TRANSCENDENTAL B-DIVISORS

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ABSTRACT. We establish an intersection theory for transcendental b-divisors, answering a question of Dang–Favre.

CONTENTS

1. INTRODUCTION

In this paper, we extend the intersection theory of b-divisors developed in [DF22] to the transcendental setting.

Let X be a connected compact Kähler manifold of dimension n. Recall that an algebraic b-divisor (class) is an assignment $\mathbb{D}_Y \in \mathrm{NS}^1(Y)_{\mathbb{R}}$ for each modification $\pi\colon Y \to X$ such that they are mutually compatible under push-forwards. An example is a so-called Cartier b-divisor, where we start with a modification $\pi\colon Y \to X$ and a class α on Y, the value of $\mathbb{D}_Z$ on any modification $Z \to X$ dominating π is the pull-back of α to Z. The Cartier b-divisor is called nef if α can be taken as nef. In general, an algebraic b-divisor is nef if it can be approximated by nef Cartier b-divisors.

The b-divisors can be seen as generalizations of divisors, taking the bimeromorphic twists into account. It is of interest to understand their intersection theory. When X is projective, an intersection theory of b-divisors is established by Dang–Favre [DF22]. Roughly speaking, they proved that in this case, a nef b-divisor can always be approximated by a *decreasing* sequence of nef Cartier b-divisors. This result reduces the general intersection theory to that of Cartier b-divisors, which is essentially the same as the classical intersection theory as in [Ful98].

In the same paper, Dang–Favre asked the question of whether one can develop a similar theory for transcendental b-divisors, namely, when X is not necessarily projective and when the $\mathbb{D}_Y$'s are just classes in $\mathrm{H}^{1,1}(Y,\mathbb{R})$. We give an affirmative answer in this paper.

The idea of the proof is already contained in my previous papers [Xia22b; Xia22a]. Let us content ourselves to the algebraic setting for the moment. In this case, the two papers give an analytic approach to the intersection theory. Let us be more specific. Suppose that L is a big line bundle on X. Then for any singular Hermitian metrics $h_1,\dots,h_n$ on L, we can construct a natural algebraic nef b-divisor $\mathbb{D}_1,\dots,\mathbb{D}_n$ on X using Siu's decomposition. The main result in these papers show that the Dang–Favre intersection of $\mathbb{D}_1,\dots,\mathbb{D}_n$ is the same as the mixed volume of $h_1,\dots,h_n$. The mixed volume of singular metrics could mean several different things: There is a non-pluripolar mixed mass, a mixed volume in the sense of [DX22; DX21], as well as a mixed volume in the sense of Cao [Cao14]. We will clarify their relations in this paper, see Theorem 3.6. In particular, we show that the mixed volumes in the latter two senses are equal and they are equal to the Dang–Favre intersection as well.

Conversely, in general, modulo some technical details, we can show that any nef b-divisor can essentially be constructed in this way. Hence the analytic intersection theory is completely equivalent to the algebraic intersection theory of Dang–Favre. In the transcendental setting, we use these analytic tools to conversely define the analytic intersection theory.

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2. Preliminaries

Let X be a connected compact Kähler manifold of dimension n.

2.1. **Modifications and cones.** In this paper, we use the word modification in a very non-standard sense.

Definition 2.1. A *modification* of X is a bimeromorphic morphism $\pi\colon Y \to X$, which is a finite composition of blow-ups with smooth centers.

Note that π is necessarily projective and Y is always a Kähler manifold.

Definition 2.2. We say a modification $\pi'\colon Z \to X$ *dominates* another $\pi\colon Y \to X$ if there is a morphism $g\colon Z \to Y$ making the following diagram commutative:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X. & \end{tikzcd} \tag{2.1}$$

The modifications of X together with the domination relation form a directed set $\mathrm{Modif}(X)$. Let us recall the behavior of several cones under modifications.

