Notes on Complex Analytic Geometry

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Last update: July 2022

Chapter 1

Basic notions of complex spaces

1.1 Notations and conventions

The following notations and conventions are assumed throughout the monograph.

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

$\mathbf{i} = \sqrt{-1}$.

$\{0\}, \mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \dots$ are called **(complex) number spaces**.

Unless otherwise stated, all vector spaces are over $\mathbb{C}$.

$\mathbb{C}\{z_1, \dots, z_n\}$ denotes $\mathscr{O}_{\mathbb{C}^n,0}$, the algebra of convergent power series of $z_1, \dots, z_n$. $\mathbb{C}[z_1, \dots, z_n]$ denotes the algebra of polynomials of $z_1, \dots, z_n$.

We assume the readers are familiar with the basic notions of sheaves and their maps (morphisms), sheafifications, image sheaves, kernels and cokernels of sheaves. (A review of these concepts can be found e.g. in [Gui22, Sec.A].) For each presheaf $\mathscr{E}$ on a topological space X, we let $\mathscr{E}_x$ denote the stalk of $\mathscr{E}_x$ at x. If $\varphi : X \to Y$ is a continuous map of topological spaces, then the **direct image** $\varphi_*\mathscr{E}$ denotes the sheaf on Y whose space of sections over any open $V \subset Y$ is $\mathscr{E}(\varphi^{-1}(V))$, i.e.

$$(\varphi_*\mathscr{E})(V) = \mathscr{E}(\varphi^{-1}(V)).$$

If $f : \mathscr{E}_1 \to \mathscr{E}_2$ is a sheaf map, then we have a canonical $\varphi_* f : \varphi_*\mathscr{E}_1 \to \varphi_*\mathscr{E}_2$.

If $\mathscr{F}$ is an $\mathscr{O}_Y$-module, the **inverse image** $\varphi^{-1}(\mathscr{F})$ is the sheafification of the presheaf on X associating to each open subsets of X:

$$U \mapsto \varinjlim_{V \supset \varphi(U)} \mathscr{Y}(V)$$

where the direct limit is over all open subset $V \subset Y$ containing $\varphi(U)$. For each $x \in X$ there is a natural equivalence

$$(\varphi^{-1}\mathscr{F})_x \simeq \mathscr{F}_{\varphi(x)}. \tag{1.1.1}$$

$\mathscr{E}_U$, $\mathscr{E}|U$, $\mathscr{E}|_U$, $\mathscr{E}\restriction_U$ all denote the restriction of an X-sheaf $\mathscr{E}$ to the open subset U. If Y is a subset of X, we define the **set theoretic restriction**

$$\mathscr{E}\restriction_Y = \iota^{-1}(\mathscr{E}). \tag{1.1.2}$$

In particular, for each $y \in Y$, we have a canonical identification

$$(\mathscr{E}\restriction_Y)_y = \mathscr{E}_y. \tag{1.1.3}$$

Warning: in the future, we will define the restriction $\mathscr{E}|_Y = \mathscr{E}|Y$ when Y is a complex subspace of a complex space X and $\mathscr{E}$ is an $\mathscr{O}_X$-module. $\mathscr{E}|Y$ will be different from $\mathscr{E}\restriction_Y$. In particular, $(\mathscr{E}|_Y)_y$ is not $\mathscr{E}_y$.

We also write $\mathscr{E}(U)$ as $H^0(U, \mathscr{E})$.

Recall that the **support of an X-sheaf** $\mathscr{E}$, denoted by $\mathrm{Supp}(\mathscr{E})$, is the subset of all $x \in X$ such that $\mathscr{E}_x = 0$.

1.2 $\mathbb{C}$-ringed spaces and sheaves of modules

1.2.1 $\mathbb{C}$-ringed spaces

Definition 1.2.1. A **$\mathbb{C}$-ringed space** is a topological space X together with a **sheaf of local $\mathbb{C}$-algebras** $\mathscr{O}_X$ on X (i.e., for each open $U \subset X$, $\mathscr{O}_X(U)$ is a $\mathbb{C}$-algebra with unity, and the additions and multiplications are compatible with the restriction to open subsets of U; each stalk $\mathscr{O}_{X,x}$ is a **local $\mathbb{C}$-algebra**).

By saying that $\mathscr{O}_{X,x}$ is a local $\mathbb{C}$-algebra, we mean that there is a unique maximal ideal $\mathfrak{m}_{X,x}$ of $\mathscr{O}_{X,x}$, and that we have an isomorphism of vector spaces

$$\mathbb{C} \xrightarrow{\simeq} \mathbb{C}_x := \mathscr{O}_{X,x}/\mathfrak{m}_{X,x}, \qquad \lambda \mapsto \lambda 1.$$

We write $\mathfrak{m}_{X,x}$ as $\mathfrak{m}_x$ when no confusion arises. For each $f \in \mathscr{O}_{X,x}$, we let $f(x) \in \mathbb{C}$ denote the residue class of f in $\mathscr{O}_{X,x}/\mathfrak{m}_x$, called the **value** of f at x. In this way, any section of $\mathscr{O}_X$ can be viewed as a function.

$\mathscr{O}_X$ is called the **structure sheaf** of X. Each open subset $U \subset X$ is automatically a $\mathbb{C}$-ringed subspace of X with structure sheaf $\mathscr{O}_U := \mathscr{O}_X|_U$. □

For the sake of brevity, we write

$$\mathscr{O}(X) = \mathscr{O}_X(X) \tag{1.2.1}$$

The following important fact is obvious:

Proposition 1.2.2. *An element $f \in \mathscr{O}_{X,x}$ is a unit (i.e. invertible in the ring $\mathscr{O}_{X,x}$) iff $f(x) \neq 0$.*

Proof. $f(x) = 0$ iff $f \in \mathfrak{m}_{X,x}$ iff f is not a unit. $\square$

Definition 1.2.3. A **morphism of $\mathbb{C}$-ringed spaces** $\varphi : X \to Y$ is a continuous map of topological spaces, together with a morphism of sheaves of $\mathbb{C}$-algebras $\varphi^\# : \mathscr{O}_Y \to \varphi_* \mathscr{O}_X$ (namely, $\varphi^\#$ is a sheaf map, and $\varphi^\# : \mathscr{O}_Y(V) \to \mathscr{O}_X(\varphi^{-1}(V))$ is a morphism of $\mathbb{C}$-algebra for each open $V \subset Y$), and for each $x \in X$ and $y = \varphi(x)$, the restriction $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a **morphism of local $\mathbb{C}$-algebras**, i.e. a morphism of $\mathbb{C}$-algebras such that

$$\varphi^\#(\mathfrak{m}_{Y,y}) \subset \mathfrak{m}_{X,x}. \tag{1.2.2}$$

The set of morphisms of $\mathbb{C}$-ringed spaces $X \to Y$ is denoted by $\mathrm{Mor}(X, Y)$. If $\varphi \in \mathrm{Mor}(X, Y)$ and $\psi \in \mathrm{Mor}(Y, Z)$, then their **composition** $\psi \circ \varphi \in \mathrm{Mor}(X, Z)$ is the usual composition of maps of sets, together with

$$(\psi \circ \varphi)^\# = \varphi^\# \circ \psi^\# : \mathscr{O}_{Z,\psi\circ\varphi(x)} \to \mathscr{O}_{X,x}$$

for all $x \in X$.

We leave it to the readers to define isomorphisms of $\mathbb{C}$-ringed spaces.

Proposition 1.2.4. *For each section $f \in \mathscr{O}_Y$ defined at $y = \varphi(x)$, we have*

$$(\varphi^\# f)(x) = f \circ \varphi(x). \tag{1.2.3}$$

Proof. This is true when $f = 1$ since $\varphi^\#$ preserves 1. It is also true when $f \in \mathfrak{m}_{Y,y}$. So it is true in general. $\square$

Thus, $\varphi^\#$ may be viewed as the transpose of φ. When studying morphisms between complex spaces, we may write $\varphi^\# f$ as $f \circ \varphi$ (cf. Rem. 1.4.2).

Example 1.2.5. A complex manifold is a $\mathbb{C}$-ringed space if we define the structure sheaf $\mathscr{O}_X$ to be the sheaf of (germs of) holomorphic functions. If X and Y are complex manifolds, then a morphism of $\mathbb{C}$-ringed spaces from X to Y is exactly a holomorphic map.

1.2.2 Modules over $\mathbb{C}$-ringed spaces

We begin this section the following general observation:

Remark 1.2.6. If $\mathscr{M}, \mathscr{N}$ are two subsheaves of an X-sheaf such that $\mathscr{M}_x = \mathscr{N}_x$ for all $x \in X$, then $\mathscr{M} = \mathscr{N}$. (For any $s \in \mathscr{M}$, $s_x \in \mathscr{M}_x = \mathscr{N}_x$ for all x on which s is defined. So $s \in \mathscr{N}$. So $\mathscr{M} \subset \mathscr{N}$, and vice versa.) Thus, we can talk about" the *unique* subsheaf of a given sheaf whose stalks are..." where the unique part is automatic.

Definition 1.2.7. A **presheaf of $\mathscr{O}_X$-modules** $\mathscr{E}$ on a $\mathbb{C}$-ringed space X is a sheaf such that for each open $U \subset X$, $\mathscr{E}(U)$ is an $\mathscr{O}(U)$-module, and that the linear combination and the action of $\mathscr{O}(U)$ on $\mathscr{E}(U)$ are compatible with the restriction to open subsets of U. If $\mathscr{E}$ is a sheaf, we call $\mathscr{E}$ an $\mathscr{O}_X$**-module**. We call the vector space

$$\mathscr{E}|x = \mathscr{E}_x/\mathfrak{m}_{X,x}\mathscr{E}_x = \mathscr{E}_x \otimes (\mathscr{O}_{X,x}/\mathfrak{m}_{X,x}) \tag{1.2.4}$$

the **fiber** of $\mathscr{E}$ at x. The right most expression of (1.2.4) will be explained in Rem. 1.9.3. The residue class of $s \in \mathscr{E}$ in $\mathscr{E}|x$ is denoted by $s(x)$ or $s|x$.

Definition 1.2.8. A **morphism of (presheaves of)** $\mathscr{O}_X$**-modules** $\varphi : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E}$ and $\mathscr{F}$ are (presheaves of) $\mathscr{O}_X$-modules, is a sheaf map intertwining the actions of $\mathscr{O}_X$. More precisely, for each open $U \subset X$, $\varphi : s \in \mathscr{E}(U) \mapsto \varphi(s) \in \mathscr{F}(U)$ is a morphism of $\mathscr{O}(U)$-modules; if $V \subset U$ is open, then $\varphi(s|_U) = \varphi(s)|_U$.

φ is called **injective** resp. **surjective** if it is so as a sheaf map, namely $\varphi : \mathscr{E}_x \to \mathscr{F}_x$ is injective resp surjective for all $x \in X$. $\mathscr{E} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{G}$ is called **exact** if the corresponding sequence of stalk map $\mathscr{E}_x \xrightarrow{\varphi} \mathscr{F}_x \xrightarrow{\psi} \mathscr{G}_x$ is exact for all $x \in X$. φ is an **isomorphism** of $\mathscr{O}_X$-modules iff φ has an inverse iff φ is both injective and surjective. $\square$

Remark 1.2.9 (Gluing construction of sheaves). Let $(V_\alpha)_{\alpha\in\mathfrak{A}}$ be an open cover of a topological space X. Suppose that for each $\alpha \in \mathfrak{A}$, we have a sheaf $\mathscr{E}^\alpha$, that for any $\alpha, \beta \in \mathfrak{A}$, we have a sheaf isomorphism $\phi_{\beta,\alpha} : \mathscr{E}^\alpha_{V_\alpha\cap V_\beta} \xrightarrow{\simeq} \mathscr{E}^\beta_{V_\alpha\cap V_\beta}$, that $\phi_{\alpha,\alpha} = \mathbf{1}$, and that $\phi_{\gamma,\alpha} = \phi_{\gamma,\beta}\phi_{\beta,\alpha}$ when restricted to $V_\alpha \cap V_\beta \cap V_\gamma$. Then we can define a sheaf $\mathscr{E}$ on X as follows. For any open $U \subset X$, $\mathscr{E}(U)$ is the set of all $(s_\alpha)_{\alpha\in\mathfrak{A}} \in \prod_{\alpha\in\mathfrak{A}} \mathscr{E}^\alpha(U \cap V_\alpha)$ (where each component s_α is in $\mathscr{E}^\alpha(U \cap V_\alpha)$) satisfying that $s_\beta|_{V_\alpha\cap V_\beta} = \phi_{\beta,\alpha}(s_\alpha|_{V_\alpha\cap V_\beta})$ for any $\alpha, \beta \in \mathfrak{A}$. If W is an open subset of U, then the restriction $\mathscr{E}(U) \to \mathscr{E}(W)$ is defined by that of $\mathscr{E}^\alpha(U\cap V_\alpha) \to \mathscr{E}^\alpha(W\cap V_\alpha)$. Then for each $\beta \in \mathfrak{A}$, we have a canonical isomorphism (trivialization) $\phi_\beta : \mathscr{E}_{V_\beta} \xrightarrow{\simeq} \mathscr{E}^\beta_{V_\beta}$ defined by $(s_\alpha)_{\alpha\in\mathfrak{A}} \mapsto s_\beta$. It is clear that for each $\alpha, \beta \in \mathfrak{A}$, we have $\phi_\beta = \phi_{\beta,\alpha}\phi_\alpha$ when restricted to $V_\alpha \cap V_\beta$.

In the case that X is a $\mathbb{C}$-ringed space, that each $\mathscr{E}^\alpha$ is an $\mathscr{O}_{U_\alpha}$-module, and that $\phi_{\beta,\alpha}$ is an equivalence of $\mathscr{O}_{U_\alpha\cap U_\beta}$-modules, then $\mathscr{E}$ is a sheaf of $\mathscr{O}_X$-modules. $\square$

Let X be a $\mathbb{C}$-ringed space.

Definition 1.2.10. A set of sections $\mathfrak{S} \subset \mathscr{O}_X(X)$ is said to **generate** the $\mathscr{O}_X$-module $\mathscr{E}$ if they generate each stalk $\mathscr{E}_x$, i.e., each element of $\mathscr{E}_x$ is an $\mathscr{O}_{X,x}$-linear combination of finitely many elements of $\mathfrak{S}$. Equivalently, this means that the $\mathscr{O}_X$-module morphism

$$\bigoplus_{s\in\mathfrak{S}} \mathscr{O}_X \to \mathscr{E}, \qquad \oplus_s f_s \mapsto \sum_s f_s \cdot s \tag{1.2.5}$$

(where $f_s \in \mathscr{O}_X$) is surjective. If it is also injective, we say $\mathfrak{S}$ **generates freely** $\mathscr{E}$.

Definition 1.2.11. We say an $\mathscr{O}_X$-module $\mathscr{E}$ is of **finite type** if each $x \in X$ is contained in a neighborhood U such that the restriction $\mathscr{E}|_U$ is generated by finitely many elements of $\mathscr{E}(U)$, or equivalently, there is a surjective $\mathscr{O}_U$-module morphism $\mathscr{O}_U^n \to \mathscr{E}|_U$.

Exercise 1.2.12. Show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, then each stalk $\mathscr{E}_x$ is a finitely generated $\mathscr{O}_{X,x}$-module, and hence each fiber $\mathscr{E}|x$ is finite-dimensional.

Definition 1.2.13. If $\mathscr{E}_1, \mathscr{E}_2$ are $\mathscr{O}_X$-submodules of an $\mathscr{O}_X$-module $\mathscr{F}$. The sheafification of the presheaf

$$(\mathscr{E}_1 + \mathscr{E}_2)^{\mathrm{pre}}(U) = \mathscr{E}_1(U) + \mathscr{E}_2(U) \tag{1.2.6}$$

is denoted by $\mathscr{E}_1 + \mathscr{E}_2$. It is the unique subsheaf of $\mathscr{F}$ (cf. Rem. 1.2.6) whose stalks are $(\mathscr{E}_1 + \mathscr{E}_2)_x = \mathscr{E}_1 + \mathscr{E}_2$. It follows that if $\mathscr{E}_1$ is generated by $s_1, s_2, \dots \in \mathscr{E}_1(X)$ and $\mathscr{E}_2$ is generated by $t_1, t_2, \dots \in \mathscr{E}_2(X)$, then $\mathscr{E}_1 + \mathscr{E}_2$ is generated by $s_1, s_2, \dots, t_1, t_2, \dots$.

