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Singularities in global pluripotential theory

– Lectures at Zhejiang University –

Updated on November 5, 2025. Fix some typos.

Preface

This book is an expanded version of my lecture notes at the Institute for Advanced Study in Mathematics (IASM) at Zhejiang university. My initial goal was to write a self-contained reference for the participants of the lectures. But I soon realized that many results have never been rigorously proved in any literature. When attempting to resolve these loose ends, the notes grew increasingly lengthy, ultimately resulting in the current book.

In this book, I would like to present my point of view towards the *global* pluripotential theories. There are three different but interrelated theories which deserve this name. They are

(1) the pluripotential theory on compact Kähler manifolds,
(2) the pluripotential theory on the Berkovich analytification of projective varieties, and
(3) the toric pluripotential theory on projective toric varieties.

We will begin by explaining the picture in the first case. Let us fix a compact Kähler manifold X. The central objects are the *quasi-plurisubharmonic functions* on X.

We are mostly interested in the *singularities* of such functions, that is, the places where a quasi-plurisubharmonic function φ tends to $-\infty$ and how it tends to $-\infty$.

Singularities occur naturally in mathematics. In geometric applications, X should be regarded as the compactified moduli space of certain geometric objects. A Zariski open subset $U \subseteq X$ would parametrize smooth objects. The natural metric on the associated polarizing line bundle is usually smooth only on U, not on X. In case we have suitable positivities, the classical Grauert–Remmert extension theorem (Theorem B.2.2) allows us to extend the metric outside U, but at the cost of introducing singularities.

The classification of singularities is a huge project. Locally near the singularities we know that quasi-plurisubharmonic functions present very complicated behaviours. There are many local invariants associated with the singularities. The most notable ones are the Lelong numbers and the multiplier ideal sheaves. These invariants only reflect the rough behaviour of a quasi-plurisubharmonic function. As an example, a quasi-plurisubharmonic function with log-log singularities have the same local invariants as a bounded one.

The situation changes drastically in the global setting, namely on compact manifolds. In the global setting, there are three different ways to classify quasi-plurisubharmonic functions according to their singularities:

(1) The singularity type characterizing the singularities up to a bounded term.
(2) The P-singularity type associated with global masses.
(3) The $\mathcal{I}$-singularity type associated with all non-Archimedean data.

The classification becomes rougher and rougher as we go downward. In the first case, we say two quasi-plurisubharmonic functions have the same singularity type if their difference lies in L^∞. The corresponding equivalence class gives us essentially the finest information of the singularities we can expect. The other two relations are more delicate, we will study them in detail in Chapter 6.

A natural idea to study the singularities would consist of the following steps:

(1) Classify the $\mathcal{I}$-singularity types.
(2) Classify the P-singularity types within a given $\mathcal{I}$-singularity class.
(3) Classify the singularity types within a given P-equivalence class.

The Step 3 is well-studied in the literature in the last decade under the name of pluripotential theory with prescribed singularities. There are numerous excellent results in this direction. In some sense, this step is already well-understood.

We will give a complete answer to Step 1 in Chapter 7, where we show that $\mathcal{I}$-singularity types can be described very explicitly.

It remains to consider Step 2. This is not an easy task. It is easy to construct examples where a given $\mathcal{I}$-equivalence class consists of a huge amount of P-equivalence classes.

On the other hand, by contrast, in the toric pluripotential theory and non-Archimedean pluripotential theory, Step 2 is essentially trivial: An $\mathcal{I}$-equivalence class consists of a single P-equivalence class. In the toric situation, an $\mathcal{I}$ or P-equivalence class is simply a sub-convex body of the Newton body, while in the non-Archimedean situation, an $\mathcal{I}$ or P-equivalence class is a homogeneous plurisubharmonic metric.

This apparent anomaly and numerous examples show that in the pluripotential theory on compact Kähler manifolds, certain singularities are pathological. Within each $\mathcal{I}$-equivalence class, we could pick up a canonical P-equivalence class, the quasi-plurisubharmonic functions in which are said to be *$\mathcal{I}$-good*. We will study the theory of $\mathcal{I}$-good singularities in Chapter 7. As we will see later on, almost all (if not all) singularities occurring naturally are $\mathcal{I}$-good.

My personal impression is that we are in a situation quite similar to the familiar one in real analysis. There are many non-measurable functions, but in real life, unless you construct a pathological function by force, you only encounter measurable functions. Similarly, although there exist many non-$\mathcal{I}$-good singularities, you would never encounter them in reality!

Having established this general principle, we could content ourselves in the framework of $\mathcal{I}$-good singularities. Then Step 2 is essentially solved, and we have a pretty good understanding of the classification of singularities.

Of course, this classification is a bit abstract. To put it into use, we will introduce two general techniques allowing us to make induction on $\dim X$. For a prime divisor Y in general position, we have the so-called analytic Bertini theorems relating quasi-plurisubharmonic functions on X and on Y. For a non-generic Y, we have the technique of trace operators. These techniques will be explained in Chapter 8.

In the toric situation, these constructions and methods are quite straightforward and are likely known to experts before I entered this field, see Chapter 5 for the toric pluripotential theory on ample line bundles.

The corresponding toric pluripotential theory on big line bundles has never been written down in the literature. We will develop the theory of partial Okounkov bodies in Chapter 10 and the general toric pluripotential theory will be developed as an application in Chapter 12.

Finally, we give applications to non-Archimedean pluripotential theory in Chapter 13 based on the theory of test curves developed in Chapter 9. We also prove the convergence of the partial Bergman kernels in Chapter 14.

The readers are only supposed to be familiar with the basic pluripotential theory. The excellent book [GZ17] is more than enough.

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in Hangzhou, March 2024

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夏铭辰，职业乳法选手

Acknowledgments

The clerks are French, and, like most French people, are in a bad temper till they have eaten their lunch.
— George Orwell (1903–1950)

I would like to express my gratitude to Bing Wang and Song Sun for their gracious invitations to China and for providing me with the opportunity to deliver a series of lectures.

Furthermore, I am indebted to the dedicated researchers and secretaries of the University of Science and Technology of China (USTC) and the Institute for Advanced Study in Mathematics (IASM) for cultivating an exceptional research environment. Their commitment to excellence has allowed me to immerse myself fully in the field of mathematics during my time in China.

I am also immensely thankful to the participants of the course, including Song Sun, Mingyang Li, Xin Fu, Jiyuan Han, Junsheng Zhang, Yifan Chen, Yueqing Feng, Minghao Miao, and Federico Giust. Their active engagement and insightful discussions have greatly enriched my lectures and enhanced my understanding of the subject matter.

Special appreciation goes to Yi Yao, Kewei Zhang and Robert Berman for their invaluable contributions to discussions on toric geometry, which ultimately inspired the theory developed in Chapter 12.

Most results in this book are developed in collaboration with Tamás Darvas and Kewei Zhang, whose insights are always crucial in the development of the theories. I would like to thank them for the collaborations over years.

A substantial part of the current book was essentially contained in my PhD thesis. I would like to thank my advisor Robert Berman for his guidance and my colleagues in Göteborg and Paris for constant discussions, especially Bo Berndtsson, David Witt Nyström, Sébastien Boucksom and Elizabeth Wulcan.

I am especially grateful to Zetian Yan and Andreas Hohl. The former helped to settle the problem of typesetting Polish, Romanian and Swedish special characters within the framework of SVMono class provided by Springer. Most of the book was typeset on the computer I inherited from my officemate Andreas at IMJ-PRG, which proved a worthy successor to the “computeroid” graciously provided by the university — a machine on which coding was, shall we say, aspirational.

This work would not have been possible without the unwavering support and encouragement of all those mentioned above. Thank you for your generosity, guidance, and camaraderie throughout this endeavor.

I want to thank the following people for pointing out mistakes or typos in the original version of the book: Vasanth Pidaparthy, Prakhar Gupta, Yi Yao, Tamás Darvas, Yan He.

Enfin, je tiens à exprimer ma gratitude à Sébastien Boucksom et Madame Natalia Hristic de la Sorbonne Université, qui m'ont aidé à contacter le ministère de l'intérieure en France. Sans leur intervention, je serais encore coincé en France, échoué par l'efficacité extraordinaire du gouvernement français, en particulier de la préfecture de Créteil et ce livre n'aurait jamais vu le jour. Merci également au comité du CNRS de m'avoir privé d'une audition deux années de suite (à savoir 2023 et 2024), me libérant ainsi à jamais de mon combat contre la bureaucratie française.

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Conventions

In the whole book, we adopt the following conventions:

- A complex space is always assumed to be *reduced*, *paracompact* and *Hausdorff*.
- A *modification* of a complex space X is proper bimeromorphic morphism $\pi\colon Y \to X$ that is locally obtained from a finite composition of blow-ups with smooth centers.
- A *subnet* of a net refers to a Kelley subnet.
- A *domain* in $\mathbb{C}^n$ refers to a connected open subset.
- A *complex manifold* is assumed to be paracompact.
- A *submanifold* of a complex manifold means a closed complex submanifold.
- A *neighborhood* is not necessarily open.
- The set $\mathbb{N}$ of natural numbers includes 0.
- *Increasing functions* and *decreasing functions* are not necessarily strictly monotone.

We will use the following notations throughout the book:

- If I is a non-empty set, then $\mathrm{Fin}(I)$ denote the net of finite non-empty subsets of I, ordered by inclusion.
- $\mathrm{dd^c}$ means $(2\pi)^{-1}\mathrm{i}\partial\overline{\partial}$.

Part I
Preliminaries

In the first two chapters Chapter 1 and Chapter 2 of this part, we recall a few preliminaries about the notion of plurisubharmonic functions and the non-pluripolar products of plurisubharmonic functions.

Most materials in these chapters are standard and are well-documented in other textbooks, so we will be rather sketchy. The readers are encouraged to consult the excellent textbook [GZ17].

In Chapter 3, we develop the techniques of envelope operators. All results in this section are known and are written in various articles.

In Chapter 4, we develop the theory of geodesics in the space of quasi-plurisubharmonic functions. Most results in this chapter are known to different degrees, but not in the fully general form as we present. Most proofs are similar to the known proofs in the literature, but the presence of singularities requires a very careful treatment.

In Chapter 5, we recall the basic results about the toric pluripotential theory on ample line bundles, which will be generalized to big line bundles in Chapter 12.

Experienced readers may safely skip the whole part.

Chapter 1
Plurisubharmonic functions

Once Frigyes Riesz[a] *gave a brilliant explanation of why scientific work is easy. "Everyone has ideas, both right ideas and wrong ideas," he said. "Scientific work consists merely of separating them."*
— Istvan Vincze

[a] Frigyes Riesz (1880–1956), known as Frédéric Riesz in French and Frederic Riesz in English was the first mathematician to define the general notion of subharmonic functions, who also gave these functions a Frenglish name from the very beginning — *fonctions subharmoniques*.

In this chapter, we recall the notion of plurisubharmonic functions and a few basic properties of these functions. The main purpose is to fix the notations for later chapters, so we refer to the literature for most of the proofs.

We give some details about the plurifine topology in Section 1.3, since the related proofs are scattered in a number of articles.

In the literature related to multiplier ideal sheaves and Lelong numbers, there are several different conventions about their normalizations. The readers can find more about the conventions we adopt throughout the book in Section 1.4.

1.1 The definition of plurisubharmonic functions

In this section, we recall the notion of plurisubharmonic functions. We will also take care of the 0-dimensional case, which makes a number of induction arguments easier to carry out. None of our references treats the 0-dimensional case, but the readers can easily verify that the results in this section hold in this exceptional case.

1.1.1 The 1-dimensional case

Let Ω be a domain (a connected open subset) in $\mathbb{C}$.

Definition 1.1.1 A *subharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) φ satisfies the *sub-mean value inequality*: For any $a \in \Omega$ and $r > 0$ such that $B_1(a,r) \Subset \Omega$, we have

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a + r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$[1].

We will denote the set of subharmonic functions on Ω as $\mathrm{SH}(\Omega)$.

Here, $B_1(a, r)$ denotes the open ball with center a and radius r. See (1.1).

In fact, for each $a \in \Omega$, in (3), it suffices to require the sub-mean value inequality for all small enough $r > 0$.

Intuitively, at a specific point $a \in \Omega$, the Condition (2) gives a lower bound of the value of $\varphi(a)$ using the nearby values of φ, while the Condition (3) gives an upper bound. This intuition leads to the following rigidity theorem:

Theorem 1.1.1 *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\Delta\varphi \ge 0$.*
(2) *φ coincides almost everywhere with a subharmonic function ψ on Ω.*

Moreover, the subharmonic function ψ in (2) is unique.

Here in Condition (1), $\Delta\varphi$ is the Laplacian in the sense of currents. This is a special case of Theorem 1.1.2 below.

This theorem gives a very useful way of constructing subharmonic functions.

1.1.2 The higher dimensional case

We will fix $n \in \mathbb{N}$ and a domain Ω (a connected open subset) in $\mathbb{C}^n$.

Definition 1.1.2 When $n \ge 1$, a *plurisubharmonic function* on Ω is a function $\varphi\colon \Omega \to [-\infty, \infty)$ satisfying the following three conditions:

(1) $\varphi \not\equiv -\infty$;
(2) φ is upper semi-continuous;
(3) for any complex line $L \subseteq \mathbb{C}^n$ and any connected component U of $L \cap \Omega$, the restriction $\varphi|_U$ is either subharmonic or constantly $-\infty$.[2]

When $n = 0$, the only domain Ω is the singleton. In this case, a *plurisubharmonic function* on Ω is a real-valued function on Ω.

The set of plurisubharmonic functions on Ω is denoted by $\mathrm{PSH}(\Omega)$.

A plurisubharmonic function is also called a psh function for short. The relevant notations are indicated in Fig. 1.1.[3]

[1] Condition (2) guarantees that φ is measurable and locally bounded from above, and hence the integral in Condition (3) makes sense.

[2] An extremely common mistake in the literature is to replace (3) by the condition that φ is locally integrable and $\mathrm{dd}^{\mathrm{c}}\varphi \ge 0$ in the sense of currents. For a concrete counterexample, consider a function φ that takes a constant value 0 at all but one single point, at which the value of φ is 1.

[3] We remind the readers that most figures in this book are somewhat misleading: We usually draw a complex dimension as a real dimension. The figures should not be read literally!

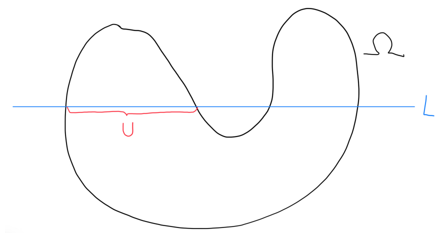

Fig. 1.1 A domain cut by a line

Example 1.1.1 When $n = 0$, we have a canonical bijection $\mathrm{PSH}(\Omega) \cong \mathbb{R}$.

Example 1.1.2 When $n = 1$, we have $\mathrm{PSH}(\Omega) = \mathrm{SH}(\Omega)$.

Similar to Theorem 1.1.1, we have a rigidity theorem for plurisubharmonic functions as well.

Theorem 1.1.2 *Let $\varphi\colon \Omega \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on Ω.*

Moreover, the plurisubharmonic function ψ is unique.

Here, the operator $\mathrm{dd^c}$ is normalized so that

$$\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}.$$

For the proof, we refer to [GZ17, Proposition 1.43].

Plurisubharmonic functions have nice functorialities:

Proposition 1.1.1 *Let $n' \in \mathbb{N}$ and $\Omega' \subseteq \mathbb{C}^{n'}$ be a domain. Given any holomorphic map $f\colon \Omega \to \Omega'$ and any $\varphi \in \mathrm{PSH}(\Omega')$ exactly one of the following cases occurs:*

(1) *$f^*\varphi \equiv -\infty$;*
(2) *$f^*\varphi \in \mathrm{PSH}(\Omega)$.*

We refer to [GZ17, Proposition 1.44] for the proof[4].

For each $n \in \mathbb{N}$, $a \in \mathbb{C}^n$ and $r > 0$, we write

$$B_n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}. \tag{1.1}$$

[4] We remind the readers that the statement of [GZ17, Proposition 1.44] is flawed. One has to reduce to the case where Case (1) does not occur before following their proof.

Proposition 1.1.2 *Let $\varphi \in \mathrm{PSH}(B_n(a, r_0))$ for some $r_0 > 0$. Then the function*

$$(-\infty, \log r_0) \to \mathbb{R}, \quad \log r \mapsto \sup_{B_n(a,r)} \varphi$$

is convex and increasing.

See [Hö07, Theorem 4.1.13] for the case $n > 1$ and [Bou17, Corollary 2.4] for the general case.

Proposition 1.1.3 *Let $a < b$ be two real numbers. Let $f : (a, b) \to [-\infty, \infty)$ be a function. Define*

$$g : \{z \in \mathbb{C} : a < \operatorname{Re} z < b\} \to [-\infty, \infty), \quad z \mapsto f(\operatorname{Re} z).$$

Suppose that g is subharmonic, then f is convex. In particular, f takes real values only.

See [HK76, Theorem 2.12] for a more general result.

1.1.3 The manifold case

Let X be a complex manifold. In the whole book, complex manifolds are assumed to be paracompact, namely, all connected components have countable bases.

Definition 1.1.3 A *plurisubharmonic function* on X is a function $\varphi : X \to [-\infty, \infty)$ such that for any $x \in X$, there exists an open neighborhood U of x in X, an integer $n \in \mathbb{N}$, a domain $\Omega \subseteq \mathbb{C}^n$ and a biholomorphic map $F : \Omega \to U$ such that $F^*(\varphi|_U) \in \mathrm{PSH}(\Omega)$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Example 1.1.3 When X is a domain in $\mathbb{C}^n$, the notions of plurisubharmonic functions in Definition 1.1.3 and in Definition 1.1.2 coincide.

Example 1.1.4 Write $\{X_i\}_{i \in I}$ for the set of connected components of X. Then we have a natural bijection

$$\mathrm{PSH}(X) \cong \prod_{i \in I} \mathrm{PSH}(X_i).$$

Here the product is in the category of sets. In particular, if $X = \emptyset$, then $\mathrm{PSH}(X)$ is a singleton.

This example allows us to reduce to the case of connected manifolds when studying general plurisubharmonic functions.

Proposition 1.1.4 *Let Y be another complex manifold and $f : Y \to X$ be a holomorphic map. Then for any $\varphi \in \mathrm{PSH}(X)$, exactly one of the following cases occurs:*

(1) *$f^*\varphi$ is identically $-\infty$ on some connected component of Y;*
(2) $f^*\varphi \in \mathrm{PSH}(Y)$.

This proposition follows easily from Proposition 1.1.1. We leave the details to the readers.

Theorem 1.1.2 implies immediately the general form of the rigidity theorem:

Theorem 1.1.3 *Let $\varphi\colon X \to [-\infty, \infty)$ be a measurable function. Then the following are equivalent:*

(1) *φ is locally integrable and $\mathrm{dd^c}\varphi \geq 0$;*
(2) *φ coincides almost everywhere with a plurisubharmonic function ψ on X.*

Moreover, the plurisubharmonic function ψ in (2) is unique.

Definition 1.1.4 A subset $E \subseteq X$ is *pluripolar* if for any $x \in X$, there is an open neighborhood U of x in X and a function $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{E\cap U} \equiv -\infty.$$

A subset $E \subseteq X$ is *non-pluripolar* if E is not pluripolar.

A subset $F \subseteq X$ is *co-pluripolar* if $X \setminus F$ is pluripolar.

When X has dimension 1, a pluripolar set is called a *polar set*. We say some property about objects on X holds *quasi-everywhere* if it holds outside a pluripolar set.

Theorem 1.1.4 (Josefson's theorem) *Let $E \subseteq \mathbb{C}^n$ be a pluripolar set. Then there is $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ such that $\varphi|_E \equiv -\infty$.*

See [GZ17, Corollary 4.41] for the proof of a more general result.

There is also a global version of Josefson's theorem:

Theorem 1.1.5 *Assume that X is a compact complex manifold and $E \subseteq X$ is a pluripolar set. Then there is a quasi-plurisubharmonic function φ on X with $\varphi|_E \equiv -\infty$.*

For a proof, see [Vu19].

1.2 Properties of plurisubharmonic functions

In this section, we explore the basic properties of plurisubharmonic functions.

Let X be a complex manifold.

Proposition 1.2.1

(1) *Assume that $(\varphi_i)_{i\in I}$ is a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. Then ${\sup_{i\in I}}^*\varphi_i \in \mathrm{PSH}(X)$.*
(2) *Assume that $(\varphi_i)_{i\in I}$ is a decreasing net in $\mathrm{PSH}(X)$ such that $\lim_{i\in I}\varphi_i$ is not identically $-\infty$ on each connected component of X, then $\lim_{i\in I}\varphi_i \in \mathrm{PSH}(X)$.*

Here $\sup^*$ denotes the upper semicontinuous regularization of the supremum. When I is a finite family, observe that

$$\sup_{i\in I}{}^*\varphi_i = \sup_{i\in I}\varphi_i.$$

When $I = \{1, \dots, m\}$, we write

$$\varphi_1 \vee \cdots \vee \varphi_m \coloneqq \sup_{i\in I}\varphi_i.$$

We refer to [GZ17, Proposition 1.28, Proposition 1.40][5].

Proposition 1.2.2 (Choquet's lemma) *Assume that X has countably many connected components. Assume that $(\varphi_i)_{i\in I}$ is a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. There exists a countable subset $J \subseteq I$ such that*

$$\sup_{i\in I}{}^*\varphi_i = \sup_{j\in J}{}^*\varphi_j.$$

Proof We may assume that X is connected. Since by our convention, the complex manifold X is paracompact, it can be covered by countably many open balls, so we can easily reduce to the case where X is an open ball. In this case, the result is proved in [GZ17, Lemma 4.31]. □

Proposition 1.2.3 *Let $\varphi \in \mathrm{PSH}(X)$, then for any $p \geq 1$, $\varphi \in L^p_{\mathrm{loc}}(X)$.*

See [GZ17, Theorem 1.46, Theorem 1.48].

Proposition 1.2.4 *A pluripolar set $E \subseteq X$ is a Lebesgue null set.*

Proof This is a trivial consequence of Proposition 1.2.3. □

Proposition 1.2.5 *Let $(\varphi_i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}(X)$ that is locally uniformly bounded from above. Then the set*

$$\left\{x \in X : \sup_{i\in I}\varphi_i < \sup_{i\in I}{}^*\varphi_i\right\}$$

is pluripolar and hence Lebesgue null.

See [GZ17, Corollary 4.28].

Proposition 1.2.6 *Suppose that $\varphi, \psi \in \mathrm{PSH}(X)$. Assume that there is a dense subset $E \subseteq X$ such that $\varphi|_E \leq \psi|_E$, then $\varphi \leq \psi$.*

[5] In [GZ17, Proposition 1.28], the second part is only stated for sequences, the net version is obvious using the sub-mean value inequality.

Proof The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$.

We may assume that $\varphi|_E = \psi|_E$ after replacing φ by $\varphi \vee \psi$. Then we need to show that $\varphi = \psi$.

By [GZ17, Theorem 4.20], φ and ψ are quasi-continuous. It follows that $\varphi = \psi$ outside a set $Y \subseteq X$ with vanishing capacity. By [GZ17, Theorem 4.40], Y is also pluripolar. In particular, $\varphi = \psi$ almost everywhere. It follows from the uniqueness statement in Theorem 1.1.3 that $\varphi = \psi$. □

Proposition 1.2.7 *Let $(E_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence of pluripolar sets in X. Then*

$$E := \bigcup_{i=1}^{\infty} E_i$$

is also pluripolar.

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain. In this case, by Theorem 1.1.4 for each $i \in \mathbb{Z}_{>0}$ we can choose $\psi_i \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\psi_i|_{E_i} \equiv -\infty, \quad \psi_i|_X \leq 0$$

for all $i > 0$. After shrinking X, we may guarantee that $\psi_i|_X \in L^1(X)$ for all $i > 0$. After rescaling, we may also assume that $\|\psi_i\|_{L^1(X)} \leq 1$ for all $i > 0$.

We then define

$$\psi = \sum_{i=1}^{\infty} 2^{-i}\psi_i|_X.$$

Then $\psi \in \mathrm{PSH}(X)$ according to Proposition 1.2.1 and $\psi|_E = -\infty$. □

Corollary 1.2.1 *Let $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{PSH}(X)$ such that $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi \in \mathrm{PSH}(X)$. Then the set*

$$\left\{x \in X : \varphi(x) \neq \varlimsup_{j\to\infty} \varphi_j(x)\right\}$$

is pluripolar.

Proof We first observe that $(\varphi_j)_j$ is locally uniformly bounded from above. This follows from [GZ17, Exercise 1.20].

For each $j \geq 1$, let

$$\psi_j = \sup_{k\geq j}{}^* \varphi_k.$$

Then $\psi_j \in \mathrm{PSH}(X)$ by Proposition 1.2.1. Moreover, $(\psi_j)_j$ is a decreasing sequence and $\psi_j \geq \varphi_j$ for all j. In particular, $\varphi \leq \psi := \inf_j$ almost everywhere. By Proposition 1.2.1 again, $\psi \in \mathrm{PSH}(X)$.

On the other hand, by Proposition 1.2.5, there exist pluripolar sets $Z_j \subseteq X$ such that

$$\psi_j = \sup_{k\geq j} \varphi_k$$

on $X \setminus Z_j$. Let

$$Z = \bigcup_{j=1}^{\infty} Z_j .$$

Then Z is a pluripolar set by Proposition 1.2.7, and for any $x \in X \setminus Z$, we have $\psi(x) = \overline{\lim}_j \varphi_j(x)$. Since $\varphi_j \xrightarrow{L^1_{\mathrm{loc}}} \varphi$, we can find a set $Y \subseteq X$ with zero Lebesgue measure such that $\varphi_j(x) \to \varphi(x)$ for all $x \in X \setminus Y$.

In particular, for any $x \in X \setminus (Y \cup Z)$, we have

$$\psi(x) = \varphi(x).$$

But thanks to Proposition 1.2.6, the equality holds everywhere. Therefore, for all $x \in X \setminus Z$,

$$\varphi(x) = \overline{\lim_{j\to\infty}} \varphi_j(x).$$

Theorem 1.2.1 (Brelot, Grauert–Remmert) *Let Z be an analytic subset in X and $\varphi \in \mathrm{PSH}(X \setminus Z)$. Then the function φ admits an extension to $\mathrm{PSH}(X)$ in the following two cases:*

(1) *The set Z has codimension at least 2 everywhere.*
(2) *The set Z has codimension at least 1 everywhere and φ is locally bounded from above on an open neighborhood of Z.*

In both cases, the extension is unique and is given by

$$\varphi(x) = \overline{\lim_{X\setminus Z \ni y \to x}} \varphi(y), \quad x \in Z. \tag{1.2}$$

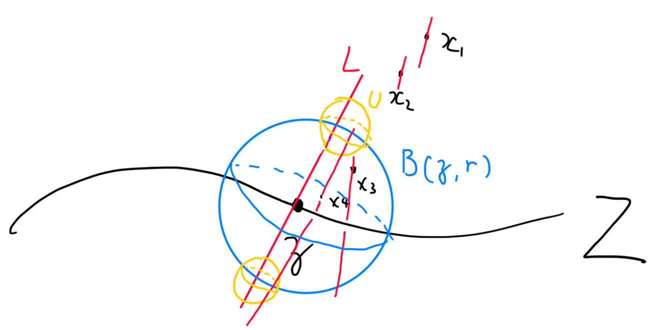

Fig. 1.2 The proof of Grauert–Remmert extension theorem

Proof The extension is unique thanks to Proposition 1.2.6.

(2) Thanks to the uniqueness of the extension, the problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ with $n > 0$ and there is a non-zero holomorphic function f vanishing identically on Z. For each $\epsilon > 0$, we claim that the function φ_ϵ defined by

$$\varphi_\epsilon(x) := \begin{cases} \varphi(x) + \epsilon \log |f(x)|^2, & x \in X \setminus Z; \\ -\infty, & x \in Z \end{cases}$$

is plurisubharmonic on X. By Definition 1.1.2, it suffices to verify the case $n = 1$. In this case, we may assume that $Z = \{0\}$, It is clear that $\varphi_\epsilon \in \mathrm{SH}(X \setminus Z)$. It suffices to verify the sub-mean value inequality at 0, which is immediate.

Next observe that the sequence φ_ϵ is increasing as $\epsilon \searrow 0$ and φ_ϵ is locally uniformly bounded from above. It follows from Proposition 1.2.1 that $\tilde{\varphi} := {\sup_{\epsilon>0}}^* \varphi_\epsilon \in \mathrm{PSH}(X)$. Moreover, $\tilde{\varphi}$ clearly extends φ. Note that (1.2) follows from the construction.

(1) We invite the readers to have a look at Fig. 1.2 for our notations in the proof.

It suffices to verify that φ is locally bounded from above near each point of Z. The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ with $n \geq 2$.

Assume that our assertion fails. Take $z \in Z$ so that there exists a sequence $(x_j)_j$ in $X \setminus Z$ converging to z such that

$$\lim_{j\to\infty} \varphi(x_j) = \infty.$$

Since Z has codimension at least 2[6], we could take a complex line L passing through z and intersects Z only on a discrete set. After shrinking X, we may assume that

$$L \cap Z = \{z\}.$$

Take an open ball $B_n(z, r) \Subset X$. After adding a constant to φ, we may guarantee that $\varphi < 0$ on $L \cap \partial B_n(z, r)$. Since φ is upper semi-continuous, we could find an open neighborhood U of $L \cap \partial B_n(z, r)$ such that

$$\varphi|_U < 0.$$

For each $j \geq 1$, take a complex line L_j passing through x_j and avoiding Z such that $L_j \to L$ as $j \to \infty$. Here we rely on the fact that Z has codimension at least 2. Here the convergence is in the obvious sense. Then for large enough j, we know have

$$L_j \cap \partial B_n(z, r) \subseteq U.$$

It follows from the sub-mean value inequality that $\varphi(x_j) < 0$ for large enough j, which is a contradiction. □

Lemma 1.2.1 *Let $\varphi \in \mathrm{PSH}((\Delta^*)^n)$ be an $(S^1)^n$-invariant plurisubharmonic function. Then φ is finite everywhere.*

---

[6] In fact, codimension at least 1 suffices for this step.

Here
$$\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

Proof It clearly suffices to handle the case $n = 1$. In this case, by [HK76, Theorem 2.12], the map
$$\log r \mapsto \int_0^1 \varphi(r \exp(2\pi \mathrm{i}\theta))\, \mathrm{d}\theta = \varphi(r)$$
is a convex function of $\log r$. So, the set $\{r \in (0,1) : \varphi(r) = -\infty\}$ is convex. But φ is almost everywhere finite by Proposition 1.2.3. Since φ is S^1-invariant, $\varphi|_{(0,1)}$ is almost everywhere finite. It follows from the convexity that it is everywhere finite.□

Proposition 1.2.8 (Kiselman's principle) *Let $\Omega \subseteq \mathbb{C}^m \times \mathbb{C}^n$ be a pseudoconvex domain. Assume that for each $z \in \mathbb{C}^m$, the set*
$$\Omega_z := \{w \in \mathbb{C}^n : (z, w) \in \Omega\}$$
has the form $E + \mathrm{i}\mathbb{R}^n$, where $E \subseteq \mathbb{R}^n$ is a subset. Let $\varphi \in \mathrm{PSH}(\Omega)$, assume that φ is independent of the imaginary part of the variable in $\mathbb{C}^n$. Let Ω' be the projection of Ω to $\mathbb{C}^m$. Define $\psi : \Omega' \to [-\infty, \infty)$ as follows:
$$\psi(z) = \inf_{w \in \Omega_z} \varphi(z, w).$$
Then either $\psi \equiv -\infty$ or $\psi \in \mathrm{PSH}(\Omega')$.

See [Dem12b, Theorem 7.5] for the proof as well as the notion of pseudoconvex domains.

Lemma 1.2.2 *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $\Omega' \subseteq \Omega$ be a subdomain. Consider $\varphi \in \mathrm{PSH}(\Omega)$ and $\psi \in \mathrm{PSH}(\Omega')$. Assume that*
$$\varliminf_{\substack{\Omega' \ni y \to x, \\ \psi(y) \neq -\infty}} (\varphi(y) - \psi(y)) \geq 0$$
for any $x \in \Omega \cap \partial\Omega'$. Define
$$\eta(z) = \begin{cases} \varphi(z) \vee \psi(z), & \text{if } z \in \Omega', \\ \varphi(z), & \text{if } z \in \Omega \setminus \Omega'. \end{cases}$$
Then $\eta \in \mathrm{PSH}(\Omega)$.

This is morally just [GZ17, Proposition 1.30]. But the statement in the reference is slightly misleading, so I reproduced the proof just for clarification.

Proof See Fig. 1.3 for the notations used in the proof.

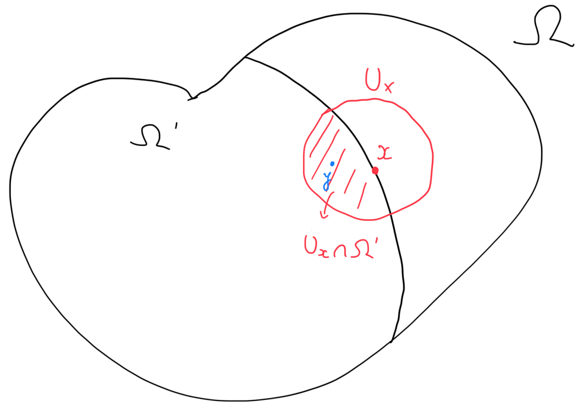

Fig. 1.3 Gluing procedure

Take $\epsilon > 0$. We first define

$$\eta_\epsilon(z) = \begin{cases} \varphi(z) \vee (\psi(z) - 2\epsilon), & \text{if } z \in \Omega', \\ \varphi(z), & \text{if } z \in \Omega \setminus \Omega'. \end{cases}$$

We claim that

$$\eta_\epsilon \in \mathrm{PSH}(\Omega).$$

By our assumption, for each $x \in \Omega \cap \partial\Omega'$, we can find an open neighborhood $U_x \subseteq \Omega$ such that for any $y \in U_x \cap \Omega'$, we have $\varphi(y) \geq \psi(y) - \epsilon$. Therefore, there is an open neighborhood U of $\Omega \cap \partial\Omega'$ such that

$$\varphi(y) \geq \psi(y) - \epsilon, \quad \forall y \in U \cap \Omega'.$$

Therefore, on the open set $(\Omega \setminus \Omega') \cup U$, we have $\eta_\epsilon = \varphi$ and hence η_ϵ is plurisubharmonic there. It is plurisubharmonic on Ω' by Proposition 1.2.1. So our claim follows.

Next we observe that as ϵ decreases to 0, the functions η_ϵ increases to η. Therefore, $\eta^* \in \mathrm{PSH}(\Omega)$ by Proposition 1.2.1. But observe that η is upper semicontinuous. This is only non-trivial on the boundary of Ω': Take $x \in \Omega \cap \partial\Omega'$ and let $(y_i)_{i>0}$ be a sequence in Ω' with limit x. Then we need to show that

$$\varlimsup_{i\to\infty} \psi(y_i) \leq \varphi(x). \tag{1.3}$$

We may assume that $\psi(y_i) \neq -\infty$ for all $i > 0$ and the left-hand side of (1.3) is not $-\infty$. Then we can compute

$$\varlimsup_{i\to\infty} \psi(y_i) \leq \varlimsup_{i\to\infty} \psi(y_i) + \varliminf_{i\to\infty} (\varphi(y_i) - \psi(y_i)) \leq \varlimsup_{i\to\infty} \varphi(y_i) \leq \varphi(x).$$

Therefore, $\eta = \eta^* \in \mathrm{PSH}(\Omega)$. □

1.3 Plurifine topology

In this section, we introduce the notion of plurifine topology. Unlike other sections in this chapter, this section contains full details. This is mainly due to the unfortunate omissions and numerous minor problems in the foundational paper [BT87]. In particular, the key theorem Theorem 1.3.2 was claimed by Bedford–Taylor without proof. Our presentation largely follows [EMW06]. We also include enough details so that this section is readable for those who are not familiar with the classical potential theory.

Very recently (after the submission of this book to Springer), El Kadiri–Fuglede published a book [EKF25][7] about classical fine potential theory, containing essentially all results in this section. The interested readers are encouraged to read their book for further details.

1.3.1 Plurifine topology on domains

Let $\Omega \subseteq \mathbb{C}^n$ $(n \in \mathbb{N})$ be a domain.

Definition 1.3.1 The *plurifine topology* on Ω is the weakest topology making all $\mathbb{R}$-valued plurisubharmonic functions on Ω continuous.

We want to distinguish the Euclidean topology from the plurifine topology. In the whole book, topological notions without adjectives refer to those with respect to the Euclidean topology. We include the symbol $\mathcal{F}$ in order to denote those with respect to the plurifine topology. For example, we will say $\mathcal{F}$-open subset, $\mathcal{F}$-neighborhood, $\mathcal{F}$-closure, etc. The $\mathcal{F}$-closure of a set $E \subseteq \Omega$ will be denoted by $\bar{E}^{\mathcal{F}}$.

We remind the readers that in the whole book, we follow Bourbaki's convention, a neighborhood is not necessarily open. Similarly, an $\mathcal{F}$-neighborhood is not necessarily $\mathcal{F}$-open.

A priori, we should include Ω into the notations as well, but as we will see shortly in Corollary 1.3.1, this is usually unnecessary.

Proposition 1.3.1 *The plurifine topology on Ω is finer than the Euclidean topology.*

Proof It suffices to show that the unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ is $\mathcal{F}$-open. This follows from the observation that this set can be written as

$$\{\psi < 0\} \text{ with } \psi(z) := (\log |z|) \vee (-1).$$

[7] An anonymous referee kindly pointed me to this reference in a rather enigmatic way — instead of revealing the title or the authors of the book, he/she merely informed me that, because of the existence of a book on this subject, my entire section was meaningless.

Example 1.3.1 Let $\varphi \in \mathrm{PSH}(\Omega)$ and $C \in \mathbb{R}$. Then the sets $\{\varphi > C\}$ and $\{\varphi < C\}$ are both $\mathcal{F}$-open.

In fact, the later case follows from Proposition 1.3.1. While the former follows from the observation

$$\{\varphi > C\} = \{\varphi \vee (C-1) > C\}.$$

Definition 1.3.2 A subset $E \subseteq \Omega$ is *thin*[8] at $x \in \Omega$ if one of the following conditions holds:

(1) $x \notin \bar{E}$;
(2) $x \in \bar{E}$ and there is an open neighborhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y \to x, y \in E \setminus \{x\}} \varphi(y) < \varphi(x).$$

We say E is *thin* if it is thin at all $x \in \Omega$.

Remark 1.3.1 In the second case, we can always arrange that

$$\varphi|_{(E\setminus\{x\})\cap U}$$

is a constant. In fact, we may assume that $\varphi \leq 0$ and $C < 0$ is such that

$$\varlimsup_{y \to x, y \in E \setminus \{x\}} \varphi(y) < C < \varphi(x).$$

We let

$$\psi = (-C)^{-1}(\varphi \vee C) + 1.$$

Then ψ satisfies our requirements for a smaller U.

In the second case, the function φ can be very much improved.

Proposition 1.3.2 (Bedford–Taylor) *Consider a set $E \subseteq \Omega$ and $x \in \bar{E}$. Assume that E is thin at x, then there is $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ with the following properties:*

(1) *φ is locally bounded outside a neighborhood of x;*
(2) *$\varphi(x) > -\infty$;*
(3) $\varlimsup_{y \to x, y \in E \setminus \{x\}} \varphi(y) = -\infty$.

Proof [9] By Remark 1.3.1, there is an open neighborhood $U \subseteq \Omega$ of x and $\psi \in \mathrm{PSH}(U)$ such that

$$\psi|_{U \cap (E \setminus \{x\})} = -1 < \psi(x) = 0.$$

Without loss of generality, we may assume that $x = 0$, U is the unit ball in $\mathbb{C}^n$.

[8] A more proper name would be *plurithin*. But since we will never need the classical notion of thin sets à la Cartan in this book, we prefer omitting the prefix *pluri-*.

[9] The original argument in [BT82, Proposition 10.2] was quite intriguing: Neither the auxiliary functions φ_j's nor the simple computations were correct. However, I believe that Bedford–Taylor had a correct proof in mind. Something more than a typo, but not yet a mistake, could be properly called a *thinkpo*, a terminology invented by R. Berman.

As ψ is upper semicontinuous, we may choose a decreasing sequence $\delta_j \in (0,1)$ such that $\psi(y) < 2^{-j-2}$ when $y \in \mathbb{C}^n$ satisfies $|y| < \delta_j$. Set

$$\gamma_j := \exp\left(2^{j+1}\log\delta_j\right) \in (0,1).$$

Observe that γ_j is also decreasing.

We let

$$\varphi_j(z) := \begin{cases} \left(\dfrac{2^{-j-1}}{|\log\delta_j|}\log|z|\right) \vee \left(\psi(z) - 2^{-j}\right), & \text{if } |z| < \delta_j, \\ \dfrac{2^{-j-1}}{|\log\delta_j|}\log|z|, & \text{if } |z| \geq \delta_j. \end{cases}$$

Observe that when $|z|$ is sufficiently close to δ_j from below (depending on j), we have

$$\frac{2^{-j-1}}{|\log\delta_j|}\log|z| > 2^{-j-2} - 2^{-j} > \psi(z) - 2^{-j}.$$

In particular, thanks to Lemma 1.2.2, $\varphi_j \in \mathrm{PSH}(\mathbb{C}^n)$ and $\varphi_j|_U \leq 0$. Moreover, we have

$$\varphi_j(0) = -2^{-j}.$$

Observe that for $z \in U \cap (E \setminus \{0\})$ with $|z| < \gamma_j$, we have $\varphi_j(z) \leq -1$.

We then define

$$\varphi := \sum_{j=1}^{\infty} \varphi_j.$$

Since

$$\varphi(0) = -\sum_{j=1}^{\infty} 2^{-j} > -\infty, \quad \sum_{j=1}^{\infty} \frac{2^{-j-1}}{|\log\delta_j|} < \infty,$$

we have $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$. Moreover, fix j, for any $z \in E \setminus \{0\}$ with $|z| < \gamma_j$, we have

$$\varphi(z) \leq \sum_{k=1}^{j} \varphi_k(z) \leq -j.$$

Therefore,

$$\varlimsup_{y\to x, y\in E\setminus\{0\}} \varphi(y) = -\infty.$$

Lemma 1.3.1 *Let $E_1, E_2 \subseteq \Omega$. Assume that E_1, E_2 are both thin at $x \in \Omega$, then so is $E_1 \cup E_2$.*

Proof We may assume that $x \in \overline{E_1} \cap \overline{E_2}$. Take an open neighborhood $U \subseteq \Omega$ of x and $\varphi_1, \varphi_2 \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y\to x, y\in E_i\setminus\{x\}} \varphi_i(y) < \varphi_i(x), \quad i = 1, 2.$$

Then $\varphi_1 + \varphi_2 \in \mathrm{PSH}(U)$ and

$$\varlimsup_{y\to x, y\in (E_1\cup E_2)\setminus\{x\}} (\varphi_1 + \varphi_2)(y) < \varphi_1(x) + \varphi_2(x).$$

In particular, $E_1 \cup E_2$ is thin at x. □

Theorem 1.3.1 (H. Cartan) *Consider $x \in \Omega$ and a set $E \subseteq \Omega$. Assume that $x \in E$. Then the following are equivalent:*

(1) *E is an $\mathcal{F}$-neighborhood of x;*
(2) *$\Omega \setminus E$ is thin at x.*

Proof (2) $\implies$ (1). We may assume that $x \in \overline{\Omega \setminus E}$. Otherwise, our assertion follows from Proposition 1.3.1.

By Proposition 1.3.2, there is an open neighborhood U of x in Ω and $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ such that

$$\varphi(x) > \sup_{y\in U\cap(\Omega\setminus E)} \varphi(y) =: \lambda.$$

Let $F = \{y \in \Omega : \varphi(y) > \lambda\}$. Then $x \in F$ and F is $\mathcal{F}$-open by Example 1.3.1. Moreover, $U \cap F \subseteq E$. By Proposition 1.3.1, we conclude (1).

(1) $\implies$ (2). We may always replace E by smaller $\mathcal{F}$-neighborhoods of x. In particular, we may assume that E has the following form

$$\{y \in U : \varphi_1(y) > \lambda_1, \dots, \varphi_m(y) > \lambda_m\},$$

where $U \subseteq \Omega$ is an open neighborhood of x, and $\varphi_1, \dots, \varphi_m$ are $\mathbb{R}$-valued psh functions on Ω, and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since a finite union of thin sets is still thin by Lemma 1.3.1, we may assume that $m = 1$. In this case, $\Omega \setminus E$ is clearly thin at x. □

Theorem 1.3.2 *A base of the plurifine topology on Ω is given by sets of the following form:*

$$\{x \in U : \varphi(x) > 0\}, \tag{1.4}$$

where $U \subseteq \Omega$ is an open subset and $\varphi \in \mathrm{PSH}(U)$.

Proof Observe that sets of the form (1.4) are $\mathcal{F}$-open.[10] By Theorem 1.3.1, it suffices to show its complement in Ω is thin at each point of (1.4), which is obvious.

Now consider $x \in \Omega$ and an $\mathcal{F}$-open neighborhood $V \subseteq \Omega$ of x. We want to find a set of the form (1.4) contained in V and containing x.

Write $E = \Omega \setminus V$. In case $x \in \operatorname{Int} V$, there is nothing to prove. So we may assume that $x \in \bar{E}$. By Theorem 1.3.1, E is thin at x. By definition, there is an open neighborhood $U \subseteq \Omega$ of x and $\varphi \in \mathrm{PSH}(U)$ such that

$$\varlimsup_{y\to x, y\in U\cap(E\setminus\{x\})} \varphi(y) < \varphi(x).$$

We may assume that $\varphi|_{E\cap U} \le 0 < \varphi(x)$, Then the set $\{y \in U : \varphi(y) > 0\}$ suffices for our purpose. □

[10] This is not entirely obvious by definition, as φ is not defined on the whole Ω.

Remark 1.3.2 We remind the readers that in general, an $\mathcal{F}$-open set is *not* a *countable* union of sets of the form (1.4). In fact, an $\mathcal{F}$-open set is not a Borel set in general. See [EK23] for a concrete example.

Corollary 1.3.1 *Let $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n$ be two non-empty open subsets. Then the plurifine topology on Ω_1 is the same as the subspace topology induced from the plurifine topology on Ω_2.*

In particular, when we talk about an $\mathcal{F}$-open set U in $\mathbb{C}^n$, we no longer have to specify the domain $\Omega \supseteq U$.

Corollary 1.3.2 *Let L be an affine subspace of $\mathbb{C}^n$, then the plurifine topology on L is the same as the subspace topology induced from the plurifine topology on $\mathbb{C}^n$.*

Proof We may assume that $L = \mathbb{C}^k \times \{0\}$ for some $k \leq n$. We write the coordinate z on $\mathbb{C}^n$ as (z', z'') with $z \in \mathbb{C}^k$ and $z'' \in \mathbb{C}^{n-k}$.

Consider an $\mathcal{F}$-open set $U \subseteq \mathbb{C}^n$ and $x = (x', 0) \in U \cap L$. We want to show that $U \cap L$ (identified with a subset of $\mathbb{C}^k$) is an $\mathcal{F}$-neighborhood of x' in L. By Theorem 1.3.2, we may assume that there are connected open subsets $U' \subseteq \mathbb{C}^k$ containing x' and $U'' \subseteq \mathbb{C}^{n-k}$ containing 0 together with a psh function ψ on $U' \times U''$ such that

$$x \in \{(z', z'') \in U' \times U'' : \psi(z', z'') > 0\} \subseteq \Omega.$$

It follows that

$$x' \in \{z' \in U' : \psi(z', 0) > 0\} \subseteq U \cap L.$$

Thanks to Proposition 1.1.1, $\psi(z', 0)$ is plurisubharmonic in z' because $\psi(x', 0) \neq -\infty$. In particular, $U \cap L$ is an $\mathcal{F}$-neighborhood of x'.

Conversely, if $U \subseteq \mathbb{C}^k$ is an $\mathcal{F}$-open subset, we claim that $U \times \mathbb{C}^{n-k}$ is $\mathcal{F}$-open in $\mathbb{C}^n$. In fact, suppose that $(x', x'') \in U \times \mathbb{C}^{n-k}$. By Theorem 1.3.1, we can find an open neighborhood $V \subseteq \mathbb{C}^k$ of x' and a psh function φ on V such that

$$x' \in \{y \in V : \varphi(y) > 0\} \subseteq U.$$

We define $\psi(z', z'') \coloneqq \varphi(z')$. Then $\psi \in \mathrm{PSH}(V \times \mathbb{C}^{n-k})$ by Proposition 1.1.1 and

$$(x', x'') \in \{y \in V \times \mathbb{C}^n : \psi(y) > 0\} \subseteq U \times \mathbb{C}^{n-k}.$$

Corollary 1.3.3 *Let $\Omega \subseteq \mathbb{C}^n$ be an $\mathcal{F}$-open subset and $x \in \Omega$. Then x has a compact $\mathcal{F}$-neighborhood contained in Ω.*

Proof By Theorem 1.3.2, we may assume that there is an open set $U \subseteq \mathbb{C}^n$ and a plurisubharmonic function φ on U such that $\Omega = \{y \in U : \varphi(y) > 0\}$.

Take a compact neighborhood K of x in U. Now $\{y \in K : \varphi(y) \geq \varphi(x)/2\}$ is a compact $\mathcal{F}$-neighborhood of x contained in Ω. □

Corollary 1.3.4 *Let $\Omega \in \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^{n'}$ be two domains and $F\colon \Omega' \to \Omega$ be a surjective holomorphic map. Then F is $\mathcal{F}$-continuous.*

Proof It suffices to show that the inverse image $F^{-1}(U)$ of each $\mathcal{F}$-open subset $U \subseteq \Omega$ is $\mathcal{F}$-open. By Theorem 1.3.2, after possibly shrinking Ω and Ω', we may assume that U has the form $\{x \in \Omega : \psi(x) > 0\}$, where $\psi \in \mathrm{PSH}(\Omega)$. Since $F^*\psi \in \mathrm{PSH}(\Omega')$ by Proposition 1.1.4, we find that

$$F^{-1}(U) = \{y \in \Omega' : F^*\psi(y) > 0\}$$

is $\mathcal{F}$-open. □

1.3.2 Plurifine topology on manifolds

Let X be a complex manifold.

Definition 1.3.3 The *plurifine topology* on X is the topology with a base consisting of sets of the form $F^{-1}(V)$, where $U \subseteq X$ is an open subset and $F\colon U \to \Omega$ is a biholomorphic morphism with $\Omega \subseteq \mathbb{C}^n$ is a domain for some $n \in \mathbb{N}$ and $V \subseteq \Omega$ is $\mathcal{F}$-open.

Note that these sets form a topological base thanks to Corollary 1.3.4. Moreover, it also follows from Corollary 1.3.4 that the plurifine topologies on domains defined in Definition 1.3.3 and in Definition 1.3.1 coincide.

We refer to Definition 1.5.1 for the notion of quasi-plurisubharmonic functions.

Proposition 1.3.3 *Let $\varphi \in \mathrm{QPSH}(X)$, then $\varphi|_{\{\varphi \neq -\infty\}}$ is $\mathcal{F}$-continuous.*

Proof The problem is local, so we may assume that $X \subseteq \mathbb{C}^n$ is a domain and $\varphi = \psi + g$, where $\psi \in \mathrm{PSH}(X)$ and $g \in C^\infty(X)$ and $|g| \leq C$ for some $C > 0$. Take an open interval $(a, b) \subseteq \mathbb{R}$, it suffices to show that

$$U := \{x \in X : a < \varphi(x) < b\} = \{x \in X : a - g(x) < \psi(x) < b - g(x)\}$$

is $\mathcal{F}$-open. Take $x \in U$, we can find an open neighborhood V of x in U such that

$$\sup_{y \in V} (a - g(y)) < \psi(x) < \inf_{y \in V} (b - g(y)) .$$

Therefore,

$$\left\{ z \in V : \sup_{y \in V} (a - g(y)) < \psi(z) < \inf_{y \in V} (b - g(y)) \right\}$$

is an $\mathcal{F}$-open neighborhood of z in U. We conclude that U is $\mathcal{F}$-open. □

Corollary 1.3.5 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Then the set*

$$\{x \in X : \varphi(x) > \psi(x)\}$$

is $\mathcal{F}$-open.

Proof It suffices to show that for any $x \in X$ such that $\varphi(x) > \psi(x)$, the same holds on an $\mathcal{F}$-neighborhood U of x. Observe that $\varphi(x) \neq -\infty$. If $\psi(x) \neq -\infty$, then it suffices to apply Proposition 1.3.3. Otherwise, take

$$U := \{y \in X : \varphi(y) > \varphi(x) - 1\} \cap \{y \in X : \psi(y) < \varphi(x) - 1\}.$$

Lemma 1.3.2 *Let $Z \subseteq X$ be a pluripolar subset. Then*

$$\overline{X \setminus Z}^{\mathcal{F}} = X.$$

Proof The problem is local, so we may assume that X is a domain in $\mathbb{C}^n$ and $Z = \{\varphi = -\infty\}$ for some $\varphi \in \mathrm{PSH}(X)$. We need to show that $\{\varphi > -\infty\}$ is $\mathcal{F}$-dense.

Let $x \in X$ be a point with $\varphi(x) = -\infty$ and $U \subseteq X$ be an $\mathcal{F}$-open neighborhood of x in X. We need to show that $U \cap \{\varphi > -\infty\} \neq \emptyset$.

Thanks to Theorem 1.3.2, after shrinking U, we may assume that there is $\psi \in \mathrm{PSH}(X)$ such that $U = \{\psi > 0\}$. Observe that U is not a pluripolar set: Otherwise, $\psi \leq 0$ almost everywhere by Proposition 1.2.4, and hence everywhere by Proposition 1.2.6. So $\varphi|_U \not\equiv -\infty$. We conclude. □

Corollary 1.3.6 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Set*

$$W = \{x \in X : \varphi(x) = -\infty\} \text{ or } W = \{x \in X : \psi(x) = -\infty\}.$$

Then for any pluripolar set $Z \subseteq X$, we have

$$\sup_{X \setminus W} (\varphi - \psi) = \sup_{X \setminus W \cup Z} (\varphi - \psi), \quad \inf_{X \setminus W} (\varphi - \psi) = \inf_{X \setminus W \cup Z} (\varphi - \psi).$$

In particular, taking $\psi = 0$, we find that

$$\sup_{X \setminus Z} \varphi = \sup_X \varphi.$$

Proof This is an immediate consequence of Lemma 1.3.2 and Proposition 1.3.3. □

In the literature about pluripotential theory, one often finds the careless expressions like $\sup_X (\varphi - \psi)$. The issue is that $\varphi - \psi$ is not defined everywhere, and hence this expression does not make sense if you read it literally. Corollary 1.3.6 tells you that you do not have to worry too much about the details on a pluripolar set. In other words, sup and inf could always be understood as a kind of essential supremum and essential infimum modulo pluripolar sets.

There is a convenient way to fix this issue in the literature — Just replace the suprema by quasi-suprema:

Definition 1.3.4 Let $Z \subseteq X$ be a pluripolar set, and $f : X \setminus Z \to [-\infty, \infty]$ be a function. We define the *quasi-supremum* and the *quasi-infimum* of f as follows:

$$\operatorname*{q\text{-}sup}_X f \coloneqq \inf\{\sup_{X\setminus W} : W \supseteq Z \text{ is pluripolar}\},$$
$$\operatorname*{q\text{-}inf}_X f \coloneqq \sup\{\inf_{X\setminus W} : W \supseteq Z \text{ is pluripolar}\}.$$

For two functions f and g equal quasi-everywhere, the quasi-suprema and quasi-infima of them are equal, as is clear from the definition.

For a quasi-plurisubharmonic function φ, we have

$$\operatorname*{q\text{-}sup}_X \varphi = \sup_X \varphi,$$

as a consequence of Corollary 1.3.6.

1.4 Lelong numbers and multiplier ideal sheaves

In this section, we briefly recall the notions of Lelong numbers and multiplier ideal sheaves. Our presentation is by no means intended to be complete. The readers are encouraged to read the textbooks [GZ17, Section 2.3] and [Dem12a].

Let X be a complex manifold.

Definition 1.4.1 Let $\varphi \in \mathrm{PSH}(X)$ and $x \in X$. The *Lelong number* $\nu(\varphi, x)$ of φ at x is defined as follows: Take an open neighborhood U of x in X and a biholomorphic map $F : U \to \Omega$, where Ω is a domain in $\mathbb{C}^n$. Then we define

$$\nu(\varphi, x) \coloneqq \sup\left\{\gamma \in \mathbb{R}_{\geq 0} : \varphi|_U(F^{-1}(y)) \leq \gamma \log|y - F(x)|^2 + O(1) \text{ as } y \to F(x)\right\}. \tag{1.5}$$

Observe that $\nu(\varphi, x)$ does not depend on the choices of U and F. Furthermore, it follows from Proposition 1.4.1 below that the supremum in (1.5) is a maximum.

Remark 1.4.1 Our definition of the Lelong number is not standard. It differs from the standard definition by a factor of 2. As a mnemonic, just remember

$$\nu\left(\log|z|^2, 0\right) = 1 \quad \text{(instead of 2)}.$$

Our convention of the Lelong numbers together with the convention of the multiplier ideal sheaves below make sure that (1.8) has no extra factors.

These normalizations together with the normalization of $\mathrm{dd^c}$ as

$$\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}$$

guarantees that in Theorem 7.4.1, there are no ugly factors.

Proposition 1.4.1 *Let $\varphi \in \mathrm{PSH}(B_n(0,1))$. Then*

$$\nu(\varphi,0) = \lim_{r\searrow 0} \frac{\sup_{B_n(0,r)} \varphi}{\log r^2} \in [0,\infty). \tag{1.6}$$

Proof It follows from Proposition 1.1.2 that the limit in (1.6) exists and is finite. We shall denote the limit by $\nu'(\varphi,0)$ for the time being.

We first observe that by Proposition 1.1.2,

$$\varphi(x) \leq \nu'(\varphi,0)\log|x|^2 + \sup_{B_n(0,1)} \varphi \tag{1.7}$$

when $x \in B_n(0,1)$. In particular, $\nu(\varphi,x) \geq \nu'(\varphi,0)$.

In order to argue the reverse inequality, we may assume that $\nu(\varphi,x) > 0$.

Next observe that by (1.5), for each small enough $\epsilon > 0$, we can find $r_0 \in (0,1)$ and $C > 0$ so that for all $x \in B_n(0,r_0)$, we have

$$\varphi(x) \leq (\nu(\varphi,0)-\epsilon)\log|x|^2 + C.$$

It follows that $\nu'(\varphi,0) \geq \nu(\varphi,0)-\epsilon$. Letting $\epsilon \to 0+$, we conclude. □

We recall Siu's semicontinuity theorem.

Theorem 1.4.1 (Siu) *Let $\varphi \in \mathrm{PSH}(X)$, then the map $X \ni x \mapsto \nu(\varphi,x)$ is upper semi-continuous with respect to the Zariski topology.*

For an elegant proof we refer to [Dem12a, Theorem 2.10].

Proposition 1.4.2 *Let $\varphi,\psi \in \mathrm{PSH}(X)$, $\lambda \in \mathbb{R}_{>0}$ and $x \in X$, then*

$$\begin{aligned}
\nu(\varphi \vee \psi, x) &= \min\{\nu(\varphi,x),\nu(\psi,x)\},\\
\nu(\varphi+\psi,x) &= \nu(\varphi,x)+\nu(\psi,x),\\
\nu(\lambda\varphi,x) &= \lambda\nu(\varphi,x).
\end{aligned}$$

Proof All properties are local, so we may assume that $X = B_n(0,1)$ for some $n \in \mathbb{N}$. All properties follow directly from Proposition 1.4.1. □

Corollary 1.4.1 *Let $(\varphi_i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}(X)$ locally uniformly bounded from above and $x \in X$, then*

$$\nu\left(\sup_{i\in I}{}^*\varphi_i, x\right) = \inf_{i\in I} \nu(\varphi_i,x).$$

Proof We may assume that X is connected. Write $\varphi = \sup_{i\in I}{}^*\varphi_i$. Then $\varphi \in \mathrm{PSH}(X)$ by Proposition 1.2.1.

We observe that the $\leq$ inequality is trivial. It remains to argue the reverse inequality.

It follows from Proposition 1.2.2 that we may assume that I is countable. When I is finite, this is already proved in Proposition 1.4.2. Otherwise, we may further assume that $I = \mathbb{Z}_{>0}$. Thanks to Proposition 1.4.2, we may further assume that $(\varphi_i)_{i\in\mathbb{Z}_{>0}}$ is an increasing sequence. Furthermore, since the problem is local, we may assume that $X = B_n(0,1)$ for some $n \in \mathbb{N}$ and $(\varphi_i)_i$ is uniformly bounded from above. In this case, by (1.7), we have

$$\varphi_i(x) \leq \nu(\varphi_i, 0) \log |x|^2 + C$$

for all $x \in B_n(0,1)$ and all $i \geq 1$ and C is a constant independent of i. In particular, thanks to Proposition 1.2.5, for almost all $x \in B_n(0,1)$, we have

$$\varphi(x) \leq \lim_{i\to\infty} \nu(\varphi_i, 0) \log |x|^2 + C.$$

Thanks of Proposition 1.2.6, the same holds for all x and hence

$$\nu(\varphi, x) \geq \lim_{i\to\infty} \nu(\varphi_i, x).$$

Definition 1.4.2 Let $F \subseteq X$ be a non-empty analytic subset. Then we define the *generic Lelong number* of φ along F as

$$\nu(\varphi, F) \coloneqq \min_{x\in F} \nu(\varphi, x).$$

Note that the minimum is attained by Theorem 1.4.1.

Definition 1.4.3 Let $\varphi \in \mathrm{PSH}(X)$. Let E be a prime divisor over X (see Definition B.1.1). Take a proper bimeromorphic morphism $\pi\colon Y \to X$ from a complex manifold Y such that E is a prime divisor on Y, then we define the *generic Lelong number* of φ along E as

$$\nu(\varphi, E) \coloneqq \nu(\pi^*\varphi, E).$$

It follows from Theorem 1.4.1 that $\nu(\varphi, E)$ does not depend on the choice of π.

Definition 1.4.4 Let $\varphi \in \mathrm{PSH}(X)$, the *multiplier ideal sheaf* $\mathcal{I}(\varphi)$ of φ is by definition the ideal sheaf given by

$$\Gamma(U, \mathcal{I}(\varphi)) = \left\{ f \in \mathcal{O}_X(U) : |f|^2 \exp(-\varphi) \in L^1_{\mathrm{loc}}(U) \right\}$$

for any open subset $U \subseteq X$.

Remark 1.4.2 This definition is different from a few references, where instead of $\exp(-\varphi)$, they use $\exp(-2\varphi)$. The conventions adopted in the current book is the most convenient one as far as I know. It simplifies a number of formulae. As a mnemonic, for any real $\lambda > 0$, we have

$$\mathcal{I}(\lambda \log |z|^2) = \mathcal{O}_{\mathbb{C}}(-\lfloor\lambda\rfloor\{0\}) \quad \text{(instead of } \mathcal{O}_{\mathbb{C}}(-\lfloor 2\lambda\rfloor\{0\})\text{)},$$

where z is a variable in $\mathbb{C}$ and $\{0\}$ is the divisor defined by $0 \in \mathbb{C}$.

Proposition 1.4.3 (Nadel) *Let $\varphi \in \mathrm{PSH}(X)$. Then $\mathcal{I}(\varphi)$ is coherent.*

See [Dem12a, Proposition 5.7].

Theorem 1.4.2 *Let $\varphi, \psi \in \mathrm{PSH}(X)$, then*

$$\mathcal{I}(\varphi + \psi) \subseteq \mathcal{I}(\varphi) \cdot \mathcal{I}(\psi).$$

See [Dem12a, Theorem 14.2].

The two invariants are related by the following simple result:

Proposition 1.4.4 *Let $\varphi \in \mathrm{PSH}(X)$ and E be a prime divisor over X. Then*

$$\nu(\varphi, E) = \lim_{k\to\infty} \frac{1}{k} \operatorname{ord}_E \mathcal{I}(k\varphi). \tag{1.8}$$

See [DX24b, Proposition 2.14].

We remind the readers that this particular form of the formula is compatible with our conventions of ν and $\mathcal{I}$. As a consistency check, consider $\varphi = \log|z|^2$ with $z \in \mathbb{C}$ and E is the divisor defined by $0 \in \mathbb{C}$. Then both sides of (1.8) are equal to 1. See Remark 1.4.1 and Remark 1.4.2.

Also observe the following simple lemma:

Lemma 1.4.1 *Let $x \in X$ and $\varphi \in \mathrm{PSH}(X)$. Let $\pi\colon Y \to X$ be the blow-up of X at x with exceptional divisor E. Then*

$$\nu(\varphi, x) = \nu(\varphi, E),$$

See [Bou02b, Corollaire 1.1.8].

Conversely, the information of the generic Lelong numbers determines the multiplier ideal sheaves:

Theorem 1.4.3 *Let $\varphi \in \mathrm{PSH}(X)$. Let $x \in X$ and $f \in \mathcal{O}_{X,x}$. Then the following are equivalent:*

(1) $f \in \mathcal{I}(\varphi)_x$;
(2) *there exists $\epsilon > 0$ such that for any prime divisor E over X such that x is contained in the center of E on X, we have*

$$\operatorname{ord}_E(f) \geq (1+\epsilon)\nu(\varphi, E) - \frac{1}{2}A_X(E). \tag{1.9}$$

In case φ has analytic singularities and $\pi\colon Y \to X$ is a log resolution of φ (see Definition 1.6.3 for the definition) with finitely many exceptional divisors $\{E_i\}$ whose centers on X contain x, one may replace (1.9) by

$$\operatorname{ord}_{E_i}(f) > \nu(\varphi, E_i) - \frac{1}{2}A_X(E_i) \quad \forall i. \tag{1.10}$$

Here A_X denotes the log discrepancy. We refer to [Bou17, Corollary 10.18, Proposition 10.12] for the proof and the precise definition of A_X. The formula (1.9) differs from that in Boucksom's notes: The coefficient $\frac{1}{2}$ in front of $A_X(E)$ arises from our convention for ν and $\mathcal{I}$.

The notion of analytic singularities is recalled in Section 1.6.

Theorem 1.4.4 (Guan–Zhou) *Let $\varphi, \psi_j \in \mathrm{PSH}(X)$ ($j \in \mathbb{Z}_{>0}$) such that ψ_j is an increasing sequence converging to φ almost everywhere. Then for any $x \in X$, the germs satisfy*

$$\mathcal{I}(\psi_j)_x = \mathcal{I}(\varphi)_x$$

when j is large enough.

See [GZ15, Hie14] for the proof.

Proposition 1.4.5 *Let $\pi\colon Y \to X$ be a smooth morphism between complex manifolds. Assume that $\varphi \in \mathrm{PSH}(X)$, then*

$$\mathcal{I}(\pi^*\varphi) = \pi^*\mathcal{I}(\varphi).$$

Proof It follows from [Gro60, Théorème 3.10] that locally π can be written as the composition of an étale morphism and a projection. It suffices to handle the two cases separately.

Recall that in the complex analytic setting, an étale morphism is locally biholomorphic, so there is nothing to prove in this case.

Next, assume that $Y = X \times U$, where $U \subseteq \mathbb{C}^n$ is a domain and π is the natural projection. It follows from Fubini's theorem that

$$\mathcal{I}(\pi^*\varphi) \subseteq \pi^*\mathcal{I}(\varphi).$$

The reverse inequality is proved in [Dem12a, Proposition 14.3][11]. □

Definition 1.4.5 Given a coherent ideal sheaf $\mathcal{I}$ on X, the *restriction* $\mathrm{Res}_Y \mathcal{I}$ is the inverse image ideal sheaf given by

$$\mathrm{Res}_Y \mathcal{I} := \mathcal{I}/(\mathcal{I} \cap \mathcal{I}_Y), \tag{1.11}$$

where $\mathcal{I}_Y$ is the ideal sheaf defining Y.

In the literature, it is common to denote this sheaf by the misleading notation $\mathcal{I}|_Y$.

There is a natural morphism

$$i_Y^*\mathcal{I} = \mathcal{I}/(\mathcal{I} \cdot \mathcal{I}_Y) \to \mathrm{Res}_Y \mathcal{I}, \tag{1.12}$$

where $i_Y\colon Y \to X$ is the inclusion.

Theorem 1.4.5 (Ohsawa–Takegoshi) *Let Y be a connected submanifold of X and $\varphi \in \mathrm{PSH}(X)$. Assume that $\varphi|_Y \not\equiv -\infty$, then*

$$\mathcal{I}(\varphi|_Y) \subseteq \mathrm{Res}_Y \mathcal{I}(\varphi).$$

[11] In [Dem12a, Proposition 14.3], Demailly used the highly non-standard notation $f^*\mathcal{I}(\varphi)$ to denote the image of $f^*\mathcal{I}(\varphi) \to \mathcal{O}_X$, even when f is not flat.

See [Dem12a, Theorem 14.1].

1.5 Quasi-plurisubharmonic functions

In practice, it is important to consider a variant of plurisubharmonic functions. We will fix a complex manifold X.

Definition 1.5.1 Let θ be a closed real smooth $(1,1)$-form on X.

A *θ-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that for each $x \in X$ and each open neighborhood U of x in X satisfying the condition that $\theta = \mathrm{dd^c} g$ for some smooth function g on U, we have $g + \varphi|_U \in \mathrm{PSH}(U)$. The set of θ-psh functions on X is denoted by $\mathrm{PSH}(X,\theta)$.

A *quasi-plurisubharmonic function* on X is a function $\varphi\colon X \to [-\infty,\infty)$ such that there exists a smooth closed real $(1,1)$-form θ' on X such that $\varphi \in \mathrm{PSH}(X,\theta')$. The set of quasi-plurisubharmonic functions on X is denoted by $\mathrm{QPSH}(X)$.

There is a natural non-strict partial order on $\mathrm{QPSH}(X)$ defined as follows:

Definition 1.5.2 Assume that X is compact. Given $\varphi,\psi \in \mathrm{QPSH}(X)$, we say that φ is *more singular* than ψ and write $\varphi \preceq \psi$[12] if there is $C \in \mathbb{R}$ such that $\varphi \leq \psi + C$. We also say ψ is less singular than φ and write $\psi \succeq \varphi$.

In case $\varphi \preceq \psi$ and $\psi \preceq \varphi$, we say φ and ψ have the same *singularity type*. We write $\varphi \sim \psi$ in this case.

When X is not compact, one can still define similar notions, but the generalization is not unique, and we shall not consider them in this book.

Remark 1.5.1 The proceeding results concerning plurisubharmonic functions can be extended *mutatis mutandis* to quasi-plurisubharmonic functions. We will apply these extensions without further explanations.

Proposition 1.5.1 *Assume that X is compact. Let θ be a closed real smooth $(1,1)$-form on X. Then for any $a,b \in \mathbb{R}$, $a \leq b$, the set*

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_X \varphi \in [a,b]\right\}$$

is compact with respect to the L^1-topology. Moreover, $\varphi \mapsto \sup_X \varphi$ is L^1-continuous for $\varphi \in \mathrm{PSH}(X,\theta)$.

This is an immediate consequence of [GZ17, Proposition 8.5, Exercise 1.20].

Remark 1.5.2 More generally, if $K \subseteq X$ is a closed non-pluripolar subset. Then

$$\left\{\varphi \in \mathrm{PSH}(X,\theta) : \sup_K \varphi \in [a,b]\right\}$$

[12] Some people write $\psi \preceq \varphi$.

is relatively compact with respect to the L^1-topology. See [GZ05, Corollary 4.3].

Proposition 1.5.2 *Assume that X is compact. Let θ be a closed real smooth $(1,1)$-form on X and E be a prime divisor over E. Then*

$$\sup\{\nu(\varphi, E) : \varphi \in \mathrm{PSH}(X,\theta)\} < \infty.$$

Proof It follows from the proof of Corollary 1.4.1 that $\nu(\bullet, E)$ is upper semi-continuous with respect to the L^1-topology on $\mathrm{PSH}(X,\theta)$. Thus, the desired upper bound follows from Proposition 1.5.1. □

Proposition 1.5.3 *Let $\pi : Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Let θ be a closed real smooth $(1,1)$-form on X. Then the pull-back gives a bijection*

$$\pi^* : \mathrm{PSH}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}(Y, \pi^*\theta).$$

This follows from a more general result Theorem B.1.1.

1.6 Analytic singularities

The simplest type of plurisubharmonic singularities is given by the so-called *analytic singularities*. The notion is fairly subtle and there are several mutually *incompatible* definitions in the literature, as we shall explain below.

Let X be a complex manifold.

Definition 1.6.1 We say $\varphi \in \mathrm{QPSH}(X)$ has *analytic singularities* if for each $x \in X$, we can find an open neighborhood U of x such that $\varphi|_U$ has the form:

$$c\log(|f_1|^2 + \cdots + |f_N|^2) + R, \tag{1.13}$$

where $f_1, \ldots, f_N$ are holomorphic functions on U, $c \in \mathbb{Q}_{>0}$ and R is a bounded function on U.

When R can be taken to be smooth[13], we say φ has *neat analytic singularities*.

Suppose that there is a coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ on X such that we can choose U so that the $f_1, \ldots, f_N$ can be chosen as the generators of $\Gamma(U, \mathcal{I})$ and c is independent of the choice of U, we say φ has analytic singularities of *type* $(c, \mathcal{I})$.

Each potential with analytic singularities has a type. The type is not uniquely determined. We refer to [Bou02b] and [Bou02a] for the details.

Some people take $c \in \mathbb{R}_{>0}$ in (1.13). But this is a bad definition because the following proposition, which is essential in constructing Demailly approximations, would then fail.

[13] The decomposition (1.13) is highly non-unique. Here we mean for any x, there is an open neighborhood U and a decomposition of the form (1.13) with R smooth. In the non-trivial cases, R cannot be smooth for all decompositions (1.13).

Proposition 1.6.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$ be potentials with analytic singularities, then so are $\lambda\varphi$ ($\lambda \in \mathbb{Q}_{>0}$), $\varphi + \psi$ and $\varphi \vee \psi$.*

Proof The $\lambda\varphi$ assertion is trivial. The $\vee$ assertion is proved in [Dem15, Proposition 4.1.8]. The addition assertion is easy and is left to the readers. □

Definition 1.6.2 Let D be an effective $\mathbb{Q}$-divisor[14] on X. We say $\varphi \in \mathrm{QPSH}(X)$ has *log singularities* (along D) on X if for each $x \in X$, there is an open neighborhood U of x such that

(1) $D|_U$ has finitely many irreducible components and can be written as

$$D|_U = \sum_{i=1}^{N} a_i D_i$$

with D_i being prime divisors on U, $a_i \in \mathbb{Q}_{>0}$ and there is a holomorphic function s_i on U defining D_i, and

(2) we have

$$\varphi|_U = a_i \sum_{i=1}^{N} \log |s_i|^2 + R, \tag{1.14}$$

where R is a bounded function on U.

By Proposition 1.6.1, φ has analytic singularities.

Lemma 1.6.1 *Suppose that θ is a closed smooth real $(1,1)$-form on X, a compact Kähler manifold and $\varphi \in \mathrm{PSH}(X, \theta)$. Suppose that φ has log singularities along an effective $\mathbb{Q}$-divisor D on X. Then the cohomology class $[\theta] - [D]$ is nef.*

Moreover, if in addition θ_φ is a Kähler current[15]*, then the cohomology class $[\theta] - [D]$ is ample.*

Here and in the sequel, we write θ_φ for $\theta + \mathrm{dd^c}\varphi$.

Proof The first assertion follows immediately from the fact that R in (1.14) has bounded coefficients.

The second assertion follows immediately from the first. □

The following proposition follows immediate from the definitions:

Proposition 1.6.2 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a complex manifold Y. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities (resp. has log singularities along an effective $\mathbb{Q}$-divisor D). Then $\pi^*\varphi$ has analytic singularities (resp. has log singularities along $\pi^* D$).*

Definition 1.6.3 Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. A *log resolution* of φ is a modification $\pi\colon Y \to X$ such that $\pi^*\varphi$ has log singularities.

---

[14] Divisors and $\mathbb{Q}$-divisors are implicitly assumed to have locally finite coefficients as usual.

[15] That is, there is a Kähler form ω on X such that $\theta_\varphi \geq \omega$ in the sense of currents.

See Definition B.1.3 for the notion of modification.

Theorem 1.6.1 *Assume that X is compact. Suppose that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then there is a log resolution of φ.*

For a proof, we refer to the arguments on [MM07, Page 104].

A general quasi-plurisubharmonic function can be nicely approximated by those with analytic singularities. We need a few preliminary definitions.

Definition 1.6.4 Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. A sequence $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$ in $\mathrm{QPSH}(X)$ is *quasi-equisingular approximation* of φ if

(1) φ_j has analytic singularities for each j;
(2) φ_j is decreasing with limit φ;
(3) there is a decreasing sequence $\epsilon_j \geq 0$ with limit 0 and a Kähler form ω on X such that $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$;
(4) for each $\lambda' > \lambda > 0$, there is $j > 0$ such that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi). \tag{1.15}$$

We also say θ_{φ_j} is a quasi-equisingular approximation of θ_φ.

Definition 1.6.5 Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf and $c \in \mathbb{Q}_{>0}$. A function $\varphi \in \mathrm{QPSH}(X)$ is said to have *gentle analytic singularities* (of type $(c,\mathcal{I})$) if

(1) φ has analytic singularities of type $(c,\mathcal{I})$;
(2) $\mathrm{e}^{\varphi/c}\colon X \to \mathbb{R}_{\geq 0}$ is a smooth function;
(3) there is a proper bimeromorphic morphism $\pi\colon \tilde{X} \to X$ from a Kähler manifold $\tilde{X}$ and an effective $\mathbb{Z}$-divisor D on $\tilde{X}$ such that one can write $\pi^*\varphi$ locally as

$$\pi^*\varphi = c\log|g|^2 + h,$$

where g is a local equation of the divisor D and h is smooth.

Theorem 1.6.2 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Then any $\varphi \in \mathrm{PSH}(X,\theta)$ admits a quasi-equisingular approximation $(\varphi_j)_{j\in\mathbb{Z}_{>0}}$.*

Moreover, we can guarantee that for all $j > 0$, φ_j has gentle analytic singularities of type $\left(2^{-j}, \mathcal{I}(2^j\varphi)\right)$.

We refer to [DPS01] for the proof.

Quasi-equisingular approximations are essentially unique in the following sense:

Proposition 1.6.3 *Let X be a compact Kähler manifold and θ be a closed real smooth $(1,1)$-form on X. Consider $\varphi \in \mathrm{PSH}(X,\theta)$. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be two quasi-equisingular approximations of φ. Then for any $\epsilon > 0$ and any $j > 0$, we can find $k_0 > 0$ such that for any $k \geq k_0$, we have*

$$\psi_k \preceq (1-\epsilon)\varphi_j.$$

See [Dem15, Corollary 4.1.7].

Definition 1.6.6 Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then we define $\mathcal{I}_\infty(\varphi)$ as the ideal sheaf consisting of germs f of holomorphic functions such that $|f|^2 \exp(-\varphi)$ is locally bounded.

By definition, $\mathcal{I}_\infty(\varphi) \subseteq \mathcal{I}(\varphi)$.

Lemma 1.6.2 *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. The sheaf $\mathcal{I}_\infty(\varphi)$ is a coherent sheaf.*

Proof By Theorem 1.6.1, we may find a modification $\pi\colon Y \longrightarrow X$ such that $\pi^*\varphi$ has log singularities. Observe that

$$\mathcal{I}_\infty(\varphi) = \pi_*\mathcal{I}(\pi^*\varphi),$$

so we may replace X and φ by Y and $\pi^*\varphi$ and assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. We decompose D into its irreducible components:

$$D = \sum_{i=1}^N a_i D_i.$$

In this case, observe that

$$\mathcal{I}_\infty(\varphi) = \mathcal{O}_X\left(-\sum_{i=1}^N (\lceil a_i \rceil D_i)\right)$$

is clearly coherent. □

The multiplier ideal sheaf $\mathcal{I}$ and the sheaf $\mathcal{I}_\infty$ are not very different in the asymptotic sense, as shown by the following lemma:

Lemma 1.6.3 *Assume that X is compact. Let $\varphi \in \mathrm{QPSH}(X)$ be a potential with analytic singularities. Then for any $\epsilon > 0$, we can find $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$\mathcal{I}(k(1+\epsilon)\varphi) \subseteq \mathcal{I}_\infty(k\varphi). \tag{1.16}$$

Proof We shall prove a more precise local result. Take $x \in X$, take an open neighborhood $U \subseteq X$ of x, on which

$$\varphi = c \log\left(|g_1|^2 + \cdots + |g_N|^2\right) + O(1),$$

where $c \in \mathbb{Q}_{>0}$ and $g_1, \ldots, g_N$ are holomorphic functions on U. Then we claim that for any $\lambda > 0$, we have

$$\mathcal{I}(\lambda\varphi)_x \subseteq \mathcal{I}_\infty\left(\left(\lambda - c^{-1}n\right)_+ \varphi\right)_x, \tag{1.17}$$

where for any $a \in \mathbb{R}$, a_+ means $a \vee 0$. Note that (1.16) is a straigtforward consequence of (1.17).

Fix a smooth volume form $\mathrm{d}V$ on X. Take $f \in \mathcal{I}(\lambda\varphi)$, by strong openness Theorem 1.4.4, we can find $\epsilon > 0$ so that $f \in \mathcal{I}((\lambda + \epsilon/c)\varphi)$. Therefore, we can find an open neighborhood $W \subseteq U$ of x so that

$$\int_W |f|^2 \left(|g_1|^2 + \cdots + |g_N|^2\right)^{-c\lambda-\epsilon} \mathrm{d}V < \infty.$$

But it follows from the Briançon–Skoda division theorem ([Dem12a, Theorem 11.17]) that

$$f_x \in \left(g_{1,x}, \ldots, g_{N,x}\right)^{\alpha},$$

where $\alpha = (\lfloor c\lambda \rfloor - n + 1)_+$. Here $\left(g_{1,x}, \ldots, g_{N,x}\right)$ denotes the ideal in $\mathcal{O}_{X,x}$ generated by the germs of $g_1, \ldots, g_N$ at x. Note that $\alpha \geq (c\lambda - n)_+$.

It follows that on a neighborhood of x, we have

$$\log |f|^2 \leq \alpha \log \left(|g_1|^2 + \cdots + |g_N|^2\right) + O(1) \leq c^{-1}\alpha\varphi + O(1).$$

Hence (1.17) follows. □

Theorem 1.6.3 *Let X be a connected compact Kähler manifold and $Y \subseteq X$ be a connected submanifold. Take a Kähler form ω on X and $\varphi \in \mathrm{PSH}(Y, \omega|_Y)$ such that $\omega|_Y + \mathrm{dd^c}\varphi$ is a Kähler current and that e^{φ} is a Hölder continuous function on V. Then there exists $\tilde{\varphi} \in \mathrm{PSH}(X, \omega)$ satisfying*

(1) *$\tilde{\varphi}|_Y = \varphi$;*
(2) *$\omega_{\tilde{\varphi}}$ is a Kähler current.*

In addition, if φ has analytic singularities, then so does $\tilde{\varphi}$.

See [DRWN⁺23, Theorem 6.1].

1.7 The space of currents

Let X be a connected compact Kähler manifold of dimension n and $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$.

Definition 1.7.1 Let Y be a complex manifold and $m \in \mathbb{N}$. We say an (m, m)-current T on Y is *positive*[16] if either $m > n$ or for any smooth $(1, 0)$-forms $\beta_1, \ldots, \beta_{n-m}$ on X, the measure

$$T \wedge \mathrm{i}\beta_1 \wedge \overline{\beta_1} \wedge \cdots \wedge \mathrm{i}\beta_{n-m} \wedge \overline{\beta_{n-m}}$$

is positive.

The basic properties of positive currents can be found in [Dem12b, Section III.1]. We remind the readers that a positive current is necessarily real.

[16] This notion is sometimes known as *weak positivity*.

Definition 1.7.2 We say α is *pseudo-effective* if there is a closed positive $(1,1)$-current in α.

We say α is *big* if there is a closed positive $(1,1)$-current T in α dominating a Kähler form. Such currents are called *Kähler currents*.

Given classes $\alpha, \beta \in \mathrm{H}^{1,1}(X, \mathbb{R})$, we say $\alpha \leq \beta$ if $\beta - \alpha$ is pseudo-effective.

Definition 1.7.3 We introduce the following notations:

(1) $\mathcal{Z}_+(X)$ denotes the space of closed positive $(1,1)$-currents on X;
(2) given a pseudo-effective $(1,1)$-class α on X, we write $\mathcal{Z}_+(X,\alpha)$ for the set of $T \in \mathcal{Z}_+(X)$ such that $[T] = \alpha$.

Here $[T]$ denotes the cohomology class represented by T.

Definition 1.5.2 has a natural analogue for currents.

Definition 1.7.4 Given $T, T' \in \mathcal{Z}_+(X)$, we write $T \preceq T'$ and say T is *more singular* than T' if when we write $T = \theta + \mathrm{dd^c}\varphi$, $T' = \theta' + \mathrm{dd^c}\varphi'$, we have $\varphi \preceq \varphi'$. We write $T \sim T'$ if $T \preceq T'$ and $T' \preceq T$. In this case, we say T and T' have the same *singularity type*.

Remark 1.7.1 Observe that

$$\mathcal{Z}_+(X)/\sim \,\cong \mathrm{QPSH}(X)/\sim$$

canonically. The correspondence sends the class of a closed positive current $\theta_\varphi = \theta + \mathrm{dd^c}\varphi$ to the class of φ.

We will adopt the following convention: Whenever we have a notion for quasi-plurisubharmonic functions which depends only on the singularity type, we use the same notation and the same definition for closed positive $(1,1)$-currents.

Example 1.7.1 An an important example of Remark 1.7.1, given $T = \theta + \mathrm{dd^c}\varphi \in \mathcal{Z}_+(X)$ and $x \in X$, we define

$$\nu(T,x) = \nu(\varphi, x). \tag{1.18}$$

Again, as Remark 1.4.1, this differs from the definitions in some literature by a factor of 2. But given our normalization

$$\mathrm{dd^c} = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial},$$

(1.18) seems to be the most natural choice.

The key example to keep in mind is the following:

$$\nu([0], 0) = 1,$$

where $[0]$ is the current of integration at $0 \in \mathbb{P}^1$. In fact, as a simple application of the Green's second identity, one can verify that

$$\frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log|z|^2 = \delta_0,$$

where the right-hand side is the Dirac delta distribution at $0 \in \mathbb{C}$.

Definition 1.7.5 Given $T \in \mathcal{Z}_+(X)$. We represent T as $\theta + \mathrm{dd^c}\varphi$ for some closed smooth real $(1,1)$-form θ on X and $\varphi \in \mathrm{PSH}(X,\theta)$, then the *polar locus* of T is defined as the set $\{\varphi = -\infty\}$.

It is clear that the polar locus of T is independent of the choices of θ and φ.

Definition 1.7.6 Assume that α is big. The *non-Kähler locus* $\mathrm{nK}(\alpha)$ of α is the intersection of the polar loci of all Kähler currents with analytic singularities in α.

Theorem 1.7.1 (Boucksom) *Assume that α is big. There is a Kähler current $T \in \alpha$ with analytic singularities, such that the polar locus of T is exactly* $\mathrm{nK}(\alpha)$. *In particular,* $\mathrm{nK}(\alpha)$ *is a proper Zariski closed subset of X.*

See [Bou02b, Théorème 2.1.20].

Definition 1.7.7 Assume that α is big. The *non-nef locus* $\mathrm{nn}(\alpha)$ of α is the following set:

$$\mathrm{nn}(\alpha) \coloneqq \{x \in X : \nu(T_{\min}, x) > 0\},$$

where $T_{\min}$ is a current with minimal singularities in α.

Note that $\mathrm{nn}(\alpha) \subseteq \mathrm{nK}(\alpha)$. Thanks to Theorem 1.4.1, $\mathrm{nn}(\alpha)$ is a countable union of proper Zariski closed subsets of X. The non-Kähler locus and non-nef locus are studied in detail in [Bou02b].

Lemma 1.7.1 (Siu's decomposition) *Let E be a prime divisor on X. Then for any closed positive $(1,1)$-current T on X, the difference $T - \nu(T,E)[E]$ is a closed positive $(1,1)$-current.*

Here $[E]$ is the current of integration associated with E.[17] See [GH94, Page 386, Example 1] for the precise definition. See [Dem12a, Lemma 2.17] for the proof.

As a consequence, for each closed positive $(1,1)$-current T on X, we can write

$$T = \operatorname{Reg} T + \sum_i c_i[E_i], \tag{1.19}$$

where $\{E_i\}$ is a countable collection of prime divisors on X, $c_i > 0$.

Definition 1.7.8 A closed positive $(1,1)$-current T on X is *non-divisorial* (resp. *divisorial*) if $T = \operatorname{Reg} T$ (resp. $\operatorname{Reg} T = 0$).

---

[17] We have also used $[E]$ to denote the cohomology class of $[E]$. Whenever there is a risk of confusion, we shall denote the cohomology class by $\{E\}$ instead.

It is helpful to check that our conventions are always consistent: There is no extra factor of 2 or $1/2$ anywhere. One could verify this using our favorite example as in Example 1.7.1.

Next we recall the notion of modified nef classes.

Definition 1.7.9 A class $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$ is *modified nef* if the following condition holds: Fix a reference Kähler metric ω on X, then for any $\epsilon > 0$, we can find a closed $(1,1)$-current $T \in \alpha$ such that

(1) $T + \epsilon\omega \geq 0$;
(2) $\nu(T + \epsilon\omega, D) = 0$ for any prime divisor D on X.

This definition is independent of the choice of ω.

These classes are called *nef en codimension* 1 in Boucksom's thesis [Bou02b], where they were introduced for the first time. Modified nef classes form a closed convex cone in $\mathrm{H}^{1,1}(X,\mathbb{R})$. Note that a modified nef class is necessarily pseudo-effective. A nef class is obviously modified nef.

Recall the multiplicity of a cohomology class as defined in [Bou02b, Section 2.1.3].

Definition 1.7.10 Let $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$ be a pseudo-effective class and D be a prime divisor on X. We define the *Lelong number* $\nu(\alpha, D)$ as follows:

(1) When α is big, define $\nu(\alpha, D) = \nu(T, D)$ for any closed positive $(1,1)$-current $T \in \alpha$ with minimal singularities.
(2) In general, define
$$\nu(\alpha, D) \coloneqq \lim_{\epsilon \to 0+} \nu(\alpha + \epsilon\{\omega\}, D).$$

When α is big, (2) is compatible with (1) and the definition is independent of the choice of ω.

By definition, a pseudo-effective class α is modified nef if and only if $\nu(\alpha, D) = 0$ for all prime divisors D on X.

Let us recall the behavior of several cones under modifications.

Proposition 1.7.1 *Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y.*

(1) *For any nef class $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$, $\pi^*\alpha$ is nef.*
(2) *For any modified nef class $\beta \in \mathrm{H}^{1,1}(Y,\mathbb{R})$, $\pi_*\beta$ is modified nef.*

Proof Only (2) requires a proof. Fix a Kähler class γ. Replacing β by $\beta + \epsilon\gamma$ for $\epsilon \in (0,1)$, we reduce immediately to the case where β is big as well. Let T (resp. S) be a current with minimal singularities in $\pi_*\beta$ (resp. in β) and D be a prime divisor on X, it suffices to show that
$$\nu(T, D) = 0,$$
by Lemma 1.7.2 below, $\nu(\pi_*S, D) = 0$, so our assertion follows. □

Recall that non-divisorial currents are introduced in Definition 1.7.8.

Lemma 1.7.2 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from Kähler manifold Y. Let T be a non-divisorial current on Y, then $\pi_* T$ is non-divisorial.*

Conversely, if S is a non-divisorial current on X, $\pi^* S$ is could have divisorial part. As a simple example, consider S on $\mathbb{P}^2$, whose local potential near $0 \in \mathbb{C}^2_{z,w}$ looks like $\log(|z|^2 + |w|^2)$.

Proof Let D be a prime divisor on X. It follows from Zariski's main theorem Theorem B.1.1 that D is not contained in the exceptional locus of π. Let D' be the strict transform of D. Thanks to Siu's semicontinuity theorem, we have

$$\nu(\pi_* T, D) = \nu(T, D') = 0.$$

Hence $\pi_* T$ is non-divisorial. □

1.8 Plurisubharmonic metrics on line bundles

A natural source of quasi-plurisubharmonic functions is the metrics on line bundles.

Let X be a connected Kähler manifold and L be a holomorphic line bundle on X. Usually, we do not distinguish L from the associated invertible sheaf $\mathcal{O}_X(L)$.

Definition 1.8.1 Let V be a 1-dimensional complex linear space. A *Hermitian form* h on V is a map $h\colon V \times V \to \mathbb{C}$ such that

(1) h is $\mathbb{C}$-linear in the second variable and conjugate linear in the first, and
(2)
$$|v|_h^2 := h(v, v) \in \mathbb{R}_{>0}$$
for each $v \in V \setminus \{0\}$.

We usually identify h with the quadratic form $V \to \mathbb{R}$ sending v to $|v|_h^2$. We write $|v|_h = \sqrt{|v|_h^2}$ for any $v \in V$.

The *singular Hermitian form* on V is the map $V \to \{0, \infty\}$ sending 0 to 0 and other elements to ∞.

Definition 1.8.2 Let V_1 and V_2 be 1-dimensional complex linear spaces. Given two maps $h_i\colon V_i \to [0, \infty]$ $(i = 1, 2)$ each of which is either a Hermitian form or a singular Hermitian form. Then we define the *tensor product* $h_1 \otimes h_2\colon V_1 \otimes V_2 \to [0, \infty]$ as follows:

(1) If either h_1 or h_2 is singular, we define $h_1 \otimes h_2$ as the singular Hermitian form;
(2) otherwise, define $h_1 \otimes h_2$ as the usual tensor product: For any $v_1 \in V_1$, $v_2 \in V_2$, set
$$h_1 \otimes h_2(v_1 \otimes v_2) = h_1(v_1) h_2(v_2).$$

Definition 1.8.3 A *Hermitian metric* h on L is a family of Hermitian forms $(h_x)_{x \in X}$, such that

(1) for each $x \in X$, h_x is a Hermitian form on L_x, and
(2) for each local section s of $\mathcal{O}_X(L)$, the map $x \mapsto |s(x)|_{h_x}$ is smooth.

The pair (L, h) is called a *Hermitian line bundle*. We shall write $\mathrm{dd^c} h = c_1(L, h)$[18] for the first Chern form of h[19], normalized so that

$$[c_1(L, h)] = c_1(L).$$

The map $x \mapsto |s(x)|_{h_x}$ will be denoted by $|s|_h$.

To be more precise, if $U \subseteq X$ is an open subset on which L admits a nowhere vanishing holomorphic section s, then we define

$$(\mathrm{dd^c} h)|_U = \mathrm{dd^c}\left(-\log |s|_h^2\right).$$

Proposition 1.8.1 (Lelong–Poincaré) *Let $s \in \mathrm{H}^0(X, L)$ be non-zero and h be a Hermitian metric on L. Then*

$$c_1(L, h) + \mathrm{dd^c} \log |s|_h^2 = [Z(s)], \tag{1.20}$$

where $Z(s)$ is the zero divisor defined by s and $[\bullet]$ denote the associated current of integration.

See [Dem12a, (3.11)]. Again, we want to check that our conventions are compatible by investigating the following simple example.

Example 1.8.1 Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. The homogeneous coordinates on $\mathbb{P}^1$ will be denoted by $[X_0 : X_1]$. At a point $x = [X_0 : X_1] \in \mathbb{P}^1$, the fiber L_x is identified with the dual of $[x]$, where $[x] \subseteq \mathbb{C}^2$ is the line represented by x.

In order to introduce the Hermitian metric h on L, we fix the standard Hermitian norm $\| \bullet \|$ on $\mathbb{C}^2$. Then given $\lambda \in L_x = [x]^\vee$, we introduce

$$|\lambda|_{h_x} = \frac{|\lambda(\tilde{x})|}{\|\tilde{x}\|},$$

where $\tilde{x}$ is an arbitrary non-zero element in $[x]$. The readers can easily verify that h is indeed a Hermitian metric on L. The Hermitian metric h is known as the *Fubini–Study metric*.

A holomorphic section $s \in \mathrm{H}^0(X, L)$ can be formally identified with a linear form $a_0X_0 + a_1X_1$: At $x \in X$, the corresponding linear form on $[x]$ is given by sending (X_0, X_1) to $a_0X_0 + a_1X_1$.

Next we compute $\mathrm{dd^c} h = c_1(L, h)$. For this purpose, we cover $\mathbb{P}^1$ by $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ and $\mathbb{P}^1 \setminus \{0\}$. Both are holomorphic coordinate charts with coordinate function $z = X_0/X_1$ and $z^{-1} = X_1/X_0$ respectively.

---

[18] The unusual notation $\mathrm{dd^c} h$ is sometimes referred to as the *Göteborg notation* because it is widely used by the complex geometriers in Göteborg (usually spelled as Gothenburg in English, the second largest (yet very poorly known) city in Sweden). As I identify myself as *Göteborgare*, I do not feel guilty about this notation.

[19] In the literature, people sometimes define the *curvature form* of (L, h) as $\Theta_h = -2\pi i \mathrm{dd^c} h$.

We claim that on $\mathbb{C}$,

$$\mathrm{dd^c} h = \mathrm{dd^c} \log(1 + |z|^2). \tag{1.21}$$

In fact, let t be the nowhere vanishing section of L on $\mathbb{C}$ corresponding to X_1. Then for $z \in \mathbb{C}$, we have an obvious lift $(z, 1) \in [z]$, so

$$|t|_h^2(z) = \frac{1}{|z|^2 + 1}.$$

So (1.21) follows.

In order to obtain a non-trivial case of the Lelong–Poincaré formula, we need to consider a section which vanishes at some points in $\mathbb{C}$. Let s be the holomorphic section of L corresponding to X_0. Then

$$\log |s|_h^2(z) = \log \frac{|z|^2}{|z|^2 + 1}$$

for any $z \in \mathbb{C}$ using the same argument as above. Therefore, we find that restricted to $\mathbb{C}$, we have

$$c_1(L, h) + \mathrm{dd^c} \log |s|_h^2 = \mathrm{dd^c} f = [0],$$

where $f(z) = \log |z|^2$. So the Lelong–Poincaré formula (1.20) is verified in this case.

The Kähler form $\mathrm{dd^c} h$ on $\mathbb{P}^1$ is also known as the *Fubini–Study metric*.

Definition 1.8.4 A (singular) *plurisubharmonic metric* (or *psh metric* for short)[20] h on L is a family $(h_x)_{x\in X}$ such that

(1) for each $x \in X$, h_x is either a Hermitian form on L_x or the singular Hermitian form on L_x, and
(2) there is a Hermitian metric h_0 on L and $\varphi \in \mathrm{PSH}(X, c_1(L, h_0))$ such that for each $x \in X$ and each $v \in L_x$, we have

$$|v|_{h_x}^2 = \begin{cases} 0, & \text{if } v = 0; \\ |v|_{h_{0,x}}^2 \mathrm{e}^{-\varphi(x)}, & \text{if } v \neq 0. \end{cases} \tag{1.22}$$

The *(first) Chern current* of h is by definition

$$\mathrm{dd^c} h = c_1(L, h) \coloneqq c_1(L, h_0) + \mathrm{dd^c}\varphi.$$

We shall write the plurisubharmonic metric defined by (1.22) as $h_0 \exp(-\varphi)$[21]. As the readers can easily verify, our conventions guarantee that $c_1(L, h)$ does not depend on the choice of h_0.

[20] In the literature, people usually refer to such metrics as *positively curved singular Hermitian metrics*. I dislike this terminology, as having positive curvature only determines a plurisubharmonic metric almost everywhere, not everywhere.

[21] Be careful, this is not $h_0^2 \exp(-\varphi)$, as I prefer to think of h_0 as a quadratic form.

Remark 1.8.1 In the literature, some people prefer the convention that in (1.22), neither side has the square. Our choice seems to be the most natural one given our normalization of $\mathrm{dd^c}$.

Observe that once a Hermitian metric h_0 on L is given, the construction in (2) gives a bijection between $\mathrm{PSH}(X, c_1(L, h_0))$ and the set of plurisubharmonic metrics on L.

Definition 1.8.5 Given two holomorphic line bundles L_1, L_2 on X and plurisubharmonic functions h_1 on L_1 and h_2 on L_2, we define the *tensor product* plurisubharmonic metric $h_1 \otimes h_2$ on $L_1 \otimes L_2$ as follows: for each $x \in X$, define

$$(h_1 \otimes h_2)_x = h_{1,x} \otimes h_{2,x}$$

in the sense of Definition 1.8.2.

We can easily verify that $h_1 \otimes h_2$ is indeed a plurisubharmonic metric on $L_1 \otimes L_2$.

Example 1.8.2 We continue with our example Example 1.8.1. Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(1)$. Let h^0 denote the Fubini–Study metric on L as defined in Example 1.8.1. Note that we have changed the notation from h to h^0. Let $\omega = \mathrm{dd^c} h^0$.

We construct $\varphi \in \mathrm{PSH}(X, \omega)$ as follows: On $\mathbb{C}$, define

$$\varphi(z) = \log \frac{|z|^2}{1 + |z|^2}. \tag{1.23}$$

Then $\varphi \in \mathrm{PSH}(\mathbb{C}, \omega|_{\mathbb{C}})$ by (1.21). Setting $\varphi(\infty) = 0$, we can easily verify that $\varphi \in \mathrm{PSH}(\mathbb{P}^1, \omega)$.[22]

We then get a plurisubharmonic metric $h^0 \exp(-\varphi)$. To be more explicit, h_0 is singular, $h_\infty = h^0_\infty$, while for $z \in \mathbb{C} \setminus \{0\}$ and $\lambda \in [z]^\vee$, we have

$$|\lambda|_{h_z} = \frac{|\lambda(z, 1)|}{|z|}.$$

In the remaining of this section, we assume that X is compact.

Definition 1.8.6 Assume that L is a pseudo-effective line bundle on X. A *Fubini–Study metric* on L is a psh metric h on L of the following form: There exists $m \in \mathbb{Z}_{>0}$, finitely many sections $s_1, \ldots, s_N \in \mathrm{H}^0(X, L^m)$ and $\lambda_1, \ldots, \lambda_N \in \mathbb{Q}$ such that for any local nowhere vanishing holomorphic section s of L, we have

$$|s|_h^2 = \min_{i=1,\ldots,N} \left| \frac{s^{\otimes m}}{\mathrm{e}^{\lambda_i/2} s_i} \right|^{2m^{-1}}.$$

We write $\mathrm{FS}(L)$ for the set of Fubini–Study metrics on L.

[22] This can also be verified using the Grauert–Remmert extension theorem Theorem 1.2.1.

If we fix a reference smooth Hermitian metric h_0 on L with $\theta = \mathrm{dd^c} h_0$, we can write $h = h_0 \exp(-\varphi)$ with

$$\varphi = \frac{1}{m} \max_{i=1,\ldots,N} \left(\log |s_i|^2_{h_0^m} + \lambda_i \right).$$

Similarly, we write $\mathrm{FS}(X, \theta)$ for the set of such functions.

Definition 1.8.7 Assume that L is a pseudo-effective line bundle on X. The set $\widetilde{\mathrm{FS}}(L)$ of *generalized Fubini metrics* is the smallest subset of $\mathrm{PSH}(L)$ containing $\mathrm{FS}(L)$ which is closed under the following two operations:

(1) $\mathbb{Q}$-convex combinations: if $h_1, h_2 \in \widetilde{\mathrm{FS}}(L)$ and $t \in (0, 1)$, then

$$h_1^t \otimes h_2^{1-t} \in \widetilde{\mathrm{FS}}(L);$$

(2) minima: if $h_1, h_2 \in \widetilde{\mathrm{FS}}(L)$, then

$$\min\{h_1, h_2\} \in \widetilde{\mathrm{FS}}(L).$$

We shall need the following Ohsawa–Takegoshi type extension theorem.

Theorem 1.8.1 *Assume that L is big and T is a holomorphic line bundle on X. Fix a Hermitian metric h_T on T. Take a Kähler form ω on X. Let $Y \subseteq X$ be a connected submanifold of dimension m. Suppose that $\varphi \in \mathrm{PSH}(X, \theta - \delta\omega)$ for some $\delta > 0$ and $\varphi|_Y \not\equiv -\infty$. Then there exists $k_0(\delta, h_T) > 0$ such that for all $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y))$* [23]*, there exists an extension $\tilde{s} \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ such that*

$$\int_X (h^k \otimes h_T)(\tilde{s}, \tilde{s}) \mathrm{e}^{-k\varphi}\, \omega^n \leq C \int_Y (h^k \otimes h_T)|_Y (s, s) \mathrm{e}^{-k\varphi|_Y}\, \omega|_Y^m,$$

where $C > 0$ is an absolute constant, independent of the data (φ, s, k).

This is a special case of [His12, Theorem 1.4].

Proposition 1.8.2 *Let (L, h) be a Hermitian line bundle on X and set $\theta = c_1(L, h)$. Let (T, h_T) be a Hermitian line bundle on X. Assume that $\varphi \in \mathrm{PSH}(X, \theta)$ is a potential with analytic singularities such that θ_φ is a Kähler current. Fix a Kähler form ω on X. For each $k \geq 1$, we let*

$$\varphi_k \coloneqq \frac{1}{k} \log \sup_{\substack{s \in \mathrm{H}^0(X, L^k \otimes T) \\ \int_X h^k \otimes h_T(s,s) \mathrm{e}^{-k\varphi} \omega^n \leq 1}} h^k \otimes h_T(s, s). \tag{1.24}$$

Then for any $k \geq 0$,

$$\varphi \preceq \varphi_k \preceq \alpha_k \varphi,$$

where $\alpha_k \in (0, 1)$ is an increasing sequence with limit 1.

[23] Here and in the sequel, we usually abbreviate $\otimes k$ in the super-index as k to save spaces.

Note that when k is large enough, $\varphi_k \in \mathrm{PSH}(X, \theta)$. We refer to [DX24b, Remark 2.9] for the proof.

Chapter 2
Non-pluripolar products

Pour exprimer d'une manière frappante que le monument que j'élève sera placé sous l'invocation de la Science, j'ai décidé d'inscrire en lettres d'or sur la grande frise du premier étage et à la place d'honneur, les noms des plus grands savants[a] *qui ont honoré la France depuis 1789 jusqu'à nos jours.*
— Gustave Eiffel, 1889

[a] Gaspard Monge, Comte de Péluse (1746–1818), known oddly by his family name instead of *de Péluse*, is one of the 72 names scribed on the Eiffel tower. He was both a mathematician and a politician, active mainly after the French Revolution.

Let X be a complex manifold and $\varphi_1, \dots, \varphi_p \in \mathrm{PSH}(X)$ ($p \in \mathbb{N}$). When the functions $\varphi_1, \dots, \varphi_p$ are all smooth, there is an obvious definition of a differential form

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p \tag{2.1}$$

by the usual differential calculus. The product is usually known as the *Monge–Ampère product*. It is of interest to extend this construction to the case where the φ_i's have worse regularities.

