PLURIPOTENTIAL THEORY ON COMPLEX ANALYTIC SPACES

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This is a preliminary version. There might be mistakes. Feel free to contact me if you have comments. For the latest version, see http://www.math.chalmers.se/~xiam/PTV.pdf.

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1. Introduction

In this note, we will develop the theory of non-pluripolar products on complex analytic spaces. To a large extent, the theory is well-known to experts, although no systematic survey has been written so far.

In principle, the theory of non-pluripolar products on singular analytic spaces can be developed in a completely parallel fashion as the classical theory on complex manifolds ([BEGZ10]). For bounded plurisubharmonic functions, this is done in [Dem85]. However, given the possibility of resolving singularities in characteristic 0, we could instead reduce the general theory to the smooth case. This, however, requires that we work in a *birational* (or bimeromorphic) framework. Classically, in the global setting, people consider quasi-plurisubharmonic function with respect to a representative in a Kähler class. For us, it is necessary to replace Kähler classes by big (or big and nef) classes. Thanks to the recent development in the smooth setting, most foundational results about non-pluripolar products are actually proved for general big classes. This note relies exactly on these results.

Let us mention some brand new features in the analytic space setting. First of all, there are two different notions of pluri-subharmonic functions (see [FN80]). Although the stronger notion of pluri-subharmonic functions works better for extension problems ([CGZ13]), the weaker notion of weakly pluri-subharmonic functions behaves better under proper bimeromorphic morphisms. Fortunately, on a unibranch space, these notions coincide, as an easy consequence of Zariski's main theorem (Theorem 2.4). So we decide to develop our theory only for unibranch spaces. This is not too restrictive for practice: otherwise the continuity of envelopes never holds, which ruins a large part of the theory.

The second difference is the lack of global regularization of pluri-subharmonic functions. The problem is, given a closed analytic subspace X of a pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$ and a psh function φ on X, there are no canonical extensions of φ to Ω. Thus after taking local regularization using mollifier technique, we do not get any control on the overlap. So the standard technique of regularization in the smooth setting fails. It is interesting to find a concrete example where no regularization exists.

This note is the first of a series of notes in preparation. We plan to develop similar results on polyhedral spaces (in the sense of [Jel16]) and Berkovich spaces in the next part.

We do not claim any originality in this work. In fact, all non-trivial proofs are already written down in the literature.

We assume that the reader is familiar with the classical pluripotential theory on smooth Kähler manifolds and the basic theory of complex analytic spaces.

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2. Preliminaries on analytic spaces

For each $N \geq 0$, we endow an open subset $\Omega \subseteq \mathbb{C}^N$ with the sheaf of holomorphic functions, so that Ω can be regarded as a locally $\mathbb{C}$-ringed space.

2.1. Complex analytic space.

In this section we recall some basic facts about complex analytic spaces. This section is by no means intended to be complete. For details, we refer to [Fis06], [CAS].

Definition 2.1. A *local model* of a complex analytic space is a locally $\mathbb{C}$-ringed space $(V, \mathcal{O}_V)$, such that there is an analytic closed immersion $V \hookrightarrow \Omega$ into a bounded pseudo-convex domain Ω in some $\mathbb{C}^N$. To be more precise, this means that there are $f_1, \dots, f_M \in H^0(\Omega, \mathcal{O}_\Omega)$, such that V is closed subset of Ω defined as the common zero locus of all f_i's, $\mathcal{O}_V$ is the quotient of $\mathcal{O}_\Omega$ by the ideal $(f_1, \dots, f_M)$, regarded as a sheaf on V.

The category of local models is a full subcategory of the category of locally $\mathbb{C}$-ringed spaces.

Definition 2.2. A *complex analytic space* is a locally $\mathbb{C}$-ringed space $(X, \mathcal{O}_X)$ such that

(1) X is a para-compact, Hausdorff space.
(2) For any $x \in X$, there is an open neighbourhood $U \subseteq X$ of x, such that $(U, \mathcal{O}_U)$ (with $\mathcal{O}_U$ being the restriction of $\mathcal{O}_X$ to U) is isomorphic to a local model as locally $\mathbb{C}$-ringed spaces.

The category is complex analytic spaces is a full subcategory of the category of locally $\mathbb{C}$-ringed spaces.

Definition 2.3. A complex analytic space $(X, \mathcal{O}_X)$ is

(1) *reduced* (resp. *normal*) at $x \in X$ if $\mathcal{O}_{X,x}$ is reduced (normal). We say X is *reduced* (resp. *normal*) if it is reduced (resp. normal) at all points.
(2) *unibranch* at $x \in X$ if $\mathcal{O}_{X,x}^{\mathrm{red}}$ is integral. We say X is *unibranch* if it is unibranch at all points.

Remark 2.1. Readers familiar with commutative algebra may find our definition of unibranchness a bit unusual. Assume that X is unibranch at $x \in X$, we prove that $\mathcal{O}_{X,x}$ is unibranch in the usual sense ([EGA IV$_1$, Section 0.23.2.1]). Let $K = \operatorname{Frac} \mathcal{O}_{X,x}^{\mathrm{red}}$. Let A be the integral closure of $\mathcal{O}_{X,x}^{\mathrm{red}}$ in K. We need to show that A is local. Recall that $\mathcal{O}_{X,x}$ is Henselian ([CAS, Page 45]), so is $\mathcal{O}_{X,x}^{\mathrm{red}}$, it follows that A is local.

Remark 2.2. We would like to persuade the readers that our notion of unibranchness is indeed natural. Let X be a scheme of finite type over $\mathbb{C}$. Let X^{an} be the complex analytification of X. Let $x \in X(\mathbb{C})$. Then we claim that X is unibranch at x iff X^{an} is unibranch at x. By GAGA, there is a natural morphism of ringed spaces $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}) \to (X, \mathcal{O}_X)$. We may assume that X is reduced. Then X^{an} is also reduced*. Then since $\mathcal{O}_{X,x}$ is excellent, X is unibranch at x iff $\widehat{\mathcal{O}_{X,x}} = \widehat{\mathcal{O}_{X^{\mathrm{an}},x}}$ is integral ([EGA IV$_4$, Théorème 18.9.1]). The latter implies that $\mathcal{O}_{X^{\mathrm{an}},x}$ is unibranch by [AC, Chapitre 5, Exercise 2.8]. Running the same argument the other way round and noting Remark 2.1, we conclude that $\mathcal{O}_{X,x}$ is unibranch iff $\mathcal{O}_{X^{\mathrm{an}},x}$ is.

Remark 2.3. For us, a ring is normal if it is an integral domain and integrally closed. So in particular, a normal analytic space is reduced. Also recall that a normal ring is always unibranch.

Remark 2.4. Note that our definition of unibranch space is different from the notion of locally irreducible space in [CAS, Page 8].

Note that X is unibranch iff X^{red} is.

Definition 2.4. A complex analytic space $(X, \mathcal{O}_X)$ is

(1) *Holomorphically separable* if for any $x, y \in X$, $x \neq y$, there is $f \in H^0(X, \mathcal{O}_X)$, such that $f(x) \neq f(y)$.
(2) *Holomorphically convex* if for any compact set $K \subseteq X$, the set
$$\hat{K} := \left\{ x \in X : |f(x)| \leq \sup_{k \in K} |f(k)|, \quad \forall f \in H^0(X, \mathcal{O}_X) \right\}$$
is compact.
(3) *Stein* if it is both holomorphically separable and holomorphically convex.

Note that all three conditions are stable under passing to closed subspaces.

Theorem 2.1. *Let $(X, \mathcal{O}_X)$ be a complex analytic space. Then X is Stein iff X^{red} is.*

*In fact, the conditions R_m and S_m are both preserved under GAGA

See [Fis06, Page 33].

Definition 2.5. Let $f : X \to S$ be a morphism of complex analytic spaces. We say the morphism f is *formally smooth* if given any solid commutative diagram

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd},$$

where $i : T \to T'$ is a first order thickening of Stein spaces, there is a dotted morphism $T' \to X$ making the whole diagram commutative.

A complex analytic space X is formally smooth if the morphism to the final object is formally smooth.

Theorem 2.2. *Let $f : X \to S$ be a morphism of complex analytic spaces. Assume that f is smooth, then f is formally smooth.*

The proof is taken from [Stacks, Tag 02H6].

Proof. Suppose that we are given a diagram as in Definition 2.5.

Consider the sheaf $\mathcal{F}$ of sets on T':

$$H^0(U', \mathcal{F}) := \{ a' : U' \to X : a'|_U = a|_U \}, \quad U = U' \cap T$$

for any open set $U' \subseteq T'$. We want to show that $\mathcal{F}$ admits a global section on T'. Let

$$\mathcal{H} := \mathcal{H}\mathrm{om}_{\mathcal{O}_T}(a^*\Omega^1_{X/S}, \mathcal{C}_{T/T'}),$$

where $\mathcal{C}_{T/T'}$ is the conormal sheaf of T in T', namely, if T is defined by a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{T'}$, then $\mathcal{C}_{T/T'}$ is $\mathcal{I}/\mathcal{I}^2$ regarded as an $\mathcal{O}_T$-sheaf. There is an obvious action of $\mathcal{H}$ on $\mathcal{F}$, making $\mathcal{F}$ a pseudo-$\mathcal{H}$-torsor. We will show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor.

First of all, let us show that $\mathcal{F}$ has non-trivial fibres. Let $t \in T$. Let $x = a(t)$, $s = f(x)$, $t' = i(t)$. We know that $f^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is smooth, hence formally smooth. Thus we can find a local homomorphism $g^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{T',t'}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_{S,s} \arrow[r, "a^\sharp"] \arrow[d, "f^\sharp"] & \mathcal{O}_{T',t'} \arrow[d, "i^\sharp"] \\ \mathcal{O}_{X,x} \arrow[r] \arrow[ur, "g^\sharp"] & \mathcal{O}_{T,t} \end{tikzcd}.$$

Now by [Fis06, Section 0.21], the homomorphism $g^\sharp$ induces a morphism of germs $g : (T', t') \to (X, x)$. This shows that $\mathcal{F}_t \neq \emptyset$.