We recall the well-known

Theorem 1.2.14 (Nakayama's lemma). *If A is a $\mathbb{C}$-local algebra with maximal ideal $\mathfrak{m}$, and if $\mathcal{M}$ is a finitely generated A-module. Then a finite set of elements $s_1, \dots, s_n \in \mathcal{M}$ generate the A-module $\mathcal{M}$ (i.e. elements of $\mathcal{M}$ are A-linear combinations of $s_1, \dots, s_n$) iff their residue classes in $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$ span the vector space $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$.*

Indeed, this is true when A is in general a local ring. In that case, $\mathcal{M}/\mathfrak{m} \cdot \mathcal{M}$ is a vector space over the field $A/\mathfrak{m}$.

Proof. [AM, Prop. 2.8] or [Gui22, Sec. A]. □

To apply Nakayama's lemma to sheaves of modules, we need the following observation:

Remark 1.2.15. Let $\mathscr{E}$ be a finite-type $\mathscr{O}_X$-module. Let $s_1, \dots, s_n$ be sections of $\mathscr{E}$ defined on a neighborhood of $x \in X$. Suppose (the germs of) $s_1, \dots, s_n$ generate the $\mathscr{O}_{X,x}$-module $\mathscr{E}_x$. Then there is a possibly smaller neighborhood U of x such that $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. In particular, "$\mathscr{E}_x$ generates $\mathscr{E}|_U$".

Proof. Since $\mathscr{E}$ is finite-type, we may find U such that $\mathscr{E}|_U$ is generated by $t_1, \dots, t_m \in \mathscr{E}(U)$. Since $s_1, \dots, s_n$ generate $\mathscr{O}_x$, the germs of $t_1, \dots, t_m$ are $\mathscr{O}_{X,x}$-linear combinations of $s_1, \dots, s_n$. Thus, on a possibly smaller U, $t_1, \dots, t_m$ are $\mathscr{O}_X(U)$-linear combinations of $s_1, \dots, s_n$. So $s_1, \dots, s_n$ generate $\mathscr{E}|_U$. □

Exercise 1.2.16. Use Nakayama's lemma and Rem. 1.2.15 to show that if $\mathscr{E}$ is a finite type $\mathscr{O}_X$-module, and if $s_1, \dots, s_n \in \mathscr{E}(U)$ (where U is a neighborhood of x) are such that $s_1(x), \dots, x_n(x)$ span the fiber $\mathscr{E}|x$, then they generate $\mathscr{E}|_V$ for a possibly smaller neighborhood V of x. (The opposite direction is obvious.) Nakayama's lemma is most often used in this form.

Definition 1.2.17. We say that an $\mathscr{O}_X$-module $\mathscr{E}$ is **free** if is is isomorphic to $\mathscr{O}_X^n$ for some $n \in \mathbb{N}$. We say $\mathscr{E}$ is **locally free** if each $x \in X$ is contained in a neighborhood U such that $\mathscr{E}|_U$ is free (or equivalently, that $\mathscr{E}|_U$ is generated freely by finitely many elements of $\mathscr{E}(U)$).

Exercise 1.2.18. Show that for a complex manifold X, locally free $\mathscr{O}_X$-modules $\mathscr{E}$ are the same as holomorphic vector bundles on X. Describe local trivializations and transition functions in terms of local free generators of $\mathscr{E}$. (See e.g. [Gui22, Sec. A] for details.)

Definition 1.2.19. An **ideal sheaf** $\mathcal{I}$ on a $\mathbb{C}$-ringed space X is an $\mathscr{O}_X$-submodule of $\mathscr{O}_X$. In particular, each stalk $\mathcal{I}_x$ is an ideal of $\mathscr{O}_{X,x}$. The **zero set** $N(\mathcal{I})$ is defined to be

$$\begin{aligned} N(\mathcal{I}) &:= \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}_x\} = \{x \in X : \mathcal{I}_x \subsetneq \mathfrak{m}_{X,x}\} \\ &=\{x \in X : \mathcal{I}_x \neq \mathscr{O}_{X,x}\} = \mathrm{Supp}(\mathscr{O}_U/\mathcal{I}). \end{aligned} \tag{1.2.7}$$

Proof. Note that $(\mathscr{O}_U/\mathcal{I})_x = \mathscr{O}_{U,x}/\mathcal{I}_x$. So $x \in \mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ iff $\mathscr{O}_{U,x}/\mathcal{I}_x \neq 0$ iff $\mathcal{I}_x \subsetneq \mathscr{O}_{U,x}$ iff $\mathcal{I}_x \subset \mathfrak{m}_x$ (as $\mathfrak{m}_x$ is the unique maximal ideal) iff $f(x) = 0$ for all $f \in \mathfrak{m}_x$. □

Remark 1.2.20. If $\mathcal{I}$ is generated by $f_1, \dots, f_n \in \mathscr{O}_X$, written as

$$\mathcal{I} = f_1\mathscr{O}_X + \cdots + f_n\mathscr{O}_X,$$

then clearly

$$N(\mathcal{I}) = \{\text{The common zeros of } f_1, \dots, f_n\}, \tag{1.2.8}$$

which is a closed subset of X. Thus, in general, if $\mathcal{I}$ is finite-type, then each $x \in X$ is contained in a neighborhood U such that $U \cap N(\mathcal{I})$ is closed in U; so $N(\mathcal{I})$ is closed in X.

$$\mathcal{I} \text{ is finite type} \Longrightarrow N(\mathcal{I}) \text{ is closed in } X$$

1.3 Complex spaces and subspaces

Definition 1.3.1. A **(complex) model space** is

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})}\big) \tag{1.3.1}$$

where U is an open subset of a number space $\mathbb{C}^n$, $\mathscr{O}_U$ is the sheaf of holomorphic functions on U, $\mathcal{I}$ is a *finite-type* ideal of $\mathscr{O}_U$. $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is called the **analytic spectrum** of the sheaf $\mathscr{O}_U/\mathcal{I}$. Its underlying topological space is $\mathrm{Supp}(\mathscr{O}_U/\mathcal{I})$ as a subset of U, and its structure sheaf is $(\mathscr{O}_U/\mathcal{I})\restriction_{N(\mathcal{I})}$, whose stalk at any $x \in N(\mathcal{I})$ is $\mathscr{O}_{U,x}/\mathcal{I}_x$ (cf. (1.1.3)). With abuse of notations, one also writes for simplicity

$$\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) := \big(N(\mathcal{I}), \mathscr{O}_U/\mathcal{I}\big). \tag{1.3.2}$$

The stalk at $x \in N(\mathcal{I})$ of the structure sheaf is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{U,x}/\mathcal{I}_x, \mathfrak{m}_{U,x}/\mathcal{I}_x\big)$$

Definition 1.3.2. A $\mathbb{C}$-ringed Hausdorff space X is called a **complex space** if each point $x \in X$ is contained in a neighborhood V such that the $\mathbb{C}$-ringed space V (whose structure sheaf is defined by $\mathscr{O}_V := \mathscr{O}_X|_V$) is isomorphic to a model space. Sections of $\mathscr{O}_X(X)$ are called **holomorphic functions on** X. $\mathscr{O}_{X,x}$ is called an **analytic local $\mathbb{C}$-algebra**. Equivalently, an analytic local $\mathbb{C}$-algebra is $\mathbb{C}\{z_1,\dots,z_n\}/I$ for some finitely generated ideal I.[1]

If X, Y are complex spaces, a morphism $\varphi : X \to Y$ of $\mathbb{C}$-ringed spaces is called a **holomorphic map**. If φ has an inverse morphism $Y \to X$, we say that φ is a **biholomorphism**. Clearly, a holomorphic map φ is a biholomorphism iff it is a homeomorphism of topological spaces and induces isomorphisms $\varphi^\# : \mathscr{O}_{Y,\varphi(x)} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ for each $x \in X$. □

Definition 1.3.3. A **morphism of (analytic) local $\mathbb{C}$-algebras** $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a homomorphism of unital algebra sending $\mathfrak{m}_{Y,y}$ into $\mathfrak{m}_{X,x}$.

Definition 1.3.4. Let X be a complex space. An **open complex subspace** is $(U, \mathscr{O}_X|_U)$ where U is an open subset of X. A **closed complex subspace** is

$$\mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) := \big(N(\mathcal{I}), (\mathscr{O}_X/\mathcal{I})\restriction_{N(\mathcal{I})}\big) \tag{1.3.3}$$

where $\mathcal{I}$ is a finite type ideal of $\mathscr{O}_X$. (The closedness is justified by Rem. 1.2.20.) The stalk of the structure sheaf at $x \in N(\mathcal{I})$ is a local $\mathbb{C}$-algebra

$$\big(\mathscr{O}_{X,x}/\mathcal{I}_x, \mathfrak{m}_x/\mathcal{I}_x\big).$$

We have an obvious inclusion map which is holomorphic:

$$\iota : \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$$

such that $\iota^\#$ is the quotient map $\mathscr{O}_X \to \mathscr{O}_X/\mathcal{I} = \iota_*\iota^{-1}(\mathscr{O}_X/\mathcal{I})$.

[1] As we shall see, $\mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian. So the condition that I is finitely generated is redundant.

Remark 1.3.5. Let $X_0 = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. The construction of $\mathscr{O}_{X_0} = (\mathscr{O}_X/\mathcal{I})\restriction_{N(\mathcal{I})}$ involves two sheafifications: one for quotient, and one for set-theoretic restriction. It would be convenient to combine these two into one: $\mathscr{O}_{X_0}$ is the sheafification of the presheaf $\mathscr{O}_{X_0}^{\mathrm{pre}}$ sending each open $U_0 \subset X_0$ (more precisely, $U_0 \subset N(\mathcal{I})$) to

$$\mathscr{O}_{X_0}^{\mathrm{pre}}(U_0) = \varinjlim_{U \supset U_0} \mathscr{O}_X(U)/\mathcal{I}(U) \tag{1.3.4}$$

where the direct limit is over all open $U \subset X$ containing U_0. Indeed, one can also take the direct limit over all open U satisfying $U \cap N(\mathcal{I}) = U_0$.

Complex spaces arise from

Remark 1.3.6 (Gluing construction of complex spaces). Suppose X is a Hausdorff space with an open cover $\mathfrak{V} = (V_\alpha)$. Suppose that for each V_α there is a homoemorphism $\varphi_\alpha : V_\alpha \to U_\alpha$ where U_α is a complex space. Suppose also that for each α, β, the homeomorphism $\varphi_\beta\varphi_\alpha^{-1} : \varphi_\alpha(V_\alpha \cap V_\beta) \to \varphi_\beta(V_\alpha \cap V_\beta)$ (where the source and the target are regarded as open subspaces of U_α and U_β respectively) can be extended to an isomorphism $\varphi_{\beta,\alpha}$ of $\mathbb{C}$-ringed spaces satisfying the **cocycle condition**: for all α, β, γ, we have $\varphi_{\alpha,\alpha} = \mathbf{1}$ and $\varphi_{\gamma,\alpha} = \varphi_{\gamma,\beta}\varphi_{\beta,\alpha}$ (from $\varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$ to $\varphi_\gamma(V_\alpha \cap V_\beta \cap V_\gamma)$). Then X is naturally a complex space such that $\varphi_\alpha : V_\alpha \to U_\alpha$ is extended to an isomorphism of $\mathbb{C}$-ringed spaces such that $\varphi_\beta = \varphi_{\beta,\alpha}\varphi_\alpha$ (from $V_\alpha \cap V_\beta$ to $\varphi_\beta(V_\alpha \cap V_\beta)$). Indeed, $\mathscr{O}_X$ is constructed by gluing all the V_α-sheaves $\varphi_\alpha^{-1}\mathscr{O}_{U_\alpha}$ (cf. Rem. 1.2.9).

Let us see some examples of complex spaces. We begin with an easier class of examples:

Definition 1.3.7. Let X be a complex space, and let $\mathscr{C}_X$ be the sheaf of complex valued continuous functions on X. Then there is a natural **morphism of sheaves of local $\mathbb{C}$-algebras** (i.e. a morphism of X-sheaves which preserve the linear structures and algebra multiplications when restricted to each open subset, and whose stalk maps send the maximal ideals into maximal ones)

$$\mathrm{red} : \mathscr{O}_X \to \mathscr{C}_X \tag{1.3.5}$$

sending each $f \in \mathscr{O}_X$ to f as a function (cf. Def. 1.2.1). If $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective, we say X is **reduced at** $x \in X$. If X is reduced everywhere, X is called a **reduced complex space** (also called a **(complex) analytic variety**).

Thus, a holomorphic function on a reduced complex space can be viewed as a genuine continuous function without losing information. (Formally speaking: $\mathscr{O}_X$ is a subsheaf of $\mathscr{C}_X$.) For non-reduced spaces, holomorphic functions cannot be viewed as genuine functions.

Remark 1.3.8. In commutative algebra, there is a notion of reducedness: $\mathscr{O}_{X,x}$ is called reduced if it has no non-zero nilpotent element. We will see later that a complex space X is reduced at x iff $\mathscr{O}_{X,x}$ is a reduced ring. This is the famous Nullstellensatz.

Example 1.3.9. Let $U \subset \mathbb{C}^m \times \mathbb{C}^n$ be open, and let $\mathcal{I} = z_1\mathscr{O}_U + \cdots + z_m\mathscr{O}_U$. Then $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally equivalent to the complex submanifold $U \cap (0 \times \mathbb{C}^n) \simeq U \cap \mathbb{C}^n$ (whose structure sheaf is the sheaf of holomorphic functions $f(\zeta_1, \ldots, \zeta_n)$).

Proof. Clearly $N(\mathcal{I}) = U \cap \mathbb{C}^n$ (cf. Rem. 1.2.20). Consider the identity map $\varphi : U \cap \mathbb{C}^n \to X$ as a homeomorphism of topological spaces. In particular, we have an isomorphism $\mathrm{red}\varphi^\# : \mathscr{C}_X \to \mathscr{C}_{U\cap\mathbb{C}^n}$. We shall construct $\varphi^\# : \mathscr{O}_X = \mathscr{O}_U/\mathcal{I} \restriction_{N(\mathcal{I})} \to \mathscr{O}_{U\cap\mathbb{C}^n}$ such that φ is an isomorphism of $\mathbb{C}$-ringed spaces.

By (1.1.3), for each $x \in U \cap \mathbb{C}^n$,

$$\mathscr{O}_{X,x} = ((\mathscr{O}_U/\mathcal{I}) \restriction_{N(\mathcal{I})})_x \simeq \mathscr{O}_{\mathbb{C}^{m+n},x}/\mathcal{I}_x \simeq \mathscr{O}_{\mathbb{C}^n,x}$$

where the last isomorphism can be seen by taking power series expansions of $f(z_\bullet, \zeta_\bullet) = f(z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_n)$ at n and throwing away every terms containing powers of $\zeta_\bullet$. Define a sheaf map

$$\varphi^\# : \mathscr{O}_X \xrightarrow{\mathrm{red}} \mathscr{C}_X \xrightarrow[\simeq]{\mathrm{red}\varphi^\#} \mathscr{C}_{U\cap\mathbb{C}^n}.$$

Its stalk map is $\mathscr{O}_{\mathbb{C}^n,x} \to \mathscr{C}_{U\cap\mathbb{C}^n}$ sending each f to the function f itself. From this we see that the stalk map is injective and has image $\mathscr{O}_{U\cap\mathbb{C}^n,x}$. This shows that $\varphi^\#$ is an injective sheaf map with image $\mathscr{O}_{U\cap\mathbb{C}^n}$. So $\varphi^\#$ restricts to an isomorphism of sheaves of local $\mathbb{C}$-algebras $\mathscr{O}_X \to \mathscr{O}_{U\cap\mathbb{C}^n}$. $\square$

Remark 1.3.10. The proof of Exp. 1.3.9 suggests a way of understanding a *reduced* model space $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$: 1. Find the underlying topological space $N(\mathcal{I})$. 2. Understand each stalk $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}/\mathcal{I}_x$ and show that $\mathrm{red} : \mathscr{O}_{X,x} \to \mathscr{C}_{X,x}$ is injective. 3. Find a familiar sheaf of local $\mathbb{C}$-subalgebras $\mathscr{A} \subset \mathscr{C}_X$ such that $\mathscr{A}_x = \mathrm{red}(\mathscr{O}_{X,x})$. Then $X \simeq (N(\mathcal{I}), \mathscr{A})$.

Exercise 1.3.11. Let U be a neighborhood of $0 \in \mathbb{C}^2$. Let z, w be the standard coordinates of $\mathbb{C}^2$. Let $\mathcal{I} = zw \cdot \mathscr{O}_U$, the ideal sheaf generated by the function zw. Show that $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is equivalent to the $\mathbb{C}$-ringed space whose underlying topological space is $N(\mathcal{I}) = \{(z, w) \in U : z = 0 \text{ or } w = 0\}$, and whose structure sheaf is the sheaf of continuous functions on open subsets of $N(\mathcal{I})$ that are holomorphic when restricted respectively to the z-axis and to the w-axis.