There are a number of different approaches to this problem. In this book, we will choose the so-called *non-pluripolar theory* due to Bedford, Taylor, Guedj, Zeriahi, Boucksom and Eyssidieux. The reason is that the non-pluripolar theory is the only known theory satisfying the following two features: It is defined for all psh singularities (at least in the global setting) and it satisfies a monotonicity theorem.

We will recall the Bedford–Taylor theory in Section 2.1 and the non-pluripolar theory in Section 2.2.

Some key properties of the non-pluripolar products are recalled in Section 2.4.

The readers who are not familiar with this notion are encouraged to read the original article [BEGZ10] as well as the survey article [DDNL23].

2.1 Bedford–Taylor theory

Let X be a complex manifold and $\varphi_1, \dots, \varphi_p \in \mathrm{PSH}(X)$ ($p \in \mathbb{N}$) be locally bounded plurisubharmonic functions on X[1]. In this case, there is a canonical definition of the Monge–Ampère type product (2.1).

[1] In the literature, some people use $\mathrm{PSH}(X) \cap L^\infty_{\mathrm{loc}}(X)$ to denote the set of such functions, which is an abuse of notation. However, this is legitimate thanks to the rigidity Theorem 1.1.3.

Definition 2.1.1 We define the closed positive (p,p)-current (2.1) on X as follows: We make an induction on $p \geq 0$. When $p = 0$, we define (2.1) as the $(0,0)$-current $[X]$. When $p > 0$, we let

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p := \mathrm{dd^c}\left(\varphi_1\, \mathrm{dd^c}\varphi_2 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p\right).$$

We call this product the *Bedford–Taylor product* .

Remark 2.1.1 There is also a slightly more general version of this construction. Given a closed positive current T, one can also define the product

$$\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p \wedge T$$

in a very similar way.

Proposition 2.1.1 *The product* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ *is a closed positive* (p,p)*-current on* X*. Moreover, the product is symmetric in the* φ_i*'s.*

See [GZ17, Proposition 3.3, Corollary 3.12]. The proof relies crucially on an important estimate, known as the *Chern–Levine–Nirenberg inequality*. See [GZ17, Theorem 3.9].

The Bedford–Taylor theory has many satisfactory properties.

Theorem 2.1.1 *Let* $(\varphi_i^j)_{j\in\mathbb{Z}_{>0}}$ *be decreasing sequences (resp. increasing sequences) of locally bounded psh functions on* X *converging (resp. converging a.e.) to locally bounded psh function* φ_i*, where* $i = 1, \ldots, p$*. Then*

$$\varphi_0^j\, \mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^j \rightharpoonup \varphi_0\, \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$$

as $j \to \infty$*. In particular, if* φ_0^j *is the constant sequence* 1*, we have*

$$\mathrm{dd^c}\varphi_1^j \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^j \rightharpoonup \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p.$$

Here the notation $\rightharpoonup$ denotes the weak-* convergence of currents.

We refer to [GZ17, Theorem 3.18, Theorem 3.23] for the proofs.

By contrast, we emphasize that the Bedford–Taylor product is not continuous with respect to the L^1_{loc}-convergence in general. A simple example can be found in [GZ17, Example 3.25].

2.2 The non-pluripolar products

The proof of all results in this section can be found in [BEGZ10].

Let X be a complex manifold.

Definition 2.2.1 Let $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$. We set

$$O_k \coloneqq \bigcap_{j=1}^{p} \{\varphi_j > -k\}, \quad k \in \mathbb{Z}_{>0}.$$

We say that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is *well-defined* if for each connected open subset $U \subseteq X$, any smooth Hermitian form ω on U, for each compact subset $K \subseteq U$, we have

$$\sup_{k\geq 0} \int_{K\cap O_k} \left(\bigwedge_{j=1}^{p} \mathrm{dd^c}(\varphi_j \vee (-k)) \right)\Bigg|_U \wedge \omega^{\dim U - p} < \infty. \tag{2.2}$$

In this case, we define the *non-pluripolar product* $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ by

$$\mathbb{1}_{O_k} \mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p = \mathbb{1}_{O_k} \bigwedge_{j=1}^{p} \mathrm{dd^c}\left(\varphi_j \vee (-k)\right) \tag{2.3}$$

on $\bigcup_{k\geq 0} O_k$ and make a zero-extension to X.

As recalled in Section 1.3, an $\mathcal{F}$-open subset means an open subset with respect to the plurifine topology.

Proposition 2.2.1 *Let $\varphi_1, \ldots, \varphi_p \in \mathrm{PSH}(X)$.*

(1) *The product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is local with respect to the plurifine topology in the following sense: Let $O \subseteq X$ be an $\mathcal{F}$-open subset and $\psi_1, \ldots, \psi_p \in \mathrm{PSH}(X)$. Assume that*

$$\varphi_j|_O = \psi_j|_O, \quad j = 1, \ldots, p,$$

and that

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \text{ and } \bigwedge_{j=1}^{p} \mathrm{dd^c}\psi_j$$

are both well-defined, then

$$\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \right|_O = \left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\psi_j \right|_O. \tag{2.4}$$

If furthermore O is open in the usual topology, then the product

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_O$$

on O is well-defined and

$$\left.\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j \right|_O = \bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_O. \tag{2.5}$$

Let $\mathcal{U}$ be an open covering of X. Then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined if and only if each of the following product is well-defined

$$\bigwedge_{j=1}^{p} \mathrm{dd^c}\varphi_j|_U, \quad U \in \mathcal{U}.$$

(2) *The current $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ and the fact that it is well-defined depend only on the currents $\mathrm{dd^c}\varphi_j$, not on the choice of the φ_j's nor on the ordering of the φ_j's.*

(3) *When $\varphi_1, \ldots, \varphi_p \in L^\infty_{\mathrm{loc}}(X)$, the product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined and is equal to the Bedford–Taylor product.*

(4) *Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ puts no mass on pluripolar sets.*

(5) *Assume that $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is well-defined, then $\bigwedge_{j=1}^p \mathrm{dd^c}\varphi_j$ is a closed positive (p,p)-current on X.*

(6) *The product is multilinear: Let $\psi_1 \in \mathrm{PSH}(X)$, $a, b > 0$ then*

$$\mathrm{dd^c}(a\varphi_1 + b\psi_1) \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j = a\mathrm{dd^c}\varphi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j + b\mathrm{dd^c}\psi_1 \wedge \bigwedge_{j=2}^{p} \mathrm{dd^c}\varphi_j \tag{2.6}$$

in the sense that left-hand side is well-defined if and only if both terms on right-hand side are well-defined, and the equality holds in that case.

In view of (3), we do not need to specify whether our product $\mathrm{dd^c}\varphi_1 \wedge \cdots \wedge \mathrm{dd^c}\varphi_p$ is the Bedford–Taylor product or the non-pluripolar product when the φ_i's are all locally bounded.

Definition 2.2.2 Let $T_1, \ldots, T_p$ be closed positive $(1,1)$-currents on X. We say that $T_1 \wedge \cdots \wedge T_p$ is *well-defined* if there exists an open covering $\mathcal{U}$ of X, such that on each $U \in \mathcal{U}$, we can find $\varphi_j^U \in \mathrm{PSH}(U)$ $(j = 1, \ldots, p)$ such that

$$\mathrm{dd^c}\varphi_j^U = T_j, \quad j = 1, \ldots, p$$

and $\mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U$ is well-defined. In this case, we define the *non-pluripolar product* $T_1 \wedge \cdots \wedge T_p$ as the closed positive (p,p)-current on X defined by

$$(T_1 \wedge \cdots \wedge T_p)\,|_U = \mathrm{dd^c}\varphi_1^U \wedge \cdots \wedge \mathrm{dd^c}\varphi_p^U, \quad U \in \mathcal{U}. \tag{2.7}$$

The product $T_1 \wedge \cdots \wedge T_p$ is independent of the choices we made thanks to Proposition 2.2.1 (1) and (2).

Proposition 2.2.1 can be formulated in terms of currents without any difficulty.

Remark 2.2.1 Similar to Remark 2.1.1, there is also an extension of the non-pluripolar theory allowing us to define

$$T_1 \wedge \cdots \wedge T_p \cap T$$

for any closed positive current T. This is the *relative non-pluripolar product* introduced by Vu [Vu21]. Unlike the relative Bedford–Taylor products, the relative non-pluripolar products present some pathological behaviors. For example, they are not linear in general.

Remark 2.2.2 Another possible generalization of the non-pluripolar products is motivated by Proposition 2.2.1. One could begin by defining of generalized notion of plurisubharmonic functions on $\mathcal{F}$-open sets, called *$\mathcal{F}$-plurisubharmonic functions* and define their non-pluripolar products. See [EKFW11, EKW14].

Proposition 2.2.2 *Let X be a compact Kähler manifold and $T_1, \dots, T_p$ are closed positive $(1,1)$-currents on X. Then $T_1 \wedge \dots \wedge T_p$ is well-defined.*

This proposition explains why we usually work in the setting of compact Kähler manifolds.

2.3 Quasi-continuous functions

Let X be a compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class.

Definition 2.3.1 Let $A \subseteq X$ be a Borel subset. The *θ-capacity* $\operatorname{Cap}_\theta(A)$ of A is defined as

$$\operatorname{Cap}_\theta(A) \coloneqq \sup\left\{\int_A \theta_\varphi^n : \varphi \in \operatorname{PSH}(X,\theta), V_\theta - 1 \le \varphi \le V_\theta\right\}.$$

The capacity is not very sensitive to the choice of θ:

Theorem 2.3.1 *Let θ' be another closed real smooth $(1,1)$-form on X representing a big cohomology class. Then there are continuous functions $f, g\colon [0,\infty) \to [0,\infty)$ such that for any Borel subset $E \subseteq X$, we have*

$$\operatorname{Cap}_\theta(E) \le f\left(\operatorname{Cap}_{\theta'}(E)\right), \quad \operatorname{Cap}_{\theta'}(E) \le g\left(\operatorname{Cap}_\theta(E)\right).$$

A more general result is proved in [Lu21]. Similar comparison results hold between θ-capacity and the classical Bedford–Taylor capacity, see [GZ17, Section 9.2] for the proof. As a consequence, we can freely apply the results in [GZ17, Section 4.2], even though capacity has a different meaning there.

Definition 2.3.2 Let U be an open subset of X. A function $f\colon U \to [-\infty,\infty]$ is quasi-continuous if it is Borel measurable and for any $\epsilon > 0$, there is an open subset $G \subseteq U$ such that

(1) $\operatorname{Cap}_\theta(G) \le \epsilon$;
(2) $f|_{U\setminus G}$ is real-valued and continuous.

Thanks to Theorem 2.3.1, the notion of quasi-continuous functions is independent of the choice of θ. Note that if $f, g\colon U \to [-\infty, \infty]$ are two Borel measurable functions equal quasi-everywhere (see Definition 1.1.4), then f is quasi-continuous if and only if g is.

Theorem 2.3.2 *Let $A \subseteq X$ be a Borel set. Then the following are equivalent:*

(1) *A is pluripolar;*
(2) $\mathrm{Cap}_\theta(A) = 0$.

See [GZ17, Theorem 4.40] for the proof.

Example 2.3.1 A quasi-plurisubharmonic function on an open subset $U \subseteq X$ is always quasi-continuous. See [GZ17, Theorem 4.20] for the proof.

More generally, if φ, ψ are two quasi-plurisubharmonic functions on U, then the following function

$$f(x) := \begin{cases} \varphi(x) - \psi(x), & \text{if } \varphi \vee \psi(x) \neq -\infty; \\ \infty, & \text{otherwise} \end{cases}$$

is quasi-continuous.[2]

Definition 2.3.3 Let U be an open subset of X. Let $(f_i)_{i\in I}$ be a net of Borel measurable functions $f_j\colon U \to [-\infty, \infty]$, and $f\colon U \to [-\infty, \infty]$ be a Borel measurable function. We say $(f_i)_{i\in I}$ converges to f *in capacity* if for any $\delta > 0$, we have

$$\lim_{i\in I} \mathrm{Cap}_\theta\left(\{f_i > f + \delta\}\right) = 0, \quad \lim_{i\in I} \mathrm{Cap}_\theta\left(\{f_i < f - \delta\}\right) = 0^{3}.$$

We sometimes write $f_i \xrightarrow{\mathrm{C}} f$.

Note that f is not uniquely determined by the net $(f_i)_{i\in I}$. Thanks to Theorem 2.3.1, the notion of quasi-continuous functions is independent of the choice of θ.

Proposition 2.3.1 *Let $(\varphi_i)_{i\in I}$ be a net in $\mathrm{PSH}(U, \theta)$ and $\varphi \in \mathrm{PSH}(U, \theta)$. Assume one of the following conditions holds:*

(1) *$(\varphi_i)_{i\in I}$ is decreasing and φ is the limit of the net;*
(2) *$(\varphi_i)_{i\in I}$ is increasing and converges almost everywhere to φ.*

Then $(\varphi_i)_{i\in I}$ converges to φ in capacity.

See [GZ17, Proposition 4.25] for the proof. The reference concerns only the sequence case, but the proof works for nets as well.

---

[2] In the literature, people usually say carelessly that $\varphi - \psi$ is quasi-continuous.

[3] It is very tempting to write $\lim_{i\in I} \mathrm{Cap}_\theta\left(\{|f_i - f| > \delta\}\right) = 0$, as in [GZ17, Definition 4.23] for example. But the set where $f_j - f$ is not defined is not a pluripolar set in general. Hence this abuse of notation is not acceptable.

2.4 Properties of non-pluripolar products

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta_1, \dots, \theta_n$ be closed real smooth $(1,1)$-forms on X.

We write

$$\mathrm{PSH}(X,\theta)_{>0} = \left\{ \varphi \in \mathrm{PSH}(X,\theta) : \int_X \theta_\varphi^n > 0 \right\}. \tag{2.8}$$

The non-pluripolar product θ_φ^n is well-defined thanks to Proposition 2.2.2.

Remark 2.4.1 Suppose that X is a connected complex manifold of dimension 0, namely, X is a single point. In this case, by definition, the non-pluripolar product θ_φ^n is given by the current of integration at the unique point. So $\mathrm{PSH}(X,\theta)_{>0} = \mathrm{PSH}(X,\theta) \cong \mathbb{R}$ in this case and $\int_X \theta_\varphi^n = 1$ for all $\varphi \in \mathrm{PSH}(X,\theta)$.

Recall the following basic result:

Proposition 2.4.1 *Assume that* $\mathrm{PSH}(X,\theta)_{>0}$ *is non-empty, then the cohomology class* $[\theta]$ *is big.*

See [BEGZ10, Proposition 1.22].

We recall a few basic facts about the non-pluripolar masses.

Proposition 2.4.2 *Let* $\pi \colon Y \to X$ *be a proper bimeromorphic morphism from a Kähler manifold* Y *and* $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ *for* $i = 1,\dots,n$. *Then*

$$\int_Y \pi^*\theta_{1,\pi^*\varphi_1} \wedge \cdots \wedge \pi^*\theta_{n,\pi^*\varphi_n} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

Proof This follows immediately from Proposition 2.2.1 (1) and (4). □

Theorem 2.4.1 *Let* $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. *Then the map*

$$[0,1] \ni t \mapsto \log \int_X \theta^n_{t\varphi_1 + (1-t)\varphi_0}$$

is concave.

See [DDNL21a] for the proof.

Remark 2.4.2 Here and in the sequel, when we write expressions like $t\varphi + (1-t)\psi$ for $\varphi, \psi \in \mathrm{QPSH}(X)$, we will follow the convention that when $t = 0$, the value is ψ and when $t = 1$, the value is φ.

We shall write

$$V_\theta = \sup\left\{ \varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq 0 \right\}. \tag{2.9}$$

It follows from Proposition 1.2.1 that $V_\theta \in \mathrm{PSH}(X,\theta)$ if $\mathrm{PSH}(X,\theta) \neq \emptyset$. The function V_θ should be regarded as a canonical representative of the least singular potentials in $\mathrm{PSH}(X,\theta)$. We recall the following result:

Theorem 2.4.2 (Di Nezza–Trapani) *We have $V_\theta \in C^{1,\bar{1}}(X \setminus \mathrm{nK}(\{\theta\}))$, and*

$$\theta^n_{V_\theta} = \mathbb{1}_{\{V_\theta=0\}}\theta^n.$$

Recall that the non-Kähler locus is defined in Definition 1.7.6. See [DNT21, DNT24] for the proof.

The non-pluripolar product has a lower semicontinuity property.

Theorem 2.4.3 (Semicontinuity theorem) *Let $\varphi_j, \varphi_j^k \in \mathrm{PSH}(X,\theta_j)$ ($k \in \mathbb{Z}_{>0}$, $j = 1,\dots,n$). Let χ_k, χ ($k \in \mathbb{Z}_{>0}$) be non-negative uniformly bounded quasi-continuous functions on X such that $(\chi_k)_k$ converges to χ in capacity. Assume that for any $j = 1,\dots,n$, as $k \to \infty$, either φ_j^k decreases to $\varphi_j \in \mathrm{PSH}(X,\theta)$ or increases to $\varphi_j \in \mathrm{PSH}(X,\theta)$ almost everywhere. Then*

$$\varliminf_{k\to\infty} \int_X \chi_k\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} \geq \int_X \chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.10}$$

If in addition,

$$\lim_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n},$$

then $\chi_k\, \theta_{1,\varphi_1^k} \wedge \cdots \wedge \theta_{n,\varphi_n^k}$ converges to $\chi\, \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}$ weakly as measures[4]. In particular, the limit in (2.10) exists and equality holds in (2.10).

See [DDNL23, Theorem 2.6] for the proof.

The non-pluripolar mass is a monotone quantity with respect to the singularity type.

Theorem 2.4.4 (Monotonicity theorem) *Let $\varphi_j, \psi_j \in \mathrm{PSH}(X,\theta_j)$ for $j = 1,\dots,n$. Assume that $\varphi_j \succeq \psi_j$[5] for every j, then*

$$\int_X \theta_{1,\varphi_1} \wedge \cdots \theta_{n,\varphi_n} \geq \int_X \theta_{1,\psi_1} \wedge \cdots \theta_{n,\psi_n}.$$

In particular, if $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ with $\varphi \succeq \psi$, then

$$\int_X \theta^n_\varphi \geq \int_X \theta^n_\psi.$$

See [DDNL18b, Theorem 1.1]. We will prove a vast extension of this theorem in Proposition 6.1.4.

Thanks to this theorem, the non-pluripolar mass $\int_X \theta^n_\varphi$ could be used as a rough measure of the singularities of $\varphi \in \mathrm{PSH}(X,\theta)$. In Section 3.1, we shall refine this measure by defining the notion of P-envelope.

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[4] We remind the readers that the weak convergence of Radon measures is stronger than the weak convergence as currents in general. When the Radon measures have uniformly bounded total variation, they are equivalent.

[5] See Definition 1.5.2 for the notation.

As a corollary, we obtain that

Corollary 2.4.1 *Fix a directed set I. For each $j = 1, \dots, n$, take an increasing net $(\varphi_j^i)_{i\in I}$ in $\mathrm{PSH}(X, \theta_j)$, uniformly bounded from above. Set*

$$\varphi_j \coloneqq \sup_{i\in I}{}^* \varphi_j^i, \quad j = 1, \dots, n.$$

Then

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{2.11}$$

Proof We may assume that I is infinite as there is nothing to prove otherwise. Thanks to Theorem 2.4.4, we already know the $\leq$ inequality in (2.11). We prove the reverse inequality. When $I \cong \mathbb{Z}_{>0}$ as directed sets, the reverse inequality follows from Theorem 2.4.3. In general, by Choquet's lemma Proposition 1.2.2, we can find a countable infinite subset $R \subseteq I$ such that

$$\sup_{r\in R}{}^* \varphi_j^r = \sup_{i\in I}{}^* \varphi_j^i$$

for all $j = 1, \dots, n$. We fix a bijection $R \cong \mathbb{Z}_{>0}$. For any $j = 1, \dots, n$, we will then denote elements φ_j^r ($r \in R$) by $\varphi_j^1, \varphi_j^2, \dots$. We shall write

$$\psi_j^a = \varphi_j^1 \vee \cdots \vee \varphi_j^a$$

for each $a \in \mathbb{Z}_{>0}$.

It follows from the fact that I is a directed set and Theorem 2.4.4 that

$$\lim_{i\in I} \int_X \theta_{1,\varphi_1^i} \wedge \cdots \wedge \theta_{n,\varphi_n^i} \geq \lim_{a\to\infty} \int_X \theta_{1,\psi_1^a} \wedge \cdots \wedge \theta_{n,\psi_n^a}.$$

From the special case mentioned above, we know that the right-hand side is exactly the right-hand side of (2.11), so we conclude. □

We prove an interesting inequality about the Monge–Ampère measure of the maximum of two potentials.

Lemma 2.4.1 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Then*

$$\theta_{\varphi\vee\psi}^n \geq \mathbb{1}_{\{\varphi\geq\psi\}} \theta_\varphi^n + \mathbb{1}_{\{\varphi<\psi\}} \theta_\psi^n. \tag{2.12}$$

In particular, if $\varphi \leq \psi$, then

$$\mathbb{1}_{\{\varphi=\psi\}} \theta_\varphi^n \leq \mathbb{1}_{\{\varphi=\psi\}} \theta_\psi^n.$$

At a first sight, (2.12) might seem trivial, and it is if we replace $\{\varphi \geq \psi\}$ by $\{\varphi > \psi\}$. The difficulty really lies on understanding the contact set $\{\varphi = \psi\}$.

Proof Recall that V_θ is defined in (2.9). For each $k \in \mathbb{N}$, we set

$$\psi_k = \psi \vee (V_\theta - k), \quad \varphi_k = \varphi \vee (V_\theta - k).$$

For each $t > 0$ and each $k \in \mathbb{N}$, we have

$$\begin{aligned}\theta^n_{\psi_k \vee (\varphi_k + t)} \geq & \mathbb{1}_{\{\psi_k > \varphi_k + t\}} \theta^n_{\psi_k \vee (\varphi_k + t)} + \mathbb{1}_{\{\psi_k < \varphi_k + t\}} \theta^n_{\psi_k \vee (\varphi_k + t)} \\ = & \mathbb{1}_{\{\psi_k > \varphi_k + t\}} \theta^n_{\psi_k} + \mathbb{1}_{\{\psi_k < \varphi_k + t\}} \theta^n_{\varphi_k},\end{aligned}$$

where the equality follows from Proposition 2.2.1(1). We observe that as $t \to 0+$, the measures $\theta^n_{\psi_k \vee (\varphi_k + t)}$ converges weakly to $\theta^n_{\psi_k \vee \varphi_k}$. In fact, as a consequence of Theorem 2.4.3 and Theorem 2.4.4.

Now letting $t \to 0+$, we conclude that

$$\theta^n_{\psi_k \vee \varphi_k} \geq \mathbb{1}_{\{\psi_k > \varphi_k\}} \theta^n_{\psi_k} + \mathbb{1}_{\{\psi_k \leq \varphi_k\}} \theta^n_{\varphi_k}.$$

In particular, multiplying both sides by $\mathbb{1}_{\{\min\{\varphi,\psi\} > V_\theta - k\}}$ and applying Proposition 2.2.1(1) again, we find that

$$\begin{aligned}\mathbb{1}_{\{\min\{\varphi,\psi\} > V_\theta - k\}} \theta^n_{\psi \vee \varphi} \geq & \mathbb{1}_{\{\min\{\varphi,\psi\} > V_\theta - k\} \cap \{\psi \leq \varphi\}} \theta^n_{\varphi} \\ & + \mathbb{1}_{\{\min\{\varphi,\psi\} > V_\theta - k\} \cap \{\psi > \varphi\}} \theta^n_{\psi}.\end{aligned}$$

Letting $k \to \infty$, we conclude (2.12). □

Corollary 2.4.2 *Let $(\varphi_j)_{j>0}$ be a sequence in $\mathrm{PSH}(X,\theta)$, and $\varphi \in \mathrm{PSH}(X,\theta)$ so that $\varphi_j \to \varphi$ in L^1. Assume that $\varphi_j \leq 0$ for all $j > 0$. Then*

$$\varlimsup_{j \to \infty} \int_{\{\varphi_j = 0\}} \theta^n_{\varphi_j} \leq \int_{\{\varphi = 0\}} \theta^n_{\varphi}. \tag{2.13}$$

Proof For each $k > 0$, let

$$\psi_k = \sup_{j \geq k}{}^* \varphi_j.$$

Then $(\psi_k)_k$ is a decreasing sequence in $\mathrm{PSH}(X,\theta)$ with limit φ. See the proof of Corollary 1.2.1 for example.

Thanks to Lemma 2.4.1, for each $k > 0$, we have

$$\mathbb{1}_{\{\varphi_k = \psi_k\}} \theta^n_{\varphi_k} \leq \mathbb{1}_{\{\varphi_k = \psi_k\}} \theta^n_{\psi_k}.$$

Multiplying both sides with $\mathbb{1}_{\{\varphi_k = 0\}}$, we find that

$$\mathbb{1}_{\{\varphi_k = 0\}} \theta^n_{\varphi_k} \leq \mathbb{1}_{\{\varphi_k = 0\}} \theta^n_{\psi_k}.$$

Take $b, C > 0$, for each $k > 0$, we have

$$\begin{aligned}& \int_{\{\varphi_k = 0\}} \theta^n_{\varphi_k} \leq \int_{\{\varphi_k = 0\}} \theta^n_{\psi_k} \leq \int_{\{\psi_k = 0\}} \theta^n_{\psi_k} \\ = & \int_{\{\psi_k = 0\}} \theta^n_{\psi_k \vee (V_\theta - C)} \leq \int_X e^{b\psi_k} \theta^n_{\psi_k \vee (V_\theta - C)},\end{aligned}$$

where the equality part follows from Proposition 2.2.1(1).

Letting $k \to \infty$ with the help of Theorem 2.4.3, we conclude that

$$\varlimsup_{k\to\infty} \int_{\{\varphi_k=0\}} \theta^n_{\varphi_k} \leq \int_X \mathrm{e}^{b\varphi}\theta^n_{\varphi\vee(V_\theta-C)}.$$

Letting $b \to \infty$ and then $C \to \infty$, we conclude (2.13). □

Next we introduce an envelope construction, which will be repeatedly used in the sequel.

Definition 2.4.1 Given a function $f\colon X \to [-\infty,\infty]$, we define $P_\theta(f)$ as follows:

$$P_\theta(f) \coloneqq \sup{}^* \{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \leq f \text{ quasi-everywhere}\}\ [6]. \tag{2.14}$$

The function $P_\theta(f)$ is either constantly $\pm\infty$ or lies in $\mathrm{PSH}(X,\theta)$. Moreover, given another function $g\colon X \to [-\infty,\infty]$, equal to f quasi-everywhere, we have $P_\theta(f) = P_\theta(g)$. In particular, it makes sense to talk about $P_\theta(f)$ even if f is only defined outside a pluripolar set.

We also observe that

$$P_\theta(f) \leq f \quad \text{quasi-everywhere}. \tag{2.15}$$

This is a consequence of Proposition 1.2.2 and Proposition 1.2.5.

Theorem 2.4.5 *Let $f\colon X \to [-\infty,\infty]$ be a quasi-continuous function.*

Assume that $P_\theta(f) \not\equiv \pm\infty$. Then $P_\theta(f) \in \mathrm{PSH}(X,\theta)$ and

$$\int_{\{P_\theta(f)<f\}} \theta^n_{P_\theta(f)} = 0. \tag{2.16}$$

Thanks to (2.15), we could rewrite (2.16) as

$$\int_{\{P_\theta(f)\neq f\}} \theta^n_{P_\theta(f)} = 0\ [7].$$

Proof **Step 1**. We first reduce to the case where f is bounded.

Step 1.1. Reduce to the case where $f \leq 0$.

Take $C \in \mathbb{R}$ so that $P_\theta(f) \leq C$. Then

[6] In the original article [DDNL23], the authors required that $\varphi \leq f$ everywhere. But in the proof of their Theorem 2.7 (Theorem 2.4.5 below), they actually relied on the current definition.

[7] It is tempting to say that $\theta^n_{P_\theta(f)}$ is supported on the contact set $\{P_\theta(f) = f\}$, as people are already doing in the literature. It should be mentioned that this is an abuse of the language, since the support of $\theta^n_{P_\theta(f)}$ (as a closed subset of X) could be much larger in general. One could probably introduce a notion of plurifine support, similar to what Fuglede did in the classical potential theory [Fug72].

$$P_\theta(f) = P_\theta\left(\min\{f, C\}\right).$$

So we could replace f by $\min\{f, C\}$. Replacing f by $f - \sup_X f$, we reduce easily to the case where $f \leq 0$.

Step 1.2. Reduce to the case where f is bounded. Assume that in this case, the theorem is known. We prove (2.16) for $f \leq 0$.

For each $j > 0$, we have

$$\int_{\{P_\theta(f_j)<f_j\}} \theta^n_{P_\theta(f_j)} = 0, \tag{2.17}$$

where $f_j = f \vee (-j)$.

Now fix $C > 0$ and two open sets $G' \Subset G \Subset X \setminus \mathrm{nK}(\{\theta\})$. Recall that the non-Kähler locus is introduced in Definition 1.7.6. Fix a smooth function $\chi\colon X \to [0,1]$ so that $\chi|_{G'} \equiv 1$ and χ is supported in G.

Define

$$U_C \coloneqq G \cap \{P_\theta(f) > V_\theta - C\}, \quad U'_C \coloneqq G' \cap \{P_\theta(f) > V_\theta - C\}.$$

Then U_C is $\mathcal{F}$-open. It follows from Proposition 2.2.1(1) that for any $j > 0$,

$$\mathbb{1}_{U_C}\, \theta^n_{P_\theta(f_j)\vee(V_\theta - C)} = \mathbb{1}_{U_C}\, \theta^n_{P_\theta(f_j)}.$$

In particular, thanks to (2.17),

$$\int_{U_C} \left(1 - \mathrm{e}^{P_\theta(f_j)-f_j}\right) \theta^n_{P_\theta(f_j)\vee(V_\theta-C)} = 0. \tag{2.18}$$

Note that the $P_\theta(f_j) \vee (V_\theta - C)$'s for various j are uniformly bounded from below, thanks to Theorem 2.4.2.

Next we claim that $P_\theta(f_j)$ is decreasing and converges to $P_\theta(f)$.

In fact, $P_\theta(f_j) \leq f_j$ quasi-everywhere. It follows that $\inf_j P_\theta(f_j) \leq f$ quasi-everywhere and hence

$$\inf_{j>0} P_\theta(f_j) \leq P_\theta(f).$$

The reverse inequality is trivial, and our assertion follows.

Since $P_\theta(f) \not\equiv -\infty$, we know that the set $\{f = -\infty\}$ is pluripolar. It follows that $P_\theta(f_j) - f_j$ converges to the following function $g\colon X \to [-\infty, \infty)$ in capacity:

$$g(x) = \begin{cases} P_\theta(f)(x) - f(x), & \text{if } f(x) \neq -\infty; \\ -\infty, & \text{otherwise.} \end{cases}$$

Therefore,

$$1 - \mathrm{e}^{P_\theta(f_j)-f_j} \xrightarrow{\mathrm{C}} 1 - \mathrm{e}^{g}.$$

Next we claim that

$$\int_{U'_C} (1-\mathrm{e}^g)\ \theta^n_{P_\theta(f)} = 0. \tag{2.19}$$

Since $g \leq 0$ quasi-everywhere, the left-hand side of (2.19) is non-negative. It suffices to prove the $\leq$ direction.

We wish to let $j \to \infty$ directly in (2.18), but since U_C is not open in general, this cannot be done directly. In the sequel, we shall slightly enlarge U_C to get an open set and then take the limit.

Fix $\epsilon > 0$, we can find an open subset $W \Subset X \setminus \mathrm{nK}(\{\theta\})$, so that

$$\mathrm{Cap}_{C^{-1}\theta}(W \setminus U_C) < \epsilon. \tag{2.20}$$

For example, we could define W in the following way: By Example 2.3.1, we can always find an open set $A \subseteq G$ so that $\mathrm{Cap}_{C^{-1}\theta}(A) < \epsilon$ and $P_\theta(f)$ is continuous on $G \setminus A$. Then it suffices to take $W = U_C \cup A$.

Thanks to (2.20) and (2.18), we have

$$\int_W \chi \left(1 - \mathrm{e}^{P_\theta(f_j)-f_j}\right)\ \theta^n_{P_\theta(f_j)\vee(V_\theta - C)} \leq C^n\epsilon.$$

Letting $j \to \infty$ and applying the convergence theorem of [GZ17, Theorem 4.26], we find that

$$\int_W \chi\, (1-\mathrm{e}^g)\ \theta^n_{P_\theta(f)\vee(V_\theta - C)} \leq C^n\epsilon.$$

Since $\epsilon > 0$ is arbitrary, (2.19) follows.

Now letting $C \to \infty$ in (2.19), we find

$$\int_{G'} (1-\mathrm{e}^g)\ \theta^n_{P_\theta(f)} = 0.$$

Since $G' \Subset X \setminus \mathrm{nK}(\{\theta\})$ is arbitrary, we finally conclude

$$\int_X (1-\mathrm{e}^g)\ \theta^n_{P_\theta(f)} = 0.$$

In other words, $P_\theta(f) = f$ almost everywhere with respect to $\theta^n_{P_\theta(f)}$. This proves (2.16).

Step 2. We now assume that $-C' \leq f \leq 0$ for some $C' > 0$.

For each $j > 0$, take an open subset $V_j \subseteq X$ so that

(1) $\mathrm{Cap}_\theta(V_j) \leq 2^{-j-1}$, and
(2) $f|_{X\setminus V_j}$ is continuous.

Without loss of generality, we may assume that the V_j's are decreasing. Take a continuous function $f_j : X \to [-C', 0]$ extending $f|_{X\setminus V_j}$. This is always possible thanks to Tietze's extension theorem. For each $j > 0$, we let

$$h_j := \sup_{k \geq j} f_k.$$

Then h_j is lower semi-continuous and h_j agrees with f outside V_j.

$$h = \inf_{j>0} h_j.$$

Then the set $\{g \neq f\}$ is contained in the intersection of the V_j's and hence is a pluripolar set, thanks to Theorem 2.3.2. In particular, $P_\theta(f) = P_\theta(g)$.

Now we can apply the *balayage* argument of [BT82, Corollary 9.2] to conclude that

$$\int_{X \setminus \mathrm{nK}(\{\theta\})} \left(1 - \mathrm{e}^{P_\theta(h_j) - h_j}\right) \theta^n_{P_\theta(h_j)} = 0$$

for each $j > 0$.

Fix two open subsets $G' \Subset G \Subset X \setminus \mathrm{nK}(\{\theta\})$. Note that $-C_0 \leq h_j \leq 0$. In particular,

$$V_\theta - C_0 \leq P_\theta(h_j) \leq V_\theta.$$

Hence,

$$\begin{aligned}
&\int_G \left(1 - \mathrm{e}^{P_\theta(h_j) - f}\right) \theta^n_{P_\theta(h_j)} \\
=&\int_{G \cap V_j} \left(1 - \mathrm{e}^{P_\theta(h_j) - f}\right) \theta^n_{P_\theta(h_j)} + \int_{G \setminus V_j} \left(1 - \mathrm{e}^{P_\theta(h_j) - h_j}\right) \theta^n_{P_\theta(h_j)} \\
\leq& 2^{-j-1} C_0^n.
\end{aligned}$$

Now we could apply the same arguments as in Step 1.2 to conclude that

$$\int_{G'} \left(1 - \mathrm{e}^{P_\theta(f) - f}\right) \theta^n_{P_\theta(f)} = 0.$$

Since $G' \Subset X \setminus \mathrm{nK}(\{\theta\})$, (2.16) follows. □

The following lemma is striking in that we begin only with an upper bound of φ, but at the end of the day, we get a lower bound almost for free. This powerful method will be employed again and again in the whole book.

Lemma 2.4.2 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, $\varphi \leq \psi$ and $\int_X \theta^n_\varphi > 0$. Then for any*

$$a \in \left(1, \left(\frac{\int_X \theta^n_\psi}{\int_X \theta^n_\psi - \int_X \theta^n_\varphi}\right)^{1/n}\right)^{8}, \tag{2.21}$$

there is $\eta \in \mathrm{PSH}(X, \theta)_{>0}$ such that

$$a^{-1}\eta + (1 - a^{-1})\psi \leq \varphi. \tag{2.22}$$

We write

[8] The fraction in (2.21) is understood as ∞ if either $\int_X \theta^n_\psi = \int_X \theta^n_\varphi$ or $n = 0$. Thanks to Theorem 2.4.4, the interval (2.21) is non-empty.

$$P_\theta(a\varphi+(1-a)\psi) \coloneqq \sup{}^*\left\{\eta\in \mathrm{PSH}(X,\theta) : a^{-1}\eta+(1-a^{-1})\psi\leq \varphi\right\} \\ \in\mathrm{PSH}(X,\theta)\cup\{-\infty\}. \tag{2.23}$$

Note that if we regard $a\varphi+(1-a)\psi$ as a function defined outside the pluripolar set $\{\varphi=\psi=-\infty\}$ on X, then (2.23) coincides with the envelope in the sense of (2.14).

Observe that

$$a^{-1}P_\theta(a\varphi+(1-a)\psi)+(1-a^{-1})\psi\leq\varphi. \tag{2.24}$$

In fact, this equation holds outside a pluripolar set by Proposition 1.2.5, hence it holds everywhere by Proposition 1.2.6.

Proof Without loss of generality, we may assume that $\varphi\leq\psi\leq 0$.

Step 1. We show the existence of $\eta\in\mathrm{PSH}(X,\theta)$ satisfying (2.22).

Step 1.1. We make a first reduction of the problem.

Define

$$\phi\coloneqq \sup{}^*\left\{\eta\in\mathrm{PSH}(X,\theta):\eta\leq\min\{\psi+C,0\}\text{ for some }C>0\right\}\,^{9}.$$

Observe that due to Corollary 2.4.1 and Theorem 2.4.4, we have

$$\int_X\theta_\phi^n=\int_X\theta_\psi^n.$$

In particular, replacing ψ by ϕ does not chance the condition on a as in (2.21).

Since $\psi\leq\phi$, it suffices to prove the existence of $\eta\in\mathrm{PSH}(X,\theta)$ so that

$$a^{-1}\eta+(1-a^{-1})\phi\leq\varphi. \tag{2.25}$$

Let us record the following observation for later use. Suppose that $\tau\in\mathrm{PSH}(X,\theta)$ and $\tau\leq\psi$. Then

$$\sup_{\{\phi\neq-\infty\}}(\tau-\phi)=\sup_X\tau. \tag{2.26}$$

Observe that $\phi\leq 0$, so on the set $\{\phi\neq-\infty\}$, we have

$$\tau-\phi\geq\tau.$$

So the $\geq$ direction in (2.26) follows from Corollary 1.3.6. Conversely, by assumption, we can find a constant $C>0$ so that

$$\tau-\sup_X\tau\leq\min\{\psi+C,0\}.$$

It follows that $\tau-\sup_X\tau\leq\phi$. Therefore, the reverse inequality follows.

For each $k>0$, we introduce

[9] In terms of the P-envelope introduced later in Definition 3.1.2, this equation says that $\phi=P_\theta[\psi]$.

$$\varphi_k = \varphi \vee (\phi - k)\,, \quad \eta_k \coloneqq P_\theta\left(a\varphi_k + (1-a)\phi\right).$$

Since $\varphi_k \sim \phi$, we have $\eta_k \in \mathrm{PSH}(X,\theta)$ and $\eta_k \sim \phi$ as well.

Note that η_k is decreasing in k. Let

$$\eta \coloneqq \inf_{k\in\mathbb{N}} \eta_k.$$

Note that η automatically satisfies (2.25). It remains to show that $\eta \not\equiv -\infty$.

Step 1.2. We prove that $\eta \not\equiv -\infty$.

Assume by contrary that $\sup_X \eta_k \to -\infty$. For each $k > 0$, let

$$\gamma_k \coloneqq a^{-1}\eta_k + (1-a^{-1})\phi, \quad D_k \coloneqq \left\{a^{-1}\eta_k + (1-a^{-1})\phi = \varphi_k\right\}.$$

We claim that

$$\theta^n_{\eta_k} \leq a^n \mathbb{1}_{D_k}\theta^n_{\varphi_k}. \tag{2.27}$$

Since $\gamma_k \leq \varphi_k$ with equality on D_k. It follows from Lemma 2.4.1 that

$$\mathbb{1}_{D_k}\theta^n_{\gamma_k} \leq \mathbb{1}_{D_k}\theta^n_{\varphi_k}.$$

Since

$$a^{-1}\eta_k + (1-a^{-1})\phi \leq \varphi_k,$$

we deduce from Theorem 2.4.4 that

$$\theta^n_{\eta_k} \leq a^n \theta^n_{\varphi_k}. \tag{2.28}$$

Finally, it follows from Theorem 2.4.5 that

$$\mathbb{1}_{D_k}\,\theta^n_{\eta_k} = \theta^n_{\eta_k}.$$

Putting these results together, we conclude (2.27).

Fix $j > k > 0$. Note that

$$\int_{\{\varphi\leq\phi-k\}} \theta^n_{\varphi_j} = \int_X \theta^n_{\varphi_j} - \int_{\{\varphi>\phi-k\}} \theta^n_{\varphi_j} = \int_X \theta^n_\phi - \int_{\{\varphi>\phi-k\}} \theta^n_\varphi, \tag{2.29}$$

where we applied Theorem 2.4.4 and Proposition 2.2.1(1) in the second inequality.

Next, we compute

$$
\begin{aligned}
\int_{\{\eta_j \le \phi - ak\}} \theta^n_{\eta_j} \le & a^n \int_{\{\eta_j \le \phi - ak\}} \mathbb{1}_{D_j} \theta^n_{\varphi_j} && \text{by (2.27)} \\
\le & a^n \int_{\{a\varphi_j + (1-a)\phi \le \phi - ak\} \cap \{\phi \neq -\infty\}} \theta^n_{\varphi_j} \\
= & a^n \int_{\{\varphi_j \le \phi - k\}} \theta^n_{\varphi_j} \\
= & a^n \int_{\{\varphi \le \phi - k\}} \theta^n_{\varphi_j} \\
\le & a^n \left(\int_X \theta^n_\phi - \int_{\{\varphi > \phi - k\}} \theta^n_\varphi \right) && \text{by (2.29).}
\end{aligned}
$$

Next thanks to (2.26),

$$
\sup_{\{\phi \neq -\infty\}} (\eta_j - \phi) = \sup_X \eta_j \longrightarrow -\infty.
$$

In particular, for a fixed k, if j is large enough, we have

$$
\{\eta_j \le \phi - ak\} = X.
$$

Therefore, for a fix k, for any large enough j,

$$
\int_X \theta^n_\phi = \int_X \theta^n_{\eta_j} \le a^n \left(\int_X \theta^n_\phi - \int_{\{\varphi > \phi - k\}} \theta^n_\varphi \right).
$$

Letting $k \longrightarrow \infty$, we find

$$
\int_X \theta^n_\phi \le a^n \left(\int_X \theta^n_\phi - \int_X \theta^n_\varphi \right).
$$

This is a contradiction with our choice of a.

Step 2. Next we argue that $P_\theta(a\varphi + (1-a)\psi) \in \mathrm{PSH}(X, \theta)_{>0}$. Choose

$$
a' \in \left(a, \left(\frac{\int_X \theta^n_\psi}{\int_X \theta^n_\psi - \int_X \theta^n_\varphi} \right)^{1/n} \right).
$$

It follows from (2.23) that

$$
P_\theta(a\varphi + (1-a)\psi) \ge \frac{a}{a'} P_\theta(a'\varphi + (1-a')\psi) + \frac{a'-a}{a'} \varphi. \tag{2.30}
$$

Therefore, by Theorem 2.4.4, we have

$$
\int_X \theta^n_{P_\theta(a\varphi + (1-a)\psi)} \ge \frac{(a'-a)^n}{a'^n} \int_X \theta^n_\varphi > 0. \tag{2.31}
$$

When φ and ψ have the same mass, we can say more:

Corollary 2.4.3 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, $\varphi \leq \psi$. Assume that $\int_X \theta_\varphi^n = \int_X \theta_\psi^n$. Then for any $\epsilon \in (0,1)$, there is $\eta \in \mathrm{PSH}(X,\theta)$ such that*

(1) $\int_X \theta_\eta^n = \int_X \theta_\varphi^n$;
(2) $\epsilon\eta + (1-\epsilon^{-1})\psi \leq \varphi$.

Note that by (2), we trivially have $\eta \leq \psi$.

Proof Fix $\epsilon \in (0,1)$, we define

$$\eta = P_\theta\left(\epsilon^{-1}\varphi + (1-\epsilon^{-1})\psi\right).$$

This is well-defined due to Theorem 2.4.4.

Thanks to (2.31), for each $a' > \epsilon^{-1}$, we have

$$\int_X \theta_\eta^n > \left(\frac{a'-\epsilon^{-1}}{a'}\right)^n \int_X \theta_\varphi^n.$$

Letting $a' \to \infty$, we conclude that

$$\int_X \theta_\eta^n \geq \int_X \theta_\varphi^n.$$

On the other hand, since $\eta \leq \psi$, using Theorem 2.4.4 we find that

$$\int_X \theta_\eta^n \leq \int_X \theta_\psi^n = \int_X \theta_\varphi^n.$$

Hence,

$$\int_X \theta_\eta^n = \int_X \theta_\varphi^n.$$

Next we prove a domination principle.

Theorem 2.4.6 (Domination principle) *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. Assume that there is $\phi \in \mathrm{PSH}(X,\theta)$ so that*

$$\varphi \leq \phi, \quad \psi \leq \phi, \quad \int_X \theta_\varphi^n = \int_X \theta_\psi^n = \int_X \theta_\phi^n, \tag{2.32}$$

and

$$\int_{\{\varphi<\psi\}} \theta_\varphi^n = 0. \tag{2.33}$$

Then $\varphi \geq \psi$.

Remark 2.4.3 Using the terminologies to be introduced in Chapter 3, we can reformulate a special case of Theorem 2.4.6 as follows: Suppose that $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ is a model potential and $\varphi, \psi \in \mathcal{E}(X,\theta;\phi)$. Assume that (2.33) holds then $\varphi \geq \psi$.

Proof Thanks to Theorem 2.4.4,

$$\int_X \theta^n_{\varphi\vee\psi} = \int_X \theta^n_{\phi}.$$

We may replace ψ by $\varphi \vee \psi$ and assume that $\varphi \leq \psi$.

Now fix $a > 1$, we define

$$\eta_a := P_\theta\left(a\varphi + (1-a)\psi\right).$$

Note that $\eta_a \in \mathrm{PSH}(X,\theta)$ by Corollary 2.4.3 and

$$\int_X \theta^n_{\eta_a} = \int_X \theta^n_{\phi}.$$

Define

$$\gamma_a := a^{-1}\eta_a + (1-a^{-1})\psi, \quad D_a := \{\gamma_a = \varphi\}.$$

Then $\gamma_a \leq \varphi$ with equality on D_a. Therefore,

$$\begin{aligned} a^{-n}\,\theta^n_{\eta_a} &\leq a^{-n}\,\theta^n_{\eta_a} + \mathbb{1}_{D_a}(1-a^{-1})^n\,\theta^n_\psi \\ &= \mathbb{1}_{D_a} a^{-n}\,\theta^n_{\eta_a} + \mathbb{1}_{D_a}(1-a^{-1})^n\,\theta^n_\psi && \text{by Theorem 2.4.5} \\ &\leq \mathbb{1}_{D_a}\theta^n_{\gamma_a} \\ &\leq \mathbb{1}_{D_a}\theta^n_{\varphi} && \text{by Lemma 2.4.1} \\ &= \mathbb{1}_{D_a\cap\{\varphi=\psi\}}\,\theta^n_\varphi && \text{by (2.33).} \end{aligned}$$

Note that on $D_a \cap \{\varphi = \psi\}$, we have $\eta_a = \varphi$. We deduce that

$$\theta^n_{\eta_a} = \mathbb{1}_{\{\eta_a=\varphi\}}\,\theta^n_{\eta_a} \leq \theta^n_\varphi,$$

where the inequality follows from Lemma 2.4.1.

But the two ends have the same mass, and hence

$$\theta^n_{\eta_a} = \theta^n_\varphi = \mathbb{1}_{\{\eta_a=\varphi\}}\,\theta^n_\varphi.$$

Therefore,

$$\int_X \mathrm{e}^{\eta_a}\,\theta^n_\varphi = \int_X \mathrm{e}^{\varphi}\,\theta^n_\varphi > 0.$$

Note that η_a is decreasing in a. The above equation shows that

$$\eta := \inf_{a>1}\eta_a \not\equiv -\infty.$$

On the other hand, if $x \in X$ is such that $\varphi(x) < \psi(x)$, we then have

$$\eta_a(x) \leq a\varphi(x) - (a-1)\psi(x) \leq \psi(x) + a\left(\varphi(x) - \psi(x)\right).$$

Letting $a \to \infty$, we find that $\eta(x) = -\infty$. Therefore, $\{\varphi \neq \psi\}$ is pluripolar, and hence empty by Proposition 1.2.6. Our assertion follows. □

Lemma 2.4.3 *For any $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, there is $\psi \in \mathrm{PSH}(X, \theta)$ such that*

(1) *θ_ψ is a Kähler current, and*
(2) *$\psi \leq \varphi$.*

In particular, there is an increasing sequence $(\varphi_i)_{i>0}$ in $\mathrm{PSH}(X, \theta)$ converging almost everywhere to φ such that θ_{φ_i} is a Kähler current for all $i \geq 1$.

Proof Using Lemma 2.4.2, we can find $\epsilon \in (0, 1)$ and $\gamma \in \mathrm{PSH}(X, \theta)$ such that

$$\epsilon V_\theta + (1 - \epsilon)\gamma \leq \varphi.$$

We observe that the cohomology class $[\theta]$ is big as a consequence of Proposition 2.4.1. Therefore, we can take $\eta \in \mathrm{PSH}(X, \theta)$ such that θ_η is a Kähler current and $\eta \leq 0$. Then we may take

$$\psi \coloneqq \epsilon\eta + (1 - \epsilon)\gamma.$$

Then ψ clearly satisfies (1) and (2).

For the latter claim, it suffices to take

$$\varphi_i = \left(1 - (i + 1)^{-1}\right) \varphi + (i + 1)^{-1}\psi$$

for each $i > 0$. □

Lemma 2.4.4 *Let L be a holomorphic line bundle on X with $\theta \in c_1(L)$. Assume that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then there exists $k_0 > 0$ such that for each $k \geq k_0$, we have*

$$\mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) \neq 0.$$

Proof By Lemma 2.4.3, we may further assume that θ_φ is a Kähler current. In this case, the result follows from Hörmander's L^2-estimate, see [Dem12a, Theorem 13.21]. □

Chapter 3
The envelope operators

Politiques et scientifiques ont le sens des réalités, mais ce ne sont pas les mêmes. Il en résulte — et ce sera là un principe que le général de Gaulle fera sien que l'activité de recherche ne peut être évaluée, quant à sa qualité propre, que par des hommes qui la pratiquent eux-mêmes.
— Pierre Lelong[a], 1999

[a] Pierre Lelong (1912–2011) was the husband of another famous mathematician Jacqueline Ferrand. During their marriage (1947–1977), the latter published under the name of Jacqueline Lelong-Ferrand.

In this chapter, we study two envelope operators lying at the heart of the whole theory. The first envelope, called the P-envelope, is defined using the non-pluripolar masses, while the second, called the $\mathcal{I}$-envelope, is defined using the multiplier ideal sheaves. The corresponding theories are developed in Section 3.1 and Section 3.2 respectively.

Later on in Chapter 6, we will develop the corresponding P and $\mathcal{I}$-partial orders associated with these envelopes, allowing us to compare the singularities.

We reproduced a large number of proofs, which are already explained in detail in the survey of Darvas–Di Nezza–Lu [DDNL23] at the strong request of the referee. Personally I would encourage the readers to skip these lengthy details, at least on a first reading. If the readers do wish to understand these techniques in detail, their survey is much more helpful.

3.1 The P-envelope

In this section, X will denote a connected compact Kähler manifold of dimension n.

3.1.1 Rooftop operator and the definition of the P-envelope

We will fix a smooth closed real $(1,1)$-form θ on X.

Definition 3.1.1 Given $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we define their *rooftop operator* as follows:

$$\varphi \wedge \psi = \sup\left\{\eta \in \mathrm{PSH}(X,\theta) : \eta \le \varphi, \eta \le \psi\right\}. \tag{3.1}$$

For the simplicity of notations, we extend the definition to the case where φ or ψ is constantly $-\infty$, in which case we simply set

$$\varphi \wedge \psi = -\infty.$$

When we want to be more specific, we could also write $\varphi \wedge_\theta \psi$.

Proposition 3.1.1 *The operator $\wedge$ is a well-defined commutative, associative binary operator*

$$\mathrm{PSH}(X,\theta) \cup \{-\infty\} \times \mathrm{PSH}(X,\theta) \cup \{-\infty\} \longrightarrow \mathrm{PSH}(X,\theta) \cup \{-\infty\}.$$

Proof We first show that the map is well-defined. For this purpose, take $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. When the set in (3.1) is empty, there is nothing to prove. So let us assume that the set is not empty.

Define

$$\gamma = \sup{}^* \{\eta \in \mathrm{PSH}(X,\theta) : \eta \leq \varphi, \eta \leq \psi\}.$$

Then by Proposition 1.2.1, we find that $\gamma \in \mathrm{PSH}(X,\theta)$ and hence γ is a candidate for the supremum in (3.1). Therefore, $\gamma \leq \varphi \wedge \psi$. The reverse inequality is trivial, so

$$\varphi \wedge \psi = \gamma \in \mathrm{PSH}(X,\theta).$$

The commutativity and the associativity of $\wedge$ are both trivial. □

Lemma 3.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi \wedge \psi \in \mathrm{PSH}(X,\theta)$. Then*

$$\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\theta^n_\varphi + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\theta^n_\psi. \tag{3.2}$$

Proof We first observe that as a consequence of Lemma 2.4.1, we have

$$\mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\,\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\,\theta^n_\varphi, \quad \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\,\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\,\theta^n_\psi.$$

Applying Theorem 2.4.5 to $\min\{\varphi,\psi\}$, we conclude that

$$\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\,\theta^n_{\varphi\wedge\psi} + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\,\theta^n_{\varphi\wedge\psi} \leq \mathbb{1}_{\{\varphi\wedge\psi=\varphi\}}\,\theta^n_\varphi + \mathbb{1}_{\{\varphi\wedge\psi=\psi\}}\,\theta^n_\psi,$$

and (3.2) is established. □

We recall that the relations $\preceq$ and $\sim$ are introduced in Definition 1.5.2.

Definition 3.1.2 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define its *$P$-envelope* as follows:

$$P_\theta[\varphi] := \sup{}^* \{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq \varphi\}. \tag{3.3}$$

Observe that by Proposition 1.2.1, we have $P_\theta[\varphi] \in \mathrm{PSH}(X,\theta)$ and $P_\theta[\varphi] \leq 0$. Moreover, the definition can be equivalently described as

$$P_\theta[\varphi] = \sup_{C\in\mathbb{Z}_{>0}}{}^*(\varphi + C) \wedge V_\theta. \tag{3.4}$$

Recall that V_θ is introduced in (2.9). For any $C \in \mathbb{R}$, we have $(\varphi+C)\wedge V_\theta \in \mathrm{PSH}(X,\theta)$ and

$$(\varphi + C) \wedge V_\theta \sim \varphi.$$

In other words, in (3.3), we may replace the condition $\psi \leq \varphi$ by $\psi \sim \varphi$.

The idea lying behind the definition of $P_\theta[\varphi]$ is that we choose the least singular element out of all potentials with the same singularity type as φ. As we shall see in Example 3.1.1 below, $P_\theta[\varphi]$ does not necessarily have the same singularity type as φ. This forces us to define a rougher equivalence relation in Definition 6.1.1.

The envelope depends on the choice of θ, but the dependence is easy to understand:

Proposition 3.1.2 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$P_\theta[\varphi] \sim P_{\theta'}[\varphi'].$$

Proof By symmetry, it suffices to show that

$$P_\theta[\varphi] \leq P_{\theta'}[\varphi'].$$

We may assume that $g \geq 0$. Then for any $\psi \in \mathrm{PSH}(X,\theta)$ with $\psi \leq \varphi$ and $\psi \leq 0$, we set $\psi' := \psi - g \in \mathrm{PSH}(X,\theta')$. Then $\psi' \leq \varphi'$ and $\psi' \leq 0$, so $\psi' \leq P_{\theta'}[\varphi']$. Since ψ is arbitrary, it follows that

$$P_\theta[\varphi] - \sup_X g \leq P_\theta[\varphi] - g \leq P_{\theta'}[\varphi'].$$

The P-envelope preserves the non-pluripolar masses:

Proposition 3.1.3 *Suppose that $\theta_1, \dots, \theta_n$ be smooth closed real $(1,1)$-forms on X. Let $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ for each $i = 1, \dots, n$. Then*

$$\int_X \theta_{1,P_{\theta_1}[\varphi_1]} \wedge \cdots \wedge \theta_{n,P_{\theta_n}[\varphi_n]} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{3.5}$$

This proposition together with Theorem 2.4.4 will be combined to a common generalization in Proposition 6.1.4 after introducing the P-partial order.

Proof For each $C \in \mathbb{Z}_{>0}$ and each $i = 1, \dots, n$, we have

$$(\varphi_i + C) \wedge V_{\theta_i} \sim \varphi_i.$$

It follows from Theorem 2.4.4 that

$$\int_X \theta_{1,(\varphi_1+C)\wedge V_{\theta_1}} \wedge \cdots \wedge \theta_{n,(\varphi_n+C)\wedge V_{\theta_n}} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}.$$

So (3.5) follows from (3.4) and Corollary 2.4.1. □

Conversely, Proposition 3.1.3 characterizes the P-envelope, as we will see in a moment in Theorem 3.1.2. We need some preparations for the proof. We first show that the P-envelope can be regarded as a concentration of the non-pluripolar mass:

Theorem 3.1.1 *Let $\varphi \in \mathrm{PSH}(X, \theta)$, then*

$$\theta^n_{P_\theta[\varphi]} \leq \mathbb{1}_{\{P_\theta[\varphi]=0\}}\, \theta^n. \tag{3.6}$$

Proof Thanks to Lemma 3.1.1, for each $C > 0$, we have

$$\begin{aligned}\theta^n_{(\varphi+C)\wedge V_\theta} &\leq \mathbb{1}_{\{(\varphi+C)\wedge V_\theta=\varphi+C\}}\, \theta^n_\varphi + \mathbb{1}_{\{(\varphi+C)\wedge V_\theta=V_\theta\}}\, \theta^n_{V_\theta} \\ &\leq \mathbb{1}_{\{\varphi+C\leq V_\theta\}}\, \theta^n_\varphi + \mathbb{1}_{\{P_\theta[\varphi]=V_\theta\}}\, \theta^n_{V_\theta}.\end{aligned}$$

We wish to let $C \to \infty$. The dominated convergence theorem assures that $\mathbb{1}_{\{\varphi+C\leq V_\theta\}}\theta^n_\varphi$ converges weakly to 0. While Theorem 2.4.3 and Proposition 3.1.3 guarantee that $\theta^n_{(\varphi+C)\wedge V_\theta}$ converges weakly to $\theta^n_{P_\theta[\varphi]}$. So we conclude that

$$\theta^n_{P_\theta[\varphi]} \leq \mathbb{1}_{\{P_\theta[\varphi]=V_\theta\}}\, \theta^n_{V_\theta}.$$

Taking Theorem 2.4.2 into consideration, we conclude (3.6). □

Using essentially the same proof, we arrive at the following conclusion:

Corollary 3.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $\varphi \leq \psi$. Let*

$$\eta \coloneqq \sup_{C>0}{}^{*} (\varphi + C) \wedge \psi.$$

Then

$$\theta^n_\eta \leq \mathbb{1}_{\{\eta=\psi\}}\, \theta^n_\psi.$$

Theorem 3.1.2 *Assume that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then*

$$P_\theta[\varphi] = \sup \left\{ \psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \varphi \leq \psi, \int_X \theta^n_\varphi = \int_X \theta^n_\psi \right\}. \tag{3.7}$$

In particular, in this case,

$$P_\theta\left[P_\theta[\varphi]\right] = P_\theta[\varphi]. \tag{3.8}$$

Note that in (3.7) and (3.3), the auxiliary function ψ lies on different sides of φ.

Proof Let ψ be a candidate of the right-hand side of (3.7). It follows from Theorem 3.1.1 that

$$\int_{\{P_\theta[\varphi]<\psi\}} \theta^n_{P_\theta[\varphi]} \leq \int_{\{P_\theta[\varphi]<\psi\}\cap\{P_\theta[\varphi]=0\}} \theta^n = 0.$$

On the other hand, we have

$$P_\theta[\varphi] \leq P_\theta[\psi], \quad \psi \leq P_\theta[\psi]$$

and all these potentials have the same mass due to Proposition 3.1.3. So the domination principle Theorem 2.4.6 is applicable and gives

$$P_\theta[\varphi] \geq \psi.$$

Hence we get the $\geq$ direction in (3.7).

Let γ denote the upper semi-continuous regularization of the right-hand side of (3.7). We also find

$$P_\theta[\varphi] \geq \gamma \geq \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \varphi \leq \psi, \int_X \theta_\varphi^n = \int_X \theta_\psi^n\right\}.$$

On the other hand, $P_\theta[\varphi]$ itself is a candidate of the right-hand side of (3.7), as a consequence of Proposition 3.1.3. Therefore, (3.7) follows.

As for (3.8), the $\geq$ direction is trivial. While for the other direction, let $\psi \in \mathrm{PSH}(X,\theta)$ be a potential satisfying

$$\psi \leq 0, \quad P_\theta[\varphi] \leq \psi, \quad \int_X \theta_{P_\theta[\varphi]}^n = \int_X \theta_\psi^n.$$

Then it follows from Proposition 3.1.3 that

$$\psi \leq 0, \quad \varphi \leq \psi, \quad \int_X \theta_\varphi^n = \int_X \theta_\psi^n.$$

In view of (3.7), we conclude the $\leq$ direction of (3.8). □

In general, we do not know if (3.8) holds when $\int_X \theta_\varphi^n = 0$. We expect it to be wrong. According to our general philosophy, the P-envelope operator is the correct object only when the non-pluripolar mass is positive. We will avoid using the degenerate case in the whole book.

Definition 3.1.3 If $\varphi = P_\theta[\varphi]$ and $\int_X \theta_\varphi^n > 0$, we say φ is a *model potential*.

We remind the readers that the notion of model potentials depends heavily on the choice of θ. When there is a risk of confusion, we also say φ is a model potential in $\mathrm{PSH}(X,\theta)$.

Remark 3.1.1 Definition 3.1.3 is different from the common definition in the literature: We impose the extra condition $\int_X \theta_\varphi^n > 0$. The author believes that this is the only case where this notion is natural. We sometimes emphasize this point by saying $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ is a model potential.

There are plenty of model potentials:

Corollary 3.1.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then $P_\theta[\varphi]$ is a model potential in $\mathrm{PSH}(X,\theta)$. Moreover,*

$$\int_X \theta_{P_\theta[\varphi]}^n = \int_X \theta_\varphi^n.$$

Proof This follows immediately from Theorem 3.1.2 and Proposition 3.1.3. □

As we have seen in the proof of Lemma 2.4.2, we have the following interesting property:

Proposition 3.1.4 *Let* $\varphi \in \mathrm{PSH}(X,\theta)$. *Consider a model potential* $\phi \in \mathrm{PSH}(X,\theta)_{>0}$. *Assume that* $\varphi \preceq \phi$. *Then*

$$\sup_{\{\phi\neq-\infty\}}(\varphi-\phi)=\sup_X\varphi.$$

Example 3.1.1 We continue our favorite example Example 1.8.1. Let $X=\mathbb{P}^1$ and ω be the Fubini–Study metric. We define $\varphi\in\mathrm{PSH}(X,\omega)$ as follows: For $z\in\mathbb{C}$, we let

$$\varphi(z)=\begin{cases}-\log(|z|^2+1)+\left(-\log\left(-\log|z|^2\right)\right)\vee\left(2+\log|z|^2\right), & \text{if } |z|<1/\sqrt{2},\\ 2+\log\dfrac{|z|^2}{|z|^2+1}, & \text{Otherwise,}\end{cases}$$

while $\varphi(\infty)=2$. The singularity of φ only occurs at $z=0$, close to which, $\varphi\sim-\log\left(-\log|z|^2\right)$. This type of singularity is therefore called the *log-log type singularity*.

We claim that

$$P_\omega[\varphi]=0. \tag{3.9}$$

In particular, we find that φ and $P_\omega[\varphi]$ have different singularity types.

Due to Theorem 3.1.2, in order to verify (3.9), it suffices to verify that

$$\int_X\omega_\varphi=1. \tag{3.10}$$

Here ω_φ is taken in the non-pluripolar sense. Since $\{0,\infty\}\subseteq\mathbb{P}^1$ is pluripolar, this reduces to show that

$$\int_{\mathbb{C}^*}\mathrm{dd^c}\psi=\frac{1}{4\pi}\int_{\mathbb{C}^*}(\Delta\psi)\,\mathrm{d}\mu=1,$$

where $\psi(z)=\varphi(z)+\log(|z|^2+1)$ and μ is the standard Lebesgue measure on $\mathbb{C}$.

Note that the Laplacian vanishes outside $\overline{B(0,0.7)}$ since $\psi(z)=2+\log|z|^2$ there, which is harmonic. Therefore,

$$\int_{\mathbb{C}^*}\mathrm{dd^c}\psi=\frac{1}{4\pi}\int_{|z|<1/\sqrt{2}}(\Delta\psi)(z)\,\mathrm{d}\mu.$$

It is an elementary exercise to see that the right-hand side is exactly equal to 1. If you are familiar with toric geometry, this is more or less trivial since

$$\nabla_r\left((-\log(-r))\vee(2+r)\right)(-\infty,-\log 2)=[-1,0).$$

Otherwise, just try to evaluate the integral using Green's identities. Therefore, (3.10) is proved and our assertion (3.9) follows.

Next we give a criterion on when the rooftop operator is not identically $-\infty$.

Proposition 3.1.5 *Assume that $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ and*

$$\int_X \theta_\varphi^n + \int_X \theta_\psi^n > \int_X \theta_{\varphi \vee \psi}^n. \tag{3.11}$$

Then $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$.

Thanks to Theorem 2.4.4, we may also replace $\varphi \vee \psi$ on the right-hand side of (3.11) by any $\gamma \in \mathrm{PSH}(X, \theta)$ such that $\varphi \vee \psi \leq \gamma$.

Proof Without loss of generality, we may assume that $\varphi, \psi \leq 0$. For simplicity, we write

$$\eta = P_\theta[\varphi \vee \psi].$$

Take $C > 0$ large enough, so that

$$\int_{\{\varphi > \eta - C\}} \theta_\varphi^n + \int_{\{\psi > \eta - C\}} \theta_\psi^n > \int_X \theta_\eta^n. \tag{3.12}$$

This is possible thanks to Proposition 2.2.1(4). Fix $C' > C$. Write

$$\gamma_{C'} := (\varphi \vee (\eta - C')) \wedge (\psi \vee (\eta - C')).$$

Then observe that $\gamma_{C'} \sim \eta$, and

$$\inf_{C' > C} \gamma_{C'} = \varphi \wedge \psi.$$

Assume by contradiction that $\varphi \wedge \psi \equiv -\infty$, then we have

$$\lim_{C' \to \infty} \sup_X \gamma_{C'} = -\infty.$$

Thanks to Proposition 3.1.4, for each $C' > C$,

$$\sup_X \gamma_{C'} = \sup_{\{\eta \neq -\infty\}} (\gamma_{C'} - \eta),$$

since η is a model potential. It follows that

$$\lim_{C' \to \infty} \sup_{\{\eta \neq -\infty\}} (\gamma_{C'} - \eta) = -\infty. \tag{3.13}$$

In particular, we could take C' large enough so that

$$\gamma_{C'} \leq \eta - C.$$

For such C', we have

$$\begin{aligned}
\int_X \theta^n_{\gamma_{C'}} &= \int_{\{\gamma_{C'}\leq \eta - C\}} \theta^n_{\gamma_{C'}} \\
&\leq \int_{\{\varphi\vee(\eta-C')\leq\eta-C\}} \theta^n_{\varphi\vee(\eta-C')} + \int_{\{\psi\vee(\eta-C')\leq\eta-C\}} \theta^n_{\psi\vee(\eta-C')} \\
&=2\int_X \theta^n_\eta - \int_{\{\varphi>\eta-C\}} \theta^n_\varphi - \int_{\{\psi>\eta-C\}} \theta^n_\psi \\
&< \int_X \theta^n_\eta,
\end{aligned}$$

where the second line follows from Lemma 3.1.1, the fourth line follows from (3.12). This contradicts the fact that $\gamma_{C'} \sim \eta$ in view of Theorem 2.4.4. □

Proposition 3.1.6 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that*

$$\varphi = P_\theta[\varphi], \quad \psi = P_\theta[\psi], \quad \varphi\wedge\psi \not\equiv -\infty. \tag{3.14}$$

Then

$$P_\theta[\varphi\wedge\psi] = \varphi\wedge\psi. \tag{3.15}$$

Proof Observe that

$$P_\theta[\varphi\wedge\psi] \leq P_\theta[\varphi] = \varphi, \quad P_\theta[\varphi\wedge\psi] \leq P_\theta[\psi] = \psi.$$

So the $\leq$ direction in (3.15) holds. The reverse direction is trivial. □

Lemma 3.1.2 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$. Assume that (3.14) holds and*

$$\int_X \theta^n_{\varphi\wedge\psi} > 0.$$

Suppose that $\gamma, \eta \in \mathrm{PSH}(X,\theta)$ and satisfy

$$\gamma \leq \varphi, \quad \eta \leq \psi, \quad \int_X \theta^n_\gamma = \int_X \theta^n_\varphi, \quad \int_X \theta^n_\eta = \int_X \theta^n_\psi.$$

Then $\gamma\wedge\eta \not\equiv -\infty$ and

$$\int_X \theta^n_{\gamma\wedge\eta} = \int_X \theta^n_{\varphi\wedge\psi}. \tag{3.16}$$

Proof Without loss of generality, we may assume that

$$\gamma \leq \varphi, \quad \eta \leq \psi.$$

Step 1. We first show that $\gamma\wedge\psi \not\equiv -\infty$ and

$$\int_X \theta^n_{\gamma\wedge\psi} = \int_X \theta^n_{\varphi\wedge\psi}. \tag{3.17}$$

For any $a > 1$, we define

$$\gamma_a \coloneqq P_\theta\left(a\gamma + (1-a)\varphi\right).$$

When $a = 1$, we simply write $\gamma_1 = \gamma$.

Thanks to Corollary 2.4.3, for any $a \geq 1$, we have $\gamma_a \leq \varphi$ and

$$\int_X \theta_{\gamma_a}^n = \int_X \theta_\varphi^n.$$

By our assumption, for any $a \geq 1$, we have

$$\int_X \theta_{\gamma_a}^n + \int_X \theta_{\varphi\wedge\psi}^n > \int_X \theta_\varphi^n, \quad \gamma_a \leq \varphi, \quad \varphi \wedge \psi \leq \varphi.$$

So Proposition 3.1.5 gives

$$\gamma_a \wedge \varphi \wedge \psi = \gamma_a \wedge \psi \in \mathrm{PSH}(X,\theta).$$

In particular, for $a = 1$, we find $\gamma \wedge \varphi \not\equiv -\infty$. It remains to verify (3.17).

For $a > 1$, we have

$$\gamma \geq a^{-1}\gamma_a + \left(1 - a^{-1}\right)\varphi,$$

thanks to the definition of γ_a (c.f. (2.24)). Hence

$$\gamma \wedge \psi \geq a^{-1}\left(\gamma_a \wedge \psi\right) + \left(1 - a^{-1}\right)\left(\varphi \wedge \psi\right).$$

Therefore, by Theorem 2.4.4,

$$\int_X \theta_{\gamma\wedge\psi}^n \geq \left(1 - a^{-1}\right)^n \int_X \theta_{\varphi\wedge\psi}^n.$$

Letting $a \to \infty$ and taking Theorem 2.4.4 into account, (3.17) follows.

Step 2. We complete the proof.

For any $a > 1$, we let

$$\eta_a \coloneqq P_\theta\left(a\eta + (1-a)\psi\right).$$

When $a = 1$, we set $\eta_1 = \eta$. Thanks to Corollary 2.4.3, for each $a \geq 1$, we have $\eta_a \leq \varphi$ and

$$\int_X \theta_{\eta_a}^n = \int_X \theta_\psi^n.$$

From Step 1, we have

$$\int_X \theta_{\gamma\wedge\psi}^n + \int_X \theta_{\eta_a}^n = \int_X \theta_{\varphi\wedge\psi}^n + \int_X \theta_\psi^n > \int_X \theta_\psi^n.$$

Applying Proposition 3.1.5 again, we find

$$\gamma \wedge \psi \wedge \eta_a = \gamma \wedge \eta_a \in \mathrm{PSH}(X,\theta).$$

When $a = 1$, we find $\gamma \wedge \eta \in \mathrm{PSH}(X, \theta)$. It remains to prove (3.16).

By definition of η_a, we have

$$\eta \geq a^{-1}\eta_a + \left(1 - a^{-1}\right)\psi.$$

Therefore,

$$\gamma \wedge \eta \geq a^{-1}(\gamma \wedge \eta_a) + \left(1 - a^{-1}\right)(\gamma \wedge \psi).$$

By Theorem 2.4.4,

$$\int_X \theta^n_{\gamma\wedge\eta} \geq \left(1 - a^{-1}\right)^n \int_X \theta^n_{\gamma\wedge\psi}.$$

Letting $a \to \infty$ and applying Theorem 2.4.4 again, we arrive at (3.16). □

There is an interesting diamond-like inequality regarding the non-pluripolar masses:

Theorem 3.1.3 *Assume that* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ *and* $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$. *Then*

$$\int_X \theta^n_\varphi + \int_X \theta^n_\psi \leq \int_X \theta^n_{\varphi\vee\psi} + \int_X \theta^n_{\varphi\wedge\psi}. \tag{3.18}$$

Proof We may assume that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$, as otherwise (3.18) follows immediately from Theorem 2.4.4.

Step 1. We claim that it suffices to prove (3.18) with $P_\theta[\varphi]$ and $P_\theta[\psi]$ in place of φ and ψ.

In fact, as $\varphi \wedge \psi \not\equiv -\infty$, we also have $P_\theta[\varphi] \wedge P_\theta[\psi] \not\equiv -\infty$. Moreover,

$$\int_X \theta^n_{P_\theta[\varphi]} = \int_X \theta^n_\varphi, \quad \int_X \theta^n_{P_\theta[\psi]} = \int_X \theta^n_\psi$$

by Proposition 3.1.3. On the other hand, as $C \to \infty$, the potentials

$$((\varphi + C) \wedge V_\theta) \vee ((\psi + C) \wedge V_\theta)$$

converge to $P_\theta[\varphi] \vee P_\theta[\psi]$ almost everywhere. Therefore, thanks to Corollary 2.4.1 and Theorem 2.4.4,

$$\int_X \theta^n_{P_\theta[\varphi]\vee P_\theta[\psi]} = \int_X \theta^n_{\varphi\vee\psi}.$$

Finally, we have

$$\int_X \theta^n_{P_\theta[\varphi]\wedge P_\theta[\psi]} = \int_X \theta^n_{\varphi\wedge\psi}$$

as well. When the left-hand side vanishes, this follows from Theorem 2.4.4. Otherwise, it follows from Lemma 3.1.2.

In particular, (3.18) is equivalent to the corresponding result with $P_\theta[\varphi]$ and $P_\theta[\psi]$ in place of φ and ψ.

Step 2. We shall assume that φ and ψ are model potentials in the sequel.

We write $\gamma = \varphi \vee \psi$ for simplicity.

For each $C > 0$, we introduce

$$\varphi_C = \varphi \vee (\gamma - C), \quad \psi_C = \psi \vee (\gamma - C).$$

Note that $\varphi_C \sim \psi_C \sim \gamma$.

For $C > 0$, we compute

$$\begin{aligned}
\theta^n_{\varphi_C} =& \mathbb{1}_{\{\varphi > \gamma - C\}}\, \theta^n_{\varphi_C} + \mathbb{1}_{\{\varphi \leq \gamma - C\}}\, \theta^n_{\varphi_C} \\
=& \mathbb{1}_{\{\varphi > \gamma - C\}}\, \theta^n_{\varphi} + \mathbb{1}_{\{\varphi \leq \gamma - C\}}\, \theta^n_{\varphi_C} && \text{by Proposition 2.2.1} \\
=& \theta^n_{\varphi} + \mathbb{1}_{\{\varphi \leq \gamma - C\}}\, \theta^n_{\varphi_C} && \text{by Theorem 3.1.1.}
\end{aligned}$$

We also get a similar formula for $\theta^n_{\psi_C}$.

Therefore, taking Theorem 2.4.4 into consideration, we find

$$\int_X \theta^n_\gamma - \int_X \theta^n_\varphi = \int_{\{\varphi \leq \gamma - C\}} \theta^n_{\varphi_C}, \quad \int_X \theta^n_\gamma - \int_X \theta^n_\psi = \int_{\{\psi \leq \gamma - C\}} \theta^n_{\psi_C}. \tag{3.19}$$

On the other hand, using Lemma 3.1.1, we find

$$\begin{aligned}
\theta^n_{\varphi_C \wedge \psi_C} \leq& \mathbb{1}_{\{\varphi_C \wedge \psi_C = \varphi_C\}}\, \theta^n_{\varphi_C} + \mathbb{1}_{\{\varphi_C \wedge \psi_C = \psi_C\}}\, \theta^n_{\psi_C} \\
\leq& \mathbb{1}_{\{\varphi_C \wedge \psi_C = \varphi_C = 0\}}\, \theta^n_{\varphi} + \mathbb{1}_{\{\varphi_C \wedge \psi_C = \psi_C = 0\}}\, \theta^n_{\psi} + \mathbb{1}_{\{\varphi \leq \gamma - C\}}\, \theta^n_{\varphi_C} \\
&+ \mathbb{1}_{\{\psi \leq \gamma - C\}}\, \theta^n_{\psi_C}.
\end{aligned}$$

In particular,

$$\mathbb{1}_{\{\varphi_C \wedge \psi_C < 0\}}\, \theta^n_{\varphi_C \wedge \psi_C} \leq \mathbb{1}_{\{\varphi \leq \gamma - C\}}\, \theta^n_{\varphi_C} + \mathbb{1}_{\{\psi \leq \gamma - C\}}\, \theta^n_{\psi_C}.$$

Taking integration, we find

$$\begin{aligned}
\int_{\{\varphi_C \wedge \psi_C = 0\}} \theta^n_{\varphi_C \wedge \psi_C} \geq& \int_X \theta^n_\gamma - \int_{\{\varphi \leq \gamma - C\}} \theta^n_{\varphi_C} - \int_{\{\psi \leq \gamma - C\}} \theta^n_{\psi_C} \\
=& \int_X \theta^n_\varphi + \int_X \theta^n_\psi - \int_X \theta^n_\gamma,
\end{aligned}$$

where on the first line we used the fact that $\varphi_C \wedge \psi_C \sim \gamma$, while on the second line, we used (3.19).

Letting $C \to \infty$, and using Corollary 2.4.2, we conclude (3.18). □

3.1.2 Properties of the P-envelope

Let $\theta, \theta_1, \theta_2$ be smooth closed real $(1,1)$-forms on X.

Proposition 3.1.7 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for any $\varphi \in \mathrm{PSH}(X, \theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi] = \pi^* P_\theta[\varphi].$$

In particular, a potential $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ *is model if and only if* $\pi^*\varphi \in \mathrm{PSH}(Y,\pi^*\theta)_{>0}$ *is model.*

Proof This follows immediately from Proposition 1.5.3. □

We have the following concavity property of the P-envelope.

Proposition 3.1.8

(1) *Suppose that* $\varphi \in \mathrm{PSH}(X,\theta)$ *and* $\lambda \in \mathbb{R}_{>0}$*, then*

$$P_{\lambda\theta}[\lambda\varphi] = \lambda P_\theta[\varphi].$$

(2) *Suppose that* $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ *and* $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$*, then*

$$P_{\theta_1+\theta_2}[\varphi_1+\varphi_2] \geq P_{\theta_1}[\varphi_1] + P_{\theta_2}[\varphi_2].$$

(3) *Suppose that* $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ *and* $\varphi \leq \psi$*, then*

$$P_\theta[\varphi] \leq P_\theta[\psi].$$

Proof (1) This is obvious by definition.

(2) Suppose that $\psi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\psi_2 \in \mathrm{PSH}(X,\theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \leq \varphi_i$$

for $i = 1, 2$. Then

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \leq \varphi_1 + \varphi_2.$$

It follows from (3.3) that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1+\varphi_2].$$

Since ψ_1 and ψ_2 are arbitrary, we conclude.

(3) This is obvious by definition. □

Proposition 3.1.9 *Let* $(\varphi_j)_{j\in I}$ *be a decreasing net of potentials in* $\mathrm{PSH}(X,\theta)$ *satisfying* $P_\theta[\varphi_j] = \varphi_j$ *for each* $j \in I$*. Set* $\varphi := \inf_{j\in I}\varphi_j$*. Then* $P_\theta[\varphi] = \varphi$*.*

Proof Since $\sup_X \varphi_j = 0$ for all $j \in I$, we know that $\varphi \not\equiv -\infty$. It follows from Proposition 1.2.1 that $\varphi \in \mathrm{PSH}(X,\theta)$. Therefore, for each $j \in I$,

$$\varphi \leq P_\theta[\varphi] \leq P_\theta[\varphi_j] = \varphi_j.$$

Therefore, $\varphi = P_\theta[\varphi]$. □

Proposition 3.1.10 *Let* $(\epsilon_j)_{j\in I}$ *be a decreasing net in* $\mathbb{R}_{\geq 0}$ *with limit* 0*. Take a Kähler form* ω *on* X*. Consider a decreasing net* $\varphi_j \in \mathrm{PSH}(X,\theta+\epsilon_j\omega)$ $(j \in I)$ *satisfying*

$$P_{\theta+\epsilon_j\omega}\left[\varphi_j\right]=\varphi_j \tag{3.20}$$

with pointwise limit φ. Then

$$\lim_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}=\int_X\theta^n_\varphi. \tag{3.21}$$

Moreover, if $\int_X\theta^n_\varphi>0$, then for any prime divisor E over X, we have

$$\lim_{j\in I}\nu\left(\varphi_j,E\right)=\nu(\varphi,E). \tag{3.22}$$

Note that both (3.21) and (3.22) fail without the assumption (3.20).

Proof Observe that $\varphi\in\operatorname{PSH}(X,\theta)$. By Theorem 2.4.4, we have

$$\varliminf_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}\geq\varliminf_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi}=\int_X\theta^n_\varphi.$$

We now argue the reverse inequality.

Fix $j_0\in I$, we have

$$\begin{aligned}\varlimsup_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}&=\varlimsup_{j\in I}\int_{\{\varphi_j=0\}}(\theta+\epsilon_j\omega)^n_{\varphi_j}\\&\leq\varlimsup_{j\in I}\int_{\{\varphi_j=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi_j}\\&\leq\int_{\{\varphi=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi},\end{aligned}$$

where in the first line we used (3.20) and Theorem 3.1.1, and in the last line we have used the fact that $\varphi_j\searrow\varphi$ and Corollary 2.4.2. Taking limit with respect to j_0, we arrive at the desired conclusion:

$$\varlimsup_{j\in I}\int_X(\theta+\epsilon_j\omega)^n_{\varphi_j}\leq\varliminf_{j_0\in I}\int_{\{\varphi=0\}}(\theta+\epsilon_{j_0}\omega)^n_{\varphi}=\int_{\{\varphi=0\}}\theta^n_\varphi\leq\int_X\theta^n_\varphi.$$

This finishes the proof of (3.21).

It remains to argue (3.22). By Lemma 2.4.2 and (3.21), for any $\epsilon\in(0,1)$ and j big enough there exists $\psi_j\in\operatorname{PSH}(X,\theta+\epsilon_j\omega)$ such that $(1-\epsilon)\varphi_j+\epsilon\psi_j\leq\varphi$. This implies that for j big enough we have

$$(1-\epsilon)\nu(\varphi_j,E)+\epsilon\nu(\psi_j,E)\geq\nu(\varphi,E)\geq\nu(\varphi_j,E).$$

On the other hand, the Lelong numbers $\nu(\psi_j,E)$ admit an upper bound for various j by Proposition 1.5.2. So taking limit with respect to j, we conclude (3.22). □

Corollary 3.1.3 *Let $\varphi\in\operatorname{PSH}(X,\theta)_{>0}$ be a model potential. Let ω be a Kähler form on X. Then*

$$\varphi = \inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi].$$

Proof Clearly, we have the $\leq$ direction and the right-hand side is non-positive. So by Theorem 3.1.2, it suffices to show that they have the same mass, which follows from Proposition 3.1.10. □

Proposition 3.1.11 *Let $(\varphi_i)_{i\in I}$ be an increasing net of potentials in* $\mathrm{PSH}(X,\theta)_{>0}$ *uniformly bounded from above. Let $\varphi \coloneqq \sup_{i\in I}{}^*\varphi_i$. Then*

$$\sup_{i\in I}{}^* P_\theta[\varphi_i] = P_\theta[\varphi].$$

In particular, if φ_i is model for all $i \in I$, then so is φ.

Proof We may assume that I is infinite since otherwise, there is nothing to prove.
We write

$$\eta \coloneqq \sup_{i\in I}{}^* P_\theta[\varphi_i].$$

Then it is clear that $\eta \leq P_\theta[\varphi]$.
By Corollary 2.4.1, we have

$$\lim_{i\in I} \int_X \theta^n_{\varphi_i} = \int_X \theta^n_\varphi > 0.$$

So by Lemma 2.4.2, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0,1)$ and $\psi_i \in \mathrm{PSH}(X,\theta)$ ($i \in I$) such that for all $i \in I$,

$$(1-\epsilon_i)\varphi + \epsilon_i\psi_i \leq \varphi_i.$$

By Proposition 3.1.8, we have

$$P_\theta[\varphi] + \epsilon_i P_\theta[\psi_i] \leq (1-\epsilon_i)P_\theta[\varphi] + \epsilon_i P_\theta[\psi_i] \leq \eta.$$

Taking limit with respect to i, we conclude that $P_\theta[\varphi] \leq \eta$. □

3.1.3 Relative full mass classes

Let θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Definition 3.1.4 We define

$$
\begin{aligned}
\operatorname{PSH}(X,\theta;\phi) &\coloneqq \{\eta \in \operatorname{PSH}(X,\theta) : \eta \leq \phi\}\,,\\
\mathcal{E}^\infty(X,\theta;\phi) &\coloneqq \{\eta \in \operatorname{PSH}(X,\theta) : \eta \sim \phi\}\,,\\
\mathcal{E}(X,\theta;\phi) &\coloneqq \left\{\eta \in \operatorname{PSH}(X,\theta;\phi) : \int_X \theta_\eta^n = \int_X \theta_\phi^n\right\},\\
\mathcal{E}^1(X,\theta;\phi) &\coloneqq \left\{\eta \in \mathcal{E}(X,\theta;\phi) : \int_X |\phi-\eta|\,\theta_\eta^n < \infty\right\}.
\end{aligned}
$$

Potentials in the last three classes are said to have *relatively minimal singularities*, *full mass* and *finite energy* relative to ϕ respectively.

We have the following inclusions:

$$
\mathcal{E}^\infty(X,\theta;\phi) \subseteq \mathcal{E}^1(X,\theta;\phi) \subseteq \mathcal{E}(X,\theta;\phi) \subseteq \operatorname{PSH}(X,\theta;\phi). \tag{3.23}
$$

The only non-trivial part is the first inclusion, which follows from Theorem 2.4.4.

Remark 3.1.2 Note that this integral

$$
\int_X |\phi-\eta|\,\theta_\eta^n
$$

is defined: The locus where $\phi-\eta$ is undefined is a pluripolar set, while the product θ_η^n puts no mass on pluripolar sets (Proposition 2.2.1).

Similar remarks apply when we talk about similar integrals in the sequel.

When $\phi = V_\theta$, we usually write

$$
\begin{aligned}
\mathcal{E}^\infty(X,\theta;V_\theta) =&\mathcal{E}^\infty(X,\theta),\\
\mathcal{E}(X,\theta;V_\theta) =&\mathcal{E}(X,\theta),\\
\mathcal{E}^1(X,\theta;V_\theta) =&\mathcal{E}^1(X,\theta).
\end{aligned}
$$

Potentials in the three classes are said to have *minimal singularities*, *full mass* and *finite energy* respectively. The relation (3.23) can be written as

$$
\mathcal{E}^\infty(X,\theta) \subseteq \mathcal{E}^1(X,\theta) \subseteq \mathcal{E}(X,\theta)
$$

in this case.

The P-envelope can be used to characterize the full mass classes:

Proposition 3.1.12 *Let $\varphi \in \operatorname{PSH}(X,\theta)$. Then the following are equivalent:*

(1) *$\varphi \in \mathcal{E}(X,\theta;\phi)$;*
(2) *$P_\theta[\varphi] = \phi$.*

Proof (2) $\implies$ (1). This follows from Proposition 3.1.3.

(1) $\implies$ (2). Note that ϕ is a candidate of $P_\theta[\varphi]$ as in (3.7). So $P_\theta[\varphi] = \phi$. □

We have the following comparison principle.

Proposition 3.1.13 (Comparison principle) *Fix $j \in \{0, \ldots, n\}$. Let $\theta_1, \ldots, \theta_j$ be closed smooth real $(1,1)$-forms on X and $\psi_i \in \mathrm{PSH}(X, \theta_i)$ for $i = 1, \ldots, j$. Suppose that $\varphi, \psi \in \mathcal{E}(X, \theta; \phi)$, then*

$$\int_{\{\varphi<\psi\}} \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \geq \int_{\{\varphi<\psi\}} \theta_{\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j}. \tag{3.24}$$

Proof Observe that $\varphi \vee \psi \in \mathcal{E}(X, \theta; \phi)$, as a consequence of Theorem 2.4.4. We compute

$$\begin{aligned}
&\int_X \theta_{\varphi\vee\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \\
\geq &\int_{\{\varphi>\psi\}} \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} + \int_{\{\varphi<\psi\}} \theta_{\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \\
= &\int_X \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} - \int_{\{\varphi\leq\psi\}} \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \\
&+ \int_{\{\varphi<\psi\}} \theta_{\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \\
= &\int_X \theta_{\varphi\vee\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} - \int_{\{\varphi\leq\psi\}} \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \\
&+ \int_{\{\varphi<\psi\}} \theta_{\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j},
\end{aligned}$$

where in the last step, we applied Proposition 3.1.3. Therefore,

$$\int_{\{\varphi\leq\psi\}} \theta_{\varphi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j} \geq \int_{\{\varphi<\psi\}} \theta_{\psi}^{n-j} \wedge \theta_{1,\psi_1} \wedge \cdots \wedge \theta_{j,\psi_j}.$$

Next we replace φ by $\varphi + \epsilon$ and let $\epsilon \searrow 0$, then we conclude (3.24). □

The full mass potentials are essential in resolving the Monge–Ampère equations. We recall the following two theorems.

Theorem 3.1.4 *Let μ be a non-pluripolar measure on X with $\mu(X) = \int_X \theta_\phi^n$. Then there is a unique $\varphi \in \mathrm{PSH}(X, \theta; \phi)$ such that*

$$\theta_\varphi^n = \mu, \quad \sup_X \varphi = 0.$$

Recall that a measure μ on X is *non-pluripolar* if it is a Radon measure and $\mu(K) = 0$ for each pluripolar set $K \subseteq X$.

Theorem 3.1.5 *Fix $\lambda > 0$. Let μ be a non-pluripolar measure on X with $\mu(X) > 0$. Then there is a unique $\varphi \in \mathcal{E}(X, \theta; \phi)$ such that*

$$\theta_\varphi^n = \mathrm{e}^{\lambda\varphi}\mu.$$

Furthermore, for any $\psi \in \mathcal{E}(X, \theta; \phi)$ satisfying

$$\theta_\psi^n \geq \mathrm{e}^{\lambda\psi}\mu,$$

we have $\varphi \geq \psi$.

For the proofs, we refer to [DDNL21a].

In order to handle the finite energy classes, it is convenient to introduce the following quantity:

Definition 3.1.5 We define the *Monge–Ampère energy* $E_\theta^\phi \colon \mathcal{E}^\infty(X,\theta;\phi) \to \mathbb{R}$ as follows

$$E_\theta^\phi(\varphi) \coloneqq \frac{1}{n+1}\sum_{j=0}^n \int_X (\varphi-\phi)\,\theta_\varphi^j \wedge \theta_\phi^{n-j}. \tag{3.25}$$

More generally, we extend E_θ^ϕ to a functional $E_\theta^\phi \colon \mathrm{PSH}(X,\theta;\phi) \to [-\infty,\infty)$ as follows

$$E_\theta^\phi(\varphi) \coloneqq \inf\left\{E_\theta^\phi(\psi) : \psi \in \mathcal{E}^\infty(X,\theta;\phi), \varphi \leq \psi\right\}. \tag{3.26}$$

We write E_θ instead of E_θ^ϕ when $\phi = V_\theta$.

Note that

$$E_\theta^\phi(\varphi + C) = E_\theta^\phi(\varphi) + C\int_X \theta_\phi^n \tag{3.27}$$

for any $\varphi \in \mathrm{PSH}(X,\theta;\phi)$ and $C \in \mathbb{R}$.

Lemma 3.1.3 *The functional* $E_\theta^\phi \colon \mathcal{E}^\infty(X,\theta;\phi) \to \mathbb{R}$ *is increasing. In particular, the extended definition* (3.26) *agrees with* (3.25) *on* $\mathcal{E}^\infty(X,\theta;\phi)$*, and is increasing as well.*

Proof Let $\varphi, \psi \in \mathcal{E}^\infty(X,\theta;\phi)$, we have

$$\begin{aligned}
&(n+1)E_\theta^\phi(\varphi) - (n+1)E_\theta^\phi(\psi) - \sum_{j=0}^n (\varphi-\psi)\theta_\varphi^j \wedge \theta_\psi^{n-j}\\
=&\sum_{j=0}^n \int_X (\varphi-\phi)\,\theta_\varphi^j \wedge \theta_\phi^{n-j} - \sum_{j=0}^n \int_X (\psi-\phi)\,\theta_\psi^j \wedge \theta_\phi^{n-j} - \sum_{j=0}^n (\varphi-\psi)\theta_\varphi^j \wedge \theta_\psi^{n-j}\\
=&\sum_{j=0}^n \int_X (\varphi-\psi)\,\left(\theta_\varphi^j \wedge \theta_\phi^{n-j} - \theta_\varphi^j \wedge \theta_\psi^{n-j}\right)\\
&+\sum_{j=0}^n \int_X (\psi-\phi)\,\left(\theta_\varphi^j \wedge \theta_\phi^{n-j} - \theta_\psi^j \wedge \theta_\phi^{n-j}\right)\\
=&\sum_{\substack{j+a+b=n-1,\\ j,a,b\geq 0}} (n-j)\int_X (\psi-\phi)\,\left(\theta_\phi^a \wedge \theta_\psi^{b+1} \wedge \theta_\varphi^j - \theta_\phi^a \wedge \theta_\psi^b \wedge \theta_\varphi^{j+1}\right)\\
&+\sum_{\substack{j+a+b=n-1,\\ j,a,b\geq 0}} (n-j)\int_X (\psi-\phi)\,\left(\theta_\phi^j \wedge \theta_\varphi^{a+1} \wedge \theta_\psi^b - \theta_\phi^j \wedge \theta_\varphi^a \wedge \theta_\psi^{b+1}\right)\\
=&0,
\end{aligned}$$

where the third equality follows from the integration by parts formula. See [Xia19, Lu21] for the proof in the context of non-pluripolar products.

In other words,

$$E^{\phi}_{\theta}(\varphi) - E^{\phi}_{\theta}(\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (\varphi-\psi)\,\theta^j_{\varphi}\wedge\theta^{n-j}_{\psi}. \tag{3.28}$$

The monotonicity follows. □

Lemma 3.1.4 *Let $\varphi,\psi \in \mathcal{E}^{\infty}(X,\theta;\phi)$. Then*

$$\int_X (\varphi-\psi)\,\theta^n_{\psi} \geq E^{\phi}_{\theta}(\varphi) - E^{\phi}_{\theta}(\psi) \geq \int_X (\varphi-\psi)\,\theta^n_{\varphi}. \tag{3.29}$$

Proof Thanks to (3.27), we may assume that $\varphi \geq \psi$.

As we have seen in (3.28), the middle term can be written as

$$\frac{1}{n+1}\sum_{j=0}^{n}\int_X (\varphi-\psi)\,\theta^j_{\varphi}\wedge\theta^{n-j}_{\psi}.$$

We claim that for each individual j, the inequality (3.29) holds:

$$\int_X (\varphi-\psi)\,\theta^n_{\psi} \geq \int_X (\varphi-\psi)\,\theta^j_{\varphi}\wedge\theta^{n-j}_{\psi} \geq \int_X (\varphi-\psi)\,\theta^n_{\varphi}.$$

Thanks to Proposition 3.1.13, we have

$$\begin{aligned}\int_X (\varphi-\psi)\,\theta^j_{\varphi}\wedge\theta^{n-j}_{\psi} &= \int_0^{\infty}\int_{\{\varphi>\psi-t\}} \theta^j_{\varphi}\wedge\theta^{n-j}_{\psi}\,\mathrm{d}t\\ &\geq \int_0^{\infty}\int_{\{\varphi>\psi-t\}} \theta^n_{\varphi}\,\mathrm{d}t\\ &= \int_X (\varphi-\psi)\,\theta^n_{\varphi}.\end{aligned}$$

The other inequality is similar. □

Proposition 3.1.14 *Let $(\varphi_i)_{i\in I}$ be a decreasing net in* $\mathrm{PSH}(X,\theta;\phi)$ *with limit $\varphi \not\equiv -\infty$. Then*

$$\lim_{i\in I} E^{\phi}_{\theta}(\varphi_i) = E^{\phi}_{\theta}(\varphi). \tag{3.30}$$

Proof Thanks to Lemma 3.1.3, we know that $E^{\phi}_{\theta}(\varphi_i)$ is decreasing in i. So the limit in (3.30) exists and the $\geq$ inequality holds. Conversely, let $\psi \in \mathcal{E}^{\infty}(X,\theta;\phi)$ and $\varphi \leq \psi$. We need to show that

$$E^{\phi}_{\theta}(\psi) \geq \lim_{i\in I} E^{\phi}_{\theta}(\varphi_i \vee \psi).$$

In particular, we have reduced to the case where $\varphi \in \mathcal{E}^{\infty}(X,\theta;\phi)$.

In this case, thanks to Lemma 3.1.4, we have

$$E_\theta^\phi(\varphi_i) - E_\theta^\phi(\varphi) \le \int_X (\varphi_i - \varphi)\, \theta_\varphi^n.$$

Our assertion follows from the dominated convergence theorem. □

Proposition 3.1.15 *Let $\varphi \in \mathcal{E}(X,\theta;\phi)$. The following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X,\theta;\phi)$;
(2) $E_\theta^\phi(\varphi) > -\infty$.

When the conditions are satisfied, (3.25) *holds.*

Given $\varphi, \psi \in \mathcal{E}^1(X,\theta;\phi)$, we have the following cocycle equality

$$E_\theta^\phi(\psi) - E_\theta^\phi(\varphi) = \frac{1}{n+1}\sum_{j=0}^n \int_X (\psi - \varphi)\, \theta_\psi^j \wedge \theta_\varphi^{n-j}. \tag{3.31}$$

As a consequence of (3.31), for $\varphi \in \mathcal{E}^1(X,\theta;\phi)$, (3.25) continues to hold.

Proof Fix $\varphi \in \mathcal{E}(X,\theta;\phi)$. Without loss of generality, we may assume that $\varphi \le 0$.

(2) $\implies$ (1). Assume (2). Observe that $\mathbb{1}_{\{\varphi > \phi - C\}}\theta^n_{\varphi\vee(\phi-C)}$ converges *strongly* to θ_φ^n as $C \to \infty$ (Namely, the convergence holds after integrating against any L^∞ function). Therefore, for a fixed $D > 0$,

$$\begin{aligned}
\int_X (\varphi \vee (\phi - D) - \phi)\ \theta_\varphi^n &= \lim_{C\to\infty} \int_{\{\varphi > \phi - C\}} (\varphi \vee (\phi - D) - \phi)\ \theta^n_{\varphi\vee(\phi-C)}\\
&\ge \varlimsup_{C\to\infty} \int_{\{\varphi > \phi - C\}} (\varphi \vee (\phi - C) - \phi)\ \theta^n_{\varphi\vee(\phi-C)}\\
&\ge \int_X (\varphi - \phi)\ \theta_\varphi^n\\
&\ge (n+1) E_\theta^\phi(\varphi).
\end{aligned}$$

Letting $D \to \infty$, we conclude (1).

(1) $\implies$ (2). Assume (1). Thanks to Lemma 3.1.4, we know that for each $C > 0$,

$$\begin{aligned}
E_\theta^\phi(\varphi \vee (\phi - C)) &\ge \int_X (\varphi \vee (\phi - C) - \phi)\, \theta^n_{\varphi\vee(\phi-C)}\\
&= \int_{\{\varphi + C > \phi\}} (\varphi + C - \phi)\, \theta_\varphi^n - C\int_X \theta_\phi^n\\
&\ge \int_X (\varphi + C - \phi)\, \theta_\varphi^n - C\int_X \theta_\phi^n\\
&= \int_X (\varphi - \phi)\, \theta_\varphi^n.
\end{aligned}$$

Due to Proposition 3.1.14, we have

$$\lim_{C\to\infty} E_\theta^\phi\,(\varphi \vee (\phi - C)) = E_\theta^\phi(\varphi),$$

so (2) holds.

Now assume that $\varphi, \psi \in \mathcal{E}^1(X, \theta; \phi)$. It remains to establish (3.31). Since the conclusion is known when $\varphi, \psi \in \mathcal{E}^\infty(X, \theta; \psi)$ (see (3.28)), it suffices to prove that for each $k = 0, \dots, n$, we have

$$\lim_{C\to\infty} \int_X (\varphi \vee (\phi - C) - \psi \vee (\phi - C))\ \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} = \int_X (\varphi - \psi)\, \theta^k_\varphi \wedge \theta^{n-k}_\psi. \tag{3.32}$$

For this purpose, we may assume that $\varphi, \psi \leq \phi$ and it suffices to establish the following:

$$\lim_{C\to\infty} \int_X (\varphi \vee (\phi - C) - \phi)\ \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} = \int_X (\varphi - \phi)\, \theta^k_\varphi \wedge \theta^{n-k}_\psi. \tag{3.33}$$

We compute the difference as follows:

$$\begin{aligned}
&\int_X (\varphi \vee (\phi - C) - \phi)\ \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} - \int_X (\varphi - \phi)\, \theta^k_\varphi \wedge \theta^{n-k}_\psi \\
=& \int_{\{\min\{\varphi,\psi\} > \phi - C\}} (\varphi - \phi)\ \theta^k_\varphi \wedge \theta^{n-k}_\psi - \int_X (\varphi - \phi)\, \theta^k_\varphi \wedge \theta^{n-k}_\psi \\
&+ \int_{\{\min\{\varphi,\psi\} \leq \phi - C\}} (\varphi \vee (\phi - C) - \phi)\ \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} \\
=& - \int_{\{\min\{\varphi,\psi\} \leq \phi - C\}} (\varphi - \phi)\ \theta^k_\varphi \wedge \theta^{n-k}_\psi + \\
&+ \int_{\{\min\{\varphi,\psi\} \leq \phi - C\}} (\varphi \vee (\phi - C) - \phi)\ \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)}
\end{aligned}$$

The first term tends to 0 as $C \to \infty$ thanks to the fact that $E_\theta^\phi(\varphi) > -\infty$, so we only have to establish the same for the second term. It then suffices to prove the following:

$$\begin{aligned}
\lim_{C\to\infty} C \int_{\{\varphi \leq \phi - C\}} \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} &= 0, \\
\lim_{C\to\infty} C \int_{\{\psi \leq \phi - C\}} \theta^k_{\varphi\vee(\phi-C)} \wedge \theta^{n-k}_{\psi\vee(\phi-C)} &= 0.
\end{aligned} \tag{3.34}$$

By symmetry, it suffices to hand the first. Observe the following inclusions:

$$\{\varphi \leq \phi - C\} \subseteq \left\{\varphi \vee (\phi - C) \leq \frac{1}{2}\,(\psi \vee (\phi - C) + \phi - C)\right\} \subseteq \{\varphi \leq \phi - C/2\}.$$

By Proposition 3.1.13 and the obvious inequality

$$\theta^{n-k}_{\psi\vee(\phi-C)} \leq 2^{n-k}\theta^{n-k}_{\frac{1}{2}(\psi\vee(\phi-C)+\phi-C)},$$

we find

$$\begin{aligned}
&C\int_{\{\varphi\leq\phi-C\}}\theta^k_{\varphi\vee(\phi-C)}\wedge\theta^{n-k}_{\psi\vee(\phi-C)}\\
\leq&C\int_{\{\varphi\vee(\phi-C)\leq\frac{1}{2}(\psi\vee(\phi-C)+\phi-C)\}}\theta^k_{\varphi\vee(\phi-C)}\wedge\theta^{n-k}_{\psi\vee(\phi-C)}\\
\leq&2^{n-k}C\int_{\{\varphi\vee(\phi-C)\leq\frac{1}{2}(\psi\vee(\phi-C)+\phi-C)\}}\theta^k_{\varphi\vee(\phi-C)}\wedge\theta^{n-k}_{\frac{1}{2}(\psi\vee(\phi-C)+\phi-C)}\\
\leq&2^{n-k}C\int_{\{\varphi\vee(\phi-C)\leq\frac{1}{2}(\psi\vee(\phi-C)+\phi-C)\}}\theta^n_{\varphi\vee(\phi-C)}\\
\leq&2^{n-k}C\int_{\{\varphi\leq\phi-C/2\}}\theta^n_{\varphi\vee(\phi-C)}\\
=&2^{n-k}C\int_{\{\varphi\leq\phi-C/2\}}\theta^n_{\varphi}.
\end{aligned}$$

On the other hand,

$$\varlimsup_{C\to\infty}C\int_{\{\varphi\leq\phi-C\}}\theta^n_\varphi\leq\varlimsup_{C\to\infty}\int_{\{\varphi\leq\phi-C\}}(\phi-\varphi)\,\theta^n_\varphi=0,$$

since $\varphi\in\mathcal{E}^1(X,\theta;\phi)$. We conclude (3.34). □

Proposition 3.1.16 *Let $\varphi,\psi\in\mathcal{E}^1(X,\theta;\phi)$. Then*

$$\int_X(\psi-\varphi)\,\theta^n_\psi\leq E^\phi_\theta(\psi)-E^\phi_\theta(\varphi)\leq\int_X(\psi-\varphi)\,\theta^n_\varphi. \tag{3.35}$$

Proof Thanks to (3.32), this can be reduced to the case where $\varphi,\psi\in\mathcal{E}^\infty(X,\theta;\phi)$, which is established in Lemma 3.1.4. □

Proposition 3.1.17 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathcal{E}^1(X,\theta;\phi)$. Assume that $\varphi\in\mathcal{E}^1(X,\theta;\phi)$ and either one of the following conditions holds:*

(1) *$(\varphi_j)_j$ is decreasing with limit φ;*
(2) *$(\varphi_j)_j$ is increasing with almost everywhere limit φ.*

Then

$$\lim_{j\in I}E^\phi_\theta(\varphi_j)=E^\phi_\theta(\varphi). \tag{3.36}$$

Proof The decreasing case is already proved in Proposition 3.1.14, so let us focus on the case where (φ_j) is increasing.