Now we prove $H^1(T, \mathcal{H}) = 0$. In fact, $\mathcal{H}$ is coherent, so we could apply Cartan's theorem B ([Fis06, Page 33]). □

Lemma 2.3. *Let $f : Y \to X$ be a proper dominant morphism of complex analytic spaces. Assume that X and Y are both irreducible and reduced. For any $x \in X$, the number of connected components of $f^{-1}(x)$ is at most the number of maximal ideals of $\bar{\mathcal{O}}_{X,x}$, where $\bar{\mathcal{O}}_{X,x}$ is the integral closure of $\mathcal{O}_{X,x}$ in $\mathcal{M}(Y)$.*

Proof. Observe that $(f_*\mathcal{O}_Y)_x$ is a subring of $\mathcal{M}(X)$ containing $\mathcal{O}_{X,x}$. As f is proper, $f_*\mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module by Grauert's coherence theorem ([Fis06, Section 1.17]). In particular, $(f_*\mathcal{O}_Y)_x$ is finite over $\mathcal{O}_{X,x}$. Hence $(f_*\mathcal{O}_Y)_x \subseteq \bar{\mathcal{O}}_{X,x}$. In particular, each maximal ideal of $(f_*\mathcal{O}_Y)_x$ is the intersection of a maximal ideal of $\bar{\mathcal{O}}_{X,x}$ with $(f_*\mathcal{O}_Y)_x$ by going-up and the incomparability property. We conclude. □

We say an analytic space X is irreducible if X^{red} is irreducible ([CAS, Section 9.1]).

Theorem 2.4 (Zariski's main theorem)**.** *Let $f : Y \to X$ be a proper bimeromorphic morphism of complex analytic spaces. Assume that X is unibranch at $x \in X$. Then the fibre $f^{-1}(x)$ is connected.*

The proof is taken from [EGA III.1, Corollaire 4.3.7].

Proof. Note that f induces a morphism $f^{\mathrm{red}} : Y^{\mathrm{red}} \to X^{\mathrm{red}}$ satisfying all assumptions of the theorem, so we may assume that X and Y are reduced. As X is unibranch at x, there is at most one irreducible component of X containing x ([CAS, Page 174]). So we may assume that X is irreducible, hence so is Y. As f is proper bimeromorphic morphism, $\mathcal{M}(X) = \mathcal{M}(Y)$ ([Fis06, Section 4.9]), so we conclude by Lemma 2.3 and Remark 2.1. □

Lemma 2.5 (Partition of unity). *Let X be a complex analytic space. Let $\{U_\alpha\}$ be an open covering of X. Then there is a smooth partition of unity subordinate to $\{U_\alpha\}$.*

Proof. Consider the covering

$$\left\{V \subseteq X : V \text{ is open}, \bar{V} \subseteq U_\alpha \text{ for some } \alpha\right\} .$$

As X is assumed to be paracompact, there is a locally finite subcover $\{W_\gamma\}$. Recall that by general topology, there exists open sets V_γ, Z_γ in X for each γ with $V_\gamma \Subset Z_\gamma \Subset W_\gamma$, such that $\{V_\gamma\}$ is a locally finite covering of X and such that each Z_γ can be embedded into a bounded open pseudo-convex domain Ω_γ in $\mathbb{C}^{N_\gamma}$. Now by usual partition of unity on Ω_γ, we can construct a smooth function $f_\gamma : \Omega_\gamma \to [0,1]$ with compact support, such that $f_\gamma = 1$ on V_γ. We regard f_γ as a smooth function with compact support on X. It is easy to see that $f := \sum_\gamma f_\gamma$ is a locally finite sum, hence smooth and non-zero. Define $g_\gamma = f_\gamma / f$ to conclude. □

2.2. **Kähler spaces.** Let X be a complex analytic space.

Definition 2.6. A *Kähler form* on X is a smooth $(1,1)$-form ω on X, such that at any point $x \in X$, there is a neighbourhood $V \subseteq X$ of x, a closed immersion $V^{\mathrm{red}} \hookrightarrow \Omega$ into some bounded pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, a Kähler form ω_Ω on Ω, such that $\omega = \omega_\Omega|_V$.

A *Kähler space* is a complex analytic space which admits a Kähler form.

This definition differs from the standard definition in [Gra62, Page 346].

Remark 2.5. Note that by definition, Kähler forms on X is the same as Kähler forms on X^{red}. One may argue that it is more natural to define Kähler forms using closed immersions of V instead of V^{red}. The problem is, we do not know if the following holds: given a Kähler form ω on X^{red}, is there a Kähler form on X (in the latter sense) inducing ω?

Example 2.1. *Any Kähler manifold is a Kähler space. In fact, when X is smooth, the notion of Kähler form in Definition 2.6 coincides with the usual one.*

Example 2.2. *Let X be a Kähler space. Let Y be a closed analytic subspace, then Y is a Kähler space.*

As a consequence, any projective analytic space is Kähler. More generally,

Lemma 2.6. *Let X be a compact Kähler space. Let $f : Y \to X$ be a projective morphism. Then Y is a Kähler space.*

For the definition of projective morphism, see [GPR94, Section V.4].

Proof. We can embed Y as a closed subspace of $X \times \mathbb{P}^N$ for some $N \geq 0$ preserving the morphism to X. Let p_1, p_2 be the projection from $X \times \mathbb{P}^N$ to two factors. Take a Kähler form ω on X. Let ω_{FS} be the Fubini–Study metric on $\mathbb{P}^N$. Then we claim that $p_1^*\omega + p_2^*\omega_{\mathrm{FS}}$ defines a Kähler form on Y. By Example 2.2, it suffices to show that $p_1^*\omega + p_2^*\omega_{\mathrm{FS}}$ is a Kähler form on $X \times \mathbb{P}^N$. We may assume that X is reduced. The problem is also local in $\mathbb{P}^N$, we could replace $\mathbb{P}^N$ by a polydisk Δ in it.

The problem is local on X, so we may assume that X is a closed subspace of a pseudo-convex domain Ω in $\mathbb{C}^M$ and there is a Kähler form ω_Ω such that $\omega = \omega_\Omega|_X$. Now $X \times \Delta \hookrightarrow \Omega \times \Delta$ and the form

$$p_1^*\omega + p_2^*\omega_{\mathrm{FS}} = (\pi_1^*\omega_\Omega + \pi_2^*\omega_{\mathrm{FS}})\,|_{X\times\Delta} ,$$

where π_1, π_2 are the two projection from $\Omega \times \mathbb{P}^N$ to the two factors. □

Corollary 2.7. *Let X be a compact Käher space. Let Y be a closed subspace. Then $\mathrm{Bl}_Y X$ is a Kähler space.*

Here $\mathrm{Bl}_Y X$ is the blowing-up of X with center Y. For its precise definition, we refer to [GPR94, Section VII.2].

Recall that we can always resolve singularities of a complex analytic space.

Theorem 2.8. *Let X be a reduced complex analytic space. Then there is a (proper) resolution of singularity of X.*

This theorem was first proved by Aroca–Hironaka–Vicente, see the book [AHV18]. Later simplifications are due to Bierstone–Milman and Villamayor. See [Wło09] for details and further references.

Corollary 2.9. *Let X be a reduced compact Kähler complex analytic space. Then there is a resolution of singularity $\pi : Y \to X$ such that Y is a compact Kähler manifold.*

Proof. Let $p : Z \to X$ be a resolution. By Hironaka's Chow lemma ([Hir75]), Z is dominated (over X) by a complex analytic space W, which is a sequence of blowing-ups of X with smooth centers. By Corollary 2.7, W admits a Kähler form. Hence Z admits a Kähler current. Hence Z is of Fujiki's class $\mathcal{C}$ ([DP04]). In particular, there is a proper bimeromorphic morphism $Y \to Z$, such that Y is a compact Kähler manifold. □

It is NOT true that a resolution of singularity of a complex analytic space is always given by a sequence of blowing-ups, in contrast to the nice situation of complex varieties or excellent schemes in characteristic 0. It seems that a lot of people are making this mistake.

In the remaining of this note, by a resolution of singularity $f : Y \to X$ of a reduced Kähler space X, we always assume that Y is Kähler.

3. Pluripotential theory

Let X be a complex analytic space.

3.1. Plurisubharmonic functions.

Definition 3.1. Let $U \subseteq X$ be an open immersion. A function $\varphi : U \to [-\infty, \infty)$ is *pluri-subharmonic* if

(1) φ is not identically $-\infty$ on any irreducible component of U.
(2) For any $x \in U$, there is an open neighbourhood V of x in U, a bounded pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$, an open set $\tilde{V} \subseteq \Omega$ with $x \in \tilde{V}$, a pluri-subharmonic function $\tilde{\varphi}$ on $\tilde{V}$, such that $\varphi|_{\tilde{V}\cap V} = \tilde{\varphi}|_{\tilde{V}\cap V}$.

The set of pluri-subharmonic functions on U is denoted by $\mathrm{PSH}(U)$.

Proposition 3.1. *Let X be a complex manifold. Then with the canonical complex analytic space structure on X, the definition of $\mathrm{PSH}(X)$ coincides with the usual one.*

Proof. It suffices to recall that a psh function on a domain restricts to a psh function on a closed analytic submanifold. □

Proposition 3.2. *Let $f : X \to Y$ be a morphism between complex analytic spaces. Let $\varphi \in \mathrm{PSH}(Y)$. Assume that $f^*\varphi$ is not identically equal to $-\infty$ on each irreducible component of X, then $f^*\varphi \in \mathrm{PSH}(X)$.*

Proof. Let $x \in X$, $y = f(x)$. We need to verify Condition (2). The problem is local, so we may assume that there is a closed immersion $X \hookrightarrow \Sigma$ and that there is an open neighbourhood $V \subseteq Y$ of y, a closed immersion $V \hookrightarrow \Omega$, an open set $\tilde{V} \subseteq \Omega$ containing y, a psh function $\tilde{\varphi}$ on $\tilde{V}$, such that $\tilde{\varphi}|_{\tilde{V}\cap V} = \varphi|_{\tilde{V}\cap V}$, where both Σ and Ω are bounded pseudoconvex domains in $\mathbb{C}^N$ and $\mathbb{C}^M$ respectively. Shrinking X, we may assume that $f(X) \subseteq \tilde{V} \cap V$. We the get a closed immersion:

$$X \hookrightarrow X \times V \hookrightarrow \Sigma \times \Omega\,,$$

where the first morphism is the base change of $\Delta_V : V \to V \times V$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X \times V \arrow[r, hook] \arrow[d] & \Sigma \times \Omega \\ V \arrow[r, hook, "\Delta_V"] & V \times V & \end{tikzcd}\,.$$

Define $\tilde{X} := \Sigma \times \tilde{V}$. Define a psh function ψ on $\tilde{X}$ as the pull-back of $\tilde{\varphi}$ from the projection onto the second variable. Then $f^*\varphi|_{\tilde{X}\cap X} = \psi|_{\tilde{X}\cap X}$. □

Proposition 3.3. *There is a canonical bijection*

$$\mathrm{PSH}(X) \xrightarrow{\sim} \mathrm{PSH}(X^{\mathrm{red}})\,. \tag{3.1}$$

Proof. Let $\varphi \in \mathrm{PSH}(X)$. We claim that $\varphi \in \mathrm{PSH}(X^{\mathrm{red}})$ as well. Condition (1) is trivially satisfied. Let us prove Condition (2). The problem is local. Fix $x \in X$. We may assume that there is a closed immersion $X \hookrightarrow \Omega$, where Ω is a bounded pseudo-convex domain in $\mathbb{C}^N$, an open set $\tilde{X} \subseteq \Omega$ with $x \in \tilde{X}$, a psh function $\tilde{\varphi}$ on $\tilde{X}$, such that $\tilde{\varphi}|_{\tilde{X}\cap X} = \varphi|_{\tilde{X}\cap X}$. Note that $X^{\mathrm{red}} \hookrightarrow X$ induces a closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$. Now we can take the same $\tilde{\varphi}$ to conclude.