Example 1.3.12. Let $k \in \mathbb{Z}_+$. Let U be a neighborhood of $0 \in \mathbb{C}$. We call $\mathrm{Specan}(\mathscr{O}_U/z^k\mathscr{O}_U) = (0, \mathbb{C}\{z\}/z^k\mathbb{C}\{z\}) = (0, \mathbb{C}[z]/z^k\mathbb{C}[z])$ the k**-fold point**. It is not reduced when $k > 1$. A single point denotes a 1-fold point, which is the same as the connected 0-dimensional complex manifold $\mathbb{C}^0$.

1.4 Holomorphic maps

In order to construct complex spaces by gluing model spaces (Rem. 1.3.6), and to understand holomorphic maps between complex spaces, we need to understand morphisms (i.e. holomorphic maps) between model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I}) \to \mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ (where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open). This is a main goal of this section.

The first step is to understand the case that target is just V. As one may expect, holomorphic maps in this case are described by an n-tuple of holomorphic functions. Recall that $\mathrm{Mor}(X, Y)$ is the set of holomorphic maps from the complex space X to Y. Let $z_1, \dots, z_n$ be the standard coordinates of $\mathbb{C}^n$.

Theorem 1.4.1. *Let X be a complex space. Then the following map is bijective:*

$$\mathrm{Mor}(X, \mathbb{C}^n) \to \mathscr{O}(X)^n, \qquad \varphi \mapsto (\varphi^\# z_1, \dots, \varphi^\# z_n). \tag{1.4.1}$$

Remark 1.4.2. Due to this theorem, if $\psi : X \to Y$ is a holomorphic map and $f \in \mathscr{O}(Y)$, then we may write

$$f \circ \psi = \psi^\# f \tag{1.4.2}$$

by viewing f as a holomorphic map $Y \to \mathbb{C}$.

The proof of Thm. 1.4.1 relies on the Noetherian property of $\mathscr{O}_{X,x}$, whose proof is deferred to the next section.

Proof that (1.4.1) *is surjective assuming* (1.4.1) *is injective.* Assume (1.4.1) is injective for all complex spaces. Fix X and $F = (f_1, \dots, f_n) \in \mathscr{O}(X)^n$. We claim that each $x \in X$ is contained in a neighborhood U_x such that $F|_{U_x} \in \mathscr{O}(U_x)^n$ corresponds to some $\varphi_x \in \mathrm{Mor}(U_x, \mathbb{C}^n)$. By the injectivity, for every $x, y \in X$, φ_x and φ_y agree on $U_x \cap U_y$. Gluing all φ_x together gives the desired φ corresponding to F.

To prove the claim, we may assume U_x is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathcal{I})$ where $V \subset \mathbb{C}^m$ is open and $\mathcal{I}$ is finite-type. Since the stalk $(\mathscr{O}_V/\mathcal{I})|_x$ equals $\mathscr{O}_{V,x}/\mathcal{I}_x$, we can further shrink U_x so that $F|_{U_x}$ can be lifted to $\widetilde{F}|_V \in \mathscr{O}(V)^n$. $\widetilde{F}$ can be viewed as a holomorphic map $V \to \mathbb{C}^n$. Its composition with the inclusion $\iota : \mathrm{Specan}(\mathscr{O}_V/\mathcal{I}) \hookrightarrow V$ gives the desired holomorphic map φ. □

Proof that (1.4.1) *is injective.* Let $\varphi_1, \varphi_2 \in \mathrm{Mor}(X, \mathbb{C}^n)$ correspond to the same element $(f_1, \dots, f_n)$ of $\mathscr{O}(X)^n$. By (1.2.3), $z_i \circ \varphi_\bullet(x) = (\varphi_\bullet^\# z_i)(x) = f_i(x)$. So φ_1 equals φ_2 as set maps, i.e. $\varphi_\bullet(x) = (f_1(x), \dots, f_n(x))$. Checking that they are equal as morphisms of $\mathbb{C}$-ringed spaces is equivalent to showing for any x that $\varphi_1^\# = \varphi_2^\#$ as maps from $\mathscr{O}_{\mathbb{C}^n, \varphi_\bullet(x)} = \mathscr{O}\{z_1 - f_1(x), \dots, z_n - f_n(x)\}$ to $\mathscr{O}_{X,x}$. We know that they both send each $z_i - f_i(x)$ to $f_i - f_i(x)$. So they are equal by the uniqueness part of the following proposition. □

The following proposition can be viewed as the infinitesimal version of Thm. 1.4.1. (This will become clear after the readers read Thm. 1.6.2.)

Proposition 1.4.3. *Let $\mathscr{O}_{X,x}$ be an analytic local $\mathbb{C}$-algebra. Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathscr{O}_{X,x}$. Then there is a unique morphism of local $\mathbb{C}$-algebras satisfying*

$$\Phi : \mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\} \to \mathscr{O}_{X,x}, \qquad z_i \mapsto f_i - f_i(x). \tag{1.4.3}$$

Note that as a morphism of *local* rings, Φ is assumed to send $\mathfrak{m}_{\mathbb{C}^n,0} = \sum_{j=1}^n z_j \mathbb{C}\{z_1, \dots, z_n\}$ into $\mathfrak{m}_{X,x}$.

Proof. Assume for simplicity that $f_\bullet(x) = 0$.

Existence: By the second paragraph of the proof that (1.4.1) is surjective (which does not rely on the injectivity of (1.4.1)), by shrinking X, we may choose a holomorphic map $\phi : X \to \mathbb{C}^n$ corresponding to $(f_1, \dots, f_n)$. Then the stalk map $\phi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ gives Φ.

Injectivity: Assume Φ_1, Φ_2 both satisfy the requirement. Then they clearly agree when restricted to the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. Now we choose $g \in \mathbb{C}\{z_\bullet\}$. For each $k \in \mathbb{N}$, we may write g as a polynomial of $z_\bullet$ plus $g_k \in \mathfrak{m}^k_{\mathbb{C}^n,0}$. So $\Phi_1(g) - \Phi_2(g)$ equals $\Phi_1(g_k) - \Phi_2(g_k)$, which belongs to $\mathfrak{m}^k_{X,x}$ since Φ_i sends $\mathfrak{m}_{\mathbb{C},0}$ into $\mathfrak{m}_{X,x}$ and hence $\mathfrak{m}^k_{\mathbb{C},0}$ into $\mathfrak{m}^k_{X,x}$. So $\Phi_1(g) - \Phi_2(g)$ belongs to $\bigcap_{k \in \mathbb{N}} \mathfrak{m}^k_{X,x}$, which is 0 due to the following theorem *and the fact that $\mathscr{O}_{X,x}$ is Noetherian.* $\square$

Theorem 1.4.4 (Krull's intersection theorem). *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{M}$ be a finitely-generated A-module. Then $\bigcap_{k \in \mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M} = 0$.*

Proof. The submodule $\mathcal{N} = \bigcap_{k \in \mathbb{N}} \mathfrak{m}^k \cdot \mathcal{M}$ is also finitely generated as A is Noetherian. Then $\mathcal{N} = 0$ will follow from $\mathfrak{m}\mathcal{N} = \mathcal{N}$ (equivalently, 0 spans the "fiber" $\mathcal{N}/\mathfrak{m}\mathcal{N}$) and Nakayama's lemma. That $\mathfrak{m}\mathcal{N} = \mathcal{N}$ is due to Artin-Rees lemma (applied to the $\mathfrak{m}$-stable filtration $(\mathfrak{m}^k\mathcal{M})_{k \in \mathbb{N}}$ to show that $(\mathcal{N} \cap \mathfrak{m}^k\mathcal{M})_{k \in \mathbb{N}} = (\mathcal{N})_{k \in \mathbb{N}}$ is $\mathfrak{m}$-stable). $\square$

Recall that if I is an ideal of a ring A, an I-**filtration** $(\mathcal{M}_n)_{n \in \mathbb{N}}$ (of $\mathcal{M}_0$) is a descending chain of A-modules $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots$ satisfying $I\mathcal{M}_n \subset \mathcal{M}_{n+1}$ for all $n \in \mathbb{N}$. It is called I-**stable** if for some $N \in \mathbb{N}$ we have $I\mathcal{M}_n = \mathcal{M}_{n+1}$ for all $n \geqslant N$.

Theorem 1.4.5 (Artin-Rees lemma). *Let I be an ideal of a Noetherian ring A. Then for any I-stable filtration of $(\mathcal{M}_n)_{n \in \mathbb{N}}$ inside a finitely-generated A-module $\mathcal{M}$, and for any submodule $\mathcal{N} \subset \mathcal{M}$, $(\mathcal{N} \cap \mathcal{M}_n)_{n \in \mathbb{N}}$ is I-stable.*

Proof. This follows from two ingredients: 1. The graded ring $A_\bullet = (A, I, I^2, \cdots)$ is a quotient of the Noetherian ring $A[z_1, \dots, z_m]$ if I is generated by m elements. So $A_\bullet$ is Noetherian. 2. An I-filtration $(\mathcal{M}_0)_{n \in \mathbb{N}}$ of finitely-generated A-modules is I-stable iff the graded $A_\bullet$-module $\mathcal{M}_\bullet = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \cdots)$ is finitely-generated. See [AM, Sec. 10.3] for details. $\square$

The uniqueness part of Thm. 1.4.1 can be formulated in the following form.

Remark 1.4.6 (Substitution rule). Let X be a complex space, let $\mathcal{I}$ be a finite type ideal of $\mathscr{O}_X$ containing $f_1-g_1,\dots,f_n-g_n$ where $f_\bullet,g_\bullet\in\mathscr{O}(X)$. Let $F=(f_1,\dots,f_n)$ and $G=(g_1,\dots,g_n)$. Let $h\in\mathscr{O}_{\mathbb{C}^n}$. Then $F^\#h$ and $G^\#h$ restrict to the same (locally defined) holomorphic function of $Y=\mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$, i.e. they are equal as elements of $\mathscr{O}_X/\mathcal{I}$.

Proof. f_i and g_i are equal as holomorphic functions of Y. So by Thm. 1.4.1, F and G are the same holomorphic map $X\to\mathbb{C}^n$. So $F^\#h$ equals $G^\#h$ as elements of $\mathscr{O}_Y$. □

Example 1.4.7. Let $U\subset\mathbb{C}^2$ be open, let $f\in\mathscr{O}(U)$, and let $\mathcal{I}$ be the ideal sheaf of $\mathscr{O}_U$ generated by $z_2-f(z_1,z_2)$. Then for each $h\in\mathscr{O}_{\mathbb{C}^2}$, $h(z_1,z_2)$ and $h(z_1,f(z_1,z_2))$ are equal as elements of $\mathscr{O}_U/\mathcal{I}$.

We have seen how a holomorphic map from a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ to $V\subset\mathbb{C}^n$ looks like. The next question is when this map "has image in $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$"? This is answered by the following theorem whose proof does not rely on the Noetherian property.

Theorem 1.4.8. *Let $\varphi:X\to Y$ be a holomorphic map of complex spaces. Let $X_0=\mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$ and $Y_0=\mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$ be closed complex subspaces of X and Y respectively. Then the following are equivalent:*

(a) There is a (necessarily unique) holomorphic map $\psi:X_0\to Y_0$ such that the following diagram commutes:

$$\begin{tikzcd} X_0 \arrow[r, "\psi"] \arrow[d, hook] & Y_0 \arrow[d, hook] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd} \tag{1.4.4}$$

(b) For each $x\in X$ and $y=\varphi(x)$, the stalk map $\varphi^\#:\mathscr{O}_{Y,y}\to\mathscr{O}_{X,x}$ satisfies

$$\varphi^\#(\mathcal{J}_y)\subset\mathcal{I}_x$$

Proof. Assume (a). If $x\in X_0$, then each $f\in\mathcal{J}_y\subset\mathscr{O}_{Y,y}$ is sent by the transpose $\iota^\#_{Y_0,Y}$ to 0. Also f is sent by $\varphi^\#$ to $\varphi^\#(f)\in\mathscr{O}_{X,x}$, and then sent by $\iota^\#_{X_0,X}$ to $\varphi^\#(f)+\mathcal{I}_x$ in $\mathscr{O}_{X_0,x}=\mathscr{O}_{X,x}/\mathcal{I}_x$, which must be 0 since (1.4.4) commutes. So $\varphi^\#(f)\in\mathcal{I}_x$.

If $x\in X\backslash X_0$, then $x\neq N(\mathcal{I})$. So some element of $\mathcal{I}_x$ has non-zero value at x, which must be invertible. So $\mathcal{I}_x=\mathscr{O}_{X,x_0}$. Then clearly $\varphi^\#(\mathcal{J}_y)\subset\mathcal{I}_x$. (b) is proved.

Now assume (b). If $y\notin N(\mathcal{J})$, then $\mathcal{J}_y=\mathscr{O}_{Y,y}$. So $1\in\mathcal{J}_y$, and so $1=\varphi^\#(1)$ belongs to $\mathcal{I}_x$. Therefore $x\notin N(\mathcal{I})$. This proves $\varphi(N(\mathcal{I}))\subset\varphi(N(\mathcal{J}))$. So ψ exists as a continuous map of topological spaces, and such a map is clearly unique.

Choose $x \in X_0$ i.e. $x \in N(\mathcal{I})$. By (b), we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X_0,x} = \mathscr{O}_{X,x}/\mathcal{I}_x & \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \arrow[l, "\psi^\#"'] \\
\mathscr{O}_{X,x} \arrow[u, two heads] & \mathscr{O}_{Y,y} \arrow[l, "\varphi^\#"] \arrow[u, two heads]
\end{tikzcd}$$

for a unique stalk map $\psi^\# : \mathscr{O}_{Y_0,y} \to \mathscr{O}_{X_0,x}$, which is clearly a morphism of local $\mathbb{C}$-algebras. It remains to show that these stalk maps can be assembled into a sheaf map.

Recall the presheaves in Rem. 1.3.5. For each open $V \subset Y$, (b) implies $\varphi^\#(\mathcal{J}(V)) \subset \mathcal{I}(\varphi^{-1}(V))$. So the map $\varphi^\# : \mathscr{O}_Y(V) \to (\varphi_*\mathscr{O}_X)(V) = \mathscr{O}_X(\varphi^{-1}(V))$ descends to

$$\mathscr{O}_Y(V)/\mathcal{J}(V) \to \mathscr{O}_X(\varphi^{-1}(V))/\mathcal{I}(\varphi^{-1}(V)).$$

By taking direct limit over all V containing a fixed open $V_0 \subset X_0$, we obtain

$$\mathscr{O}_{Y_0}^{\mathrm{pre}}(V_0) \to \mathscr{O}_{X_0}^{\mathrm{pre}}(\varphi^{-1}(V_0)) \xrightarrow{\text{restriction}} \mathscr{O}_{X_0}^{\mathrm{pre}}(\psi^{-1}(V_0))$$

(note that $\psi^{-1}(V_0)$ is precisely $\varphi^{-1}(V_0) \cap X_0$). Its composition with

$$\mathscr{O}_{X_0}^{\mathrm{pre}}(\psi^{-1}(V_0)) \to \mathscr{O}_{X_0}(\psi^{-1}(V_0)) = (\psi_*\mathscr{O}_{X_0})(V_0)$$

gives a presheaf map $\mathscr{O}_{Y_0}^{\mathrm{pre}} \to \psi_*\mathscr{O}_{X_0}$ whose sheafification is the desired $\psi^\# : \mathscr{O}_{Y_0} \to \psi_*\mathscr{O}_{X_0}$. $\square$

1.5 Weierstrass division theorem and Noetherian property of $\mathscr{O}_{X,x}$

1.5.1 Main results

Now that we have seen the importance of the Noetherian property, we prove this in this section. Since $\mathscr{O}_{X,x}$ is a quotient of $\mathscr{O}_{\mathbb{C}^n,0}$, it suffices to prove that $\mathscr{O}_{\mathbb{C}^n,0}$ is Noetherian. The proof relies on Weierstrass division theorem, which we state below.

Definition 1.5.1. We say that $f(z) \in \mathbb{C}\{z\}$ has **order** $k \in \mathbb{N} \cup \{\infty\}$ if $f(z) = z^k(a_k + a_{k+1}z + a_{k+2}z^2 + \cdots)$ and $a_k \neq 0$; f has order ∞ iff $f = 0$. More generally, for $m \in \mathbb{N}$, we say that $f(w_\bullet, z) = f(w_1, \ldots, w_m, z) \in \mathbb{C}\{w_\bullet, z\}$ has **order** k **(in** z**)** if $f(0, z) \in \mathbb{C}\{z\}$ has order k. Equivalently, $f(w_\bullet, z) = \sum_{i=0}^\infty a_k(w_\bullet)z^k$ where

$$a_0(0) = \cdots = a_{k-1}(0) = 0, \qquad a_k(0) \neq 0. \tag{1.5.1}$$

That f has order ∞ in z means $a_i(0) = 0$ for all i.

