6 and Lemma 9.4.5. □

We prove a version of Choquet's lemma.

Proposition 9.4.6 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.39). *Then there is a countable subset $I' \subseteq I$ such that*

$$\sup_{i\in I}{}^*\Gamma^i = \sup_{i\in I'}{}^*\Gamma^i.$$

Proof We may assume that I is infinite.

It follows from Proposition 1.2.2 that we can find a countable subset $I' \subseteq I$ such that for each

$$\tau \in \left(-\infty, \sup_{i\in I}{}^*\Gamma^i_{\max}\right) \cap \mathbb{Q},$$

we have

$$\sup_{i\in I}{}^*\Gamma^i_\tau = \sup_{i\in I'}{}^*\Gamma^i_\tau.$$

Let $\Gamma' = \sup_{i\in I'}{}^*\Gamma^i$. Then clearly, $\Gamma' \leq \Gamma$. We claim that they are actually equal. For this purpose, it suffices to show that for any $\tau < \sup_{i\in I}{}^*\Gamma^i_{\max}$, we have

$$\int_X \left(\theta + \mathrm{dd^c}\Gamma'_\tau\right)^n = \int_X \left(\theta + \mathrm{dd^c}\Gamma_\tau\right)^n.$$

Since we know that this holds on a dense subset of τ, the same holds everywhere by Proposition 9.1.1. □

Proposition 9.4.7 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.39). *Let $C \in \mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i + C) = \sup_{i\in I}{}^*\Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another family in $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.39). *Suppose that $\Gamma^i \leq \Gamma'^i$ for all $i \in I$, then*

$$\sup_{i\in I}{}^*\Gamma^i \leq \sup_{i\in I}{}^*\Gamma'^i.$$

Proof This is immediate by definition. □

Definition 9.4.8 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\lambda > 0$, we define $\lambda\Gamma \in \mathrm{TC}(X,\lambda\theta)_{>0}$ as follows:

(1) We set
$$(\lambda\Gamma)_{\max} = \lambda\Gamma_{\max};$$

(2) for any $\tau < \lambda\Gamma_{\max}$, we set
$$(\lambda\Gamma)_\tau = \lambda\Gamma_{\lambda^{-1}\tau}.$$

Proposition 9.4.8 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\lambda > 0$, then $\lambda\Gamma$ as defined in Definition 9.4.8 lies in* $\mathrm{TC}(X,\lambda\theta)_{>0}$. *Moreover, if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then $\lambda\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)_{>0}$.*

We have

$$(\lambda\Gamma)_{-\infty} = \lambda\Gamma_{-\infty}. \tag{9.44}$$

Proof This is immediate by definition. □

Proposition 9.4.9 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, $C \in \mathbb{R}$ *and* $\lambda, \lambda' > 0$, *we have*

$$\begin{aligned}\lambda(\Gamma+\Gamma') =& \lambda\Gamma+\lambda\Gamma',\\ (\lambda\lambda')\Gamma =& \lambda(\lambda'\Gamma),\\ \lambda(\Gamma+C) =& \lambda\Gamma+\lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ *is a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.39), *then*

$$\lambda\left(\sup_{i\in I}{}^{*}\Gamma^i\right) = \sup_{i\in I}{}^{*}(\lambda\Gamma^i).$$

Proof This is immediate by definition. □

Lemma 9.4.8 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ *and* $\lambda > 0$. *Then for any closed smooth positive* $(1,1)$*-form* ω *on* X, *we have*

$$P_{\lambda(\theta+\omega)}[\lambda\Gamma] = \lambda P_{\theta+\omega}[\Gamma].$$

Proof This is clear by definition. □

Definition 9.4.9 Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{TC}(X,\theta)_{>0}$. Assume that

$$\sup_{i\in I}\Gamma^i_{\max} < \infty, \tag{9.45}$$

and

$$\inf_{i\in I}\int_X \left(\theta+\mathrm{dd^c}\Gamma^i_{-\infty}\right)^n > 0. \tag{9.46}$$

Then we define $\inf_{i\in I}\Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set

$$\left(\inf_{i\in I}\Gamma^i\right)_{\max} = \inf_{i\in I}\Gamma^i_{\max};$$

(2) for any $\tau < \inf_{i\in I}\Gamma^i_{\max}$, we let

$$\left(\inf_{i\in I}\Gamma^i\right)_{\tau} := \inf_{i\in I}\Gamma^i_{\tau}.$$

Chapter 10
The theory of Okounkov bodies

It is very fortunate that, unlike people who dig for gold, mathematicians can freely share their precious treasures with everybody. Once you understand something really well, it feels great to explain it to all.
— Andrei Okounkov

In this chapter, we apply our theory of singularities to the study of Okounkov bodies. We establish the theory of partial Okounkov bodies, which are convex bodies constructed from a given plurisubharmonic singularity. These objects allow us to reduce many problems in pluripotential theory to problems in convex geometry, which are usually simpler.

We will establish two related theories. One in the algebraic setting in Section 10.2 and one in the transcendental setting in Section 10.3.

10.1 Flags and valuations

10.1.1 The algebraic setting

Let X be an irreducible normal projective variety of dimension n.

Definition 10.1.1 An *admissible flag* $Y_\bullet$ on X is a flag of subvarieties

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

such that Y_i is irreducible of codimension i and is smooth at the point Y_n.

Given any admissible flag $Y_\bullet$, we can define a rank n valuation $\nu_{Y_\bullet} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$. Here we consider $\mathbb{Z}^n$ as a totally ordered Abelian group with the lexicographic order. We sometimes write $\mathbb{Z}^n_{\mathrm{lex}}$ to emphasize this point.

The automorphism group $\mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ of $\mathbb{Z}^n_{\mathrm{lex}}$ is then identified with the subgroup of $\mathrm{GL}(n,\mathbb{Z})$ consisting of matrices of the form $\mathrm{I} + U$, where I is the identity matrix and U is a strictly upper triangular matrix with elements in $\mathbb{Z}$.

We recall the definition of $\nu_{Y_\bullet}$: Let $s \in \mathbb{C}(X)^\times$. Let $\nu(s)_1 = \mathrm{ord}_{Y_1} s$. After localization around Y_n, we can take a local defining equation t^1 of Y_1, set $s_1 = (s(t^1)^{-\nu_1(s)})|_{Y_1}$. Then $s_1 \in \mathbb{C}(Y_1)^\times$. We can repeat this construction with Y_2 in place of Y_1 to get $\nu(s)_2$ and s_2. Repeating this construction n times, we get

$$\nu_{Y_\bullet}(s) = (\nu(s)_1, \nu(s)_2, \dots, \nu(s)_n) \in \mathbb{Z}^n.$$

It is easy to verify that $\nu_{Y_\bullet}$ is indeed a rank n valuation.

The same construction can be applied to define $\nu_{Y_\bullet}(s)$ when $s \in \mathrm{H}^0(X, L)$ or $\nu_{Y_\bullet}(D)$ when D is an effective divisor on X.

Remark 10.1.1 Conversely, by a theorem of Abhyankar, any valuation of $\mathbb{C}(X)$ with Noetherian valuation ring of rank n is equivalent to a valuation taking value in $\mathbb{Z}^n$, see [FK18, Chapter 0, Theorem 6.5.2]. As shown in [CFK+17, Theorem 2.9], any such valuation is equivalent[1] to (but not necessarily equal to) a valuation induced by an admissible flag on a modification of X.

10.1.2 The transcendental setting

Let X be a connected compact Kähler manifold of dimension n.

Definition 10.1.2 A *smooth flag* $Y_\bullet$ on X consists of a flag of connected submanifolds of X:

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n,$$

where Y_i has dimension $n - i$.

In this section, we will fix a smooth flag $Y_\bullet$ on X.

Definition 10.1.3 Let T be a closed positive $(1,1)$-current on X. We define the *valuation* of T along $Y_\bullet$ as

$$\nu_{Y_\bullet}(T) = \left(\nu_{Y_\bullet}(T)_1, \dots, \nu_{Y_\bullet}(T)_n\right) \in \mathbb{R}^n_{\geq 0}$$

by induction on n. When $n = 0$, we define $\nu_{Y_\bullet}(T)$ as the unique point in $\mathbb{R}^0$. When $n > 1$, we define

$$\nu_{Y_\bullet}(T)_1(T) = \nu(T, Y_1);$$

Then for $i = 2, \dots, n$, we define

$$\nu_{Y_\bullet}(T)_i = \nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])\right)_{i-1}.$$

Proposition 10.1.1 *Let T be a closed positive $(1,1)$-current on X. Then $\nu_{Y_\bullet}(T) \in \mathbb{R}^n_{\geq 0}$ defined in Definition 10.1.3 is independent of the choices of the trace operators in the definition. Moreover, $\nu_{Y_\bullet}(T)$ depends only on the $\mathcal{I}$-equivalence class of T.*

Proof We will prove both statements at the same time by induction on $n \geq 0$. The case $n = 0$ is trivial.

[1] Two valuations ν, ν' with value in $\mathbb{Z}^n$ are equivalent if one can find a matrix G of the form $\mathrm{I} + N$, where N is strictly upper triangular with integral entries, such that $\nu' = \nu G$.

Let us consider the case $n > 0$ and assume that the result is known in dimension $n-1$. We first observe that $\nu_{Y_\bullet}(T)$ is independent of the choice of the trace operator: different choices of $\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])$ are $\mathcal{I}$-equivalent by Proposition 8.1.2. Therefore, by induction, its valuation is well-defined.

Next, let T' be another closed positive $(1,1)$-current such that $T \sim_{\mathcal{I}} T'$. Using Proposition 3.2.1, we know that $\nu(T, Y_1) = \nu(T', Y_1)$. Therefore,

$$T - \nu(T, Y_1)[Y_1] \sim_{\mathcal{I}} T' - \nu(T', Y_1)[Y_1].$$

It follows by induction that

$$\nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])\right) = \nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T' - \nu(T', Y_1)[Y_1])\right).$$

Example 10.1.1 When X is projective, we have

$$\nu_{Y_\bullet}([D]) = \nu_{Y_\bullet}(D),$$

where the right-hand side is defined in Section 10.1.1.

Proposition 10.1.2 *Let T, S be closed positive $(1,1)$-currents on X, $\lambda \in \mathbb{R}_{\geq 0}$. Then*

(1) *if $T \preceq_{\mathcal{I}} S$, we have*
$$\nu_{Y_\bullet}(T) \geq_{\mathrm{lex}} \nu_{Y_\bullet}(S). \tag{10.1}$$

(2) *We have the following additivity property:*

$$\nu_{Y_\bullet}(T+S) = \nu_{Y_\bullet}(T) + \nu_{Y_\bullet}(S), \quad \nu_{Y_\bullet}(\lambda T) = \lambda \nu_{Y_\bullet}(T). \tag{10.2}$$

Proof (1) We make an induction on $n \geq 0$. The case $n = 0, 1$ is trivial. Assume that $n \geq 2$ and the case $n-1$ is known. Observe that $\nu(T, Y_1) \geq \nu(S, Y_1)$, if the inequality is strict, we are done. So let us assume that $\nu(T, Y_1) = \nu(S, Y_1)$. By Proposition 8.2.1, we find that

$$\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1]) \preceq_{\mathcal{I}} \mathrm{Tr}_{Y_1}(S - \nu(T, Y_1)[Y_1]).$$

By the inductive hypothesis, we conclude (10.1).

(2) We make an induction on $n \geq 0$. The cases $n = 0, 1$ are trivial. Assume that $n \geq 2$ and the case $n-1$ is known. By Proposition 1.4.2, we have

$$\nu(T+S, Y_1) = \nu(T, Y_1) + \nu(S, Y_1), \quad \nu(\lambda T, Y_1) = \lambda \nu(T, Y_1).$$

By Proposition 8.2.1, we have

$$\begin{aligned} \mathrm{Tr}_{Y_1}(T + S - \nu(T+S, Y_1)[Y_1]) &\sim_P \mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1]) + \mathrm{Tr}_{Y_1}(S - \nu(S, Y_1)[Y_1]), \\ \mathrm{Tr}_{Y_1}(\lambda T - \nu(\lambda T, Y_1)[Y_1]) &\sim_P \lambda\, \mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1]). \end{aligned}$$

By the inductive hypothesis, we conclude (10.2).

Definition 10.1.4 Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. We say that a smooth flag $W_\bullet$ on Z is a *lifting* of $Y_\bullet$ to Z if the restriction of π to $W_i \to Y_i$ is defined and bimeromorphic for each $i = 0, \dots, n$.

In this case, we define $\operatorname{cor}(Y_\bullet, \pi) \in \operatorname{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ inductively as follows:

$$\operatorname{cor}(Y_\bullet, \pi) \coloneqq \begin{bmatrix} 1 & -\nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) \\ 0 & \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi|_{W_1} : W_1 \to Y_1) \end{bmatrix}. \tag{10.3}$$

We observe that a lifting $W_\bullet$ of $Y_\bullet$ on Z is unique if it exists. For each $i = 0, \dots, n-1$, the component W_{i+1} is necessarily the strict transform of Y_{i+1} with respect to the bimeromorphic morphism $W_i \to Y_i$. We shall also say that $(W_\bullet, \operatorname{cor}(Y_\bullet, \pi))$ is *the lifting* of $Y_\bullet$ to Z.

Proposition 10.1.3 *Let $\pi\colon Z \to X$, $p\colon Z' \to Z$ be proper bimeromorphic morphisms with Z and Z' being Kähler manifolds. Assume that $Y_\bullet$ admits a lifting $W_\bullet$ (resp. $W'_\bullet$) to Z (resp. Z'). Then*

$$\operatorname{cor}(Y_\bullet, \pi \circ p) = \operatorname{cor}(Y_\bullet, \pi) \operatorname{cor}(W_\bullet, p). \tag{10.4}$$

Proof We let $\pi' = \pi \circ p$:

$$\begin{tikzcd} Z' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Z \arrow[dl, "\pi"] \\ & X & \end{tikzcd}.$$

We make induction on $n \geq 1$. The case $n = 1$ is trivial. Assume that $n \geq 2$ and the case $n-1$ has been solved. Then by (10.3), the desired formula (10.4) can be reformulated as

$$\begin{aligned} &\begin{bmatrix} 1 & -\nu_{W'_1 \supseteq \cdots \supseteq W'_n}((\pi'^*[Y_1] - [W'_1])|_{W'_1}) \\ 0 & \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi'|_{W'_1} : W'_1 \to Y_1) \end{bmatrix} = \\ &\begin{bmatrix} 1 & -\nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) \\ 0 & \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi|_{W_1} : W_1 \to Y_1) \end{bmatrix} \cdot \\ &\begin{bmatrix} 1 & -\nu_{W'_1 \supseteq \cdots \supseteq W'_n}((p^*[W_1] - [W'_1])|_{W'_1}) \\ 0 & \operatorname{cor}(W_1 \supseteq \cdots \supseteq W_n, p|_{W'_1} : W'_1 \to W_1) \end{bmatrix} \end{aligned}$$

By the inductive hypothesis, this is equivalent to

$$\begin{aligned} \nu_{W'_1 \supseteq \cdots \supseteq W'_n}((\pi'^*[Y_1] - [W'_1])|_{W'_1}) &= \nu_{W'_1 \supseteq \cdots \supseteq W'_n}((p^*[W_1] - [W'_1])|_{W'_1}) + \\ \nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) &\operatorname{cor}(W_1 \supseteq \cdots \supseteq W_n, p|_{W'_1} : W'_1 \to W_1), \end{aligned}$$

which can be further rewritten as

$$\nu_{W_1'\supseteq\cdots\supseteq W_n'}((\pi'^*[Y_1]-[W_1'])|_{W_1'})=\nu_{W_1'\supseteq\cdots\supseteq W_n'}((p^*[W_1]-[W_1'])|_{W_1'})+\nu_{W_1'\supseteq\cdots\supseteq W_n'}(p|_{W_1'}^*(\pi^*[Y_1]-[W_1])|_{W_1}).$$

This follows from Proposition 10.1.2. □

Proposition 10.1.4 *Let $\pi\colon Z\to X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. Let $W_\bullet$ be a lifting of $Y_\bullet$, then for any closed positive $(1,1)$-current T on X, we have*

$$\nu_{W_\bullet}(\pi^*T)=\nu_{Y_\bullet}(T)\operatorname{cor}(Y_\bullet,\pi). \tag{10.5}$$

Proof We make induction on $n\geq 0$. The case $n=0$ is trivial. In general, assume that $n\geq 1$ and the result is proved in dimension $n-1$.

For simplicity, we write $\nu=\nu_{Y_\bullet}$ and $\nu'=\nu_{W_\bullet}$. Let μ (resp. μ') be the valuation of currents defined by the truncated flag $Y_1\supseteq\cdots\supseteq Y_n$ (resp. $W_1\supseteq\cdots\supseteq W_n$). Then we need to show that

$$\begin{aligned}&\left[\nu'(\pi^*T)_1\ \mu'(\operatorname{Tr}_{W_1}(\pi^*T-\nu'(\pi^*T)_1[W_1]))\right]\\ =&\left[\nu(T)_1\ \mu(\operatorname{Tr}_{Y_1}(T-\nu(T)_1[Y_1]))\right]\operatorname{cor}(Y_\bullet,\pi).\end{aligned} \tag{10.6}$$

By Zariski's main theorem,

$$\nu'(\pi^*T)_1=\nu(T)_1=:c.$$

By the inductive hypothesis, we have

$$\mu'(\Pi^*\operatorname{Tr}_{Y_1}(T-c[Y_1]))=\mu(\operatorname{Tr}_{Y_1}(T-c[Y_1]))\operatorname{cor}(Y_1\supseteq\cdots\supseteq Y_n,\Pi), \tag{10.7}$$

where $\Pi\colon W_1\to Y_1$ is the restriction of π. By Lemma 8.2.1 and Proposition 8.2.1,

$$\begin{aligned}&\Pi^*\operatorname{Tr}_{Y_1}(T-c[Y_1])\sim_P\operatorname{Tr}_{W_1}(\pi^*(T-c[Y_1]))\\ \sim_P&\operatorname{Tr}_{W_1}(\pi^*T-c[W_1])+c\operatorname{Tr}_{W_1}(\pi^*[Y_1]-[W_1]).\end{aligned}$$

So

$$\mu'(\Pi^*\operatorname{Tr}_{Y_1}(T-c[Y_1]))=\mu'(\operatorname{Tr}_{W_1}(\pi^*T-c[W_1]))+c\mu'(\operatorname{Tr}_{W_1}(\pi^*[Y_1]-[W_1])).$$

Combining the above with (10.7), we see that (10.6) follows. □

Theorem 10.1.1 *Let $\pi\colon Z\to X$ be a proper bimeromorphic morphism from a reduced complex space Z. Then there is a modification $W\to X$ dominating $Z\to X$ such that $Y_\bullet$ admits a lifting to W.*

Proof By Hironaka's Chow lemma, we may assume that π is a modification.

We begin by setting $W_0=Z$. We will construct W_i inductively for each i. Assume that for $0\leq i<n$ a smooth partial flag $W_0\supset\cdots\supset W_i$ has been constructed on a modification $\pi_i:Z_i\to Z$ so that $\pi\circ\pi_i$ restricts to bimeromorphic morphisms $W_j\to Y_j$ for each $j=0,\ldots,i$.

By Zariski's main theorem, $W_i \to Y_i$ is an isomorphism outside a codimension 2 subset of Y_i. We let W_{i+1} be the strict transform of Y_{i+1} in W_i. The problem is that W_{i+1} is not necessarily smooth.

We will further modify Z_i and lift $W_1, \ldots, W_{i+1}$ in order to make the flag smooth. Take the embedded resolution of (W_j, W_{i+1}), say $W'_j \to W_j$ for each $j = 0, \ldots, i$.

We have canonical embeddings $W'_i \hookrightarrow W'_{i-1} \hookrightarrow \cdots \hookrightarrow W'_0$ making the following diagram commutative:

$$\begin{tikzcd} W'_i \arrow[r, hook] \arrow[d] & W'_{i-1} \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] \arrow[d, phantom, "\vdots"] & W'_0 \arrow[d] \\ W_i \arrow[r, hook] & W_{i-1} \arrow[r, hook] & \cdots \arrow[r, hook] & W_0 \end{tikzcd}$$

Let W'_{i+1} be the strict transform of W_{i+1} in W'_i. It suffices to define π_{i+1} as the morphism $W'_0 \to Z_i \to Z$ and replace $W_0 \supset \cdots \supset W_{i+1}$ by $W'_0 \supset \cdots \supset W'_{i+1}$. □

Remark 10.1.2 Suppose that X is a normal projective variety. Consider a rank n (surjective) valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ and a closed positive $(1,1)$-current T on X. Then we can always define $\nu(T) \in \mathbb{R}^n$ as follows: Take a resolution $\pi\colon Y \to X$ such that there is a smooth flag $Y_\bullet$ on Y and $g \in \mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ such that

$$\nu = \nu_{Y_\bullet} g.$$

Then we define

$$\nu(T) \coloneqq \nu_{Y_\bullet}(\pi^* T) g.$$

This definition does not depend on the choice of π, as a consequence of Proposition 10.1.4.

10.2 Algebraic partial Okounkov bodies

Let X be a connected smooth complex projective variety of dimension n and (L, h) be a Hermitian big line bundle on X.

Let h_0 be a smooth Hermitian metric on L. Let $\theta = c_1(L, h_0)$. Then we can identify h with a function $\varphi \in \mathrm{PSH}(X, \theta)$. We will use interchangeably the notations (θ, φ) and (L, h).

Fix a rank n valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$, which without loss of generality can be assumed to be surjective.

We will adopt the notations of Appendix C.2.

10.2.1 The spaces of sections

Definition 10.2.1 We will write

$$\Gamma(\theta,\varphi) := \left\{(\nu(s),k) : k\in\mathbb{N}, s\in \mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))^\times\right\},$$
$$\Delta_k(\theta,\varphi) :=\mathrm{Conv}\left\{k^{-1}\nu(f) : f\in \mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))^\times\right\}\subseteq\mathbb{R}^n,\quad k\geq 0.$$

When $\theta = V_\theta$, we simply write $\Gamma(L)$ and $\Delta_k(L)$ instead.

Here Conv denotes the convex hull. For large enough k, $\Delta_k(\theta,\varphi)$ is non-empty thanks to Theorem 7.3.1.

Definition 10.2.2 Assume that φ has analytic singularities. We define

$$\Gamma^\infty(\theta,\varphi) := \left\{(\nu(s),k) : k\in\mathbb{N}, s\in \mathrm{H}^0(X,L^k\otimes\mathcal{I}_\infty(k\varphi))^\times\right\}. \tag{10.8}$$

For later use, we introduce a twisted version as well.

Definition 10.2.3 If T is a holomorphic line bundle on X, we introduce

$$\Delta_{k,T}(\theta,\varphi) :=\mathrm{Conv}\left\{k^{-1}\nu(f) : f\in \mathrm{H}^0(X,T\otimes L^k\otimes\mathcal{I}(k\varphi))^\times\right\}\subseteq\mathbb{R}^n,$$
$$\Delta_{k,T}(L) :=\mathrm{Conv}\left\{k^{-1}\nu(f) : f\in \mathrm{H}^0(X,T\otimes L^k)^\times\right\}\subseteq\mathbb{R}^n.$$

10.2.2 Algebraic Okounkov bodies

Proposition 10.2.1 *There is a convex body $\Delta\in\mathcal{K}_n$ such that $\Gamma(L)\in\mathcal{S}'(\Delta)$.*

Proof **Step 1**. We first show that there is $\Delta\in\mathcal{K}_n$ such that $\Delta_k(L)\subseteq\Delta$. For this purpose, using Remark 10.1.1, we may assume that ν is induced by an admissible flag $Y_\bullet$ on X.

Fix $s\in \mathrm{H}^0(X,L^k)^\times$ for some $k\in\mathbb{Z}_{>0}$. Assume that $s\neq 0$. We need to show that for each $i=1,\dots,n$, $\nu(s)_i\leq Ck$ for some constant $C>0$, independent of the choices of k and s.

Fix an ample divisor H on X. Take a large enough integer $b_1>0$ such that

$$(L-b_1Y_1)\cdot H^{n-1}<0.$$

Then $\nu(s)_1\leq b_1k$. Next take a large enough integer b_2 such that

$$\left((L-aY_1)|_{Y_1}-b_2Y_2\right)\cdot H^{n-2}<0.$$

It follows that $\nu(s)_2\leq b_2k$. Continue in this manner, we conclude that $\nu(s)_i/k$ is bounded for each i.

Step 2. Observe that $\Gamma(L)$ is clearly a semigroup. It remains to show that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$ as an Abelian group.

For this purpose, take two very ample divisors A and B so that $L = \mathcal{O}_X(A - B)$. After choosing A and B ample enough, we may guarantee that there exist sections $s_0 \in \mathrm{H}^0(X, A)$, $t_i \in \mathrm{H}^0(X, B)$ for $i = 0, \dots, n$ such that

$$\nu(s_0) = \nu(t_0) = 0$$

and $\nu(t_i)$ is the i-th unit vector $e_i \in \mathbb{R}^n$ for $i = 1, \dots, n$.

Since L is big, we can find $m_0 > 0$ such that for any $m \geq m_0$ we can find an effective divisor F_m on X linearly equivalent to $mL - B$. Let $f_m = \nu([F_m])$. Then we find that

$$(f_m, m), (f_m + e_1, m), \dots, (f_m + e_n, m) \in \Gamma(L).$$

Since $(m+1)L$ is linearly equivalent to $A + F_m$, so

$$(f_m, m+1) \in \Gamma(L).$$

It follows that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$. □

Thanks to Proposition 10.2.1, we can introduce the next definition.

Definition 10.2.4 We define the *Okounkov body* of L with respect to the valuation ν as

$$\Delta_\nu(L) \coloneqq \Delta(\Gamma(L)).$$

Proposition 10.2.2 *The Okounkov body $\Delta_\nu(L)$ depends only on the numerical class of L.*

See [LM09, Proposition 4.1] for the elegant proof.

Corollary 10.2.1 *We have*

$$\operatorname{vol} \Delta_\nu(L) = \frac{1}{n!} \operatorname{vol} L. \tag{10.9}$$

Proof This follows immediately from Proposition 10.2.1 and Theorem C.2.1. □

Proposition 10.2.3 *Assume that φ has analytic singularities and θ_φ is a Kähler current. Then we have*

$$\Gamma^\infty(\theta, \varphi) \in \mathcal{S}'(X, \theta)$$

and

$$\operatorname{vol} \Gamma^\infty(\theta, \varphi) = \frac{1}{n!} \int_X \theta_\varphi^n.$$

Proof Replacing X by a modification, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. See Theorem 1.6.1.

In this case,

$$\Gamma^\infty(\theta, \varphi) = \left\{(\nu(s), k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{O}_X(-\lfloor kD \rfloor)).\right\}$$

Since $L - D$ is ample by Lemma 1.6.1, our assertion follows from the same argument as Proposition 10.2.1. □

We first extend Theorem C.2.1 to the twisted case.

Proposition 10.2.4 *For any holomorphic line bundle T on X, as $k \to \infty$*

$$\Delta_{k,T}(L) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(L).$$

Proof As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0, and
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$\mathrm{H}^0(X, L^{k-k_0}) \xrightarrow{\times s_1} \mathrm{H}^0(X, T \otimes L^k) \xrightarrow{\times s_0} \mathrm{H}^0(X, L^{k+k_0}).$$

It follows that

$$(k - k_0)\Delta_{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_{k,T}(L) \subseteq (k + k_0)\Delta_{k+k_0}(L) - \nu(s_0).$$

Using Theorem C.2.1, we conclude. □

Proposition 10.2.5 *Let L' be another big line bundle on X. Then*

$$\Delta_\nu(L) + \Delta_\nu(L') \subseteq \Delta_\nu(L \otimes L').$$

Proof Observe that for each $k \in \mathbb{N}$, we have

$$\Delta_k(L) + \Delta_k(L') \subseteq \Delta_k(L \otimes L').$$

So our assertion follows immediately from Theorem C.2.1. □

Proposition 10.2.6 *For any $a \in \mathbb{Z}_{>0}$, we have*

$$\Delta_\nu(L^a) = a\Delta_\nu(L).$$

Proof This is an immediate consequence of Theorem C.2.1. □

10.2.3 Construction of partial Okounkov bodies

Theorem 10.2.1 *We have*

$$\Gamma(\theta, \varphi) \in \overline{\mathcal{S}'(\Delta_\nu(L))}_{>0}.$$

This theorem allows us to give the following definition:

Definition 10.2.5 The *partial Okounkov body* of (L,h) is defined as

$$\Delta_\nu(L,h)=\Delta_\nu(\theta,\varphi)\coloneqq\Delta\left(\Gamma(\theta,\varphi)\right). \tag{10.10}$$

When ν is induced by an admissible flag $Y_\bullet$ on X (see Definition 10.1.1), we also say that $\Delta_\nu(\theta,\varphi)$ the *partial Okounkov body* of (L,h) or of (θ,φ) with respect to $Y_\bullet$. In this case, we also write $\Delta_{Y_\bullet}$ instead of Δ_ν.

Corollary 10.2.2 *We have*

$$\operatorname{vol}\Delta_\nu(\theta,\varphi)=\frac{1}{n!}\operatorname{vol}\theta_\varphi. \tag{10.11}$$

Proof This follows immediately from Theorem 10.2.1, Theorem 7.3.1 and Theorem C.2.2. □

We will prove Theorem 10.2.1 and Corollary 10.2.2 at the same time.

Proof **Step 1**. We first assume that φ has analytic singularities and θ_φ is a Kähler current.

We claim that

$$d_{\mathrm{sg}}(\Gamma^\infty(\theta,\varphi),\Gamma(\theta,\varphi))=0. \tag{10.12}$$

Observe that for each $\epsilon\in\mathbb{Q}_{>0}$, we have

$$\mathrm{H}^0(X,L^k\otimes\mathcal{I}_\infty(k\varphi))\subseteq\mathrm{H}^0(X,L^k\otimes\mathcal{I}(k\varphi))\subseteq\mathrm{H}^0(X,L^k\otimes\mathcal{I}_\infty(k(1-\epsilon)\varphi))$$

for all large enough k. This is a consequence of Lemma 1.6.3. Therefore, it suffices to show that

$$\lim_{\mathbb{Q}\ni\epsilon\to 0+}\operatorname{vol}\Gamma^\infty(\theta,(1-\epsilon)\varphi)=\operatorname{vol}\Gamma^\infty(\theta,\varphi).$$

This follows from the explicit formula in Proposition 10.2.3.

Step 2. We next handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. Then $\varphi_j\xrightarrow{d_S}P_\theta[\varphi]_{\mathcal{I}}$ by Corollary 7.1.2.

In this case, it suffices to prove that

$$\Gamma(\theta,\varphi_j)\xrightarrow{d_{\mathrm{sg}}}\Gamma(\theta,\varphi). \tag{10.13}$$

In fact, by Theorem 7.3.1, we have

$$\begin{aligned}
&d_{\mathrm{sg}}(\Gamma(\theta,\varphi_j),\Gamma(\theta,\varphi))\\
=&\varlimsup_{k\to\infty}k^{-n}\left(h^0(X,L^k\otimes\mathcal{I}(k\varphi_j))-h^0(X,L^k\otimes\mathcal{I}(k\varphi))\right)\\
=&\lim_{k\to\infty}k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi_j))-\lim_{k\to\infty}k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi))\\
=&\frac{1}{n!}\operatorname{vol}\theta_{\varphi_j}-\frac{1}{n!}\operatorname{vol}\theta_\varphi.
\end{aligned}$$

Letting $j \to \infty$, we conclude (10.13) by Theorem 6.2.5.

Step 3. Now we only assume that $\operatorname{vol}\theta_\varphi > 0$. We may replace φ with $P_\theta[\varphi]_{\mathcal{I}}$ and then assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$.

Take a potential $\psi \in \mathrm{PSH}(X,\theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. The existence of ψ is proved in Lemma 2.3.2. For each $\epsilon \in (0,1)$, let $\varphi_\epsilon = (1-\epsilon)\varphi + \epsilon\psi$. It suffices to show that

$$\Gamma(\theta,\varphi_\epsilon) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta,\varphi)$$

as $\epsilon \to 0+$. We compute using Theorem 7.3.1:

$$\begin{aligned}
&d_{\mathrm{sg}}\left(\Gamma(\theta,\varphi_\epsilon),\Gamma(\theta,\varphi)\right)\\
=&\varlimsup_{k\to\infty} k^{-n}\left(h^0(X,L^k\otimes\mathcal{I}(k\varphi)) - h^0(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon))\right)\\
=&\lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi)) - \lim_{k\to\infty} k^{-n}h^0(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon))\\
=&\frac{1}{n!}\operatorname{vol}\theta_\varphi - \frac{1}{n!}\operatorname{vol}\theta_{\varphi_\epsilon}\\
\to&0
\end{aligned}$$

by Theorem 6.2.5, as $\epsilon \to 0+$. □

Remark 10.2.1 It follows from the proof that if φ has analytic singularities and θ_φ is a Kähler current, then (10.12) holds.

If we take a modification $\pi\colon Y \to X$ such that $\pi^*\varphi$ has log singularities along an effective $\mathbb{Q}$-divisor D on Y, then

$$\Delta_\nu(\theta,\varphi) = \Delta_\nu(\pi^*L - D) + \nu(D). \tag{10.14}$$

10.2.4 Basic properties of partial Okounkov bodies

Proposition 10.2.7 *The partial Okounkov body $\Delta_\nu(L,h)$ depends only on $\mathrm{dd^c}h$, not on the explicit choices of L, h_0, h.*

Thanks to this result, given a closed positive $(1,1)$-current $T \in c_1(L)$ on X with $\int_X T^n > 0$, we can write

$$\Delta_\nu(T) := \Delta_\nu(\theta,\varphi)$$

if $T = \theta + \mathrm{dd^c}\varphi$ for some $\varphi \in \mathrm{PSH}(X,\theta)$.

Proof There are two different claims to prove, as detailed in the two steps below.

Step 1. Let h_0' be another Hermitian metric on L. Set $\theta' = c_1(L,h_0')$. Write $\mathrm{dd^c} f = \theta - \theta'$. Let $\varphi' = \varphi + f \in \mathrm{PSH}(X,\theta')$. Then

$$\Delta_\nu(\theta,\varphi) = \Delta_\nu(\theta',\varphi'). \tag{10.15}$$

This is obvious since $\Gamma(\theta,\varphi) = \Gamma(\theta',\varphi')$.

Step 2. Let L' be another big line bundle on X. By Step 1, we may assume that the reference Hermitian metric h'_0 on L' is such that $c_1(L', h'_0) = \theta$.

Let h' be a plurisubharmonic metric on L' with $c_1(L, h) = c_1(L', h')$. Then

$$\Delta_\nu(L, h) = \Delta_\nu(L', h').$$

From our construction, we may assume that $c_1(L, h)$ has analytic singularities. After taking a birational resolution, it suffices to deal with the case where $c_1(L, h)$ has analytic singularities along an effective $\mathbb{Q}$-divisors D. By rescaling, we may also assume that D is a divisor. By Remark 10.2.1, we further reduce to the case where $c_1(L, h)$ is not singular.

In this case, the assertion is proved in Proposition 10.2.2. □

Proposition 10.2.8 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. Assume that $\varphi \preceq_{\mathcal{I}} \psi$, then*

$$\Delta_\nu(\theta, \varphi) \subseteq \Delta_\nu(\theta, \psi). \tag{10.16}$$

Proof This follows from Corollary C.2.2. □

Theorem 10.2.2 *The Okounkov body map*

$$\Delta_\nu(\theta, \bullet) : (\mathrm{PSH}(X, \theta)_{>0}, d_S) \to (\mathcal{K}_n, d_{\mathrm{Haus}})$$

is continuous.

Proof Let $\varphi_j \to \varphi$ be a d_S-convergent sequence in $\mathrm{PSH}(X, \theta)_{>0}$. We want to show that

$$\Delta_\nu(\theta, \varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta, \varphi). \tag{10.17}$$

By Proposition 10.2.8, we may assume that all φ_j's and φ are model potentials.

By Theorem C.1.1 and Proposition 6.2.3, we may assume that $(\varphi_j)_j$ is either decreasing or increasing. By Theorem 6.2.3, we may further assume that the φ_j's are $\mathcal{I}$-model. In both cases, we claim that

$$\Gamma(\theta, \varphi_j) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta, \varphi)$$

as $j \to \infty$. In fact, using Theorem 7.3.1, we can compute

$$\begin{aligned} d_{\mathrm{sg}}\left(\Gamma(\theta, \varphi_j), \Gamma(\theta, \varphi)\right) &= \varlimsup_{k\to\infty} k^{-n} \left|\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi_j)) - \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))\right| \\ &= \frac{1}{n!}\left|\operatorname{vol}\theta_{\varphi_j} - \operatorname{vol}\theta_\varphi\right|, \end{aligned}$$

which converges to 0 by Theorem 6.2.5. □

Proposition 10.2.9 *Let $\pi \colon Y \to X$ be a modification. Then*

$$\Delta_\nu(\pi^* L, \pi^* h) = \Delta_\nu(L, h).$$

Proof Thanks to Proposition 3.2.5, we may assume that φ is $\mathcal{I}$-model. By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ with analytic singularities in $\mathrm{PSH}(X,\theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi$. It is clear that $\pi^*\varphi_j \xrightarrow{d_S} \pi^*\varphi$. By Theorem 10.2.2, we may then reduce to the case where φ has analytic singularities. In this case, it suffices to apply Remark 10.2.1. $\square$

Proposition 10.2.10 *Let (L',h') be another Hermitian big line bundle on X. Then*

$$\Delta_\nu(L,h) + \Delta_\nu(L',h') \subseteq \Delta_\nu(L\otimes L', h\otimes h').$$

Proof Take a smooth metric h'_0 on L' and let $\theta' = c_1(L',h'_0)$. We identify h' with $\varphi' \in \mathrm{PSH}(X,\theta')$. Then we need to show

$$\Delta_\nu(\theta,\varphi) + \Delta_\nu(\theta',\varphi') \subseteq \Delta_\nu(\theta+\theta',\varphi+\varphi'). \tag{10.18}$$

By Theorem 7.1.1, we can find sequences $(\varphi_j)_j$ and $(\varphi'_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ and $\mathrm{PSH}(X,\theta')_{>0}$ respectively such that

(1) φ_j and φ'_j both have analytic singularities for all $j \geq 1$, and

(2) $\varphi_j \xrightarrow{d_S} \varphi$, $\varphi'_j \xrightarrow{d_S} \varphi'$.

Then $\varphi_j + \varphi'_j \in \mathrm{PSH}(X,\theta+\theta')_{>0}$ and $\varphi_j + \varphi'_j \xrightarrow{d_S} \varphi + \varphi'$ by Theorem 6.2.2. Thus, by Theorem 10.2.2, we may assume that φ and ψ both have analytic singularities. Taking a birational resolution, we may further assume that they have log singularities. By Remark 10.2.1, we reduce to the case without singularities, in which case the result is just Proposition 10.2.5. $\square$

Theorem 10.2.3 *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$. Then for any $t \in (0,1)$,*

$$\Delta_\nu(\theta, t\varphi + (1-t)\psi) \supseteq t\Delta_\nu(\theta,\varphi) + (1-t)\Delta_\nu(\theta,\psi). \tag{10.19}$$

Proof We may assume that t is rational as a consequence of Theorem 10.2.2. Similarly, as in the proof of Proposition 10.2.10, we could reduce to the case where both φ and ψ have analytic singularities. In this case, let $N > 0$ be an integer such that Nt is an integer. Then for any $s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi))$ and $r \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\psi))$, we have

$$s^{tN} \otimes r^{N-tN} \in \mathrm{H}^0(X, L^{kN} \otimes \mathcal{I}_\infty(Nt\varphi + (N-Nt)\psi)).$$

By Theorem C.2.1 and Remark 10.2.1, (10.19) follows. $\square$

Proposition 10.2.11 *For any $a \in \mathbb{Z}_{>0}$,*

$$\Delta_\nu(a\theta, a\varphi) = a\Delta_\nu(\theta,\varphi).$$

Proof As in the proof of Proposition 10.2.10, we may assume that φ has log singularities. Using Remark 10.2.1, we reduce to the case without the singularity φ, which is proved in Proposition 10.2.6. $\square$

In particular, if T is a closed positive $(1,1)$-current on X with $\int_X T^n > 0$ and such that

$$[T] \in \mathrm{NS}^1(X)_{\mathbb{Q}},$$

we can define

$$\Delta_\nu(T) := a^{-1}\Delta_\nu(aT) \tag{10.20}$$

for a sufficiently divisible positive integer a.

We also need the following perturbation. Let A be an ample line bundle on X. Fix a Hermitian metric h_A on A such that $\omega := c_1(A, h_A)$ is a Kähler form on X.

Proposition 10.2.12 *As $\delta \searrow 0$, the convex bodies $\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi)$ are decreasing and*

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta_\varphi).$$

Proof Let $0 \leq \delta < \delta'$ be two rational numbers. Take $C \in \mathbb{N}_{>0}$ divisible enough, so that $C\delta$ and $C\delta'$ are both integers. Then by Proposition 10.2.10,

$$\Delta_\nu(C\theta + C\delta\omega + C\mathrm{dd^c}\varphi) \subseteq \Delta_\nu(C\theta + C\delta'\omega + C\mathrm{dd^c}\varphi).$$

It follows that

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \subseteq \Delta_\nu(\theta + \delta'\omega + \mathrm{dd^c}\varphi).$$

On the other hand,

$$\mathrm{vol}\,\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) = \frac{1}{n!}\,\mathrm{vol}(\theta + \delta\omega)_\varphi = \frac{1}{n!}\int_X (\theta + \delta\omega)^n_{P_\theta[\varphi]_{\mathcal{I}}},$$

where we applied Example 7.1.2. As $\delta \to 0+$, the right-hand side converges to

$$\mathrm{vol}\,\Delta_\nu(\theta,\varphi) = \frac{1}{n!}\,\mathrm{vol}\,\theta_\varphi.$$

Our assertion therefore follows. □

10.2.5 The Hausdorff convergence property

Let T be a holomorphic line bundle on X.

Theorem 10.2.4 *As $k \to \infty$, we have $\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi)$.*

Although we are only interested in the untwisted case, the proof given below requires twisted case.

Lemma 10.2.1 *Assume that φ has analytic singularities and θ_φ is a Kähler current, then as $k \to \infty$,*

$$\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi).$$

Proof Up to replacing X by a birational model and twisting T accordingly, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D, see Proposition 10.2.9 and Theorem 1.6.1.

Take a small enough $\epsilon \in \mathbb{Q}_{>0}$. In this case, for large enough $k \in \mathbb{Z}_{>0}$ we have

$$\mathrm{H}^0(X, T\otimes L^k\otimes\mathcal{I}_\infty(k\varphi)) \subseteq \mathrm{H}^0(X, T\otimes L^k\otimes\mathcal{I}(k\varphi)) \subseteq \mathrm{H}^0(X, T\otimes L^k\otimes\mathcal{I}_\infty(k(1-\epsilon)\varphi)).$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$, say the sequence defined by the indices $k_1, k_2, \dots$. We want to show that $\Delta' = \Delta(\theta,\varphi)$.

There exists $t \in \{0, 1, \dots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing k_i by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$. Then for large enough i, we have

$$\begin{aligned}\mathrm{H}^0(X, T\otimes L^{-N+t}\otimes L^{N(g_i+1)}\otimes\mathcal{I}_\infty(N(g_i+1)\varphi)) &\subseteq \mathrm{H}^0(X, T\otimes L^{k_i}\otimes\mathcal{I}(k_i\varphi))\\ &\subseteq \mathrm{H}^0(X, T\otimes L^t\otimes L^{Ng_i}\otimes\mathcal{I}_\infty(g_iN(1-\epsilon)\varphi)).\end{aligned}$$

So

$$\begin{aligned}(g_i+1)\Delta_{g_i+1, T\otimes L^{-N+t}}(NL-ND) + N(g_i+1)\nu(D) &\subseteq (Ng_i+t)\Delta_{k,T}(\theta,\varphi)\\ &\subseteq g_i\Delta_{g_i, T\otimes L^t}(NL - N(1-\epsilon)D) + Ng_i(1-\epsilon)\nu(D).\end{aligned}$$

Letting $i \to \infty$, by Proposition 10.2.4,

$$\Delta_\nu(L-D) + \nu(D) \subseteq \Delta' \subseteq \Delta_\nu(L-(1-\epsilon)D) + (1-\epsilon)\nu(D).$$

Letting $\epsilon \to 0+$, we find that

$$\Delta_\nu(L-D) + \nu(D) = \Delta'.$$

It follows from Theorem C.1.1 that

$$\Delta_{k,T}(\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(L-D) + \nu(D) = \Delta_\nu(\theta,\varphi)$$

as $k \to \infty$. □

Lemma 10.2.2 *Assume that θ_φ is a Kähler current, then as $\mathbb{Q} \ni \beta \to 0+$, we have*

$$\Delta_\nu((1-\beta)\theta,\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta,\varphi).$$

Here and in the sequel, $\Delta_\nu((1-\beta)\theta,\varphi) = \Delta_\nu((1-\beta)\theta + \mathrm{dd^c}\varphi)$.

Proof By Proposition 10.2.10, we have

$$\Delta_\nu\,((1-\beta)\theta,\varphi) + \beta\Delta_\nu(L) \subseteq \Delta_\nu(\theta,\varphi).$$

In particular, if Δ' is the Hausdorff limit of a subsequence of $(\Delta((1-\beta)\theta,\varphi))_\beta$, then $\Delta' \subseteq \Delta_\nu(\theta,\varphi)$. But

$$\operatorname{vol}\Delta' = \lim_{\beta\to 0+}\Delta_\nu((1-\beta)\theta,\varphi) = \lim_{\beta\to 0+}\int_X ((1-\beta)\theta + \mathrm{dd^c} P_{(1-\beta)\theta}[\varphi]_{\mathcal{I}})^n$$
$$= \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n,$$

where the last step follows easily from Theorem 11.2.1. It follows that $\Delta' = \Delta_\nu(\theta,\varphi)$. We conclude by Theorem C.1.1. □

Proof (Proof of Theorem 10.2.4) Fix a Kähler form $\omega \geq \theta$ on X.

Step 1. We first handle the case where θ_φ is a Kähler current, say $\theta_\varphi \geq 2\delta\omega$ for some $\delta \in (0,1)$. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\operatorname{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j} \geq \delta\omega$ for all $j \geq 1$.

Let Δ' be a limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$. Let us say the indices of the subsequence are $k_1 < k_2 < \cdots$. By Theorem C.1.1, it suffices to show that $\Delta' = \Delta_\nu(\theta,\varphi)$.

Observe that for each $j \geq 1$, we have $\Delta' \subseteq \Delta_\nu(\theta,\varphi_j)$ by Lemma 10.2.1. Letting $j \to \infty$, we find $\Delta' \subseteq \Delta_\nu(\theta,\varphi)$. Therefore, it suffices to prove that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu(\theta,\varphi). \tag{10.21}$$

Fix an integer $N > \delta^{-1}$. Observe that for any $j \geq 1$, we have $\varphi_j \in \operatorname{PSH}(X,(1-N^{-1})\theta)$. Similarly, $\varphi \in \operatorname{PSH}(X,(1-N^{-1})\theta)$. By Lemma 10.2.2, it suffices to argue that

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta_\nu((1-N^{-1})\theta,\varphi). \tag{10.22}$$

For this purpose, we are free to replace k_i's by a subsequence, so we may assume that $k_i \equiv a$ modulo q for all $i \geq 1$, where $a \in \{0,1,\ldots,q-1\}$. We write $k_i = g_i q + a$. Observe that for each $i \geq 1$,

$$\mathrm{H}^0(X, T\otimes L^{k_i}\otimes\mathcal{I}(k_i\varphi)) \supseteq \mathrm{H}^0\left(X, T\otimes L^{-q+a}\otimes L^{g_iq+q}\otimes\mathcal{I}((g_iq+q)\varphi)\right).$$

Up to replacing T by $T\otimes L^{-q+a}$, we may therefore assume that $a = 0$.