Step 1. We first prove the result when $\varphi_j,\varphi\in\mathcal{E}^\infty(X,\theta;\phi)$ for each $j\in I$.

In fact, in this case, by (3.28), we have

$$\begin{aligned}
0\leq&E^\phi_\theta(\varphi)-E^\phi_\theta(\varphi_j)\\
=&\frac{1}{n+1}\sum_{k=0}^n\int_X(\varphi-\varphi_j)\,\theta^k_\varphi\wedge\theta^{n-k}_{\varphi_j}.
\end{aligned}$$

We then apply Theorem 2.4.3 to conclude that the right-hand side converges to 0.

Step 2. We prove the general case. The $\leq$ direction in (3.36) follows from the monotonicity of E^{ϕ}_{θ}, as proved in Lemma 3.1.3. It remains to prove the $\geq$ direction.

Without loss of generality, we may assume that $\varphi \leq 0$. For each $C > 0$ and $j \in I$, we let

$$\varphi_C \coloneqq \varphi \vee (\phi - C), \quad \varphi_{j,C} \coloneqq \varphi_j \vee (\phi - C).$$

By Step 1, for each $C > 0$, we have

$$\lim_{j\in I} E^{\phi}_{\theta}(\varphi_{j,C}) = E^{\phi}_{\theta}(\varphi_C) \geq E^{\phi}_{\theta}(\varphi).$$

It suffices therefore to show that

$$\lim_{C\to\infty} E^{\phi}_{\theta}(\varphi_{j,C}) - E^{\phi}_{\theta}(\varphi_j) = 0$$

uniformly in $j \in I$.

Fix $i \in I$. Fix $j \in I$ such that $j \geq i$, we compute

$$\begin{aligned}
& E^{\phi}_{\theta}(\varphi_{j,C}) - E^{\phi}_{\theta}(\varphi_j) \\
\leq & \int_X \left(\varphi_{j,C} - \varphi_j\right) \theta^n_{\varphi_j} && \text{by Proposition 3.1.16} \\
\leq & \int_{\{\varphi_j \leq \phi - C\}} (\phi - C - \varphi_j)\, \theta^n_{\varphi_j} \\
= & \int_C^\infty \int_{\{\varphi_j \leq \phi - C\}} \theta^n_{\varphi_j}\, \mathrm{d}t \\
\leq & \int_C^\infty \int_{\{\varphi_i \leq (\varphi_j + \phi - C)/2\}} \theta^n_{\varphi_j}\, \mathrm{d}t \\
\leq & 2^n \int_C^\infty \int_{\{\varphi_i \leq (\varphi_j + \phi - C)/2\}} \theta^n_{(\varphi_j + \phi - C)/2}\, \mathrm{d}t \\
\leq & 2^n \int_C^\infty \int_{\{\varphi_i \leq (\varphi_j + \phi - C)/2\}} \theta^n_{\varphi_i}\, \mathrm{d}t && \text{by Proposition 3.1.13} \\
\leq & 2^n \int_C^\infty \int_{\{\varphi_i \leq \phi - C/2\}} \theta^n_{\varphi_i}\, \mathrm{d}t \\
= & 2^{n+1} \int_{\{\varphi_i < \phi - C/2\}} (\phi - \varphi_i - C/2)\, \theta^n_{\varphi_i} \\
= & 2^{n+1} \int_{\{\varphi_i < \phi - C/2\}} (\phi - \varphi_i)\, \theta^n_{\varphi_i} - C2^{-n} \int_{\{\varphi_i < \phi - C/2\}} \theta^n_{\varphi_i}.
\end{aligned}$$

Both terms converge to 0 as $C \to \infty$ as we have seen in the proof of Proposition 3.1.15. □

Next we want to prove that $\mathcal{E}^1(X, \theta; \phi)$ is closed under the rooftop operator. For this purpose, we shall need a few preliminary results.

We shall approximate the rooftop operator by solutions of certain Monge–Ampère equations.

Lemma 3.1.5 *Let $\varphi, \psi \in \mathcal{E}^\infty(X, \theta; \phi)$. Then there is $\gamma \in \mathcal{E}^\infty(X, \theta; \phi)$ such that*

$$\theta_\gamma^n = e^{\gamma - \varphi}\, \theta_\varphi^n + e^{\gamma - \psi}\, \theta_\psi^n. \tag{3.37}$$

It is not clear if $\int_X e^{-\varphi}\theta_\varphi^n < \infty$, hence we cannot say that $e^{-\varphi}\theta_\varphi^n + e^{-\psi}\theta_\psi^n$ is a Radon measure. Hence Theorem 3.1.5 is not directly applicable.

Proof For each $j \geq 1$, let $\varphi_j \coloneqq \varphi \vee (-j)$, $\psi_j \coloneqq \psi \vee (-j)$. Let

$$\mu_j = e^{-\varphi_j}\theta_\varphi^n + e^{-\psi_j}\theta_\psi^n.$$

By Theorem 3.1.5, we can find $\gamma_j \in \mathcal{E}(X, \theta; \phi)$ such that

$$\theta_{\gamma_j}^n = e^{\gamma_j}\mu_j. \tag{3.38}$$

Take a constant $C > 0$ so that $\psi - 2C \leq \varphi \leq \psi + 2C$. Let

$$\eta \coloneqq \frac{\varphi + \psi}{2} - C - n\log 2.$$

Then $\eta \in \mathcal{E}^\infty(X, \theta; \phi)$ and a simple computation shows that

$$\theta_\eta^n \geq e^\eta \mu_j.$$

Hence, $\gamma_j \geq \eta$ by Theorem 3.1.5. By Theorem 3.1.5 again, γ_j is decreasing in j, let $\gamma = \inf_{j>0}\gamma_j$. Then $\gamma \geq \eta$, hence $\gamma \in \mathcal{E}^\infty(X, \theta; \phi)$.

Now observe that as $j \to \infty$, $\theta_{\gamma_j}^n$ converges weakly to θ_γ^n, as a consequence of Theorem 2.4.3. Finally observe that there is a constant $C' > 0$ so that

$$\gamma_1 \leq \varphi + C', \quad \gamma_1 \leq \psi + C'.$$

Therefore, for each $j > 0$,

$$\gamma_j \leq \varphi_j + C', \quad \gamma_j \leq \psi_j + C'.$$

Hence, (3.37) follows from (3.38) by letting $j \to \infty$. □

We also need a few integral estimates.

Lemma 3.1.6 *Let $\varphi, \psi, \gamma \in \mathcal{E}(X, \theta; \phi)$. Assume that $\gamma \geq \varphi \vee \psi$. Then*

$$\int_X (\gamma - \varphi)\, \theta_\psi^n \leq 2\int_X (\gamma - \varphi)\, \theta_\varphi^n + 2\int_X (\gamma - \psi)\, \theta_\psi^n.$$

Proof Observe that

$$\{\gamma > \varphi + 2t\} \subseteq \{\gamma > \psi + t\} \cup \{\psi > \varphi + t\}. \tag{3.39}$$

So

$$\begin{aligned}\int_X(\gamma-\varphi)\,\theta_\psi^n =&2\int_0^\infty\int_{\{\gamma>\varphi+2t\}}\theta_\psi^n\,\mathrm{d}t\\ \leq&2\int_0^\infty\int_{\{\gamma>\psi+t\}}\theta_\psi^n\,\mathrm{d}t+2\int_0^\infty\int_{\{\psi>\varphi+t\}}\theta_\psi^n\,\mathrm{d}t\\ \leq&2\int_X(\gamma-\psi)\,\theta_\psi^n+2\int_0^\infty\int_{\{\psi>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t\\ \leq&2\int_X(\gamma-\psi)\,\theta_\psi^n+2\int_0^\infty\int_{\{\gamma>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t\\ =&2\int_X(\gamma-\psi)\,\theta_\psi^n+2\int_X(\gamma-\varphi)\,\theta_\varphi^n,\end{aligned}$$

where the second line follows from (3.39), while the third line follows from Proposition 3.1.13. □

Lemma 3.1.7 *Let $\varphi,\psi,\gamma\in\mathcal{E}(X,\theta;\phi)$. Assume that $\varphi\leq\psi\leq\gamma$. Then*

$$\int_X(\gamma-\psi)\,\theta_\psi^n\leq 2^{n+1}\int_X(\gamma-\varphi)\,\theta_\varphi^n.$$

Proof Observe that for any $t\geq 0$,

$$\{\gamma>\psi+2t\}\subseteq\{(\gamma+\psi)/2>\psi+t\}\subseteq\{(\gamma+\psi)/2>\varphi+t\}\subseteq\{\gamma>\varphi+t\}. \tag{3.40}$$

So

$$\begin{aligned}\int_X(\gamma-\psi)\,\theta_\psi^n =&2\int_0^\infty\int_{\{\gamma-\psi>2t\}}\theta_\psi^n\,\mathrm{d}t\\ \leq&2\int_0^\infty\int_{\{(\gamma+\psi)/2>\psi+t\}}\theta_\psi^n\,\mathrm{d}t && \text{by (3.40)}\\ \leq&2^{n+1}\int_0^\infty\int_{\{(\gamma+\psi)/2>\varphi+t\}}\theta_{(\gamma+\psi)/2}^n\,\mathrm{d}t\\ \leq&2^{n+1}\int_0^\infty\int_{\{(\gamma+\psi)/2>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t && \text{by Proposition 3.1.13}\\ \leq&2^{n+1}\int_0^\infty\int_{\{\gamma>\varphi+t\}}\theta_\varphi^n\,\mathrm{d}t && \text{by (3.40)}\\ =&2^{n+1}\int_X(\gamma-\varphi)\,\theta_\varphi^n && \text{by Proposition 3.1.13.}\end{aligned}$$

Lemma 3.1.8 *Let $\varphi_j,\gamma\in\mathcal{E}^1(X,\theta;\phi)$ ($j\in\mathbb{Z}_{>0}$). Assume that $\varphi_j\leq\gamma$ for each j and that φ_j converges to $\varphi\in\mathrm{PSH}(X,\theta)$ with respect to the L^1-topology. Assume that there is a constant $A>0$ such that for any $j>0$,*

$$\int_X(\varphi_j-\gamma)\,\theta_{\varphi_j}^n\geq -A.$$

Then $\varphi \in \mathcal{E}^1(X,\theta;\phi)$ *and*

$$\int_X (\varphi-\gamma)\,\theta_\varphi^n \geq -2^{n+3}A. \tag{3.41}$$

Proof **Step 1**. Assume that $(\varphi_j)_j$ is decreasing. In this case, we prove

$$\int_X (\varphi-\gamma)\,\theta_\varphi^n \geq -4A.$$

By Lemma 3.1.6, for any $j,k>0$,

$$\int_X (\varphi_j-\gamma)\,\theta_{\varphi_k}^n \geq -4A.$$

For any $C>0$,

$$\int_X \left(\varphi_j \vee (\gamma-C)-\gamma\right)\,\theta_{\varphi_k}^n \geq \int_X (\varphi_j-\gamma)\,\theta_{\varphi_k}^n \geq -4A.$$

Letting $k\to\infty$, by Theorem 2.4.3, we find

$$\int_X \left(\varphi_j \vee (\gamma-C)-\gamma\right)\,\theta_{\varphi}^n \geq -4A.$$

Letting $j\to\infty$, by the monotone convergence theorem, we get

$$\int_X \left(\varphi \vee (\gamma-C)-\gamma\right)\,\theta_{\varphi}^n \geq -4A.$$

Then we let $C\to\infty$, again by the monotone convergence theorem,

$$\int_X \left(\varphi-\gamma\right)\,\theta_{\varphi}^n \geq -4A.$$

Step 2. In general, let

$$\psi_j = \sup_{k\geq j}{}^*\varphi_k.$$

Then $\varphi=\inf_{j>0}\psi_j$.

For each $C>0$, let

$$\psi_{j,C}=\psi_j\vee(\gamma-C),\quad \varphi_C=\varphi\vee(\gamma-C).$$

Observe that $\psi_{j,C}$ decreases to φ_C as $j\to\infty$. Moreover,

$$\gamma\geq\psi_{j,C}\geq\psi_j\geq\varphi_j.$$

By Lemma 3.1.7,

$$\int_X (\psi_{j,C}-\gamma)\,\theta_{\psi_{j,C}}^n \geq -2^{n+1}A.$$

By Step 1,

$$\int_X (\varphi_C - \gamma)\,\theta^n_{\varphi_C} \geq -2^{n+3}A. \tag{3.42}$$

In particular,

$$\int_{\{\varphi > \gamma - C\}} (\varphi - \gamma)\,\theta^n_{\varphi} \geq -2^{n+3}A.$$

Letting $C \to \infty$, we conclude (3.41).

In order to conclude that $\varphi \in \mathcal{E}^1(X, \theta; \phi)$, we still have to prove that $\varphi \in \mathcal{E}(X, \theta; \phi)$. In fact, by (3.42),

$$\int_{\{\varphi \leq \gamma - C\}} \theta^n_{\varphi_C} \leq \frac{1}{C}\int_X (\gamma - \varphi_C)\,\theta^n_{\varphi_C} \leq 2^{n+3}C^{-1}A.$$

Using Theorem 2.4.4, we find

$$\int_X \theta^n_{\phi} = \int_X \theta^n_{\varphi_C} = \int_{\{\varphi \leq \gamma - C\}} \theta^n_{\varphi_C} + \int_{\{\varphi > \gamma - C\}} \theta^n_{\varphi}.$$

Letting $C \to \infty$, we conclude that

$$\int_X \theta^n_{\varphi} = \int_X \theta^n_{\phi}.$$

Proposition 3.1.18 *Assume that $\varphi, \psi \in \mathcal{E}(X, \theta; \phi)$ (resp. $\mathcal{E}^1(X, \theta; \phi)$, $\mathcal{E}^\infty(X, \theta; \phi)$), then so is $\varphi \wedge \psi$.*

Proof The case of $\mathcal{E}^\infty(X, \theta; \phi)$ is trivial.

We consider the case $\mathcal{E}(X, \theta; \phi)$. It follows from Proposition 3.1.5 that $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$. By Theorem 3.1.3, we have

$$\int_X \theta^n_{\varphi \wedge \psi} \geq \int_X \theta^n_{\phi}.$$

By Theorem 2.4.4, equality holds. By Theorem 3.1.2, we conclude that

$$P_\theta[\varphi \wedge \psi] = \phi.$$

Finally, assume that $\varphi, \psi \in \mathcal{E}^1(X, \theta; \phi)$. We may assume that $\varphi, \psi \leq \phi$. For each $j \geq 1$, consider the approximations:

$$\varphi_j := \varphi \vee (\phi - j), \quad \psi_j := \psi \vee (\phi - j).$$

By Lemma 3.1.5 below, we can take $\gamma_j \in \mathcal{E}^\infty(X, \theta; \phi)$ solving the following equation:

$$\theta^n_{\gamma_j} = \mathrm{e}^{\gamma_j - \varphi_j}\,\theta^n_{\varphi_j} + \mathrm{e}^{\gamma_j - \psi_j}\,\theta^n_{\psi_j}.$$

It follows from Theorem 3.1.5 that $\gamma_j \leq \varphi_j \wedge \psi_j$. We claim that

$$\int_X (\gamma_j - \phi)\, \theta^n_{\gamma_j} > -C \tag{3.43}$$

for some C independent of j.

Assume the claim is true for now. We get immediately that

$$\sup_X \gamma_j = \sup_{X\setminus\{\phi=-\infty\}} (\gamma_j - \phi) \geq -C/\int_X \theta^n_\phi,$$

where the first equality follows from Proposition 3.1.4. By Proposition 1.5.1, after possibly subtracting a subsequence, we may assume that $\gamma_j \to \gamma \in \mathrm{PSH}(X,\theta)$ in L^1-topology. Then $\gamma \in \mathcal{E}^1(X,\theta;\phi)$ by Lemma 3.1.8. Moreover, since $\gamma_j \leq \varphi_j \wedge \psi_j$, we know that $\gamma \leq \varphi \wedge \psi$. In particular, $\varphi \wedge \psi \in \mathrm{PSH}(X,\theta)$. Now by Lemma 3.1.7, $\varphi \wedge \psi \in \mathcal{E}^1(X,\theta;\phi)$.

Now we prove the claim (3.43). By symmetry, it suffices to prove

$$\int_X (\phi - \gamma_j)\mathrm{e}^{\gamma_j - \varphi_j}\, \theta^n_{\varphi_j} \leq C.$$

But note that

$$\int_X (\phi - \gamma_j)\mathrm{e}^{\gamma_j - \varphi_j}\, \theta^n_{\varphi_j} = \int_X (\phi - \varphi_j)\mathrm{e}^{\gamma_j - \varphi_j}\, \theta^n_{\varphi_j} + \int_X (\varphi_j - \gamma_j)\mathrm{e}^{\gamma_j - \varphi_j}\, \theta^n_{\varphi_j}.$$

But $xe^{-x} \leq C$ when $x \geq 0$, so the second term is bounded, it remains to prove

$$\int_X (\phi - \varphi_j)\mathrm{e}^{\gamma_j - \varphi_j}\, \theta^n_{\varphi_j} \leq C.$$

As $\gamma_j \leq \varphi_j$, it suffices to prove

$$\int_X (\phi - \varphi_j)\, \theta^n_{\varphi_j} \leq C. \tag{3.44}$$

We compute

$$\int_X (\varphi_j - \phi)\, \theta^n_{\varphi_j} \geq (n+1) E^\phi_\theta(\varphi_j)$$

Thus, (3.44) follows from Proposition 3.1.14. ◻

Proposition 3.1.19 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ be potentials such that $\varphi \leq \psi \leq \phi$. Assume that $\varphi \in \mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is ψ.*

Proof We may assume that $\varphi \leq \psi$.

The case $\mathcal{E}^\infty(X,\theta;\phi)$ is trivial. The case $\mathcal{E}(X,\theta;\phi)$ follows from Theorem 2.4.4. The case $\mathcal{E}^1(X,\theta;\phi)$ follows from the characterization of $\mathcal{E}^1(X,\theta;\phi)$ in Proposition 3.1.15 and the monotonicity of E^ϕ_θ proved in Lemma 3.1.3. ◻

Proposition 3.1.20 *Let $(\varphi_i)_{i\in I}$ be a uniformly bounded from above non-empty family in $\mathcal{E}(X,\theta;\phi)$ (resp. $\mathcal{E}^1(X,\theta;\phi)$, $\mathcal{E}^\infty(X,\theta;\phi)$), then so is $\sup^*_{i\in I}\varphi_i$.*

Proof Thanks to Proposition 3.1.19, it suffices to show that

$$\sup_{i\in I}{}^{*}\varphi_i \preceq \phi.$$

Since ϕ is model and $\varphi_i \preceq \phi$, we know that

$$\varphi_i - \sup_X \varphi_i \le \phi$$

for any $i \in I$. By assumption $(\varphi_i)_{i\in I}$ is uniformly bounded from above, our assertion follows. □

Proposition 3.1.21 *Let $\varphi, \psi \in \mathcal{E}(X, \theta; \phi)$. Then*

$$\sup_{C>0}{}^{*}(\varphi + C) \wedge \psi = \psi.$$

Proof Since for each $C \ge 0$,

$$(\varphi \wedge \psi + C) \wedge \psi \le (\varphi + C) \wedge \psi \le \psi,$$

we may replace φ by $\varphi \wedge \psi$ (c.f. Proposition 3.1.18) and assume that $\varphi \le \psi$.

Let

$$\gamma := \sup_{C>0}{}^{*}(\varphi + C) \wedge \psi.$$

Observe that $\gamma \le \psi$. Then Corollary 3.1.1 guarantees that

$$\int_{\{\gamma<\psi\}} \theta_\gamma^n = 0.$$

Therefore, we could apply Theorem 2.4.6 to conclude that $\gamma = \psi$. □

Lemma 3.1.9 *Let $\varphi, \psi \in \mathcal{E}^\infty(X, \theta; \phi)$. Define*

$$\eta_t := ((1-t)\varphi + t\psi) \wedge \psi, \quad t \in [0, 1].$$

Then $E_\theta^\phi(\eta_t)$ is differentiable for $t \in [0, 1]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\theta^\phi(\eta_t) = \int_X (\psi - \min\{\varphi, \psi\})\ \theta_{\eta_t}^n. \tag{3.45}$$

Proof Let us prove (3.45) with right-derivative instead of derivative, and $t \in [0, 1)$. The left-derivative case is completely parallel.

For each $t \in [0, 1]$, we let

$$f_t = \min\{(1-t)\varphi + t\psi, \psi\}.$$

Fix $t \in [0, 1)$ and $s > 0$ small enough so that $t+s < 1$. Thanks to Proposition 3.1.16, we have

$$
\begin{aligned}
E_\theta^\phi(\eta_{t+s}) - E_\theta^\phi(\eta_t) \le & \int_X (\eta_{t+s} - \eta_t)\, \theta_{\eta_t}^n \\
= & \int_X (\eta_{t+s} - f_t)\, \theta_{\eta_t}^n \qquad \text{by Lemma 3.1.1} \\
\le & \int_X (f_{t+s} - f_t)\, \theta_{\eta_t}^n \\
= & s \int_X (\psi - \min\{\varphi, \psi\})\, \theta_{\eta_t}^n .
\end{aligned}
$$

Similarly, we get

$$
E_\theta^\phi(\eta_{t+s}) - E_\theta^\phi(\eta_t) \ge s \int_X (\psi - \min\{\varphi, \psi\})\, \theta_{\eta_{t+s}}^n .
$$

Observe that η_{t+s} converges in capacity (in fact, even uniformly) to η_t as $s \to 0+$. The desired result (3.45) is then just a consequence of Theorem 2.4.3. □

In particular, we have a diamond-like inequality.

Corollary 3.1.4 *Let $\varphi, \psi \in \mathcal{E}^1(X, \theta; \phi)$. Then*

$$
E_\theta^\phi(\varphi \vee \psi) + E_\theta^\phi(\varphi \wedge \psi) \ge E_\theta^\phi(\psi) + E_\theta^\phi(\varphi). \tag{3.46}
$$

Proof **Step 1**. We first reduce to the case where $\varphi, \psi \in \mathcal{E}^\infty(X, \theta; \phi)$.

Assume that (3.46) is known in that case. For each $C > 0$, we let

$$
\varphi_C = \varphi \vee (\phi - C), \quad \psi_C = \psi \vee (\phi - C).
$$

Then

$$
E_\theta^\phi(\varphi_C \vee \psi_C) + E_\theta^\phi(\varphi_C \wedge \psi_C) \ge E_\theta^\phi(\psi_C) + E_\theta^\phi(\varphi_C).
$$

Letting $C \to \infty$ and applying Proposition 3.1.14, we conclude (3.46).

Step 2. We assume that $\varphi, \psi \in \mathcal{E}^\infty(X, \theta; \phi)$.

Thanks to (3.1.9), we have

$$
\begin{aligned}
E_\theta^\phi(\varphi \vee \psi) - E_\theta^\phi(\varphi) = & \int_0^1 \int_X (\varphi \vee \psi - \varphi)\, \theta_{(1-t)\varphi + t(\varphi \vee \psi)}^n \,\mathrm{d}t \\
= & \int_0^1 \int_{\{\psi > \varphi\}} (\psi - \varphi)\, \theta_{(1-t)\varphi + t\psi}^n \,\mathrm{d}t .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& E_\theta^\phi(\psi) - E_\theta^\phi(\varphi \wedge \psi) \\
= & \int_0^1 \int_X (\psi - \min\{\varphi, \psi\})\, \theta_{((1-t)\varphi + t\psi) \wedge \psi}^n \,\mathrm{d}t \\
\le & \int_0^1 \int_{\{\varphi < \psi\}} (\psi - \varphi)\, \theta_{(1-t)\varphi + t\psi}^n \,\mathrm{d}t \qquad \text{by Lemma 3.1.1.}
\end{aligned}
$$

Adding these equations together, we conclude (3.46). □

3.2 The $\mathcal{I}$-envelope

From the algebraic point of view, a more natural envelope operator is given by the $\mathcal{I}$-envelope.

In this section, X will denote a connected compact Kähler manifold of dimension n.

3.2.1 $\mathcal{I}$-equivalence

Proposition 3.2.1 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *For any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi);$$
(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) = \mathcal{I}(\lambda\psi);$$
(3) *for any modification $\pi\colon Y \to X$ and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y);$$
(4) *for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) = \nu(\pi^*\psi, y);$$
(5) *for any prime divisor E over X, we have*
$$\nu(\varphi, E) = \nu(\psi, E).$$

See Definition B.1.1 for the definition of prime divisors over X. We remind the readers that in the whole book, a *modification* of a compact complex space means a finite composition of blow-ups with smooth centers. This terminology is highly non-standard.

Proof (4) $\iff$ (5). This follows from Lemma 1.4.1.

(3) $\iff$ (5). This follows from Corollary B.1.1.

(1) $\implies$ (5). This follows from Proposition 1.4.4.

(5) $\implies$ (2). This follows from Theorem 1.4.3.

(2) $\implies$ (1). This is trivial. □

Definition 3.2.1 Given $\varphi, \psi \in \mathrm{QPSH}(X)$, we say they are $\mathcal{I}$*-equivalent* and write $\varphi \sim_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 3.2.1 are satisfied.

Clearly, $\sim_{\mathcal{I}}$ is an equivalence relation on $\mathrm{QPSH}(X)$.

Proposition 3.2.2 *Let $\pi : Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for $\varphi, \psi \in \mathrm{QPSH}(X)$, then the following are equivalent:*

(1) $\varphi \sim_{\mathcal{I}} \psi$;
(2) $\pi^*\varphi \sim_{\mathcal{I}} \pi^*\psi$.

Proof (1) $\implies$ (2). This follows from Proposition 3.2.1(4).

(2) $\implies$ (1). This follows from the simple fact that

$$\mathcal{I}(k\varphi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\varphi)\right), \quad \mathcal{I}(k\psi) = \pi_*\left(\omega_{Y/X} \otimes \mathcal{I}(k\pi^*\psi)\right)$$

for any $k \in \mathbb{Z}_{>0}$. □

Proposition 3.2.3 *Let $\varphi, \varphi', \psi, \psi' \in \mathrm{QPSH}(X)$ and $\lambda > 0$. Assume that $\varphi \sim_{\mathcal{I}} \psi$ and $\varphi' \sim_{\mathcal{I}} \psi'$, then*

$$\varphi \vee \varphi' \sim_{\mathcal{I}} \psi \vee \psi', \quad \varphi + \varphi' \sim_{\mathcal{I}} \psi + \psi', \quad \lambda\varphi \sim_{\mathcal{I}} \lambda\psi.$$

Similarly, if $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ are two non-empty uniformly bounded from above families in $\mathrm{PSH}(X,\theta)$ for some closed smooth real $(1,1)$-form θ on X such that $\varphi_i \sim_{\mathcal{I}} \psi_i$ for all $i \in I$, then

$$\sup_{i\in I}{}^*\varphi_i \sim_{\mathcal{I}} \sup_{i\in I}{}^*\psi_i.$$

Proof This follows from Proposition 1.4.2 and Corollary 1.4.1. □

3.2.2 The definition of the $\mathcal{I}$-envelope

We will fix a smooth closed real $(1,1)$-form θ on X.

Definition 3.2.2 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define its *$\mathcal{I}$-envelope* as follows:

$$P_\theta[\varphi]_{\mathcal{I}} := \sup{}^*\{\psi \in \mathrm{PSH}(X,\theta) : \psi \le 0, \psi \sim_{\mathcal{I}} \varphi\}. \tag{3.47}$$

If $\varphi = P_\theta[\varphi]_{\mathcal{I}}$, we say φ is an *$\mathcal{I}$-model potential* (in $\mathrm{PSH}(X,\theta)$).

Note that by Proposition 1.2.1, $P_\theta[\varphi]_{\mathcal{I}} \in \mathrm{PSH}(X,\theta)$.

Proposition 3.2.4 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$P_\theta[\varphi]_{\mathcal{I}} \sim P_{\theta'}[\varphi']_{\mathcal{I}}.$$

The proof is similar to that of Proposition 3.1.2, so we omit it.

Proposition 3.2.5 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X. Then for $\varphi \in \mathrm{PSH}(X,\theta)$, we have*

$$P_{\pi^*\theta}[\pi^*\varphi]_{\mathcal{I}} = \pi^* P_\theta[\varphi]_{\mathcal{I}}.$$

Proof The proof is similar to that of Proposition 3.1.7 in view of Proposition 3.2.2. □

Proposition 3.2.6 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\varphi \sim_{\mathcal{I}} P_\theta[\varphi]_{\mathcal{I}}.$$

In particular,

$$P_\theta\left[P_\theta[\varphi]_{\mathcal{I}}\right]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}$$

and the upper semicontinuous regularization in (3.47) is not necessary.

Proof In view of Proposition 3.2.1, it suffices to show that for $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}}). \tag{3.48}$$

By Proposition 1.2.2, we can find $\psi_i \in \mathrm{PSH}(X,\theta)$ ($i \in \mathbb{Z}_{>0}$) such that $\psi_i \leq 0$, $\psi_i \sim_{\mathcal{I}} \varphi$ for all $i \geq 1$ and

$$\sup_{i>0}{}^*\psi_i = P_\theta[\varphi]_{\mathcal{I}}.$$

By Proposition 3.2.3, we may replace ψ_i by $\psi_1 \vee \cdots \vee \psi_i$ and assume that the sequence ψ_i is increasing. In this case, it follows from the strong openness theorem Theorem 1.4.4 that for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}(k\psi_j) = \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}})$$

for j large enough. □

Definition 3.2.3 Let $\varphi \in \mathrm{PSH}(X,\theta)$, we define the *volume*[1] $\mathrm{vol}(\theta,\varphi)$ as

$$\mathrm{vol}(\theta,\varphi) = \int_X \left(\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}}\right)^n.$$

Proposition 3.2.7 *Let $\theta' = \theta + \mathrm{dd^c} g$ for some $g \in C^\infty(X)$. Then for any $\varphi \in \mathrm{PSH}(X,\theta)$, we have $\varphi' = \varphi - g \in \mathrm{PSH}(X,\theta')$ and*

$$\mathrm{vol}(\theta,\varphi) = \mathrm{vol}(\theta',\varphi').$$

Proof This follows immediately from Proposition 3.2.4 and Theorem 2.4.4. □

In view of Proposition 3.2.7, the volume $\mathrm{vol}(\theta,\varphi)$ depends only on the current θ_φ, and we could write

---

[1] We choose to call this quantity the *volume* instead of the $\mathcal{I}$-*volume* so that the terminology is consistent with the line bundle case.

$$\operatorname{vol}\theta_\varphi = \operatorname{vol}(\theta,\varphi). \tag{3.49}$$

Definition 3.2.4 Let $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$ be a pseudo-effective class. The *volume* $\operatorname{vol}\alpha$ of α is defined as

$$\operatorname{vol}\alpha = \operatorname{vol}T_{\min},$$

where $T_{\min}$ is a current with minimal singularities in α. Note that $\operatorname{vol}\alpha$ is independent of the choice of $T_{\min}$, thanks to Theorem 2.4.4.

Let us recall the following elementary result for latter use.

Proposition 3.2.8 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y.*

(1) *For any big class $\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R})$, $\pi^*\alpha$ is big. Moreover,* $\operatorname{vol}\pi^*\alpha = \operatorname{vol}\alpha$.
(2) *For any big class $\beta \in \mathrm{H}^{1,1}(Y,\mathbb{R})$, $\pi_*\beta$ is big. Moreover,* $\operatorname{vol}\pi_*\beta \geq \operatorname{vol}\beta$.

Proof (1) Take a current $T_{\min}$ with minimal singularities in α, then $\pi^*T_{\min}$ is a current with minimal singularities in $\pi^*\alpha$. Our assertion follows.

(2) Take a current $S_{\min}$ with minimal singularities in β, then $\pi_*S_{\min}$ is a current in $\pi_*\beta$, so our assertion follows from Theorem 2.4.4. □

The $\mathcal{I}$-envelope and the P-envelope are related in a simple manner.

Proposition 3.2.9 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}, \quad \varphi \sim_{\mathcal{I}} P_\theta[\varphi].$$

Proof It suffices to show that $\varphi \sim_{\mathcal{I}} P_\theta[\varphi]$. Namely, for each $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(k\varphi) = \mathcal{I}\left(kP_\theta[\varphi]\right). \tag{3.50}$$

Fix k for now. It follows from (3.4) and the strong openness theorem Theorem 1.4.4 that

$$\mathcal{I}\left(kP_\theta[\varphi]\right) = \mathcal{I}\left((k\varphi + C) \wedge kV_\theta\right),$$

when C is large enough. Since $(k\varphi + C) \wedge kV_\theta \sim k\varphi$, we have

$$\mathcal{I}\left((k\varphi + C) \wedge kV_\theta\right) = \mathcal{I}(k\varphi)$$

and (3.50) follows. □

In particular, we obtain an interesting relation between the non-pluripolar mass and the volume.

Corollary 3.2.1 *Let $\varphi \in \mathrm{PSH}(X,\theta)$, then*

$$\int_X \theta_\varphi^n \leq \operatorname{vol}\theta_\varphi.$$

The reverse inequality fails in general, see Example 6.1.3.

Proof This follows from Proposition 3.2.9, Theorem 2.4.4 and Proposition 3.1.3. □

We note the following special case:

Proposition 3.2.10 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that φ has analytic singularities, then*

$$\varphi \sim P_\theta[\varphi] \sim P_\theta[\varphi]_{\mathcal{I}}. \tag{3.51}$$

In particular,

$$\int_X \theta_\varphi^n = \operatorname{vol} \theta_\varphi. \tag{3.52}$$

Proof First observe that (3.52) follows from (3.51) and Theorem 2.4.4. It remains to establish (3.51).

In view of Proposition 3.2.9, it suffices to show that

$$P_\theta[\varphi]_{\mathcal{I}} \leq \varphi. \tag{3.53}$$

By Proposition 3.2.5, Proposition 3.1.7 and Theorem 1.6.1, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. By rescaling using Proposition 3.2.11, we may assume that D is a divisor. Take quasi-equisingular approximations $(\eta_j)_j$ and $(\varphi_j)_j$ of $P_\theta[\varphi]_{\mathcal{I}}$ and of φ respectively. Recall that by Theorem 1.6.2, we can guarantee that η_j and φ_j both have the singularity type $(2^{-j}, \mathcal{I}(2^j\varphi))$ and hence $\eta_j \sim \varphi_j$ for all large enough j. On the other hand, it is clear that $\varphi_j \sim \varphi$ for all $j \geq 1$. So (3.53) follows. □

3.2.3 Properties of the $\mathcal{I}$-envelope

Let $\theta, \theta_1, \theta_2$ be smooth closed real $(1,1)$-forms on X.

We have the following properties of the $\mathcal{I}$-envelope.

Proposition 3.2.11

(1) *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ and $\lambda \in \mathbb{R}_{>0}$, then*

$$P_{\lambda\theta}[\lambda\varphi]_{\mathcal{I}} = \lambda P_\theta[\varphi]_{\mathcal{I}}.$$

(2) *Suppose that $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1+\varphi_2]_{\mathcal{I}} \geq P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(3) *Suppose that $\varphi_1 \in \mathrm{PSH}(X,\theta_1)$ and $\varphi_2 \in \mathrm{PSH}(X,\theta_2)$, then*

$$P_{\theta_1+\theta_2}[\varphi_1+\varphi_2]_{\mathcal{I}} \sim_{\mathcal{I}} P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_2}[\varphi_2]_{\mathcal{I}}.$$

(4) *Suppose that $\varphi_1, \varphi_2 \in \mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi_1 \vee \varphi_2]_{\mathcal{I}} \sim_{\mathcal{I}} P_\theta[\varphi_1]_{\mathcal{I}} \vee P_\theta[\varphi_2]_{\mathcal{I}}.$$

Proof (1) This is obvious by definition.

(2) Suppose that $\psi_1 \in \mathrm{PSH}(X, \theta_1)$ and $\psi_2 \in \mathrm{PSH}(X, \theta_2)$ satisfy

$$\psi_i \leq 0, \quad \psi_i \sim_{\mathcal{I}} \varphi_i$$

for $i = 1, 2$. Then thanks to Proposition 3.2.3,

$$\psi_1 + \psi_2 \leq 0, \quad \psi_1 + \psi_2 \sim_{\mathcal{I}} \varphi_1 + \varphi_2.$$

It follows that

$$\psi_1 + \psi_2 \leq P_{\theta_1+\theta_2}[\varphi_1 + \varphi_2]_{\mathcal{I}}.$$

Since ψ_1 and ψ_2 are arbitrary, we conclude.

(3) and (4) These follow easily from Proposition 3.2.6 and Proposition 3.2.3. □

Lemma 3.2.1 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Assume that $\varphi \leq \psi$, then*

$$P_\theta[\varphi]_{\mathcal{I}} \leq P_\theta[\psi]_{\mathcal{I}}.$$

Proof It suffices to observe that $P_\theta[\varphi]_{\mathcal{I}} \vee \psi \sim_{\mathcal{I}} \psi$ as a consequence of Proposition 1.4.2 and Proposition 3.2.6. □

Proposition 3.2.12 *Consider a decreasing net $(\varphi_i)_{i\in I}$ of model potentials in $\mathrm{PSH}(X, \theta)_{>0}$. Suppose that $\varphi := \inf_{i\in I} \varphi_i \not\equiv -\infty$ and $\int_X \theta_\varphi^n > 0$. Then*

$$\inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

Proof Let $\eta = \inf_{i\in I} P_\theta[\varphi_i]_{\mathcal{I}}$. We have $\eta \geq P_\theta[\varphi]_{\mathcal{I}}$ as a consequence of Lemma 3.2.1.

By Proposition 3.1.10, we have

$$\lim_{i\in I} \int_X \theta_{\varphi_i}^n = \int_X \theta_\varphi^n > 0.$$

So by Lemma 2.4.2, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0, 1)$ and $\psi_i \in \mathrm{PSH}(X, \theta)$ such that for all $i \in I$,

$$(1 - \epsilon_i)\varphi_i + \epsilon_i\psi_i \leq \varphi.$$

By Proposition 3.2.11 and Lemma 3.2.1, we have

$$\eta + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq (1-\epsilon_i)\eta + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq (1-\epsilon_i)P_\theta[\varphi_i]_{\mathcal{I}} + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq P_\theta[\varphi]_{\mathcal{I}}.$$

Taking limit with respect to i, we conclude that $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$. □

Proposition 3.2.13 *Let $(\varphi_i)_{i\in I}$ be a decreasing net of $\mathcal{I}$-model potentials in $\mathrm{PSH}(X, \theta)$. Set $\varphi := \inf_{i\in I} \varphi_i$, then φ is also $\mathcal{I}$-model in $\mathrm{PSH}(X, \theta)$.*

Proof Observe that $\varphi \leq 0$. Let $\eta \in \mathrm{PSH}(X, \theta)$ with $\eta \sim_{\mathcal{I}} \varphi$ and $\eta \leq 0$. Then for each $i \in I$, using Proposition 3.2.3, we have $\eta \vee \varphi_i \sim_{\mathcal{I}} \varphi_i$. Therefore,

$$\eta \leq \eta \vee \varphi_i \leq \varphi_i.$$

It follows that $\eta \leq \varphi$. Hence $\varphi = P_\theta[\varphi]_{\mathcal{I}}$. □

Proposition 3.2.14 *Let $(\varphi_i)_{i \in I}$ be an increasing net in $\mathrm{PSH}(X, \theta)_{>0}$ uniformly bounded from above. Let $\varphi := \sup_{i \in I}{}^* \varphi_i$. Then*

$$\sup_{i \in I}{}^* P_\theta[\varphi_i]_{\mathcal{I}} = P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, if the φ_i's are all $\mathcal{I}$-model, then so is φ.

Proof Let $\eta = \sup_{i \in I}{}^* P_\theta[\varphi_i]_{\mathcal{I}}$. Then $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$ as a consequence of Lemma 3.2.1.
By Corollary 2.4.1, we have

$$\lim_{i \in I} \int_X \theta_{\varphi_i}^n = \int_X \theta_\varphi^n > 0.$$

So by Lemma 2.4.2, we can find a decreasing net $\epsilon_i \searrow 0$ ($i \in I$) with $\epsilon_i \in (0, 1)$ and $\psi_i \in \mathrm{PSH}(X, \theta)$ such that for all $i \in I$,

$$(1 - \epsilon_i)\varphi + \epsilon_i \psi_i \leq \varphi_i.$$

By Proposition 3.2.11 and Lemma 3.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq (1 - \epsilon_i) P_\theta[\varphi]_{\mathcal{I}} + \epsilon_i P_\theta[\psi_i]_{\mathcal{I}} \leq P_\theta[\varphi_i]_{\mathcal{I}} \leq \eta.$$

Taking limit with respect to i, we conclude that $\eta \geq P_\theta[\varphi]_{\mathcal{I}}$. □

Remark 3.2.1 One could also define the following interpolation between the $\mathcal{I}$-envelope and the P-envelope: Suppose $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, $j \in \{0, \ldots, n\}$. Then we let

$$\begin{aligned} P_{\theta,j}[\varphi] := \sup{}^* \Bigg\{ \psi \in \mathrm{PSH}(X, \theta) : \psi \leq 0, \varphi \leq \psi, \int_X \theta_\varphi^j \wedge \theta_{P_\theta[\varphi]_{\mathcal{I}}}^{n-j} \\ = \int_X \theta_\psi^j \wedge \theta_{P_\theta[\psi]_{\mathcal{I}}}^{n-j} \Bigg\}. \end{aligned}$$

Based on the techniques developed in Chapter 6, one could show that $P_{\theta,j}[\bullet]$ is a projection operator. When $j = n$, this operator reduces to the P-envelope, while when $j = 0$, this operator reduces to the $\mathcal{I}$-envelope.

Chapter 4
Geodesic rays in the space of potentials

In den Dreißiger Jahren besuchte ich regelmäßig die Schweiz, teils um mich auch auf den Viertausendern zu tummeln, zum großen Teil aber auch, um Emigrantenblätter zu lesen und mich mit Kollegen über Naziverbrechen zu unterhalten. Aber auch die Schweizer schauten sich, wenn sie offen reden wollten, ebenso ängstlich um wie das bei uns üblich war.[a]
— Oskar Perron[b]

[a] The recent policy of ETH against Chinese students makes me feel that nothing has changed in Switzerland after the collapsing of Nazi for almost 80 years.

[b] Oskar Perron (1880–1975), after earning an *Eisernes Kreuz* during WWI, obtained a position in München in 1922, initiating the glorious period there. Among his colleagues were Carathéodory, Tietze and Sommerfield.

In this chapter, we study subgeodesics and geodesics in the space of quasi-plurisubharmonic functions. Unlike what one usually finds in the literature, here we are carrying out the constructions in the space of Kähler potentials with prescribed singularities. Therefore, it is impossible to reduce to the case of geodesics with regular boundary points.

4.1 Subgeodesics

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

Definition 4.1.1 Let us fix $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. A *subgeodesic* from φ_0 to φ_1 is a family $(\varphi_t)_{t\in(0,1)}$ in $\mathrm{PSH}(X,\theta)$ such that

(1) if we define
$$\Phi\colon X \times \{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\} \to [-\infty,\infty), \quad (x,z) \mapsto \varphi_{\operatorname{Re} z}(x),$$
then Φ is $p_1^*\theta$-psh, where $p_1\colon X \times \{z \in \mathbb{C} : \operatorname{Re} z \in (0,1)\} \to X$ is the natural projection;

(2) when $t \to 0+$ (resp. to $1-$), φ_t converges to φ_0 (resp. φ_1) with respect to the L^1-topology.

We also say $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

We call Φ the *complexification* of the subgeodesic $(\varphi_t)_t$.

When we do not want to specify φ_0 and φ_1, we shall say $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic.

More generally, a family $(\psi_t)_{t\in[a,b]}$ in $\mathrm{PSH}(X,\theta)$ for some $a \leq b$ is called a *subgeodesic* if $(\psi_{tb+(1-t)a})_{t\in[0,1]}$ is a subgeodesic.

Remark 4.1.1 In the literature, people sometimes regard Φ as a function defined on $X \times \{z \in \mathbb{C} : \mathrm{e}^{-1} < |z| < 1\}$, with $\Phi(x,z) = \varphi_{-\log|z|^2}(x)$. We sometimes also use this definition without explicit explanation. It should not be difficult to tell which definition we are using from the context.

In general, there are no subgeodesics from φ_0 to φ_1. In fact, the existence of a subgeodesic implies that $\varphi_0 \wedge \varphi_1 \not\equiv -\infty$ by Proposition 4.1.2 below, which does not always hold as we show in Example 5.2.3.

We first note that the subgeodesics are well-behaved under the change of θ:

Proposition 4.1.1 *Let g be a smooth real function on X. Let $\theta' = \theta + \mathrm{dd^c} g$. Suppose that $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$. Then $(\varphi_t - g)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta')$.*

Proof This follows trivially by definition. □

Example 4.1.1 Let $\varphi_0 \in \mathrm{PSH}(X,\theta)$, $C \in \mathbb{R}$. Let

$$\varphi_t = \varphi_0 + tC, \quad t \in (0,1].$$

Then $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

For this purpose, it suffices to observe that $\mathrm{Re}\, z$ is a harmonic function in z.

As a consequence, the constant $(\varphi_0)_{t\in[0,1]}$ is a subgeodesic, called the *constant subgeodesic* at φ_0.

A more general version is as follows: Suppose that $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$, $C_1, C_2 \in \mathbb{R}$, then $(\varphi_t + C_1 t + C_2)_{t\in[0,1]}$ is also a subgeodesic.

Proposition 4.1.2 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$ and $(\varphi_t)_{t\in(0,1)}$ be a subgeodesic from φ_0 to φ_1. Then for each $x \in X$, $[0,1] \ni t \mapsto \varphi_t(x)$ is a convex function. In particular,*

$$\inf_{t\in(0,1)} \varphi_t \in \mathrm{PSH}(X,\theta), \qquad \inf_{t\in(0,1)} \varphi_t \leq \varphi_0 \wedge \varphi_1.$$

Proof Let Φ be the complexification of $(\varphi_t)_{t\in(0,1)}$.

For each $x \in X$, the map

$$\{z \in \mathbb{C} : \mathrm{Re}\, z \in (0,1)\} \to [-\infty, \infty), \quad z \mapsto \Phi(x,z)$$

is either subharmonic or constantly $-\infty$, as follows from Definition 4.1.1 (1) and Proposition 1.1.4. In the latter case, the convexity of $[0,1] \ni t \mapsto \varphi_t(x)$ is trivial. In the former case, the convexity on the interval $(0,1)$ follows from Proposition 1.1.3.

In order to verify the convexity at the boundary, let us fix $s \in (0,1)$. We need to show that

$$\varphi_s(x) \leq s\varphi_1(x) + (1-s)\varphi_0(x) \tag{4.1}$$

for all $x \in X$. Thanks to Proposition 1.2.6, it suffices to prove this for almost all x.

Take a set $Z \subseteq X$ with zero Lebesgue measure such that for all $x \in X \setminus Z$, we have

(1) $\varphi_t(x) \neq -\infty$ for all $t \in [0,1] \cap \mathbb{Q}$;
(2) $\varphi_t(x) \to \varphi_0(x)$ as $t \to 0+$ and $\varphi_t(x) \to \varphi_1(x)$ as $t \to 1-$.

For all such x, the convexity of $\varphi_t(x)$ for $t \in (0,1)$ guarantees that $\varphi_t(x) \neq -\infty$ for all $t \in [0,1]$ and $t \mapsto \varphi_t(x)$ is convex for $t \in [0,1]$. In particular, (4.1) holds.

Let us prove the last assertion. Let

$$\varphi := \inf_{t\in(0,1)} \varphi_t .$$

By Kiselman's principle Proposition 1.2.8, we know that $\varphi \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$. Take $x \in X$ so that

$$\lim_{t\to 0+} \varphi_t(x) = \varphi_0(x) \neq -\infty, \qquad \lim_{t\to 1-} \varphi_t(x) = \varphi_1(x) \neq -\infty.$$

Then $\varphi(x) \neq -\infty$. Hence we conclude that $\varphi \in \mathrm{PSH}(X,\theta)$. For any $t \in (0,1)$, using the convexity established above, we have

$$\varphi \leq (1-t)\varphi_1 + t\varphi_0 .$$

It follows that $\varphi \leq \varphi_0$ and $\varphi \leq \varphi_1$ almost everywhere and hence everywhere by Proposition 1.2.6. Our assertion follows. □

Proposition 4.1.3 *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two non-empty uniformly bounded from above families in* $\mathrm{PSH}(X,\theta)$. *Let $(\varphi_t^i)_{t\in(0,1)}$ be subgeodesics from φ_0^i to φ_1^i for each $i \in I$. Then*

$$\left(\sup_{i\in I}{}^{*}\varphi_t^i\right)_{t\in(0,1)}$$

is a subgeodesic from $\sup_i{}^*\varphi_0^i$ to $\sup_i{}^*\varphi_1^i$.

Proof We may assume that $\varphi_0^i, \varphi_1^i \leq 0$ for all $i \in I$. Then it follows that $\varphi_t^i \leq 0$ for all $t \in (0,1)$ and all $i \in I$ by Proposition 4.1.2.

We define

$$\varphi_t := \sup_{i\in I}{}^{*}\varphi_t^i \in \mathrm{PSH}(X,\theta)$$

for all $t \in [0,1]$. Observe that $[0,1] \ni t \mapsto \varphi_t$ is convex by the same argument leading to (4.1).

Let $(\psi_t)_{t\in(0,1)}$ be the subgeodesic whose complexification Φ_ψ corresponds to $\sup_i{}^*\Phi_{\varphi^i}$, where Φ_{φ^i} is the complexification of $(\varphi_t^i)_{t\in(0,1)}$. Then clearly, $\varphi_t \leq \psi_t$ for each $t \in (0,1)$. On the other hand, by Proposition 1.2.5,

$$\psi_t = \sup_{i\in I} \varphi_t^i = \varphi_t \quad \text{almost everywhere}$$

for almost all $t \in (0,1)$. Therefore, using Proposition 1.2.6, we find $\psi_t = \varphi_t$ for almost all $t \in (0,1)$. Since both functions are convex in t, we conclude that $\psi_t = \varphi_t$ for all $t \in (0,1)$.

It remains to argue that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$ and $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. By symmetry, it suffices to argue the former.

Thanks to Proposition 1.2.2, we may further assume that I is a countable set. We know that for any $t \in (0,1)$ and any $j \in I$,

$$\varphi_t^j \leq \varphi_t \leq t\varphi_1 + (1-t)\varphi_0.$$

Letting $t \to 0+$, we find that

$$\varphi_0^j \leq \varlimsup_{t\to 0+} \varphi_t \leq \varphi_0$$

almost everywhere. Since I is countable, we conclude that

$$\varphi_0 = \varlimsup_{t\to 0+} \varphi_t \tag{4.2}$$

almost everywhere.

Fix $i_0 \in I$. Recall that by Proposition 4.1.2, for each $t \in (0,1)$, we have

$$\inf_{t\in(0,1)} \sup_X \varphi_t \geq \inf_{t\in(0,1)} \sup_X \varphi_t^{i_0} \geq \sup_X \left(\varphi_0^{i_0} \wedge \varphi_1^{i_0}\right) > -\infty,$$

so the set $\{\varphi_t\}_{t\in(0,1)}$ is relatively compact with respect to the L^1-topology by Proposition 1.5.1. Let ψ be a cluster point as $t \to 0+$. It suffices to show that $\psi = \varphi_0$. By Corollary 1.2.1 and (4.2), this holds almost everywhere. Therefore, it holds everywhere by Proposition 1.2.6. □

Proposition 4.1.4 *Let $(\varphi_t)_{t\in[0,1]}$ be a subgeodesic in $\mathrm{PSH}(X,\theta)$. Then for any $0 \leq a \leq b \leq 1$, the segment $(\varphi_t)_{t\in[a,b]}$ is a subgeodesic.*

Proof It suffices to show that

$$\varphi_{tb+(1-t)a} \xrightarrow{L^1} \varphi_a, \quad \varphi_{tb+(1-t)a} \xrightarrow{L^1} \varphi_b$$

as $t \to 0+$ and $t \to 1-$ respectively. In other words, we need to show that for any $c \in (0,1)$, we have

$$\varphi_t \xrightarrow{L^1} \varphi_c$$

as $t \to c$. For this purpose, observe that by Proposition 4.1.2,

$$\sup_X \inf_{s\in(0,1)} \varphi_s \leq \sup_X \varphi_t \leq \left(\sup_X \varphi_0\right) \vee \left(\sup_X \varphi_1\right)$$

for any $t \in (0,1)$. Therefore, $\{\varphi_t\}_{t\in(0,1)}$ is a relatively compact family with respect to the L^1-topology on $\mathrm{PSH}(X,\theta)$ by Proposition 1.5.1. It suffices to show that any cluster point ψ of φ_t as $t \to c$ is equal to φ_c. By Corollary 1.2.1 and the convexity Proposition 4.1.2, we have $\varphi_c = \psi$ almost everywhere and hence everywhere by Proposition 1.2.6. □

Definition 4.1.2 A ray $\ell = (\ell_t)_{t\geq 0}$ is a *subgeodesic ray* in $\mathrm{PSH}(X,\theta)$ if for any $0 \leq a \leq b$, the segment $(\varphi_t)_{t\in[a,b]}$ is a subgeodesic in $\mathrm{PSH}(X,\theta)$. We say ℓ *emanates* from ℓ_0.

The *complexification* of a subgeodesic ray ℓ is defined as the potential

$$\Phi\colon X \times \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} \to [-\infty, \infty), \quad (x, z) \mapsto \ell_z(x).$$

Note that Φ is $p_1^*\theta$-psh, where $p_1\colon X \times \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\} \to X$ is the natural projection.

Remark 4.1.2 Similar to Remark 4.1.1, we could also define the complexification as a function $X \times \{z \in \mathbb{C} : 0 < |z| < 1\} \to [-\infty, \infty)$.

4.2 Geodesics in the space of potentials

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$. See Definition 3.1.3 for the definition.

Definition 4.2.1 Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. The *geodesic* $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 is the family of potentials $\varphi_t \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$ such that

$$\begin{aligned}\varphi_t = \sup{}^* \{\psi_t : (\psi_s)_s \text{ is a subgeodesic from } \psi_0 \text{ to } \psi_1,\\ \psi_0, \psi_1 \in \mathrm{PSH}(X,\theta),\ \psi_0 \leq \varphi_0, \psi_1 \leq \varphi_1\}\,.\end{aligned} \tag{4.3}$$

More generally, let $(\varphi_t)_{t\in[a,b]}$ ($a, b \in \mathbb{R}$, $a \leq b$) be a curve in $\mathrm{PSH}(X,\theta)$. We say $(\varphi_t)_{t\in[a,b]}$ is a *geodesic* if the curve $(\varphi_{tb+(1-t)a})_{t\in(0,1)}$ is a geodesic from φ_a to φ_b.

We also say $(\varphi_t)_{t\in(a,b)}$ is a *geodesic* in $\mathrm{PSH}(X,\theta)$ from φ_a to φ_b.

The envelopes of the form (4.3) are usually referred to as the *Perron envelopes*. In general, the geodesic defined by (4.3) fails to have the correct limit when $t \to 0+$ or $t \to 1-$. Therefore, *a priori* it is not a subgeodesic from φ_0 to φ_1. Although geodesics are defined in this case, we shall always avoid using this terminology.

Example 4.2.1 Let $\varphi_0 \in \mathrm{PSH}(X,\theta)$ and $C \in \mathbb{R}$. Then the subgeodesic $(\varphi_0 + tC)_{t\in[0,1]}$ studied in Example 4.1.1 is a geodesic. This follows easily from Proposition 4.1.2.

In particular, when $C = 0$, we find that the constant subgeodesic at φ_0 is indeed a geodesic, which we call the *constant geodesic* at φ.

More generally, suppose that $(\varphi_t)_{t\in[0,1]}$ is a geodesic and $C_1, C_2 \in \mathbb{R}$, then $(\varphi_t + C_1 t + C_2)_{t\in[0,1]}$ is also a geodesic. This follows immediately from Example 4.1.1.

Next we want to show that under mild assumptions, there exists subgeodesics between two potentials. The assumption below turns out to be necessary as well, as we shall prove in Theorem 6.1.1 below.

Proposition 4.2.1 *Given $\varphi_0, \varphi_1 \in \mathcal{E}(X, \theta; \phi)$, the geodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1 defined by (4.3) is a subgeodesic from φ_0 to φ_1 and $\varphi_t \in \mathcal{E}(X, \theta; \phi)$ for each $t \in (0, 1)$.*

Moreover, for any $0 \leq a \leq b \leq 1$, the restriction $(\varphi_t)_{t\in[a,b]}$ is a geodesic.

If furthermore $\varphi_0, \varphi_1 \in \mathcal{E}^1(X, \theta; \phi)$ (resp. $\mathcal{E}^\infty(X, \theta; \phi)$), then $\varphi_t \in \mathcal{E}^1(X, \theta; \phi)$ (resp. $\mathcal{E}^\infty(X, \theta; \phi)$) for all $t \in (0, 1)$.

We refer to Section 3.1.3 for the definition of $\mathcal{E}(X, \theta; \phi)$. Our assumption means that $P_\theta[\varphi_0] = P_\theta[\varphi_1] = \phi$.

Proof Without loss of generality, we may assume that $\varphi_0, \varphi_1 \leq \phi$. It follows from Proposition 4.1.2 that $\varphi_t \leq \phi$ for all $t \in (0, 1)$. In fact, we have the stronger estimate

$$\varphi_t \leq t\varphi_1 + (1 - t)\varphi_0, \quad t \in (0, 1). \tag{4.4}$$

We first observe that when $\varphi_0, \varphi_1 \in \mathcal{E}(X, \theta; \phi)$, so is $\varphi_0 \wedge \varphi_1$, see Proposition 3.1.18. In particular, the constant subgeodesic $t \mapsto \varphi_0 \wedge \varphi_1$ is a candidate in (4.3). So

$$\varphi_t \geq \varphi_0 \wedge \varphi_1, \quad t \in (0, 1). \tag{4.5}$$

By Proposition 4.1.3, $(\varphi_t)_{t\in(0,1)}$ is a subgeodesic.[1] It follows from Proposition 3.1.19 that $\varphi_t \in \mathcal{E}(X, \theta; \phi)$ for all $t \in (0, 1)$.

Next, we show that as $t \to 0+$, we have $\varphi_t \xrightarrow{L^1} \varphi_0$. The corresponding result at $t = 1$ is similar.

We first argue the special case where $\varphi_0 \leq \varphi_1$. Take a constant $C > 0$ such that

$$\varphi_0 - C \leq \varphi_1.$$

Then $(\varphi_0 - Ct)_{t\in(0,1)}$ is clearly a candidate in (4.3), see Example 4.1.1. Therefore, for all $t \in (0, 1)$,

$$\varphi_0 - Ct \leq \varphi_t \leq t\varphi_1 + (1 - t)\varphi_0. \tag{4.6}$$

It follows that $\varphi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$.

Let us come back to the general case. By (4.4) and (4.5), we know that for all $t \in (0, 1)$,

$$\sup_X \varphi_0 \wedge \varphi_1 \leq \sup_X \varphi_t \leq \left(\sup_X \varphi_0\right) \vee \left(\sup_X \varphi_1\right).$$

It follows from Proposition 1.5.1 that $\{\varphi_t : t \in (0, 1)\}$ is a relatively compact subset of $\mathrm{PSH}(X, \theta)$ with respect to the L^1-topology.

Let ψ be an L^1-cluster point of φ_t as $t \searrow 0$, it suffices to show that $\psi = \varphi_0$.

For each $M \in \mathbb{N}$, we write

$$\varphi_0^M = \varphi_0 \wedge (\varphi_1 + M).$$

[1] Be careful, here $t \in (0, 1)$ instead of $[0, 1]$.

Observe that $\varphi_0^M \in \mathcal{E}(X,\theta;\phi)$ by Proposition 3.1.18. Let $(\varphi_t^M)_{t\in(0,1)}$ be the geodesic from φ_0^M to φ_1. Then it is clear that $\varphi_t^M \leq \varphi_t$ for all $t \in (0,1)$. Therefore,

$$\psi \geq \varphi_0 \wedge (\varphi_1 + M)$$

almost everywhere hence everywhere by Proposition 1.2.6. On the other hand, by (4.4), $\psi \leq \varphi_0$. So it suffices to show that

$$\varphi_0 \wedge (\varphi_1 + M) \xrightarrow{L^1} \varphi_0$$

as $M \to \infty$, which is shown in Proposition 3.1.21.

Now we have shown that $(\varphi_t)_{t\in[0,1]}$ is a subgeodesic.

Next, take $0 \leq a \leq b \leq 1$. We want to show that the restriction $(\varphi_t)_{t\in[a,b]}$ is the geodesic from φ_a to φ_b. We may assume that $a < b$. The argument is the standard *balayage* argument.

Let $(\psi_t)_{t\in(a,b)}$ be the geodesic from φ_a to φ_b. Since $(\varphi_t)_{t\in[a,b]}$ is a subgeodesic by Proposition 4.1.4. we have $\psi_t \geq \varphi_t$ for all $t \in (a,b)$.

We define

$$\eta_t = \begin{cases} \psi_t, & \text{if } t \in (a,b), \\ \varphi_t, & \text{if } t \in (0,1) \setminus (a,b). \end{cases}$$

We claim that $(\eta_t)_{t\in(0,1)}$ is a subgeodesic from φ_0 to φ_1. This is clear by Lemma 1.2.2 when neither $a = 0$ nor $b = 1$. Next we handle the case where $a = 0$. By the previous part of the proof, we know that $\psi_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$. But $\psi_t = \eta_t$ for $t \in (0,b)$. Hence $\eta_t \xrightarrow{L^1} \varphi_0$ as $t \to 0+$. The case $b = 1$ is handled similarly.

Therefore, for all $t \in (0,1)$, we have

$$\varphi_t \geq \eta_t.$$

In particular, for $t \in (a,b)$, we have

$$\varphi_t \geq \eta_t = \psi_t \geq \varphi_t.$$

In other words, $(\varphi_t)_{t\in(a,b)} = (\psi_t)_{t\in(a,b)}$ is the geodesic from φ_a to φ_b.

Finally, assume furthermore that $\varphi_0, \varphi_1 \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$). Thanks to (4.5), Proposition 3.1.18 and Proposition 3.1.19, we find $\varphi_t \in \mathcal{E}^1(X,\theta;\phi)$ (resp. $\mathcal{E}^\infty(X,\theta;\phi)$) for all $t \in (0,1)$. $\square$

Proposition 4.2.2 *Let $\varphi_1, \varphi_0 \in \mathcal{E}(X,\theta;\phi)$ with $\varphi_1 \leq \varphi_0$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$s \sup_{\{\varphi_0\neq-\infty\}} (\varphi_1 - \varphi_0) = \sup_{\{\varphi_0\neq-\infty\}} (\varphi_s - \varphi_0) \tag{4.7}$$

for all $s \in [0,1]$.

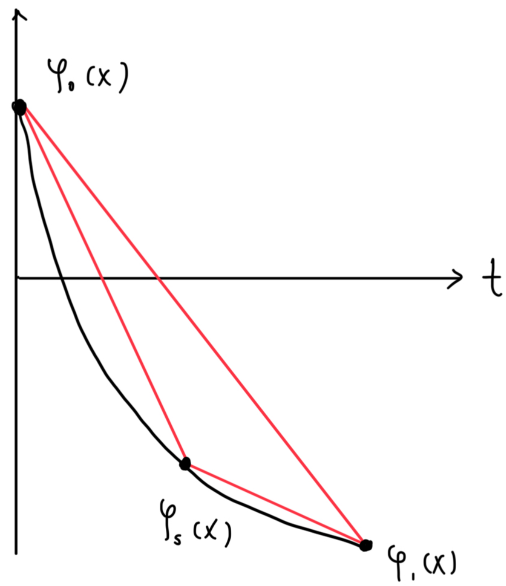

Fig. 4.1 The typical behavior of $\varphi_t(x)$

Proof The notations in the proof are indicated in Fig. 4.1.[2]

We may assume that $s \in [0, 1)$ since there is nothing to prove when $s = 1$.

After replacing φ_t by $\varphi_t - C't$ for some large enough $C' > 0$, we may assume that $\varphi_1 \leq \varphi_0$. This procedure preserves the geodesic property by Example 4.2.1.

Since the constant geodesic at φ_1 is a candidate in (4.3), it follows that $\varphi_1 \leq \varphi_t$ for all $t \in [0, 1]$. Similarly, $[0, 1] \ni t \mapsto \varphi_t$ is decreasing.

Let

$$C = \sup_{\{\varphi_1 \neq -\infty\}} (\varphi_1 - \varphi_0) \leq 0. \tag{4.8}$$

Then by Proposition 1.2.6, we have

$$\varphi_1 \leq \varphi_0 + C.$$

So $(\varphi_1 - C(1-t))_{t \in (0,1)}$ is a candidate in (4.3) and hence

$$\varphi_1 - C(1-t) \leq \varphi_t, \quad t \in (0, 1). \tag{4.9}$$

By Proposition 4.2.1, we have $\varphi_t \xrightarrow{L^1} \varphi_1$ as $t \to 1-$. Since φ_t is decreasing in $t \in (0, 1)$. It follows that $\varphi_1 = \inf_{t \in (0,1)} \varphi_t$. Therefore, we can find a pluripolar set $Z \subseteq X$ such that $\varphi_t(x) \to \varphi_1(x) > -\infty$ as $t \to 1-$ for all $x \in X \setminus Z$.

---

[2] When dealing with convex functions, drawing a picture is the easiest way to keep track of the directions of inequalities.

Similarly, since $\varphi_0 = \sup_{t\in(0,1)}{}^*\varphi_t$, after enlarging Z, we may also guarantee that $\varphi_t(x) \to \varphi_0(x) > -\infty$ as $t \to 0+$ for all $x \in X \setminus Z$ by Proposition 1.2.5.

For any such $x \in X \setminus Z$, the function $t \mapsto \varphi_t(x)$ is a real-valued continuous convex function on $[0,1]$. In particular, $t \mapsto \varphi_t(x)$ is absolutely continuous on $[0,1]$. Hence, for any $s \in [0,1)$, we have

$$\varphi_1(x) - \varphi_s(x) = \int_s^1 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x)\,\mathrm{d}t \leq (1-s)\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t} \leq (1-s)C, \tag{4.10}$$

where the second inequality follows from (4.9).

Taking supremum in (4.10), we find that

$$\sup_{X\setminus Z}(\varphi_1 - \varphi_s) \leq (1-s)\sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t} \leq (1-s)C. \tag{4.11}$$

When $s = 0$, we deduce from Corollary 1.3.6 and (4.8) that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0) = \sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t}.$$

But this equality works equally well for the geodesic $(\varphi_{(1-s)t+s})_{t\in[0,1]}$. It follows that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_s) = (1-s)\sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t} = (1-s)C.$$

Therefore, invoking Corollary 1.3.6 again, we deduce that all inequalities in (4.11) are in fact equalities. In other words,

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0) = \sup_{x\in X\setminus Z}\lim_{t\to 1-}\frac{\varphi_1(x)-\varphi_t(x)}{1-t} = \sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_1-\varphi_s}{1-s}. \tag{4.12}$$

On the other hand, we have the trivial inequality

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0) \leq s\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s} + (1-s)\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_1-\varphi_s}{1-s}.$$

Together with (4.12), we find that

$$\sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0) \leq \sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s}.$$

The reverse inequality follows from the convexity,

$$\sup_{\{\varphi_1\neq-\infty\}}\frac{\varphi_s-\varphi_0}{s} = \sup_{\{\varphi_1\neq-\infty\}}(\varphi_1 - \varphi_0).$$

Using Corollary 1.3.6, we conclude (4.7). □

With an almost identical proof, we find

Proposition 4.2.3 *Let $\varphi_1, \varphi_0 \in \mathcal{E}^\infty(X,\theta;\phi)$. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then*

$$t \inf_{\{\phi\neq-\infty\}} (\varphi_1 - \varphi_0) = \inf_{\{\phi\neq-\infty\}} (\varphi_t - \varphi_0)$$

for all $t \in (0,1]$.

Definition 4.2.2 Let $\ell = (\ell_t)_{t\geq 0}$ be a curve in $\mathcal{E}(X,\theta;\phi)$. We say ℓ is a *geodesic ray* in $\mathcal{E}(X,\theta;\phi)$ emanating from ℓ_0 if for each $0 \leq a \leq b$, the restriction $(\ell_t)_{t\in[a,b]}$ is a geodesic.

The set of geodesic rays in $\mathcal{E}(X,\theta;\phi)$ emanating from ϕ is denoted by $\mathcal{R}(X,\theta;\phi)$.

We say a geodesic ray $\ell \in \mathcal{R}(X,\theta;\phi)$ has *finite energy* if $\ell_t \in \mathcal{E}^1(X,\theta;\phi)$ for all $t > 0$. The set of geodesic rays with finite energy is denoted by $\mathcal{R}^1(X,\theta;\phi)$.

We say a geodesic ray $\ell \in \mathcal{R}(X,\theta;\phi)$ is *bounded* if $\ell_t \in \mathcal{E}^\infty(X,\theta;\phi)$ for all $t \geq 0$. The set of bounded geodesic rays is denoted by $\mathcal{R}^\infty(X,\theta;\phi)$.

Given $\ell, \ell' \in \mathcal{R}(X,\theta;\phi)$, we write $\ell \leq \ell'$ if $\ell_t \leq \ell'_t$ for each $t \geq 0$.

When $\phi = V_\theta$, we usually omit it from the notations and write $\mathcal{R}(X,\theta)$, $\mathcal{R}^1(X,\theta)$ and $\mathcal{R}^\infty(X,\theta)$ respectively.

Proposition 4.2.4 *Let $\ell \in \mathcal{R}(X,\theta;\phi)$. Then there is a constant $C \in \mathbb{R}$ such that*

$$\sup_X \ell_t = Ct, \quad t \geq 0.$$

Proof It follows from Proposition 4.2.2 that

$$\sup_{\{\phi\neq-\infty\}} (\ell_t - \phi) = t \sup_X (\ell_1 - \phi)$$

for all $t \geq 0$.

It suffices to show that for any $t \geq 0$,

$$\sup_{\{\phi\neq-\infty\}} (\ell_t - \phi) = \sup_X \ell_t.$$

This was already proved in Proposition 3.1.4. □

4.3 The metrics on the spaces of potentials and rays

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

We first study a natural metric on $\mathcal{E}^1(X,\theta;\phi)$.

Definition 4.3.1 Let $\varphi, \psi \in \mathcal{E}^1(X,\theta;\phi)$, we define

$$d_1(\varphi,\psi) = E^\phi_\theta(\varphi) + E^\phi_\theta(\psi) - 2E^\phi_\theta(\varphi \wedge \psi).$$

Note that by Proposition 3.1.18, $\varphi \wedge \psi \in \mathcal{E}^1(X,\theta;\phi)$.

Recall that E^{ϕ}_{θ} is defined in Definition 3.1.5.

In particular, if $\varphi \leq \psi$, we have

$$d_1(\varphi,\psi) = E^{\phi}_{\theta}(\psi) - E^{\phi}_{\theta}(\varphi). \tag{4.13}$$

We wish to show that d_1 is a complete metric. We first prove a contraction property:

Proposition 4.3.1 *Let $\varphi,\psi,\gamma \in \mathcal{E}^1(X,\theta;\phi)$. Then*

$$d_1(\varphi,\psi) \geq d_1(\varphi\wedge\gamma,\psi\wedge\gamma). \tag{4.14}$$

Proof **Step 1**. We first assume that $\varphi \geq \psi$. Then

$$\begin{aligned} d_1(\varphi,\psi) =& E^{\phi}_{\theta}(\varphi) - E^{\phi}_{\theta}(\psi) \\ \geq& E^{\phi}_{\theta}((\varphi\wedge\gamma)\vee\psi) - E^{\phi}_{\theta}(\psi) \qquad \text{by Lemma 3.1.3} \\ \geq& E^{\phi}_{\theta}(\varphi\wedge\gamma) - E^{\phi}_{\theta}(\psi\wedge\gamma) \qquad \text{by Corollary 3.1.4.} \end{aligned}$$

Step 2. We prove the general case.

By Step 1, we have

$$d_1(\varphi,\varphi\wedge\psi) \geq d_1(\varphi\wedge\gamma,\varphi\wedge\psi\wedge\gamma), \quad d_1(\psi,\varphi\wedge\psi) \geq d_1(\psi\wedge\gamma,\varphi\wedge\psi\wedge\gamma).$$

Adding the two inequalities together, we conclude (4.14). □

Lemma 4.3.1 *The function d_1 is a metric on $\mathcal{E}^1(X,\theta;\phi)$.*

Proof There are two facts to prove, as in the two steps below.

Step 1. Let $\varphi,\psi \in \mathcal{E}^1(X,\theta;\phi)$. Assume that $d_1(\varphi,\psi) = 0$, then we will show that $\varphi = \psi$.

We may assume that $\psi \leq \varphi$, thanks to the definition of d_1. Then it follows from (4.13) and Proposition 3.1.15 that

$$\int_X (\psi - \varphi)\,\theta^n_{\varphi} = 0.$$

We conclude that $\psi = \varphi$ using Theorem 2.4.6.

Step 2. Let $\varphi,\psi,\gamma \in \mathcal{E}^1(X,\theta;\phi)$. We prove the triangle inequality:

$$d_1(\varphi,\psi) \leq d_1(\varphi,\gamma) + d_1(\psi,\gamma).$$

This can be translated to

$$E^{\phi}_{\theta}(\varphi\wedge\gamma) - E^{\phi}_{\theta}(\varphi\wedge\psi) \leq E^{\phi}_{\theta}(\gamma) - E^{\phi}_{\theta}(\psi\wedge\gamma).$$

We just have to compute using Proposition 4.3.1:

$$
\begin{aligned}
E_\theta^\phi(\gamma) - E_\theta^\phi(\psi \wedge \gamma) \geq & E_\theta^\phi(\varphi \wedge \gamma) - E_\theta^\phi(\varphi \wedge \psi \wedge \gamma) \\
\geq & E_\theta^\phi(\varphi \wedge \gamma) - E_\theta^\phi(\varphi \wedge \psi),
\end{aligned}
$$

where the second line follows from Lemma 3.1.3. □

We introduce an auxiliary functional.

Definition 4.3.2 Define $I_\theta : \mathcal{E}^1(X, \theta; \phi) \times \mathcal{E}^1(X, \theta; \phi) \to \mathbb{R}$ as follows:

$$
I_\theta(\varphi, \psi) := \int_X |\varphi - \psi| \left(\theta_\varphi^n + \theta_\psi^n\right). \tag{4.15}
$$

We observe the following elementary equality.

Lemma 4.3.2 *Let $\varphi, \psi \in \mathcal{E}^1(X, \theta; \phi)$, then*

$$
I_\theta(\varphi, \psi) = I_\theta(\varphi \vee \psi, \varphi) + I_\theta(\varphi \vee \psi, \psi).
$$

Proof It suffices to write

$$
\begin{aligned}
I_\theta(\varphi, \psi) &= \int_{\{\varphi<\psi\}} (\psi - \varphi) \left(\theta_\varphi^n + \theta_\psi^n\right) + \int_{\{\varphi>\psi\}} (\varphi - \psi) \left(\theta_\varphi^n + \theta_\psi^n\right), \\
I_\theta(\varphi \vee \psi, \varphi) &= \int_{\{\psi>\varphi\}} (\psi - \varphi) \left(\theta_\psi^n + \theta_\varphi^n\right), \\
I_\theta(\varphi \vee \psi, \psi) &= \int_{\{\psi<\varphi\}} (\varphi - \psi) \left(\theta_\psi^n + \theta_\varphi^n\right).
\end{aligned}
$$

We have an interesting relation between d_1 and I_θ defined in Definition 4.3.2.

Theorem 4.3.1 *Let $\varphi, \psi \in \mathcal{E}^1(X, \theta; \phi)$. Then*

$$
\frac{1}{C_n} I_\theta(\varphi, \psi) \leq d_1(\varphi, \psi) \leq I_\theta(\varphi, \psi), \tag{4.16}
$$

where $C_n = 3 \cdot 2^{n+2}(n+1)$.

Proof **Step 1**. We first prove the right-hand part of (4.16).

Thanks to Proposition 3.1.16 and Lemma 3.1.1, we have

$$
\begin{aligned}
E_\theta^\phi(\varphi) - E_\theta^\phi(\varphi \wedge \psi) &\leq \int_X (\varphi - \varphi \wedge \psi)\, \theta_{\varphi\wedge\psi}^n \\
&\leq \int_{\{\psi=\varphi\wedge\psi\}} (\varphi - \psi)\, \theta_\psi^n \\
&\leq \int_X |\varphi - \psi|\, \theta_\psi^n.
\end{aligned}
$$

Similarly,

$$E_\theta^\phi(\psi) - E_\theta^\phi(\varphi\wedge\psi) \leq \int_X |\varphi-\psi|\,\theta_\varphi^n.$$

Adding these inequalities up, we find

$$d_1(\varphi,\psi) \leq I_\theta(\varphi,\psi).$$

Step 2. We prove the left-hand part of (4.16).

We claim that

$$d_1\left(\varphi, \frac{\varphi+\psi}{2}\right) \leq \frac{3(n+1)}{2} d_1(\varphi,\psi). \tag{4.17}$$

For this purpose, we compute directly

$$\begin{aligned}
& d_1\left(\varphi, \frac{\varphi+\psi}{2}\right) \\
=& d_1\left(\varphi, \varphi\wedge\frac{\varphi+\psi}{2}\right) + d_1\left(\frac{\varphi+\psi}{2}, \varphi\wedge\frac{\varphi+\psi}{2}\right) \\
\leq& d_1\left(\varphi,\varphi\wedge\psi\right) + d_1\left(\frac{\varphi+\psi}{2}, \varphi\wedge\psi\right) && \text{by Lemma 3.1.3} \\
\leq& \int_X (\varphi-\varphi\wedge\psi)\,\theta^n_{\varphi\wedge\psi} + \int_X \left(\frac{\varphi+\psi}{2} - \varphi\wedge\psi\right)\theta^n_{\varphi\wedge\psi} && \text{by Proposition 3.1.16} \\
=& \frac{3}{2}\int_X (\varphi-\varphi\wedge\psi)\,\theta^n_{\varphi\wedge\psi} + \frac{1}{2}\int_X (\psi-\varphi\wedge\psi)\,\theta^n_{\varphi\wedge\psi} \\
\leq& \frac{3(n+1)}{2} d_1\left(\varphi,\varphi\wedge\psi\right) + \frac{n+1}{2} d_1(\psi,\varphi\wedge\psi) && \text{by (3.31)} \\
\leq& \frac{3(n+1)}{2} d_1(\varphi,\psi),
\end{aligned}$$

and (4.17) follows.

We now estimate the left-hand side of (4.17):

$$\begin{aligned}
d_1\left(\varphi, \frac{\varphi+\psi}{2}\right) \geq & d_1\left(\varphi, \varphi\wedge\frac{\varphi+\psi}{2}\right) \\
\geq & \int_X \left(\varphi - \varphi\wedge\frac{\varphi+\psi}{2}\right)\theta_\varphi^n,
\end{aligned}$$

as a consequence of Proposition 3.1.16.

Similarly,

$$\begin{aligned}
d_1\left(\varphi, \frac{\varphi+\psi}{2}\right) \geq & d_1\left(\frac{\varphi+\psi}{2}, \varphi\wedge\frac{\varphi+\psi}{2}\right) \\
\geq & \int_X \left(\frac{\varphi+\psi}{2} - \varphi\wedge\frac{\varphi+\psi}{2}\right)\theta^n_{(\varphi+\psi)/2} \\
\geq & 2^{-n}\int_X \left(\frac{\varphi+\psi}{2} - \varphi\wedge\frac{\varphi+\psi}{2}\right)\theta^n_{\varphi}.
\end{aligned}$$

Adding these estimates up, we find

$$3 \cdot 2^n (n+1) d_1(\varphi, \psi) \geq \frac{1}{2} \int_X |\varphi - \psi| \, \theta_\varphi^n .$$

By symmetry, we also get a similar expression after exchanging φ and ψ. Adding these inequalities together, we find

$$3 \cdot 2^{n+2} (n+1) d_1(\varphi, \psi) \geq I_\theta(\varphi, \psi).$$

The left-hand part of (4.16) then follows, in view of (4.17). □

Lemma 4.3.3 *There is $A, B > 0$ so that for any $\varphi \in \mathcal{E}^1(X, \theta; \phi)$, we have*

$$d_1(\varphi, \phi) \geq -\int_X \theta_\phi^n \cdot \sup_X \varphi \geq -A d_1(\varphi, \phi) - B. \tag{4.18}$$

Proof When $\sup_X \varphi \leq 0$, the right-hand part of (4.18) is trivial. While

$$d_1(\phi, \varphi) = -E_\theta^\phi(\varphi) \geq -\int_X \theta_\phi^n \cdot \sup_{\{\phi \neq -\infty\}} (\varphi - \phi) = -\int_X \theta_\phi^n \cdot \sup_X \varphi,$$

where the last equality follows from Proposition 3.1.4.

We can therefore assume that $\sup_X \varphi > 0$. In this case, the left-hand part of (4.18) is trivial. Take a Kähler form $\omega \geq \theta$ on X. Then thanks to Theorem 3.1.1, we can find a constant $C > 0$ so that

$$\theta_\phi^n \leq C \omega^n .$$

Thanks to Proposition 1.5.1, there is a constant $C' > 0$, independent of the choice of φ, so that

$$\int_X \left(\phi - \varphi + \sup_X \varphi \right) \theta_\phi^n \leq C' .$$

We estimate

$$\begin{aligned} I_\theta(\phi, \varphi) &\geq \int_X |\varphi - \phi| \, \theta_\phi^n \\ &\geq \left(\sup_X \varphi \right) \int_X \theta_\phi^n - \int_X \left(\phi - \varphi + \sup_X \varphi \right) \theta_\phi^n \\ &\geq \left(\sup_X \varphi \right) \int_X \theta_\phi^n - C' . \end{aligned}$$

The right-hand part of (4.18) then follows from Theorem 4.3.1. □

Next we handle the completeness of d_1. The completeness can be proved in a more general framework.

Definition 4.3.3 Let E be a set. A *pre-rooftop structure* on E is a binary operator $\wedge : E \times E \to E$, satisfying the following axioms: For $x, y, z \in E$,

(1) $x \wedge y = y \wedge x$.
(2) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$.
(3) $x \wedge x = x$.

We call $(E, \wedge)$ a *pre-rooftop space*.

A pre-rooftop structure $\wedge$ defines a partial order $\leq$ on E as follows:

$$x \leq y \quad \text{if and only if} \quad x \wedge y = x.$$

Here by abuse of notation, we use $\leq$ to denote the partial order.

In particular, it makes sense to talk about an increasing and decreasing sequences in E.

Definition 4.3.4 Let (E, d) be a metric space. A *pre-rooftop structure* on (E, d) is a pre-rooftop structure $\wedge$ on E. We say $(E, d, \wedge)$ is a *pre-rooftop metric space*.

A *rooftop structure* on (E, d) is a pre-rooftop structure $\wedge$ on E such that

$$d(x \wedge z, y \wedge z) \leq d(x, y), \quad \forall x, y, z \in E. \tag{4.19}$$

We call $(E, d, \wedge)$ a *rooftop metric space*.

Lemma 4.3.4 *Let $(E, d, \wedge)$ be a rooftop metric space. Let $x, y, x', y' \in E$, then*

$$d(x \wedge y, x' \wedge y') \leq d(x, x') + d(y, y'). \tag{4.20}$$

Proof We compute

$$d(x \wedge y, x' \wedge y') \leq d(x \wedge y, x \wedge y') + d(x \wedge y', x' \wedge y') \leq d(x, x') + d(y, y').$$

Proposition 4.3.2 *Let $(E, d, \wedge)$ be a rooftop metric space. Then (E, d) is complete if and only if both of the followings hold:*

(1) *Each increasing Cauchy sequence converges.*
(2) *Each decreasing Cauchy sequence converges.*

Proof The direct implication is trivial.

Conversely, assume that both conditions are true. Let $(x_j)_{j>0}$ be a Cauchy sequence in E. We want to prove that $(x_j)_j$ converges. By passing to a subsequence, we may assume that

$$d(x_j, x_{j+1}) \leq 2^{-j}.$$

For $k, j \geq 1$, let

$$y_j^k := x_k \wedge \cdots \wedge x_{k+j}.$$

Then $(y_j^k)_j$ is decreasing, and

$$d(y_j^k, y_{j+1}^k) \leq d(x_{k+j}, x_{k+j+1}) \leq 2^{-k-j}.$$

So $(y_k^j)_j$ is a decreasing Cauchy sequence. Define

$$y^k := \lim_{j\to\infty} y_j^k.$$

Then

$$d(y^k, y^{k+1}) = \lim_{j\to\infty} d(y_{j+1}^k, y_j^{k+1}) \le d(x_k, x_{k+1}) \le 2^{-k}.$$

So y^k is an increasing Cauchy sequence. Let

$$y := \lim_{k\to\infty} y^k.$$

Then

$$d(y^k, x_k) = \lim_{j\to\infty} d(y_j^k, x_k) \le \lim_{j\to\infty} d(y_{j-1}^{k+1}, x_k).$$

Note that

$$d(y_{j-1}^{k+1}, x_k) \le d(y_{j-1}^{k+1}, x_{k+1}) + d(x_{k+1}, x_k) \le d(y_{j-1}^{k+2}, x_{k+1}) + 2^{-k}.$$

Hence

$$d(y^k, x_k) \le 2^{-k} + \lim_{j\to\infty} d(y_{j-1}^{k+2}, x_{k+1}) \le \lim_{j\to\infty} \sum_{r=k}^{j+k} d(x_r, x_{r+1}) \le 2^{1-k}.$$

So $(x_k)_k$ converges to y. □

Theorem 4.3.2 *The metric space $(\mathcal{E}^1(X,\theta;\phi), d_1)$ is complete.*

It follows from the proof below that if $(\varphi_j)_{j>0}$ is a monotone sequence in $\mathcal{E}^1(X,\theta;\phi)$ with d_1-limit φ, then φ is the almost everywhere limit of $(\varphi_j)_{j>0}$. We will use this result without further explanation in the sequel.

Proof As we have seen in Lemma 4.3.1 and Proposition 4.3.1, the triple

$$\left(\mathcal{E}^1(X,\theta;\phi), d_1, \wedge\right)$$

is a rooftop metric space. Hence thanks to Proposition 4.3.2, it remains to show that each increasing or decreasing Cauchy sequence in $\mathcal{E}^1(X,\theta;\phi)$ converges.

We first consider an increasing sequence $(\varphi_i)_{i>0}$ in $\mathcal{E}^1(X,\theta;\phi)$. The Cauchy property simply means that

$$\sup_{i>0} E_\theta^\phi(\varphi_i) < \infty.$$

We claim that

$$\varphi := \sup_{i>0}{}^* \varphi_i$$

is the d_1-limit of the sequence. We first observe that $(\sup_X \varphi_i)_{i>0}$ is bounded, as a consequence of Lemma 4.3.3. Therefore, Proposition 3.1.20 guarantees that $\varphi \in \mathcal{E}^1(X,\theta;\phi)$, and hence our assertion follows from Proposition 3.1.17.

Next we consider a decreasing sequence $(\varphi_i)_{i>0}$ in $\mathcal{E}^1(X,\theta;\phi)$. The Cauchy property simply means that

$$\inf_{i>0} E^{\phi}_{\theta}(\varphi_i) > -\infty.$$

We claim that

$$\varphi \coloneqq \inf_{i>0} \varphi_i$$

is the d_1-limit of the sequence. We first observe that $(\sup_X \varphi_i)_{i>0}$ is bounded, as a consequence of Lemma 4.3.3. Therefore, $\varphi \in \mathrm{PSH}(X,\theta;\phi)$. Our assertion then follows from Proposition 3.1.14. □

Lemma 4.3.5 *Let $(\varphi_i)_{i>0}$ be a sequence in $\mathcal{E}^1(X,\theta;\phi)$ with d_1-limit $\varphi \in \mathcal{E}^1(X,\theta;\phi)$. Then after replacing $(\varphi_i)_{i>0}$ by a subsequence, we can find two sequences $(\psi_i)_{i>0}$ and $(\eta_i)_{i>0}$ in $\mathcal{E}^1(X,\theta;\phi)$ such that*

(1) *$(\psi_i)_{i>0}$ is decreasing with d_1 and pointwise limit φ;*
(2) *$(\eta_i)_{i>0}$ is increasing with d_1 and almost everywhere limit φ.*

Proof We first note that as a consequence of Lemma 4.3.3, $(\sup_X \varphi_i)_{i>0}$ is bounded.

We first construct $(\psi_i)_{i>0}$. For this purpose, it suffices to define

$$\psi_i \coloneqq \sup_{j\geq i}{}^{*} \varphi_j$$

for each $i>0$. It follows from Proposition 3.1.20 that $\psi_i \in \mathcal{E}^1(X,\theta;\phi)$ for each $i>0$. Furthermore, φ is the limit of the decreasing sequence $(\psi_i)_{i>0}$ as we have seen in the proof of Corollary 1.2.1. Then φ is the d_1-limit of $(\psi_j)_{j>0}$ as a consequence of Proposition 3.1.14.

Next we construct $(\eta_i)_{i>0}$. For this purpose, we may replace $(\varphi_i)_{i>0}$ by a subsequence and assume that

$$d_1(\varphi_i,\varphi_{i+1}) \leq 2^{-i}.$$

For each $i>0$ and $k\geq 0$, we let

$$\eta_i^k \coloneqq \varphi_i \wedge \varphi_{i+1} \wedge \cdots \wedge \varphi_{i+k}.$$

Then

$$\begin{aligned} d_1(\varphi_i,\eta_i^k) &\leq \sum_{j=0}^{k-1} d_1\left(\eta_i^j,\eta_i^{j+1}\right) \\ &\leq \sum_{j=0}^{k-1} d_1\left(\varphi_{i+1},\varphi_{i+j+1}\right) \quad \text{by Proposition 4.3.1} \\ &\leq \sum_{j=0}^{k-1} 2^{-i-j} \\ &\leq 2^{1-i}. \end{aligned}$$

Therefore, Theorem 4.3.2 shows that

$$\eta_i \coloneqq \sup_{k\geq 0}{}^* \eta_i^k$$

is the d_1-limit of $(\eta_i^k)_{k\geq 0}$ and Proposition 3.1.17 shows that

$$d_1(\varphi_i, \eta_i) \leq 2^{1-i}.$$

Therefore, φ is the d_1-limit of the increasing sequence $(\eta_i)_{i>0}$. As we have seen in the proof of Theorem 4.3.2, this implies that φ is also the almost everywhere limit of $(\eta_i)_{i>0}$. □

Theorem 4.3.3 *The functional $E_\theta^\phi \colon \mathcal{E}^1(X,\theta;\phi) \to \mathbb{R}$ is continuous.*

Proof Let $(\varphi_j)_{j>0}$ be a sequence in $\mathcal{E}^1(X,\theta;\phi)$ with d_1-limit $\varphi \in \mathcal{E}^1(X,\theta;\phi)$. We wish to show that

$$\lim_{j\to\infty} E_\theta^\phi(\varphi_j) = E_\theta^\phi(\varphi). \tag{4.21}$$

For this purpose, we may freely replace $(\varphi_j)_{j>0}$ by a subsequence. In particular, thanks to Lemma 4.3.5 and Lemma 3.1.3, we may assume that $(\varphi_j)_{j>0}$ is a monotone sequence. In this case, (4.21) follows from Proposition 3.1.17. □

Next we recall two particular properties when $\phi = V_\theta$.[3]

Proposition 4.3.3 *Let $(\varphi_t)_{t\in[a,b]}$ be a geodesic in $\mathcal{E}^1(X,\theta)$, then $t \mapsto E_\theta(\varphi_t)$ is a linear function of $t \in [a,b]$.*

See [DDNL18c, Theorem 3.12].

Proposition 4.3.4 *Let $(\varphi_0^i)_{i\in I}$, $(\varphi_1^i)_{i\in I}$ be two uniformly bounded from above increasing nets in $\mathcal{E}^\infty(X,\theta)$. Let $(\varphi_t^i)_{t\in(0,1)}$ be the geodesic from φ_0^i to φ_1^i for each $i \in I$. Then*

$$\left(\sup_{i\in I}{}^* \varphi_t^i\right)_{t\in(0,1)}$$

is the geodesic from $\sup_i{}^*\varphi_0^i$ to $\sup_i{}^*\varphi_0^i$.

Proof By Proposition 1.2.2 and Proposition 4.1.3, we may assume that I is countable. In this case, the assertion follows from [DDNL18c, Proposition 3.3] and Theorem 2.1.1. □

Proposition 4.3.5 *Let $(\varphi_t)_{t\in[0,1]}$, $(\psi_t)_{t\in[0,1]}$ be geodesics in $\mathcal{E}^1(X,\theta)$. Then the distance $d_1(\varphi_t, \psi_t)$ is a convex function of $t \in [0,1]$.*

Proof By definition of d_1, it suffices to show the concavity of

$$[0,1] \ni t \mapsto E_\theta(\varphi_t \wedge \psi_t).$$

---

[3] I expect that these assertions hold even when $\phi \neq V_\theta$. But I am unable to prove them in full generality.

Let $(\eta_t)_{t\in[0,1]}$ be the geodesic from $\varphi_0 \wedge \psi_0$ to $\varphi_1 \wedge \psi_1$. Then for each $t \in [0,1]$, we have $\eta_t \leq \varphi_t \wedge \psi_t$. Then thanks to Proposition 4.3.6 and Proposition 3.1.17, we have

$$E_\theta(\varphi_t \wedge \psi_t) \geq E_\theta(\eta_t) = tE_\theta(\varphi_0 \wedge \psi_0) + (1-t)E_\theta(\varphi_1 \wedge \psi_1)$$

for each $t \in [0,1]$. Our assertion follows. □

In particular, we can introduce:

Definition 4.3.5 Let $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. We define

$$d_1(\ell,\ell') := \lim_{t\to\infty} \frac{1}{t} d_1(\ell_t,\ell'_t).$$

Theorem 4.3.4 *The function d_1 defined in Definition 4.3.5 is a metric.*

One can actually show that $(\mathcal{R}^1(X,\theta), d_1)$ is a complete metric space. We do not need this fact in the sequel, so we omit the proof. See [DDNL21b, Theorem 2.14].

Proof We first observe that $d_1(\ell,\ell') < \infty$ for any $\ell, \ell' \in \mathcal{R}^1(X,\theta)$. In fact, for each $t > 0$, we have

$$d_1(\ell_t,\ell'_t) \leq d_1(\ell_t,\phi) + d_1(\ell'_t,\phi) = -E^\phi_\theta(\ell_t) - E^\phi_\theta(\ell'_t) = -tE^\phi_\theta(\ell_1) - tE^\phi_\theta(\ell'_1)$$

by Proposition 4.3.3.

In view of Lemma 4.3.1, in order to prove that d_1 is a metric on $\mathcal{R}^1(X,\theta)$, it suffices to prove the following assertion: Suppose that $\ell, \ell' \in \mathcal{R}^1(X,\theta)$ and $d_1(\ell,\ell') = 0$, then $\ell = \ell'$.

Fix $s > 0$, then it follows from Proposition 4.3.5 that

$$\frac{d_1(\ell_s,\ell'_s)}{s} \leq \lim_{t\to\infty} \frac{d_1(\ell_t,\ell'_t)}{t} = d_1(\ell,\ell') = 0.$$

Therefore, $\ell_s = \ell'_s$ and hence $\ell = \ell'$.

Definition 4.3.6 We define the *radial Monge–Ampère energy* $\mathbf{E}^\phi : \mathcal{R}(X,\theta;\phi) \to \mathbb{R} \cup \{\infty\}$ as follows:

$$\mathbf{E}^\phi(\ell) := \varlimsup_{t\to\infty} \frac{E^\phi_\theta(\ell_t)}{t}.$$

When $\phi = V_\theta$, we write $\mathbf{E}$ instead of $\mathbf{E}^{V_\theta}$.

Proposition 4.3.6 *Let $\ell, \ell' \in \mathcal{R}^1(X,\theta)$ and $\ell \leq \ell'$. Then*

$$d_1(\ell,\ell') = \mathbf{E}(\ell') - \mathbf{E}(\ell). \tag{4.22}$$

Proof This is a direct consequence of (4.13). □

Next we recall that $\vee$ operator at the level of geodesic rays.

Definition 4.3.7 Let $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$. We define $\ell \vee \ell'$ as the minimal ray in $\mathcal{R}^\infty(X,\theta)$ lying above both ℓ and ℓ'.

Proposition 4.3.7 *Given $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$. Then $\ell \vee \ell' \in \mathcal{R}^\infty(X,\theta)$ exists, and*

$$\mathbf{E}(\ell \vee \ell') = \lim_{t\to\infty} \frac{1}{t} E_\theta\left(\ell_t \vee \ell'_t\right). \tag{4.23}$$

Proof For each $t > 0$, let $(\ell''^t_s)_{s\in[0,t]}$ be the geodesic from V_θ to $\ell_t \vee \ell'_t$.

Step 1. We first show that for each fixed $s \geq 0$, ℓ''^t_s is increasing in $t \in [s, \infty)$.

To see this, fix $s \geq 0$ and choose $t' > t \geq s$. We need to show that

$$\ell''^{t'}_s \geq \ell''^t_s. \tag{4.24}$$

Since $(\ell''^{t'}_a)_{a\in[0,t]}$ is a geodesic. It suffices to show that $(\ell''^t_a)_{a\in[0,t]}$ is a candidate in the Perron envelope defining the former geodesic. In other words, in verifying (4.24), we may assume that either $s = 0$ or $s = t$. The case $s = 0$ is of course trivial. So it remains to prove the following:

$$\ell''^{t'}_t \geq \ell_t \vee \ell'_t.$$

By symmetry, it suffices to prove

$$\ell''^{t'}_t \geq \ell_t.$$

But since $(\ell_a)_{a\in[0,t']}$ is a candidate in the Perron envelope defining $\ell''^{t'}$, this inequality follows.

Step 2. Next, observe that for a fixed $s \geq 0$, we have

$$\sup_X \ell''^t_s \leq \frac{s}{t}\sup_X \ell''^t_t + \frac{t-s}{t}\sup_X \ell''^t_0 = \frac{s}{t}\left(\sup_X \ell_t\right) \vee \left(\sup_X \ell'_t\right)$$

for all $t > s$. The right-hand side is bounded from above by a constant independent of $t \geq s$ by Proposition 4.2.4. Let

$$(\ell \vee \ell')_s := \sup_{t>s}{}^* \ell''^t_s. \tag{4.25}$$

Then Proposition 4.3.4 guarantees that $\ell \vee \ell' \in \mathcal{R}^\infty(X,\theta)$.

Step 3. We need to show that $\ell \vee \ell'$ defined in this way is indeed the minimal ray lying above ℓ and ℓ'.

First, by Step 1, we have

$$\ell''^t_s \geq \ell''^s_s \geq \ell_s$$

for any $t \geq s \geq 0$. Therefore,

$$(\ell \vee \ell')_s \geq \ell_s$$

for all $s \geq 0$. In other words, $\ell \vee \ell' \geq \ell$. Similarly, $\ell \vee \ell' \geq \ell'$.

Next, let $L \in \mathcal{R}^\infty(X,\theta)$ be a ray lying above both ℓ and ℓ'. Then we have

$$L_t \geq \ell_t \vee \ell'_t$$

for all $t \geq 0$. In particular,

$$L_s \geq \ell''^t_s$$

for all $t \geq s \geq 0$. It follows that

$$L_s \geq (\ell \vee \ell')_s$$

for all $s \geq 0$.

Step 4. It remains to argue (4.23):

$$\mathbf{E}(\ell \vee \ell') = E_\theta(\ell \vee \ell')_1 = \lim_{t\to\infty} E_\theta\left(\ell''^t_1\right) = \lim_{t\to\infty} \frac{1}{t} E_\theta\left(\ell_t \vee \ell'_t\right),$$

where we applied Proposition 3.1.17 and Proposition 4.3.3. □

Lemma 4.3.6 *For any $\ell, \ell' \in \mathcal{R}^\infty(X,\theta)$, we have*

$$d_1(\ell,\ell') \leq d_1(\ell,\ell \vee \ell') + d_1(\ell',\ell \vee \ell') \leq C_n d_1(\ell,\ell'), \tag{4.26}$$

where $C_n = 3(n+1)2^{n+2}$.

Proof The first inequality is trivial. As for the second, we estimate

$$\begin{aligned} d_1(\ell,\ell \vee \ell') =& \mathbf{E}(\ell \vee \ell') - \mathbf{E}(\ell) && \text{by Proposition 4.3.6} \\ =& \lim_{t\to\infty} \frac{1}{t}\mathbf{E}\left(\ell_t \vee \ell'_t\right) - \mathbf{E}(\ell) && \text{by (4.23)} \\ =& \lim_{t\to\infty} \frac{1}{t} d_1\left(\ell_t \vee \ell'_t, \ell_t\right) && \text{by Proposition 4.3.6.} \end{aligned}$$

By symmetry, we find

$$d_1(\ell,\ell \vee \ell') + d_1(\ell',\ell \vee \ell') \leq \lim_{t\to\infty} \frac{1}{t}\left(d_1\left(\ell_t \vee \ell'_t, \ell_t\right) + d_1\left(\ell_t \vee \ell'_t, \ell'_t\right)\right).$$

By Theorem 4.3.1 and Lemma 4.3.2, for each $t > 0$,

$$d_1\left(\ell_t \vee \ell'_t, \ell_t\right) + d_1\left(\ell_t \vee \ell'_t, \ell'_t\right) \leq 3(n+1)2^{n+2} d_1(\ell_t,\ell'_t).$$

Now (4.26) follows. □

Example 4.3.1 Let $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi \leq 0$. For each $C > 0$, let $(\ell^{\varphi,C}_t)_{t\in[0,C]}$ be the geodesic from V_θ to $(V_\theta - C) \vee \varphi$. For each $t \geq 0$, there is $\ell^\varphi_t \in \mathcal{E}^\infty(X,\theta)$ such that

$$\ell^{\varphi,C}_t \xrightarrow{d_1} \ell^\varphi_t \tag{4.27}$$

as $C \to \infty$. Then $\ell^\varphi \in \mathcal{R}^\infty(X,\theta)$ and

$$\mathbf{E}(\ell^\varphi) = \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_{V_\theta}^n\right). \tag{4.28}$$

From (4.28), we see that $\ell^{\varphi+C} = \ell^\varphi$ for any $C \in \mathbb{R}$. Therefore, for a general $\varphi \in \mathrm{PSH}(X,\theta)$, we could simply define

$$\ell^\varphi := \ell^{\varphi - \sup_X \varphi}.$$

Then the conclusions of this example continue to hold.

Proof We first show that for each fixed $t \geq 0$, $\ell_t^{\varphi,C}$ is increasing in $C \geq t$.

To see this, choose $t \leq C_1 < C_2$. We need to show that

$$\ell_t^{\varphi,C_1} \leq \ell_t^{\varphi,C_2}.$$

Since both sides are geodesics for $t \in [0, C_1]$, it suffices to show that

$$(V_\theta - C_1) \vee \varphi \leq \ell_{C_1}^{\varphi,C_2}. \tag{4.29}$$

Now $((V_\theta - t) \vee \varphi)_{t\in[0,C_2]}$ is a subgeodesic from V_θ to $(V_\theta - C_2) \vee \varphi$ by Proposition 4.1.3.[4] At $t = 0$ and $t = C_1$, it is dominated by the geodesic ℓ_t^{φ,C_2}, hence we conclude that the same holds at $t = C_1$, which is exactly (4.29).

From Proposition 4.1.2, we know that for any $C > t > 0$, we have

$$\ell_t^{\varphi,C} \leq \frac{t}{C}\left((V_\theta - C) \vee \varphi\right) + \frac{C-t}{C}\cdot V_\theta \leq 0,$$

so by Proposition 1.2.1,

$$\ell_t^\varphi := \sup_{C>t}{}^* \ell_t^{\varphi,C} \in \mathcal{E}^\infty(X,\theta) \tag{4.30}$$

for all $t \geq 0$. Thanks to Proposition 3.1.17, we have

$$\ell_t^{\varphi,C} \xrightarrow{d_1} \ell_t^\varphi$$

as $C \to \infty$ for all $t \geq 0$. It follows from Proposition 4.3.4 that $\ell^\varphi \in \mathcal{R}^\infty(X,\theta)$.

It remains to compute the energy of ℓ^φ. We first fix $C \geq t > 0$ and compute using Proposition 4.3.3:

$$E_\theta\left(\ell_t^{\varphi,C}\right) = \frac{t}{C}E_\theta\left((V_\theta - C) \vee \varphi\right).$$

Letting $C \to \infty$ and applying Proposition 3.1.17, we find that

$$E_\theta(\ell_t^\varphi) = \lim_{C\to\infty}\frac{t}{C}E_\theta\left((V_\theta - C)\vee\varphi\right)$$

for any $t \geq 0$. It follows that

[4] Here we need $\varphi \leq 0$.

$$\mathbf{E}(\ell^{\varphi}) = \lim_{C\to\infty} \frac{1}{C} E_{\theta}\left((V_{\theta} - C) \vee \varphi\right).$$

Using the definition of E_{θ}, in order to obtain (4.28), it suffices to show that for each $j = 0, \ldots, n$, we have

$$\lim_{C\to\infty} \int_X \frac{(V_{\theta} - C) \vee \varphi - V_{\theta}}{C} \theta^{j}_{(V_{\theta}-C)\vee\varphi} \wedge \theta^{n-j}_{V_{\theta}} = \int_X \theta^{j}_{\varphi} \wedge \theta^{n-j}_{V_{\theta}} - \int_X \theta^{n}_{V_{\theta}}. \quad (4.31)$$

For this purpose, for each $C > 0$, we decompose X as $\{\varphi > V_{\theta} - C\}$ and $\{\varphi \leq V_{\theta} - C\}$. We have

$$\begin{aligned} &\int_{\{\varphi > V_{\theta}-C\}} \frac{(V_{\theta} - C) \vee \varphi - V_{\theta}}{C} \theta^{j}_{(V_{\theta}-C)\vee\varphi} \wedge \theta^{n-j}_{V_{\theta}} \\ =& \int_{\{\varphi > V_{\theta}-C\}} \frac{\varphi - V_{\theta}}{C} \theta^{j}_{\varphi} \wedge \theta^{n-j}_{V_{\theta}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{\{\varphi \leq V_{\theta}-C\}} \frac{(V_{\theta} - C) \vee \varphi - V_{\theta}}{C} \theta^{j}_{(V_{\theta}-C)\vee\varphi} \wedge \theta^{n-j}_{V_{\theta}} \\ =& -\int_{\{\varphi \leq V_{\theta}-C\}} \theta^{j}_{(V_{\theta}-C)\vee\varphi} \wedge \theta^{n-j}_{V_{\theta}} \\ =& -\int_X \theta^{n}_{V_{\theta}} + \int_{\{\varphi > V_{\theta}-C\}} \theta^{j}_{\varphi} \wedge \theta^{n-j}_{V_{\theta}}. \end{aligned}$$

Observe that for $C > 0$, the functions $\mathbb{1}_{\{\varphi > V_{\theta}-C\}} C^{-1}(\varphi - V_{\theta})$ is defined quasi-everywhere and is bounded. When $C \to \infty$, these functions converge to 0 almost everywhere. Therefore, (4.31) follows.