Now let $\varphi \in \mathrm{PSH}(X^{\mathrm{red}})$. We want to show $\varphi \in \mathrm{PSH}(X)$. Again, it suffices to prove Condition (2). The problem is local. We may assume that X is Stein. Take $x \in X$. We may assume that there is a closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$, where Ω is a bounded pseudo-convex domain in $\mathbb{C}^N$, an open set $\tilde{X} \subseteq \Omega$ with $x \in \tilde{X}$, a psh function $\tilde{\varphi}$ on $\tilde{X}$, such that $\tilde{\varphi}|_{\tilde{X}\cap X} = \varphi|_{\tilde{X}\cap X}$.

By Theorem 2.2, after possibly shrinking X, we can lift the closed immersion $X^{\mathrm{red}} \hookrightarrow \Omega$ to a morphism $j : X \to \Omega$:

$$\begin{tikzcd} X^{\mathrm{red}} \arrow[r, hook] \arrow[d, hook] & \Omega \\ X \arrow[ur, "j"] & \end{tikzcd} .$$

By Proposition 3.2, $j^*\varphi$ is psh. Now φ is the image of $j^*\varphi$ under (3.1). □

By this proposition, we could always restrict our attention to reduced analytic spaces.

Theorem 3.4 (Fornaess–Narasimhan)**.** *Let $\varphi : X \to [-\infty, \infty)$ be a function. Assume that φ is not identically $-\infty$ on any irreducible component of X, then the following are equivalent:*

(1) φ is psh.

(2) φ is usc and for any morphism $f : \Delta \to X$ from the open unit disk Δ in $\mathbb{C}$ to X such that $f^*\varphi$ is not identically $-\infty$, then $f^*\varphi$ is psh.

Moreover, assume that X is unibranch, then the conditions are equivalent to

(3) $\varphi|_{X\setminus \mathrm{Sing}\, X^{\mathrm{red}}}$ is psh, φ is locally bounded from above near $\mathrm{Sing}\, X^{\mathrm{red}}$ and φ is strongly usc in the following sense:

$$\varphi(x) = \varlimsup_{y\to x, y\in X\setminus \mathrm{Sing}\, X^{\mathrm{red}}} \varphi(y)\,, \quad x \in X\,. \tag{3.2}$$

(4) φ is locally integrable, locally bounded from above, strongly usc and $\mathrm{dd^c}\varphi \geq 0$.

The proof is taken from [FN80].

Proof. (1) implies (2): This follows from Proposition 3.2.

(2) implies (1): Note that functions satisfying either conditions are invariant under the change X to X^{red}: For the former, this is Proposition 3.3, for the later, one simply applies the universal property of $\bullet^{\mathrm{red}}$.

The problem is local, so we may assume that X is a closed analytic subspace of a bounded pseudo-convex domain Ω in $\mathbb{C}^N$. Let φ be a function satisfying Condition (2). Let

$$\tilde{\Omega} := \{\, (x,z) \in X \times \mathbb{C} : \varphi(x) + \log|z| < 0 \,\}\,.$$

We claim that $\tilde{\Omega}$ is Stein.

Assume this claim for now. By [Siu76], there is a Stein open set $\Sigma \subseteq \Omega \times \mathbb{C}$, such that $\Sigma \cap (X \times \mathbb{C}) = \tilde{\Omega}$.

Let

$$\Sigma' := \mathrm{Int} \bigcap_{\tau\in\mathbb{C}, |\tau|\geq 1} \{\, (x,z) \in \Omega \times \mathbb{C} : (x, \tau^{-1}z) \in \Sigma \,\}\,.$$

Then Σ' is also Stein, $\Sigma' \cap (X \times \mathbb{C}) = \tilde{\Omega}$.

Let

$$\Sigma'' := \{\, (x,z) \in \Omega \times \mathbb{C} : (x,0) \in \Sigma' \,\}\,.$$

Observe that Σ'' is a Hartogs domain and can be written as

$$\Sigma'' = \{\, (x,z) \in D \times \mathbb{C} : |z| < \exp(-\tilde{\varphi}(x)) \,\}\,,$$

where $D \subseteq \Omega$ is an open subset containing X, $\tilde{\varphi} : D \to [-\infty, \infty)$ is a function. As Σ'' is Stein, D is Stein and $\tilde{\varphi}$ is psh. Note that $\tilde{\varphi}|_X = \varphi$. We conclude.

Now it remains to prove the claim. We omit the details and refer to [FN80, Page 64].

Now assume that X is unibranch. Without loss of generality, we may assume X is reduced.

(1) implies (3): That $\varphi|_{U\setminus \operatorname{Sing} X}$ is psh follows from Proposition 3.1 and Proposition 3.2.

Now we prove that φ is strongly usc. Let $\pi : Y \to X$ be a resolution of singularity that is an isomorphism over $X \setminus \operatorname{Sing} X$. Note that resolution of singularity exists by Theorem 2.8. Then $\pi^*\varphi \in \operatorname{PSH}(Y)$ by Proposition 3.2. It is a classical fact that

$$\pi^*\varphi(x) = \varlimsup_{y\to x, y\in Y\setminus \pi^{-1}\operatorname{Sing} X} \pi^*\varphi(y)\,.$$

Hence (3.2) follows.

(3) implies (2): Assume that φ satisfies Condition (3). Now we show that φ is psh. We take a resolution of singularity $\pi : Y \to X$ that is an isomorphism over $X \setminus \operatorname{Sing} X$. By Theorem 2.4, for each $x \in X$, $\pi^{-1}(x)$ is connected as X is unibranch. Note that φ is usc by Condition (3), so φ is locally bounded from above, hence so is $\pi^*\varphi|_{\pi^{-1}(X\setminus \operatorname{Sing} X)}$. So $\pi^*\varphi|_{\pi^{-1}(X\setminus \operatorname{Sing} X)}$ extends uniquely to a psh function ψ on Y. Note that ψ is constant along $\pi^*(x)$ for any $x \in X$, so we may descend ψ to a function on X, which clearly equals φ. Let $f : (\Delta, 0) \to (X, x)$ be a holomorphic morphism of germs, where $x \in X$. We may assume that f is non-constant. Then there is a germ of curve (Γ, y) lying above $(f(\Delta), x)$. So there is a morphism of germs $g : (\Delta, 0) \to (Y, y)$, such that

$$f(t^k) = \pi(f(t))\,, \quad t \in \Delta\,,$$

where k is the ramification index. Observe that

$$f^*\varphi(t^k) = g^*\psi(t)\,, \quad t \in \Delta\,,$$

we conclude that $f^*\varphi$ is psh.

(4) implies (3): Obvious.

(1) implies (4): The only non-trivial point is to show that φ is locally integrable. See [Dem85, Section 1.8]. □

Corollary 3.5. *Assume that X is unibranch. Let $\varphi \in \operatorname{PSH}(X \setminus \operatorname{Sing} X^{\mathrm{red}})$. Assume that φ is locally bounded from above near $x \in \operatorname{Sing} X^{\mathrm{red}}$, then there is a unique extension $\varphi \in \operatorname{PSH}(X)$.*

Proposition 3.6.

(1) Assume that X is unibranch. Let φ_θ be a family in $\operatorname{PSH}(X)$, locally uniformly bounded from above. Then $\sup^*_\theta \varphi_\theta$ is also psh.

(2) Let φ_θ be a decreasing net in $\operatorname{PSH}(X)$, such that $\inf_\theta \varphi_\theta$ is not identically $-\infty$ on each irreducible component of X, then $\inf_\theta \varphi_\theta$ is psh.

Here

$$\sup{}^* f_\theta := (\sup_\theta f_\theta)^*$$

$$f^*(x) := \varlimsup_{y\to x, y\in X\setminus \operatorname{Sing} X^{\mathrm{red}}} f(y)\,.$$

Proof. (1) Observe that

$$(\sup_\theta{}^* \varphi_\theta)|_{X\setminus \operatorname{Sing} X^{\mathrm{red}}} = \sup_\theta{}^* \varphi_\theta|_{X\setminus \operatorname{Sing} X^{\mathrm{red}}}\,,$$

where the right-hand side is psh by the classical theory. Now $\sup^*_\theta \varphi_\theta$ is clearly locally bounded from above, (3.2) also clearly holds.

(2) Note that $\inf_\theta \varphi_\theta$ is usc. Condition (2) of Theorem 3.4 clearly holds. □

3.2. Extension theorem.

Theorem 3.7 (Extension theorem)**.** *Let M be a Stein manifold. Let $N \subseteq M$ be a closed reduced complex analytic subvariety. Let φ be a psh function on N. Assume that ψ is a continuous psh exhaustion function on M, such that $\varphi \le \psi|_N$. Let $c > 1$. Then there is a psh extension of φ to M, such that $\varphi \le c\max\{\psi, 0\}$.*

For the proof, we refer to [CGZ13].