By Lemma 2.3.1, we can find $k' \in \mathbb{Z}_{>0}$ such that for all $k \geq k'$, there is $\psi \in \operatorname{PSH}(X,\theta)_{>0}$ satisfying

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1-N^{-1})\varphi_k + N^{-1}\psi_k.$$

Fix $k \geq k'$. It suffices to show that

$$\Delta_\nu((1-N^{-1})\theta,\varphi_k) + \nu' \subseteq \Delta' \tag{10.23}$$

for some $\nu' \in \mathbb{R}^n$. In fact, if this is true, we have

$$\operatorname{vol}\Delta' \geq \operatorname{vol}\Delta((1-N^{-1})\theta,\varphi_k).$$

Letting $k \to \infty$ and applying Theorem 10.2.2, we conclude (10.22).

It remains to prove (10.23). By the proof of Theorem 7.3.1, there is $j_0 > 0$ such that for any $j \geq j_0$, we can find a non-zero section $s_j \in \mathrm{H}^0(X, L^j\otimes\mathcal{I}(j\psi_k))$ such that we get an injective linear map

$$\mathrm{H}^0(X, T\otimes L^{(N-1)j}\otimes \mathcal{I}(jN\varphi_k)) \xrightarrow{\times s_j} \mathrm{H}^0(X, T\otimes L^{jN}\otimes \mathcal{I}(jN\varphi)).$$

In particular, when $j = k_i$ for some i large enough, we then find

$$\Delta_{k_i,T}((N-1)\theta, N\varphi_k) + (k_i)^{-1}\nu(s_{k_i}) \subseteq N\Delta_{k_i,T}(\theta,\varphi).$$

We observe that $(k_i)^{-1}\nu(s_{k_i})$ is bounded as both convex bodies appearing in this equation are bounded when i varies. Then by Lemma 10.2.1, there is a vector $\nu' \in \mathbb{R}^n$ such that (10.23) holds.

Step 2. Next we handle the general case.

Let Δ' be the Hausdorff limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$, say the subsequence with indices $k_1 < k_2 < \cdots$. By Theorem C.1.1, it suffices to prove that $\Delta' = \Delta_\nu(\theta,\varphi)$.

Take $\psi \in \mathrm{PSH}(X,\theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. The existence of ψ follows from Lemma 2.3.2.

Then for any $\epsilon \in \mathbb{Q}\cap(0,1)$,

$$\Delta_{k,T}(\theta,\varphi) \supseteq \Delta_{k,T}(\theta,(1-\epsilon)\varphi+\epsilon\psi)$$

for all $k \geq 1$. It follows from Step 1 that

$$\Delta' \supseteq \Delta_\nu(\theta,(1-\epsilon)\varphi+\epsilon\psi).$$

Letting $\epsilon \to 0$ and applying Theorem 10.2.2, we have $\Delta' \supseteq \Delta_\nu(\theta,\varphi)$. It remains to establish that

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(\theta,\varphi). \tag{10.24}$$

For this purpose, we are free to replace $k_1 < k_2 < \cdots$ by a subsequence. Fix $q > 0$, we may then assume that $k_i \equiv a$ modulo q for all $i \geq 1$ for some $a \in \{0,1,\ldots,q-1\}$. We write $k_i = g_i q + a$. Observe that

$$\mathrm{H}^0(X, T\otimes L^{k_i}\otimes\mathcal{I}(k_i\varphi)) \subseteq \mathrm{H}^0(X, T\otimes L^a\otimes L^{g_iq}\otimes\mathcal{I}(g_iq\varphi)).$$

Up to replacing T by $T\otimes L^a$, we may assume that $a = 0$.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in \mathrm{H}^0(X,H)$.

We have an injective linear map

$$\mathrm{H}^0(X, T\otimes L^{jq}\otimes\mathcal{I}(jq\varphi)) \xrightarrow{\times s^j} \mathrm{H}^0(X, T\otimes H^j\otimes L^{jq}\otimes\mathcal{I}(jq\varphi))$$

for each $j \geq 1$. In particular, for each $i \geq 1$,

$$k_i\Delta_{k_i,T}(q\theta,q\varphi) + k_i\nu(s) \subseteq k_i\Delta_{k_i,T}(\omega+q\theta,q\varphi).$$

Letting $i \to \infty$, by Step 1, we have

$$q\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + q\theta, q\varphi).$$

So

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(q^{-1}\omega + \theta, \varphi) = \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P_{q^{-1}\omega+\theta}[\varphi]_{\mathcal{I}})^n.$$

By Example 7.1.2,

$$\operatorname{vol}\Delta' \leq \int_X (q^{-1}\omega + \theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

Letting $q \to \infty$, we conclude (10.24). □

10.2.6 Recover Lelong numbers from partial Okounkov bodies

Theorem 10.2.5 *Let E be a prime divisor on X. Let $Y_\bullet$ be an admissible flag with $E = Y_1$. Then*

$$\nu(\varphi, E) = \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1. \tag{10.25}$$

Here x_1 denotes the first component of x.

Proof Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good.

Step 1. We first reduce to the case where φ has analytic singularities.

By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$. It follows from Theorem 10.2.2 that

$$\Delta_{Y_\bullet}(\theta, \varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta, \varphi).$$

Therefore,

$$\lim_{j\to\infty} \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi_j)} x_1 = \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1.$$

In view of Theorem 6.2.4, it suffices to prove (10.25) with φ_j in place of φ.

Step 2. Assume that φ has analytic singularities. In view of Proposition 10.2.9 and Theorem 1.6.1, after replacing X by a birational model, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor F.

Perturbing L by an ample $\mathbb{Q}$-line bundle by Proposition 10.2.12, we may assume that θ_φ is a Kähler current. Therefore, $L - F$ is ample by Lemma 1.6.1. Finally, by rescaling, we may assume that F is a divisor and L is a line bundle.

By Theorem 10.2.4, we know that

$$\min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1 = \lim_{k\to\infty} \min_{x\in\Delta_k(\theta,\varphi)} x_1.$$

By definition,

$$\min_{x\in\Delta_k(\theta,\varphi)} x_1 = k^{-1}\operatorname{ord}_E \mathrm{H}^0(X, L^k\otimes \mathcal{I}(k\varphi)).$$

It remains to show that

$$\lim_{k\to\infty} k^{-1}\operatorname{ord}_E \mathrm{H}^0(X, L^k\otimes \mathcal{I}(k\varphi)) = \lim_{k\to\infty} k^{-1}\operatorname{ord}_E \mathcal{I}(k\varphi). \tag{10.26}$$

The $\geq$ direction is trivial, we prove the converse. Observe that

$$\mathrm{H}^0(X, L^k\otimes \mathcal{I}(k\varphi)) = \mathrm{H}^0(X, L^k\otimes \mathcal{O}_X(-kF)), \quad \mathcal{I}(k\varphi) = \mathcal{O}(-kF).$$

As $L-F$ is ample, for large enough k, we have

$$\operatorname{ord}_E \mathrm{H}^0(X, L^k\otimes \mathcal{O}_X(-kF)) = \operatorname{ord}_E(kF).$$

Thus, (10.26) is clear. □

Corollary 10.2.3 *Let $\varphi,\psi\in \mathrm{PSH}(X,\theta)_{>0}$. If*

$$\Delta_{W_\bullet}(\pi^*\theta,\pi^*\varphi)\subseteq \Delta_{W_\bullet}(\pi^*\theta,\pi^*\psi)$$

for all birational models $\pi: Y\longrightarrow X$ and all admissible flags $W_\bullet$ on Y, then $\varphi\preceq_{\mathcal{I}}\psi$.

Proof This follows immediately from Theorem 10.2.5. □

Corollary 10.2.4 *Let E be a prime divisor over X. Then*

$$\nu(V_\theta, E) = \lim_{k\to\infty}\frac{1}{k}\operatorname{ord}_E \mathrm{H}^0(X, L^k). \tag{10.27}$$

Proof This follows from Theorem 10.2.5 and the fact that $\Delta_{Y_\bullet}(\theta, V_\theta) = \Delta_{Y_\bullet}(L)$ for any admissible flag $Y_\bullet$ on X. □

10.3 Transcendental partial Okounkov bodies

Let X be a connected compact Kähler manifold of dimension n. Fix a smooth flag $Y_\bullet$ on X.

10.3.1 The traditional approach to the Okounkov body problem

Definition 10.3.1 Let α be a big cohomology class on X. We define the *Okounkov body* of α as

$$\Delta_{Y_\bullet}(\alpha) \coloneqq \overline{\left\{\nu_{Y_\bullet}(S) : S\in \mathcal{Z}_+(X,\alpha), S \text{ has gentle analytic singularities}\right\}}. \tag{10.28}$$

See Definition 1.6.5 for the definition of gentle analytic singularities.

The results of [DRWN+23] can be summarized as follows:

Theorem 10.3.1 *For any big cohomology class α on X, the set $\Delta_{Y_\bullet}(\alpha) \subseteq \mathbb{R}^n$ is a convex body satisfying the following properties:*

(1) *we have*
$$\operatorname{vol} \Delta_{Y_\bullet}(\alpha) = \frac{1}{n!} \operatorname{vol} \alpha;$$
(2) *Given another big cohomology class α' on X, we have*
$$\Delta_{Y_\bullet}(\alpha) + \Delta_{Y_\bullet}(\alpha') \subseteq \Delta_{Y_\bullet}(\alpha + \alpha');$$
(3) *Let $\pi \colon Y \longrightarrow X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that $(W_\bullet, g)$ is the lifting of $Y_\bullet$ to Y, then*
$$\Delta_{W_\bullet}(\pi^*\alpha) = \Delta_{Y_\bullet}(\alpha) g;$$
(4) *The map $\alpha \mapsto \Delta_{Y_\bullet}(\alpha)$ is continuous in the big cone with respect to the Hausdorff metric;*
(5) *For any small enough $t > 0$, we have*
$$\left\{ y \in \mathbb{R}^{n-1} : (t, y) \in \Delta_{Y_\bullet}(\beta) \right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}((\beta - t[Y_1])|_{Y_1}).$$

10.3.2 Definitions of partial Okounkov bodies

Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class α.

Let $T = \theta_\varphi \in \mathcal{Z}_+(X, \alpha)$. We shall define a convex body $\Delta_{Y_\bullet}(T) \subseteq \mathbb{R}^n$, which is also written as $\Delta_{Y_\bullet}(\theta, \varphi)$. This convex body is called the *partial Okounkov body* of T with respect to the flag $Y_\bullet$.

10.3.2.1 The case of analytic singularities

Definition 10.3.2 When T is a Kähler current with analytic singularities, we take a modification $\pi \colon Y \longrightarrow X$ so that

(1)
$$\pi^* T = [D] + R, \tag{10.29}$$
where D is an effective $\mathbb{Q}$-divisor on Y and R is a closed positive $(1,1)$-current with bounded potential, and

(2) the lifting $(Z_\bullet, g)$ of $Y_\bullet$ to Y exists.

Define
$$\Delta_{Y_\bullet}(T) \coloneqq \Delta_{Z_\bullet}([R]) g^{-1} + \nu_{Z_\bullet}([D]) g^{-1}.$$

The existence of π is guaranteed by Theorem 1.6.1 and Theorem 10.1.1.

Lemma 10.3.1 *The convex body $\Delta_{Y_\bullet}(T)$ defined in Definition 10.3.2 is independent of the choice of π.*

Proof Take another map $\pi' \colon Y' \to X$ with the same properties. We want to show that π and π' defines the same $\Delta_{Y_\bullet}(T)$. We may assume that π' dominates π through $p \colon Y' \to Y$, so that we have a commutative diagram

$$\begin{tikzcd} Y' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

We take D and R as in (10.29). Then

$$\pi'^* T = [p^* D] + p^* R.$$

Write $(Z_\bullet, g)$ and $(Z'_\bullet, g')$ for the liftings of $Y_\bullet$ to Y and Y' respective. We need to prove that

$$\Delta_{Z_\bullet}([R])g^{-1} + \nu_{Z_\bullet}([D])g^{-1} = \Delta_{Z'_\bullet}([p^* R])g'^{-1} + \nu_{Z'_\bullet}([p^* D])g'^{-1}.$$

This follows Theorem 10.3.1, Proposition 10.1.4 and Proposition 10.1.3. □

Note that from the above proof, we could describe the bimeromorphic behaviour of $\Delta_{Y_\bullet}(T)$ as follows:

Lemma 10.3.2 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current with analytic singularities. Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism and $(W_\bullet, g)$ be the lifting of $Y_\bullet$ to Y. Then*

$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T)g.$$

Lemma 10.3.3 *Assume that $T, S \in \mathcal{Z}_+(X, \alpha)$ are two Kähler currents with analytic singularities and $T \preceq S$, then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Moreover,

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \int_X T^n. \tag{10.30}$$

Proof We first show that

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S).$$

Using Lemma 10.3.2, we may assume that T and S have log singularities along effective $\mathbb{Q}$-divisors E and F respectively. By assumption, $E \geq F$. Replacing T and S by $T - [F]$ and $S - [F]$ respectively, we may assume that $F = 0$.

In this case, we need to show that

$$\Delta_{Y_\bullet}(\alpha) \supseteq \Delta_{Y_\bullet}(\alpha - [E]) + \nu_{Y_\bullet}([E]),$$

which is obvious.

Next we prove that

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

By Lemma 10.3.2 and Theorem 10.3.1 again, we may assume that T has log singularities. We take D and β as in (10.29). We need to show that

$$\Delta_{Y_\bullet}(\alpha - [D]) + \nu_{Y_\bullet}([D]) \subseteq \Delta_{Y_\bullet}(\alpha),$$

which is again obvious.

Finally, (10.30) follows immediately from Theorem 10.3.1. □

10.3.2.2 The case of Kähler currents

Definition 10.3.3 Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. Then we define

$$\Delta_{Y_\bullet}(T) \coloneqq \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T_j).$$

Lemma 10.3.4 *The convex body $\Delta_{Y_\bullet}(T)$ in Definition 10.3.3 is independent of the choices of the T_j's.*

In particular, if T also has analytic singularities, then the $\Delta_{Y_\bullet}(T)$'s defined in Definition 10.3.3 and in Definition 10.3.2 coincide.

Proof Let $(S_j)_j$ be another quasi-equisingular approximation of T in $\mathcal{Z}_+(X, \alpha)$. By Proposition 1.6.3, for any small rational $\epsilon > 0$, $j > 0$, we can find $k > 0$ so that

$$S_k \preceq (1 - \epsilon) T_j.$$

It is more convenient to use the language of θ-psh functions at this point. Let ψ_k (resp. φ_k) denote the potentials in $\mathrm{PSH}(X, \theta)$ corresponding to S_k (resp. T_k) for each $k \geq 1$. Note that ψ_k and φ_k are unique up to additive constants.

By Lemma 10.3.3,

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \Delta_{Y_\bullet}(\theta, (1 - \epsilon)\varphi_j).$$

On the other hand, observe that

$$\bigcap_{\epsilon \in \mathbb{Q}_{>0} \text{ small enough}} \Delta_{Y_\bullet}(\theta, (1 - \epsilon)\varphi_j) = \Delta_{Y_\bullet}(\theta, \varphi_j).$$

In fact, the $\supseteq$ direction follows from Lemma 10.3.3, so it suffices to show that the two sides have the same volume, which follows from (10.30).

It follows that

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\theta, \varphi_j).$$

The other inclusion follows by symmetry. □

The same argument shows that

Corollary 10.3.1 *Suppose that $T, S \in \mathcal{Z}_+(X, \alpha)$ are two Kähler currents satisfying $T \preceq_{\mathcal{I}} S$. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proposition 10.3.1 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Then*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T. \tag{10.31}$$

Proof Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. Note that $\Delta_{Y_\bullet}(T_j)$ is decreasing in j, as follows from Lemma 10.3.3. Our assertion follows from (10.30) and Theorem 6.2.5. □

Lemma 10.3.5 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current and ω be a Kähler form on X. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T + \omega). \tag{10.32}$$

Moreover,

$$\Delta_{Y_\bullet}(T) = \bigcap_{\epsilon > 0} \Delta_{Y_\bullet}(T + \epsilon\omega). \tag{10.33}$$

Proof We first prove (10.32). Taking quasi-equisingular approximations, we reduce immediately to the case where T has analytic singularities. By Lemma 10.3.2, we may assume that T has log singularities. Take D and R as in (10.29). By definition again, it suffices to show that

$$\Delta_{Y_\bullet}([\beta]) \subseteq \Delta_{Y_\bullet}([\beta + \omega]),$$

which is clear by definition.

Next we prove (10.33). Thanks to (10.32), it remains to prove that both sides have the same volume:

$$\lim_{\epsilon \to 0+} \operatorname{vol}(T + \epsilon\omega) = \operatorname{vol} T.$$

This is proved in Proposition 7.2.3. □

10.3.2.3 The general case

Definition 10.3.4 Let $T \in \mathcal{Z}_+(X, \alpha)$. Take a Kähler form ω on X, we define

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega). \tag{10.34}$$

The same definition makes sense when α is only pseudo-effective.

This definition is clearly independent of the choice of ω by Lemma 10.3.5. Moreover, it extends Definition 10.3.3 and Definition 10.3.2 as a result of Lemma 10.3.5.

Remark 10.3.1 When α is pseudoeffective but not big and T has minimal singularities, Definition 10.3.4 differs from all known definitions of $\Delta_{Y_\bullet}(\alpha)$ in the literature. But in view of Lemma 10.3.7, our definition seems to be the most natural one.

The main properties of $\Delta_{Y_\bullet}(T)$ are summarized as follows:

Theorem 10.3.2 *The convex bodies $\Delta_{Y_\bullet}(T)$'s satisfies the following properties:*

(1) *Suppose that $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, We have*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T. \tag{10.35}$$

(2) *For $T, S \in \mathcal{Z}_+(X,\alpha)$ satisfying $T \preceq_{\mathcal{I}} S$, we have*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

(3) *For any current $T \in \mathcal{Z}_+(X,\alpha)$ with minimal singularities, we have*

$$\Delta_{Y_\bullet}(T) = \Delta_{Y_\bullet}(\alpha).$$

(4) *The map $\mathcal{Z}_+(X,\alpha)_{>0} \to \mathcal{K}_n$ given by $T \mapsto \Delta_{Y_\bullet}(T)$ is continuous, where we endow the d_S-pseudometric on $\mathcal{Z}_+(X,\alpha)_{>0}$ and the Hausdorff topology on $\mathcal{K}_n$.*

(5) *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that the lifting $(W_\bullet, g)$ of $Y_\bullet$ to Y exists, then for any $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, we have*

$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T) g.$$

(6) *For $T, S \in \mathcal{Z}_+(X,\alpha)$, we have*

$$\Delta_{Y_\bullet}(T) + \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(T+S). \tag{10.36}$$

Proof (1) By (10.34) and (10.31), for any Kähler form ω on X,

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \lim_{j\to\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega) = \frac{1}{n!} \lim_{j\to\infty} \operatorname{vol}(T + j^{-1}\omega).$$

The right-hand side is computed in Proposition 7.2.3. Hence, (10.35) follows.

(2) Fix a Kähler form ω on X. By Corollary 10.3.1, for each $j \geq 1$,

$$\Delta_{Y_\bullet}(T + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(S + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

It remains to show that

$$\Delta_{Y_\bullet}(\alpha) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

The $\subseteq$ direction is clear. Comparing the volumes using Theorem 10.3.1, we conclude that equality holds.

(3) This follows from (1) and (2).

(4) Let $(T_j)_j$ be a sequence in $\mathcal{Z}_+(X,\alpha)_{>0}$ converging to $T \in \mathcal{Z}_+(X,\alpha)_{>0}$ with respect to d_S. We want to show that $\Delta_{Y_\bullet}(T_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(T)$. By Proposition 6.2.3 and (2), we may assume that the singularity type of T_j is either increasing or decreasing. In both cases, the continuity follows from (1).

(5) We may assume that T is $\mathcal{I}$-good. It follows from (4) and Theorem 7.1.1 that we could reduce to the case where T has analytic singularities. Our assertion follows from Lemma 10.3.2.

(6) By (10.34), in order to prove (10.36), we may assume that T and S are both Kähler currents. Take quasi-equisingular approximations $(T_j)_j$ and $(S_j)_j$ of T and S respectively. By Theorem 6.2.2, $T_j + S_j \xrightarrow{d_S} T + S$. By (4), we may therefore assume that T and S have analytic singularities. Replacing X by a suitable modification, we may assume that T and S both have log singularities, say

$$T = [D] + R, \quad S = [D'] + R',$$

where D and D' are $\mathbb{Q}$-divisors on X and β and β' are closed positive $(1,1)$-currents with bounded potentials. We need to show that

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) + \nu_{Y_\bullet}([D]) + \nu_{Y_\bullet}([D']) \subseteq \Delta_{Y_\bullet}([R+R']) + \nu_{Y_\bullet}([D+D']).$$

By Proposition 10.1.2, this is equivalent to

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) \subseteq \Delta_{Y_\bullet}([R+R']),$$

which is already proved in Theorem 10.3.1. □

Corollary 10.3.2 *Assume that L is a big line bundle on X and h is a plurisubharmonic metric on L with positive volume. Then*

$$\Delta_{Y_\bullet}(\mathrm{dd^c} h) = \Delta_{Y_\bullet}(L, h). \tag{10.37}$$

Similarly, the definition (10.20) is compatible with the definition in Definition 10.3.4.

Proof We may assume that $\mathrm{dd^c} h$ has positive mass and is $\mathcal{I}$-good. By the d_S-continuity of both sides of (10.37) as proved in Theorem 10.3.2 and Theorem 10.2.2, together with Theorem 7.1.1, we may assume that $\mathrm{dd^c} h$ has analytic singularities.

In this case, using the birational invariance of both sides of (10.37) as proved in Proposition 10.2.9 and Theorem 10.3.2, we may assume that $\mathrm{dd^c} h$ has log singularities. Finally, after all these reductions, the equality (10.37) holds by construction. □

10.3.3 The valuative characterization

In this section, we will characterize the partial Okounkov bodies using valuations of currents.

Lemma 10.3.6 *Let β be a nef class on X. Then*

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1}). \tag{10.38}$$

Proof **Step 1**. We first reduce to the case where β is a Kähler class.

Take a Kähler class α on X. It follows from the volume formula in Theorem 10.3.1 that

$$\Delta_{Y_\bullet}(\beta) = \bigcap_{\epsilon>0} \Delta_{Y_\bullet}(\beta + \epsilon\alpha), \quad \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1}) = \bigcap_{\epsilon>0} \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1} + \epsilon\alpha|_{Y_1}).$$

So it suffices to prove (10.38) with $\beta + \epsilon\alpha$ in place of β.

Step 2. Assume that α is a Kähler class. The $\supseteq$ direction in (10.38) follows from the extension theorem Theorem 1.6.3. To prove the other direction, recall that by Theorem 10.3.1, for $t > 0$ small enough, we have

$$\left\{y \in \mathbb{R}^{n-1} : (t, y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left((\beta - t[Y_1])|_{Y_1}\right).$$

As $t \to 0+$, the right-hand side converges to $\Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\beta|_{Y_1})$ with respect to the Hausdorff metric as a consequence of Theorem 10.3.1, while the left-hand side converges to

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(\beta)\right\}$$

by Lemma C.1.2. We conclude our assertion. □

Lemma 10.3.7 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Assume that $\nu(T, Y_1) = 0$, then*

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(T)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right). \tag{10.39}$$

More generally, if $T \in \mathcal{Z}_+(X, \alpha)$ and $\nu(T, Y_1) = 0$, suppose in addition that $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$ is defined, then (10.39) still holds.

See Remark 8.1.1 for the definition of $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$. Note that $\Delta_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right)$ is independent of the choice of the representative $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$.

Remark 10.3.2 More generally, the same argument shows the following result: Let $k = 0, \ldots, n$ and $T \in \mathcal{Z}_+(X, \alpha)$ such that $\nu(T, Y_k) = 0$. Assume that $\mathrm{Tr}_{Y_k}^{\alpha|_{Y_k}}(T)$ is defined, then

$$\left\{y \in \mathbb{R}^{n-k} : (0, \ldots, 0, y) \in \Delta_{Y_\bullet}(T)\right\} = \Delta_{Y_k \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_k}^{\alpha|_{Y_k}}(T)\right). \tag{10.40}$$

Also note that this result extends [Jow10, Theorem 3.4] and hence gives simpler proofs of [Jow10, Theorem A, Theorem B].

Proof Let ω be a Kähler form on X. The last assertion follows from the first by perturbing θ to $\theta + \epsilon\omega$.

Step 1. We first handle the case where T has analytic singularities. Let $\pi\colon Z \to X$ be a modification such that

(1) $Y_\bullet$ admits a lifting $(W_\bullet, g)$, and
(2) $\pi^*T = [D] + R$, where D is an effective $\mathbb{Q}$-divisor on Z and R is closed positive $(1,1)$-current with bounded potential.

This is possible by Theorem 1.6.1 and Theorem 10.1.1.

By Lemma 8.2.1,

$$\Pi^* \operatorname{Tr}_{Y_1}(T) \sim_P \operatorname{Tr}_{W_1}(\pi^*T),$$

where $\Pi\colon W_1 \to Y_1$ is the restriction of π. It follows from Theorem 10.3.2 that

$$\begin{aligned}\Delta_{W_1\supseteq\cdots\supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^*T)) =&\Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\operatorname{Tr}_{Y_1}(T))\operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \Pi),\\ \Delta_{W_\bullet}(\pi^*T) =&\Delta_{Y_\bullet}(T)g.\end{aligned}$$

Taking (10.3) into account, we find that it suffices to show that

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{W_\bullet}(\pi^*T)\right\} = \Delta_{W_1\supseteq\cdots\supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^*T)).$$

We may assume that π is the identity map. Then we have

$$T = [D] + R, \quad T|_{Y_1} = [D]|_{Y_1} + R|_{Y_1}.$$

Note that $[D]|_{Y_1}$ is the current of integration along an effective $\mathbb{Q}$-divisor on Y_1.

In particular,

$$\begin{aligned}\Delta_{Y_\bullet}(T) =&\Delta_{Y_\bullet}([R]) + \nu_{Y_\bullet}([D]),\\ \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(T|_{Y_1}) =&\Delta_{Y_1\supseteq\cdots\supseteq Y_n}([R]|_{Y_1}) + \nu_{Y_1\supseteq\cdots\supseteq Y_n}([D]|_{Y_1}).\end{aligned}$$

So it suffices to show that

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}([R])\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}([R]|_{Y_1}),$$

which is exactly Lemma 10.3.6.

Step 2. Next we consider the case where T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. From Step 1, we know that for large $j \geq 1$,

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(T_j)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\operatorname{Tr}_{Y_1}(T_j)).$$

Letting $j \to \infty$ and applying Theorem 10.3.2 and Proposition 8.2.2, we conclude (10.39). □

Theorem 10.3.3 *Assume that $T \in \mathcal{Z}_+(X, \alpha)_{>0}$ is a Kähler current. We have*

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.41}$$

Here the minimum is with respect to the lexicographic order.

Proof We make induction on $n \geq 0$. The case $n = 0$ is of course trivial. Let us assume that $n > 0$ and the case $n-1$ has been proved.

We first observe that by Theorem 10.3.2,

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \dots, 0) \subseteq \Delta_{Y_\bullet}(T).$$

Comparing the volumes of both sides using Theorem 10.3.2 and Proposition 7.2.3, we find that equality holds:

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \dots, 0) = \Delta_{Y_\bullet}(T).$$

Replacing T by $T - \nu(T, Y_1)[Y_1]$, we may therefore assume that $\nu(T, Y_1) = 0$. It suffices to apply Lemma 10.3.7 and the inductive hypothesis. □

Corollary 10.3.3 *For any $T \in \mathcal{Z}_+(X, \alpha)$,*

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proof When T is a Kähler current, this follows from Theorem 10.3.3.

In general, by definition, $\nu_{Y_\bullet}(T) = \nu_{Y_\bullet}(T + \omega)$ for any Kähler form ω on X. It follows that

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T + \omega)$$

for any Kähler form ω. It follows that $\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T)$. □

Theorem 10.3.4 *For any $T \in \mathcal{Z}_+(X, \alpha)_{>0}$,*

$$\Delta_{Y_\bullet}(T) = \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \preceq_{\mathcal{I}} T\right\}}. \tag{10.42}$$

In particular,

$$\Delta_{Y_\bullet}(\alpha) = \overline{\left\{\nu_{Y_\bullet}(T) : T \in \mathcal{Z}_+(X, \alpha)\right\}}.$$

Remark 10.3.3 We expect that the closure operation in (10.42) is not necessary. This problem is closely related to the Dirichlet problem of the trace operator, see Page 308 for more details.

Proof The $\supseteq$ direction in (10.42) follows from Corollary 10.3.3 and Theorem 10.3.2(2).

Let us write

$$D_{Y_\bullet}(T) = \left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \preceq_{\mathcal{I}} T\right\}$$

for the time being.

Step 1. Assume that T has analytic singularities. We have

$$\begin{aligned}\Delta_{Y_\bullet}(T) \supseteq & \overline{D_{Y_\bullet}(T)} \\ \supseteq & \overline{\left\{\nu_{Y_\bullet}(S) : \mathcal{Z}_+(X, \alpha) \ni S \text{ has gentle analytic singularities}, S \preceq T\right\}}.\end{aligned}$$

It follows easily from Theorem 10.3.1 that the volume of the right-hand side is equal to the volume of $\Delta_{Y_\bullet}(T)$, so (10.42) holds.

Step 2. Assume that T is a Kähler current. Take a quasi-equisingular approximation $T_j \in \mathcal{Z}_+(X,\alpha)$ of T. Next we use the language of psh functions. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ be the potentials corresponding to T_j, T for each $j \geq 1$.

Fix an integer $N > 0$. For large enough $j \geq 1$, we can find $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ such that

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1-N^{-1})\varphi_j + N^{-1}\psi_j.$$

The existence of ψ_j follows from Lemma 2.3.1. It follows that

$$\begin{aligned} D_{Y_\bullet}(T) \supseteq & D_{Y_\bullet}\left(\theta + \mathrm{dd^c}\left((1-N^{-1})\varphi_j + N^{-1}\psi_j\right)\right) \\ \supseteq & (1-N^{-1})D_{Y_\bullet}(T_j) + N^{-1}D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j). \end{aligned}$$

By Theorem C.1.1, up to replacing T_j by a subsequence, we may guarantee that $\overline{D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j)}$ admits a Hausdorff limit contained in $\Delta_{Y_\bullet}(\alpha)$ as $j \to \infty$. Let $j \to \infty$ and $N \to \infty$ then it follows that

$$\overline{D_{Y_\bullet}(T)} \supseteq \bigcap_{j=1}^{\infty} D_{Y_\bullet}(T_j).$$

By Lemma C.1.3,

$$\overline{D_{Y_\bullet}(T)} \supseteq \overline{\bigcap_{j=1}^{\infty} D_{Y_\bullet}(T_j)} = \bigcap_{j=1}^{\infty} \overline{D_{Y_\bullet}(T_j)}.$$

Therefore, by Step 1, we conclude that

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^{\infty} \overline{\Delta_{Y_\bullet}(T_j)} = \bigcap_{j=1}^{\infty} \overline{D_{Y_\bullet}(T_j)} \subseteq \overline{D_{Y_\bullet}(T)}.$$

The reverse direction is already known.

Step 3. Finally, consider the general case. Take a Kähler current $T' \in \mathcal{Z}_+(X,\alpha)$ more singular than T. For each $\epsilon \in (0,1)$. The existence of T' is proved in Lemma 2.3.2. We know that

$$\Delta_{Y_\bullet}((1-\epsilon)T + \epsilon T') = \overline{D_{Y_\bullet}((1-\epsilon)T + \epsilon T')} \subseteq \overline{D_{Y_\bullet}(T)}.$$

Letting $\epsilon \to 0+$ and using Proposition 7.2.3, we find that

$$\Delta_{Y_\bullet}(T) \subseteq \overline{D_{Y_\bullet}(T)}.$$

As the other inclusion is already known, we conclude. □

Corollary 10.3.4 *Assume that $T \in \mathcal{Z}_+(X,\alpha)_{>0}$. We have*

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.43}$$

Proof By Theorem 10.3.4, it is clear that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \leq_{\text{lex}} \nu_{Y_\bullet}(T).$$

On the other hand, we clearly have

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T+\omega)$$

for any Kähler form ω on X. It follows that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \geq_{\text{lex}} \min_{\text{lex}} \Delta_{Y_\bullet}(T+\omega).$$

By Theorem 10.3.3, the right-hand side is just $\nu_{Y_\bullet}(T+\omega) = \nu_{Y_\bullet}(T)$. We conclude the proof. □

10.4 Okounkov test curves

Fix $n \in \mathbb{N}$. Let $\Delta, \Delta' \subseteq \mathbb{R}^n$ be convex bodies with positive volumes. The standard Lebesgue measure on $\mathbb{R}^n$ is denoted by vol.

We refer to Appendix C for the notations $\mathcal{K}_n$ and d_{Haus}.

Definition 10.4.1 An *Okounkov test curve* relative to Δ consists of

(1) a number $\Delta_{\max} \in \mathbb{R}$ and
(2) an assignment $(-\infty, \Delta_{\max}) \ni \tau \mapsto \Delta_\tau \in \mathcal{K}_n$ satisfying

  a. the assignment $\tau \mapsto \Delta_\tau$ is a decreasing and concave[2];
  b. we have $\Delta_\tau \xrightarrow{d_{\text{Haus}}} \Delta$ as $\tau \to -\infty$.

The set of Okounkov test curves relative to Δ is denoted by $\mathrm{TC}(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ relative to Δ is *bounded* if $\Delta_\tau = \Delta$ when τ is small enough. The subset of bounded Okounkov test curves is denoted by $\mathrm{TC}^\infty(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ relative to Δ is said to have *finite energy* if

$$\mathbf{E}(\Delta_\bullet) := n!\Delta_{\max} \operatorname{vol} \Delta + n! \int_{-\infty}^{\Delta_{\max}} (\operatorname{vol} \Delta_\tau - \operatorname{vol} \Delta) \, \mathrm{d}\tau > -\infty. \tag{10.44}$$

The subset of Okounkov test curves with finite energy is denoted by $\mathrm{TC}^1(\Delta)$.

Given $\Delta_\bullet \in \mathrm{TC}(\Delta)$ and $\Delta'_\bullet \in \mathrm{TC}(\Delta')$, we say $\Delta_\bullet \leq \Delta'_\bullet$ if $\Delta_{\max} \leq \Delta'_{\max}$ and for any $\tau < \Delta_{\max}$, we have $\Delta_\tau \subseteq \Delta'_\tau$.

Sometimes it is convenient to introduce

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[2] Here concavity refers to the concavity with respect to the Minkowski sum.

$$\Delta_{\Delta_{\max}} = \bigcap_{\tau < \Delta_{\max}} \Delta_\tau \in \mathcal{K}_n. \tag{10.45}$$

We shall always make this extension in the sequel when we talk about $\Delta_{\Delta_{\max}}$. Observe that $(-\infty, \Delta_{\max}] \ni \tau \mapsto \Delta_\tau$ is still concave.

Proposition 10.4.1 *Any Okounkov test curve $(\Delta_\tau)_{\tau<\Delta_{\max}}$ relative to Δ is continuous in τ. Moreover,* $\operatorname{vol}\Delta_\tau > 0$ *for all* $\tau < \Delta_{\max}$.

Proof We first claim that $\operatorname{vol}\Delta_{\tau'} > 0$ for all $\tau' < \Delta_{\max}$. By Condition (2b) in Definition 10.4.1 and Theorem C.1.2, we know that $\operatorname{vol}\Delta_{\tau''} > 0$ when τ'' is small enough. Fix one such τ''. We may assume that $\tau'' \leq \tau'$ since otherwise there is nothing to prove. Next take $\tau''' \in (\tau', \Delta_{\max})$. Take $t \in (0,1)$ such that $\tau' = t\tau''' + (1-t)\tau''$. It follows that

$$\operatorname{vol}\Delta_{\tau'} \geq \operatorname{vol}\left(t\Delta_{\tau'''} + (1-t)\Delta_{\tau''}\right) \geq (1-t)^n \operatorname{vol}\Delta_{\tau''} > 0.$$

Next we claim that $\operatorname{vol}\Delta_\tau$ is continuous for $\tau < \Delta_{\max}$. In fact, it follows from Theorem C.1.4 that $(-\infty, \Delta_{\max}) \ni \tau \mapsto \log\operatorname{vol}\Delta_\tau$ is concave, but we have already known that it is finite, hence the continuity follows.

Next we show that

$$\Delta_\tau = \bigcap_{\tau'<\tau} \Delta_{\tau'}.$$

The $\subseteq$ direction is obvious. By the continuity of the volume, both sides have the same volume and the volume is positive, we therefore obtain the equality.

Similarly, we have

$$\Delta_\tau = \overline{\bigcup_{\tau'>\tau} \Delta_{\tau'}}.$$

The continuity of Δ_τ at $\tau < \Delta_{\max}$ is proved. □

Definition 10.4.2 A *test function* on Δ is a function $G\colon \Delta \to [-\infty, \infty)$ such that

(1) G is concave,
(2) G is finite on $\operatorname{Int}\Delta$, and
(3) G is upper semicontinuous.

A test function G is *bounded* if G is bounded from below.

A test function G has *finite energy* if

$$\mathbf{E}(G) := n! \int_\Delta G \, \mathrm{d}\lambda > -\infty. \tag{10.46}$$

Definition 10.4.3 Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We define its *Legendre transform* as

$$G[\Delta_\bullet]\colon \Delta \to [-\infty, \infty), \quad a \mapsto \sup\left\{\tau < \Delta_{\max} : a \in \Delta_\tau\right\}.$$

Given a test function $G\colon \Delta \to [-\infty, \infty)$, we define its *inverse Legendre transform* $\Delta[G]_\bullet$ as the Okounkov test curve relative to Δ defined as follows:

(1) $\Delta[G]_{\max} = \sup_\Delta G$, and
(2) for each $\tau < \sup_\Delta G$, we set

$$\Delta[G]_\tau = \{x \in \Delta : G \geq \tau\}.$$

We observe that

$$G[\Delta_\bullet](a) = \max\{\tau \leq \Delta_{\max} : a \in \Delta_\tau\}, \text{ if } G[\Delta_\bullet](a) > -\infty. \tag{10.47}$$

Lemma 10.4.1 *Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. Then $G[\Delta_\bullet]$ defined in Definition 10.4.3 is a test function.*

Similar, if $G\colon \Delta \to [-\infty, \infty)$ is a test function, then $\Delta[G]_\bullet$ is an Okounkov test curve.

Proof First suppose that $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We want to verify that $G[\Delta_\bullet]$ satisfies the conditions in Definition 10.4.2.

We first verify the concavity. Take $a, b \in \Delta$. We want to prove that for any $t \in (0, 1)$,

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b). \tag{10.48}$$

There is nothing to prove if $G[\Delta_\bullet](a)$ or $G[\Delta_\bullet](b)$ is $-\infty$. So we assume that both are finite. In this case, by (10.47),

$$a \in \Delta_{G[\Delta_\bullet](a)}, \quad b \in \Delta_{G[\Delta_\bullet](b)}.$$

Thus,

$$ta + (1-t)b \in t\Delta_{G[\Delta_\bullet](a)} + (1-t)\Delta_{G[\Delta_\bullet](b)} \subseteq \Delta_{tG[\Delta_\bullet](a)+(1-t)G[\Delta_\bullet](b)}.$$

We deduce that

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b).$$

Therefore, (10.48) follows.

It is clear that $G[\Delta_\bullet]$ is finite on the interior of Δ. It remains to argue that $G[\Delta_\bullet]$ is upper semicontinuous.

Let $(a_i)_{i\geq 1}$ be a sequence in Δ with limit $a \in \Delta$. Define $\tau_i = G[\Delta_\bullet](a_i)$. Let $\tau = \overline{\lim}_i \tau_i$. We need to show that

$$G[\Delta_\bullet](a) \geq \tau. \tag{10.49}$$

There is nothing to prove if $\tau = -\infty$. We assume that it is not this case. Up to subtracting a subsequence we may assume that $\tau_i \to \tau$. In particular, we can assume that $\tau_i \neq -\infty$ for all $i \geq 1$. It follows from (10.47) that $a_i \in \Delta_{\tau_i}$ for all $i \geq 1$. Since $\Delta_{\tau_i} \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$. By Theorem C.1.3 it follows that $a \in \Delta_\tau$. Thus,(10.49) follows.

Conversely, suppose that $G\colon \Delta \to [-\infty, \infty)$ is a test function. We argue that $\Delta[G]_\bullet$ is an Okounkov test curve. We verify the conditions in Definition 10.4.1.

Firstly, for each $\tau < \sup_\Delta G$, the set $\Delta[G](\tau)$ is a convex body as G is concave and usc. Moreover, $\Delta[G]_\tau$ is clearly decreasing in τ.

Secondly, for each $a \in \Delta$, we can write $a = \lim_i a_i$ with $a_i \in \operatorname{Int}\Delta$. By assumption, G is finite at a_i. Thus,

$$a \in \overline{\{G > -\infty\}} = \overline{\bigcup_{\tau < \sup_\Delta G} \Delta[G]_\tau}.$$

By Theorem C.1.3, $\Delta[G]_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta$ as $\tau \to -\infty$.

Thirdly, $\Delta[G]$ is concave. To see, take $\tau, \tau' < \Delta_{\max}$, we need to prove that for any $t \in (0,1)$,

$$\Delta[G]_{t\tau+(1-t)\tau'} \supseteq t\Delta[G]_\tau + (1-t)\Delta[G]_{\tau'}. \tag{10.50}$$

Let $a \in \Delta[G]_\tau$ and $b \in \Delta[G]_{\tau'}$. We have $G(a) \geq \tau$ and $G(b) \geq \tau'$. As G is concave, we have $G(ta + (1-t)b) \geq t\tau + (1-t)\tau'$. Thus,

$$ta + (1-t)b \in \Delta[G]_{t\tau+(1-t)\tau'}$$

and (10.50) follows. □

Theorem 10.4.1 *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between* $\mathrm{TC}(\Delta)$ *and the set of test functions on* Δ.

Under this bijection, $\mathrm{TC}^1(\Delta)$ *corresponds to test functions on* Δ *with finite energy and* $\mathrm{TC}^\infty(\Delta)$ *corresponds to bounded test functions on* Δ.

Proof Thanks to Lemma 10.4.1, in order to prove the first assertion, it only remains to see that the Legendre transform and the inverse Legendre transform are inverse to each other, which is immediate by definition.

It is obvious that $\mathrm{TC}^\infty(\Delta)$ corresponds to bounded test curves. Moreover, a direct computation shows that if $\Delta_\bullet \in \mathrm{TC}(\Delta)$, then

$$\mathbf{E}(\Delta_\bullet) = \mathbf{E}(G[\Delta_\bullet]),$$

concluding the $\mathrm{TC}^1(\Delta)$ case. □

Proposition 10.4.2 *Let* $(\Delta^i)_{i\in I}$ *be a decreasing net in* $\mathcal{K}_n$*. Consider a decreasing net* $(\Delta^i_\bullet)_{i\in I}$ *with* $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ *for all* $i \in I$ *such that there is* $\Delta_\bullet \in \mathrm{TC}(\Delta)$ *satisfying the following properties:*

(1) $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$*;*

(2) *for any* $\tau < \Delta_{\max}$*, we have* $\Delta^i_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$*.*

Then for any $a \in \Delta$*, we have*

$$\lim_{i\in I} G[\Delta^i_\bullet](a) = G[\Delta_\bullet](a). \tag{10.51}$$

Note that in general,

$$\Delta \subsetneq \bigcap_{i\in I} \Delta^i .$$

Proof Fix $a \in \Delta$. It follows immediately from the definition of G that the net $(G[\Delta^i_\bullet](a))_{i\in I}$ is decreasing and the $\geq$ direction in (10.51) holds. Let us prove the reverse inequality. Let τ denote the left-hand side of (10.51) for the moment. By definition, for any $\epsilon > 0$ and any $i \in I$, we have $a \in \Delta^i_{\tau-\epsilon}$. It follows that

$$a \in \Delta^\infty_{\tau-\epsilon}.$$

Therefore,

$$\tau \leq G[\Delta_\bullet](a).$$

Similarly, for increasing nets, we have:

Proposition 10.4.3 *Let $(\Delta^i)_{i\in I}$ be an increasing net in $\mathcal{K}_n$ with Hausdorff limit Δ such that* $\operatorname{vol}\Delta^i > 0$ *for all $i \in I$. Consider an increasing net $(\Delta^i_\bullet)_{i\in I}$ with $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ for all $i \in I$. Let $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$. For any $\tau < \Delta_{\max}$, let Δ_τ be the Hausdorff limit of Δ^i_τ. Then $\Delta_\bullet \in \mathrm{TC}(\Delta)$ and*

$$\lim_{i\in I} G[\Delta^i_\bullet](a) = G[\Delta_\bullet](a) \tag{10.52}$$

for any $a \in \operatorname{Int}\Delta$.

Proof It is obvious that $\Delta_\bullet \in \mathrm{TC}(\Delta)$.

Fix $a \in \operatorname{Int}\Delta$. Then up to replacing I by a subnet, we may assume that $a \in \Delta^i$ for all $i \in I$. By definition, the net $(G[\Delta^i_\bullet](a))_{i\in I}$ is increasing and the $\leq$ direction in (10.52) holds. Let us write $\tau = G[\Delta_\bullet](a)$ for the time being. By definition of G, for any $\epsilon > 0$, we have

$$a \in \Delta_{\tau-\epsilon/2}.$$

The concavity of $\Delta_\bullet$ guarantees that

$$a \in \operatorname{Int}\Delta_{\tau-\epsilon}.$$

It follows that there is a subnet J in I such that for all $j \in J$,

$$a \in \Delta^j_{\tau-\epsilon}.$$

Therefore,

$$\tau - \epsilon \leq G[\Delta^j_\bullet](a).$$

Taking the limit with respect to j and then with respect to ϵ, we conclude the desired inequality. □

Definition 10.4.4 Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) \coloneqq G[\Delta_\bullet]_*(\mathrm{vol}).$$

It is a Radon measure on $\mathbb{R}$.

In other words, $\mathrm{DH}(\Delta_\bullet)$ is the distribution of the random variable $G[\Delta_\bullet]$.