Chapter 5
Toric pluripotential theory on ample line bundles

There are two principal ways to formulate mathematical assertions (problems, conjectures, theorems, . . .): Russian and French. The Russian way is to choose the most simple and specific case (so that nobody could simplify the formulation preserving the main point). The French way is to generalize the statement as far as nobody could generalize it further.
— Vladimir Arnold[a]

[a] Vladimir Igorevich Arnold (1937–2010), who became a professor at l'Université Paris IX after the dissolution of USSR, was always sick of France (so am I!). In the public lecture entitled "Sur l'éducation mathématique" in 1997, he invented the famous joke "Combien font $2 + 3$?" to question the french education system.

In this chapter, we briefly recall the toric pluripotential theory relative to an ample line bundle. The general case of big line bundles will be handled in Chapter 12 after developing the powerful machinery of partial Okounkov bodies in Chapter 10. The main new result is Theorem 5.2.2 computing the L^2-sections of a Hermitian big line bundle in the toric setting.

We assume that the readers are familiar with basic toric geometry, such as the materials in [CLS11]. If not, this section can be safely skipped.

Some basic facts about convex functions and convex bodies are recalled in Appendix A.

5.1 Toric setup

Let T be a complex torus of dimension n[1] and $T_c \subset T(\mathbb{C})$ denotes the corresponding compact torus. Write M for the character lattice of T, which is a free Abelian group of rank n. Similarly, let N be cocharacter lattice of T, which is the dual lattice of M. Given $m \in M$, the corresponding character of M is denoted by χ^m. Write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is denoted by $\langle \bullet, \bullet \rangle$.

Let $P \subseteq M_{\mathbb{R}}$ be a full-dimensional *smooth*[2] lattice polytope[3].

Given any (closed) facet F of P, let $u_F \in N$ denote the unique ray generator (the first non-zero integral element) of the inward normal ray of F. Then P can be

[1] Namely, an algebraic group defined over $\mathbb{C}$, which is isomorphic to $\mathbb{G}_{\mathrm{m}}^n$.

[2] Recall that *smooth* means that for every vertex $v \in P$, if we take the first lattice point w_E apart from v as one transverses each edge E of P containing v from v, then $\{w_E - v\}_E$ forms a basis of M. See [CLS11, Definition 2.4.2]. We also say P is a *Delzant polytope* in this case.

[3] A *lattice polytope* in $M_{\mathbb{R}}$ is the convex hull of finitely many points in M.

represented as

$$P = \{m \in M_{\mathbb{R}} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \text{ of } P\} \tag{5.1}$$

for some uniquely determined integers a_F. The presentation is called the *facet presentation* of P.

Given any (closed) face Q of P, we let $\sigma_Q \subseteq N_{\mathbb{R}}$ be the closed convex cone generated by the u_F's, where F runs over all facets of P containing Q. When $Q = P$, σ_P is understood as $\{0\}$.

Let Σ be the *(inner) normal fan* of P. Namely,

$$\Sigma = \left\{\sigma_Q : Q \text{ is a face of } P\right\}.$$

The notation $\Sigma(1)$ denotes the set of rays in Σ. Note that $\Sigma(1)$ is in bijective correspondence with the set of facets of P. In fact, given any facet F of P, the cone σ_F is just the ray generated by u_F, namely, the inward normal ray of F.

For any $\rho \in \Sigma(1)$, let $u_\rho \in N$ denote the ray generator of ρ, namely the first non-zero element in $N \cap \rho$. If $\rho = \sigma_F$ for some facet F of P, then $u_\rho = u_F$.

Now the facet presentation (5.1) can be equivalently rewritten as

$$P = \left\{m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\right\}.$$

Let $\operatorname{Supp}_P : N_{\mathbb{R}} \to \mathbb{R}$ denote the *support function* of P. Recall that the support function (Example A.1.2) of P is defined as

$$\operatorname{Supp}_P(n) = \max\left\{\langle m, n \rangle : m \in P\right\}.$$

Note that our support function differs from [CLS11, Proposition 4.2.14], where instead of a maximum, they took the minimum.

Recall that the *characteristic function* $\chi_P : N_{\mathbb{R}} \to \{0, \infty\}$ of P is defined as in Example A.1.1:

$$\chi_P(n) := \begin{cases} 0, & n \in P; \\ \infty, & n \notin P. \end{cases}$$

Let $X = X_\Sigma$ be the smooth projective toric variety corresponding to Σ. See [CLS11, Theorem 3.1.5] for the construction of X and [CLS11, Theorem 3.1.19] for the smoothness of X. There is a canonical embedding $T \subseteq X$ as a dense Zariski open subset.

Let D be the Cartier divisor on X defined by P:

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where D_ρ is the toric prime divisor defined by ρ under the orbit-cone correspondence [CLS11, Theorem 3.2.6].

Let L be the toric line bundle induced by P, namely $L = \mathcal{O}_X(D)$. Since P has full dimension, L^k is very ample for each $k \geq n-1$ by [CLS11, Corollary 2.2.19], we actually know that L is ample.

We will choose the base e for the logarithm map

$$\mathbb{C}^* \to \mathbb{R}, \quad z \mapsto \log |z|^2.^{4} \tag{5.2}$$

This choice will be fixed throughout the whole book. Since we have a canonical identification $T(\mathbb{C}) \cong N \otimes_{\mathbb{Z}} \mathbb{C}^*$, the logarithm map then induces a tropicalization map after tensoring with N:

$$\mathrm{Trop}\colon T(\mathbb{C}) \to N_{\mathbb{R}}. \tag{5.3}$$

Before proceeding, it is always helpful to understand everything in our favorite example.

Example 5.1.1 We take $n = 1$ and $P = [0, 1] \subseteq M_{\mathbb{R}} = \mathbb{R}$. In this case, the facet representation (5.1) becomes

$$P = \left\{ m \in \mathbb{R} : \langle m, 1 \rangle \geq 0, \langle m, -1 \rangle \geq -1 \right\},$$

with $u_{\{0\}} = 1$, $u_{\{1\}} = -1$, $a_{\{0\}} = 0$ and $a_{\{1\}} = 1$. The normal fan Σ is

$$\Sigma = \left\{ (-\infty, 0], \{0\}, [0, \infty) \right\}.$$

See Fig. 5.1.

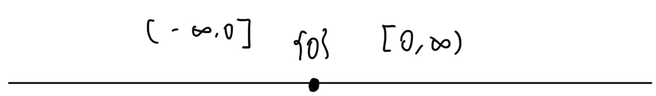

Fig. 5.1 The fan Σ of $\mathbb{P}^1$.

The corresponding toric variety is just $X = \mathbb{P}^1$. Under the orbit-cone correspondence, we have

$$D_{\{[0,\infty)\}} = [0], \quad D_{\{(-\infty,0]\}} = [\infty].$$

The associated divisor $D = [\infty]$ and therefore,

$$L = \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}(1).$$

---

[4] Be careful when you compare with other references, some people prefer $\log |z|$, $-\log |z|$ or $-\log |z|^2$ instead.

5.2 Toric plurisubharmonic functions

We continue to use the notations of Section 5.1.

Lemma 5.2.1 *Let $F\colon N_{\mathbb{R}} \to [-\infty, \infty]$ be a function. Then the following are equivalent:*

(1) *F is convex and takes values in $\mathbb{R}$, and*
(2) *$\mathrm{Trop}^* F$ is plurisubharmonic on $T(\mathbb{C})$.*

Proof We may choose an identification $N \cong \mathbb{Z}^n$ so that we have an identification $T(\mathbb{C}) \cong \mathbb{C}^{*n}$. Then Trop is identified with the map

$$\mathrm{Trop}\colon \mathbb{C}^{*n} \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto \left(\log |z_1|^2, \ldots, \log |z_n|^2\right).$$

(1) $\implies$ (2). Let $F_k \in C^\infty(\mathbb{R}^n) \cap \mathrm{Conv}(\mathbb{R}^n)$ be a decreasing sequence with limit F (see Proposition A.3.3). It follows from a straightforward computation that

$$\mathrm{dd^c}\, \mathrm{Trop}^* F_k(z_1, \ldots, z_n) = \frac{\mathrm{i}}{2\pi} \sum_{i,j=1}^{n} \partial_{ij} F_k \left(\log |z_1|^2, \ldots, \log |z_n|^2\right) z_i^{-1} \overline{z_j}^{-1} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j}. \tag{5.4}$$

So $\mathrm{Trop}^* F_k$ is plurisubharmonic. It follows from Proposition 1.2.1 that $\mathrm{Trop}^* F$ is plurisubharmonic.

(2) $\implies$ (1). It follows from Lemma 1.2.1 that F is finite. Moreover, take a radial mollifier, we may find a decreasing sequence φ_k of $(S^1)^n$-invariant smooth psh functions on $\mathbb{C}^{*n}$ with limit $\mathrm{Trop}^* F$. Write $\varphi_k = \mathrm{Trop}^* F_k$ for some function $F_k\colon \mathbb{R}^n \to \mathbb{R}$, it follows from (5.4) that F_k is convex for all k. Therefore, F is convex by Lemma A.1.2. □

Next we define a canonical Kähler form in $c_1(L)$.

Let $G_0\colon M_{\mathbb{R}} \to (-\infty, \infty]$ be defined as

$$G_0(m) := \begin{cases} \sum_{\rho \in \Sigma(1)} \left(\langle m, u_\rho \rangle + a_\rho\right) \log \left(\langle m, u_\rho \rangle + a_\rho\right)^{5}, & \text{if } m \in P, \\ \infty, & \text{otherwise.} \end{cases} \tag{5.5}$$

This is a closed proper convex function and $G_0 \sim \chi_P$, where $\sim$ is the relation defined in Definition A.1.8.

Let

$$F_0 = G_0^* \in \mathcal{E}^\infty(N_{\mathbb{R}}, P). \tag{5.6}$$

Here G_0^* is the Legendre transform of G_0, as recalled in Definition A.2.1. The set $\mathcal{E}^\infty(N_{\mathbb{R}}, P)$ is defined in Definition A.3.1.

By Guillemin's theorem [Gui94, CDG03], $\mathrm{dd^c}\, \mathrm{Trop}^* F_0$ can be extended to a unique Kähler form ω in $c_1(L)$. The Kähler form ω is clearly T_c-invariant.

[5] We understand that $0 \log 0 = 0$ in this expression.

For each $\rho \in \Sigma(1)$, we write

$$r_\rho(m) = \log\left(\langle m, u_\rho\rangle + a_\rho\right) + 1, \quad m \in P.$$

It follows from (5.5) that

$$\nabla G_0(m) = \sum_{\rho\in\Sigma(1)} r_\rho(m) u_\rho, \quad m \in \operatorname{Int} P. \tag{5.7}$$

Example 5.2.1 Let us move on with our favorite example Example 5.1.1. We continue to use the same notations. In this case,

$$G_0(m) = \begin{cases} m\log m + (1-m)\log(1-m), & \text{if } m \in [0,1], \\ \infty, & \text{otherwise.} \end{cases}$$

The Legendre transform is given[6] by

$$F_0(n) = \log\left(1 + \mathrm{e}^n\right).$$

Composing with the tropicalization map, we find that

$$\omega|_{\mathbb{C}^*}(z) = \mathrm{dd^c} \log\left(1 + |z|^2\right).$$

This is exactly the Fubini–Study metric as we have seen in Example 1.8.1.

Now we could explain one subtlety: In our expression (5.5), there is no factor $1/2$ before the sum, this is due to the presence of the square in our choice of the tropicalization map (5.2).

Let $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ denote the set of T_c-invariant ω-psh functions.

Theorem 5.2.1 *There are canonical bijections between the following three sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$,
(2) *the set* $\mathcal{P}(N_{\mathbb{R}}, P)$ *in Definition A.3.1, namely, the set of convex functions* $F\colon N_{\mathbb{R}} \to \mathbb{R}$ *satisfying* $F \leq \mathrm{Supp}_P$, *and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G|_{M_{\mathbb{R}}\setminus P} \equiv \infty.$$

For the notion of closeness and properness, we refer to Definition A.1.2 and Definition A.1.7.

Proof The bijection between (2) and (3) is the classical Legendre duality. Given F as in (2), we construct $G = F^*$ and *vice versa*, see Proposition A.2.5.

[6] While reading an advanced mathematical textbook/paper, I usually tend to trust the authors for their elementary computations. A few years ago, I was asked to present the result of a landmark paper written by two respected mathematicians on a conference. After spending a few days on the elementary integrals, I found out that all non-trivial constants in that paper were wrong. So I ask the readers to really verify this expression, if it is not obvious to you.

The map from (1) to (2) is given as follows: Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, since φ is T_c-invariant, we can find $f\colon N_{\mathbb{R}} \to [-\infty,\infty)$ such that

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^* f. \tag{5.8}$$

We then define $F = f + F_0$. Then $\mathrm{Trop}^* F \in \mathrm{PSH}(T(\mathbb{C}))$. By Lemma 5.2.1, $F(n)$ is finite for any $n \in N_{\mathbb{R}}$ and F is convex. Moreover, $F \preceq \mathrm{Supp}_P$ since this holds for F_0.

Conversely, given a map $F \in \mathcal{P}(N_{\mathbb{R}}, P)$, then

$$\mathrm{Trop}^*(F - F_0) \in \mathrm{PSH}\left(T(\mathbb{C}), \omega|_{T(\mathbb{C})}\right).$$

It follows from Theorem 1.2.1 that this function can be extended uniquely to an ω-psh function on X. The uniqueness of the extension guarantees its T_c-invariance.

The two maps are clearly inverse to each other. □

Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$, we will write F_φ and G_φ for the convex functions given by Theorem 5.2.1. From the proof, we have the following relations:

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^*\left(F_\varphi - F_0\right), \quad G_\varphi = F_\varphi^*. \tag{5.9}$$

Example 5.2.2 Let us take our favorite example Example 5.2.1 again. We will continue to use the same notations.

Recall that in Example 1.8.2 and Example 3.1.1, we constructed two S^1-invariant functions in $\mathrm{PSH}(X,\omega)$.

We begin with the function φ in Example 1.8.2. Recall that

$$\varphi(z) = \log \frac{|z|^2}{|z|^2+1}$$

for $z \in \mathbb{C}$. The function $f\colon \mathbb{R} \to \mathbb{R}$ in (5.8) is therefore

$$f(n) = \log \frac{\mathrm{e}^n}{1+\mathrm{e}^n}.$$

Therefore, $F_\varphi\colon \mathbb{R} \to \mathbb{R}$ is

$$F_\varphi(n) = n.$$

Correspondingly, $G_\varphi\colon \mathbb{R} \to \mathbb{R}$ is

$$G_\varphi(m) = \begin{cases} 0, & \text{if } m = 1; \\ \infty, & \text{otherwise.} \end{cases}$$

Similarly, if ψ denote the function in Example 3.1.1, then the function f in (5.8) is

$$f(n) = \begin{cases} -\log\left(\mathrm{e}^n + 1\right) + (-\log(-n)) \vee (n+2), & \text{if } n < -\log 2; \\ 2 + \log \frac{\mathrm{e}^n}{1+\mathrm{e}^n}, & \text{otherwise.} \end{cases}$$

Therefore,

$$F_\psi(n) = \begin{cases} (-\log(-n)) \vee (n+2), & \text{if } n < -\log 2; \\ 2+n, & \text{otherwise.} \end{cases}$$

The Legendre transform is tricky to compute. Let λ be the large solution of $\log x = x-2$. So $\lambda \approx 3.146$. The smaller solution is around $0.159 < \log 2 \approx 0.693$. It might be helpful to have a look at the poorly drawn picture Fig. 5.2.

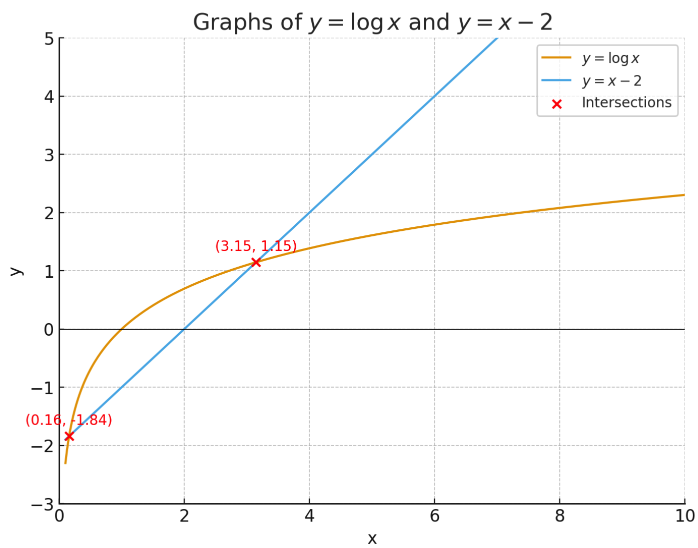

Fig. 5.2 The graphs of $\log x$ and $x - 2$.

It is immediate that $G_\psi(m) = -\infty$ unless $m \in [0,1]$. Let us assume that $m \in [0,1]$. Then

$$\begin{aligned} G_\psi(m) &= \sup_{n\in\mathbb{R}} \left(mn - F_\psi(n)\right) \\ &= \sup_{n<-\log 2} (mn - (-\log(-n)) \vee (n+2)) \vee \sup_{n\geq -\log 2} (mn - n - 2) \\ &= \sup_{n>\log 2} (-mn + (\log n) \wedge (n-2)) \vee ((1-m)\log 2 - 2)\,. \end{aligned}$$

Let us focus on the first part, which can be decomposed further into

$$\begin{aligned} &\sup_{n>\log 2} (-mn + (\log n) \wedge (n-2)) \\ =& \sup_{n\in(\log 2,\lambda]} (n - 2 - mn) \vee \sup_{n>\lambda}(\log n - mn) \\ =& ((1-m)\lambda - 2) \vee \sup_{n>\lambda} (\log n - mn)\,. \end{aligned}$$

The latter part can be computed easily:

$$\sup_{n>\lambda} (\log n - mn) = \begin{cases} -\log m - 1, & \text{if } m \in \left[0, \lambda^{-1}\right]; \\ \log \lambda - m\lambda, & \text{if } m \in \left(\lambda^{-1}, 1\right]. \end{cases}$$

Putting everything together, we find

$$G_\psi(m) = \begin{cases} (-\log m - 1) \vee ((1-m)\lambda - 2), & \text{if } m \in \left[0, \lambda^{-1}\right]; \\ (\log \lambda - m\lambda) \vee ((1-m)\lambda - 2), & \text{if } m \in \left(\lambda^{-1}, 1\right]. \end{cases}$$

This can be further simplified, the final result is

$$G_\psi(m) = \begin{cases} -\log m - 1, & \text{if } m \in \left[0, \lambda^{-1}\right]; \\ (1-m)\lambda - 2, & \text{if } m \in \left(\lambda^{-1}, 1\right]; \\ \infty, & \text{otherwise.} \end{cases}$$

The graph of G_ψ on $(0, 1]$ is sketched in Fig. 5.3.

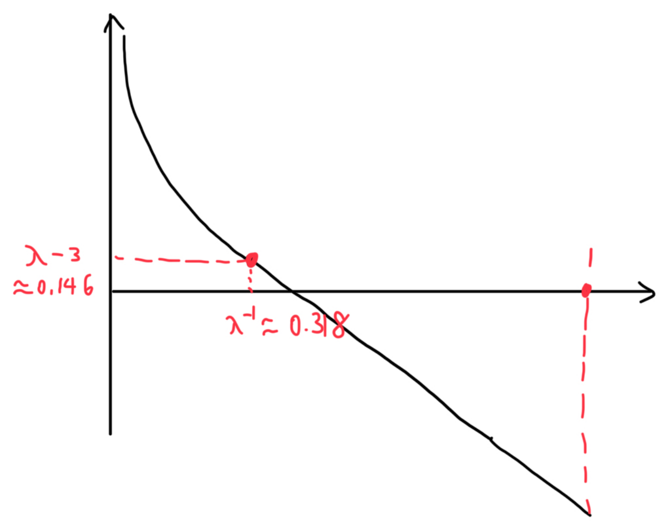

Fig. 5.3 The graph of G_ψ.

We observe a few elementary facts.

Proposition 5.2.1 *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \omega)$. The following are equivalent:*

(1) $\varphi \leq \psi$,
(2) $F_\varphi \leq F_\psi$, *and*

(3) $G_\psi \preceq G_\varphi$.

The same holds if we replace all $\preceq$'s by $\leq$.

Proof The equivalence between (1) and (2) follows from the definition (5.9). The equivalence between (2) and (3) follows from the definition of the Legendre transform. □

Similarly, we have

Proposition 5.2.2 *Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and $C \in \mathbb{R}$. We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

Proposition 5.2.3 *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ with $\varphi \wedge \psi \not\equiv -\infty$, then $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and*

$$F_{\varphi\wedge\psi} = F_\varphi \wedge F_\psi, \quad G_{\varphi\wedge\psi} = G_\varphi \vee G_\psi.$$

The operators $\wedge$ and $\vee$ are defined in Definition A.1.5 and Definition A.1.6.

Proof It is clear that $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. So $\varphi \wedge \psi$ is the biggest element in $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ which id dominated by both φ and ψ. In view of Theorem 5.2.1 and Proposition 5.2.1, $G_{\varphi\wedge\psi}$ is the smallest closed proper convex function G on $M_{\mathbb{R}}$ dominating both G_φ and G_ψ, which is just $G_\varphi \vee G_\psi$.

The claim for F follows from Proposition A.2.3. □

Example 5.2.3 Now we can give an example of $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ with $\varphi\wedge\psi \equiv -\infty$.

We take $P = [0,1]$ so that $X = \mathbb{P}^1$ and ω is the Fubini–Study metric. Let $\varphi \in \mathrm{PSH}(X,\omega)$ be such that

$$\varphi(z) = \log \frac{|z|^2}{|z|^2+1}$$

for $z \in \mathbb{C}$. We have computed that G_φ in Example 5.2.2:

$$G_\varphi(m) = \begin{cases} 0, & \text{if } m = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Now we define $\psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ as the unique function such that

$$\psi(z) = \log \frac{1}{|z|^2+1}$$

for $z \in \mathbb{C}$. Then a similar computation shows that

$$G_\psi(m) = \begin{cases} 0, & \text{if } m = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Now we claim that $\varphi \wedge \psi \equiv -\infty$. Otherwise, we would have

$$G_{\varphi\vee\psi} = G_\varphi \vee G_\psi \equiv \infty,$$

which is not proper.

Proposition 5.2.4 *Let* $\{\varphi_i\}_{i\in I}$ *be a non-empty family in* $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *uniformly bounded from above. Then* $\sup_{i\in I}{}^*\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and*

$$F_{\sup_{i\in I}{}^*\varphi_i} = \bigvee_{i\in I} F_{\varphi_i}, \quad G_{\sup_{i\in I}{}^*\varphi_i} = \mathrm{cl} \bigwedge_{i\in I} G_{\varphi_i}.$$

Moreover, if I *is finite, then*

$$G_{\max_{i\in I}\varphi_i} = \bigwedge_{i\in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i\in I}$ *is a decreasing net in* $\mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *such that* $\inf_{i\in I}\varphi_i \not\equiv -\infty$*, then* $\inf_{i\in I}\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and*

$$F_{\inf_{i\in I}\varphi_i} = \inf_{i\in I} F_{\varphi_i}, \quad G_{\inf_{i\in I}\varphi_i} = \bigvee_{i\in I} G_{\varphi_i}.$$

Recall that the closure cl is defined in Definition A.1.7.

Proof Thanks to Lemma A.1.2 and Proposition A.1.1, in both cases, the statement for F is clear. The corresponding statement for G is obtained via Proposition A.2.3. □

The complex Monge–Ampère operator is closely related to the real one:

Proposition 5.2.5 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$*, then*

$$\mathrm{Trop}_* \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi). \tag{5.10}$$

In particular,

$$\int_X \omega_\varphi^n = \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F_\varphi) = n!\,\mathrm{vol}\,\overline{\{G_\varphi < \infty\}}$$

and

$$\int_X \omega^n = n!\,\mathrm{vol}\,P.$$

Here the real Monge–Ampère operator is defined in Definition A.4.1. The normalization of the Lebesgue measure vol on $M_{\mathbb{R}}$ is such that the fundamental lattice cube as measure 1.[7]

Proof We only need to prove (5.10). By Proposition A.3.3, we can find a decreasing sequence of smooth convex functions F_j on $N_{\mathbb{R}}$ with limit F_φ. We write $F_j = F_{\varphi_j}$ for some $\varphi_j \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. By Theorem 2.1.1 and Theorem A.4.1, it suffices to establish (5.10) for the φ_j's. We may therefore reduce to the case where F_φ is smooth.

[7] In some references like [GKZ08], the normalization is so that the fundamental lattice cube has measure $n!$. Be careful when making comparisons with these references.

We write $F = F_\varphi$ to simplify the notations. The notations $a_i = \log |z_i|^2$ will be used, where $i = 1, \dots, n$.

Next we fix an identification $N = \mathbb{Z}^n$. Fix a test function $f \in C_c^0(N_\mathbb{R})$, we need to show that

$$\int_{\mathbb{C}^{*n}} f(a_1, \dots, a_n)\, (\mathrm{dd^c}\, \mathrm{Trop}^* F(z_1, \dots, z_n))^n = \int_{\mathbb{R}^n} f\, \mathrm{MA}_\mathbb{R}(F).$$

Using Proposition A.4.1 and (5.4), this reduces to

$$\left(\frac{\mathrm{i}}{2\pi}\right)^n \int_{\mathbb{C}^{*n}} f(a_1, \dots, a_n) \left(\sum_{i,j=1}^n \partial_{i,j} F(a_1, \dots, a_n) z_i^{-1} \overline{z_j}^{-1} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j} \right)^n = \\ n! \int_{\mathbb{R}^n} f \det \nabla^2 F \,\mathrm{d}\,\mathrm{vol}\,. \tag{5.11}$$

Expanding the bracket, we get

$$\left(\sum_{i,j=1}^n \partial_{i,j} F z_i^{-1} \overline{z_j}^{-1} \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j} \right)^n = \sum_{i_1, \dots, i_n = 1}^n \sum_{j_1, \dots, j_n = 1}^n \partial_{i_1 j_1} F \cdots \partial_{i_n j_n} F \cdot \\ \mathrm{d}\log z_{i_1} \wedge \mathrm{d}\log \overline{z_{j_1}} \wedge \cdots \wedge \mathrm{d}\log z_{i_n} \wedge \mathrm{d}\log \overline{z_{j_n}},$$

where $\mathrm{d}\log z_i = z_i^{-1}\mathrm{d}z_i$ and $\mathrm{d}\log \overline{z_i} = \overline{z_i}^{-1}\mathrm{d}\overline{z_i}$ are understood.

Using the apparent symmetry, the expression on the right-hand side becomes

$$\begin{aligned}
&\sum_{\sigma,\tau \in \mathfrak{S}_n} \prod_{k=1}^n \partial_{\sigma(k)\tau(k)} F \,\mathrm{d}\log z_{\sigma(1)} \wedge \mathrm{d}\log \overline{z_{\tau(1)}} \wedge \cdots \wedge \mathrm{d}\log z_{\sigma(n)} \wedge \mathrm{d}\log \overline{z_{\tau(n)}}, \\
=& n! \sum_{\tau \in \mathfrak{S}_n} \prod_{k=1}^n \partial_{k\tau(k)} F \,\mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_{\tau(1)}} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_{\tau(n)}} \\
=& n! \sum_{\tau \in \mathfrak{S}_n} (-1)^{\mathrm{Sign}\,\tau} \prod_{k=1}^n \partial_{k\tau(k)} F \,\mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_1} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_n} \\
=& n! \det \nabla^2 F \mathrm{d}\log z_1 \wedge \mathrm{d}\log \overline{z_1} \wedge \cdots \wedge \mathrm{d}\log z_n \wedge \mathrm{d}\log \overline{z_n},
\end{aligned}$$

where $\mathfrak{S}_n$ is the permutation group on $\{1, \dots, n\}$ and $\mathrm{Sign}(\tau)$ is the sign of τ.

Next, switch to polar coordinates for each z_i: Let $z_i = r_i \exp(\mathrm{i}\theta_i)$ and recall that $r_i = \exp(a_i/2)$, then the left-hand side of (5.11) becomes

$$\begin{aligned}
&\frac{n!}{(2\pi)^n} \int_{\mathbb{R}^n \times [0,2\pi)^n} f \det \nabla^2 F \,\mathrm{d}a_1 \wedge \mathrm{d}\theta_1 \wedge \cdots \wedge \mathrm{d}a_n \wedge \mathrm{d}\theta_n \\
=& n! \int_{\mathbb{R}^n} f \det \nabla^2 F \,\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n,
\end{aligned}$$

which is exactly what we have expected. □

Next we study the envelope operators developed in Chapter 3 in the toric setting.

Definition 5.2.1 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. We define its *Newton body* as

$$\Delta(\omega,\varphi) := \overline{\{G_\varphi < \infty\}} \subseteq P.$$

Note that $\Delta(\omega,\varphi)$ is a convex body.

By Proposition A.2.2, we have

$$\Delta(\omega,\varphi) = \overline{\nabla F_\varphi(N_{\mathbb{R}})}.$$

Example 5.2.4 By (5.5), we have

$$\Delta(\omega,0) = P.$$

In the case of Example 5.2.2, we have

$$\Delta(\omega,\varphi) = \{1\}, \quad \Delta(\omega,\psi) = [0,1].$$

Observe that in the latter case,

$$\{G_\varphi < \infty\} \subsetneq P.$$

Proposition 5.2.6 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Then $P_\omega[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ and*

$$G_{P_\omega[\varphi]}(x) = \begin{cases} G_0(x), & \text{if } x \in \Delta(\omega,\varphi); \\ \infty, & \text{otherwise.} \end{cases} \tag{5.12}$$

Proof By (3.4), we have

$$P_\omega[\varphi] = \sup_{C\in\mathbb{R}}{}^*\big((\varphi + C)\wedge 0\big).$$

It follows from Proposition 5.2.2, Proposition 5.2.3 and Proposition 5.2.4 that $P_\omega[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Moreover, by the same propositions, we have

$$G_{P_\omega[\varphi]} = \mathrm{cl}\inf_{C\in\mathbb{R}}\left(G_0 \vee (G_\varphi - C)\right),$$

which is clearly equal to the right-hand side of (5.12).

Recall that $\mathrm{H}^0(X,L)$ can be identified with the vector space generated by χ^m for all $m \in P\cap M$, see [CLS11, Proposition 4.3.3]. In other words, a character χ^m of T can be extended to a regular function on X if and only if $m \in P$. This gives a beautiful characterization of the lattice points in P. The following theorem of Yi Yao gives an analogous characterization of the lattice points in the Newton body.

Theorem 5.2.2 (Yao) *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. Let $m \in M$.*

(1) *Suppose that $m \in \Delta(\omega,\varphi)$, then $\chi^m \in \mathrm{H}^0(X, L\otimes \mathcal{I}(\varphi))$.*

(2) *There is a constant* $C_0 > 0$ *such that if there is* $\rho \in \Sigma(1)$ *with*

$$\langle m, -u_\rho \rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(-u_\rho) > C_0, \tag{5.13}$$

then $\chi^m \notin \mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$.
Moreover, the constant C_0 *does not change if we replace* P *by a positive integer multiple of* P.

Proof It is convenient to use explicit coordinates. We will identify N with $\mathbb{Z}^n$ after choosing a basis. In this way, we get an identification $M = \mathbb{Z}^n$ and $T(\mathbb{C}) = \mathbb{C}^{*n}$. In this case, we have

$$\chi^m(z) = z^m$$

with the multi-index notation.

Observe that $\mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$ is a $\mathbb{C}^{*n}$-invariant subspace of $\mathrm{H}^0(X, L)$, it follows that $\mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$ is the direct sum of suitable $\mathbb{C}\chi^m$'s. Due to Proposition 3.2.9, we may replace φ by $P_\omega[\varphi]$ and thanks to Proposition 5.2.6, we may assume that G_φ has the following form:

$$G_\varphi(x) = \begin{cases} G_0(x), & \text{if } x \in \Delta(\omega, \varphi); \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, $F_\varphi \sim \mathrm{Supp}_{\Delta(\omega,\varphi)}$.

Now given $m \in M \cap P$, we need to know whether the following expression is finite or not:

$$\int_{\mathbb{C}^{*n}} |\chi^m|^2 \exp(-\operatorname{Trop}^* F_0 - \varphi)\, \omega^n. \tag{5.14}$$

By Proposition 5.2.5, (5.14) is finite if and only if the following integral is finite:

$$\int_{\mathbb{R}^n} \exp\left(\langle m, n\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(n)\right) \mathrm{MA}_{\mathbb{R}}(F_0)(n).$$

By a change of variable, this integral is finite if and only if the following integral is:

$$\int_P \exp\left(\langle m, \nabla G_0(m')\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(\nabla G_0(m'))\right) \mathrm{d}m'. \tag{5.15}$$

Suppose that $m \in \Delta(\omega, \varphi)$, then the integrand in (5.15) is bounded from above by 1, so (1) follows.

Next we consider (2). Fix the standard norm on $N_{\mathbb{R}} = \mathbb{R}^n$.

Suppose that m satisfies the assumptions of (2). Take $\rho \in \Sigma(1)$ so that (5.13) holds. The condition on $C_0 > 0$ will be clarified later on. Take an open subset U of P which satisfies the following two conditions:

- The intersection $U \cap Q$ has dimension $n - 1$, where Q is the face of P defined by $\langle \bullet, \rho \rangle = -a_\rho$;
- U does not intersect other faces of P.

See Fig. 5.4 for the visualization of U.

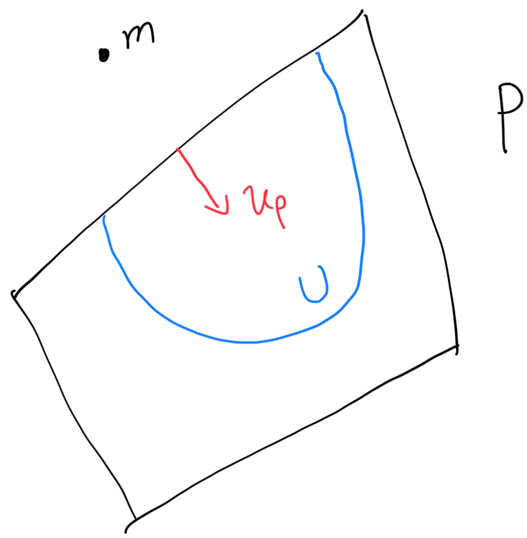

Fig. 5.4 The choice of U.

Then by (5.7),

$$(\nabla G_0)\,|_U = -r_\rho |u_\rho| + O(1). \tag{5.16}$$

We claim that

$$\int_U \exp\Big(\langle m, \nabla G_0(m')\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(\nabla G_0(m'))\Big)\,\mathrm{d}m' = \infty.$$

In view of (5.13) and (5.7), after slightly shrinking U, we may guarantee that the direction of $\nabla G_0(m')$ is close enough to that of $-u_\rho$, so that

$$\langle m, \nabla G_0(m')\rangle - \mathrm{Supp}_{\Delta(\omega,\varphi)}(\nabla G_0(m')) > \frac{1}{2} C_0 |\nabla G_0(m')|.$$

It suffices therefore to establish the following assertion:

$$\int_U \exp\Big(-2^{-1} C_0 r_\rho(m') |u_\rho(m')|\Big)\,\mathrm{d}m' = \infty.$$

Taking the definition of r_ρ into account, this is further equivalent to the following:

$$\int_U \big(\langle m', u_\rho\rangle + a_0\big)^{-2^{-1}C_0|u_\rho|}\,\mathrm{d}m' = \infty.$$

This holds as long as $C_0|u_\rho| > 2$. Since there are only finitely many $\rho \in \Sigma(1)$, the constant C_0 can be chosen so that it is independent of the choice of ρ. Furthermore, since replace P by kP for some $k \in \mathbb{Z}_{>0}$ does not change the condition on C_0, we conclude the final assertion. □

Corollary 5.2.1 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$ *and* $\int_X \omega_\varphi^n > 0$*, then*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0(X, L^k \otimes \mathcal{I}(k\varphi)) = n! \operatorname{vol}\Delta(\omega,\varphi).$$

Example 5.2.5 In general, in the setup of Theorem 5.2.2, there exists $m \in M \cap (P \setminus \Delta(\omega,\varphi))$ such that $\chi^m \in \mathrm{H}^0(X, L \otimes \mathcal{I}(\varphi))$.

As a concrete example, let us take $P = [0,1]$. Take φ so that $\Delta(\omega,\varphi) = [0,1/2]$. We claim that χ^1 is L^2-integrable.

It suffices to verify the convergence of (5.15). Recall that

$$\nabla G_0(m') = \log \frac{m'}{1-m'}, \quad m' \in [0,1],$$

while

$$\mathrm{Supp}_{[0,1/2]}(a) = \begin{cases} a/2, & \text{if } a > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, (5.15) becomes

$$\int_0^{1/2} \frac{m'}{1-m'}\,\mathrm{d}m' + \int_{1/2}^1 \left(\frac{m'}{1-m'}\right)^{1/2} \mathrm{d}m' < \infty.$$

We interpret various classes of potentials studied in Section 3.1.3 in the toric setting.

Proposition 5.2.7 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$*. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^\infty(X,\omega)$*;*
(2) $F_\varphi \in \mathcal{E}^\infty(N_\mathbb{R}, P)$*;*
(3) $G_\varphi \sim G_0$.

The notation $\mathcal{E}^\infty(N_\mathbb{R}, P)$ is defined in Definition A.3.1.

Proof This follows immediately from Proposition 5.2.1. □

Proposition 5.2.8 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$*. Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}(X,\omega)$*;*
(2) $F_\varphi \in \mathcal{E}(N_\mathbb{R}, P)$*;*
(3) $\overline{\operatorname{Dom} G_\varphi} = P$.

The notation $\mathcal{E}(N_\mathbb{R}, P)$ is defined in Definition A.3.1.

Proof (1) $\iff$ (3). By Proposition 5.2.5

$$\int_X \omega_\varphi^n = \int_{T(\mathbb{C})} \left(\omega|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = n!\,\mathrm{vol}\,\overline{\mathrm{Dom}\,G_\varphi}, \quad \int_X \omega^n = n!\,\mathrm{vol}\,P.$$

Therefore, (1) and (3) are equivalent.

(2) $\iff$ (3). This follows from Proposition A.2.2. □

Proposition 5.2.9 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$*, then*

$$E_\omega(\varphi) = n! \int_P \left(G_0 - G_\varphi\right) \,\mathrm{d}\,\mathrm{vol}\,.$$

Proof It suffices to consider the case where φ is bounded. In this case, one could apply [BB13, Proposition 2.9]. □

Corollary 5.2.2 *Let* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\omega)$. *Then the following are equivalent:*

(1) $\varphi \in \mathcal{E}^1(X,\omega)$*;*
(2) $F_\varphi \in \mathcal{E}^1(N_{\mathbb{R}},P)$*;*
(3) $G_\varphi \in L^1(P)$.

The notation $\mathcal{E}^1(N_{\mathbb{R}},P)$ is defined in Definition A.3.1.

Definition 5.2.2 We define

$$\begin{aligned}\mathcal{E}^\infty_{\mathrm{tor}}(X,\omega) =& \mathcal{E}^\infty(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega),\\ \mathcal{E}^1_{\mathrm{tor}}(X,\omega) =& \mathcal{E}^1(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega),\\ \mathcal{E}_{\mathrm{tor}}(X,\omega) =& \mathcal{E}(X,\omega) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\omega).\end{aligned}$$

Corollary 5.2.3 *Let* $\varphi,\psi \in \mathcal{E}^1_{\mathrm{tor}}(X,\omega)$*, then*

$$d_1(\varphi,\psi) = n! \int_P \left(2\left(G_\varphi \vee G_\psi\right) - G_\varphi - G_\psi\right) \,\mathrm{d}\,\mathrm{vol}\,.$$

Proof This follows from (5.2.9), Proposition 5.2.3 and Definition 4.3.1. □

Part II
The theory of $\mathcal{I}$-good singularities

This part is the technical core of the whole book. We will develop the theory of $\mathcal{I}$-good singularities.

We first develop some general techniques to compare the singularities in Chapter 6: The P-partial order, the $\mathcal{I}$-partial order and the d_S-pseudometric.

The P-partial order seems to be new. Some basic properties of the d_S-pseudometric have never appeared in the literature either.

Then in Chapter 7, we introduce the notion of $\mathcal{I}$-good singularities and characterize $\mathcal{I}$-good singularities in different ways. Then we establish the asymptotic Riemann–Roch formula for Hermitian pseudoeffective line bundles.

In Chapter 8, we develop two key techniques in the inductive study of singularities: The trace operator and the analytic Bertini theorem. Roughly speaking, the latter tells us the behavior of a quasi-plurisubharmonic function along a general divisor, while the former handles the case of special divisors. We will establish a relative version of the asymptotic Riemann–Roch formula as well.

In Chapter 9, we develop the theory of test curves. These are curves of model potentials. The key technique is the Ross–Witt Nyström correspondence, which relates test curves to geodesic rays. The complete proof of the most general form of this correspondence has never appeared in the literature, so we will give the full details.

In Chapter 10, we develop the theory of partial Okounkov bodies, in both algebraic and transcendental setting. The partial Okounkov bodies can be regarded as non-toric extensions of the Newton bodies. It turns out that even in the toric setting, our techniques give non-trivial new results.

In Chapter 11, we develop the theory of b-divisors. We establish their intersection theory. We also relate the theory of partial Okounkov bodies to b-divisors.

These chapters are supposed to be read linearly, but after finishing Chapter 7 and Chapter 8, the readers could also choose to proceed to any following chapters in this book.

Chapter 6
Comparison of singularities

Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.
— Michael Atiyah[a]

[a] Sir Michael Francis Atiyah (1929–2019) wrote the influential *Introduction to commutative algebra* together with I. G. MacDonald, a poor guy whose name is often omitted or misspelled.

In this chapter, we study several ways of comparing the singularities of quasi-plurisubharmonic functions. In Section 6.1, we will introduce the P and $\mathcal{I}$-partial orders, closely related to the P and $\mathcal{I}$-equivalence relations introduced in Chapter 3.

In Section 6.2, we introduce and study the d_S-pseudometric characterizing the differences between singularities. We will prove that a number of continuity results with respect to d_S.

6.1 The P and $\mathcal{I}$-partial orders

Let X be a connected compact Kähler manifold of dimension n.

Recall that we have defined a (non-strict) partial order on $\mathrm{QPSH}(X)$ in Definition 1.5.2 to compare the singularity types of quasi-plurisubharmonic functions. The problem with this partial order is that it is too fine. In general, for our interest, it is helpful to consider rougher relations.

6.1.1 The definitions of the partial orders

Recall that the P-envelope is defined in Definition 3.1.2.

Definition 6.1.1 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is *P-more singular than* ψ and write $\varphi \preceq_P \psi$ if for some closed smooth real $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, we have

$$P_\theta[\varphi] \leq P_\theta[\psi]. \tag{6.1}$$

Suppose that $\varphi \preceq_P \psi$ and $\psi \preceq_P \varphi$, we shall write $\varphi \sim_P \psi$ and say φ and ψ have the same *P-singularity type*.

Note that if $\varphi \preceq \psi$, then $\varphi \preceq_P \psi$. So the P-partial order is *coarser* than $\preceq$.

The condition (6.1) is independent of the choice of θ:

Lemma 6.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. For any Kähler form ω on X, the following are equivalent:*

(1) $P_\theta[\varphi] \leq P_\theta[\psi]$;
(2) $P_{\theta+\omega}[\varphi] \leq P_{\theta+\omega}[\psi]$.

In particular, $\leq_P$ defines a non-strict partial order on $\mathrm{QPSH}(X)$.

Proof (1) $\implies$ (2). Observe that

$$\varphi \leq P_\theta[\varphi] \leq P_{\theta+\omega}[\varphi].$$

It follows from Theorem 3.1.2 that

$$P_{\theta+\omega}[\varphi] = P_{\theta+\omega}[P_\theta[\varphi]]. \tag{6.2}$$

A similar formula holds for ψ. So we see that (2) holds.

(2) $\implies$ (1). By (6.2), we may assume that φ and ψ are both model potentials in $\mathrm{PSH}(X,\theta)_{>0}$.

Observe that $\varphi \vee \psi \leq P_{\theta+\omega}[\psi]$. It follows that $P_{\theta+\omega}[\varphi\vee\psi] \leq P_{\theta+\omega}[\psi]$. The reverse inequality is trivial, so

$$P_{\theta+\omega}[\varphi\vee\psi] = P_{\theta+\omega}[\psi].$$

From the direction we have proved, for any $C \geq 1$,

$$P_{\theta+C\omega}[\varphi\vee\psi] = P_{\theta+C\omega}[\psi].$$

So by Proposition 3.1.3,

$$\int_X \left(\theta + C\omega + \mathrm{dd^c}(\varphi\vee\psi)\right)^n = \int_X \left(\theta + C\omega + \mathrm{dd^c}\psi\right)^n.$$

Since both sides are polynomials in C, the equality extends to $C = 0$, namely,

$$\int_X \theta^n_{\varphi\vee\psi} = \int_X \theta^n_\psi.$$

In particular, $\varphi\vee\psi \leq P_\theta[\psi] = \psi$ by (3.7). So (1) follows. □

As a consequence of Lemma 6.1.1, we can define the P-partial order at the level of currents. Given closed positive $(1,1)$-currents $T = \theta_\varphi$, $S = \theta'_\psi$, we write $T \leq_P S$ (resp. $T \sim_P S$) if $\varphi \leq_P \psi$ (resp. $\varphi \sim_P \psi$). This definition is independent of the decompositions of T and S.

As a first example of P-equivalence, we have:

Example 6.1.1 Let θ be a closed smooth real $(1,1)$-form on X and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then

$$\varphi \sim_P P_\theta[\varphi]^{1}.$$

This follows immediately from Theorem 3.1.2.

We give a very useful criterion of the P-equivalence in terms of the non-pluripolar masses.

Proposition 6.1.1 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \leq \psi$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) *for each $j = 0, \ldots, n$, we have*

$$\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}. \tag{6.3}$$

Assume furthermore that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, then these conditions are equivalent to the following:

(3) *We have*

$$\int_X \theta_\varphi^n = \int_X \theta_\psi^n.$$

Recall that V_θ is introduced in (2.9).

Proof We first prove the equivalence between (1) and (3) when $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$.

(1) $\implies$ (3). Assume that $\varphi \sim_P \psi$. By Lemma 6.1.1, we have

$$P_\theta[\varphi] = P_\theta[\psi].$$

So (3) follows from Proposition 3.1.3.

(3) $\implies$ (1). It follows from Theorem 3.1.2 that $P_\theta[\varphi] = P_\theta[\psi]$, so (1) follows.

Let us come back to the general case.

(1) $\implies$ (2). Fix $j \in \{0, \ldots, n\}$, we argue (6.3).

Take a Kähler form ω on X. By Lemma 6.1.1, for each $\epsilon > 0$, we have

$$P_{\theta+\epsilon\omega}[\varphi] = P_{\theta+\epsilon\omega}[\psi].$$

It follows from Proposition 3.1.3 that

$$\begin{aligned}\int_X (\theta + \epsilon\omega + \mathrm{dd^c}\psi)^j \wedge \theta_{V_\theta}^{n-j} &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\psi])^j \wedge \theta_{V_\theta}^{n-j} \\ &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi])^j \wedge \theta_{V_\theta}^{n-j} \\ &= \int_X (\theta + \epsilon\omega + \mathrm{dd^c}\varphi)^j \wedge \theta_{V_\theta}^{n-j}.\end{aligned}$$

Since the two extremes are both polynomials in ϵ, we conclude that the same holds when $\epsilon = 0$, that is, (6.3) holds.

(2) $\implies$ (1). Assume (6.3) holds for all $j = 0, \ldots, n$. For each $t \in (0,1)$, we have

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[1] I do not know if the same holds when φ has vanishing mass.

$$\int_X \theta^n_{t\varphi+(1-t)V_\theta} = \int_X \theta^n_{t\psi+(1-t)V_\theta}$$

by the binomial expansion. By the implication (3) $\implies$ (1), we have

$$t\varphi + (1-t)V_\theta \sim_P t\psi + (1-t)V_\theta$$

for each $t \in (0,1)$.

Fix a Kähler form ω on X. From the implication (1) $\implies$ (3), we have

$$\int_X (\theta+\omega)^n_{t\varphi+(1-t)V_\theta} = \int_X (\theta+\omega)^n_{t\psi+(1-t)V_\theta}.$$

Since both sides are polynomials in t, the same holds when $t = 1$. From the implication (3) $\implies$ (1) again, we have $\varphi \sim_P \psi$. □

Next we introduce a different partial order.

Proposition 6.1.2 *Given $\varphi, \psi \in \mathrm{QPSH}(X)$, the following are equivalent:*

(1) *For any $k \in \mathbb{Z}_{>0}$, we have*
$$\mathcal{I}(k\varphi) \subseteq \mathcal{I}(k\psi);$$
(2) *for any $\lambda \in \mathbb{R}_{>0}$, we have*
$$\mathcal{I}(\lambda\varphi) \subseteq \mathcal{I}(\lambda\psi);$$
(3) *for any modification $\pi\colon Y \to X$ and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y);$$
(4) *for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a Kähler manifold and any $y \in Y$, we have*
$$\nu(\pi^*\varphi, y) \geq \nu(\pi^*\psi, y);$$
(5) *for any prime divisor E over X, we have*
$$\nu(\varphi, E) \geq \nu(\psi, E).$$

Proof The proof is almost identical to that of Proposition 3.2.1. □

Definition 6.1.2 Let $\varphi, \psi \in \mathrm{QPSH}(X)$, we say φ is *$\mathcal{I}$-more singular than* ψ and write $\varphi \preceq_{\mathcal{I}} \psi$ if the equivalent conditions in Proposition 6.1.2 are satisfied.

It is clear that $\preceq_{\mathcal{I}}$ is a non-strict partial order on $\mathrm{QPSH}(X)$.

Note that $\varphi \preceq_{\mathcal{I}} \psi$ and $\psi \preceq_{\mathcal{I}} \varphi$ both hold if and only if $\varphi \sim_{\mathcal{I}} \psi$ in the sense of Definition 3.2.1.

Given closed positive $(1,1)$-currents $T = \theta_\varphi$, $S = \theta'_\psi$, we write $T \preceq_{\mathcal{I}} S$ (resp. $T \sim_{\mathcal{I}} S$) if $\varphi \preceq_{\mathcal{I}} \psi$ (resp. $\varphi \sim_{\mathcal{I}} \psi$). This definition is independent of the decompositions of T and S.

Lemma 6.1.2 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$, then*

$$P_\theta[\varphi \vee \psi] = P_\theta\left[P_\theta[\varphi] \vee P_\theta[\psi]\right]. \tag{6.4}$$

Proof Since $\varphi \vee \psi \leq P_\theta[\varphi] \vee P_\theta[\psi]$, the $\leq$ direction of (6.4) follows. Conversely, it suffices to show that

$$P_\theta[\varphi \vee \psi] \geq P_\theta[\varphi] \vee P_\theta[\psi],$$

which is obvious. □

Lemma 6.1.3 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Then the following are equivalent:*

(1) *$\varphi \preceq_P \psi$ (resp. $\varphi \preceq_{\mathcal{I}} \psi$);*
(2) *$\varphi \vee \psi \sim_P \psi$ (resp. $\varphi \vee \psi \sim_{\mathcal{I}} \psi$).*

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. We only prove the P case, the $\mathcal{I}$ case is similar.

(2) $\implies$ (1). By (2) and Example 6.1.1, $P_\theta[\varphi \vee \psi] = P_\theta[\psi] \sim_P \psi$. But $\varphi \leq P_\theta[\varphi \vee \psi]$, so (1) follows.

(1) $\implies$ (2). We may assume that φ, ψ are both model by Lemma 6.1.2. Then $\varphi \leq \psi$ and (2) follows. □

Corollary 6.1.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Assume that $\varphi \preceq_P \psi$, then $\varphi \preceq_{\mathcal{I}} \psi$.*

Proof This follows from Lemma 6.1.3 and Proposition 3.2.9. □

Next we give a few extra characterizations of the P-envelope.

Corollary 6.1.2 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then*

$$\begin{aligned} P_\theta[\varphi] &= \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim_P \varphi\right\} \\ &= \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq_P \varphi\right\}. \end{aligned}$$

Just for comparison, let us recall a few other characterizations of the P-envelope for $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$:

$$\begin{aligned} P_\theta[\varphi] &= \sup{}^*\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \preceq \varphi\right\} \\ &= \sup{}^*\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \psi \sim \varphi\right\} \\ &= \sup_{C \in \mathbb{Z}_{>0}}{}^*(\varphi + C) \wedge V_\theta \\ &= \sup\left\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0, \varphi \preceq \psi, \int_X \theta_\varphi^n = \int_X \theta_\psi^n\right\}. \end{aligned}$$

Proof Note that $\psi \sim_P \varphi$ implies that $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ by Proposition 6.1.4. We observe that

$$\begin{aligned}&\sup\{\psi\in\mathrm{PSH}(X,\theta):\psi\le 0,\psi\sim_P\varphi\}\\=&\sup\{\psi\in\mathrm{PSH}(X,\theta):\psi\le 0,\varphi\le\psi,\psi\sim_P\varphi\}\end{aligned}$$

by Lemma 6.1.3. So the first equality is a direct consequence of Proposition 6.1.1 and Theorem 3.1.2.

Next we prove the second equality. We only need to show that for any $\psi\in\mathrm{PSH}(X,\theta)$ with $\psi\le 0$ and $\psi\preceq_P\varphi$, we have $\psi\le P_\theta[\varphi]$.

By Lemma 6.1.3 and Example 6.1.1, we know that $P_\theta[\varphi]\vee\psi\sim_P\varphi$ and $P_\theta[\varphi]\vee\psi\le 0$. It follows from the first equality that $\psi\le P_\theta[\varphi]$. □

Similarly, we have a new characterization of the $\mathcal{I}$-envelope.

Corollary 6.1.3 *Assume that $\varphi\in\mathrm{PSH}(X,\theta)$, then*

$$P_\theta[\varphi]_{\mathcal{I}}=\sup\{\psi\in\mathrm{PSH}(X,\theta):\psi\le 0,\psi\preceq_{\mathcal{I}}\varphi\}\,.$$

Proof It suffices to show that for any $\psi\in\mathrm{PSH}(X,\theta)$ with $\psi\le 0$ and $\psi\preceq_{\mathcal{I}}\varphi$, we have $\psi\le P_\theta[\varphi]_{\mathcal{I}}$. By Lemma 6.1.3 and Proposition 3.2.6, we know that $P_\theta[\varphi]_{\mathcal{I}}\vee\psi\sim_{\mathcal{I}}\varphi$. Therefore,

$$\psi\le P_\theta[\varphi]_{\mathcal{I}}\vee\psi\le P_\theta[\varphi]_{\mathcal{I}}\,.$$

Proposition 6.1.3 *Suppose that $\varphi,\psi\in\mathrm{QPSH}(X)$ and θ is a closed real smooth $(1,1)$-form on X such that $\varphi,\psi\in\mathrm{PSH}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi\preceq_{\mathcal{I}}\psi$;
(2) $P_\theta[\varphi]_{\mathcal{I}}\le P_\theta[\psi]_{\mathcal{I}}$.

Proof (1) $\implies$ (2). This follows immediately from Corollary 6.1.3.

(2) $\implies$ (1). This follows from Proposition 3.2.6. □

Example 6.1.2 Let us continue our example Example 3.1.1, where $X=\mathbb{P}^1$, ω is the Fubini–Study metric and $\varphi\in\mathrm{PSH}(X,\omega)$ has log-log singularity at 0. We have shown that $P_\omega[\varphi]=0$ in (3.9), so $\varphi\sim_P 0$ and hence $\varphi\sim_{\mathcal{I}} 0$. In particular, P-equivalence is not equivalent to the equivalence of singularity types.

On the other hand, consider a potential $\psi\in\mathrm{PSH}(X,\omega)$ with log singularity at 0, as in Example 1.8.2. We know that $\nu(\psi,0)=1$ from the explicit expression (1.23). So $\psi\not\sim_{\mathcal{I}} 0$ and hence $\psi\not\sim_P 0$.

Moreover, $\psi\preceq_P\varphi$ and hence $\psi\preceq_{\mathcal{I}}\varphi$.

We give an example showing that P-equivalence is not equivalent to $\mathcal{I}$-equivalence.

Example 6.1.3 Let $X=\mathbb{P}^1$ and ω be the Fubini–Study metric. Let $K\subseteq\mathbb{P}^1$ be a polar Cantor sets carrying an atom free probability measure μ supported on K (see [Car83, Page 31]). Write $\mu=\omega+\mathrm{dd^c}\varphi$ for some ω-subharmonic function φ. Since μ is atom free, we know that all Lelong numbers of φ are 0. On the other hand, φ has 0 non-pluripolar mass since K is pluripolar.

Then observe that $\varphi\sim_{\mathcal{I}} 0$ while $\varphi\not\sim_P 0$.

For later use, we give the following definition.

Definition 6.1.3 Let L be a pseudo-effective line bundle on X. An *elementary metric* on L is a psh metric h on L such that there is a generalized Fubini–Study metric h' on L such that

$$\mathrm{dd^c} h \sim_P \mathrm{dd^c} h'.$$

The set of elementary metrics on L is denoted by $\mathrm{Ele}(L)$.

We also say $\mathrm{dd^c} h$ is elementary. If we have fixed a Hermitian metric h_0 on L, and if we represent h as $h_0 \exp(-\varphi)$, we also say the quasi-psh function φ is elementary.

Recall that the generalized Fubini–Study metrics are defined in Definition 1.8.7.

6.1.2 Properties of the partial orders

Now we state a more natural version of the monotonicity theorem Theorem 2.4.4.

Proposition 6.1.4 *Let $\theta_1, \dots, \theta_n$ be closed real smooth $(1,1)$-forms on X. Let $\varphi_i, \psi_i \in \mathrm{PSH}(X, \theta_i)$ for $i = 1, \dots, n$. Assume that $\varphi_i \preceq_P \psi_i$ for each i. Then*

$$\int_X \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n} \leq \int_X \theta_{1,\psi_1} \wedge \dots \wedge \theta_{n,\psi_n}.$$

Proof Fix a Kähler form ω on X. For each $i = 1, \dots, n$, since $\varphi_i \preceq_P \psi_i$, we have

$$P_{\theta_i + \epsilon\omega}[\varphi_i] \leq P_{\theta_i + \epsilon\omega}[\psi_i]$$

for all $\epsilon > 0$. Therefore, by Proposition 3.1.3 and Theorem 2.4.4, we have

$$\int_X (\theta_1 + \epsilon\omega)_{\varphi_1} \wedge \dots \wedge (\theta_n + \epsilon\omega)_{\varphi_n} \leq \int_X (\theta_1 + \epsilon\omega)_{\psi_1} \wedge \dots \wedge (\theta_n + \epsilon\omega)_{\psi_n}.$$

Letting $\epsilon \to 0+$, we find the desired inequality. □

Next we show that the P and $\mathcal{I}$-partial orders are preserved by some natural operations.

Lemma 6.1.4 *Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y. Given two quasi-plurisubharmonic functions φ, ψ on X, then the following are equivalent:*

- *$\varphi \preceq_P \psi$;*
- *$\pi^*\varphi \preceq_P \pi^*\psi$.*

The same holds with $\mathcal{I}$ in place of P.

Proof In the P-case, this follows from Proposition 3.1.7, while in the $\mathcal{I}$-case, this follows from Proposition 3.2.5. □

Proposition 6.1.5 *Let $\varphi, \psi, \varphi', \psi' \in \mathrm{QPSH}(X)$. Assume that*

$$\varphi \preceq_P \psi, \quad \varphi' \preceq_P \psi'.$$

Then

$$\varphi + \varphi' \preceq_P \psi + \psi'.$$

The same holds with $\preceq_{\mathcal{I}}$ in place of $\preceq_P$.

Proof Take a Kähler form ω on X such that $\varphi, \psi, \varphi', \psi' \in \mathrm{PSH}(X, \omega)_{>0}$. The statement for $\preceq_{\mathcal{I}}$ is a simple consequence of Proposition 1.4.2. We only need to handle the case of $\preceq_P$.

Step 1. We first show that

$$P_\omega[\varphi] + P_\omega[\varphi'] \sim_P \varphi + \varphi'.$$

In fact, we clearly have

$$P_\omega[\varphi] + P_\omega[\varphi'] \geq \varphi + \varphi'.$$

So by Proposition 6.1.1, it suffices to show that they have the same mass. We compute

$$\begin{aligned}
&\int_X \left(2\omega + \mathrm{dd^c} P_\omega[\varphi] + \mathrm{dd^c} P_\omega[\varphi']\right)^n \\
=& \sum_{j=0}^n \binom{n}{j} \int_X (\omega + \mathrm{dd^c} P_\omega[\varphi])^j \wedge (\omega + \mathrm{dd^c} P_\omega[\varphi'])^{n-j} \\
=& \sum_{j=0}^n \binom{n}{j} \int_X \omega_\varphi^j \wedge \omega_{\varphi'}^{n-j} \\
=& \int_X (2\omega + \varphi + \varphi')^n ,
\end{aligned}$$

where we applied Proposition 3.1.3 on the third line.

Step 2. By Step 1, we may assume that $\varphi, \psi, \varphi', \psi'$ are all model potentials. So $\varphi \leq \psi$ and $\varphi' \leq \psi'$. Our assertion follows. □

Proposition 6.1.6 *Let $(\varphi_i)_{i \in I}$, $(\psi_i)_{i \in I}$ be uniformly bounded from above non-empty families in $\mathrm{QPSH}(X)$. Assume that there exists a closed smooth real $(1,1)$-form θ such that $\varphi_i, \psi_i \in \mathrm{PSH}(X, \theta)$ and $\varphi_i \preceq_P \psi_i$ for all $i \in I$. Then*

$$\sup_{i \in I}{}^* \varphi_i \preceq_P \sup_{i \in I}{}^* \psi_i.$$

The same holds with $\preceq_{\mathcal{I}}$ in place of $\preceq_P$.

Proof By increasing θ, we may assume that $\varphi_i, \psi_i \in \mathrm{PSH}(X, \theta)_{>0}$ for all $i \in I$. The statement for $\preceq_{\mathcal{I}}$ is a simple consequence of Corollary 1.4.1, we only have to consider the statement for $\preceq_P$.

Step 1. We first handle the case where I is a directed set and $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are increasing nets.

In this case, our assertion follows simply from Proposition 3.1.11.

Step 2. We handle the case where I is finite. We may assume that $I = \{0, 1\}$. It suffices to show that

$$P_\theta[\varphi_0] \vee P_\theta[\varphi_1] \sim_P \varphi_0 \vee \varphi_1,$$

which follows from Lemma 6.1.2.

Step 3. The general case can be reduced to the two cases handled in Step 1 and Step 2. More precisely, by Proposition 1.2.2, we could find a countable subset $J \subseteq I$ such that

$$\sup_{j\in J}{}^*\varphi_j = \sup_{i\in I}{}^*\varphi_i, \quad \sup_{i\in I}{}^*\psi_j = \sup_{i\in I}{}^*\psi_i.$$

We may replace I by J and assume that I is countable. We may assume that I is infinite, as otherwise, we could apply Step 2 directly. So let us assume that $J = \mathbb{Z}_{>0}$. In this case, by Step 2 again, we may assume that both $(\varphi_i)_i$ and $(\psi_i)_i$ are increasing, which is the situation of Step 1.

Proposition 6.1.7 *Let $\varphi, \psi, \varphi', \psi' \in \mathrm{PSH}(X, \theta)_{>0}$ for some closed smooth real $(1, 1)$-form θ on X. Assume that*

$$\varphi \sim_P \varphi', \quad \psi \sim_P \psi', \quad \varphi' \wedge \psi' \in \mathrm{PSH}(X, \theta)_{>0}.$$

Then

$$\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)_{>0}, \quad \varphi \wedge \psi \sim_P \varphi' \wedge \psi'.$$

Proof We first observe that $\varphi, \varphi', \psi, \psi' \in \mathrm{PSH}(X, \theta)_{>0}$ by assumption. Let

$$\phi \coloneqq P_\theta[\varphi] = P_\theta[\varphi'], \quad \gamma = P_\theta[\psi] = P_\theta[\psi'].$$

Then $\phi \wedge \gamma \in \mathrm{PSH}(X, \theta)_{>0}$ since this holds for $\varphi' \wedge \psi'$. It follows from Lemma 3.1.2 that

$$\int_X \theta^n_{\varphi'\wedge\psi'} = \int_X \theta^n_{\phi\wedge\gamma}.$$

Next, we apply Lemma 3.1.2 again to conclude that $\varphi \wedge \psi \in \mathrm{PSH}(X, \theta)$ and

$$\int_X \theta^n_{\varphi\wedge\psi} = \int_X \theta^n_{\phi\wedge\gamma} = \int_X \theta^n_{\varphi'\wedge\psi'} > 0.$$

The P-equivalence relation characterizes when a subgeodesic exists. See Definition 4.1.1 for the notion of subgeodesics.

Theorem 6.1.1 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X, \theta)_{>0}$. Then the following are equivalent:*

(1) *There is a subgeodesic from φ_0 to φ_1;*
(2) $\varphi_0 \sim_P \varphi_1$.

Proof (2) $\implies$ (1). This follows from Proposition 4.2.1.

(1) $\implies$ (2). Let $(\varphi_t)_{t\in(0,1)}$ be a subgeodesic from φ_0 to φ_1.

Step 1. We assume that $\varphi_0 \geq \varphi_1$.

Without loss of generality, we may assume that $(\varphi_t)_{t\in(0,1)}$ is the geodesic from φ_0 to φ_1. Then $t \mapsto \varphi_t$ is decreasing as we have seen in the proof of Proposition 4.2.2.

Let $\varphi_t = \varphi_1$ for all $t > 1$. Then by the gluing lemma Lemma 1.2.2, we find that $(\varphi_t)_{t\geq 0}$ is a subgeodesic ray.

Next, we consider the Legendre transform

$$\Gamma_\tau \coloneqq \inf_{t\geq 0}(\varphi_t - t\tau), \quad \tau \in \mathbb{R}.$$

It follows from Kiselman's principle Proposition 1.2.8 that $\Gamma_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$. Note that for $\tau > 0$, we clearly have $\Gamma_\tau \equiv -\infty$. On the other hand, for $\tau \leq 0$,

$$\Gamma_\tau = \inf_{t\in[0,1]}(\varphi_t - t\tau) \in \mathrm{PSH}(X,\theta).$$

By Legendre inversion, for $t > 0$,

$$\varphi_t = \sup_{\tau\in\mathbb{R}}(\Gamma_\tau + t\tau).$$

Fix a Kähler form ω on X. It follows from Proposition 6.1.6 that for each $t > 0$,

$$\varphi_t \sim_P \sup_{\tau<0}{}^* P_{\theta+\omega}[\Gamma_\tau].$$

The right-hand side is independent of t. Here by adding ω, we no longer have to worry about the possibility where Γ_τ has vanishing mass.

Write

$$\varphi_0 = \sup_{t\in(0,1)}{}^*\varphi_t.$$

By Proposition 6.1.6 again, we find that

$$\varphi_0 \sim_P \sup_{\tau<0}{}^* P_{\theta+\omega}[\Gamma_\tau]$$

as well. So $\varphi_0 \sim_P \varphi_1$.

Step 2. We prove the general case.

Observe that $(\varphi_t \vee \varphi_1)_{t\in(0,1)}$ is a subgeodesic from $\varphi_0 \vee \varphi_1$ to φ_1. By Step 1,

$$\varphi_0 \vee \varphi_1 \sim_P \varphi_1.$$

Hence $\varphi_0 \preceq_P \varphi_1$ by Proposition 6.1.3. The converse is proved similarly. Hence (2) follows.

Restating the (1) $\implies$ (2) direction in the theorem in terms of the complexification, we find the following interesting result.

Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. We write $p_1 \colon X \times S \to X$ for the natural projection.

Corollary 6.1.4 *Let* $\Phi \in \mathrm{PSH}(X \times S, p_1^*\theta)$. *Assume that for any* $c \in \mathbb{R}$, $x \in X$ *and* $z \in S$, *we have*

$$\Phi(x, z) = \Phi(x, z + \mathrm{i}c).$$

Then $\int_X (\theta + \mathrm{dd^c}\Phi_z)^n$ *is independent of* $z \in S$, *where* $\Phi_z \in \mathrm{PSH}(X, \theta)$ *is given by* $\Phi_z(x) = \Phi(x, z)$.

This seems to be the first non-trivial result concerning the variation of non-pluripolar masses.

6.2 The d_S-pseudometric

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1, 1)$-form on X representing a big cohomology class. The goal of this section is to study a pseudometric on the space $\mathrm{PSH}(X, \theta)$.

6.2.1 The definition of the d_S-pseudometric

Recall that for any $\varphi \in \mathrm{PSH}(X, \theta)$, the geodesic ray $\ell^\varphi \in \mathcal{R}^1(X, \theta)$ is defined in Example 4.3.1.

Definition 6.2.1 For $\varphi, \psi \in \mathrm{PSH}(X, \theta)$, we define

$$d_S(\varphi, \psi) \coloneqq d_1(\ell^\varphi, \ell^\psi).$$

When we want to be more specific, we write $d_{S,\theta}$ instead of d_S.

The d_1 distance of geodesic rays is defined in Definition 4.3.5.

Proposition 6.2.1 *The function* d_S *defined in Definition 6.2.1 is a pseudometric on* $\mathrm{PSH}(X, \theta)$.

Proof This follows immediately from Theorem 4.3.4. □

When studying a pseudometric, the first thing is to understand when the distance between two elements vanishes.

We first prove a preparation:

Lemma 6.2.1 *Let* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. *Then*

$$d_S(\varphi, \psi) \leq d_S(\varphi, \varphi \vee \psi) + d_S(\psi, \varphi \vee \psi) \leq C_n d_S(\varphi, \psi),$$

where $C_n = 3(n + 1)2^{n+2}$.

We shall use the notations introduced in Example 4.3.1.

Proof We claim that

$$\ell^{\varphi} \vee \ell^{\psi} = \ell^{\varphi \vee \psi}. \tag{6.5}$$

Recall that $\vee$ is defined in Definition 4.3.7. Note that this assertion implies our desired inequality by Lemma 4.3.6.

In proving this assertion, we may assume that $\varphi, \psi \leq 0$ since

$$\ell^{\varphi+C} = \ell^{\varphi}, \quad \ell^{\psi+C} = \ell^{\psi}, \quad \ell^{(\varphi+C)\vee(\psi+C)} = \ell^{\varphi\vee\psi}$$

for any $C \in \mathbb{R}$.

In fact, it is clear that

$$\ell^{\varphi} \leq \ell^{\varphi\vee\psi}, \quad \ell^{\psi} \leq \ell^{\varphi\vee\psi},$$

so the $\leq$ direction in (6.5) holds.

Conversely, if $\ell' \in \mathcal{R}^1(X, \theta)$ and $\ell' \geq \ell^{\varphi} \vee \ell^{\psi}$, then for each $t \geq 0$,

$$\ell'_t \geq ((V_\theta - t) \vee \varphi) \vee ((V_\theta - t) \vee \psi) = (V_\theta - t) \vee (\varphi \vee \psi).$$

Therefore,

$$\ell'_s \geq \ell_s^{\varphi\vee\psi, t}$$

for any $0 \leq s \leq t$. It follows from (4.30) that $\ell'_s \geq \ell_s^{\varphi\vee\psi}$ for any $s \geq 0$. □

Proposition 6.2.2 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Then the following are equivalent:*

(1) $\varphi \sim_P \psi$;
(2) $d_S(\varphi, \psi) = 0$.

In particular, $d_S(\varphi, P_\theta[\varphi]) = 0$ for all $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$.

Proof By Lemma 6.1.3, we have $\varphi \sim_P \psi$ if and only if $\varphi \sim_P \varphi \vee \psi$ and $\psi \sim_P \varphi \vee \psi$. By Lemma 6.2.1, $d_S(\varphi, \psi) = 0$ if and only if $d_S(\varphi, \varphi \vee \psi) = 0$ and $d_S(\psi, \varphi \vee \psi) = 0$. So it suffices to prove the assertion when $\varphi \leq \psi$. Assuming this, by Proposition 4.3.6 we have that (2) holds if and only if

$$\mathbf{E}(\ell^{\varphi}) = \mathbf{E}(\ell^{\psi}),$$

where $\mathbf{E}$ is introduced in Definition 4.3.6. But by (4.28), this holds if and only if

$$\sum_{j=0}^{n} \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \sum_{j=0}^{n} \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}.$$

Thanks to Theorem 2.4.4, this holds if and only if for all $j = 0, \dots, n$,

$$\int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j},$$

which is equivalent to (1) by Proposition 6.1.1. □

Lemma 6.2.2 *Suppose that $\varphi, \psi \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi$, then*

$$d_S(\varphi,\psi) = \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}\right).$$

Proof This follows trivially from (4.28). □

Corollary 6.2.1 *Suppose that $\varphi, \psi, \eta \in \mathrm{PSH}(X,\theta)$ and $\varphi \preceq_P \psi \preceq_P \eta$. Then*

$$d_S(\varphi,\eta) \geq d_S(\varphi,\psi), \quad d_S(\varphi,\eta) \geq d_S(\psi,\eta).$$

Proof This is an immediate consequence of Lemma 6.2.2 and Proposition 6.1.4. □

Corollary 6.2.2 *For any $\varphi, \psi \in \mathrm{PSH}(X,\theta)$, we have*

$$\begin{aligned} d_S(\varphi,\psi) \leq & \frac{1}{n+1}\sum_{j=0}^{n}\left(2\int_X \theta_{\varphi\vee\psi}^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} - \int_X \theta_\psi^j \wedge \theta_{V_\theta}^{n-j}\right) \\ \leq & C_n d_S(\varphi,\psi), \end{aligned} \tag{6.6}$$

where $C_n = 3(n+1)2^{n+2}$.

In particular, if $(\varphi_i)_{i\in I}$ is a net in $\mathrm{PSH}(X,\theta)$ with d_S-limit φ, then for each $j = 0,\dots,n$,

$$\lim_{i\in I}\int_X \theta_{\varphi_i}^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j} = \lim_{i\in I}\int_X \theta_{\varphi_i\vee\varphi}^j \wedge \theta_{V_\theta}^{n-j}.$$

Proof The estimates (6.6) follows from the combination of Lemma 6.2.2 and Lemma 6.2.1.

Suppose that $\varphi_i \xrightarrow{d_S} \varphi$, then $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$ by Lemma 6.2.1. Therefore, Theorem 2.4.4 and Lemma 6.2.2 imply that

$$\lim_{i\in I}\int_X \theta_{\varphi_i\vee\varphi}^j \wedge \theta_{V_\theta}^{n-j} = \int_X \theta_\varphi^j \wedge \theta_{V_\theta}^{n-j}$$

for any $j = 0,\dots,n$. The last assertion now follows from (6.6) and Theorem 2.4.4. □

Corollary 6.2.3 *Suppose that $\varphi_i \in \mathrm{PSH}(X,\theta)$ ($i \in I$) be an increasing net, uniformly bounded from above. Then*

$$\varphi_i \xrightarrow{d_S} \sup_{j\in I}{}^* \varphi_j.$$

If the φ_i's are all model potentials in $\mathrm{PSH}(X,\theta)_{>0}$, then so is $\sup_{j\in I}{}^*\varphi_j$, as we have seen in Proposition 3.1.11.

Proof Write $\varphi = \sup_{j\in I}{}^*\varphi_j$. Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X,\theta)$. By Lemma 6.2.2, it suffices to show that for each $k = 0,\dots,n$, we have

$$\lim_{j\in I}\int_X \theta_{\varphi_j}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}.$$

The latter follows from Corollary 2.4.1. □

Corollary 6.2.4 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)$. Then*

$$\left|\int_X \theta_\varphi^n - \int_X \theta_\psi^n\right| \leq D_n d_S(\varphi, \psi),$$

where $D_n = 3(n+1)C_n$ with C_n being the same constant as in Lemma 6.2.1.

Proof We compute

$$\begin{aligned}\left|\int_X \theta_\varphi^n - \int_X \theta_\psi^n\right| \leq & \left|2\int_X \theta_{\varphi\vee\psi}^n - \int_X \theta_\varphi^n - \int_X \theta_\psi^n\right| + 2\left|\int_X \theta_{\varphi\vee\psi}^n - \int_X \theta_\varphi^n\right| \\ \leq & (n+1)C_n d_S(\varphi, \psi) + 2(n+1) d_S(\varphi, \varphi \vee \psi) \\ \leq & (n+1)C_n d_S(\varphi, \psi) + 2(n+1)C_n d_S(\varphi, \psi),\end{aligned}$$

where the first line is just the triangle inequality, the second line follows from Corollary 6.2.2 and the third line follows from Lemma 6.2.1. □

By contrast, for decreasing nets, the situation is different:

Corollary 6.2.5 *Suppose that $(\varphi_i)_{i\in I}$ is a decreasing net in $\mathrm{PSH}(X, \theta)$ such that $\varphi := \inf_{i\in I} \varphi_i \not\equiv -\infty$. Then the following are equivalent:*

(1) *We have*

$$\varphi_i \xrightarrow{d_S} \varphi;$$

(2) *for each $k = 0, \ldots, n$, we have*

$$\lim_{j\in I} \int_X \theta_{\varphi_j}^k \wedge \theta_{V_\theta}^{n-k} = \int_X \theta_\varphi^k \wedge \theta_{V_\theta}^{n-k}. \tag{6.7}$$

If we assume furthermore that $\int_X \theta_\varphi^n > 0$, then the above conditions are equivalent to the following:

(3) *We have*

$$\lim_{j\in I} \int_X \theta_{\varphi_j}^n = \int_X \theta_\varphi^n.$$

In the latter case, we also have

$$P_\theta[\varphi] = \inf_{j\in I} P_\theta[\varphi_j]. \tag{6.8}$$

Proof Recall that by Proposition 1.2.1, $\varphi \in \mathrm{PSH}(X, \theta)$.

(1) $\iff$ (2). This follows immediately from Lemma 6.2.2.

Assume that $\int_X \theta_\varphi^n > 0$.

(2) $\implies$ (3). This is trivial.

(3) $\implies$ (2). Let $(b_j)_{j\in I}$ be a net converging to ∞ such that

$$b_j \in \left(1, \left(\frac{\int_X \theta^n_{\varphi_j}}{\int_X \theta^n_{\varphi_j} - \int_X \theta^n_{\varphi}}\right)^{1/n}\right).$$

By Lemma 2.4.2, for each $j \in I$, we can find $\eta_j \in \mathrm{PSH}(X, \theta)$ such that

$$b_j^{-1}\eta_j + (1 - b_j^{-1})\varphi_j \leq \varphi.$$

It follows from Theorem 2.4.4 that for any $k = 0, \dots, n$,

$$\int_X \theta^k_{\varphi} \wedge \theta^{n-k}_{V_\theta} \geq \left(1 - b_j^{-1}\right)^k \int_X \theta^k_{\varphi_j} \wedge \theta^{n-k}_{V_\theta}.$$

Taking the limit, we conclude the $\leq$ direction in (6.7). The $\geq$ direction follows from Theorem 2.4.4.

Finally, we argue (6.8). We may assume that $\varphi_j \leq 0$ for all $j \in I$. Let $\psi_j = P_\theta[\varphi_j] \geq \varphi_j$. It follows from Corollary 3.1.2 that ψ_j is a model potential. Let

$$\psi = \inf_{j \in I} \psi_j \geq \varphi.$$

It follows from Proposition 3.1.3 and Proposition 3.1.10 that

$$\int_X \theta^n_{\psi} = \lim_{j \in I} \int_X \theta^n_{\psi_j} = \lim_{j \in I} \int_X \theta^n_{\varphi_j} = \int_X \theta^n_{\varphi}.$$

By Proposition 3.1.9, ψ is a model potential. Hence $\psi = P_\theta[\varphi]$ by Theorem 3.1.2. □

Having understood the increasing and decreasing cases, we shall handle more general convergent sequences. In fact, since d_S is a pseudometric, the topology is completely determined by convergent sequences, so we do not need to consider nets in general.

Proposition 6.2.3 *Let $\varphi_j, \varphi \in \mathrm{PSH}(X, \theta)$ $(j \geq 1)$, $\varphi_j \xrightarrow{d_S} \varphi$. Assume that there is $\delta > 0$ such that*

$$\int_X \theta^n_{\varphi_j} \geq \delta$$

for all j and the φ_j's and φ are all model potentials. Then up to replacing $(\varphi_j)_j$ by a subsequence, there is a decreasing sequence $(\psi_j)_j$ and an increasing sequence $(\eta_j)_j$ in $\mathrm{PSH}(X, \theta)$ such that

(1) $\psi_j \xrightarrow{d_S} \varphi$, $\eta_j \xrightarrow{d_S} \varphi$;
(2) $\psi_j \geq \varphi_j \geq \eta_j$ for all j.

In fact, for any $j \geq 1$, we will take

$$\eta_j = \inf_{k \in \mathbb{N}} \varphi_j \wedge \varphi_{j+1} \wedge \dots \wedge \varphi_{j+k}, \quad \psi_j = \sup_{k \geq j}{}^* \varphi_k.$$

Proof We are free to replace $(\varphi_j)_j$ by a subsequence. So we may assume that

$$d_S(\varphi_j,\varphi_{j+1}) \leq C_n^{-2j}, \quad d_S(\varphi,\varphi_j) \leq \frac{2^{-j}}{D_n}, \tag{6.9}$$

where C_n is the constant in Corollary 6.2.2, D_n is the constant in Corollary 6.2.4.

In particular, by Corollary 6.2.4,

$$\left| \int_X \theta^n_{\varphi_j} - \int_X \theta^n_{\varphi} \right| \leq 2^{-j}. \tag{6.10}$$

Step 1. We handle the ψ_j's. For each $j \geq 1$ and $k \geq 1$, by Lemma 6.2.1 we have

$$\begin{aligned} d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \leq & C_n d_S(\varphi_j, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \\ \leq & C_n d_S(\varphi_j,\varphi_{j+1}) + C_n d_S(\varphi_{j+1}, \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}). \end{aligned}$$

By iteration, we find

$$\begin{aligned} d_S(\varphi_j, \varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k}) \leq & \sum_{a=j}^{j+k-1} C_n^{a+1-j} d_S(\varphi_a,\varphi_{a+1}) \\ \leq & \sum_{a=j}^{j+k-1} C_n^{a+1-j} C_n^{-2a} \leq \frac{C_n^{1-2j}}{1-C_n^{-1}}. \end{aligned}$$

Using Corollary 6.2.3, we have

$$\varphi_j \vee \varphi_{j+1} \vee \cdots \vee \varphi_{j+k} \xrightarrow{d_S} \psi_j$$

as $k \to \infty$. Hence

$$d_S(\varphi_j,\psi_j) \leq \frac{C_n^{1-2j}}{1-C_n^{-1}}. \tag{6.11}$$

We conclude that $\psi_j \xrightarrow{d_S} \varphi$.

Moreover, we observe that

$$\varphi = \inf_{j\geq 1} P_\theta[\psi_j] \tag{6.12}$$

by Corollary 6.2.5.

Step 2. We consider the η_j's.

For each $j \geq 1$ and $k \geq 0$, we let

$$\eta_j^k \coloneqq \varphi_j \wedge \cdots \wedge \varphi_{j+k}.$$

Using (6.11) and Corollary 6.2.4, we have

$$\left|\int_X \theta^n_{\psi_j} - \int_X \theta^n_{\varphi}\right| \leq 2^{-j-1}$$

when $j \geq j_0$ for some large j_0. Taking (6.10), we have

$$\left|\int_X \theta^n_{\varphi_j} - \int_X \theta^n_{\psi_{j-1}}\right| \leq 2^{1-j} \tag{6.13}$$

for $j > j_0$. Take $j_1 > j_0$ so that for $j \geq j_1$, $2^{1-j} < \delta$.

Step 2.1. We claim that for a fixed $j \geq j_1$, for any $k \in \mathbb{N}$, we have $\eta^k_j \in \mathrm{PSH}(X, \theta)$ and

$$\int_X \theta^n_{\eta^k_j} \geq \int_X \theta^n_{\varphi_j} - \sum_{a=1}^{k} 2^{1-j-a}. \tag{6.14}$$

We argue by induction on $k \geq 0$. The case $k = 0$ is trivial. When $k > 0$, assume that the case $k - 1$ is known. Then

$$\begin{aligned}\int_X \theta^n_{\eta^{k-1}_j} + \int_X \theta^n_{\varphi_{j+k}} &\geq \int_X \theta^n_{\varphi_j} - \sum_{a=1}^{k-1} 2^{1-j-a} + \int_X \theta^n_{\psi_{j+k-1}} - 2^{1-j-k}\\ &> \int_X \theta^n_{\varphi_j} - 2^{1-j} + \int_X \theta^n_{\psi_{j+k-1}} > \int_X \theta^n_{\psi_{j+k-1}},\end{aligned}$$

where the first inequality follows from the inductive hypothesis and (6.13).

Observe that

$$\eta^{k-1}_j \vee \varphi_{j+k} \leq \psi_{j+k-1},$$

it follows from Proposition 3.1.5 that $\eta^k_j \in \mathrm{PSH}(X, \theta)$. By Theorem 3.1.3, we deduce that

$$\begin{aligned}\int_X \theta^n_{\eta^k_j} &\geq \int_X \theta^n_{\varphi_{j+k}} + \int_X \theta^n_{\eta^{k-1}_j} - \int_X \theta^n_{\psi_{j+k-1}}\\ &\geq \int_X \theta^n_{\varphi_j} - \sum_{a=1}^{k} 2^{1-j-a},\end{aligned}$$

where the second inequality follows from the inductive hypothesis and (6.13). Therefore, (6.14) follows.

Step 2.2. It follows from Proposition 3.1.6 that for any $j \geq j_1$, $k \geq 0$,

$$P_\theta\left[\eta^k_j\right] = \eta^j_k.$$

By Proposition 3.1.10, we have

$$\lim_{k\to\infty} \int_X \theta^n_{\eta^k_j} = \int_X \theta^n_{\eta_j}$$

for any $j \geq j_1$. Letting $k \to \infty$ in (6.14), we find that

$$\int_X \theta^n_{\eta_j} \geq \int_X \theta^n_{\varphi_j} - 2^{1-j} > 0$$

for $j \geq j_1$. Observe that we also have

$$\int_X \theta^n_{\eta_j} \leq \int_X \theta^n_{\varphi_j} \leq \int_X \theta^n_{\psi_j}$$

for $j \geq j_1$ by Theorem 2.4.4. It follows from Corollary 2.4.1 that

$$\int_X \theta^n_{\eta} = \lim_{j\to\infty} \int_X \theta^n_{\varphi_j} = \lim_{j\to\infty} \int_X \theta^n_{\psi_j} = \int_X \theta^n_{\varphi},$$

where $\eta = \sup_{j\geq j_1}{}^*\eta_j$. Since $\eta_j \leq \varphi_j \leq \psi_j \leq 0$, we also have that $\eta_j \leq P_\theta[\psi_j]$. Therefore, by (6.12), we also have $\eta \leq \varphi$. It follows from Proposition 6.1.1 that $\eta \sim_P \varphi$. By Corollary 6.2.3 and Proposition 6.2.2, we have $\eta_j \xrightarrow{d_S} \varphi$. □

Corollary 6.2.6 *Let $(\varphi_j)_{j\in I}$ be a Cauchy net (with respect to d_S) in* $\mathrm{PSH}(X,\theta)$*. Assume that there is $\delta > 0$ such that $\int_X \theta^n_{\varphi_j} \geq \delta$ for all $j \in I$. Then $(\varphi_j)_{j\in I}$ converges with respect to d_S.*

In particular, if $(\varphi_j)_{j\in I}$ is a decreasing net such that $\int_X \theta^n_{\varphi_j} \geq \delta > 0$ for all $j \in I$, then $(\varphi_j)_{j\in I}$ converges with respect to d_S.

We can obviously relax the decreasing condition to the following: the P-singularity types of $(\varphi_j)_{j\in I}$ are decreasing.

Proof If the net $(\varphi_j)_{j\in I}$ is decreasing, then it is convergent by Corollary 6.2.5 and Proposition 3.1.10.

It remains to prove the first assertion. Since the space of $\varphi \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta^n_\varphi \geq \delta$ is a pseudometric space, its completeness can be characterized using sequences instead of nets. So we may assume that $(\varphi_j)_{j\in I}$ is a sequence and $I = \mathbb{Z}_{>0}$.

Replacing $(\varphi_j)_{j>0}$ by a subsequence, we may assume that (6.9) holds. Define

$$\psi_j = \sup_{k\geq j}{}^*\varphi_k$$

for each $j > 0$. As in the proof of Proposition 6.2.3 Step 1, especially (6.11), we know that

$$\lim_{j\to\infty} d_S(\varphi_j, \psi_j) = 0.$$

It suffices to prove our assertion for $(\psi_j)_j$ in place of $(\varphi_j)_j$. But since $(\psi_j)_j$ is decreasing, this case has already been handled at the beginning of the proof. □

Lemma 6.2.3 *There is a constant $C > 0$ depending only on X and θ such that for any $\varphi \in \mathrm{PSH}(X,\theta)$ satisfying that θ_φ is a Kähler current, we have*

$$d_{S,\theta}((1-\epsilon)\varphi, \varphi) \leq C\epsilon$$

for $\epsilon > 0$ such that $(1-\epsilon)\varphi \in \mathrm{PSH}(X,\theta)$.

Proof By Lemma 6.2.2, we can compute

$$
\begin{aligned}
d_{S,\theta}((1-\epsilon)\varphi,\varphi) =& \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X \theta^j_{(1-\epsilon)\varphi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}\right)\\
=& \frac{1}{n+1}\sum_{j=0}^{n}\left(\int_X (1-\epsilon)^j\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}\right)\\
&+\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j-1}\binom{j}{k}(1-\epsilon)^k\epsilon^{j-k}\int_X \theta^{j-k}\wedge\theta^k_{\varphi}\wedge\theta^{n-j}_{V_\theta}.
\end{aligned}
$$

Both terms are of the order of $O(\epsilon)$. □

6.2.2 Convergence theorems

Next we establish some important convergence theorems, allowing us to effectively manipulate the d_S-convergence.

Lemma 6.2.4 *Let $(\varphi_i)_{i\in I}$ be a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi\in\mathrm{PSH}(X,\theta)$. *Assume that* $\varphi_i \xrightarrow{d_S} \varphi$. *Then for any* $t\in(0,1]$,

$$
(1-t)\varphi_i + tV_\theta \xrightarrow{d_S} (1-t)\varphi + tV_\theta.
$$

When $t=1$, the sum is understood as in Remark 2.4.2.

Proof Fix $t\in(0,1]$, we write

$$
\varphi_{i,t} = (1-t)\varphi_i + tV_\theta,\quad \varphi_t = (1-t)\varphi + tV_\theta
$$

for any $i\in I$.

By Corollary 6.2.2, it suffices to show that for each $j=0,\dots,n$,

$$
2\int_X \theta^j_{\varphi_{i,t}\vee\varphi_t}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_{i,t}}\wedge\theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi_t}\wedge\theta^{n-j}_{V_\theta} \to 0. \tag{6.15}
$$

Observe that

$$
\varphi_{i,t}\vee\varphi_t = (1-t)(\varphi\vee\varphi_i) + tV_\theta.
$$

So after binomial expansion, (6.15) follows from Corollary 6.2.2. □

Lemma 6.2.5 *Let* $\varphi\in\mathrm{PSH}(X,\theta)$. *For each* $t\in(0,1)$, *let* $\varphi_t = (1-t)\varphi + tV_\theta$. *Then*

$$
\varphi_t \xrightarrow{d_S} \varphi
$$

as $t\to 0+$.

Proof By Lemma 6.2.2, we need to show that for each $j = 1, \dots, n$, we have

$$\lim_{t\to 0+} \int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}.$$

For this purpose, we compute

$$\begin{aligned}&\int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} - \int_X \theta^j_{\varphi} \wedge \theta^{n-j}_{V_\theta}\\ =&\sum_{i=0}^{j-1} \binom{j}{i} (1-t)^i t^{j-i} \int_X \theta^i_{\varphi} \wedge \theta^{n-i}_{V_\theta}.\end{aligned}$$

As $t \to 0+$, the right-hand side clearly tends to 0. □

The following convergent theorem lies at the heart of the whole theory.

Theorem 6.2.1 *Let $\theta_1, \dots, \theta_n$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi^k_j)_{k\in I}$ are nets in $\mathrm{PSH}(X, \theta_j)$ and $\varphi_j \in \mathrm{PSH}(X, \theta_j)$ for $j = 1, \dots, n$. We assume that $\varphi^k_j \xrightarrow{d_S} \varphi_j$ for each $j = 1, \dots, n$. Then*

$$\lim_{k\in I} \int_X \theta_{1,\varphi^k_1} \wedge \cdots \wedge \theta_{n,\varphi^k_n} = \int_X \theta_{1,\varphi_1} \wedge \cdots \wedge \theta_{n,\varphi_n}. \tag{6.16}$$

Proof Since d_S is a pseudometric, in order to establish the continuity of mixed masses, it suffices to consider sequences instead of nets. So we may assume that $I = \mathbb{Z}_{>0}$ as ordered sets.

Step 1. We reduce to the case where φ^k_j, φ_j all have positive masses and there is a constant $\delta > 0$, such that for all j and k,

$$\int_X \theta^n_{j,\varphi^k_j} > \delta.$$

Take $t \in (0,1)$. By Lemma 6.2.4, we have

$$(1-t)\varphi^k_j + tV_{\theta_j} \xrightarrow{d_S} (1-t)\varphi_j + tV_{\theta_j}$$

as $k \to \infty$ for each j. Assume that we have proved the special case of the theorem, we have

$$\begin{aligned}&\lim_{k\to\infty} \int_X \theta_{1,(1-t)\varphi^k_1+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi^k_n+tV_{\theta_n}}\\ =&\int_X \theta_{1,(1-t)\varphi_1+tV_{\theta_1}} \wedge \cdots \wedge \theta_{n,(1-t)\varphi_n+tV_{\theta_n}}.\end{aligned}$$

Since both sides are polynomials in t, by Lagrange interpolation formula, the limit exists at $t = 0$ as well and the same formula holds at $t = 0$. From this, (6.16) follows.

Step 2. Next we may assume that φ_j^k, φ_j are model potentials for all $j = 1, \dots, n$, $k > 0$ by Proposition 6.2.2 and Corollary 3.1.2.

It suffices to prove that any subsequence of $\int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k}$ has a converging subsequence with limit $\int_X \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n}$. Thus, by Proposition 6.2.3 and Theorem 2.4.4, we may assume that for each fixed i, $(\varphi_i^k)_k$ is either increasing or decreasing. We may assume that there is $i_0 \in \{0, \dots, n\}$ such that for $i \le i_0$, the sequence is decreasing and for $i > i_0$, the sequence is increasing.

Thanks to Corollary 6.2.5, Corollary 6.2.3 and Proposition 3.1.11, we have

$$\varphi_i = \inf_{k>0} \varphi_i^k, \quad i \le i_0$$

and

$$\varphi_i = \sup_{k>0}{}^* \varphi_j^k, \quad i > i_0.$$

Therefore, for each $k > 0$, using Theorem 2.4.4, we have

$$\int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \ge \int_X \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{i_0,\varphi_{i_0}} \wedge \theta_{i_0+1,\varphi_n^{i_0+1}} \wedge \dots \wedge \theta_{n,\varphi_n^k}.$$

Using Corollary 2.4.1, we therefore conclude that

$$\varliminf_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \ge \int_X \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n}.$$

It remains to prove

$$\varlimsup_{k\to\infty} \int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \le \int_X \theta_{1,\varphi_1} \wedge \dots \wedge \theta_{n,\varphi_n}. \tag{6.17}$$

By Theorem 2.4.4, for each $k > 0$, we have

$$\int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{n,\varphi_n^k} \le \int_X \theta_{1,\varphi_1^k} \wedge \dots \wedge \theta_{i_0,\varphi_{i_0}^k} \wedge \theta_{i_0+1,\varphi_{i_0+1}} \wedge \dots \wedge \theta_{n,\varphi_n}.$$

When proving (6.17), we may replace φ_j^k by φ_j whenever $j > i_0$, $k > 0$. Thus, we are reduced to the case where for all i, $(\varphi_i^k)_k$ is decreasing.

Thanks to Lemma 2.4.2, for each $i = 1, \dots, n$, we may take an increasing sequence $(b_i^k)_k$ tending to ∞ satisfying

$$b_i^k \in \left(1, \left(\frac{\int_X \theta_{i,\varphi_i^k}^n}{\int_X \theta_{i,\varphi_i^k}^n - \int_X \theta_{i,\varphi_i}^n}\right)^{1/n}\right)$$

and a sequence $(\psi_i^k)_k$ in $\mathrm{PSH}(X, \theta_i)$ such that

$$(b_i^k)^{-1}\psi_i^k + \left(1 - (b_i^k)^{-1}\right)\varphi_i^k \le \varphi_i.$$

Then by Theorem 2.4.4 again,

$$\prod_{i=1}^{n}\left(1-(b_i^k)^{-1}\right)\int_X \theta_{1,\varphi_1^k}\wedge\cdots\wedge\theta_{n,\varphi_n^k}\leq\int_X \theta_{1,\varphi_1}\wedge\cdots\wedge\theta_{n,\varphi_n}.$$

Letting $k\to\infty$, we conclude (6.17). □

Corollary 6.2.7 *Suppose that* $(\varphi_i)_{i\in I}$ *is a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi\in\mathrm{PSH}(X,\theta)$. *Then the following are equivalent:*

(1) $\varphi_i\xrightarrow{d_S}\varphi$;
(2) $\varphi_i\vee\varphi\xrightarrow{d_S}\varphi$ *and*

$$\lim_{i\in I}\int_X \theta_{\varphi_i}^j\wedge\theta_{V_\theta}^{n-j}=\int_X \theta_{\varphi}^j\wedge\theta_{V_\theta}^{n-j} \tag{6.18}$$

for each $j=0,\ldots,n$;
(3) *for each* $j=0,\ldots,n$, (6.18) *holds and*

$$\lim_{i\in I}\int_X \theta_{\varphi_i\vee\varphi}^j\wedge\theta_{V_\theta}^{n-j}=\int_X \theta_{\varphi}^j\wedge\theta_{V_\theta}^{n-j}. \tag{6.19}$$

The corollary allows us to reduce a number of convergence problems related to d_S to the case $\varphi_i\geq\varphi$. This is the most handy way of establishing d_S-convergence in practice.

Proof The equivalence between (2) and (3) follows directly from Lemma 6.2.2.

(1) $\implies$ (2). That $\varphi_i\vee\varphi\xrightarrow{d_S}\varphi$ follows from Corollary 6.2.2. While (6.18) follows from Theorem 6.2.1.

(2) $\implies$ (1). By (6.6), we need to show that for each $j=0,\ldots,n$, we have

$$2\int_X \theta_{\varphi_i\vee\varphi}^j\wedge\theta_{V_\theta}^{n-j}-\int_X \theta_{\varphi}^j\wedge\theta_{V_\theta}^{n-j}-\int_X \theta_{\varphi_i}^j\wedge\theta_{V_\theta}^{n-j}\to 0.$$

This follows from Theorem 6.2.1 and (6.18). □

Corollary 6.2.8 *Let* $(\varphi_i)_{i\in I}$ *be a net in* $\mathrm{PSH}(X,\theta)$ *and* $\varphi\in\mathrm{PSH}(X,\theta)$. *Let* ω *be a closed smooth positive* $(1,1)$*-form on* X. *Then the following are equivalent:*

(1) $\varphi_i\xrightarrow{d_{S,\theta}}\varphi$;
(2) $\varphi_i\xrightarrow{d_{S,\theta+\omega}}\varphi$.

In particular, there is no risk when we simply write $\varphi_i\xrightarrow{d_S}\varphi$.

Proof (1) $\implies$ (2). It suffices to show that for each $j=0,\ldots,n$, we have

$$2\int_X (\theta+\omega)^j_{\varphi_i\vee\varphi}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X(\theta+\omega)^j_{\varphi_i}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\-\int_X(\theta+\omega)^j_{\varphi}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\to 0.$$

Note that this quantity is a linear combination of terms of the following form:

$$2\int_X \theta^r_{\varphi_i\vee\varphi}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}-\int_X\theta^r_{\varphi_i}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}}\\-\int_X\theta^r_{\varphi}\wedge\omega^{j-r}\wedge(\theta+\omega)^{n-j}_{V_{\theta+\omega}},$$

where $r=0,\dots,j$. By Theorem 6.2.1, it suffices to show that $\varphi\vee\varphi_i\xrightarrow{d_S}\varphi$. But this follows from Corollary 6.2.7.

(2) $\implies$ (1). From the direction we already proved, for each $C\geq 1$, we have that

$$\varphi_i\xrightarrow{d_{S,\theta+C\omega}}\varphi.$$

By Theorem 6.2.1, it follows that

$$\lim_{i\in I}\int_X(\theta+C\omega)^j_{\varphi_i}\wedge\theta^{n-j}_{V_\theta}=\int_X(\theta+C\omega)^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}$$

for all $j=0,\dots,n$. It follows that

$$\lim_{i\in I}\int_X\theta^j_{\varphi_i}\wedge\theta^{n-j}_{V_\theta}=\int_X\theta^j_{\varphi}\wedge\theta^{n-j}_{V_\theta}. \tag{6.20}$$

By Corollary 6.2.7, it remains to show that $\varphi_i\vee\varphi\xrightarrow{d_{S,\theta}}\varphi$. By Corollary 6.2.7 again, we know that $\varphi_i\vee\varphi\xrightarrow{d_{S,\theta+\omega}}\varphi$. So it suffices to apply (6.20) to $\varphi_i\vee\varphi$ instead of φ_i, and we conclude by Lemma 6.2.2. □

We sometimes need a slightly more general form.

Corollary 6.2.9 *Let $(\varphi_j)_{j\in I}$, $(\psi_j)_{j\in I}$ be nets in $\mathrm{PSH}(X,\theta)$. Consider a closed smooth positive $(1,1)$-form ω on X. Then the following are equivalent:*

(1) $d_{S,\theta}(\varphi_i,\psi_i)\to 0$*;*
(2) $d_{S,\theta+\omega}(\varphi_i,\psi_i)\to 0$.

In particular, we can write $d_S(\varphi_i,\psi_i)\to 0$ without ambiguity.

Proof The proof is similar to that of Corollary 6.2.8, which is therefore left to the readers. □

Corollary 6.2.10 *Let $\varphi_0,\varphi_1\in\mathrm{PSH}(X,\theta)$. Define $\varphi_t=t\varphi_1+(1-t)\varphi_0$ for $t\in(0,1)$. Then*

$$\varphi_t\xrightarrow{d_S}\varphi_0$$

as $t\to 0+$.

Proof First note that for each $j = 0, \dots, n$,

$$\lim_{t\to 0+} \int_X \theta^j_{\varphi_t} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi_0} \wedge \theta^{n-j}_{V_\theta}.$$

So thanks to Corollary 6.2.7, it remains to argue that for all $j = 0, \dots, n$,

$$\lim_{t\to 0+} \int_X \theta^j_{\varphi_t \vee \varphi_0} \wedge \theta^{n-j}_{V_\theta} = \int_X \theta^j_{\varphi_0} \wedge \theta^{n-j}_{V_\theta}.$$

Observe that for $t \in (0,1)$, we have

$$\varphi_t \vee \varphi_0 = t(\varphi_1 \vee \varphi_0) + (1-t)\varphi_0,$$

so the desired inequality follows. □

We have the following sandwich criterion:

Corollary 6.2.11 *Let $(\varphi_i)_{i\in I}, (\psi_i)_{i\in I}, (\eta_i)_{i\in I}$ be three nets in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that*

(1) *$\psi_i \preceq_P \varphi_i \preceq_P \eta_i$ for each $i \in I$;*
(2) *$\eta_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \varphi$.*

Then $\varphi_i \xrightarrow{d_S} \varphi$.

Proof By Corollary 6.2.8, we may replace θ by $\theta + \omega$, where ω is a Kähler form on X. In particular, we may assume that $\varphi_i, \psi_i, \eta_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$. By Proposition 6.2.2, we may assume that $\varphi_i, \psi_i, \eta_i$ are model potentials for all $i \in I$ and hence $\varphi_i \le \psi_i \le \eta_i$ for all $i \in I$.

It follows from Theorem 2.4.4 that for each $k = 0, \dots, n$, we have

$$\int_X \theta^k_{\psi_i} \wedge \theta^{n-k}_{V_\theta} \le \int_X \theta^k_{\varphi_i} \wedge \theta^{n-k}_{V_\theta} \le \int_X \theta^k_{\eta_i} \wedge \theta^{n-k}_{V_\theta}$$

for all $i \in I$. By Theorem 6.2.1, the limits with respect to $i \in I$ of the both ends are $\int_X \theta^k_\varphi \wedge \theta^{n-k}_{V_\theta}$. It follows that

$$\lim_{i\in I} \int_X \theta^k_{\varphi_i} \wedge \theta^{n-k}_{V_\theta} = \int_X \theta^k_{\varphi} \wedge \theta^{n-k}_{V_\theta}. \tag{6.21}$$

By Corollary 6.2.7, it remains to prove that $\varphi_i \vee \varphi \xrightarrow{d_S} \varphi$. By Corollary 6.2.7 and Proposition 6.1.6, up to replacing ψ_i (resp. φ_i, η_i) by $\psi_i \vee \varphi$ (resp. $\varphi_i \vee \varphi$, $\eta_i \vee \varphi$), we may assume from the beginning that $\psi_i, \varphi_i, \eta_i \ge \varphi$. Now $\varphi_i \xrightarrow{d_S} \varphi$ by (6.21) and Lemma 6.2.2. □

Proposition 6.2.4 *Let $(\varphi_i)_{i\in I}$, $(\psi_i)_{i\in I}$ be nets in $\mathrm{PSH}(X,\theta)$ such that $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$ and $\psi_i \xrightarrow{d_S} \psi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi_i \preceq_P \psi_i$ for all $i \in I$. Then $\varphi \preceq_P \psi$.*

Proof It follows from Proposition 6.2.5 that

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

By Lemma 6.1.3, we have $\varphi_i \vee \psi_i \sim_P \psi_i$ for all $i \in I$. In particular, by Proposition 6.2.2,

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \psi.$$

By Proposition 6.2.2 again, $\varphi \vee \psi \sim_P \psi$ and hence $\varphi \leq_P \psi$ by Lemma 6.1.3. □

Proposition 6.2.5 *Let $(\varphi_i)_{i\in I}$ (resp. $(\psi_i)_{i\in I}$) be a net in* $\mathrm{PSH}(X,\theta)$ *such that $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$ (resp. $\varphi_i \xrightarrow{d_S} \psi \in \mathrm{PSH}(X,\theta)$). Then*

$$\varphi_i \vee \psi_i \xrightarrow{d_S} \varphi \vee \psi.$$

Proof Since d_S is a pseudometric, we may assume that both nets are actually sequences and $I = \mathbb{Z}_{>0}$. By Corollary 6.2.8, we may assume that the masses $\int_X \theta_\varphi^n > 0$, $\int_X \theta_\psi^n > 0$.