3.3. **Local regularization.** For each $N \geq 0$, we fix a Friedrichs kernel $\rho = \rho_N : [0,\infty) \to [0,\infty)$, such that ρ is smooth, $\rho(r) = 0$ for $r \geq 1$ and

$$\int_{\mathbb{C}^N} \rho(|z|)\,\mathrm{d}\lambda(z) = 1\,.$$

Let $U \subseteq \mathbb{C}^N$ be an open subset. For any locally integrable function $u : U \to [-\infty,\infty)$ and any $\delta > 0$, define

$$\varphi_\delta(x) := \int_{\mathbb{C}^N} u(x-\delta y)\rho(|y|)\,\mathrm{d}\lambda(y)\,, \quad x \in U_\delta\,,$$

where

$$U_\delta := \{\, x \in U : B(x,\delta) \subseteq U \,\}\,.$$

Lemma 3.8. *Let $\Omega \subseteq \mathbb{C}^N$ be a bounded pseudo-convex domain. Let $V \hookrightarrow \Omega$ be a closed analytic subspace. Let $W \Subset V$ be an open subset. Let $\varphi \in \mathrm{PSH}(V) \cap L^\infty(V)$. There there exists a decreasing sequence φ_i of smooth psh functions on W, converging pointwisely to φ on W.*

Proof. By Theorem 3.7, φ can be extended to a psh function on Ω. Define $\varphi_i = \varphi_{1/i}|_W$. □

3.4. **Bedford–Taylor product.** Proofs in this section are mostly taken from [Dem85] and the book [GZ17].

Fix a complex analytic space X.

Definition 3.2 (Bedford–Taylor)**.** Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1,\dots,k)$. Let T be a closed positive current of bidimension (m,m) on X. We define

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T := \mathrm{dd^c}\left(\varphi_1 \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T\right)\,. \tag{3.3}$$

Remark 3.1. Unless X is equi-dimensional, the bidegree of a current is not well-defined. So we only talk about bidimensions.

Proposition 3.9. *Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1,\dots,k)$. Let T be a closed positive current of bidimension (m,m) on X. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T$ is a closed positive current of bidimension $(m-k,m-k)$.*

Proof. We prove by induction. When $k = 0$, there is nothing to prove. Assume that the result is known for $k-1$, namely, assume that $\mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T$ is closed positive. Then for any smooth psh function φ,

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \mathrm{dd^c}\varphi_k \wedge T$$

is clearly closed positive. As our problem is local, we may assume that there is a decreasing sequence of smooth psh functions φ^i converging pointwisely to φ_1, then

$$\varphi^i \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \rightharpoonup \varphi_1 \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T\,,$$

so

$$\mathrm{dd^c}\varphi^i \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T\,,$$

this proves our result. □

Now we prove the functoriality of this product.

Proposition 3.10 (Projection formula)**.** *Let $\pi : Y \to X$ be a proper morphism of complex analytic spaces. Let $\varphi_0,\dots,\varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Let T be a closed positive current of bidimension (m,m) on Y. Then*

$$\pi_*(\pi^*\varphi_0 \mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k \wedge T) = \varphi_0 \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge \pi_*T\,.$$

Proof. By induction on k, we may assume that $k = 1$. The result simply follows from the fact that d and $\mathrm{d^c}$ are functorial and the classical projection formula. □

In particular, let Y be an irreducible component of X of dimension n, let $i : Y \to X$ be the inclusion, $T = [Y]$ is a closed positive current of bidimension (n,n) (see [GH78, Section 0.2] for example). Then we conclude

Corollary 3.11. *Let Y be an irreducible component of X of dimension n, let $i : Y \to X$ be the inclusion. Let $\varphi_1,\dots,\varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then*

$$i_*(\mathrm{dd^c}\varphi_1|_Y \wedge \cdots \wedge \mathrm{dd^c}\varphi_k|_Y) = \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge [Y]\,.$$

Corollary 3.12. *Let $\pi : Y \to X$ be a proper bimeromorphic morphism between complex analytic spaces. Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then*

$$\pi_*(\pi^*\varphi_0 \mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k) = \varphi_0 \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k .$$

Corollary 3.13. *Let $\varphi_1, \dots, \varphi_k \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Let $i : X^{\mathrm{red}} \to X$ be the canonical inclusion. Then*

$$i_*(\mathrm{dd^c}\varphi_1|_{X^{\mathrm{red}}} \wedge \cdots \wedge \mathrm{dd^c}\varphi_k|_{X^{\mathrm{red}}}) = \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k .$$

Definition 3.3. Let $\varphi \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$, let T be a closed positive current of bidimension (p,p) on X, then on each open set $U \Subset X$,

$$\mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge T := \frac{1}{2}\mathrm{dd^c}(\varphi - \inf_U \varphi)^2 \wedge T - (\varphi - \inf_U \varphi)\mathrm{dd^c}\varphi \wedge T .$$

This gives a well-defined $(k+1, k+1)$-current on X.

Theorem 3.14 (Chern–Levine–Nirenberg)**.** *Let $U \subseteq X$ be an open subset. Let $\varphi_i \in \mathrm{PSH}(U) \cap L^\infty_{\mathrm{loc}}(U)$ $(i = 1, \dots, k)$. Let T be a closed positive current of bidimension (k,k) on U. Let $W \Subset V \Subset U$ be two open sets, then there is a constant $C = C(W, V) > 0$, such that for any compact set $K \subset W$,*

$$\int_K \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \leq C \|\varphi_1\|_{L^\infty(E)} \cdots \|\varphi_k\|_{L^\infty(E)} \|T\|_E \tag{3.4}$$

and

$$\int_K \mathrm{d}\varphi_1 \wedge \mathrm{d^c}\varphi_1 \wedge \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \wedge T \leq C \|\varphi_1\|^2_{L^\infty(E)} \|\varphi_2\|_{L^\infty(E)} \cdots \|\varphi_k\|_{L^\infty(E)} \|T\|_E ,$$

where $E = \mathrm{Supp}\, T \cap (V \setminus W)$.

Here $\|T\|_E$ is a semi-norm, defined as follows: take finitely many open sets $U_i \subseteq U$, open subsets $V_i \Subset U_i$, such that $\{V_i\}$ covers U and such that there are Kähler forms ω_i on each U_i. Then

$$\|T\|_E := \sum_i \int_{V_i} \omega_i^k \wedge T .$$

Proof. We first prove (3.4). By induction, it suffices to treat the case where $k = 1$. We omit the index and write $\varphi = \varphi_1$. We may assume that $\varphi|_V \leq 0$. Let χ be a compactly supported smooth function on V, equal to 1 on W. Then

$$\int_W \mathrm{dd^c}\varphi_1 \wedge T = \int_V \chi \mathrm{dd^c}\varphi_1 \wedge T = \int_V \varphi_1 \mathrm{dd^c}\chi \wedge T = \int_{V \setminus W} \varphi_1 \mathrm{dd^c}\chi \wedge T ,$$

where we have applied Lemma 3.15. Now (3.4) is obvious.

The other part is similar. □

Lemma 3.15. *Let $U \subseteq X$ be an open subset. Let T be a closed positive current of bidimension $(1,1)$ on U. Let $\varphi, \psi \in \mathrm{PSH}(U) \cap L^\infty_{\mathrm{loc}}(U)$, $\varphi, \psi \leq 0$. Assume $\lim_{x \to \partial U} \varphi(x) = 0$ and $\int_U \mathrm{dd^c}\psi \wedge T < \infty$, then*

$$\int_U \psi\, \mathrm{dd^c}\varphi \wedge T \leq \int_U \varphi\, \mathrm{dd^c}\psi \wedge T .$$

Proof. For $\epsilon > 0$, let $\varphi_\epsilon := \varphi \vee (-\epsilon)$. By monotone convergence theorem,

$$\int_U \varphi\, \mathrm{dd^c}\psi \wedge T = \lim_{\epsilon \to 0+} \int_U (\varphi - \varphi_\epsilon)\, \mathrm{dd^c}\psi \wedge T .$$

Let K_ϵ be the integral closure of $\{\varphi < -\epsilon\}$. Fix $\epsilon > 0$. Let $D_1 \Subset U$ be a domain close to U and such that $K \subseteq D_1$. Consider a standard mollifier ρ_η. Then

$$\int_{D_1} (\varphi - \varphi_\epsilon)\, \mathrm{dd^c}\psi \wedge T = \lim_{\eta \to 0+} \int_{D_1} (\varphi - \varphi_\epsilon) * \rho_\eta\, \mathrm{dd^c}\psi \wedge T$$

$$= \lim_{\eta \to 0+} \int_{D_1} \psi\, \mathrm{dd^c}((\varphi - \varphi_\epsilon) * \rho_\eta) \wedge T \geq \lim_{\eta \to 0+} \int_{D_1} \psi\, \mathrm{dd^c}(\varphi * \rho_\eta) \wedge T .$$

When ψ is continuous in a neighbourhood of $\bar{D}_1$. We could assume that ∂D_1 is null with respect to $-\psi \mathrm{dd^c}\varphi \wedge T$. Thus the right-hand side is equal to $\int_{D_1} \psi \mathrm{dd^c}\varphi \wedge T$. Hence

$$\int_{D_1} \varphi \, \mathrm{dd^c}\psi \wedge T \geq \int_{D_1} \psi \, \mathrm{dd^c}\varphi \wedge T - \epsilon \int_{D_1} \mathrm{dd^c}\psi \wedge T \,.$$

In general, take a continuous non-positive psh functions ψ_j on Ω converging to ψ on a neighbourhood of $\bar{D}_1$. Choose a domain $D_2 \Subset D_1$, let L be the closure of D_2. We get

$$\begin{aligned}\int_L \varphi \, \mathrm{dd^c}\psi \wedge T \geq \varlimsup_{j\to\infty} \int_L \varphi \, \mathrm{dd^c}\psi_j \wedge T \geq \lim_{j\to\infty} \int_{D_1} \psi_j \, \mathrm{dd^c}\varphi \wedge T - \epsilon \varliminf_{j\to\infty} \int_{D_1} \mathrm{dd^c}\psi_j \wedge T \\ \geq \int_{D_1} \psi \, \mathrm{dd^c}\varphi \wedge T - \epsilon \int_{D_1} \mathrm{dd^c}\psi \wedge T \,.\end{aligned}$$

Take limit in D_1, D_2 and ϵ, we conclude. $\square$

Definition 3.4. A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighbourhood $V \subseteq X$ of x, a psh function φ on V, such that $E \cap V \subseteq \{x \in V : \varphi(x) = -\infty\}$.

Definition 3.5. The σ-algebra of *quasi-Borel sets* is the σ-algebra generated by all Borel sets and all pluripolar set. A set in this σ-algebra is called a *quasi-Borel set*.

Definition 3.6. Assume that X is unibranch. A set $E \subseteq X$ is *negligible* if for any $x \in X$, there is an open neighbourhood $V \subseteq X$ of x, a bounded from above family $\{\psi_\theta\}$ of psh functions on V, such that

$$E \cap V \subseteq \{\, x \in V : \sup{}^*\{\psi_\theta\}(x) > \sup\{\psi_\theta\}(x) \,\} \,.$$

Remark 3.2. There are different definitions in the literature about both pluripolar sets and negligible sets. In some literature, our notion of pluri-polar sets is called locally pluripolar. However, we want to emphasis that what is actually proved in [BT82] is that pluri-polarity is equivalent to negligiblility *in our sense*.