Proposition 10.4.4 *Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. Let $m \in \mathbb{Z}_{>0}$. Then the m-th moment of the $\mathrm{DH}(\Delta_\bullet)$ is given by*

$$\int_{\mathbb{R}} x^m \,\mathrm{DH}(\Delta_\bullet)(x) = \Delta_{\max}^m \operatorname{vol}\Delta + m\int_{-\infty}^{\Delta_{\max}} \tau^{m-1}(\operatorname{vol}\Delta_\tau - \operatorname{vol}\Delta)\,\mathrm{d}\tau \tag{10.53}$$

and

$$\int_{\mathbb{R}} \mathrm{DH}(\Delta_\bullet) = \operatorname{vol}\Delta. \tag{10.54}$$

Proof In fact, (10.54) follows immediately from the definition, while (10.53) follows form a straightforward computation:

$$\begin{aligned}
&\int_{\mathbb{R}} x^m \,\mathrm{DH}(\Delta_\bullet)(x)\\
=&\int_{\Delta} G[\Delta_\bullet](a)^m \,\mathrm{d}\operatorname{vol}(a)\\
=&\int_{\Delta}\left(\Delta_{\max}^m - \int_{G[\Delta_\bullet](a)}^{\Delta_{\max}} m\tau^{m-1}\,\mathrm{d}\tau\right)\mathrm{d}\operatorname{vol}(a)\\
=&\Delta_{\max}^m\operatorname{vol}\Delta - m\int_{\mathbb{R}}\int_{\Delta}\mathbb{1}_{[G(\Delta_\bullet](a),\Delta_{\max}]}(\tau)\tau^{m-1}\,\mathrm{d}\operatorname{vol}(a)\,\mathrm{d}\tau\\
=&\Delta_{\max}^m\operatorname{vol}\Delta - m\int_{-\infty}^{\Delta_{\max}}\int_{\Delta\setminus\Delta_\tau}\tau^{m-1}\,\mathrm{d}\operatorname{vol}(a)\,\mathrm{d}\tau\\
=&\Delta_{\max}^m\operatorname{vol}\Delta - m\int_{-\infty}^{\Delta_{\max}}\tau^{m-1}\left(\operatorname{vol}\Delta - \operatorname{vol}\Delta_\tau\right)\,\mathrm{d}\tau.
\end{aligned}$$

Lemma 10.4.2 *Let $(\Delta^i)_{i\in I}$ be a decreasing net in $\mathcal{K}_n$ with limit Δ. Suppose that $(\Delta^i_\bullet)_{i\in I}$ is a decreasing net with $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$. Suppose that there is $\Delta_\bullet \in \mathrm{TC}(\Delta)$ such that*

(1) $\Delta_{\max} = \lim_{i\in I}\Delta^i_{\max}$;

(2) *for any $\tau < \Delta_{\max}$, we have $\Delta^i_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$.*

Then $\mathrm{DH}(\Delta^i_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$.

Proof It follows from Proposition 10.4.2 that

$$G[\Delta^i_\bullet] \to G[\Delta_\bullet]$$

pointwisely on Δ. Our assertion then follows from the dominated convergence theorem. ◻

Similarly, we have

Lemma 10.4.3 *Let $(\Delta^i)_{i\in I}$ be an increasing net in $\mathcal{K}_n$ with Hausdorff limit Δ such that $\operatorname{vol}\Delta^i > 0$ for all $i \in I$. Consider an increasing net $(\Delta^i_\bullet)_{i\in I}$ with $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ for all $i \in I$. Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$ be defined as*

(1) $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$;
(2) *for any $\tau < \Delta_{\max}$, Δ_τ is the Hausdorff limit of Δ^i_τ.*

Then we have

$$\mathrm{DH}(\Delta^i_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet).$$

Proof It follows from Proposition 10.4.3 that

$$G[\Delta^i_\bullet] \to G[\Delta_\bullet]$$

almost everywhere on Δ. Our assertion then follows from the dominated convergence theorem. □

The main source of Okounkov test curves is the following:

Theorem 10.4.2 *Let X be a connected compact Kähler manifold and θ be a closed smooth real $(1,1)$-form on X representing a big cohomology class α. Let $Y_\bullet$ be a smooth flag on X and $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. Then the map*

$$(-\infty, \Gamma_{\max}) \ni \tau \mapsto \Delta_{Y_\bullet}(\theta,\Gamma)_\tau := \Delta_{Y_\bullet}(\theta,\Gamma_\tau)$$

defines an Okounkov test curve relative to $\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty})$.

If furthermore $\Gamma \in \mathrm{TC}^1(X,\theta;\Gamma_{-\infty})$ (resp. $\mathrm{TC}^\infty(X,\theta;\Gamma_{-\infty})$), then we have $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$ (resp. $\mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$).

See Definition 9.1.1 and Definition 9.1.2 for the relevant definitions.

Proof Consider $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. We need to verify that $\Delta_{Y_\bullet}(\theta,\Gamma)$ is an Okounkov test curve relative to $\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty})$.

First observe that $\tau \mapsto \Delta_{Y_\bullet}(\theta,\Gamma_\tau)$ is concave and decreasing for $\tau < \Gamma_{\max}$. This is a direct consequence of Theorem 10.3.4.

Next we show that as $\tau \to -\infty$, we have

$$\Delta_{Y_\bullet}(\theta,\Gamma_\tau) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}).$$

It suffices to compute

$$\begin{aligned}\lim_{\tau\to-\infty} \operatorname{vol}\Delta_{Y_\bullet}(\theta,\Gamma_\tau) = \frac{1}{n!}\lim_{\tau\to-\infty}\operatorname{vol}(\theta+\mathrm{dd^c}\Gamma_\tau) &= \frac{1}{n!}\operatorname{vol}(\theta+\mathrm{dd^c}\Gamma_{-\infty})\\ &= \operatorname{vol}\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}),\end{aligned}$$

where we applied Theorem 10.3.2 and Theorem 6.2.5.

When $\Gamma \in \mathrm{TC}^\infty(X,\theta;\Gamma_{-\infty})$, it is clear that $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$.

When $\Gamma \in \mathrm{TC}^1(X,\theta;\Gamma_{-\infty})$, by Theorem 10.3.2(1), (9.3) and (10.44), we have

$$\mathbf{E}^{\Gamma_{-\infty}}(\Gamma) = \mathbf{E}(\Delta_{Y_\bullet}(\theta,\Gamma)).$$

So $\Gamma \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\theta, \Gamma_{-\infty}))$. □

Remark 10.4.1 As a special case of this construction, suppose that Γ is the test curve induced by a test configuration as in Example 9.3.1 and Remark 9.3.1, then for any $\tau < \Gamma_{\max}$, $\Delta_{Y_\bullet}(\theta, \Gamma_\tau)$ is the Okounkov body of a graded linear series

$$\bigoplus_{k=0}^{\infty} \mathcal{F}_k^{k\tau},$$

where $\mathcal{F}$ is the filtration induced by the test configuration. See [Xia21, Theorem 5.28] for the details. In particular, in this case, our theory of partial Okounkov bodies recovers the Okounkov bodies of the filtered linear series in the sense of [BC11].

Chapter 11
The theory of b-divisors

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: There is no permanent place in this world for ugly mathematics.
— Godfrey Harold Hardy

In this chapter, we study the theory of b-divisors. In Section 11.2, we prove a Chern–Weil type formula, which relates volumes of currents to intersection numbers.

In Section 11.3, we prove that the algebraic partial Okounkov bodies constructed in Chapter 10 have natural interpretations in terms of the b-divisors.

11.1 The intersection theory of b-divisors

In this section, we briefly recall the intersection theory of Dang–Favre [DF22].

Let X be a connected smooth projective variety of dimension n.

Definition 11.1.1 A *birational model* of X is a projective birational morphism $\pi\colon Y \to X$ from a *smooth* variety Y. A morphism between two birational models $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$ is a morphism $Y \to Y'$ over X.

We write $\mathrm{Bir}(X)$ for the isomorphism classes of birational models of X. It is a directed set under the partial ordering of domination.

We will usually be sloppy by omitting π and say Y is a birational model of X.

We write $\mathrm{NS}^1(X)$ for the Néron–Severi group of X and $\mathrm{NS}^1(X)_K$ for $\mathrm{NS}^1(X)\otimes_{\mathbb{Z}} K$ for any subfield K of $\mathbb{R}$. Given $\alpha, \beta \in \mathrm{NS}^1(X)_K$, we write $\alpha \leq \beta$ if $\beta - \alpha$ is pseudo-effective.

Definition 11.1.2 A *Weil b-divisor* $\mathbb{D}$ on X is an assignment that associates with each $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ a class $\mathbb{D}_Y = \mathbb{D}_\pi \in \mathrm{NS}^1(Y)_{\mathbb{R}}$ such that when $\pi' : Y' \to X$ dominates π through $p : Y' \to Y$, we have

$$p_*\mathbb{D}_{Y'} = \mathbb{D}_Y.$$

The set of Weil b-divisors on X is denoted by $\mathrm{bWeil}(X)$.

A Weil b-divisor $\mathbb{D}$ on X is *Cartier* if there is $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ such that for any $(\pi' : Y' \to X) \in \mathrm{Bir}(X)$ which dominates π through $p : Y' \to Y$, we have

$$\mathbb{D}_{Y'} = p^*\mathbb{D}_Y.$$

In this case we say $\mathbb{D}$ is *determined* on Y or $\mathbb{D}$ has an *incarnation* $\mathbb{D}_Y$ on Y and write $\mathbb{D} = \mathbb{D}(\mathbb{D}_Y)$. We also say $\mathbb{D}$ is a Cartier b-divisor. The linear space of Cartier b-divisors is denoted by $\mathrm{bCart}(X)$.

Our definition simply means

$$\begin{aligned} \mathrm{bWeil}(X) &= \varprojlim_{(\pi\colon Y\to X)\in \mathrm{Bir}(X)} \mathrm{NS}^1(Y)_{\mathbb{R}}, \\ \mathrm{bCart}(X) &= \varinjlim_{(\pi\colon Y\to X)\in \mathrm{Bir}(X)} \mathrm{NS}^1(Y)_{\mathbb{R}}, \end{aligned} \tag{11.1}$$

in the category of vector spaces.

We endow $\mathrm{bWeil}(X)$ with the projective limit topology, then the first equation in (11.1) becomes a projective limit in the category of locally convex linear spaces. Clearly, $\mathrm{bCart}(X)$ is dense in $\mathrm{bWeil}(X)$.

Definition 11.1.3 A Cartier b-divisor $\mathbb{D}$ on X is *nef* (resp. *big*) if some incarnation is (equivalently all incarnations are) nef (resp. big).

A Weil b-divisor $\mathbb{D}$ on X is *nef* if it lies in the closure of the set of nef Cartier b-divisors.

Write $\mathrm{bWeil}_{\mathrm{nef}}(X)$ for the set of nef Weil b-divisors on X.

A Weil b-divisor $\mathbb{D}$ on X is *pseudo-effective* if for all $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$, $\mathbb{D}_Y \geq 0$.

We introduce a partial ordering on $\mathrm{bWeil}(X)$:

$$\mathbb{D} \leq \mathbb{D}' \text{ if and only if } \mathbb{D}_Y \leq \mathbb{D}'_Y \text{ for all } (\pi\colon Y \to X) \in \mathrm{Bir}(X).$$

We summarise Dang–Favre's results:

Theorem 11.1.1 ([DF22, Theorem 2.1]) *Let $\mathbb{D} \in \mathrm{bWeil}(X)$ be a nef Weil b-divisor. Then there is a* decreasing *net $(\mathbb{D}_i)_{i\in I}$ of nef Cartier b-divisors such that*

$$\mathbb{D} = \lim_{i\in I} \mathbb{D}_i.$$

Definition 11.1.4 Let $\mathbb{D}_i \in \mathrm{bWeil}(X)$ $(i = 1, \ldots, n)$ be nef Cartier b-divisors on X. We define $(\mathbb{D}_1, \ldots, \mathbb{D}_n) \in \mathbb{R}$ as follows: take $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$ such that all $\mathbb{D}_i's$ are determined on Y. Then define

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) \coloneqq (\mathbb{D}_{1,Y}, \ldots, \mathbb{D}_{n,Y}). \tag{11.2}$$

The intersection number $(\mathbb{D}_1, \ldots, \mathbb{D}_n)$ does not depend on the choice of Y.

Theorem 11.1.2 ([DF22, Proposition 3.1,Theorem 3.2]) *There is a unique pairing*

$$(\mathrm{bWeil}_{\mathrm{nef}}(X))^n \to \mathbb{R}_{\geq 0}$$

extending the pairing in Definition 11.1.4 such that

(1) *The pairing is monotonically increasing in each variable.*
(2) *The pairing is continuous along decreasing nets in each variable.*

Moreover, this pairing has the following properties:

(1) *It is symmetric, multilinear.*
(2) *It is usc in each variable.*

Definition 11.1.5 We define the *volume* of $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$ by

$$\operatorname{vol} \mathbb{D} = (\mathbb{D}, \dots, \mathbb{D}). \tag{11.3}$$

We say $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$ is *big* if $\operatorname{vol} \mathbb{D} > 0$.

Note that the definition of bigness is compatible with the definition in Definition 11.1.3 in the case of Cartier b-divisors.

Lemma 11.1.1 *Let* $\mathbb{D} \in \mathrm{bWeil}_{\mathrm{nef}}(X)$*, then*

$$\operatorname{vol} \mathbb{D} = \inf_{(Y \to X) \in \mathrm{Bir}(X)} \operatorname{vol} \mathbb{D}_Y = \lim_{(Y \to X) \in \mathrm{Bir}(X)} \operatorname{vol} \mathbb{D}_Y .$$

Proof By Theorem 11.1.1, we can find a decreasing net $\mathbb{D}^\alpha$ of nef Cartier b-divisors on X converging to $\mathbb{D}$. Clearly,

$$\operatorname{vol} \mathbb{D}^\alpha = \inf_{Y \to X} \operatorname{vol} \mathbb{D}^\alpha_Y .$$

It follows from Theorem 11.1.2 and the continuity of the volume functional [ELM+05, Corollary 2.6] that

$$\operatorname{vol} \mathbb{D} = \inf_{\alpha} \inf_{Y \to X} \operatorname{vol} \mathbb{D}^\alpha_Y = \inf_{Y \to X} \operatorname{vol} \mathbb{D}_Y .$$

On the other hand, as in general push-forward will increase the volume, we see that $\operatorname{vol} \mathbb{D}_Y$ is decreasing in Y, so we conclude. □

11.2 The singularity b-divisors

Let X be a connected smooth projective variety of dimension n. Let $\alpha \in \mathrm{NS}^1(X)_{\mathbb{R}}$ be a big class and T be a closed positive $(1,1)$-current in α.

Fix a closed real smooth $(1,1)$-form θ in $[T]$ and we can write $T = \theta_\varphi$ for some $\varphi \in \mathrm{PSH}(X, \theta)$.

Definition 11.2.1 Define the *singularity divisor* $\mathrm{Sing}_X T$ of T as the formal sum

$$\mathrm{Sing}_X T := \sum_E \nu(T, E) E, \tag{11.4}$$

where E runs over all prime divisors contained in X.

The singularity divisor is *not* a Weil divisor in general.

Note that this is a countable sum by Siu's semicontinuity theorem Theorem 1.4.1. Although $\mathrm{Sing}_X T$ is not a divisor in general, it does define a closed positive $(1,1)$-current due to Siu's decomposition Lemma 1.7.1. Moreover, the numerical class $[\mathrm{Sing}_X T]$ in $\mathrm{NS}^1(X)_{\mathbb{R}}$ is also well-defined by treating the sum in (11.4) as a sum of numerical classes [BFJ09, Proposition 1.3].

Definition 11.2.2 The *singularity b-divisor* $\mathrm{Sing}\, T$ of T is the b-divisor over X defined by

$$(\mathrm{Sing}\, T)_Y := [\mathrm{Sing}_Y \pi^* T],$$

where $(\pi : Y \to X) \in \mathrm{Bir}(X)$.

Define

$$\mathbb{D}(T) := \mathbb{D}(\alpha) - \mathrm{Sing}\, T.$$

Here $\mathbb{D}(\alpha)$ is the Cartier b-divisor determined by α on X.

We are ready to derive the first version of the Chern–Weil formula.

Theorem 11.2.1 *The b-divisor $\mathbb{D}(T)$ is a nef b-divisor and if in addition $\mathrm{vol}\, T > 0$,*

$$\mathrm{vol}\, \mathbb{D}(T) = \mathrm{vol}\, T. \tag{11.5}$$

Proof **Step 1**. We first handle the case where T has analytic singularities. After replacing X by a modification, we may assume that T has log singularities along an effective $\mathbb{Q}$-divisor D on X. Namely, we can write

$$T = [D] + R,$$

where R is a closed positive $(1,1)$-current with bounded potential. In this case, $\mathbb{D}(T) = \mathbb{D}(\alpha - D)$, which is nef. In order to prove (11.5), it suffices to show that

$$\int_X T^n = ((\alpha - D)^n), \tag{11.6}$$

which is obvious.

Step 2. Assume that T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X,\theta)$. By Theorem 6.2.5, we have

$$\lim_{j\to\infty} \mathrm{vol}\, T_j = \mathrm{vol}\, T.$$

In view of Step 1 and Theorem 11.1.2, it remains to show that $\mathbb{D}(T_j) \to \mathbb{D}(T)$ as $j \to \infty$. In more concrete terms, this means that for any $(\pi : Y \to X) \in \mathrm{Bir}(X)$,

$$[\mathrm{Sing}_Y(\pi^* T_j)] \to [\mathrm{Sing}_Y(\pi^* T)]$$

in $\mathrm{NS}^1(Y)_{\mathbb{R}}$. This obviously follows from Theorem 6.2.4 if $\mathrm{Sing}(\pi^* T)$ has only finitely many components. In general, fix an ample class ω in $\mathrm{NS}^1(Y)$. We want to show that for any $\epsilon > 0$, we can find $j_0 > 0$ so that when $j \geq j_0$,

$$[\mathrm{Sing}_Y(\pi^* T_j)] \geq [\mathrm{Sing}_Y(\pi^* T)] - \epsilon\omega. \tag{11.7}$$

Write

$$[\mathrm{Sing}_Y(\pi^* T)] = \sum_{i=1}^{\infty} a_i E_i, \quad [\mathrm{Sing}(\pi^* T_j)] = \sum_{i=1}^{\infty} a_i^j E_i.$$

Then $a_i^j \leq a_i$. We can find $N > 0$ large enough, so that

$$[\mathrm{Sing}_Y(\pi^* T)] \leq \sum_{i=1}^{N} a_i E_i + \frac{\epsilon}{2}\omega.$$

By Theorem 6.2.4, we can take j_0 large enough so that for $j > j_0$,

$$(a_i - a_i^j)E_i \leq \frac{\epsilon}{2N}\omega, \quad i = 1, \dots, N.$$

Then (11.7) follows.

Step 3. Assume that $\operatorname{vol} T > 0$.

By Lemma 2.3.2, we can take a Kähler current $S \in \alpha$ such that $S \preceq T$. Consider $\epsilon S + (1-\epsilon)T$ for $\epsilon \in (0,1)$. When $\epsilon \to 0+$, we have $\epsilon S + (1-\epsilon)T \xrightarrow{d_S} T$. Using Theorem 6.2.5, we reduce immediately to the situation of Step 2.

Step 4. We handle the general case.

Take a Kähler form ω on X From Step 3, we know that for any $\epsilon > 0$, $\mathbb{D}(T) + \epsilon\mathbb{D}(\omega)$ is a nef b-divisor. It follows immediately that $\mathbb{D}(T)$ is nef. □

Corollary 11.2.1 *Assume that* $\operatorname{vol} T > 0$, *then* T *is* $\mathcal{I}$*-good if and only if*

$$\operatorname{vol} \mathbb{D}(T) = \int_X T^n.$$

Proof This follows from Theorem 11.2.1 and Theorem 7.3.1. □

Theorem 11.2.2 *The map* $\mathbb{D}\colon \mathrm{PSH}(X,\theta) \to \mathrm{bWeil}(X)$ *is continuous. Here on* $\mathrm{PSH}(X,\theta)$ *we take the* d_S*-pseudometric.*

Proof Let $\varphi_i \in \mathrm{PSH}(X,\theta)$ be a sequence converging to $\varphi \in \mathrm{PSH}(X,\theta)$ with respect to d_S. We want to show that

$$\mathbb{D}(\theta + \mathrm{dd^c}\varphi_i) \to \mathbb{D}(T).$$

As $\varphi_i \xrightarrow{d_S} \varphi$ implies that $\pi^*\varphi_i \xrightarrow{d_S} \pi^*\varphi$ for any $(\pi\colon Y \to X) \in \mathrm{Bir}(X)$, it suffices to prove

$$[\mathrm{Sing}_X \varphi_i] \to [\mathrm{Sing}_X \varphi] \quad \text{in } \mathrm{NS}^1(X)_{\mathbb{R}}. \tag{11.8}$$

Write

$$\mathrm{Sing}_X \varphi_i = \sum_E a_i^E E, \quad \mathrm{Sing}_X \varphi = \sum_E a^E E,$$

where E runs over all prime divisors on X. By Theorem 6.2.4, $a_i^E \to a^E$ as $i \to \infty$. When the number of E's is finite, (11.8) follows trivially. Otherwise, we write the prime divisors on X having positive coefficients in either $\mathrm{Sing}_X \varphi_i$ or $\mathrm{Sing}_X \varphi$ as $E_1, E_2, \ldots$.

We fix a basis $e_1, \ldots, e_N$ of the finite-dimensional vector space $\mathrm{NS}^1(X)_{\mathbb{R}}$, so that the pseudo-effective cone is contained in the cone $\sum_d \mathbb{R}_{\geq 0} e_d$. Write

$$E_i = \sum_{d=1}^{N} f_i^d e_d, \quad i = 1, 2, \ldots.$$

Then we need to show that for any $d = 1, \ldots, N$,

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a_i^{E_j} f_j^d = \sum_{j=1}^{\infty} a^{E_j} f_j^d.$$

This follows from the dominated convergence theorem, since

$$\sum_{j=1}^{\infty} a_i^{E_j} [E_j] \leq \alpha, \quad \sum_{j=1}^{\infty} a^{E_j} [E_j] \leq \alpha.$$

A mixed version of Theorem 11.2.1 is also true:

Theorem 11.2.3 *Let $T_1, \ldots, T_n \in \mathcal{Z}_+(X)$ such that $\mathrm{vol}\, T_i > 0$ for each $i = 1, \ldots, n$. Then*

$$(\mathbb{D}(T_1), \ldots, \mathbb{D}(T_n)) \geq \int_X T_1 \wedge \cdots \wedge T_n. \tag{11.9}$$

If the T_i's are $\mathcal{I}$-good, then equality holds.

Proof This follows from Theorem 11.2.1 and Proposition 7.2.1. □

11.3 Okounkov bodies of b-divisors

Let X be a connected projective manifold of dimension n and (L, h) be a Hermitian pseudoeffective line bundle on X with $\mathrm{vol}\, \mathrm{dd^c} h > 0$.

Fix a smooth flag $Y_\bullet$ on X. Let $\nu = \nu_{Y_\bullet} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$ be the valuation associated with $Y_\bullet$.

Theorem 11.3.1 *The partial Okounkov body $\Delta_{Y_\bullet}(L, h)$ admits the following expression:*

$$\Delta_{Y_\bullet}(L, h) = \nu_{Y_\bullet}(\mathrm{dd^c} h) + \lim_{\pi : Z \to X} \Delta_{Y_\bullet}\left(c_1(\pi^* L) - [\mathrm{Sing}_Z(\pi^* h)]\right), \tag{11.10}$$

where π runs over the directed set of projective birational morphisms to X with Z normal.

Here the limit is a Hausdorff limit. Recall that $\nu_{Y_\bullet}(\mathrm{dd^c}h)$ is defined in Definition 10.1.3.

This theorem suggests that we define

$$\Delta_{Y_\bullet}(\mathbb{D}(\mathrm{dd^c}h)) := \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet}\left(c_1(\pi^*L) - [\mathrm{Sing}_Z(\pi^*h)]\right). \tag{11.11}$$

Then one could rewrite (11.10) as

$$\Delta_{Y_\bullet}(L,h) = \Delta_{Y_\bullet}(\mathbb{D}(\mathrm{dd^c}h)) + \nu_{Y_\bullet}(\mathrm{dd^c}h),$$

which formally resembles (10.14).

Remark 11.3.1 The formula (11.11) shows that the partial Okounkov bodies are *algebraic* objects in nature (modulo the transcendental term $\nu_{Y_\bullet}(\mathrm{dd^c}h)$ of course).

One should be able to prove the existence of the limits like (11.11) over other base fields, at least after assuming the existence of resolution of singularities. If so, one would get an interesting extension of the theory of partial Okounkov bodies.

Lemma 11.3.1 *Let T be a closed positive $(1,1)$-current on X. Then we have*

$$\lim_{\pi\colon Z\to X} \nu_{Y_\bullet}(\mathrm{Sing}_Z(\pi^*T)) = \nu_{Y_\bullet}(T), \tag{11.12}$$

where π runs over the directed set of projective birational morphisms to X with Z normal.

Here the valuation of currents is defined in Remark 10.1.2.

Proof Let us write $\nu = \nu_{Y_\bullet}$.

Given $\pi\colon Z \to X$, we let W_1 denote the strict transform of Y_1 in Z. The restriction $\pi_1\colon W_1 \to Y_1$ is necessarily birational due to Zariski's main theorem. Let $\widetilde{W_1}$ be the normalization of W_1. Let $\widetilde{\pi_1}$ denote the normalization of π_1 so that we have a commutative diagram

$$\begin{tikzcd} \widetilde{W_1} \arrow[r] \arrow[d, "\widetilde{\pi_1}"] & W_1 \arrow[r, hook] \arrow[d, "\pi_1"] & Z \arrow[d, "\pi"] \\ Y_1 \arrow[r, equal] & Y_1 \arrow[r, hook] & X. \end{tikzcd}$$

We will argue by induction. The case $n = 0$ is trivial. Assume that $n > 0$ and the case $n-1$ is known.

We may clearly assume that $\nu(T, Y_1) = 0$. By definition, we have

$$\nu(T) = \left(0, \mu(\mathrm{Tr}_{Y_1}(T))\right),$$

where μ denotes the valuation induced by the flag $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$.

Observe that birational morphisms of the form $\pi_1\colon \widetilde{W_1} \to Y_1$ are cofinal in the directed set of projective birational morphisms of Y_1. This is obvious since the modifications given by compositions of blow-ups with smooth centers on Y_1 are cofinal. It suffices to blow-up X with the same centers.[1]

Therefore, by the inductive hypothesis applied to $\mathrm{Tr}_{Y_1} T$, we find

$$\mu\left(\mathrm{Tr}_{Y_1}(T)\right) = \lim_{\pi\colon Z\to X} \mu\left(\mathrm{Sing}_{\widetilde{W_1}}(\widetilde{\pi_1}^* \mathrm{Tr}_{Y_1} T)\right).$$

It suffices to argue that for a fixed $\pi\colon Z \to X$,

$$\nu(\mathrm{Sing}_Z(\pi^*T)) = \left(0, \mu\left(\mathrm{Sing}_{\widetilde{W_1}} \widetilde{\pi_1}^*(\mathrm{Tr}_{Y_1}(T))\right)\right). \tag{11.13}$$

From Lemma 8.2.1, we know that

$$\widetilde{\pi_1}^* \mathrm{Tr}_{Y_1}(T) \sim_P \mathrm{Tr}_{W_1}(\pi^*T).$$

So we only need to prove

$$\nu(\mathrm{Sing}_Z(\pi^*T)) = \left(0, \mu(\mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{W_1}(\pi^*T))\right),$$

This is reduced to the following statement:

$$\mathrm{Tr}_{W_1} \mathrm{Sing}_Z(\pi^*T) \sim_P \mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{W_1}(\pi^*T)). \tag{11.14}$$

In order to prove this, we may add a Kähler form to T and assume that T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T. Then $(\pi^*T_j)_j$ is a quasi-equisingular approximation of π^*T. Thanks to Proposition 8.2.2, we have

$$\mathrm{Tr}_{W_1}(\pi^*T_j) \xrightarrow{d_S} \mathrm{Tr}_{W_1}(\pi^*T)$$

Therefore, as in the proof of Theorem 11.2.2, we find that Sing_Z and $\mathrm{Sing}_{\widetilde{W_1}}$ are both continuous along this sequence as well. So we finally reduce to the case where T has analytic singularities.

In this case, arguing as before, we may assume replace π by a modification dominating it so that $\pi^*T \sim_P [D]$ for an effective $\mathbb{Q}$-divisor D on Z, in which case (11.14) is clear. □

Proof (The proof of Theorem 11.3.1) It would be more convenient to use the language of currents. We shall write $T = \mathrm{dd^c} h$ and $\nu = \nu_{Y_\bullet}$.

Instead of arguing (11.10), we shall argue a slightly more general version: For any $\alpha \in \mathrm{NS}^1(X)_{\mathbb{R}}$, we have

$$\Delta_{Y_\bullet}(T) = \nu(T) + \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet}(\alpha - [\mathrm{Sing}_Z(\pi^*T)]). \tag{11.15}$$

We argue by induction on n. The case $n = 0$ is of course trivial. Let us assume that $n > 0$ and the result is known in dimension $n - 1$.

[1] It is in this inductive step that we are forced to introduce singularities, as W_1 is not smooth in general.

We may replace T by $T - \nu(T, Y_1)[Y_1]$ and α by $\alpha - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$.

For any projective birational morphism $\pi\colon Z \to X$ with Z normal, it follows from Theorem 10.3.4 (which also holds for a normal variety, as can be seen after passing to a resolution) that we have

$$\Delta_{Y_\bullet}\left(\pi^*\alpha - [\mathrm{Sing}_Z(\pi^*T)]\right) = \overline{\left\{\nu(S) : S \in \pi^*\alpha - [\mathrm{Sing}_Z(\pi^*T)]\right\}}.$$

Therefore,

$$\Delta_{Y_\bullet}\left(\pi^*\alpha - [\mathrm{Sing}_Z(\pi^*T)]\right) + \nu(\mathrm{Sing}_Z(\pi^*T)) \subseteq \overline{\left\{\nu(S) : S \in \alpha, \pi^*S \geq \mathrm{Sing}_Z(\pi^*T)\right\}}.$$

We observe that the right-hand side is decreasing with respect to π, which together with Lemma 11.3.1 implies that the net of convex bodies $\Delta_{Y_\bullet}(c_1(\pi^*L) - [\mathrm{Sing}_Z(\pi^*T)])$ for various Z is uniformly bounded. Suppose that Δ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in c_1(L), S \leq_{\mathcal{I}} T\}}.$$

As shown in Theorem 10.3.4, the right-hand side is exactly $\Delta_{Y_\bullet}(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_{Y_\bullet}(T).$$

But observe that both sides have the same volume, as computed in Theorem 10.3.2 and Theorem 11.2.1. So equality holds.

It follows from the Blaschke selection theorem Theorem C.1.1 that the limit in (11.15) exists and (11.15) holds. □

Part III
Applications

In this part, we explain a few applications of the theory developed in this book.

In Chapter 12, we develop the pluripotential theory on big line bundles on toric varieties. This theory depends crucially on the theory of partial Okounkov bodies developed in Chapter 10.

In Chapter 13, we develop the transcendental theory of non-Archimedean metrics based on the theory of test curves developed in Chapter 9.

In Chapter 14, we prove the convergence of partial Bergman measures.

Chapter 12
Toric pluripotential theory on big line bundles

C'est l'harmonie des diverses parties, leur symétrie, leur heureux balancement; c'est en un mot tout ce qui y met de l'ordre, tout ce qui leur donne de l'unité, ce qui nous permet par conséquent d'y voir clair et d'en comprendre l'ensemble en même temps que les détails.
— *Henri Poincaré,* L'avenir des mathématiques

In this chapter, we develop the toric pluripotential theory on big line bundles. Our development here is based on the theory of partial Okounkov bodies developed in Chapter 10. We will deduce two non-trivial consequences from the general theory: Corollary 12.2.2 and Theorem 12.2.2. The author does not know how to prove either result without relying on partial Okounkov bodies.

12.1 Toric setup

Let T be a complex torus of dimension n with character lattice M and cocharacter lattice N. Some basic terminologies are recalled in Section 5.1. Recall that T_c is the compact torus contained in $T(\mathbb{C})$.

Consider a rational polyhedral fan Σ in $N_{\mathbb{R}}$ corresponding to an n-dimensional smooth toric variety X.

Let
$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$$
be a T-invariant big divisor on X. Then $P_D \subseteq M_{\mathbb{R}}$ be the polytope[1] generated by $m \in M$ such that
$$D + \operatorname{div} \chi^m \geq 0.$$
In view of [CLS11, Proposition 4.1.2], we have
$$P_D = \left\{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \quad \forall \rho \in \Sigma(1) \right\}. \tag{12.1}$$
Since we have assumed that D is big, P_D is n-dimensional.

Let $L = \mathcal{O}_X(D)$. Note that replacing D by a linearly equivalent divisor amounts to replacing D by an integral translation.

[1] Note that P_D is not necessarily a lattice polytope, see [CLS11, Example 10.5.4].

Recall that for each $\rho \in \Sigma(1)$, u_ρ denotes the ray generator of ρ. Let $\{m_\sigma\}_{\sigma\in\Sigma}$ denote the *Cartier data* associated with D. In other words, for each $\sigma \in \Sigma$, $m_\sigma \in M$ satisfies that

$$\langle m_\sigma, u_\rho\rangle = -a_\rho, \quad \forall \rho \in \sigma(1).$$

The element $m_\sigma \in M$ is well-defined modulo

$$M(\sigma) \coloneqq \sigma^\perp \cap M, \tag{12.2}$$

where

$$\sigma^\perp \coloneqq \{m \in M_\mathbb{R} : \langle m, u\rangle = 0 \quad \forall u \in \sigma\}.$$

Moreover, if τ is a face of σ, then

$$m_\sigma \equiv m_\tau \mod M(\tau). \tag{12.3}$$

See [CLS11, Theorem 4.2.8]. In particular, for an n-dimensional $\sigma \in \Sigma$, the element m_σ is uniquely determined.

Note that for any n-dimensional face σ in Σ and any $\rho \in \sigma(1)$, we have

$$\langle m - m_\sigma, u_\rho\rangle \geq 0, \quad \forall m \in P, \tag{12.4}$$

as a consequence of (12.4) and (12.1).

Recall that

$$D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma} \tag{12.5}$$

for all $\sigma \in \Sigma$, where U_σ is the affine subvariety of X corresponding to σ. See [CLS11, Proposition 4.1.2].

Next consider a T-invariant irreducible subvariety $Y \subseteq X$. Since X is smooth, so is Y. Let σ be the cone in Σ corresponding to Y. We observe that σ corresponds to a face Q_σ of P_D:

$$Q_\sigma = \left\{m \in P_D : \langle m, u_\rho\rangle = -a_\rho \quad \forall \rho \in \sigma(1)\right\}. \tag{12.6}$$

The dimension of σ is not necessarily equal to the codimension of Q as we will see in Example 12.1.2.

We will keep two examples in mind.

Example 12.1.1 In this case, Σ is the fan in Fig. 12.1 consisting of three 2-dimensional cones σ_0, σ_1 and σ_2; three 1-dimensional cones σ_4, σ_5 and σ_6; one 0-dimensional cone σ_0.

The fan Σ is just the fan of $X = \mathbb{P}^2$. Under the orbit-cone correspondence, we have

$$\begin{aligned}
D_{\sigma_1} &= \{[1:0:0]\}, \quad D_{\sigma_2} = \{[0:1:0]\}, \quad D_{\sigma_3} = \{[0:0:1]\},\\
D_{\sigma_4} &= \{[0:X_1:X_2] : X_1X_2 \neq 0\}, \quad D_{\sigma_5} = \{[X_0:0:X_2] : X_0X_2 \neq 0\},\\
D_{\sigma_6} &= \{[X_0:X_1:0] : X_0X_1 \neq 0\}, \quad D_{\sigma_0} = \mathbb{P}^2.
\end{aligned}$$

In particular, $\Sigma(1) = \{\sigma_4, \sigma_5, \sigma_6\}$. We shall take

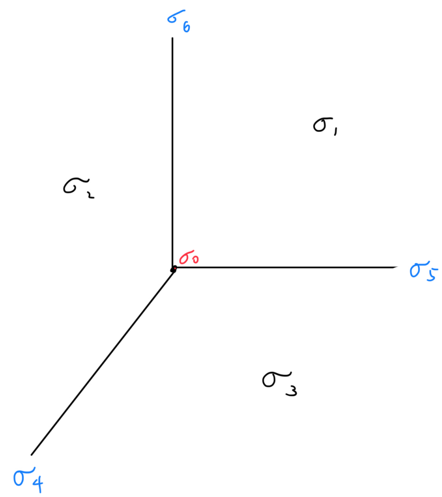

Fig. 12.1 The fan of $\mathbb{P}^2$

$$D = D_{\sigma_4}.$$

In other words,

$$a_{\sigma_5} = a_{\sigma_6} = 0, \quad a_{\sigma_4} = 1.$$

Note that the ray generators are given by

$$u_{\sigma_4} = (-1,-1), \quad u_{\sigma_5} = (1,0), \quad u_{\sigma_6} = (0,1).$$

It follows that

$$P_D = \{m = (m_1, m_2) \in \mathbb{R}^2 : m_1 + m_2 \leq 1, m_1 \geq 0, m_2 \geq 0\}.$$

Therefore, P_D is just the polytope in Fig. 12.2 . In this case, the Cartier data for 2-dimensional cones are given as follows:

$$m_{\sigma_1} = (0,0), \quad m_{\sigma_2} = (1,0), \quad m_{\sigma_3} = (0,1);$$

while the remaining Cartier data are determined by (12.3).

In this case, $L = \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^2}(1)$. Hence the line bundle L is ample.

We also observe that

$$\begin{aligned}
Q_{\sigma_1} =& \{(0,0)\}, \quad Q_{\sigma_2} = \{(1,0)\}, \quad Q_{\sigma_3} = \{(0,1)\}, \\
Q_{\sigma_4} =& \{(m_1, m_2) : m_1 \geq 0, m_2 \geq 0, m_1 + m_2 = 1\}, \\
Q_{\sigma_5} =& \{0\} \times [0,1], \quad Q_{\sigma_6} = [0,1] \times \{0\}, \\
Q_{\sigma_0} =& P_D.
\end{aligned}$$

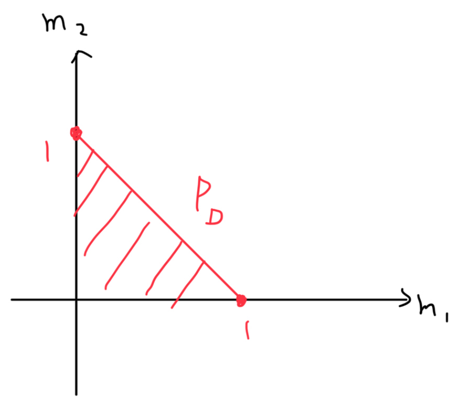

Fig. 12.2 The polytope P_D

Next we give a non-ample example.

Example 12.1.2 Let Σ be the fan shown in Fig. 12.3. Comparing with our previous

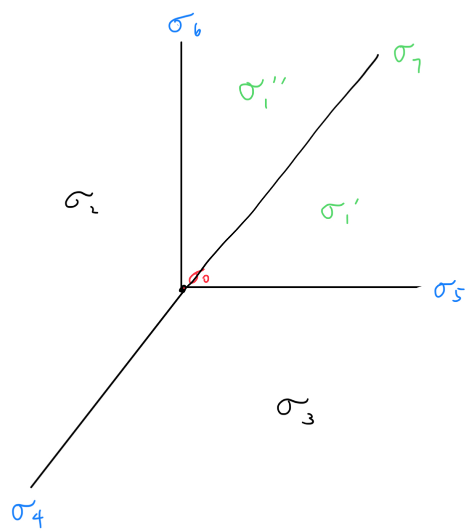

Fig. 12.3 The fan of $\mathbb{P}^2$ blown-up at the origin

example Fig. 12.1, we have divided σ_1 from the middle, giving rise to two additional 2-dimensional cones σ_1' and σ_1'', and one additional 1-dimensional cone σ_7.

The corresponding $X = \mathrm{Bl}_0\mathbb{P}^2$ is just the blow-up of $\mathbb{P}^2$ at the origin 0 and hence $L = \pi^* O_{\mathbb{P}^2}(1)$. Let $\pi \colon X \longrightarrow \mathbb{P}^2$ denote the blow-up morphism. Let

$$D = D_{\sigma_4}.$$

Then D is the pull-back of the divisor D in Example 12.1.1. Note that D is not ample, since it has degree 0 on the exceptional divisor.

In this case, we have

$$\Sigma(1) = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7\},$$

and D_{σ_7} is just the exceptional divisor.

The corresponding ray generators are

$$u_{\sigma_4} = (-1, -1), \quad u_{\sigma_5} = (1, 0), \quad u_{\sigma_6} = (0, 1), \quad u_{\sigma_7} = (1, 1),$$

while

$$m_{\sigma_1'} = m_{\sigma_1''} = (0, 0), \quad m_{\sigma_2} = (1, 0), \quad m_{\sigma_3} = (0, 1).$$

Therefore, P_D is the same as in Fig. 12.2.

We also observe that

$$\begin{aligned}
Q_{\sigma_1'} =& \{(0,0)\}, \quad Q_{\sigma_1''} = \{(0,0)\}, \quad Q_{\sigma_7} = \{(0,0)\} \\
Q_{\sigma_2} =& \{(1,0)\}, \quad Q_{\sigma_3} = \{(0,1)\}, \\
Q_{\sigma_4} =& \{(m_1, m_2) : m_1 \geq 0, m_2 \geq 0, m_1 + m_2 = 1\}, \\
Q_{\sigma_5} =& \{0\} \times [0, 1], \quad Q_{\sigma_6} = [0, 1] \times \{0\}, \\
Q_{\sigma_0} =& P_D.
\end{aligned}$$

12.2 Toric partial Okounkov bodies

We continue to use the notations in Section 12.1.

We shall fix a T_c-invariant Hermitian metric h on L. Let $\theta = c_1(L, h)$. Fix a smooth function $F_\theta : N_{\mathbb{R}} \to \mathbb{R}$ such that

$$\theta = \mathrm{dd^c}\, \mathrm{Trop}^* F_\theta.$$

Note that F_θ is well-defined up to a linear term.

Next, we make an additional requirement on F_θ to fix the linear term. Let s_D be a rational section of L corresponding to D. Then s_D is well-defined up to a non-zero multiple. By Proposition 1.8.1, we have

$$\mathrm{dd^c}\left(\mathrm{Trop}^* F_\theta + \log |s_D|_h^2\right) = 0$$

on $T(\mathbb{C})$. Therefore, this function is the tropicalization of a linear function. Therefore, after adding a linear function to F_θ, we can guarantee that

$$\mathrm{Trop}^* F_\theta + \log |s_D|_h^2 = 0 \tag{12.7}$$

from now on. Note that a different choice of s_D means adding a constant to F_θ.

12.2.1 Newton bodies

Let $\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ be the set of T_c-invariant functions in $\mathrm{PSH}(X,\theta)$.

Definition 12.2.1 A function $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ can be written as

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^* f$$

for some unique $f\colon N_{\mathbb{R}} \to [-\infty,\infty)$. Then we define $F_\varphi\colon N_{\mathbb{R}} \to \mathbb{R}$ as follows:

$$F_\varphi = F_\theta + f. \tag{12.8}$$

Observe that F_φ is a convex function and takes finite values by Lemma 5.2.1. In particular, f is also real-valued. Once h and D are fixed, F_φ is well-defined up to a constant since F_θ is.

Definition 12.2.2 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$, we define its *Newton body* as

$$\Delta(\theta,\varphi) \coloneqq \overline{\nabla F_\varphi(N_{\mathbb{R}})} \subseteq M_{\mathbb{R}}.$$

Note that $\Delta(\theta,\varphi)$ is independent of the choice of s_D. It depends on the choice of D: A different choice of D corresponds to a translation of $\Delta(\theta,\varphi)$. We will see in a while (Theorem 12.2.1) that once D is fixed $\Delta(\theta,\varphi)$ depends only on the current θ_φ. Hence, the choice of h is irrelevant.

Proposition 12.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$, then*

$$\mathrm{Trop}_* \left(\theta|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi). \tag{12.9}$$

In particular,

$$\int_X \theta_\varphi^n = n!\,\mathrm{vol}\,\Delta(\theta,\varphi) \tag{12.10}$$

Proof Let F_0 be a smooth convex function on $N_{\mathbb{R}}$ such that $\mathrm{dd^c}\,\mathrm{Trop}^* F_0$ can be extended to a Kähler form on X. For example, Guillemin's construction (5.5) with respect to a suitable Delzant polytope gives such an example.

Then for any large enough $C>0$, $\theta + C\omega$ is a Kähler form. So we conclude from Proposition 5.2.5 that

$$\mathrm{Trop}_* \left((\theta + C\omega)|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi + CF_0).$$

Since both sides are polynomials in C, we conclude that the same holds for $C = 0$. Therefore, (12.9) follows.

(12.10) is a direct consequence. □

12.2.2 Partial Okounkov bodies

There are some canonical choices of smooth flags in the toric setting.

Since X is smooth and projective, we could choose a full-dimensional cone σ in Σ with rays $\rho_1, \dots, \rho_n \in \Sigma(1)$ such that $u_{\rho_1}, \dots, u_{\rho_n}$ form a basis of N. Define

$$Y_i = D_{\rho_1} \cap \cdots \cap D_{\rho_i}, \quad i = 1, \dots, n.$$

Then $Y_\bullet$ is a smooth flag on X. Let

$$\Phi\colon M \to \mathbb{Z}^n, \quad m \mapsto \left(\langle m - m_\sigma, u_{\rho_1}\rangle, \dots, \langle m - m_\sigma, u_{\rho_n}\rangle\right). \tag{12.11}$$

Then Φ is an isomorphism of lattices. It induces an $\mathbb{Z}$-affine isomorphism

$$\Phi_\mathbb{R}\colon M_\mathbb{R} \to \mathbb{R}^n.$$

Proposition 12.2.2 *We have*

$$k^{-1}\nu_{Y_\bullet}\left(\mathrm{H}^0(X, L^k)^\times\right) = \Phi_\mathbb{R}\left(P_D \cap k^{-1}M\right) \tag{12.12}$$

for any $k \in \mathbb{Z}_{>0}$. In particular,

$$\Delta_{Y_\bullet}(L) = \Phi_\mathbb{R}(P_D). \tag{12.13}$$

Proof We first reduce to the case where $D|_{U_\sigma} = 0$. In fact, replacing D by $D + \operatorname{div}\chi^{m_\sigma}$ would result in changing P_D to $P_D - m_\sigma$. So in view of (12.5), we may assume that $D|_{U_\sigma} = 0$ and hence $m_\sigma = 0$.

Fix $k \in \mathbb{Z}_{>0}$. Let $s \in \mathrm{H}^0(X, L^k)$ be a non-zero section, say χ^m for some $m \in kP_D \cap M$. The zero-locus of s on U_σ is given by

$$kD + \sum_{i=1}^{n} \langle m, u_{\rho_i}\rangle D_{\rho_i},$$

see [CLS11, Proposition 4.1.2]. Therefore,

$$\nu_{Y_\bullet}(s) = \left(\langle m, u_{\rho_1}\rangle, \dots, \langle m, u_{\rho_n}\rangle\right) = \Phi(m).$$

So (12.12) follows. □

Example 12.2.1 Let us continue the example of $\mathbb{P}^2$ in Example 12.1.1. We use the same notations. Take σ_1 as our reference cone, and $\rho_1 = \sigma_5, \rho_2 = \sigma_6$. Then

$$Y_1 = \{[X_0 : 0 : X_2] : X_0X_2 \neq 0\}, \quad Y_2 = \{[X_0 : 0 : 0] : X_0 \neq 0\}.$$

The map Φ is given by

$$\Phi(m_1, m_2) = (m_1, m_2).$$

In this case, we see easily

$$\Delta_{Y_\bullet}(\mathcal{O}_{\mathbb{P}^2}(1)) = P_D$$

is the polytope in Fig. 12.2.