Using Proposition 6.2.3, we may assume that both sequences are monotone and lie in $\mathrm{PSH}(X,\theta)_{>0}$.

Thanks to Proposition 6.1.6, we may assume that the φ_j's, the ψ_j's, φ and ψ are all model. In particular, $(\varphi_j)_j$ (resp. $(\psi_j)_j$) converges to φ (rest. ψ) almost everywhere.

We handle three cases separately.

Step 1. Assume that both sequences are increasing.

In this case, we have $\varphi_j \vee \psi_j \nearrow \varphi \vee \psi$ almost everywhere. Therefore, $\varphi_j \vee \psi_j \xrightarrow{d_S} \varphi \vee \psi$ by Corollary 6.2.3.

Step 2. Assume that one sequence, say $(\varphi_j)_j$ is increasing while the other is decreasing. Then we have

$$\varphi_j \vee \psi \leq \varphi_j \vee \psi_j \leq \varphi \vee \psi_j.$$

Thanks to Corollary 6.2.11, it suffices to show that both sides converge to $\varphi \vee \psi$ with respect to d_S. So we reduce to the case where both sequences are decreasing.

Step 3. Assume that both sequences are decreasing.

In this case, due to Corollary 6.2.5, it suffices to show that

$$\lim_{j\to\infty} \int_X \theta^n_{\varphi_j \vee \psi_j} = \int_X \theta^n_{\varphi\vee\psi}. \tag{6.22}$$

The $\geq$ direction follows from Theorem 2.4.4, it remains to argue the $\leq$ direction.

Thanks to Lemma 2.4.2, we may find a sequence $(\epsilon_j)_j$ in $(0,1)$ with limit 0 and a sequences $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ such that

$$(1-\epsilon_j)\varphi_j + \epsilon_j \eta_j \leq \varphi, \quad \eta_j \leq \varphi_j.$$

It follows that for each $j \geq 1$, we have

$$(1-\epsilon_j)(\varphi_j \vee \psi_j) + \epsilon_j \eta_j \leq \varphi \vee \psi_j.$$

Therefore by Theorem 2.4.4,

$$(1-\epsilon_j)^n \int_X \theta^n_{\varphi_j \vee \psi_j} \leq \int_X \theta^n_{\varphi \vee \psi_j}.$$

Letting $j \to \infty$, we find that

$$\lim_{j\to\infty} \int_X \theta^n_{\varphi_j \vee \psi_j} \leq \lim_{j\to\infty} \int_X \theta^n_{\varphi \vee \psi_j}.$$

Therefore, in order to prove (6.22), we may assume that one of the sequences is constant, let us say $\psi_j = \psi$ for all j. Repeating the same argument as before and constructing $(\epsilon_j)_j$, $(\eta_j)_j$ as above, we get

$$(1-\epsilon_j)^n \int_X \theta^n_{\varphi_j \vee \psi} \leq \int_X \theta^n_{\varphi \vee \psi}.$$

Letting $j \to \infty$, we conclude (6.22). □

Theorem 6.2.2 *Let θ_1, θ_2 be smooth real closed $(1,1)$-forms on X representing big cohomology classes. Suppose that $(\varphi_i)_{i\in I}$ (resp. $(\psi_i)_{i\in I}$) be a net in $\mathrm{PSH}(X,\theta_1)$ (resp. $\mathrm{PSH}(X,\theta_2)$) and $\varphi \in \mathrm{PSH}(X,\theta_1)$ (resp. $\psi \in \mathrm{PSH}(X,\theta_2)$). Consider the following three conditions:*

(1) $\varphi_i \xrightarrow{d_S} \varphi$;

(2) $\psi_i \xrightarrow{d_S} \psi$;

(3) $\varphi_i + \psi_i \xrightarrow{d_S} \varphi + \psi$.

Then any two of these conditions imply the third.

Proof By Corollary 6.2.8, we may assume that θ_1, θ_2 are both Kähler forms. We denote them by ω_1, ω_2 instead. Let $\omega = \omega_1 + \omega_2$.

(1)+(2) $\implies$ (3). It suffices to show that for each $r = 0, \dots, n$,

$$2\int_X \omega^r_{(\varphi_j+\psi_j)\vee(\varphi+\psi)} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi_j+\psi_j} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi+\psi} \wedge \omega^{n-r} \to 0.$$

Observe that for each $j \in I$,

$$(\varphi_j + \psi_j) \vee (\varphi + \psi) \leq \varphi_j \vee \varphi + \psi_j \vee \psi.$$

Thus, it suffices to show that

$$2\int_X \omega^r_{\varphi_j\vee\varphi+\psi_j\vee\psi} \wedge \omega - \int_X \omega^r_{\varphi_j+\psi_j} \wedge \omega^{n-r} - \int_X \omega^r_{\varphi+\psi} \wedge \omega^{n-r} \to 0.$$

The left-hand side is a linear combination of

$$2\int_X \omega_{1,\varphi_j\vee\varphi}^a \wedge \omega_{2,\psi_j\vee\psi}^{r-a} \wedge \omega^{n-r} - \int_X \omega_{1,\varphi_j}^a \wedge \omega_{2,\psi_j}^{r-a} \wedge \omega^{n-r} - \int_X \omega_{1,\varphi}^a \wedge \omega_{2,\psi}^{r-a} \wedge \omega^{n-r}$$

with $a = 0, \dots, r$. Observe that $\varphi_j \vee \varphi \xrightarrow{d_S} \varphi$ and $\psi_j \vee \psi \xrightarrow{d_S} \psi$ by Corollary 6.2.2, each term tends to 0 by Theorem 6.2.1.

(1)+(3) $\implies$ (2). For each $C \geq 1$, from the direction we already proved,

$$C\varphi_i + \psi_i \xrightarrow{d_S} C\varphi + \psi.$$

By Theorem 6.2.1, for each $j = 0, \dots, n$,

$$\begin{aligned}&\lim_{i\in I}\int_X (C\omega_1 + \omega_2 + \mathrm{dd^c}(C\varphi_i + \psi_i))^j \wedge \omega_2^{n-j}\\ =&\int_X (C\omega_1 + \omega_2 + \mathrm{dd^c}(C\varphi + \psi))^j \wedge \omega_2^{n-j}.\end{aligned}$$

It follows that

$$\lim_{i\in I}\int_X \omega_{2,\psi_i}^j \wedge \omega_2^{n-j} = \int_X \omega_{2,\psi}^j \wedge \omega_2^{n-j}. \tag{6.23}$$

Therefore, (2) follows if $\psi_i \geq \psi$ for each i by Lemma 6.2.2.

Next we prove the general case. By the direction that we already proved, we know that $\varphi_i + \psi \xrightarrow{d_S} \varphi + \psi$. By Proposition 6.2.5, we have that

$$\varphi_i + \psi_i \vee \psi \xrightarrow{d_S} \varphi + \psi.$$

It follows from the special case above that $\psi_i \vee \psi \xrightarrow{d_S} \psi$. It follows from (6.23) and Corollary 6.2.7 that (2) holds.

(2)+(3) $\implies$ (1). This is similar.

Theorem 6.2.3 *The map*

$$P_\theta[\bullet]_\mathcal{I} : \mathrm{PSH}(X,\theta)_{>0} \to \mathrm{PSH}(X,\theta)_{>0}$$

is continuous with respect to d_S.

Proof Let $(\varphi_i)_{i\in\mathbb{Z}_{>0}}$ be a sequence in $\mathrm{PSH}(X,\theta)_{>0}$ such that $\varphi_i \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)_{>0}$. We want to show that

$$P_\theta[\varphi_i]_\mathcal{I} \xrightarrow{d_S} P_\theta[\varphi]_\mathcal{I}. \tag{6.24}$$

We may assume that the φ_i's and φ are all model potentials by Proposition 6.2.2.

By Proposition 6.2.3 and Corollary 6.2.11, we may assume that $(\varphi_i)_i$ is either increasing or decreasing. In the increasing case, we apply Proposition 3.2.14 and Corollary 6.2.3, while in the decreasing case, we apply Proposition 3.2.12, Proposition 3.1.10 and Corollary 6.2.5. □

We record the following result for later use.

Lemma 6.2.6 *Fix a Kähler form ω on X. As $\epsilon \to 0$[2], we have*

$$V_{\theta+\epsilon\omega} \xrightarrow{d_S} V_\theta. \tag{6.25}$$

Proof There are two assertions to prove, as detailed in the two steps.

Step 1. We first handle the case where $\epsilon \to 0+$.

In this case (6.25) means

$$V_{\theta+\epsilon\omega} \xrightarrow{d_{S,\omega}} V_\theta$$

as $\epsilon \to 0+$. So thanks to Corollary 6.2.5 and Proposition 3.1.10, it suffices to prove the following:

$$\inf_{\epsilon>0} P_{\theta+\omega}\left[V_{\theta+\epsilon\omega}\right] = P_{\theta+\omega}\left[V_\theta\right]. \tag{6.26}$$

First observe that

$$V_\theta = \inf_{\epsilon>0} V_{\theta+\epsilon\omega}.$$

In fact, the $\leq$ direction is trivial. As for the reverse inequality, it suffices to observe that the right-hand side lies in $\mathrm{PSH}(X,\theta)$. Therefore, due to Proposition 3.1.10,

$$\lim_{\epsilon\to 0+} \int_X (\theta + \epsilon\omega + \mathrm{dd^c} V_{\theta+\epsilon\omega})^n = \int_X \theta_{V_\theta}^n.$$

Therefore, for all $\epsilon > 0$ small enough, we can find $\eta_\epsilon \in \mathrm{PSH}(X, \theta+\epsilon\omega)$ and $a_\epsilon > 0$ decreasing to 0 so that

$$(1-a_\epsilon)V_{\theta+\epsilon\omega} + a_\epsilon \eta_\epsilon \leq V_\theta.$$

Therefore, thanks to Proposition 3.1.8,

$$(1-a_\epsilon)P_{\theta+\omega}\left[V_{\theta+\epsilon\omega}\right] + a_\epsilon P_{\theta+\omega}\left[\eta_\epsilon\right] \leq P_{\theta+\omega}\left[V_\theta\right].$$

Letting $\epsilon \to 0$, we conclude (6.26).

Step 2. We then handle the case where $\epsilon \to 0-$.

In this case, (6.25) simply means

$$V_{\theta-\epsilon\omega} \xrightarrow{d_{S,\theta}} V_\theta$$

as $\epsilon \to 0+$. But this follows from Corollary 6.2.3 if we can prove

$$\sup_{\epsilon>0}{}^{*} V_{\theta-\epsilon\omega} = V_\theta, \tag{6.27}$$

where we understand that ϵ is small enough so that $\theta - \epsilon\omega$ represents a big cohomology class. Since $\{\theta\}$ is big, we can find $\varphi \in \mathrm{PSH}(X,\theta)$ so that

[2] This is not a typo, we mean $\epsilon \to 0$ from two sides.

$$\varphi \leq 0, \quad \theta_\varphi \geq \delta\omega$$

for some $\delta > 0$. For a small $\epsilon > 0$, we then have

$$\theta + \mathrm{dd^c}\left(\left(1 - \frac{\epsilon}{\delta}\right) V_\theta + \frac{\epsilon}{\delta}\varphi\right) \geq \epsilon\omega.$$

Therefore,

$$\left(1 - \frac{\epsilon}{\delta}\right) V_\theta + \frac{\epsilon}{\delta}\varphi \leq V_{\theta-\epsilon\omega}.$$

Letting $\epsilon \longrightarrow 0+$, we then find

$$V_\theta \leq \sup_{\epsilon>0}{}^* V_{\theta-\epsilon\omega}.$$

The reverse inequality is trivial, hence (6.27) is established. □

6.2.3 Continuity of invariants

In this section, we prove the continuity of a few invariants of the singularities with respect to d_S.

Theorem 6.2.4 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{PSH}(X,\theta)$. Then for any prime divisor E over X, we have*

$$\lim_{j\in I} \nu(\varphi_j, E) = \nu(\varphi, E). \tag{6.28}$$

Proof First observe that since d_S is a pseudometric, it suffices to prove (6.28) when $I = \mathbb{Z}_{>0}$ as partially ordered sets.

By Corollary 6.2.8, we may assume that the masses of φ_j and of φ are bounded from below by a positive constant.

By Theorem 6.2.3, we may assume that φ_i and φ are both $\mathcal{I}$-model and hence model. When proving (6.28), we are free to pass to subsequences.

By Proposition 6.2.3, we may assume that the sequence (φ_i) is either increasing or decreasing. In the increasing case, there is nothing to prove. In the decreasing case, (6.28) follows from Proposition 3.1.10. □

Theorem 6.2.5 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Assume that $\varphi_j \xrightarrow{d_S} \varphi$, then*

$$\operatorname{vol}\theta_{\varphi_j} \longrightarrow \operatorname{vol}\theta_\varphi, \quad \int_X \theta_{\varphi_j}^n \longrightarrow \int_X \theta_\varphi^n. \tag{6.29}$$

Recall the volume is defined in Definition 3.2.3. In fact, we do not have to assume the positivity of the mass of φ. The proof of the general statement is slightly more involved. See Corollary 7.3.1 below.

Proof The latter part of (6.29) is just a special case of Theorem 6.2.1. It remains to prove the former part.

We may therefore assume that $\int_X \theta^n_{\varphi_j} > 0$ for all $j \in I$. Then by Theorem 6.2.3, we have

$$P_\theta[\varphi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}.$$

Therefore, the first part of (6.29) follows again from Theorem 6.2.1. □

Next we show that d_S-convergent sequences have a sort of quasi-equisingular property (c.f. (1.15)).

Theorem 6.2.6 *Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ ($j \in \mathbb{Z}_{>0}$). Assume that $\varphi_j \xrightarrow{d_S} \varphi$. Then for each $\lambda' > \lambda > 0$, there is $j_0 > 0$ so that for $j \geq j_0$,*

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi). \tag{6.30}$$

Proof Fix $\lambda' > \lambda > 0$, we want to find $j_0 > 0$ so that for $j \geq j_0$, (6.30) holds.

Step 1. We first assume that φ has analytic singularities.

Let $\pi\colon Y \to X$ be a log resolution of φ and let $E_1, \dots, E_N$ be all prime divisors in the polar locus of φ on Y. Recall that by Theorem 1.4.3, a local holomorphic function f lies in the right-hand side of (6.30) if and only if

$$\mathrm{ord}_{E_i}(f) > \lambda\nu(\varphi, E_i) - \frac{1}{2}A_X(E_i) \tag{6.31}$$

whenever they make sense. Here A_X denotes the log discrepancy. Similarly, f lies in the left-hand side of (6.30) implies that there is $\epsilon > 0$ so that

$$\mathrm{ord}_{E_i}(f) \geq (1+\epsilon)\lambda'\nu(\varphi_j, E_i) - \frac{1}{2}A_X(E_i).$$

As Lelong numbers are continuous with respect to d_S by Theorem 6.2.4, we can find $j_0 > 0$ so that when $j \geq j_0$, $\lambda'\nu(\varphi_j, E_i) \geq \lambda\nu(\varphi, E_i)$ for all i. In particular, (6.31) follows.

Step 2. We handle the general case.

By Corollary 6.2.8, we are free to increase θ and assume that θ_φ is a Kähler current.

Take a quasi-equisingular approximation $(\psi_k)_k$ of φ in $\mathrm{PSH}(X,\theta)$. The existence is guaranteed by Theorem 1.6.2. Take $\lambda'' \in (\lambda, \lambda')$, then by definition, we can find $k > 0$ so that

$$\mathcal{I}(\lambda''\psi_k) \subseteq \mathcal{I}(\lambda\varphi).$$

Observe that $\varphi_j \vee \psi_k \xrightarrow{d_S} \psi_k$ as $j \to \infty$ by Proposition 6.2.5. By Step 1, we can find $j_0 > 0$ so that for $j \geq j_0$,

$$\mathcal{I}(\lambda'(\varphi_j \vee \psi_k)) \subseteq \mathcal{I}(\lambda''\psi_k).$$

It follows that for $j \geq j_0$,

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

Chapter 7
$\mathcal{I}$-good singularities

Le but de cette thèse est de munir son auteur du titre de Docteur.[a]
— Adrien Douady[b]*, at the beginning of his thesis*

[a] Similarly, the purpose of the current book is to make my complaints about France in the acknowledgments published.

[b] Adrien Douady (1935–2006) was a French mathematician known for his pioneering work in complex dynamics and fractal geometry. Along with John H. Hubbard, he proved important results about the Mandelbrot set and developed renormalization theory for polynomial mappings. He discovered the *Douady Rabbit*, a famous fractal Julia set.

Douady studied at École Normale Supérieure (the place where I began to hate France, thanks to Claude Viterbo) and taught at several French universities. He was also a member of the Bourbaki group.

Tragically, he died in a swimming accident in 2006.

In this chapter, we study the key notion in the whole theory: The $\mathcal{I}$-good singularities. We will give several useful characterizations of $\mathcal{I}$-good singularities. The key result is the asymptotic Riemann–Roch formula for Hermitian pseudo-effective line bundles Theorem 7.4.1.

7.1 The notion of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold of dimension n.

Theorem 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class, and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) *There exists a sequence $(\varphi_j)_{j>0}$ in $\mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$;*

(2) *we have*
$$\int_X \theta_\varphi^n = \operatorname{vol}\theta_\varphi; \tag{7.1}$$

(3) *we have*
$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}. \tag{7.2}$$

In (1), we could in addition require that each θ_{φ_j} is a Kähler current.

Moreover, if θ_φ is a Kähler current, the sequence in (1) can be taken as any quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

Since $(\mathrm{PSH}(X,\theta), d_S)$ is a pseudometric space, in (1) we could also replace the word *sequence* by *net*.

Recall that according to Corollary 3.2.1 and Proposition 3.2.9, one direction of (7.1) and (7.2) always holds:

$$\int_X \theta_\varphi^n \leq \operatorname{vol}\theta_\varphi, \quad P_\theta[\varphi] \leq P_\theta[\varphi]_{\mathcal{I}}.$$

Proof (1) $\implies$ (2). By Theorem 6.2.1, we have

$$\lim_{j\to\infty}\int_X \theta_{\varphi_j}^n = \int_X \theta_\varphi^n > 0.$$

We may therefore assume that $\int_X \theta_{\varphi_j}^n > 0$ for all $j \geq 1$. It follows from Proposition 3.2.10 that

$$\int_X \theta_{\varphi_j}^n = \operatorname{vol}\theta_{\varphi_j}$$

for any $j \geq 1$. Using Theorem 6.2.5, we conclude (7.1).

(2) $\iff$ (3). This follows from Theorem 3.1.2.

(3) $\implies$ (1). Note that the condition in (1) characterizes the closure of analytic singularities in $\mathrm{PSH}(X,\theta)$.

Step 1. We first assume that θ_φ is a Kähler current. We will prove the following more general result in this case: Without assuming (3), $P_\theta[\varphi]_{\mathcal{I}}$ always lies in the closure of analytic singularities.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We will show that $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$. Let

$$\psi = \inf_{j\in\mathbb{Z}_{>0}} P_\theta[\varphi_j].$$

We know that $\varphi_j \xrightarrow{d_S} \psi$ by Proposition 6.2.2, Proposition 3.1.10 and Corollary 6.2.5.

Moreover, observe that ψ is $\mathcal{I}$-model by Proposition 3.2.12 and Proposition 3.2.10. So it suffices to show that $\varphi \sim_{\mathcal{I}} \psi$.

First observe that since for all $j > 0$, $\varphi \leq \varphi_j$, we have

$$\varphi - \sup_X \varphi \leq P_\theta[\varphi_j].$$

Therefore,

$$\varphi - \sup_X \varphi \leq \psi.$$

Conversely, it remains to argue that $\psi \preceq_{\mathcal{I}} \varphi$. For this purpose, take $\lambda > 0$, we need to show that

$$\mathcal{I}(\lambda\psi) \subseteq \mathcal{I}(\lambda\varphi).$$

By the strong openness Theorem 1.4.4, we may take $\lambda' > \lambda$ such that $\mathcal{I}(\lambda\psi) = \mathcal{I}(\lambda'\psi)$, then it follows from the definition of the quasi-equisingular approximation that

$$\mathcal{I}(\lambda'\psi) \subseteq \mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi)$$

for large enough j. Our assertion follows.

It follows from the proof that we may take φ_j so that θ_{φ_j} is a Kähler current for all $j \geq 1$.

Step 2. We handle the general case.

Assume (3) holds. By Lemma 2.4.3, we can find $\psi \in \mathrm{PSH}(X, \theta)$ so that θ_ψ is a Kähler current and $\psi \leq \varphi$. We let

$$\psi_j = (1 - j^{-1})\varphi + j^{-1}\psi$$

for each $j \in \mathbb{Z}_{>1}$. Then $(\psi_j)_j$ is an increasing sequence converging almost everywhere to φ. Then

$$P_\theta[\psi_j]_{\mathcal{I}} \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} = P_\theta[\varphi]$$

by Proposition 3.2.14, Corollary 6.2.3. From Step 1, we know that each $P_\theta[\psi_j]_{\mathcal{I}}$ lies in the closure of analytic singularities, hence so is $P_\theta[\varphi] \sim_P \varphi$. Therefore, (1) follows. □

Definition 7.1.1 We say a potential $\varphi \in \mathrm{QPSH}(X)$ is *$\mathcal{I}$-good* if for some smooth closed real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, we have

$$P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}. \tag{7.3}$$

Remark 7.1.1 In view of Theorem 7.1.1 and Corollary 3.2.1, the failure of $\mathcal{I}$-goodness of a given $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$ can be characterized using the difference between the volume and the mass. We therefore introduce

$$\mathrm{Macron}(\theta_\varphi) \coloneqq \mathrm{vol}\,\theta_\varphi - \int_X \theta_\varphi^n.$$

As we mentioned in the introduction, all potentials in practice are expected to be $\mathcal{I}$-good. The evil guy Macron is bound to be eliminated[1].

An immediate question is to verify that Definition 7.1.1 is in dependent of the choice of θ.

Lemma 7.1.1 *Let $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$ for some smooth closed real $(1,1)$-form θ on X. Take a Kähler form ω on X. Then the following are equivalent:*

(1) $P_\theta[\varphi] = P_\theta[\varphi]_{\mathcal{I}}$;
(2) $P_{\theta+\omega}[\varphi] = P_{\theta+\omega}[\varphi]_{\mathcal{I}}$.

Proof (1) $\Longrightarrow$ (2). By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X, \theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_{S,\theta}} \varphi$. By Corollary 6.2.8, we have $\varphi_j \xrightarrow{d_{S,\theta+\omega}} \varphi$. Therefore, by Theorem 7.1.1 again, (2) holds.

[1] I learned the following folklore claim at the math department of Chalmers university: If you hate someone, you should name an extremely trivial mathematical object after him/her.

(2) $\implies$ (1). Suppose that (1) fails, so that

$$\int_X (\theta + \mathrm{dd^c}\varphi)^n < \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

It follows that

$$\begin{aligned}\int_X (\theta + \omega + \mathrm{dd^c}\varphi)^n &= \sum_{i=0}^n \binom{n}{i} \int_X \theta_\varphi^i \wedge \omega^{n-i} \\ &< \sum_{i=0}^n \binom{n}{i} \int_X \theta^i_{P_\theta[\varphi]_{\mathcal{I}}} \wedge \omega^{n-i} \\ &= \int_X (\theta + \omega + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n \\ &\leq \int_X (\theta + \omega + \mathrm{dd^c} P_{\theta+\omega}[\varphi]_{\mathcal{I}})^n.\end{aligned}$$

So (2) fails as well. □

Corollary 7.1.1 *Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class. Let $(\varphi_j)_{j\in I}$ be a net of $\mathcal{I}$-good potentials in* $\mathrm{PSH}(X,\theta)$ *such that $\varphi_j \xrightarrow{d_S} \varphi$. Then φ is $\mathcal{I}$-good.*

Note that we do not need to assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proof By Corollary 6.2.8, we may assume that $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)_{>0}$ for all $j \in I$. It follows from Theorem 7.1.1 that

$$\int_X \theta^n_{\varphi_j} = \mathrm{vol}\,\theta_{\varphi_j}$$

for all $j \in I$. Taking limit with respect to j with the help of Theorem 6.2.5, we conclude that

$$\int_X \theta^n_\varphi = \mathrm{vol}\,\theta_\varphi.$$

Therefore, by Theorem 7.1.1 again, we find that φ is $\mathcal{I}$-good. □

Example 7.1.1 Assume that $\varphi \in \mathrm{QPSH}(X)$ has analytic singularities. Then φ is $\mathcal{I}$-good. This is proved in Proposition 3.2.10.

In particular, the potential in Example 1.8.2 is $\mathcal{I}$-good.

Example 7.1.2 Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class, and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$[2] is an $\mathcal{I}$-model potential for some closed real smooth $(1,1)$-form θ on X. Then φ is $\mathcal{I}$-good.

[2] I do not know whether the same holds when φ has vanishing mass.

Example 7.1.3 Let θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class, and $\varphi \in \mathcal{E}(X,\theta)$. Then φ is $\mathcal{I}$-good. In fact, since $P_\theta[\varphi] = V_\theta$, we deduce that $P_\theta[\varphi]_{\mathcal{I}} = V_\theta$ as well.

In particular, the potential in Example 3.1.1 is $\mathcal{I}$-good.

A further class of examples of $\mathcal{I}$-good singularities will be given in Example 7.4.1 below.

On the other hand, there do exist non-$\mathcal{I}$-good potentials.

Example 7.1.4 The potential in Example 6.1.3 is not $\mathcal{I}$-good. In fact, since φ has no non-vanishing Lelong numbers, we know that $\varphi \sim_{\mathcal{I}} 0$, hence

$$P_{2\omega}[\varphi] = 0.$$

On the other hand,

$$\int_X (2\omega + \mathrm{dd^c}\varphi) = \int_X \omega < \int_X (2\omega),$$

where $2\omega + \mathrm{dd^c}\varphi$ is understood in the non-pluripolar sense.

Quasi-equisingular approximations and d_S-convergent sequences are related in the following manner:

Corollary 7.1.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ and $(\epsilon_j)_j$ be a decreasing sequence in $\mathbb{R}_{\geq 0}$ with limit 0. Fix a Kähler form ω on X. Consider a decreasing sequence $(\varphi_j)_{j>0}$ with $\varphi_j \in \mathrm{PSH}(X, \theta + \epsilon_j\omega)$ being a potential with analytic singularities. Assume that $\varphi = \inf_j \varphi_j$. Then the following are equivalent:*

(1) *$\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$* [3]*, and*
(2) *$(\varphi_j)_j$ is a quasi-equisingular approximation of φ.*

Proof By Corollary 6.2.8 and Example 7.1.2, we may replace θ by $\theta + C\omega$ for some large constant $C > 0$ and assume that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta - \omega)$ for all $j \geq 1$.

(2) $\implies$ (1). This is already proved in the proof of Theorem 7.1.1.

(1) $\implies$ (2). This follows from Theorem 6.2.6. □

7.2 Properties of $\mathcal{I}$-good singularities

Let X be a connected compact Kähler manifold of dimension n.

We show that $\mathcal{I}$-goodness is preserved by a number of natural operations.

Proposition 7.2.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$ be $\mathcal{I}$-good and $\lambda > 0$. Then the following potentials are all $\mathcal{I}$-good:*

(1) *$\varphi + \psi$;*

---

[3] Just to be sure, this means $\varphi_j \xrightarrow{d_{S,\theta+\epsilon\omega}} P_\theta[\varphi]_{\mathcal{I}}$ for any $\epsilon > 0$. The choice of ϵ is irrelevant due to Corollary 6.2.8.

(2) $\varphi \vee \psi$;
(3) $\lambda\varphi$.

Proof Take a closed real smooth $(1,1)$-form θ on X such that $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. It follows from Theorem 7.1.1 that there are sequences $(\varphi_j)_j, (\psi_j)_j$ in $\mathrm{PSH}(X,\theta)$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$ and $\psi_j \xrightarrow{d_S} \psi$.

By Theorem 6.2.2, Proposition 6.2.5, we have

$$\varphi_j + \psi_j \xrightarrow{d_S} \varphi + \psi, \quad \varphi_j \vee \psi_j \xrightarrow{d_S} \varphi \vee \psi.$$

On the other hand, it is clear that

$$\lambda\varphi_j \xrightarrow{d_S} \lambda\varphi.$$

Therefore, our assertions follow from Theorem 7.1.1. □

Example 7.2.1 Let L be a pseudo-effective line bundle on X. Elementary metrics on L are defined in Definition 6.1.3. Let h be an elementary metric on L, then $\mathrm{dd^c} h$ is $\mathcal{I}$-good.

This is a direct consequence of Proposition 7.2.1 and Example 7.1.1.

Proposition 7.2.2 *Let $(\varphi_j)_{j\in I}$ be a non-empty family of $\mathcal{I}$-good potentials in* $\mathrm{PSH}(X,\theta)$ *for some closed real smooth* $(1,1)$*-form θ on X. Then* $\sup_{j\in I}{}^*\varphi_j$ *is $\mathcal{I}$-good.*

Proof After adding a Kähler form to θ, we may assume that $\varphi_j \in \mathrm{PSH}(X,\theta)_{>0}$ for all $j \in I$.

When I is finite, this result follows from Proposition 7.2.1. When I is infinite, we may assume that $I = \mathbb{Z}_{>0}$ by Proposition 1.2.2. By Proposition 7.2.1, we may assume that the sequence $(\varphi_j)_j$ is increasing. In this case, as shown in Corollary 6.2.3,

$$\varphi_j \xrightarrow{d_S} \sup_{i\in\mathbb{Z}_{>0}}{}^*\varphi_i.$$

Therefore, $\sup_{i\in\mathbb{Z}_{>0}}{}^*\varphi_i$ is $\mathcal{I}$-good by Corollary 7.1.1. □

7.3 Mixed volumes

We first extend the notion of volume in Definition 3.2.3 to the mixed case. Let $\theta_1, \dots, \theta_n$ be smooth closed real $(1,1)$-forms on X representing pseudo-effective classes.

Definition 7.3.1 Let $\varphi_i \in \mathrm{PSH}(X,\theta_i)$ for $i = 1, \dots, n$. Write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for each $i = 1, \dots, n$. We define the *mixed volume* $\mathrm{vol}(T_1, \dots, T_n)$ as follows:

(1) Suppose that $\operatorname{vol} T_i > 0$ for all $i = 1, \ldots, n$, then we let

$$\operatorname{vol}(T_1, \ldots, T_n) = \int_X \left(\theta_1 + \mathrm{dd^c} P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c} P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right); \tag{7.4}$$

(2) in general, take a Kähler form ω on X, we define

$$\operatorname{vol}(T_1, \ldots, T_n) = \lim_{\epsilon \to 0+} \operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega). \tag{7.5}$$

Note that $\operatorname{vol}(T_1, \ldots, T_n)$ does not depend on the choice of ω.

We first make a few observations: When $\operatorname{vol} T_i > 0$ for each $i = 1, \ldots, n$, the definition (7.4) does not depend on how we represent T_i as $T_i = \theta_i + \mathrm{dd^c}\varphi_i$, this is a consequence of Theorem 2.4.4 and Proposition 3.2.4.

Next, when $\operatorname{vol} T_i > 0$ for each i, the definition (7.5) coincides with (7.4). In fact, in this case, for each i and each $\epsilon > 0$, we have

$$P_{\theta_i}[\varphi_i]_{\mathcal{I}} \sim_P P_{\theta_i + \epsilon\omega}\left[P_\theta[\varphi_i]_{\mathcal{I}}\right]_{\mathcal{I}} = P_{\theta_i + \epsilon\omega}[\varphi_i]_{\mathcal{I}}$$

as a consequence of Example 7.1.2. Hence using Proposition 6.1.4,

$$\begin{aligned}
&\lim_{\epsilon \to 0+} \operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega)\\
=&\lim_{\epsilon \to 0+} \int_X \left(\theta_1 + \epsilon\omega + P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right) \wedge \cdots \wedge \left(\theta_n + \epsilon\omega + P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right)\\
=&\operatorname{vol}(T_1, \ldots, T_n).
\end{aligned}$$

Finally, for any closed positive $(1,1)$-current T on X, we have

$$\operatorname{vol}(T, \ldots, T) = \operatorname{vol} T. \tag{7.6}$$

Write $T = \theta_\varphi$. In more concrete terms, we need to show that

$$\lim_{\epsilon \to 0+} \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

We may replace φ by $P_\theta[\varphi]_{\mathcal{I}}$ and assume that φ is $\mathcal{I}$-model in $\mathrm{PSH}(X, \theta)$. Then we claim that

$$\varphi = \inf_{\epsilon > 0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}.$$

From this, our assertion follows from Proposition 3.1.10.

The $\leq$ direction is clear. For the converse, it suffices to show that for each prime divisor E over X, we have

$$\nu(\varphi, E) \leq \nu\left(\inf_{\epsilon > 0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right).$$

We simply compute

$$\nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) \geq \sup_{\epsilon>0} \nu\left(P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) = \nu(\varphi, E).$$

Proposition 7.3.1

(1) *The mixed volume is symmetric in its n-variables.*
(2) *Consider closed positive* $(1,1)$*-currents* $T_1, \ldots, T_n, T_1'$ *on X, then*

$$\mathrm{vol}(T_1 + T_1', T_2, \ldots, T_n) = \mathrm{vol}(T_1, T_2, \ldots, T_n) + \mathrm{vol}(T_1', T_2, \ldots, T_n). \tag{7.7}$$

(3) *Consider closed positive* $(1,1)$*-currents* $T_1, \ldots, T_n$ *on X and* $\lambda \geq 0$*, then*

$$\mathrm{vol}(\lambda T_1, T_2, \ldots, T_n) = \lambda \, \mathrm{vol}(T_1, T_2, \ldots, T_n).$$

(4) *Suppose that* $T_1, \ldots, T_n, S_1, \ldots, S_n$ *are closed positive* $(1,1)$*-currents on X such that* $T_i \preceq_{\mathcal{I}} S_i$ *and* $\{T_i\} = \{S_i\}$ *for each* $i = 1, \ldots, n$*. Then*

$$\mathrm{vol}(T_1, \ldots, T_n) \leq \mathrm{vol}(S_1, \ldots, S_n). \tag{7.8}$$

(5) *Suppose that* $T_1, \ldots, T_n$ *are closed positive* $(1,1)$*-currents on X, then*

$$\mathrm{vol}\left(T_1, \ldots, T_n\right) = \mathrm{vol}\left(\mathrm{Reg}\, T_1, \ldots, \mathrm{Reg}\, T_n\right). \tag{7.9}$$

The notation Reg is defined in (1.19).

Proof (1) This is obvious.

(2) By definition of the mixed volume, we may assume that the relevant currents $T_1, \ldots, T_n, T_1'$ are all Kähler currents. We write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ as before for each i and $T_1' = \theta_1' + \mathrm{dd^c}\varphi_1'$. Then thanks to Proposition 7.2.1,

$$P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_1'}[\varphi_1']_{\mathcal{I}} \sim_P P_{\theta_1+\theta_1'}\left[P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_1'}[\varphi_1']_{\mathcal{I}}\right]_{\mathcal{I}} = P_{\theta_1+\theta_1'}\left[\varphi_1 + \varphi_1'\right]_{\mathcal{I}}$$

Thus by Proposition 6.1.4, we have

$$\begin{aligned}
&\mathrm{vol}(T_1 + T_1', T_2, \ldots, T_n) \\
={}& \int_X \left(\theta_1 + \theta_1' + \mathrm{dd^c} P_{\theta_1+\theta_1'}[\varphi_1 + \varphi_1']_{\mathcal{I}}\right) \wedge \left(\theta_2 + \mathrm{dd^c} P_{\theta_2}[\varphi_2]_{\mathcal{I}}\right) \wedge \cdots \\
&\wedge \left(\theta_n + \mathrm{dd^c} P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right) \\
={}& \int_X \left(\theta_1 + \theta_1' + \mathrm{dd^c} P_{\theta_1}[\varphi_1]_{\mathcal{I}} + \mathrm{dd^c} P_{\theta_1'}[\varphi_1']_{\mathcal{I}}\right) \wedge \left(\theta_2 + \mathrm{dd^c} P_{\theta_2}[\varphi_2]_{\mathcal{I}}\right) \wedge \cdots \\
&\wedge \left(\theta_n + \mathrm{dd^c} P_{\theta_n}[\varphi_n]_{\mathcal{I}}\right) \\
={}& \mathrm{vol}(T_1, T_2, \ldots, T_n) + \mathrm{vol}(T_1', T_2, \ldots, T_n).
\end{aligned}$$

(3) This is obvious.

(4) Thanks to the definition of the mixed volume, we may assume that $T_1, \ldots, T_n$ and $S_1, \ldots, S_n$ are all Kähler currents. In this case, our assertion follows from Proposition 6.1.4.

(5) Using (2) and (3), it suffices to establish the following: Suppose that

$$T = \sum_i c_i [E_i] \tag{7.10}$$

is a closed positive $(1,1)$-current on X, where $\{E_i\}$ is a countable collection of prime divisors on X and $c_i > 0$. Then

$$\operatorname{vol}(T, T_2, \dots, T_n) = 0 \tag{7.11}$$

for any closed positive $(1,1)$-currents $T_2, \dots, T_n$ on X.

Step 1. We first assume that (7.10) is a finite sum. Fix a Kähler form ω on X.

In this case, thanks to (2) and (3) again, we may assume that $T = [E]$ for some prime divisor E on X. Write $E = \theta_\varphi$, then φ has analytic singularities thanks to Proposition 1.8.1. Therefore, $P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}} \sim \varphi$ for any $\epsilon > 0$ due to Proposition 3.2.10. Therefore, writing $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for $i = 2, \dots, n$, and take $C > 0$ so that $C\omega + \theta_i$ are Kähler forms for each $i = 2, \dots, n$, then we have

$$\begin{aligned}
&\operatorname{vol}(T, T_2, \dots, T_n)\\
=&\lim_{\epsilon\to 0+} \int_X ([E] + \epsilon\omega) \wedge \left(\theta_2 + \epsilon\omega + \mathrm{dd^c} P_{\theta_2+\epsilon\omega}[\varphi_2]_{\mathcal{I}}\right) \wedge \cdots\\
&\wedge \left(\theta_n + \epsilon\omega + \mathrm{dd^c} P_{\theta_n+\epsilon\omega}[\varphi_n]_{\mathcal{I}}\right)\\
=&\lim_{\epsilon\to 0+} \epsilon \int_X \omega \wedge \left(\theta_2 + \epsilon\omega + \mathrm{dd^c} P_{\theta_2+\epsilon\omega}[\varphi_2]_{\mathcal{I}}\right) \wedge \cdots\\
&\wedge \left(\theta_n + \epsilon\omega + \mathrm{dd^c} P_{\theta_n+\epsilon\omega}[\varphi_n]_{\mathcal{I}}\right)\\
\leq&\lim_{\epsilon\to 0+} \epsilon \int_X \omega \wedge (\theta_2 + C\omega) \wedge \cdots \wedge (\theta_n + C\omega)\\
=&0.
\end{aligned}$$

Step 2. We prove the case where $\{E_i\}$ is infinite. We may assume that i runs over $\mathbb{Z}_{>0}$.

Write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for $i = 2, \dots, n$ as before. Fix a ω on X so that $\theta_i + \omega$ is a Kähler form for each $i = 2, \dots, n$. Fix $\epsilon > 0$.

We can find $N_0 > 0$ so that for any $N \geq N_0$, the class of

$$\epsilon\omega + \sum_{i=N+1}^{\infty} c_i [E_i]$$

is Kähler. Take a Kähler form ω_N in this class. Then the currents

$$\sum_{i=1}^{N} c_i [E_i] + \omega_N, \quad \sum_{i=1}^{\infty} c_i [E_i] + \epsilon\omega$$

all lie in the same cohomology class.

We claim that

$$\sum_{i=1}^{N} c_i[E_i] + \omega_N \xrightarrow{d_S} \sum_{i=1}^{\infty} c_i[E_i] + \epsilon\omega. \tag{7.12}$$

In fact, it suffices to show the convergence of the non-pluripolar masses, due to Corollary 6.2.5. In other words, we need to show that

$$\lim_{N\to\infty} \int_X \omega_N^n = \int_X (\epsilon\omega)^n,$$

which follows from the convergence $\{\omega_N\} \to \{\epsilon\omega\}$ as $N \to \infty$. Our claim (7.12) follows.

Then thanks to Theorem 6.2.1,

$$\begin{aligned}
\operatorname{vol}(T, T_2, \ldots, T_n) &\leq \operatorname{vol}(T + \epsilon\omega, T_2, \ldots, T_n) \\
&= \lim_{N\to\infty} \operatorname{vol}\left(\sum_{i=1}^{N} c_i[E_i] + \omega_N, T_2, \ldots, T_n\right) \\
&= \lim_{N\to\infty} \operatorname{vol}(\omega_N, T_2, \ldots, T_n) \\
&\leq \lim_{N\to\infty} \int_X \omega_N \wedge (\theta_2 + \omega) \wedge \cdots \wedge (\theta_n + \omega) \\
&= \epsilon \int_X \omega \wedge (\theta_2 + \omega) \wedge \cdots \wedge (\theta_n + \omega),
\end{aligned}$$

where the third line follows from Step 1. Since $\epsilon > 0$ is arbitrary, we conclude (7.11). □

Lemma 7.3.1 *Let ω be a Kähler form on X. Then there is a constant $C > 0$ depending only on $X, \omega, \{T_1\}, \ldots, \{T_n\}$ such that*

$$0 \leq \operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega) - \operatorname{vol}(T_1, \ldots, T_n) \leq C\epsilon$$

for any $\epsilon \in [0, 1]$.

Proof By linearity, we can write

$$\operatorname{vol}(T_1 + \epsilon\omega, \ldots, T_n + \epsilon\omega) - \operatorname{vol}(T_1, \ldots, T_n)$$

as a linear combination of the mixed volumes between the T_i's and ω with coefficients ϵ^j for some $j \geq 1$. It suffices to show that

$$\operatorname{vol}(\omega, T_2, \ldots, T_n) \leq C,$$

where C depends only on $X, \omega, \{T_2\}, \ldots, \{T_n\}$. Represent T_i as $T_i = \theta_i + \mathrm{dd^c}\varphi_i$. Take a constant $D > 0$ so that $D\omega + \theta_i$ is a Kähler form for each $i = 2, \ldots, n$. Then

$$\operatorname{vol}(\omega, T_2, \ldots, T_n) \leq \operatorname{vol}(\omega, D\omega + \theta_2, \ldots, D\omega + \theta_n).$$

Our assertion follows. □

Next we show that Theorem 6.2.5 continues to hold even when φ has vanishing mass. We prove a slightly more general result:

Theorem 7.3.1 *Suppose that $(\varphi_j^k)_{k\in I}$ are nets in $\mathrm{PSH}(X,\theta_j)$ and $\varphi_j \in \mathrm{PSH}(X,\theta_j)$ for $j=1,\dots,n$. We assume that $\varphi_j^k \xrightarrow{d_S} \varphi_j$ for each $j=1,\dots,n$. Then*

$$\lim_{k\in I} \mathrm{vol}\left(\theta_{1,\varphi_1^k},\dots,\theta_{n,\varphi_n^k}\right) = \mathrm{vol}\left(\theta_{1,\varphi_1},\dots,\theta_{n,\varphi_n}\right). \tag{7.13}$$

Proof Fix a Kähler form ω, then for any $\epsilon>0$, we have

$$\lim_{k\in I} \mathrm{vol}\left((\theta_1+\epsilon\omega)_{\varphi_1^k},\dots,(\theta_n+\epsilon\omega)_{\varphi_n^k}\right) = \mathrm{vol}\left((\theta_1+\epsilon\omega)_{\varphi_1},\dots,(\theta_n+\epsilon\omega)_{\varphi_n}\right)$$

as a consequence of Theorem 6.2.1 and Theorem 6.2.3.

Now thanks to Lemma 7.3.1, there is $C>0$ so that for each $k\in I$,

$$\begin{aligned}
&0 \le \mathrm{vol}\left((\theta_1+\epsilon\omega)_{\varphi_1^k},\dots,(\theta_n+\epsilon\omega)_{\varphi_n^k}\right) - \mathrm{vol}\left(\theta_{1,\varphi_1^k},\dots,\theta_{n,\varphi_n^k}\right) \le C\epsilon,\\
&0 \le \mathrm{vol}\left((\theta_1+\epsilon\omega)_{\varphi_1},\dots,(\theta_n+\epsilon\omega)_{\varphi_n}\right) - \mathrm{vol}\left(\theta_{1,\varphi_1},\dots,\theta_{n,\varphi_n}\right) \le C\epsilon.
\end{aligned}$$

Therefore, (7.13) follows. □

Corollary 7.3.1 *Let $(\varphi_j)_{j\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi\in\mathrm{PSH}(X,\theta)$. Assume that $\varphi_j \xrightarrow{d_S} \varphi$, then*

$$\mathrm{vol}\,\theta_{\varphi_j} \to \mathrm{vol}\,\theta_\varphi, \quad \int_X \theta_{\varphi_j}^n \to \int_X \theta_\varphi^n. \tag{7.14}$$

Proof The first part of (7.14) is a special case of Theorem 7.3.1, while the second part of (7.14) is a special case of Theorem 6.2.1. □

The mixed volume has a log-concavity property:

Proposition 7.3.2 *Let $T_1,\dots,T_n$ be closed positive $(1,1)$-currents on X, then*

$$\mathrm{vol}(T_1,\dots,T_n) \ge \prod_{i=1}^n (\mathrm{vol}\,T_i)^{1/n}.$$

Proof We may assume that $\mathrm{vol}\,T_i>0$ for each $i=1,\dots,n$ since there is nothing to prove otherwise. In this case, we need to show that

$$\begin{aligned}
&\int_X \left(\theta_1+\mathrm{dd^c}P_{\theta_1}[\varphi_1]_{\mathcal{I}}\right)\wedge\cdots\wedge\left(\theta_n+\mathrm{dd^c}P_{\theta_1}[\varphi_n]_{\mathcal{I}}\right)\\
\ge &\prod_{i=1}^n \left(\int_X \left(\theta_i+\mathrm{dd^c}P_{\theta_i}[\varphi_i]_{\mathcal{I}}\right)^n\right)^{1/n}.
\end{aligned}$$

This is a special case of Theorem 2.4.1. □

Next, we also study how the mixed volume behaves under bimeromorphic transforms.

Proposition 7.3.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y to X, then*

$$\operatorname{vol}(\pi^*T_1, \dots, \pi^*T_n) = \operatorname{vol}(T_1, \dots, T_n)\,.$$

Proof Using the definition of the mixed volume, we may easily reduce to the case where $\operatorname{vol} T_i > 0$ for each $i = 1, \dots, n$. By 3.2.5, we know that if we write $T_i = \theta_i + \mathrm{dd^c}\varphi_i$, then

$$\pi^* P_{\theta_i}[\varphi_i]_{\mathcal{I}} = P_{\pi^*\theta_i}[\pi^*\varphi_i]_{\mathcal{I}}$$

for each $i = 1, \dots, n$. In particular,

$$\operatorname{vol} \pi^*T_i = \operatorname{vol} T_i > 0.$$

Our assertion follows from the bimeromorphic invariance of the non-pluripolar product Proposition 2.4.2. □

As for pushforward, we also have a similar result. We need a preliminary result:

Lemma 7.3.2 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y. Then for any non-divisorial closed positive $(1,1)$-current T on X, we have*

$$\pi^*\pi_*T = T + \sum_{i=1}^{N} c_i[E_i]$$

for finitely many π-exceptional divisors E_i and $c_i > 0$.

Proof Let E be the exceptional locus of π. Then

$$T = \mathbb{1}_{Y\setminus E}\pi^*\pi_*T.$$

Therefore,

$$\pi^*\pi_*T - T = \mathbb{1}_E\pi^*\pi_*T,$$

which has the stated form, due to the support theorems, see [Dem12b, Section 8]. □

Corollary 7.3.2 *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Then*

$$\operatorname{vol}(T_1, \dots, T_n) = \operatorname{vol}(\pi_*T_1, \dots, \pi_*T_n). \tag{7.15}$$

Proof Observe that we may assume that $T_i = \operatorname{Reg} T_i$ for all $i = 1, \dots, n$. In fact, since the pushforward of the divisorial part of T_i is divisorial as well, hence by Proposition 7.3.1(5), they do not contribute to the volumes.

Now by Proposition 7.3.3, it remains to show that

$$\operatorname{vol}(T_1, \dots, T_n) = \operatorname{vol}(\pi^*\pi_*T_1, \dots, \pi^*\pi_*T_n).$$

By Lemma 7.3.2, the difference $\pi^*\pi_* T_i - T_i$ is divisorial, hence our desired equality follows from Proposition 7.3.1(5). □

A particular corollary of Corollary 7.3.2 will be useful later.

Corollary 7.3.3 *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Assume that T is an $\mathcal{I}$-good closed positive $(1,1)$-current on X, then so is $\pi_* T$.*

Proof We may assume that $\int_X T^n > 0$. Then by Corollary 7.3.2,

$$\operatorname{vol} \pi_* T = \operatorname{vol} T > 0$$

as well. Since T is $\mathcal{I}$-good, we have

$$\operatorname{vol} T = \int_X T^n.$$

But $\int_X T^n = \int_Z (\pi_* T)^n$, so

$$\operatorname{vol} \pi_* T = \int_Z (\pi_* T)^n > 0.$$

It follows that $\pi_* T$ is $\mathcal{I}$-good. □

Finally we establish a semicontinuity property of the mixed volumes.

Theorem 7.3.2 *Let $(\varphi_i^j)_{j\in J}$ $(i = 1, \ldots, n)$ be nets in $\mathrm{PSH}(X, \theta_i)$. Assume that for each prime divisor E over X, we have*

$$\lim_{j\in J} \nu\left(\varphi_i^j, E\right) = \nu\left(\varphi_i, E\right), \quad i = 1, \ldots, n.$$

Then

$$\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1 + \mathrm{dd^c}\varphi_1^j, \ldots, \theta_n + \mathrm{dd^c}\varphi_n^j\right) \leq \operatorname{vol}\left(\theta_1 + \mathrm{dd^c}\varphi_1, \ldots, \theta_n + \mathrm{dd^c}\varphi_n\right).$$

Proof **Step 1**. We first assume that $\operatorname{vol}(\theta_i + \mathrm{dd^c}\varphi_i^j) > 0$ and $\operatorname{vol}(\theta_i + \mathrm{dd^c}\varphi_i) > 0$ for all $i = 1, \ldots, n$ and $j \in J$.

Without loss of generality, we may assume that the φ_i^j's and the φ_i's are $\mathcal{I}$-model for all $i = 1, \ldots, n$ and $j \in J$. Our assertion becomes

$$\varlimsup_{j\in J} \int_X \left(\theta_1 + \mathrm{dd^c}\varphi_1^j\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}\varphi_n^j\right) \leq \int_X \left(\theta_1 + \mathrm{dd^c}\varphi_1\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}\varphi_n\right). \tag{7.16}$$

For each $j \in J$, define

$$\psi_i^j \coloneqq \sup_{k\geq j}{}^* \varphi_i^k, \quad i = 1, \ldots, n.$$

Observe that ψ_i^j is $\mathcal{I}$-good thanks to Proposition 7.2.2. It follows from Corollary 1.4.1 and our assumption that

$$\lim_{j\in J} \nu\left(\psi_i^j, E\right) = \nu\left(\varphi_i, E\right), \quad i = 1, \ldots, n.$$

For each $i = 1, \ldots, n$, we define

$$\psi_i = \inf_{j\in J} P_{\theta_i}[\psi_i^j].$$

Due to Proposition 3.2.13, ψ_i is $\mathcal{I}$-model. Thanks to Proposition 3.1.10, we know

$$\nu(\psi_i, E) = \nu(\varphi_i, E)$$

for any $i = 1, \ldots, n$ and any prime divisor E over X. In other words, $\psi_i \sim_{\mathcal{I}} \varphi_i$ for $i = 1, \ldots, n$. But both φ_i and ψ_i are $\mathcal{I}$-good, therefore,

$$\psi_i \sim_P \varphi_i, \quad i = 1, \ldots, n.$$

By Proposition 6.1.4, we have

$$\int_X (\theta_1 + \mathrm{dd^c}\psi_1) \wedge \cdots \wedge (\theta_n + \mathrm{dd^c}\psi_n) = \int_X (\theta_1 + \mathrm{dd^c}\varphi_1) \wedge \cdots \wedge (\theta_n + \mathrm{dd^c}\varphi_n).$$

Next by Proposition 6.1.4 again,

$$\begin{aligned}&\varlimsup_{j\in J} \int_X \left(\theta_1 + \mathrm{dd^c}\varphi_1^j\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}\varphi_n^j\right)\\ \leq &\varlimsup_{j\in J} \int_X \left(\theta_1 + \mathrm{dd^c}\psi_1^j\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}\psi_n^j\right).\end{aligned}$$

On the other hand, due to Proposition 3.1.10 and Corollary 6.2.5, for each $i = 1, \ldots, n$, we have

$$\psi_i^j \xrightarrow{d_S} \psi_i.$$

We conclude from Theorem 6.2.1 that

$$\varlimsup_{j\in J} \int_X \left(\theta_1 + \mathrm{dd^c}\psi_1^j\right) \wedge \cdots \wedge \left(\theta_n + \mathrm{dd^c}\psi_n^j\right) = \int_X (\theta_1 + \mathrm{dd^c}\psi_1) \wedge \cdots \wedge (\theta_n + \mathrm{dd^c}\psi_n).$$

Putting these equations together, (7.16) follows.

Step 2. Next we handle the general case.

Fix a Kähler form ω on X. For any $\epsilon \in (0, 1]$, from Step 1, we know that

$$\begin{aligned}&\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1 + \epsilon\omega + \mathrm{dd^c}\varphi_1^j, \ldots, \theta_n + \epsilon\omega + \mathrm{dd^c}\varphi_n^j\right)\\ \leq &\operatorname{vol}\left(\theta_1 + \epsilon\omega + \mathrm{dd^c}\varphi_1, \ldots, \theta_n + \epsilon\omega + \mathrm{dd^c}\varphi_n\right).\end{aligned}$$

Using Lemma 7.3.1, we have

$$\begin{aligned}
&\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1 + \mathrm{dd^c}\varphi_1^j, \dots, \theta_n + \mathrm{dd^c}\varphi_n^j\right)\\
\leq &\varlimsup_{j\in J} \operatorname{vol}\left(\theta_1 + \epsilon\omega + \mathrm{dd^c}\varphi_1^j, \dots, \theta_n + \epsilon\omega + \mathrm{dd^c}\varphi_n^j\right)\\
\leq &\operatorname{vol}\left(\theta_1 + \epsilon\omega + \mathrm{dd^c}\varphi_1, \dots, \theta_n + \epsilon\omega + \mathrm{dd^c}\varphi_n\right)\\
\leq &\operatorname{vol}\left(\theta_1 + \mathrm{dd^c}\varphi_1, \dots, \theta_n + \mathrm{dd^c}\varphi_n\right) + C\epsilon.
\end{aligned}$$

But since ϵ is arbitrary, our assertion follows. □

7.4 The volumes of Hermitian pseudo-effective line bundles

Let X be a connected compact Kähler manifold of dimension n.

Definition 7.4.1 A *Hermitian pseudo-effective line bundle* (L, h) on a complex manifold Y consists of a holomorphic line bundle L on Y together with a plurisubharmonic metric h on L.

Theorem 7.4.1 *Let (L, h) be a Hermitian pseudo-effective line bundle on X and T be a holomorphic line bundle on X. We have*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(h^k)\right) = \operatorname{vol}\left(\mathrm{dd^c} h\right). \tag{7.17}$$

In particular, the limit exists.

For the proof, let us fix a smooth Hermitian metric h_0 on L with $\theta = c_1(L, h_0)$. We identify h with $h_0 \exp(-\varphi)$ for some $\varphi \in \operatorname{PSH}(X, \theta)$. See Section 1.8 for the relevant notations.

Recall that when X admits a big line bundle, it is necessarily projective. See [MM07, Theorem 2.2.26].

We first handle the case where φ has analytic singularities.

Proposition 7.4.1 *Under the assumptions of Theorem 7.4.1, assume furthermore that φ has analytic singularities, then (7.17) holds.*

Proof **Step 1**. Reduce to the case of log singularities.

Let $\pi\colon Y \to X$ be a log resolution of φ. In this case, for each $k \in \mathbb{Z}_{>0}$, we have

$$h^0(X, T \otimes L^k \otimes \mathcal{I}(kh)) = h^0(Y, K_{Y/X} \otimes \pi^*T \otimes \pi^*L^k \otimes \mathcal{I}(k\pi^*h)).$$

By Proposition 3.2.5, we have

$$\operatorname{vol}(\mathrm{dd^c} h) = \operatorname{vol}(\mathrm{dd^c}\pi^* h).$$

Therefore, it suffices to argue (7.17) with $K_{Y/X} \otimes \pi^*T$, π^*L and π^*h in place of T, L and h.

Step 2. Assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D, we decompose D into irreducible components, say

$$D = \sum_{i=1}^{N} a_i D_i.$$

In this case, we can easily compute

$$\mathcal{I}(k\varphi) = \mathcal{O}_X\left(-\sum_{i=1}^{N} \lfloor k a_i \rfloor D_i\right)$$

for each $k \in \mathbb{Z}_{>0}$. Observe that $L - D$ is nef (see Lemma 1.6.1), so we could apply the asymptotic Riemann–Roch theorem [Laz04, Corollary 1.4.41][4] to conclude that

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{O}_X\left(-\sum_{i=1}^{N} \lfloor k a_i \rfloor D_i\right)\right) = (L-D)^n.$$

Observe that by Proposition 1.8.1,

$$\theta_\varphi = [D] + T,$$

where T is a closed positive $(1,1)$-current with bounded potential. Therefore,

$$(L-D)^n = \int_X T^n = \int_X \theta_\varphi^n.$$

By Example 7.1.1, we know that the right-hand side is exactly $\operatorname{vol}\theta_\varphi$. □

Proof (Proof of Theorem 7.4.1) **Step 1**. We first handle the case where θ_φ is a Kähler current. Fix a Kähler form $\omega \geq \theta$ on X such that $\theta_\varphi \geq 2\delta\omega$ for some $\delta \in (0,1)$.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j} \geq \delta\omega$ for all j. From Proposition 7.4.1, we know that for each $j \geq 1$,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq \lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_j)\right) = \operatorname{vol}\theta_{\varphi_j}.$$

It follows from Theorem 7.1.1 and Theorem 6.2.5 that the right-hand side converges to $\operatorname{vol}\theta_\varphi$ as $j \to \infty$. Therefore,

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq \operatorname{vol}\theta_\varphi.$$

Conversely, fix an integer $N > \delta^{-1}$. From Theorem 7.1.1 and Theorem 6.2.1, we know that

[4] Please try to complete the full details if it is not completely clear to you how to apply the Riemann–Roch theorem of *integral* divisors in this setup.

$$\lim_{j\to\infty}\int_X \theta_{\varphi_j}^n = \int_X \theta_{P_\theta[\varphi]_{\mathcal{I}}}^n > 0. \tag{7.18}$$

Therefore, by Lemma 2.4.2, we can find $j_0 > 0$ such that for $j \geq j_0$, there is $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ (depending on j) with

$$(1-N^{-1})\varphi_j + N^{-1}\psi \leq P_\theta[\varphi]_{\mathcal{I}}. \tag{7.19}$$

For each $k > 0$, we write $k = k'N - r$, where $k' \in \mathbb{N}$ and $r \in \{0,1,\ldots,N-1\}$. Then we compute for $j \geq j_0$ and large enough k (to be specified shortly) that

$$\begin{aligned}
&h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right)\\
\geq &h^0\left(X, T\otimes L^{-r}\otimes L^{k'N} \otimes \mathcal{I}(k'N\varphi)\right)\\
\geq &h^0\left(X, T\otimes L^{-r}\otimes L^{k'N} \otimes \mathcal{I}\left(k'(\psi+(N-1)\varphi_j)\right)\right)\\
\geq &h^0\left(X, T\otimes L^{-r}\otimes L^{k'(N-1)} \otimes \mathcal{I}\left(k'N\varphi_j\right)\right),
\end{aligned}$$

where the third line follows from (7.19), the fourth line can be argued as follows: For large enough k, there is a non-zero section $s \in \mathrm{H}^0(X, L^{k'}\otimes\mathcal{I}(k'\psi))$ by Lemma 2.4.4. It follows from Lemma 1.6.3 that for large enough k,

$$\mathcal{I}\left(k'N\varphi_j\right) \subseteq \mathcal{I}_\infty\left(k'(N-1)\varphi_j\right).$$

It follows that multiplication by s gives an injective map

$$\begin{aligned}
&\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'(N-1)}\otimes \mathcal{I}\left(k'N\varphi_j\right)\right) \hookrightarrow\\
&\mathrm{H}^0\left(X, T\otimes L^{-r}\otimes L^{k'N}\otimes \mathcal{I}\left(k'\psi + k'(N-1)\varphi_j\right)\right).
\end{aligned}$$

Next observe that

$$(N-1)\theta + N\mathrm{dd^c}\varphi_j \geq 0.$$

So Proposition 7.4.1 is applicable. We let $k\to\infty$ to conclude that

$$\begin{aligned}
\varliminf_{k\to\infty}\frac{n!}{k^n}h^0(X, T\otimes L^k\otimes\mathcal{I}(k\varphi)) \geq & N^{-n}\int_X \left((N-1)\theta + N\mathrm{dd^c}\varphi_j\right)^n\\
= & \int_X \left((1-N^{-1})\theta + \mathrm{dd^c}\varphi_j\right)^n\\
\geq & \int_X (\theta+\mathrm{dd^c}\varphi_j)^n - CN^{-1},
\end{aligned}$$

where C is a constant independent of N and j. Letting $j\to\infty$ and then $N\to\infty$ and using (7.18), we find that

$$\varliminf_{k\to\infty} h^0\left(X, T\otimes L^k\otimes\mathcal{I}(k\varphi)\right) \geq \int_X \theta^n_{P_\theta[\varphi]_{\mathcal{I}}}.$$

Therefore, (7.17) follows.

Step 2. We handle the case where $\operatorname{vol}\theta_\varphi > 0$. We may assume that φ is $\mathcal{I}$-model.

Fix observe that L is big by Proposition 2.4.1. Hence X is projective. Take a very ample line bundle A on X and a Kähler form ω in $c_1(A)$. Take a Hermitian metric h_A on A with $\mathrm{dd^c} h_A = \omega$.

Fix $N \in \mathbb{Z}_{>0}$, we decompose any $k > 0$ as $k = k'N + r$ with $k' \in \mathbb{N}$ and $r \in \{0, 1, \ldots, N-1\}$. Then

$$h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq h^0\left(X, T \otimes L^r \otimes L^{k'N} \otimes \mathcal{I}(k'N\varphi)\right).$$

Therefore,

$$\begin{aligned}
&\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right)\\
\leq{}& \max_{r=0,\ldots,N-1} \varlimsup_{k'\to\infty} \frac{n!}{k'^n N^n} h^0\left(X, T \otimes L^r \otimes L^{k'N} \otimes \mathcal{I}(k'N\varphi)\right)\\
\leq{}& \max_{r=0,\ldots,N-1} \varlimsup_{k'\to\infty} \frac{n!}{k'^n N^n} h^0\left(X, T \otimes L^r \otimes L^{k'N} \otimes A^{k'} \otimes \mathcal{I}(k'N\varphi)\right)\\
={}& \int_X \left(N^{-1}\omega + \theta + \mathrm{dd^c} P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}}\right)^n,
\end{aligned}$$

where we have applied Step 1 to the Hermitian pseudo-effective line bundle $(L^N \otimes A, h^{\otimes N} \otimes h_A)$ on the fourth line. On the other hand, since φ is $\mathcal{I}$-good by Example 7.1.2, we have

$$P_{\theta+N^{-1}\omega}[\varphi]_{\mathcal{I}} = P_{\theta+N^{-1}\omega}[\varphi].$$

It follows from Proposition 3.1.3 that

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq \int_X \left(N^{-1}\omega + \theta + \mathrm{dd^c}\varphi\right)^n.$$

Letting $N \to \infty$, we conclude

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq \int_X \theta_\varphi^n.$$

It remains to argue the reverse inequality.

Choose $\psi \in \operatorname{PSH}(X, \theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. The existence of ψ is guaranteed by Lemma 2.4.3. Then for any $t \in (0, 1)$, we set

$$\varphi_t = (1-t)\varphi + t\psi.$$

It follows again from Step 1 that

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) \geq \lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi_t)\right) = \operatorname{vol}\theta_{\varphi_t}.$$

On the other hand, by Proposition 7.3.1,

$$\lim_{t\to 0+} \operatorname{vol}\theta_{\varphi_t} = \operatorname{vol}\theta_{\varphi}.$$

So we find

$$\varliminf_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right) \geq \operatorname{vol}\theta_{\varphi}.$$

We conclude (7.17) in this case.

Step 3. We finally handle the case where $\operatorname{vol}\theta_\varphi = 0$. Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-model.

Assume that (7.17) fails. That is,

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k\right) \geq \varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right) > 0,$$

then L is a big line bundle and hence X is projective.

Fix a very ample line bundle A on X and a Kähler form $\omega \in c_1(A)$. Take a decreasing sequence $(\epsilon_j)_j$ of rational numbers with limit 0 and a quasi-equisingular approximation $(\varphi_j)_j$ of φ with $\varphi_j \in \operatorname{PSH}(X, \theta+\epsilon_j\omega)_{>0}$.

We claim that as $j\to\infty$, the sequence $P_{\theta+\epsilon_j\omega}[\varphi_j]$ is decreasing with limit φ.

It is clear that this sequence is decreasing. Let ψ denote its limit for the moment. It is also clear that $\psi \geq \varphi$. Since φ is $\mathcal{I}$-model, it remains to show that $\psi \leq_{\mathcal{I}} \varphi$. But the argument is exactly as in the proof of Theorem 7.1.1. So we conclude.

By our claim and Proposition 3.1.10, we find that

$$\lim_{j\to\infty} \int_X \left(\theta + \epsilon_j\omega + \mathrm{dd^c}\varphi_j\right)^n = \int_X \theta_\varphi^n = 0. \tag{7.20}$$

Fix $j > 0$, take an integer $N > 0$ so that $N\epsilon_j$ is an integer. Then we compute

$$\begin{aligned}
&\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right)\\
\leq &\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi_j)\right)\\
\leq &\max_{a=0,\dots,N-1} \varlimsup_{k'\to\infty} \frac{n!}{(k'N)^n} h^0\left(X, T\otimes L^a \otimes L^{Nk'} \otimes \mathcal{I}(Nk'\varphi_j)\right)\\
\leq &\max_{a=0,\dots,N-1} \varlimsup_{k'\to\infty} \frac{n!}{(k'N)^n} h^0\left(X, T\otimes L^a \otimes L^{Nk'} \otimes A^{k'N\epsilon_j} \otimes \mathcal{I}(Nk'\varphi_j)\right)\\
= &\frac{1}{N^n}\int_X \left(N\theta + \epsilon_j N\omega + N\mathrm{dd^c}\varphi_j\right)^n,
\end{aligned}$$

where the third line follows by writing $k = Nk' + a$ as before, we applied Step 2 on the last line. Letting $N\to\infty$, we find that

$$\varlimsup_{k\to\infty} \frac{n!}{k^n} h^0\left(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)\right) \leq \int_X \left(\theta + \epsilon_j\omega + \mathrm{dd^c}\varphi_j\right)^n.$$

Since we know (7.20), letting $j \to \infty$, we conclude that

$$\overline{\lim_{k\to\infty}} \frac{n!}{k^n} h^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) = 0,$$

which is a contradiction. Hence (7.17) is established in full generality. □

Corollary 7.4.1 *Let L be a pseudo-effective line bundle on X, h be a Hermitian metric on L with $\theta = c_1(L, h)$. Then we have*

$$\lim_{k\to\infty} \frac{n!}{k^n} h^0\left(X, L^k\right) = \int_X \theta^n_{V_\theta}. \tag{7.21}$$

This common quantity is the *volume* of L, usually denoted by $\operatorname{vol} L$. In view of Definition 3.2.4, we have

$$\operatorname{vol} L = \operatorname{vol} c_1(L). \tag{7.22}$$

Example 7.4.1 If X is a toric smooth projective variety and θ is invariant under the action of the compact torus. Then any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ is $\mathcal{I}$-good.

Proof Thanks to Lemma 7.1.1, we may assume that $\theta \in c_1(L)$ for some toric invariant ample line bundle L. In this case, the result follows from Theorem 7.1.1, Theorem 7.4.1 and Theorem 5.2.2. □

Chapter 8
The trace operator

The difference between mathematicians and physicists is that after physicists prove a big result they think it is fantastic but after mathematicians prove a big result they think it is trivial.
— Lucien Szpiro[a]

[a] Lucien Szpiro (1941–2020) was a French mathematician known for his significant contributions to number theory and arithmetic geometry. His work often focused on problems related to Diophantine equations and the arithmetic of elliptic curves.

Szpiro is perhaps best known for Szpiro's Conjecture, which has deep connections to the famous abc conjecture in number theory, an important open problem with wide-ranging implications.

In this chapter, we develop the theory of trace operators and prove the analytic Bertini theorem. These techniques allow us to make induction on the dimension while studying the singularities. Roughly speaking, the analytic Bertini theorem allows us to study generic restrictions, while the trace operator handles the remaining cases.

In Section 8.4, we establish a relative version of the Theorem 7.4.1.

The name trace operator comes from the familiar situation in the theory of Sobolev spaces. Let me take this opportunity to explain a general analogy which I had in mind for years.

Pluripotential theory	Real analysis
Quasi-psh functions	Functions
Quasi-psh functions with analytic singularities	Smooth functions
$\mathcal{I}$-good singularities	Measurable functions
$\mathcal{I}$-equivalence	Almost everywhere equality

In real analysis, people wish to study all functions, while in pluripotential theory, people wish to study all quasi-psh functions. In general, neither is realistic: In real analysis, we only have a good function theory for measurable functions, for example, measurability is the key property underlying Littlewood's three famous principles. Similarly, in pluripotential theory, a reasonable theory is only established for $\mathcal{I}$-good singularities.

Smooth functions, as a special class of measurable functions, enjoy much better properties compared to general ones. For example, the precise pointwise value is meaningful. For general measurable functions, we only care about their properties modulo almost everywhere equality. In pluripotential theory, quasi-psh functions with analytic singularities play the role of smooth functions. General quasi-psh functions are limits of those with analytic singularities, just as a measurable function can be approximated by smooth functions.

Coming back to the theme of this chapter, our notion of trace operator is motivated by this analogy. Recall that given a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, we have the famous trace operator of Sobolev:

$$\mathrm{Tr}\colon W^{1,p}(\Omega) \to L^p(\partial\Omega).$$

For a smooth function f on $\bar{\Omega}$, $\mathrm{Tr}\, f$ is nothing but the pointwise restriction of f to $\partial\Omega$, while for a general Sobolev function f, the trace $\mathrm{Tr}\, f$ is only defined up to almost everywhere equality.

Similarly, in pluripotential theory, we wish to restrict a quasi-psh function φ to a subvariety. When φ has analytic singularities, the restriction has a definite value, while for general singularities, the restriction is only defined up to $\mathcal{I}$-equivalence.

8.1 The definition of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset.

The trace operator gives a way to restrict a quasi-plurisubharmonic function on X to $\tilde{Y}$, the normalization of Y. It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space. We refer to Appendix B for the pluripotential theory on unibranch Kähler spaces.

For later applications, we need this generality even if initially we are only interested in the smooth case.

Before diving into the trace operators, let us try to understand what goes wrong with the naive way of doing the restriction: Just take $\varphi|_{\tilde{Y}}$. Both examples below are local, but can easily be globalized since the singularities are isolated.

Example 8.1.1 Consider the case where φ has log-log singularities as in Example 3.1.1. Locally we can take $\varphi(z) = -\log(-\log|z|^2)$. We have $\nu(\varphi, 0) = 0$ but $\varphi|_0 = -\infty$. So the naive restriction is not defined in this case.

Even if the naive restriction is defined, it does not behave well.

Example 8.1.2 Consider a psh function φ in two variables, say

$$\varphi(z,w) = \left(-\log(-\log|z|^2)\right) \vee \left(\log|w|^2\right).$$

Take an arbitrary quasi-equisingular approximation $(\varphi_j)_j$. Then each φ_j is locally bounded since $\nu(\varphi,(0,0)) = 0$. Let us consider the restrictions to $H = \{z = 0\}$. Then $\varphi_j|_H$ is still bounded, while

$$\varphi|_H(w) = \log|w|^2.$$

In other words, the restrictions $\varphi_j|_H$ fail to be a quasi-equisingular approximation of $\varphi|_H$.

Our trace operator gives an elegant way to solve both problems.

We first observe that given $\varphi \in \mathrm{QPSH}(X)$ with analytic singularities such that $\nu(\varphi, Y) = 0$, then $\varphi|_Y \not\equiv -\infty$. This observation will be crucial in the sequel.

Proposition 8.1.1 *Let $\varphi \in \mathrm{QPSH}(X)$ be a function such that $\nu(\varphi, Y) = 0$. Let $(\varphi_i)_i$, $(\psi_i)_i$ be quasi-equisingular approximations of φ. Then*

$$\lim_{i\to\infty} d_S\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) = 0. \tag{8.1}$$

The meaning of (8.1) is explained in Corollary 6.2.9.

Proof Take a Kähler form ω on X such that $\varphi_i, \psi_i \in \mathrm{PSH}(X, \omega/2)$ for all $i \geq 1$. By Corollary 6.2.9, we need to show that

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) = 0.$$

Assume that this fails, then up to replacing the sequences by subsequences, we may assume that the following limit exists and

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) > 0.$$

Take a Kähler form $\tilde{\omega}$ on $\tilde{Y}$, then

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\varphi_i|_{\tilde{Y}}, \psi_i|_{\tilde{Y}}\right) > 0$$

by Corollary 6.2.9.

Replacing φ by $P_\omega[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good. In particular, $\varphi_i \xrightarrow{d_S} \varphi$, $\psi_i \xrightarrow{d_S} \varphi$. Therefore,

$$\varphi_i \vee \psi \xrightarrow{d_S} \varphi$$

due to Proposition 6.2.5. We may replace $(\psi_i)_i$ with $(\varphi_i \vee \psi_i)_i$ and assume that $\varphi_i \leq \psi_i$ for all $i \geq 1$.

Take a decreasing sequence $(\epsilon_j)_j$ in $\mathbb{R}_{>0}$ with limit 0 such that $(1-\epsilon_j)\varphi_j \in \mathrm{PSH}(X, \omega)$. We first observe that

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}}\left(\varphi_i|_{\tilde{Y}}, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}\right) = 0.$$

This is a consequence of Lemma 6.2.3. Hence, by Corollary 6.2.9, we find

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\varphi_i|_{\tilde{Y}}, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}\right) = 0.$$

But thanks to Corollary 6.2.6, there is $\psi \in \mathrm{PSH}(\tilde{Y}, \omega|_{\tilde{Y}} + \tilde{\omega})$ such that

$$\varphi_i|_{\tilde{Y}} \xrightarrow{d_S} \psi.$$

Hence,

$$\lim_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\epsilon, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}\right) = 0.$$

Next by Proposition 1.6.3, we could find a subsequence $(\psi_{j_i})_{i\in\mathbb{Z}_{>0}}$ of $(\psi_j)_j$ such that for each $i \geq 1$,

$$\varphi_{j_i} \preceq \psi_{j_i} \preceq (1-\epsilon_i)\varphi_i.$$

Hence,

$$\varphi_{j_i}|_{\tilde{Y}} \preceq \psi_{j_i}|_{\tilde{Y}} \preceq (1-\epsilon_i)\varphi_i|_{\tilde{Y}}.$$

Therefore, by Corollary 6.2.1,

$$\begin{aligned}\varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\varphi_{j_i}|_{\tilde{Y}}, \psi_{j_i}|_{\tilde{Y}}\right) \leq& \varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\varphi_{j_i}|_{\tilde{Y}}, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}\right)\\ =& \varlimsup_{i\to\infty} d_{S,\omega|_{\tilde{Y}}+\tilde{\omega}}\left(\psi, (1-\epsilon_i)\varphi_i|_{\tilde{Y}}\right)\\ =&0,\end{aligned}$$

which is a contradiction. □

Definition 8.1.1 Let $\varphi \in \mathrm{QPSH}(X)$ be a function such that $\nu(\varphi, Y) = 0$. We say a potential $\psi \in \mathrm{QPSH}(\tilde{Y})$ is a[1] *trace operator* of φ along Y if there is a quasi-equisingular approximation $(\varphi_j)_{j>0}$ of φ such that

$$\varphi_j|_{\tilde{Y}} \xrightarrow{d_S} \psi\text{[2]}. \tag{8.2}$$

By Corollary 6.2.6, the trace operator is always defined. Observe that by Proposition 8.1.1, the condition (8.2) is independent of the choice of $(\varphi_j)_j$.

Later on in Theorem 12.3.2, we shall prove that the trace operator corresponds to the natural way of restricting convex bodies in the toric setting.

Proposition 8.1.2 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Suppose that ψ and ψ' are trace operators of φ along Y. Then ψ and ψ' are $\mathcal{I}$-good and $\psi \sim_P \psi'$.*

Proof That ψ and ψ' are $\mathcal{I}$-good follows from Theorem 7.1.1. The fact that $\psi \sim_P \psi'$ follows from Proposition 8.1.1 and Proposition 6.2.2. □

Example 8.1.3 As a trivial example, when Y is just a single point, then $\mathrm{QPSH}(Y)$ is canonically identified with $\mathbb{R}$. Any constant $c \in \mathbb{R}$ is a trace operator of a function $\varphi \in \mathrm{QPSH}(X)$ satisfying $\nu(\varphi, Y) = 0$.

---

[1] Let us resume our analogy in the introduction of this chapter. In real analysis, instead of saying that a function on $\partial\Omega$ modulo almost everywhere equality is *the* trace operator of a Sobolev function f on Ω, we say the function is *a* trace operator of f. Similarly, here we sat ψ is *a* trace operator of φ instead of the P-equivalence class of ψ is *the* trace operator of φ.

[2] To be more precise, what we mean is the following: We can find a closed smooth real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X,\theta)$. Then there is a Kähler form such that $\omega + \theta + \mathrm{dd^c}\varphi_j \geq 0$ for all $j \geq 1$. Take a Kähler form $\tilde{\omega}$ on $\tilde{Y}$ so that $\tilde{\omega} \geq (\theta+\omega)|_{\tilde{Y}}$ and that $\psi \in \mathrm{PSH}(\tilde{Y}, \tilde{\omega})$. Then our condition means that $\varphi_j|_{\tilde{Y}} \xrightarrow{d_{S,\tilde{\omega}}} \psi$. This condition is independent of the choices of θ, ω and $\tilde{\omega}$ by Corollary 6.2.8.

Definition 8.1.2 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. We write $\mathrm{Tr}_Y(\varphi)$ for any trace operator of φ along Y.

Given a closed smooth real $(1,1)$-form θ on X. When $\mathrm{Tr}_Y(\varphi)$ can be chosen to lie in $\mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})_{>0}$, we write

$$\mathrm{Tr}_Y^\theta(\varphi) := P_{\theta|_{\tilde{Y}}}\left[\mathrm{Tr}_Y(\varphi)\right] = P_{\theta|_{\tilde{Y}}}\left[\mathrm{Tr}_Y(\varphi)\right]_{\mathcal{I}}.$$

The trace operator $\mathrm{Tr}_Y(\varphi)$ is therefore well-defined only up to P-equivalence by Proposition 8.1.2. Also observe that if $\varphi \in \mathrm{PSH}(X, \theta)$ for some smooth closed real $(1,1)$-form θ on X, then for any Kähler form ω on X, the trace operator $\mathrm{Tr}_Y^{\theta+\omega}(\varphi)$ is always defined. In particular, if θ_φ is a Kähler current, $\mathrm{Tr}_Y^\theta(\varphi)$ is always defined.

Remark 8.1.1 As in Remark 1.7.1, the trace operator could also be applied to closed positive $(1,1)$-currents on X. If $T \in \mathcal{Z}_+(X, \alpha)$ for some pseudo-effective class α on X (see Definition 1.7.3) and $\beta \in \mathrm{H}^{1,1}(\tilde{Y}, \mathbb{R})$, then we write

$$\mathrm{Tr}_Y^\beta(T)$$

for any (if exists) closed positive $(1,1)$-current in β representing $\mathrm{Tr}_Y(T)$ when $\nu(T, Y) = 0$.

Proposition 8.1.3 *Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that $\varphi|_Y \not\equiv -\infty$. Then*

$$\varphi|_{\tilde{Y}} \preceq_P \mathrm{Tr}_Y(\varphi).$$

Proof Take a Kähler form ω such that ω_φ is a Kähler current. Let $(\varphi_j)_{j>0}$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X, \omega)_{>0}$. We may assume that $\varphi_j \leq 0$ for all $j \geq 1$.

Then

$$\varphi_j|_{\tilde{Y}} \leq P_{\omega|_{\tilde{Y}}}\left[\varphi_j|_{\tilde{Y}}\right] \tag{8.3}$$

for all $j \geq 1$. In particular,

$$\varphi|_{\tilde{Y}} \leq \inf_{j\geq 1} P_{\omega|_{\tilde{Y}}}\left[\varphi_j|_{\tilde{Y}}\right].$$

Thanks to Corollary 6.2.5,

$$\mathrm{Tr}_Y(\varphi) \sim_P \inf_{j\geq 1} P_{\omega|_{\tilde{Y}}}[\varphi_j|_{\tilde{Y}}]. \tag{8.4}$$

We conclude our assertion. □

Example 8.1.4 Let $\varphi \in \mathrm{QPSH}(X)$ such that $\nu(\varphi, Y) = 0$. Assume that φ has analytic singularities, then

$$\mathrm{Tr}_Y(\varphi) \sim_P \varphi|_{\tilde{Y}}.$$

Example 8.1.5 Let $\varphi \in \mathrm{QPSH}(X)$. Take a closed real smooth $(1,1)$-form θ on X such that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$, then

$$\mathrm{Tr}_X(\varphi) \sim_P P_\theta[\varphi]_{\mathcal{I}}, \quad \mathrm{Tr}_X^\theta(\varphi) = P_\theta[\varphi]_{\mathcal{I}}.$$

In particular, the trace operator can be regarded as a generalization of the $\mathcal{I}$-envelope.

Example 8.1.6 Assume that $\varphi \in \mathrm{PSH}(X,\theta)$ for some closed smooth real $(1,1)$-form θ on X, $\nu(\varphi, Y) = 0$ and

$$\lim_{\epsilon \searrow 0} \int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right)^{\dim Y} > 0 \tag{8.5}$$

for any arbitrary choice of a Kähler form ω on X. Then it follows from Proposition 3.1.10 that $\mathrm{Tr}_Y^\theta(\varphi)$ is defined, and its mass is exact the above limit.

As a consequence, we have the following formula:

$$\int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^{\dim Y} = \lim_{\epsilon \searrow 0} \int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right)^{\dim Y}, \tag{8.6}$$

where the left-hand side is understood as 0 if $\mathrm{Tr}_Y^\theta(\varphi)$ is not defined.

Remark 8.1.2 The trace operator allows us to introduce the following extension of the moving Seshadri constant: Let $T \in \mathcal{Z}_+(X,\alpha)$ and $x \in X$, we define

$$\epsilon(T,x) \coloneqq \inf_{V \ni x} \left(\frac{\mathrm{vol}\,\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T}{\mathrm{mult}_x V} \right)^{\frac{1}{\dim V}},$$

where $\mathrm{vol}\,\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T = 0$ if $\mathrm{Tr}_V^{\alpha|_{\tilde{V}}} T$ is not defined. Here V runs over all positive-dimensional closed irreducible analytic subsets of X containing x.

These moving Seshadri constants seem to be new. But since I do not have particularly good applications in mind, I will not study these objects in this book.

8.2 Properties of the trace operator

Let X be a connected compact Kähler manifold and $Y \subseteq X$ be an irreducible analytic subset.

We prove a few elementary properties of the trace operator.

Proposition 8.2.1 *Let $\varphi, \psi \in \mathrm{QPSH}(X)$, $\lambda > 0$. Assume that $\nu(\varphi, Y) = \nu(\psi, Y) = 0$. Then we have the following:*

(1) *Suppose that $\varphi \preceq_{\mathcal{I}} \psi$, then $\mathrm{Tr}_Y(\varphi) \preceq_P \mathrm{Tr}_Y(\psi)$.*
(2) *We have*
$$\mathrm{Tr}_Y(\varphi+\psi) \sim_P \mathrm{Tr}_Y(\varphi) + \mathrm{Tr}_Y(\psi).$$
(3) *We have*
$$\mathrm{Tr}_Y(\lambda\varphi) \sim_P \lambda\,\mathrm{Tr}_Y(\varphi).$$

(4) *We have*

$$\mathrm{Tr}_Y(\varphi \vee \psi) \sim_P \mathrm{Tr}_Y(\varphi) \vee \mathrm{Tr}_Y(\psi).$$

Proof Take a closed smooth real $(1,1)$-form θ on X such that $\theta_\varphi, \theta_\psi$ are both Kähler currents. Let $(\varphi_j)_j$ and $(\psi_j)_j$ be quasi-equisingular approximations of φ and ψ in $\mathrm{PSH}(X,\theta)$ respectively. We may assume that $\varphi_j \leq 0$ and $\psi_j \leq 0$ for all $j \geq 1$.

(1) By Corollary 7.1.2 and Proposition 6.2.5, we may assume that $\varphi_j \leq \psi_j$ for all j. Then our assertion follows from Proposition 6.2.4.

(2) It follows from Theorem 6.2.2 that $\varphi_j + \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}}$. However, by Proposition 3.2.11 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi + \psi]_{\mathcal{I}}.$$

Therefore, by Proposition 6.2.2, Corollary 7.1.2 and Proposition 1.6.1, $(\varphi_j + \psi_j)_j$ is a quasi-equisingular approximation of $\varphi + \psi$. We conclude using Theorem 6.2.2.

(3) Let $(\lambda_j)_j$ be an increasing sequence of positive rational numbers with limit λ. Then $(\lambda_j \varphi_j)_j$ is a quasi-equisingular approximation of φ. Our assertion follows Lemma 6.2.3.

(4) By Proposition 6.2.5, we have

$$\varphi_j \vee \psi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}}.$$

By Proposition 3.2.11 and Proposition 7.2.1, we have

$$P_\theta[\varphi]_{\mathcal{I}} \vee P_\theta[\psi]_{\mathcal{I}} \sim_P P_\theta[\varphi \vee \psi]_{\mathcal{I}}.$$

Therefore, our assertion follows exactly as in the proof of (2). □

The trace operator is continuous along d_S-convergent decreasing sequences.

Proposition 8.2.2 *Let $(\varphi_j)_{j\in I}$ be a decreasing net in* $\mathrm{QPSH}(X)$. *Assume that there exists a closed real smooth $(1,1)$-form θ such that $\varphi_j \in \mathrm{PSH}(X,\theta)$ for each $j \in I$. Assume that $\varphi_j \xrightarrow{d_S} \varphi \in \mathrm{QPSH}(X)$ and $\nu(\varphi, Y) = 0$. Then*

$$\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \mathrm{Tr}_Y(\varphi).$$

In view of Corollary 7.1.2, the trace operator preserves the property of being a quasi-equisingular approximation, hence solving the problem in Example 8.1.1.

Proof By Corollary 6.2.8, we may assume that there is a Kähler form ω on X such that $\varphi, \varphi_j \in \mathrm{PSH}(X, \theta - \omega)$ for all $j \in I$. Thanks to Proposition 8.2.1, for each $j \geq 1$,

$$\mathrm{Tr}_Y(\varphi_{j+1}) \leq_P \mathrm{Tr}_Y(\varphi_j).$$

It follows from Proposition 8.2.1 and Corollary 6.2.6 that there exists $\psi \in \mathrm{PSH}(\tilde{Y}, \theta|_{\tilde{Y}})$ such that $\mathrm{Tr}_Y(\varphi_j) \xrightarrow{d_S} \psi$.

For each $j \geq 1$, we take a quasi-equisingular approximation $(\varphi_j^k)_k$ in $\mathrm{PSH}(X,\theta)$ of φ_j. Using Theorem 1.6.2, we may guarantee that

$$\varphi_{j+1}^k \leq \varphi_j^k$$

for each $j, k \geq 1$. In particular, $(\varphi_j^j)_j$ is a quasi-equisingular approximation of φ. By Proposition 6.2.4, we have $\psi \leq_P \mathrm{Tr}_Y(\varphi)$.

Conversely, by Proposition 8.2.1, $\mathrm{Tr}_Y(\varphi_j) \geq_P \mathrm{Tr}_Y(\varphi)$. It follows again from Proposition 6.2.4 that $\mathrm{Tr}_Y(\varphi) \leq_P \psi$. □

Example 8.2.1 The trace operator is not continuous along increasing sequences. Let us consider the case $X = \mathbb{P}^2$ with coordinates (z_1, z_2) on $\mathbb{C}^2 \subseteq X$. Let ω_{FS} denote the Fubini–Study metric. The subvariety $Y \cong \mathbb{P}^1$ is defined by $z_2 = 0$. Consider an increasing sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X, \omega_{\mathrm{FS}})$, whose potentials near $(0,0)$ are given by

$$\log |z_1|^2 \vee \left(k^{-1} \log |z_2|^2\right) + O(1).$$

The pointwise restriction of these potentials to Y are given locally by

$$\log |z_1|^2 + O(1).$$

On the other hand, locally

$$\log |z_1|^2 \vee \left(k^{-1} \log |z_2|^2\right) \to 0$$

almost everywhere as $k \to \infty$. So the trace operator is not continuous along the sequence $(\varphi_j)_j$.

Lemma 8.2.1 *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a connected Kähler manifold. Assume that W (resp. Y) be analytic subsets in Z (resp. X) of codimension 1 such that the restriction $\Pi\colon W \to Y$ of π is defined and is bimeromorphic, so that we have the following commutative diagram*

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d, "\tilde{\Pi}"] & W \arrow[r, hook] \arrow[d, "\Pi"] & Z \arrow[d, "\pi"] \\
\tilde{Y} \arrow[r] & Y \arrow[r, hook] & X.
\end{tikzcd}$$

Then for any $\varphi \in \mathrm{QPSH}(X)$ with $\nu(\varphi, Y) = 0$, we have

$$\tilde{\Pi}^* \mathrm{Tr}_Y(\varphi) \sim_P \mathrm{Tr}_W(\pi^*\varphi). \tag{8.7}$$

Proof We first observe that by Zariski's main theorem, $\nu(\pi^*\varphi, W) = 0$. So the right-hand side of (8.7) makes sense.

Step 1. Assume that φ has analytic singularities. It suffices to apply Example 8.1.4 to reformulate (8.7) as

$$\tilde{\Pi}^*(\varphi|_{\tilde{Y}}) \sim_P (\pi^*\varphi)|_{\tilde{W}}.$$

In fact, the strict equality holds, which is nothing but the functoriality of pullbacks.

Step 2. Next we handle the general case. Choose a smooth closed real $(1,1)$-form θ such that θ_φ is a Kähler current. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\mathrm{PSH}(X,\theta)$. By Corollary 7.1.2, $(\pi^*\varphi_j)_j$ is a quasi-equisingular approximation of $\pi^*\varphi$. From Step 1, we know that for each j,

$$\tilde{\Pi}^* \operatorname{Tr}_Y(\varphi_j) \sim_P \operatorname{Tr}_W(\pi^*\varphi_j).$$

Letting $j \to \infty$, we conclude (8.7) using Proposition 8.2.2. □

Proposition 8.2.3 *Let $\varphi \in \mathrm{QPSH}(X)$ with $\nu(\varphi, Y) = 0$. Assume that Y is smooth. Then for any $\lambda > 0$, we have*

$$\mathcal{I}(\lambda \operatorname{Tr}_Y(\varphi)) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi). \tag{8.8}$$

See Definition 1.4.5 for the definition of Res_Y.

Proof Take a Kähler form ω on X such that ω_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\omega)$.

By definition, for each $j \geq 1$, we get that

$$\operatorname{Tr}_Y(\varphi) \preceq_P \varphi_j|_Y.$$

For any $\lambda' > \lambda > 0$, we can find $j > 0$ so that

$$\mathcal{I}(\lambda'\varphi_j) \subseteq \mathcal{I}(\lambda\varphi).$$

By Theorem 1.4.5, we have

$$\mathcal{I}(\lambda' \operatorname{Tr}_Y(\varphi)) \subseteq \mathcal{I}(\lambda'\varphi_j|_Y) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda'\varphi_j) \subseteq \operatorname{Res}_Y \mathcal{I}(\lambda\varphi).$$

Thanks to Theorem 1.4.4, we conclude (8.8). □

Lastly, we turn our attention to global sections. For this we will need the following global Ohsawa–Takegoshi extension theorem for the trace operator:

Theorem 8.2.1 *Let L be a big line bundle on X and θ is a closed real smooth $(1,1)$-form on X representing $c_1(L)$. Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ and θ_φ is a Kähler current. Assume that $\nu(\varphi, Y) = 0$. Let T be a holomorphic line bundle on X. Then there exists k_0 such that for all $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k \operatorname{Tr}_Y^\theta(\varphi)))$, there exists an extension $\tilde{s} \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$.*

It is of interest to know if one could control the L^2-norm of $\tilde{s}$ in the above result.

Proof Fix a Kähler form ω on X. We may assume that $Y \neq X$ and that $\theta_\varphi \geq 3\delta\omega$ for some $\delta > 0$. Let $(\varphi_j)_j$ be the decreasing quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We can assume that $\theta_{\varphi_j} \geq 2\delta\omega$ for all $j \geq 1$. Also, there exists $\epsilon_0 > 0$ such that $\theta_{(1+\epsilon)\varphi_j} \geq \delta\omega$ for any $\epsilon \in (0, \epsilon_0)$. Take $k_0 = k_0(\delta)$ as in Theorem 1.8.1.

We fix $k \geq k_0$ and $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k \operatorname{Tr}_Y^\theta(\varphi)))$. By Theorem 1.4.4, there exists $\epsilon \in (0, \epsilon_0)$ such that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon) \operatorname{Tr}_Y^\theta(\varphi)))$.

Since $\operatorname{Tr}_Y^\theta(\varphi) \leq \varphi_j|_Y$, we obtain that $s \in \mathrm{H}^0(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j|_Y))$. Due to Theorem 1.8.1 there exists $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k(1+\epsilon)\varphi_j))$ such that $\tilde{s}_j|_Y = s$, for all j.

But by definition of quasi-equisingular approximation, we obtain that for high enough j the inclusion $\mathcal{I}(k(1+\epsilon)\varphi_j) \subseteq \mathcal{I}(k\varphi)$ holds. As a result, $\tilde{s}_j \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))$ for high enough j, finishing the argument. □

8.3 Relation to the classical restricted volumes

Let X be a connected compact Kähler manifold of dimension n and Y be a connected submanifold of dimension m. Fix a big class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$. Take a closed smooth real $(1,1)$-form $\theta \in \alpha$. Fix a Kähler form ω on X.

Recall that the notions of non-Kähler locus and non-nef locus was defined in Definition 1.7.6 and Definition 1.7.7.

When $Y \not\subseteq \mathrm{nK}(\alpha)$, Matsumura ([Mat13, Definition 1.4]) defines the restricted volume of $\{\theta\}$ to Y in the following manner:

$$\operatorname{vol}_{X|Y}(\alpha) \coloneqq \sup_{\varphi} \int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m, \tag{8.9}$$

where φ runs over elements in $\operatorname{PSH}(X, \theta)$ with analytic singularities such that $\varphi|_Y \not\equiv -\infty$. This definition is independent of the choice of θ.

In case $Y \not\subseteq \mathrm{nn}(\alpha)$, Collins–Tosatti [CT22] extend the above definition of restricted volume:

$$\operatorname{vol}_{X|Y}(\alpha) \coloneqq \lim_{\epsilon \searrow 0} \sup_{\varphi} \int_Y (\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\varphi|_Y)^m. \tag{8.10}$$

where φ runs over elements in $\operatorname{PSH}(X, \theta + \epsilon\omega)$ with analytic singularities such that $\varphi|_Y \not\equiv -\infty$. This definition is independent of the choice of θ.

These definitions extend the more classical definition in the algebraic setting due to [ELM+09].

Proposition 8.3.1 *Assume that $Y \not\subseteq \mathrm{nK}(\alpha)$, then*

$$\operatorname{vol}_{X|Y}(\alpha) = \int_Y \left(\theta|_Y + \mathrm{dd^c} \operatorname{Tr}_Y^\theta(V_\theta)\right)^m = \int_Y (\theta|_Y + \mathrm{dd^c} V_\theta|_Y)^m. \tag{8.11}$$

Proof We start with the first equality of (8.11). Since $Y \not\subseteq \mathrm{nK}(\alpha)$, $V_\theta|_Y \not\equiv -\infty$ as a consequence of Theorem 2.4.2, hence also $\nu(V_\theta, Y) = 0$.

Take a quasi-equisingular approximation $(\varphi_j)_{j>0}$ of V_θ with $\varphi_j \in \operatorname{PSH}(X, \theta + \epsilon_j\omega)$. By Theorem 2.4.4, we have

$$\int_Y (\theta|_Y + \epsilon_j\omega + \mathrm{dd^c}\varphi_j|_Y)^m \geq \int_Y (\theta|_Y + \mathrm{dd^c}\varphi|_Y)^m.$$

Letting $j \to \infty$ and applying Example 8.1.6, we conclude that the $\geq$ direction in the first equality of (8.11).

For the reverse direction, by definition, for any fixed $\epsilon > 0$, we have

$$\int_Y (\theta|_Y + \epsilon_j \omega|_Y + \mathrm{dd^c}\varphi_j|_Y)^m \leq \int_Y (\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\varphi_j|_Y)^m \leq \mathrm{vol}_{X|Y}(\{\theta\} + \epsilon\{\omega\})$$

for all large enough j. Letting $j \to \infty$ and $\epsilon \searrow 0$, using the continuity of $\mathrm{vol}_{X|Y}$ ([Mat13, Corollary 4.11]) together with Example 8.1.6, we conclude the first equality of (8.11).

Now we address the second equality. Due to Theorem 2.4.4, the defining formula (8.9), and the definition of V_θ, we obtain that

$$\mathrm{vol}_{X|Y}(\{\theta\}) \leq \int_Y (\theta|_Y + \mathrm{dd^c}V_\theta|_Y)^m.$$

The reverse equality now follows from the first equality of (8.11), Theorem 2.4.4 and the fact that $V_\theta|_Y \preceq_P \mathrm{Tr}_Y^\theta(V_\theta)$ as proved in Proposition 8.1.3. □

Theorem 8.3.1 *If $Y \not\subseteq \mathrm{nn}(\alpha)$, then*

$$\begin{aligned}\mathrm{vol}_{X|Y}(\alpha) &= \lim_{\epsilon\to 0+} \int_Y (\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}V_{\theta+\epsilon\omega}|_Y)^m \\ &= \lim_{\epsilon\to 0+} \int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(V_{\theta+\epsilon\omega})\right)^m \\ &= \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(V_\theta)\right)^m .\end{aligned} \tag{8.12}$$

Proof Since $\nu(V_\theta, Y) = 0$, we have $Y \not\subseteq \mathrm{nK}(\alpha + \epsilon\{\omega\})$ for all $\epsilon > 0$. As a result, due to (8.10) and (8.11) only the last equality of (8.12) needs to be argued.

Thanks to Lemma 6.2.6, we have

$$V_{\theta+\epsilon\omega} \xrightarrow{d_S} V_\theta$$

as $\epsilon \to 0+$.

Therefore, using Proposition 8.2.2, we find

$$\mathrm{Tr}_Y(V_{\theta+\epsilon\omega}) \xrightarrow{d_S} \mathrm{Tr}_Y(V_\theta).$$

Therefore, thanks to Theorem 6.2.1, for any $\epsilon_0 > 0$, we have

$$\lim_{\epsilon\to 0+} \int_Y (\theta|_Y + \epsilon_0\omega|_Y)^m_{\mathrm{Tr}_Y^{\theta+\epsilon_0\omega}(V_{\theta+\epsilon\omega})} = \int_Y (\theta|_Y + \epsilon_0\omega|_Y)^m_{\mathrm{Tr}_Y^{\theta+\epsilon_0\omega}(V_\theta)} \geq \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^\theta(V_\theta)}$$

On the other hand,

$$\begin{aligned}\lim_{\epsilon_0\to 0+}\int_Y (\theta|_Y+\epsilon_0\omega|_Y)^m_{\mathrm{Tr}_Y^{\theta+\epsilon_0\omega}(V_\theta)} &= \lim_{\epsilon_0\to 0+}\int_Y (\theta|_Y+\epsilon_0\omega|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}\\ &= \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}.\end{aligned}$$

These two equations together imply the existence of $(\epsilon_j)_{j>0}$ so that $0<\epsilon_j<1/j$ we have that

$$\lim_{j\to\infty}\int_Y \left(\theta|_Y + j^{-1}\omega|_Y\right)^m_{\mathrm{Tr}_Y^{\theta+j^{-1}\omega}(V_{\theta+\epsilon_j\omega})} = \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}. \tag{8.13}$$

Moreover, for each $j>0$,

$$\begin{aligned}\int_Y \left(\theta|_Y + j^{-1}\omega|_Y\right)^m_{\mathrm{Tr}_Y^{\theta|_Y+j^{-1}\omega|_Y}(V_{\theta+\epsilon_j\omega})} &\geq \int_Y \left(\theta|_Y + \epsilon_j\omega|_Y\right)^m_{\mathrm{Tr}_Y^{\theta+\epsilon_j\omega}(V_{\theta+\epsilon_j\omega})}\\ &\geq \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}.\end{aligned} \tag{8.14}$$

Putting (8.13) and (8.14) together, it results that

$$\lim_{j\to\infty}\int_Y \left(\theta|_Y + \epsilon_j\omega|_Y\right)^m_{\mathrm{Tr}_Y^{\theta+\epsilon_j\omega}(V_{\theta+\epsilon_j\omega})} = \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}.$$

Finally, since $\int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(V_{\theta+\epsilon\omega})\right)^m$ depends monotonically on $\epsilon>0$, we conclude that

$$\lim_{\epsilon\to 0+}\int_Y \left(\theta|_Y + \epsilon\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+\epsilon\omega}(V_{\theta+\epsilon\omega})\right)^m = \int_Y (\theta|_Y)^m_{\mathrm{Tr}_Y^{\theta}(V_\theta)}.$$

8.4 Restricted volumes of line bundles

Let X be a connected projective manifold of dimension n and $Y\subseteq X$ be a connected submanifold of dimension m. Consider a big line bundle L on X, a Hermitian metric h_0 on L with $\theta = c_1(L,h_0)$. Let A be a very ample line bundle on X. Take a Hermitian metric h_A on A such that $\omega = \mathrm{dd^c} h_A$ is a Kähler form.

Using the trace operator, one could prove the following generalization of Theorem 7.4.1.

Theorem 8.4.1 *Let h be a singular plurisubharmonic metric on L with $\nu(\mathrm{dd^c}h, Y)=0$. Assume that*

$$\lim_{\epsilon\to 0+}\left(\mathrm{Tr}_Y^{c_1(L|_Y)+\epsilon\omega}(c_1(L,h))\right)^m > 0. \tag{8.15}$$

Then for any holomorphic line bundle T on X we have that

$$\int_Y \left(\operatorname{Tr}_Y^{c_1(L|_Y)}(c_1(L,h))\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \operatorname{Res}_Y\left(\mathcal{I}(h^k)\right)\right). \quad (8.16)$$

Recall that Res_Y is defined in Definition 1.4.5. Observe that by Example 8.1.6, (8.15) implies that $\operatorname{Tr}_Y^{c_1(L|_Y)}(c_1(L,h))$ is defined. So (8.16) is defined.

We will identify h with $\varphi \in \operatorname{PSH}(X,\theta)$ as in (1.22).

We only need to consider the case $Y \neq X$, since otherwise, the result is proved in Theorem 7.4.1. We will always assume $Y \neq X$ in the sequel.

Lemma 8.4.1 *There is $\psi_Y \in \operatorname{QPSH}(X)$ with neat analytic singularities such that $\{\psi_Y = -\infty\} = Y$ and in an open neighborhood of Y, we have*

$$\psi_Y(x) = 2(n-m)\log \operatorname{dist}(x, Y) \quad (8.17)$$

for some Riemannian distance function $\operatorname{dist}(\cdot, Y)$.

See Definition 1.6.1 for the definition of neat analytic singularities.

See [Fin22a, Lemma 2.3] for the proof.

Lemma 8.4.2 *The multiplier ideal sheaf of ψ_Y can be calculated as*

$$\mathcal{I}(\psi_Y) = \mathcal{I}_Y. \quad (8.18)$$

Moreover, given $y \in Y$ and $\epsilon > 0$, for any germ $f \in \mathcal{I}_{Y,y}$ we have

$$\int_U |f|^\epsilon e^{-\psi_Y}\omega^n < \infty, \quad (8.19)$$

where U is an open neighborhood of y in X.

In other words, ψ_Y has *log canonical singularities*.

Proof Since ψ_Y is locally bounded away from Y, it suffices to prove (8.18) along Y. Fix $y \in Y$, and we will verify (8.18) germ-wise at y.

Take an open neighbourhood $U \subset X$ of y and a biholomorphic map $F: U \to V \times W$, where V is an open neighbourhood of y in Y and W is a connected open subset in $\mathbb{C}^{n-m}$ containing 0, such that $F(Y \cap U) = V \times \{0\}$. For any $x \in U$, write x_V, x_W for the two components of $F(x)$ in V and W respectively. We denote the coordinates in $\mathbb{C}^{n-m}$ as $w_1, \dots, w_{n-m}$.

Due to (8.17), after possibly shrinking U, we may assume that

$$\exp(-\psi_Y(x)) = |x_W|^{2m-2n} + O(1)$$

for any $x \in U \setminus Y$.

Given $f \in \mathcal{I}_{Y,y}$, after shrinking U, we may assume that there exists $g_1, \dots, g_{n-m} \in \mathrm{H}^0(V \times W, \mathcal{O}_{V\times W})$ such that

$$f = \sum_{i=1}^{n-m} w_i g_i.$$

In order to verify $f \in \mathcal{I}(\psi_Y)_y$, it suffices to show $w_i g_i \in \mathcal{I}\left((\sum_{i=1}^{n-m}|w_i|^2)^{m-n}\right)_{F(y)}$, which follows from Fubini's theorem. The proof of (8.19) is similar.

Conversely, take $f \in \mathcal{I}(\psi_Y)$, the similar application of Fubini's theorem shows that after possible shrinking U, we have $f|_Y = 0$. By Rückert's Nullstellensatz [GR84, Page 67], it follows that $f \in \mathcal{I}_Y$. □

Lemma 8.4.3 *Assume that φ has analytic singularity type and θ_u is a Kähler current. Suppose that $\varphi|_Y \not\equiv -\infty$. Then*

$$\int_Y (\theta|_Y + \mathrm{dd}^c\varphi|_Y)^m = \lim_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))\right\}. \quad (8.20)$$

Recall that $\mathcal{I}_\infty$ is defined in Definition 1.6.6.