Recall that the **degree** of a polynomial $p(w_\bullet, z) \in \mathbb{C}\{w_\bullet\}[z]$ is the smallest power of z whose coefficient is a non-zero element of $\mathbb{C}\{w_\bullet\}$. The degree of zero polynomial is set to be $-\infty$. □

Remark 1.5.2. Let $f(w_\bullet, z)$ have order $k < \infty$ in z, defined on a neighborhood of 0. Then inside this neighborhood we can find a smaller one $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ such that $f(0, z)$ has one zero in V (namely $z = 0$) with multiplicity k. By Rouché's theorem, we may shrink U such that for each fixed $w_\bullet \in U$, the holomorphic function $f(w_\bullet, z)$ of z has k zeros in V counting multiplicities; see Fig. 1.5.1

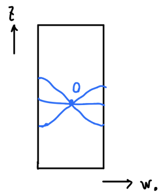

Figure 1.5.1

In the following, we suppress the variable $w_\bullet$ when necessary.

Theorem 1.5.3 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\}$ has order $k < \infty$ in z. Then for each $f \in \mathbb{C}\{w_\bullet, z\}$, there exist unique $q \in \mathbb{C}\{w_\bullet, z\}$ and $r \in \mathbb{C}\{w_\bullet\}[z]$ with degree $< k$ such that $f = gq + r$.*

We shall prove the Noetherian property using the following equivalent form of WDT.

Theorem 1.5.4 (Weierstrass division theorem (WDT)). *Suppose $g \in \mathbb{C}\{w_\bullet, z\} = \mathscr{O}_{\mathbb{C}^{m+1}}$ has order $k < \infty$ in z. Then $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a rank-k free $\mathscr{O}_{\mathbb{C}^m}$-module. $1, z, \dots, z^{k-1}$ are a set of free generators.*

Theorem 1.5.5. *Every analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$ is Noetherian.*

Proof. It suffices to discuss $\mathscr{O}_{\mathbb{C}^n,0}$. We prove this by induction on n. The case $n = 0$ is trivial. Suppose the case $m = n - 1$ is known. We prove the case $m + 1$. Choose any ideal non-zero $I \subset \mathscr{O}_{\mathbb{C}^{m+1},0}$. Choose $0 \neq g \in I$. Then on a complex line passing through 0, 0 must be an isolated zero of h. (Otherwise, on each line, g vanishes on a neighborhood of 0. So g vanishes on each line (and hence each domain containing 0) by complex analysis.) By choosing new coordinates, we may assume the last coordinate axis is that line. Namely, writing $g = g(w_1, \dots, w_m, z)$, g has finite order in z.

By WDT, $\mathscr{O}_{\mathbb{C}^{m+1},0}/g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is a finitely-generated $\mathscr{O}_{\mathbb{C}^m}$-module. Its submodule $I/I \cap g\mathscr{O}_{\mathbb{C}^{m+1},0}$ is generated by finitely many elements $f_1, \dots, f_N \in I$, thanks to the assumption that $\mathscr{O}_{\mathbb{C}^m}$ is Noetherian. So elements of I are $\mathscr{O}_{\mathbb{C}^{m+1}}$-linear combinations of $f_1, \dots, f_N, g$. □

1.5.2 Proof of WDT

We prove the first version of WDT following [GR].

Proof of the uniqueness. Let $f = gq_1+r_1 = gq_2+r_2$. Then $g(q_1-q_2) = r_2-r_1$. Choose a small enough neighborhood $U \times V \subset \mathbb{C}^m \times \mathbb{C}$ as in Rem. 1.5.2 such that for all fixed $w_\bullet \in U$, $g(z)$ has k zeros in V (counting multiplicities). So $g(q_1-q_2)$ has $\geqslant k$ zeros in z. Since $r_2 - r_1$ has degree $< k$ in z, for the fixed $w_\bullet$, the number of zeros of $r_2 - r_1$ is either $< k$ (which is impossible), or is ∞. Since the latter is the only possible case, we conclude $(r_1 - r_2)(z) = 0$ for all $w_\bullet$. And $(q_1-q_2)(z) = 0$ since it is so outside the (finitely many) zeros of g. □

Discussion. We now discuss the proof of the existence part. Let $\widehat{f}, \widehat{g}$ be the first k terms in their power series expansions of z. So

$$g(w_\bullet, z) = \underbrace{a_0 + a_1 z + \cdots + a_{k-1}z^{k-1}}_{\widehat{g}} + z^k(a_k + a_{k+1}z + a_{k+2}z^2 + \cdots)$$

where all $a_i = a_i(w_\bullet) \in \mathbb{C}\{w_\bullet\}$ and $a_0(0) = \cdots = a_{k-1}(0) = 0$, $a_k(0) \neq 0$. So $(g-\widehat{g})z^{-k}$ and similarly $(f-\widehat{f})z^{-k}$ are naturally elements of $\mathbb{C}\{w_\bullet, z\}$. Moreover, $(g-\widehat{g})z^{-k}$ is a unit.

A naïve attempt to find the decomposition $f = gq + r$ is to write

$$f = g \cdot \frac{f - \widehat{f}}{g} + \widehat{f}$$

since clearly $\widehat{f} \in \mathbb{C}\{w_\bullet\}[z]$ has degree $< k$ in z. This certainly works for single-variable functions. However, when $m > 0$, the expression $(f-\widehat{f})/g$ might not be continuous at the origin. (Take for instance the quotient to be $z^2/(wz+z^2)$.) We can only divide $f - \widehat{f}$ by $g - \widehat{g}$, which gives an element of $\mathbb{C}\{w_\bullet, z\}$. So we write

$$f = (g-\widehat{g}) \cdot \frac{f-\widehat{f}}{g-\widehat{g}} + \widehat{f} = g \cdot \frac{f-\widehat{f}}{g-\widehat{g}} + \widehat{f} + \underbrace{\left(-\widehat{g} \cdot \frac{f-\widehat{f}}{g-\widehat{g}}\right)}_{f_1}$$

We then decompose f_1, find f_2, and then repeat this procedure again and again to produce an infinite series, which we hope would converge to the expected decomposition. Namely, we let $f_0 = f$. So the above defines f_1 in terms of f_0. We define in a similar way f_{n+1} in terms of f_n:

$$f_n = g \cdot \frac{f_n - \widehat{f}_n}{g-\widehat{g}} + \widehat{f}_n + f_{n+1}. \tag{1.5.2}$$

Substituting $f_0, f_1, \ldots, f_n$ into f, we get

$$\begin{aligned} f &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0\right) + f_1 \\ &= \left(g \cdot \frac{f_0 - \widehat{f}_0}{g - \widehat{g}} + \widehat{f}_0\right) + \left(g \cdot \frac{f_1 - \widehat{f}_1}{g - \widehat{g}} + \widehat{f}_1\right) + f_2 = \cdots \\ &= g \cdot \sum_{i=0}^{n} \frac{f_i - \widehat{f}_i}{g - \widehat{g}} + \sum_{i=0}^{n} \widehat{f}_i + f_{n+1}. \end{aligned} \tag{1.5.3}$$

In the following formal proof, we give careful analysis when $n \to \infty$. □

Finishing the proof of WDT. For each $(r_\bullet, \rho) = (r_1, \ldots, r_m, \rho) \in \mathbb{R}_{>0}^m \times \mathbb{R}_{>0}$, define a norm $\|\cdot\|_{r_\bullet,\rho}$ on $\mathbb{C}\{w_\bullet, z\}$ as follows: if $h = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} b_{i_\bullet,j} w_1^{i_1} \cdots w_m^{i_m} z^j$ then

$$\|h\|_{r_\bullet,\rho} = \sum_{i_1,\ldots,i_m,j\in\mathbb{N}} |b_{i_\bullet,j}| r_1^{i_1} \cdots r_m^{i_m} \rho^j,$$

which might take value ∞. We have

$$\|h_1 h_2\|_{r_\bullet,\rho} \leqslant \|h_1\|_{r_\bullet,\rho} \cdot \|h_2\|_{r_\bullet,\rho} \qquad \|h - \widehat{h}\|_{r_\bullet,\rho} \leqslant \|h\|_{r_\bullet,\rho}. \tag{1.5.4}$$

We write (1.5.2) as

$$\begin{aligned} -f_{n+1} &= \frac{\widehat{g}}{(g - \widehat{g})} \cdot (f_n - \widehat{f}_n) \\ &= \frac{\widehat{g}}{z^{-k}(g - \widehat{g})} \cdot z^{-k}(f_n - \widehat{f}_n) =: \beta \cdot \alpha_n. \end{aligned} \tag{1.5.5}$$

Recall the first paragraph in the previous *Discussion*: $\beta, \alpha_n \in \mathbb{C}\{w_\bullet, z\}$ and β is a unit of the ring. Choose $r_\bullet, \rho$ such that f, g are defined (and holomorphic) and $g - \widehat{g}$ has no zeros in the polydisc D with multiradii $r_\bullet, \rho$ except at the origin. Then (1.5.5) shows that all f_n are defined in this domain.

Slightly shrink ρ so that $C := \|f\|_{r_\bullet,\rho} < \infty$. *Now we use the condition that g has order k in z in full power*: it tells us that $\beta(0, z) = 0$. So we may shrink $r_\bullet$ such that $\|\beta\|_{r_\bullet,\rho} < \frac{1}{2}\rho^k$. Clearly $\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} = \rho^k \|\alpha_n\|_{r_\bullet,\rho}$. So by (1.5.4),

$$\|f_{n+1}\|_{r_\bullet,\rho} < \frac{1}{2}\|f_n - \widehat{f}_n\|_{r_\bullet,\rho} \leqslant \frac{1}{2}\|f_n\|_{r_\bullet,\rho}.$$

Thus $\|f_n\|_{r_\bullet,\rho} < 2^{-n}C$. So $\|z^{-k}(f_n - \widehat{f}_n)\|_{r_\bullet,\rho} < 2^{-n}\rho^{-k}C$ and $\|\widehat{f}_n\|_{r_\bullet,\rho} < 2^{-n}C$.

The uniform norm on the polydisc with multi-radii $(r_\bullet, \rho)$ is clearly $\leqslant \|\cdot\|_{r_\bullet,\rho}$. So $f_n \to 0$ uniformly on the polydisc D. The infinite series $\sum_{i=0}^{\infty} \frac{z^{-k}(f_i - \widehat{f}_i)}{z^{-k}(g - \widehat{g})}$ converges uniformly to a continuous function q on any compact subset of D. q

is holomorphic, since it is so on each variable by Morera's theorem. Similarly, $\sum_{i=0}^{\infty} \widehat{f}_i$ converges uniformly to a holomorphic r. Residue theorem and the fact that contour integrals commute with (uniformly convergent) infinite sum show that r does not have $\geqslant k$ powers of z (since each $\widehat{f}_n$ does not). Thus, we obtain the decomposition $f = gq + r$ by letting $n \to \infty$ in (1.5.3). □

1.6 Germs of complex spaces

Definition 1.6.1. The **category of germs of complex spaces** denotes the one whose objects are (X, x) where X is a complex space and x is a marked point. A **morphism of germs** from (X, x) to (Y, y) is a holomorphic map $\varphi : U \to Y$ where $U \subset X$ is a neighborhood of x such that $\varphi(x) = y$. Two morphisms $\varphi_1, \varphi_2 : (X, x) \to (Y, y)$ are regarded equal if there is a neighborhood U of x such that $\varphi_1|_U$ equals $\varphi_2|_U$ as holomorphic maps $U \to Y$. Composition of morphisms are the usual one for holomorphic functions (i.e. for $\mathbb{C}$-ringed spaces).

An **isomorphism of germs of complex spaces** $\varphi : (X, x) \to (Y, y)$ is a morphism of germs with inverses, namely, there is a morphism $\psi : (Y, y) \to (X, x)$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are **1** on neighborhoods of x and y respectively. Equivalently, there are neighborhoods $U \ni x$ and $V \ni y$ such that $\varphi : U \to V$ is a biholomorphism, and that $\varphi(x) = y$. □

The category of analytic local $\mathbb{C}$-algebras is understood in the obvious way: the morphisms are (homo)morphisms of $\mathbb{C}$-algebras sending maximal ideals into maximal ones.

Theorem 1.6.2. *The contravariant function $\mathfrak{F}$ from the category of germs of complex spaces to the category of analytic local $\mathbb{C}$-algebras, sending (X, x) to $\mathscr{O}_{X,x}$ and sending $\varphi : (X, y) \to (Y, y)$ to $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$, is an* ***antiequivalence of categories****. Namely:*

(1) For each (X, x) and (Y, y), the following map is bijective

$$\mathfrak{F} : \mathrm{Mor}\big((X, x), (Y, y)\big) \to \mathrm{Mor}\big(\mathscr{O}_{Y,y}, \mathscr{O}_{X,x}\big), \qquad \varphi \mapsto \varphi^\#. \tag{1.6.1}$$

(2) Each analytic local $\mathbb{C}$-algebra is isomorphic to $\mathfrak{F}((X, x))$ for some germ of complex space (X, x).

Part (2) is obvious. Let us prove part (1).

Proof. Assume without loss of generality that Y is a model space $\mathrm{Specan}(\mathscr{O}_V/\mathscr{J})$ where $V \subset \mathbb{C}^n$ is open and $y = 0$.

Suppose $\varphi_1^\#, \varphi_2^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ are equal. Then for each $j = 1, \dots, n$, $\varphi_1^\# z_j$ equals $\varphi_2^\# z_j$ as elements of $\mathscr{O}_{X,x}$. So they are equal on X if we shrink

X to a smaller neighborhood of x. By Thm. 1.4.1, φ_1 and φ_2 are equal as holomorphic maps $X \to V$, and hence are equal as $X \to Y$. So the map $\mathfrak{F}$ in (1.6.1) is injective.

Next, we choose a morphism $\Phi : \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$. Let $f_1 = \Phi(z_1), \dots, f_n = \Phi(z_n)$, which are elements of $\mathscr{O}(X)$ if we shrink X to a smaller neighborhood of x. View $F = (f_1, \dots, f_n) \in \mathscr{O}(X)^n$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$. Replace X by $\varphi^{-1}(V)$ such that $\varphi : X \to V$. Note that $\varphi(x) = 0$. So $h \in \mathscr{O}_{\mathbb{C}^n,0} \mapsto h \circ \varphi = \varphi^\# h \in \mathscr{O}_{X,x}$ is a morphism of local $\mathbb{C}$-algebras. It agrees with $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\Phi} \mathscr{O}_{X,x}$ on $z_1, \dots, z_n$ by the very definition of F. So they agree on any element of $\mathscr{O}_{\mathbb{C}^n,0}$ due to Prop. 1.4.3. We conclude $\varphi^\#(h) = \Phi(h)$ for all $h \in \mathscr{O}_{\mathbb{C}^n,0}$ (where the h in Φ denotes the residue class of h). In particular, as Φ vanishes on $\mathscr{J}_0$, we have $\varphi^\# \mathscr{J}_0 = 0$.

Shrink V and choose $g_1, \dots, g_k \in \mathscr{O}_{\mathbb{C}^n}(V)$ generating the ideal $\mathscr{J}_0$ and killed by $\varphi^\#$ (in $\mathscr{O}_{\mathbb{C}^n}(V)$). Since $\mathscr{J}$ is finite-type, by Rem. 1.2.15, we can shrink V such that $g_1, \dots, g_k$ generate $\mathscr{J}$. Thus $\varphi^\# \mathscr{J}$=0. By Thm. 1.4.8, φ restricts to a holomorphic map $\widetilde{\varphi} : X \to Y$. $\widetilde{\varphi}^\# : \mathscr{O}_{Y,y} = \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \to \mathscr{O}_{X,x}$ equals Φ since $\varphi^\# : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{X,x}$ factors as $\mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{\mathbb{C}^n,0}/\mathscr{J}_0 \xrightarrow{\widetilde{\varphi}^\#} \mathscr{O}_{X,x}$. This proves that $\mathfrak{F}$ is surjective. □

Corollary 1.6.3. *Let X, Y be complex spaces, $x \in Y, y \in Y$, and $\Phi : \mathscr{O}_{Y,y} \xrightarrow{\simeq} \mathscr{O}_{X,x}$ be an isomorphism of local algebras. Then there are neighborhoods $U \ni x, V \ni y$ and a biholomorphism $\varphi : U \xrightarrow{\simeq} V$ whose transpose $\varphi^\# : \mathscr{O}_{V,y} \to \mathscr{O}_{U,x}$ equals Φ.*

Definition 1.6.4. An analytic local $\mathbb{C}$-algebra is called **regular** if it is isomorphic to $\mathscr{O}_{\mathbb{C}^n,0} = \mathbb{C}\{z_1, \dots, z_n\}$ for some n.