Example 12.2.2 Let us continue the example of $\mathrm{Bl}_0\mathbb{P}^2$ in Example 12.1.2. This time, let us take σ_1' as our reference cone and $\rho_1 = \sigma_5$, $\rho_2 = \sigma_7$. Then Y_1 is just the strict transform of the line $\{[X_0 : 0 : X_2] : X_0X_2 \neq 0\}$ in $\mathbb{P}^2$, while Y_2 is the point $Y_1 \cap E$, where E is the exceptional divisor.

In this case, the map Φ is given by

$$\Phi(m_1, m_2) = (m_1, m_1 + m_2).$$

We find that

$$\Delta_{Y_\bullet}(\mathrm{Bl}_0\mathbb{P}^2, \pi^*\mathcal{O}_{\mathbb{P}^2}(1))$$

is the polytope in Fig. 12.4.

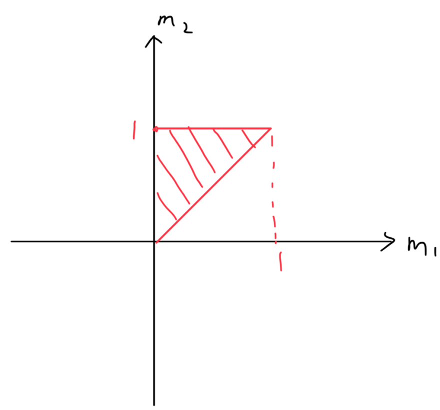

Fig. 12.4 The Okounkov body $\Delta_{Y_\bullet}(\mathrm{Bl}_0\mathbb{P}^2, \pi^*\mathcal{O}_{\mathbb{P}^2}(1))$

Note that it differs from the polytope in Example 12.2.1.[2]

[2] Although these examples are almost trivial, they did confuse me a lot at the beginning of 2023, when Kewei Zhang, Tamás Darvas and I were collaborating on [DRWN+23]. At that time, Kewei himself already proved the main theorem for a generic flag. I realized that some simple birational geometry would suffice to prove the same result for general flags. I persuaded myself and Kewei that the Okounkov bodies are always birationally invariant, and deduced some apparently wrong conclusions. I got no clue for a couple of weeks, then one day, on the noisy metro line 7 of Paris, I got nothing to do, so I said to myself: Why not compute the simplest toric examples? Then after a few minutes, all of a sudden, the whole picture became completely clear.

Theorem 12.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$, then*

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) = \Delta_{Y_\bullet}(\theta,\varphi). \tag{12.14}$$

In particular, once D is fixed, the Newton body $\Delta(\theta,\varphi)$ depends only on the current θ_φ, not on the specific choice of h and φ.

Proof We first reduce to the case where $D|_{U_\sigma} = 0$. In fact, changing D to $D+\operatorname{div}\chi^{m_\sigma}$ would result in changing F_θ to $F_\theta - m_\sigma$. Hence, F_φ changes to $F_\varphi - m_\sigma$. Therefore, $\Delta(\theta,\varphi)$ becomes $\Delta(\theta,\varphi) - m_\sigma$. Taking (12.5) into consideration, we may assume that $m_\sigma = 0$.

Step 1. We first reduce to the case where θ_φ is a Kähler current.

By Lemma 2.3.2, we can find $\psi \in \mathrm{PSH}(X,\theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. Taking the average along T_c, we may assume that ψ is T_c-invariant.

For each $t \in (0,1)$, we let

$$\varphi_t = (1-t)\psi + t\varphi.$$

Suppose that Kähler current case is known. Then we get

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_t)\right) = \Delta_{Y_\bullet}(\theta,\varphi_t)$$

for any $t \in (0,1)$. It follows from Theorem A.4.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \supseteq \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_t)\right) = \Delta_{Y_\bullet}(\theta,\varphi_t)$$

for any $t \in (0,1)$. Thanks to Theorem 10.2.2, we have

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \supseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1 and (10.11), we find that

$$n!\,\mathrm{vol}\,\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) = \int_X \theta_\varphi^n = \mathrm{vol}\,\theta_\varphi = n!\,\mathrm{vol}\,\Delta_{Y_\bullet}(\theta,\varphi).$$

In particular, we conclude (12.14).

Step 2. We handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

We may assume that φ_j is T_c-invariant for each $j \geq 1$ from the construction of [Dem12a, Theorem 13.21].

Now assume that the result is known for each φ_j. Then

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_j)\right) = \Delta_{Y_\bullet}(\theta,\varphi_j).$$

In particular, by Proposition 12.2.1 again,

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi_j)$$

for each $j \geq 1$. It follows from Theorem 10.2.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1, (10.11) and Theorem 5.2.2, we conclude (12.14).

Step 3. It remains to handle the case where φ has analytic singularities and θ_φ is a Kähler current. In fact, we may assume that φ has the form

$$\varphi = \log \sum_{i=1}^{a} |s_i|_h^2 + O(1),$$

where $s_1, \ldots, s_a \in \mathrm{H}^0(X, L)$ are toric invariant. This follows from the proof of Step 2 and the construction of [Dem12a, Theorem 13.21].

Let $m_1, \ldots, m_a \in P_D \cap M$ be the lattice points corresponding to $s_1, \ldots, s_a$. Observe that

$$\begin{aligned}
\Delta(\theta,\varphi) = \overline{\nabla F_\varphi(N_{\mathbb{R}})} &= \left\{ m \in M_{\mathbb{R}} : F_\varphi(n) - \langle m, n\rangle \text{ is bounded from below} \right\} \\
&= \left\{ m \in M_{\mathbb{R}} : \log \sum_{i=1}^{a} \mathrm{e}^{(m_i,n)} - \langle m, n\rangle \text{ is bounded from below} \right\} \\
&= \mathrm{Conv}\{m_1, \ldots, m_a\},
\end{aligned}$$

where we have applied (12.7) on the second line and Lemma A.5.2 on the third line. In particular, by Lemma A.5.1, let $k \in \mathbb{Z}_{>0}$, given any $m \in k\Delta(\theta,\varphi) \cap M$, we have

$$|\chi^m|^2 \mathrm{e}^{-k\varphi}$$

is bounded from above on $T(\mathbb{C})$. In other words, the section s of L defined by m satisfies

$$s \in \mathrm{H}^0\left(X, L^k \otimes \mathcal{I}_\infty(k\varphi)\right).$$

Therefore,

$$\nu_{Y_\bullet}(s) = \Phi(m) \in k\Delta_k(\theta,\varphi),$$

where Δ_k is defined Section 10.2. Hence,

$$\Phi\left(k\Delta(\theta,\varphi) \cap M\right) \subseteq k\Delta_k(\theta,\varphi).$$

Letting $k \to \infty$ and applying Theorem 10.2.4, we find that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta(\theta,\varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1 and (10.11), we conclude that the equality holds and (12.14) follows. □

The following two consequences are both due to Yi Yao.

Corollary 12.2.1 *Let E be a T-invariant prime divisor on X corresponding to a ray $\rho \in \Sigma(1)$. Then for any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$, we have*

$$\nu(\varphi, E) = \inf\left\{\langle m - m_\rho, u_\rho\rangle : m \in \Delta(\theta, \varphi)\right\}.$$

Proof This follows immediately from Theorem 12.2.1 and Theorem 10.2.5. In fact, since X is projective and smooth, there is always a T-invariant smooth flag $Y_\bullet$ with $Y_1 = E$. □

Corollary 12.2.2 *For any T-invariant subvariety $Y \subsetneq X$ corresponding to a cone σ in Σ and any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$. Then the following are equivalent:*

(1) $\nu(\varphi, Y) = 0$*;*
(2) *there is a point $m \in \Delta(\theta, \varphi)$ such that $(m - m_\rho) \cdot u_\rho = 0$ for any $\rho \in \sigma(1)$;*
(3) *we have*

$$\Delta(\theta, \varphi) \cap Q_\sigma \neq \varnothing.$$

Proof (2) $\iff$ (3). This follows from the definition of Q_σ in (12.6).

(1) $\iff$ (2). Let $\rho_1, \dots, \rho_r$ be the rays of σ. Up to replacing D by a translation, we may assume that $m_\sigma = 0$. Hence, we may take $m_{\rho_i} = 0$ for all i.

Let $\pi\colon Z \to X$ be the blow-up of X along Y. See [CLS11, Page 132] for the basic properties of the toric blow-up. Take the divisor π^*D on Z. We choose the pull-back metric π^*h on π^*L. Then $F_{\pi^*\theta}$ can be taken as π^*F_θ by (12.7). It follows $\Delta(\theta, \varphi) = \Delta(\pi^*\theta, \pi^*\varphi)$. On the other hand, the ray corresponding to the exceptional divisor E is generated by $u_{\rho_1} + \cdots + u_{\rho_r}$. Since X is smooth, this vector is primitive.

Recall that the support function of π^*D is the same as the support function of D, see [CLS11, Proposition 6.2.7]. In particular, we can take the Cartier datum $m_\rho = m_\sigma$ mod $M(\rho)$, where ρ is the ray corresponding to E.

It follows from Corollary 12.2.1 and [Bou02a, Corollaire 1.1.8] that

$$\nu(\varphi, Y) = \nu(\pi^*\varphi, E) = \inf\{(m - m_\sigma, u_{\rho_1} + \cdots + u_{\rho_r}) : m \in \Delta(\theta, \varphi)\}. \tag{12.15}$$

Our assertion follows in view of (12.4). □

It follows from (12.15) that

$$\nu(\varphi, Y) \geq \sum_{i=1}^{a} \nu(\varphi, E_i),$$

where the E_i's are the prime divisors corresponding to the rays of σ. This inequality seems to be new as well.

The following consequence of Theorem 12.2.1 is the key to the development of the toric pluripotential theory.

Theorem 12.2.2 *We have*

$$F_{V_\theta} \in \mathcal{E}(N_{\mathbb{R}}, P_D).$$

In particular,

$$\int_X \theta_{V_\theta}^n = n! \operatorname{vol} P. \tag{12.16}$$

Proof Take $\varphi = V_\theta$ in Theorem 12.2.1, we find

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, V_\theta)\right) = \Delta_{Y_\bullet}(\theta, V_\theta) = \Delta_{Y_\bullet}(L) = \Phi_{\mathbb{R}}(P_D),$$

where we applied Proposition 12.2.2 in the last equality. Therefore,

$$\Delta(\theta, V_\theta) = P_D.$$

Finally, (12.16) follows from Proposition 12.2.1. □

12.3 The pluripotential theory

We continue to use the notations in Section 12.1.

Theorem 12.3.1 *There is a canonical bijection between the following sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$*;*
(2) *the set of* $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$ *satisfying* $F \leq F_{V_\theta}$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G \geq F^*_{V_\theta}.$$

The set $\mathcal{P}(N_{\mathbb{R}}, P_D)$ is defined in Definition A.3.1. As before, we write F_φ, G_φ for the functions determined by this construction.

Proof The proof is similar to that of Theorem 5.2.1, but due to its importance, we give the proof. Again, the correspondence between (2) and (3) follows easily from Proposition A.2.5.

Given φ, we can construct F_φ in (2) as explained earlier in (12.8). Conversely, suppose that $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$ is such that $F \leq F_{V_\theta}$. Then

$$\mathrm{Trop}^*(F - F_\theta) \in \mathrm{PSH}(T(\mathbb{C}), \theta|_{T(\mathbb{C})})$$

by Lemma 5.2.1. Since $F \leq F_{V_\theta}$, we see that $\mathrm{Trop}^*(F - F_\theta)$ is bounded from above. It follows that Grauert–Remmert's extension theorem Theorem 1.2.1 is applicable, and this function extends to a unique θ-psh function φ. The uniqueness of the extension guarantees that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$.

The two maps are clearly inverse to each other. □

We fix a model potential $\phi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$ with Newton body $\Delta(\theta, \phi)$.

A similar argument guarantees the following:

Corollary 12.3.1 *There is a canonical bijection between the following sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta; \phi)$*,*
(2) *the set of* $F \in \mathcal{P}(N_{\mathbb{R}}, \Delta(\theta, \phi))$ *satisfying* $F \leq F_{V_\theta}$*, and*

(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G \geq F^*_{V_\theta}, \quad G|_{M_{\mathbb{R}} \setminus \Delta(\theta,\phi)} = \infty.$$

Moreover, under these correspondences, we have the following bijections:

(1) *The set* $\mathcal{E}_{\mathrm{tor}}(X, \theta; \phi)$*,*
(2) *the set of* $F \in \mathcal{E}(N_{\mathbb{R}}, \Delta(\theta, \phi))$ *satisfying* $F \preceq F_{V_\theta}$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G \geq F^*_{V_\theta}, \quad G|_{M_{\mathbb{R}} \setminus \Delta(\theta,\phi)} = \infty, \quad G|_{\mathrm{Int}\, \Delta(\theta,\phi)} < \infty.$$

Here the notations are defined as follows:

$$\begin{aligned} \mathrm{PSH}_{\mathrm{tor}}(X, \theta; \phi) &:= \{\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta) : \varphi \preceq \phi\}, \\ \mathcal{E}_{\mathrm{tor}}(X, \theta; \phi) &:= \mathcal{E}(X, \theta; \phi) \cap \mathrm{PSH}_{\mathrm{tor}}(X, \theta). \end{aligned}$$

The proofs of the following results are similar to the ample case studied in Chapter 5. We omit the details.

Proposition 12.3.1 *Given* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *and* $C \in \mathbb{R}$*. We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

Proposition 12.3.2 *Given* $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$*, assume that* $\varphi \land \psi \not\equiv -\infty$*, then* $\varphi \land \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *and*

$$F_{\varphi \land \psi} = F_\varphi \land F_\psi, \quad G_{\varphi \land \psi} = G_\varphi \lor G_\psi.$$

Proposition 12.3.3 *Let* $(\varphi_i)_{i \in I}$ *be a family in* $\mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *uniformly bounded from above. Then* ${\sup_{i\in I}}^* \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *and*

$$F_{{\sup_{i\in I}}^* \varphi_i} = \bigvee_{i \in I} F_{\varphi_i}, \quad G_{{\sup_{i\in I}}^* \varphi_i} = \mathrm{cl} \bigwedge_{i \in I} G_{\varphi_i}.$$

Moreover, if I *is finite, then*

$$G_{\bigvee_{i\in I} \varphi_i} = \bigwedge_{i \in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i \in I}$ *is a decreasing net in* $\mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *such that* $\inf_{i\in I} \varphi_i \not\equiv -\infty$*, then* $\inf_{i \in I} \varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *and*

$$F_{\inf_{i\in I} \varphi_i} = \bigwedge_{i \in I} F_{\varphi_i}, \quad G_{\inf_{i\in I} \varphi_i} = \bigvee_{i \in I} G_{\varphi_i}.$$

Proposition 12.3.4 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$*. Then* $P_\theta[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ *and*

$$G_{P_\theta[\varphi]}(x) = \begin{cases} G_{V_\theta}(x), & \text{if } x \in \Delta(\theta, \varphi); \\ \infty, & \text{otherwise.} \end{cases} \tag{12.17}$$

As a consequence, we have

Corollary 12.3.2 *Let $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$. Then the following are equivalent:*

(1) $\varphi \preceq_P \psi$;
(2) $\varphi \preceq_{\mathcal{I}} \psi$;
(3) $\Delta(\theta, \varphi) \subseteq \Delta(\theta, \psi)$.

Proof (1) $\iff$ (3). This follows from Proposition 12.3.4.
(1) $\iff$ (2). This follows from Example 7.3.1. □

Next we handle subgeodesics.

Proposition 12.3.5 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$. There is a canonical bijection between the following sets:*

(1) *The set of T_c-invariant subgeodesics from φ_0 to φ_1;*
(2) *the set of convex functions $F\colon N_{\mathbb{R}} \times (0,1) \to \mathbb{R}$ such that for each $r \in (0,1)$, the function*

$$F_r\colon N_{\mathbb{R}} \to \mathbb{R}, \quad n \mapsto F(n, r) \tag{12.18}$$

satisfies $F_r \to F_{\varphi_1}$ (resp. $F_r \to F_{\varphi_0}$) everywhere as $r \to 1-$ (resp. $r \to 0+$).

Proof We begin with a subgeodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1. Then we define $F\colon N_{\mathbb{R}} \times (0,1) \to \mathbb{R}$ as follows:

$$F(n,t) = F_{\varphi_t}(n).$$

Define F_t as in (12.18), we have

$$\mathrm{Trop}^* F_t - \mathrm{Trop}^* F_\theta = \varphi_t, \quad t \in (0,1).$$

By definition, as $t \to 0+$, $\varphi_t \to \varphi_0$ almost everywhere. By Fubini's theorem, $F_t \to F_0$ almost everywhere, hence everywhere by Theorem A.1.2. Similarly, $F_t \to F_1$ everywhere as $t \to 1-$.

Next we show that F is convex. Let $p_1\colon X \times S \to X$ be the projection, where

$$S := \{z \in \mathbb{C} : \mathrm{e}^{-1} < |z|^2 < 1\}.$$

Since F is a subgeodesic, its complexification Φ is $p_1^*\theta$-psh. Recall that Φ is defined as

$$\Phi(x,z) = \mathrm{Trop}^* \left(F_{-\log|z|^2} - F_\theta\right)(x). \tag{12.19}$$

In particular, $\Psi\colon T(\mathbb{C}) \times S \to \mathbb{R}$ defined by

$$\Psi(x,z) := \Phi(x,z) + \mathrm{Trop}^* F_\theta(x) = \mathrm{Trop}^* F_{-\log|z|^2}(x)$$

is plurisubharmonic and $T_c \times S^1$-invariant. Fix a small enough $\epsilon > 0$, we could find a decreasing sequence of $T_c \times S^1$-invariant plurisubharmonic functions Ψ_i on $T(\mathbb{C}) \times S_\epsilon$ converging to Ψ everywhere, where

$$S_\epsilon := \{z \in \mathbb{C} : \mathrm{e}^{-1+\epsilon} + \epsilon < |z|^2 < \mathrm{e}-\epsilon\}.$$

Let us write

$$\Psi_i(x, z) = \mathrm{Trop}^* F_{i,-\log|z|^2}(x)$$

for some $F_i : X \times S_\epsilon \to \mathbb{R}$.

The same computation as in Lemma 5.2.1 shows that F_i is convex. It follows that F, as the decreasing limit of F_i, is also convex on $X \times (\epsilon, 1 - \epsilon)$. Since $\epsilon > 0$ is arbitrary, we conclude that F is convex on $X \times (0, 1)$.

Conversely, suppose that we are given F in (2). We define $\Phi : T(\mathbb{C}) \times S \to \mathbb{R}$ using (12.19). The arguments in the previous part can be reversed to show that Φ is $p_1^*\theta|_{T(\mathbb{C})\times S}$-psh.

By our assumption, for each $t \in (0, 1)$, we have

$$F_t \le tF_{\varphi_1} + (1 - t)F_{\varphi_0} \le F_{V_\theta} + C \tag{12.20}$$

for some constant $C \in \mathbb{R}$ independent of the choice of t. Therefore, Φ is bounded from above and hence by Theorem 1.2.1, we conclude that Φ admits a unique extension to a $p_1^*\theta$-psh extension to $X \times S$, which we still denote by Φ. We let

$$\varphi_t(x) = \Phi(x, \mathrm{e}^{-t/2})$$

for all $t \in (0, 1)$ and $x \in X$. We claim that (φ_t) is a subgeodesic from φ_0 to φ_1.

For this purpose, we only need to show that $(\varphi_t)_{t\in(0,1)}$ has the correct boundary value. But from our assumption in (2), we know that as $t \to 0+$ (resp. $t \to 1-$), $\varphi_t \to \varphi_0$ (resp. $\varphi_t \to \varphi_1$) almost everywhere. In particular, $\sup_X \varphi_t \ge -C'$ for some large constant $C' > 0$ independent of $t \in (0, 1)$. Therefore, together with (12.20), we deduce from Proposition 1.5.1 that $\{\varphi_t\}_{t\in(0,1)}$ is a relatively compact family with respect to the L^1-topology. We need to show that each cluster point ψ as $t \to 0+$ is equal to φ_0. But we already know that $\psi = \varphi_0$ almost everywhere. Hence we deduce $\psi = \varphi_0$ from Proposition 1.2.6. As $t \to 0+$, we have $\varphi_t \xrightarrow{L^1} \varphi_0$. Similarly, as $t \to 1-$, we have $\varphi_t \xrightarrow{L^1} \varphi_1$.

The two constructions are clearly inverse to each other.

Corollary 12.3.3 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$. Then there is a canonical bijection between the following sets:*

(1) *The set of T_c-invariant subgeodesics from ψ_0 to ψ_1, where $\psi_0, \psi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ and $\psi_0 \le \varphi_0$, $\psi_1 \le \varphi_1$;*
(2) *the set of closed proper convex functions Ψ on $M_\mathbb{R} \times \mathbb{R}$ such that there is a closed proper convex function $G \in \mathrm{Conv}(M_\mathbb{R})$ such that*

$$G(m) + (s \vee 0) \geq \Psi(m,s) \geq G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right). \tag{12.21}$$

Proof Let us begin with a subgeodesic $(\psi_t)_{t\in(0,1)}$ as in (1). Let F be the convex function as in Proposition 12.3.5. We extend F to a function $F\colon N_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ as follows: For any $n \in N_{\mathbb{R}}$, we define

$$F(n,t) = \begin{cases} F_{\psi_0}(n), \text{ if } t = 0, \\ F_{\psi_1}(n), \text{ if } t = 1, \\ \quad \infty, \text{ otherwise.} \end{cases}$$

Then F is a proper closed convex function on $N_{\mathbb{R}} \times \mathbb{R}$. Let Ψ be the Legendre transform of F. Then Ψ is a proper closed convex function on $M_{\mathbb{R}} \times \mathbb{R}$ by Theorem A.2.1. By (A.2), for any $m \in M_{\mathbb{R}}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned} \Psi(m,s) &= \sup_{n\in N_{\mathbb{R}}, t\in[0,1]} \left(\langle m,n\rangle + ts - F(n,t)\right) \\ &= \sup_{t\in[0,1]} \left(ts + F_t^*(m)\right). \end{aligned}$$

Therefore, the latter half of (12.21) follows. Next recall that

$$\eta \coloneqq \inf_{t\in(0,1)} \psi_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta),$$

as follows from Proposition 4.1.2. Therefore,

$$\begin{aligned} \Psi(m,s) &= \sup_{n\in N_{\mathbb{R}}, t\in[0,1]} \left(\langle m,n\rangle + ts - F(n,t)\right) \\ &\leq \sup_{n\in N_{\mathbb{R}}, t\in[0,1]} \left(\langle m,n\rangle + ts - F_\eta\right) \\ &= \sup_{t\in[0,1]} ts + G_\eta(m) \\ &= (s \vee 0) + G_\eta(m). \end{aligned}$$

Conversely, let us begin with a function Ψ as in (2). Let F be the Legendre transform of Ψ. We first observe that $F(n,t) = \infty$ for all $n \in N_{\mathbb{R}}$ and $t \notin [0,1]$.

In fact,

$$\begin{aligned} F(n,t) &= \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} \left(\langle m,n\rangle + ts - \Psi(m,s)\right) \\ &\leq \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} \left(\langle m,n\rangle + ts - G_{\varphi_0}(m)\right) \\ &= \sup_{s\in\mathbb{R}} \left(ts + F_{\varphi_0}(n)\right) \\ &= \begin{cases} F_{\varphi_0}(n), \text{ if } t = 0, \\ \quad \infty, \text{ otherwise.} \end{cases} \end{aligned}$$

Similarly,

$$
\begin{aligned}
F(n,t) &= \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} (\langle m,n\rangle + ts - \Psi(m,s)) \\
&\leq \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} \left(\langle m,n\rangle + ts - G_{\varphi_1}(m) - s\right) \\
&= \sup_{s\in\mathbb{R}} \left(ts - s + F_{\varphi_1}(n)\right) \\
&= \begin{cases} F_{\varphi_0}(n), \text{ if } t=1, \\ \infty, \text{ otherwise.} \end{cases}
\end{aligned}
$$

Therefore, we conclude that

$$
F(n,t) \leq tF_{\varphi_1} + (1-t)F_{\varphi_0}
$$

for all $t \in [0,1]$ and $n \in N_{\mathbb{R}}$. Let $(\psi_t)_{t\in(0,1)}$ be the subgeodesic defined by Proposition 12.3.5, then $(\psi_t)_{t\in(0,1)}$ satisfies (1). Next observe that

$$
\begin{aligned}
F(n,t) &= \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} (\langle m,n\rangle + ts - \Psi(m,s)) \\
&\geq \sup_{m\in M_{\mathbb{R}}, s\in\mathbb{R}} (\langle m,n\rangle + ts - G(m) - s\vee 0) \\
&= G^*(n) + \sup_{s\in\mathbb{R}} (ts - (s\vee 0)) \\
&= \begin{cases} G^*(n), \text{ if } t\in[0,1] \\ \infty, \text{ otherwise.} \end{cases}
\end{aligned}
$$

The two operations are clearly inverse to each other. □

As an immediate corollary,

Corollary 12.3.4 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta) \cap \mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) $\varphi_0 \sim_P \varphi_1$*;*
(2) *there is a subgeodesic from φ_0 to φ_1;*
(3) *there is a geodesic from φ_0 to φ_1.*

If these conditions are satisfied, let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then $\varphi_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ for all $t\in(0,1)$ and

$$
G_{\varphi_t} = (1-t)G_{\varphi_1} + tG_{\varphi_0}. \tag{12.22}
$$

Proof We first observe that (2) $\iff$ (3) follows from the definition of geodesics Definition 4.2.1 and the fact that a geodesic is a subgeodesic Proposition 4.2.1. Also (1) $\implies$ (3) follows from Proposition 4.2.1.

Let us assume for the moment that (3) holds. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. It is clear that $\varphi_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ for all $t\in(0,1)$. Let Ψ' be the proper convex function on $M_{\mathbb{R}} \times \mathbb{R}$ defined by Corollary 12.3.3. Then Ψ' is the minimum of all Ψ satisfying (12.21). We claim that

$$\Psi'(m,s) = G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right). \tag{12.23}$$

It suffices to show that the right-hand side is proper, namely, $G_{\varphi_0} \vee G_{\varphi_1}$ is not identically ∞. But recall that by Proposition 4.1.2, we have $\varphi_0 \wedge \varphi_1 \in \mathrm{PSH}(X,\theta)$. Therefore, by Proposition 12.3.2,

$$G_{\varphi_0} \vee G_{\varphi_1} = G_{\varphi_0 \wedge \varphi_1} \not\equiv \infty.$$

In particular, (12.23) follows.

Now by construction,

$$G_{\varphi_t}(m) = \sup_{s\in\mathbb{R}} (st - \Psi'(m,s)) = (1-t)G_{\varphi_1}(m) + tG_{\varphi_0}(m)$$

for all $t \in (0,1)$. So (12.22) follows. It remains to establish (1). We first reduce to the case where $\varphi_0 \leq \varphi_1$. In fact, we know that $(\varphi_0 \vee \varphi_t)_{t\in[0,1]}$ is a T_c-invariant subgeodesic by Proposition 4.1.3 and Example 4.1.1. It we manage to prove the special case, then we would know that $\varphi_0 \sim_P \varphi_1 \vee \varphi_0$. Hence $\varphi_1 \leq_P \varphi_0$ by Lemma 6.1.3. The converse follows similarly.

Next assume that $\varphi_0 \leq \varphi_1$. Then $(F_{\varphi_{1-m^{-1}}})_{m>1}$ is an increasing sequence with limit F_{φ_1}. Hence by Proposition A.2.3, we have

$$G_{\varphi_1}(m) = \mathrm{cl} \bigwedge_{m=2}^{\infty} G_{\varphi_{1-m^{-1}}} = \begin{cases} \infty, \text{ if } G_{\varphi_0}(m) = \infty, \\ G_{\varphi_1}(m), \text{ otherwise.} \end{cases}$$

Hence

$$\overline{\{G_{\varphi_0} = \infty\}} = \overline{\{G_{\varphi_1} = \infty\}}.$$

We conclude that $\varphi_0 \sim_P \varphi_1$ by Corollary 12.3.2. □

Next we consider the trace operator. For this purpose, we will need to fix a T-invariant subvariety $Y \subsetneq X$. Let σ be the corresponding cone in Σ and Q be the corresponding face of P_D. The cocharacter lattice of Y is given by

$$N(\sigma) := N/N \cap \langle\sigma\rangle,$$

where $\langle\sigma\rangle$ is the linear span of σ. See [CLS11, (3.2.6)]. In particular, we have a canonical identification of the character lattice $M(\sigma)$ of Y:

$$M(\sigma) = \sigma^\perp \cap M,$$

which is compatible with our previous notation (12.2). Let $i_\sigma \colon M(\sigma) \to M$ be the inclusion map. Let T_Y be the torus of Y. Then we have a natural surjection $q_T \colon T \to T_Y$. In particular, then tropicalization map

$$\mathrm{Trop}\colon T(\mathbb{C}) \to N_{\mathbb{R}}$$

descends to the tropicalization map of Y:

$$\mathrm{Trop}_Y : T_Y(\mathbb{C}) \to N(\sigma)_{\mathbb{R}}.$$

We let

$$D_Y = \sum_{\substack{\rho \in \Sigma(1) \\ \rho \not\leq \sigma}} a_\rho D_\rho|_Y,$$

where $\rho \not\leq \sigma$ means that ρ is not a face of σ. Then $\mathcal{O}_Y(D_Y) = L|_Y$.

Theorem 12.3.2 *There is a canonical choice of the Cartier datum $m_\sigma \in M$ such that for any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ with $\nu(\varphi, Y) = 0$, $\mathrm{Tr}_Y^\theta(\varphi)$ is defined and $\mathrm{vol}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) > 0$*[3]*, we have*

$$\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) = \Delta(\theta, \varphi) \cap Q - m_\sigma$$

as subsets of $M(\sigma)_{\mathbb{R}}$.

Observe that the condition $\nu(\varphi, Y) = 0$ means exactly that $\Delta(\theta, \varphi) \cap Q \neq \emptyset$ by Corollary 12.2.2.

Since Y itself is a smooth toric variety, the proceeding constructions of X all apply to Y. We briefly summarize the situation in Table 12.1.

Notions for X	Notions for Y
N	$N(\sigma)$
M	$M(\sigma)$
Σ	$\mathrm{Star}(\sigma)$
D	D_Y
L	$L\|_Y$
h	$h\|_Y$
θ	$\theta\|_Y$
Trop	Trop_Y
P_D	Q
s_D	s_{D_Y}

Table 12.1 The correspondence between X and Y

Recall that $\mathrm{Star}(\sigma)$ is the fan in $N(\sigma)_{\mathbb{R}}$ consisting of $\bar{\tau}$ for all faces $\tau \in \Sigma$ containing σ, where $\bar{\tau}$ is the image of τ in $N(\sigma)_{\mathbb{R}}$. See [CLS11, Proposition 3.2.7].

Proof The idea of the proof is that since we know how the partial Okounkov bodies behave under restrictions by Lemma 10.3.7 and Remark 10.3.2, and know how to compare partial Okounkov bodies and Newton bodies Theorem 12.2.1, we should be able to deduce the behavior of Newton bodies under restriction as well.

First we note that by our assumption, $L|_Y$ is a big line bundle. In particular, if we set $r = \dim \sigma$, then $\dim Y = n - r$.

For this purpose, let σ^0 be an n-dimensional face of Σ containing σ. The image $\overline{\sigma^0}$ in $N(\sigma)$ is then an r-dimensional face of $\mathrm{Star}(\sigma)$. We shall use these faces as the reference faces while defining the partial Okounkov bodies.

[3] Note that $\mathrm{Tr}_Y^\theta \in \mathrm{PSH}_{\mathrm{tor}}(Y, \theta|_Y)$.

We list the rays in $\sigma^0(1)$ as follows:

$$\rho_1, \dots, \rho_n, \tag{12.24}$$

where $\rho_1, \dots, \rho_r \in \sigma(1)$ and hence $\rho_{r+1}, \dots, \rho_n \notin \sigma(1)$. In particular, the images

$$\overline{\rho_{r+1}}, \dots, \overline{\rho_n} \tag{12.25}$$

of the latter give a list of $\overline{\sigma^0}(1)$.

We construct the flag $Y_\bullet$ on X using the rays (12.24) and the flag $Z_\bullet$ on Y using the rays (12.25). Note that

$$Z_i = Y_{r+i},$$

where $i = 1, \dots, n-r$.

Next we compute the Cartier data associated with $\overline{\sigma^0}$. By definition, $m_{\overline{\sigma_0}} \in M(\sigma)$ is the unique element satisfying

$$m_{\overline{\sigma_0}} \cdot u_{\rho_j} = -a_{\rho_j}$$

for all $j = r+1, \dots, n$.

Let $\Phi\colon M \to \mathbb{Z}^n$ and $\Psi\colon M(\sigma) \to \mathbb{Z}^{n-r}$ be defined as

$$\begin{aligned}
\Phi(m) &= \left(\langle m - m_{\sigma^0}, u_{\rho_1}\rangle, \dots, \langle m - m_{\sigma^0}, u_{\rho_n}\rangle\right)\\
&= \left(\langle m, u_{\rho_1}\rangle, \dots, \langle m, u_{\rho_n}\rangle\right) + (a_{\rho_1}, \dots, a_{\rho_n}),\\
\Psi(m) &= \left(\langle m - m_{\overline{\sigma_0}}, u_{\overline{\rho_1}}\rangle, \dots, \langle m - m_{\overline{\sigma_0}}, u_{\overline{\rho_{n-r}}}\rangle\right)
\end{aligned}$$

Observe that for $i = r+1, \dots, n$, we have

$$u_{\overline{\rho_i}} = u_{\rho_i} \mod N \cap \langle \sigma \rangle,$$

so

$$\Psi(m) = \left(\langle m, u_{\rho_{r+1}}\rangle, \dots, \langle m, u_{\rho_n}\rangle\right) + (a_{\rho_{r+1}}, \dots, a_{\rho_n})$$

for $m \in M(\sigma)$. Therefore, we have a commutative diagram

$$\begin{tikzcd}
M_{\mathbb{R}} \arrow[r, "\Phi_{\mathbb{R}}"] & \mathbb{R}^n \arrow[d, "L"] \\
M(\sigma)_{\mathbb{R}} \arrow[u, hook] \arrow[r, "\Psi_{\mathbb{R}}"] & \mathbb{R}^{n-r},
\end{tikzcd}$$

where $L\colon \mathbb{R}^n \to \mathbb{R}^{n-r}$ is the map

$$(b_1, \dots, b_n) \mapsto (b_{r+1}, \dots, b_n).$$

By Theorem 12.2.1, we have

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi)\right) = \Delta_{Y_\bullet}(\theta, \varphi), \quad \Psi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi))\right) = \Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)).$$

The latter can be written as

$$\Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}^\theta_Y(\varphi)) = L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}^\theta_Y(\varphi))\right).$$

While by Lemma 10.3.7 and Remark 10.3.2,

$$\begin{aligned}\Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}^\theta_Y(\varphi)) =& L\left(\Delta_{Y_\bullet}(\theta,\varphi) \cap (\{0\}^r \times \mathbb{R}^{n-r})\right)\\ =& L\left(\Phi_{\mathbb{R}}(\Delta(\theta),\varphi) \cap (\{0\}^r \times \mathbb{R}^{n-r})\right)\\ =& L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi) \cap Q\right).\end{aligned}$$

Hence,

$$L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}^\theta_Y(\varphi))\right) = L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi) \cap Q\right).$$

It follows that

$$\Delta(\theta|_Y, \mathrm{Tr}^\theta_Y(\varphi)) + m_{\sigma^0} - m_{\overline{\sigma^0}} = \Delta(\theta,\varphi) \cap Q.$$

Finally, observe that $m_{\sigma^0} - m_{\overline{\sigma^0}}$ represents m_σ. Our assertion follows. □

Chapter 13
Non-Archimedean pluripotential theory

A good theorem lasts forever. Once proved, it will always stay proved, and other mathematicians are free to use it and build on it as they please, sometimes to great effect.
— John Tate

In this chapter, we will establish the non-Archimedean pluripotential theory using the theory of $\mathcal{I}$-good singularities. We show that our theory extends the algebraic theory à la Boucksom–Jonsson in Section 13.4.

We also construct the Duistermaat–Heckman measure of a non-Archimedean metric in Section 13.3.

13.1 The definition of non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n. Let $\mathrm{Käh}(X)$ be the set of Kähler forms on X with the partial order given as follows: We say $\omega \leq \omega'$ if $\omega \geq \omega'$. Note that the ordered set $\mathrm{Käh}(X)$ is a directed set.

Let θ be a closed smooth real $(1,1)$-form.

Definition 13.1.1 We define

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) = \varprojlim_{\omega\in\mathrm{Käh}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}$$

in the category of sets, where the transition maps are given as follows: Suppose that $\omega, \omega' \in \mathrm{Käh}$ and $\omega \geq \omega'$, then the transition map is defined in Proposition 9.3.4:

$$P_{\theta+\omega'}[\bullet]_{\mathcal{I}} : \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega')_{>0} \to \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}. \tag{13.1}$$

Recall that $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined in Definition 9.3.1.

In general, when we denote an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ by Γ, its component in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}$ ($\omega \in \mathrm{Käh}(X)$) will be written as either Γ^{ω} or $P_{\theta+\omega'}[\Gamma]_{\mathcal{I}}$.

Note that $\Gamma^{\omega}_{\max}$ is independent of the choice of $\omega \in \mathrm{Käh}(X)$. We denote this common value by $\Gamma_{\max}$.

Remark 13.1.1 Thanks to Proposition 9.3.2, for any other θ' representing $[\theta]$, we have a canonical bijection

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(X,\theta').$$

Moreover, these bijections satisfy the cocycle condition. If we view the set of closed real smooth $(1,1)$-forms representing $[\theta]$ as a category with a unique morphism between any two objects, then we can define

$$\mathrm{PSH}^{\mathrm{NA}}(X,[\theta]) = \varprojlim_{\theta} \mathrm{PSH}^{\mathrm{NA}}(X,\theta).$$

This definition is independent of the choice of the explicit representative of the cohomology class $[\theta]$.

However, given the fact that our notations are already quite heavy, we decide to stick to the set $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. The readers should verify that all constructions below are independent of the choice of θ within its cohomology class.

Proposition 13.1.1 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Then given $\omega,\omega' \in \mathrm{K\ddot{a}h}(X)$ with $\omega \geq \omega'$, we have*

$$P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{-\infty}\right] = P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{-\infty}\right]_{\mathcal{I}} = \Gamma^{\theta+\omega}_{-\infty}.$$

Proof Since for any $\tau < \Gamma_{\max}$, the potential $\Gamma^{\theta+\omega'}_{\tau}$ is $\mathcal{I}$-good by Example 7.1.2, it follows that

$$P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{\tau}\right] = P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{\tau}\right]_{\mathcal{I}} = \Gamma^{\theta+\omega}_{\tau}$$

for all $\tau < \Gamma_{\max}$. Our assertion follows from Proposition 3.1.10 and Proposition 3.2.13. □

Proposition 13.1.2 *There is a natural injective map*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \hookrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta), \quad \Gamma \mapsto \left(P_{\theta+\omega}[\Gamma]_{\mathcal{I}}\right)_{\omega\in\mathrm{K\ddot{a}h}(X)}.$$

In the sequel, we will not distinguish an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ with its image in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Note that given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, the value of $\Gamma_{\max}$ does not depend on if we view it as an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ or in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

Proof It is obvious that this map is well-defined. It suffices to argue its injectivity. Suppose that $\Gamma,\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} = P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$$

for some Kähler form ω on X. Then $\Gamma_{\max} = \Gamma'_{\max}$ and for any $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \sim_{\mathcal{I}} \Gamma'_\tau$$

by Proposition 6.1.3. It follows again from Proposition 6.1.3 that

$$\Gamma_\tau = \Gamma'_\tau.$$

Definition 13.1.2 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define its *volume* as follows:

$$\operatorname{vol}\Gamma := \lim_{\omega\in\mathrm{K\ddot{a}h}(X)} \int_X \left(\theta+\omega+\mathrm{dd^c}\Gamma_{-\infty}^{\theta+\omega}\right)^n \in [0,\infty).$$

Observe that the net is decreasing, so the limit exists.

Proposition 13.1.3 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. *Then*

$$\operatorname{vol}\Gamma = \int_X \left(\theta+\mathrm{dd^c}\Gamma_{-\infty}\right)^n .$$

Proof This follows from Proposition 3.1.9, Corollary 3.1.2 and Proposition 9.4.1. □

Lemma 13.1.1 *The image of the canonical injection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \hookrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$$

is given by the set of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *with positive volume.*

Proof By Proposition 13.1.3, it is clear that the image of an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ has positive volume.

Conversely, take $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with positive volume. We want to construct $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ representing Γ.

Fix a Kähler form ω on X. Using the same arguments as Lemma 9.1.1, we find that

$$\lim_{k\to\infty} \int_X \left(\theta+k^{-1}\omega+\mathrm{dd^c}\Gamma_\tau^{\theta+k^{-1}\omega}\right)^n > 0$$

for any $\tau < \Gamma_{\max}$. We define

$$\Gamma'_\tau := \lim_{k\to\infty} \Gamma_\tau^{\theta+k^{-1}\omega}$$

for any $\tau < \Gamma_{\max}$. It follows from Proposition 3.1.9 that Γ' represents Γ. □

Example 13.1.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we can define an associated $\Gamma^\varphi \in \mathrm{PSH}(X,\theta)$ as follows:

(1) $\Gamma^\varphi_{\max} = 0$;
(2) for any $\omega \in \mathrm{K\ddot{a}h}(X)$ and any $\tau < 0$, we set

$$\Gamma_\tau^{\varphi,\theta+\omega} = P_{\theta+\omega}[\varphi]_{\mathcal{I}}.$$

Definition 13.1.3 Let ω be a closed real smooth positive $(1,1)$-form on X. We define the map

$$P_{\theta+\omega}[\bullet]_{\mathcal{I}} : \mathrm{PSH}^{\mathrm{NA}}(X,\theta) \to \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$$

as follows: Given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define $P_{\theta+\omega}[\Gamma]_{\mathcal{I}}$ as the element such that for any $\omega' \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}}^{\theta+\omega+\omega'} = \Gamma^{\theta+\omega+\omega'}.$$

It is straightforward to check that under the identification of Proposition 13.1.2, the map $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ extends the map (13.1).

Proposition 13.1.4 *The maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 together induce a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega\in\mathrm{K\ddot{a}h}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega). \tag{13.2}$$

Proof It is a tautology that the maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 are compatible with the transition maps. So the map (13.2) is well-defined. It is injective by the same argument as Proposition 13.1.2. We argue the surjectivity.

By unfolding the definitions, an object in the target of (13.2) is an assignment: With each $\omega \in \mathrm{K\ddot{a}h}(X)$, we associate a family $(\Gamma^{\omega,\omega'})_{\omega'\in\mathrm{K\ddot{a}h}(X)}$ satisfying:

(1) $\Gamma^{\omega,\omega'} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega+\omega')_{>0}$ for each $\omega,\omega' \in \mathrm{K\ddot{a}h}(X)$;
(2) for each $\omega,\omega',\omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega'' \geq \omega'$, we have

$$P_{\theta+\omega+\omega''}\left[\Gamma^{\omega,\omega'}\right]_{\mathcal{I}} = \Gamma^{\omega,\omega''};$$

(3) for each $\omega,\omega',\omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega \leq \omega'$, we have

$$P_{\theta+\omega'+\omega''}\left[\Gamma^{\omega,\omega''}\right]_{\mathcal{I}} = \Gamma^{\omega',\omega''}.$$

The preimage of such an object is given by the family $(\Gamma^{\omega})_{\omega\in\mathrm{K\ddot{a}h}(X)}$ given by

$$\Gamma^{\omega} = \Gamma^{\omega/2,\omega/2}.$$

The fact that the image of Γ is as expected is a tautology, which we leave to the readers. □

With an almost identical argument involving Proposition 3.1.9, we get

Proposition 13.1.5 *The maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 and the injective maps Proposition 13.1.2 together induce bijections*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0} \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega), \tag{13.3}$$

where ω runs over either the partially ordered set of all smooth closed real positive $(1,1)$-forms with positive volume[1] *on X or $\mathrm{K\ddot{a}h}(X)$.*

Corollary 13.1.1 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then π^* induces a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta).$$

[1] This partially ordered set is not directed.

Proof This follows immediately from Proposition 13.1.5. □

It is immediate to verify that π^* in Corollary 13.1.1 extends the map Proposition 9.3.3.

13.2 Operations on non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta', \theta''$ be closed real smooth $(1,1)$-forms on X representing big cohomology classes.

Definition 13.2.1 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. We say $\Gamma \leq \Gamma'$ if for some $\omega \in \mathrm{Käh}(X)$, we have

$$\Gamma^{\theta+\omega} \geq \Gamma'^{\theta+\omega}.$$

This notion is independent of the choice of ω thanks to Lemma 9.4.1.

Moreover, we have the following:

Proposition 13.2.1 *Let* $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *and* ω *be a closed smooth positive* $(1,1)$*-form on X, then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \leq P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$.

Proof This follows immediately from Lemma 9.4.1. □

Observe that this definition coincides with the corresponding definition in Definition 9.4.2 when $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

Definition 13.2.2 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. Then we define $\Gamma + \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$(\Gamma+\Gamma')^{\theta+\theta'+2\omega} = \Gamma^{\theta+\omega} + \Gamma'^{\theta'+\omega}.$$

This definition yields an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')$ by Lemma 9.4.3 and it extends the definition in Definition 9.4.3 by Lemma 9.4.3 as well.

Proposition 13.2.2 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *and* $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. *Suppose that* ω, ω' *are two smooth closed positive* $(1,1)$*-forms on X. Then*

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma+\Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + P_{\theta'+\omega'}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.3. □

Proposition 13.2.3 *The operation + is commutative and associative: For any* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$*,* $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ *and* $\Gamma'' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta'')$*, we have*

$$\Gamma+\Gamma' = \Gamma'+\Gamma, \quad (\Gamma+\Gamma')+\Gamma'' = \Gamma+(\Gamma'+\Gamma'').$$

Proof This is a direct consequence of Proposition 9.4.2. □

Definition 13.2.3 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $C \in \mathbb{R}$. We define $\Gamma + C \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$(\Gamma + C)^{\theta+\omega} = \Gamma^{\theta+\omega} + C.$$

It is obvious from Definition 9.4.4 that $\Gamma + C \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. It is also obvious that this definition extends Definition 9.4.4.

Proposition 13.2.4 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $C \in \mathbb{R}$. Suppose that ω is a smooth closed positive $(1, 1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + C = P_{\theta+\omega}[\Gamma + C]_{\mathcal{I}}.$$

Proof This is clear by definition. □

Proposition 13.2.5 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta')$ and $C, C' \in \mathbb{R}$, then*

(1) $(\Gamma + \Gamma') + C = \Gamma + (\Gamma' + C) = (\Gamma + C) + \Gamma'$;
(2) $\Gamma + (C + C') = (\Gamma + C) + C'$.

Proof This is a direct consequence of Proposition 9.4.3. □

Definition 13.2.4 Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, we define $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$(\Gamma \vee \Gamma')^{\theta+\omega} = \Gamma^{\theta+\omega} \vee \Gamma'^{\theta+\omega}.$$

It follows from Lemma 9.4.5 that $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and this definition extends the corresponding definition in Definition 9.4.5.

Proposition 13.2.6 *Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and ω be a closed smooth positive $(1, 1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma \vee \Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \vee P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.5. □

Proposition 13.2.7 *The operation $\vee$ is commutative and associative.*

In particular, given a finite non-empty family $(\Gamma^i)_{i\in I}$ in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, we then define $\bigvee_{i\in I} \Gamma^i$ in the obvious way.