Proposition 2.3. *Let $\pi\colon Y \to X$ be a modification.*

(1) For any nef class $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$, $\pi^*\alpha$ is nef.
(2) For any modified nef class $\beta \in \mathrm{H}^{1,1}(Y,\mathbb{R})$, $\pi_*\beta$ is modified nef.
(3) For any big class $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$, $\pi^*\alpha$ is big. Moreover, $\operatorname{vol}\pi^*\alpha = \operatorname{vol}\alpha$.
(4) For any big class $\beta \in \mathrm{H}^{1,1}(Y,\mathbb{R})$, $\pi_*\beta$ is big. Moreover, $\operatorname{vol}\pi_*\beta \geq \operatorname{vol}\beta$.

2.2. **The convergences of quasi-plurisubharmonic functions.** We first recall the notion of P and $\mathcal{I}$-equivalences. The latter is introduced in [DX22] based on [BFJ08]. The former was introduced in [Xia] based on [RWN14].

Definition 2.4. Let φ, ψ be quasi-plurisubharmonic functions on X. We say $\varphi \sim_P \psi$ if there is a closed positive $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$ and

$$P_\theta[\varphi] = P_\theta[\psi].$$

Here $\mathrm{PSH}(X,\theta)$ denotes the space of θ-plurisubharmonic functions on X and $\mathrm{PSH}(X,\theta)_{>0}$ denotes the subset consisting of $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_\varphi^n > 0$. Here and in the sequel, the Monge–Ampère type product θ_φ^n is always understood in the non-pluripolar sense of [BT87; GZ07; BEGZ10]. The envelope P_θ is defined as follows:

$$P_\theta[\varphi] \coloneqq \sup_{C\in\mathbb{R}}{}^*(\varphi + C) \wedge 0,$$

where $(\varphi + C) \wedge 0$ is the maximal element in $\mathrm{PSH}(X,\theta)$ dominated by both $\varphi + C$ and 0.

Definition 2.5. Let φ, ψ be quasi-plurisubharmonic functions on X. We say $\varphi \sim_{\mathcal{I}} \psi$ if $\mathcal{I}(\lambda\varphi) = \mathcal{I}(\lambda\psi)$ for all real $\lambda > 0$.

Here $\mathcal{I}$ denotes the multiplier ideal sheaf in the sense of Nadel.

It is shown in [DDNL21] that there is a pseudometric d_S on $\mathrm{PSH}(X,\theta)$ satisfying the following inequality: For any $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we have

$$\begin{aligned} d_S(\varphi,\psi) \leq& \frac{1}{n+1}\sum_{j=0}^{n}\left(2\int_X \theta_{\varphi\vee\psi}^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}\right) \\ \leq& C_n d_S(\varphi,\psi), \end{aligned} \tag{2.2}$$

where $C_n = 3(n+1)2^{n+2}$. Here $V_\theta = \max\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0\}$. Moreover, $d_S(\varphi,\psi) = 0$ if and only if $\varphi \sim_P \psi$. See [Xia, Proposition 6.2.2]. In particular, the d_S-pseudometric descends to a pseudometric (still denoted by d_S) on the space of closed positive $(1,1)$-currents in $\{\theta\}$.

Given a net of closed positive $(1,1)$-currents T_i in $\{\theta\}$, and another closed positive $(1,1)$-current T in $\{\theta\}$. It is shown in [Xia21] that $T_i \xrightarrow{d_S} T$ if and only if $T_i + \omega \xrightarrow{d_S} T + \omega$ for any Kähler form ω on X. See [Xia, Corollary 6.2.8].

3. Mixed volumes

Let X be a connected compact Kähler manifold of dimension n. Let $T_1,\dots,T_n$ be closed positive $(1,1)$-currents on X. Let $\theta_1,\dots,\theta_n$ be closed real smooth $(1,1)$-forms on X in the cohomology classes of $T_1,\dots,T_n$ respectively. Consider $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ so that $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for each $i = 1,\dots,n$. Fix a reference Kähler form ω on X.

3.1. **The different definitions.** For each $i = 1,\dots,n$, let $(\varphi_i^j)_j$ be a quasi-equisingular approximation of φ_i. Let $(\epsilon_j)_j$ be a decreasing sequence in $\mathbb{R}_{\geq 0}$ with limit 0 so that

$$\varphi_i^j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$$

for each $i = 1,\dots,n$ and $j > 0$.