Corollary 1.6.5. *Let X be a complex space and $x \in X$. If $\mathscr{O}_{X,x}$ is regular, then there is a neighborhood U of x biholomorphic to an open subset of $\mathbb{C}^n$.*

1.7 Immersions and closed embeddings; generating $\mathscr{O}_{X,x}$ analytically

Definition 1.7.1. A holomorphic map $\varphi : X \to Y$ is called an **immersion at** $x \in X$ if $\varphi^\# : \mathscr{O}_{Y,\varphi(y)} \to \mathscr{O}_{X,x}$ is surjective. φ is called an **immersion** if it is an immersion at every $x \in X$. φ is called a **closed (resp. open) embedding** if there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\simeq"'] & & Y \\ & Y_0 \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{1.7.1}$$

where Y_0 is a closed (resp. open) complex subspace of Y and $X \xrightarrow{\simeq} Y_0$ is a biholomorphic map.

A closed embedding is clearly an immersion. Moreover, an immersion is locally a closed embedding:

Proposition 1.7.2. *Let $\varphi : X \to Y$ be an immersion at x. Then there are neighborhoods U of x and V of $y = \varphi(x)$ such that $\varphi : U \to V$ is a closed embedding. In particular, φ is an immersion on U.*

Proof. By assumption, $\varphi^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective. Let J be its kernel, and choose generating elements $g_1, \dots, g_k \in J$. By shrinking Y to a neighborhood of y (and shrink X accordingly), we assume $g_1, \dots, g_k \in \mathscr{O}_Y(Y)$. Let $\mathcal{J} = g_1\mathscr{O}_Y + \cdots + g_k\mathscr{O}_Y$. Then $\mathcal{J}_x = J$. Define a closed subspace $Z = \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$ of Y. Then φ factors as

$$\varphi^\# : \mathscr{O}_{Y,y} \twoheadrightarrow \mathscr{O}_{Y,y}/J = \mathscr{O}_{Z,y} \xrightarrow[\simeq]{\Psi} \mathscr{O}_{X,x}.$$

By Cor. 1.6.3, we may shrink X so that there is an open embedding $\widetilde{\varphi} : X \to Z$, $\widetilde{\varphi}(x) = y$, such that $\widetilde{\varphi}^\# : \mathscr{O}_{Z,y} \to \mathscr{O}_{X,x}$ equals Ψ. Let $\iota : Z \to Y$ be the inclusion. Then $(\iota\widetilde{\varphi})^\# = \widetilde{\varphi}^\#\iota^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ equals $\varphi^\#$. By Thm. 1.6.2, we may find open $U \ni x$ such that $\varphi = \iota\widetilde{\varphi}$. Since $\widetilde{\varphi}(U)$ is an open subset of Z, we may find open $V \subset Y$ such that $\widetilde{\varphi}(U) = V \cap Z = V \cap N(\mathcal{J})$. So φ restricts to the biholomorphism $\widetilde{\varphi} : U \to \widetilde{\varphi}(U)$ where $\widetilde{\varphi}(U)$ is a closed subspace of V. $\square$

We now discuss when an immersion is a closed embedding and give some examples.

Proposition 1.7.3. *Let X be complex spaces and $\varphi : X \to Y$ a holomorphic immersion. Assume that φ is an injective and closed map[2] of topological spaces. Suppose we have a finite type ideal $\mathcal{J}$ of $\mathscr{O}_Y$ such that $N(\mathcal{J})$ equals the image of φ, and that*

$$\mathcal{J}_y = \ker(\mathscr{O}_{Y,y} \xrightarrow{\varphi^\#} \mathscr{O}_{X,x}) \tag{1.7.2}$$

for all $x \in X$ and $y = \varphi(x)$. Then φ is a closed embedding. More precisely, φ restricts to a biholomorphism

$$\widetilde{\varphi} : X \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J}). \tag{1.7.3}$$

Proof. Let $Y_0 := \mathrm{Specan}(\mathscr{O}_Y/\mathcal{J})$. By Thm. 1.4.8, the restriction (1.7.3) as a holomorphic map exists, i.e., we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & Y_0 \arrow[d, hook] \\ & Y \end{tikzcd}$$

[2] φ is called closed if it maps closed subsets to closed subsets.

The underlying topological space of $Y_0 := \mathrm{Specan}(\mathscr{O}_X/\mathcal{J})$ is $N(\mathcal{J})$. So $\widetilde{\varphi}$ is a continuous closed bijection from X to $N(\mathcal{J})$, which is therefore a homeomorphism. For each $x \in X, y = \varphi(x)$, the stalk map $\widetilde{\varphi}^\# : \mathscr{O}_{Y_0,y} = \mathscr{O}_{Y,y}/\mathcal{J}_y \to \mathscr{O}_{X,x}$ is surjective since φ is an immersion, and is injective by (1.7.2). So $\widetilde{\varphi}$ is a biholomorphism. $\square$

Example 1.7.4. The holomorphic map $\iota : 0 \times \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{C}^n$ is an immersion and a closed injective map, and the kernels of $\iota^\#$ at the level of stalks are the stalks of the ideal $\mathcal{I} = z_1\mathscr{O}_{\mathbb{C}^{m+n}} + \cdots + z_m\mathscr{O}_{\mathbb{C}^{m+n}}$. Thus, by Prop. 1.7.3, ι restricts to a biholomorphism $0 \times \mathbb{C}^n \xrightarrow{\simeq} \mathrm{Specan}(\mathscr{O}_{\mathbb{C}^{m+n}}/\mathcal{I})$. This reproves Exp. 1.3.9.

Example 1.7.5. Let X be a complex space, and let $\mathcal{I}, \mathcal{J}$ be finite-type ideals of $\mathscr{O}_X$. Let $Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. So $\mathscr{O}_Y = (\mathscr{O}_X/\mathcal{I})|_{N(\mathcal{I})}$. Then

$$\widetilde{\mathcal{J}} = \big((\mathcal{I} + \mathcal{J})/\mathcal{I}\big) \restriction_{N(\mathcal{I})}$$

is a finite-type ideal of $\mathscr{O}_Y$, and is the unique ideal whose stalk at each $x \in N(\mathcal{I})$ equals $(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x$. Then there is a biholomorphism

$$\mathrm{Specan}(\mathscr{O}_X/(\mathcal{I} + \mathcal{J})) \xrightarrow[\simeq]{\varphi} \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}). \tag{1.7.4}$$

which equals $N(\mathcal{I}+\mathcal{J}) \xrightarrow{=} N(\mathcal{I}) \cap N(\mathcal{J})$ as maps of topological spaces, and whose stalk maps are

$$\mathscr{O}_{Y,x}/\widetilde{\mathcal{J}}_x = \frac{\mathscr{O}_{X,x}/\mathcal{I}_x}{(\mathcal{I}_x + \mathcal{J}_x)/\mathcal{I}_x} \quad \xrightarrow{\simeq} \quad \mathscr{O}_{X,x}/(\mathcal{I}_x + \mathcal{J}_x).$$

Proof. The key point is to show that the above stalk isomorphisms can be assembled into a sheaf isomorphism. Consider the diagrams

$$\begin{tikzcd} & \mathrm{Specan}(\mathscr{O}_Y/\widetilde{\mathcal{J}}) \arrow[d, hook] \\ \mathrm{Specan}(\mathscr{O}_X/(\mathcal{I} + \mathcal{J})) \arrow[ur, "\varphi"] \arrow[r, "\alpha"] \arrow[dr, hook] & Y \arrow[d, hook] \\ & X \end{tikzcd} \tag{1.7.5}$$

By Thm. 1.4.8, there is a holomorphic map α such that the lower triangle commutes. The stalk maps are $\alpha^\# : \mathscr{O}_{X,x}/\mathcal{I}_x \to \mathscr{O}_{X,x}/(\mathcal{I}_x+\mathcal{J}_x)$, with kernel $(\mathcal{I}_x+\mathcal{J}_x/\mathcal{I}_x)$. These kernels can be assembled into the ideal sheaf $\widetilde{\mathcal{J}}$ on $N(\mathcal{I})$. Thus, Prop. 1.7.3 guarantees that there is a biholomorphism making the upper triangle in (1.7.5) commutes. $\square$

Exp. 1.7.5 shows that a closed complex subspace of a closed subspace is again a closed subspace of the original space. Thus, we have more generally:

Corollary 1.7.6. *If $\alpha : X \to Y$ and $\beta : Y \to Z$ are closed embeddings, then so is the composition $\beta \circ \alpha : X \to Z$.*

Let us consider the special case $\varphi : X \to \mathbb{C}^n$, where φ is represented by $(f_1, \dots, f_n) \in \mathscr{O}_X^n$ (cf. Thm. 1.4.1). Assume for simplicity that $\varphi(x) = 0$. Then φ is an immersion at x iff the morphism of analytic local $\mathbb{C}$-algebras defined in Prop. 1.4.3, namely $\mathbb{C}\{z_\bullet\} \to \mathscr{O}_{X,x}$ sending z_j to f_j, is surjective. This actually mean that $f_1, \dots, f_n$ *generate (analytically) the analytic local $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$*. (They certainly do not generate the ring $\mathscr{O}_{X,x}$ algebraically. But one can imagine that the subalgebra generated algebraically by $f_\bullet$ is "dense" in $\mathscr{O}_{X,x}$, where the density means approximation by power series of $f_1, \dots, f_n$.) The situation is similar to the case of a surjective morphism of $\mathbb{C}$-algebras $\mathbb{C}[z_\bullet] \to A$, whose algebro-geometric meaning is that the affine scheme $\mathrm{Spec}(A)$ is embedded into the affine plane $\mathbb{C}^n$.

We must find a criterion on whether $f_1, \dots, f_n$ generate $\mathscr{O}_{X,x}$ (analytically). At first sight, this problem seems not easy even if X is smooth. (For instance, take $f_1, \dots, f_n$ to be some arbitrary holomorphic functions and deduce whether they generate $\mathscr{O}_{X,x}$.) There is indeed a simple criterion, which is proved using the (holomorphic version of) inverse function theorem. To begin with, we define:

Definition 1.7.7. If X is a complex space and $x \in X$, the vector space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is called the **cotangent space** of X at x, and its dual space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the **tangent space**. Since $\mathscr{O}_{X,x}$ is Noetherian, $\mathfrak{m}_{X,x}$ is finitely-generated, and hence $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ is finite-dimensional.

It is inspiring to write the residue class of $f - f(x)$ (where $f \in \mathscr{O}(X)$) in the cotangent space $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ as df_x.

Theorem 1.7.8. *Let X be a complex space and $x \in X$. Let $f_1, \dots, f_n \in \mathscr{O}(X)$. Consider $(f_1, \dots, f_n)$ as a holomorphic map $\varphi : X \to \mathbb{C}^n$ (cf. Thm. 1.4.1). The following are equivalent.*

(1) φ is an immersion at x.

(2) The morphism of analytic local $\mathbb{C}$-algebras $\Phi : \mathbb{C}\{z_1, \dots, z_n\} \to \mathscr{O}_{X,x}$ sending z_i to $f_i - f_i(x)$ (cf. Prop. 1.4.3) is surjective.

(3) (The residue classes of) $f_1 - f_1(x), \dots, f_n - f_n(x)$ span $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$.

(4) (The germs of) $f_1 - f_1(x), \dots, f_n - f_n(x)$ generate the ideal $\mathfrak{m}_{X,x}$.

*If any of these conditions holds, we say that $f_1, \dots, f_n$ **generate (the algebra) $\mathscr{O}_{X,x}$ analytically**.*

Proof. Assume for simplicity that $\varphi(x) = 0$. Clearly (1)⇔(2) and (3)⇔(4). (Note that (3)⇒(4) follows from Nakayama's lemma.) It remains to prove (2)⇔(3).

Assume (2). Choose any $g \in \mathfrak{m}_{X,x}$. Then there is $h(z_\bullet) \in \mathscr{O}_{\mathbb{C}^n,0}$ sent by Φ to g. We may write $h(z_\bullet) = \sum_i a_i z_i$ + an element of $\mathfrak{m}^2_{\mathbb{C}^n,0}$ where $a_i \in \mathbb{C}$. Since $\Phi(z_i) = f_i$ and $\Phi(\mathfrak{m}^2_{\mathbb{C}}) \subset \mathfrak{m}^2_{X,x}$, we have $g \in \sum_i a_i f_i + \mathfrak{m}^2_{X,x}$. This proves (3).

Asume (3). By discarding some elements, we may assume that $f_1, \dots, f_n$ form a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$. Assume X is a model space $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ where $U \subset \mathbb{C}^N$ is open and $x = 0$. So $\mathscr{O}_{X,x} = \mathscr{O}_{\mathbb{C}^N,0}/\mathcal{I}_0$, $\mathfrak{m}_{X,x} = \mathfrak{m}_{\mathbb{C}^N,0}/\mathcal{I}_0$, and hence

$$\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} = \mathfrak{m}_{\mathbb{C}^N,0}/(\mathfrak{m}^2_{\mathbb{C}^N,0} + \mathcal{I}_0).$$

Lift $f_\bullet$ to elements of $\mathcal{I}_{\mathbb{C}^N,0}$, still denoted by $f_\bullet$. Then we can extend $f_1, \dots, f_n$ to a $f_1, \dots, f_N$ whose residue classes form a basis of $\mathfrak{m}_{\mathbb{C}^N,0}/\mathfrak{m}^2_{\mathbb{C}^N,0}$ such that $f_{n+1}, \dots, f_N \in \mathcal{I}_0$. By the inverse function theorem, we may assume $x = 0$ and $f_1, \dots, f_N$ are the standard coordinates $z_1, \dots, z_N$ of $\mathbb{C}^N$. By shrinking U, we may assume $z_{n+1}, \dots, z_N \in \mathcal{I}(U)$.

Assume for simplicity that $\mathcal{I}$ is generated by $z_{n+1}, \dots, z_N$ together with $g_1, \dots, g_k \in \mathcal{I}(U)$. Let $\mathcal{I}_1 = z_{n+1}\mathscr{O}_U + \cdots + z_N\mathscr{O}_U$. Then by Exp. 1.7.5, $X = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ is naturally a closed subspace of $X_1 = \mathrm{Specan}(\mathscr{O}_U/\mathcal{I}_1)$ (defined by $g_1, \dots, g_k$). By Exp. 1.7.4, X_1 is naturally equivalent to $U \cap (\mathbb{C}^n \times 0)$. So the map $(z_1, \dots, z_n) : X_1 \to \mathbb{C}^n$ is an open embedding. φ is its restriction to X, which is therefore an immersion at 0. This proves (1) and hence (2). $\square$

We give an application of analytically generating elements.

Proposition 1.7.9.

Let $\Phi, \Psi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ be morphisms of analytic local $\mathbb{C}$-algebras. Assume $f_1, \dots, f_n \in \mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically.

(1) *If $\Phi(f_i) = \Psi(f_i)$ for all $i = 1, \dots, n$, then $\Phi = \Psi$.*

(2) *Let I be the ideal of $\mathscr{O}_{X,x}$ generated by $\Phi(f_i) - \Psi(f_i)$ for all i. Then I contains $\Phi(h) - \Psi(h)$ for every $h \in \mathscr{O}_{Y,y}$.*

Proof. (1): By Prop. 1.4.3, we have a (unique) morphism $\Upsilon : \mathscr{O}_{\mathbb{C}^n,0} \to \mathscr{O}_{Y,y}$ sending z_i to f_i. So $\Phi \circ \Upsilon$ and $\Psi \circ \Upsilon$ agree at $z_1, \dots, z_n$. So $\Phi \circ \Upsilon = \Psi \circ \Upsilon$ by Prop. 1.4.3. By assumption, Υ is surjective. So $\Phi = \Psi$.

(2): Apply (1) to the restriction $\Phi, \Psi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}/I$. $\square$

Prop. 1.7.9-(2) is the stalk version of a geometric construction called equalizer.

1.8 Equalizers of $X \rightrightarrows Y$

Definition 1.8.1. Let $\varphi, \psi : X \to Y$ be holomorphic maps of complex spaces. A **kernel** or an **equalizer of the double arrow** $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ is a complex space E and a holomorphic map $\iota : E \to X$ such that $\varphi \circ \iota = \psi \circ \iota$, and that for every complex space S and holomorphic map $\mu : S \to X$ satisfying $\varphi \circ \mu = \psi \circ \mu$ there is a unique holomorphic $\widetilde{\mu} : S \to E$ such that $\mu = \iota \circ \widetilde{\mu}$.

$$\begin{tikzcd} S \arrow[d, dashed, "\widetilde{\mu}"'] \arrow[dr, "\mu"] & & \\ E \arrow[r, "\iota"'] & X \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & Y \end{tikzcd} \tag{1.8.1}$$

It is easy to see that equalizers are unique up to isomorphism.