Proof This is a direct consequence of Corollary 9.4.1. □

Definition 13.2.5 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. Assume that

$$\sup_{i\in I} \Gamma^i_{\max} < \infty. \tag{13.4}$$

Then we define $\sup^*_{i\in I} \Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$\left(\sup_{i\in I}{}^* \Gamma^i\right)^{\theta+\omega} = \sup_{i\in I}{}^* \Gamma^{i,\theta+\omega}.$$

It follows immediately from Lemma 9.4.7 that $\sup_{i\in I}{}^*\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and this definition extends Definition 9.4.7. Moreover, this definition clearly extends Definition 13.2.4 as well.

Proposition 13.2.8 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right]_{\mathcal{I}} = \sup_{i\in I}{}^* P_{\theta+\omega}\left[\Gamma^i\right]_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.7. □

We also have a non-Archimedean version of Choquet's lemma.

Proposition 13.2.9 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Then there exists a countable subfamily $I'\subseteq I$ such that*

$$\sup_{i\in I}{}^*\Gamma^i = \sup_{i\in I'}{}^*\Gamma^i.$$

Proof For any fixed $\omega\in \mathrm{K\ddot{a}h}(X)$, thanks to Proposition 9.4.6, we could find a countable subfamily $I'\subseteq I$ such that

$$\sup_{i\in I}{}^* P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}} = \sup_{i\in I'}{}^* P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}}.$$

It suffices to show that for any other $\omega'\in \mathrm{K\ddot{a}h}(X)$, we have

$$\sup_{i\in I}{}^* P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}} = \sup_{i\in I'}{}^* P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}}.$$

This is an immediate consequence of Proposition 6.1.6. □

Proposition 13.2.10 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Let $C\in\mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i + C) = \sup_{i\in I}{}^*\Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.4). Suppose that $\Gamma^i\leq\Gamma'^i$ for all $i\in I$, then

$$\sup_{i\in I}{}^*\Gamma^i \leq \sup_{i\in I}{}^*\Gamma'^i.$$

Proof This is an immediate consequence of Proposition 9.4.7. □

Definition 13.2.6 Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that

$$\inf_{i\in I}\Gamma^i_{\max} > -\infty, \tag{13.5}$$

then we define $\inf_{i\in I}\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for each $\omega \in \mathrm{Käh}(X)$, the component

$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega}$$

is defined as follows:

(1) We set

$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega}_{\max} = \inf_{i\in I}\Gamma^i_{\max};$$

(2) for any $\tau < \inf_{i\in I}\Gamma^i_{\max}$, we define

$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega}_{\tau} = \inf_{i\in I}\Gamma^{i,\theta+\omega}_{\tau}. \tag{13.6}$$

We observe that

$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}.$$

This follows from Proposition 3.2.12 and Proposition 3.1.9. Moreover, by Proposition 3.2.11, we have $\inf_{i\in I}\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

Proposition 13.2.11 *Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.5). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\inf_{i\in I}\Gamma^i\right]_{\mathcal{I}} = \inf_{i\in I} P_{\theta+\omega}\left[\Gamma^i\right]_{\mathcal{I}}.$$

Proof This follows from Proposition 3.2.11. □

Proposition 13.2.12 *Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.5). Let $C\in\mathbb{R}$. Then*

$$\inf_{i\in I}(\Gamma^i + C) = \inf_{i\in I}\Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.5). Suppose that $\Gamma^i \leq \Gamma'^i$ for all $i\in I$, then

$$\inf_{i\in I}\Gamma^i \leq \inf_{i\in I}\Gamma'^i.$$

Proof This is clear by definition. □

Definition 13.2.7 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\lambda\in\mathbb{R}_{>0}$, then we define $\lambda\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ as the unique element such that for any $\omega\in\mathrm{Käh}(X)$, we have

$$(\lambda\Gamma)^{\lambda\theta+\omega} = \lambda\Gamma^{\theta+\lambda^{-1}\omega}.$$

It follows immediately from Lemma 9.4.8 that $\lambda\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ and this definition extends Definition 9.4.8.

Proposition 13.2.13 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *and* $\lambda \in \mathbb{R}_{>0}$. *Then for any closed smooth positive* $(1,1)$*-form* ω *on* X, *we have*

$$P_{\lambda\theta+\omega}[\lambda\Gamma]_{\mathcal{I}} = \lambda P_{\theta+\lambda^{-1}\omega}[\Gamma]_{\mathcal{I}}.$$

Proof This follows immediately from Lemma 9.4.8. □

Proposition 13.2.14 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$, $C \in \mathbb{R}$ *and* $\lambda, \lambda' > 0$, *we have*

$$\begin{aligned}\lambda(\Gamma+\Gamma') =& \lambda\Gamma + \lambda\Gamma',\\ (\lambda\lambda')\Gamma =& \lambda(\lambda'\Gamma),\\ \lambda(\Gamma+C) =& \lambda\Gamma+\lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ *is a non-empty family in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.4), *then*

$$\lambda\left(\sup_{i\in I}{}^{*}\Gamma^i\right) = \sup_{i\in I}{}^{*}(\lambda\Gamma^i).$$

If $(\Gamma^i)_{i\in I}$ *is a decreasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.5), *then*

$$\lambda\left(\inf_{i\in I}\Gamma^i\right) = \inf_{i\in I}(\lambda\Gamma^i).$$

Proof Everything except the last assertion follows from Proposition 9.4.9. The last assertion is obvious by definition. □

Definition 13.2.8 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Let $Y \subseteq X$ be an irreducible analytic subset. We say that the trace operator of Γ along Y is *well-defined* if

$$\nu\left(\Gamma_\tau^{\theta+\omega}, Y\right) = 0$$

for small enough τ and any $\omega \in \mathrm{Käh}(X)$. We define

$$(\mathrm{Tr}_Y(\Gamma))_{\max} := \sup\left\{\tau < \Gamma_{\max} : \nu\left(\Gamma_\tau^{\theta+\omega}, Y\right) = 0\right\}.$$

In this case, we define $\mathrm{Tr}_Y(\Gamma) \in \mathrm{PSH}^{\mathrm{NA}}(\tilde{Y}, \theta|_{\tilde{Y}})$[2] as the unique element such that for any $\omega \in \mathrm{Käh}(\tilde{Y})$, the component

$$\mathrm{Tr}_Y(\Gamma)^{\theta|_{\tilde{Y}}+\omega} \in \mathrm{PSH}^{\mathrm{NA}}(Y, \theta|_{\tilde{Y}}+\omega)_{>0}$$

is defined as follows:

(1) We let

$$\left(\mathrm{Tr}_Y(\Gamma)^{\theta|_{\tilde{Y}}+\omega}\right)_{\max} = (\mathrm{Tr}_Y(\Gamma))_{\max}; \tag{13.7}$$

[2] Here $\tilde{Y} \to Y$ is the normalization of Y.

(2) for each $\tau \in \mathbb{R}$ less than the common value (13.7), we define

$$\mathrm{Tr}_Y(\Gamma)_\tau^{\theta|_{\tilde{Y}}+\omega} := P_{\theta|_{\tilde{Y}}+\omega}\left[\mathrm{Tr}_Y^{\theta+\tilde{\omega}}\left(\Gamma_\tau^{\theta+\tilde{\omega}}\right)\right],$$

where $\tilde{\omega}$ is an arbitrary Kähler form on X such that $\omega \geq \tilde{\omega}|_{\tilde{Y}}$.

It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space. We observe that the choice of the trace operator $\mathrm{Tr}_Y^{\theta+\tilde{\omega}}\left(P_{\theta+\tilde{\omega}}[\Gamma]_{\mathcal{I},\tau}\right)$ is irrelevant since two different choice are $\mathcal{I}$-equivalent. Moreover,

$$P_{\theta|_{\tilde{Y}}+\omega}\left[\mathrm{Tr}_Y(\Gamma)\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(Y, \theta|_{\tilde{Y}}+\omega)_{>0}$$

as a consequence of Proposition 8.1.2 and Proposition 8.2.1. It is therefore clear that $\mathrm{Tr}_Y(\Gamma) \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$.

Proposition 13.2.15 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then all definitions in this section are invariant under pulling-back to Y.*

The meaning is clear in most cases. In the case of the trace operator, this means the following: Suppose that $Z \subseteq X$ is an analytic subset and $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ has non-trivial restriction to Z. Suppose that Z is not contained in the non-isomorphism locus of π so that the strict transform W of Z is defined. If we write $\Pi\colon W \to Z$ for the restriction of π and $\tilde{\Pi}\colon \tilde{W} \to \tilde{Z}$ the strict transform of Π, then we have

$$\tilde{\Pi}^* \mathrm{Tr}_Z(\Gamma) = \mathrm{Tr}_W(\pi^*\Gamma).$$

Proof We only prove the assertion for the trace operator, as the other proofs are similar.

We shall use the notations above. Observe that for any closed positive smooth $(1,1)$-form ω on X with positive mass, we have

$$\left(\tilde{\Pi}^* \mathrm{Tr}_Z(\Gamma)\right)_{\max} = \left(\mathrm{Tr}_Z(\Gamma)\right)_{\max} = \sup\left\{\tau < \Gamma_{\max} : \nu(\Gamma_\tau^{\theta+\omega}, Z) = 0\right\}$$

and

$$\begin{aligned}(\mathrm{Tr}_W(\pi^*\Gamma))_{\max} &= \sup\left\{\tau < \Gamma_{\max} : \nu((\pi^*\Gamma_\tau)^{\pi^*\theta+\pi^*\omega}, W) = 0\right\}\\ &= \sup\left\{\tau < \Gamma_{\max} : \nu(\pi^*\Gamma_\tau^{\theta+\omega}, W) = 0\right\}\\ &= \sup\left\{\tau < \Gamma_{\max} : \nu(\Gamma_\tau^{\theta+\omega}, Z) = 0\right\}.\end{aligned}$$

Here we applied implicitly Proposition 13.1.5. Therefore,

$$\left(\tilde{\Pi}^* \mathrm{Tr}_Z(\Gamma)\right)_{\max} = \left(\mathrm{Tr}_W(\pi^*\Gamma)\right)_{\max}.$$

Let $\tau \in \mathbb{R}$ be less than this common value. Take a Kähler form ω (resp. ω') on $\tilde{Z}$ (resp. $\tilde{W}$). We may assume that $\omega' \geq \tilde{\Pi}^*\omega$. Take a Kähler form $\tilde{\omega}$ on Y (resp. $\tilde{\omega}'$ on X) such that

$$\omega' \geq \tilde{\omega}'|_{\tilde{W}}, \quad \omega \geq \tilde{\omega}|_{\tilde{Z}}.$$

Without loss of generality, we may assume that

$$\tilde{\omega}' \geq \pi^* \tilde{\omega}.$$

It suffices to show that

$$\operatorname{Tr}_W^{\pi^*\theta+\tilde{\omega}'} \left((\pi^*\Gamma)_\tau^{\pi^*\theta+\tilde{\omega}'} \right) \sim_P \tilde{\Pi}^* \operatorname{Tr}_Z^{\theta+\tilde{\omega}} \left[\Gamma_\tau^{\theta+\tilde{\omega}} \right].$$

Using Proposition 8.2.1, this is equivalent to

$$\operatorname{Tr}_W \left((\pi^*\Gamma)_\tau^{\pi^*\theta+\pi^*\omega} \right) \sim_P \tilde{\Pi}^* \operatorname{Tr}_Z \left[\Gamma_\tau^{\theta+\tilde{\omega}} \right].$$

This is a consequence of Lemma 8.2.1. □

13.3 Duistermaat–Heckman measures

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class.

Definition 13.3.1 Assume that X admits a smooth flag $Y_\bullet$. Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. The *Duistermaat–Heckman measure* $\mathrm{DH}(\Gamma)$ of Γ is defined as

$$\mathrm{DH}(\Gamma) := n! \cdot \mathrm{DH}\left(\Delta_{Y_\bullet}(\theta,\Gamma)\right).$$

Recall that $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$ is the Okounkov test curve defined in Theorem 10.4.2. See Definition 10.4.4 for the definition of the Duistermaat–Heckman measure of an Okounkov test curve.

Theorem 13.3.1 *Assume that X admits a smooth flag $Y_\bullet$. The Duistermaat–Heckman measure $\mathrm{DH}(\Gamma)$ of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ in Definition 13.3.1 is independent of the choice of the smooth flag $Y_\bullet$. Furthermore, for any $m \in \mathbb{Z}_{>0}$, the m-th moment of $\mathrm{DH}(\Gamma)$ is given by*

$$\int_{\mathbb{R}} x^m \,\mathrm{DH}(\Gamma)(x) = \Gamma_{\max}^m \operatorname{vol}\Gamma + m \int_{-\infty}^{\Gamma_{\max}} \tau^{m-1} \left(\operatorname{vol}(\theta + \mathrm{dd^c}\Gamma_\tau) - \operatorname{vol}\Gamma\right) \mathrm{d}\tau \quad (13.8)$$

if $m > 0$ and

$$\int_{\mathbb{R}} \mathrm{DH}(\Gamma) = \operatorname{vol}\Gamma. \quad (13.9)$$

Proof We observe that the moments of the random variable $G[\Delta_{Y_\bullet}(\theta,\Gamma)]$ as computed in Proposition 10.4.4 are independent of the choice of the flag: In fact, they are given by (13.8) and (13.9) thanks to Theorem 10.3.2(1).

Assume first that Γ is bounded. Since the Duistermaat–Heckman measure has bounded support in this case (c.f. Theorem 10.4.1), we conclude that $\mathrm{DH}(\Gamma)$ is uniquely determined.

In general, we may assume that $\Gamma_{\max} = 0$. For each $\epsilon > 0$, we define $\Gamma^{\epsilon} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ as follows:

(1) Let $\Gamma^{\epsilon}_{\max} = 0$, and
(2) we set

$$\Gamma^{\epsilon}_{\tau} = \begin{cases} \phi, \text{ if } \tau \leq -\epsilon^{-1}, \\ P_{\theta}\left[(1+\epsilon\tau)\Gamma_{\tau} - \epsilon\tau\phi\right], \text{ if } \tau \in \left(-\epsilon^{-1}, 0\right). \end{cases}$$

Then it follows from the argument of Theorem 9.2.1 Step 3.3 that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of $\Delta_{Y_\bullet}(\Gamma^{\epsilon})_\tau$ for any $\tau < \Gamma_{\max}$ as $\epsilon \to 0+$. So $\mathrm{DH}(\Gamma^{\epsilon}) \rightharpoonup \mathrm{DH}(\Gamma)$ by Lemma 10.4.2. It follows that $\mathrm{DH}(\Gamma)$ is independent of the choice of the flag. □

More generally, when X does not admit a smooth flag, we could make a modification $\pi\colon Y \to X$ so that Y admits a flag. We define

$$\mathrm{DH}(\Gamma) := \mathrm{DH}(\pi^*\Gamma). \tag{13.10}$$

It follows from Theorem 10.3.2(5) that this measure is independent of the choice of π.

Proposition 13.3.1 *Let $(\Gamma^i)_{i\in I}$ be a net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Assume one of the following conditions holds:*

(1) *The net $(\Gamma^i)_{i\in I}$ is decreasing and $\Gamma = \inf_{i\in I}\Gamma^i$. Assume that*

$$\operatorname{vol}\Gamma = \lim_{i\in I} \operatorname{vol}\Gamma^i. \tag{13.11}$$

(2) *The net $(\Gamma^i)_{i\in I}$ is increasing and $\Gamma = \sup_{i\in I}{}^*\Gamma^i$.*

Then

$$\mathrm{DH}(\Gamma^i) \rightharpoonup \mathrm{DH}(\Gamma). \tag{13.12}$$

Proof We may assume that X admits a smooth flag $Y_\bullet$.

Assume (1). Note that (13.11) implies that

$$\Gamma_{-\infty} = \inf_{i\in I} \Gamma^i_{-\infty}.$$

We want to derive (13.12) from Lemma 10.4.2. It boils down to prove the following: For any $\tau < \Gamma_{\max}$, we have

$$\Delta_{Y_\bullet}(\theta, \Gamma^i_\tau) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta, \Gamma_\tau).$$

This follows immediately from Theorem 10.3.2(1) and Proposition 3.1.9.

The proof under the assumption (2) is similar. We only need to apply Lemma 10.4.3 instead of Lemma 10.4.2. □

Definition 13.3.2 When $[\theta]$ is a Hodge class and Γ is induced by a test configuration as in Example 9.3.1 and Remark 9.3.1, our Duistermaat–Heckman measure coincides with the more traditional definition of [BHJ17, Section 3.2]. This is explained in [Xia21, Remark 7.17].

13.4 Comparison with Boucksom–Jonsson's theory

13.4.1 A brief recap of Boucksom–Jonsson's theory

In this section, we briefly recall the non-Archimedean global pluripotential theory à la Boucksom–Jonsson [BJ22a]. As our presentation is far from being complete, the readers are strongly recommended to read their original paper before reading the current section.

13.4.1.1 Valuations

Let X be an irreducible reduced variety over $\mathbb{C}$ of dimension n. We recall the notion of Berkovich analytification X^{An} of X with respect to the trivial valuation on $\mathbb{C}$.

Definition 13.4.1 A (real-valued) *valuation* on X (or a *valuation* of $\mathbb{C}(X)$) is a map $v\colon \mathbb{C}(X) \to (-\infty, \infty]$ satisfying the following conditions:

(1) For $f \in \mathbb{C}(X)$, $v(f) = \infty$ if and only if $f = 0$;
(2) For $f, g \in \mathbb{C}(X)$, $v(fg) = v(f) + v(g)$;
(3) For $f, g \in \mathbb{C}(X)$, $v(f + g) \geq v(f) \wedge v(g)$.

The set of valuations on X is denoted by X^{val}. The center of a valuation v is the scheme-theoretic point $c = c(v)$ of X such that $v \geq 0$ on $\mathcal{O}_{X,c}$ and $v > 0$ on the maximal ideal $\mathfrak{m}_{X,c}$ of $\mathcal{O}_{X,c}$. The center is unique if exists. It exists if X is proper.

In the remaining of this section, we assume that X is projective.

As a set, X^{An} is the set of *semi-valuations* on X, in other words, real-valued valuations v on irreducible reduced subvarieties Y in X that is trivial on $\mathbb{C}$. We call Y the *support* of the semi-valuation v. In other words,

$$X^{\mathrm{An}} = \coprod_Y Y^{\mathrm{val}}.$$

We will write $v_{\mathrm{triv}} \in X^{\mathrm{An}}$ for the trivial valuation on X: $v_{\mathrm{triv}}(f) = 0$ for any $f \in \mathbb{C}(X)^{\times}$.

We endow X^{An} with the coarsest topology such that

(1) for any Zariski open subset $U \subseteq X$, the subset U^{An} of X^{An} consisting of semi-valuations whose supports meet U is open;

(2) for each Zariski open subset $U \subseteq X$ and each $f \in \mathrm{H}^0(U, O_X)$ (here O_X is the sheaf of regular functions), the map $|f| : U^{\mathrm{An}} \to \mathbb{R}$ sending v to $\exp(-v(f))$ is continuous.

See [Ber93] for more details.

We will be most interested in divisorial valuations. Recall that a *divisorial valuation* on X is a valuation of the form $t \operatorname{ord}_E$, where $t \in \mathbb{Q}_{>0}$ and E is a prime divisor over X. The set of divisorial valuations on X is denoted by X^{div}. When $\mathbb{Q}_{>0}$ is replaced by $\mathbb{R}_{>0}$, we can similarly define a space $X^{\mathrm{div}}_{\mathbb{R}}$.

Given any coherent ideal $\mathfrak{a}$ on X and any $v \in X^{\mathrm{An}}$, we define

$$v(\mathfrak{a}) \coloneqq \min\{v(f) : f \in \mathfrak{a}_{c(v)}\} \in [0, \infty], \tag{13.13}$$

where $c(v)$ is the center of the valuation v on X.

Given any valuation v on X, the Gauss extension of v is a valuation $\sigma(v)$ on $X \times \mathbb{A}^1$:

$$\sigma(v)\left(\sum_i f_i t^i\right) \coloneqq \min_i (v(f_i) + i). \tag{13.14}$$

Here t is the standard coordinate on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. The key property is that when v is a divisorial valuation, then so it $\sigma(v)$. See [BHJ17, Lemma 4.2].

13.4.1.2 Non-Archimedean plurisubharmonic functions

Let X be an irreducible complex projective variety of dimension n and L be a holomorphic pseudoeffective $\mathbb{Q}$-line bundle on X. Through the GAGA morphism $X^{\mathrm{An}} \to X$ of ringed spaces, L can be pulled-back to an analytic line bundle L^{An} on X. The purpose of this section is to study the psh metrics on L^{An}. We will follow the approach of [BJ22a], which avoids the direct treatment of L^{An} itself.

Following [BJ22a, Definition 2.18], we define $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L^{\mathrm{An}})$, the set of *(rational) generically finite Fubini–Study functions* $\phi : X^{\mathrm{An}} \to [-\infty, \infty)$, that are of the following form:

$$\phi = \frac{1}{m} \max_j \{\log |s_j| + \lambda_j\}. \tag{13.15}$$

Here $m \in \mathbb{Z}_{>0}$ is an integer such that L^m is a line bundle, the s_j's are a finite collection of non-vanishing sections in $\mathrm{H}^0(X, L^m)$, and $\lambda_j \in \mathbb{Q}$. We followed the convention of Boucksom–Jonsson by writing $\log |s_j|(v) = -v(s_j)$.

Definition 13.4.2 ([BJ22a, Definition 4.1]) A *plurisubharmonic metric* (or *psh metric* for short) on L^{An} is a function $\phi : X^{\mathrm{An}} \to [-\infty, \infty)$ that is not identically $-\infty$, and is the pointwise limit of a decreasing net $(\phi_i)_{i \in I}$, where $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L_i^{\mathrm{An}})$ for some $\mathbb{Q}$-line bundles L_i on X satisfying $c_1(L_i) \to c_1(L)$ in $\mathrm{NS}^1(X)_{\mathbb{R}}$.

The set of psh metrics on L^{An} is denoted by $\mathrm{PSH}(L^{\mathrm{An}})$. We endow $\mathrm{PSH}(L^{\mathrm{An}})$ with the topology of pointwise convergence on X^{div}. This topology is Hausdorff as functions in $\mathrm{PSH}(L^{\mathrm{An}})$ are completely determined by their restriction on X^{div}:

Theorem 13.4.1 ([BJ22a, Theorem 4.22]) *Let $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$ and $\psi : X^{\mathrm{An}} \to [-\infty, \infty)$ be an usc function. Assume that $\phi \leq \psi$ on X^{div}, then the same holds on X^{An}.*

Proposition 13.4.1 ([BJ22a, Theorem 4.7]) *Let $\phi, \phi' \in \mathrm{PSH}(L^{\mathrm{An}})$, then so is their pointwise maximum $\phi \vee \phi'$.*

Proposition 13.4.2 *Let H be an ample line bundle on X. Consider $\phi \in \mathrm{PSH}((L + H)^{\mathrm{An}})$. Assume that for each $m \in \mathbb{Z}_{>0}$, we have $\phi \in \mathrm{PSH}((L + m^{-1}H)^{\mathrm{An}})$, then $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$.*

This is a special case of [BJ22a, (PSH2)] on Page 45.

Next we note that we may use sequences instead of nets in the definition of $\mathrm{PSH}(L^{\mathrm{An}})$:

Theorem 13.4.2 ([BJ22a, Corollary 12.18]) *Let S be an ample line bundle on X. Let $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$. Then there is a sequence of rational numbers $\varepsilon_i \searrow 0$ and a decreasing sequence $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L + \varepsilon_i S)^{\mathrm{An}})$ such that ϕ is the pointwise limit of ϕ_i, as $i \to \infty$.*

The space $\mathrm{PSH}(L^{\mathrm{An}})$ inherits most of the expected properties of (Archimedean) psh functions ([BJ22a, Theorem 4.7]). However, the following compactness result is not known:

Conjecture 13.4.1 ([BJ22a, §5]) Assume that X is unibranch, then every bounded from above increasing net of elements in $\mathrm{PSH}(L^{\mathrm{An}})$ converges in $\mathrm{PSH}(L^{\mathrm{An}})$.

This prediction is equivalent to so-called envelope conjecture [BJ22a, Conjecture 5.14]: the regularized supremum of a bounded from above family of functions in $\mathrm{PSH}(L^{\mathrm{An}})$ lies in $\mathrm{PSH}(L^{\mathrm{An}})$. See [BJ22a, Theorem 5.11] for the proof of the equivalence. This conjecture is proved when X is smooth and L is nef in [BJ22a]. More recently, in [BJ22b], Boucksom–Jonsson further established the case when X is smooth and L is pseudoeffective.

13.4.2 The analytifications

Let X be a connected projective manifold of dimension n. Let θ be a closed smooth real $(1, 1)$-form on X representing a pseudoeffective cohomology class.

13.4.2.1 The transcendental setting

Definition 13.4.3 For $\varphi \in \mathrm{PSH}(X, \theta)$, we define the *analytification* $\varphi^{\mathrm{An}} : X^{\mathrm{An}} \to [-\infty, 0]$ as follows:

$$\varphi^{\mathrm{An}}(\nu) \coloneqq -\nu(\varphi) = -\lim_{k\to\infty} \frac{1}{k} \nu\left(\mathcal{I}(k\varphi)\right). \tag{13.16}$$

By Theorem 1.4.2 and Fekete's lemma, the limit in (13.16) exists.

Note that we can also write

$$\varphi^{\mathrm{An}}(\nu) = \inf_{k\in\mathbb{Z}_{>0}} -2^{-k}\nu\left(\mathcal{I}(2^k\varphi)\right). \tag{13.17}$$

When $\nu = t\operatorname{ord}_E$ for some prime divisor E over X, $\varphi^{\mathrm{An}}(\nu) = -t\nu(\varphi, E)$ by Proposition 1.4.4.

Definition 13.4.4 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. We define the *analytification* $\Gamma^{\mathrm{An}}\colon X^{\mathrm{div}} \to [-\infty,\infty)$ of Γ as follows: For any $\omega \in \mathrm{Käh}(X)$, we define

$$\Gamma^{\mathrm{An}}(\nu) \coloneqq \sup_{\tau\leq\Gamma_{\max}} \left(\Gamma_\tau^{\omega,\mathrm{An}}(\nu) + \tau\right). \tag{13.18}$$

Clearly, (13.18) is independent of the choice of ω.

Note that (13.18) can be equivalently written as

$$\Gamma^{\mathrm{An}}(\nu) = \sup_{\tau\leq\Gamma_{\max}} \left(\Gamma_\tau^{\omega,\mathrm{An}}(\nu) + \tau\right) = \sup_{\tau\in\mathbb{R}} \left(\Gamma_\tau^{\omega,\mathrm{An}}(\nu) + \tau\right)$$

with $(-\infty)^{\mathrm{An}}(\nu) = -\infty$ understood.

Proposition 13.4.3 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ with $\Gamma_{\max} \leq 0$. Let Ψ be the complexification of Γ^*. Then*

$$\Gamma^{\mathrm{An}}(\nu) = -\sigma(\nu)(\Psi) \quad \forall \nu \in X^{\mathrm{div}}. \tag{13.19}$$

See Definition 4.1.2 for the definition of the complexification $\Psi \in \mathrm{QPSH}(X\times\Delta)$. Note that since $\Gamma_{\max} \leq 0$, by Corollary 9.2.3 and Theorem 1.2.1, Ψ extends uniquely to a quasi-psh function on $X \times \Delta$.

Proof Recall that

$$\Psi(x,\delta) = \sup_{\tau\leq\Gamma_{\max}} \left(\psi_\tau(x) - \log|\delta|^2\tau\right) \quad \text{for } x\in X, \delta\in\Delta^*.$$

By (13.14), we have $\sigma(\nu)(\log|\delta|^2) = 1$ and $\sigma(\nu)(\Gamma_\tau) = \nu(\Gamma_\tau)$ for all $\tau \leq \Gamma_{\max}$. So we have that

$$\sigma(\nu)(\Gamma_\tau(x) - \log|\delta|^2\tau) = \nu(\Gamma_\tau) - \tau.$$

Lastly, since $\sigma(\nu)$ is a divisorial valuation on $X\times\Delta$, by Corollary 1.4.1, we conclude (13.19). ◻

Definition 13.4.5 Let $N \in \mathbb{N}$, and $A_0,\ldots,A_N$ be a finite collection of elements in $\mathrm{PSH}(X,\theta)$, and $\tau_0 > \tau_1 > \cdots > \tau_N$ be finitely many real numbers. Then the *piecewise linear curve* $A = (A_\tau)_{\tau\in\mathbb{R}}$ in $\mathrm{PSH}(X,\theta)\cup\{-\infty\}$ associated with these data is the affine interpolation of these data:

(1) $A_{\tau_i} = A_i$ for $i = 0, \dots, N$;
(2) $A_\tau = A_{\tau_N}$ for $\tau \leq \tau_N$;
(3) for $t \in (0,1)$ and $i = 0, \dots, N-1$, we have

$$A_{(1-t)\tau_i + t\tau_{i+1}} = (1-t)A_{\tau_i} + tA_{\tau_{i+1}};$$

(4) $A_\tau \equiv -\infty$ for $\tau > \tau_0$.

The *analytification* of A is the function $A^{\mathrm{An}} \colon X^{\mathrm{An}} \to [-\infty, \infty)$ defined as follows:

$$A^{\mathrm{An}}(v) := \sup_{\tau \leq \tau_0} \left(A^{\mathrm{An}}(v) + \tau\right) = \max_{i=0,\dots,N} \left(A^{\mathrm{An}}_{\tau_i}(v) + \tau_i\right) \quad \forall v \in X^{\mathrm{An}}. \tag{13.20}$$

We also say $A = (A_\tau)_{\tau \leq \tau_0}$ is a piecewise linear curve in $\mathrm{PSH}(X, \theta)$.

Remark 13.4.1 Note that $\tau \mapsto A_\tau$ is upper semicontinuous, but not necessarily concave. Let $(A'_\tau)_{\tau \in \mathbb{R}}$ be the upper concave envelope of $\tau \mapsto A_\tau$. Then it can be inductively constructed as follows:

(1) For $\tau \in (\tau_0, \infty)$, we let $A'_\tau \equiv -\infty$;
(2) we set $A'_{\tau_0} = A_{\tau_0}$;
(3) define inductively for $j = 0, \dots, N-1$ the following: For $\tau \in [\tau_{j+1}, \tau_j)$, we set

$$A'_\tau = \max_{i=j+1,\dots,N} \left(\frac{\tau_j - \tau}{\tau_j - \tau_i} A_{\tau_i} + \frac{\tau_j - \tau}{\tau_j - \tau_i} A'_{\tau_j}\right) \vee A'_{\tau_j};$$

(4) for $\tau \in (-\infty, \tau_N)$, we set $A'_\tau = A_{\tau_N}$.

This construction is just a reformulation of the general formula Proposition A.1.2.

In particular, $A'_\tau \in \mathrm{PSH}(X, \theta)$ for all $\tau \leq \tau_0$.

Note that A' is not necessarily piecewise linear.

Lemma 13.4.1 *Let A be a piecewise linear curve in* $\mathrm{PSH}(X, \theta)$. *Let $(A'_\tau)_{\tau \in \mathbb{R}}$ be the upper concave envelope of $\tau \mapsto A_\tau$. Then $\tilde{A} := (P_\theta[A'_\tau]_{\mathcal{I}})_{\tau < \tau_0} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. Moreover,*

$$A^{\mathrm{An}} = \tilde{A}^{\mathrm{An}} \quad \textit{on } X^{\mathrm{div}}. \tag{13.21}$$

Here τ_0 is as in Definition 13.4.5.

Proof We continue to use the notations in Definition 13.4.5. The fact that $\tilde{A} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ follows from Remark 13.4.1. In order to prove (13.21), we fix $v \in X^{\mathrm{div}}$. By Remark 13.4.1,

$$\tau \mapsto \left(P_\theta[A'_\tau]_{\mathcal{I}}\right)^{\mathrm{An}}(v) = \left(A'_\tau\right)^{\mathrm{An}}(v)$$

is just the upper concave envelope of

$$\tau \mapsto A^{\mathrm{An}}_\tau(v).$$

Therefore, (13.21) follows. $\square$

13.4.2.2 The algebraic setting

Let L be a $\mathbb{Q}$-line bundle on X and h be a Hermitian metric on L with $\theta = c_1(L, \theta)$.

Lemma 13.4.2 *For any $\varphi \in \mathrm{PSH}(X, \theta)$ we have that $\varphi^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$.*

Proof After replacing L with a sufficiently high power, we may assume that L is a line bundle. Take a very ample line bundle H on X. By Siu's uniform global generation theorem [Siu98], [Dem12a, Theorem 6.27] there exists $b > 0$ large enough so that $H^b \otimes L^k \otimes \mathcal{I}(k\varphi)$ is globally generated for all $k > 0$. Let $\{s_i\}_i$ be a finite set of global sections that generate the sheaf $H^b \otimes L^k \otimes \mathcal{I}(k\varphi)$. Then

$$v(\mathcal{I}(k\varphi)) = \min_i v(s_i).$$

It follows that $v \mapsto -k^{-1} v\,(\mathcal{I}(k\varphi))$ lies in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L + \frac{b}{k}H)^{\mathrm{An}})$. Using (13.17), we conclude that $\varphi^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$. □

Lemma 13.4.3 *Let Γ be a piecewise linear curve in $\mathrm{PSH}(X, \theta)$. Then $\Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$.*

Proof The result follows from (13.20), Proposition 13.4.1 and Lemma 13.4.2. □

Lemma 13.4.4 *Let R be a commutative $\mathbb{C}$-algebra of finite type and I be an ideal of $R[t]$. If for any $a \in S^1$, $a^* I \subseteq I$, then I is stable under the $\mathbb{C}^*$-action. Moreover, there are ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m$ in R so that*

$$I = I_0 + I_1 t + \cdots + I_m(t^m), \tag{13.22}$$

Proof It suffices to argue that I can be expanded as in (13.22). To see this, assume that $a \in I$. We can write $a = a_0 + a_1 t + \cdots + a_m t^m$ with $a_i \in R$. Then our assumption implies that $\sum_i a_i \rho^i t^i \in I$ as well for all $\rho \in S^1$. So by the Lagrange interpolation formula, $a_i t^i \in I$ for all i. Therefore, we can write I as $I_0 + I_1 t + I_2 t^2 + \dots$ for some ideals $I_0 \subseteq I_1 \subseteq \dots$ in R. But as R is noetherian, there is $m \geq 0$ so that $I_{m'} = I_m$ for $m' > m$. (13.22) follows. □

Lemma 13.4.5 *Let X be a complex projective variety and $p : X \times \mathbb{C} \to X$ be the natural projection. Assume that $\mathcal{I}$ is an analytic coherent ideal sheaf on $X \times \mathbb{C}$. Assume that $\mathcal{I}|_{X \times \mathbb{C}^*} = p^* \mathcal{J}$ for some coherent ideal sheaf $\mathcal{J}$ on X. Then $\mathcal{I}$ is the analytification of an algebraic coherent ideal sheaf.*

Proof Let $q : X \times (\mathbb{P}^1 \setminus \{0\}) \to X$ be the natural projection. As $\mathbb{C}^* \subset \mathbb{P}^1 \setminus \{0\}$ we can glue $q^* \mathcal{J}$ with $\mathcal{I}$ to get an analytic coherent ideal sheaf on $X \times \mathbb{P}^1$. By the GAGA principle, this ideal sheaf is necessarily algebraic, hence so is its restriction to $X \times \mathbb{C}$. □

Next we point out a version of Siu's uniform global generatedness lemma [Siu88] that we will need in the proof of our next theorem:

Lemma 13.4.6 *Let L be a big line bundle on X such that $c_1(L) = \{\theta\}$ and $\Phi \in \mathrm{PSH}(X \times \Delta, p_1^*\theta)$, where Δ is the unit disk. Let G be an ample line bundle on X. Then there exists $k > 0$, only dependent on X and G such that $p_1^*(G^k \otimes L^m) \otimes \mathcal{I}(m\Phi)$ is globally generated for all $m \in \mathbb{N}$.*

Proof The argument for this is exactly the same as the one in [BBJ21, Lemma 5.6] with Nadal's vanishing replaced by the family version proved by Matsumura in [Mat18, Theorem 1.7]. □

Proposition 13.4.4 *Let $\phi \in \mathrm{PSH}(X, \theta)_{>0}$ be a model potential and $\ell \in \mathcal{R}(X, \theta; \phi)$ with $\sup_X \ell_1 \leq 0$. Let Φ be the complexification of ℓ. Then the function*

$$v \mapsto -\sigma(v)(\Phi) \quad \textit{for } v \in X^{\mathrm{div}}$$

admits a unique extension to an element in $\mathrm{PSH}(L^{\mathrm{An}})$.

Proof We may assume that L is a line bundle. Observe that the extension is unique if it exists by Theorem 13.4.1.

Let $p_1 : X \times \mathbb{C} \to X$ be the projection. Thanks to Proposition 1.4.5 and Lemma 8.4.3, for each $m \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(m\Phi)|_{X \times \Delta^*} = p_1^* \mathcal{I}(m\phi)|_{X \times \Delta^*}.$$

In particular, $\mathcal{I}(m\Phi)$ admits a $\mathbb{C}^*$-invariant extension to a coherent ideal sheaf on $X \times \mathbb{C}$, namely $\mathcal{I}(mp_1^*\phi)$.

From Lemma 13.4.4 and Lemma 13.4.5, we get that

$$\mathcal{I}(m\Phi) = \mathfrak{a}_0 + \mathfrak{a}_1 t + \cdots + \mathfrak{a}_{N-1} t^{N-1} + \mathfrak{a}_N (t^N), \tag{13.23}$$

where the $\mathfrak{a}_i$'s are coherent ideal sheaves on X.

Using Lemma 13.4.6, there exists $T \to X$ ample such that $p_1^* T \otimes L^m \otimes \mathcal{I}(m\Phi)$ is globally generated, which is equivalent to $T \otimes L^m \otimes \mathfrak{a}_i$ being globally generated for all i.[3]

We define

$$\varphi_m(v) := -\frac{1}{m}\sigma(v)(\mathcal{I}(m\Phi)) = -\frac{1}{m}\min_i(v(\mathfrak{a}_i) + i), \quad v \in X^{\mathrm{div}}.$$

From the right-hand side of the formula, φ_m can be extended to an element in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L + m^{-1}T)^{\mathrm{An}})$, which we denote by the same symbol.

For $v \in X^{\mathrm{div}}$,

$$-\sigma(v)(\Phi) = \lim_{m \to \infty} -\frac{1}{2^m}\sigma(v)(\mathcal{I}(2^m\Phi)) = \lim_{m \to \infty} \varphi_{2^m}(v)$$

and the right-hand side defines an element in $\mathrm{PSH}(L^{\mathrm{An}})$ by definition, since $\{\varphi_{2^m}\}_m$ is decreasing. □

[3] In contrast with the case where ϕ is bounded, explored in [BBJ21], $\mathfrak{a}_N \neq \mathcal{O}_X$ in general.

Corollary 13.4.1 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. *Then* Γ^{An} *defined in Definition 13.4.4 admits a unique extension to* $\mathrm{PSH}(L^{\mathrm{An}})$.

The extension will be denoted by the same notation Γ^{An}.

Proof Observe that the extension is unique if it exists by Theorem 13.4.1. We may assume that $\Gamma_{\max} = 0$ without loss of generality.

When $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, our assertion follows from Proposition 13.4.4 and Proposition 13.4.3.

In general, fix an ample line bundle H on X and a Kähler form $\omega \in c_1(H)$. Then we know that

$$\Gamma^{\mathrm{An}} = \left(\Gamma^{m^{-1}\omega}\right)^{\mathrm{An}} \in \mathrm{PSH}((L+m^{-1}H)^{\mathrm{An}})$$

for any $m \in \mathbb{Z}_{>0}$. Therefore, $\Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$ by Proposition 13.4.2. □

13.4.3 The comparison theorem

Let X be a connected projective manifold of dimension n. Let L be a pseudoeffective $\mathbb{Q}$-line bundle on X and h be a Hermitian metric on L with $\theta = c_1(L,h)$.

Thanks to Corollary 13.4.1, we already have a map

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \to \mathrm{PSH}(L^{\mathrm{An}}), \quad \Gamma \mapsto \Gamma^{\mathrm{An}}. \tag{13.24}$$

We observe that for $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and a Kähler form ω on X, we have

$$\left(P_{\theta+\omega}\left[\Gamma\right]_{\mathcal{I}}\right)^{\mathrm{An}} = \Gamma^{\mathrm{An}}.$$

Also observe that

$$\Gamma_{\max} = \Gamma^{\mathrm{An}}(v_{\mathrm{triv}}), \tag{13.25}$$

Lemma 13.4.7 *The map* (13.24) *is order preserving. Moreover, suppose that* $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfies that* $\Gamma^{\mathrm{An}} \leq \Gamma'^{\mathrm{An}}$, *then* $\Gamma \leq \Gamma'$.

In particular, the map (13.24) *is injective.*

Proof The map (13.24) is order preserving by definition. Let us take $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with $\Gamma^{\mathrm{An}} \leq \Gamma'^{\mathrm{An}}$. Fix a Kähler form ω on X.

Let $v \in X^{\mathrm{div}}$ and $t \in \mathbb{Q}_{>0}$. Then, using (13.18) we notice that

$$t\Gamma^{\mathrm{An}}\left(t^{-1}v\right) = \sup_{\tau\in\mathbb{R}}\left(\left(\Gamma^{\omega}_{\tau}\right)^{\mathrm{An}}(v) + t\tau\right). \tag{13.26}$$

A similar equality holds for Γ'. Therefore, by Corollary A.2.1, we have

$$\left(\Gamma^{\omega}_{\tau}\right)^{\mathrm{An}} \leq \left(\Gamma'^{\omega}_{\tau}\right)^{\mathrm{An}}$$

for all $\tau \in \mathbb{R}$. It follows that

$$\Gamma^{\omega}_{\tau} \leq \Gamma'^{\omega}_{\tau}$$

for all $\tau \in \mathbb{R}$. Our assertion follows. □

Lemma 13.4.8 *Let $\phi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L^{\mathrm{An}})$. Then there is a piecewise linear curve A in $\mathrm{PSH}(X,\theta)$ with $\phi = A^{\mathrm{An}}$. In particular, ϕ is in the image of* (13.24).

Note that from the proof below, the test curve Γ corresponding to ϕ satisfies the following: For any $\tau \leq \Gamma_{\max}$, Γ_τ is elementary. See Definition 6.1.3 for the definition of elementary metrics.

Proof Let us write

$$\phi = \frac{1}{m} \max_{i=1,\dots,M} \left(\log |s_i| + \lambda_i\right), \tag{13.27}$$

where $m \in \mathbb{Z}_{>0}$, $s_1, \dots, s_M$ are a finite number of sections of L^m and $\lambda_1, \dots, \lambda_M \in \mathbb{Q}$.

Write I_λ for the set of i such that $\lambda_i = \lambda$. We denote the finitely many λ so that I_λ is non-empty as $\tau_0 > \cdots > \tau_N$. For each $i = 0, \dots, N$, we write

$$A_{\tau_i} = \frac{1}{m} \max_{j \in I_{\tau_i}} \left(\log |s_j|^2_{h^m} + \tau_i\right).$$

We define A as the piecewise linear curve associated with the A_{τ_i}'s and the τ_i's. Then clearly $\phi = A^{\mathrm{An}}$.

The final assertion follows from Lemma 13.4.1. □

Proposition 13.4.5 *Let $(\Gamma_i)_{i \in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that* (13.5) *is satisfied. Then*

$$\left(\inf_{i \in I} \Gamma_i\right)^{\mathrm{An}} = \inf_{i \in I} \Gamma_i^{\mathrm{An}}.$$

Proof Take a Kähler form ω on X. We need to show that

$$\left(\inf_{i \in I} \Gamma_{i,\omega}\right)^{\mathrm{An}} = \inf_{i \in I} \Gamma_i^{\mathrm{An}}.$$

Therefore, after replacing θ by $\theta + \omega$, we may assume that $\Gamma_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$ and $\inf_{i \in I} \Gamma_i \in \mathrm{PSH}(X,\theta)_{>0}$. Fix $v \in X^{\mathrm{div}}$. By Theorem 13.4.1, it suffices to prove that

$$\sup_{\tau \in \mathbb{R}} \left(\left(\inf_{i \in I} \Gamma_{i,\tau}\right)^{\mathrm{An}} (v) + \tau \right) = \inf_{i \in I} \sup_{\tau \in \mathbb{R}} \left(\Gamma^{\mathrm{An}}_{i,\tau}(v) + \tau\right). \tag{13.28}$$

But thanks to Proposition 3.1.9, we have

$$\left(\inf_{i \in I} \Gamma_{i,\tau}\right)^{\mathrm{An}} (v) = \inf_{i \in I} \Gamma^{\mathrm{An}}_{i,\tau}(v),$$

so (13.28) is a consequence of Proposition A.2.3. □

Theorem 13.4.3 *The map* (13.24) *is an order preserving bijection.*

Proof The map (13.24) is an order preserving injection by Lemma 13.4.7. It remains to prove that it is surjective. Let $\phi \in \mathrm{PSH}(L^{\mathrm{NA}})$. We want to construct $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with $\Gamma^{\mathrm{An}} = \phi$.

Let H be an ample line bundle and $(\epsilon_i)_i$ be a decreasing sequence of rational numbers with limit 0, $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L+\epsilon_i H)^{\mathrm{An}})$ such that

$$\phi = \inf_{i>0} \phi_i .$$

The existence of these data is guaranteed by Theorem 13.4.2. Fix a Kähler form $\omega \in c_1(H)$,

Thanks to Lemma 13.4.8, we can find $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\epsilon_i\omega)$ with $(\Gamma^i)^{\mathrm{An}} = \phi_i$. It follows from Lemma 13.4.7 that

$$\Gamma^i \geq P_{\theta+\epsilon_i\omega}\left[\Gamma^{i+1}\right]_{\mathcal{I}} \geq \Gamma^{i+1}.$$

Therefore, for any $\omega' \in \mathrm{K\ddot{a}h}(X)$, the sequence $(P_{\theta+\omega'}\left[\Gamma^i\right]_{\mathcal{I}})_i$ is decreasing. We let

$$\Gamma^{\omega'} = \inf_{i>0} P_{\theta+\omega'}\left[\Gamma^i\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega').$$

Note that the infimum is defined thanks to (13.25). It follows from Proposition 13.4.5 that

$$\left(\Gamma^{\omega'}\right)^{\mathrm{An}} = \phi.$$

From this, it is clear that for $\omega', \omega'' \in \mathrm{K\ddot{a}h}(X)$ with $\omega' \leq \omega''$, we have

$$P_{\theta+\omega''}\left[\Gamma^{\omega'}\right]_{\mathcal{I}} = \Gamma^{\omega''}.$$

It follows that $(\Gamma^{\omega'})_{\omega'\in\mathrm{K\ddot{a}h}(X)}$ defines an element Γ in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\Gamma^{\mathrm{An}} = \phi$.□

Theorem 13.4.4 *Under the bijection Lemma 13.4.7, the operations on* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *defined in Section 13.2 all correspond to the corresponding operations on* $\mathrm{PSH}(L^{\mathrm{An}})$ *in Boucksom–Jonsson's theory.*

The meaning should be clear for all operations except for the trace operator, and the proofs are elementary, as we have seen in Proposition 13.4.5 in the case of infimum operator. We shall only restate and prove the case of trace operators, and leave the remaining arguments to the readers.[4]

Theorem 13.4.5 *Let $Y \subseteq X$ be an irreducible analytic subset. Consider an element $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with well-defined restriction to Y. Then*

$$\mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}|_{Y^{\mathrm{div}}} = \Gamma^{\mathrm{An}}|_{Y^{\mathrm{div}}}. \tag{13.29}$$

[4] In case you find any of the arguments non-trivial, please refer to [Xia23b] for the full details.