Now we study the pluri-fine topology on X.

Classical analogues of the remaining results in this section are proved in [BT82] and [BT87].

Assume that X has pure dimension n. Let $\varphi_1, \dots, \varphi_n \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_n$ is a Borel measure. We extend this measure by taking its completion. Then all quasi-Borel sets are measurable.

Theorem 3.16. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $E \subseteq X$ be a subset. Then E is negligible iff E is pluripolar.*

Proof. We may assume that X is reduced.

Assume that E is pluripolar. We show that E is negligible. The problem is local, so we may assume that there is a psh function $\varphi \leq 0$ on X, such that $E \subseteq \{\varphi = -\infty\}$. We may assume that equality actually holds. For each $i \geq 1$, let $\varphi_i := i^{-1}\varphi$. Observe that E has empty interior by the classical theory $E \cap (X \setminus \operatorname{Sing} X)$ has empty interior. Now $\sup_i \varphi_i = 0$ on $X \setminus E$, while $\sup_i \varphi_i|_E = -\infty$, so $\sup^* \varphi_i = 0$. Thus

$$E = \left\{ \sup_i \varphi_i < \sup_i{}^* \varphi_i \right\} .$$

Now assume that E is negligible. We show that E is pluripolar. We may assume that X is a reduced, unibranch Kähler space with a Kähler form ω.

Let $\pi : Y \to X$ be a resolution of singularity. Observe that $\pi^{-1}(E)$ is negligible, as the union of two negligible sets. Let

$$u^*_{E,\omega} := \sup{}^* \{\, \varphi \in \mathrm{PSH}(X, \omega) : \varphi \leq 0, \varphi|_E \leq -1 \,\} \in \mathrm{PSH}(X, \omega) \,.$$

Observe that

$$\pi^* u^*_{E,\omega} = u^*_{\pi^{-1}E, \pi^*\omega} = 0 \,,$$

where the second equality follows from the classical theory (see [Lu20, Lemma 2.3] for example). In particular, $u^*_{E,\omega} = 0$. So by Choquet's lemma, we may take an increasing sequence of ω-psh functions ψ_i, $\psi_i \leq 0$, $\psi_i|_E \leq -1$ such that the L^1-norm of ψ_i is bounded from above by 2^{-i}. Then take $\psi = \sum_i \psi_i$. We find $E \subseteq \{\psi = -\infty\}$. $\square$

Theorem 3.17. *Assume that X is an open subspace of a compact unibranch Kähler space. An arbitrary union of pluri-fine open subsets differs from a countable subunion by at most a pluripolar set.*

Proof. Let $U_\theta \subseteq X$ ($\theta \in I$) be a family of pluri-fine open subsets in X. Let $U = \cup_\theta U_\theta$. Take a countable basis B_i of the topology of X. We may assume that each U_θ is of the form $B_i \cap \{\varphi_\theta > -1\}$ for some B_i, where $\varphi_\theta \leq 0$ is a bounded psh function on B_i. It suffices to prove that for each fixed i,

$$\bigcup_\theta B_i \cap \{\varphi_\theta > -1\}$$

with θ running through the subset set J_i of I consisting of all θ such that U_θ is of the form $B_i \cap \{\varphi_\theta > -1\}$. So we may assume that there are bounded psh functions φ_θ defined on X, such that $-1 \leq \varphi_\theta \leq -1$, $U_\theta = \{\varphi_\theta > 0\}$. Now by Choquet's lemma, there is a countable subset $J \subseteq I$, such that

$$\sup_{\theta\in I}{}^* \varphi_\theta = \sup_{\theta\in J}{}^* \varphi_\theta \,.$$

So by Theorem 3.17,

$$\sup_{\theta\in I} \{\varphi_\theta > 0\} = \bigcup_{\theta\in I} U_\theta$$

differs from

$$\sup_{\theta\in J} \{\varphi_\theta > 0\} = \bigcup_{\theta\in J} U_\theta$$

by at most a pluripolar set. This proves our theorem. □

This is a corollary of the previous theorem.

Corollary 3.18. *Assume that X is an open subspace of a compact unibranch Kähler space, then all pluri-fine Borel sets are quasi-Borel.*

Theorem 3.19. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_0^j, \ldots, \varphi_k^j$ be $(n+1)$-sequences of psh functions on X. Assume that the sequences are all uniformly bounded and converging a.e. monotonely (either increasing or decreasing) to psh functions $\varphi_0, \ldots, \varphi_k$ on X. Then*

(1)
$$\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \,.$$

(2)
$$\varphi_0^j\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \varphi_0\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \,.$$

(3)
$$\mathrm{d}\varphi_0^j \wedge \mathrm{d^c}\varphi_0^j \wedge \mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_k^j \xrightarrow{\text{p.f.}} \mathrm{d}\varphi_0 \wedge \mathrm{d^c}\varphi_0 \wedge \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k \,.$$

Here p.f. *means that the weak convergence is with respect to the pluri-fine topology.*

Proof. It suffices to prove (2).

We may assume that X is reduced. Let $\pi : Y \to X$ be a resolution of singularity. From the classical theory, we know that

$$\pi^*\varphi_1^j\mathrm{dd^c}\pi^*\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k^j \xrightarrow{\text{p.f.}} \pi^*\varphi_1\mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_k \,.$$

We conclude by Proposition 3.10. □

Corollary 3.20. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_i \in \mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ $(i = 1, \ldots, k)$. Then for any $\sigma \in \mathcal{S}_k$,*

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_k = \mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(k)} \,.$$

This is a standard consequence of Theorem 3.19, we omit the details.

In particular, we may use the following notation for any of these products:

$$\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j \,.$$

Lemma 3.21. *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi_1,\dots,\varphi_k \in \mathrm{PSH}(X)$. Let E be a pluripolar set. Then $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k$ does not charge E in the following sense: For each irreducible component Y of X, $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k\wedge[Y]$ does not charge $E\cap Y$.*

Proof. We may assume that X is irreducible of dimension n. The problem is local, by Theorem 3.16, we may assume that there is a bounded from above sequence $\psi_j \in \mathrm{PSH}(X)$, such that $E\subseteq N$, where $N := \{\sup_j \psi_j < \sup^*{}_j \psi_j\}$. We want to show that N is null with respect to $\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k$. By taking a cut-off, we may assume that ψ_j are uniformly bounded. Take a compact set $K\subseteq X$, a smooth positive $(n-k,n-k)$-form θ on X. Then it suffices to show that

$$\int_{K\cap N} \mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k\wedge\theta = 0\,.$$

Pick a smooth function $\chi : X\to[0,1]$ with compact support, such that χ is equal to 1 near K. Then by Theorem 3.19,

$$\begin{aligned}\int_X \chi\sup_j\psi_j\,\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k\wedge\theta &= \lim_{j\to\infty}\int_X \chi\psi_j\,\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k\wedge\theta\\ &= \int_X \chi\sup_j{}^*\psi_j\,\mathrm{dd^c}\varphi_1\wedge\cdots\wedge\mathrm{dd^c}\varphi_k\wedge\theta\,.\end{aligned}$$

We conclude. □

Lemma 3.22. *Assume that X is an open subspace of a compact unibranch Kähler space. Assume that X is of pure dimension n. Let $\varphi,\psi\in\mathrm{PSH}(X)\cap L^\infty_{\mathrm{loc}}(X)$. Let $O := \{\varphi>\psi\}$. Then*

$$(\mathrm{dd^c}(\varphi\vee\psi))^n|_O = (\mathrm{dd^c}\varphi)^n|_O\,. \tag{3.5}$$

Proof. The problem is local. We may assume that X is a closed analytic subspace of a bounded pseudo-convex domain Ω in some $\mathbb{C}^N$. By Lemma 3.8, we may assume that there is a decreasing sequence of smooth psh functions φ_k on X converging pointwisely to φ. It is obvious that (3.5) holds for φ_k in place of φ. Hence we conclude by Lemma 3.23. □

Lemma 3.23. *Let $\Omega\subseteq\mathbb{C}^N$ be a bounded pseudo-convex domain. Let X be a closed subspace of Ω. Assume that X can be realized as an open subspace of a compact unibranch Kähler space. Assume that X is of pure dimension n. Let $O\subseteq X$ be a pluri-fine open subset. Let φ be a bounded psh function on X. Let φ_k be a decreasing sequence of bounded psh functions on X converging pointwisely to φ. Let $\psi\in\mathrm{PSH}(X)\cap L^\infty(X)$. Assume that for any k,*

$$(\mathrm{dd^c}\varphi_k)^n|_O = (\mathrm{dd^c}\psi)^n|_O\,,$$

then

$$(\mathrm{dd^c}\varphi)^n|_O = (\mathrm{dd^c}\psi)^n|_O\,,$$

Proof. By Theorem 3.17, we may assume that $O = B\cap\{\eta>0\}$, where $B\cap X$ is an open ball in $\mathbb{C}^n$ and η is a bounded psh function on B. Replace η with a non-negative pluri-fine continuous function with compact support in X such that $O=\{\eta>0\}$. Then by Theorem 3.19, for any continuous function f on X,

$$\int_X f\eta\,(\mathrm{dd^c}\psi)^n = \lim_{k\to\infty}\int_X f\eta\,(\mathrm{dd^c}\varphi_k)^n = \int_X f\eta\,(\mathrm{dd^c}\varphi)^n\,.$$

We conclude. □

Theorem 3.24 (Plurilocality)**.** *Assume that X is an open subspace of a compact unibranch Kähler space. Let $\varphi^i,\psi^i\in\mathrm{PSH}(X)\cap L^\infty_{\mathrm{loc}}(X)$ $(i=1,\dots,n)$. Let $W\subseteq X$ be an open set with respect to the plurifine topology. Assume that $\varphi^i|_W=\psi^i|_W$ for all i, then*

$$\mathrm{dd^c}\varphi^1\wedge\cdots\wedge\mathrm{dd^c}\varphi^n|_W = \mathrm{dd^c}\psi^1\wedge\cdots\wedge\mathrm{dd^c}\psi^n|_W\,.$$

Proof. By polarization, we may assume that all φ^i are equal and all ψ^i are equal. We omit the superindex. Then we want to show

$$(\mathrm{dd^c}\varphi)^n = (\mathrm{dd^c}\psi)^n\,.$$

Note that $\varphi = \varphi\vee(\psi-\epsilon)$ for any $\epsilon>0$. It suffices to apply Lemma 3.22. □

Finally, we introduce the notion of quasi-psh functions.