The main result of this section is:

Theorem 1.8.2. *Every double arrow* $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ *of holomorphic maps has an equalizer which is the inclusion map of a closed subspace* $\iota : E = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$. *This is called the* ***canonical equalizer***. *The finite-type ideal* $\mathcal{I}$ *is uniquely determined by the fact that for all* $x \in X$:

(a) If $\varphi(x) \neq \psi(x)$, *then* $\mathcal{I}_x = \mathscr{O}_{X,x}$.

(b) If $\varphi(x) = \psi(x)$, *then by considering* $\varphi^\#, \psi^\#$ *as stalk maps* $\mathscr{O}_{Y,\varphi(x)} \to \mathscr{O}_{X,x}$, $\mathcal{I}_x$ *is the ideal of* $\mathscr{O}_{X,x}$ *generated by all* $\varphi^\#(f) - \psi^\#(f)$ *(where* $f \in \mathscr{O}_{Y,\varphi(x)}$*).*

Moreover, $N(\mathcal{I})$, *the underlying set of* E, *is* $\Delta = \{x \in X : \varphi(x) = \psi(x)\}$.

From Prop. 1.7.9, it is clear that $\mathcal{I}_x$ is generated by $\varphi^\#(f_i) - \psi^\#(f_i)$ if $f_1, \dots, f_n \in \mathscr{O}_{Y,y}$ generate the algebra $\mathscr{O}_{Y,y}$ analytically, e.g. $z_1, \dots, z_n$ if Y is a model space in $\mathbb{C}^n$.

Construction of E. We define a finite-type ideal $\mathcal{I}$ satisfying (a) and (b). We shall first define it locally and then glue the pieces. Then $\mathcal{I}$ gives E.

Let $\Omega = X \backslash \Delta$ which is open. We set $\mathcal{I}_\Omega = \mathscr{O}_X|_\Omega$. For each $x \in \Delta$, we choose a neighborhood $V_y \subset Y$ of $y = \varphi(x)$ biholomorphic to a model space. So we can choose finitely many $f_1, \dots f_n \in \mathscr{O}_Y(V_y)$ embedding V_y onto a closed subspace of an open subset of $\mathbb{C}^n$. $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$ is a neighborhood of x, and we set $\mathcal{I}_{U_x}$ to be the ideal of $\mathscr{O}_{U_x}$ generated by $\varphi^\#(f_1) - \psi^\#(f_1), \dots, \varphi^\#(f_n) - \psi^\#(f_n)$ (defined on U_x).

We claim that these locally defined finitely-generated ideals are compatible. If $p \in U_x \cap \Delta$ then, as $\varphi(p) = \psi(p)$, by Prop. 1.7.9 or by substitution rule (Rem. 1.4.6), the stalk $(\mathcal{I}_{U_x})_p$ is the ideal generated by all $\varphi^\#(f) - \psi^\#(f) \in \mathscr{O}_{X,p}$ where $f \in \mathscr{O}_{Y,\varphi(p)}$. If $p \in U_x \cap \Omega$, then as $\varphi(p) \neq \psi(p)$ and $(f_1, \ldots, f_n)$ is an embedding, there is some f_i among $f_1, \ldots, f_n$ such that $\varphi^\#(f_i) - \psi^\#(f_i)$ has non-zero value at p, and hence its germ at p is not in $\mathfrak{m}_{X,p}$. This proves $(\mathcal{I}_{U_x})_p = \mathscr{O}_{X,p}$. Combining these two cases together, we see that $\mathcal{I}_\Omega$ and $\mathcal{I}_{U_x}$ (for all $x \in \Delta$) are compatible. This defines $\mathcal{I}$.

If $\varphi(x) \neq \psi(x)$, then $\mathcal{I}_x = \mathscr{O}_x$ shows $x \notin N(\mathcal{I})$. If $\varphi(x) = \psi(x)$, then $\varphi^\#(f) - \psi^\#(f)$ clearly vanishes on x. So $\mathcal{I}_x$ vanishes on x. So $x \in N(\mathcal{I})$. This proves $\Delta = N(\mathcal{I})$. □

Proof that E is an equalizer. It is easy to check $\varphi \circ \iota = \psi \circ \iota$. Choose any holomorphic $\mu : S \to X$ such that $\varphi \circ \mu = \psi \circ \mu$. For any $s \in S$, let $x = \mu(s)$. Then $\varphi(x) = \psi(x)$. Choose any $f \in \mathscr{O}_{Y,\varphi(x)}$. Then $\varphi \circ \mu = \psi \circ \mu$ shows that $\mu^\#$ sends $\varphi^\#(f) - \psi^\#(f)$ to $0 \in \mathscr{O}_{S,s}$. Thus $\mu^\# : \mathscr{O}_{X,x} \to \mathscr{O}_{S,s}$ vanishes on $\mathcal{I}_x$. Thus, by Thm. 1.4.8, there is a unique holomorphic $\widetilde{\mu} : S \to E$ such that the triangle in (1.8.1) commutes. □

The proof of Thm. 1.8.2 is finished. From the proof, we know:

Remark 1.8.3. Assume the setting of Thm. 1.8.2. Assume $\varphi(x) = \psi(x) =: y$. Let V_y be a neighborhood of y biholomorphic to a model space. More precisely, we choose $(f_1, \ldots, f_n) \in \mathscr{O}_Y(V_y)^n$ which, considered as a holomorphic map $V_y \to \mathbb{C}^n$, is a closed embedding of V_y into an open subset of $\mathbb{C}^n$. Let $U_x = \varphi^{-1}(V_y) \cap \psi^{-1}(V_y)$. Then the ideal sheaf $\mathcal{I}|_{U_x}$ is generated by $\varphi^\#(f_1) - \psi^\#(f_1), \ldots, \varphi^\#(f_n) - \psi^\#(f_n) \in \mathscr{O}(U_x)$.

1.9 $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$, $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$, and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$

We fix a complex space X.

1.9.1 Tensor product

Definition 1.9.1. Let $\mathscr{E}$ and $\mathscr{F}$ be $\mathscr{O}_X$-modules. Consider the presheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules defined by $\mathscr{G}(U) = \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$. The tensor product of restriction maps $\mathscr{E}(U) \to \mathscr{E}(V)$ and $\mathscr{F}(U) \to \mathscr{F}(V)$ gives the restriction map $\mathscr{G}(U) \to \mathscr{G}(V)$. The sheafification of $\mathscr{G}$ is denoted by $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ or simply $\mathscr{E} \otimes \mathscr{F}$ and called the **tensor product** of $\mathscr{E}$ and $\mathscr{F}$.

Remark 1.9.2. Let A be a commutative ring, and fix an A-module $\mathcal{N}$. Recall the following basic facts:

1. **Tensor products commute with direct limits**. More precisely, let $(\mathcal{M}_\alpha)$ be a direct system of A-modules. Then the canonical map $\mathcal{M}_\beta \otimes_A \mathcal{N} \to (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}$ (for each fixed β) defines, by passing to the direct limit, an isomorphism

$$\varinjlim_\alpha (\mathcal{M}_\alpha \otimes_A \mathcal{N}) \xrightarrow{\simeq} (\varinjlim_\alpha \mathcal{M}_\alpha) \otimes_A \mathcal{N}. \tag{1.9.1}$$

(Proof: Construct the inverse map explicitly.)

2. **The tensor product functor $-\otimes \mathcal{N}$ is right exact**. Namely, if

$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0$$

is an exact sequence of A-modules, then so is

$$\mathcal{M}_1 \otimes \mathcal{N} \xrightarrow{f\otimes \mathbf{1}} \mathcal{M}_2 \otimes \mathcal{N} \xrightarrow{g\otimes \mathbf{1}} \mathcal{M}_3 \otimes \mathcal{N} \to 0.$$

Identify $\mathcal{M}_3$ with $\mathrm{coker} f = \mathcal{M}_2/f(\mathcal{M}_1)$. Then the right exactness of tensor product is equivalent to that **tensor products commute with cokernels**: we have an equivalence of A-modules

$$\mathrm{coker}\big(\mathcal{M}_1 \otimes_A \mathcal{N} \xrightarrow{f\otimes \mathbf{1}} \mathcal{M}_2 \otimes_A \mathcal{N}\big) \quad \xrightarrow{\simeq} \quad \mathrm{coker}\big(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2\big) \otimes_A \mathcal{N} \tag{1.9.2}$$

descended from the canonical morphism

$$\mathcal{M}_2 \otimes_A \mathcal{N} \longrightarrow \frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \otimes_A \mathcal{N}. \tag{1.9.3}$$

□

Proof. We have a well-defined map sending $\frac{\mathcal{M}_2}{f(\mathcal{M}_1)} \times \mathcal{N}$ to $\frac{\mathcal{M}_2 \otimes_A \mathcal{N}}{(f\otimes\mathbf{1})(\mathcal{M}_1 \otimes_A \mathcal{N})}$ (i.e. the LHS of (1.9.2)) sending $[\xi] \times \eta$ to $[\xi \otimes_A \eta]$, where $[\cdots]$ stands for the residue classes, and $\xi \in \mathcal{M}_2, \eta \in \mathcal{N}$. This map is clearly A-biinvariant. So it gives an A-module morphism from the RHS to the LHS of (1.9.2), which is clearly the inverse of the map in (1.9.2) from LHS to RHS. So (1.9.2) is an isomorphism. □

Remark 1.9.3. We can now use (1.9.2) to explain the last equality of (1.2.4):

$$\begin{aligned} &\mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x}/\mathfrak{m}_x) = \mathscr{E}_x \otimes \mathrm{coker}(\mathfrak{m}_x \hookrightarrow \mathscr{O}_{X,x}) \\ =&\mathrm{coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x \otimes \mathscr{O}_{X,x}) = \mathrm{coker}(\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x) = \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \end{aligned}$$

since the image of the multiplication map $\mathscr{E}_x \otimes \mathfrak{m}_x \to \mathscr{E}_x$ is $\mathfrak{m}_x\mathscr{E}_x$.

Proposition 1.9.4. *The canonical morphism of $\mathscr{O}(U)$-modules*

$$\mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \to \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$$

(where $U \ni x$ is open and the map is the tensor product of $\mathscr{E}(U) \to \mathscr{E}_x$ and $\mathscr{F}(U) \to \mathscr{F}_x$) induces an isomorphism

$$(\mathscr{E} \otimes \mathscr{F})_x = \varinjlim_{U \ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U) \quad \xrightarrow{\simeq} \quad \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x. \tag{1.9.4}$$

Proof. Define a canonical map from $\mathscr{E}_x \times \mathscr{F}_x$ to $\varinjlim_{U \ni x} \mathscr{E}(U) \otimes_{\mathscr{O}(U)} \mathscr{F}(U)$ and show that it is $\mathscr{O}_{X,x}$-biinvariant. This descends to the inverse map of (1.9.4). □

Corollary 1.9.5. *For each $\mathscr{O}_X$-module $\mathscr{F}$, the functor $- \otimes \mathscr{F}$ on the abelian category of $\mathscr{O}_X$-modules is right exact: if*

$$\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$$

is exact, then so is

$$\mathscr{E}_1 \otimes \mathscr{F} \to \mathscr{E}_2 \otimes \mathscr{F} \to \mathscr{E}_3 \otimes \mathscr{F} \to 0.$$

Proof. Exactness of sheaves can be checked at the level of stalks. Then this follows from the isomorphism (1.9.4) and the right exactness of $- \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$. □

1.9.2 Hom

We leave it to the readers to check the following easy facts:

Remark 1.9.6. Let A be a commutative ring, and fix an A-module $\mathcal{N}$:

1. $\mathrm{Hom}_A(\mathcal{N}, -)$ **is a left exact functor**. Namely, for any exact sequence of A-modules

$$0 \to \mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3, \tag{1.9.5}$$

we have an exact sequence

$$0 \to \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_1) \xrightarrow{f_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)$$

where f_* sends T to $f \circ T$ and g_* is defined similarly. Equivalently, $\mathrm{Hom}_A(\mathcal{N}, -)$ **commutes with kernels**: there is a canonical equivalence

$$\mathrm{Hom}_A\big(\mathcal{N}, \ker(\mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3)\big) \simeq \ker\big(\mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_2) \xrightarrow{g_*} \mathrm{Hom}_A(\mathcal{N}, \mathcal{M}_3)\big). \tag{1.9.6}$$

2. $\mathrm{Hom}_A(-,\mathcal{N})$ **is a left exact contravariant functor**. for any exact sequence of A-modules

$$\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3 \to 0 \tag{1.9.7}$$

we have an exact sequence

$$0 \to \mathrm{Hom}_A(\mathcal{M}_3,\mathcal{N}) \xrightarrow{g^*} \mathrm{Hom}_A(\mathcal{M}_2,\mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1,\mathcal{N})$$

where f^* sends T to $T \circ f$ and g^* is defined similarly. Equivalently, $\mathrm{Hom}_A(-,\mathcal{N})$ **turns cokernels into kernels**: there is a canonical equivalence

$$\mathrm{Hom}_A\big(\mathrm{coker}(\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2),\mathcal{N}\big) \simeq \ker\big(\mathrm{Hom}_A(\mathcal{M}_2,\mathcal{N}) \xrightarrow{f^*} \mathrm{Hom}_A(\mathcal{M}_1,\mathcal{N})\big). \tag{1.9.8}$$

Definition 1.9.7. Let $\mathscr{E},\mathscr{F}$ be $\mathscr{O}_X$-modules. The **hom space** $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ is defined to be the space of all $\mathscr{O}_X$-module morphims from $\mathscr{E}$ to $\mathscr{F}$.

The presheaf of $\mathscr{O}_X$-modules sending each open $U \subset X$ to the $\mathscr{O}(U)$-module $\mathrm{Hom}_{\mathscr{O}_U}(\mathscr{E}_U,\mathscr{F}_U)$, and whose restriction map is the obvious restriction of sheaf morphisms, is automatically a sheaf of $\mathscr{O}_X$-modules. It is called the **hom sheaf** and denoted by $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$.

The dual and the double dual of $\mathscr{E}$ is defined by

$$\mathscr{E}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{O}_X), \qquad \mathscr{E}^{\vee\vee} = (\mathscr{E}^\vee)^\vee. \tag{1.9.9}$$

□

Exercise 1.9.8. Describe canonical equivalences

$$\mathscr{E} \simeq \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{E} \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{E}). \tag{1.9.10}$$

In general, the stalks of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{F})$ cannot be identified with $\mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{E}_x,\mathscr{F}_x)$. But good things happen when $\mathscr{E},\mathscr{F}$ are coherent, as we will see in the next chapter.

1.10 $(\mathscr{O}_X\mathrm{-mod}) \otimes_{\mathscr{O}_S} (\mathscr{O}_S\mathrm{-mod})$; pullback sheaves

Definition 1.10.1. Let $\varphi : X \to S$ be a holomorphic map of complex spaces. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module and $\mathscr{M}$ an $\mathscr{O}_S$-module. Then $\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M} = \mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{E}$ denotes the sheafification of the presheaf of $\mathscr{O}_X$-modules sending each open $U \subset X$ to

$$(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})^{\mathrm{pre}}(U) = \varinjlim_{V \supset \varphi(U)} \mathscr{E}(U) \otimes_{\mathscr{O}_S(V)} \mathscr{M}(V) \tag{1.10.1}$$

where the direct limit is over all open subset $V \subset S$ containing $\varphi(U)$, and $g \in \mathscr{O}_S(V)$ acts on $\varsigma \in \mathscr{E}(U)$ as

$$g \cdot \varsigma := \varphi^{\#}(g) \cdot \varsigma. \tag{1.10.2}$$

For each $x \in X$, we have a canonical equivalence

$$(\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M})_x \simeq \mathscr{E}_x \otimes_{\mathscr{O}_{S,\varphi(x)}} \mathscr{M}_{\varphi(x)}. \tag{1.10.3}$$

Thus $\mathscr{M} \mapsto \mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}$ is a right exact functor.