Observe that there is a canonical identification $Y^{\mathrm{div}} = \tilde{Y}^{\mathrm{div}}$. Recall that a generalized Fubini–Study metric is defined in Definition 1.8.7.

Proof We may assume that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Let $\phi = \Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$. By Lemma 13.4.9, $\phi(v_{Y,\mathrm{triv}}) \neq -\infty$.

Take an ample line bundle S on X, a Kähler form ω in $c_1(S)$. Write ϕ as the decreasing limit of a sequence ϕ^i of elements in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L+i^{-1}S)^{\mathrm{An}})$ as in Theorem 13.4.2.

Take $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+i^{-1}\omega)$ such that $\Gamma^{i,\mathrm{An}} = \phi^i$. Note that by Lemma 13.1.1, $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+i^{-1}\omega)_{>0}$.

It follows from Proposition 13.4.5 (applied to the images of Γ^i in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$) that for any $\tau < \Gamma_{\max}$, we have

$$\inf_{i\to\infty} \Gamma^i_\tau = \Gamma_\tau.$$

In particular, $\Gamma^i_\tau \xrightarrow{d_{S,\theta+\omega}} \Gamma_\tau$ for all $\tau < \Gamma_{\max}$.

By Lemma 13.4.9 again, each Γ^i has non-trivial restriction to E. By Proposition 8.2.2, for any Kähler form ω' on $\tilde{Y}$ satisfying $\omega' \geq \omega|_{\tilde{Y}}$ we have

$$\mathrm{Tr}_Y\left(\Gamma_\tau^{i,\theta|_{\tilde{Y}}+\omega'}\right) \xrightarrow{d_S} \mathrm{Tr}_Y\left(\Gamma_\tau^{\theta|_{\tilde{Y}}+\omega'}\right)$$

for any $\tau < (\mathrm{Tr}_Y(\Gamma))_{\max}$. Thanks to Theorem 6.2.4,

$$\mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}(c\,\mathrm{ord}_F) = \inf_{i\geq 1} \mathrm{Tr}_Y(\Gamma)^{i,\mathrm{An}}(c\,\mathrm{ord}_F)$$

for any $c\,\mathrm{ord}_F \in Y^{\mathrm{div}}$. In particular, it suffices to prove (13.29) with Γ^i in place of Γ.

In other words, we have reduced to the case where $\phi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ and L is big.

Let $\Gamma \in \mathrm{PSH}(X,\theta)_{>0}$ with $\Gamma^{\mathrm{An}} = \phi$. By Lemma 13.4.8, we can find a concave curve $(\Gamma'_\tau)_{\tau\leq\Gamma_{\max}}$ with Γ'_τ being a generalized Fubini–Study metric for each $\tau \leq \Gamma_{\max}$ and that

$$\Gamma_\tau = P_\theta[\Gamma'_\tau].$$

It follows that for any $c\,\mathrm{ord}_F \in E^{\mathrm{div}}$,

$$\begin{aligned}
\phi|_{Y^{\mathrm{An}}}(c\,\mathrm{ord}_F) &= \sup_{\tau<\Gamma_{\max}} \left({\Gamma'_\tau}^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\left(\Gamma'_\tau|_{\tilde{Y}}\right)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\mathrm{Tr}_Y(\Gamma'_\tau)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\mathrm{Tr}_Y(\Gamma_\tau)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}(c\,\mathrm{ord}_F).
\end{aligned}$$

The third equality follows from Proposition 8.2.1. It remains to justify the second line. Namely, we want to show that for any generalized Fubini–Study metric φ, we have

$$\varphi^{\mathrm{An}}(c\,\mathrm{ord}_F) = \left(\varphi|_{\tilde{Y}}\right)(c\,\mathrm{ord}_F). \tag{13.30}$$

We could immediately reduce to the case where φ is a Fubini–Study metric, and then to the case

$$\varphi = \log|s|^2_{h_0},$$

where s is a holomorphic section of L, not vanishing identically on Y, in which case (13.30) is obvious. □

Lemma 13.4.9 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $Y \subseteq X$ be an irreducible analytic subset. Then the following are equivalent:*

(1) $\Gamma^{\mathrm{An}}(v_{Y,\mathrm{triv}}) \neq -\infty$;
(2) $\Gamma^{\mathrm{An}}|_{Y^{\mathrm{An}}} \not\equiv -\infty$;
(3) Γ *has a well-defined restriction to Y.*

Here $v_{Y,\mathrm{triv}}$ denotes the trivial valuation of $\mathbb{C}(Y)$.

Proof The equivalence between (1) and (2) is a simple consequence of the maximum principle [BJ22a, Lemma 1.4(i)].

To see the equivalence between (1) and (3), it suffices to observe that for any $\varphi \in \mathrm{PSH}(X, \theta)$,

$$\varphi^{\mathrm{An}}(v_{Y,\mathrm{triv}}) = \begin{cases} -\infty, & \text{if } \nu(\varphi, Y) > 0; \\ 0, & \text{if } \nu(\varphi, Y) = 0. \end{cases}$$

Chapter 14
Partial Bergman kernels

I speak twelve languages: English is the bestest.
— Stefan Bergman

In this chapter, we prove the convergence of the partial Bergman kernels.

14.1 Partial envelopes

In this section, let X be a connected compact Kähler manifold of dimension n and $K \subseteq X$ be a closed non-pluripolar set. Let θ be a smooth closed real $(1,1)$-form on X representing a pseudoeffective cohomology class. Fix $\varphi \in \mathrm{PSH}(X,\theta)$.

Definition 14.1.1 Given a function $v\colon K \to [-\infty, \infty)$, we introduce the *relative P-envelope* of φ (with respect to K, v, θ) as

$$P_{\theta,K}[\varphi](v) \coloneqq \sup{}^* \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \leq \varphi\} . \tag{14.1}$$

Similarly, we define the *relative $\mathcal{I}$-envelope* of φ (with respect to K, v, θ) as

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) \coloneqq \sup{}^* \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \preceq_{\mathcal{I}} \varphi\} . \tag{14.2}$$

Observe that when v is bounded, neither envelope is identically $-\infty$. When $K = X$ and $v = 0$, these definitions reduce to the usual P-envelope and $\mathcal{I}$-envelope of φ studied in Chapter 3.

It would be helpful to consider the following auxiliary functions:

$$P'_{\theta,K}[\varphi](v) \coloneqq \sup \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \leq \varphi\} ,$$
$$P'_{\theta,K}[\varphi]_{\mathcal{I}}(v) \coloneqq \sup \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \preceq_{\mathcal{I}} \varphi\} .$$

We note the following maximum principles, that follow from the above definitions:

Lemma 14.1.1 *Let $v \in C^0(K)$. Let $\eta \in \mathrm{PSH}(X,\theta)$. Assume that $\eta \leq \varphi$, then*

$$\sup_K(\eta - v) = \sup_{\{\eta\neq-\infty\}}(\eta - P'_{\theta,K}[\varphi](v)) = \sup_{\{P'_{\theta,K}[\varphi](v)\neq-\infty\}}(\eta - P'_{\theta,K}[\varphi](v)). \quad (14.3)$$

Proof We prove the first equality at first. We write $S = \{\eta = -\infty\}$.

By definition, $P'_{\theta,K}[\varphi](v)|_K \leq v$, so

$$\left(h - P'_{\theta,K}[\varphi](v)\right)\Big|_{K\setminus S} \geq \eta|_{K\setminus S} - v|_{K\setminus S}\,.$$

This implies that

$$\sup_K(\eta - v) \leq \sup_{X\setminus S}(\eta - P'_{\theta,K}[\varphi](v)).$$

Conversely, observe that $\sup_K(\eta - v) > -\infty$ as K is non-pluripolar. Let $\eta' := \eta - \sup_K(\eta - v)$, then η' is a candidate in the definition of $P'_{\theta,K}[\varphi](v)$, hence $\eta' \leq P'_{\theta,K}[\varphi](v)$, namely,

$$\eta - \sup_K(\eta - v) \leq P'_{\theta,K}[\varphi](v),$$

the latter implies that

$$\sup_K(\eta - v) \geq \sup_{X\setminus S}(\eta - P'_{\theta,K}[\varphi](v)),$$

finishing the proof of the first identity.

We have $\{P'_{\theta,K}[\varphi](v) = -\infty\} \subseteq S$, and we notice that points in $S\setminus\{P'_{\theta,K}[\varphi](v) = -\infty\}$ do not contribute to the supremum of $\eta - P'_{\theta,K}[\varphi](v)$ on $X\setminus\{P'_{\theta,K}[\varphi](v) = -\infty\}$, hence the last equality of (14.3) also follows. □

Next, we make the following observations about the singularity types of our envelopes:

Lemma 14.1.2 *For any $v \in C^0(K)$ we have*

$$P_{\theta,K}[\varphi](v) \sim P_\theta[\varphi], \quad P_{\theta,K}[\varphi]_{\mathcal{I}}(v) \sim P_\theta[\varphi]_{\mathcal{I}}.$$

If φ has analytic singularities, we have

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[\varphi]_{\mathcal{I}}(v). \quad (14.4)$$

Proof Let $C > 0$ such that $-C \leq v \leq C$. Then

$$P_\theta[\varphi] - C \leq P_{\theta,K}[\varphi](v).$$

Since K is non-pluripolar, for $\eta \in \mathrm{PSH}(X,\theta)$ the condition $\eta|_K \leq v \leq C$ implies that $\eta \leq \tilde{C}$ on X for some $\tilde{C} := \tilde{C}(C,K) > 0$ by Remark 1.5.2. This implies that

$$P_{\theta,K}[\varphi](v) \leq P_\theta[\varphi] + \tilde{C},$$

giving

$$P_{\theta,K}[\varphi](v) \sim P_\theta[\varphi].$$

The exact same argument applies in case of the relative $\mathcal{I}$-envelope.

Next assume that φ has analytic singularities, then we have that

$$\varphi \sim P_\theta[\varphi]_{\mathcal{I}}$$

by Proposition 3.2.9. In particular, for $\eta \in \mathrm{PSH}(X,\theta)$, $\eta \leq \varphi$ if and only if $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$. So (14.4) follows. □

Corollary 14.1.1 *Let $v \in C^0(X)$. Then*

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) = P_{\theta,X}[P_{\theta,K}[\varphi]_{\mathcal{I}}(v)]_{\mathcal{I}}(v).$$

Proof By definition, we have

$$\begin{aligned}
&P_{\theta,X}[P_{\theta,K}[\varphi]_{\mathcal{I}}(v)]_{\mathcal{I}}(v)\\
=&\sup{}^*\left\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v, \eta \leq_{\mathcal{I}} P_{\theta,K}[\varphi]_{\mathcal{I}}(v)\right\}\\
=&\sup{}^*\left\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v, \eta \leq_{\mathcal{I}} \varphi\right\}\\
=&P_{\theta,K}[\varphi]_{\mathcal{I}}(v),
\end{aligned}$$

where we applied Lemma 14.1.2 on the third line. □

Lemma 14.1.3 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Let $v \in C^0(K)$. Let $S \subseteq X$ be a pluripolar set and $\eta \in \mathrm{PSH}(X,\theta)_{>0}$ with $\eta \leq \varphi$. Assume that $\eta|_{K\setminus S} \leq v|_{K\setminus S}$, then $\eta \leq P_{\theta,K}[\varphi](v)$.*

Proof By Theorem 1.1.5, there is $\chi \in \mathrm{PSH}(X,\theta)$, such that $\chi|_S \equiv -\infty$. We claim that we can choose χ so that

$$\chi \leq \eta.$$

In fact, since $\int_X \theta_\eta^n > 0$, fixing some χ and $\epsilon \in (0,1)$ small enough, we have

$$\int_X \theta^n_{\epsilon\chi+(1-\epsilon)V_\theta} + \int_X \theta^n_\eta > \int_X \theta^n_{V_\theta}.$$

Thus, by Proposition 3.1.4, we have

$$(\epsilon\chi + (1-\epsilon)V_\theta) \wedge \eta \in \mathrm{PSH}(X,\theta).$$

It suffices to replace χ by $(\epsilon\chi + (1-\epsilon)V_\theta) \wedge \eta$.

Fix $\chi \leq \eta$ as above. For any $\delta \in (0,1)$, we have

$$(1-\delta)\eta|_K + \delta\chi|_K \leq v, \quad (1-\delta)\eta + \delta\chi \leq \varphi.$$

Hence,

$$(1-\delta)\eta + \delta\chi \leq P_{\theta,K}[\varphi](v).$$

Letting $\delta \to 0+$, we conclude that $\eta \leq P_{\theta,K}[\varphi](v)$. □

Corollary 14.1.2 *Assume that* $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. *Let* $v \in C^0(K)$. *Then*

$$P_{\theta,K}[\varphi](v) = P_{\theta,X}[\varphi]\left(P_{\theta,K}[V_\theta](v)\right).$$

Proof It is clear that

$$P_{\theta,K}[\varphi](v) \leq P_{\theta,X}[\varphi]\left(P_{\theta,K}[V_\theta](v)\right).$$

For the reverse direction, it suffices to prove that any $\eta \in \mathrm{PSH}(X,\theta)$ such that

$$\eta \leq \varphi, \quad \eta \leq P_{\theta,K}[V_\theta](v),$$

we have

$$\eta \leq P_{\theta,K}[\varphi](v). \tag{14.5}$$

As φ has positive mass, we can assume that η has positive mass as well. Let

$$S = \left\{P_{\theta,K}[V_\theta](v) > P'_{\theta,K}[V_\theta](v)\right\}.$$

By Proposition 1.2.5, S is a pluripolar set. Observe that

$$\eta|_{K\setminus S} \leq v|_{K\setminus S}.$$

Hence, (14.5) follows from Lemma 14.1.3. □

The next result motivates our terminology to call the measures $\theta^n_{P_{\theta,K}[\varphi](v)}$ the *partial equilibrium measures* of our context:

Lemma 14.1.4 *Let* $v \in C^0(K)$. *Then*

$$\int_{X\setminus K} \theta^n_{P_{\theta,K}[\varphi](v)} = 0.$$

Moreover, $P_{\theta,K}[\varphi](v)|_K = v$ *almost everywhere with respect to* $\theta^n_{P_{\theta,K}[\varphi](v)}$. *More precisely, we have*

$$\theta^n_{P_{\theta,K}[\varphi](v)} \leq \mathbb{1}_{K\cap\{P_{\theta,K}[\varphi](v)=P_{\theta,K}[V_\theta](v)=v\}}\,\theta^n_{P_{\theta,K}[V_\theta](v)}. \tag{14.6}$$

Proof **Step 1**. We address the case where $\varphi = V_\theta$.

Let $S \subseteq X$ be a closed pluripolar set, such that V_θ is locally bounded on $X \setminus S$. This is possible because we can always find a Kähler current with analytic singularities in the cohomology class $[\theta]$, as a consequence of Theorem 1.6.2.

For the first assertion, it suffices to show that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge any open ball $B \Subset X \setminus (S \cup K)$.

By Proposition 1.2.2, we can take an increasing sequence $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)$ such that

$$\eta_j \to P_{\theta,K}[V_\theta](v) \text{ almost everywhere}, \quad \eta_j|_K \leq v \text{ for all } j \geq 1.$$

By [BT82, Proposition 9.1], for each $j \geq 1$, we can find $\gamma_j \in \mathrm{PSH}(X,\theta)$, such that $(\theta + \mathrm{dd^c}\gamma_j|_B)^n = 0$ and w_j agrees with η_j outside B. Note that $(\gamma_j)_j$ is clearly increasing and

$$\gamma_j \geq \eta_j, \quad \gamma_j|_K \leq v.$$

for all $j \geq 1$.

It follows that γ_j converges to $P_{\theta,K}[V_\theta](v)$ almost everywhere as well. By Theorem 2.3.1, we find that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge B, as desired.

For the second assertion, let $x \in (X \setminus S) \cap K$ be a point such that $P_{\theta,K}[V_\theta](v)(x) < v(x) - \epsilon$ for some $\epsilon > 0$. Let B be a ball centered at x, small enough so that θ has a local potential on B, allowing us to identify θ-psh functions with psh functions (on B). By shrinking B, we can further guarantee

(1) $\overline{B} \subseteq X \setminus S$.
(2) $P_{\theta,K}[V_\theta](v)|_{\overline{B}} < v(x) - \epsilon$.
(3) $v|_{\overline{B}\cap K} > v(x) - \epsilon$.

Construct the sequences η_j, γ_j as above. On B, by choosing a local potential of θ, we may identify η_j, γ_j with the corresponding psh functions in a neighborhood of $\overline{B}$. By (2), we have $\gamma_j \leq v(x) - \epsilon$ on ∂B, hence by the comparison principle, $\gamma_j|_B \leq v(x) - \epsilon$. By (3), we have $\gamma_j|_{B\cap K} \leq v|_{B\cap K}$. Thus, we conclude that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge B, as in the previous paragraph.

Step 2. We handle the general case. We can assume $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Indeed, due to Lemma 14.1.2 and Theorem 2.3.2, we have that

$$\int_X \theta^n_{P_{\theta,K}[\varphi](v)} = \int_X \theta^n_\varphi.$$

Hence, there is nothing to prove if $\int_X \theta^n_\varphi = 0$.

By Corollary 14.1.2,

$$P_{\theta,K}[\varphi](v) = P_{\theta,X}[\varphi](P_{\theta,K}[V_\theta](v)).$$

Now [DDNL18b, Theorem 3.8] gives

$$\begin{aligned} \theta^n_{P_{\theta,K}[\varphi](v)} &\leq \mathbb{1}_{\{P_{\theta,K}[\varphi](v)=P_{\theta,K}[V_\theta](v)\}}\theta^n_{P_{\theta,K}[V_\theta](v)} \\ &\leq \mathbb{1}_{\{P_{\theta,K}[\varphi](v)=v\}}\theta^n_{P_{\theta,K}[V_\theta](v)}, \end{aligned}$$

where in the second inequality we have used Step 1. □

Corollary 14.1.3 *Let $v \in C^0(K)$.*

$$\begin{aligned} \int_{(X\setminus K)\cup\{P_{\theta,K}[\varphi](v)<v\}} \theta^n_{P_{\theta,K}[\varphi](v)} &= 0, \\ \int_{(X\setminus K)\cup\{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)<v\}} \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} &= 0. \end{aligned} \tag{14.7}$$

Proof The first equation in (14.7) follows from Lemma 14.1.4. For the second, we can assume that

$$\int_X \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} > 0, \tag{14.8}$$

otherwise there is nothing to prove. By definition, we have

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) = P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v).$$

Next we show that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) = P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}](v).$$

The $\geq$ direction is trivial. It remains to prove the reverse inequality. By Lemma 14.1.2, we get that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \sim P_\theta[\varphi]_{\mathcal{I}}.$$

Due to Proposition 1.2.5, we get that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \leq v$$

on $K \setminus S$, where $S \subseteq X$ is a pluripolar set. As a result, due to (14.8), Lemma 14.1.3 allows to conclude that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \leq P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}](v).$$

Since

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) = P_{\theta,K}[\varphi]_{\mathcal{I}}(v),$$

we get that the second equation in (14.7), using the first. □

Proposition 14.1.1 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Let $v \in C^0(K)$. Then*

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_\theta[\varphi]](v). \tag{14.9}$$

In particular,

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_{\theta,K}[\varphi](v)](v).$$

Proof The $\leq$ direction in (14.9) is obvious. We to prove the reverse inequality. As $P_{\theta,K}[\varphi](v)$ and $P_{\theta,K}[P_\theta[\varphi]](v)$ have the same singularity types by Lemma 14.1.2, by the domination principle [DDNL18b, Corollary 3.10], it suffices to show that

$$P_{\theta,K}[\varphi](v) \geq P_{\theta,K}[P_\theta[\varphi]](v) \text{ almost everywhere with respect to } \theta^n_{P_{\theta,K}[\varphi](v)}. \tag{14.10}$$

By (14.6),

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[V_\theta](v) = v$$

almost everywhere with respect to $\theta^n_{P_{\theta,K}[\varphi](v)}$. Hence,

$$P_{\theta,K}[P_\theta[\varphi]](v) = v$$

almost everywhere with respect to $\theta^n_{P_{\theta,K}[\varphi](v)}$. We conclude that

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_\theta[\varphi]](v).$$

Finally, (14.10) follows from Lemma 14.1.2 and (14.9). □

Definition 14.1.2 Given $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, the *partial equilibrium energy* functional $\mathcal{E}^\theta_{[\varphi],K} : C^0(K) \to \mathbb{R}$ of $v \in C^0(K)$ as follows

$$\mathcal{E}^\varphi_{\theta,K}(v) \coloneqq E_\theta^{P_\theta[\varphi]_\mathcal{I}}(P_{\theta,K}[\varphi]_\mathcal{I}(v)). \tag{14.11}$$

Recall that the energy $E_\theta^{P_\theta[\varphi]_\mathcal{I}}$ functional is defined in Definition 3.1.5.

Note that by Lemma 14.1.2, we have

$$P_{\theta,K}[\varphi]_\mathcal{I}(v) \in \mathcal{E}^\infty(X,\theta;P_\theta[\varphi]_\mathcal{I}),$$

so $\mathcal{E}^\varphi_{\theta,K}(v) \in \mathbb{R}$.

Proposition 14.1.2 *Let $K \subseteq X$ be a closed non-pluripolar set, $v, f \in C^0(K)$ and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Then $\mathbb{R} \ni t \mapsto \mathcal{E}^\varphi_{\theta,K}(v+tf)$ is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varphi_{\theta,K}(v+tf) = \int_K f\,\theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)} \tag{14.12}$$

for all $t \in \mathbb{R}$.

Proof We may assume that φ is $\mathcal{I}$-model by replacing φ by $P_\theta[\varphi]_\mathcal{I}$.

Note that it suffices to prove (14.12) at $t = 0$, which is equivalent to

$$\lim_{t\to 0}\frac{E^\varphi_\theta(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)) - E^\varphi_\theta(P_{\theta,K}[\varphi]_\mathcal{I}(v))}{t} = \int_K f\,\theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}. \tag{14.13}$$

By switching f to $-f$, we may assume that $t > 0$ in the above limit.

By the comparison principle [DDNL18b, Proposition 3.5] and Proposition 3.1.12, we find

$$\begin{aligned}
&E^\varphi_\theta(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)) - E^\varphi_\theta(P_{\theta,K}[\varphi]_\mathcal{I}(v))\\
=&\frac{1}{n+1}\sum_{i=0}^n\int_X(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\,\theta^i_{P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)}\wedge\theta^{n-i}_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}\\
\leq&\int_X(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\,\theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}.
\end{aligned}$$

By Lemma 14.1.4,

$$\int_X(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\,\theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)} \leq t\int_K f\,\theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}.$$

Thus, we get the inequality,

$$\overline{\lim_{t\to 0+}} \frac{E_\theta^\varphi(P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)) - E_\theta^\varphi(P_{\theta,K}[\varphi]_{\mathcal{I}}(v))}{t} \le \int_K f\,\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}.$$

Similarly, we have

$$\begin{aligned}
&E_\theta^\varphi(P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)) - E_\theta^\varphi(P_{\theta,K}[\varphi]_{\mathcal{I}}(v))\\
\ge &\int_X (P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf) - P_{\theta,K}[\varphi](v))\,\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)}\\
\ge &t\int_K f\,\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)}.
\end{aligned}$$

Together with the above, this implies (14.13). □

Lemma 14.1.5 *Fix a Kähler form ω on X. For $v \in C^0(K)$ there exists an increasing bounded sequence $(v_j^-)_j$ in $C^\infty(X)$ and a decreasing bounded sequence $(v_j^+)_j$ in $C^\infty(X)$, such that for all $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ and $\delta \in [0,1]$ we have*

(1) $P_{\theta+\delta\omega,X}[\varphi](v_j^+) \searrow P_{\theta+\delta\omega,K}[\varphi](v)$,
(2) $P_{\theta+\delta\omega,X}[\varphi](v_j^-) \nearrow P_{\theta+\delta\omega,K}[\varphi](v)$ *almost everywhere,*
(3) $\sup_X |v_j^-| \le C$, $\sup_X |v_j^+| \le C$ *for some constant C depending only on $\|v\|_{C^0(K)}$, K and $\theta+\omega$, and*
(4)
$$\lim_{j\to\infty} \mathcal{E}^\varphi_{\theta,K}(v_j^-) = \mathcal{E}^\varphi_{\theta,K}(v), \quad \lim_{j\to\infty} \mathcal{E}^\varphi_{\theta,K}(v_j^+) = \mathcal{E}^\varphi_{\theta,K}(v).$$

Proof We fix $\delta \in [0,1]$. First we prove the existence of $(v_j^-)_j$. Let

$$C_{K,v} := \sup\left\{\sup_X \eta : \eta \in \mathrm{PSH}(X,\theta+\omega), \eta|_K \le v\right\}.$$

Since K is non-pluripolar, we have that $C_{K,v} \in \mathbb{R}$. Now define $\tilde{v}\colon X \to \mathbb{R}$ as

$$\tilde{v}(x) = \begin{cases} v(x), & x \in K;\\ C_{k,v}+1, & x \in X \setminus K.\end{cases}$$

Since $\tilde{v}$ is lower semicontinuous, there exists an increasing and uniformly bounded sequence $(v_j^-)_j$ in $C^\infty(X)$, such that $v_j^- \nearrow \tilde{v}$.

Observe that $P_{\theta+\delta\omega,X}[\varphi](v_j^-)$ is increasing in $j \ge 1$, and

$$P_{\theta+\delta\omega,X}[\varphi](v_j^-) \le P_{\theta+\delta\omega,K}[\varphi](v).$$

To prove that

$$P_{\theta+\delta\omega,X}[\varphi](v_j^-) \nearrow P_{\theta+\delta\omega,K}[\varphi](v)$$

almost everywhere, let η be a candidate for $P_{\theta+\delta\omega,K}[\varphi](v)$ such that $\sup_K(\eta - v) < 0$. Then, since η is upper semicontinuous and $\eta < \tilde{v}$, by Dini's lemma there exists $j_0 > 0$

such that $\eta < v_j^-$ for $j \geq j_0$, i.e.

$$\eta \leq P_{\theta+\delta\omega,X}[\varphi](v_j^-),$$

proving existence of $(v_j^-)_j$.

Next, we prove the existence of $(v_j^+)_j$. Since

$$h \coloneqq P_{\theta+\omega,K}[V_{\theta+\omega}](v) \vee (\inf_K v - 1)$$

is usc, there exists a decreasing and uniformly bounded sequence $(v_j^+)_j$ in $C^\infty(X)$, such that $v_j^+ \searrow h$. Trivially,

$$\chi \coloneqq \lim_{j\to\infty} P_{\theta+\delta\omega,X}[\varphi](v_j^+) \geq P_{\theta+\delta\omega,K}[\varphi](v).$$

In particular, χ has positive mass, since it has the same singularity types as $P_{\theta+\delta\omega,K}[\varphi](v)$ by Lemma 14.1.2. We introduce

$$S \coloneqq \left\{P'_{\theta+\omega,K}[V_{\theta+\omega}](v) < P_{\theta+\omega,K}[V_{\theta+\omega}](v)\right\}.$$

By Proposition 1.2.5, S is a pluripolar set. Observe that

$$P_{\theta+\delta\omega,X}[\varphi](v_j^+) \leq v_j^+$$

for all $j \geq 1$. Thus, $\chi \leq h$. On the other hand, $h \leq v$ on $K \setminus S$. So in particular, $\chi|_{K\setminus S} \leq v|_{K\setminus S}$. By Lemma 14.1.2 we also have that $\chi \sim P_{\theta+\delta\omega,K}[\varphi](v)$. Hence, by Lemma 14.1.3,

$$\chi \leq P_{\theta+\delta\omega,K}[P_{\theta+\delta\omega,K}[\varphi](v)](v) = P_{\theta+\delta\omega,K}[\varphi](v),$$

where we also used the last statement of Proposition 14.1.1.

Finally observe that (4) follows from Lemma 14.1.2, Lemma 14.1.5 and Theorem 2.3.1. □

Proposition 14.1.3 *Let $K \subseteq X$ be a compact and non-pluripolar subset. Let $v \in C^0(K)$. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)_{>0}$ ($j \geq 1$) with $\varphi_j \xrightarrow{d_S} \varphi$. Then the following hold:*

(1) *If $\varphi_j \searrow \varphi$, then $P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) \searrow P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$ and $P_{\theta,K}[\varphi_j](v) \searrow P_{\theta,K}[u](v)$.*

(2) *If $\varphi_j \nearrow \varphi$ almost everywhere then $P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) \nearrow P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$ almost everywhere, and $P_{\theta,K}[\varphi_j](v) \nearrow P_{\theta,K}[\varphi](v)$ almost everywhere.*

Proof (1) By Theorem 6.2.1, we have

$$\lim_{j\to\infty}\int_X \theta^n_{\varphi_j} = \int_X \theta^n_\varphi.$$

It follows from Lemma 2.3.1 that there is a decreasing sequence $\epsilon_j \searrow 0$ with $\epsilon_j \in (0,1)$ and $\eta_j \in \mathrm{PSH}(X,\theta)$ such that

$$(1-\epsilon_j)\varphi_j + \epsilon_j\eta_j \le \varphi.$$

By the concavity similar to Proposition 3.2.10, we get

$$\begin{aligned}(1-\epsilon_j)P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) + \epsilon_j P_{\theta,K}[\eta_j]_{\mathcal{I}}(v) \le & P_{\theta,K}[(1-\epsilon_j)\varphi_j + \epsilon_j\eta_j]_{\mathcal{I}}(v) \\ \le & P_{\theta,K}[\varphi]_{\mathcal{I}}(v).\end{aligned}$$

Since $(\varphi_j)_j$ is decreasing, so is $(P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v))_j$, hence

$$\psi \coloneqq \lim_{j\to\infty} P_\theta[\varphi_j]_{\mathcal{I}}(v) \ge P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$$

exists. Since $\epsilon_j \to 0$ and $\sup_X P_{\theta,K}[\eta_j]_{\mathcal{I}}(v)$ is bounded, we can let $j \to \infty$ in the above estimate to conclude that

$$\psi = P_{\theta,K}[\varphi]_{\mathcal{I}}(v).$$

The same ideas yield that

$$P_{\theta,K}[\varphi_j](v) \searrow P_{\theta,K}[\varphi](v).$$

The proof of (2) is similar and is left to the readers. □

14.2 Quantization of partial equilibrium measures

Let X be a connected compact Kähler manifold of dimension n and L be a pseudoeffective line bundle on X. Let h be a Hermitian metric on L and set $\theta = c_1(L,h)$. Let (T, h_T) be a Hermitian line bundle on X. Take a Kähler form ω on X so that

$$\int_X \omega^n = 1.$$

14.2.1 Bernstein–Markov measures

Let $K \subseteq X$ be a closed non-pluripolar subset. Let v be a measurable function on K and let μ be a positive Borel probability measure on K. We introduce the following functions on $\mathrm{H}^0(X, L^k \otimes T)$ $(k \ge 1)$, with values possibly equaling ∞:

$$\begin{aligned}N^k_{v,\upsilon}(s) &\coloneqq \left(\int_K h^k \otimes h_T(s,s)\mathrm{e}^{-kv}\,\mathrm{d}\upsilon\right)^{1/2}, \\ N^k_{v,K}(s) &\coloneqq \sup_{K\setminus\{v=-\infty\}} \left(h^k \otimes h_T(s,s)\mathrm{e}^{-kv}\right)^{1/2}.\end{aligned}$$

We start with the following elementary observation:

Lemma 14.2.1 *Let $v_1 \leq v_2$ be two measurable functions on X. Assume that $\{v_1 = -\infty\} = \{v_2 = -\infty\}$. Then for any $s \in \mathrm{H}^0(X, L^k \otimes T)$ ($k \geq 1$), we have*

$$N^k_{v_1,K}(s) \geq N^k_{v_2,K}(s).$$

If υ puts no mass on $\{v = -\infty\}$ then we always have

$$N^k_{v,\upsilon}(s) \leq N^k_{v,K}(s). \tag{14.14}$$

Definition 14.2.1 A *weighted subset* of X is a pair (K, v) consisting of a closed non-pluripolar subset $K \subseteq X$ and a function $v \in C^0(K)$.

Definition 14.2.2 Let (K, v) be a weighted subset of X. A positive Borel probability measure υ on K is *Bernstein–Markov* with respect to (K, v) if for each $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$N^k_{v,K}(s) \leq C_\epsilon \mathrm{e}^{\epsilon k} N^k_{v,\upsilon}(s) \tag{14.15}$$

for any $s \in \mathrm{H}^0(X, L^k \otimes T)$ and any $k \in \mathbb{N}$. We write $\mathrm{BM}(K, v)$ for the set of Bernstein–Markov measures with respect to (K, v).

As pointed out in [BBWN11], any volume form on X is Bernstein–Markov with respect to (X, v), with $v \in C^\infty(X)$.

Proposition 14.2.1 *Assume that (K, v) is a weighted subset of X, then*

(1) *$N^k_{v,K}$ is a norm on $\mathrm{H}^0(X, L^k \otimes T)$.*
(2) *For any $\upsilon \in \mathrm{BM}(K, v)$, $N^k_{v,\upsilon}$ is a norm on $\mathrm{H}^0(X, L^k \otimes T)$.*

Proof (1) As v is bounded, $N^k_{v,K}$ is clearly finite on $\mathrm{H}^0(X, L^k \otimes T)$. In order to show that it is a norm, it suffices to show that for any $s \in \mathrm{H}^0(X, L^k \otimes T)$, $N^k_{v,K}(s) = 0$ implies that $s = 0$. In fact, we have $s|_K = 0$, hence $s = 0$ by the connectedness of X.

(2) As v is bounded, clearly $N^k_{v,\upsilon}$ is finite and satisfies the triangle inequality. Non-degeneracy follows from the fact that $N^k_{v,K}$ is a norm and (14.15). □

14.2.2 Partial Bergman kernels

In this section, we fix a weighted subset (K, v) of X and $\upsilon \in \mathrm{BM}(K, v)$.

Definition 14.2.3 For any $\varphi \in \mathrm{PSH}(X, \theta)$, we introduce the *partial Bergman kernels* of φ (with respect to (K, v)) as follows: For any $k \geq 0$, we introduce

$$\begin{aligned} B^k_{v,\varphi,\upsilon}(x) := \sup \Big\{ & h^k \otimes h_T(s, s)\mathrm{e}^{-kv}(x) : N^k_{v,\upsilon}(s, s) \leq 1, \\ & s \in \mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi)) \Big\}, \quad x \in K. \end{aligned} \tag{14.16}$$

We extend $B^k_{v,\varphi,\upsilon}$ to the whole X by setting it to be 0 outside K.

The *partial Bergman measures* of φ (with respect to (K,v)) are defined as

$$\beta^k_{v,\varphi,\upsilon} \coloneqq \frac{n!}{k^n} B^k_{v,\varphi,\upsilon}\,\mathrm{d}\upsilon \tag{14.17}$$

for each $k \geq 0$.

Observe that

$$\int_K \beta^k_{v,\varphi,\upsilon} = \frac{n!}{k^n} h^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)). \tag{14.18}$$

The goal of this section is to prove the following theorem:

Theorem 14.2.1 *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Let (K,v) be a weighed subset of X, let $\upsilon \in \mathrm{BM}(K,v)$. Then*

$$\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} \tag{14.19}$$

as $k \to \infty$.

Proposition 14.2.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. If $v \in C^\infty(X)$, then*

$$\beta^k_{v,\varphi,\omega^n} \rightharpoonup \theta^n_{P_{\theta,X}[\varphi]_{\mathcal{I}}(v)} = \theta^n_{P_{\theta,X}[\varphi](v)} \tag{14.20}$$

as $k \to \infty$.

Proof The equality part in (14.20) follows from Lemma 14.1.2. We start with noticing that as $k \to \infty$,

$$\beta^k_{v,\varphi,\omega^n} \leq \beta^k_{v,V_\theta,\omega^n} \rightharpoonup \theta^n_{P_{\theta,X}[V_\theta](v)} = \mathbb{1}_{\{v=P_{\theta,X}[V_\theta](v)\}}\theta^n_v,$$

where the convergence follows from [Ber09, Theorem 1.2], and the last identity is due to [DNT21, Corollary 3.4]. Let μ be the weak limit of a subsequence of $\beta^k_{v,\varphi,\omega^n}$, then we obtain that

$$\mu \leq \mathbb{1}_{\{v=P_{\theta,X}[V_\theta](v)\}}\theta^n_v\,. \tag{14.21}$$

Let $k \geq 0$, $s \in \mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))$ be a section such that $N^k_{v,\omega^n}(s,s) \leq 1$. Then by [Ber09, Lemma 4.1], there exists $C > 0$ such that

$$h^k \otimes h_T(s,s)\mathrm{e}^{-kv} \leq B^k_{v,\varphi,\omega^n} \leq B^k_{v,V_\theta,\omega^n} \leq k^n C.$$

This implies that

$$\frac{1}{k}\log h^k \otimes h_T(s,s) \leq v + \frac{\log C}{k} + n\frac{\log k}{k}.$$

We define φ_k as in Proposition 1.8.2. Take $\alpha_k \nearrow 1$ as in Proposition 1.8.2. Then

$$\frac{1}{k}\log h^k \otimes h_T(s,s) \leq \varphi_k \leq \alpha_k\varphi.$$

Let $\epsilon > 0$. We notice that $\frac{1}{k}\log h^k \otimes h_T(s,s) \in \mathrm{PSH}(X, \theta + \epsilon\omega)$ for all $k \geq k_0(\epsilon)$. In particular,

$$\frac{1}{k}\log h^k \otimes h_T(s,s) - \frac{\log C}{k} - n\frac{\log k}{k} \leq P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](\nu).$$

Now taking supremum over all candidates s, we obtain that

$$B^k_{\nu,\varphi,\omega^n} \leq Ck^n \mathrm{e}^{k(P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](\nu)-\nu)}, \quad k \geq k_0. \tag{14.22}$$

We claim that μ does not put mass on $\{P_{\theta+\epsilon\omega,X}[\varphi](\nu) < \nu\}$ for any $\epsilon > 0$. Since

$$P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](\nu) \searrow P_{\theta+\epsilon\omega,X}[\varphi](\nu)$$

by Proposition 14.1.3, we get that

$$\{P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](\nu) < \nu\} \nearrow \{P_{\theta+\epsilon\omega,X}[\varphi](\nu) < \nu\}.$$

As a result, to argue the claim, it suffices to show that μ does not put mass on the set $\{P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](\nu) < \nu\}$ for any k. Note that the latter set is open, hence (14.22) implies our claim.

Since φ has analytic singularities, we have that

$$P_{\theta+\epsilon\omega,X}[\varphi](\nu) \sim \varphi$$

for all $\epsilon \geq 0$ by Lemma 14.1.2 and Proposition 3.2.9. As a result,

$$P_{\theta+\epsilon\omega,X}[\varphi](\nu) \searrow P_{\theta,X}[\varphi](\nu),$$

and we can let $\epsilon \searrow 0$ to conclude that μ does not put mass on $\{P_{\theta,X}[\varphi](\nu) < \nu\} = \bigcup_{\epsilon>0}\{P_{\theta+\epsilon\omega,X}[\varphi](\nu) < \nu\}$. Putting this together with (14.21), we obtain that

$$\mu \leq \mathbb{1}_{\{P_{\theta,X}[\varphi](\nu)=\nu\}}\theta^n_\nu = \theta^n_{P_{\theta,X}[\varphi](\nu)},$$

where the last equality is due to [DNT21, Corollary 3.4]. Comparing total masses via (14.18) and Theorem 7.3.1, we conclude that $\mu = \theta^n_{P_{\theta,X}[\varphi](\nu)}$. As μ is an arbitrary cluster point of $\beta^k_{\nu,\varphi,\omega^n}$, we conclude that $\beta^k_{\nu,\varphi,\omega^n}$ converges weakly to $\theta^n_{P_{\theta,X}[\varphi](\nu)}$, as $k \to \infty$. □

Definition 14.2.4 Take $k \geq 0$ and $\varphi \in \mathrm{PSH}(X,\theta)$, let $\mathrm{Norm}(\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi)))$ be the space of Hermitian norms on the vector space $\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))$.

Let $\mathcal{L}_{k,\varphi}\colon \mathrm{Norm}(\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))) \to \mathbb{R}$ be the *partial Donaldson functional*:

$$\mathcal{L}_{k,\varphi}(H) = \frac{n!}{k^{n+1}}\log\frac{\mathrm{vol}\{s : H(s) \leq 1\}}{\mathrm{vol}\{s : N^k_{0,\omega^n}(s) \leq 1\}}, \tag{14.23}$$

where vol is simply the Euclidean volume.

Proposition 14.2.3 *Let $w, w' \in C^0(X)$ and $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current, then*

$$\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n})\right) = \mathcal{E}^\varphi_{\theta,X}(w) - \mathcal{E}^\varphi_{\theta,X}(w'). \tag{14.24}$$

In particular,

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) = \mathcal{E}^\varphi_{\theta,X}(w)\,. \tag{14.25}$$

Proof First observe that by Proposition 14.2.1, for any $k \geq 0$, N^k_{w,ω^n} and N^k_{w',ω^n} are both norms, hence the expressions inside the limit in (14.24) make sense.

To start, we make the following observation:

$$\begin{aligned}\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{k,\varphi}(N^k_{w+t(w'-w),\omega^n})\,\mathrm{d}t \\ &= \int_0^1 \int_X (w'-w)\,\beta^k_{w+t(w'-w),\varphi,\omega^n}\,\mathrm{d}t.\end{aligned}$$

By Proposition 14.2.2, we have

$$\lim_{k\to\infty} \int_X (w'-w)\,\beta^k_{w+t(w'-w),\varphi,\omega^n} = \int_X (w'-w)\,\theta^n_{P_{\theta,X}[\varphi](w+t(w'-w))}.$$

By Theorem 7.3.1, we have $|\int_X (w'-w)\beta^k_{w+t(w'-w),u,\omega^n}| \leq C\sup_X |w-w'|$. Hence, by the dominated convergence theorem we obtain that

$$\begin{aligned}\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n})\right) &= \int_0^1 \int_X (w'-w)\,\theta^n_{P_{\theta,X}[\varphi](w+t(w'-w))}\,\mathrm{d}t \\ &= \mathcal{E}^\varphi_{\theta,X}(w) - \mathcal{E}^\varphi_{\theta,X}(w'),\end{aligned}$$

where in the last line we have used Proposition 14.1.2.

Finally, (14.25) is just a special case of (14.24) with $w' = 0$. □

Lemma 14.2.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Let (K,ν) be a weighted subset of X. Let $\upsilon \in \mathrm{BM}(K,\nu)$. Then*

$$\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{\nu,K}) - \mathcal{L}_{k,\varphi}(N^k_{\nu,\upsilon})\right) = 0\,. \tag{14.26}$$

Proof This is a direct consequence of the definition of Bernstein–Markov measures (14.15). □

Corollary 14.2.1 *Let $w \in C^0(X)$, $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Then*

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{w,X}) = \mathcal{E}^\varphi_{\theta,X}(w).$$

Proof This follows from Lemma 14.2.2 and Proposition 14.2.3 and the fact that $\omega^n \in \mathrm{BM}(X, 0)$. □

Proposition 14.2.4 *Let $\varphi \in \mathrm{PSH}(X, \theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K, v), (K', v') be two weighted subsets of X. Then*

$$\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{v',K'})\right) = \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K'}(v'). \tag{14.27}$$

In particular,

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{v,K}) = \mathcal{E}^{\varphi}_{\theta,K}(v). \tag{14.28}$$

Proof First observe that by Proposition 14.2.1, for any $k > 0$, $N^k_{v,K}$ and $N^k_{v',K'}$ are both norms, hence the expressions inside the limit in (14.27) make sense. Moreover, (14.28) is just a special case of (14.27) for $K' = X$ and $v' = 0$.

To prove (14.27) it is enough to show that for any fixed $w \in C^\infty(X)$ we have

$$\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,\omega^n})\right) = \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,X}(w)\,. \tag{14.29}$$

For $\epsilon \in (0, 1)$ small enough we have that $\theta_{(1-\epsilon)\varphi}$ is still a Kähler current. Let us fix such ϵ, along with an arbitrary $\epsilon' \in (0, 1)$.

Let $(v_j^-)_j$, $(v_j^+)_j$ be the sequences of smooth functions constructed in Lemma 14.1.5 for the data (K, v).

By Proposition 1.8.2 there exists $k_0(\epsilon, \epsilon') \in \mathbb{N}$ such that

$$\frac{1}{k} \log h^k \otimes h_T(s, s) \le (1-\epsilon)u,$$

and $\frac{1}{k} \log h^k \otimes h_T(s, s) \in \mathrm{PSH}(X, \theta + \epsilon'\omega)$ for any $s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$, as long as $k \ge k_0(\epsilon, \epsilon')$.

In particular, Lemma 14.1.1 gives that

$$\begin{aligned}
N^k_{P'_{\theta+\epsilon'\omega,K}[(1-\epsilon)\varphi](v),X}(s) &= N^k_{v,K}(s),\\
N^k_{P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^-),X}(s) &= N^k_{v_j^-,X}(s),\\
N^k_{P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^+),X}(s) &= N^k_{v_j^+,X}(s).
\end{aligned}$$

As

$$P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^-) \le P'_{\theta+\epsilon'\omega,K}[(1-\epsilon)\varphi](v) \le P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^+),$$

by Lemma 14.2.1 we have

$$N^k_{v_j^+,X}(s) \le N^k_{v,K}(s) \le N^k_{v_j^-,X}(s), \quad s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)), k \ge k_0(\epsilon, \epsilon').$$

Composing with $\mathcal{L}_{k,\varphi}$ we arrive at

$$\mathcal{L}_{k,\varphi}(N^k_{v_j^-,X}) \leq \mathcal{L}_{k,\varphi}(N^k_{v,K}) \leq \mathcal{L}_{k,\varphi}(N^k_{v_j^+,X}),\ k \geq k_0(\epsilon, \epsilon').$$

For any $j > 0$, by Corollary 14.2.1 we get

$$\begin{aligned}
\mathcal{E}^{\varphi}_{\theta,X}(v_j^-) - \mathcal{E}^{\varphi}_{\theta,X}(w) &= \lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v_j^+,X}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&\leq \varliminf_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&\leq \varlimsup_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&\leq \lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v_j^-,X}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&=\mathcal{E}^{\varphi}_{\theta,X}(v_j^+) - \mathcal{E}^{\varphi}_{\theta,X}(w)\,.
\end{aligned}$$

Using Lemma 14.1.5, we can let $j \to \infty$ to arrive at

$$\begin{aligned}
\mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K}(w) &\leq \varliminf_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&\leq \varlimsup_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\right) \\
&\leq \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K}(w)\,.
\end{aligned}$$

Hence, (14.29) follows. □

Corollary 14.2.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K, v) be a weighted subset of X. Assume that $\upsilon \in \mathrm{BM}(K, v)$. Then*

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{v,\upsilon}) = \mathcal{E}^{\varphi}_{\theta,K}(v).$$

Proof Our claim follows from Proposition 14.2.4 and Lemma 14.2.2. □

Proposition 14.2.5 *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K, v) be a weighted subset of X. Let $\upsilon \in \mathrm{BM}(K, v)$. Then*

$$\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} = \theta^n_{P_{\theta,K}[\varphi](v)}$$

weakly as $k \to \infty$.