Definition 3.7. Let θ be a smooth strongly closed real $(1,1)$-form on X (in the sense that locally it is the pull-back of a smooth closed real $(1,1)$-form $\tilde{\theta}$ on pseudo-convex domains). Then the set $\mathrm{PSH}(X,\theta)$ consists of all functions $\varphi : X \to [-\infty,\infty)$, such that on each open set $V \subseteq X$, embedded as a closed analytic subspace in a pseudo-convex domain $\Omega \subseteq \mathbb{C}^N$, such that for any smooth function a on U with $\mathrm{dd^c} a = \theta$, $a|_V + \varphi$ is psh.

3.5. **Non-pluripolar product.** Assume that X is an open subset of a compact unibranch Kähler space of pure dimension n.

Definition 3.8. Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Let

$$O_k = \bigcap_{j=1}^{p} \{\varphi_j > -k\} \,.$$

We say that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is *well-defined* if for each open subset $U \subseteq X$ such that there is a Kähler form ω on U, each compact subset $K \subseteq U$, we have

$$\sup_{k\geq 0} \int_{K\cap O_k} \left(\bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\} \right)\Bigg|_U \wedge \omega^{n-p} < \infty. \tag{3.6}$$

In this case, we define $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ by

$$\mathbb{1}_{O_k} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\}. \tag{3.7}$$

on $\bigcup_{k\geq 0} O_k$ and make a zero-extension to X.

Remark 3.3. The condition (3.6) is clearly independent of the choice of U and ω.

Proposition 3.25. *Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Let $\sigma \in \mathcal{S}_p$. By definition, $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined iff $\mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(p)}$ is. Moreover, in this case,*

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathrm{dd^c}\varphi_{\sigma(1)} \wedge \cdots \wedge \mathrm{dd^c}\varphi_{\sigma(p)} \,.$$

In particular, we may use the following notation for any of these products:

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \,.$$

Proof. This is a direct consequence of Corollary 3.20. ☐

Lemma 3.26. *Let $\varphi_1,\dots,\varphi_p \in \mathrm{PSH}(X)$. Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined. Let $E_k \subseteq O_k$ ($k \geq 0$) be Borel sets such that $X - \cup_k E_k$ is pluripolar. Let Ω be a $(n-p, n-p)$-form with measurable coefficients. Assume that the following conditions are satisfied:*

1. *$\mathrm{Supp}\,\Omega$ is compact.*
2. *For each open subset $U \subseteq X$, each Kähler form ω on U, there is a constant $C > 0$ such that*
$$-C\omega^{n-p} \leq \Omega \leq C\omega^{n-p}$$
holds on $\mathrm{Supp}\,\Omega \cap U$.

Then

$$\lim_{k\to\infty} \int_X \mathbb{1}_{E_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\} \wedge \Omega = \int_X \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \wedge \Omega.$$

In particular,

$$\mathbb{1}_{E_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k\} \rightharpoonup \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j, \quad k \to \infty$$

as currents and the convergence is strong on each compact subset of X.

Proof. Since the problem is local, we may assume that $\operatorname{Supp}\Omega \subseteq U$, where $U \subseteq X$ is an open subset with a Kähler form ω. Take $C > 0$ so that

$$-C\omega^{n-p} \leq \Omega \leq C\omega^{n-p}.$$

Then observe that

$$\begin{aligned} 0 \leq \int_X \mathbb{1}_{O_k} \bigwedge_{j=1}^p \mathrm{dd^c} \max\{\varphi_j, -k\} \wedge \Omega - \int_X \mathbb{1}_{E_k} \bigwedge_{j=1}^p \mathrm{dd^c} \max\{\varphi_j, -k\} \wedge \Omega \\ \leq \int_{\operatorname{Supp}\Omega} (1 - \mathbb{1}_{E_k}) \bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j \wedge \Omega. \end{aligned}$$

The right-hand side tends to 0 by dominated convergence theorem. So it suffices to prove the theorem for $E_k = O_k$. In this case, the theorem again follows from dominated convergence theorem. □

Proposition 3.27. *Let $\varphi_1, \dots, \varphi_m$ be psh functions on X. Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity. Then $\varphi_1 \wedge \cdots \wedge \varphi_m$ is well-defined iff $\pi^*\varphi_1 \wedge \cdots \wedge \pi^*\varphi_m$ is. In this case,*

$$\pi_* \left(\mathrm{dd^c}\pi^*\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\pi^*\varphi_m\right) = \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_m\,.$$

Proof. This follows directly from Proposition 3.10. □

This proposition allows us to generalize directly known facts about non-pluripolar products in the smooth setting to the current setting.

Proposition 3.28. *Let $\varphi_1, \dots, \varphi_p \in \mathrm{PSH}(X)$.*

(1) The product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is local in pluri-fine topology. In the following sense: let $O \subseteq X$ be a pluri-fine open subset, let $v_1, \dots, v_p \in \mathrm{PSH}(X)$, assume that

$$\varphi_j|_O = v_j|_O, \quad j = 1, \dots, p\,.$$

Assume that

$$\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j, \quad \bigwedge_{j=1}^p \mathrm{dd^c}v_j$$

are both well-defined, then

$$\left.\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j\right|_O = \left.\bigwedge_{j=1}^p \mathrm{dd^c}v_j\right|_O\,. \tag{3.8}$$

If O is open in the usual topology, then the product

$$\bigwedge_{j=1}^p \mathrm{dd^c}(\varphi_j|_O)$$

on O is well-defined and

$$\left.\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j\right|_O = \bigwedge_{j=1}^p \mathrm{dd^c}(\varphi_j|_O)\,. \tag{3.9}$$

Let $\mathcal{U}$ be an open covering of X. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined iff each of the following product is well-defined

$$\bigwedge_{j=1}^p \mathrm{dd^c}(\varphi_j|_U)\,, \quad U \in \mathcal{U}\,.$$

(2) The current $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ and the fact that it is well-defined depend only on the currents $\mathrm{dd^c}\varphi_j$, not on specific φ_j.

(3) When $\varphi_1, \dots, \varphi_p \in L^\infty_{\mathrm{loc}}(X)$, $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined and is equal to the Bedford–Taylor product.

(4) Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ puts not mass on pluripolar sets.

(5) *Assume that* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is well-defined, then*

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j$$

is a closed positive current of bidegree (p,p) *on* X.

(6) *The product is multi-linear: let* $v_1 \in \mathrm{PSH}(X)$, *then*

$$\mathrm{dd^c}(\varphi_1 + v_1) \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j = \mathrm{dd^c}\varphi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j + \mathrm{dd^c}v_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j \tag{3.10}$$

in the sense that the left-hand side is well-defined iff both terms on the right-hand side are well-defined, and the equality holds in that case.

Proof. The only non-trivial part is (2).

(2) By (1), we may assume that there is a Kähler form ω on X. Let w_j $(j = 1, \dots, p)$ be pluri-harmonic functions on X. Assume that $\langle \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p \rangle$ is well-defined. We want to prove that $\langle \mathrm{dd^c}(w_1 + \varphi_1) \wedge \cdots \wedge \mathrm{dd^c}(w_p + \varphi_p) \rangle$ is also well-defined and

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j = \bigwedge_{j=1}^{p} \mathrm{dd^c}(w_j + \varphi_j)\,. \tag{3.11}$$

By further shrinking X, we may assume that w_j are bounded from above on X, say

$$w_j \leq C, \quad j = 1, \dots, p.$$

Then for any $k \geq 0$, on the pluri-fine open set

$$V_k := \bigcap_{j=1}^{p} \{\varphi_j + w_j > -k\}\,,$$

we have $\varphi_j > -k - C$, so by (1),

$$\max\{\varphi_j + w_j, -k\} = \max\{\varphi_j, -k - C\} + w_j\,.$$

Let $K \subseteq X$ be a compact subset, then

$$\int_K \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j + w_j, -k\} \wedge \omega^{n-p} = \int_K \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c} \max\{\varphi_j, -k - C\} \wedge \omega^{n-p}\,.$$

The right-hand side is bounded by assumption. So the right-hand side of (3.11) is well-defined and (3.11) follows. □

Let $\theta_1, \dots, \theta_p$ be smooth strongly closed $(1,1)$-forms on X. Let $\varphi_i \in \mathrm{PSH}(X, \theta_i)$. Let $T_i = \theta_i + \mathrm{dd^c}\varphi_i$. Then $T_1 \wedge \cdots \wedge T_p$ can be defined in the obvious way.

Definition 3.9. We say a closed positive $(1,1)$-current T on X is *good* if for any $x \in X$, there is a neighbourhood $V \subseteq X$ of x, such that there exists a smooth strongly closed $(1,1)$-form θ on V, a function $\varphi \in \mathrm{PSH}(V, \theta)$, such that $T = \theta + \mathrm{dd^c}\varphi$ on V.

Let $T_1, \dots, T_p$ be good closed positive $(1,1)$-currents on X, we can define $T_1 \wedge \cdots \wedge T_p$ in the obvious way. Namely,

Definition 3.10. Let $T_1, \dots, T_p$ be good closed positive $(1,1)$-currents on X. Locally on a small enough open set $V \subseteq X$, write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$, where there are closed immersions $X \hookrightarrow \Omega_i$ with $\Omega_i \subseteq \mathbb{C}^{N_i}$ being bounded pseudo-convex domains, smooth closed real forms $\tilde{\theta}_i$ on Ω_i, such that $\theta_i|_V = \tilde{\theta}_i|_V$, $\varphi_i \in \mathrm{PSH}(V, \theta_i)$. Then writing $\tilde{\theta}_i = \mathrm{dd^c}a_i$ for some smooth functions a_i on Ω_i, we define

$$T_1 \wedge \cdots \wedge T_p := \bigwedge_{i=1}^{p} \mathrm{dd^c}(a_i + \varphi_i)\,.$$

Proposition 3.29. *Let* $T_1, \dots, T_p$ *be closed positive currents of bidegree* $(1,1)$ *on* X. *Assume that all* T_i*'s are good.*