Definition 1.10.2. The **pullback sheaf** of $\mathscr{M}$ along φ is the $\mathscr{O}_X$-module defined by

$$\varphi^* \mathscr{M} := \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{M} \tag{1.10.4}$$

whose stalk at x is $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,\varphi(x)}} \mathscr{M}_x$. It can be viewed as the induced representation of $\mathscr{M}$. Thus we may write

$$\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M} = \mathscr{E} \otimes_{\mathscr{O}_X} \varphi^* \mathscr{M}. \tag{1.10.5}$$

If $V \subset S$ is open and $\sigma \in \mathscr{M}(V)$, then the **pullback section** $\varphi^*(\sigma) \in \varphi^* \mathscr{M}(\varphi^{-1}(V))$ is the image of

$$1 \otimes \sigma \in \mathscr{O}(\varphi^{-1}(V)) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to (\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{M})(\varphi^{-1}(V)) = (\varphi_* \varphi^* \mathscr{M})(V). \tag{1.10.6}$$

This gives a canonical morphism of $\mathscr{O}_S$-modules

$$\mathscr{M} \to \varphi_* \varphi^* \mathscr{M}. \tag{1.10.7}$$

If $g : \mathscr{M}_1 \to \mathscr{M}_2$ is a morphism of $\mathscr{O}_S$-modules, we define an $\mathscr{O}_X$-module morphism

$$\varphi^* g := \mathbf{1} \otimes g : \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_1 \to \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{M}_2, \tag{1.10.8}$$

called the **pullback of** g. If $\sigma \in \mathscr{M}_1(V)$, then

$$\varphi^* g(\varphi^* \sigma) = \varphi^*(g(\sigma)) \qquad \in \mathscr{M}_2(\varphi^{-1}(V)). \tag{1.10.9}$$

□

The notation $\mathscr{E} \otimes_{\mathscr{O}_S} \mathscr{M}$ is a generalization of $\mathscr{E} \otimes_{\mathbb{C}} W$ for a ($\mathbb{C}$-)vector space W by viewing $\mathbb{C}$ as the structure sheaf of the single reduced point $\{0\}$, and by viewing the holomorphic map as the obvious one $X \to \{0\}$.

Proposition 1.10.3. *(φ^*, φ_*) is a pair of* ***adjoint functors*** *between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_S$-modules (with φ^* the left adjoint and φ_* the right one). Namely, there is a functorial isomphism*

$$\mathrm{Hom}_{\mathscr{O}_X}(\varphi^*\mathscr{M}, \mathscr{E}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}, \varphi_*\mathscr{E}). \tag{1.10.10}$$

The word ***functorial*** *(also called* ***natural****) means that for any morphisms $g : \mathscr{M}_2 \to \mathscr{M}_1$ of $\mathscr{O}_S$-modules and $f : \mathscr{E}_1 \to \mathscr{E}_2$ of $\mathscr{O}_X$-modules, φ^*g and φ_*f induce a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{O}_X}(\varphi^*\mathscr{M}_1, \mathscr{E}_1) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_1, \varphi_*\mathscr{E}_1) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathscr{O}_X}(\varphi^*\mathscr{M}_2, \mathscr{E}_2) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}_2, \varphi_*\mathscr{E}_2) \end{array} \tag{1.10.11}$$

Proof. Given a morphism $F : \varphi^*\mathscr{M} \to \mathscr{E}$, the composition of $\mathscr{M} \to \varphi_*\varphi^*\mathscr{M}$ with $\varphi_*F : \varphi_*\varphi^*\mathscr{M} \to \varphi_*\mathscr{E}$ gives a morphism $G : \mathscr{M} \to \varphi_*\mathscr{E}$. Informally,

$$G(\sigma) = F(1 \otimes \sigma). \tag{1.10.12}$$

We leave it to the readers to check that $F \mapsto G$ is functorial.

Conversely, given $G : \mathscr{M} \to \varphi_*\mathscr{E}$. The $\mathscr{O}(U)$-module morphisms

$$\mathscr{O}(U) \otimes_{\mathscr{O}(V)} \mathscr{M}(V) \to \mathscr{E}(U), \qquad 1 \otimes \sigma \mapsto G(\sigma)|_U$$

for all open $U \subset X$ and $V \supset U$ pass to $F : \varphi^*\mathscr{M} \to \mathscr{E}$. This gives the inverse of the above construction. □

Definition 1.10.4. Let $\iota : Y = \mathrm{Specan}(\mathscr{O}_X/\mathcal{I}) \hookrightarrow X$ be a closed subspace of X. Let $\mathscr{E}$ be an $\mathscr{O}_X$-module. Then the **(sheaf theoretic) restriction of $\mathscr{E}$ to Y**, denoted by $\mathscr{E}|_Y$ or $\mathscr{E}|Y$ is

$$\mathscr{E}|_Y = \iota^*\mathscr{E} = (\mathscr{O}_X/\mathcal{I}) \restriction_{N(\mathcal{I})} \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.13}$$

Remark 1.10.5. If $\iota : Y \to X$ is an embedding of closed complex subspace, one may view an $\mathscr{O}_Y$-module $\mathscr{F}$ as the corresponding $\mathscr{O}_X$-module $\iota_*\mathscr{F}$. In particular, we have natural isomorphisms

$$\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{F}_1, \mathscr{F}_2) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\iota_*\mathscr{F}_1, \iota_*\mathscr{F}_2),$$
$$\iota_*(\mathscr{F}_1 \otimes_{\mathscr{O}_Y} \mathscr{F}_2) \simeq (\iota_*\mathscr{F}_1) \otimes_{\mathscr{O}_X} (\iota_*\mathscr{F}_2).$$

Note that since $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$ is surjective (if $y \in Y$), we have

$$\mathscr{F}_{1,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{F}_{2,y} \simeq \mathscr{F}_{1,y} \otimes_{\mathscr{O}_{X,y}} \mathscr{F}_{2,y}. \tag{1.10.14}$$

If $\mathscr{E}$ is an $\mathscr{O}_X$-module, we also have a natural isomorphism

$$\iota_*(\mathscr{E}|_Y) \simeq (\mathscr{O}_X/\mathcal{I}) \otimes_{\mathscr{O}_X} \mathscr{E}. \tag{1.10.15}$$

Thus, the study of the restriction $\mathscr{E}|_Y$ can be turned into the study of an $\mathscr{O}_X$-module.

1.11 Fiber products

Definition 1.11.1. Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. A **fiber product** of these two maps, also called the **pullback/base change** of φ along ψ, is a complex space $X \times_S Y$ together with holomorphic maps $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$ satisfying:

(1) $\varphi \circ \mathrm{pr}_X = \psi \circ \mathrm{pr}_Y$.

(2) For each complex space Z and holomorphic maps $\alpha : Z \to X$ and $\beta : Z \to Y$ satisfying $\varphi \circ \alpha = \psi \circ \beta$ there is a unique holomorphic map $\alpha \vee \beta : Z \to X \times_S Y$ such that $\alpha = \mathrm{pr}_X \circ (\alpha \vee \beta)$ and that $\beta = \mathrm{pr}_Y \circ (\alpha \vee \beta)$.

$$\begin{tikzcd} & & Z \arrow[lld, bend right, "\alpha"'] \arrow[ld, dashed, "\alpha\vee\beta"'] \arrow[ldd, bend left, "\beta"] \\ X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & \\ S & Y \arrow[l, "\psi"] & \end{tikzcd} \tag{1.11.1}$$

The commutative square diagram above involving $S, X, Y, X \times_S Y$ is called a **Cartesian square**. □

The following is easy to check:

Proposition 1.11.2. *In Def. 1.11.1, let $\gamma : Z' \to Z$ be a holomorphic map. Then*

$$(\alpha \vee \beta) \circ \gamma = (\alpha \circ \gamma) \vee (\beta \circ \gamma) : Z' \to X \times_S Y. \tag{1.11.2}$$

Fiber products are clearly unique up to isomorphisms. The following is easy to check.

Remark 1.11.3. Suppose that the following two small commuting square diagrams are both Cartesian, then the largest rectangular square is also Cartesian.

$$\begin{tikzcd} X \arrow[d] & X \times_S Y \arrow[l] \arrow[d] & (X \times_S Y) \times_Y Z \arrow[l] \arrow[d] \\ S & Y \arrow[l] & Z \arrow[l] \end{tikzcd}$$

Namely, $(X \times_S Y) \times_Y Z$, together with its maps to X and Z, is a pullback of $X \to S$ along $Z \to S$. This can be generalized to more complicated situations. For instance, if the following 4 small cells are Cartesian squares, then so is the largest square diagram.

$$\begin{tikzcd}
X_1 & Z_1 \arrow[l] & Z_3 \arrow[l] \\
X \arrow[from=1-1, d] & Z \arrow[l] \arrow[from=1-2, d] & Z_2 \arrow[l] \arrow[from=1-3, d] \\
S \arrow[from=2-1] & Y \arrow[l] \arrow[from=2-2] & Y_1 \arrow[l] \arrow[from=2-3]
\end{tikzcd}$$

Example 1.11.4. Let U, V be open subsets of a complex space X. Then $U \cap V$ is a fiber product $U \times_X V$: we have Cartesian square

$$\begin{tikzcd}
U \arrow[d, hook] & U \cap V \arrow[l, hook'] \arrow[d, hook] \\
X & V \arrow[l, hook']
\end{tikzcd}$$

Definition 1.11.5. Let $\varphi : X \to S$, $\psi : Y \to S$, $\alpha : X' \to X$, $\beta : Y' \to Y$ be holomorphic maps of complex spaces. Assume $X \times_S Y$ exists. Assume we have a fiber product $X' \times_S Y'$ of $\varphi \circ \alpha : X' \to S$ and $\psi \circ \beta : Y' \to S$. Then

$$\alpha \times \beta : X' \times_S Y' \to X \times_S Y \tag{1.11.3}$$

denotes $(\alpha \circ \mathrm{pr}_{X'}) \vee (\beta \circ \mathrm{pr}_{Y'})$, the unique holomorphic map making the following diagrams commute.

$$\begin{tikzcd}
 & & X' \arrow[dll, "\alpha"'] & X' \times_S Y' \arrow[l, "\mathrm{pr}_{X'}"'] \arrow[dll, "\alpha\times\beta"'] \arrow[d, "\mathrm{pr}_{Y'}"] \\
X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & & Y' \arrow[dll, "\beta"] \\
S & Y \arrow[l, "\psi"] & &
\end{tikzcd} \tag{1.11.4}$$

The following is easy to check:

Proposition 1.11.6. *In Def. 1.11.5, let $\mu : Z \to X'$, $\nu : Z \to Y'$ holomorphic maps of complex spaces such that $\varphi \circ \alpha \circ \mu = \psi \circ \beta \circ \nu$. Then we have equality*

$$(\alpha \times \beta) \circ (\mu \vee \nu) = (\alpha \circ \mu) \vee (\beta \circ \nu) : Z \to X \times_S Y. \tag{1.11.5}$$

Let $\widetilde{\alpha} : X'' \to X'$, $\widetilde{\beta} : Y'' \to Y'$ be holomorphic maps of complex spaces, and assume that a fiber product $X'' \times_S Y''$ exists for $\varphi \circ \alpha \circ \widetilde{\alpha} : X'' \to S$ and $\psi \circ \beta \circ \widetilde{\beta} : Y'' \to S$. Then

$$(\widetilde{\alpha} \times \widetilde{\beta}) \circ (\alpha \times \beta) = (\widetilde{\alpha} \circ \alpha) \times (\widetilde{\beta} \circ \beta) : X'' \times_S Y'' \to X \times_S Y. \tag{1.11.6}$$

Remark 1.11.7. There are no canonical fiber products of give holomorphic $\varphi : X \to S$, $\psi : Y \to S$. But suppose that a fiber product $X \times_S Y$ exists and is fixed. Then for each open $U \subset X$ and $X \subset Y$, there is a unique (open) **fiber product** $U \times_S V$ **inside** $X \times_S Y$. which is the open complex subspace

$$U \times_S V := \mathrm{pr}_X^{-1}(U) \cap \mathrm{pr}_Y^{-1}(V)$$

of $X \times_S Y$. The projections $\mathrm{pr}_U : U \times_S V \to U$ and $\mathrm{pr}_V : U \times_S V \to V$ are defined respectively by the restrictions of $\mathrm{pr}_X, \mathrm{pr}_Y$.

Moreover, assume that $\alpha : X' \to X$, $\beta : Y' \to Y$ are holomorphic, and a fiber product $X' \times_S Y'$ is fixed. Let $U' \subset X'$ and $V' \subset Y'$ be open such that $\alpha(U') \subset U$, $\beta(V') \subset V$. Let $U' \times_S V'$ be the fiber product inside $X' \times_S Y'$. The we have a commutative diagram

$$\begin{tikzcd} X' \times_S Y' \arrow[r, "\alpha \times \beta"] & X \times_S Y \\ U' \times_S V' \arrow[u, hook] \arrow[r, "\alpha|_{U'} \times \beta|_{V'}"] & U \times_S V \arrow[u, hook] \end{tikzcd} \tag{1.11.7}$$

□

Proof. Show that the inclusion $U \times_S V \hookrightarrow X \times_S Y$ is the product of $U \hookrightarrow X$ and $V \hookrightarrow Y$ and $U' \times_S V' \hookrightarrow X' \times_S Y'$ similarly. Then apply Prop. 1.11.6. □

With the help of the above observation, we can prove:

Lemma 1.11.8 (Gluing fiber products). *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic maps of complex spaces. Let $(U_i)_{i \in \mathfrak{I}}$ and $(V_t)_{t \in \mathfrak{T}}$ be open covers of X and Y respectively. Suppose that for each $i \in \mathfrak{I}$ and $t \in \mathfrak{T}$ there exists a fiber product $U_i \times_S V_t$. Then a fiber product $X \times_S Y$ exists.*

Proof. It suffices to assume (V_t) has only one member, which is Y. So each $U_i \times_S Y$ exists. To simplify notations, for each $i, j, k \in \mathfrak{I}$ we set $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. We let $U_{ij} \times_i Y$ and $U_{ijk} \times_i Y$ denote corresponding open fiber products inside $U_i \times_S Y$. So $U_{ij} \times_i Y$ and $U_{ij} \times_j Y$ are isomorphism but not identical.

We now apply the gluing construction Rem. 1.3.6 to construct $X \times Y$ by gluing all $U_i \times Y$ together. As gluing of topological spaces the process is trivial. To glue the structures of complex spaces, we must assign an isomorphism $\pi_{j,i} : U_{ij} \times_i Y \xrightarrow{\simeq} U_{ij} \times_j Y$ for all i, j. This is chosen to be $\mathbf{1}_{U_{ij}} \times_{j,i} \mathbf{1}_Y$ defined by Def. 1.11.5. (Note that this is not an identity map since the source does not equal the target. The symbol $\times_{j,i}$ reflects the fact that this product relies on both i and j.)

Clearly $\pi_{i,i}$ is the identity. To finish checking the cocycle condition, we must show that the holomorphic maps $\pi_{k,i}$ and $\pi_{k,j} \circ \pi_{j,i}$ are equal when restricted to open subsets $U_{ijk} \times_i Y \to U_{ijk} \times_k Y$. By Rem. 1.11.7, $\pi_{k,i}$ restricts to $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$, and $\pi_{k,j} \circ \pi_{j,i}$ restricts to $(\mathbf{1}_{U_{ijk}} \times_{k,j} \mathbf{1}_Y) \circ (\mathbf{1}_{U_{ijk}} \times_{j,i} \mathbf{1}_Y)$, which equals $\mathbf{1}_{U_{ijk}} \times_{k,i} \mathbf{1}_Y$ by Prop. 1.11.6.

Thus the complex space $X \times_S Y$ is constructed. We leave it to the readers to define pr_X and pr_Y. □

1.12 Fiber products and inverse images of subspaces

Proposition 1.12.1. *Let $\varphi : X \to S$ be a holomorphic map of complex spaces, and let $\mathscr{J}$ be a finite type ideal of $\mathscr{O}_S$. Then we have a Cartesian product*

$$\begin{tikzcd} X \arrow[d, "\varphi"'] & \varphi^{-1}(S_0) := \mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X) \arrow[l, hook'] \arrow[d, "\tilde{\varphi}"'] \\ S & S_0 := \mathrm{Specan}(\mathscr{O}_S/\mathscr{J}) \arrow[l, hook'] \end{tikzcd} \tag{1.12.1}$$

where $\mathscr{J}\mathscr{O}_X$ is the (necessarily finite-type) ideal of $\mathscr{O}_X$ generated by $\mathscr{J}$ (more precisely, by $\varphi^\#(\mathscr{J})$). $\varphi^{-1}(S_0) := \mathrm{Specan}(\mathscr{O}_X/\mathscr{J}\mathscr{O}_X)$ is called the ***inverse image of*** *$S_0 = \mathrm{Specan}(\mathscr{O}_S/\mathscr{J})$ along φ.*

Proof. Use Thm. 1.4.8. □

Remark 1.12.2. As an $\mathscr{O}_X$-module, $\mathscr{O}_{\varphi^{-1}(S_0)}$ has a natural equivalence

$$\mathscr{O}_{\varphi^{-1}(S_0)} = \mathscr{O}_X/\mathscr{J}\mathscr{O}_X \simeq \mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathscr{J}) = \varphi^*(\mathscr{O}_{S_0}). \tag{1.12.2}$$

Proof. Using the right exactness of $\mathscr{O}_X \otimes_{\mathscr{O}_S} -$, we have

$$\begin{aligned} &\mathscr{O}_X \otimes_{\mathscr{O}_S} (\mathscr{O}_S/\mathscr{J}) = \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathrm{coker}(\mathscr{J} \hookrightarrow \mathscr{O}_S) \\ \simeq &\mathrm{coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{J} \to \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{O}_S) \simeq \mathrm{coker}(\mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{J} \to \mathscr{O}_X) \end{aligned}$$

which equals $\mathscr{O}_X/\mathscr{J}\mathscr{O}_X$ since the term insider the last coker is the multiplication map. (Compare Rem. 1.7.3.) □

Example 1.12.3. Let $\mathscr{I}, \mathscr{J}$ be finite-type ideals of $\mathscr{O}_S$. Using Thm. 1.4.8 again, one easily checks that there is a Cartesian square that breaks into two commuting triangles.

$$\begin{tikzcd} X = \mathrm{Specan}(\mathscr{O}_S/\mathscr{I}) \arrow[d, hook] & X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathscr{I}+\mathscr{J})) \arrow[l] \arrow[d] \arrow[dl, dashed, hook] \\ S & Y = \mathrm{Specan}(\mathscr{O}_S/\mathscr{J}) \arrow[l, hook'] \end{tikzcd} \tag{1.12.3}$$

Thus, the inverse image of Y along X is naturally equivalent to the closed subspace $X \cap Y := \mathrm{Specan}(\mathscr{O}_S/(\mathcal{I} + \mathcal{J}))$ of S, called the **intersection of** X **and** Y. (Compare this with Exp. 1.7.5.) In view of this equivalence, we shall view $X \cap Y$ as closed subspaces of X and Y in the future.