Definition 3.1. The mixed volume of $T_1,\dots,T_n$ in the sense of Cao is defined as follows:

$$\langle T_1,\dots,T_n\rangle_C \coloneqq \lim_{j\to\infty} \int_X (\theta_1 + \epsilon_j\omega + \mathrm{dd^c}\varphi_1^j) \wedge \cdots \wedge (\theta_n + \epsilon_j\omega + \mathrm{dd^c}\varphi_n^j).$$

Here the product is understood in the non-pluripolar sense.

It is shown in [Cao14] Section 2 that this definition is independent of the choices of the θ_i's, the ϵ_j's, the φ_i's, φ_i^j's and ω.

A different definition relies on the $\mathcal{I}$-envelope technique studied in [DX21; DX22]. Recall that the pure volume of a current is defined in [Xia, Definition 3.2.3]:

$$\mathrm{vol}(\theta + \mathrm{dd^c}\varphi) = \int_X \left(\theta + \mathrm{dd^c}P_\theta[\varphi]_{\mathcal{I}}\right)^n.$$

Definition 3.2. Assume that $\mathrm{vol}\,T_i > 0$ for all $i = 1,\dots,n$. The mixed volume of $T_1,\dots,T_n$ in the sense of Darvas–Xia is defined as follows:

$$\mathrm{vol}(T_1,\dots,T_n) = \int_X \left(\theta_1 + \mathrm{dd^c}P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right). \tag{3.1}$$

In general, define

$$\mathrm{vol}(T_1,\dots,T_n) = \lim_{\epsilon\to 0+} \mathrm{vol}(T_1 + \epsilon\omega,\dots,T_n + \epsilon\omega). \tag{3.2}$$

This definition is again independent of the choices of ω, the θ_i's and the φ_i's, using the same proof as [Xia, Proposition 3.2.7].

Remark 3.3. When $\mathrm{vol}\,T_i > 0$ for all $i = 1,\dots,n$, the definition (3.2) is compatible with (3.1), thanks to [Xia, Example 7.1.2].

When $T_1 = \cdots = T_n = T$, the above definition is compatible with pure case:

Proposition 3.4. *We always have*

$$\mathrm{vol}(T,\dots,T) = \mathrm{vol}\,T.$$

Proof. Write $T = \theta_\varphi$. In more concrete terms, we need to show that

$$\lim_{\epsilon\to 0+} \int_X (\theta + \epsilon\omega + \mathrm{dd^c}P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c}P_\theta[\varphi]_{\mathcal{I}})^n.$$

We may replace φ by $P_\theta[\varphi]_{\mathcal{I}}$ and assume that φ is $\mathcal{I}$-model in $\mathrm{PSH}(X,\theta)$. Then we claim that

$$\varphi = \inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}.$$

From this, our assertion follows from [Xia, Proposition 3.1.9].

The $\leq$ direction is clear. For the converse, it suffices to show that for each prime divisor E over X, we have

$$\nu(\varphi, E) \leq \nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right).$$

We simply compute

$$\nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) \geq \sup_{\epsilon>0} \nu\left(P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) = \nu(\varphi, E).$$

□

Proposition 3.5. *Both volumes are symmetric. The mixed volume in the sense of Cao is multi-$\mathbb{Q}_{\geq 0}$-linear, while the mixed volume in the sense of Darvas–Xia is multi-$\mathbb{R}_{\geq 0}$-linear.*

The multi-$\mathbb{Q}_{\geq 0}$-linearity means two things:

(1) For each $\lambda \in \mathbb{Q}_{\geq 0}$, we have

$$\langle \lambda T_1, T_2, \ldots, T_n \rangle_C = \lambda \langle T_1, T_2, \ldots, T_n \rangle_C.$$

(2) If T_1' is anther closed positive $(1,1)$-current, then

$$\langle T_1 + T_1', T_2, \ldots, T_n \rangle_C = \langle T_1, T_2, \ldots, T_n \rangle_C + \langle T_1', T_2, \ldots, T_n \rangle_C. \tag{3.3}$$

Multi-$\mathbb{R}_{\geq 0}$-linearity is defined similarly.

Proof. We first handle the mixed volumes in the sense of Cao. Only the property (3.3) needs a proof. But this follows from the fact that the sum of two quasi-equisingular approximations is again a quasi-equisingular approximation. See [Xia, Theorem 6.2.2, Corollary 7.1.2].