(1) *The product $T_1\wedge\cdots\wedge T_p$ is local in pluri-fine topology in the following sense: let $O\subseteq X$ be a pluri-fine open subset, let $S_1,\ldots,S_p$ be closed positive currents of bidegree $(1,1)$ on X. Assume that all S_i's are good. Assume that*
$$T_j|_O=S_j|_O,\quad j=1,\ldots,p.$$
Assume that
$$T_1\wedge\cdots\wedge T_p,\quad S_1\wedge\cdots\wedge S_p.$$
are both well-defined, then
$$T_1\wedge\cdots\wedge T_p\big|_O=S_1\wedge\cdots\wedge S_p\big|_O\,. \tag{3.12}$$
If O is open in the usual topology, then the product
$$T_1\wedge\cdots\wedge T_p|_O$$
on O is well-defined and
$$T_1\wedge\cdots\wedge T_p\big|_O=T_1\wedge\cdots\wedge T_p|_O. \tag{3.13}$$
Let $\mathcal{U}$ be an open covering of X. Then $T_1\wedge\cdots\wedge T_p$ is well-defined iff each of the following product is well-defined
$$T_1\wedge\cdots\wedge T_p|_U,\quad U\in\mathcal{U}.$$
(2) *Assume that $T_1\wedge\cdots\wedge T_p$ is well-defined, then the product $T_1\wedge\cdots\wedge T_p$ puts not mass on pluripolar sets.*
(3) *Assume that $T_1\wedge\cdots\wedge T_p$ is well-defined, then $T_1\wedge\cdots\wedge T_p$ is a closed positive current of bidegree (p,p).*
(4) *The product $T_1\wedge\cdots\wedge T_p$ is symmetric.*
(5) *The product is multi-linear: let T_1' be a good closed positive current of bidegree $(1,1)$, then*
$$(T_1+T_1')\wedge T_2\wedge\cdots\wedge T_p=T_1\wedge T_2\wedge\cdots\wedge T_p+T_1'\wedge T_2\wedge\cdots\wedge T_p$$
in the sense that left-hand side is well-defined iff both terms on right-hand side are well-defined, and the equality holds in that case.

Proposition 3.30. *Let X be a compact unibranch Kähler space of pure dimension n. Let $T_1,\ldots,T_p$ be good closed positive currents of bidegree $(1,1)$ on X. Then $T_1\wedge\cdots\wedge T_p$ is well-defined.*

Proof. We may assume that X is reduced.

Fix a Kähler form ω on X. In this case, write $T_j=(T_j+C\omega)-C\omega$ for $C>0$ large enough and apply Proposition 3.29 (5), we may assume that T_j is in a Kähler class. So we can write
$$T_j=\omega_j+\mathrm{dd^c}\varphi_j,$$
where ω_j is a Kähler form and φ_j is ω_j-psh. Let U be an open subset on which we can write
$$\omega_j=\mathrm{dd^c}\psi_j$$
with psh functions $\psi_j\leq 0$ on U. Now on U, for each $k\geq 0$,
$$\{\psi_j+\varphi_j>-k\}\subseteq\{\varphi_j>-k\},$$
so for each compact subset $K\subseteq U$,
$$\begin{aligned}&\int_K \mathbb{1}_{\cap_{j=1}^p\{\psi_j+\varphi_j>-k\}}\bigwedge_{j=1}^p\mathrm{dd^c}\max\{\psi_j+\varphi_j,-k\}\wedge\omega^{n-p}\\
=&\int_K \mathbb{1}_{\cap_{j=1}^p\{\psi_j+\varphi_j>-k\}}\bigwedge_{j=1}^p\left(\omega_j+\mathrm{dd^c}\max\{\varphi_j,-k\}\right)\wedge\omega^{n-p}\\
\leq&\int_X\bigwedge_{j=1}^p\left(\omega_j+\mathrm{dd^c}\max\{\varphi_j,-k\}\right)\wedge\omega^{n-p}\\
=&\int_X\bigwedge_{j=1}^p\omega_j\wedge\omega^{n-p}.\end{aligned}$$
The last step follows from the corresponding result on a resolution. □

3.6. **Bimeromorphic behaviour.** Let X be a compact unibranch Kähler complex analytic space of pure dimension n.

Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity that is an isomorphism over $X \setminus \operatorname{Sing} X^{\mathrm{red}}$.

Let θ be a strongly closed smooth $(1,1)$-form on X. Assume that $[\theta]$ is big: for all proper bimeromorphic morphism $f : Y \to X$ from a smooth manifold Y, $f^*\theta$ is big. In this case, define $\operatorname{vol}\theta = \operatorname{vol} f^*\theta$.

We set

$$V_\theta := \sup{}^* \{ \varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0 \} .$$

Definition 3.11. Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Define

$$\varphi \wedge \psi := \sup{}^* \{ \eta \in \mathrm{PSH}(X,\theta) : \eta \leq \varphi , \eta \leq \psi \} .$$

When the set is empty, we just define $\sup^* \emptyset = -\infty$.

Definition 3.12. Let $U \subseteq X$ be an open subset. Let $\varphi \in \mathrm{PSH}(\pi^{-1}U)$, define $\pi_*\varphi \in \mathrm{PSH}(U)$ as the unique psh extension of $\varphi|_{\pi^{-1}(U \setminus \operatorname{Sing} X)}$. We call $\pi_*\varphi$ the psh pushforward of φ.

Definition 3.13. Let

$$\mathcal{E}(X,\theta) := \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n = \operatorname{vol}\theta \right\} .$$

For any $p \geq 1$,

$$\mathcal{E}^p(X,\theta) := \left\{ \varphi \in \mathcal{E}(X,\theta) : \int_X |\varphi|^p \theta_\varphi^n < \infty \right\} .$$

Let

$$\mathcal{E}^\infty(X,\theta) := \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \sup_X |\varphi - V_\theta| < \infty \right\} .$$

Definition 3.14. Let $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a map $\varphi : (a,b) \to \mathcal{E}^1(X,\theta)$, such that

(1) The potential Φ on $X \times \{z \in \mathbb{C} : e^{-b} < |z| < e^{-a}\}$ defined by $\Phi(x,z) := \varphi_{-\log|z|}(x)$ is $\pi_1^*\theta$-psh, where $\pi_1 : X \times \{z \in \mathbb{C} : e^{-b} < |z| < e^{-a}\} \to X$ is the natural projection.

(2)
$$\lim_{t\to a+} \varphi_t = \varphi_a , \quad \lim_{t \to b-} \varphi_t = \varphi_b .$$

Definition 3.15. A potential $\varphi \in \mathrm{PSH}(X,\theta)$ is *model* if

$$\varphi = \sup_{C>0}{}^* (\varphi + C) \wedge V_\theta .$$

Definition 3.16. For any $\varphi \in \mathcal{E}^1(X,\theta)$, define $\mathcal{R}^1(X,\theta)$ as the set of all geodesic rays in $\mathcal{E}^1$ emanating from φ.

Definition 3.17. The *geodesic* between $\varphi_0, \varphi_1 \in \mathcal{E}^1$ is the maximal subgeodesic between them.

Proposition 3.31. *The functions π^* and π_* are inverse to each other. Under this correspondence, we get a bijection between*

(1) $\mathrm{PSH}(X,\theta)$ *and* $\mathrm{PSH}(Y,\pi^*\theta)$*. This bijection preserves the pre-rooftop structures.*
(2) $\mathcal{E}(X,\theta)$ *and* $\mathcal{E}(Y,\pi^*\theta)$*.*
(3) $\mathcal{E}^p(X,\theta)$ *and* $\mathcal{E}^p(Y,\pi^*\theta)$*.*
(4) Subgeodesics in $\mathcal{E}^1(X,\theta)$ *and subgeodesics in* $\mathcal{E}^1(Y,\pi^*\theta)$*.*
(5) Geodesics in $\mathcal{E}^1(X,\theta)$ *and geodesics in* $\mathcal{E}^1(Y,\pi^*\theta)$*.*
(6) $\mathcal{R}^1(X,\theta)$ *and* $\mathcal{R}^1(X,\pi^*\theta)$*.*
(7) Model potentials in $\mathrm{PSH}(X,\theta)$ *and in* $\mathrm{PSH}(Y,\pi^*\theta)$*.*

Proof. Only (4) needs a proof. Let Φ be a subgeodesic in $\mathcal{E}^1(Y,\pi^*\theta)$, regarded as a potential on $Y \times A$, where $A = \{z \in \mathbb{C} : e^{-1} < |z| < 1\}$. It is easy to verify then that the psh pushforward of Φ is the same as the ensemble of all psh pushforwards for fixed $z \in A$. $\square$

In particular, d_p metric is defined on $\mathcal{E}^p(X,\theta)$. This result partially generalizes [Dar17].

Let $\varphi, \phi \in \mathrm{PSH}(X,\theta)$. We define

$$[\varphi] \wedge \psi := \sup_{C>0}{}^* \left((\varphi + C) \wedge \psi \right) .$$

Lemma 3.32. *Let $\varphi, \phi \in \mathrm{PSH}(X,\theta)$. Then*

$$[\pi^*\varphi] \wedge \pi^*\psi = \pi^*\left([\varphi] \wedge \psi\right) \,.$$

Proof. By Proposition 3.31, for each $C > 0$,

$$(\pi^*\varphi + C) \wedge \pi^*\psi = \pi^*\left((\varphi + C) \wedge \psi\right) \,.$$

As $[\varphi] \wedge \psi$ is by definition the minimal θ-psh function lying above all $(\varphi + C) \wedge \psi$, by Proposition 3.31 again, $\pi^*([\varphi] \wedge \psi)$ is the minimal $\pi^*\theta$-psh function lying above all $\pi^*((\varphi + C) \wedge \psi) = (\pi^*\varphi + C) \wedge \pi^*\psi$. Hence we conclude. $\square$

3.7. **Basic properties.** Let X be a compact Kähler unibranch complex analytic space of pure dimension n.

The classical analogues of the following theorems are proved in [DDNL18b], [DDNL18a], [DDNL18d], [DDNL18c], [Xia19], [BB10], [WN19].

All results can be proved by passing to a resolution.

Let θ, $\theta_1, \dots, \theta_n$ be big strongly closed smooth $(1,1)$-form on X.

Theorem 3.33 (Comparison principles)**.**

(1) *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume one of the following conditions hold:*
   (a) $[\psi] \preceq [\varphi] \wedge V_\theta$.
   (b) $\varphi, \psi \in \mathcal{E}(X,\theta)$.