Proof For $w \in C^0(X)$, let

$$f_k(t) := \mathcal{L}_{k,\varphi}(N^k_{v+tw,\upsilon}), \quad g(t) := \mathcal{E}^{\varphi}_{\theta,K}(v + tw).$$

By Corollary 14.2.2 $\varliminf_{k\to\infty} f_k(t) = g(t)$. Note that f_k is concave by Hölder's inequality (see [BBWN11, Proposition 2.4]), so by [BB10, Lemma 7.6], $\lim_{k\to\infty} f_k'(0) = g'(0)$, which is equivalent to $\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi](v)}$, by Proposition 14.1.2. □

Proposition 14.2.6 *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ such that θ_φ is a Kähler current. Let (K,ν) be a weighted subset of X and $\upsilon \in \mathrm{BM}(K,\nu)$. Then*

$$\beta^k_{\nu,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)} \tag{14.30}$$

as $k \to \infty$.

Proof Let μ be the weak limit of a subsequence of $\beta^k_{\nu,\varphi,\upsilon}$. We claim that

$$\mu \leq \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)}. \tag{14.31}$$

Observe that this claim implies the conclusion. In fact, by Theorem 7.3.1, we have equality of the total masses, so equality holds in (14.31). As μ is an arbitrary cluster point of the sequence $(\beta^k_{\nu,\varphi,\upsilon})_k$, we get (14.30).

It remains to prove (14.31). Let (φ_j) be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We may assume that θ_{φ_j} is a Kähler current for all $j \geq 1$. By Lemma 14.1.2, Corollary 7.1.2, we know that

$$\varphi_j \xrightarrow{d_S} P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu).$$

In particular,

$$\lim_{j\to\infty} \int_X \theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(\nu)} = \int_X \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)}. \tag{14.32}$$

Observe that

$$\beta^k_{\nu,\varphi,\upsilon} \leq \beta^k_{\nu,\varphi_j,\upsilon}$$

for any $k \geq 1$. As $\upsilon \in \mathrm{BM}(K,\nu)$, by Proposition 14.2.5,

$$\mu \leq \theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(\nu)},$$

for any $j \geq 1$ fixed. By Proposition 14.1.3,

$$P_{\theta,K}[\varphi_j]_{\mathcal{I}}(\nu) \searrow P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)$$

as $j \to \infty$. Hence, by (14.32) and Theorem 2.3.1, (14.31) follows. □

Proof (Proof of Theorem 14.2.1) By Lemma 14.1.2, we have that

$$\begin{aligned}\mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(k\varphi)\right) &= \mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}})\right)\\ &= \mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(kP_{\theta,K}[\varphi]_{\mathcal{I}}(\nu))\right).\end{aligned}$$

This allows us to replace φ with $P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)$.

By Lemma 2.3.2, there exists $\varphi_j \in \mathrm{PSH}(X,\theta)$, such that $\varphi_j \nearrow \varphi$ a.e. and θ_{φ_j} is a Kähler current for each $j \geq 1$. This gives

$$\beta^k_{\nu,\varphi_j,\upsilon} \leq \beta^k_{\nu,\varphi,\upsilon}.$$

Let μ be the weak limit of a subsequence of $(\beta^k_{v,\varphi,\nu})_k$. Then by Proposition 14.2.6,

$$\theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)} \leq \mu.$$

By Proposition 14.1.3 and Theorem 2.3.1 we have that

$$\theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)} \nearrow \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}.$$

Hence,

$$\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} \leq \mu. \tag{14.33}$$

A comparison of total masses using (14.18) and Theorem 7.3.1 gives that equality holds in (14.33). As μ is an arbitrary cluster limit of the weak compact sequence $(\beta^k_{v,\varphi,\mu})_k$, we obtain (14.19). □

Remark 14.2.1 The results in this chapter could also be reformulated as the large deviation principle of a determinantal point process on X using the Gärtner–Ellis theorem exactly as in [Ber14]. We leave the details to the readers.

Comments

A brief history

Here we recall the origin of various results.

Chapter 1.

The notion of plurisubharmonic functions was introduced by Lelong [Lel45], based on F. Riesz's theory of subharmonic functions [Rie26]. See [Bre72] for an excellent introduction to the early history of the subject. We refer to [Bre65] for the foundations of potential theory and [GZ17] for the pluripotential theory.

The global Josefson theorem Theorem 1.1.5 was due to Vu [Vu19]. In the projective setting, it was due to Dinh–Sibony [DS06] and in the Kähler setting, it was established by Guedj–Zeriahi [GZ05].

The extension theorem Theorem 1.2.1 was proved in [GR56]. In fact, they proved a more general version for complex spaces, see Theorem B.2.2. For some related important extension theorems, see [Shi72, Wan22].

The plurifine topology was introduced by Fuglede during the Séminaire d'analyse de Lelong–Dolbeault–Skoda of the year 1983/1984 [LDS86] based on H. Cartan's works on the fine topology. The key result Theorem 1.3.2 was claimed in Bedford–Taylor's work [BT87, Theorem 2.3] without proof. The first rigorous proof was given by El Marzguioui–Wiegerinck [EMW06]. A weaker result was proved earlier in [Kli91, Theorem 4.8.7].

Results in Section 1.3.2 are certainly well-known and are already implicitly used in the literature. I could not find the proofs in the literature and hence all details are presented.

The strong openness was first established by Guan–Zhou [GZ15]. A more elegant proof was due to Hiep [Hie14].

The idea of Theorem 1.4.3 first appeared in the ground-breaking work of Boucksom–Favre–Jonsson [BFJ08].

Proposition 1.2.8 was due to Kiselman [Kis78].

The semicontinuity theorem was due to Siu [Siu74]

Chapter 2

The Monge–Ampère operators for bound plurisubharmonic functions were introduced by Bedford–Taylor [BT76, BT82]. The non-pluripolar product is due to Bedford–Taylor [BT87], Guedj–Zeriahi [GZ07] and Boucksom–Eyssidieux–Guedj–Zeriahi [BEGZ10].

Chapter 3

The notion of the P-envelope is due to Ross–Witt Nyström [RWN14] based on the ideas of Rashkovskii–Sigurdsson [RS05].

The $\mathcal{I}$-envelope was introduced by Darvas–Xia [DX22], inspired by the works of Dano Kim [Kim15] and Boucksom–Favre–Jonsson [BFJ08]. The notion of $\mathcal{I}$-model singularities was first formulated in the explicit way in [DX22] in 2020, although it was already essentially known in Boucksom–Jonsson's work. In fact, they correspond exactly to the homogeneous non-Archimedean potentials assuming that the relevant masses do not vanish. A less explicit equivalent formulation of $\mathcal{I}$-model potentials also appeared in [Dem15]. A few months later, the same notion was rediscovered by Trusiani [Tru22].

Proposition 3.1.4 was first proved in [DDNL21b].

Chapter 4

The notion of weak geodesics was studied in detail by Darvas [Dar17] in the Kähler case.

The case of general big classes was partly handled in [DDNL18c], [DDNL18a]. However, the key fact that the geodesics between two full mass potentials have the correct limit at the end points does not seem to have been proved in any references. We give a proof in Proposition 4.2.1. We also extend the relevant results to the relative setting.

Previously, Proposition 4.2.2 and Proposition 4.2.4 were only known in the Kähler case.

Chapter 5

The toric framework was first written down by Coman–Guedj–Sahin–Zeriahi in [CGSZ19].

The beautiful theorem Theorem 5.2.2 was first proved by Yi Yao, who did not publish the result. Later on, a new proof was found by Botero–Burgos Gil–Holmes–de Jong [BBGHdJ21]. We chose to present the approach of Yao, which integrates naturally with our framework.

Chapter 6

The notion of P-partial order is new, as well as most results in Section 6.1.

The d_S-pseudometric was introduced in [DDNL21b]. The basic properties are proved in [DDNL21b] and [Xia21].

Example 6.1.3 was due to Berman–Boucksom–Jonsson [BBJ21].

Theorem 6.2.4 is proved in [Xia22b]. Theorem 6.2.6 and Theorem 6.2.5 appear to be new. These results appeared previously in the form of lecture notes.

Chapter 7

The notion of $\mathcal{I}$-good singularities was due to [DX21]. The name *$\mathcal{I}$-good* was chosen in [Xia22b].

Theorem 7.1.1 and Theorem 7.3.1 are due to [DX21, DX22].

There are some further examples of $\mathcal{I}$-good singularities provided by [BBGHdJ21] with applications in the theory of modular forms in [BBGHdJ22].

Chapter 8

The trace operator was introduced in [DX24]. Here we present a different point of view. Theorem 8.3.1 was proved in [DX24].

The analytic Bertini theorem Theorem 8.4.1 was proved in [Xia22a], based on the works of Matsumura–Fujino [FM21] and [Fuj23]. A weaker result was established by Meng–Zhou [MZ23].

Chapter 9

The technique of test curves originates from [RWN14]. It was generalized by Darvas–Di Nezza–Lu [DDNL18a], [DX21], [DZ22] and [DXZ23]. We give the full details of the proofs.

Test curves in Definition 9.1.1 are called *maximal test curves* in the literature, a terminology which I do not like. I prefer to call the usual notion of test curves in the literature *sub-test curves*.

Proposition 9.2.2 was first proved by He–Testorf–Wang in [HTW23].

Results in Section 9.4 are easy generalizations of the results proved in [Xia23b].

Chapter 10

The algebraic theory of partial Okounkov bodies was developed in [Xia21]. The transcendental Okounkov body was first defined by Deng [Den17] as suggested by Demailly. The volume identity was proved in [DRWN+23]. The transcendental theory of partial Okounkov bodies is new. Results in Section 11.3 are also new.

Chapter 11

The applications of b-divisors in pluripotential theory began with [BFJ09]. The intersection theory of nef b-divisors was introduced by Dang–Favre [DF22]. The technique of singularity b-divisors was introduced in [Xia23c] in 2020. The general form first appeared in [Xia22b]. One year later, a special case was rediscovered in [BBGHdJ21]. In 2023, another special case was rediscovered by Trusiani [Tru23].

Chapter 12

The whole chapter appears to be new. The study of toric pluripotential theory on big line bundles was made possible by the development of partial Okounkov bodies. The key result is Theorem 12.2.2.

Most results in this chapter resulted from discussions with Yi Yao.

Chapter 13

Most results from this chapter are from [Xia23b]. Results from Section 13.3 are new, although the main idea was already contained in [Xia21].

Theorem 13.4.3 is due to [DXZ23]. An alternative approach to the transcendental theory is due to Mesquita-Piccione [MP24].

Special cases of the results in this section have been applied to study K-stability, see [Xia23c], [DZ22], [DXZ23] and [DR22]. In [DX22], we established the bijective correspondence between a class of $\mathcal{I}$-model test curves with the maximal geodesic rays in the sense of [BBJ21].

Chapter 14

The special case of Theorem 14.2.1 without the prescribed singularity φ was due to Berman–Boucksom–Witt Nyström, see [BB10], [BBWN11]. The general case is due to [DX21].

Open problems

We give a list of important open problem in this theory.

We do not repeat the conjectures mentioned in the main text.

Conjecture 14.2.1 Let X be a connected compact Kähler manifold and Y be a submanifold. Fix a Kähler class α on X. For each Kähler current $S \in \alpha|_Y$, we can find a Kähler current $T \in \alpha$ such that

$$\mathrm{Tr}_Y(T) \sim_{\mathcal{I}} S.$$

If we formally view Tr_Y as an analogue of the trace operator in the theory of Sobolev spaces, then this conjecture corresponds exactly to the Dirichlet problem.

Using Proposition 8.2.2, one could also reduce this conjecture to a strong version of the extension theorem Theorem 1.6.3.

Conjecture 14.2.2 Let X be a connected compact Kähler manifold and Y be a submanifold. Fix a Kähler class α on X. Consider Kähler currents $R \in \alpha$, $S \in \alpha|_Y$ with analytic singularities such that $S \preceq R|_Y$. Assume in addition that S has gentle analytic singularities. Then there is a Kähler current $T \in \alpha$ with analytic singularities such that

$$\mathrm{Tr}_Y(T) \sim_{\mathcal{I}} S, \quad T \preceq R.$$

This conjecture was proposed by Darvas for different purposes.

Conjecture 14.2.3 Let X be a connected smooth projective variety of dimension n. Assume that (L_i, h_i) is a Hermitian big line bundle on X for each $i = 1, \ldots, n$ with the h_i's being $\mathcal{I}$-good. Then

$$\int_X c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_n, h_n) = \sup_{\nu} \mathrm{vol}\left(\Delta_\nu(L_1, h_1), \ldots, \Delta_\nu(L_n, h_n)\right),$$

where $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ runs over all (surjective) valuation of rank n.

See [Sch93, Section 5.1] for the notion of mixed volumes.

This conjecture seems reasonable in view of Corollary 10.2.3 and Corollary 10.2.2.

Even when $h_1, \ldots, h_n$ have minimal singularities, this conjecture remains open:

Conjecture 14.2.4 Let X be a connected smooth projective variety of dimension n. Assume that $L_1, \ldots, L_n$ are big line bundles on X. Then

$$\langle L_1, \ldots, L_n \rangle = \sup_{\nu} \mathrm{vol}\left(\Delta_\nu(L_1), \ldots, \Delta_\nu(L_n)\right),$$

where $\nu\colon \mathbb{C}(X)^{\times} \to \mathbb{Z}^n$ runs over all (surjective) valuation of rank n.

Here on the left-hand side, we are using the movable intersection theory [BDPP13].

Problem 14.2.1 Is it possible to extend the definition of the trace operator Tr_Y to the case where the ambient variety is only unibranch?

The difficulty lies in the lack of Demailly type regularization theorems.

Problem 14.2.2 What is the relation between the Duistermaat–Heckman measure in Section 13.3 and the definition in [Ino22]?

Problem 14.2.3 Is there a natural definition of the transcendental Okounkov body of a closed positive $(1,1)$-current T with 0-mass so that its dimension is equal to the numerical dimension of T?

See [Cao14] for the definition of the numerical dimension of a current.

The following two problems are proposed by Witt Nyström.

Problem 14.2.4 Consider a compact Kähler manifold X and a connected submanifold Y. We have defined the trace operator Tr_Y from a subset of $\mathrm{QPSH}(X)/\sim_{\mathcal{I}}$ to $\mathrm{QPSH}(Y)/\sim_{\mathcal{I}}$. Is it possible to refine this operator to one from a subset of $\mathrm{QPSH}(X)/\sim_P$ to $\mathrm{QPSH}(Y)/\sim_P$?

Problem 14.2.5 Consider a connected compact Kähler manifold X of dimension n and a smooth flag $Y_\bullet$ on X. Consider closed smooth real $(1,1)$-form θ on X representing a big cohomology class and $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\varphi^n > 0$.

Can one define a refined notion of partial Okounkov bodies $\Delta'_{Y_\bullet}(\theta+\mathrm{dd^c}\varphi)$ contained in $\Delta_{Y_\bullet}(\theta+\mathrm{dd^c}\varphi)$ with volume given by $\frac{1}{n!}\int_X \theta_\varphi^n$?

We also look for generalizations of our theory to more general settings.

Problem 14.2.6 To what extent can the results in the current book be generalized to the non-Kähler setting?

The non-pluripolar products in the non-Kähler setting was recently studied by Boucksom–Guedj–Lu in [BGL24]. See also the references therein.

Problem 14.2.7 To what extent can the results in the current book about closed positive $(1,1)$-currents be generalized to closed positive currents of higher bidegree?

A fundamental issue is the lack of a strong enough Demailly type approximation for general currents. The regularization theorem of Dinh–Sibony [DS04] seems too weak for our purposes.

Appendix A
Convex functions and convex bodies

We recall some basic facts about convex functions in this section. Our basic reference is [Roc70]. The results in this appendix can be applied to concave functions after considering their negatives.

A.1 The notion of convex functions

Let N be a real vector space of finite dimension.

Definition A.1.1 Let $F\colon N \to [-\infty, \infty]$ be a function. The *epigraph* of F is defined as the following set

$$\operatorname{epi} F := \{(n, r) \in N \times \mathbb{R} : r \geq F(n)\} .$$

Definition A.1.2 A *convex function* on N is a function $F\colon N \to [-\infty, \infty]$ such that the epigraph $\operatorname{epi} F$ is a convex subset of $N \times \mathbb{R}$.

The *effective domain* of F is the set

$$\operatorname{Dom} F := \{n \in N : F(n) < \infty\} .$$

A convex function F on N such that $\operatorname{Dom} F \neq \varnothing$ and $F(n) \neq -\infty$ for all $n \in N$ is said to be *proper*.

The set of convex functions on N is denoted by $\operatorname{Conv}(N)$. The subset set of proper convex functions is denoted by $\operatorname{Conv}^{\mathrm{prop}}(N)$.

The following characterization of convex functions is well-known.

Lemma A.1.1 *Let $F\colon N \to [-\infty, \infty]$. Then F is convex if and only if the following condition holds: suppose that $n, r \in N$ and $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$, then for any $t \in (0, 1)$, we have*

$$F(tn + (1-t)r) < ta + (1-t)b.$$

See [Roc70, Theorem 4.2] for the proof.

Example A.1.1 Let $A \subseteq N$ be a convex subset. Then the *characteristic function* $\chi_A \colon N \to \{0, \infty\}$ of A is defined by

$$\chi_A(n) \coloneqq \begin{cases} 0, & n \in A; \\ \infty, & n \notin A. \end{cases}$$

The function χ_A lies in $\mathrm{Conv}(N)$.

Example A.1.2 Let M be the dual vector space of N and $P \subseteq M$ be a convex subset. The *support function* $\mathrm{Supp}_P \in \mathrm{Conv}(N)$ of P is defined as follows:

$$\mathrm{Supp}_P(n) \coloneqq \sup\{\langle m, n\rangle : m \in P\}.$$

It is well-known that convexity is preserved by a number of natural operations. We recall a few to fix the notation.

Definition A.1.3 Let $F_1, \ldots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$ ($m \in \mathbb{Z}_{>0}$). We define their *infimal convolution* $F_1 \square \cdots \square F_m \in \mathrm{Conv}(N)$ as follows:

$$F_1 \square \cdots \square F_m(n) \coloneqq \inf\left\{\sum_{i=1}^m F_i(n_i) : n_i \in N, \sum_{i=1}^m n_i = n\right\}.$$

The fact $F_1 \square \cdots \square F_m \in \mathrm{Conv}(N)$ is proved in [Roc70, Theorem 5.4]. One should note that $F_1 \square \cdots \square F_m$ is not always proper.

Proposition A.1.1 *Let* $\{F_i\}_{i \in I}$ *be a non-empty family in* $\mathrm{Conv}(N)$*. Then* $\sup_{i\in I} F_i \in \mathrm{Conv}(N)$.

This follows from [Roc70, Theorem 5.5]. In particular, this allows us to introduce

Definition A.1.4 Let $f \colon N \to [-\infty, \infty]$. The *lower convex envelope* of f is defined as

$$\mathrm{CE}\, f \coloneqq \sup\{F \in \mathrm{Conv}(N) : F \leq f\}.$$

It follows from Proposition A.1.1 that $\mathrm{CE}\, f \in \mathrm{Conv}(N)$.

Definition A.1.5 Given a non-empty family $\{F_i\}_{i\in I}$ in $\mathrm{Conv}(N)$, we define

$$\bigwedge_{i \in I} F_i \coloneqq \mathrm{CE}\left(\inf_{i \in I} F_i\right).$$

When the family I is finite, say $I = \{1, \ldots, m\}$, we also write

$$F_1 \wedge \cdots \wedge F_m = \bigwedge_{i \in I} F_i.$$

Definition A.1.6 Given a non-empty family $\{F_i\}_{i\in I}$ in $\mathrm{Conv}(N)$, we define

$$\bigvee_{i\in I} F_i \coloneqq \sup_{i\in I} F_i .$$

When the family I is finite, say $I = \{1, \dots, m\}$, we also write

$$F_1 \vee \cdots \vee F_m = \bigvee_{i\in I} F_i .$$

Recall that $\bigvee_{i\in I} F_i \in \mathrm{Conv}(N)$ by Proposition A.1.1.

Proposition A.1.2 *Let $F_1, \dots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$, then*

$$F_1 \wedge \cdots \wedge F_m(x) = \inf \left\{ \sum_{i=1}^m \lambda_i F_i(x_i) : x_i \in \mathrm{Dom}(F_i), \right.$$
$$\left. \lambda_i \in [0,1], \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \right\} .$$

See [Roc70, Theorem 5.6] for the more general result.

Lemma A.1.2 *Let $\{F_i\}_{i\in I}$ be a decreasing net in* $\mathrm{Conv}(N)$*. Then* $\inf_{i\in I} F_i \in \mathrm{Conv}(N)$.

Proof Write $F = \inf_{i\in I} F_i$. We shall apply the characterization in Lemma A.1.1. Take $n, r \in N$, $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$ and $t \in (0,1)$. We need to show that

$$F(tn + (1-t)r) < ta + (1-t)b. \tag{A.1}$$

By definition, there exists $j \in I$ such that for any $i \geq I$ with $i \geq j$, we have

$$a > F_i(n), \quad b > F_i(r).$$

It follows from Lemma A.1.1 that

$$F_i(tn + (1-t)r) < ta + (1-t)b$$

for any $i \geq j$. Since F_i is decreasing in i, we conclude (A.1). □

Definition A.1.7 Let $F \in \mathrm{Conv}(N)$. The *closure* $\mathrm{cl}\, F \in \mathrm{Conv}(N)$ of F is defined as follows: If $F(n) = -\infty$ for some $n \in N$, then $\mathrm{cl}\, F \coloneqq -\infty$. Otherwise, we define $\mathrm{cl}\, F$ as the lower semicontinuity regularization of F.

A convex function $F \in \mathrm{Conv}(N)$ is *closed* if $F = \mathrm{cl}\, F$. In other words, $F \in \mathrm{Conv}(N)$ if one of the following conditions hold:

(1) $F \equiv -\infty$;
(2) $F \equiv \infty$;
(3) F is proper and lower semi-continuous.

Proposition A.1.3 *Let $F \in \operatorname{Conv}(N)$ be a closed convex function. Then F is the supremum of all affine functions lying below F.*

See [Roc70, Theorem 12.1].

Theorem A.1.1 *Let $F \in \operatorname{Conv}^{\mathrm{prop}}(N)$. Then $\operatorname{cl} F$ is a closed proper convex function. Moreover, $\operatorname{cl} F$ agrees with F except possibly on the relative boundary of $\operatorname{Dom} F$.*

See [Roc70, Theorem 7.4].

Definition A.1.8 Given $F, F' \in \operatorname{Conv}(N)$, we write $F \preceq F'$ if there is $C \in \mathbb{R}$ such that

$$F \leq F' + C.$$

We say $F \sim F'$ if $F \preceq F'$ and $F' \preceq F$ both hold.

Theorem A.1.2 *Let $C \subseteq N$ be an open subset. Let $(f_i)_{i>0}$ be a sequence of real-valued convex functions on C. Suppose that the sequence converges on a dense subset of C and the limit is finite, then the limit*

$$f(x) := \lim_{i \to \infty} f_i(x)$$

exists for all $x \in C$ and is convex on C. Moreover, the sequence $(f_i)_i$ converges uniformly to f on each compact subset of C.

This is a special case of [Roc70, Theorem 10.8].

A.2 Legendre transform

Let N be a real vector space of finite dimension and M be the dual vector space. The pairing $M \times N \to \mathbb{R}$ will be denoted by $\langle \bullet, \bullet \rangle$.

Definition A.2.1 Let $F \in \operatorname{Conv}(N)$ be a convex function. We define the *Legendre transform* of F as the function $F^* \in \operatorname{Conv}(M)$:

$$F^*(m) := \sup_{n \in N} \left(\langle m, n \rangle - F(n) \right) = \sup_{n \in \operatorname{RelInt} \operatorname{Dom} F} \left(\langle m, n \rangle - F(n) \right). \tag{A.2}$$

The latter equality follows from [Roc70, Corollary 12.2.2].

Recall the well-known Legendre–Fenchel duality [Roc70, Theorem 12.2].

Theorem A.2.1 *Let $F \in \operatorname{Conv}(N)$. Then F^* is a closed convex function. The function F^* is proper if and only if F is.*

Moreover, we have $(\operatorname{cl} F)^* = F^*$ and

$$F^{**} = \operatorname{cl} F.$$

Example A.2.1 Let $P \subseteq M$ be a closed convex subset. Then

$$\mathrm{Supp}_P^* = \chi_P, \quad \chi_P^* = \mathrm{Supp}_P\,.$$

See [Roc70, Theorem 13.2].

The following special case will be useful to us in the sequel.

Corollary A.2.1 *Let $F\colon (0,\infty) \to [-\infty,\infty)$ be a convex function. If we define $G\colon \mathbb{R} \to (-\infty,\infty]$ by*

$$G(\tau) = \sup_{t>0} \left(t\tau - F(t)\right),$$

then G is a convex function and

$$F(t) = G^*(t), \quad \forall t > 0. \tag{A.3}$$

Moreover,

$$G(\tau) = \sup_{t\in\mathbb{Q}_{>0}} \left(t\tau - F(t)\right). \tag{A.4}$$

Proof We distinguish two cases.

First suppose that $F(t) = -\infty$ for some $t > 0$. Then $F(t) = -\infty$ for all $t > 0$ by the convexity of F. Our assertions are clear in this case.

Next assume that $F(t) \neq -\infty$ for all $t > 0$. In this case, Theorem A.1.1 guarantees that F admits a closed proper extension $\tilde{F} \in \mathrm{Conv}(\mathbb{R})$ with

$$\tilde{F}(t) = \infty, \quad \forall t < 0.$$

It follows from (A.2) that

$$G(\tau) = \tilde{F}^*(\tau), \quad \forall \tau \in \mathbb{R}.$$

Now Theorem A.2.1 implies (A.3). Finally (A.4) follows from the continuity of F.□

Proposition A.2.1 *Let $F\colon N \to [-\infty,\infty]$, then the function $F^*\colon M \to [-\infty,\infty]$ defined by*

$$F^*(m) \coloneqq \sup_{n\in N} \left(\langle m,n\rangle - F(n)\right).$$

Then

$$F^* = (\mathrm{cl}\,\mathrm{CE}\, f)^*.$$

See [Roc70, Corollary 12.1.1].

Definition A.2.2 Let $F \in \mathrm{Conv}(N)$ and $n \in N$. An element $m \in M$ is a *subgradient* of F at n if

$$F(n') \geq F(n) + \langle n' - n, m\rangle, \quad \forall n' \in N. \tag{A.5}$$

The set of subgradients of F at n is denoted by $\nabla F(n)$.

More generally, for any subset $E \subseteq N$, we write

$$\nabla F(E) = \bigcup_{n \in E} \nabla F(n).$$

Definition A.2.3 Given $F, F' \in \mathrm{Conv}(N)$, we write $F \leq_P F'$ if

$$\overline{\nabla F(N)} \subseteq \overline{\nabla F'(N)}.$$

We write $F \sim_P F'$ if $F \leq_P F'$ and $F' \leq_P F$.

Theorem A.2.2 *Suppose that $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. Then the following hold:*

(1) *For any $n \notin \mathrm{Dom}\, F$, $\nabla F(n) = \varnothing$;*
(2) *for any $n \in \mathrm{RelInt}\,\mathrm{Dom}\, F$, $\nabla F(n) \neq \varnothing$; Moreover, for any $n' \in N$, we have*

$$\partial_{n'} F(n) = \sup \{\langle n', m\rangle : m \in \nabla F(n)\};$$

(3) *for $n \in N$, the set $\nabla F(n)$ is bounded if and only if $n \in \mathrm{Int}\,\mathrm{Dom}\, F$.*

For the proof, we refer to [Roc70, Theorem 23.4].

Proposition A.2.2 *Let $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. Then*

$$\nabla F(N) \subseteq \mathrm{Dom}\, F^*.$$

If moreover F is closed, we have

$$\mathrm{RelInt}\,\mathrm{Dom}\, F^* \subseteq \nabla F(N). \tag{A.6}$$

In particular, if F is a proper closed convex function on N, then

$$\overline{\nabla F(N)} = \overline{\mathrm{Dom}\, F^*}.$$

Proof Suppose that $m \in \nabla F(n)$ for some $n \in N$, it follows that (A.5) holds. In particular,

$$\langle m, n'\rangle - F(n') \leq \langle m, n\rangle - F(n).$$

It follows that

$$F^*(m) \leq \langle m, n\rangle - F(n) < \infty.$$

(A.6) is proved in [Roc70, Corollary 23.5.1]. For the last assertion, it suffices to observe that $\overline{\mathrm{RelInt}\,\mathrm{Dom}\, F^*} = \overline{\mathrm{Dom}\, F^*}$. □

Proposition A.2.3 *Let $\{F_i\}_{i\in I}$ be a non-empty family in $\mathrm{Conv}^{\mathrm{prop}}(N)$. Then*

$$\left(\bigwedge_{i\in I} F_i\right)^* = \bigvee_{i\in I} F_i^*, \quad \left(\bigvee_{i\in I} \mathrm{cl}\, F_i\right)^* = \mathrm{cl} \bigwedge_{i\in I} F_i^*.$$

If I is finite and $\overline{\mathrm{Dom}\, F_i}$ is independent of the choice of $i \in I$, then

$$\left(\bigvee_{i\in I} F_i\right)^* = \bigwedge_{i\in I} F_i^*.$$

Recall that $\wedge$ is defined in Definition A.1.5 and $\vee$ in Definition A.1.6. See [Roc70, Theorem 16.5] for the proof.

Proposition A.2.4 *Let $F_1, \ldots, F_r \in \mathrm{Conv}^{\mathrm{prop}}(N)$ $(r \in \mathbb{Z}_{>0})$. Assume that*

$$\bigcap_{i=1}^{r} \mathrm{RelInt}\,\mathrm{Dom}(F_i) \neq \emptyset,$$

then

$$\left(\sum_{i=1}^{r} F_i\right)^{*}(m) = \inf\left\{\sum_{i=1}^{r} F_i^{*}(m_i) : m_1, \ldots, m_r \in M, \sum_{i=1}^{r} m_i = m\right\}.$$

Proposition A.2.5 *Let $P \subseteq M$ be a convex body*[1] *and $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. The following are equivalent:*

(1) *$F \preceq \mathrm{Supp}_P$;*
(2) *$\mathrm{Dom}\,F = N$ and $F^*|_{M \setminus P} \equiv \infty$;*
(3) *$\mathrm{Dom}\,F = N$ and $\nabla F(N) \subseteq P$.*

Moreover, under these conditions,

$$F(n) - \mathrm{Supp}_P(n) \leq F(0), \quad \forall n \in N. \tag{A.7}$$

Proof (1) $\Longrightarrow$ (2). It is clear that $\mathrm{Dom}\,F = N$ since $\mathrm{Dom}\,\mathrm{Supp}_P = N$. From $F \preceq \mathrm{Supp}_P$ and Example A.2.1, we know that

$$\chi_P = \mathrm{Supp}_P^{*} \preceq F^{*}.$$

So ii follows.

(2) $\Longrightarrow$ (3). This follows from Proposition A.2.2.

(3) $\Longrightarrow$ (1). Taken $n \in N$, we know that F is locally Lipschitz [Roc70, Theorem 10.4], so we can compute

$$\begin{aligned} F(n) - F(0) &= \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(tn)\,\mathrm{d}t = \int_0^1 \langle \nabla F(tn), n \rangle\,\mathrm{d}t \\ &\leq \int_0^1 \mathrm{Supp}_P(n)\,\mathrm{d}t = \mathrm{Supp}_P(n). \end{aligned}$$

In particular, (A.7) also follows. ◻

[1] Here a convex body refers to a non-empty closed convex subset, not necessarily having non-empty interior.

A.3 Classes of convex functions

Let N be a real vector space of finite dimension and M be the dual vector space.

We shall fix a convex body $P \subseteq M$.

The following classes are introduced in [BB13].

Definition A.3.1 We define the set $\mathcal{P}(N, P)$ as the set of proper convex functions $F \in \mathrm{Conv}(N)$ such that $F \preceq \mathrm{Supp}_P$.

We define the set $\mathcal{E}^\infty(N, P)$ as the set of closed convex functions $F \in \mathrm{Conv}(N)$ such that $F \sim \mathrm{Supp}_P$.

We define the set $\mathcal{E}(N, P)$ as follows: Suppose that $\mathrm{Int}\, P = \emptyset$, then $\mathcal{E}(N, P) := \mathcal{P}(N, P)$; otherwise, let

$$\mathcal{E}(N, P) = \left\{ F \in \mathcal{P}(N, P) : P = \overline{\nabla F(N)} \right\}.$$

We define the set $\mathcal{E}^1(N, P)$ as the subset of $\mathcal{E}(N, P)$ consisting of $F \in \mathcal{E}(N, P)$ with

$$\int_P F^* \,\mathrm{d}\,\mathrm{vol} < \infty,$$

where $\mathrm{d}\,\mathrm{vol}$ is any Lebesgue measure on N.

Observe that for any $F \in \mathcal{P}(N, P)$, we have $\mathrm{Dom}\, F = N$ and F is necessarily closed.

Proposition A.3.1 *We have*

$$\mathcal{E}^\infty(N, P) \subseteq \mathcal{E}^1(N, P) \subseteq \mathcal{E}(N, P) \subseteq \mathcal{P}(N, P).$$

Proof When $\mathrm{Int}\, P = \emptyset$, the assertion is clear. We assume that $\mathrm{Int}\, P \neq \emptyset$. The second inclusion follows from definition. We only hand the first inequality. Take $F \in \mathcal{E}^\infty(N, P)$. By definition, $F \sim \mathrm{Supp}_P$ and hence $F^* \sim \chi_P$. It follows that $P = \mathrm{Dom}\, F^*$.

By Proposition A.2.5, we already know that

$$\nabla F(N) \subseteq P = \mathrm{Dom}\, F^*.$$

On the other hand, by Proposition A.2.2, we have

$$\mathrm{Int}\, P \subseteq \nabla F(N).$$

So it follows that

$$P = \overline{\nabla F(N)}.$$

It is clear that $F^* \sim \chi_P$ is integrable. □

Proposition A.3.2 *For any $F \in \mathcal{E}^\infty(N, P)$, we have $F^*|_{M \setminus P} \equiv \infty$ and F^* is bounded on P.*

Proof From $F \sim \mathrm{Supp}_P$, we take the Legendre transform to get $F^* \sim \mathrm{Supp}_P^* = \chi_P$, where we applied Example A.2.1. □

Definition A.3.2 We endow the topology of pointwise convergence on $\mathcal{P}(N, P)$. Note that this topology coincides with the compact-open topology.

Proposition A.3.3 *Let $F \in \mathcal{P}(N, P)$. Then there is a decreasing sequence $F_j \in \mathcal{E}^\infty(N, P) \cap C^\infty(N)$ converging to F.*

See [BB13, Lemma 2.2].

We observe that the point $0 \in N$ plays a special role since it does in the definition of the support function.

Proposition A.3.4 *For any $F \in \mathrm{Conv}(N, P)$, we have*

$$\max_N (F - \mathrm{Supp}_P) = F(0).$$

Proof It follows from (A.7) that

$$\sup_N (F - \mathrm{Supp}_P) \leq F(0).$$

The equality is clearly obtained at $0 \in N$. □

A.4 Monge–Ampère measures

Let N be a free Abelian group of finite rank (i.e. a lattice) and M be its dual lattice. There is a canonical Lebesgue type measure on $M_\mathbb{R}$, denoted by $\mathrm{d\,vol}$, normalized so that the smallest cubes in M have volume 1. Similarly, the canonical measure on $N_\mathbb{R}$ is normalized in the same way and is denoted by $\mathrm{d\,vol}$ as well.

We will write

$$N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}, \quad M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}.$$

Definition A.4.1 Let $F \in \mathrm{Conv}(N_\mathbb{R})$, we define the *real Monge–Ampère measure* $\mathrm{MA}_\mathbb{R} F$ as the Borel measure on $N_\mathbb{R}$ given as follows: for each Borel measurable set $E \subseteq N_\mathbb{R}$, define

$$\mathrm{MA}_\mathbb{R} F(E) := n! \int_{\nabla F(E)} \mathrm{d\,vol}\,.$$

Proposition A.4.1 *Suppose that $F \in C^{1,1}(N_\mathbb{R}) \cap \mathrm{Conv}(N_\mathbb{R})$, fix an identification $N = \mathbb{Z}^n$, then*

$$\mathrm{MA}_\mathbb{R} F = n! \cdot \det \nabla^2 F \,\mathrm{d\,vol}\,.$$

See [Fig17, Example 2.2].

Proposition A.4.2 *Let $P \in M_\mathbb{R}$ be a convex body and $F \in \mathcal{P}(N_\mathbb{R}, P)$. Then $F \in \mathcal{E}(N_\mathbb{R}, P)$ if and only if*

$$\int_{M_\mathbb{R}} \mathrm{MA}_\mathbb{R} F = n! \,\mathrm{vol}\, P. \tag{A.8}$$

Proof By definition of $\mathrm{MA}_{\mathbb{R}}$, (A.8) is equivalent to

$$\operatorname{vol} \overline{\nabla F(N_{\mathbb{R}})} = \operatorname{vol} P.$$

We first handle the case where $\operatorname{Int} P \neq \varnothing$. By Proposition A.2.5, the latter is equivalent to

$$\overline{\nabla F(N_{\mathbb{R}})} = P.$$

Now assume that $\operatorname{Int} P = \varnothing$, then $\operatorname{vol} \overline{\nabla F(N)} = \operatorname{vol} P = 0$ by Proposition A.2.5. The assertion is clear. □

Theorem A.4.1 *Let $F, F_j \in \mathcal{P}(N_{\mathbb{R}}, P)$ $(j \in \mathbb{Z}_{>0})$. Assume that $F_j \to F$, then $\mathrm{MA}_{\mathbb{R}}(F_j)$ converges to $\mathrm{MA}_{\mathbb{R}}(F)$ weakly.*

See [Fig17, Proposition 2.6].

There is a well-known comparison principle.

Theorem A.4.2 *Let $F, F' \in \mathcal{P}(N_{\mathbb{R}}, P)$. Assume that $F \leq F'$, then*

$$\overline{\nabla F(N_{\mathbb{R}})} \subseteq \overline{\nabla F'(N_{\mathbb{R}})}.$$

$$\int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F) \leq \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F').$$

See [BB13, Lemma 2.5].

A.5 Separation lemmata

Lemma A.5.1 *Let $\alpha, \beta_1, \dots, \beta_m \in \mathbb{Z}^n$. Let Δ be the polytope generated by $\beta_1, \dots, \beta_m$. Then the following are equivalent:*

(1)
$$|z^{\alpha}|^2 \left(\sum_{i=1}^{m} |z^{\beta_i}|^2 \right)^{-1} \tag{A.9}$$

is a bounded function on $\mathbb{C}^{*n}$.

(2) $\alpha \in \Delta$.

Proof (2) $\implies$ (1). Write $\alpha = \sum_i t_i \beta_i$, where $t_i \in [0, 1]$, $\sum_i t_i = 1$. Then

$$\begin{aligned} |z^{\alpha}|^2 \left(\sum_{i=1}^{m} |z^{\beta_i}|^2 \right)^{-1} &= \prod_i |z^{\beta_i}|^{2t_i} \left(\sum_{i=1}^{m} |z^{\beta_i}|^2 \right)^{-1} \\ &\leq \prod_i \sum_j |z^{\beta_j}|^{2t_i} \left(\sum_{i=1}^{m} |z^{\beta_i}|^2 \right)^{-1} \leq 1. \end{aligned}$$

(1) $\implies$ (2). Assume that $\alpha \notin \Delta$. Let H be a hyperplane that separates α and Δ. Say H is defined by $a_1 x_1 + \cdots + a_n x_n = C$. Set

$$z(t) := (t^{a_1}, \ldots, t^{a_n}).$$

Then clearly (A.9) evaluated at $z(t)$ is not bounded. □

Lemma A.5.2 *Let $\beta_1, \ldots, \beta_m \in \mathbb{N}^n$ and $\beta \in \mathbb{R}^n$. Then the following are equivalent*

(1) $\log \sum_{i=1}^{m} \mathrm{e}^{x \cdot \beta_i} - (x, \beta)$ *is bounded from below.*
(2) β *is in the convex hull of the* β_i*'s.*

Proof The proof follows the same pattern as Lemma A.5.1. □

Appendix B
Pluripotential theory on unibranch spaces

In this appendix, we extend the theory in the book to compact unibranch Kähler spaces.

B.1 Complex spaces

A complex space is assumed to be reduced, Hausdorff and paracompact in the whole book.

Definition B.1.1 A *prime divisor* over an irreducible complex space Z is a connected smooth hypersurface $E \subseteq X'$, where $X' \to Z$ is a proper bimeromorphic morphism with X' smooth. Such a morphism $X' \to Z$ is also called a *resolution* of Z. The *center* of the prime divisor is defined as the image of E in Z.

Two prime divisors $E_1 \subseteq X'_1$ and $E_2 \subseteq X'_2$ over Z are *equivalent* if there is a common resolution $X'' \to X$ dominating both X'_1 and X'_2 such that the strict transforms of E_1 and E_2 coincide.

The set Z^{div} is the set of pairs (c, E), where $c \in \mathbb{Q}_{>0}$ and E is an equivalence class of a prime divisor over Z. For simplicity, we will denote the pair (c, E) by $c \operatorname{ord}_E$, although one should not really think of this object as a valuation unless Z is projective and irreducible.

Note that a prime divisor on Z does not always define a prime divisor over Z if Z is singular.

Definition B.1.2 A complex space X is *unibranch* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is unibranch.

It is shown in the arXiv version of [Xia23a, Remark 2.7] that when X is a projective variety, this notion coincides with the corresponding algebraic notion of unibranchness.

Theorem B.1.1 (Zariski's main theorem) *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism between complex spaces. Assume that X is unibranch, then π has connected fibers.*

We refer to [Dem85, Proof of Théorème 1.7].

Definition B.1.3 A *modification* of a compact complex space X is a finite composition of blow-ups with smooth centers.

Theorem B.1.2 (Hironaka's Chow lemma) *Suppose that X is a compact complex space. Then every proper bimeromorphic morphism to X can be dominated by a modification.*

This follows from the proof of [Hir75, Corollary 2].

Theorem B.1.3 *Let X be a compact complex space. Then there is a modification $\pi\colon Y \to X$ such that Y is smooth.*

See [BM97, W l09].

Corollary B.1.1 *Let X be a compact complex space and E be a prime divisor over X. Then there is a modification $\pi\colon Y \to X$ such that Y is smooth and E can be realized as a prime divisor on Y.*

B.2 Plurisubharmonic functions

Let X be a complex space.

Definition B.2.1 A function $\varphi\colon X \to [-\infty, \infty)$ is *plurisubharmonic* if

(1) φ is not identically $-\infty$ on any irreducible component of X, and
(2) for any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a plurisubharmonic function $\tilde{\varphi} \in \mathrm{PSH}(\Omega)$ such that $\varphi|_{\Omega\cap V} = \tilde{\varphi}|_{\Omega\cap V}$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Similarly, if θ is a smooth closed[1] real $(1,1)$-form on X, then a function $\varphi\colon X \to [-\infty, \infty)$ is *θ-plurisubharmonic* if for any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a smooth function g on Ω such that $\theta = (\mathrm{dd^c} g)|_{V\cap\Omega}$ and $g + \varphi|_V \in \mathrm{PSH}(V)$.

Theorem B.2.1 (Fornaess–Narasimhan) *Let $\varphi\colon X \to [-\infty, \infty)$ be a function. Assume that φ is not identically $-\infty$ on any irreducible component of X, then the following are equivalent:*

(1) *φ is psh;*

[1] Here *closed* means that locally θ is defined by a closed form under a local embedding.

(2) *φ is usc and for any morphism $f\colon \Delta \to X$ from the open unit disk Δ in $\mathbb{C}$ to X such that $f^*\varphi$ is not identically $-\infty$, the pull-back $f^*\varphi$ is psh.*

See [FsN80].

Theorem B.2.2 (Grauert–Remmert) *Let X be a unibranch*[2] *complex space and Z be an analytic subset in X and $\varphi \in \mathrm{PSH}(X \setminus Z)$. Then the function φ admits an extension to $\mathrm{PSH}(X)$ in the following two cases:*

(1) *The set Z has codimension at least 2 everywhere.*
(2) *The set Z has codimension at least 1 everywhere and is locally bounded from above on an open neighborhood of Z.*

In both cases, the extension is unique and is given by

$$\varphi(x) = \varlimsup_{X\setminus Z \ni y \to x} \varphi(y), \quad x \in X. \tag{B.1}$$

The proof given below combines [Dem85, Théorème 1.7] and [GR56]. [3]

Proof We first prove the uniqueness, which is a local problem on X. Let ψ denote the function defined by the right-hand side of (B.1). Since any extension φ has to be upper semicontinuous, we know that $\varphi \geq \psi$. Conversely, take $z \in Z$, we take a holomorphic map $f\colon \Delta \to X$ such that $f(0) = z$ and $f(\Delta) \not\subset Z$. From the subharmonicity of $f^*\varphi$ and (1.2), we find that

$$\varphi(z) = f^*\varphi(0) = \varlimsup_{\Delta^* \ni w \to 0} f^*\varphi(w) \leq \varlimsup_{X\setminus Z \ni y \to x} \varphi(y).$$

So (B.1) follows.

Having established the uniqueness of the extension, the existence also becomes a local problem. So we are going to use the same descriptions as in the first paragraph above.

(2) Let $\pi\colon Y \to X$ be a resolution of singularities. By Theorem 1.2.1, we know that $\pi^*\varphi$ admits a unique extension to a psh function on Y, which we denote by η.

---

[2] Unibranchness is very important here. Otherwise, consider the case where X is the union of two copies of $\mathbb{C}$ intersecting only at their origins, Z is the common origin. If we set $\varphi \equiv 0$ on one punctured plane and $\varphi \equiv 1$ on the other, then it is clear that φ cannot be extended to X. This leads to a few misleading statements in the modern literature. The problem is that in the early German literature, *komplexer Raum* is assumed to be either normal or unibranch!

[3] This theorem has a quite entangled history. The corresponding results for subharmonic functions are generally attributed to Brelot. In [GR56], they cited a paper of Brelot written 1934, which I cannot find. But in 1949, on the very first issue of *Annales de l'Institut Fourier*, Brelot published a paper [Bre49] with a very similar title, studying the behavior of a subharmonic function on the punctured neighborhood of a point. The general theorem was due to Grauert and Remmert, see [GR56]. Their original proof was quite complicated, due to the fact that resolution of singularities was not available at that time. Later on, in 1985, Demailly published the influential paper [Dem85] and gave a simpler proof. Oddly enough, Demailly did not cite either Grauert–Remmert or Brelot. He did not even mention that this result was already proved by Grauert–Remmert. The paper [Dem85] is so influential that in France few people know the existence of [GR56].

Note that all fibers of π are connected since X is unibranch. Hence η is constant along the fibers of π. It therefore descends to an upper semicontinuous function η on X.

We verify that φ is plurisubharmonic using Theorem B.2.1. Let $f : \Delta \to X$ be a holomorphic map. We assume that $f^*\varphi \not\equiv -\infty$. It suffices to show that $f^*\varphi$ it is subharmonic at $0 \in \Delta$. The germ of f lifts to Y, say represented by $f' : \Delta \to Y$ so that

$$f(t^k) = \pi(f'(t))$$

for all t close to 0, where k is an integer. Therefore, $\psi(f(t^k)) = \eta(f(t))$ near 0. It follows that $f^*\varphi$ is subharmonic near 0.