Next we handle the case of mixed volumes in the sense of Darvas–Xia. We only need to show that

$$\operatorname{vol}(T_1 + T_1', T_2, \ldots, T_n) = \operatorname{vol}(T_1, T_2, \ldots, T_n) + \operatorname{vol}(T_1', T_2, \ldots, T_n).$$

Thanks to the definition (3.2), we may assume that $\operatorname{vol} T_i > 0$ for each i and $\operatorname{vol} T_1' > 0$. Write $T_1' = \theta_1' + \mathrm{dd^c}\varphi_1'$. Then

$$P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_1'}[\varphi_1']_{\mathcal{I}} \sim_P P_{\theta_1+\theta_1'}[\varphi_1 + \varphi_1']_{\mathcal{I}}$$

as a consequence of [Xia, Example 7.1.2, Proposition 7.2.1]. Our assertion follows. □

3.2. The equivalence.

Theorem 3.6. *We have*

$$\langle T_1, \ldots, T_n \rangle_C = \operatorname{vol}(T_1, \ldots, T_n). \tag{3.4}$$

Proof. **Step 1**. We reduce to the case where $T_1 = \cdots = T_n$.

Suppose this special case has been proved. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}_{>0}$ be some numbers. Then

$$\langle \sum_{i=1}^n \lambda_i T_i, \ldots, \sum_{i=1}^n \lambda_i T_i \rangle_C = \operatorname{vol}(\sum_{i=1}^n \lambda_i T_i).$$

It follows from Proposition 3.5 that both sides are polynomials in the λ_i's. Comparing the coefficients of $\lambda_1 \cdots \lambda_n$, we conclude (3.4).

From now on, we assume that $T_1 = \cdots = T_n = T$. Write $T = \theta_\varphi$.

Step 2. We reduce to the case where T is a Kähler current. For this purpose, it suffices to show that

$$\lim_{\epsilon\to 0+} \langle T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega \rangle_C = \langle T_1, \ldots, T_n \rangle_C,$$

which is obvious by definition.

Step 3. Let $(\varphi^j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X, \theta)$. We need to show that

$$\lim_{j\to\infty} \int_X (\theta + \mathrm{dd^c}\varphi^j)^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

This follows from [Xia, Corollary 7.1.2].

□

4. Transcendental b-divisors

Let X be a connected compact Kähler manifold of dimension n.

4.1. **The definitions.** The b-divisors defined in this section are sometimes known as b-divisor classes. We always omit the word *classes* to save space.

Definition 4.1. A *(Weil) b-divisor* over X is an assignment $(\alpha_\pi)_{\pi\colon Y\to X}$, where $\pi\colon Y \to X$ runs over all modifications of X such that

(1) $\alpha_\pi \in \mathrm{H}^{1,1}(Y,\mathbb{R})$;
(2) The classes are compatible under push-forwards: If $\pi'\colon Z \to X$ and $\pi\colon Y \to X$ are both in $\mathrm{Modif}(X)$ and π' dominates π through $g\colon Z \to Y$ (namely, g makes the diagram (2.1) commutative), then $g_*\alpha_{\pi'} = \alpha_\pi$.

We also write $\alpha_Y = \alpha_\pi$ if there is no risk of confusion.

Definition 4.2. The *volume* of a Weil b-divisor $\mathbb{D}$ over X is

$$\operatorname{vol}\mathbb{D} := \lim_{\pi\colon Y\to X} \operatorname{vol}\mathbb{D}_Y.$$

The right-hand side is a decreasing net due to Proposition 2.3, hence the limit always exists.

We say $\mathbb{D}$ is *big* if $\operatorname{vol}\mathbb{D} > 0$.

:volbdivnet **Lemma 4.3.** *Let $\mathbb{D}_i$ be a net of b-divisors converging to $\mathbb{D}$. Then*

$$\varlimsup_i \operatorname{vol}\mathbb{D}_i \leq \operatorname{vol}\mathbb{D}.$$

If the net is decreasing, then

$$\lim_i \operatorname{vol}\mathbb{D}_i \leq \operatorname{vol}\mathbb{D}.$$

Definition 4.4. A *Cartier b-divisor* $\mathbb{D}$ over X is a Weil b-divisor $\mathbb{D}$ over X such that there exists a modification $\pi\colon Y \to X$ and a class $\alpha_Y \in \mathrm{H}^{1,1}(Y,\mathbb{R})$ so that for each $\pi'\colon Z \to X$ dominating π, the class α_Z is the pull-back of α_Y. Any such (π, α_Y) is called a *realization* of $\mathbb{D}$.