Proposition 1.12.4. *Let $\varphi : X \to S$ and $\psi : Y \to S$ be holomorphic, and let X_0 and Y_0 be complex subspaces of X, Y respectively. Assume that $X \times_S Y$ is a fiber product of φ and ψ. Recall $\mathrm{pr}_X : X \times_S Y \to X$ and $\mathrm{pr}_Y : X \times_S Y \to Y$. Then the intersection*

$$X_0 \times_S Y_0 := \mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$$

*is a fiber product of $X_0 \hookrightarrow X \xrightarrow{\varphi} S$ and $Y_0 \hookrightarrow Y \xrightarrow{\psi} S$, called the **(closed) fiber product inside** $X \times_S Y$. The projections of $\mathrm{pr}_X^{-1}(X_0) \cap \mathrm{pr}_Y^{-1}(Y_0)$ to X_0 and Y_0 are respectively the restrictions of pr_X and pr_Y. Moreover, the inclusion $X_0 \times_S Y_0 \hookrightarrow X \times_S Y$ equals the product of $X_0 \hookrightarrow X$ and $Y_0 \hookrightarrow Y$.*

Proof. The four cells are Cartesian squares. So is the largest one (Rem. 1.11.3).

$$\begin{tikzcd}
X_0 \arrow[d, hook] & \mathrm{pr}_X^{-1}(X_0) \arrow[l] \arrow[d, hook] & X_0 \times_S Y_0 \arrow[l, hook'] \arrow[d, hook] \\
X \arrow[d, "\varphi"'] & X \times_S Y \arrow[l, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"'] & \mathrm{pr}_Y^{-1}(Y_0) \arrow[l, hook'] \arrow[d] \\
S & Y \arrow[l, "\psi"'] & Y_0 \arrow[l, hook']
\end{tikzcd} \tag{1.12.4}$$

The claim about inclusions is obvious. □

Remark 1.12.5. The closed fiber product $X_0 \times_S Y_0 \subset X \times_S Y$ can be written more explicitly. Choose finite-type ideals $\mathcal{I} \subset \mathscr{O}_X$ and $\mathcal{J} \subset \mathscr{O}_Y$ defining X_0, Y_0 respectively. Then $X_0 \times_S Y_0$ is defined by the ideal $\mathcal{K} \subset \mathscr{O}_{X \times_S Y}$ generated by $\mathrm{pr}_X^\#(\mathcal{I})$ and $\mathrm{pr}_Y^\#(\mathcal{J})$.

In practice, we may assume X and Y are small enough such that $\mathcal{I}$ is generated by $f_1, \dots, f_m \in \mathscr{O}(X)$ and $\mathcal{J}$ is generated by $g_1, \dots, g_n \in \mathscr{O}(Y)$. Then all $\mathrm{pr}_X^\#(f_i)$ and $\mathrm{pr}_Y^\#(g_j)$ generate $\mathcal{K}$. □

Remark 1.12.6. Similar to Rem. 1.11.7, suppose we have holomorphic $\alpha : X' \to X$, $\beta : Y' \to Y$, $\varphi : X \to S$, $\psi : Y \to S$. Let $X_0 \subset X, Y_0 \subset Y, X'_0 \subset X', Y'_0 \subset Y'$ be closed subspaces such that α restricts to $\alpha : X'_0 \to X_0$ and β restricts to $\beta : Y'_0 \to Y_0$ (in the sense of Thm. 1.4.8). Then for the closed fiber products $X_0 \times_S Y_0 \subset X \times_S Y$ and X'_0, the following diagram commutes.

$$\begin{tikzcd}
X' \times_S Y' \arrow[r, "\alpha \times \beta"] & X \times_S Y \\
X'_0 \times_S Y'_0 \arrow[u, hook] \arrow[r, "\alpha|_{X'_0} \times \beta|_{Y'_0}"] & X_0 \times_S Y_0 \arrow[u, hook]
\end{tikzcd} \tag{1.12.5}$$

1.13 Fiber products, direct products, and equalizers

Definition 1.13.1. Let X, Y be complex spaces. A **direct product** of X, Y, or simply a **product** of X, Y, is a fiber product of $X \to 0$ and $Y \to 0$ and denoted by $X \times Y$ (together with the projections $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$).

To spell out the definition: For each complex space Z and holomorphic $\alpha : Z \to X, \beta : Z \to Y$, there is a unique holomorphic map $\alpha \vee \beta : Z \to X \times Y$ such that the following diagrams commute.

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[d, "\alpha\vee\beta"'] \arrow[dr, "\beta"] & \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

If $f \in \mathscr{O}_X$ and $g \in \mathscr{O}_Y$, we write

$$f \otimes 1 := \mathrm{pr}_X^\#(f), \qquad 1 \otimes g := \mathrm{pr}_Y^\#(g), \qquad f \otimes g := \mathrm{pr}_X^\#(f)\mathrm{pr}_Y^\#(g).$$

If $x \in X$ and $y \in Y$, we define the **completed tensor product**

$$\mathscr{O}_{X,x} \widehat{\otimes} \mathscr{O}_{Y,y} := \mathscr{O}_{X \times Y, x \times y}$$

which is well-defined up to isomorphisms by Cor. 1.6.3. □

Remark 1.13.2. One can also view $\mathscr{O}_{X \times_S Y, x \times y}$ as $\mathscr{O}_{X,x} \widehat{\otimes}_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$ (if $s = \varphi(x) = \psi(y)$), a completed tensor product over $\mathscr{O}_{S,s}$. In the case that either φ or ψ is "finite", the stalk $\mathscr{O}_{X \times_S Y, x \times y}$ is actually equal to the usual tensor product $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{S,s}} \mathscr{O}_{Y,y}$. This will be studied in the next chapter.

Example 1.13.3. $\mathbb{C}^{m+n}$ is naturally a product of $\mathbb{C}^m$ and $\mathbb{C}^n$.

Proof. Apply Thm. 1.4.1. □

Lemma 1.13.4. *For every complex spaces X, Y there is a product $X \times Y$.*

Proof. We know this is true when X, Y are number spaces, and hence when X, Y are open subspaces of number spaces (cf. Exp. 1.11.4), and hence if X, Y are model spaces (due to Prop. 1.12.4), and hence for all complex spaces (by Lemma 1.11.8). □

Remark 1.13.5. If X and Y are model spaces $\mathrm{Specan}(\mathscr{O}_U/\mathcal{I})$ and $\mathrm{Specan}(\mathscr{O}_V/\mathcal{J})$ where $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open, $\mathcal{I}$ is generated by $f_1, f_2, \dots \in \mathcal{I}(U)$, and $\mathcal{J}$ is generated by $g_1, g_2, \dots \in \mathcal{J}(V)$, then $X \times Y$ as a closed direct product inside $U \times V$ can be written down explicitly with the help of Rem. 1.12.5: it is the model space $\mathrm{Specan}(\mathscr{O}_{U \times V}/\mathcal{K})$ where $\mathcal{K}$ is the ideal generated by all $f_i \otimes 1$ and $1 \otimes g_j$.

In the following, we give two proves that fiber products always exist. We need the following notion:

Proposition 1.13.6. *Let $\varphi : X \to Y$ be a holomorphic map. Then $\mathbf{1}_X \vee \varphi : X \to X \times Y$ is an equalizer:*

$$X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y \underset{\mathrm{pr}_Y}{\overset{\varphi\circ\mathrm{pr}_X}{\rightrightarrows}} Y \tag{1.13.1}$$

The image of $\mathbf{1} \vee \varphi$ as a closed subspace of $X \times Y$ (namely, the canonical equalizer of $X \times Y \rightrightarrows Y$) is called the ***graph of*** *φ.*

Proof. Let Z be a complex space. Any holomorphic map $Z \to X \times Y$ is $\alpha \vee \beta$ for some $\alpha : Z \to X$ and $\beta : Z \to Y$. Suppose that the compositions of $\alpha \vee \beta$ with $\varphi \circ \mathrm{pr}_X$ and with pr_Y are equal. Then $\varphi \circ \alpha = \beta$. Then we may find a holomorphic map $Z \to X$ such that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[d] \arrow[dr, "\alpha\vee\beta"] & \\ X \arrow[r, "\mathbf{1}\vee\varphi"] & X \times Y \end{tikzcd}$$

Indeed, we can choose this map to be α. Then by Prop. 1.11.2, $(\mathbf{1} \vee \varphi) \circ \alpha = \alpha \vee (\varphi \circ \alpha) = \alpha \vee \beta$. On the other hand, if we have another such holomorphic map $\psi : Z \to X$. Composing the above triangle with $\mathrm{pr}_X : X \times Y \to X$ shows that $\psi = \mathrm{pr}_X \circ (\mathbf{1} \vee \varphi) \circ \psi$ equals $\mathrm{pr}_X \circ (\alpha \vee \beta) = \alpha$. This proves the uniqueness of such ψ. $\square$

Remark 1.13.7. Using Thm. 1.8.2, one can give a more explicit description of the graph of $\varphi : X \to Y$. We write it as $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ for a finite-type ideal $\mathscr{J}$. Let $x \in X, y \in Y$. If $y \neq \varphi(x)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y,x\times y}$. If $y = \varphi(x)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y,x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes f \tag{1.13.2}$$

for all $f \in \mathscr{O}_{Y,y}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{Y,y}$ analytically). The underlying topological space of the graph is $\{x \times y \in X \times Y : y = \varphi(x)\}$.

Remark 1.13.8. The graph construction shows that every holomorphic map $\varphi : X \to Y$ is the composition of a closed embedding $X \xrightarrow{\mathbf{1}\vee\varphi} X \times Y$ and a projection of direct product $X \times Y \xrightarrow{\mathrm{pr}_Y} Y$. Thus, very often, the study of general holomorphic maps reduce to the studies of these two special types of maps. As an application of this observation, we prove:

Theorem 1.13.9. *For any holomorphic maps of complex spaces $\varphi : X \to S, \psi : Y \to S$, there exists a fiber product $X \times_S Y$.*

Proof. We want to show that the pullback of φ along ψ exists. We know it exists when ψ is a closed embedding due to Prop. 1.12.1. It also exists when ψ is a projection $S \times Y_1 \to S$: in that case $X \times_S Y$ is given by the Cartesian square

$$\begin{tikzcd} X & X \times Y_1 \arrow[l] \arrow[d, "\varphi\times\mathbf{1}"'] \\ S \arrow[u, leftarrow, "\varphi"'] & S \times Y_1 \arrow[l] \end{tikzcd} \tag{1.13.3}$$

(We leave it to the readers to check that this commutative diagram is indeed Cartesian.) The general case follows from Rem. 1.13.8 and the fact that the pullback of a pullback is a pullback (Rem. 1.11.3). □

We now give another way of constructing fiber products. This construction is very explicit when X and Y are model spaces.

Proposition 1.13.10. *Let $\varphi : X \to S, \varphi : Y \to S$ be holomorphic maps of complex spaces. Let $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ be the projections of $X \times Y$. Then the canonical equalizer E of the following double arrow is a fiber product $X \times_S Y$:*

$$E \overset{\iota}{\hookrightarrow} X \times Y \underset{\psi\circ\mathrm{pr}_Y}{\overset{\varphi\circ\mathrm{pr}_X}{\rightrightarrows}} S \tag{1.13.4}$$

The projections of E to X, Y are $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ respectively. We call E the (closed) ***fiber product of** X, Y **inside the direct product** $X \times Y$.*

Proof. That E is an equalizer means that $\varphi \circ (\mathrm{pr}_X \circ \iota) = \psi \circ (\mathrm{pr}_Y \circ \iota)$, and that for every holomorphic $\alpha \vee \beta : Z \to X \times Y$ whose compositions with $\varphi \circ \mathrm{pr}_X$ and with $\psi \circ \mathrm{pr}_Y$ are the same (namely, $\varphi \circ \alpha = \psi \circ \beta$) there is a unique holomorphic $\gamma : Z \to E$ such that $\iota \circ \gamma = \alpha \vee \beta$ (namely, $(\mathrm{pr}_X \circ \iota) \circ \gamma = \alpha$ and $(\mathrm{pr}_Y \circ \iota) \circ \gamma = \beta$). This means precisely that E equipped with $\mathrm{pr}_X \circ \iota$ and $\mathrm{pr}_Y \circ \iota$ is a fiber product. □

Remark 1.13.11. Using Thm. 1.8.2, we can describe the fiber product $X \times_S Y$ inside a given $X \times Y$ easily: It is $\mathrm{Specan}(\mathscr{O}_{X\times Y}/\mathscr{J})$ where $\mathscr{J}$ is a finite-type ideal. Let $x \in X, y \in Y$. If $\varphi(x) \neq \psi(y)$ then $\mathscr{J}_{x\times y} = \mathscr{O}_{X\times Y, x\times y}$. If $\varphi(x) = \psi(y)$ then $\mathscr{J}_{x\times y}$ is the ideal of $\mathscr{O}_{X\times Y, x\times y}$ generated by

$$(f \circ \varphi) \otimes 1 - 1 \otimes (f \circ \psi) \tag{1.13.5}$$

for all $f \in \mathscr{O}_{S,\varphi(x)}$ (equivalently, for a set of f generating the algebra $\mathscr{O}_{S,\varphi(x)}$ analytically). The underlying topological space of $X \times_S Y$ is $\{x \times y \in X \times Y : \varphi(x) = \psi(y)\}$.

Exercise 1.13.12. Show that the pullback of $\varphi \times \psi : X \times Y \to S \times S$ along the **diagonal map** Δ_S defined by $\mathbf{1}_S \vee \mathbf{1}_S : S \to S \times S$ is a fiber product $X \times_S Y$.

We have seen that fiber products can be constructed from equalizers. Conversely, equalizers can also be viewed as special cases of fiber products:

Proposition 1.13.13. *Let $\varphi, \psi : X \to Y$ be holomorphic maps, and let $\Delta_Y : Y \to Y \times Y$ be the diagonal map of Y with image $\widetilde{Y}$ being a closed subspace of $Y \times Y$, called the* ***diagonal of*** *$Y \times Y$. Then the inverse image E of $\widetilde{Y}$ along $\varphi \vee \psi : X \to Y \times Y$ is the canonical equalizer of $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Y$.*

Proof. Write $\widetilde{Y}$ as $\mathrm{Specan}(\mathscr{O}_{Y\times Y}, \mathcal{J})$. Then by Rem. 1.13.7, $\mathcal{J}_{y,y'} = \mathscr{O}_{Y\times Y, y\times y'}$ if $y \neq y'$, and $\mathcal{J}_{y,y'}$ is generated by all $f \otimes 1 - 1 \otimes f$ where $f \in \mathscr{O}_{Y,y}$.

Write E as $\mathrm{Specan}(\mathscr{O}_X/\mathcal{I})$. Then by Prop. 1.12.1, if $\varphi(x) \neq \psi(x)$ then $\mathcal{I}_x$ equals $\mathscr{O}_{X,x}$ (since $\mathcal{J}_{\varphi(x),\psi(x)} = \mathscr{O}_{Y\times Y,\varphi(x)\times\psi(x)}$); if $\varphi(x) = \psi(x)$ then $\mathcal{I}_x$ is generated by $(f \otimes 1 - 1 \otimes f) \circ (\varphi \vee \psi)$ (i.e. by $f \circ \varphi - f \circ \psi$) for all $f \in \mathscr{O}_{Y,\varphi(x)}$. Comparing this description with Thm. 1.8.2, we see that E is the canonical equalizer. $\square$

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[Gui22] B. Gui, Lectures on Vertex Operator Algebras and Conformal Blocks, 2022

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