Proof Suppose that $\epsilon \in (0,1)$ is small enough so that $(1-\epsilon)u \in \mathrm{PSH}(X,\theta)$.

Using Theorem 7.4.1 we can start to write the following sequence of inequalities:

$$\begin{aligned}
&\frac{1}{m!}\int_Y (\theta|_Y + \mathrm{dd}^c\varphi|_Y)^m \\
=&\lim_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi|_Y)\right)\\
\leq& \varliminf_{k\to\infty} \frac{1}{k^m} \dim \left\{ s|_Y : s\in \mathrm{H}^0\left(X, T\otimes L^k\otimes \mathcal{I}(k\varphi)\right)\right\} \quad \text{by Theorem 1.8.1}\\
\leq& \varlimsup_{k\to\infty} \frac{1}{k^m} \dim \left\{ s|_Y : s\in \mathrm{H}^0\left(X, T\otimes L^k\otimes \mathcal{I}(k\varphi)\right)\right\}\\
\leq& \varlimsup_{k\to\infty} \frac{1}{k^m} \dim \left\{ s|_Y : s\in \mathrm{H}^0\left(X, T\otimes L^k\otimes \mathcal{I}_\infty((1-\epsilon)k\varphi)\right)\right\} \quad \text{by Lemma 1.6.3}\\
\leq& \varlimsup_{k\to\infty} \frac{1}{k^m} \dim \left\{ s\in \mathrm{H}^0\left(Y, T|_Y\otimes L|_Y^k\right) : \log h^k(s,s) \leq (1-\epsilon)k\varphi|_Y\right\}\\
\leq& \varlimsup_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}((1-\epsilon)k\varphi|_Y)\right)\\
=&\frac{1}{m!}\int_Y (\theta|_Y + (1-\epsilon)\mathrm{dd}^c\varphi|_Y)^m \quad \text{by Theorem 7.4.1.}
\end{aligned}$$

Letting $\epsilon \to 0$, (8.20) follows from multi-linearity of the non-pluripolar product. □

Proposition 8.4.1 *In the setting of Theorem 8.4.1, assume that* $\mathrm{dd}^c h$ *is a Kähler current. Then* (8.16) *holds.*

Proof Let $(\varphi_j)_j$ a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. After possibly replacing $(\varphi_j)_j$ by a subsequence, there exists $\epsilon_0 \in (0,1)\cap\mathbb{Q}$ such that $\theta_{(1-\epsilon)^2\varphi_j}$ and $\theta_{(1-\epsilon)\varphi_j}$ are also Kähler currents for any $\epsilon \in (0,\epsilon_0)$.

We claim that for any $j \geq 1$ and $k \in \mathbb{N}$, we have

$$\mathcal{I}_\infty\left((1-\epsilon)k\varphi_j\right) \cap \mathcal{I}(\psi_Y) \subseteq \mathcal{I}\left((1-\epsilon)^2 k\varphi_j + \psi_Y\right). \quad (8.21)$$

Take $x \in X$, and it suffices to argue (8.21) along the germ of x. Since ψ_Y is locally bounded outside Y, we may assume that $x \in Y$. Recall that by Lemma 8.4.2, $\mathcal{I}(\psi_Y) = \mathcal{I}_Y$.

Let $f \in \mathcal{I}_\infty((1-\epsilon)k\varphi_j)_x \cap \mathcal{I}(\psi_Y)_x$. Then there is an open neighborhood U of x in X such that $|f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j} \leq C$ holds on $U \setminus \{\varphi_j = -\infty\}$ for some $C > 0$, hence

$$\begin{aligned}\int_U |f|^2\mathrm{e}^{-k(1-\epsilon)^2\varphi_j-\psi_Y}\,\omega^n =& \int_U |f|^{2(1-\epsilon)}\mathrm{e}^{-k(1-\epsilon)^2\varphi_j}|f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n\\ \leq& C\int_U |f|^{2\epsilon}\mathrm{e}^{-\psi_Y}\,\omega^n < \infty,\end{aligned}$$

where the last inequality follows from Lemma 8.4.2. We have proved the claim (8.21).

Next we consider the following composition morphism of coherent sheaves on Y:

$$\mathrm{Res}_Y\,\mathcal{I}_\infty((1-\epsilon)k\varphi_j) \hookrightarrow \frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}_\infty((1-\epsilon)k\varphi_j)\cap\mathcal{I}_Y} \to \frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}. \quad (8.22)$$

Here we have identified the coherent $\mathcal{O}_X$-modules supported on Y with coherent $\mathcal{O}_Y$-modules. Note that the target of (8.22) is also supported on Y as ψ_Y is locally bounded outside Y. We denote the coherent $\mathcal{O}_Y$-module whose pushforward to X gives $\frac{\mathcal{I}((1-\epsilon)^2k\varphi_j)}{\mathcal{I}((1-\epsilon)^2k\varphi_j+\psi_Y)}$ by $\mathcal{I}_{k,j}$.

In (8.22), the first map is the inclusion and the second one is the obvious projection induced by (8.21). Although in general the second map fails to be injective, we observe that the composition is still injective as

$$\mathcal{I}\left((1-\epsilon)^2k\varphi_j+\psi_Y\right) \subseteq \mathcal{I}(\psi_Y) = \mathcal{I}_Y.$$

Therefore, for any $k \in \mathbb{N}$, we have an injective morphism of coherent $\mathcal{O}_Y$-modules:

$$L|_Y^k \otimes T|_Y \otimes \mathrm{Res}_Y\,\mathcal{I}_\infty\left((1-\epsilon)k\varphi_j\right) \hookrightarrow L|_Y^k \otimes T|_Y \otimes \mathcal{I}_{k,j}. \quad (8.23)$$

Using Theorem 7.4.1 we can start the following inequalities:

$$
\begin{aligned}
&\frac{1}{m!}\int_Y \left(\theta|_Y + dd^c \operatorname{Tr}_Y^\theta(\varphi)\right)^m \\
&= \lim_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k \operatorname{Tr}_Y^\theta(\varphi)\right)\right) \quad \text{by Theorem 7.4.1} \\
&\le \varliminf_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \operatorname{Res}_Y\left(\mathcal{I}(k\varphi)\right)\right) \quad \text{by Theorem 1.4.5} \\
&\le \varlimsup_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \operatorname{Res}_Y\left(\mathcal{I}(k\varphi)\right)\right) \\
&\le \varlimsup_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k\varphi_j\right)\big|_Y\right) \\
&\le \varlimsup_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}_\infty\left((1-\epsilon)k\varphi_j\right)\big|_Y\right) \quad \text{by Lemma 1.6.3} \\
&\le \varlimsup_{k\to\infty} \frac{1}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}_{k,j}\right) \quad \text{by (8.23)} \\
&\le \varlimsup_{k\to\infty} \frac{1}{k^m} \dim\left\{ s|_Y : s \in \mathrm{H}^0\left(X, T\otimes L^k \otimes \frac{\mathcal{I}((1-\epsilon)^2 k\varphi_j)}{\mathcal{I}((1-\epsilon)^2 k\varphi_j + \psi_Y)}\right)\right\} \\
&= \varlimsup_{k\to\infty} \frac{1}{k^m} \dim\left\{ s|_Y : s \in \mathrm{H}^0\left(X, T\otimes L^k \otimes \mathcal{I}\left((1-\epsilon)^2 k\varphi_j\right)\right)\right\} \\
&= \frac{1}{m!}\int_Y \left(\theta|_Y + (1-\epsilon)^2 dd^c \varphi_j|_Y\right)^m \quad \text{by Lemma 8.4.3,}
\end{aligned}
$$

where in the penultimate line we used [CDM17, Theorem 1.1(6)] for $q = 0$. Letting $\epsilon \to \infty$ and then $j \to \infty$ the result follows. □

Proof (Proof of Theorem 8.4.1) Using Proposition 8.2.3 and Theorem 7.4.1 we obtain that

$$
\begin{aligned}
\int_Y \left(\theta|_Y + dd^c \operatorname{Tr}_Y^\theta(\varphi)\right)^m &= \lim_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k \operatorname{Tr}_Y^\theta(\varphi)\right)\right) \\
&\le \varliminf_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \operatorname{Res}_Y\left(\mathcal{I}(k\varphi)\right)\right).
\end{aligned}
$$

Now we address the other direction in (8.16). Let $\phi \in \mathrm{H}^0(X, A)$ be a section that does not vanish identically on Y. Such ϕ exists since A is very ample.

We fix $k_0 \in \mathbb{N}$. For any $k \ge 0$, we have that $k = qk_0 + r$ with $q, r \in \mathbb{N}$ and $r \in \{0, \dots, k_0 - 1\}$. Also, we have an injective linear map

$$
\mathrm{H}^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k\varphi|_Y\right)\right) \xrightarrow{\cdot\phi^{\otimes q}} \mathrm{H}^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}\left(k\varphi|_Y\right)\right).
$$

Therefore,

$$
\begin{aligned}
&\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k\varphi|_Y\right)\right)\\
\leq &\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes A|_Y^q \otimes \mathcal{I}\left(k\varphi|_Y\right)\right)\\
=&\frac{1}{k_0^m}\varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}\left(k\varphi|_Y\right)\right)\\
\leq&\frac{1}{k_0^m}\varlimsup_{q\to\infty} \frac{m!}{q^m} h^0\left(Y, T|_Y \otimes L|_Y^{qk_0} \otimes A|_Y^q \otimes L|_Y^r \otimes \mathcal{I}\left(k_0 q\varphi|_Y\right)\right)\\
=&\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta+k_0^{-1}\omega}(\varphi)\right)^m\\
=&\int_Y \left(\theta|_Y + k_0^{-1}\omega|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m,
\end{aligned}
$$

where in the fourth line we have used that $k_0 q \leq k$ and in the last line we have used Proposition 8.4.1 for the big line bundle $L^{k_0} \otimes A$, the Kähler current $k_0\theta_u - \mathrm{dd^c}\log g = k_0\theta_u + \omega$, and twisting bundle $T \otimes L^r$. Letting $k_0 \to \infty$, we conclude that

$$
\varlimsup_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k\varphi|_Y\right)\right) \leq \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m .
$$

Theorem 8.4.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$ such that $\nu(\varphi, Y) = 0$. Assume that θ_φ is a Kähler current. Then*

$$
\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m = \lim_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right)\right\}.
$$

Proof This is a consequence of Theorem 7.4.1, Theorem 8.2.1 and Theorem 8.4.1:

$$
\begin{aligned}
\int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m &= \lim_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}\left(k\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)\right)\\
&\leq \varliminf_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right)\right\}\\
&\leq \varlimsup_{k\to\infty} \frac{m!}{k^m} \dim_{\mathbb{C}} \left\{ s|_Y : s \in \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right)\right\}\\
&\leq \lim_{k\to\infty} \frac{m!}{k^m} h^0\left(Y, T|_Y \otimes L|_Y^k \otimes \mathcal{I}(k\varphi)|_Y\right)\\
&= \int_Y \left(\theta|_Y + \mathrm{dd^c}\,\mathrm{Tr}_Y^{\theta}(\varphi)\right)^m .
\end{aligned}
$$

Remark 8.4.1 One could also show that when (8.15) fails, the right-hand side of (8.16) is 0. See [DX24a].

8.5 Analytic Bertini theorems

Let X be a connected projective manifold of dimension $n \geq 1$.

The analytic Bertini theorem handles the restriction along a generic subvariety.

Theorem 8.5.1 *Let $\varphi \in \mathrm{QPSH}(X)$. Let $p\colon X \to \mathbb{P}^N$ be a morphism ($N \geq 1$). Define*

$$\mathcal{G} := \left\{ H \in |\mathcal{O}_{\mathbb{P}^N}(1)| : H' := H \cap X \text{ is smooth and } \mathcal{I}(\varphi|_{H'}) = \mathrm{Res}_{H'}\big(\mathcal{I}(\varphi)\big) \right\}.$$

Then $\mathcal{G} \subseteq |\mathcal{O}_{\mathbb{P}^N}(1)|$ is co-pluripolar.

Recall that co-pluripolar sets are defined in Definition 1.1.4. We adopt the convention that $\mathcal{I}(-\infty) = 0$.

Remark 8.5.1 Here and in the sequel, we slightly abuse the notation by writing $H \cap X$ for $p^{-1}H$, the scheme-theoretic inverse image of H. In other words, $H \cap X := H \times_{\mathbb{P}^N} X$.

By definition, any $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ such that $p^{-1}H = \emptyset$ lies in $\mathcal{G}$.

Proof Take an ample line bundle L with a smooth Hermitian metric h such that $c_1(L,h) + \mathrm{dd^c}\varphi \geq 0$, where $c_1(L,h)$ is the first Chern form of (L,h), namely the curvature form of h. We introduce $\Lambda := |\mathcal{O}_{\mathbb{P}^N}(1)|$ to simplify our notations.

Step 1. We prove that the following set is co-pluripolar:

$$\begin{aligned}\mathcal{G}_L := \big\{ H \in \Lambda : H \cap X \text{ is smooth and } \mathrm{H}^0\left(H\cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})\right) = \\ \mathrm{H}^0\left(H\cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathrm{Res}_{H\cap X}(\mathcal{I}(\varphi))\right)\big\}.\end{aligned}$$

Here $\omega_{H\cap X}$ denotes the dualizing sheaf of $H \cap X$.

Let $U \subseteq \Lambda \times X$ be the closed subvariety whose $\mathbb{C}$-points correspond to pairs $(H,x) \in \Lambda \times X$ with $p(x) \in H$. Let $\pi_1\colon U \to \Lambda$ be the natural projection. We may assume that π_1 is surjective, as otherwise there is nothing to prove.

Observe that U is a local complete intersection scheme by *Krulls Hauptidealsatz* and *a fortiori* a Cohen–Macaulay scheme. It follows from miracle flatness [Mat89, Theorem 23.1] that the natural projection $\pi_2\colon U \to X$ is flat. As the fibers of π_2 over closed points of X are isomorphic to $\mathbb{P}^{N-1}$, it follows that π_2 is smooth. Thus, U is smooth as well. Moreover, observe that

$$\mathcal{I}(\pi_2^*\varphi) = \pi_2^*\mathcal{I}(\varphi) \tag{8.24}$$

by Proposition 1.4.5.

In the following, we will construct pluripolar sets $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \Lambda$ such that the behaviour of π_1 is improved successively on the complement of Σ_i.

Step 1.1. The usual Bertini theorem shows that there is a proper Zariski closed set $\Sigma_1 \subseteq \Lambda$ such that π_1 has smooth fibres outside Σ_1. Enlarging Σ_1, we could guarantee that π_1 and $\mathcal{I}(\pi_2^*\varphi)$ are both flat outside Σ_1. See [DG65, Théorème 6.9.1]. Then after further enlarging Σ_1 so that H avoids all associated points of $\mathcal{O}_X/\mathcal{I}(\varphi)$, for all $H \in \Lambda \setminus \Sigma_1$. Let $\pi_{1,H}$ denote the fibre of π_1 at H and write $i_H\colon \pi_{1,H} \to U$ for the inclusion morphism. We arrive at

$$\mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi)) = i_H^*\mathcal{I}(\pi_2^*\varphi)$$

for all $H \in \Lambda \setminus \Sigma_1$.[3]

Step 1.2. By Grauert's coherence theorem,

$$\mathcal{F}^i := R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L \otimes \mathcal{I}(\pi_2^*\varphi)\right)$$

is coherent for all i. Here $\omega_{U/\Lambda}$ denotes the relative dualizing sheaf of the morphism $U \to \Lambda$. Thus, there is a proper Zariski closed set $\Sigma_2 \subsetneq \Lambda$ such that

(1) $\Sigma_2 \supseteq \Sigma_1$.
(2) The $\mathcal{F}^i$'s are locally free outside Σ_2.

We write $\mathcal{F} = \mathcal{F}^0$. By cohomology and base change [Har77, Theorem III.12.11], for any $H \in \Lambda \setminus \Sigma_2$, the fibre $\mathcal{F}|_H$ of $\mathcal{F}$ is given by

$$\mathcal{F}|_H = \mathrm{H}^0\left(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \pi_2^*L|_{\pi_{1,H}} \otimes \mathrm{Res}_{\pi_{1,H}}(\mathcal{I}(\pi_2^*\varphi))\right).$$

Step 1.3. In order to proceed, we need to make use of the Hodge metric $h_{\mathcal{H}}$ on $\mathcal{F}$ defined in [HPS18]. We briefly recall its definition in our setting. By [HPS18, Section 22], we can find a proper Zariski closed set $\Sigma_3 \subsetneq \Lambda$ such that

(1) $\Sigma_3 \supseteq \Sigma_2$,
(2) π_1 is smooth outside Σ_3,
(3) both $\mathcal{F}$ and $\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)/\mathcal{F}$ are locally free outside Σ_3, and
(4) for each i,
$$R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*L\right)$$
is locally free outside Σ_3.

Then for any $H \in \Lambda \setminus \Sigma_3$,

$$\mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \subseteq \mathcal{F}|_H \subseteq \mathrm{H}^0(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X}).$$

See [HPS18, Lemma 22.1].

Now we can give the definition of the Hodge metric on $\Lambda\setminus\Sigma_3$. Given any $H \in \Lambda\setminus\Sigma_3$, any $\alpha \in \mathcal{F}|_H$, the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha, \alpha) := \int_{X\cap H} |\alpha|_h^2 \mathrm{e}^{-\varphi} \in [0, \infty].$$

Observe that $h_{\mathcal{H}}(\alpha, \alpha) < \infty$ if and only if $\alpha \in \mathrm{H}^0(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes\mathcal{I}(\varphi|_{H\cap X}))$. Moreover, $h_{\mathcal{H}}(\alpha, \alpha) > 0$ if $\alpha \neq 0$. It is shown in [HPS18] (c.f. [PT18, Theorem 3.3.5]) that $h_{\mathcal{H}}$ is indeed a singular Hermitian metric, and it extends to a positive metric on $\mathcal{F}$.

Step 1.4. The determinant $\det h_{\mathcal{H}}$ is singular at all $H \in \Lambda \setminus \Sigma_3$ such that

$$\mathrm{H}^0\left(H \cap X, \omega_{H\cap X} \otimes L|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})\right) \neq \mathcal{F}|_H.$$

[3] This subtle point was overlooked in the proof of [Xia22a].

As the map π_2 is smooth, we have $\pi_2^*\mathcal{I}(\varphi) = \mathcal{I}(\pi_2^*\varphi)$ by Proposition 1.4.5. Under the identification $\pi_{1,H} \cong H \cap X$, we have

$$\operatorname{Res}_{\pi_{1,H}}\left(\pi_2^*\mathcal{I}(\varphi)\right) \cong \operatorname{Res}_{H\cap X}\left(\mathcal{I}(\varphi)\right).$$

Thus, we have the following inclusions:

$$\begin{aligned}&\mathrm{H}^0\Big(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes \mathcal{I}\left(\varphi|_{H\cap X}\right)\Big)\\ \subseteq&\mathrm{H}^0\Big(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes \operatorname{Res}_{H\cap X}\big(\mathcal{I}(\varphi)\big)\Big),\end{aligned}$$

the right-hand side being $\mathcal{F}|_H$.

Recall that the first inclusion follows from Theorem 1.4.5. Hence, $\det h_{\mathcal{H}}$ is singular at all $H \in |\mathcal{O}_{\mathbb{P}^N}(1)| \setminus \Sigma_3$ such that

$$\begin{aligned}&\mathrm{H}^0\Big(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes \mathcal{I}\left(\varphi|_{H\cap X}\right)\Big)\\ \neq&\mathrm{H}^0\Big(H\cap X, \omega_{H\cap X}\otimes L|_{H\cap X}\otimes \operatorname{Res}_{H\cap X}\big(\mathcal{I}(\varphi)\big)\Big).\end{aligned}$$

Let Σ_4 be the union of Σ_3 and the set of all such H. Since the Hodge metric $h_{\mathcal{H}}$ is positive ([PT18, Theorem 3.3.5] and [HPS18, Theorem 21.1]), its determinant $\det h_{\mathcal{H}}$ is also positive ([Rau15, Proposition 1.3] and [HPS18, Proposition 25.1]), it follows that Σ_4 is pluripolar. As a consequence, $\mathcal{G}_L$ is co-pluripolar.

Step 2.

Fix an ample invertible sheaf S on X. The same result holds with $L\otimes S^{\otimes a}$ in place of L. Thus, the set

$$A := \bigcap_{a=0}^{\infty} \mathcal{G}_{L\otimes S^{\otimes a}}$$

is co-pluripolar. For each $H \in W$ such that $X\cap H$ is smooth and $\mathcal{I}(\varphi|_{X\cap H}) \neq \operatorname{Res}_{H\cap X}(\mathcal{I}(\varphi))$, let $\mathcal{K}$ be the following cokernel:

$$0 \to \mathcal{I}\left(\varphi|_{X\cap H}\right) \to \operatorname{Res}_{H\cap X}\big(\mathcal{I}(\varphi)\big) \to \mathcal{K} \to 0.$$

By Serre vanishing theorem, taking a large enough, we may guarantee that

$$\mathrm{H}^1\Big(X\cap H, \omega_{X\cap H}\otimes (L\otimes S^{\otimes a})|_{X\cap H}\otimes \mathcal{I}(\varphi|_{X\cap H})\Big) = 0$$

and

$$\mathrm{H}^0\Big(X\cap H, \omega_{X\cap H}\otimes (L\otimes S^{\otimes a})|_{X\cap H}\otimes \mathcal{K}\Big) \neq 0.$$

Then

$$\begin{aligned}&\mathrm{H}^0\Big(X\cap H, \omega_{X\cap H}\otimes (L\otimes S^{\otimes a})|_{X\cap H}\otimes \mathcal{I}\left(\varphi|_{X\cap H}\right)\Big) \neq\\ &\mathrm{H}^0\Big(X\cap H, \omega_{X\cap H}\otimes (L\otimes S^{\otimes a})|_{X\cap H}\otimes \operatorname{Res}_{H\cap X}\left(\mathcal{I}(\varphi)\right)\Big).\end{aligned}$$

Thus, $H \notin A$. We conclude that $\mathcal{G}$ is co-pluripolar. □

Remark 8.5.2 More generally, the same technique implies the following general result: Let $f : X \to Y$ be a projective morphism between complex manifolds and (L, h) be a Hermitian pseudo-effective line bundle on X. Then for quasi-every[4] $y \in Y$, the fiber X_y is smooth and

$$\mathcal{I}(\lambda h|_{X_y}) = \mathrm{Res}_{X_y}(\mathcal{I}(\lambda h)).$$

In the sequel of this section, we fix a base-point free linear system Λ on X.

Corollary 8.5.1 *Let $\varphi \in \mathrm{QPSH}(X)$. Then for quasi-every $H \in \Lambda$, we have $\varphi|_H \not\equiv -\infty$.*

Proof This follows immediately from Theorem 8.5.1. □

Corollary 8.5.2 *Assume that $n \geq 2$. Let $\varphi \in \mathrm{QPSH}(X)$. Then quasi-every $H \in \Lambda$ is connected and smooth, satisfies $\nu(\varphi, H) = 0$ and we have*

$$\mathrm{Tr}_H(\varphi) \sim_{\mathcal{I}} \varphi|_H.$$

The assumption $n \geq 2$ is only to guarantee that a general element $H \in \Lambda$ is connected, since we developed most of our theories only in this case.

Proof First observe that the set $\{x \in X : \nu(\varphi, x) > 0\}$ is a countable union of proper analytic subsets by Theorem 1.4.1. It follows that a very general element in Λ is not contained in this set.

Fix an ample line bundle L so that there is a smooth psh metric h_L such that $c_1(L, h_L) + \mathrm{dd^c}\varphi$ is a Kähler current. Thanks to Theorem 8.5.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that each $H \in \Lambda'$ satisfies the following:

(1) H is smooth;
(2) $\nu(\varphi, H) = 0$;
(3) $\mathcal{I}(k\varphi|_H) = \mathrm{Res}_H(\mathcal{I}(k\varphi))$ for all $k > 0$.

It follows from Theorem 8.4.1 and Theorem 7.4.1 that

$$\int_H \left(c_1(L, h_L)|_H + \mathrm{dd^c}\,\mathrm{Tr}_Y^{c_1(L,h_L)}(\varphi)\right)^{n-1} = \int_H \left(c_1(L, h_L)|_H + \mathrm{dd^c}\varphi|_H\right)^{n-1}.$$

Since $\varphi|_H \preceq_P \mathrm{Tr}_Y(\varphi)$ by Proposition 8.1.3, our assertion follows. □

Lemma 8.5.1 *Assume that $n \geq 2$. Let T be a closed positive $(1,1)$-current on X with $\int_X T^n > 0$. Then quasi-every $H \in \Lambda$ is connected and smooth, $T|_H$ is well-defined and satisfies*

$$\int_H T|_H^{n-1} > 0.$$

Proof Write $T = \theta_\varphi$ for some smooth closed real $(1,1)$-form θ on X and $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$. Thanks to Lemma 2.4.3, we can find $\psi \in \mathrm{PSH}(X, \theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. By Corollary 8.5.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that each $H \in \Lambda'$ satisfies:

[4] That is, for all y outside a pluripolar subset of Y.

(1) H is smooth and connected;
(2) the restriction $\psi|_H$ is not identically $-\infty$.

Therefore, $\psi|_H \leq \varphi|_H$ are two potentials in $\mathrm{PSH}(H, \theta|_H)$ for any $H \in \Lambda'$. Our assertion follows from Theorem 2.4.4. □

Corollary 8.5.3 *Assume that $n \geq 2$. Let T be a closed positive $(1,1)$-current on X with $\operatorname{vol} T > 0$. Then quasi-every $H \in \Lambda$ is connected and smooth, and $\operatorname{Tr}_H^{[T]|_H}(T)$ is well-defined.*

Proof This follows from Example 8.1.6, Corollary 8.5.2 and Lemma 8.5.1. □

Proposition 8.5.1 *Assume that $n \geq 2$. Let $\varphi, \psi \in \mathrm{QPSH}(X)$. Assume that $\varphi \leq_P \psi$. Then quasi-every $H \in \Lambda$ is connected and smooth, and $\varphi|_H \leq_P \psi|_H$.*

Proof Thanks to Lemma 6.1.3, we may replace φ by $\varphi \vee \psi$ and assume that $\varphi \sim_P \psi$. It suffices to show that $\varphi|_H \sim_P \psi|_H$ for quasi-every $H \in \Lambda$.

Take a smooth closed real $(1,1)$-form θ on X so that $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. It suffices to compare φ and ψ with $P_\theta[\varphi]$, so without loss of generality, we may assume that ψ is a model potential in $\mathrm{PSH}(X, \theta)_{>0}$. Up to adding a constant to φ, we may then assume that $\varphi \leq \psi$. It follows from Lemma 2.4.2 that we can find a sequence $(\eta_j)_j$ in $\mathrm{PSH}(X, \theta)_{>0}$ such that

$$j^{-1}\eta_j + \left(1 - j^{-1}\right)\psi \leq \varphi$$

for all $j \geq 2$. By Corollary 8.5.1, Lemma 8.5.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that any $H \in \Lambda'$ satisfies:

(1) H is smooth and connected;
(2) $\eta_j|_H \in \mathrm{PSH}(H, \theta|_H)_{>0}$ for all $j \geq 2$ and $\psi|_H \in \mathrm{PSH}(H, \theta|_H)_{>0}$.

Therefore, taking Proposition 3.1.8 into account, we arrive at

$$j^{-1}P_{\theta|_H}\left[\eta_j|_H\right] + \left(1 - j^{-1}\right)P_{\theta|_H}\left[\psi|_H\right] \leq P_{\theta|_H}\left[\varphi|_H\right]$$

for all $j \geq 2$. Letting $j \to \infty$, we conclude that

$$P_{\theta|_H}[\psi|_H] \leq P_{\theta|_H}[\varphi|_H]$$

and hence $\psi|_H \leq_P \varphi|_H$. □

Lemma 8.5.2 *Assume that $n \geq 2$. Let θ be a closed smooth $(1,1)$-form on X representing a big cohomology class and $(\varphi_j)_j$ be a decreasing sequence in $\mathrm{PSH}(X, \theta)$. Assume that $\varphi \in \mathrm{PSH}(X, \theta)$ and $\varphi_j \xrightarrow{d_S} \varphi$. Then quasi-every $H \in \Lambda$ is connected and smooth, $\varphi_j|_H \not\equiv -\infty$ for all $j \geq 1$, $\varphi|_H \not\equiv -\infty$, and*

$$\varphi_j|_H \xrightarrow{d_S} \varphi|_H.$$

Proof By Corollary 6.2.8, we may assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Using Lemma 2.4.2, we could find a decreasing sequence $(\epsilon_j)_j$ in $(0,1)$ with limit 0 and $\eta_j \in \mathrm{PSH}(X,\theta)_{>0}$ such that $\eta_j \leq \varphi_j$ and

$$\epsilon_j\eta_j + (1-\epsilon_j)\varphi_j \leq \varphi.$$

By Corollary 8.5.1, Lemma 8.5.1, we can find a co-pluripolar set $\Lambda' \subseteq \Lambda$ such that any $H \in \Lambda'$ satisfies:

(1) H is smooth and connected;
(2) $\eta_j|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$ for all $j \geq 1$ and $\varphi|_H \in \mathrm{PSH}(H,\theta|_H)_{>0}$.

Therefore, taking Proposition 3.1.8 into account, we arrive at

$$\epsilon_j P_{\theta|_H}\left[\eta_j|_H\right] + (1-\epsilon_j)\, P_{\theta|_H}\left[\varphi_j|_H\right] \leq P_{\theta|_H}\left[\varphi|_H\right].$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} P_{\theta|_H}\left[\varphi_j|_H\right] \leq P_{\theta|_H}\left[\varphi|_H\right].$$

By Theorem 2.4.4 and Proposition 3.1.10, we conclude that

$$\lim_{j\to\infty}\int_H \left(\theta|_H + \mathrm{dd^c}\varphi_j|_H\right)^{n-1} = \int_H \left(\theta|_H + \mathrm{dd^c}\varphi|_H\right)^{n-1}.$$

Therefore, using Corollary 6.2.5, we conclude that $\varphi_j|_H \xrightarrow{d_S} \varphi|_H$. □

Corollary 8.5.4 *Assume that $n \geq 2$. Let $\varphi \in \mathrm{QPSH}(X)$ be an $\mathcal{I}$-good potential. Then quasi-every $H \in \Lambda$ satisfies:*

(1) *H is connected and smooth;*
(2) *$\varphi|_H \in \mathrm{PSH}(X,\theta|_H)$ is $\mathcal{I}$-good;*
(3) *$\nu(\varphi, H) = 0$;*
(4) *$\mathrm{Tr}_H\,\varphi \sim_P \varphi|_H$.*

Furthermore, if θ is a closed smooth real $(1,1)$-form on X such that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, then we could further guarantee that $\mathrm{Tr}_H(\varphi)$ has a representative $\mathrm{Tr}_H(\varphi) \in \mathrm{PSH}(H,\theta|_H)_{>0}$ for all $H \in \Lambda'$.

Proof This is a consequence of Lemma 8.5.2, Theorem 7.1.1, Corollary 8.5.2 and Corollary 8.5.3. □

For later use, let us also prove a reverse Bertini theorem herem.

Lemma 8.5.3 (Reverse Bertini theorem) *Let X be a complex manifold, $f\colon X \to \Delta^*$ be a projective surjective morphism to the punctured unit disk Δ^*. Let (L,h), (L,h') be Hermitian pseudo-effective line bundles on X with the same underlying line bundle. Assume that there is a biholomorphic S^1-action on (X,L) making f equivariant and such that h and h' are invariant under this action. Assume that for quasi-every $z \in \Delta^*$, X_z is smooth and $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$, then $h \sim_{\mathcal{I}} h'$.*

Proof We need to show that $\mathcal{I}(kh) = \mathcal{I}(kh')$ for all positive integer k. Clearly, it suffices to prove the case $k = 1$. We will therefore prove $\mathcal{I}(h) = \mathcal{I}(h')$. First observe that it suffices to prove that

$$f_*(K_X \otimes L \otimes \mathcal{I}(h)) = f_*(K_X \otimes L \otimes \mathcal{I}(h')) \tag{8.25}$$

as subsheaves of $f_*(K_X \otimes L)$. In fact, suppose that (8.25) holds. Let H be a f-ample invertible sheaf on X, then (8.25) also holds with $L \otimes H^m$ in place of L. It follows from Grauert–Remmert's version of Serre vanishing theorem [BS76, Theorem 2.1(A)] that $\mathcal{I}(h) = \mathcal{I}(h')$.

It remains to prove (8.25). Observe that both sides of (8.25) are locally free by [Mat22, Corollary 1.5]. We claim that it suffices to show that

$$f_* \left(K_X \otimes L \otimes \mathcal{I}(h)\right)_z = f_* \left(K_X \otimes L \otimes \mathcal{I}(h')\right)_z \tag{8.26}$$

for one $z \in \Delta^*$. In fact, this implies that the same holds outside a countable subset of Δ^*. But the set where (8.26) fails has to be S^1-invariant, it has to be empty.

In fact, we will prove (8.26) for quasi-every $z \in \Delta^*$. By cohomology and base change together with the analytic Bertini theorem Remark 8.5.2, for quasi-every $z \in \Delta^*$, we have

$$\begin{aligned} f_* \left(K_X \otimes L \otimes \mathcal{I}(h)\right)_z &= \mathrm{H}^0 \left(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h|_{X_z})\right), \\ f_* \left(K_X \otimes L \otimes \mathcal{I}(h')\right)_z &= \mathrm{H}^0 \left(X_z, K_X|_{X_z} \otimes L|_{X_z} \otimes \mathcal{I}(h'|_{X_z})\right). \end{aligned}$$

But we assumed that for quasi-every z, $h|_{X_z} \sim_{\mathcal{I}} h|'_{X_z}$, it follows that for quasi-every $z \in \Delta^*$, (8.26) holds. The proof is complete. □

Chapter 9
Test curves

Comment se fait-il que M. Gauss ait osé vous faire dire que la plupart de vos théorèmes lui étaient connus et qu'il en avait fait la découverte dès 1808. Cet excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertes des autres.
— Adrien-Marie Legendre[a], in a letter to Jacobi in 1827

[a] Adrien-Marie Legendre (1752–1833) was a French mathematician known for his foundational contributions to number theory, statistics, and mathematical analysis. Apart from his mathematical contributions, he also helped formalize the metric system during the French Revolution.

In this chapter, we develop the theory of test curves. Roughly speaking, a test curve is a concave curve of model potentials. In Section 9.2, we will prove the Ross–Witt Nyström[1] correspondence, through which the test curves are related to geodesic rays in the space of quasi-plurisubharmonic functions. Our version of the correspondence here is more general than all similar results in the literature. In Section 9.4, we define operations on test curves, anticipating applications in non-Archimedean pluripotential theory in Chapter 13.

We shall freely apply all results in Appendix A. The results in that appendix are all about convex functions. When we apply those results to concave functions, we always apply to their negatives.

9.1 The notion of test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class.

Recall that the notion of model potentials is defined in Definition 3.1.3.

Definition 9.1.1 A *test curve* Γ in $\mathrm{PSH}(X,\theta)$ consists of a real number $\Gamma_{\max}$ together with a map $(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta)$ denoted by $\tau \mapsto \Gamma_\tau$ satisfying the following conditions:

(1) The map $\tau \mapsto \Gamma_\tau$ is concave and decreasing;
(2) each Γ_τ is a model potential;
(3) the potential

$$\Gamma_{-\infty} \coloneqq \sup_{\tau<\Gamma_{\max}}{}^{*}\Gamma_\tau \tag{9.1}$$

[1] Witt and Nyström are both family names of a single person. Some Swedes have double family names. It should not be spelled as Witt-Nyström as some people do in the literature.

satisfies

$$\int_X (\theta + \mathrm{dd^c}\Gamma_{-\infty})^n > 0.$$

Let $\phi \in \mathrm{PSH}(X, \theta)_{>0}$ be a model potential. The set of test curves Γ with $\Gamma_{-\infty} = \phi$ is denoted by $\mathrm{TC}(X, \theta; \phi)$.

The union of all $\mathrm{TC}(X, \theta; \phi)$'s for various model potentials $\phi \in \mathrm{PSH}(X, \theta)_{>0}$ is denoted by $\mathrm{TC}(X, \theta)_{>0}$.

By (2), $\sup_X \Gamma_\tau = 0$ for each $\tau < \Gamma_{\max}$. So $\Gamma_{-\infty} \in \mathrm{PSH}(X, \theta)$ by Proposition 1.2.1. Moreover, $\Gamma_{-\infty}$ is a model potential by Proposition 3.1.11.

Remark 9.1.1 Sometimes it is convenient to extend Γ_τ to $\tau \geq \Gamma_{\max}$ as well. This can be done as follows: For $\tau > \Gamma_{\max}$, we set $\Gamma_\tau \equiv -\infty$. For $\tau = \Gamma_{\max}$, we set

$$\Gamma_\tau \coloneqq \inf_{\tau' < \Gamma_{\max}} \Gamma_{\tau'} \in \mathrm{PSH}(X, \theta).$$

We will always make this extension in the sequel.

Recall that according to our general principle, we only talk about model potentials when a potential has positive mass. Fortunately, this principle is not violated in the above definition, as shown below:

Lemma 9.1.1 *Assume that $\Gamma \in \mathrm{TC}(X, \theta)_{>0}$. Then for each $\tau < \Gamma_{\max}$, we have*

$$\int_X (\theta + \mathrm{dd^c}\Gamma_\tau)^n > 0. \tag{9.2}$$

Proof The notations in the proof below are summarized in Fig. 9.1.

Fix $\tau \in (-\infty, \Gamma_{\max})$.

By assumption, $\Gamma_{-\infty}$ has positive mass. By Corollary 2.4.1, we have

$$\int_X \theta^n_{\Gamma_{-\infty}} = \lim_{\tau \to -\infty} \int_X \theta^n_{\Gamma_\tau}.$$

In particular, for a sufficiently small $\tau_0 < \tau$, we have

$$\int_X \theta^n_{\Gamma_{\tau_0}} > 0.$$

Now take $\tau' \in (\tau, \Gamma_{\max})$ and $t \in (0, 1)$ so that

$$\tau = (1 - t)\tau' + t\tau_0.$$

From the concavity of Γ, we find that

$$\Gamma_\tau \geq (1 - t)\Gamma_{\tau'} + t\Gamma_{\tau_0}.$$

By Theorem 2.4.4,

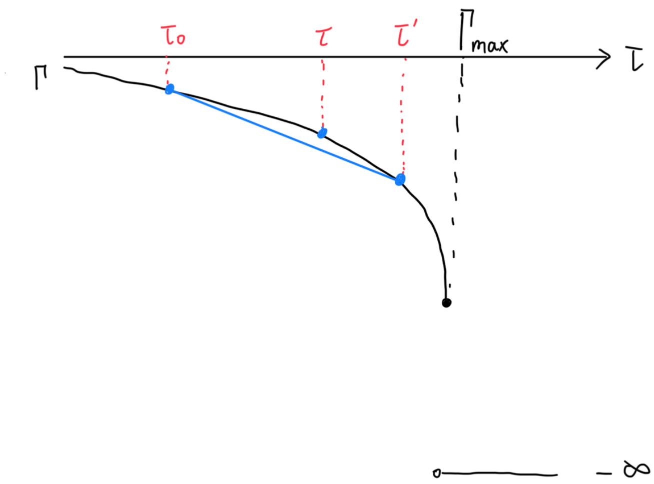

Fig. 9.1 The test curve Γ.

$$\int_X \theta^n_{\Gamma_\tau} \geq \int_X \theta^n_{(1-t)\Gamma_{\tau'}+t\Gamma_{\tau_0}} \geq t^n \int_X \theta^n_{\Gamma_{\tau_0}} > 0$$

and (9.2) follows. □

Proposition 9.1.1 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. *Then the map*

$$[-\infty, \Gamma_{\max}) \longrightarrow \mathbb{R}, \quad \tau \mapsto \log \int_X \theta^n_{\Gamma_\tau}$$

is concave and continuous.

Proof The concavity of this function follows from Theorem 2.4.1 and Theorem 2.4.4. The continuity at $-\infty$ is a consequence of Corollary 2.4.1. □

Definition 9.1.2 Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential.

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to be *bounded* if for τ small enough, $\Gamma_\tau = \phi$. The subset of bounded test curves in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathrm{TC}^\infty(X,\theta;\phi)$. In this case, we write

$$\Gamma_{\min} := \max\{\tau \in \mathbb{R} : \Gamma_\tau = \phi\}. \tag{9.3}$$

A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is said to have *finite energy* if

$$\mathbf{E}^\phi(\Gamma) := \Gamma_{\max} \int_X \theta^n_\phi + \int_{-\infty}^{\Gamma_{\max}} \left(\int_X \theta^n_{\Gamma_\tau} - \int_X \theta^n_\phi \right) \mathrm{d}\tau > -\infty. \tag{9.4}$$

When $\phi = V_\theta$, we write $\mathbf{E}$ instead of $\mathbf{E}^\phi$.

The subset of test curves with finite energy in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathrm{TC}^1(X,\theta;\phi)$.

Example 9.1.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, there is a canonically associated test curve $\Gamma^\varphi \in \mathrm{TC}^\infty(X,\theta;V_\theta)$: Set $\Gamma^\varphi_{\max} = 0$ and

$$\Gamma^\varphi_\tau = \begin{cases} V_\theta, & \text{if } \tau \leq -1; \\ P_\theta\left[(1+\tau)\varphi - \tau V_\theta\right], & \text{if } -1 < \tau < 0. \end{cases}$$

Note that Γ^φ is indeed a test curve, as follows from Proposition 3.1.8.

We first observe that the notion of test curves does not really depend on the choice of θ within its cohomology class.

Proposition 9.1.2 *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. Let $\phi' \in \mathrm{PSH}(X,\theta')_{>0}$ be the unique model potential satisfying $\phi \sim \phi'$.*

Then there is a canonical bijection

$$\mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(X,\theta';\phi').$$

This bijection induces the following bijections:

$$\mathrm{TC}^1(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^1(X,\theta';\phi'), \quad \mathrm{TC}^\infty(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}^\infty(X,\theta';\phi').$$

These bijections satisfy the obvious cocycle conditions.

Proof Choose $g \in C^\infty(X)$ such that $\theta' = \theta + \mathrm{dd^c} g$. Given any $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we observe that $\Gamma' \colon (-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta')$ defined as

$$\tau \mapsto P_{\theta'}[\Gamma_\tau - g]$$

lies in $\mathrm{TC}(X,\theta';\phi')$. Moreover, the choice of g is irrelevant since for any other choice of g, say g', we have

$$\Gamma_\tau - g \sim \Gamma_\tau - g'$$

for all $\tau < \Gamma_{\max}$. All assertions follow directly from the definition. □

Proposition 9.1.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold Y. Then the pointwise pull-back induces a bijection*

$$\pi^*\colon \mathrm{TC}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{TC}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This follows immediately from Proposition 3.1.7. □

Next we verify the closedness of a test curve as a family of concave functions, so that no pathologies are presented in the Legendre transforms which we will consider shortly. The notion of closeness is recalled in Definition A.1.7.

Proposition 9.1.4 *Let Γ be a test curve in $\mathrm{PSH}(X,\theta)$. For each $x \in X$, the map $\mathbb{R} \ni \tau \mapsto \Gamma_\tau(x)$ is a closed concave function. Moreover, the map is proper as long as $\Gamma_{\Gamma_{\max}}(x) \neq -\infty$.*

Proof We argue the closeness. Fix $x \in X$. Assume that $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. We only need to argue the upper-semicontinuity of $\tau \mapsto \Gamma_\tau(x)$. The upper semi-continuity is clear at $\tau \geq \Gamma_{\max}$, so we are reduced to prove the following:

$$\Gamma_\tau = \inf_{\tau'<\tau} \Gamma_{\tau'} \tag{9.5}$$

for any $\tau < \Gamma_{\max}$. Take $\tau'' \in (\tau, \Gamma_{\max})$. Outside the polar locus of $\Gamma_{\tau''}$, we know that (9.5) holds by continuity of real-valued concave functions. So (9.5) holds everywhere by Proposition 1.2.6.

The final assertion is trivial. □

Definition 9.1.3 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a smooth closed real positive $(1,1)$-form. Then we define $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$ as follows:

(1) Define

$$P_{\theta+\omega}[\Gamma]_{\max} = \Gamma_{\max};$$

(2) for each $\tau < \Gamma_{\max}$, define

$$P_{\theta+\omega}[\Gamma]_\tau = P_{\theta+\omega}[\Gamma_\tau].$$

It follows form Proposition 3.1.8 that $P_{\theta+\omega}[\Gamma] \in \mathrm{TC}(X,\theta+\omega)_{>0}$.

Proposition 9.1.5 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a closed real smooth semipositive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma]_{-\infty} = P_{\theta+\omega}[\Gamma_{-\infty}].$$

Proof This follows from Proposition 3.1.11. □

9.2 Ross–Witt Nyström correspondence

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Proposition 9.1.4 allows us to talk about the Legendre transforms of test curves in the expected way.

The general definition of the Legendre transform Definition A.2.1 can be translated as follows:

Definition 9.2.1 Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$. We define its *Legendre transform* as $\Gamma^*\colon (0,\infty) \to \mathrm{PSH}(X,\theta)$ given by

$$\Gamma^*_t = \sup_{\tau\in\mathbb{R}} (t\tau + \Gamma_\tau)\,[2]. \tag{9.6}$$

[2] There is no usc regularization in the following formula. This is not a typo.

Thanks to Remark 9.1.1, (9.6) can be equivalently written as

$$\Gamma_t^* = \sup_{\tau<\Gamma_{\max}} (t\tau + \Gamma_\tau) = \sup_{\tau\leq\Gamma_{\max}} (t\tau + \Gamma_\tau) .$$

It is sometimes handy to *define*

$$\Gamma_0^* := \phi \tag{9.7}$$

at $t = 0$. But it is important to remember by doing so, (9.6) is not true at $t = 0$ in general.

Remark 9.2.1 Here we do not talk about the case $t < 0$ because its behavior is pretty trivial: Take $x \in X$, if $\Gamma_\tau(x) = -\infty$ for all $\tau < \Gamma_{\max}$, then $\Gamma_t^*(x) = -\infty$; otherwise, $\Gamma_t^*(x) = \infty$.

The information about $t > 0$ suffices to characterize Γ.

Proposition 9.2.1 *Let* $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. *Then*

$$\Gamma_\tau = \inf_{t>0} \left(\Gamma_t^* - t\tau\right) \tag{9.8}$$

for all $\tau \in \mathbb{R}$.

Due to our convention (9.7), in (9.8) we could as well take $t \geq 0$.

Proof Fix $x \in X$. We want to establish (9.8) at x. We distinguish two cases. First suppose that $\Gamma_\tau(x) = -\infty$ for all $\tau < \Gamma_{\max}$ and hence all $\tau \in \mathbb{R}$. In this case, we have $\Gamma_t^*(x) = -\infty$ for all $t > 0$. Therefore, (9.8) follows trivially.

Otherwise, by Remark 9.2.1, we know that $\Gamma_t^*(x) = \infty$ for all $t < 0$. The relative interior of the domain of $t \mapsto \Gamma_t^*(x)$ is contained in $(0, \infty)$. Therefore, (9.8) follows from Theorem A.2.1, Proposition 9.1.4. □

In Definition 9.2.1, we have made a non-trivial claim that $\Gamma_t^* \in \mathrm{PSH}(X, \theta)$ for all $t > 0$. Let us prove this.

Lemma 9.2.1 *Let* $\Gamma \in \mathrm{TC}(X, \theta; \phi)$. *Then* $\Gamma_t^* \in \mathrm{PSH}(X, \theta)$ *for all* $t > 0$. *In fact,* Γ *is upper semicontinuous as a function of* $X \times (0, \infty)$.

Proof We first observe that for each $x \in X$, we have

$$\Gamma_t^*(x) \leq t\Gamma_{\max} < \infty.$$

Let $R = \{a + \mathrm{i}b \in \mathbb{C} : a > 0, b \in \mathbb{R}\}$. We consider

$$F : X \times R \to [-\infty, \infty), \quad (x, a + \mathrm{i}b) \mapsto \Gamma_a^*(x).$$

Let $\pi : X \times R \to X$ be the natural projection. Observe that the upper-semicontinuous regularization G of F is $\pi^*\theta$-psh by Proposition 1.2.1. It suffices to show that $F = G$. We let

$$E := \{(x,z) \in X \times R : F(x,z) < G(x,z)\}.$$

We want to argue that $E = \varnothing$. Clearly, E can be written as $B \times \mathrm{i}\mathbb{R}$ for some set $B \subseteq X \times (0,\infty)$. Since E is a pluripolar set by Proposition 1.2.5, it has zero Lebesgue measure. Hence, B has zero Lebesgue measure. For each $x \in X$, write

$$B_x = \{t \in (0,\infty) : (t,x) \in B\}.$$

By Fubini's theorem, B_x has vanishing 1-dimensional Lebesgue measure for all $x \in X \setminus Z$, where $Z \subseteq X$ is a subset of measure 0. We may assume that $Z \supseteq \{\Gamma_{\Gamma_{\max}} = -\infty\}$ so that for $x \in X \setminus Z$, $\Gamma_t(x) \neq -\infty$ for all $t > 0$.

For any $x \in X \setminus Z$, both $t \mapsto F(x,t)$ and $G(x,t)$ are convex functions with values in $\mathbb{R}$ on $(0,\infty)$. They agree almost everywhere, hence everywhere by their continuity. It follows that for $x \in X \setminus Z$, we have $B_x = \varnothing$.

By Proposition 9.2.1, for any $x \in X$, we have

$$\Gamma_\tau(x) = \inf_{t>0}(F(x,t) - t\tau), \quad \tau < \Gamma_{\max}.$$

On the other hand, let

$$\chi_\tau(x) = \inf_{t>0}(G(x,t) - t\tau), \quad \tau < \Gamma_{\max}, \quad x \in X. \tag{9.9}$$

By Kiselman's principle Proposition 1.2.8, $\chi_\tau \in \mathrm{PSH}(X,\theta)$. But on $X \setminus Z$, we already know that $\Gamma_\tau = \chi_\tau$ for all $\tau < \Gamma_{\max}$. By Proposition 1.2.6,

$$\Gamma_\tau = \chi_\tau, \quad \tau < \Gamma_{\max}.$$

Now we conclude that $F(x,t) = G(x,t)$ by Corollary A.2.1. □

Corollary 9.2.1 *Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$. Then $\Gamma_t^* \in \mathcal{E}(X,\theta;\phi)$ for all $t > 0$.*

Proof Fix $t > 0$. We already know that $\Gamma_t^* \in \mathrm{PSH}(X,\theta)$ by Lemma 9.2.1. It suffices to show that

$$\Gamma_t^* \sim_P \phi.$$

From (9.6) and Proposition 6.1.6, we know that

$$\Gamma_t^* \sim_P \sup_{\tau<\Gamma_{\max}}{}^*\Gamma_\tau = \phi.$$

Lemma 9.2.2 *Let $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, then*

$$\sup_X \Gamma_t^* = t\Gamma_{\max}$$

for all $t > 0$.

In particular, $t \mapsto \Gamma_t^* - t\Gamma_{\max}$ is a decreasing function in $t > 0$.

Proof Choose $x \in X$ such that $\Gamma_{\Gamma_{\max}}(x) = 0$. Then $\Gamma_\tau(x) = 0$ for all $\tau < \Gamma_{\max}$, and hence for all $t > 0$,

$$\Gamma_t^*(x) = t\Gamma_{\max}$$

by definition. On the other hand, since $\Gamma_\tau \leq 0$ for all $\tau < \Gamma_{\max}$, we have

$$\sup_X \Gamma_t^* \leq t\Gamma_{\max}$$

for all $t > 0$. □

Lemma 9.2.3 *Given* $\Gamma \in \mathrm{TC}(X, \theta; \phi)$*, we have* $\Gamma^* \in \mathcal{R}(X, \theta; \phi)$.

See Definition 4.2.2 for the notation $\mathcal{R}(X, \theta; \phi)$.

Proof It follows from Lemma 9.2.1, (9.6) and Proposition 1.2.1 that Γ^* is a subgeodesic ray. By Corollary 9.2.1, for any $t > 0$, $\Gamma_t^* \in \mathcal{E}(X, \theta; \phi)$.

First observe that as $t \to 0+$, we have

$$\Gamma_t^* \xrightarrow{L^1} \phi. \tag{9.10}$$

By Lemma 9.2.2 and Proposition 1.5.1, it suffices to show each L^1-cluster point $\psi \in \mathrm{PSH}(X, \theta)$ as Γ_t^* as $t \to 0$ is equal to ϕ.

To see this, first observe that by (9.6), for any fixed $t > 0$,

$$\Gamma_t^* \leq t\Gamma_{\max} + \phi.$$

Therefore, $\psi \leq \phi$. On the other hand, for any fixed $\tau < \Gamma_{\max}$, by (9.6), we have

$$\Gamma_t^* \geq \Gamma_\tau + t\tau$$

for any $t > 0$. So $\psi \geq \Gamma_\tau$ almost everywhere and hence everywhere by Proposition 1.2.6. It follows that $\psi \geq \phi$. Therefore, $\psi = \phi$.

Assume that Γ^* is not a geodesic ray. Then we can find $0 \leq a < b$ such that $(\Gamma_t^*)_{t \in (a,b)}$ differs from the geodesic $(\eta_t)_{t \in (a,b)}$ from Γ_a^* to Γ_b^*. The existence of $(\eta_t)_t$ is guaranteed by Proposition 4.2.1. We consider the subgeodesic $(\ell_t)_{t>0}$ given by $\ell_t = \eta_t$ for $t \in (a, b)$ and $\ell_t = \Gamma_t^*$ otherwise. Note that ℓ is a subgeodesic due to Lemma 1.2.2.

Consider the Legendre transform

$$\Gamma'_\tau = \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}.$$

Then $\Gamma'_\tau \geq \Gamma_\tau$ and $\Gamma'_\tau \in \mathrm{PSH}(X, \theta) \cup \{-\infty\}$ by Proposition 1.2.8 for all $\tau \in \mathbb{R}$.

We claim that

$$\Gamma'_\tau \leq \Gamma_\tau + (b - a)(\Gamma_{\max} - \tau), \quad \tau \in \mathbb{R}. \tag{9.11}$$

Observe that $\Gamma'_\tau \equiv -\infty$ when $\tau > \Gamma_{\max}$ by Lemma 9.2.2. So it suffices to consider $\tau \leq \Gamma_{\max}$. In this case, we compute

$$\inf_{t\in[a,b]}(\ell_t - t\tau) \leq \Gamma^*_b - b\tau \leq (b-a)(\Gamma_{\max} - \tau) + \inf_{t\in[a,b]}(\Gamma^*_t - t\tau),$$

where we applied Lemma 9.2.2. Therefore, (9.11) follows. In particular, for any $\tau < \Gamma_{\max}$, we have $\Gamma'_\tau \sim \Gamma_\tau$. On the other hand, by definition of Γ'_τ, we clearly have $\Gamma'_\tau \leq 0$ for all $\tau < \Gamma_{\max}$. It follows from the fact that Γ_τ is a model potential that $\Gamma_\tau = \Gamma'_\tau$ for all $\tau < \Gamma_{\max}$. Therefore, by Theorem A.2.1, we have $\Gamma^*_t = \ell'_t$ for all $t > 0$, which is a contradiction. □

Given $\ell \in \mathcal{R}(X,\theta;\phi)$, define its Legendre transform

$$\ell^*_\tau := \inf_{t>0}(\ell_t - t\tau), \quad \tau \in \mathbb{R}. \tag{9.12}$$

Lemma 9.2.4 *Given $\ell \in \mathcal{R}(X,\theta;\phi)$, then $\ell^* = (\ell^*_\tau)_{\tau<\sup_X \ell_1} \in \mathrm{TC}(X,\theta)$.*

Proof Note that it follows from Proposition 1.2.8 that $\ell^*_\tau \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}$ for all $\tau \in \mathbb{R}$. It is clear that $\mathbb{R} \ni \tau \mapsto \ell^*_\tau$ is a decreasing and concave function.

By Proposition 4.2.4,

$$\sup_X \ell_t = t \sup_X \ell_1 \quad \forall t \geq 0.$$

Observe that $(0,\infty) \ni t \mapsto \ell_t - t\sup_X \ell_1$ is a decreasing net in $\mathrm{PSH}(X,\theta)$ with $\sup_X(\ell_t - t\sup_X \ell_1) = 0$. It follows that

$$\ell^*_{\sup_X \ell_1} = \inf_{t>0}\left(\ell_t - t\sup_X \ell_1\right) \in \mathrm{PSH}(X,\theta).$$

On the other hand, for $\tau > \sup_X \ell_1$, the same argument shows that

$$\ell^*_\tau \equiv -\infty.$$

Therefore, $\ell^*_\tau \in \mathrm{PSH}(X,\theta)$ if and only if $\tau \leq \ell^*_{\max} := \sup_X \ell_1$.

We claim that $(\ell^*_\tau)_{\tau<\ell^*_{\max}}$ is a test curve. We first observe that for $\tau < \ell^*_{\max}$, we have

$$\ell^*_\tau \leq \ell_1 - \tau \sim_P \phi.$$

Therefore,

$$\ell^*_\tau \leq_P \phi, \quad \forall \tau < \ell^*_{\max}. \tag{9.13}$$

Also observe that for any $\tau \leq \ell^*_{\max}$ and any $t > 0$, we have

$$\sup_X \ell^*_\tau \leq \sup_X \ell_t - t\tau = \ell^*_{\max} t - t\tau.$$

Letting $t \to 0+$, we find that for any $\tau \leq \ell^*_{\max}$, we have

$$\sup_X \ell^*_\tau \leq 0. \tag{9.14}$$

Fix $\tau < \ell^*_{\max}$, we want to argue that

$$P_\theta\left[\ell^*_\tau\right]=\ell^*_\tau. \tag{9.15}$$

First we claim that for any $C>0$, we have

$$(\ell^*_\tau+C)\wedge\phi=(\ell^*_\tau+C)\wedge V_\theta. \tag{9.16}$$

The $\leq$ direction is trivial. We argue the reverse inequality, which reduces to

$$\phi\geq(\ell^*_\tau+C)\wedge V_\theta.$$

Since ϕ is model and $(\ell^*_\tau+C)\wedge V_\theta\leq 0$, it suffices to show that

$$\phi\geq_P(\ell^*_\tau+C)\wedge V_\theta,$$

which follows from (9.13). Therefore, (9.16) is established. Thanks to (9.14), we have the obvious inequality

$$(\ell^*_\tau+C)\wedge V_\theta\geq\ell^*_\tau$$

for any $C>0$. Therefore, in order to prove (9.15), it remains to argue that for any $C>0$,

$$(\ell^*_\tau+C)\wedge\phi\leq\ell^*_\tau. \tag{9.17}$$

For this purpose, let us consider the following geodesics: For any $M>0$ and $t\in[0,1]$, let

$$\ell^{1,M}_t=\ell_{tM}-tM\tau,\quad \ell^{2,M}_t=(\ell^*_\tau+C)\wedge\phi-Ct.$$

It is clear that at $t=0,1$, we have $\ell^{2,M}_t\leq\ell^{1,M}_t$. Hence, the same holds for all $t\in[0,1]$. In particular, for any fixed $s\in(0,1]$, we have

$$(\ell^*_\tau+C)\wedge\phi-Cs\leq\ell_{sM}-sM\tau$$

for all $M>0$. Taking infimum with respect to $M>0$, we find

$$(\ell^*_\tau+C)\wedge\phi-Cs\leq\ell^*_\tau.$$

Since $s\in(0,1]$ is arbitrary, we conclude (9.17). □

Theorem 9.2.1 *The Legendre transform in Definition 9.2.1 is a bijection*

$$\mathrm{TC}(X,\theta;\phi)\xrightarrow{\sim}\mathcal{R}(X,\theta;\phi). \tag{9.18}$$

Moreover, this bijection restricts to the following bijections:

$$\mathrm{TC}^1(X,\theta;\phi)\xrightarrow{\sim}\mathcal{R}^1(X,\theta;\phi),\quad \mathrm{TC}^\infty(X,\theta;\phi)\xrightarrow{\sim}\mathcal{R}^\infty(X,\theta;\phi). \tag{9.19}$$

For any $\Gamma\in\mathrm{TC}^1(X,\theta;\phi)$, we have

$$\mathbf{E}^\phi(\Gamma)=\mathbf{E}^\phi(\Gamma^*). \tag{9.20}$$

Recall that the two energy functionals in (9.20) are defined in (9.4) and Definition 4.3.6 respectively.

The correspondence (9.18) will be referred to as the *Ross–Witt Nyström correspondence*.

To appreciate this result, just consider the simple case where θ is a Kähler form and $\phi = 0$. In this case, elements in $\mathcal{R}^\infty(X,\theta)$ are rays of *bounded* potentials, while elements in $\mathrm{TC}^\infty(X,\theta)$ are rays of *singular* potentials. This result establishes a bridge between the pluripotential theory of regular potentials and that of singular potentials!

Proof **Step 1**. We first establish (9.18).

It follows from Lemma 9.2.3 that the forward map is well-defined. The inverse map is given by (9.12). We show that the inverse map is also well-defined. Given $\ell \in \mathcal{R}(X,\theta;\phi)$, we know from Lemma 9.2.4 that $\ell^* \in \mathrm{TC}(X,\theta)$. We need to show that $\ell^* \in \mathrm{TC}(X,\theta;\phi)$.

By Corollary A.2.1 and Lemma 9.2.3, we know that

$$\ell = (\ell^*)^* \in \mathcal{R}(X,\theta;\ell^*_{-\infty}).$$

So it follows that $\ell^*_{-\infty} = \phi$. Therefore, $\ell^* \in \mathrm{TC}(X,\theta;\phi)$ as expected.

The two operations are inverse to each other thanks to Corollary A.2.1. Hence, (9.18) is established.

Step 2. Next we consider the bounded situation. Namely, we want to establish the second half of (9.19).

Suppose that $\Gamma \in \mathrm{TC}^\infty(X,\theta;\phi)$. Take $\tau_0 \in \mathbb{R}$ so that $\Gamma_\tau = \phi$ for all $\tau \le \tau_0$. It follows from (9.6) that

$$\Gamma^*_t \ge \phi + t\tau_0$$

for all $t > 0$. Therefore, $\Gamma^*_t \sim \phi$ for all $t > 0$ and hence $\Gamma^* \in \mathcal{R}^\infty(X,\theta;\phi)$.

Conversely, suppose that $\ell \in \mathcal{R}^\infty(X,\theta;\phi)$. Thanks to Proposition 4.2.3, there is a constant $C > 0$ such that

$$\ell_t \ge \phi - Ct.$$

Therefore, according to (9.12), we have

$$\ell^*_\tau \ge \inf_{t>0}\left(\phi - (C+\tau)t\right) = \phi$$

if $\tau \le -C$. Therefore, $\ell^*_\tau = \phi$ for all $\tau \le -C$.

Step 3. We establish (9.20) and the first half of (9.19).

Step 3.1. We reduce to the case where $\Gamma_{\max} = 0$.

Suppose that we define

$$\Gamma'_\tau = \Gamma_{\tau+\Gamma_{\max}}, \quad \forall \tau < 0.$$

Then $\Gamma' \in \mathrm{TC}(X,\theta;\phi)$ as well and for all $t > 0$,

$$\Gamma'^*_t = \sup_{\tau<0}\left(t\tau + \Gamma'_\tau\right) = \sup_{\tau<\Gamma_{\max}}\left(t\tau + \Gamma_\tau\right) - t\Gamma_{\max} = \Gamma^*_t - t\Gamma_{\max}.$$

Therefore,

$$\mathbf{E}^{\phi}\left(\Gamma'^{*}\right)=\mathbf{E}^{\phi}\left(\Gamma^{*}\right)-\Gamma_{\max}\int_X \theta_{\phi}^n.$$

by (3.27). Using (9.4), we also have

$$\begin{aligned}\mathbf{E}^{\phi}(\Gamma') &= \int_{-\infty}^{0}\left(\int_X \theta^n_{\Gamma'_\tau}-\int_X \theta^n_{\phi}\right)\mathrm{d}\tau\\ &=\int_{-\infty}^{\Gamma_{\max}}\left(\int_X \theta^n_{\Gamma_\tau}-\int_X \theta^n_{\phi}\right)\mathrm{d}\tau\\ &=\mathbf{E}^{\phi}(\Gamma)-\Gamma_{\max}\int_X\theta^n_{\phi}.\end{aligned}$$

Therefore, it suffices to establish (9.20) for Γ' in place of Γ.

Step 3.2. We assume that $\Gamma_{\max}=0$ and $\Gamma\in\mathrm{TC}^{\infty}(X,\theta;\phi)$. We prove (9.20).

For $N\in\mathbb{Z}_{>0}$, $M\in\mathbb{Z}$, we introduce the following:

$$\Gamma^{*,N,M}_t := \max_{\substack{k\in\mathbb{Z}\\ k\leq M}}\left(\Gamma_{k/2^N}+tk/2^N\right)\in\mathcal{E}^{\infty}(X,\theta;\phi),\quad t>0.$$

We first claim that for all $t>0$, $N\in\mathbb{Z}_{>0}$ and $M\in\mathbb{Z}$,

$$\frac{t}{2^N}\int_X\theta^n_{\Gamma_{(M+1)/2^N}}\leq E^{\phi}_{\theta}\left(\Gamma^{*,N,M+1}_t\right)-E^{\phi}_{\theta}\left(\Gamma^{*,N,M}_t\right)\leq\frac{t}{2^N}\int_X\theta^n_{\Gamma_{M/2^N}}. \tag{9.21}$$

Assuming this, let us prove (9.20).

Fixing N, let $M=\lfloor 2^N\Gamma_{\min}\rfloor$. Recall that $\Gamma_{\min}$ is defined in (9.3). Then repeated applications of (9.21) yield

$$\sum_{j=M+1}^{0}\frac{t}{2^N}\int_X\theta^n_{\Gamma_{j/2^N}}\leq E^{\phi}_{\theta}\left(\Gamma^{*,N,0}_t\right)-E^{\phi}_{\theta}\left(E^{*,N,M}_t\right)\leq\sum_{j=M}^{-1}\frac{t}{2^N}\int_X\theta^n_{\Gamma_{j/2^N}}.$$

Since $M\leq 2^N\Gamma_{\min}$, we have that

$$\Gamma^{*,N,M}_t=\phi+tM/2^N.$$

Using (3.27), we can continue to write

$$\sum_{j=M+1}^{0}\frac{t}{2^N}\left(\int_X\theta^n_{\Gamma_{j/2^N}}-\int_X\theta^n_{\phi}\right)\leq E^{\phi}_{\theta}\left(\Gamma^{*,N,0}_t\right)\leq\sum_{j=M}^{-1}\frac{t}{2^N}\left(\int_X\theta^n_{\Gamma_{j/2^N}}-\int_X\theta^n_{\phi}\right).$$

We now notice that we have Riemann sums on both the left and right of the above inequality. Using Proposition 9.1.1, it is possible to let $N\to\infty$ and obtain

$$E^{\phi}_{\theta}(\Gamma^*_t)=t\mathbf{E}^{\phi}(\Gamma) \tag{9.22}$$

So (9.20) follows as desired.

It remains to argue (9.21). Fix $t > 0$, $N \in \mathbb{Z}_{>0}$ and $M \in \mathbb{Z}$. By Proposition 3.1.16,

$$\begin{aligned}\int_X \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M+1}} \leq & E^\phi_\theta\left(\Gamma_t^{*,N,M+1}\right) - E^\phi_\theta\left(\Gamma_t^{*,N,M}\right) \\ \leq & \int_X \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M}} .\end{aligned} \tag{9.23}$$

Clearly $\Gamma_t^{*,N,M+1} \geq \Gamma_t^{*,N,M}$. Moreover, since $\mathbb{R} \ni \tau \mapsto \Gamma_\tau + t\tau$ is concave, we notice that

$$U_t := \left\{\Gamma_t^{*,N,M+1} > \Gamma_t^{*,N,M}\right\} = \left\{\Gamma_{(M+1)/2^N} + 2^{-N} t > \Gamma_{M/2^N}\right\},$$

and on U_t we have

$$\Gamma_t^{*,N,M+1} = \Gamma_{(M+1)/2^N} + t(M+1)/2^N, \quad \Gamma_t^{*,N,M} = \Gamma_{M/2^N} + tM/2^N. \tag{9.24}$$

We also note that U_t is $\mathcal{F}$-open by Corollary 1.3.5. So from the lower bound in (9.23), we have

$$\begin{aligned}E^\phi_\theta\left(\Gamma_t^{*,N,M+1}\right) - E^\phi_\theta\left(\Gamma_t^{*,N,M}\right) \geq & \int_{U_t} \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M+1}} \\ = & \int_{U_t} \left(\Gamma_{(M+1)/2^N} - \Gamma_{M/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{(M+1)/2^N}} \\ \geq & \int_{\left\{\Gamma_{(M+1)/2^N} = 0\right\}} t2^{-N}\, \theta^n_{\Gamma_{(M+1)/2^N}},\end{aligned}$$

where on the second line, we applied (9.24) and Proposition 2.2.1, on the third line, we applied the fact that $\theta^n_{\Gamma_{(M+1)/2^N}}$ is supported on the set

$$\left\{\Gamma_{(M+1)/2^N} = 0\right\} \subseteq U_t \cap \left\{\Gamma_{M/2^N} = 0\right\},$$

see Theorem 3.1.1. We have deduced the first inequality in (9.21). Next, we apply the upper bound part in (9.23) and compute similarly

$$\begin{aligned}E^\phi_\theta(\Gamma_t^{*,N,M+1}) - E^\phi_\theta(\Gamma_t^{*,N,M}) \leq & \int_X \left(\Gamma_t^{*,N,M+1} - \Gamma_t^{*,N,M}\right) \theta^n_{\Gamma_t^{*,N,M}} \\ = & \int_{U_t} \left(\Gamma_{(M+1)/2^N} - \Gamma_{M/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{M/2^N}} \\ \leq & \int_{\{\Gamma_{M/2^N}=0\} \cap U_t} \left(\Gamma_{(M+1)/2^N} + t2^{-N}\right) \theta^n_{\Gamma_{M/2^N}} \\ \leq & \int_{\{\Gamma_{M/2^N}=0\} \cap U_t} t2^{-N}\, \theta^n_{\Gamma_{M/2^N}} .\end{aligned}$$

We conclude the latter half of (9.21).

Step 3.3. We assume that $\Gamma_{\max} = 0$. Now $\Gamma \in \mathrm{TC}(X, \theta; \phi)$ only.

For each $\epsilon > 0$, we introduce $\Gamma^\epsilon \in \mathrm{TC}^\infty(X,\theta;\phi)$ as follows:

(1) Let $\Gamma^\epsilon_{\max} = 0$, and
(2) we set

$$\Gamma^\epsilon_\tau = \begin{cases} \phi, & \text{if } \tau \leq -\epsilon^{-1}; \\ P_\theta\left[(1+\epsilon\tau)\Gamma_\tau - \epsilon\tau\phi\right], & \text{if } \tau \in \left(-\epsilon^{-1}, 0\right). \end{cases}$$

It follows from Corollary 6.2.10 and Corollary 6.2.5 that for each $\tau < 0$, the sequence Γ^ϵ_τ is a decreasing sequence with limit Γ_τ as $\epsilon \searrow 0$. Therefore, by Proposition 3.1.10, we have

$$\lim_{\epsilon\to 0+} \int_X \left(\theta + \mathrm{dd^c}\Gamma^\epsilon_\tau\right)^n = \int_X \left(\theta + \mathrm{dd^c}\Gamma_\tau\right)^n$$

for all $\tau < 0$. Hence, by the monotone convergence theorem and Step 3.2, we find

$$\mathbf{E}^\phi(\Gamma) = \lim_{\epsilon\to 0+} \mathbf{E}^\phi(\Gamma^\epsilon) = \lim_{\epsilon\to 0+} \mathbf{E}^\phi(\Gamma^{\epsilon *}) = \lim_{\epsilon\to 0+} E^\phi_\theta(\Gamma^{\epsilon *}_1), \tag{9.25}$$

where the last equality follows from (9.22). Furthermore, according to Proposition A.2.3, we have

$$\Gamma^*_t = \inf_{\epsilon>0} \Gamma^{\epsilon *}_t$$

for all $t > 0$. Note that we do not have to take the closure of the right-hand side since it is automatically upper semicontinuous in t.

Now suppose that $\Gamma \in \mathrm{TC}^1(X,\theta;\phi)$. Then by (9.25), as $\epsilon \to 0+$, $(\Gamma^{\epsilon *}_t)_\epsilon$ is a decreasing Cauchy net in $\mathcal{E}^1(X,\theta;\phi)$ and hence by Theorem 4.3.3 for each $t > 0$,

$$E^\phi_\theta\left(\Gamma^*_t\right) = \lim_{\epsilon\to 0+} E^\phi_\theta\left(\Gamma^{\epsilon,*}_t\right) = t\mathbf{E}^\phi(\Gamma) > -\infty,$$

where we have applied (9.22) and (9.25). Hence, $\Gamma^* \in \mathcal{E}^1(X,\theta;\phi)$. Moreover, (9.20) follows.

Conversely, suppose that $\Gamma^* \in \mathcal{R}^1(X,\theta;\phi)$. Then (9.25) implies that

$$\mathbf{E}^\phi(\Gamma) = \lim_{\epsilon\to 0+} E^\phi_\theta\left(\Gamma^{\epsilon *}_1\right) \geq E^\phi_\theta\left(\Gamma^*_1\right) > -\infty.$$

Hence, $\Gamma \in \mathrm{TC}^1(X,\theta;\phi)$. □

Remark 9.2.2 One could also consider geodesic rays emanating from another potential $\Phi \in \mathcal{E}(X,\theta;\phi)$. In this case, one can show that these geodesic rays are in bijection with Φ-twisted test curves: In Definition 9.1.1, we replace (2) by the following condition:

$$\sup_{C>0}{}^* \left(\Gamma_\tau + C\right) \wedge \Phi = \Gamma_\tau.$$

Furthermore, we require that $\Gamma_{-\infty} = \Phi$.

The above results equally work in the twisted setting. The proofs are almost identical to the untwisted case.

As an immediate consequence of the proof, we have

Corollary 9.2.2 *Let $\ell \in \mathcal{R}^1(X,\theta;\phi)$, then $[0,\infty) \ni t \mapsto E^{\phi}_{\theta}(\ell_t)$ is linear.*

Proof This follows from the same argument as that of (9.25). □

Corollary 9.2.3 *Let $\ell \in \mathcal{R}(X,\theta;\phi)$. Then $\sup_X \ell_t = \ell^*_{\max} t$ for any $t \geq 0$.*
In particular, $\ell_t - \ell^*_{\max} t$ is a decreasing function of $t \geq 0$.

Proof This follows from Lemma 9.2.2 and Theorem 9.2.1. □

Example 9.2.1 Let us see what the test curve in Example 9.1.1 correspond to under the Ross–Witt Nyström correspondence. Fix $\varphi \in \mathrm{PSH}(X,\theta)$. We claim that

$$\ell^{\varphi} = \Gamma^{\varphi *}, \tag{9.26}$$

where ℓ^{φ} is as in Example 4.3.1. We may assume that $\varphi \leq 0$ since both sides are invariant after adding a constant to φ.

We first prove the easy direction $\ell^{\varphi} \geq \Gamma^{\varphi *}$, which is equivalent to $\ell^{\varphi *} \geq \Gamma^{\varphi}$. Since $\ell^{\varphi *}$ is a test curve, the latter is equivalent to

$$\ell^{\varphi *}_{\tau} \geq (1+\tau)\varphi - \tau V_\theta$$

for all $\tau \in (-1,0)$. By Legendre duality, this is equivalent to

$$\ell^{\varphi}_t \geq \sup_{\tau \in (-1,0)} \left((1+\tau)\varphi - \tau V_\theta + t\tau\right) = \varphi \vee (V_\theta - t) \tag{9.27}$$

for all $t > 0$.

Using the notations of Example 4.3.1, we find easily that

$$\ell^{\varphi,C}_t \geq \varphi \vee (V_\theta - t)$$

for any $C > 0$ and $t \in [0,C]$, since it holds at $t=0$ and $t=C$. Letting $C \to \infty$, we find (9.27) Therefore, $\ell^{\varphi} \geq \Gamma^{\varphi *}$ follows.

In order to prove the equality in (9.26), it suffices to show that the two sides have the same energy, as a consequence of (4.22). So we compute

$$\begin{aligned}
\mathbf{E}\left(\Gamma^{\varphi *}\right) &= \mathbf{E}\left(\Gamma^{\varphi}\right) \\
&= \int_{-1}^{0} \left(\int_X \theta^n_{(1+\tau)V_\theta - \tau\varphi} - \int_X \theta^n_{V_\theta} \right) \mathrm{d}\tau \\
&= \sum_{j=0}^{n} \binom{n}{j} \int_X \theta^j_{V_\theta} \wedge \theta^{n-j}_{\varphi} \int_0^1 \tau^j (1-\tau)^{n-j} \,\mathrm{d}\tau - \int_X \theta^n_{V_\theta} \\
&= \sum_{j=0}^{n} \binom{n}{j} \frac{j!(n-j)!}{(n+1)!} \int_X \theta^j_{V_\theta} \wedge \theta^{n-j}_{\varphi} - \int_X \theta^n_{V_\theta} \\
&= \mathbf{E}(\ell^{\varphi}),
\end{aligned}$$

where we used the value of the β-function[3] on the fourth line, and the last line is just (4.28).

The multiplier ideal sheaves of a test curve can be characterized using the corresponding geodesic ray in a very simple manner.

Proposition 9.2.2 (He–Testorf–Wang) *Let $\ell \in \mathcal{R}(X, \theta; \phi)$. Given any $\tau < \ell^*_{\max}$ and $x \in X$, we have*

$$\mathcal{I}\left(\ell^*_\tau\right)_x = \left\{ f \in \mathcal{O}_{X,x} : |f|^2 \int_0^\infty \exp(-\ell_t + t\tau)\,\mathrm{d}t \text{ is integrable near } x \right\}. \tag{9.28}$$

Proof Fix $x \in X$, $\tau < \ell^*_{\max}$ and $f \in \mathcal{O}_{X,x}$. Fix a Kähler form ω on X.

Step 1. We first assume that f lies in the right-hand side of (9.28).

Given any $y \in X$, it follows from (9.12) that there is $t_0 > 0$ with

$$\ell^*_\tau(y) + 1 \geq \ell_{t_0}(y) - t_0\tau.$$

Observe that $t \mapsto \ell_t - t\ell^*_{\max}$ is decreasing in t by Corollary 9.2.3, it follows that for $t \in [t_0, t_0 + 1]$, we have

$$\ell^*_\tau(y) + 1 - t_0(\ell^*_{\max} - \tau) \geq \ell_{t_0}(y) - t_0\ell^*_{\max} \geq \ell_t(y) - t\ell^*_{\max}.$$

Since $\tau < \ell^*_{\max}$ and $t_0 \geq t - 1$, we deduce that

$$\ell^*_\tau(y) + 1 + \ell^*_{\max} - \tau \geq \ell_t(y) - t\tau, \quad t \in [t_0, t_0 + 1]. \tag{9.29}$$

Take a sufficiently small open neighborhood U of x such that

$$\int_U |f|^2 \int_0^\infty \exp(-\ell_t + t\tau)\,\mathrm{d}t\,\omega^n < \infty.$$

Applying (9.29), we deduce that

$$\int_U |f|^2 \exp\left(-\ell^*_\tau\right)\,\omega^n < \infty.$$

Therefore, $f \in \mathcal{I}(\ell^*_\tau)_x$.

Step 2. Assume that $f \in \mathcal{I}(\ell^*_\tau)_x$.

It follows from Theorem 1.4.4 that $f \in \mathcal{I}(\ell^*_{\tau+\epsilon})_x$ for some small enough $\epsilon > 0$ with $\tau + \epsilon < \ell^*_{\max}$. Take a sufficiently small open neighborhood U of x such that

$$\int_U |f|^2 \exp\left(-\ell^*_{\tau+\epsilon}\right)\,\omega^n < \infty.$$

We compute

[3] Also known as Euler integral of the first kind.

$$\begin{aligned}\int_U |f|^2 \int_0^\infty \exp\left(-\ell_t + t\tau\right) \mathrm{d}t\, \omega^n \leq & \int_U |f|^2 \int_0^\infty \exp\left(-\ell^*_{\tau+\epsilon} - t\epsilon\right) \mathrm{d}t\, \omega^n \\ = & \frac{1}{\epsilon} \int_U |f|^2 \exp\left(-\ell^*_{\tau+\epsilon}\right) \omega^n \\ < & \infty.\end{aligned}$$

Therefore, f lies in the right-hand side of (9.28). □

The masses of potentials on a test curve can also be expressed in terms of the corresponding geodesic ray.

Proposition 9.2.3 (Hisamoto) *Let $\ell \in \mathcal{R}(X, \theta; \phi)$. Given any $\tau < \ell^*_{\max}$, we have*

$$\int_X \left(\theta + \mathrm{dd^c} \ell^*_\tau\right)^n = \int_{\{\dot{\ell}_0 \geq \tau\}} \theta^n_\phi. \tag{9.30}$$

Here $\dot{\ell}_0$ denotes the right-derivative of ℓ_t with respect to t at $t = 0$. It is well-defined quasi-everywhere, and hence the right-hand side of (9.30) makes sense.

Proof Fix $\tau < \ell^*_{\max}$. We first observe that

$$\int_{\{\dot{\ell}_0 \geq \tau\}} \theta^n_\phi = \int_{\{\ell^*_\tau = \phi\}} \theta^n_\phi. \tag{9.31}$$

From this, (9.30) follows from [DNT21, Corollary 3.4], since both sides of (9.30) can then be written as

$$\int_{\{\ell^*_\tau = 0\}} \theta^n.$$

In order to prove (9.31), it suffices to show that

$$\left\{\dot{\ell}_0 \geq \tau\right\} = \left\{\ell^*_\tau = \phi\right\} \quad \text{outside a pluripolar set.} \tag{9.32}$$

Take $x \in X$ so that $(\ell_t(x))_{t \geq 0}$ is finite and right-differentiable at $t = 0$. Note that this condition holds quasi-everywhere. Suppose that $\ell^*_\tau(x) = \phi(x)$, then

$$\dot{\ell}_0(x) = \inf_{t>0} \frac{\ell_t(x) - \phi(x)}{t} \geq \inf_{t>0} \frac{\ell^*_\tau(x) + t\tau - \phi(x)}{t} = \tau.$$

Therefore, the $\supseteq$ direction in (9.32) follows. Conversely, suppose that $\dot{\ell}_0(x) \geq \tau$, then

$$\ell^*_\tau(x) = \inf_{t>0} \left(\ell_t(x) - t\tau\right) \geq \inf_{t>0} \left(\phi(x) + t\dot{\ell}_0(x) - t\tau\right) \geq \phi(x).$$

Therefore, the $\subseteq$ direction in (9.32) follows as well. □

9.3 $\mathcal{I}$-model test curves

Let X be a connected compact Kähler manifold of dimension n and θ be a smooth closed real $(1,1)$-form on X representing a big cohomology class. Fix a model potential $\phi \in \mathrm{PSH}(X,\theta)_{>0}$.

Definition 9.3.1 A test curve $\Gamma \in \mathrm{TC}(X,\theta;\phi)$ is *$\mathcal{I}$-model* if for any $\tau < \Gamma_{\max}$, the potential Γ_τ is $\mathcal{I}$-model.

The subset of $\mathcal{I}$-model test curves in $\mathrm{TC}(X,\theta;\phi)$ is denoted by $\mathcal{E}^{\mathrm{NA}}(X,\theta;\phi)$. When $\phi = V_\theta$, we omit ϕ and write $\mathcal{E}^{\mathrm{NA}}(X,\theta)$ instead.

The union of the sets of $\mathcal{I}$-model test curves in $\mathrm{PSH}(X,\theta)$ for all model potentials $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ is denoted by $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

Note that $\Gamma_{\Gamma_{\max}}$ is automatically $\mathcal{I}$-model by Proposition 3.2.13.

Here we write NA with non-Archimedean in mind. The precise relation with non-Archimedean pluripotential theory will be clear in Chapter 13. The readers are encouraged to skip this section and the next, and consult the necessary results only when reading Chapter 13.

Proposition 9.3.1 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Then $\Gamma_{-\infty}$ is an $\mathcal{I}$-model potential.*

Proof This follows from Proposition 3.2.14. □

Proposition 9.3.2 *Let θ' be another smooth closed real $(1,1)$-form on X representing the same cohomology class as θ. Then there is a canonical bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}.$$

This bijection satisfies the obvious cocycle condition.

Proof This is an immediate consequence of Proposition 9.1.2 and Example 7.1.2. □

Proposition 9.3.3 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a Kähler manifold. Then the pointwise pull-back induces a bijection*

$$\pi^*\colon \mathrm{PSH}^{\mathrm{NA}}(X,\theta;\phi) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta;\pi^*\phi).$$

Proof This is an immediate consequence of Proposition 9.1.3 and Proposition 3.2.5. □

Definition 9.3.2 Given $\Gamma \in \mathrm{TC}(X,\theta;\phi)$, we define its *$\mathcal{I}$-envelope* $P_\theta[\Gamma]_{\mathcal{I}}$ as the map

$$(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta), \quad \tau \mapsto P_\theta\left[\Gamma_\tau\right]_{\mathcal{I}}.$$

More generally, for any closed real smooth positive $(1,1)$-form ω on X, we define $P_{\theta+\omega}[\Gamma]_{\mathcal{I}}$ as the map

$$(-\infty,\Gamma_{\max}) \to \mathrm{PSH}(X,\theta), \quad \tau \mapsto P_{\theta+\omega}\left[\Gamma_\tau\right]_{\mathcal{I}}.$$

Proposition 9.3.4 *Let* $\Gamma \in \mathrm{TC}(X, \theta; \phi)$*, then*

$$P_\theta[\Gamma]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta; P_\theta[\phi]_{\mathcal{I}}).$$

More generally, for any closed real smooth positive $(1,1)$*-form* ω *on* X*, we have*

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta+\omega; P_{\theta+\omega}[\phi]_{\mathcal{I}}).$$

Proof The only non-trivial point is to show that

$$\sup_{\tau<\Gamma_{\max}}{}^{*} P_\theta[\Gamma_\tau]_{\mathcal{I}} = P_\theta[\phi]_{\mathcal{I}}, \qquad \sup_{\tau<\Gamma_{\max}}{}^{*} P_{\theta+\omega}[\Gamma_\tau]_{\mathcal{I}} = P_{\theta+\omega}[\phi]_{\mathcal{I}}.$$

These follow from Proposition 3.2.14. □

Definition 9.3.3 Let $\phi \in \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. A geodesic ray $\ell \in \mathcal{R}(X, \theta; \phi)$ is *maximal* if ℓ^* is $\mathcal{I}$-model.

An important class of $\mathcal{I}$-model test curves is given by filtrations. We briefly recall the corresponding terminology.

Definition 9.3.4 Let L be a big line bundle. We write

$$R(X, L) = \bigoplus_{k=0}^{\infty} \mathrm{H}^0(X, L^k)$$

for the section ring[4] of L.

A *filtration* on $R(X,L)$ is a decreasing family of graded linear subspaces $(\mathcal{F}^\lambda)_{\lambda\in\mathbb{R}}$ of $R(X,L)$ with graded pieces

$$\mathcal{F}^\lambda = \bigoplus_{k=0}^{\infty} \mathcal{F}^\lambda_k,$$

such that the following conditions are satisfied:

- The filtration is left-continuous: For any $\lambda \in \mathbb{R}$, we have
$$\mathcal{F}^\lambda = \bigcap_{\lambda'<\lambda} \mathcal{F}^{\lambda'};$$
- the filtration is multiplicative: For any $\lambda, \lambda' \in \mathbb{R}$ and any $k, k' \in \mathbb{N}$, we have
$$\mathcal{F}^\lambda_k \cdot \mathcal{F}^{\lambda'}_{k'} \subseteq \mathcal{F}^{\lambda+\lambda'}_{k+k'};$$

[4] Personally I hate the notion of section rings: We never consider inhomogeneous elements. So it is more natural to replace the direct sum by a disjoint union. This leads to the notion of ringoids (*annénoïdes* in French), introduced by Ducros in [Duc21] in the context of Temkin's graded reduction of Berkovich germs.

- there is an integer $C > 0$ such that
$$\mathcal{F}_m^{Cm} = 0, \quad \mathcal{F}_m^{-Cm} = \mathrm{H}^0(X, L^k) \tag{9.33}$$
for all $m \in \mathbb{N}$.

Given a filtration $\mathcal{F}$ on $R(X, L)$, we define
$$\tau_k(\mathcal{F}) = \max\left\{\lambda \in \mathbb{R} : \mathcal{F}_k^\lambda \neq 0\right\}.$$
By Fekete's lemma, we can introduce
$$\tau(\mathcal{F}) = \lim_{k\to\infty} \frac{1}{k}\tau_k(\mathcal{F}) = \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\tau_k(\mathcal{F}).$$
Note that $\tau(\mathcal{F})$ is bounded from above by the constant C in (9.33), hence finite.

Example 9.3.1 Let L be a big line bundle on X and $\mathcal{F}$ be a filtration on $R(X, L)$. Fix a smooth Hermitian metric h on L and write $\theta = c_1(L, h)$.

We introduce a few auxiliary functions. For each $k \in \mathbb{Z}_{>0}$, we introduce
$$\Gamma_\tau^{\mathcal{F},k} := \sup{}^* \left\{\log |s|_{h^k}^2 : s \in \mathcal{F}_k^{k\tau}, |s|_{h^k}^2 \leq 1\right\}.$$
When $k\tau \leq \tau_k(\mathcal{F})$, we know that $\mathcal{F}_k^{k\tau} \neq 0$. Moreover, Proposition 1.8.1 and Proposition 1.2.1 imply that
$$\Gamma_\tau^{\mathcal{F},k} \in \mathrm{PSH}(X, k\theta), \quad \tau \leq k^{-1}\tau_k(\mathcal{F}).$$
Observe that for $k, k' \in \mathbb{Z}_{>0}$, we have
$$\Gamma_\tau^{\mathcal{F},k+k'} \geq \Gamma_\tau^{\mathcal{F},k} + \Gamma_\tau^{\mathcal{F},k'}.$$
In particular, by Fekete's lemma,
$$\lim_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_\tau^{\mathcal{F},k} = \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma_\tau^{\mathcal{F},k} \tag{9.34}$$
exists for any $\tau < \tau(\mathcal{F})$.

We define $(\Gamma_\tau^{\mathcal{F}})_{\tau<\tau(\mathcal{F})}$ as follows:
$$\Gamma_\tau^{\mathcal{F}} := P_\theta\left[\sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma_\tau^{\mathcal{F},k}\right]^{5}.$$
We claim that $\Gamma^{\mathcal{F}} \in \mathcal{E}^{\mathrm{NA}}(X, \theta)$ and is bounded.

[5] It is not clear if $P_\theta[\bullet]$ is necessary here. When L is ample, it is shown in [RWN14, Proposition 7.11] that it is not necessary. The proof in the reference relies on a Skoda division theorem [RWN14, Theorem 7.10], which is not known in the case of big line bundles.

It is clear that $(-\infty, \tau(\mathcal{F})) \ni \tau \mapsto \Gamma^{\mathcal{F}}_{\tau}$ is decreasing. We prove its concavity. By Proposition 3.1.8, it suffices to show that

$$(-\infty, \tau(\mathcal{F})) \ni \tau \mapsto \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma^{\mathcal{F},k}_{\tau}$$

is concave. In other words, we need to prove the following: Given $\tau_0 < \tau_1 < \tau(\mathcal{F})$ and $t \in (0,1)$, we have

$$\sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma^{\mathcal{F},k}_{t\tau_1+(1-t)\tau_0} \geq t \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma^{\mathcal{F},k}_{\tau_1} + (1-t) \sup_{k\in\mathbb{Z}_{>0}}{}^* \frac{1}{k}\Gamma^{\mathcal{F},k}_{\tau_0}.$$

But thanks to Proposition 1.2.6 and Proposition 1.2.5, it suffices to show that

$$\sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma^{\mathcal{F},k}_{t\tau_1+(1-t)\tau_0} \geq t \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma^{\mathcal{F},k}_{\tau_1} + (1-t) \sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma^{\mathcal{F},k}_{\tau_0}$$

for all $t \in (0,1)$. Take $s_i \in \mathcal{F}^{k_i\tau_i}_{k_i}$ for $i = 0,1$ with $|s|^2_{h^{k_i}} \leq 1$, where $k_0, k_1 \in \mathbb{Z}_{>0}$. We need to prove that

$$\sup_{k\in\mathbb{Z}_{>0}} \frac{1}{k}\Gamma^{\mathcal{F},k}_{t\tau_1+(1-t)\tau_0} \geq \frac{1-t}{k_0}\log|s_0|^2_{h^{k_0}} + \frac{t}{k_1}\log|s_1|^2_{h^{k_1}}. \tag{9.35}$$

Approximate t by rational number from above, we may reduce to the case where $t \in \mathbb{Q}$. Write $t = p/q$ with $p, q \in \mathbb{Z}_{>0}$. Then

$$s := s_0^{k_1(q-p)} \otimes s_1^{k_0 p} \in \mathcal{F}^{k_0k_1\tau_0(q-p)+k_0k_1\tau_1 p}_{k_0k_1q},$$

and

$$\begin{aligned}
&\frac{1}{k_0k_1q}\log|s|^2_{h^{k_0k_1q}}\\
=&\frac{1}{k_0k_1q}\left(k_1(q-p)\log|s_0|^2 + k_0p\log|s_1|^2\right)\\
=&\frac{1-t}{k_0}\log|s_0|^2_{h^{k_0}} + \frac{t}{k_1}\log|s_1|^2_{h^{k_1}}.
\end{aligned}$$

So (9.35) follows.

Note that for each $k \in \mathbb{Z}_{>0}$, $\tau \leq k^{-1}\tau_k(\mathcal{F})$, we know that $\Gamma^{\mathcal{F},k}_{\tau}$ is $\mathcal{I}$-good by Proposition 7.2.2. It follows from the same proposition that for each $\tau < \tau(\mathcal{F})$, the potential $\Gamma^{\mathcal{F}}_{\tau}$ is also $\mathcal{I}$-good.

It remains to show that the test curve $\Gamma^{\mathcal{F}}$ is bounded and lies in $\mathcal{E}^{\mathrm{NA}}(X,\theta)$. Fix $\tau \leq -C$, where C is as in (9.33), we will show that

$$\Gamma^{\mathcal{F}}_{\tau} = V_{\theta}. \tag{9.36}$$

Of course, this follows from the Bergman kernel technique. But based on the theory we have developed so far, we could give an elegant and elementary argument.

Fix $k > 0$. Observe that for any $s \in \mathrm{H}^0(X, L^k)$, we have

$$s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\Gamma^{\mathcal{F}}_\tau)).$$

In fact, by definition of $\Gamma^{\mathcal{F}}_\tau$, it suffices to show that

$$s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(\Gamma^{\mathcal{F},k}_\tau)),$$

which is clear by definition. Therefore, by Theorem 7.4.1,

$$\operatorname{vol}\left(\theta + \mathrm{dd^c}\Gamma^{\mathcal{F}}_\tau\right) = \operatorname{vol} L.$$

But since $\Gamma^{\mathcal{F}}_\tau$ is $\mathcal{I}$-model, this implies (9.36).

Remark 9.3.1 There is an important special case of Example 9.3.1: Suppose that L is ample and $\mathcal{F}$ is the filtration induced by a smooth test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L). Then the geodesic ray $\Gamma^{\mathcal{F}*}$ is exactly the Phong–Sturm geodesic ray associated with $(\mathcal{X}, \mathcal{L})$. See [RWN14, Section 9].

Remark 9.3.2 We deduce from Example 9.3.1 that the ray $\Gamma^{\mathcal{F}*}$ induced by a filtration $\mathcal{F}$ is maximal.

9.4 Operations on test curves

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta', \theta''$ be smooth closed real $(1,1)$-forms on X representing big cohomology classes.

In this section, we develop several general operations on test curves, anticipating the applications in non-Archimedean geometry in Chapter 13. The readers are encouraged to read Chapter 13 first and consult this section when necessary.

Definition 9.4.1 Given $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, we say $\Gamma \leq \Gamma'$ if for all $\Gamma_{\max} \leq \Gamma'_{\max}$ and for all $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \leq_P \Gamma'_\tau. \tag{9.37}$$

Observe that (9.37) actually holds for all $\tau \in \mathbb{R}$ if $\theta = \theta'$. It is easy to verify that $\leq$ defines a partial order on $\mathrm{TC}(X,\theta)_{>0}$.

Lemma 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ and ω be a closed real smooth positive $(1,1)$-form on X. Then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma] \leq P_{\theta+\omega}[\Gamma']$.

Proof This follows from Example 6.1.1. □

Definition 9.4.2 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then we define $\Gamma+\Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$ as follows:

(1) We set
$$(\Gamma+\Gamma')_{\max} \coloneqq \Gamma_{\max}+\Gamma'_{\max};$$

(2) for any $\tau < (\Gamma+\Gamma')_{\max}$, we define[6]
$$(\Gamma+\Gamma')_\tau \coloneqq P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}}\left(\Gamma_\delta+\Gamma'_{\tau-\delta}\right)\right]. \tag{9.38}$$

Lemma 9.4.2 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then for any $\tau < (\Gamma+\Gamma')_{\max}$, we have*
$$\sup_{\delta\in\mathbb{R}}\left(\Gamma_\delta+\Gamma'_{\tau-\delta}\right) \in \mathrm{PSH}(X,\theta).$$
This potential is $\mathcal{I}$-good if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

In particular, (9.38) in Definition 9.4.2 makes sense.

Proof Let
$$\eta_\tau = \sup_{\delta\in\mathbb{R}}\left(\Gamma_\delta+\Gamma'_{\tau-\delta}\right) = \sup_{\tau-\Gamma'_{\max}<\delta<\Gamma_{\max}}\left(\Gamma_\delta+\Gamma'_{\tau-\delta}\right)$$
for all $\tau\in\mathbb{R}$. Set
$$Z = \{x\in X : \Gamma_\upsilon(x) = -\infty\ \forall \upsilon\in\mathbb{R}\} \cup \left\{x\in X : \Gamma'_\upsilon(x) = -\infty\ \forall \upsilon\in\mathbb{R}\right\}.$$
It follows from Proposition A.2.4 that for any $x \in X\setminus Z$, we have
$$\eta^*_t(x) = \Gamma^*_t(x)+\Gamma'^*_t(x)$$
for all $t>0$. The same trivially holds when $x\in Z$, so the equation holds everywhere. In particular, by Corollary A.2.1 and Proposition 1.2.8, we have
$$\eta_\tau = (\Gamma^*+\Gamma'^*)^*_\tau \in \mathrm{PSH}(X,\theta+\theta')$$
when $\tau < \Gamma_{\max}+\Gamma'_{\max}$.

Next, assume that Γ and Γ' are $\mathcal{I}$-model. We need to argue that so is $\Gamma+\Gamma'$. Fix $\tau<\Gamma_{\max}+\Gamma'_{\max}$. Then for each $\delta\in\mathbb{R}$ such that $\delta<\Gamma_{\max}$ and $\tau-\delta<\Gamma'_{\max}$, we know that $\Gamma_\delta \in \mathrm{PSH}(X,\theta)_{>0}$ and $\Gamma'_{\tau-\delta}\in\mathrm{PSH}(X,\theta')_{>0}$ by Lemma 9.1.1. It follows from Example 7.1.2 that Γ_t and $\Gamma'_{\tau-t}$ are both $\mathcal{I}$-good, hence so is $\Gamma_t+\Gamma'_{\tau-t}\in\mathrm{PSH}(X,\theta+\theta')_{>0}$ by Proposition 7.2.1. Therefore, η_τ is $\mathcal{I}$-good by Proposition 7.2.2. Therefore, $\Gamma+\Gamma'$ is $\mathcal{I}$-model. □

Proposition 9.4.1 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, then $\Gamma+\Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$. Moreover,*
$$(\Gamma+\Gamma')_{-\infty} = P_{\theta+\theta'}\left[\Gamma_{-\infty}+\Gamma'_{-\infty}\right]. \tag{9.39}$$

[6] There is no usc regularization in the formula. It is not a typo.

When $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *and* $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$*, we have* $\Gamma+\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$.

The operation $+$ *is commutative and associative.*

Proof It follows immediately from Lemma 9.4.2 that $\Gamma+\Gamma' \in \mathrm{TC}(X,\theta+\theta')_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\theta')_{>0}$ if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')_{>0}$.

We argue (9.39). By definition, for any small enough τ, we have

$$(\Gamma+\Gamma')_{-\infty} \geq (\Gamma+\Gamma')_{2\tau} \geq_P \Gamma_\tau + \Gamma'_\tau .$$

Letting $\tau \to -\infty$ and applying Proposition 6.2.4 and Theorem 6.2.2, we find that

$$(\Gamma+\Gamma')_{-\infty} \geq_P \Gamma_{-\infty} + \Gamma'_{-\infty} .$$

On the other hand, for each small enough τ, we have

$$(\Gamma+\Gamma')_\tau \sim_P \sup_{\delta\in\mathbb{R}} \left(\Gamma_\delta + \Gamma'_{\tau-\delta}\right) \leq_P \Gamma_{-\infty} + \Gamma'_{-\infty}$$

by Proposition 6.1.5 and Proposition 6.2.4. We apply Proposition 6.2.4 again, we conclude that

$$(\Gamma+\Gamma')_{-\infty} \leq_P \Gamma_{-\infty} + \Gamma'_{-\infty} .$$

So (9.39) follows.

Finally, let us show that $+$ is commutative and associative. Commutativity is obvious. Let $\Gamma'' \in \mathrm{TC}(X,\theta'')_{>0}$. Then we want to show that

$$(\Gamma+\Gamma')+\Gamma'' = \Gamma+(\Gamma'+\Gamma'').$$

First observe that

$$((\Gamma+\Gamma')+\Gamma'')_{\max} = (\Gamma+(\Gamma'+\Gamma''))_{\max} .$$

Fix τ less than this common value. We compute that

$$\begin{aligned} &((\Gamma+\Gamma')+\Gamma'')_\tau \\ =&P_\theta\left[\sup_{\delta_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{\delta_1}+\Gamma''_{\tau-\delta_1}\right)\right] \\ \sim_P& \sup_{\delta_1\in\mathbb{R}}\left((\Gamma+\Gamma')_{\delta_1}+\Gamma''_{\tau-\delta_1}\right) \\ \sim_P& \sup_{\delta_1,\delta_2\in\mathbb{R}}\left(\Gamma_{\delta_2}+\Gamma'_{\delta_1-\delta_2}+\Gamma''_{\tau-\delta_1}\right), \end{aligned}$$

where in the last line, we applied Proposition 6.2.4 and Proposition 6.1.5. Similarly, for $(\Gamma+(\Gamma'+\Gamma''))_\tau$, we get the same expression. The associativity follows. □

Lemma 9.4.3 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ *and* $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$*, then for any closed smooth positive* $(1,1)$*-forms* ω *and* ω' *on* X*, we have*

$$P_{\theta+\theta'+\omega+\omega'}[\Gamma+\Gamma'] = P_{\theta+\omega}[\Gamma] + P_{\theta'+\omega'}[\Gamma].$$

Proof Observe that

$$\begin{aligned} P_{\theta+\theta'+\omega+\omega'}[\Gamma+\Gamma']_{\max} =& (P_{\theta+\omega}[\Gamma] + P_{\theta'+\omega'}[\Gamma])_{\max} \\ =& \Gamma_{\max} + \Gamma_{\max}. \end{aligned}$$

Take $\tau \in \mathbb{R}$ less than this common value, we need to verify that

$$(\Gamma+\Gamma')_\tau \sim_P (P_{\theta+\omega}[\Gamma] + P_{\theta'+\omega'}[\Gamma])_\tau .$$

By definition, this means that

$$\sup_{\tau-\Gamma'_{\max}<\delta<\Gamma_{\max}} \left(\Gamma_\delta + \Gamma'_{\tau-\delta}\right) \sim_P \sup_{\tau-\Gamma'_{\max}<\delta<\Gamma_{\max}} \left(P_{\theta+\omega}[\Gamma_\delta] + P_{\theta'+\omega'}[\Gamma'_{\tau-\delta}]\right).$$

This is a consequence of Proposition 6.1.5 and Proposition 6.1.6. □

Definition 9.4.3 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $C \in \mathbb{R}$, we define $\Gamma + C \in \mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set
$$(\Gamma+C)_{\max} \coloneqq \Gamma_{\max} + C;$$
(2) for any $\tau < (\Gamma+C)_{\max}$, we set
$$(\Gamma+C)_\tau \coloneqq \Gamma_{\tau-C}.$$

It is obvious that if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then so is $\Gamma + C$.

Proposition 9.4.2 *Let* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma \in \mathrm{TC}(X,\theta')_{>0}$ *and* $C, C' \in \mathbb{R}$, *then*

(1) $(\Gamma+\Gamma')+C = \Gamma + (\Gamma'+C) = (\Gamma+C)+\Gamma'$;
(2) $\Gamma + (C+C') = (\Gamma+C)+C'$.

Proof (1) We first observe that

$$((\Gamma+\Gamma')+C)_{\max} = (\Gamma+(\Gamma'+C))_{\max} = ((\Gamma+C)+\Gamma')_{\max} = \Gamma_{\max} + \Gamma'_{\max} + C.$$

Take any $\tau \in \mathbb{R}$ less than this common value. We compute

$$\begin{aligned} ((\Gamma+\Gamma')+C)_\tau =& (\Gamma+\Gamma')_{\tau-C} = P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}} \left(\Gamma_\delta + \Gamma'_{\tau-C-\delta}\right)\right], \\ (\Gamma+(\Gamma'+C))_\tau =& P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}} \left(\Gamma_\delta + (\Gamma'+C)_{\tau-\delta}\right)\right] = P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}} \left(\Gamma_\delta + \Gamma'_{\tau-C-\delta}\right)\right], \\ ((\Gamma+C)+\Gamma')_\tau =& P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}} \left((\Gamma+C)_{C+\delta} + \Gamma'_{\tau-C-\delta}\right)\right] \\ =& P_{\theta+\theta'}\left[\sup_{\delta\in\mathbb{R}} \left(\Gamma_\delta + \Gamma'_{\tau-C-\delta}\right)\right]. \end{aligned}$$

(2) Observe that

$$(\Gamma + (C + C'))_{\max} = ((\Gamma + C) + C')_{\max} = \Gamma_{\max} + C + C'.$$

For any $\tau \in \mathbb{R}$ less than this value, we have

$$(\Gamma + (C + C'))_\tau = \Gamma_{\tau - C - C'} = ((\Gamma + C) + C')_\tau .$$

Definition 9.4.4 Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$. We define $\Gamma \vee \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$ as follows:

(1) We set
$$(\Gamma \vee \Gamma')_{\max} \coloneqq \Gamma_{\max} \vee \Gamma'_{\max};$$

(2) for any $\tau < (\Gamma \vee \Gamma')_{\max}$, we define

$$(\Gamma \vee \Gamma')_\tau \coloneqq P_\theta \left[\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)\right]. \tag{9.40}$$

Recall that the upper concave envelope CE is defined in Definition A.1.4[7]. Trivially, we have $\Gamma \vee \Gamma' \geq \Gamma$ and $\Gamma \vee \Gamma' \geq \Gamma'$.