By abuse of language, we also say (Y, α_Y) is a realization of $\mathbb{D}$. The realization is not unique in general.

Definition 4.5. A Cartier b-divisor $\mathbb{D}$ over X is *nef* if there exists a realization $(\pi\colon Y \to X, \alpha_Y)$ such that α_Y is nef.

Definition 4.6. A Weil b-divisor $\mathbb{D}$ over X is *nef* if there is a net of nef Cartier b-divisors $\mathbb{D}_i$ over X such that for each modification $\pi\colon Y \to X$, we have $\mathbb{D}_{i,Y} \to \mathbb{D}_Y$.

Note that each $\mathbb{D}_Y$ is necessarily modified nef, but it is not nef in general.

4.2. **The b-divisors of currents.** Let T be a closed positive $(1,1)$-current on X. Then we define the *regular part* $\operatorname{Reg} T$ of T as the regular part of T with respect to the Siu's decomposition.

Given any modification $\pi\colon Y \to X$, we define

$$\mathbb{D}(T)_Y := \{\operatorname{Reg}\pi^* Y\}.$$

Lemma 4.7. *Let T be a closed positive $(1,1)$-current on X. Then $\mathbb{D}(T)$ is nef. Moreover,*

$$\operatorname{vol} T = \operatorname{vol}\mathbb{D}(T).$$

Note that when T has analytic singularities, $\mathbb{D}(T)$ is Cartier.

Proof. Let ω be a Kähler form on X. Let $\mathbb{D}(\omega)$ be the Cartier b-divisor realized by $(X, \{\omega\})$. We could always approximate $\mathbb{D}(T)$ by $\mathbb{D}(T+\epsilon\omega) = \mathbb{D}(T) + \epsilon\mathbb{D}(\omega)$. Hence we may assume that T is a Kähler current.

Next, we take a closed smooth real $(1,1)$-form θ cohomologous to T and write $T = \theta_\varphi$ for some $\varphi \in \mathrm{PSH}(X,\theta)$. Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ be a quasi-equisingular approximation of φ. Then it is easy to see that $\mathbb{D}(\theta + \mathrm{dd^c}\varphi_j) \to \mathbb{D}(\theta + \mathrm{dd^c}\varphi)$.

So we may assume that φ has analytic singularities. After a resolution, we reduce to the case where T has log singularities along an effective $\mathbb{Q}$-divisor D. The assertion is clear in this case.

The final assertion follows from the same reductions. $\square$

There is a different possibility: Replace Reg by the non-pluripolar part. Given T as above, we define

$$\mathbb{D}'(T)_\pi := [\langle \pi^* T \rangle].$$

It is easy to see that

$$\mathbb{D}'(T) = \mathbb{D}'(\langle T \rangle).$$

Conversely, we want to realize nef b-divisors as $\mathbb{D}(T)$.

:Cartrealiz **Lemma 4.8.** *Let $\pi\colon Y \to X$ be a modification and $\alpha \in \mathrm{H}^{1,1}(Y,\mathbb{R})$ be a nef class. Then there is a current $T \in \pi_*\alpha$ such that $\mathbb{D}(T)$ is the Cartier b-divisor realized by (π, α).*

Proof. Let T' be a current with minimal singularities in α. Then T' has vanishing Lelong numbers everywhere. Therefore,

$$\pi^*\pi_* T' = T' + \sum_i c_i [E_i],$$

where $c_i > 0$ and E_i is a finite collection of exceptional divisors. We just take $T = \pi_* T'$. □

:dscontbdiv **Proposition 4.9.** *Let $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$ be a pseudoeffective class. Let T_i be a net of closed positive $(1,1)$-currents in α with d_S limit $T \in \alpha$. Then*

$$\mathbb{D}(T_i) \to \mathbb{D}(T).$$

The proof is the same as that in the algebraic case.