(1) By the local description of complex spaces [GR84, Section 3.4], we may assume that there is a domain $\Omega \subseteq \mathbb{C}^n$, a finite s-sheet branched covering $\Phi\colon X \to \Omega$ with branched locus contained in a proper analytic subset $V \subseteq \Omega$. We may assume that X is connected, $n \geq 1$ and $Z \subseteq \Phi^{-1}(V)$.

If suffices to show that φ is locally bounded from above near Z. Suppose that this fails. Then by (2) we can find $z \in Z$ and $x_i \in X \setminus (\Phi^{-1}(\Phi(Z) \cup V))$ $(i \geq 1)$ converging to z such that

$$\lim_{i\to\infty} \varphi(x_i) = \infty.$$

Let L be a complex line passing through $\Phi(z)$ intersecting $(\Phi(Z) \cup V) \cap \bar{B}$ only at $\Phi(z)$, where $B \Subset B'$ are two small open balls centered at $\Phi(z)$. We can find a sequence of lines L_i passing through $\Phi(x_i)$ converging to L such that $L_i \cap (B' \cap \Phi(Z)) = \varnothing$[4] while $L_i \cap (B' \cap V)$ is discrete. Then Φ restricts to a branched covering over $B' \cap L_i$ for all $i \geq 1$. Adding a constant to φ, we may assume that $\varphi|_{\Phi^{-1}(L\cap\partial B)} < 0$. We can then find an open neighborhood U of $\Phi^{-1}(L \cap \partial B)$ so that $\varphi|_U < 0$. For large i we have $\Phi^{-1}(L_i \cap \partial B) \subseteq U$, it follows from the maximum principle that $\varphi(x_i) \leq 0$, which is a contradiction. □

Corollary B.2.1 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism between compact Kähler spaces. Let θ be a smooth closed real $(1,1)$-form on X. Assume that X is unibranch, then the pull-back induces a bijection*

$$\pi^*\colon \mathrm{PSH}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}(Y,\pi^*\theta).$$

B.3 Extensions of the results in the smooth setting

Let X be an irreducible unibranch compact Kähler space of dimension n. Let θ be a closed real smooth $(1,1)$-form on X. We say *the cohomology class* $[\theta]$ is big if for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a compact Kähler manifold Y, $[\pi^*\theta]$ is big.

The non-pluripolar products can be defined exactly as in Chapter 2 and the results in that chapter hold *mutadis mutandis*.

[4] Here we need the assumption that Z has codimension at least 2.

The results in Chapter 3 can be also be easily extended. The definition of the P-envelope remains unchanged. As for the $\mathcal{I}$-envelope, we define

Definition B.3.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define $P_\theta[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$ as the unique element with the following property: If $\pi\colon Y \to X$ is a proper bimeromorphic morphism from a compact Kähler manifold Y, then

$$\pi^* P_\theta[\varphi]_{\mathcal{I}} = P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}}.$$

It follows from Corollary B.2.1 and Proposition 3.2.5 that $P_\theta[\varphi]_{\mathcal{I}}$ is independent of the choice of π and is well-defined. The other results can be easily extended.

Chapter 4 and Chapter 6 can be extended without big changes. The only exception is Theorem 6.2.6, where we do not have the notion of multiplier ideal sheaves. So we do not know how to extend this theorem.

Chapter 7 can be extended except for Section 7.3 for the same reason as above.

The trace operator defined in Chapter 8 can be extended as long as Y is not contained in X^{Sing} using the embedded resolution. In general, due to the lack of Demailly regularization, we do not know how to define the trace operator.

Chapter 9 can be easily extended.

Chapter 10 is easy to extend since the partial Okounkov bodies are bimeromorphically invariant in the sense of Theorem 10.3.2.

Chapter 11 is unchanged, since we always take projective limits with respect to all models in that section.

Chapter 13 can be extended except for the parts involving the trace operator.

Chapter 14 can be easily extended by considering a resolution.

I do not know how to extend the results in Chapter 5 and Chapter 12 to the singular setting.

Appendix C
Almost semigroups

We introduce and study almost semigroups. In particular, we will define the Okounkov bodies of almost semigroups.

C.1 Convex bodies

Fix $n \in \mathbb{N}$.

Definition C.1.1 A *convex body* in $\mathbb{R}^n$ is a non-empty compact convex set.

We allow a convex body to have empty interior.

We write $\mathcal{K}_n$ for the set of convex bodies in $\mathbb{R}^n$.

Definition C.1.2 The *Hausdorff metric* between $K_1, K_2 \in \mathcal{K}_n$ is given by

$$d_{\mathrm{Haus}}(K_1, K_2) := \max\left\{ \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

It is well-known that the metric space $(\mathcal{K}_n, d_{\mathrm{Haus}})$ is complete. We will need the following fundamental theorem:

Theorem C.1.1 (Blaschke selection theorem) *The metric space* $(\mathcal{K}_n, d_{\mathrm{Haus}})$ *is locally compact.*

We refer to [Sch93, Theorem 1.8.7] for details.

Theorem C.1.2 *The Lebesgue volume* $\mathrm{vol}\colon \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ *is continuous.*

See [Sch93, Theorem 1.8.20].

Theorem C.1.3 *Let* $K_i, K \in \mathcal{K}_n$ *(*$i \in \mathbb{N}$*). Then* $K_i \xrightarrow{d_{\mathrm{Haus}}} K$ *if and only if the following conditions hold:*

(1) *each point* $x \in K$ *is the limit of a sequence* $x_i \in K_i$*, and*

(2) *the limit of any convergent sequence* $(x_{i_j})_{j\in\mathbb{N}}$ *with* $x_{i_j} \in K_{i_j}$ *lies in* K*, where* i_j *is a strictly increasing sequence in* $\mathbb{Z}_{>0}$*.*

See [Sch93, Theorem 1.8.8].

Lemma C.1.1 *Let* $K \in \mathcal{K}_n$ *be a convex body with positive volume and* $K' \in \mathcal{K}_n$*. Assume that for some large enough* $k \in \mathbb{Z}_{>0}$*,* K' *contains* $K \cap (k^{-1}\mathbb{Z})^n$*, then* $K' \supseteq K^{n^{1/2}k^{-1}}$*.*

Proof Let $x \in K^{n^{1/2}k^{-1}}$, by assumption, the closed ball B with center x and radius $n^{1/2}k^{-1}$ is contained in K. Observe that x can be written as a convex combination of points in $B \cap (k^{-1}\mathbb{Z})^n$, which are contained in K' by assumption. It follows that $x \in K'$. □

Given a sequence of convex bodies K_i $(i \in \mathbb{N})$, we set

$$\varliminf_{i\to\infty} K_i = \overline{\bigcup_{i=0}^{\infty} \bigcap_{j\geq i} K_j}.$$

Suppose K is the limit of a subsequence of K_i, we have

$$\varliminf_{i\to\infty} K_i \subseteq K. \tag{C.1}$$

This is a simple consequence of Theorem C.1.3.

Lemma C.1.2 *Let* $K \subseteq \mathbb{R}^n$ *be a convex body. Let*

$$t_{\min} := \min\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}, \quad t_{\max} := \max\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}.$$

Then for $t \in [t_{\min}, t_{\max}]$*, the map*

$$t \mapsto \{x_1 = t\} \cap K$$

is continuous with respect to the Hausdorff metric.

Here x_1 denotes the first coordinate in $\mathbb{R}^n$.

Proof We may assume that $t_{\min} < t_{\max}$ as otherwise there is nothing to prove.

For each $t \in [t_{\min}, t_{\max}]$, we write $K_t = \{x_1 = t\} \cap K$. Let $t_j \to t$ be a convergent sequence in $[t_{\min}, t_{\max}]$, we want to show that K_{t_j} converges to K_t with respect to the Hausdorff metric. Recall that this amounts to the following two assertions:

(1) For each convergent sequence $x_j \in K_{t_j}$ with limit x, we have $x \in K_t$;
(2) Given any $x \in K_t$, up to replacing t_j by a subsequence, we can find $x_j \in K_{t_j}$ converging to x. □

The first assertion is obvious. Let us prove the second. Take $x = (t, x') \in K_t$. Up to replacing t_j by a subsequence and taking the symmetry into account, we may assume that $t_j > t$ for all t. In particular, $t < t_{\max}$.

We can find a point $y = (y^1, y') \in K$ such that $y^1 > t$ (for example, there is always such a point with $y^1 = t_{\max}$). Replacing t_j by a subsequence, we may assume that $t_j \in (t, y^1)$ for all j. Then it suffices to take

$$x_j = \frac{y^1 - t_j}{y^1 - t} x + \frac{t_j - t}{y^1 - t} y.$$

Lemma C.1.3 *Let $D_j \subseteq \mathbb{R}^n$ $(j \geq 1)$ be a decreasing sequence of convex sets. Assume that* vol $\bigcap_j D_j > 0$*, then*

$$\overline{\bigcap_{j=1}^{\infty} D_j} = \bigcap_{j=1}^{\infty} \overline{D_j}.$$

Proof The $\subseteq$ direction is clear. By convexity, it suffices to show that both sides have the same positive volume. As the boundary of convex sets has zero Lebesgue measure, it follows that the volumes of both sides are equal to $\lim_{j\to\infty} \operatorname{vol} D_j$. □

Definition C.1.3 Let $K, K' \in \mathcal{K}_n$, their *Minkowski sum* is given by

$$K + K' := \{x + x' : x \in K, x' \in K'\}.$$

Proposition C.1.1 *The Minkowski sum $\mathcal{K}_n \times \mathcal{K}_n \to \mathcal{K}_n$ is continuous.*

See [Sch93, Page 139].

Theorem C.1.4 (Brunn–Minkowski) *Let $K, K' \in \mathcal{K}_n$, then for any $t \in (0, 1)$, we have*

$$\operatorname{vol}((1-t)K' + tK) \geq (\operatorname{vol} K')^{(1-t)} (\operatorname{vol} K)^t.$$

In other words, the volume is log concave. See [Sch93, Page 372].

C.2 The Okounkov bodies of almost semigroups

Fix an integer $n \geq 0$. Fix a closed convex cone $C \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ such that $C \cap \{x_{n+1} = 0\} = \{0\}$. Here x_{n+1} is the last coordinate of $\mathbb{R}^{n+1}$.

C.2.1 Generalities on semigroups

Write $\hat{\mathcal{S}}(C)$ for the set of subsets of $C \cap \mathbb{Z}^{n+1}$ and $\mathcal{S}(C)$ for the set of sub-semigroups $S \subseteq C \cap \mathbb{Z}^{n+1}$. For each $k \in \mathbb{N}$ and $S \in \hat{\mathcal{S}}(C)$, we write

$$S_k := \{x \in \mathbb{Z}^n : (x, k) \in S\}.$$

Note that S_k is a finite set by our assumption on C.

We introduce a pseudometric on $\hat{\mathcal{S}}(C)$ as follows:

$$d_{\mathrm{sg}}(S,S') := \varlimsup_{k\to\infty} k^{-n}(|S_k| + |S'_k| - 2|(S\cap S')_k|).$$

Here $|\bullet|$ denotes the cardinality of a finite set.

Lemma C.2.1 *The above defined d_{sg} is a pseudometric on $\hat{\mathcal{S}}(C)$.*

Proof Only the triangle inequality needs to be argued. Take $S, S', S'' \in \hat{\mathcal{S}}(C)$. We claim that for any $k \in \mathbb{N}$,

$$|S_k| + |S'_k| - 2|S_k \cap S'_k| + |S''_k| + |S'_k| - 2|S''_k \cap S'_k| \geq |S_k| + |S''_k| - 2|S_k \cap S''_k|.$$

From this the triangle inequality follows. To argue the claim, we rearrange it to the following form:

$$|S'_k| - |S_k \cap S'_k| \geq |S'_k \cap S''_k| - |S_k \cap S''_k|,$$

which is obvious. □

Given $S, S' \in \hat{\mathcal{S}}(C)$, we say S is equivalent to S' and write $S \sim S'$ if $d_{\mathrm{sg}}(S,S') = 0$. This is an equivalence relation by Lemma C.2.1.

Lemma C.2.2 *Given $S, S', S'' \in \hat{\mathcal{S}}(C)$, we have*

$$d_{\mathrm{sg}}(S\cap S'', S'\cap S'') \leq d_{\mathrm{sg}}(S,S').$$

In particular, if $S^i, S'^i \in \hat{\mathcal{S}}(C)$ ($i \in \mathbb{N}$) and $S^i \to S$, $S'^i \to S'$, then

$$S^i \cap S'^i \to S \cap S'.$$

Proof Observe that for any $k \in \mathbb{N}$,

$$|S_k \cap S''_k| - |S_k \cap S'_k \cap S''_k| \leq |S_k| - |S_k \cap S'_k|.$$

The same holds if we interchange S with S'. It follows that

$$|S_k \cap S''_k| + |S'_k \cap S''_k| - 2|S_k \cap S'_k \cap S''_k| \leq |S_k| + |S'_k| - 2|S_k \cap S'_k|.$$

The first assertion follows.

Next we compute

$$\begin{aligned} d_{\mathrm{sg}}(S^i \cap S'^i, S\cap S') \leq & d_{\mathrm{sg}}(S^i \cap S'^i, S^i \cap S') + d_{\mathrm{sg}}(S^i \cap S', S\cap S') \\ \leq & d_{\mathrm{sg}}(S'^i, S') + d_{\mathrm{sg}}(S^i, S) \end{aligned}$$

and the second assertion follows. □

The volume of $S \in \mathcal{S}(C)$ is defined as

$$\operatorname{vol} S := \lim_{k\to\infty} (ka)^{-n}|S_{ka}| = \varlimsup_{k\to\infty} k^{-n}|S_k|,$$

where a is a sufficiently divisible positive integer. The existence of the limit and its independence from a both follow from the more precise result [KK12, Theorem 2].

Lemma C.2.3 *Let $S, S' \in \mathcal{S}(C)$, then*

$$|\operatorname{vol} S - \operatorname{vol} S'| \leq d_{\mathrm{sg}}(S, S').$$

Proof By definition, we have

$$d_{\mathrm{sg}}(S, S') \geq \operatorname{vol} S + \operatorname{vol} S' - 2\operatorname{vol}(S \cap S').$$

It follows that $\operatorname{vol} S - \operatorname{vol} S' \leq d_{\mathrm{sg}}(S, S')$ and $\operatorname{vol} S' - \operatorname{vol} S \leq d_{\mathrm{sg}}(S, S')$. ◻

We define $\overline{\mathcal{S}}(C)$ as the closure of $\mathcal{S}(C)$ in $\hat{\mathcal{S}}(C)$ with respect to the topology defined by the pseudometric d. By Lemma C.2.3, $\operatorname{vol}\colon \mathcal{S}(C) \to \mathbb{R}$ admits a unique 1-Lipschitz extension to

$$\operatorname{vol}\colon \overline{\mathcal{S}}(C) \to \mathbb{R}. \tag{C.2}$$

Lemma C.2.4 *Suppose that $S, S' \in \overline{\mathcal{S}}(C)$ and $S \subseteq S'$. Then*

$$\operatorname{vol} S \leq \operatorname{vol} S'.$$

Proof Take sequences S^j, S'^j in $\mathcal{S}(C)$ such that $S^j \to S$, $S'^j \to S'$. By Lemma C.2.2, after replacing S^j by $S^j \cap S'^j$, we may assume that $S^j \subseteq S'^j$ for each j. Then our assertion follows easily. ◻

C.2.2 Okounkov bodies of semigroups

Given $S \in \hat{\mathcal{S}}(C)$, we will write $C(S) \subseteq C$ for the closed convex cone generated by $S \cup \{0\}$. Moreover, for each $k \in \mathbb{Z}_{>0}$, we define

$$\Delta_k(S) := \operatorname{Conv}\left\{k^{-1}x \in \mathbb{R}^n : x \in S_k\right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the convex hull.

Definition C.2.1 Let $\mathcal{S}'(C)$ be the subset of $\mathcal{S}(C)$ consisting of semigroups S such that S generates $\mathbb{Z}^{n+1}$ (as an Abelian group).

Note that for any $S \in \mathcal{S}'(C)$, the cone $C(S)$ has full dimension (i.e. the topological interior is non-empty). Given a full-dimensional subcone $C' \subseteq C$, it is clear that $C' \cap \mathbb{Z}^{n+1} \in \mathcal{S}'(C)$.

This class behaves well under intersections:

Lemma C.2.5 *Let $S, S' \in \mathcal{S}'(C)$. Assume that* $\operatorname{vol}(S \cap S') > 0$*, then $S \cap S' \in \mathcal{S}'(C)$.*

The lemma obviously fails if $\operatorname{vol}(S \cap S') = 0$.

Proof We first observe that the cone $C(S)\cap C(S')$ has full dimension since otherwise $\mathrm{vol}(S\cap S')=0$. Take a full-dimensional subcone C' in $C(S)\cap C(S')$ such that C' intersects the boundary of $C(S)\cap C(S')$ only at 0. It follows from [KK12, Theorem 1] that there is an integer $N>0$ such that for any $x\in\mathbb{Z}^{n+1}\cap C'$ with Euclidean norm no less than N lies in $S\cap S'$. Therefore, $S\cap S'\in\mathcal{S}'(C)$. □

We recall the following definition from [KK12].

Definition C.2.2 Given $S\in\mathcal{S}'(C)$, its *Okounkov body* is defined as follows

$$\Delta(S) \coloneqq \{x\in\mathbb{R}^n : (x,1)\in C(S)\}\,.$$

Theorem C.2.1 *For each $S\in\mathcal{S}'(C)$, we have*

$$\mathrm{vol}\, S = \lim_{k\to\infty} k^{-n}|S_k| = \mathrm{vol}\,\Delta(S) > 0. \tag{C.3}$$

Moreover, as $k\to\infty$,

$$\Delta_k(S) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S). \tag{C.4}$$

This is essentially proved in [WN14, Lemma 4.8], which itself follows from a theorem of Khovanskii [Kho92]. We remind the readers that (C.3) fails for a general $W\in\mathcal{S}(C)$, see [KK12, Theorem 2].

Proof The equalities (C.3) follow from the general theorem [KK12, Theorem 2].

It remains to prove (C.4). By the argument of [WN14, Lemma 4.8], for any compact set $K\subseteq\mathrm{Int}\,\Delta(S)$, there is $k_0>0$ such that for any $k\geq k_0$, $\alpha\in K\cap(k^{-1}\mathbb{Z})^n$ implies that $\alpha\in\Delta_k(S)$.

In particular, taking $K=\Delta(S)^{\delta}$ for any $\delta>0$ and applying Lemma C.1.1, we find

$$d_{\mathrm{Haus}}(\Delta(S),\Delta_k(S)) \leq n^{1/2}k^{-1}+\delta$$

when k is large enough. This implies (C.4). □

Corollary C.2.1 *Let $S,S'\in\mathcal{S}'(C)$. Assume that* $\mathrm{vol}(S\cap S')>0$*, then we have*

$$d_{\mathrm{sg}}(S,S') = \mathrm{vol}(S)+\mathrm{vol}(S')-2\,\mathrm{vol}(S\cap S').$$

Proof This is a direct consequence of Lemma C.2.5 and (C.3). □

Lemma C.2.6 *Given $S\in\mathcal{S}'(C)$, we have $S\sim$* $\mathrm{Reg}(S)$.

Recall that the regularization $\mathrm{Reg}(S)$ of S is defined as $C(S)\cap\mathbb{Z}^{n+1}$.

Proof Since S and $\mathrm{Reg}(S)$ have the same Okounkov body, we have $\mathrm{vol}\,S=\mathrm{vol}\,\mathrm{Reg}(S)$ by Theorem C.2.1. By Corollary C.2.1 again,

$$d_{\mathrm{sg}}(\mathrm{Reg}(S),S) = \mathrm{vol}\,\mathrm{Reg}(S)-\mathrm{vol}\,S=0.$$

Lemma C.2.7 *Let $S,S'\in\mathcal{S}'(C)$. Assume that $d_{\mathrm{sg}}(S,S')=0$, then* $\Delta(S)=\Delta(S')$.

Proof Observe that $\operatorname{vol}(S \cap S') > 0$, as otherwise

$$d_{\mathrm{sg}}(S, S') \geq \operatorname{vol} S + \operatorname{vol} S' > 0,$$

which is a contradiction.

It follows from Lemma C.2.5 that $S \cap S' \in \mathcal{S}'(C)$. It suffices to show that $\Delta(S) = \Delta(S \cap S')$. In fact, suppose that this holds, since $\operatorname{vol}\Delta(S') = \operatorname{vol} S' = \operatorname{vol} S = \operatorname{vol}\Delta(S)$, the inclusion $\Delta(S') \supseteq \Delta(S \cap S') = \Delta(S)$ is an equality.

By Lemma C.2.2, we can therefore replace S' by $S \cap S'$ and assume that $S \supseteq S'$. Then clearly $\Delta(S) \supseteq \Delta(S')$. By (C.3),

$$\operatorname{vol}\Delta(S) = \operatorname{vol}\Delta(S') > 0.$$

Thus, $\Delta(S) = \Delta(S')$. □

Lemma C.2.8 *Suppose that $S^i \in \mathcal{S}'(C)$ is a decreasing sequence such that*

$$\lim_{i\to\infty} \operatorname{vol} S^i > 0.$$

Then there is $S \in \mathcal{S}'(C)$ such that $S^i \to S$.

In general, one cannot simply take $S = \bigcap_i S^i$. For example, consider the sequence $S^i = S^1 \cap \{x_{n+1} \geq i\}$.

Proof By Lemma C.2.6, we may replace S^i by its regularization and assume that $S^i = C(S^i) \cap \mathbb{Z}^{n+1}$. We define

$$S = \left(\bigcap_{i=1}^{\infty} C(S^i)\right) \cap \mathbb{Z}^{n+1}.$$

Since $\bigcap_{i=1}^{\infty} C(S^i)$ is a full-dimensional cone by assumption, we have $S \in \mathcal{S}'(C)$. By Corollary C.2.1 and Theorem C.2.1, we can compute the distance

$$d_{\mathrm{sg}}(S, S^i) = \operatorname{vol} S^i - \operatorname{vol} S = \operatorname{vol}\Delta(S^i) - \operatorname{vol}\Delta(S),$$

which tends to 0 by construction. □

C.2.3 Okounkov bodies of almost semigroups

Definition C.2.3 We define $\overline{\mathcal{S}'(C)}_{>0}$ as elements in the closure of $\mathcal{S}'(C)$ in $\hat{\mathcal{S}}(C)$ with positive volume. An element in $\overline{\mathcal{S}'(C)}_{>0}$ is called an *almost semigroup* in C.

Recall that the volume here is defined in (C.2).

Our goal is to prove the following theorem:

Theorem C.2.2 *The Okounkov body map* $\Delta\colon \mathcal{S}'(C) \to \mathcal{K}_n$ *as defined in Definition C.2.2 admits a unique continuous extension*

$$\Delta\colon \overline{\mathcal{S}'(C)}_{>0} \to \mathcal{K}_n. \tag{C.5}$$

Moreover, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, *we have*

$$\operatorname{vol} S = \operatorname{vol} \Delta(S). \tag{C.6}$$

Proof The uniqueness of the extension is clear as long as it exists. Moreover, (C.6) follows easily from Theorem C.2.1 and Theorem C.1.2 by continuity. It remains to argue the existence of the continuous extension. We first construct an extension and prove its continuity.

Step 1. We construct the desired map (C.5). Let $S \in \overline{\mathcal{S}'(C)}_{>0}$. We wish to construct a convex body $\Delta(S) \in \mathcal{K}_n$.

Let $S^i \in \mathcal{S}'(C)$ be a sequence that converges to S such that

$$d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}.$$

For each $i, j \geq 0$, we introduce

$$S^{i,j} = S^i \cap S^{i+1} \cdots \cap S^{i+j}.$$

Then by Lemma C.2.2,

$$d_{\mathrm{sg}}(S^{i,j}, S^{i,j+1}) \leq 2^{-i-j}.$$

Take $i_0 > 0$ large enough so that for $i \geq i_0$, $\operatorname{vol} S^i > 2^{-1} \operatorname{vol} S$ and $2^{2-i} < \operatorname{vol} S$ and hence

$$\operatorname{vol} S^i - \operatorname{vol} S^{i,j} \leq d_{\mathrm{sg}}(S^{i,0}, S^{i,1}) + d_{\mathrm{sg}}(S^{i,1}, S^{i,2}) + \cdots + d_{\mathrm{sg}}(S^{i,j-1}, S^{i,j}) \leq 2^{1-i}.$$

It follows that $\operatorname{vol} S^{i,j} > 2^{-1} \operatorname{vol} S - 2^{1-i} > 0$ whenever $i \geq i_0$. In particular, by Lemma C.2.5, $S^{i,j} \in \mathcal{S}'(C)$ for $i \geq i_0$.

By Lemma C.2.8, for $i \geq i_0$, there exists $T^i \in \mathcal{S}'(C)$ such that $S^{i,j} \to T^i$ as $j \to \infty$. Moreover,

$$d_{\mathrm{sg}}(T^i, S) = \lim_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S) \leq \varlimsup_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S^i) + d_{\mathrm{sg}}(S^i, S) \leq 2^{1-i} + d_{\mathrm{sg}}(S^i, S).$$

Therefore, $T^i \to S$. We then define

$$\Delta(S) \coloneqq \overline{\bigcup_{i=i_0}^{\infty} \Delta(T^i)}.$$

In other words, we have defined

$$\Delta(S) \coloneqq \varliminf_{i\to\infty} \Delta(S^i).$$

This is an honest limit: if Δ is the limit of a subsequence of $\Delta(S^i)$, then $\Delta(S) \subseteq \Delta$ by (C.1). Comparing the volumes, we find that equality holds. So by Theorem C.1.1,

$$\Delta(S) = \lim_{i\to\infty} \Delta(S^i). \tag{C.7}$$

Next we claim that $\Delta(S)$ as defined above does not depend on the choice of the sequence S^i. In fact, suppose that $S'^i \in \mathcal{S}'(C)$ is another sequence satisfying the same conditions as S^i. The same holds for $R^i := S^{i+1} \cap S'^{i+1}$. It follows that

$$\lim_{i\to\infty} \Delta(R^i) \subseteq \lim_{i\to\infty} \Delta(S^i).$$

Comparing the volumes, we find that equality holds. The same is true with S'^i in place of S^i. So we conclude that $\Delta(S)$ as in (C.7) does not depend on the choices we made.

Step 2. It remains to prove the continuity of Δ defined in Step 1. Suppose that $S^i \in \overline{\mathcal{S}'(C)}_{>0}$ is a sequence with limit $S \in \overline{\mathcal{S}'(C)}_{>0}$. We want to show that

$$\Delta(S^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S). \tag{C.8}$$

We first reduce to the case where $S^i \in \mathcal{S}'(C)$. By (C.7), for each i, we can choose $T^i \in \mathcal{S}'(C)$ such that $d_{\mathrm{sg}}(S^i, T^i) < 2^{-i}$ and $d_{\mathrm{Haus}}(\Delta(S^i), \Delta(T^i)) < 2^{-i}$. If we have shown $\Delta(T^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S)$, then (C.8) follows immediately.

Next we reduce to the case where $d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}$. In fact, thanks to Theorem C.1.1, in order to prove (C.8), it suffices to show that each subsequence of $\Delta(S^i)$ admits a subsequence that converges to $\Delta(S)$. Hence, we easily reduce to the required case.

After these reductions, (C.8) is nothing but (C.7). □

Remark C.2.1 As the readers can easily verify from the proof, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, there is $S' \in \mathcal{S}'(C)$ such that $S \sim S'$.

Corollary C.2.2 *Suppose that $S, S' \in \overline{\mathcal{S}'(C)}_{>0}$ with $S \subseteq S'$, then*

$$\Delta(S) \subseteq \Delta(S'). \tag{C.9}$$

Proof Let $S^j, S'^j \in \mathcal{S}'(C)$ be elements such that $S^j \to S$, $S'^j \to S'$. Then it follows from Lemma C.2.2 that $S^j \cap S'^j \to S$. Since vol is continuous, for large j, $S^j \cap S'^j$ has positive volume and hence lies in $\mathcal{S}'(C)$ by Lemma C.2.5. We may therefore replace S^j by $S^j \cap S'^j$ and assume that $S^j \subseteq S'^j$. Hence, (C.9) follows from the continuity of Δ proved in Theorem C.2.2. □

Remark C.2.2 As the readers can easily verify, the construction of Δ is independent of the choice of C in the following sense: Suppose that C' is another cone satisfying the same assumptions as C and $C' \supseteq C$, then the Okounkov body map $\Delta\colon \overline{\mathcal{S}'(C')}_{>0} \to \mathcal{K}_n$ is an extension of the corresponding map (C.5). We will constantly use this fact without further explanations.

Index

Symbols

References

BB10. Robert Berman and Sébastien Boucksom. Growth of balls of holomorphic sections and energy at equilibrium. *Invent. Math.*, 181(2):337–394, 2010.

BB13. Robert J. Berman and Bo Berndtsson. Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. *Ann. Fac. Sci. Toulouse Math. (6)*, 22(4):649–711, 2013.

BBGHdJ21. A. Botero, J. I. Burgos Gil, D. Holmes, and R. de Jong. Chern–Weil and Hilbert–Samuel formulae for singular hermitian line bundles, 2021.

BBGHdJ22. A. Botero, J. I. Burgos Gil, D. Holmes, and R. de Jong. Rings of Siegel–Jacobi forms of bounded relative index are not finitely generated, 2022.

BBJ21. Robert J. Berman, Sébastien Boucksom, and Mattias Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. *J. Amer. Math. Soc.*, 34(3):605–652, 2021.

BBWN11. Robert Berman, Sébastien Boucksom, and David Witt Nyström. Fekete points and convergence towards equilibrium measures on complex manifolds. *Acta Math.*, 207(1):1–27, 2011.

BC11. Sébastien Boucksom and Huayi Chen. Okounkov bodies of filtered linear series. *Compos. Math.*, 147(4):1205–1229, 2011.

BDPP13. Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013.

BEGZ10. Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi. Monge-Ampère equations in big cohomology classes. *Acta Math.*, 205(2):199–262, 2010.

Ber93. Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, pages 5–161 (1994), 1993.

Ber09. Robert J. Berman. Bergman kernels and equilibrium measures for line bundles over projective manifolds. *Amer. J. Math.*, 131(5):1485–1524, 2009.

Ber14. Robert J. Berman. Determinantal point processes and fermions on complex manifolds: large deviations and bosonization. *Comm. Math. Phys.*, 327(1):1–47, 2014.

BFJ08. Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Valuations and plurisubharmonic singularities. *Publ. Res. Inst. Math. Sci.*, 44(2):449–494, 2008.

BFJ09. Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Differentiability of volumes of divisors and a problem of Teissier. *J. Algebraic Geom.*, 18(2):279–308, 2009.

BGL24. Sébastien Boucksom, Vincent Guedj, and Chinh H. Lu. Volumes of Bott–Chern classes, 2024.

BHJ17. Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs. *Ann. Inst. Fourier (Grenoble)*, 67(2):743–841, 2017.

BJ22a. Sébastien Boucksom and Mattias Jonsson. Global pluripotential theory over a trivially valued field. *Ann. Fac. Sci. Toulouse Math. (6)*, 31(3):647–836, 2022.

BJ22b. Sébastien Boucksom and Mattias Jonsson. Addendum to the article global pluripotential theory over a trivially valued field, 2022.

BM97. Edward Bierstone and Pierre D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, 128(2):207–302, 1997.

Bou02a. S. Boucksom. *Cônes positifs des variétés complexes compactes*. PhD thesis, Université Joseph-Fourier-Grenoble I, 2002.

Bou02b. Sébastien Boucksom. On the volume of a line bundle. *Internat. J. Math.*, 13(10):1043–1063, 2002.

Bou17. Sébastien Boucksom. Singularities of plurisubharmonic functions and multiplier ideals. http://sebastien.boucksom.perso.math.cnrs.fr/notes/L2.pdf, 2017.

Bre49. Marcel Brelot. étude des fonctions sous-harmoniques au voisinage d'un point singulier. *Ann. Inst. Fourier (Grenoble)*, 1:121–156, 1949.

Bre65. Marcel Brelot. *Éléments de la théorie classique du potentiel*. "Les cours de Sorbonne", 3e cycle. Centre de Documentation Universitaire, Paris, 1965. 3e édition.

Bre72. Marcel Brelot. Les étapes et les aspects multiples de la théorie du potentiel. *Enseign. Math. (2)*, 18:1–36, 1972.

BS76. Constantin Bănică and Octavian Stănăşilă. *Algebraic methods in the global theory of complex spaces*. Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976. Translated from the Romanian.

BT76. Eric Bedford and B. A. Taylor. The Dirichlet problem for a complex Monge-Ampère equation. *Invent. Math.*, 37(1):1–44, 1976.

BT82. Eric Bedford and B. A. Taylor. A new capacity for plurisubharmonic functions. *Acta Math.*, 149(1-2):1–40, 1982.

BT87. Eric Bedford and B. A. Taylor. Fine topology, Šilov boundary, and $(dd^c)^n$. *J. Funct. Anal.*, 72(2):225–251, 1987.

Cao14. Junyan Cao. Numerical dimension and a Kawamata-Viehweg-Nadel-type vanishing theorem on compact Kähler manifolds. *Compos. Math.*, 150(11):1869–1902, 2014.

Car83. Lennart Carleson. Selected problems on exceptional sets. In *Selected reprints*, Wadsworth Math. Ser., pages iv+100. Wadsworth, Belmont, CA, 1983.

CDG03. David M. J. Calderbank, Liana David, and Paul Gauduchon. The Guillemin formula and Kähler metrics on toric symplectic manifolds. *J. Symplectic Geom.*, 1(4):767–784, 2003.

CDM17. JunYan Cao, Jean-Pierre Demailly, and Shin-ichi Matsumura. A general extension theorem for cohomology classes on non reduced analytic subspaces. *Sci. China Math.*, 60(6):949–962, 2017.

CFK+17. Ciro Ciliberto, Michal Farnik, Alex Küronya, Victor Lozovanu, Joaquim Roé, and Constantin Shramov. Newton-Okounkov bodies sprouting on the valuative tree. *Rend. Circ. Mat. Palermo (2)*, 66(2):161–194, 2017.

CGSZ19. Dan Coman, Vincent Guedj, Sibel Sahin, and Ahmed Zeriahi. Toric pluripotential theory. *Ann. Polon. Math.*, 123(1):215–242, 2019.

CLS11. David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

Dar17. Tamás Darvas. Weak geodesic rays in the space of Kähler potentials and the class $\mathcal{E}(X, \omega)$. *J. Inst. Math. Jussieu*, 16(4):837–858, 2017.

DDNL18a. Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. L^1 metric geometry of big cohomology classes. *Ann. Inst. Fourier (Grenoble)*, 68(7):3053–3086, 2018.

DDNL18b. Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. Monotonicity of nonpluripolar products and complex Monge-Ampère equations with prescribed singularity. *Anal. PDE*, 11(8):2049–2087, 2018.

DDNL18c. Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. On the singularity type of full mass currents in big cohomology classes. *Compos. Math.*, 154(2):380–409, 2018.

DDNL21a. Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. Log-concavity of volume and complex Monge-Ampère equations with prescribed singularity. *Math. Ann.*, 379(1-2):95–132, 2021.

DDNL21b. Tamás Darvas, Eleonora Di Nezza, and Hoang-Chinh Lu. The metric geometry of singularity types. *J. Reine Angew. Math.*, 771:137–170, 2021.

DDNL23. Tamás Darvas, Eleonora Di Nezza, and Chinh H. Lu. Relative pluripotential theory on compact kähler manifolds, 2023.

Dem85. Jean-Pierre Demailly. Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines. *Mém. Soc. Math. France (N.S.)*, page 124, 1985.

Dem12a. Jean-Pierre Demailly. *Analytic methods in algebraic geometry*, volume 1 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, 2012.

Dem12b. Jean-Pierre Demailly. Complex analytic and differential geometry, 2012. Available on personal website, link.

Dem15. Jean-Pierre Demailly. On the cohomology of pseudoeffective line bundles. In *Complex geometry and dynamics*, volume 10 of *Abel Symp.*, pages 51–99. Springer, Cham, 2015.

Den17. Ya Deng. Transcendental Morse inequality and generalized Okounkov bodies. *Algebr. Geom.*, 4(2):177–202, 2017.

DF22. Nguyen-Bac Dang and Charles Favre. Intersection theory of nef b-divisor classes. *Compos. Math.*, 158(7):1563–1594, 2022.

DG65. Jean Dieudonné and Alexendre Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, volume 24. Institut des hautes études scientifiques, 1965.

DNT21. Eleonora Di Nezza and Stefano Trapani. Monge-Ampère measures on contact sets. *Math. Res. Lett.*, 28(5):1337–1352, 2021.

DPS01. Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Pseudo-effective line bundles on compact Kähler manifolds. *Internat. J. Math.*, 12(6):689–741, 2001.

DR22. Ruadhai Dervan and Rémi Reboulet. Ding stability and Kähler–Einstein metrics on manifolds with big anticanonical class, 2022.

DRWN+23. Tamás Darvas, Rémi Reboulet, David Witt Nyström, Mingchen Xia, and Kewei Zhang. Transcendental okounkov bodies, 2023.

DS04. Tien-Cuong Dinh and Nessim Sibony. Regularization of currents and entropy. *Ann. Sci. École Norm. Sup. (4)*, 37(6):959–971, 2004.

DS06. Tien-Cuong Dinh and Nessim Sibony. Distribution des valeurs de transformations méromorphes et applications. *Comment. Math. Helv.*, 81(1):221–258, 2006.

DX21. Tamás Darvas and Mingchen Xia. The volume of pseudoeffective line bundles and partial equilibrium. *Geometry & Topology (to appear)*, 2021.

DX22. Tamás Darvas and Mingchen Xia. The closures of test configurations and algebraic singularity types. *Adv. Math.*, 397:Paper No. 108198, 56, 2022.

DX24. Tamás Darvas and Mingchen Xia. The trace operator of quasi-plurisubharmonic functions on compact Kähler manifolds, 2024.

DXZ23. Tamás Darvs, Mingchen Xia, and Kewei Zhang. A transcendental approach to non-Archimedean metrics of pseudoeffective classes. *Commentarii Mathematici Helvetici(to appear)*, 2023.

DZ22. T. Darvas and K. Zhang. Twisted kähler–einstein metrics in big classes, 2022.

EK23. Mohamed El Kadiri. A note on the structure of plurifinely open sets and the equality of some complex Monge–Ampère measures, 2023.

EKFW11. Mohamed El Kadiri, Bent Fuglede, and Jan Wiegerinck. Plurisubharmonic and holomorphic functions relative to the plurifine topology. *J. Math. Anal. Appl.*, 381(2):706–723, 2011.

EKW14. Mohamed El Kadiri and Jan Wiegerinck. Plurifinely plurisubharmonic functions and the Monge Ampère operator. *Potential Anal.*, 41(2):469–485, 2014.

ELM+05. Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Asymptotic invariants of line bundles. *Pure Appl. Math. Q.*, 1(2):379–403, 2005.

EMW06. Said El Marzguioui and Jan Wiegerinck. The pluri-fine topology is locally connected. *Potential Anal.*, 25(3):283–288, 2006.

Fig17. Alessio Figalli. *The Monge-Ampère equation and its applications*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017.

Fin22. Siarhei Finski. On the metric structure of section ring, 2022.

FK18. Kazuhiro Fujiwara and Fumiharu Kato. *Foundations of rigid geometry. I*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2018.

FM21. Osamu Fujino and Shin-ichi Matsumura. Injectivity theorem for pseudo-effective line bundles and its applications. *Trans. Amer. Math. Soc. Ser. B*, 8:849–884, 2021.

FsN80. John Erik Fornæ ss and Raghavan Narasimhan. The Levi problem on complex spaces with singularities. *Math. Ann.*, 248(1):47–72, 1980.

Fuj23. Osamu Fujino. Relative Bertini type theorem for multiplier ideal sheaves. *Osaka J. Math.*, 60(1):207–226, 2023.

GH94. Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

GK20. Patrick Graf and Tim Kirschner. Finite quotients of three-dimensional complex tori. *Ann. Inst. Fourier (Grenoble)*, 70(2):881–914, 2020.

GR56. Hans Grauert and Reinhold Remmert. Plurisubharmonische Funktionen in komplexen Räumen. *Math. Z.*, 65:175–194, 1956.

GR84. Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.

Gro60. Alexander Grothendieck. Techniques de construction en géométrie analytique. VI. étude locale des morphismes: germes d'espaces analytiques, platitude, morphismes simples. *Séminaire Henri Cartan*, 13(1):1–13, 1960.

Gui94. Victor Guillemin. Kaehler structures on toric varieties. *J. Differential Geom.*, 40(2):285–309, 1994.

GZ05. Vincent Guedj and Ahmed Zeriahi. Intrinsic capacities on compact Kähler manifolds. *J. Geom. Anal.*, 15(4):607–639, 2005.

GZ07. Vincent Guedj and Ahmed Zeriahi. The weighted Monge-Ampère energy of quasi-plurisubharmonic functions. *J. Funct. Anal.*, 250(2):442–482, 2007.

GZ15. Qi'an Guan and Xiangyu Zhou. Effectiveness of Demailly's strong openness conjecture and related problems. *Invent. Math.*, 202(2):635–676, 2015.

GZ17. Vincent Guedj and Ahmed Zeriahi. *Degenerate complex Monge-Ampère equations*, volume 26 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2017.

Har77. Robin Hartshorne. *Algebraic geometry*, volume No. 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

Hie14. Pham Hoang Hiep. The weighted log canonical threshold. *C. R. Math. Acad. Sci. Paris*, 352(4):283–288, 2014.

Hir75. Heisuke Hironaka. Flattening theorem in complex-analytic geometry. *Amer. J. Math.*, 97:503–547, 1975.

His12. Tomoyuki Hisamoto. Restricted Bergman kernel asymptotics. *Trans. Amer. Math. Soc.*, 364(7):3585–3607, 2012.

HK76. W. K. Hayman and P. B. Kennedy. *Subharmonic functions. Vol. I*, volume No. 9 of *London Mathematical Society Monographs*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976.

HPS18. C. Hacon, M. Popa, and C. Schnell. Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun. In *Local and global methods in algebraic geometry*, volume 712 of *Contemp. Math.*, pages 143–195. Amer. Math. Soc., [Providence], RI, 2018.

HTW23. Yan He, Johannes Testorf, and Xu Wang. Ross–Witt Nyström correspondence and Ohsawa–Takegoshi extension, 2023.

Ino22. Eiji Inoue. Entropies in μ-framework of canonical metrics and K-stability, II – Non-archimedean aspect: non-archimedean μ-entropy and μK-semistability, 2022.

Jow10. Shin-Yao Jow. Okounkov bodies and restricted volumes along very general curves. *Adv. Math.*, 223(4):1356–1371, 2010.

Kho92. A. G. Khovanskiĭ. The Newton polytope, the Hilbert polynomial and sums of finite sets. *Funktsional. Anal. i Prilozhen.*, 26(4):57–63, 96, 1992.

Kim15. Dano Kim. Equivalence of plurisubharmonic singularities and Siu-type metrics. *Monatsh. Math.*, 178(1):85–95, 2015.

Kis78. Christer O. Kiselman. The partial Legendre transformation for plurisubharmonic functions. *Invent. Math.*, 49(2):137–148, 1978.

KK12. Kiumars Kaveh and A. G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. of Math. (2)*, 176(2):925–978, 2012.

Kli91. Maciej Klimek. *Pluripotential theory*, volume 6 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.

LDS86. P. Lelong, P. Dolbeault, and H. Skoda, editors. *Séminaire d'analyse P. Lelong-P. Dolbeault-H. Skoda. Années 1983/1984*, volume 1198 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

Lel45. Pierre Lelong. Les fonctions plurisousharmoniques. *Ann. Sci. École Norm. Sup. (3)*, 62:301–338, 1945.

LM09. Robert Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.

Mat89. Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

Mat18. S. Matsumura. Injectivity theorems with multiplier ideal sheaves for higher direct images under Kähler morphisms, 2018.

MM07. Xiaonan Ma and George Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

MP24. Pietro Mesquita-Piccione. A non-Archimedean theory of complex spaces and the cscK problem, 2024.

MZ23. Xiankui Meng and Xiangyu Zhou. On the restriction formula. *J. Geom. Anal.*, 33(12):Paper No. 369, 30, 2023.

PT18. Mihai Păun and Shigeharu Takayama. Positivity of twisted relative pluricanonical bundles and their direct images. *J. Algebraic Geom.*, 27(2):211–272, 2018.

Rau15. Hossein Raufi. Singular hermitian metrics on holomorphic vector bundles. *Ark. Mat.*, 53(2):359–382, 2015.

Rie26. Frédéric Riesz. Sur les Fonctions Subharmoniques et Leur Rapport à la Théorie du Potentiel. *Acta Math.*, 48(3-4):329–343, 1926.

Roc70. R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

RS05. Alexander Rashkovskii and Ragnar Sigurdsson. Green functions with singularities along complex spaces. *Internat. J. Math.*, 16(4):333–355, 2005.

RWN14. Julius Ross and David Witt Nyström. Analytic test configurations and geodesic rays. *J. Symplectic Geom.*, 12(1):125–169, 2014.

Sch93. Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

Shi72. Bernard Shiffman. Extension of positive line bundles and meromorphic maps. *Invent. Math.*, 15(4):332–347, 1972.

Siu74. Yum Tong Siu. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.*, 27:53–156, 1974.

Siu88. Yum Tong Siu. The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. *Ann. of Math. (2)*, 127(3):585–627, 1988.

Siu98. Yum-Tong Siu. Invariance of plurigenera. *Invent. Math.*, 134(3):661–673, 1998.

Tru22. Antonio Trusiani. Kähler-Einstein metrics with prescribed singularities on Fano manifolds. *J. Reine Angew. Math.*, 793:1–57, 2022.

Tru23. Antonio Trusiani. A relative Yau–Tian–Donaldson conjecture and stability thresholds, 2023.

Vu19. Duc-Viet Vu. Locally pluripolar sets are pluripolar. *Internat. J. Math.*, 30(13):1950029, 13, 2019.

Vu21. Duc-Viet Vu. Relative non-pluripolar product of currents. *Ann. Global Anal. Geom.*, 60(2):269–311, 2021.

Wan22. Xieping Wang. Complete pluripolar sets and removable singularities of plurisubharmonic functions, 2022.

WN14. David Witt Nyström. Transforming metrics on a line bundle to the Okounkov body. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(6):1111–1161, 2014.

W l09. Jaros law W lodarczyk. Resolution of singularities of analytic spaces. In *Proceedings of Gökova Geometry-Topology Conference 2008*, pages 31–63. Gökova Geometry/Topology Conference (GGT), Gökova, 2009.

Xia21. Mingchen Xia. Partial Okounkov bodies and Duistermaat–Heckman measures of non-Archimedean metrics. *Geometry&Topology (to appear)*, 2021.

Xia22a. Mingchen Xia. Analytic Bertini theorem. *Math. Z.*, 302(2):1171–1176, 2022.

Xia22b. Mingchen Xia. Non-pluripolar products on vector bundles and Chern–Weil formulae. *Math. Ann.*, 2022.

Xia23a. Mingchen Xia. Mabuchi geometry of big cohomology classes. *J. Reine Angew. Math.*, 798:261–292, 2023.

Xia23b. Mingchen Xia. Operations on transcendental non-Archimedean metrics, 2023.

Xia23c. Mingchen Xia. Pluripotential-theoretic stability thresholds. *Int. Math. Res. Not. IMRN*, pages 12324–12382, 2023.