Lemma 9.4.4 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$. Then for any $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$, we have*

$$\mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau \in \mathrm{PSH}(X, \theta).$$

This potential is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$.

In particular, (9.40) *in Definition 9.4.4 makes sense.*

Proof To simply the notations, we write

$$\psi_\tau = \mathrm{CE}\left(\rho \mapsto \Gamma_\rho \vee \Gamma'_\rho\right)_\tau$$

for all $\tau \in \mathbb{R}$. Thanks to Proposition A.2.3, we have

$$\psi^*_t(x) = \Gamma^*_t(x) \vee \Gamma'^*_t(x) \tag{9.41}$$

for all $t > 0$ as long as $\Gamma_\tau(x) \neq -\infty$ and $\Gamma_\tau(x) \neq -\infty$ for some $\tau \in \mathbb{R}$. Otherwise, assume that $x \in X$ is such that $\Gamma_\tau = -\infty$ for all $\tau \in \mathbb{R}$, then by definition, $\psi_\tau(x) = \Gamma'_\tau(x)$ for all $\tau \in \mathbb{R}$. Therefore, $\Gamma^*_t(x) = -\infty$ for all $t > 0$ and hence (9.41) continues to hold. Therefore, we have shown that

$$\psi^*_t = \Gamma^*_t \vee \Gamma'^*_t \in \mathrm{PSH}(X, \theta).$$

It follows from Proposition 4.1.3 that $(\psi^*_t)_{t \in [a,b]}$ is a subgeodesic for any $0 < a < b$.

Next we observe that $\psi_\bullet$ is closed by definition. So it follows from Proposition A.2.3 and Proposition 1.2.8 that

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[7] In Definition A.1.4, we define the convex analogue, the *lower convex envelope*. This can be translated into concave functions in the obvious manner.

$$\psi_\tau = (\psi_\bullet^*)_\tau^* \in \mathrm{PSH}(X,\theta) \cup \{-\infty\}.$$

Due to Proposition 9.1.4 and Proposition A.1.2, there is a pluripolar set $Z \subseteq X$ such that for $x \in X \setminus Z$, we have

$$\psi_\tau(x) = \sup\left\{\lambda\Gamma_\rho(x) + (1-\lambda)\Gamma'_{\rho'}(x) : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\}$$

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$. It follows from Proposition 1.2.6 that

$$\psi_\tau = \sup{}^*\left\{\lambda\Gamma_\rho + (1-\lambda)\Gamma'_{\rho'} : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau\right\} \tag{9.42}$$

for all $\tau < \Gamma_{\max} \vee \Gamma'_{\max}$.

It follows from (9.42) that ψ_τ is $\mathcal{I}$-good if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, thanks to Proposition 7.2.1 and Proposition 7.2.2. □

Corollary 9.4.1 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$. Then $\Gamma \vee \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$ and*

$$(\Gamma \vee \Gamma')_{-\infty} = P_\theta\left[\Gamma_{-\infty} \vee \Gamma'_{-\infty}\right]. \tag{9.43}$$

If $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

For each $\Gamma'' \in \mathrm{TC}(X,\theta)_{>0}$ and each $\Gamma'' \geq \Gamma$ and $\Gamma'' \geq \Gamma'$, we have $\Gamma'' \geq \Gamma \vee \Gamma'$.

Moreover, the operation $\vee$ is associative and commutative.

In particular, given a finite family $\{\Gamma_i\}_{i\in I}$ in $\mathrm{TC}(X,\theta)_{>0}$, we can define

$$\bigvee_{i\in I} \Gamma_i$$

without ambiguity.

Proof It follows immediately from Lemma 9.4.4 that $\Gamma \vee \Gamma' \in \mathrm{TC}(X,\theta)_{>0}$, and it lies in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ if $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$.

The argument of (9.43) is very similar to that of (9.39), which we leave to the readers.

Take Γ'' as in the statement of the proposition. First observe that

$$\Gamma''_{\max} \geq \Gamma_{\max} \vee \Gamma'_{\max} = (\Gamma \vee \Gamma')_{\max}.$$

Take $\tau < (\Gamma \vee \Gamma')_{\max}$, we argue that

$$\Gamma''_\tau \geq (\Gamma \vee \Gamma')_\tau.$$

By the concavity of Γ'', this is equivalent to

$$\Gamma''_\tau \geq \Gamma_\tau \vee \Gamma'_\tau.$$

Therefore,

$$\Gamma'' \geq \Gamma \vee \Gamma'.$$

The commutativity and associativity of $\vee$ are trivial. □

Lemma 9.4.5 *Let $\Gamma, \Gamma' \in \mathrm{TC}(X, \theta)_{>0}$ and ω be a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}[\Gamma \vee \Gamma'] = P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'].$$

Proof We first observe that

$$(P_{\theta+\omega}[\Gamma \vee \Gamma'])_{\max} = (P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'])_{\max} = \Gamma_{\max} \vee \Gamma'_{\max}.$$

Let $\tau \in \mathbb{R}$ be less than this common value. We need to show that

$$(\Gamma \vee \Gamma')_\tau \sim_P (P_{\theta+\omega}[\Gamma] \vee P_{\theta+\omega}[\Gamma'])_\tau .$$

We need the formula (9.42) proved in the proof of Lemma 9.4.4:

$$(\Gamma \vee \Gamma')_\tau = \sup{}^* \left\{ \lambda \Gamma_\rho + (1-\lambda)\Gamma'_{\rho'} : \lambda \in (0,1), \rho, \rho' \in \mathbb{R}, \lambda\rho + (1-\lambda)\rho' = \tau \right\}.$$

A similar result holds with $P_{\theta+\omega}[\Gamma]$ and $P_{\theta+\omega}[\Gamma']$ in place of Γ and Γ'. So our assertion is a direct consequence of Proposition 6.1.5 and Proposition 6.1.6. □

Definition 9.4.5 Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X, \theta)_{>0}$. Assume that

$$\sup_{i\in I} \Gamma^i_{\max} < \infty. \tag{9.44}$$

Then we define $\sup_{i\in I}{}^* \Gamma^i \in \mathrm{TC}(X, \theta)_{>0}$ as follows:

(1) We set

$$\left(\sup_{i\in I}{}^* \Gamma^i \right)_{\max} = \sup_{i\in I} \Gamma^i_{\max};$$

(2) for any $\tau < \sup_{i\in I} \Gamma^i_{\max}$, we let

$$\left(\sup_{i\in I}{}^* \Gamma^i \right)_\tau := \sup_{i\in I}{}^* \Gamma^i_\tau.$$

Proposition 9.4.3 *Let $(\Gamma^i)_{i\in I}$ be an increasing net in $\mathrm{TC}(X, \theta)_{>0}$ satisfying (9.44). Then $\sup_{i\in I}{}^* \Gamma^i$ as defined in Definition 9.4.5 lies in $\mathrm{TC}(X, \theta)_{>0}$. Moreover, if $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$ for all $i \in I$, then $\sup_{i\in I}{}^* \Gamma^i$ lies in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$ as well.*

Moreover, we have

$$\left(\sup_{i\in I}{}^* \Gamma^i \right)_{-\infty} = \sup_{i\in I}{}^* \Gamma^i_{-\infty}. \tag{9.45}$$

Proof The first assertion follows easily from Proposition 3.1.11, while the second follows from Proposition 3.2.14.

It remains to argue (9.45). Without loss of generality, we may assume that I contains a minimal element i_0.

By Proposition 1.2.5, there is a pluripolar set $Z \subseteq X$ such that for any $x \in X \setminus Z$,

$$\left(\sup_{i\in I}{}^{*}\Gamma^i\right)_{-\infty}(x) = \sup_{\mathbb{Q}\ni\tau<\Gamma^{i_0}_{\max}} \left(\sup_{i\in I}{}^{*}\Gamma^i_\tau\right)(x) = \sup_{\mathbb{Q}\ni\tau<\Gamma^{i_0}_{\max}, i\in I} \Gamma^i_\tau(x) = \sup_{i\in I}\Gamma^i_{-\infty}(x).$$

So they are equal everywhere by Proposition 1.2.6. □

Lemma 9.4.6 *Let* $(\Gamma^i)_{i\in I}$ *be an increasing net in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.44). *Assume that* ω *is a closed smooth positive* $(1,1)$*-form on* X. *Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^{*}\Gamma^i\right] = \sup_{i\in I}{}^{*}P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof Observe that

$$\left(P_{\theta+\omega}\left[\sup_{i\in I}{}^{*}\Gamma^i\right]\right)_{\max} = \left(\sup_{i\in I}{}^{*}P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max}.$$

Fix $\tau\in\mathbb{R}$ less than this common value.

It suffices to show that

$$\left(\sup_{i\in I}{}^{*}\Gamma^i\right)_\tau \sim_P \left(\sup_{i\in I}{}^{*}P_{\theta+\omega}\left[\Gamma^i\right]\right)_\tau.$$

This is an immediate consequence of Proposition 6.1.6. □

Definition 9.4.6 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.44). Then we define

$$\sup_{i\in I}{}^{*}\Gamma^i := \sup_{J\in\mathrm{Fin}(I)}{}^{*}\left(\bigvee_{j\in J}\Gamma^j\right). \tag{9.46}$$

Recall that $\mathrm{Fin}(I)$ is the net of non-empty finite subsets of I, ordered by inclusion.

Observe that by Definition 9.4.4, we have

$$\sup_{J\in\mathrm{Fin}(I)}\left(\bigvee_{j\in J}\Gamma^j\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max} < \infty.$$

So (9.46) makes sense. In particular,

$$\left(\sup_{i\in I}\Gamma^i\right)_{\max} = \sup_{i\in I}\Gamma^i_{\max}. \tag{9.47}$$

It is clear that Definition 9.4.6 extends both Definition 9.4.5 and Definition 9.4.4.

Proposition 9.4.4 *Let* $(\Gamma^i)_{i\in I}$ *be a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.44). *Then* $\sup_{i\in I}{}^{*}\Gamma^i\in\mathrm{TC}(X,\theta)_{>0}$. *Moreover, if* $\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *for all* $i\in I$, *then so is* $\sup_{i\in I}{}^{*}\Gamma^i$.

Finally, we have

$$\left(\sup_{i\in I}{}^{*}\Gamma^i\right)_{-\infty} = \sup_{i\in I}{}^{*}\Gamma^i_{-\infty}. \tag{9.48}$$

Proof The first assertion and the second follow from Proposition 9.4.3 and Corollary 9.4.1.

It remains to argue (9.48). For this purpose, it suffices to show that

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)_{-\infty} \sim_P \sup_{i\in I}{}^*\Gamma^i_{-\infty}.$$

For any $J \in \mathrm{Fin}(I)$, it follows from Corollary 9.4.1 and Proposition 6.1.6 that

$$\left(\bigvee_{j\in J}\Gamma^j\right)_{-\infty} \sim_P \bigvee_{j\in J}\Gamma^j_{-\infty}.$$

From this, applying Proposition 3.1.11, Proposition 6.1.6 and Proposition 9.4.3, we conclude our assertion. □

Lemma 9.4.7 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.44). *Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right] = \sup_{i\in I}{}^*P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof This is a direct consequence of Lemma 9.4.6 and Lemma 9.4.5. □

We prove a version of Choquet's lemma.

Proposition 9.4.5 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.44). *Then there is a countable subset $I' \subseteq I$ such that*

$$\sup_{i\in I}{}^*\Gamma^i = \sup_{i\in I'}{}^*\Gamma^i.$$

Proof We may assume that I is infinite.

It follows from Proposition 1.2.2 that we can find a countable subset $I' \subseteq I$ such that for each

$$\tau \in \left(-\infty, \sup_{i\in I}{}^*\Gamma^i_{\max}\right)\cap\mathbb{Q},$$

we have

$$\sup_{i\in I}{}^*\Gamma^i_\tau = \sup_{i\in I'}{}^*\Gamma^i_\tau.$$

Let $\Gamma' = \sup_{i\in I'}{}^*\Gamma^i$. Then clearly, $\Gamma' \leq \Gamma$. We claim that they are actually equal.

Thanks to Proposition 6.1.1 and Lemma 9.1.1, it suffices to show that for any $\tau < \sup_{i\in I}{}^*\Gamma^i_{\max}$, we have

$$\int_X \left(\theta + \mathrm{dd^c}\Gamma'_\tau\right)^n = \int_X \left(\theta + \mathrm{dd^c}\Gamma_\tau\right)^n.$$

Since we know that this holds for τ lying in a dense subset, the same holds everywhere by Proposition 9.1.1. □

Proposition 9.4.6 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.44). Let $C \in \mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i + C) = \sup_{i\in I}{}^*\Gamma^i + C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another family in $\mathrm{TC}(X,\theta')_{>0}$ satisfying (9.44). Suppose that $\Gamma^i \leq \Gamma'^i$ for all $i \in I$, then

$$\sup_{i\in I}{}^*\Gamma^i \leq \sup_{i\in I}{}^*\Gamma'^i.$$

Proof This is immediate by definition. □

Definition 9.4.7 Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\lambda > 0$, we define $\lambda\Gamma \in \mathrm{TC}(X,\lambda\theta)_{>0}$ as follows:

(1) We set
$$(\lambda\Gamma)_{\max} = \lambda\Gamma_{\max};$$

(2) for any $\tau < (\lambda\Gamma)_{\max}$, we set
$$(\lambda\Gamma)_\tau = \lambda\Gamma_{\lambda^{-1}\tau}.$$

Proposition 9.4.7 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\lambda > 0$, then $\lambda\Gamma$ as defined in Definition 9.4.7 lies in $\mathrm{TC}(X,\lambda\theta)_{>0}$. Moreover, if $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, then $\lambda\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)_{>0}$.*

We have
$$(\lambda\Gamma)_{-\infty} = \lambda\Gamma_{-\infty}. \tag{9.49}$$

Proof This is immediate by definition. □

Proposition 9.4.8 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$, $\Gamma' \in \mathrm{TC}(X,\theta')_{>0}$, $C \in \mathbb{R}$ and $\lambda, \lambda' > 0$, we have*
$$\begin{aligned}\lambda(\Gamma+\Gamma') &= \lambda\Gamma + \lambda\Gamma',\\ (\lambda\lambda')\Gamma &= \lambda(\lambda'\Gamma),\\ \lambda(\Gamma + C) &= \lambda\Gamma + \lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ is a non-empty family in $\mathrm{TC}(X,\theta)_{>0}$ satisfying (9.44), then

$$\lambda\left(\sup_{i\in I}{}^*\Gamma^i\right) = \sup_{i\in I}{}^*(\lambda\Gamma^i).$$

Proof This is immediate by definition. □

Lemma 9.4.8 *Let $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$ and $\lambda > 0$. Then for any closed smooth positive $(1,1)$-form ω on X, we have*

$$P_{\lambda\theta+\lambda\omega}[\lambda\Gamma] = \lambda P_{\theta+\omega}[\Gamma].$$

Proof This is clear by definition. □

Definition 9.4.8 Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{TC}(X,\theta)_{>0}$. Assume that

$$\inf_{i\in I} \Gamma^i_{\max} > -\infty, \tag{9.50}$$

and

$$\inf_{i\in I} \int_X \left(\theta + \mathrm{dd^c}\Gamma^i_\tau\right)^n > 0, \quad \text{for some } \tau < \inf_{i\in I} \Gamma^i_{\max}. \tag{9.51}$$

Then we define $\inf_{i\in I} \Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$ as follows:

(1) We set

$$\left(\inf_{i\in I} \Gamma^i\right)_{\max} = \inf_{i\in I} \Gamma^i_{\max};$$

(2) for any $\tau < \left(\inf_{i\in I} \Gamma^i\right)_{\max}$, we let

$$\left(\inf_{i\in I} \Gamma^i\right)_\tau := \inf_{i\in I} \Gamma^i_\tau.$$

Proposition 9.4.9 *Let* $(\Gamma^i)_{i\in I}$ *be a decreasing net in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.50) *and* (9.51)*, then* $\inf_{i\in I} \Gamma^i \in \mathrm{TC}(X,\theta)_{>0}$.

Moreover, if $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *for all* $i \in I$*, then so is* $\inf_{i\in I} \Gamma^i$.

Proof The first assertion is an immediate consequence of Proposition 3.1.9 and Proposition 3.1.10. The last assertion follows from Proposition 3.2.13. □

In general, it is not true that

$$\left(\inf_{i\in I} \Gamma^i\right)_{-\infty} = \inf_{i\in I} \Gamma^i_{-\infty}.$$

Lemma 9.4.9 *Let* $(\Gamma^i)_{i\in I}$ *be a decreasing net in* $\mathrm{TC}(X,\theta)_{>0}$ *satisfying* (9.50) *and* (9.51)*. Assume that* ω *is a closed smooth positive* $(1,1)$*-form on* X*. Then*

$$P_{\theta+\omega}\left[\inf_{i\in I} \Gamma^i\right] = \inf_{i\in I} P_{\theta+\omega}\left[\Gamma^i\right].$$

Proof First observe that

$$\left(P_{\theta+\omega}\left[\inf_{i\in I} \Gamma^i\right]\right)_{\max} = \left(\inf_{i\in I} P_{\theta+\omega}\left[\Gamma^i\right]\right)_{\max}.$$

Let $\tau \in \mathbb{R}$ be less than this common value. Then we need to show the following:

$$P_{\theta+\omega}\left[\inf_{i\in I} \Gamma^i_\tau\right] \sim_P \inf_{i\in I} P_{\theta+\omega}\left[\Gamma^i_\tau\right]. \tag{9.52}$$

It follows from Proposition 3.1.10 and Corollary 6.2.5 that $\Gamma^i_\tau \xrightarrow{d_S} \inf_{j\in I} \Gamma^j_\tau$. Thanks to Corollary 6.2.8 and Corollary 6.2.5, we have

$$P_{\theta+\omega}\left[\Gamma_\tau^i\right] \xrightarrow{d_S} \inf_{j\in I}\Gamma_\tau^j, \quad P_{\theta+\omega}\left[\Gamma_\tau^i\right] \xrightarrow{d_S} \inf_{j\in I} P_{\theta+\omega}\left[\Gamma_\tau^j\right].$$

Hence, (9.52) follows from Proposition 6.2.2. □

Chapter 10
The theory of Okounkov bodies

It is very fortunate that, unlike people who dig for gold, mathematicians can freely share their precious treasures with everybody. Once you understand something really well, it feels great to explain it to all.
— Andrei Okounkov[a]

[a] Andrei Yuryevich Okounkov (1969–) is a Russian-American mathematician renowned for his contributions to representation theory. He was one of the key organizers of the ICM 2022 in St. Petersburg, which was unfortunately canceled under the indiscriminate discrimination against Russian citizens by the virtue signalers all over the western world after the war waged by the ruling class.

In this chapter, we apply our theory of singularities to the study of Okounkov bodies. We establish the theory of partial Okounkov bodies, which are convex bodies constructed from a given plurisubharmonic singularity. These objects allow us to reduce many problems in pluripotential theory to problems in convex geometry, which are usually simpler.

We will establish two related theories — One in the algebraic setting in Section 10.3 and one in the transcendental setting in Section 10.4.

The readers are assumed to have some knowledge in the classical Okounkov bodies. The original papers of Lazarsfeld–Mustață [LM09] and Kaveh–Khovanskii[KK12] are highly recommended. We give a rather brief introduction here.

Let X be an irreducible smooth projective variety of dimension n and L be a big holomorphic line bundle on X. Given any admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ on X (see Definition 10.2.1 for the precise definition), one can attach a natural convex body $\Delta(L)$ of dimension n to L, generalizing the classical Newton polytope construction in toric geometry as we recalled in Definition 5.2.1. This construction was first considered by Okounkov [Oko96, Oko03] and then extended by Lazarsfeld–Mustață [LM09] and Kaveh–Khovanskii [KK12]. The convex body $\Delta(L)$ is known as the *Okounkov body* or *Newton–Okounkov body* associated with L (with respect to the given flag). In fact, by taking the successive order of vanishing along the flag, we can define a valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$. Consider the semigroup

$$\Gamma(L) \coloneqq \left\{(\nu(s), k) \in \mathbb{Z}^{n+1} : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k)^\times\right\}.$$

Then $\Delta(L)$ is the intersection of the closed convex cone in $\mathbb{R}^{n+1}$ generated by $\Gamma(L)$ with the hyperplane $\{(x, 1) : x \in \mathbb{R}^n\}$.

In [LM09], Lazarsfeld–Mustață showed moreover that $\Delta(L)$ depends only on the numerical class of L. Conversely, it is shown by Jow [Jow10] that the information of all Okounkov bodies with respect to various flags actually determines the numerical class of L. In other words, Okounkov bodies can be regarded as universal numerical invariants of big line bundles.

This chapter concerns a similar problem. Assume that L is equipped with a psh metric ϕ. We will construct universal invariants of the singularity type of ϕ. We call these universal invariants the *partial Okounkov bodies* of (L, ϕ). The name *partial* refers to the fact that these convex bodies are contained in $\Delta(L)$.

We define a set

$$\Gamma(L,\phi) := \left\{ (\nu(s), k) \in \mathbb{Z}^{n+1} : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\phi))^\times \right\}$$

similar to $\Gamma(L)$. However, a key difference here is that $\Gamma(L, \phi)$ is not a semigroup in general. Thus, the constructions in both [LM09] and [KK12] break down. We will show that in this case, there is still a canonical construction of Okounkov bodies.

As we will see shortly, although $\Gamma(L, \phi)$ fails to be a semigroup, it is not very far away from a semigroup. It is an instance of the *almost semigroups* that we are about to define in Section 10.1. We will show that the Okounkov body construction Δ admits a continuous extension to almost semigroups. In particular, we can define

$$\Delta(L,\phi) := \Delta\left(\Gamma(L,\phi)\right).$$

We will prove that $\Delta(L, \phi)$ can be regarded as universal invariants of the singularities of ϕ, see Corollary 10.3.3.

10.1 Almost semigroups

We give a brief presentation of the theory of almost semigroups. The proofs will be presented in Appendix C.

Fix an integer $n \geq 0$. Fix a closed convex cone $C \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ such that $C \cap \{x_{n+1} = 0\} = \{0\}$. Here x_{n+1} is the last coordinate of $\mathbb{R}^{n+1}$.

Write $\hat{\mathcal{S}}(C)$ for the set of subsets of $C \cap \mathbb{Z}^{n+1}$ and $\mathcal{S}(C)$ for the set of sub-semigroups $S \subseteq C \cap \mathbb{Z}^{n+1}$. For each $k \in \mathbb{N}$ and $S \in \hat{\mathcal{S}}(C)$, we write

$$S_k := \{x \in \mathbb{Z}^n : (x, k) \in S\}.$$

Note that S_k is a finite set by our assumption on C.

We introduce a pseudometric on $\hat{\mathcal{S}}(C)$ as follows:

$$d_{\mathrm{sg}}(S, S') := \varlimsup_{k\to\infty} k^{-n} \left(|S_k| + |S'_k| - 2|(S \cap S')_k|\right). \tag{10.1}$$

Here $|\bullet|$ denotes the cardinality of a finite set. The above defined d_{sg} is a pseudometric on $\hat{\mathcal{S}}(C)$. Given $S, S' \in \hat{\mathcal{S}}(C)$, we say S is equivalent to S' and write $S \sim S'$ if $d_{\mathrm{sg}}(S, S') = 0$. This is an equivalence relation.

The volume of $S \in \mathcal{S}(C)$ is defined as

$$\operatorname{vol} S := \lim_{k\to\infty} (ka)^{-n}|S_{ka}| = \overline{\lim_{k\to\infty}}\, k^{-n}|S_k|,$$

where a is a sufficiently divisible positive integer. The existence of the limit and its independence from a both follow from the more precise result [KK12, Theorem 2].

We define $\overline{\mathcal{S}}(C)$ as the closure of $\mathcal{S}(C)$ in $\hat{\mathcal{S}}(C)$ with respect to the topology defined by the pseudometric d. The function $\operatorname{vol}\colon \mathcal{S}(C) \to \mathbb{R}$ admits a unique 1-Lipschitz extension to

$$\operatorname{vol}\colon \overline{\mathcal{S}}(C) \to \mathbb{R}. \tag{10.2}$$

Given $S \in \hat{\mathcal{S}}(C)$, we will write $C(S) \subseteq C$ for the closed convex cone generated by $S \cup \{0\}$. Moreover, for each $k \in \mathbb{Z}_{>0}$, we define

$$\Delta_k(S) := \operatorname{Conv}\left\{k^{-1}x \in \mathbb{R}^n : x \in S_k\right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the convex hull.

Definition 10.1.1 Let $\mathcal{S}'(C)$ be the subset of $\mathcal{S}(C)$ consisting of semigroups S such that S generates $\mathbb{Z}^{n+1}$ (as an Abelian group).

Note that for any $S \in \mathcal{S}'(C)$, the cone $C(S)$ has full dimension (i.e. the topological interior is non-empty). Given a full-dimensional subcone $C' \subseteq C$, it is clear that $C' \cap \mathbb{Z}^{n+1} \in \mathcal{S}'(C)$.

We recall the following definition from [KK12].

Definition 10.1.2 Given $S \in \mathcal{S}'(C)$, its *Okounkov body* is defined as follows

$$\Delta(S) := \{x \in \mathbb{R}^n : (x, 1) \in C(S)\}.$$

Theorem 10.1.1 *For each $S \in \mathcal{S}'(C)$, we have*

$$\operatorname{vol} S = \lim_{k\to\infty} k^{-n}|S_k| = \operatorname{vol}\Delta(S) > 0. \tag{10.3}$$

Moreover, as $k \to \infty$,

$$\Delta_k(S) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S). \tag{10.4}$$

Corollary 10.1.1 *Let $S, S' \in \mathcal{S}'(C)$. Assume that* $\operatorname{vol}(S \cap S') > 0$*, then we have*

$$d_{\mathrm{sg}}(S, S') = \operatorname{vol}(S) + \operatorname{vol}(S') - 2\operatorname{vol}(S \cap S').$$

Definition 10.1.3 We define $\overline{\mathcal{S}'(C)}_{>0}$ as elements in the closure of $\mathcal{S}'(C)$ in $\hat{\mathcal{S}}(C)$ with positive volume. An element in $\overline{\mathcal{S}'(C)}_{>0}$ is called an *almost semigroup* in C.

Recall that the volume here is defined in (10.2).

Theorem 10.1.2 *The Okounkov body map $\Delta\colon \mathcal{S}'(C) \to \mathcal{K}_n$ as defined in Definition 10.1.2 admits a unique continuous extension*

$$\Delta\colon \overline{\mathcal{S}'(C)}_{>0} \to \mathcal{K}_n. \tag{10.5}$$

Moreover, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$*, we have*

$$\operatorname{vol} S = \operatorname{vol} \Delta(S). \tag{10.6}$$

Corollary 10.1.2 *Suppose that* $S, S' \in \overline{\mathcal{S}'(C)}_{>0}$ *with* $S \subseteq S'$*, then*

$$\Delta(S) \subseteq \Delta(S'). \tag{10.7}$$

10.2 Flags and valuations

10.2.1 The algebraic setting

Let X be an irreducible normal projective variety of dimension n.

Definition 10.2.1 An *admissible flag* $Y_\bullet$ on X is a flag of subvarieties

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

such that Y_i is irreducible of codimension i and is smooth at the point Y_n.

Given any admissible flag $Y_\bullet$, we can define a rank n valuation $\nu_{Y_\bullet} : \mathbb{C}(X)^\times \to \mathbb{Z}^n$. Here we consider $\mathbb{Z}^n$ as a totally ordered Abelian group with the lexicographic order. We sometimes write $\mathbb{Z}^n_{\mathrm{lex}}$ to emphasize this point.

If we identify the elements in $\mathbb{Z}^n$ with a row vector, the automorphism group $\mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ of $\mathbb{Z}^n_{\mathrm{lex}}$ is then identified with the subgroup of $\mathrm{GL}(n, \mathbb{Z})$ consisting of matrices of the form $\mathrm{I} + U$, where I is the identity matrix and U is a strictly upper triangular matrix with elements in $\mathbb{Z}$.

We recall the definition of $\nu_{Y_\bullet}$: Let $s \in \mathbb{C}(X)^\times$. Let $\nu(s)_1 = \operatorname{ord}_{Y_1} s$. After localization around Y_n, we can take a local defining equation t^1 of Y_1, set $s_1 = (s(t^1)^{-\nu_1(s)})|_{Y_1}$. Then $s_1 \in \mathbb{C}(Y_1)^\times$. We can repeat this construction with Y_2 in place of Y_1 to get $\nu(s)_2$ and s_2. Repeating this construction n times, we get

$$\nu_{Y_\bullet}(s) = (\nu(s)_1, \nu(s)_2, \ldots, \nu(s)_n) \in \mathbb{Z}^n.$$

It is easy to verify that $\nu_{Y_\bullet}$ is indeed a rank n valuation.

The same construction can be applied to define $\nu_{Y_\bullet}(s)$ when $s \in \mathrm{H}^0(X, L)$ or $\nu_{Y_\bullet}(D)$ when D is an effective divisor on X.

Remark 10.2.1 Conversely, by a theorem of Abhyankar, any valuation of $\mathbb{C}(X)$ with Noetherian valuation ring of rank n is equivalent to a valuation taking value in $\mathbb{Z}^n$, see [FK18, Chapter 0, Theorem 6.5.2]. As shown in [CFK+17, Theorem 2.9], any such valuation is equivalent[1] to (but not necessarily equal to) a valuation induced by an admissible flag on a modification of X.

[1] Two valuations ν, ν' with value in $\mathbb{Z}^n$ are equivalent if one can find a matrix G of the form $\mathrm{I} + N$, where N is strictly upper triangular with integral entries, such that $\nu' = \nu G$.

10.2.2 The transcendental setting

Let X be a connected compact Kähler manifold of dimension n.

Definition 10.2.2 A *smooth flag* $Y_\bullet$ on X consists of a flag of connected closed submanifolds of X:

$$X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n,$$

where Y_i has dimension $n - i$.

In this section, we will fix a smooth flag $Y_\bullet$ on X.

Definition 10.2.3 Let T be a closed positive $(1,1)$-current on X. We define the *valuation* of T along $Y_\bullet$ as

$$\nu_{Y_\bullet}(T) = \left(\nu_{Y_\bullet}(T)_1, \ldots, \nu_{Y_\bullet}(T)_n\right) \in \mathbb{R}^n_{\geq 0}$$

by induction on n. When $n = 0$, we define $\nu_{Y_\bullet}(T)$ as the unique point in $\mathbb{R}^0$. When $n \geq 1$, we define

$$\nu_{Y_\bullet}(T)_1(T) = \nu(T, Y_1);$$

Then for $i = 2, \ldots, n$, we define

$$\nu_{Y_\bullet}(T)_i = \nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])\right)_{i-1}.$$

Proposition 10.2.1 *Let T be a closed positive $(1,1)$-current on X. Then $\nu_{Y_\bullet}(T) \in \mathbb{R}^n_{\geq 0}$ defined in Definition 10.2.3 is independent of the choices of the trace operators in the definition. Moreover, $\nu_{Y_\bullet}(T)$ depends only on the $\mathcal{I}$-equivalence class of T.*

Proof We will prove both statements at the same time by induction on $n \geq 0$. The cases $n = 0, 1$ are trivial.

Let us consider the case $n > 1$ and assume that the result is known in dimension $n - 1$. We first observe that $\nu_{Y_\bullet}(T)$ is independent of the choice of the trace operator: Different choices of $\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])$ are $\mathcal{I}$-equivalent by Proposition 8.1.2. Therefore, by induction, its valuation is well-defined.

Next, let T' be another closed positive $(1,1)$-current such that $T \sim_{\mathcal{I}} T'$. Using Proposition 3.2.1, we know that $\nu(T, Y_1) = \nu(T', Y_1)$. Therefore,

$$T - \nu(T, Y_1)[Y_1] \sim_{\mathcal{I}} T' - \nu(T', Y_1)[Y_1].$$

It follows by induction that

$$\nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1])\right) = \nu_{Y_1 \supseteq \cdots \supseteq Y_n}\left(\mathrm{Tr}_{Y_1}(T' - \nu(T', Y_1)[Y_1])\right).$$

Example 10.2.1 When X is projective, we have

$$\nu_{Y_\bullet}([D]) = \nu_{Y_\bullet}(D),$$

where the right-hand side is defined in Section 10.2.1.

Proposition 10.2.2 *Let T, S be closed positive $(1,1)$-currents on X, $\lambda \in \mathbb{R}_{\geq 0}$. Then*

(1) *if $T \preceq_{\mathcal{I}} S$, we have*

$$\nu_{Y_\bullet}(T) \geq_{\mathrm{lex}} \nu_{Y_\bullet}(S). \tag{10.8}$$

(2) *We have the following additivity property:*

$$\nu_{Y_\bullet}(T+S) = \nu_{Y_\bullet}(T) + \nu_{Y_\bullet}(S), \quad \nu_{Y_\bullet}(\lambda T) = \lambda \nu_{Y_\bullet}(T). \tag{10.9}$$

Proof (1) We make an induction on $n \geq 0$. The case $n = 0, 1$ is trivial. Assume that $n \geq 2$ and the case $n-1$ is known. Observe that $\nu(T, Y_1) \geq \nu(S, Y_1)$, if the inequality is strict, we are done. So let us assume that $\nu(T, Y_1) = \nu(S, Y_1)$. By Proposition 8.2.1, we find that

$$\mathrm{Tr}_{Y_1}(T - \nu(T, Y_1)[Y_1]) \preceq_{\mathcal{I}} \mathrm{Tr}_{Y_1}(S - \nu(T, Y_1)[Y_1]).$$

By the inductive hypothesis, we conclude (10.8).

(2) We make an induction on $n \geq 0$. The cases $n = 0, 1$ are trivial. Assume that $n \geq 2$ and the case $n-1$ is known. By Proposition 1.4.2, we have

$$\nu(T+S, Y_1) = \nu(T, Y_1) + \nu(S, Y_1), \quad \nu(\lambda T, Y_1) = \lambda \nu(T, Y_1).$$

By Proposition 8.2.1, we have

$$\begin{aligned} \mathrm{Tr}_{Y_1}\left(T + S - \nu(T+S, Y_1)[Y_1]\right) \sim_P{} & \mathrm{Tr}_{Y_1}\left(T - \nu(T, Y_1)[Y_1]\right) \\ & + \mathrm{Tr}_{Y_1}\left(S - \nu(S, Y_1)[Y_1]\right), \\ \mathrm{Tr}_{Y_1}\left(\lambda T - \nu(\lambda T, Y_1)[Y_1]\right) \sim_P{} & \lambda\, \mathrm{Tr}_{Y_1}\left(T - \nu(T, Y_1)[Y_1]\right). \end{aligned}$$

By the inductive hypothesis, we conclude (10.9).

Assume that $n > 0$ for the remaining of this section.

Definition 10.2.4 Let $\pi \colon Z \to X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. We say that a smooth flag $W_\bullet$ on Z is a *lifting* of $Y_\bullet$ to Z if the restriction of π to $W_i \to Y_i$ is defined and is bimeromorphic for each $i = 0, \ldots, n$.

In this case, we define $\mathrm{cor}(Y_\bullet, \pi) \in \mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ inductively as follows: When $n = 1$, we define $\mathrm{cor}(Y_\bullet, \pi) = [1]$; when $n > 1$, we set

$$\mathrm{cor}(Y_\bullet, \pi) := \begin{bmatrix} 1 & -\nu_{W_1 \supseteq \cdots \supseteq W_n}((\pi^*[Y_1] - [W_1])|_{W_1}) \\ 0 & \mathrm{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi|_{W_1} \colon W_1 \to Y_1) \end{bmatrix}. \tag{10.10}$$

We observe that a lifting $W_\bullet$ of $Y_\bullet$ on Z is unique if it exists: For each $i = 0, \ldots, n-1$, the component W_{i+1} is necessarily the strict transform of Y_{i+1} with respect to the bimeromorphic morphism $W_i \to Y_i$. We shall also say that $(W_\bullet, \mathrm{cor}(Y_\bullet, \pi))$ is *the lifting* of $Y_\bullet$ to Z.

Proposition 10.2.3 *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism with Z being a Kähler manifold. Let $W_\bullet$ be a lifting of $Y_\bullet$, then for any closed positive $(1,1)$-current T on X, we have*

$$\nu_{W_\bullet}(\pi^* T) = \nu_{Y_\bullet}(T)\,\mathrm{cor}(Y_\bullet, \pi). \tag{10.11}$$

Proof We make induction on $n \geq 0$. The case $n = 0$ is trivial. In general, assume that $n \geq 1$ and the result is proved in dimension $n-1$.

For simplicity, we write $\nu = \nu_{Y_\bullet}$ and $\nu' = \nu_{W_\bullet}$. Let μ (resp. μ') be the valuation of currents defined by the truncated flag $Y_1 \supseteq \cdots \supseteq Y_n$ (resp. $W_1 \supseteq \cdots \supseteq W_n$). Then we need to show that

$$\begin{aligned} &\left[\nu'(\pi^*T)_1 \;\; \mu'(\mathrm{Tr}_{W_1}(\pi^*T - \nu'(\pi^*T)_1[W_1]))\right] \\ ={}& \left[\nu(T)_1 \;\; \mu(\mathrm{Tr}_{Y_1}(T - \nu(T)_1[Y_1]))\right] \mathrm{cor}(Y_\bullet, \pi). \end{aligned} \tag{10.12}$$

By Zariski's main theorem,

$$\nu'(\pi^*T)_1 = \nu(T)_1 =: c.$$

By the inductive hypothesis, we have

$$\mu'(\Pi^* \mathrm{Tr}_{Y_1}(T - c[Y_1])) = \mu(\mathrm{Tr}_{Y_1}(T - c[Y_1]))\,\mathrm{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \Pi), \tag{10.13}$$

where $\Pi\colon W_1 \to Y_1$ is the restriction of π. By Lemma 8.2.1 and Proposition 8.2.1,

$$\begin{aligned} &\Pi^* \mathrm{Tr}_{Y_1}(T - c[Y_1]) \sim_P \mathrm{Tr}_{W_1}(\pi^*(T - c[Y_1])) \\ \sim_P{}& \mathrm{Tr}_{W_1}(\pi^*T - c[W_1]) + c\,\mathrm{Tr}_{W_1}(\pi^*[Y_1] - [W_1]). \end{aligned}$$

So

$$\mu'(\Pi^* \mathrm{Tr}_{Y_1}(T - c[Y_1])) = \mu'(\mathrm{Tr}_{W_1}(\pi^*T - c[W_1])) + c\mu'(\mathrm{Tr}_{W_1}(\pi^*[Y_1] - [W_1])).$$

Combining the above with (10.13), we see that (10.12) follows. □

Proposition 10.2.4 *Let $\pi\colon Z \to X$, $p\colon Z' \to Z$ be proper bimeromorphic morphisms with Z and Z' being Kähler manifolds. Assume that $Y_\bullet$ admits a lifting $W_\bullet$ (resp. $W'_\bullet$) to Z (resp. Z'). Then*

$$\mathrm{cor}(Y_\bullet, \pi \circ p) = \mathrm{cor}(Y_\bullet, \pi)\,\mathrm{cor}(W_\bullet, p). \tag{10.14}$$

Proof We let $\pi' = \pi \circ p$:

$$\begin{tikzcd} Z' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Z \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

We make induction on $n \geq 1$. The case $n = 1$ is trivial. Assume that $n \geq 2$ and the case $n - 1$ has been solved. Then by (10.10), the desired formula (10.14) can be reformulated as

$$\begin{bmatrix} 1 & -\nu_{W_1'\supseteq\cdots\supseteq W_n'}((\pi'^*[Y_1]-[W_1'])|_{W_1'}) \\ 0 & \mathrm{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi'|_{W_1'} : W_1' \to Y_1) \end{bmatrix} =$$
$$\begin{bmatrix} 1 & -\nu_{W_1\supseteq\cdots\supseteq W_n}((\pi^*[Y_1]-[W_1])|_{W_1}) \\ 0 & \mathrm{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \pi|_{W_1} : W_1 \to Y_1) \end{bmatrix} \cdot \quad .$$
$$\begin{bmatrix} 1 & -\nu_{W_1'\supseteq\cdots\supseteq W_n'}((p^*[W_1]-[W_1'])|_{W_1'}) \\ 0 & \mathrm{cor}(W_1 \supseteq \cdots \supseteq W_n, p|_{W_1'} : W_1' \to W_1) \end{bmatrix}$$

By the inductive hypothesis, this is equivalent to

$$\nu_{W_1'\supseteq\cdots\supseteq W_n'}\left((\pi'^*[Y_1]-[W_1'])|_{W_1'}\right) = \nu_{W_1'\supseteq\cdots\supseteq W_n'}\left((p^*[W_1]-[W_1'])|_{W_1'}\right) +$$
$$\nu_{W_1\supseteq\cdots\supseteq W_n}\left((\pi^*[Y_1]-[W_1])|_{W_1}\right) \mathrm{cor}\left(W_1 \supseteq \cdots \supseteq W_n, p|_{W_1'} : W_1' \to W_1\right),$$

which, thanks to Proposition 10.2.3, can be further rewritten as

$$\nu_{W_1'\supseteq\cdots\supseteq W_n'}\left((\pi'^*[Y_1]-[W_1'])|_{W_1'}\right) = \nu_{W_1'\supseteq\cdots\supseteq W_n'}\left((p^*[W_1]-[W_1'])|_{W_1'}\right) +$$
$$\nu_{W_1'\supseteq\cdots\supseteq W_n'}\left(p|_{W_1'}^*\left((\pi^*[Y_1]-[W_1])|_{W_1}\right)\right).$$

This follows from Proposition 10.2.2. □

Theorem 10.2.1 *Let $\pi\colon Z \to X$ be a proper bimeromorphic morphism from a reduced complex space Z. Then there is a modification $W \to X$ dominating $Z \to X$ such that $Y_\bullet$ admits a lifting to W.*

We remind the readers that in this book, a modification means a finite composition of blowing-ups with smooth centers.

Proof By Hironaka's Chow lemma Theorem B.1.2, we may assume that π is a modification.

We begin by setting $W_0 = Z$. We will construct W_i inductively for each i. Assume that for $0 \leq i < n$ a smooth partial flag $W_0 \supseteq \cdots \supseteq W_i$ has been constructed on a modification $\pi_i : Z_i \to Z$ so that $\pi \circ \pi_i$ restricts to bimeromorphic morphisms $W_j \to Y_j$ for each $j = 0, \ldots, i$.

By Zariski's main theorem, $W_i \to Y_i$ is an isomorphism outside a codimension 2 subset of Y_i. We let W_{i+1} be the strict transform of Y_{i+1} in W_i. The problem is that W_{i+1} is not necessarily smooth.

We will further modify Z_i and lift $W_1, \ldots, W_{i+1}$ in order to make the flag smooth. Take the embedded resolution of (W_j, W_{i+1}), say $W_j' \to W_j$ for each $j = 0, \ldots, i$.

We have canonical embeddings $W_i' \hookrightarrow W_{i-1}' \hookrightarrow \cdots \hookrightarrow W_0'$ making the following diagram commutative:

$$\begin{array}{ccccccc} W'_i & \hookrightarrow & W'_{i-1} & \hookrightarrow & \cdots & \hookrightarrow & W'_0 \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ W_i & \hookrightarrow & W_{i-1} & \hookrightarrow & \cdots & \hookrightarrow & W_0 \end{array}$$

Let W'_{i+1} be the strict transform of W_{i+1} in W'_i. It suffices to define π_{i+1} as the morphism $W'_0 \to Z_i \to Z$ and replace $W_0 \supseteq \cdots \supseteq W_{i+1}$ by $W'_0 \supseteq \cdots \supseteq W'_{i+1}$. □

Remark 10.2.2 Suppose that X is a normal projective variety. Consider a rank n (surjective) valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ and a closed positive $(1,1)$-current T on X. Then we can always define $\nu(T) \in \mathbb{R}^n$ as follows: Take a resolution $\pi\colon Y \to X$ such that there is a smooth flag $Y_\bullet$ on Y and $g \in \mathrm{Aut}(\mathbb{Z}^n_{\mathrm{lex}})$ such that

$$\nu = \nu_{Y_\bullet} g.$$

Then we define

$$\nu(T) := \nu_{Y_\bullet}(\pi^* T) g.$$

This definition does not depend on the choice of π, as a consequence of Proposition 10.2.3.

10.3 Algebraic partial Okounkov bodies

Let X be a connected smooth complex projective variety of dimension n and (L,h) be a Hermitian big line bundle on X.

Let h_0 be a smooth Hermitian metric on L. Let $\theta = c_1(L,h_0)$. Then we can identify h with a function $\varphi \in \mathrm{PSH}(X,\theta)$. We will use interchangeably the notations (θ,φ) and (L,h). We assume that $\mathrm{vol}\,\theta_\varphi > 0$ in this section.

Fix a rank n valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$, which without loss of generality can be assumed to be surjective.

We will adopt the notations of Section 10.1.

10.3.1 The spaces of sections

Definition 10.3.1 We will write

$$\begin{aligned} \Gamma(\theta,\varphi) &:= \left\{ (\nu(s),k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))^\times \right\}, \\ \Delta_k(\theta,\varphi) &:= \mathrm{Conv}\left\{ k^{-1}\nu(s) : s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi))^\times \right\} \subseteq \mathbb{R}^n, \quad k \in \mathbb{Z}_{>0}. \end{aligned}$$

When $\theta = V_\theta$, we simply write $\Gamma(L)$ and $\Delta_k(L)$ instead.

Here and in the sequel, the cross notation means excluding 0. Here Conv denotes the convex hull. For large enough k, $\Delta_k(\theta,\varphi)$ is non-empty thanks to Theorem 7.4.1.

Definition 10.3.2 Assume that φ has analytic singularities. We define

$$\Gamma^{\infty}(\theta,\varphi) \coloneqq \left\{(\nu(s),k) : k \in \mathbb{N}, s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_{\infty}(k\varphi))^{\times}\right\}. \tag{10.15}$$

Recall that $\mathcal{I}_{\infty}$ is introduced in Definition 1.6.6.

For later use, we introduce a twisted version as well.

Definition 10.3.3 If T is a holomorphic line bundle on X, we introduce

$$\begin{aligned}\Delta_{k,T}(\theta,\varphi) \coloneqq& \mathrm{Conv}\left\{k^{-1}\nu(s) : s \in \mathrm{H}^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi))^{\times}\right\} \subseteq \mathbb{R}^n,\\ \Delta_{k,T}(L) \coloneqq& \mathrm{Conv}\left\{k^{-1}\nu(s) : s \in \mathrm{H}^0(X, T \otimes L^k)^{\times}\right\} \subseteq \mathbb{R}^n\end{aligned}$$

for all $k \in \mathbb{Z}_{>0}$.

10.3.2 Algebraic Okounkov bodies

Proposition 10.3.1 *There is a convex body $\Delta \in \mathcal{K}_n$ such that $\Gamma(L) \in \mathcal{S}'(\Delta)$.*

Recall that the notations $\mathcal{K}_n$ and $\mathcal{S}'(\Delta)$ are introduced in Appendix C.

Proof **Step 1**. We first show that there is $\Delta \in \mathcal{K}_n$ such that $\Delta_k(L) \subseteq \Delta$. For this purpose, using Remark 10.2.1, we may assume that ν is induced by an admissible flag $Y_\bullet$ on X.

Fix $s \in \mathrm{H}^0(X, L^k)^{\times}$ for some $k \in \mathbb{Z}_{>0}$. Assume that $s \neq 0$. We need to show that for each $i = 1, \dots, n$, $\nu(s)_i \leq Ck$ for some constant $C > 0$, independent of the choices of k and s.

Fix an ample divisor H on X. Take a large enough integer $b_1 > 0$ such that

$$(L - b_1 Y_1) \cdot H^{n-1} < 0.$$

Then $\nu(s)_1 \leq b_1 k$. Next take a large enough integer b_2 such that

$$\left((L - aY_1)|_{Y_1} - b_2 Y_2\right) \cdot H^{n-2} < 0.$$

It follows that $\nu(s)_2 \leq b_2 k$. Continue in this manner, we conclude that $\nu(s)_i/k$ is bounded for each i.

Step 2. Observe that $\Gamma(L)$ is clearly a semigroup. It remains to show that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$ as an Abelian group.

For this purpose, take two very ample divisors A and B so that $L = \mathcal{O}_X(A - B)$. After choosing A and B ample enough, we may guarantee that there exist sections $s_0 \in \mathrm{H}^0(X, A)$, $t_i \in \mathrm{H}^0(X, B)$ for $i = 0, \dots, n$ such that

$$\nu(s_0) = \nu(t_0) = 0$$

and $\nu(t_i)$ is the i-th unit vector $e_i \in \mathbb{R}^n$ for $i = 1, \dots, n$.

Since L is big, we can find $m_0 > 0$ such that for any $m \geq m_0$ we can find an effective divisor F_m on X linearly equivalent to $mL - B$. Let $f_m = \nu([F_m])$. Then we find that

$$(f_m, m), (f_m + e_1, m), \dots, (f_m + e_n, m) \in \Gamma(L).$$

Since $(m+1)L$ is linearly equivalent to $A + F_m$, so

$$(f_m, m+1) \in \Gamma(L).$$

It follows that $\Gamma(L)$ generates $\mathbb{Z}^{n+1}$. □

Thanks to Proposition 10.3.1, we can introduce the next definition.

Definition 10.3.4 We define the *Okounkov body* of L with respect to the valuation ν as

$$\Delta_\nu(L) := \Delta(\Gamma(L)).$$

When ν is induced by a smooth flag $Y_\bullet$ on X, we also write $\Delta_{Y_\bullet}(L)$ instead. The same convention applies to the partial Okounkov bodies studied below as well.

Proposition 10.3.2 *The Okounkov body $\Delta_\nu(L)$ depends only on the numerical class of L.*

See [LM09, Proposition 4.1] for the elegant proof.

Corollary 10.3.1 *We have*

$$\operatorname{vol} \Delta_\nu(L) = \frac{1}{n!} \operatorname{vol} L. \tag{10.16}$$

Proof This follows immediately from Proposition 10.3.1 and Theorem 10.1.1. □

Proposition 10.3.3 *Assume that φ has analytic singularities and θ_φ is a Kähler current. Then we have*

$$\Gamma^\infty(\theta, \varphi) \in \mathcal{S}'(X, \theta)$$

and

$$\operatorname{vol} \Gamma^\infty(\theta, \varphi) = \frac{1}{n!} \int_X \theta_\varphi^n.$$

Proof Replacing X by a modification, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D. See Theorem 1.6.1.

In this case,

$$\Gamma^\infty(\theta, \varphi) = \left\{ (\nu(s), k) : k \in \mathbb{N}, s \in \mathrm{H}^0\left(X, L^k \otimes \mathcal{O}_X(-\lceil kD \rceil)\right). \right\}$$

Since $L - D$ is ample by Lemma 1.6.1, our assertion follows from the same argument as Proposition 10.3.1. □

We first extend Theorem 10.1.1 to the twisted case.

Proposition 10.3.4 *For any holomorphic line bundle T on X, as $k \to \infty$*

$$\Delta_{k,T}(L) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(L).$$

Proof As L is big, we can take $k_0 \in \mathbb{Z}_{>0}$ so that

(1) $T^{-1} \otimes L^{k_0}$ admits a non-zero global holomorphic section s_0, and
(2) $T \otimes L^{k_0}$ admits a non-zero global holomorphic section s_1.

Then for $k \in \mathbb{Z}_{>k_0}$, we have injective linear maps

$$\mathrm{H}^0(X, L^{k-k_0}) \xrightarrow{\times s_1} \mathrm{H}^0(X, T \otimes L^k) \xrightarrow{\times s_0} \mathrm{H}^0(X, L^{k+k_0}).$$

It follows that

$$(k-k_0)\Delta_{k-k_0}(L) + \nu(s_1) \subseteq k\Delta_{k,T}(L) \subseteq (k+k_0)\Delta_{k+k_0}(L) - \nu(s_0).$$

Using Theorem 10.1.1, we conclude. □

Proposition 10.3.5 *Let L' be another big line bundle on X. Then*

$$\Delta_\nu(L) + \Delta_\nu(L') \subseteq \Delta_\nu(L \otimes L').$$

Proof Observe that for each $k \in \mathbb{N}$, we have

$$\Delta_k(L) + \Delta_k(L') \subseteq \Delta_k(L \otimes L').$$

So our assertion follows immediately from Theorem 10.1.1. □

Proposition 10.3.6 *For any $a \in \mathbb{Z}_{>0}$, we have*

$$\Delta_\nu(L^a) = a\Delta_\nu(L).$$

Proof This is an immediate consequence of Theorem 10.1.1. □

10.3.3 Construction of partial Okounkov bodies

Theorem 10.3.1 *We have*

$$\Gamma(\theta, \varphi) \in \overline{\mathcal{S}'(\Delta_\nu(L))}_{>0}.$$

We refer to Definition 10.1.3 for the definition of $\overline{\mathcal{S}'(\Delta_\nu(L))}_{>0}$.

This theorem allows us to give the following definition:

Definition 10.3.5 The *partial Okounkov body* of (L, h) is defined as

$$\Delta_\nu(L,h) = \Delta_\nu(\theta,\varphi) := \Delta\left(\Gamma(\theta,\varphi)\right). \tag{10.17}$$

When ν is induced by an admissible flag $Y_\bullet$ on X (see Definition 10.2.1), we also say that $\Delta_\nu(\theta,\varphi)$ the *partial Okounkov body* of (L,h) or of (θ,φ) with respect to $Y_\bullet$. In this case, we also write $\Delta_{Y_\bullet}$ instead of Δ_ν.

Note that when h has minimal singularities, we have

$$\Delta_\nu(L,h) = \Delta_\nu(L).$$

So partial Okounkov bodies generalize Okounkov bodies.

Corollary 10.3.2 *We have*

$$\operatorname{vol}\Delta_\nu(\theta,\varphi) = \frac{1}{n!}\operatorname{vol}\theta_\varphi. \tag{10.18}$$

We will prove Theorem 10.3.1 and Corollary 10.3.2 at the same time. The proof relies on the pseudometric d_{sg} introduced in (10.1).

Proof **Step 1**. We first assume that φ has analytic singularities and θ_φ is a Kähler current.

We claim that

$$d_{\mathrm{sg}}(\Gamma^\infty(\theta,\varphi),\Gamma(\theta,\varphi)) = 0. \tag{10.19}$$

Observe that for each $\epsilon \in \mathbb{Q}_{>0}$, we have

$$\mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi)) \subseteq \mathrm{H}^0(X, L^k \otimes \mathcal{I}(k\varphi)) \subseteq \mathrm{H}^0\left(X, L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi)\right)$$

for all large enough k. This is a consequence of Lemma 1.6.3. Therefore, it suffices to show that

$$\lim_{\mathbb{Q}\ni\epsilon\to 0+} \operatorname{vol}\Gamma^\infty(\theta,(1-\epsilon)\varphi) = \operatorname{vol}\Gamma^\infty(\theta,\varphi).$$

This follows from the explicit formula in Proposition 10.3.3.

Step 2. We next handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. Then $\varphi_j \xrightarrow{d_S} P_\theta[\varphi]_{\mathcal{I}}$ by Corollary 7.1.2.

In this case, it suffices to prove that

$$\Gamma(\theta,\varphi_j) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta,\varphi). \tag{10.20}$$

In fact, by Theorem 7.4.1, we have

$$
\begin{aligned}
&d_{\mathrm{sg}}\left(\Gamma(\theta,\varphi_j),\Gamma(\theta,\varphi)\right)\\
=&\varlimsup_{k\to\infty} k^{-n}\left(h^0\left(X,L^k\otimes\mathcal{I}(k\varphi_j)\right)-h^0\left(X,L^k\otimes\mathcal{I}(k\varphi)\right)\right)\\
=&\lim_{k\to\infty} k^{-n}h^0\left(X,L^k\otimes\mathcal{I}(k\varphi_j)\right)-\lim_{k\to\infty} k^{-n}h^0\left(X,L^k\otimes\mathcal{I}(k\varphi)\right)\\
=&\frac{1}{n!}\operatorname{vol}\theta_{\varphi_j}-\frac{1}{n!}\operatorname{vol}\theta_{\varphi}.
\end{aligned}
$$

Letting $j\to\infty$, we conclude (10.20) by Theorem 6.2.5.

Step 3. Now we only assume that $\operatorname{vol}\theta_\varphi>0$. We may replace φ with $P_\theta[\varphi]_{\mathcal{I}}$ and then assume that $\varphi\in\mathrm{PSH}(X,\theta)_{>0}$.

Take a potential $\psi\in\mathrm{PSH}(X,\theta)$ such that $\psi\le\varphi$ and θ_ψ is a Kähler current. The existence of ψ is proved in Lemma 2.4.3. For each $\epsilon\in(0,1)$, let $\varphi_\epsilon=(1-\epsilon)\varphi+\epsilon\psi$. It suffices to show that

$$\Gamma(\theta,\varphi_\epsilon)\xrightarrow{d_{\mathrm{sg}}}\Gamma(\theta,\varphi)$$

as $\epsilon\to 0+$. We compute using Theorem 7.4.1:

$$
\begin{aligned}
&d_{\mathrm{sg}}\left(\Gamma(\theta,\varphi_\epsilon),\Gamma(\theta,\varphi)\right)\\
=&\varlimsup_{k\to\infty} k^{-n}\left(h^0\left(X,L^k\otimes\mathcal{I}(k\varphi)\right)-h^0\left(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon)\right)\right)\\
=&\lim_{k\to\infty} k^{-n}h^0\left(X,L^k\otimes\mathcal{I}(k\varphi)\right)-\lim_{k\to\infty} k^{-n}h^0\left(X,L^k\otimes\mathcal{I}(k\varphi_\epsilon)\right)\\
=&\frac{1}{n!}\operatorname{vol}\theta_{\varphi}-\frac{1}{n!}\operatorname{vol}\theta_{\varphi_\epsilon}\\
\to&0
\end{aligned}
$$

by Theorem 6.2.5, as $\epsilon\to 0+$. □

Remark 10.3.1 It follows from the proof that if φ has analytic singularities and θ_φ is a Kähler current, then (10.19) holds.

If we take a modification $\pi\colon Y\to X$ such that $\pi^*\varphi$ has log singularities along an effective $\mathbb{Q}$-divisor D on Y, then

$$\Delta_\nu(\theta,\varphi)=\Delta_\nu(\pi^*L-D)+\nu(D). \tag{10.21}$$

This is a very special case of Theorem 11.3.1.

10.3.4 Basic properties of partial Okounkov bodies

Proposition 10.3.7 *The partial Okounkov body $\Delta_\nu(L,h)$ depends only on $\mathrm{dd^c}h$, not on the explicit choices of L,h_0,h.*

Thanks to this result, given a closed positive $(1,1)$-current $T\in c_1(L)$ on X with $\int_X T^n>0$, we can write

$$\Delta_\nu(T) \coloneqq \Delta_\nu(\theta, \varphi)$$

if $T = \theta + \mathrm{dd^c}\varphi$ for some $\varphi \in \mathrm{PSH}(X, \theta)$.

Proof There are two different claims to prove, as detailed in the two steps below.

Step 1. Let h_0' be another Hermitian metric on L. Set $\theta' = c_1(L, h_0')$. Write $\mathrm{dd^c} f = \theta - \theta'$. Let $\varphi' = \varphi + f \in \mathrm{PSH}(X, \theta')$. Then

$$\Delta_\nu(\theta, \varphi) = \Delta_\nu(\theta', \varphi'). \tag{10.22}$$

This is obvious since $\Gamma(\theta, \varphi) = \Gamma(\theta', \varphi')$.

Step 2. Let L' be another big line bundle on X. By Step 1, we may assume that the reference Hermitian metric h_0' on L' is such that $c_1(L', h_0') = \theta$.

Let h' be a plurisubharmonic metric on L' with $c_1(L, h) = c_1(L', h')$. Then

$$\Delta_\nu(L, h) = \Delta_\nu(L', h').$$

From our construction, we may assume that $c_1(L, h)$ has analytic singularities. After taking a birational resolution, it suffices to deal with the case where $c_1(L, h)$ has analytic singularities along an effective $\mathbb{Q}$-divisors D. By rescaling, we may also assume that D is a divisor. By Remark 10.3.1, we further reduce to the case where $c_1(L, h)$ is not singular.

In this case, the assertion is proved in Proposition 10.3.2. □

Proposition 10.3.8 *Let $\varphi, \psi \in \mathrm{PSH}(X, \theta)_{>0}$. Assume that $\varphi \preceq_{\mathcal{I}} \psi$, then*

$$\Delta_\nu(\theta, \varphi) \subseteq \Delta_\nu(\theta, \psi). \tag{10.23}$$

In particular, we always have

$$\Delta_\nu(\theta, \varphi) \subseteq \Delta_\nu(L).$$

Proof This follows from Corollary 10.1.2. □

Theorem 10.3.2 *The Okounkov body map*

$$\Delta_\nu(\theta, \bullet) : (\mathrm{PSH}(X, \theta)_{>0}, d_S) \to (\mathcal{K}_n, d_{\mathrm{Haus}})$$

is continuous.

Proof Let $\varphi_j \to \varphi$ be a d_S-convergent sequence in $\mathrm{PSH}(X, \theta)_{>0}$. We want to show that

$$\Delta_\nu(\theta, \varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta, \varphi). \tag{10.24}$$

By Proposition 10.3.8, we may assume that all φ_j's and φ are model potentials.

By Theorem C.1.1 and Proposition 6.2.3, we may assume that $(\varphi_j)_j$ is either decreasing or increasing. By Theorem 6.2.3, we may further assume that the φ_j's are $\mathcal{I}$-model. In both cases, we claim that

$$\Gamma(\theta,\varphi_j) \xrightarrow{d_{\mathrm{sg}}} \Gamma(\theta,\varphi)$$

as $j \to \infty$. In fact, using Theorem 7.4.1, we can compute

$$\begin{aligned} d_{\mathrm{sg}}\left(\Gamma(\theta,\varphi_j),\Gamma(\theta,\varphi)\right) &= \varlimsup_{k\to\infty} k^{-n}\left|h^0\left(X, L^k\otimes\mathcal{I}(k\varphi_j)\right) - h^0\left(X, L^k\otimes\mathcal{I}(k\varphi)\right)\right| \\ &= \frac{1}{n!}\left|\operatorname{vol}\theta_{\varphi_j} - \operatorname{vol}\theta_\varphi\right|, \end{aligned}$$

which converges to 0 by Theorem 6.2.5. □

Proposition 10.3.9 *Let $\pi\colon Y \to X$ be a modification. Then*

$$\Delta_\nu(\pi^*L,\pi^*h) = \Delta_\nu(L,h).$$

Proof Thanks to Proposition 3.2.5, we may assume that φ is $\mathcal{I}$-model. By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ with analytic singularities in $\mathrm{PSH}(X,\theta)$ such that $\varphi_j \xrightarrow{d_S} \varphi$. It is clear that $\pi^*\varphi_j \xrightarrow{d_S} \pi^*\varphi$. By Theorem 10.3.2, we may then reduce to the case where φ has analytic singularities. In this case, it suffices to apply Remark 10.3.1. □

Proposition 10.3.10 *Let (L',h') be another Hermitian big line bundle on X. Then*

$$\Delta_\nu(L,h) + \Delta_\nu(L',h') \subseteq \Delta_\nu(L\otimes L', h\otimes h').$$

Proof Take a Hermitian metric h'_0 on L' and let $\theta' = c_1(L',h'_0)$. We identify h' with $\varphi' \in \mathrm{PSH}(X,\theta')$. Then we need to show

$$\Delta_\nu(\theta,\varphi) + \Delta_\nu(\theta',\varphi') \subseteq \Delta_\nu(\theta+\theta',\varphi+\varphi'). \tag{10.25}$$

We observe that

$$P_\theta[\varphi]_{\mathcal{I}} + P_\theta[\varphi']_{\mathcal{I}} \sim_{\mathcal{I}} \varphi+\varphi'.$$

Thus, after replacing φ and φ' by their $\mathcal{I}$-envelopes, in view of Proposition 10.3.8, we may assume that φ and φ' are $\mathcal{I}$-good.

By Theorem 7.1.1, we can find sequences $(\varphi_j)_j$ and $(\varphi'_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ and $\mathrm{PSH}(X,\theta')_{>0}$ respectively such that

(1) φ_j and φ'_j both have analytic singularities for all $j \geq 1$, and

(2) $\varphi_j \xrightarrow{d_S} \varphi$, $\varphi'_j \xrightarrow{d_S} \varphi'$.

Then $\varphi_j + \varphi'_j \in \mathrm{PSH}(X,\theta+\theta')_{>0}$ and $\varphi_j + \varphi'_j \xrightarrow{d_S} \varphi+\varphi'$ by Theorem 6.2.2. Thus, by Theorem 10.3.2, we may assume that φ and ψ both have analytic singularities. Taking a birational resolution, we may further assume that they have log singularities. By Remark 10.3.1, we reduce to the case without singularities, in which case the result is just Proposition 10.3.5. □

Theorem 10.3.3 *Let $\varphi,\psi \in \mathrm{PSH}(X,\theta)_{>0}$. Then for any $t \in (0,1)$,*

$$\Delta_\nu\left(\theta, t\varphi + (1-t)\psi\right) \supseteq t\Delta_\nu(\theta,\varphi) + (1-t)\Delta_\nu(\theta,\psi). \tag{10.26}$$

Proof We may assume that t is rational as a consequence of Theorem 10.3.2. Similarly, as in the proof of Proposition 10.3.10, we could reduce to the case where both φ and ψ have analytic singularities. In this case, let $N > 0$ be an integer such that Nt is an integer. Then for any $s \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\varphi))$ and $r \in \mathrm{H}^0(X, L^k \otimes \mathcal{I}_\infty(k\psi))$, we have

$$s^{tN} \otimes r^{N-tN} \in \mathrm{H}^0\left(X, L^{kN} \otimes \mathcal{I}_\infty\left(Nt\varphi + (N-Nt)\psi\right)\right).$$

By Theorem 10.1.1 and Remark 10.3.1, (10.26) follows. □

Proposition 10.3.11 *For any $a \in \mathbb{Z}_{>0}$,*

$$\Delta_\nu(a\theta, a\varphi) = a\Delta_\nu(\theta,\varphi).$$

Proof As in the proof of Proposition 10.3.10, we may assume that φ has log singularities. Using Remark 10.3.1, we reduce to the case without the singularities, which is proved in Proposition 10.3.6. □

In particular, if S is a closed positive $(1,1)$-current on X with $\int_X S^n > 0$ and such that

$$[S] \in \mathrm{NS}^1(X)_{\mathbb{Q}},$$

we can define

$$\Delta_\nu(S) := a^{-1}\Delta_\nu(aS) \tag{10.27}$$

for a sufficiently divisible positive integer a. This definition is independent of the choice of a thanks to Proposition 10.3.11.

We also need the following perturbation. Let A be an ample line bundle on X. Fix a Hermitian metric h_A on A such that $\omega := c_1(A, h_A)$ is a Kähler form on X.

Proposition 10.3.12 *As $\delta \searrow 0$, the convex bodies $\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi)$ are decreasing and*

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta_\varphi).$$

Proof Let $0 \leq \delta < \delta'$ be two rational numbers. Take $C \in \mathbb{N}_{>0}$ divisible enough, so that $C\delta$ and $C\delta'$ are both integers. Then by Proposition 10.3.10,

$$\Delta_\nu(C\theta + C\delta\omega + C\mathrm{dd^c}\varphi) \subseteq \Delta_\nu(C\theta + C\delta'\omega + C\mathrm{dd^c}\varphi).$$

It follows that

$$\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) \subseteq \Delta_\nu(\theta + \delta'\omega + \mathrm{dd^c}\varphi).$$

On the other hand,

$$\operatorname{vol}\Delta_\nu(\theta + \delta\omega + \mathrm{dd^c}\varphi) = \frac{1}{n!}\operatorname{vol}(\theta + \delta\omega)_\varphi.$$

As $\delta \to 0+$, the right-hand side converges to

$$\operatorname{vol} \Delta_\nu(\theta, \varphi) = \frac{1}{n!} \operatorname{vol} \theta_\varphi,$$

thanks to Proposition 7.3.1. Our assertion therefore follows. □

10.3.5 The Hausdorff convergence property

Let T be a holomorphic line bundle on X. The goal of this section is to prove the following:

Theorem 10.3.4 *As $k \to \infty$, we have*

$$\Delta_{k,T}(\theta, \varphi) \xrightarrow{d_{\mathrm{Haus}}} \Delta_\nu(\theta, \varphi). \tag{10.28}$$

Recall that $\Delta_{k,T}(\theta, \varphi)$ is define in Definition 10.3.3.

Although we are only interested in the untwisted case, the proof given below requires twisted case.

We first observe that the sequence $\Delta_{k,T}(\theta, \varphi)$ is uniformly bounded: This follows easily from Proposition 10.3.4. So Blaschke's selection theorem Theorem C.1.1 is applicable. We will apply this observation without further comments.

Lemma 10.3.1 *Assume that φ has analytic singularities and θ_φ is a Kähler current, then (10.28) holds.*

Proof Up to replacing X by a modification and twisting T accordingly, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor D, see Proposition 10.3.9 and Theorem 1.6.1.

Take$\epsilon \in \mathbb{Q} \cap (0, 1)$. In this case, for large enough $k \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}_\infty(k\varphi)\right) \subseteq & \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)\right) \\ \subseteq & \mathrm{H}^0\left(X, T \otimes L^k \otimes \mathcal{I}_\infty(k(1-\epsilon)\varphi)\right). \end{aligned}$$

Take an integer $N \in \mathbb{Z}_{>0}$ so that ND is a divisor and $N\epsilon$ is an integer.

Let Δ' be the limit of a subsequence of $(\Delta_{k,T}(\theta, \varphi))_k$, say the sequence defined by the indices $k_1, k_2, \ldots$. Thanks to Theorem C.1.1, it suffices to show that $\Delta' = \Delta_\nu(\theta, \varphi)$.

There exists $t \in \{0, 1, \ldots, N-1\}$ such that $k_i \equiv t$ modulo N for infinitely many i, up to replacing $(k_i)_i$ by a subsequence, we may assume that $k_i \equiv t$ modulo N for all i. Write $k_i = Ng_i + t$. Then for large enough i, we have

$$\begin{aligned} \mathrm{H}^0\left(X, T \otimes L^{-N+t} \otimes L^{N(g_i+1)} \otimes \mathcal{I}_\infty\left(N(g_i+1)\varphi\right)\right) \subseteq & \mathrm{H}^0\left(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi)\right) \\ \subseteq & \mathrm{H}^0\left(X, T \otimes L^t \otimes L^{Ng_i} \otimes \mathcal{I}_\infty\left(g_i N(1-\epsilon)\varphi\right)\right). \end{aligned}$$

So

$$
\begin{aligned}
(g_i+1)\Delta_{g_i+1,T\otimes L^{-N+t}}(NL-ND)+N(g_i+1)\nu(D) &\subseteq (Ng_i+t)\Delta_{k_i,T}(\theta,\varphi)\\
&\subseteq g_i\Delta_{g_i,T\otimes L^t}(NL-N(1-\epsilon)D)+Ng_i(1-\epsilon)\nu(D).
\end{aligned}
$$

Letting $i\to\infty$, by Proposition 10.3.4,

$$\Delta_\nu(L-D)+\nu(D)\subseteq\Delta'\subseteq\Delta_\nu(L-(1-\epsilon)D)+(1-\epsilon)\nu(D).$$

Letting $\epsilon\to 0+$, we find that

$$\Delta'=\Delta_\nu(L-D)+\nu(D)=\Delta_\nu(\theta,\varphi),$$

where we applied Remark 10.3.1 as well. Our assertion follows. □

Lemma 10.3.2 *Assume that θ_φ is a Kähler current, then as $\mathbb{Q}\ni\beta\to 0+$, we have*

$$\Delta_\nu\left((1-\beta)\theta,\varphi\right)\xrightarrow{d_{\mathrm{Haus}}}\Delta_\nu(\theta,\varphi).$$

Here and in the sequel, $\Delta_\nu((1-\beta)\theta,\varphi)=\Delta_\nu((1-\beta)\theta+\mathrm{dd^c}\varphi)$ is defined in (10.27).

Proof By Proposition 10.3.10, we have

$$\Delta_\nu\left((1-\beta)\theta,\varphi\right)+\beta\Delta_\nu(L)\subseteq\Delta_\nu(\theta,\varphi).$$

In particular, if Δ' is the Hausdorff limit of a subnet of $(\Delta((1-\beta)\theta,\varphi))_\beta$, then $\Delta'\subseteq\Delta_\nu(\theta,\varphi)$. But

$$
\begin{aligned}
\operatorname{vol}\Delta'=\lim_{\beta\to0+}\Delta_\nu((1-\beta)\theta,\varphi)&=\lim_{\beta\to0+}\int_X\left((1-\beta)\theta+\mathrm{dd^c}P_{(1-\beta)\theta}[\varphi]_{\mathcal{I}}\right)^n\\
&=\int_X(\theta+\mathrm{dd^c}P_\theta[\varphi]_{\mathcal{I}})^n.
\end{aligned}
$$

Since we have not developed the theory of nef b-divisors yet, we give a direct proof as well. Take a Kähler form ω so that $\theta_\varphi\geq\omega$. Let $\psi=P_{\theta-\omega}[\varphi]_{\mathcal{I}}$. Then $\varphi\sim_{\mathcal{I}}\psi$. In order to establish the last equality, we may replace φ by ψ and hence assuming that φ is $\mathcal{I}$-good. In this case, the desired equality becomes

$$\lim_{\beta\to0+}\int_X((1-\beta)\theta+\mathrm{dd^c}\varphi)^n=\int_X\theta_\varphi^n,$$

which is obvious.

It follows that $\Delta'=\Delta_\nu(\theta,\varphi)$. We conclude by Theorem C.1.1. □

Proof (Proof of Theorem 10.3.4) Fix a Kähler form $\omega\geq\theta$ on X.

Step 1. We first handle the case where θ_φ is a Kähler current, say $\theta_\varphi\geq 2\delta\omega$ for some $\delta\in(0,1)$. Take a quasi-equisingular approximation $(\varphi_j)_j$ of φ in $\mathrm{PSH}(X,\theta)$. We may assume that $\theta_{\varphi_j}\geq\delta\omega$ for all $j\geq 1$.

Let Δ' be a limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$. Let us say the indices of the subsequence are $k_1<k_2<\cdots$. By Theorem C.1.1, it suffices to show that $\Delta'=\Delta_\nu(\theta,\varphi)$.

Observe that for each $j \geq 1$, we have $\Delta' \subseteq \Delta_\nu(\theta, \varphi_j)$ by Lemma 10.3.1. Letting $j \to \infty$, we find $\Delta' \subseteq \Delta_\nu(\theta, \varphi)$ as a consequence of Theorem 10.3.2. Therefore, it suffices to prove that

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu(\theta, \varphi).$$

Fix an integer $N > \delta^{-1}$. Observe that for any $j \geq 1$, we have $\varphi_j \in \mathrm{PSH}(X, (1-N^{-1})\theta)$. Similarly, $\varphi \in \mathrm{PSH}(X, (1-N^{-1})\theta)$. By Lemma 10.3.2, it suffices to argue that

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu \left((1 - N^{-1})\theta, \varphi \right). \tag{10.29}$$

Step 1.1. We first reduce to the case where $N | k_i$ for all i.

We are free to replace $(k_i)_i$ by a subsequence, so we may assume that $k_i \equiv a$ modulo N for all $i \geq 1$, where $a \in \{0, 1, \dots, N-1\}$. We write $k_i = g_i N + a$. Observe that for each $i \geq 1$,

$$\mathrm{H}^0 \left(X, T \otimes L^{k_i} \otimes \mathcal{I}(k_i\varphi) \right) \supseteq \mathrm{H}^0 \left(X, T \otimes L^{-N+a} \otimes L^{g_i N + N} \otimes \mathcal{I}\left((g_i N + N)\varphi \right) \right).$$

Up to replacing T by $T \otimes L^{-N+a}$, we may therefore assume that $a = 0$, so that $k_i = g_i N$.

Step 1.2. Write $k_i = g_i N$ for all i. We prove (10.29).

By Lemma 2.4.2, we can find $j' \in \mathbb{Z}_{>0}$ such that for all $j \geq j'$, there is $\psi \in \mathrm{PSH}(X, \theta)_{>0}$ satisfying

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1 - N^{-1})\varphi_j + N^{-1}\psi_j.$$

Fix $j \geq j'$. It suffices to show that

$$\Delta_\nu \left((1 - N^{-1})\theta, \varphi_j \right) + \nu' \subseteq \Delta' \tag{10.30}$$

for some $\nu' \in \mathbb{R}^n$. In fact, if this is true, we have

$$\operatorname{vol} \Delta' \geq \operatorname{vol} \Delta_\nu \left((1 - N^{-1})\theta, \varphi_j \right).$$

Letting $j \to \infty$ and applying Theorem 10.3.2, we conclude (10.29).

It remains to prove (10.30). As in the proof of Theorem 7.4.1, there is $g_0 > 0$ such that for any $g \geq g_0$, we can find a non-zero section $s_g \in \mathrm{H}^0(X, L^g \otimes \mathcal{I}(g\psi_j))$ such that we get an injective linear map

$$\mathrm{H}^0 \left(X, T \otimes L^{g(N-1)} \otimes \mathcal{I}(gN\varphi_j) \right) \xrightarrow{\times s_g} \mathrm{H}^0 \left(X, T \otimes L^{gN} \otimes \mathcal{I}(gN\varphi) \right).$$

In particular, when $g = g_i$ for some i large enough, we then find

$$\Delta_{g_i, T} \left((N-1)\theta, N\varphi_j \right) + (g_i)^{-1} \nu(s_{k_i}) \subseteq N \Delta_{k_i, T}(\theta, \varphi).$$

We observe that the $(g_i)^{-1}\nu(s_{g_i})$'s are bounded as both convex bodies appearing in this equation are bounded when i varies. Then by Lemma 10.3.1, there is a vector $\nu' \in \mathbb{R}^n$ such that (10.30) holds.

Step 2. Next we handle the general case.

Let Δ' be the Hausdorff limit of a subsequence of $(\Delta_{k,T}(\theta,\varphi))_k$, say the subsequence with indices $k_1 < k_2 < \cdots$. By Theorem C.1.1, it suffices to prove that $\Delta' = \Delta_\nu(\theta,\varphi)$.

Take $\psi \in \mathrm{PSH}(X,\theta)$ such that θ_ψ is a Kähler current and $\psi \leq \varphi$. The existence of ψ follows from Lemma 2.4.3.

Then for any $\epsilon \in \mathbb{Q} \cap (0,1)$,

$$\Delta_{k,T}(\theta,\varphi) \supseteq \Delta_{k,T}\left(\theta,(1-\epsilon)\varphi+\epsilon\psi\right)$$

for all $k \geq 1$. It follows from Step 1 that

$$\Delta' \supseteq \Delta_\nu\left(\theta,(1-\epsilon)\varphi+\epsilon\psi\right).$$

Letting $\epsilon \to 0$ and applying Theorem 10.3.2, we have $\Delta' \supseteq \Delta_\nu(\theta,\varphi)$. It remains to establish that

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu(\theta,\varphi). \tag{10.31}$$

Fix an integer $N > 0$, it suffices to argue that

$$\operatorname{vol}\Delta' \leq \frac{1}{n!}\int_X \left(N^{-1}\omega+\theta+\mathrm{dd^c}P_{N^{-1}\omega+\theta}[\varphi]_{\mathcal{I}}\right)^n. \tag{10.32}$$

Assuming this, letting $N \to \infty$, we conclude (10.31), thanks to Proposition 7.3.1.

Step 2.1. We first reduce to the case $N|k_i$ for all i.

For this purpose, we are free to replace $k_1 < k_2 < \cdots$ by a subsequence. We may then assume that $k_i \equiv a$ modulo N for all $i \geq 1$ for some $a \in \{0,1,\dots,N-1\}$. We write $k_i = g_iN + a$. Observe that

$$\mathrm{H}^0\left(X, T\otimes L^{k_i}\otimes\mathcal{I}(k_i\varphi)\right) \subseteq \mathrm{H}^0\left(X, T\otimes L^a\otimes L^{g_iN}\otimes\mathcal{I}(g_iN\varphi)\right).$$

Up to replacing T by $T\otimes L^a$, we may assume that $a = 0$.

Step 2.2. We write $k_i = g_iN$ for all i.

Take a very ample line bundle H on X and fix a Kähler form $\omega \in c_1(H)$, take a non-zero section $s \in \mathrm{H}^0(X,H)$.

We have an injective linear map

$$\mathrm{H}^0\left(X, T\otimes L^{gN}\otimes\mathcal{I}(gN\varphi)\right) \xrightarrow{\times s^g} \mathrm{H}^0\left(X, T\otimes H^g\otimes L^{gN}\otimes\mathcal{I}(gN\varphi)\right)$$

for each $g \geq 1$. In particular, for each $i \geq 1$,

$$k_i\Delta_{k_i,T}(\theta,\varphi) + g_i\nu(s) \subseteq g_i\Delta_{g_i,T}\left(\omega+N\theta, N\varphi\right).$$

Letting $i \to \infty$, by Step 1, we have

$$N\Delta' + \nu(s) \subseteq \Delta_\nu(\omega + N\theta, N\varphi).$$

So

$$\operatorname{vol}\Delta' \leq \operatorname{vol}\Delta_\nu\left(N^{-1}\omega + \theta, \varphi\right) = \frac{1}{n!}\int_X \left(N^{-1}\omega + \theta + \mathrm{dd^c} P_{N^{-1}\omega+\theta}[\varphi]_{\mathcal{I}}\right)^n.$$

10.3.6 Recover Lelong numbers from partial Okounkov bodies

Theorem 10.3.5 *Let E be a prime divisor on X. Let $Y_\bullet$ be an admissible flag with $E = Y_1$. Then*

$$\nu(\varphi, E) = \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1. \tag{10.33}$$

Here x_1 denotes the first component of x.

Proof Replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$, we may assume that φ is $\mathcal{I}$-good.

Step 1. We first reduce to the case where φ has analytic singularities.

By Theorem 7.1.1, we can find a sequence $(\varphi_j)_j$ in $\mathrm{PSH}(X,\theta)_{>0}$ with analytic singularities such that $\varphi_j \xrightarrow{d_S} \varphi$. It follows from Theorem 10.3.2 that

$$\Delta_{Y_\bullet}(\theta,\varphi_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta,\varphi).$$

Therefore,

$$\lim_{j\to\infty} \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi_j)} x_1 = \min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1.$$

In view of Theorem 6.2.4, it suffices to prove (10.33) with φ_j in place of φ.

Step 2. Assume that φ has analytic singularities. In view of Proposition 10.3.9 and Theorem 1.6.1, after replacing X by a modification, we may assume that φ has log singularities along an effective $\mathbb{Q}$-divisor F.

Perturbing L by an ample $\mathbb{Q}$-line bundle by Proposition 10.3.12, we may assume that θ_φ is a Kähler current. Therefore, $L - F$ is ample by Lemma 1.6.1. Finally, by rescaling, we may assume that F is a divisor and L is a line bundle.

By Theorem 10.3.4, we know that

$$\min_{x\in\Delta_{Y_\bullet}(\theta,\varphi)} x_1 = \lim_{k\to\infty} \min_{x\in\Delta_k(\theta,\varphi)} x_1.$$

By definition,

$$\min_{x\in\Delta_k(\theta,\varphi)} x_1 = k^{-1}\operatorname{ord}_E \mathrm{H}^0\left(X, L^k\otimes\mathcal{I}(k\varphi)\right).$$

In view of Proposition 1.4.4, it remains to show that

$$\lim_{k\to\infty} k^{-1}\operatorname{ord}_E \mathrm{H}^0\left(X, L^k\otimes\mathcal{I}(k\varphi)\right) = \lim_{k\to\infty} k^{-1}\operatorname{ord}_E \mathcal{I}(k\varphi). \tag{10.34}$$

The $\geq$ direction is trivial, we prove the converse. Observe that

$$\mathrm{H}^0\left(X, L^k \otimes \mathcal{I}(k\varphi)\right) = \mathrm{H}^0\left(X, L^k \otimes \mathcal{O}_X(-kF)\right), \quad \mathcal{I}(k\varphi) = \mathcal{O}(-kF).$$

As $L - F$ is ample, for large enough k, we have

$$\mathrm{ord}_E \mathrm{H}^0\left(X, L^k \otimes \mathcal{O}_X(-kF)\right) = \mathrm{ord}_E(kF).$$

Thus, (10.34) follows. □

Corollary 10.3.3 *Let $\varphi, \psi \in \mathrm{PSH}(X,\theta)_{>0}$. If*

$$\Delta_{W_\bullet}(\pi^*\theta, \pi^*\varphi) \subseteq \Delta_{W_\bullet}(\pi^*\theta, \pi^*\psi)$$

for all modifications $\pi\colon Y \to X$ and all admissible flags $W_\bullet$ on Y, then $\varphi \preceq_{\mathcal{I}} \psi$.

Proof This follows immediately from Theorem 10.3.5. □

Corollary 10.3.4 *Let E be a prime divisor over X. Then*

$$\nu(V_\theta, E) = \lim_{k\to\infty} \frac{1}{k} \mathrm{ord}_E \mathrm{H}^0(X, L^k). \tag{10.35}$$

Proof This follows from Theorem 10.3.5 and the fact that $\Delta_{Y_\bullet}(\theta, V_\theta) = \Delta_{Y_\bullet}(L)$ for any admissible flag $Y_\bullet$ on X. □

10.4 Transcendental partial Okounkov bodies

Let X be a connected compact Kähler manifold of dimension $n > 0$. Fix a smooth flag $Y_\bullet$ on X. We will extend the theory of partial Okounkov bodies in the previous section to the transcendental setting.

10.4.1 The traditional approach to the Okounkov body problem

The following definition is essentially due to Ya Deng's thesis [Den17].

Definition 10.4.1 Let α be a big cohomology class on X. We define the *Okounkov body* of α with respect to the flag $Y_\bullet$ as

$$\Delta_{Y_\bullet}(\alpha) := \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X,\alpha), S \text{ has gentle analytic singularities}\right\}}. \tag{10.36}$$

See Definition 1.6.5 for the definition of gentle analytic singularities.

The results of [DRWN+23] can be summarized as follows:

Theorem 10.4.1 *For any big cohomology class α on X, the set $\Delta_{Y_\bullet}(\alpha) \subseteq \mathbb{R}^n$ is a convex body satisfying the following properties:*

(1) *We have*
$$\operatorname{vol} \Delta_{Y_\bullet}(\alpha) = \frac{1}{n!} \operatorname{vol} \alpha;$$
(2) *given another big cohomology class α' on X, we have*
$$\Delta_{Y_\bullet}(\alpha) + \Delta_{Y_\bullet}(\alpha') \subseteq \Delta_{Y_\bullet}(\alpha + \alpha');$$
(3) *let $\pi\colon Y \to X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that $(W_\bullet, g)$ is the lifting of $Y_\bullet$ to Y, then*
$$\Delta_{W_\bullet}(\pi^*\alpha) = \Delta_{Y_\bullet}(\alpha) g;$$
(4) *the map $\alpha \mapsto \Delta_{Y_\bullet}(\alpha)$ is continuous in the big cone with respect to the Hausdorff metric;*
(5) *for any small enough $t > 0$, we have*
$$\left\{ y \in \mathbb{R}^{n-1} : (t, y) \in \Delta_{Y_\bullet}(\beta) \right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n} \left((\beta - t[Y_1])|_{Y_1} \right).$$

See Definition 10.2.4 for the notion of lifting. The proof requires some techniques not covered in the current book. The readers could either read the original paper or regard this theorem as a black box.

10.4.2 Definitions of partial Okounkov bodies

Let θ be a closed real smooth $(1, 1)$-form on X representing a big cohomology class α.

Let $T = \theta_\varphi \in \mathcal{Z}_+(X, \alpha)$. We shall define a convex body $\Delta_{Y_\bullet}(T) \subseteq \mathbb{R}^n$, which is also written as $\Delta_{Y_\bullet}(\theta, \varphi)$. This convex body is called the *partial Okounkov body* of T with respect to the flag $Y_\bullet$.

10.4.2.1 The case of analytic singularities

Definition 10.4.2 When T is a Kähler current with analytic singularities, we take a modification $\pi\colon Y \to X$ so that

(1)
$$\pi^* T = [D] + R, \tag{10.37}$$
where D is an effective $\mathbb{Q}$-divisor on Y and R is a closed positive $(1, 1)$-current with bounded potential, and

(2) the lifting $(Z_\bullet, g)$ of $Y_\bullet$ to Y exists.

Define
$$\Delta_{Y_\bullet}(T) \coloneqq \Delta_{Z_\bullet}([R])g^{-1} + \nu_{Z_\bullet}([D])g^{-1}.$$
The existence of π is guaranteed by Theorem 1.6.1 and Theorem 10.2.1.

Lemma 10.4.1 *The convex body $\Delta_{Y_\bullet}(T)$ defined in Definition 10.4.2 is independent of the choice of π.*

Proof Take another map $\pi' : Y' \longrightarrow X$ with the same properties. We want to show that π and π' defines the same $\Delta_{Y_\bullet}(T)$. We may assume that π' dominates π through $p : Y' \longrightarrow Y$, so that we have a commutative diagram

$$\begin{tikzcd} Y' \arrow[rr, "p"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

We take D and R as in (10.37). Then
$$\pi'^* T = [p^* D] + p^* R.$$
Write $(Z_\bullet, g)$ and $(Z'_\bullet, g')$ for the liftings of $Y_\bullet$ to Y and Y' respective. We need to prove that
$$\Delta_{Z_\bullet}([R])g^{-1} + \nu_{Z_\bullet}([D])g^{-1} = \Delta_{Z'_\bullet}([p^* R])g'^{-1} + \nu_{Z'_\bullet}([p^* D])g'^{-1}.$$
This follows Theorem 10.4.1, Proposition 10.2.3 and Proposition 10.2.4. □

Note that from the above proof, we could describe the bimeromorphic behaviour of $\Delta_{Y_\bullet}(T)$ as follows:

Lemma 10.4.2 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current with analytic singularities. Let $\pi : Y \longrightarrow X$ be a proper bimeromorphic morphism and $(W_\bullet, g)$ be the lifting of $Y_\bullet$ to Y. Then*
$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T)g.$$

Lemma 10.4.3 *Assume that $T, S \in \mathcal{Z}_+(X, \alpha)$ are two Kähler currents with analytic singularities and $T \preceq S$, then*
$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$
Moreover,
$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \int_X T^n. \tag{10.38}$$

Proof We first show that
$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S).$$
Using Lemma 10.4.2, we may assume that T and S have log singularities along effective $\mathbb{Q}$-divisors E and F respectively. By assumption, $E \geq F$. Replacing T and S by $T - [F]$ and $S - [F]$ respectively, we may assume that $F = 0$.

In this case, we need to show that

$$\Delta_{Y_\bullet}(\alpha) \supseteq \Delta_{Y_\bullet}(\alpha - [E]) + \nu_{Y_\bullet}([E]),$$

which is obvious.

Next we prove that

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

By Lemma 10.4.2 and Theorem 10.4.1 again, we may assume that T has log singularities. We take D and β as in (10.37). We need to show that

$$\Delta_{Y_\bullet}(\alpha - [D]) + \nu_{Y_\bullet}([D]) \subseteq \Delta_{Y_\bullet}(\alpha),$$

which is again obvious.

Finally, (10.38) follows immediately from Theorem 10.4.1. □

10.4.2.2 The case of Kähler currents

Definition 10.4.3 Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. Then we define

$$\Delta_{Y_\bullet}(T) := \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T_j).$$

Lemma 10.4.4 *The convex body $\Delta_{Y_\bullet}(T)$ in Definition 10.4.3 is independent of the choices of the T_j's.*

In particular, if T also has analytic singularities, then the $\Delta_{Y_\bullet}(T)$'s defined in Definition 10.4.3 and in Definition 10.4.2 coincide.

Proof Let $(S_j)_j$ be another quasi-equisingular approximation of T in $\mathcal{Z}_+(X, \alpha)$. By Proposition 1.6.3, for any small rational $\epsilon > 0$, $j > 0$, we can find $k > 0$ so that

$$S_k \preceq (1 - \epsilon) T_j.$$

It is more convenient to use the language of θ-psh functions at this point. Let ψ_k (resp. φ_k) denote the potentials in $\mathrm{PSH}(X, \theta)$ corresponding to S_k (resp. T_k) for each $k \geq 1$. Note that ψ_k and φ_k are unique up to additive constants.

By Lemma 10.4.3,

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \Delta_{Y_\bullet}(\theta, (1 - \epsilon)\varphi_j).$$

On the other hand, observe that

$$\bigcap_{\epsilon \in \mathbb{Q}_{>0} \text{ small enough}} \Delta_{Y_\bullet}(\theta, (1 - \epsilon)\varphi_j) = \Delta_{Y_\bullet}(\theta, \varphi_j).$$

In fact, the $\supseteq$ direction follows from Lemma 10.4.3, so it suffices to show that the two sides have the same volume, which follows from (10.38).

It follows that

$$\bigcap_{k=1}^{\infty} \Delta_{Y_\bullet}(\theta, \psi_k) \subseteq \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\theta, \varphi_j).$$

The other inclusion follows by symmetry. □

The same argument shows that

Corollary 10.4.1 *Suppose that $T, S \in \mathcal{Z}_+(X, \alpha)$ are two Kähler currents satisfying $T \preceq_{\mathcal{I}} S$. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proposition 10.4.1 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current. Then*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T. \tag{10.39}$$

Proof Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. Note that $\Delta_{Y_\bullet}(T_j)$ is decreasing in j, as follows from Lemma 10.4.3. Our assertion follows from (10.38) and Theorem 6.2.5. □

Lemma 10.4.5 *Let $T \in \mathcal{Z}_+(X, \alpha)$ be a Kähler current and ω be a Kähler form on X. Then*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T + \omega). \tag{10.40}$$

Moreover,

$$\Delta_{Y_\bullet}(T) = \bigcap_{\epsilon > 0} \Delta_{Y_\bullet}(T + \epsilon\omega). \tag{10.41}$$

Proof We first prove (10.40). Taking quasi-equisingular approximations, we reduce immediately to the case where T has analytic singularities. By Lemma 10.4.2, we may assume that T has log singularities. Take D and R as in (10.37). By definition again, it suffices to show that

$$\Delta_{Y_\bullet}([\beta]) \subseteq \Delta_{Y_\bullet}([\beta + \omega]),$$

which is clear by definition.

Next we prove (10.41). Thanks to (10.40), it remains to prove that both sides have the same volume:

$$\lim_{\epsilon \to 0+} \operatorname{vol}(T + \epsilon\omega) = \operatorname{vol} T.$$

This is proved in Proposition 7.3.1. □

10.4.2.3 The general case

Definition 10.4.4 Let $T \in \mathcal{Z}_+(X, \alpha)$. Take a Kähler form ω on X, we define

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega). \tag{10.42}$$

The same definition makes sense when α is only pseudo-effective.

This definition is clearly independent of the choice of ω by Lemma 10.4.5. Moreover, it extends Definition 10.4.3 and Definition 10.4.2 as a result of Lemma 10.4.5.

The main properties of $\Delta_{Y_\bullet}(T)$ are summarized as follows:

Theorem 10.4.2 *The convex bodies $\Delta_{Y_\bullet}(T)$'s satisfies the following properties:*

(1) *Suppose that $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, We have*

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \frac{1}{n!} \operatorname{vol} T. \tag{10.43}$$

(2) *For $T, S \in \mathcal{Z}_+(X,\alpha)$ satisfying $T \preceq_{\mathcal{I}} S$, we have*

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(\alpha).$$

(3) *For any current $T \in \mathcal{Z}_+(X,\alpha)$ with minimal singularities, we have*

$$\Delta_{Y_\bullet}(T) = \Delta_{Y_\bullet}(\alpha).$$

(4) *The map $\mathcal{Z}_+(X,\alpha)_{>0} \longrightarrow \mathcal{K}_n$ given by $T \mapsto \Delta_{Y_\bullet}(T)$ is continuous, where we endow the d_S-pseudometric on $\mathcal{Z}_+(X,\alpha)_{>0}$ and the Hausdorff topology on $\mathcal{K}_n$.*

(5) *Let $\pi: Y \longrightarrow X$ be a proper bimeromorphic morphism with Y being a Kähler manifold. Assume that the lifting $(W_\bullet, g)$ of $Y_\bullet$ to Y exists, then for any $T \in \mathcal{Z}_+(X,\alpha)_{>0}$, we have*

$$\Delta_{W_\bullet}(\pi^* T) = \Delta_{Y_\bullet}(T)g.$$

(6) *For $T, S \in \mathcal{Z}_+(X,\alpha)$, we have*

$$\Delta_{Y_\bullet}(T) + \Delta_{Y_\bullet}(S) \subseteq \Delta_{Y_\bullet}(T+S). \tag{10.44}$$

Proof (1) By (10.42) and (10.39), for any Kähler form ω on X,

$$\operatorname{vol} \Delta_{Y_\bullet}(T) = \lim_{j\to\infty} \Delta_{Y_\bullet}(T + j^{-1}\omega) = \frac{1}{n!} \lim_{j\to\infty} \operatorname{vol}(T + j^{-1}\omega).$$

The right-hand side is computed in Proposition 7.3.1. Hence, (10.43) follows.

(2) Fix a Kähler form ω on X. By Corollary 10.4.1, for each $j \geq 1$,

$$\Delta_{Y_\bullet}(T + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(S + j^{-1}\omega) \subseteq \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

It remains to show that

$$\Delta_{Y_\bullet}(\alpha) = \bigcap_{j=1}^{\infty} \Delta_{Y_\bullet}(\alpha + j^{-1}[\omega]).$$

The $\subseteq$ direction is clear. Comparing the volumes using Theorem 10.4.1, we conclude that equality holds.

(3) This follows from (1) and (2).

(4) Let $(T_j)_j$ be a sequence in $\mathcal{Z}_+(X,\alpha)_{>0}$ converging to $T \in \mathcal{Z}_+(X,\alpha)_{>0}$ with respect to d_S. We want to show that $\Delta_{Y_\bullet}(T_j) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(T)$. By Proposition 6.2.3 and (2), we may assume that the singularity type of T_j is either increasing or decreasing. In both cases, the continuity follows from (1).

(5) We may assume that T is $\mathcal{I}$-good. It follows from (4) and Theorem 7.1.1 that we could reduce to the case where T has analytic singularities. Our assertion follows from Lemma 10.4.2.

(6) By (10.42), in order to prove (10.44), we may assume that T and S are both Kähler currents. Take quasi-equisingular approximations $(T_j)_j$ and $(S_j)_j$ of T and S respectively. By Theorem 6.2.2, $T_j + S_j \xrightarrow{d_S} T + S$. By (4), we may therefore assume that T and S have analytic singularities. Replacing X by a suitable modification, we may assume that T and S both have log singularities, say

$$T = [D] + R, \quad S = [D'] + R',$$

where D and D' are $\mathbb{Q}$-divisors on X and β and β' are closed positive $(1,1)$-currents with bounded potentials. We need to show that

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) + \nu_{Y_\bullet}([D]) + \nu_{Y_\bullet}([D']) \subseteq \Delta_{Y_\bullet}([R+R']) + \nu_{Y_\bullet}([D+D']).$$

By Proposition 10.2.2, this is equivalent to

$$\Delta_{Y_\bullet}([R]) + \Delta_{Y_\bullet}([R']) \subseteq \Delta_{Y_\bullet}([R+R']),$$

which is already proved in Theorem 10.4.1. □

Corollary 10.4.2 *Assume that L is a big line bundle on X and h is a plurisubharmonic metric on L with positive volume. Then*

$$\Delta_{Y_\bullet}(\mathrm{dd^c} h) = \Delta_{Y_\bullet}(L,h). \tag{10.45}$$

Similarly, the definition (10.27) is compatible with the definition in Definition 10.4.4.

Proof We may assume that $\mathrm{dd^c} h$ has positive mass and is $\mathcal{I}$-good. By the d_S-continuity of both sides of (10.45) as proved in Theorem 10.4.2 and Theorem 10.3.2, together with Theorem 7.1.1, we may assume that $\mathrm{dd^c} h$ has analytic singularities.

In this case, using the birational invariance of both sides of (10.45) as proved in Proposition 10.3.9 and Theorem 10.4.2, we may assume that $\mathrm{dd^c} h$ has log singularities. Finally, after all these reductions, the equality (10.45) holds by construction. □

10.4.3 The valuative characterization

In this section, we will characterize the partial Okounkov bodies using valuations of currents.

Lemma 10.4.6 *Let β be a nef class on X. Then*

$$\left\{y \in \mathbb{R}^{n-1} : (0,y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\beta|_{Y_1}). \tag{10.46}$$

Proof **Step 1**. We first reduce to the case where β is a Kähler class.

Take a Kähler class α on X. It follows from the volume formula in Theorem 10.4.1 that

$$\Delta_{Y_\bullet}(\beta) = \bigcap_{\epsilon>0} \Delta_{Y_\bullet}(\beta+\epsilon\alpha), \quad \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\beta|_{Y_1}) = \bigcap_{\epsilon>0} \Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\beta|_{Y_1}+\epsilon\alpha|_{Y_1}).$$

So it suffices to prove (10.46) with $\beta+\epsilon\alpha$ in place of β.

Step 2. Assume that α is a Kähler class. The $\supseteq$ direction in (10.46) follows from the extension theorem Theorem 1.6.3. To prove the other direction, recall that by Theorem 10.4.1, for $t>0$ small enough, we have

$$\left\{y \in \mathbb{R}^{n-1} : (t,y) \in \Delta_{Y_\bullet}(\beta)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}\left((\beta - t[Y_1])|_{Y_1}\right).$$

As $t\to 0+$, the right-hand side converges to $\Delta_{Y_1\supseteq\cdots\supseteq Y_n}(\beta|_{Y_1})$ with respect to the Hausdorff metric as a consequence of Theorem 10.4.1, while the left-hand side converges to

$$\left\{y \in \mathbb{R}^{n-1} : (0,y) \in \Delta_{Y_\bullet}(\beta)\right\}$$

by Lemma C.1.2. We conclude our assertion. □

Lemma 10.4.7 *Let $T \in \mathcal{Z}_+(X,\alpha)$ be a Kähler current. Assume that $\nu(T,Y_1)=0$, then*

$$\left\{y \in \mathbb{R}^{n-1} : (0,y) \in \Delta_{Y_\bullet}(T)\right\} = \Delta_{Y_1\supseteq\cdots\supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right). \tag{10.47}$$

More generally, if $T \in \mathcal{Z}_+(X,\alpha)$ and $\nu(T,Y_1)=0$, suppose in addition that $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$ is defined, then (10.47) still holds.

See Remark 8.1.1 for the definition of $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$. Note that $\Delta_{Y_1\supseteq\cdots\supseteq Y_n}\left(\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)\right)$ is independent of the choice of the representative $\mathrm{Tr}_{Y_1}^{\alpha|_{Y_1}}(T)$.

Remark 10.4.1 More generally, the same argument shows the following result: Let $k=0,\ldots,n$ and $T\in\mathcal{Z}_+(X,\alpha)$ such that $\nu(T,Y_k)=0$. Assume that $\mathrm{Tr}_{Y_k}^{\alpha|_{Y_k}}(T)$ is defined, then

$$\left\{y \in \mathbb{R}^{n-k} : (0,\ldots,0,y) \in \Delta_{Y_\bullet}(T)\right\} = \Delta_{Y_k\supseteq\cdots\supseteq Y_n}\left(\mathrm{Tr}_{Y_k}^{\alpha|_{Y_k}}(T)\right). \tag{10.48}$$

Also note that this result extends [Jow10, Theorem 3.4] and hence gives simpler proofs of [Jow10, Theorem A, Theorem B].

Proof Let ω be a Kähler form on X. The last assertion follows from the first by perturbing θ to $\theta + \epsilon\omega$.

Step 1. We first handle the case where T has analytic singularities. Let $\pi\colon Z \to X$ be a modification such that

(1) $Y_\bullet$ admits a lifting $(W_\bullet, g)$, and
(2) $\pi^* T = [D] + R$, where D is an effective $\mathbb{Q}$-divisor on Z and R is closed positive $(1,1)$-current with bounded potential.

This is possible by Theorem 1.6.1 and Theorem 10.2.1.

By Lemma 8.2.1,

$$\Pi^* \operatorname{Tr}_{Y_1}(T) \sim_P \operatorname{Tr}_{W_1}(\pi^* T),$$

where $\Pi\colon W_1 \to Y_1$ is the restriction of π. It follows from Theorem 10.4.2 that

$$\begin{aligned}\Delta_{W_1 \supseteq \cdots \supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^* T)) =& \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\operatorname{Tr}_{Y_1}(T)) \operatorname{cor}(Y_1 \supseteq \cdots \supseteq Y_n, \Pi),\\ \Delta_{W_\bullet}(\pi^* T) =& \Delta_{Y_\bullet}(T) g.\end{aligned}$$

Taking (10.10) into account, we find that it suffices to show that

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{W_\bullet}(\pi^* T)\right\} = \Delta_{W_1 \supseteq \cdots \supseteq W_n}(\operatorname{Tr}_{W_1}(\pi^* T)).$$

We may assume that π is the identity map. Then we have

$$T = [D] + R, \quad T|_{Y_1} = [D]|_{Y_1} + R|_{Y_1}.$$

Note that $[D]|_{Y_1}$ is the current of integration along an effective $\mathbb{Q}$-divisor on Y_1.

In particular,

$$\begin{aligned}\Delta_{Y_\bullet}(T) =& \Delta_{Y_\bullet}([R]) + \nu_{Y_\bullet}([D]),\\ \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(T|_{Y_1}) =& \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}([R]|_{Y_1}) + \nu_{Y_1 \supseteq \cdots \supseteq Y_n}([D]|_{Y_1}).\end{aligned}$$

So it suffices to show that

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}([R])\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}([R]|_{Y_1}),$$

which is exactly Lemma 10.4.6.

Step 2. Next we consider the case where T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T in $\mathcal{Z}_+(X, \alpha)$. From Step 1, we know that for large $j \geq 1$,

$$\left\{y \in \mathbb{R}^{n-1} : (0, y) \in \Delta_{Y_\bullet}(T_j)\right\} = \Delta_{Y_1 \supseteq \cdots \supseteq Y_n}(\operatorname{Tr}_{Y_1}(T_j)).$$

Letting $j \to \infty$ and applying Theorem 10.4.2 and Proposition 8.2.2, we conclude (10.47). □

Theorem 10.4.3 *Assume that $T \in \mathcal{Z}_+(X, \alpha)_{>0}$ is a Kähler current. We have*

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.49}$$

Here the minimum is with respect to the lexicographic order.

Proof We make induction on $n \geq 0$. The case $n = 0$ is of course trivial. Let us assume that $n > 0$ and the case $n-1$ has been proved.

We first observe that by Theorem 10.4.2,

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \ldots, 0) \subseteq \Delta_{Y_\bullet}(T).$$

Comparing the volumes of both sides using Theorem 10.4.2 and Proposition 7.3.1, we find that equality holds:

$$\Delta_{Y_\bullet}(T - \nu(T, Y_1)[Y_1]) + (\nu(T, Y_1), 0, \ldots, 0) = \Delta_{Y_\bullet}(T).$$

Replacing T by $T - \nu(T, Y_1)[Y_1]$, we may therefore assume that $\nu(T, Y_1) = 0$. It suffices to apply Lemma 10.4.7 and the inductive hypothesis. □

Corollary 10.4.3 *For any $T \in \mathcal{Z}_+(X, \alpha)$,*

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(\alpha).$$

Proof When T is a Kähler current, this follows from Theorem 10.4.3.