Proposition 4.10. *Each big and nef b-divisor $\mathbb{D}$ can be realized as $\mathbb{D}(T)$ for some $T \in \mathbb{D}_X$. Moreover, T and $\mathbb{D}$ have the same volume.*

Note that T is unique up to $\mathcal{I}$-equivalence.

Proof. Let $\mathbb{D}_i$ be a net of nef Cartier b-divisors converging to $\mathbb{D}$.

Thanks to Lemma 4.8, we can realize each $\mathbb{D}_i$ as $\mathbb{D}(T_i)$, where $T_i \in \mathbb{D}_{i,X}$. Moreover,

$$\operatorname{vol} T_i = \operatorname{vol} \mathbb{D}_i.$$

Fix a Kähler form ω on X. Take $\epsilon > 0$. Since by our assumption, $\mathbb{D}_{i,X} \to \mathbb{D}_X$, we may find an index i_0 so that

$$\mathbb{D}_X + 2^{-1}\epsilon\{\omega\} - \mathbb{D}_{i,X}$$

is Kähler for all $i \geq i_0$. Take a Kähler form ω_i in $\mathbb{D}_X + \epsilon\{\omega\} - \mathbb{D}_{i,X}$. Then by the completeness DDNLmetric [DDNL21], we can find a closed positive $(1,1)$-current $S_\epsilon \in \mathbb{D}_X + \epsilon\{\omega\}$ so that $T_i + \omega_i \xrightarrow{d_S} S_\epsilon$. It follows from Proposition 4.9 that

$$\mathbb{D}(S_\epsilon) = \mathbb{D} + \epsilon\mathbb{D}(\{\omega\}).$$

Observe that S_ϵ is a Kähler current. Let $S^j \in \mathbb{D}_X + \epsilon\{\omega\}$ be a quasi-equisingular approximation of S_ϵ. Then we know that

$$\operatorname{vol} S_\epsilon = \lim_{j\to\infty} \operatorname{vol} S^j = \lim_{j\to\infty} \operatorname{vol} \mathbb{D}(S^j) = \lim_{j\to\infty} \operatorname{vol} \mathbb{D}(S_\epsilon) = \operatorname{vol}\left(\mathbb{D} + \epsilon\mathbb{D}(\{\omega\})\right).$$

Now let θ be a closed smooth real $(1,1)$-form in $\mathbb{D}_X$. Let $\varphi_\epsilon \in \mathrm{PSH}(X, \theta + \epsilon\omega)$ be the $\mathcal{I}$-model potential with $\theta + \epsilon\omega + \mathrm{dd^c}\varphi_\epsilon \sim_{\mathcal{I}} S_\epsilon$. Take $\varphi = \inf_{\epsilon>0} \varphi_\epsilon \in \mathrm{PSH}(X,\theta)$. It follows from Xiabook [Xia, Proposition 3.1.9] that

$$\operatorname{vol} \theta_\varphi = \lim_{\epsilon\to 0} \operatorname{vol} S_\epsilon = \operatorname{vol} \mathbb{D} > 0.$$

By the same proposition, we also have

$$\mathbb{D}(\theta_\varphi) = \mathbb{D}.$$

Our assertion follows. □

Definition 4.11. Let $\mathbb{D}_1, \dots, \mathbb{D}_n$ be big and nef b-divisors over X. Then we define their intersection as

$$(\mathbb{D}_1, \dots, \mathbb{D}_n) \coloneqq \operatorname{vol}(T_1, \dots, T_n),$$

where $T_1, \dots, T_n$ are currents in $\mathbb{D}_{1,X}, \dots, \mathbb{D}_{n,X}$ such that $\mathbb{D}(T_i) = \mathbb{D}_i$.

In general, if the $\mathbb{D}_i$'s are only nef, we define

$$(\mathbb{D}_1, \dots, \mathbb{D}_n) \coloneqq \lim_{\epsilon \to 0+} (\mathbb{D}_1 + \epsilon\omega, \dots, \mathbb{D}_n + \epsilon\omega)\,.$$

As shown in [Xia22b; Xia22a], this intersection theory coincides with the Dang–Favre theory if X is projective and $\mathbb{D}_1, \dots, \mathbb{D}_n$ are algebraic.

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