   Then

$$\int_{\{\varphi<\psi\}} \theta_\psi^n \leq \int_{\{\varphi<\psi\}} \theta_\varphi^n. \tag{3.14}$$

(2) *Let $\varphi_k, \psi_k \in \mathrm{PSH}(X,\theta_k)$ for $k = 1, \dots, j$, where $j \leq n$. Let $u, v, \varphi \in \mathrm{PSH}(X,\theta)$. Assume*
   (a)
$$[u] \preceq [\varphi], \quad [\psi_k] \preceq [\varphi_k], \quad [v] \preceq [\varphi].$$
   (b)
$$\int_X \theta_u^{n-j} \wedge \theta_{1,\psi_1} \wedge \dots \wedge \theta_{j,\psi_j} = \int_X \theta_v^{n-j} \wedge \theta_{1,\psi_1} \wedge \dots \wedge \theta_{j,\psi_j} = \int_X \theta_\varphi^{n-j} \wedge \theta_{1,\psi_1} \wedge \dots \wedge \theta_{j,\psi_j}$$

   Then

$$\int_{\{u<v\}} \theta_v^{n-j} \wedge \theta_{1,\psi_1} \wedge \dots \wedge \theta_{j,\psi_j} \leq \int_{\{u<v\}} \theta_u^{n-j} \wedge \theta_{1,\psi_1} \wedge \dots \wedge \theta_{j,\psi_j}.$$

Theorem 3.34 (Domination principles)**.** *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume one of the following conditions hold:*

(1) $\varphi \in \mathcal{E}^\infty(X,\theta)$
(2) $[\psi] \preceq [\varphi]$, $\int_X \theta_\psi^n > 0$.
(3) $\varphi, \psi \in \mathcal{E}(X,\theta)$.

Furthermore assume that

$$\psi \leq \varphi \,, \quad \theta_\varphi^n - a.e. \,, \tag{3.15}$$

Then $\psi \leq \varphi$.

Theorem 3.35 (Integration by parts)**.** *Let θ_j $(j = 0, \dots, n)$ be big cohomology classes on X. Let θ_j $(j = 0, \dots, n)$ be smooth representatives in θ_j. Let $\gamma_j \in \mathrm{PSH}(X,\theta_j)$ $(j = 2, \dots, n)$. Let $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta_0)$, $\psi_1, \psi_2 \in \mathrm{PSH}(X,\theta_1)$. Let $u = \varphi_1 - \varphi_2$, $v = \psi_1 - \psi_2$. Assume that*

$$[\varphi_1] = [\varphi_2], \quad [\psi_1] = [\psi_2].$$

Then

$$\int_X u \,\mathrm{dd^c} v \wedge \theta_{2,\gamma_2} \wedge \dots \wedge \dots \wedge \theta_{n,\gamma_n} = \int_X v \,\mathrm{dd^c} u \wedge \theta_{2,\gamma_2} \wedge \dots \wedge \dots \wedge \theta_{n,\gamma_n}. \tag{3.16}$$

Theorem 3.36 (Semi-continuity)**.** *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X,\theta_j)$ ($k \in \mathbb{Z}_{>0}$, $j = 1,\dots,n$). Let $\chi \geq 0$ be a bounded quasi-continuous function on X. Assume that for any $j = 1,\dots,n$, $i = 1,\dots,m$, as $k \to \infty$, φ_j^k converges to φ_j monotonely. Then for any pluri-fine open set $U \subseteq X$, we have*

$$\varliminf_{k\to\infty} \int_U \chi\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_U \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n} . \tag{3.17}$$

Theorem 3.37 (Monotonicity)**.** *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X,\theta_j)$. Assume that $[\varphi_j] \succeq [\psi_j]$ for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n} .$$

Theorem 3.38 (Berman–Boucksom differentiablity theorem)**.** *Let $\varphi \in \mathcal{E}^1(X,\theta)$. Let $v \in C^0(X)$. Then $E(P(\varphi + tv))$ is differentiable at $t = 0$ and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E\left(P(\varphi+tv)\right) = \int_X v\, \theta_\varphi^n . \tag{3.18}$$

Theorem 3.39. *Let μ be a positive Radon measure on X with mass $\operatorname{vol}\theta$. Assume that μ is non-pluripolar, $E^*(\mu) < \infty$. Then there exists a unique $\varphi \in \mathcal{E}(X,\theta)$ up to constant, such that*

$$\theta_\varphi^n = \mu .$$

3.8. **Test curves.** Let X be a compact unibranch Kähler space of pure dimension n. Let θ be a smooth strongly closed real $(1,1)$-form on X, representing a big class. Let $V = \operatorname{vol}(\theta)$.

Definition 3.18. A *test curve* on (X,θ) is a map $\psi = \psi_\bullet : \mathbb{R} \to \mathrm{PSH}^{\mathrm{Model}}(X,\theta) \cup \{-\infty\}$, such that

1. $\psi_\bullet$ is concave in $\bullet$.
2. $\psi_\bullet$ is usc in $\bullet$.
3. $\lim_{\tau\to-\infty} \psi_\tau = V_\theta$ in L^1.
4. $\psi_\tau = -\infty$ for τ large enough.

Let $\tau^+ := \inf\{\tau \in \mathbb{R} : \psi_\tau = -\infty\}$. We say ψ is *normalized* if $\tau^+ = 0$. The test curve is called *bounded* if $\psi_\tau = 0$ for τ small enough. Let $\tau^- := \sup\{\tau \in \mathbb{R} : \psi_\tau = V_\theta\}$ in this case.

The set of bounded test curves is denoted by $\mathcal{TC}^\infty(X,\omega)$.

Definition 3.19. The *energy* of a test curve $\psi_\bullet$ is defined as

$$\mathbf{E}(\psi_\bullet) := \frac{1}{V}\tau^+ + \frac{1}{V}\int_{-\infty}^{\tau^+} \left(\int_X \theta_{\psi_\tau}^n - V \right) \mathrm{d}\tau . \tag{3.19}$$

A test curve ψ is said to be of *finite energy* if $\mathbf{E}(\psi) > -\infty$. We denote the set of finite energy test curves by $\mathcal{TC}^1(X,\omega)$.

Definition 3.20. Let $\ell \in \mathcal{R}^1(X,\theta)$. The *Legendre transform* of ℓ is defined as

$$\hat{\ell}_\tau := \inf_{t\geq 0} \left(\ell_t - t\tau\right) , \quad \tau \in \mathbb{R} .$$

Let $\psi \in \mathcal{TC}^1(X,\theta)$, the *inverse Legendre transform* of ψ is defined as

$$\check{\psi}_t := \sup_{\tau\in\mathbb{R}} \left(\psi_\tau + t\tau\right) , \quad t \geq 0 .$$

Definition 3.21. A test curve $\psi_\bullet$ is *maximal* if for any τ, ψ_τ is either $-\infty$ or $\mathcal{I}$-model.

Lemma 3.40. *Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity. Then pull-back induces a bijection*

$$\pi^* : \mathcal{TC}^1(X,\theta) \to \mathcal{TC}^1(Y,\pi^*\theta) .$$

Moreover,

$$\mathbf{E}(\psi_\bullet) = \mathbf{E}(\pi^*\psi_\bullet) .$$

Proof. This follows immediately from Proposition 3.31. □

Theorem 3.41. *The Legendre transform and inverse Legendre transform establish a bijection from $\mathcal{R}^1(X,\theta)$ to $\mathcal{TC}^1(X,\theta)$. For $\ell \in \mathcal{R}^1(X,\theta)$, We have $\sup_X \ell_1 = \tau^+$ and $\mathbf{E}(\ell) = \mathbf{E}(\hat{\ell})$.*

Moreover, under this correspondence, $\mathcal{R}^\infty$ corresponds to the set of bounded test curves. When $\ell \in \mathcal{R}^\infty$,$\inf_X \ell_1 = \tau^-$.

Proof. By Lemma 3.40, this reduces immediately to the smooth case, in which case, the proof of [DX20, Theorem 3.7] can be generalized directly. ☐

3.9. **Potentials with prescribed singularities.** Let X be a compact unibranch Kähler space of pure dimension n. Let θ be a smooth strongly closed real $(1,1)$-form on X, representing a big class. Let $V = \mathrm{vol}(\theta)$. Let $\phi \in \mathrm{PSH}(X,\theta)$ be a model potential with positive mass. Let $\pi : Y \to X^{\mathrm{red}}$ be a resolution of singularity.

Proposition 3.42. *The following set is relatively compact in L^1-topology.*

$$\left\{ \varphi \in \mathrm{PSH}(X,\theta) : \sup_X (\varphi - \phi) = 0 \right\} \subseteq \mathrm{PSH}(X,\theta) .$$

Proof. We may assume that X is reduced. Let $\varphi_j \in \mathrm{PSH}(X,\theta)$ be a net such that $\sup_X(\varphi_j - \phi) = 0$. Then

$$\pi^*\varphi_j \in \left\{ \psi \in \mathrm{PSH}(Y,\pi^*\theta) : \sup_Y (\psi - \pi^*\phi) = 0 \right\} .$$

By [DDNL18b, Lemma 2.2], up to subtracting a subnet, we may assume that $\pi^*\varphi_j$ converges to some $\pi^*\varphi$, $\varphi \in \mathrm{PSH}(X,\theta)$. Clearly, $\sup_X(\varphi - \phi) = 0$. Note that $\pi^*\varphi_j \to \pi^*\varphi$ in L^1 implies that $\varphi_j \to \varphi$ in L^1, we conclude. ☐

Definition 3.22. Define the relative full mass class as

$$\mathcal{E}(X,\theta;[\phi]) := \left\{ \varphi \in \mathrm{PSH}(X,\theta) : [\varphi] \preceq [\phi], \int_X \theta_\varphi^n = \int_X \theta_\phi^n \right\} .$$

Proposition 3.43. *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Then $\varphi \in \mathcal{E}(X,\theta;[\phi])$ iff $[\varphi] \wedge V_\theta = \phi$.*

Proof. We may assume that X is reduced. By Proposition 3.31, Lemma 3.32, it suffices to prove the corresponding result on Y, in which case, this is exactly [DDNL18b, Theorem 2.1]. ☐

Proposition 3.44. *Assume that $\gamma_j \in \mathcal{E}(X,\theta;[\phi])$ $(j = 1,\dots,j_0 \le n)$. Let $\varphi,\psi \in \mathcal{E}(X,\theta;[\phi])$. Then*

$$\int_{\{\varphi<\psi\}} \theta_\psi^{n-j_0} \wedge \theta_{\gamma_1} \wedge \dots \wedge \theta_{\gamma_{j_0}} \le \int_{\{\varphi<\psi\}} \theta_\varphi^{n-j_0} \wedge \theta_{\gamma_1} \wedge \dots \wedge \theta_{\gamma_{j_0}} .$$

Proof. We may assume that X is reduced. By Proposition 3.31, it suffices to prove the corresponding result on Y, in which case, this is [DDNL18c, Corollary 3.16]. ☐

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