In general, by definition, $\nu_{Y_\bullet}(T) = \nu_{Y_\bullet}(T + \omega)$ for any Kähler form ω on X. It follows that

$$\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T + \omega)$$

for any Kähler form ω. It follows that $\nu_{Y_\bullet}(T) \in \Delta_{Y_\bullet}(T)$. □

Theorem 10.4.4 *For any $T \in \mathcal{Z}_+(X, \alpha)_{>0}$,*

$$\Delta_{Y_\bullet}(T) = \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \preceq_{\mathcal{I}} T\right\}}. \tag{10.50}$$

In particular, [2]

$$\Delta_{Y_\bullet}(\alpha) = \overline{\left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha)\right\}}. \tag{10.51}$$

Remark 10.4.2 We expect that the closure operation in (10.50) is not necessary. This problem is closely related to the Dirichlet problem of the trace operator, see Page 384 for more details.

Proof The $\supseteq$ direction in (10.50) follows from Corollary 10.4.3 and Theorem 10.4.2(2).

Let us write

$$D_{Y_\bullet}(T) = \left\{\nu_{Y_\bullet}(S) : S \in \mathcal{Z}_+(X, \alpha), S \preceq_{\mathcal{I}} T\right\}$$

[2] According to Ya Deng, the definition (10.51) of $\Delta_{Y_\bullet}(\alpha)$ was what Demailly originally proposed for Deng's thesis. Due to the lack of the techniques of the trace operators, Deng had to work with analytic singularities instead. As a consequence, the transcendental analogue of Proposition 10.3.9 is not obvious. This is one of the two main technical difficulties of Theorem 10.4.1. This problem also led me to finally develop the theory of trace operators, a notion I had in mind for several years.

for the time being.

Step 1. Assume that T has analytic singularities. We have

$$\begin{aligned}\Delta_{Y_\bullet}(T) \supseteq & \overline{D_{Y_\bullet}(T)} \\ \supseteq & \overline{\left\{\nu_{Y_\bullet}(S) : \mathcal{Z}_+(X,\alpha) \ni S \text{ has gentle analytic singularities}, S \preceq T\right\}}.\end{aligned}$$

It follows easily from Theorem 10.4.1 that the volume of the right-hand side is equal to the volume of $\Delta_{Y_\bullet}(T)$, so (10.50) holds.

Step 2. Assume that T is a Kähler current. Take a quasi-equisingular approximation $T_j \in \mathcal{Z}_+(X,\alpha)$ of T. Next we use the language of psh functions. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)$ be the potentials corresponding to T_j, T for each $j \geq 1$.

Fix an integer $N > 0$. For large enough $j \geq 1$, we can find $\psi \in \mathrm{PSH}(X,\theta)_{>0}$ such that

$$P_\theta[\varphi]_{\mathcal{I}} \geq (1-N^{-1})\varphi_j + N^{-1}\psi_j.$$

The existence of ψ_j follows from Lemma 2.4.2. It follows that

$$\begin{aligned}D_{Y_\bullet}(T) \supseteq & D_{Y_\bullet}\left(\theta + \mathrm{dd^c}\left((1-N^{-1})\varphi_j + N^{-1}\psi_j\right)\right) \\ \supseteq & (1-N^{-1})D_{Y_\bullet}(T_j) + N^{-1}D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j).\end{aligned}$$

By Theorem C.1.1, up to replacing T_j by a subsequence, we may guarantee that $\overline{D_{Y_\bullet}(\theta + \mathrm{dd^c}\psi_j)}$ admits a Hausdorff limit contained in $\Delta_{Y_\bullet}(\alpha)$ as $j \to \infty$. Let $j \to \infty$ and $N \to \infty$ then it follows that

$$\overline{D_{Y_\bullet}(T)} \supseteq \bigcap_{j=1}^\infty D_{Y_\bullet}(T_j).$$

By Lemma C.1.3,

$$\overline{D_{Y_\bullet}(T)} \supseteq \overline{\bigcap_{j=1}^\infty D_{Y_\bullet}(T_j)} = \bigcap_{j=1}^\infty \overline{D_{Y_\bullet}(T_j)}.$$

Therefore, by Step 1, we conclude that

$$\Delta_{Y_\bullet}(T) = \bigcap_{j=1}^\infty \overline{\Delta_{Y_\bullet}(T_j)} = \bigcap_{j=1}^\infty \overline{D_{Y_\bullet}(T_j)} \subseteq \overline{D_{Y_\bullet}(T)}.$$

The reverse direction is already known.

Step 3. Finally, consider the general case. Take a Kähler current $T' \in \mathcal{Z}_+(X,\alpha)$ more singular than T. For each $\epsilon \in (0,1)$. The existence of T' is proved in Lemma 2.4.3. We know that

$$\Delta_{Y_\bullet}((1-\epsilon)T + \epsilon T') = \overline{D_{Y_\bullet}((1-\epsilon)T + \epsilon T')} \subseteq \overline{D_{Y_\bullet}(T)}.$$

Letting $\epsilon \to 0+$ and using Proposition 7.3.1, we find that

$$\Delta_{Y_\bullet}(T) \subseteq \overline{D_{Y_\bullet}(T)}.$$

As the other inclusion is already known, we conclude. □

Corollary 10.4.4 *Assume that $T \in \mathcal{Z}_+(X, \alpha)_{>0}$. We have*

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T). \tag{10.52}$$

Proof By Theorem 10.4.4, it is clear that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \leq_{\text{lex}} \nu_{Y_\bullet}(T).$$

On the other hand, we clearly have

$$\Delta_{Y_\bullet}(T) \subseteq \Delta_{Y_\bullet}(T + \omega)$$

for any Kähler form ω on X. It follows that

$$\min_{\text{lex}} \Delta_{Y_\bullet}(T) \geq_{\text{lex}} \min_{\text{lex}} \Delta_{Y_\bullet}(T + \omega).$$

By Theorem 10.4.3, the right-hand side is just $\nu_{Y_\bullet}(T + \omega) = \nu_{Y_\bullet}(T)$. We conclude the proof. □

10.5 Okounkov test curves

Fix $n \in \mathbb{N}$. Let $\Delta, \Delta' \subseteq \mathbb{R}^n$ be convex bodies with positive volumes. The standard Lebesgue measure on $\mathbb{R}^n$ is denoted by vol.

Recall that $\mathcal{K}_n$ denotes the set of convex bodies in $\mathbb{R}^n$ and d_{Haus} denotes the Hausdorff metric. We refer to Appendix C for the basic properties of these objects.

We will study the notion of Okounkov test curves in this section, which leads to the definition of the Duistermaat–Heckman measure of a non-Archimedean metric in Section 13.3 below. We encourage the readers to skip this section on a first reading.

Definition 10.5.1 An *Okounkov test curve* relative to Δ consists of

(1) a number $\Delta_{\max} \in \mathbb{R}$ and
(2) an assignment $(-\infty, \Delta_{\max}) \ni \tau \mapsto \Delta_\tau \in \mathcal{K}_n$ satisfying

   a. the assignment $\tau \mapsto \Delta_\tau$ is a decreasing and concave[3];
   b. we have $\Delta_\tau \xrightarrow{d_{\text{Haus}}} \Delta$ as $\tau \to -\infty$.

[3] Here concavity refers to the concavity with respect to the Minkowski sum.

The set of Okounkov test curves relative to Δ is denoted by $\mathrm{TC}(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ relative to Δ is *bounded* if $\Delta_\tau = \Delta$ when τ is small enough. The subset of bounded Okounkov test curves is denoted by $\mathrm{TC}^\infty(\Delta)$.

An Okounkov test curve $\Delta_\bullet$ relative to Δ is said to have *finite energy* if

$$\mathbf{E}(\Delta_\bullet) \coloneqq n!\Delta_{\max} \operatorname{vol}\Delta + n! \int_{-\infty}^{\Delta_{\max}} (\operatorname{vol}\Delta_\tau - \operatorname{vol}\Delta)\, \mathrm{d}\tau > -\infty. \tag{10.53}$$

The subset of Okounkov test curves with finite energy is denoted by $\mathrm{TC}^1(\Delta)$.

Given $\Delta_\bullet \in \mathrm{TC}(\Delta)$ and $\Delta'_\bullet \in \mathrm{TC}(\Delta')$, we say $\Delta_\bullet \leq \Delta'_\bullet$ if $\Delta_{\max} \leq \Delta'_{\max}$ and for any $\tau < \Delta_{\max}$, we have $\Delta_\tau \subseteq \Delta'_\tau$.

Sometimes it is convenient to introduce

$$\Delta_{\Delta_{\max}} = \bigcap_{\tau < \Delta_{\max}} \Delta_\tau \in \mathcal{K}_n. \tag{10.54}$$

We shall always make this extension in the sequel when we talk about $\Delta_{\Delta_{\max}}$. Observe that $(-\infty, \Delta_{\max}] \ni \tau \mapsto \Delta_\tau$ is still concave.

Proposition 10.5.1 *Any Okounkov test curve* $(\Delta_\tau)_{\tau<\Delta_{\max}}$ *relative to* Δ *is continuous in* τ*. Moreover,* $\operatorname{vol}\Delta_\tau > 0$ *for all* $\tau < \Delta_{\max}$.

Proof We first claim that $\operatorname{vol}\Delta_{\tau'} > 0$ for all $\tau' < \Delta_{\max}$. By Condition (2b) in Definition 10.5.1 and Theorem C.1.2, we know that $\operatorname{vol}\Delta_{\tau''} > 0$ when τ'' is small enough. Fix one such τ''. We may assume that $\tau'' \leq \tau'$ since otherwise there is nothing to prove. Next take $\tau''' \in (\tau', \Delta_{\max})$. Take $t \in (0,1)$ such that $\tau' = t\tau''' + (1-t)\tau''$. It follows that

$$\operatorname{vol}\Delta_{\tau'} \geq \operatorname{vol}\left(t\Delta_{\tau'''} + (1-t)\Delta_{\tau''}\right) \geq (1-t)^n \operatorname{vol}\Delta_{\tau''} > 0.$$

Next we claim that $\operatorname{vol}\Delta_\tau$ is continuous for $\tau < \Delta_{\max}$. In fact, it follows from Theorem C.1.4 that $(-\infty, \Delta_{\max}) \ni \tau \mapsto \log\operatorname{vol}\Delta_\tau$ is concave, but we have already known that it is finite, hence the continuity follows.

Next we show that

$$\Delta_\tau = \bigcap_{\tau'<\tau} \Delta_{\tau'}.$$

The $\subseteq$ direction is obvious. By the continuity of the volume, both sides have the same volume and the volume is positive, we therefore obtain the equality.

Similarly, we have

$$\Delta_\tau = \overline{\bigcup_{\tau'>\tau} \Delta_{\tau'}}.$$

The continuity of Δ_τ at $\tau < \Delta_{\max}$ is proved. □

Definition 10.5.2 A *test function* on Δ is a function $G\colon \Delta \to [-\infty, \infty)$ such that

(1) G is concave,

(2) G is finite on $\operatorname{Int}\Delta$, and
(3) G is upper semicontinuous.

A test function G is *bounded* if G is bounded from below.
A test function G has *finite energy* if

$$\mathbf{E}(G) := n! \int_\Delta G \,\mathrm{d}\lambda > -\infty. \tag{10.55}$$

Definition 10.5.3 Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We define its *Legendre transform* as

$$G[\Delta_\bullet]\colon \Delta \to [-\infty, \infty), \quad a \mapsto \sup\{\tau < \Delta_{\max} : a \in \Delta_\tau\}.$$

Given a test function $G\colon \Delta \to [-\infty,\infty)$, we define its *inverse Legendre transform* $\Delta[G]_\bullet$ as the Okounkov test curve relative to Δ defined as follows:

(1) $\Delta[G]_{\max} = \sup_\Delta G$, and
(2) for each $\tau < \sup_\Delta G$, we set

$$\Delta[G]_\tau = \{x \in \Delta : G \geq \tau\}.$$

We observe that

$$G[\Delta_\bullet](a) = \max\{\tau \leq \Delta_{\max} : a \in \Delta_\tau\}, \text{ if } G[\Delta_\bullet](a) > -\infty. \tag{10.56}$$

Lemma 10.5.1 *Let $\Delta_\bullet \in \mathrm{TC}(\Delta)$. Then $G[\Delta_\bullet]$ defined in Definition 10.5.3 is a test function.*

Similar, if $G\colon \Delta \to [-\infty,\infty)$ is a test function, then $\Delta[G]_\bullet$ is an Okounkov test curve.

Proof First suppose that $\Delta_\bullet \in \mathrm{TC}(\Delta)$. We want to verify that $G[\Delta_\bullet]$ satisfies the conditions in Definition 10.5.2.

We first verify the concavity. Take $a, b \in \Delta$. We want to prove that for any $t \in (0,1)$,

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b). \tag{10.57}$$

There is nothing to prove if $G[\Delta_\bullet](a)$ or $G[\Delta_\bullet](b)$ is $-\infty$. So we assume that both are finite. In this case, by (10.56),

$$a \in \Delta_{G[\Delta_\bullet](a)}, \quad b \in \Delta_{G[\Delta_\bullet](b)}.$$

Thus,

$$ta + (1-t)b \in t\Delta_{G[\Delta_\bullet](a)} + (1-t)\Delta_{G[\Delta_\bullet](b)} \subseteq \Delta_{tG[\Delta_\bullet](a)+(1-t)G[\Delta_\bullet](b)}.$$

We deduce that

$$G[\Delta_\bullet](ta + (1-t)b) \geq tG[\Delta_\bullet](a) + (1-t)G[\Delta_\bullet](b).$$

Therefore, (10.57) follows.

It is clear that $G[\Delta_\bullet]$ is finite on the interior of Δ. It remains to argue that $G[\Delta_\bullet]$ is upper semicontinuous.

Let $(a_i)_{i\geq 1}$ be a sequence in Δ with limit $a \in \Delta$. Define $\tau_i = G[\Delta_\bullet](a_i)$. Let $\tau = \overline{\lim}_i \tau_i$. We need to show that

$$G[\Delta_\bullet](a) \geq \tau. \tag{10.58}$$

There is nothing to prove if $\tau = -\infty$. We assume that it is not this case. Up to subtracting a subsequence we may assume that $\tau_i \to \tau$. In particular, we can assume that $\tau_i \neq -\infty$ for all $i \geq 1$. It follows from (10.56) that $a_i \in \Delta_{\tau_i}$ for all $i \geq 1$. Since $\Delta_{\tau_i} \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$. By Theorem C.1.3 it follows that $a \in \Delta_\tau$. Thus,(10.58) follows.

Conversely, suppose that $G\colon \Delta \to [-\infty, \infty)$ is a test function. We argue that $\Delta[G]_\bullet$ is an Okounkov test curve. We verify the conditions in Definition 10.5.1.

Firstly, for each $\tau < \sup_\Delta G$, the set $\Delta[G](\tau)$ is a convex body as G is concave and usc. Moreover, $\Delta[G]_\tau$ is clearly decreasing in τ.

Secondly, for each $a \in \Delta$, we can write $a = \lim_i a_i$ with $a_i \in \operatorname{Int}\Delta$. By assumption, G is finite at a_i. Thus,

$$a \in \overline{\{G > -\infty\}} = \overline{\bigcup_{\tau < \sup_\Delta G} \Delta[G]_\tau}.$$

By Theorem C.1.3, $\Delta[G]_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta$ as $\tau \to -\infty$.

Thirdly, $\Delta[G]$ is concave. To see, take $\tau, \tau' < \Delta_{\max}$, we need to prove that for any $t \in (0, 1)$,

$$\Delta[G]_{t\tau+(1-t)\tau'} \supseteq t\Delta[G]_\tau + (1-t)\Delta[G]_{\tau'}. \tag{10.59}$$

Let $a \in \Delta[G]_\tau$ and $b \in \Delta[G]_{\tau'}$. We have $G(a) \geq \tau$ and $G(b) \geq \tau'$. As G is concave, we have $G(ta + (1-t)b) \geq t\tau + (1-t)\tau'$. Thus,

$$ta + (1-t)b \in \Delta[G]_{t\tau+(1-t)\tau'}$$

and (10.59) follows. □

Theorem 10.5.1 *The Legendre transform and inverse Legendre transform are inverse to each other, defining a bijection between* $\mathrm{TC}(\Delta)$ *and the set of test functions on* Δ*.*

Under this bijection, $\mathrm{TC}^1(\Delta)$ *corresponds to test functions on* Δ *with finite energy and* $\mathrm{TC}^\infty(\Delta)$ *corresponds to bounded test functions on* Δ*.*

Proof Thanks to Lemma 10.5.1, in order to prove the first assertion, it only remains to see that the Legendre transform and the inverse Legendre transform are inverse to each other, which is immediate by definition.

It is obvious that $\mathrm{TC}^\infty(\Delta)$ corresponds to bounded test curves. Moreover, a direct computation shows that if $\Delta_\bullet \in \mathrm{TC}(\Delta)$, then

$$\mathbf{E}(\Delta_\bullet) = \mathbf{E}(G[\Delta_\bullet]),$$

concluding the $\mathrm{TC}^1(\Delta)$ case. $\square$

Proposition 10.5.2 *Let $(\Delta^i)_{i\in I}$ be a decreasing net in $\mathcal{K}_n$. Consider a decreasing net $(\Delta^i_\bullet)_{i\in I}$ with $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ for all $i \in I$ such that there is $\Delta_\bullet \in \mathrm{TC}(\Delta)$ satisfying the following properties:*

(1) $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$;

(2) *for any $\tau < \Delta_{\max}$, we have $\Delta^i_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$.*

Then for any $a \in \Delta$, we have

$$\lim_{i\in I} G[\Delta^i_\bullet](a) = G[\Delta_\bullet](a). \tag{10.60}$$

Note that in general,

$$\Delta \subsetneq \bigcap_{i\in I} \Delta^i.$$

Proof Fix $a \in \Delta$. It follows immediately from the definition of G that the net $(G[\Delta^i_\bullet](a))_{i\in I}$ is decreasing and the $\geq$ direction in (10.60) holds. Let us prove the reverse inequality. Let τ denote the left-hand side of (10.60) for the moment. By definition, for any $\epsilon > 0$ and any $i \in I$, we have $a \in \Delta^i_{\tau-\epsilon}$. It follows that

$$a \in \Delta^\infty_{\tau-\epsilon}.$$

Therefore,

$$\tau \leq G[\Delta_\bullet](a).$$

Similarly, for increasing nets, we have:

Proposition 10.5.3 *Let $(\Delta^i)_{i\in I}$ be an increasing net in $\mathcal{K}_n$ with Hausdorff limit Δ such that $\operatorname{vol}\Delta^i > 0$ for all $i \in I$. Consider an increasing net $(\Delta^i_\bullet)_{i\in I}$ with $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ for all $i \in I$. Let $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$. For any $\tau < \Delta_{\max}$, let Δ_τ be the Hausdorff limit of Δ^i_τ. Then $\Delta_\bullet \in \mathrm{TC}(\Delta)$ and*

$$\lim_{i\in I} G[\Delta^i_\bullet](a) = G[\Delta_\bullet](a) \tag{10.61}$$

for any $a \in \operatorname{Int}\Delta$.

Proof It is obvious that $\Delta_\bullet \in \mathrm{TC}(\Delta)$.

Fix $a \in \operatorname{Int}\Delta$. Then up to replacing I by a subnet, we may assume that $a \in \Delta^i$ for all $i \in I$. By definition, the net $(G[\Delta^i_\bullet](a))_{i\in I}$ is increasing and the $\leq$ direction in (10.61) holds. Let us write $\tau = G[\Delta_\bullet](a)$ for the time being. By definition of G, for any $\epsilon > 0$, we have

$$a \in \Delta_{\tau-\epsilon/2}.$$

The concavity of $\Delta_\bullet$ guarantees that

$$a \in \operatorname{Int}\Delta_{\tau-\epsilon}.$$

It follows that there is a subnet J in I such that for all $j \in J$,

$$a \in \Delta^{j}_{\tau-\epsilon}.$$

Therefore,

$$\tau - \epsilon \leq G[\Delta^{j}_{\bullet}](a).$$

Taking the limit with respect to j and then with respect to ϵ, we conclude the desired inequality. □

Definition 10.5.4 Let $\Delta_\bullet$ be an Okounkov test curve relative to Δ. We define the *Duistermaat–Heckman measure* $\mathrm{DH}(\Delta_\bullet)$ as

$$\mathrm{DH}(\Delta_\bullet) \coloneqq G[\Delta_\bullet]_*(\mathrm{vol}).$$

It is a Radon measure on $\mathbb{R}$.

In other words, $\mathrm{DH}(\Delta_\bullet)$ is the distribution of the random variable $G[\Delta_\bullet]$.

Proposition 10.5.4 *Let* $\Delta_\bullet \in \mathrm{TC}(\Delta)$. *Let* $m \in \mathbb{Z}_{>0}$. *Then the* m*-th moment of the* $\mathrm{DH}(\Delta_\bullet)$ *is given by*

$$\int_{\mathbb{R}} x^m \,\mathrm{DH}(\Delta_\bullet)(x) = \Delta_{\max}^m \operatorname{vol}\Delta + m\int_{-\infty}^{\Delta_{\max}} \tau^{m-1}(\operatorname{vol}\Delta_\tau - \operatorname{vol}\Delta)\,\mathrm{d}\tau \tag{10.62}$$

and

$$\int_{\mathbb{R}} \mathrm{DH}(\Delta_\bullet) = \operatorname{vol}\Delta. \tag{10.63}$$

Proof In fact, (10.63) follows immediately from the definition, while (10.62) follows form a straightforward computation:

$$\begin{aligned}
&\int_{\mathbb{R}} x^m \,\mathrm{DH}(\Delta_\bullet)(x)\\
=&\int_{\Delta} G[\Delta_\bullet](a)^m \,\mathrm{d}\operatorname{vol}(a)\\
=&\int_{\Delta}\left(\Delta_{\max}^m - \int_{G[\Delta_\bullet](a)}^{\Delta_{\max}} m\tau^{m-1}\,\mathrm{d}\tau\right)\mathrm{d}\operatorname{vol}(a)\\
=&\Delta_{\max}^m \operatorname{vol}\Delta - m\int_{\mathbb{R}}\int_{\Delta} \mathbb{1}_{[G(\Delta_\bullet](a),\Delta_{\max}]}(\tau)\tau^{m-1}\,\mathrm{d}\operatorname{vol}(a)\,\mathrm{d}\tau\\
=&\Delta_{\max}^m \operatorname{vol}\Delta - m\int_{-\infty}^{\Delta_{\max}}\int_{\Delta\setminus\Delta_\tau} \tau^{m-1}\,\mathrm{d}\operatorname{vol}(a)\,\mathrm{d}\tau\\
=&\Delta_{\max}^m \operatorname{vol}\Delta - m\int_{-\infty}^{\Delta_{\max}} \tau^{m-1}\left(\operatorname{vol}\Delta - \operatorname{vol}\Delta_\tau\right)\,\mathrm{d}\tau.
\end{aligned}$$

Lemma 10.5.2 *Let* $(\Delta^i)_{i\in I}$ *be a decreasing net in* $\mathcal{K}_n$ *with limit* Δ. *Suppose that* $(\Delta^i_\bullet)_{i\in I}$ *is a decreasing net with* $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$. *Suppose that there is* $\Delta_\bullet \in \mathrm{TC}(\Delta)$ *such that*

(1) $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$;
(2) *for any* $\tau < \Delta_{\max}$, *we have* $\Delta^i_\tau \xrightarrow{d_{\mathrm{Haus}}} \Delta_\tau$.

Then $\mathrm{DH}(\Delta^i_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet)$.

Proof It follows from Proposition 10.5.2 that

$$G[\Delta^i_\bullet] \to G[\Delta_\bullet]$$

pointwisely on Δ. Our assertion then follows from the dominated convergence theorem. □

Similarly, we have

Lemma 10.5.3 *Let* $(\Delta^i)_{i\in I}$ *be an increasing net in* $\mathcal{K}_n$ *with Hausdorff limit* Δ *such that* $\operatorname{vol}\Delta^i > 0$ *for all* $i \in I$. *Consider an increasing net* $(\Delta^i_\bullet)_{i\in I}$ *with* $\Delta^i_\bullet \in \mathrm{TC}(\Delta^i)$ *for all* $i \in I$. *Let* $\Delta_\bullet \in \mathrm{TC}(\Delta)$ *be defined as*

(1) $\Delta_{\max} = \lim_{i\in I} \Delta^i_{\max}$;
(2) *for any* $\tau < \Delta_{\max}$, Δ_τ *is the Hausdorff limit of* Δ^i_τ.

Then we have

$$\mathrm{DH}(\Delta^i_\bullet) \rightharpoonup \mathrm{DH}(\Delta_\bullet).$$

Proof It follows from Proposition 10.5.3 that

$$G[\Delta^i_\bullet] \to G[\Delta_\bullet]$$

almost everywhere on Δ. Our assertion then follows from the dominated convergence theorem. □

The main source of Okounkov test curves is the following:

Theorem 10.5.2 *Let* X *be a connected compact Kähler manifold and* θ *be a closed smooth real* $(1,1)$*-form on* X *representing a big cohomology class* α. *Let* $Y_\bullet$ *be a smooth flag on* X *and* $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. *Then the map*

$$(-\infty, \Gamma_{\max}) \ni \tau \mapsto \Delta_{Y_\bullet}(\theta,\Gamma)_\tau := \Delta_{Y_\bullet}(\theta,\Gamma_\tau)$$

defines an Okounkov test curve relative to $\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty})$.

If furthermore $\Gamma \in \mathrm{TC}^1(X,\theta;\Gamma_{-\infty})$ *(resp.* $\mathrm{TC}^\infty(X,\theta;\Gamma_{-\infty})$*), then we have* $\Delta_{Y_\bullet}(\theta,\Gamma) \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$ *(resp.* $\mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$*).*

See Definition 9.1.1 and Definition 9.1.2 for the relevant definitions.

Proof Consider $\Gamma \in \mathrm{TC}(X,\theta)_{>0}$. We need to verify that $\Delta_{Y_\bullet}(\theta,\Gamma)$ is an Okounkov test curve relative to $\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty})$.

First observe that $\tau \mapsto \Delta_{Y_\bullet}(\theta,\Gamma_\tau)$ is concave and decreasing for $\tau < \Gamma_{\max}$. This is a direct consequence of Theorem 10.4.4.

Next we show that as $\tau \to -\infty$, we have

$$\Delta_{Y_\bullet}(\theta, \Gamma_\tau) \xrightarrow{d_{\mathrm{Haus}}} \Delta_{Y_\bullet}(\theta, \Gamma_{-\infty}).$$

It suffices to compute

$$\begin{aligned}\lim_{\tau \to -\infty} \operatorname{vol} \Delta_{Y_\bullet}(\theta, \Gamma_\tau) = \frac{1}{n!} \lim_{\tau \to -\infty} \operatorname{vol}(\theta + \mathrm{dd^c}\Gamma_\tau) &= \frac{1}{n!} \operatorname{vol}(\theta + \mathrm{dd^c}\Gamma_{-\infty}) \\ &= \operatorname{vol} \Delta_{Y_\bullet}(\theta, \Gamma_{-\infty}),\end{aligned}$$

where we applied Theorem 10.4.2 and Theorem 6.2.5.

When $\Gamma \in \mathrm{TC}^\infty(X, \theta; \Gamma_{-\infty})$, it is clear that $\Delta_{Y_\bullet}(\theta, \Gamma) \in \mathrm{TC}^\infty(\Delta_{Y_\bullet}(\theta, \Gamma_{-\infty}))$.

When $\Gamma \in \mathrm{TC}^1(X, \theta; \Gamma_{-\infty})$, by Theorem 10.4.2(1), (9.4) and (10.53), we have

$$\mathbf{E}^{\Gamma_{-\infty}}(\Gamma) = \mathbf{E}(\Delta_{Y_\bullet}(\theta, \Gamma)).$$

So $\Gamma \in \mathrm{TC}^1(\Delta_{Y_\bullet}(\theta, \Gamma_{-\infty}))$. □

Remark 10.5.1 As a special case of this construction, suppose that Γ is the test curve induced by a test configuration as in Example 9.3.1 and Remark 9.3.1, then for any $\tau < \Gamma_{\max}$, $\Delta_{Y_\bullet}(\theta, \Gamma_\tau)$ is the Okounkov body of a graded linear series

$$\bigoplus_{k=0}^{\infty} \mathcal{F}_k^{k\tau},$$

where $\mathcal{F}$ is the filtration induced by the test configuration. See [Xia25b, Theorem 5.28] for the details. In particular, in this case, our theory of partial Okounkov bodies recovers the Okounkov bodies of the filtered linear series in the sense of [BC11].

Chapter 11
The theory of b-divisors

The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: There is no permanent place in this world for ugly mathematics.
— Godfrey Harold Hardy[a]

[a] Godfrey Harold Hardy (1877–1947) was a British mathematician famous for his work in number theory and mathematical analysis. Apart from his research, Hardy was a strong advocate for pure mathematics and believed that mathematics should be pursued for its own beauty, not just for practical use.

He remained lifelong unmarried and dedicated much of his life entirely to mathematics, fitting into the common stereotype of a mathematician.

In this chapter, we study the theory of nef b-divisors. In particular, we establish their intersection theory. Our main theorem Theorem 11.1.3 says roughly speaking that the closed positive $(1,1)$-currents (modulo $\mathcal{I}$-equivalence), which are analytic objects by nature are equivalent to the purely cohomological notion of nef b-divisors.

In Section 11.3, we prove that the partial Okounkov bodies constructed in Chapter 10 have natural interpretations in terms of the b-divisors.

In this section, we shall denote the current of integration associated with a prime divisor D as $[D]$, while the corresponding cohomology class will be denoted by $\{D\}$. This convention is simply to avoid any potential confusions.

11.1 The notions of b-divisors

The b-divisors defined in this section are sometimes known as b-divisor classes. We always omit the word *classes* to save space.

Let X be a connected compact Kähler manifold of dimension n.

Let us recall the following elementary result regarding how cohomology behaves under blow-up.

Proposition 11.1.1 *Let $\pi\colon Y \to X$ be a blow-up with* connected *smooth center of codimension at least* 2 *with exceptional divisor E. Then there is a natural identification*

$$\mathrm{H}^{1,1}(Y,\mathbb{R}) = \mathrm{H}^{1,1}(X,\mathbb{R}) \oplus \mathbb{R}\{E\}. \tag{11.1}$$

See [RYY19] for a much more general result.

11.1.1 Nef b-divisors

Definition 11.1.1 A *(Weil) b-divisor* $\mathbb{D}$ over X is an assignment $(\mathbb{D}_\pi)_{\pi\colon Y\to X}$, where $\pi\colon Y\to X$ runs over all modifications of X such that

(1) $\mathbb{D}_\pi \in \mathrm{H}^{1,1}(Y,\mathbb{R})$;
(2) The classes are compatible under push-forwards: If $\pi'\colon Z\to X$ and $\pi\colon Y\to X$ are both in modifications of X and π' dominates π through $g\colon Z\to Y$, then $g_*\mathbb{D}_{\pi'}=\mathbb{D}_\pi$.

We also write $\mathbb{D}_Y=\mathbb{D}_\pi$ if there is no risk of confusion.

Given two Weil b-divisors $\mathbb{D}$ and $\mathbb{D}'$ over X, we say $\mathbb{D}\leq\mathbb{D}'$ if for each modification π of X, we have $\mathbb{D}_\pi\leq\mathbb{D}'_\pi$. Recall that by definition, this means the class $\mathbb{D}'_\pi-\mathbb{D}_\pi$ is pseudo-effective.

The class $\mathbb{D}_X$ is called the *root* of $\mathbb{D}$. The set of Weil b-divisors over X has the obvious structure of real vector spaces.

Definition 11.1.2 The *volume* of a Weil b-divisor $\mathbb{D}$ over X is

$$\operatorname{vol}\mathbb{D} := \lim_{\pi\colon Y\to X}\operatorname{vol}\mathbb{D}_Y.$$

The right-hand side is a decreasing net due to Proposition 3.2.8, hence the limit always exists.

We say $\mathbb{D}$ is *big* if $\operatorname{vol}\mathbb{D}>0$.

Lemma 11.1.1 *Let $(\mathbb{D}_i)_{i\in I}$ be a net of b-divisors converging to $\mathbb{D}$. Then*

$$\varlimsup_{i\in I}\operatorname{vol}\mathbb{D}_i\leq\operatorname{vol}\mathbb{D}. \tag{11.2}$$

If the net is decreasing, then

$$\lim_{i\in I}\operatorname{vol}\mathbb{D}_i=\operatorname{vol}\mathbb{D}.$$

Here we say $(\mathbb{D}_i)_{i\in I}$ converges to $\mathbb{D}$ if for any modification $\pi\colon Y\to X$, we have $\mathbb{D}_{i,Y}\to\mathbb{D}_Y$ with respect to the Euclidean topology.

In general, we cannot expect equality in (11.2), as shown by [DF22, Example 3.3].

Proof Let $\pi\colon Y\to X$ be a modification. Then

$$\operatorname{vol}\mathbb{D}_Y=\lim_{i\in I}\operatorname{vol}\mathbb{D}_{i,Y}\geq\varlimsup_{i\in I}\operatorname{vol}\mathbb{D}_i.$$

The inequality (11.2) follows. As for the decreasing case, it suffices to observe that both sides of (11.2) can be written as

$$\inf_i\inf_{\pi\colon Y\to X}\operatorname{vol}\mathbb{D}_{i,Y}.$$

Definition 11.1.3 A *Cartier b-divisor* $\mathbb{D}$ over X is a Weil b-divisor $\mathbb{D}$ over X such that there exists a modification $\pi\colon Y \to X$ and a class $\alpha_Y \in \mathrm{H}^{1,1}(Y,\mathbb{R})$ so that for each $\pi'\colon Z \to X$ dominating π, the class $\mathbb{D}_Z$ is the pull-back of α_Y. Any such (π, α_Y) is called a *realization* of $\mathbb{D}$.

By abuse of language, we also say (Y, α_Y) is a realization of $\mathbb{D}$. The realization is not unique in general.

Definition 11.1.4 A Cartier b-divisor $\mathbb{D}$ over X is *nef* if there exists a realization $(\pi\colon Y \to X, \alpha_Y)$ of $\mathbb{D}$ such that α_Y is nef.

Definition 11.1.5 A Weil b-divisor $\mathbb{D}$ over X is *nef* if there is a net of nef Cartier b-divisors $(\mathbb{D}_i)_i$ over X converging to $\mathbb{D}$.

In other words, for each modification $\pi\colon Y \to X$, we have $\mathbb{D}_{i,Y} \to \mathbb{D}_Y$.

Note that thanks to Proposition 1.7.1, each $\mathbb{D}_Y$ is necessarily modified nef, but it is not nef in general. The notion of modified nef classes is defined in Definition 1.7.9.

A priori, for a Cartier b-divisor, nefness could mean two different things, either defined by Definition 11.1.4 or by Definition 11.1.5. We will show in Corollary 11.1.5 that they are actually equivalent. Before that, by a nef Cartier b-divisor, we always mean in the sense of Definition 11.1.4.

Our definition Definition 11.1.5 amounts defining the set of Weil b-divisors as the closure of the set of Cartier b-divisors in $\varprojlim_\pi \mathrm{H}^{1,1}(Y,\mathbb{R})$ with respect to the projective limit topology. In particular, the limit of a converging net of nef b-divisors is still nef.

11.1.2 The b-divisors of currents

Let T be a closed positive $(1,1)$-current on X.

Given any modification $\pi\colon Y \to X$, we define

$$\mathbb{D}(T)_Y := \{\operatorname{Reg} \pi^* T\} \in \mathrm{H}^{1,1}(Y,\mathbb{R}). \tag{11.3}$$

We observe that if T' is another closed positive $(1,1)$-current on X and $\lambda \geq 0$, then

$$\mathbb{D}(T+T') = \mathbb{D}(T) + \mathbb{D}(T'), \quad \mathbb{D}(\lambda T) = \lambda \mathbb{D}(T).$$

We shall use these identities implicitly in the sequel.

Note that when T has analytic singularities, $\mathbb{D}(T)$ is Cartier.

Theorem 11.1.1 *Let T be a closed positive $(1,1)$-current on X. Then $\mathbb{D}(T)$ is nef. Moreover,*

$$\operatorname{vol} T = \operatorname{vol} \mathbb{D}(T). \tag{11.4}$$

Proof Let ω be a Kähler form on X.

Step 1. Reduce to the case where T is a Kähler current.

Note that $\mathbb{D}(\omega)$ is the Cartier b-divisor realized by $(X, \{\omega\})$. We could always approximate $\mathbb{D}(T)$ by $\mathbb{D}(T+\epsilon\omega) = \mathbb{D}(T) + \epsilon\mathbb{D}(\omega)$. Moreover, we can find a constant $C > 0$ so that for any $\epsilon > 0$,

$$0 \le \operatorname{vol}\left(\mathbb{D}(T) + \epsilon\mathbb{D}(\omega)\right) - \operatorname{vol}\mathbb{D}(T) \le C\epsilon. \tag{11.5}$$

Hence it suffices to prove our assertion with $T + \epsilon\omega$ in place of T.

Step 2. We prove the assertion under the additional assumption that T has analytic singularities.

Let $\pi\colon Y \to X$ be a modification so that

$$\pi^* T = [D] + R,$$

where D is an effective $\mathbb{Q}$-divisor on Y and R is a closed positive $(1,1)$-current with locally bounded potentials. See Theorem 1.6.1 for the existence of π. Then $\mathbb{D}(T)$ is the nef Cartier b-divisor realized by $(\pi, \{R\})$. Note that (11.4) is obvious in this case.

Step 3. We prove the assertion for a general Kähler current T. Next, we take a closed smooth real $(1,1)$-form θ in the cohomology class of T and write $T = \theta_\varphi$ for some $\varphi \in \mathrm{PSH}(X,\theta)$. Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. Then we claim that

$$\mathbb{D}(\theta + \mathrm{dd^c}\varphi_j) \to \mathbb{D}(\theta + \mathrm{dd^c}\varphi). \tag{11.6}$$

By definition of this convergence, we need to establish the following: Suppose that $\pi\colon Y \to X$ is a modification, then

$$\left\{\operatorname{Reg}\pi^*\theta + \mathrm{dd^c}\pi^*\varphi_j\right\} \to \left\{\operatorname{Reg}\pi^*\theta + \mathrm{dd^c}\pi^*\varphi\right\}.$$

This obviously follows from Theorem 6.2.4 if $\operatorname{Sing}(\pi^*T)$ has only finitely many components.

We want to show that for any $\epsilon > 0$, we can find $j_0 > 0$ so that when $j \ge j_0$,

$$\left\{\operatorname{Reg}\pi^*\theta + \mathrm{dd^c}\pi^*\varphi_j\right\} \le \left\{\operatorname{Reg}\pi^*\theta + \mathrm{dd^c}\pi^*\varphi\right\} + \epsilon\omega. \tag{11.7}$$

Write the divisorial part of $\pi^*\theta + \mathrm{dd^c}\pi^*\varphi_j$ and $\pi^*\theta + \mathrm{dd^c}\pi^*\varphi$ as

$$\sum_{i=1}^{\infty} a_i^j [E_i], \quad \sum_{i=1}^{\infty} a_i [E_i].$$

Then $a_i^j \le a_i$.

We can find $N > 0$ large enough, so that

$$\sum_{i=1}^{\infty} a_i [E_i] \le \sum_{i=1}^{N} a_i [E_i] + \frac{\epsilon}{2}\omega.$$

By Theorem 6.2.4, we can take j_0 large enough so that for $j > j_0$,

$$(a_i - a_i^j)E_i \leq \frac{\epsilon}{2N}\omega, \quad i = 1, \dots, N.$$

Then (11.7) follows, and (11.6) is established.

As a consequence, $\mathbb{D}(T)$ is nef and the volume can be computed using Lemma 11.1.1:

$$\operatorname{vol} \mathbb{D}(T) = \lim_{j\to\infty} \operatorname{vol} \mathbb{D}\left(\theta + \mathrm{dd^c}\varphi_j\right) = \lim_{j\to\infty} \operatorname{vol}\left(\theta + \mathrm{dd^c}\varphi_j\right) = \operatorname{vol} T.$$

Hence, (11.4) follows. □

Conversely, we want to realize nef b-divisors as $\mathbb{D}(T)$. We first prove a continuity result.

Proposition 11.1.2 *Let θ be a closed real smooth $(1,1)$-form on X and $(\varphi_i)_{i\in I}$ be a net in $\mathrm{PSH}(X,\theta)$ and $\varphi \in \mathrm{PSH}(X,\theta)$. Assume that $\varphi_i \xrightarrow{d_S} \varphi$, then*

$$\mathbb{D}\left(\theta_{\varphi_i}\right) \to \mathbb{D}\left(\theta_{\varphi}\right). \tag{11.8}$$

Proof Fix a modification $\pi\colon Y \to X$. It suffices to establish the following:

$$\mathbb{D}\left(\theta_{\varphi_i}\right)_Y \to \mathbb{D}\left(\theta_{\varphi}\right)_Y. \tag{11.9}$$

As a map from the pseudometric space $\mathrm{PSH}(X,\theta)$ to the finite-dimensional Euclidean space $\mathrm{H}^{1,1}(Y,\mathbb{R})$, the continuity of $\mathbb{D}(\bullet)_Y$ can be tested on sequences. So without loss of generality, we may assume that $(\varphi_i)_i$ is a sequence and $I = \mathbb{Z}_{>0}$.

Since $\pi^*\varphi_i \xrightarrow{d_S} \pi^*\varphi$, when proving (11.9), we may assume without loss of generality that π is the identity map on X. Therefore, we are reduced to the following assertion:

$$\left\{\operatorname{Reg}\theta_{\varphi_i}\right\} \to \left\{\operatorname{Reg}\theta_{\varphi}\right\}. \tag{11.10}$$

After adding a Kähler form to θ, we may also assume that θ_φ is a Kähler current. In proving (11.10), we may freely replace $\{\varphi_i\}_i$ by a subsequence. In particular, with the help of Proposition 6.2.3, we may further assume that $(\varphi_i)_i$ is either increasing or decreasing.

The decreasing case can be proved *verbatim* from the proof of (11.7), and the increasing case is very similar. □

Lemma 11.1.2 *Let $\pi\colon X \to Z$ be a proper bimeromorphic morphism from X to a Kähler manifold Z. Consider non-divisorial closed positive $(1,1)$ currents T, S on X in the same cohomology class. Assume that $T \preceq_{\mathcal{I}} S$, then $\pi_*T \preceq_{\mathcal{I}} \pi_*S$.*

Proof We may assume that π is a modification thanks to Hironaka's Chow lemma Theorem B.1.2 and Lemma 6.1.4.

By Lemma 7.3.2,

$$\pi^*\pi_*T = T + \sum_{i=1}^{N} c_i[E_i],$$

where $c_i > 0$ and the E_i's are π-exceptional divisors. It follows that

$$T + \sum_{i=1}^{N} c_i[E_i] \preceq_{\mathcal{I}} S + \sum_{i=1}^{N} c_i[E_i].$$

Replacing T and S by $T + \sum_{i=1}^{N} c_i[E_i]$ and $S + \sum_{i=1}^{N} c_i[E_i]$ respectively, we may assume that $T = \pi^*\pi_*T$. In particular, S and $\pi^*\pi_*S$ lie in the same cohomology class, and hence $S = \pi^*\pi_*S$. Our assertion then follows from Lemma 6.1.4. □

Theorem 11.1.2 *Each big and nef b-divisor $\mathbb{D}$ over X can be realized as $\mathbb{D}(T)$ for some $T \in \mathbb{D}_X$. Furthermore, we may always assume that T is $\mathcal{I}$-good.*

Note that T is not unique. The current T is necessarily non-divisorial.

Proof Fix a big and nef b-divisor $\mathbb{D}$ over X.

For each $\pi\colon Y \to X$, we take a current with minimal singularities T_Y in $\mathbb{D}_Y$. We claim that $\mathbb{D}(\pi_*T_Y)$ coincides with $\mathbb{D}$ up to the level of Y: For any modification $\pi'\colon Z \to X$ dominated by π through a morphism $g\colon Y \to Z$, we have

$$\mathbb{D}_Z = \mathbb{D}(\pi_*T_Y)_Z.$$

The notations are summarized in the following commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "\pi"'] & & Z \arrow[dl, "\pi'"] \\ & X. & \end{tikzcd} \tag{11.11}$$

After unfolding the definitions, this means

$$\mathrm{Reg}(\pi'^*\pi_*T_Y) \in \mathbb{D}_Z.$$

Note that

$$\mathrm{Reg}(\pi'^*\pi_*T_Y) = \mathrm{Reg}(\pi'^*\pi'_*g_*T_Y).$$

Due to Proposition 1.7.1 and Proposition 3.2.8, we know that $\mathbb{D}_Y$ is modified nef and big. In particular, T_Y is non-divisorial, hence so is g_*T_Y by Lemma 1.7.2. It follows from Lemma 7.3.2 that

$$\mathrm{Reg}(\pi'^*\pi'_*g_*T_Y) = \mathrm{Reg}(g_*T_Y) = g_*T_Y \in \mathbb{D}_Z.$$

Note that

$$\mathrm{vol}\, T_Y \geq \mathrm{vol}\, \mathbb{D} > 0. \tag{11.12}$$

Next we claim that the P-singularity types of the net $(\pi_*T_Y)_Y$ is decreasing.

To see this, let us fix a diagram as (11.11). We need to show that

$$\pi_*T_Y \preceq_P \pi'_*T_Z.$$

Since T_Z has minimal singularities, it is clear that $g_* T_Y \preceq_{\mathcal{I}} T_Z$. In particular, Lemma 11.1.2 guarantees that $\pi_* T_Y \preceq_{\mathcal{I}} \pi'_* T_Z$. But thanks to Corollary 7.3.3, both $\pi_* T_Y$ and $\pi'_* T_Z$ are $\mathcal{I}$-good, so there is no difference between the P-partial order and the $\mathcal{I}$-partial order in this case. Our assertion follows.

Next observe that the net $(\pi_* T_Y)_Y$ has a d_S-limit as a consequence of (11.12) and Corollary 6.2.6. Take a closed positive $(1,1)$-current $T \in \mathbb{D}_X$ such that

$$\pi_* T_Y \xrightarrow{d_S} T.$$

It follows from Proposition 11.1.2 that

$$\mathbb{D}(\pi_* T_Y) \to \mathbb{D}(T).$$

Therefore, we conclude that

$$\mathbb{D}(T) = \mathbb{D}.$$

Thanks to Theorem 11.1.1, $\operatorname{vol} T > 0$. Write $T = \theta + \mathrm{dd^c}\varphi$ for some $\varphi \in \operatorname{PSH}(X, \theta)$, then

$$T' := \theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}}$$

is $\mathcal{I}$-good, non-divisorial and $\mathbb{D}(T') = \mathbb{D}(T)$. □

Corollary 11.1.1 *Let $\mathbb{D}$ be a b-divisor over X. Then the following are equivalent:*

(1) *$\mathbb{D}$ is nef;*
(2) *for each modification $\pi \colon Y \to X$, the class $\mathbb{D}_Y$ is modified nef.*

Proof (1) $\implies$ (2). Suppose that (1) holds. Fix a modification $\pi \colon Y \to X$. Then we need to show that $\mathbb{D}_Y$ is modified nef. Since modified nefness is a closed condition, after approximating $\mathbb{D}$ by nef Cartier b-divisors, we may assume that $\mathbb{D}$ itself is a nef Cartier b-divisor. We can then find a modification $\pi' \colon Z \to X$ dominating π so that $\mathbb{D}$ is realized by a nef class α on Z. Then $\mathbb{D}_Y$ is nothing but the pushforward of α, and hence modified nef thanks to Proposition 1.7.1.

(2) $\implies$ (1). Suppose that (2) holds. We need to show that $\mathbb{D}$ is nef. Fix a Kähler form ω on X. It suffices to show that $\mathbb{D} + \epsilon\mathbb{D}(\omega)$ is nef for each $\epsilon > 0$. After replacing $\mathbb{D}$ by the latter, we may further assume that $\mathbb{D}_Y$ is big for each modification $\pi \colon Y \to X$, and $\operatorname{vol} \mathbb{D}_Y$ has a uniform positive lower bound. In this case, the argument of (11.1.2) shows that $\mathbb{D} = \mathbb{D}(T)$ for some closed positive $(1,1)$-current in $\mathbb{D}_X$. Therefore, $\mathbb{D}$ is nef, thanks to Theorem 11.1.1. □

Let α be a modified nef class on X. We write $\mathcal{G}(\alpha)$ for the set of closed positive $(1,1)$-currents T on X with $T = \operatorname{Reg} T \in \alpha$ and $\operatorname{vol} T > 0$.

Theorem 11.1.3 *There is a natural bijection from $\mathcal{G}(\alpha)/\sim_{\mathcal{I}}$ to the set of big and nef b-divisors $\mathbb{D}$ over X with $\mathbb{D}_X = \alpha$.*

Proof Given $T \in \mathcal{G}(\alpha)$, we associate the b-divisor $\mathbb{D}(T)$. It is big and nef due to Theorem 11.1.1. This map clearly descends to $\mathcal{G}(\alpha)/\sim_{\mathcal{I}}$.

This map is surjective by Theorem 11.1.2. Now we show that it is injective. Let $T, T' \in \mathcal{G}(\alpha)$. Assume that $\mathbb{D}(T) = \mathbb{D}(T')$, we want to show that $T \sim_{\mathcal{I}} T'$.

Let E be a prime divisor over X, it suffices to show that

$$\nu(T, E) = \nu(T', E). \tag{11.13}$$

We may assume that E is not a prime divisor on X, as otherwise both sides vanish.

Choose a sequence of blow-ups with smooth *connected* centers

$$Y \coloneqq X_k \longrightarrow X_{k-1} \longrightarrow \cdots \longrightarrow X_0 \coloneqq X$$

so that E is a prime divisor on Y, exceptional with respect to $X_k \longrightarrow X_{k-1}$. Denote the composition by $\pi\colon Y \longrightarrow X$. Thanks to Proposition 11.1.1,

$$\mathrm{H}^{1,1}(X_k, \mathbb{R}) = \mathrm{H}^{1,1}(X_{k-1}, \mathbb{R}) \oplus \mathbb{R}\{E_k\},$$

where $E_k = E$ is the exceptional divisor of $X_k \longrightarrow X_{k-1}$.

By induction,

$$\mathrm{H}^{1,1}(Y, \mathbb{R}) = \mathrm{H}^{1,1}(X, \mathbb{R}) \oplus \bigoplus_{i=1}^{k} \mathbb{R}\{E_i\},$$

where E_i is the exceptional divisor of $X_i \longrightarrow X_{i-1}$. Now by Lemma 7.3.2,

$$\operatorname{Reg} \pi^* T = \pi^* T - \sum_{i=1}^{k} \nu(T, E_i)[E_i]. \tag{11.14}$$

In particular, the cohomology class of $\operatorname{Reg} \pi^* T$ determines $\nu(T, E)$. Hence, (11.13) follows. □

Corollary 11.1.2 *The set of nef b-divisors over X with root α can be naturally identified with*

$$\varprojlim_{\omega} \left(\mathcal{G}(\alpha + \omega)/\!\sim_{\mathcal{I}} \right),$$

where ω runs over the directed set of Kähler forms on X (with respect to the partial order of reverse domination), and given two Kähler forms $\omega \leq \omega'$ the transition map

$$\mathcal{G}(\alpha + \omega)/\!\sim_{\mathcal{I}} \longrightarrow \mathcal{G}(\alpha + \omega')/\!\sim_{\mathcal{I}}$$

is induced by the map $\mathcal{G}(\alpha + \omega) \longrightarrow \mathcal{G}(\alpha + \omega')$ sending T to $T + \omega' - \omega$.

It is tempting to extend these results to more general currents, not necessarily non-divisorial ones. For this purpose, we introduce the following definition:

Definition 11.1.6 An *augmented nef b-divisor* over X is a pair $(\mathbb{D}, D)$, where

(1) $\mathbb{D}$ is a nef b-divisor over X;
(2) D is a formal sum

$$D = \sum_E c_E E, \quad c_E \in \mathbb{R}_{\geq 0},$$

where E runs over the set of prime divisors on X,

such that the following condition is satisfied:

$$\sum_E c_E \{E\}$$

is convergent as a sum in $\mathrm{H}^{1,1}(X, \mathbb{R})$.

The *cohomology class* of $(\mathbb{D}, D)$ is defined as

$$\mathbb{D}_X + \sum_E c_E \{E\} \in \mathrm{H}^{1,1}(X, \mathbb{R}).$$

The *volume* of $(\mathbb{D}, D)$ is defined as

$$\mathrm{vol}(\mathbb{D}, D) = \mathrm{vol}\,\mathbb{D}.$$

Recall that we defined $\mathcal{Z}_+(X, \alpha)$ as the set of closed positive $(1,1)$-currents $T \in \alpha$ in Definition 1.7.3. We introduce a further notation here:

$$\mathcal{Z}_+(X, \alpha)_{>0} := \{T \in \mathcal{Z}_+(X, \alpha) : \mathrm{vol}\,T > 0\}.$$

Corollary 11.1.3 *There is a canonical bijection between the following sets:*

(1) *The set $\mathcal{Z}_+(X, \alpha)_{>0}/\sim_{\mathcal{I}}$;*
(2) *the set of augmented nef b-divisors over X with positive volume with cohomology class α.*

More precisely, given a current $T \in \mathcal{Z}_+(X, \alpha)_{>0}$, we associate $(\mathbb{D}, D)$ as follows:

$$\mathbb{D} = \mathbb{D}(T), \quad D = \sum_E \nu(T, E) E.$$

Proof This is a direct consequence of Theorem 11.1.3 and Siu's decomposition Lemma 1.7.1. □

In fact, Corollary 11.1.3 can also be reformulated in an elementary manner, without referring to b-divisors at all. Suppose that we are given a big cohomology class α on X and a non-negative real number c_E for each prime divisor E over X. A natural question is: When is there a closed positive $(1,1)$-current $T \in \alpha$ with positive volume such that $\nu(T, E) = c_E$ for every E? Then Corollary 11.1.3 says that T exists if and only if the following two conditions hold:

(1)

$$\lim_{\pi\colon Y\to X} \operatorname{vol}\left(\pi^*\alpha - \sum_{E\subseteq Y} c_E\{E\}\right) > 0^{1},$$

where π runs over all modifications of X;

(2) for each modification $\pi\colon Y \to X$, the class $\pi^*\alpha - \sum_{E\subseteq Y} c_E\{E\}$ is modified nef.

Similarly, we have the following generalization of Corollary 11.1.2.

Corollary 11.1.4 *There is a canonical bijection between the following two sets:*

(1) *The set of*

$$\varprojlim_{\omega} \left(\mathcal{Z}_+(X, \alpha+\omega)_{>0}/\sim_{\mathcal{I}}\right),$$

where ω runs over the directed set of Kähler forms on X;

(2) *the set of augmented nef b-divisors over X with cohomology class α.*

Corollary 11.1.5 *Let $\mathbb{D}$ be a Cartier b-divisor over X. Then $\mathbb{D}$ is nef in the sense of Definition 11.1.4 if and only if it is nef in the sense of Definition 11.1.5.*

Proof We only handle the non-trivial implication. Assume that $\mathbb{D}$ is nef in the sense of Definition 11.1.5. We want to show that $\mathbb{D}$ is nef in the sense of Definition 11.1.4. We may clearly assume that $\mathbb{D}$ is big. Take a non-divisorial closed positive $(1,1)$-current T on X such that $\mathbb{D} = \mathbb{D}(T)$.

Without loss of generality, we may also assume that $\mathbb{D}$ is realized by (X, α) for some cohomology class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$. Now $\mathbb{D} = \mathbb{D}(T)$ means that for each modification $\pi\colon Y \to X$, the current π^*T is non-divisorial. In particular, T has vanishing generic Lelong number along each prime divisor over X, see (11.14). That means, T has vanishing Lelong number everywhere. It follows that $\alpha = \{T\}$ is nef. □

Corollary 11.1.6 *Let T and T' be non-divisorial closed positive $(1,1)$-currents on X. Suppose that $\{T\} = \{T'\}$, then the following are equivalent:*

(1) $\mathbb{D}(T) \leq \mathbb{D}(T')$*;*

(2) $T \preceq_{\mathcal{I}} T'$.

Proof This follows from (11.14). □

Corollary 11.1.7 *Let $\mathbb{D}$ be a nef b-divisor over X. Then there is a decreasing sequence of nef and big Cartier b-divisors $\mathbb{D}_i$ over X with limit $\mathbb{D}$.*

Proof Take a Kähler form ω on X. By Theorem 11.1.2, for each $i > 0$, we can find a non-divisorial Kähler current $T_i \in \mathbb{D}_X + i^{-1}\{\omega\}$ such that

$$\mathbb{D}(T_i) = \mathbb{D} + i^{-1}\mathbb{D}(\omega).$$

We observe that

$$T_{i+1} \sim_{\mathcal{I}} T_i.$$

This follows from applying Corollary 11.1.6 to T_i and $T_{i+1} + (i^{-1} - (i+1)^{-1})\omega$. Let $(T_i^j)_j$ be quasi-equisingular approximations of T_i such that

[1] In particular, implicitly, the sum $\sum_{E\subseteq Y} c_E\{E\}$ converges in $\mathrm{H}^{1,1}(Y, \mathbb{R})$.

(1) T_i^j is a Kähler current in $\mathbb{D}_X + i^{-1}\{\omega\}$ for $j \geq j_0(i)$, and
(2) the singularity types of $(T_i^j)_i$ is constant.

Note that (2) is possible by the using the Bergman kernel construction of the quasi-equisingular approximations.

It suffices to take $\mathbb{D}_i = \mathbb{D}(T_i^{j_i})$, where j_i is a strictly increasing sequence of positive integers with $j_i \geq j_0(i)$. □

11.2 Properties of the intersection product

Let X be a connected compact Kähler manifold of dimension n.

Definition 11.2.1 Let $\mathbb{D}_1, \ldots, \mathbb{D}_n$ be big and nef b-divisors over X. Then we define their intersection as

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) \coloneqq \operatorname{vol}(T_1, \ldots, T_n),$$

where $T_1, \ldots, T_n$ are closed positive $(1,1)$-currents in $\mathbb{D}_{1,X}, \ldots, \mathbb{D}_{n,X}$ respectively such that $\mathbb{D}(T_i) = \mathbb{D}_i$.

In general, if the $\mathbb{D}_i$'s are only nef, we define

$$(\mathbb{D}_1, \ldots, \mathbb{D}_n) \coloneqq \lim_{\epsilon \to 0+} (\mathbb{D}_1 + \epsilon\mathbb{D}(\omega), \ldots, \mathbb{D}_n + \epsilon\mathbb{D}(\omega)),$$

where ω is a Kähler form on X.

The definition makes sense thanks to Theorem 11.1.2. It does not depend on the choices of $T_1, \ldots, T_n$ since they are uniquely defined up to $\mathcal{I}$-equivalence, as proved in Theorem 11.1.3.

When $\mathbb{D}_1, \ldots, \mathbb{D}_n$ are big and nef, the two definitions coincide as follows from Lemma 11.2.1 below.

We first note that even when the T_i's have vanishing volumes, the two intersection products still agree.

Proposition 11.2.1 *Let $T_1, \ldots, T_n$ be a closed positive $(1,1)$-currents on X. Then*

$$(\mathbb{D}(T_1), \ldots, \mathbb{D}(T_n)) = \operatorname{vol}(T_1, \ldots, T_n).$$

This is a trivial consequence of the definitions.

Proposition 11.2.2

(1) *The product in Definition 11.2.1 is symmetric in its n-variable.*
(2) *Let $\mathbb{D}_1, \ldots, \mathbb{D}_n, \mathbb{D}'_1$ be nef b-divisors over X. Then*

$$\left(\mathbb{D}_1 + \mathbb{D}'_1, \ldots, \mathbb{D}_n\right) = (\mathbb{D}_1, \ldots, \mathbb{D}_n) + \left(\mathbb{D}'_1, \ldots, \mathbb{D}_n\right).$$

(3) *Let $\mathbb{D}_1, \ldots, \mathbb{D}_n$ be nef b-divisors over X and $\lambda \geq 0$. Then*

$$(\lambda\mathbb{D}_1, \ldots, \mathbb{D}_n) = \lambda\,(\mathbb{D}_1, \ldots, \mathbb{D}_n).$$

Proof This follows immediately from Proposition 7.3.1. □

Proposition 11.2.3 *The product in Definition 11.2.1 is monotonically increasing in each variable.*

Proof Let $\mathbb{D}_1, \dots, \mathbb{D}_n$ and $\mathbb{D}'$ be nef b-divisors over X so that $\mathbb{D}_1 \leq \mathbb{D}'$. We want to show that

$$(\mathbb{D}_1, \dots, \mathbb{D}_n) \leq (\mathbb{D}', \mathbb{D}_2, \dots, \mathbb{D}_n)\,.$$

We can easily reduce to the case where $\mathbb{D}_1, \dots, \mathbb{D}_n$ and $\mathbb{D}'$ are all big. In this case, take $\mathcal{I}$-good non-divisorial closed positive $(1,1)$-currents $T_1, \dots, T_n$ and T' so that $\mathbb{D}(T_i) = \mathbb{D}_i$ for all $i = 1, \dots, n$ and $\mathbb{D}(T') = \mathbb{D}'$. Furthermore, we may assume that the T_i's and T' are Kähler currents by the perturbation argument.

Let $(T_i^j)_j$ be a quasi-equisingular approximation of T_i for $i = 2, \dots, n$. It follows from Theorem 6.2.1 that

$$\int_X T_1 \wedge \dots \wedge T_n = \lim_{j\to\infty} \int_X T_1 \wedge T_2^j \wedge \dots \wedge T_n^j\,.$$

It suffices to show that for all $j \geq 1$,

$$\int_X T_1 \wedge T_2^j \wedge \dots \wedge T_n^j \leq \int_X T' \wedge T_2^j \wedge \dots \wedge T_n^j\,.$$

Therefore, we have reduced to the case where $T_2, \dots, T_n$ have analytic singularities. After a resolution, we may assume that they have log singularities along $\mathbb{Q}$-divisors. By Proposition 7.3.1(5), we can further reduce to the case where $T_2, \dots, T_n$ have bounded local potentials. Perturbing $T_2, \dots, T_n$ by a Kähler form, we may further assume that $\{T_2\}, \dots, \{T_n\}$ are Kähler classes. By Proposition 7.3.1(4), we finally reduce to the case where $T_2, \dots, T_n$ are Kähler forms. In this case, our assertion is obvious. □

Lemma 11.2.1 *Let ω be a Kähler form on X. Fix a compact set $K \subseteq \mathrm{H}^{1,1}(X, \mathbb{R})$. Let $\mathbb{D}_1, \dots, \mathbb{D}_n$ be nef b-divisors over X such that $\mathbb{D}_{i,X} \in K$ for each $i = 1, \dots, n$. Then there is a constant C depending only on $X, K, \{\omega\}$ such that for any $\epsilon \in [0, 1]$, we have*

$$0 \leq (\mathbb{D}_1 + \epsilon\mathbb{D}(\omega), \dots, \mathbb{D}_n + \epsilon\mathbb{D}(\omega)) - (\mathbb{D}_1, \dots, \mathbb{D}_n) \leq C\epsilon.$$

Proof This is a simple consequence of the linearity Proposition 11.2.2. □

We first make a consistency check.

Proposition 11.2.4 *Suppose that $\mathbb{D}$ is a nef b-divisor over X, then*

$$(\mathbb{D}, \dots, \mathbb{D}) = \operatorname{vol} \mathbb{D}.$$

Proof Using Lemma 11.2.1 and (11.5), we may easily reduce to the case where $\mathbb{D}$ is nef and big. In this case, take a non-divisorial closed positive $(1,1)$-current T in $\mathbb{D}_X$ such that $\mathbb{D}(T) = \mathbb{D}$. Then we need to show that

$$\operatorname{vol} \mathbb{D} = \operatorname{vol} T,$$

which is proved in Theorem 11.1.1. □

Proposition 11.2.5 *Let $\mathbb{D}_1, \dots, \mathbb{D}_n$ be nef b-divisors over X. Then*

$$(\mathbb{D}_1, \dots, \mathbb{D}_n) \geq \prod_{i=1}^{n} (\operatorname{vol} \mathbb{D}_i)^{1/n} .$$

Proof We may assume that $\operatorname{vol} \mathbb{D}_i > 0$ for each $i = 1, \dots, n$ since there is nothing to prove otherwise. In this case, our assertion follows from Proposition 7.3.2. □

Proposition 11.2.6 *The product in Definition 11.2.1 is upper semicontinuous in the following sense. Suppose that $(\mathbb{D}_i^j)_{j\in J}$ are nets of nef b-divisors over X with limits $\mathbb{D}_i$ for each $i = 1, \dots, n$. Then*

$$\varlimsup_{j\in J} \left(\mathbb{D}_1^j, \dots, \mathbb{D}_n^j\right) \leq (\mathbb{D}_1, \dots, \mathbb{D}_n) .$$

Proof **Step 1**. We first assume that the $\mathbb{D}_i^j$'s and the $\mathbb{D}_i$'s are all big.

Take $\mathcal{I}$-good non-divisorial closed positive $(1,1)$-currents T_i^j and T_i so that $\mathbb{D}(T_i^j) = \mathbb{D}_i^j$ and $\mathbb{D}(T_i) = \mathbb{D}_i$. Note that by our assumption and the proof of Theorem 11.1.3, for any prime divisor E over X, we have

$$\lim_{j\in J} \nu(T_i^j, E) = \nu(T_i, E).$$

So our assertion follows from Theorem 7.3.2.

Step 2. Next we handle the general case.

Take a Kähler form ω on X. Then by Lemma 11.2.1, for any $\epsilon \in (0,1]$, we have

$$\begin{aligned}
\varlimsup_{j\in J} \left(\mathbb{D}_1^j, \dots, \mathbb{D}_n^j\right) &\leq \varlimsup_{j\in J} \left(\mathbb{D}_1^j + \epsilon\mathbb{D}(\omega), \dots, \mathbb{D}_n^j + \epsilon\mathbb{D}(\omega)\right) \\
&\leq (\mathbb{D}_1 + \epsilon\mathbb{D}(\omega), \dots, \mathbb{D}_n + \epsilon\mathbb{D}(\omega)) \\
&\leq (\mathbb{D}_1, \dots, \mathbb{D}_n) + C\epsilon.
\end{aligned}$$

Since ϵ is arbitrary, our assertion follows. □

Proposition 11.2.7 *The product in Definition 11.2.1 is continuous along decreasing nets in each variable. In other words, if $(\mathbb{D}_i^j)_{j\in J}$ ($i = 1, \dots, n$) are decreasing nets of nef b-divisors over X with limits $\mathbb{D}_i$. Then*

$$\lim_{j\in J} \left(\mathbb{D}_1^j, \dots, \mathbb{D}_n^j\right) = (\mathbb{D}_1, \dots, \mathbb{D}_n) .$$

Proof This is a straightforward consequence of Proposition 11.2.3 and Proposition 11.2.6. □

Remark 11.2.1 The algebraic nef b-divisors are introduced by Dang–Favre [DF22], as their intersection theory. It is a straightforward application of the results proved in this section that our transcendental intersection theory coincides with theirs in the algebraic setting.

11.3 Okounkov bodies of b-divisors

Let X be a connected compact Kähler manifold of dimension n, let T be a closed positive $(1,1)$-current on X with $\operatorname{vol} T > 0$.

Fix a smooth flag $Y_\bullet$ on X.

Theorem 11.3.1 *The partial Okounkov body $\Delta_{Y_\bullet}(T)$ admits the following expression:*

$$\Delta_{Y_\bullet}(T) = \nu_{Y_\bullet}(T) + \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet}\left(\{\operatorname{Reg}\pi^*T\}\right), \tag{11.15}$$

where π runs over the directed set of projective birational morphisms to X with Z normal.

Here the limit is a Hausdorff limit. Recall that $\nu_{Y_\bullet}(T)$ is defined in Definition 10.2.3. The notation $\Delta_{Y_\bullet}(\{\operatorname{Reg}\pi^*T\})$ requires an explanation: Take a modification $\pi'\colon Z'\to X$ dominating $\pi\colon Z\to X$ through a map $h\colon Z'\to Z$, so that $Y_\bullet$ admits a lifting $(W_\bullet, g)$ to Z', such that Z' exists as we proved in Theorem 10.2.1, then we define

$$\Delta_{Y_\bullet}\left(\{\operatorname{Reg}\pi^*T\}\right) \coloneqq \Delta_{W_\bullet}\left(h^*\{\operatorname{Reg}\pi^*T\}\right) g^{-1}.$$

It follows from Theorem 10.4.1(3) that this definition is independent of the choice of π'. Similarly, given a current S on Z, we define

$$\nu_{Y_\bullet}(S) \coloneqq \nu_{W_\bullet}(h^*S) g^{-1}.$$

This theorem suggests that we define

$$\Delta_{Y_\bullet}\left(\mathbb{D}(T)\right) \coloneqq \lim_{\pi\colon Z\to X} \Delta_{Y_\bullet}\left(\{\operatorname{Reg}\pi^*T\}\right). \tag{11.16}$$

Then one could rewrite (11.15) as

$$\Delta_{Y_\bullet}(T) = \Delta_{Y_\bullet}\left(\mathbb{D}(T)\right) + \nu_{Y_\bullet}(T),$$

which formally resembles and extends (10.21).

Remark 11.3.1 One should be able to prove the existence of the limits like (11.16) over other base fields, at least after assuming the existence of resolution of singularities. If so, one would get an interesting extension of the theory of partial Okounkov bodies.

Lemma 11.3.1 *Let T be a closed positive $(1,1)$-current on X. Then we have*

$$\lim_{\pi\colon Z\to X} \nu_{Y_\bullet}(\mathrm{Sing}_Z(\pi^*T)) = \nu_{Y_\bullet}(T), \tag{11.17}$$

where π runs over the directed set of projective bimeromorphic morphisms to X with Z normal.

Here $\mathrm{Sing}_Z(\pi^*T)$ denotes the divisorial part of π^*T in Siu's decomposition, namely

$$\mathrm{Sing}_Z(\pi^*T) = \pi^*T - \mathrm{Reg}(\pi^*T).$$

Proof Let us write $\nu = \nu_{Y_\bullet}$ for simplicity. For the purpose of the proof, let us write $\mathcal{D}_X$ for the directed set of projective bimeromorphic morphisms $\pi\colon Z \to X$ with Z normal.

Given $\pi\colon Z \to X$, we let W_1 denote the strict transform of Y_1 in Z. The restriction $\pi_1\colon W_1 \to Y_1$ is necessarily bimeromorphic due to Zariski's main theorem Theorem B.1.1. Let $\widetilde{W_1}$ be the normalization of W_1. Let $\widetilde{\pi_1}$ denote the normalization of π_1 so that we have a commutative diagram

$$\begin{tikzcd}
\widetilde{W_1} \arrow[r] \arrow[d, "\widetilde{\pi_1}"] & W_1 \arrow[r, hook] \arrow[d, "\pi_1"] & Z \arrow[d, "\pi"] \\
Y_1 \arrow[r, equal] & Y_1 \arrow[r, hook] & X.
\end{tikzcd}$$

We will argue by induction on $n \geq 0$. The case $n = 0$ is trivial. Assume that $n > 0$ and the case $n - 1$ is known.

We may clearly assume that $\nu(T, Y_1) = 0$. By definition, we have

$$\nu(T) = \left(0, \mu(\mathrm{Tr}_{Y_1}(T))\right),$$

where μ denotes the valuation induced by the flag $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$.

Observe that bimeromorphic morphisms of the form $\pi_1\colon \widetilde{W_1} \to Y_1$ are cofinal in the directed set $\mathcal{D}_{Y_1}$. This is obvious since the modifications given by compositions of blow-ups with smooth centers on Y_1 are cofinal, and it suffices to blow-up X with the same centers.[2]

Therefore, by the inductive hypothesis applied to $\mathrm{Tr}_{Y_1} T$, we find

$$\mu\left(\mathrm{Tr}_{Y_1}(T)\right) = \lim_{\pi\colon Z\to X} \mu\left(\mathrm{Sing}_{\widetilde{W_1}}(\widetilde{\pi_1}^* \mathrm{Tr}_{Y_1} T)\right).$$

It suffices to argue that for a fixed $\pi\colon Z \to X$,

$$\nu(\mathrm{Sing}_Z(\pi^*T)) = \left(0, \mu\left(\mathrm{Sing}_{\widetilde{W_1}}\left(\widetilde{\pi_1}^* \mathrm{Tr}_{Y_1}(T)\right)\right)\right). \tag{11.18}$$

From Lemma 8.2.1, we know that

[2] It is in this inductive step that we are forced to introduce singularities, as W_1 is not smooth in general.

$$\widetilde{\pi_1}^* \operatorname{Tr}_{Y_1}(T) \sim_P \operatorname{Tr}_{W_1}(\pi^* T).$$

So we only need to prove

$$\nu(\operatorname{Sing}_Z(\pi^* T)) = \left(0, \mu(\operatorname{Sing}_{\widetilde{W_1}}(\operatorname{Tr}_{W_1}(\pi^* T))\right),$$

This is reduced to the following statement:

$$\operatorname{Tr}_{W_1} \operatorname{Sing}_Z(\pi^* T) \sim_P \operatorname{Sing}_{\widetilde{W_1}}(\operatorname{Tr}_{W_1}(\pi^* T)). \tag{11.19}$$

In order to prove this, we may add a Kähler form to T and assume that T is a Kähler current. Take a quasi-equisingular approximation $(T_j)_j$ of T. Then $(\pi^* T_j)_j$ is a quasi-equisingular approximation of $\pi^* T$. Thanks to Proposition 8.2.2, we have

$$\operatorname{Tr}_{W_1}(\pi^* T_j) \xrightarrow{d_S} \operatorname{Tr}_{W_1}(\pi^* T)$$

Using the same argument as (11.6), we finally reduce to the case where T has analytic singularities.

In this case, arguing as before, we may assume replace π by a modification dominating it so that $\pi^* T \sim_P [D]$ for an effective $\mathbb{Q}$-divisor D on Z, in which case (11.19) is clear. □

Proof (The proof of Theorem 11.3.1) We shall write $\nu = \nu_{Y_\bullet}$.

We argue by induction on n. The case $n = 0$ is of course trivial. Let us assume that $n > 0$ and the result is known in dimension $n - 1$.

We may replace T by $T - \nu(T, Y_1)[Y_1]$ and α by $\alpha - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$.

For any projective bimeromorphic morphism $\pi\colon Z \longrightarrow X$ with Z normal, it follows from Theorem 10.4.4 (which also holds for a normal variety, as can be seen after passing to a resolution) that we have

$$\Delta_{Y_\bullet}(\{\operatorname{Reg} \pi^* T\}) = \overline{\{\nu(S) : S \in \{\operatorname{Reg} \pi^* T\}\}}.$$

Here S is assumed closed and positive.

Therefore,

$$\Delta_{Y_\bullet}(\{\operatorname{Reg} \pi^* T\}) + \nu(\operatorname{Sing}_Z(\pi^* T)) \subseteq \overline{\left\{\nu(S) : S \in \{T\}, \pi^* S \geq \operatorname{Sing}_Z(\pi^* T)\right\}}.$$

We observe that the right-hand side is decreasing with respect to π, which together with Lemma 11.3.1 implies that the net of convex bodies $\Delta_{Y_\bullet}(\{\operatorname{Reg} \pi^* T\})$ for various Z is uniformly bounded. Suppose that Δ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in \{T\}, S \preceq_I T\}}.$$

As shown in Theorem 10.4.4, the right-hand side is exactly $\Delta_{Y_\bullet}(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_{Y_\bullet}(T).$$

But observe that both sides have the same volume, as computed in Theorem 10.4.2 and Theorem 11.1.1. So equality holds.

It follows from the Blaschke selection theorem Theorem C.1.1 that the limit in (11.15) exists and (11.15) holds. □

Part III
Applications

In this part, we explain a few applications of the theory developed in this book.

In Chapter 12, we develop the pluripotential theory on big line bundles on toric varieties. This theory depends crucially on the theory of partial Okounkov bodies developed in Chapter 10.

In Chapter 13, we develop the transcendental theory of non-Archimedean metrics based on the theory of test curves developed in Chapter 9.

In Chapter 14, we prove the convergence of partial Bergman measures, which relies crucially on the Riemann–Roch formula proved in Chapter 7.

The three chapters are independent of each other. Each chapter requires some prerequisites in a specific domain. More specifically, Chapter 12 requires some knowledge in toric geometry, the books [CLS11] or [Ful93] should be enough. As for Chapter 13, some knowledge in Boucksom–Jonsson's non-Archimedean pluripotential theory is highly recommended, although not logically compulsory. The long article [BJ22a] is the best reference so far. The final chapter Chapter 14 requires some knowledge in the paper [BBWN11].

Chapter 12
Toric pluripotential theory on big line bundles

C'est l'harmonie des diverses parties, leur symétrie, leur heureux balancement; c'est en un mot tout ce qui y met de l'ordre, tout ce qui leur donne de l'unité, ce qui nous permet par conséquent d'y voir clair et d'en comprendre l'ensemble en même temps que les détails.
— *Henri Poincaré*[a] , L'avenir des mathématiques

[a] Henri Poincaré (1854–1912) was a French mathematician, physicist, and philosopher of science. He is considered one of the greatest mathematicians of all time and a pioneer of several modern mathematical fields. He also played a key role in the development of special relativity, and was one of the first to understand the deep connection between mathematics and physics.

In this chapter, we develop the toric pluripotential theory on big line bundles. Our development here is based on the theory of partial Okounkov bodies developed in Chapter 10. We will deduce two non-trivial consequences from the general theory: Corollary 12.2.2 and Theorem 12.2.2.

12.1 Toric setup

Let T be a complex torus of dimension n with character lattice M and cocharacter lattice N. Some basic terminologies are recalled in Section 5.1. Recall that T_c is the compact torus contained in $T(\mathbb{C})$.

Consider a rational polyhedral fan Σ in $N_{\mathbb{R}}$ corresponding to an n-dimensional smooth projective toric variety X.

Let

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$$

be a T-invariant big divisor on X. Let $P_D \subseteq M_{\mathbb{R}}$ be the following polytope[1]

$$P_D = \left\{ m \in M_{\mathbb{R}} : \langle m, u_\rho \rangle \geq -a_\rho \quad \forall \rho \in \Sigma(1) \right\} . \tag{12.1}$$

Since we have assumed that D is big, P_D is n-dimensional.

Let $L = \mathcal{O}_X(D)$. Note that replacing D by a linearly equivalent divisor amounts to replacing D by an integral translation.

[1] Note that P_D is not necessarily a lattice polytope, see [CLS11, Example 10.5.4]. In fact, any rational polytope with positive volume can be realized in this manner.

Recall that for each $\rho \in \Sigma(1)$, u_ρ denotes the ray generator of ρ. Let $\{m_\sigma\}_{\sigma\in\Sigma}$ denote the *Cartier data* associated with D. In other words, for each $\sigma \in \Sigma$, $m_\sigma \in M$ satisfies that

$$\langle m_\sigma, u_\rho\rangle = -a_\rho, \quad \forall \rho \in \sigma(1).$$

The element $m_\sigma \in M$ is well-defined modulo

$$M(\sigma) \coloneqq \sigma^\perp \cap M, \tag{12.2}$$

where

$$\sigma^\perp \coloneqq \{m \in M_{\mathbb{R}} : \langle m, u\rangle = 0 \quad \forall u \in \sigma\}.$$

Moreover, if τ is a face of σ, then

$$m_\sigma \equiv m_\tau \mod M(\tau). \tag{12.3}$$

See [CLS11, Theorem 4.2.8]. In particular, for an n-dimensional $\sigma \in \Sigma$, the element m_σ is uniquely determined. We remind the readers that in general for a $\sigma \in \Sigma(n)$, $m_\sigma \notin P_D$. In fact, $m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$ if and only if D is base-point free. See [CLS11, Theorem 6.1.7].

Note that for any n-dimensional face σ in Σ and any $\rho \in \sigma(1)$, we have

$$\langle m - m_\sigma, u_\rho\rangle \geq 0, \quad \forall m \in P, \tag{12.4}$$

as a consequence of (12.4) and (12.1).

Recall that

$$D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma} \tag{12.5}$$

for all $\sigma \in \Sigma$, where U_σ is the affine subvariety of X corresponding to σ. See [CLS11, Proposition 4.1.2].

Next consider a T-invariant irreducible subvariety $Y \subseteq X$. Since X is smooth, so is Y. Let σ be the cone in Σ corresponding to Y. We observe that σ corresponds to a face Q_σ of P_D:

$$Q_\sigma = \left\{m \in P_D : \langle m, u_\rho\rangle = -a_\rho \quad \forall \rho \in \sigma(1)\right\}. \tag{12.6}$$

The dimension of σ is not necessarily equal to the codimension of Q as we will see in Example 12.1.2.

We have the following characterization of the base locus of D. This result is definitely known, but I am unable to find a reference.

Proposition 12.1.1 *The base locus* $\operatorname{Bs}(D)$ *(with the reduced complex structure) of D is a toric-invariant (possibly reducible) subvariety given by the union of $V(\tau)$, where τ runs over elements in Σ satisfying the following condition:*

$$a_\rho + \langle m, u_\rho\rangle > 0 \quad \text{for some } \rho \in \tau(1) \tag{12.7}$$

for each $m \in M \cap P_D$.

Here $V(\tau)$ denotes the toric subvariety of X corresponding to τ. In other words,

$$V(\tau) = \bigcap_{\rho \in \tau(1)} D_\rho.$$

Proof Recall that

$$\mathrm{H}^0(X, L) \cong \bigoplus_{m \in M \cap P_D} \mathbb{C}\chi^m.$$

See [CLS11, Proposition 4.3.3] for example. So we only need to understand the common zeros of $D + \operatorname{div}\chi^m$ for all $m \in M \cap P_D$. But we know that

$$D + \operatorname{div}\chi^m = \sum_{\rho \in \Sigma(1)} \left(a_\rho + \langle m, u_\rho \rangle\right) D_\rho.$$

Our assertion follows. □

Corollary 12.1.1 *The stable base locus (with the reduced complex structure) of D is a toric-invariant (possibly reducible) subvariety given by the union of the $V(\tau)$'s, where τ runs over elements in Σ satisfying (12.7) for each $m \in P_D$.*

Geometrically, the condition means $Q_\tau = \varnothing$.

Proof It follows from Proposition 12.1.1 that the stable base locus of D is given by the union of V_τ, where τ runs over elements Σ satisfying (12.7) for all $m \in M_{\mathbb{Q}} \cap P_D$. That is,

$$M_{\mathbb{Q}} \cap \left\{m \in P_D : a_\rho + \langle m, u_\rho \rangle = 0 \quad \forall \rho \in \tau(1)\right\} = \varnothing.$$

Since the latter part is a rational polytope, this statement is equivalent to

$$\left\{m \in P_D : a_\rho + \langle m, u_\rho \rangle = 0 \quad \forall \rho \in \tau(1)\right\} = \varnothing. \tag{12.8}$$

Our first assertion follows. □

We will keep two examples in mind.

Example 12.1.1 In this case, Σ is the fan in Fig. 12.1 consisting of three 2-dimensional cones σ_0, σ_1 and σ_2; three 1-dimensional cones σ_4, σ_5 and σ_6; one 0-dimensional cone σ_0.

The fan Σ is just the fan of $X = \mathbb{P}^2$. Under the orbit-cone correspondence, we have

$$\begin{aligned} D_{\sigma_1} &=\{[1:0:0]\}, \quad D_{\sigma_2} = \{[0:1:0]\}, \quad D_{\sigma_3} = \{[0:0:1]\},\\ D_{\sigma_4} &=\{[0:X_1:X_2] : X_1X_2 \neq 0\}, \quad D_{\sigma_5} = \{[X_0:0:X_2] : X_0X_2 \neq 0\},\\ D_{\sigma_6} &=\{[X_0:X_1:0] : X_0X_1 \neq 0\}, \quad D_{\sigma_0} = \mathbb{P}^2. \end{aligned}$$

In particular, $\Sigma(1) = \{\sigma_4, \sigma_5, \sigma_6\}$. We shall take

$$D = D_{\sigma_4}.$$

In other words,

$$a_{\sigma_5} = a_{\sigma_6} = 0, \quad a_{\sigma_4} = 1.$$

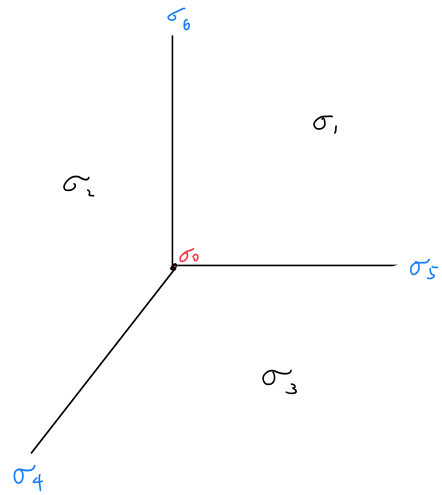

Fig. 12.1 The fan of $\mathbb{P}^2$

Note that the ray generators are given by

$$u_{\sigma_4} = (-1, -1), \quad u_{\sigma_5} = (1, 0), \quad u_{\sigma_6} = (0, 1).$$

It follows that

$$P_D = \{m = (m_1, m_2) \in \mathbb{R}^2 : m_1 + m_2 \leq 1, m_1 \geq 0, m_2 \geq 0\}.$$

Therefore, P_D is just the polytope in Fig. 12.2 . In this case, the Cartier data for

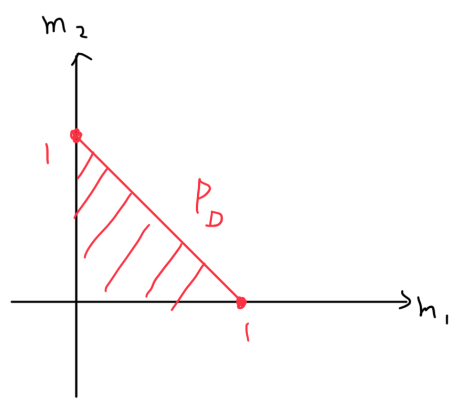

Fig. 12.2 The polytope P_D

2-dimensional cones are given as follows:

$$m_{\sigma_1} = (0,0), \quad m_{\sigma_2} = (1,0), \quad m_{\sigma_3} = (0,1);$$

while the remaining Cartier data are determined by (12.3).

In this case, $L = \mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^2}(1)$. Hence the line bundle L is ample.

We also observe that

$$\begin{aligned} Q_{\sigma_1} =& \{(0,0)\}, \quad Q_{\sigma_2} = \{(1,0)\}, \quad Q_{\sigma_3} = \{(0,1)\}, \\ Q_{\sigma_4} =& \{(m_1, m_2) : m_1 \geq 0, m_2 \geq 0, m_1 + m_2 = 1\}, \\ Q_{\sigma_5} =& \{0\} \times [0,1], \quad Q_{\sigma_6} = [0,1] \times \{0\}, \\ Q_{\sigma_0} =& P_D. \end{aligned}$$

Next we give a non-ample example.

Example 12.1.2 Let Σ be the fan shown in Fig. 12.3. Comparing with our previous

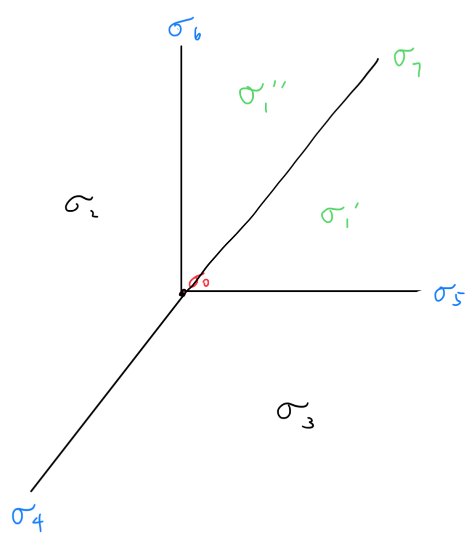

Fig. 12.3 The fan of $\mathbb{P}^2$ blown-up at the origin

example Fig. 12.1, we have divided σ_1 from the middle, giving rise to two additional 2-dimensional cones σ_1' and σ_1'', and one additional 1-dimensional cone σ_7.

The corresponding $X = \mathrm{Bl}_0\mathbb{P}^2$ is just the blow-up of $\mathbb{P}^2$ at the origin 0 and hence $L = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Let $\pi \colon X \to \mathbb{P}^2$ denote the blow-up morphism. Let

$$D = D_{\sigma_4}.$$

Then D is the pull-back of the divisor D in Example 12.1.1. Note that D is not ample, since it has degree 0 on the exceptional divisor.

In this case, we have

$$\Sigma(1) = \{\sigma_4, \sigma_5, \sigma_6, \sigma_7\},$$

and D_{σ_7} is just the exceptional divisor.

The corresponding ray generators are

$$u_{\sigma_4} = (-1,-1), \quad u_{\sigma_5} = (1,0), \quad u_{\sigma_6} = (0,1), \quad u_{\sigma_7} = (1,1),$$

while

$$m_{\sigma_1'} = m_{\sigma_1''} = (0,0), \quad m_{\sigma_2} = (1,0), \quad m_{\sigma_3} = (0,1).$$

Therefore, P_D is the same as in Fig. 12.2.

We also observe that

$$\begin{aligned}
Q_{\sigma_1'} =& \{(0,0)\}, \quad Q_{\sigma_1''} = \{(0,0)\}, \quad Q_{\sigma_7} = \{(0,0)\}\\
Q_{\sigma_2} =& \{(1,0)\}, \quad Q_{\sigma_3} = \{(0,1)\},\\
Q_{\sigma_4} =& \{(m_1,m_2) : m_1 \geq 0, m_2 \geq 0, m_1 + m_2 = 1\},\\
Q_{\sigma_5} =& \{0\} \times [0,1], \quad Q_{\sigma_6} = [0,1] \times \{0\},\\
Q_{\sigma_0} =& P_D.
\end{aligned}$$

12.2 Toric partial Okounkov bodies

We continue to use the notations in Section 12.1.

In order to study the toric-invariant singular plurisubharmonic metrics on L, we need to fix a reference toric-invariant smooth Hermitian metric, so that the psh metrics can be identified with quasi-psh functions. Unlike the ample case studied in Chapter 5, in the case of big line bundles, there does not seem to be a natural choice of a smooth Hermitian metric on L similar to Guillemin's metric.

We shall fix a T_c-invariant Hermitian metric h on L so that $\theta = c_1(L,h)$.

The first observation is the following:

Lemma 12.2.1 *There is a smooth function $F_\theta \colon N_{\mathbb{R}} \to \mathbb{R}$ such that*

$$\theta = \mathrm{dd^c}\,\mathrm{Trop}^* F_\theta \quad \text{on } T(\mathbb{C}).$$

Proof **Step 1**. We first prove the existence of F_θ for a specific choice of h.

Write D as the difference of two toric-invariant ample divisors, say $D_1 - D_2$. Let h_1 and h_2 be the associated Guillemin's metrics on D_1 and D_2 respectively. We define $h = h_1 \otimes h_2^{-1}$.

In this case, our assertion follows since it holds in the case of Guillemin's metrics.

Step 2. In general, fix h_0 as in Step 1. Then the general h can be written as $h_0 \exp(-g)$ for some smooth function g on X, but g is clearly toric-invariant. We may write $g = \mathrm{Trop}^* r$ for some smooth function r on $N_{\mathbb{R}}$. Hence it suffices to modify F_θ in Step 1 by r to conclude. □

Note that F_θ is well-defined up to a linear term.

Next, we make an additional requirement on F_θ to fix the linear term. Let s_D be a rational section of L corresponding to D. Then s_D is well-defined up to a non-zero multiple. By Lelong–Poincaré formula Proposition 1.8.1, we have

$$\mathrm{dd^c}\left(\mathrm{Trop}^* F_\theta + \log |s_D|_h^2\right) = 0$$

on $T(\mathbb{C})$. Therefore, $\mathrm{Trop}^* F_\theta + \log |s_D|_h^2$ is the tropicalization of a linear function. Hence, after adding a linear function to F_θ, we can guarantee that

$$\mathrm{Trop}^* F_\theta + \log |s_D|_h^2 = 0 \tag{12.9}$$

from now on. Note that a different choice of s_D means adding a constant to F_θ.

Summarizing the situation, we have chosen the following data in addition to the fan Σ so far: A divisor D, a Hermitian metric h on L and a rational section s_D of L subject to various conditions.

12.2.1 Newton bodies

Let $\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ be the set of T_c-invariant functions in $\mathrm{PSH}(X,\theta)$.

Definition 12.2.1 A function $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ can be written as

$$\varphi|_{T(\mathbb{C})} = \mathrm{Trop}^* f$$

for some unique function $f\colon N_\mathbb{R} \to [-\infty,\infty)$. Then we define $F_\varphi\colon N_\mathbb{R} \to \mathbb{R}$ as follows:

$$F_\varphi = F_\theta + f. \tag{12.10}$$

Observe that F_φ is a convex function and takes finite values by Lemma 5.2.1. In particular, f is also real-valued. Once D and h are fixed, F_φ is well-defined up to a constant since F_θ is.

Definition 12.2.2 Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$, we define its *Newton body* as

$$\Delta(\theta,\varphi) \coloneqq \overline{\nabla F_\varphi(N_\mathbb{R})} \subseteq M_\mathbb{R}.$$

Note that $\Delta(\theta,\varphi)$ is independent of the choice of s_D.

The Newton body $\Delta(\theta,\varphi)$ depends on the choice of D, not only on the associated line bundle L: A different choice of D inducing the same line bundle corresponds to a translation of $\Delta(\theta,\varphi)$. We will see in a while (Theorem 12.2.1) that once D is fixed $\Delta(\theta,\varphi)$ depends only on the current θ_φ. Hence, the choice of h is irrelevant.

For the moment, it is not clear if $\Delta(\theta,\varphi) \subseteq P_D$ or not. We shall prove this result using the theory of partial Okounkov bodies in the next section.

Proposition 12.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$, then*

$$\mathrm{Trop}_*\left(\theta|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}\left(F_\varphi\right). \tag{12.11}$$

In particular,

$$\int_X \theta_\varphi^n = n!\,\mathrm{vol}\,\Delta(\theta,\varphi) \tag{12.12}$$

Proof Let F_0 be a smooth convex function on $N_{\mathbb{R}}$ such that $\mathrm{dd^c}\,\mathrm{Trop}^* F_0$ can be extended to a Kähler form on X. For example, Guillemin's construction (5.5) with respect to a suitable Delzant polytope gives such an example.

Then for any large enough $C > 0$, $\theta + C\omega$ is a Kähler form. So we conclude from Proposition 5.2.5 that

$$\mathrm{Trop}_*\left((\theta + C\omega)|_{T(\mathbb{C})} + \mathrm{dd^c}\varphi|_{T(\mathbb{C})}\right)^n = \mathrm{MA}_{\mathbb{R}}(F_\varphi + CF_0).$$

Since both sides are polynomials in C, we conclude that the same holds for $C = 0$. Therefore, (12.11) follows.

(12.12) is a direct consequence of (12.11). □

12.2.2 Partial Okounkov bodies

There are some canonical choices of smooth flags in the toric setting.

Since X is smooth and projective, we could choose a full-dimensional cone σ in Σ with rays $\rho_1,\dots,\rho_n \in \sigma(1)$ such that $u_{\rho_1},\dots,u_{\rho_n}$ form a basis of N. Define

$$Y_i = D_{\rho_1} \cap \dots \cap D_{\rho_i}, \quad i = 1,\dots,n.$$

Then $Y_\bullet$ is a smooth flag on X. Let

$$\Phi\colon M \to \mathbb{Z}^n, \quad m \mapsto \left(\langle m - m_\sigma, u_{\rho_1}\rangle,\dots,\langle m - m_\sigma, u_{\rho_n}\rangle\right). \tag{12.13}$$

Then Φ is an isomorphism of lattices. It induces an $\mathbb{Z}$-affine isomorphism

$$\Phi_{\mathbb{R}}\colon M_{\mathbb{R}} \to \mathbb{R}^n.$$

Proposition 12.2.2 *We have*

$$k^{-1}\nu_{Y_\bullet}\left(\mathrm{H}^0(X, L^k)^\times\right) = \Phi_{\mathbb{R}}\left(P_D \cap k^{-1}M\right) \tag{12.14}$$

for any $k \in \mathbb{Z}_{>0}$. In particular,

$$\Delta_{Y_\bullet}(L) = \Phi_{\mathbb{R}}(P_D). \tag{12.15}$$

Recall that $\Delta_{Y_\bullet}(L) \subseteq \mathbb{R}^n$ is the Okounkov body defined in Definition 10.3.4.

Proof We first reduce to the case where $D|_{U_\sigma} = 0$. In fact, replacing D by $D + \operatorname{div} \chi^{m_\sigma}$ would result in changing P_D to $P_D - m_\sigma$. So in view of (12.5), we may assume that $D|_{U_\sigma} = 0$ and hence $m_\sigma = 0$.

Fix $k \in \mathbb{Z}_{>0}$. Let $s \in \mathrm{H}^0(X, L^k)$ be a non-zero toric-invariant section, say χ^m for some $m \in kP_D \cap M$. The zero-divisor of s on U_σ is given by

$$\sum_{i=1}^{n} \langle m, u_{\rho_i} \rangle D_{\rho_i},$$

see [CLS11, Proposition 4.1.2]. Therefore,

$$\nu_{Y_\bullet}(s) = \big(\langle m, u_{\rho_1} \rangle, \dots, \langle m, u_{\rho_n} \rangle\big) = \Phi(m).$$

So (12.14) follows. □

Example 12.2.1 Let us continue the example of $\mathbb{P}^2$ in Example 12.1.1. We use the same notations. Take σ_1 as our reference cone, and $\rho_1 = \sigma_5$, $\rho_2 = \sigma_6$. Then

$$Y_1 = \{[X_0 : 0 : X_2] : X_0 X_2 \neq 0\}, \quad Y_2 = \{[X_0 : 0 : 0] : X_0 \neq 0\}.$$

The map Φ is given by

$$\Phi(m_1, m_2) = (m_1, m_2).$$

In this case, we see easily

$$\Delta_{Y_\bullet}(\mathcal{O}_{\mathbb{P}^2}(1)) = P_D$$

is the polytope in Fig. 12.2.

Example 12.2.2 Let us continue the example of $\mathrm{Bl}_0\mathbb{P}^2$ in Example 12.1.2. This time, let us take σ_1' as our reference cone and $\rho_1 = \sigma_5$, $\rho_2 = \sigma_7$. Then Y_1 is just the strict transform of the line $\{[X_0 : 0 : X_2] : X_0 X_2 \neq 0\}$ in $\mathbb{P}^2$, while Y_2 is the point $Y_1 \cap E$, where E is the exceptional divisor.

In this case, the map Φ is given by

$$\Phi(m_1, m_2) = (m_1, m_1 + m_2).$$

We find that

$$\Delta_{Y_\bullet}(\mathrm{Bl}_0\mathbb{P}^2, \pi^* \mathcal{O}_{\mathbb{P}^2}(1))$$

is the polytope in Fig. 12.4.

Note that it differs from the polytope in Example 12.2.1.[2]

[2] Although these examples are almost trivial, they did confuse me a lot at the beginning of 2023, when Kewei Zhang, Tamás Darvas and I were collaborating on [DRWN+23]. At that time, Kewei himself already proved the main theorem for a generic flag. I realized that some simple birational geometry would suffice to prove the same result for general flags. I persuaded myself and Kewei that the Okounkov bodies are always birationally invariant, and deduced some apparently wrong conclusions. I got no clue for a couple of weeks, then one day, on the noisy metro line 7 of Paris, I

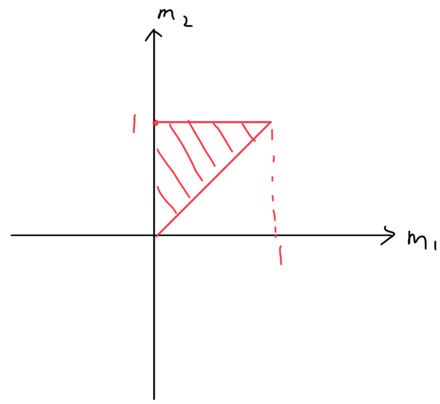

Fig. 12.4 The Okounkov body $\Delta_{Y_\bullet}(\mathrm{Bl}_0\mathbb{P}^2, \pi^*\mathcal{O}_{\mathbb{P}^2}(1))$

Theorem 12.2.1 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$, then*

$$\Phi_{\mathbb{R}}(\Delta(\theta,\varphi)) = \Delta_{Y_\bullet}(\theta,\varphi). \tag{12.16}$$

In particular,

$$n!\,\mathrm{vol}\,\Delta(\theta,\varphi) = \mathrm{vol}\,\theta_\varphi. \tag{12.17}$$

In particular, once D is fixed, the Newton body $\Delta(\theta,\varphi)$ depends only on the current θ_φ, not on the specific choices of h, φ and s_D. It makes sense to write

$$\Delta\left(\theta_\varphi\right) = \Delta(\theta,\varphi).$$

Proof We first reduce to the case where $D|_{U_\sigma} = 0$. In fact, changing D to $D + \mathrm{div}\,\chi^{m_\sigma}$ would result in changing F_θ to $F_\theta - m_\sigma$. Hence, F_φ changes to $F_\varphi - m_\sigma$. Therefore, $\Delta(\theta,\varphi)$ becomes $\Delta(\theta,\varphi) - m_\sigma$. Taking (12.5) into consideration, we may assume that $m_\sigma = 0$.

Step 1. We first reduce to the case where θ_φ is a Kähler current.

By Lemma 2.4.3, we can find $\psi \in \mathrm{PSH}(X,\theta)$ such that $\psi \leq \varphi$ and θ_ψ is a Kähler current. Taking the average along T_c, we may assume that ψ is T_c-invariant.

For each $t \in (0,1)$, we let

$$\varphi_t = (1-t)\psi + t\varphi.$$

Suppose that Kähler current case is known. Then we get

$$\Phi_{\mathbb{R}}(\Delta(\theta,\varphi_t)) = \Delta_{Y_\bullet}(\theta,\varphi_t)$$

got nothing to do, so I said to myself: Why not compute the simplest toric examples? Then after a few minutes, all of a sudden, the whole picture became completely clear.

for any $t \in (0,1)$. It follows from Theorem A.4.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \supseteq \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_t)\right) = \Delta_{Y_\bullet}(\theta,\varphi_t)$$

for any $t \in (0,1)$. Thanks to Theorem 10.3.2, we have

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \supseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1 and (10.18), we find that

$$n!\,\mathrm{vol}\,\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) = \int_X \theta_\varphi^n = \mathrm{vol}\,\theta_\varphi = n!\,\mathrm{vol}\,\Delta_{Y_\bullet}(\theta,\varphi).$$

In particular, we conclude (12.16).

Step 2. We handle the case where θ_φ is a Kähler current.

Let $(\varphi_j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$.

We may assume that φ_j is T_c-invariant for each $j \geq 1$ from the construction of [Dem12a, Theorem 13.21].

Now assume that the result is known for each φ_j. Then

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi_j)\right) = \Delta_{Y_\bullet}(\theta,\varphi_j).$$

In particular, by Proposition 12.2.1 again,

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi_j)$$

for each $j \geq 1$. It follows from Theorem 10.3.2 that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta,\varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1, (10.18) and Theorem 5.2.2, we conclude (12.16).

Step 3. It remains to handle the case where φ has analytic singularities and θ_φ is a Kähler current. In fact, we may assume that φ has the form

$$\varphi = \log\sum_{i=1}^{a} |s_i|_h^2 + O(1),$$

where $s_1,\ldots,s_a \in \mathrm{H}^0(X,L)$ are toric invariant. This follows from the proof of Step 2 and the construction of [Dem12a, Theorem 13.21].

Let $m_1,\ldots,m_a \in P_D \cap M$ be the lattice points corresponding to $s_1,\ldots,s_a$. Observe that

$$\begin{aligned}\Delta(\theta,\varphi) = \overline{\nabla F_\varphi(N_{\mathbb{R}})} &= \left\{m \in M_{\mathbb{R}} : F_\varphi(n) - \langle m,n\rangle \text{ is bounded from below}\right\}\\ &= \left\{m \in M_{\mathbb{R}} : \log\sum_{i=1}^{a} \mathrm{e}^{\langle m_i,n\rangle} - \langle m,n\rangle \text{ is bounded from below}\right\}\\ &=\mathrm{Conv}\{m_1,\ldots,m_a\},\end{aligned}$$

where we have applied (12.9) on the second line and Lemma A.5.2 on the third line. In particular, by Lemma A.5.1, let $k \in \mathbb{Z}_{>0}$, given any $m \in k\Delta(\theta, \varphi) \cap M$, we have

$$|\chi^m|^2 e^{-k\varphi}$$

is bounded from above on $T(\mathbb{C})$. In other words, the section s of L defined by m satisfies

$$s \in \mathrm{H}^0\left(X, L^k \otimes \mathcal{I}_\infty(k\varphi)\right).$$

Therefore,

$$\nu_{Y_\bullet}(s) = \Phi(m) \in k\Delta_k(\theta, \varphi),$$

where Δ_k is defined Section 10.3. Hence,

$$\Phi\left(k\Delta(\theta, \varphi) \cap M\right) \subseteq k\Delta_k(\theta, \varphi).$$

Letting $k \longrightarrow \infty$ and applying Theorem 10.3.4, we find that

$$\Phi_{\mathbb{R}}\left(\Delta(\theta, \varphi)\right) \subseteq \Delta_{Y_\bullet}(\theta, \varphi).$$

Comparing the volumes of both sides using Proposition 12.2.1 and (10.18), we conclude that the equality holds and (12.16) follows. □

The following two consequences are both due to Yi Yao.

Corollary 12.2.1 *Let E be a T-invariant prime divisor on X corresponding to a ray $\rho \in \Sigma(1)$. Then for any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$, we have*

$$\nu(\varphi, E) = \inf\left\{\langle m - m_\rho, u_\rho\rangle : m \in \Delta(\theta, \varphi)\right\}.$$

Proof This follows immediately from Theorem 12.2.1 and Theorem 10.3.5. In fact, since X is projective and smooth, there is always a T-invariant smooth flag $Y_\bullet$ with $Y_1 = E$. □

This result seems new even in the ample setting. Intuitively, after taking Theorem 12.2.2 into consideration as well, in the ample case the generic Lelong number $\nu(\varphi, E)$ is the rescaled "distance" from $\Delta(\theta, \varphi)$ to the facet of P_D corresponding to E.[3]

Corollary 12.2.2 *For any T-invariant subvariety $Y \subseteq X$ corresponding to a cone σ in Σ and any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$. Then the following are equivalent:*

(1) $\nu(\varphi, Y) = 0$;
(2) *there is a point $m \in \Delta(\theta, \varphi)$ such that $(m - m_\rho) \cdot u_\rho = 0$ for any $\rho \in \sigma(1)$;*
(3) *we have*

$$\Delta(\theta, \varphi) \cap Q_\sigma \neq \varnothing.$$

Recall that Q_σ is defined in (12.6).

[3] Be cautious! In the big setting, in general, the condition $\langle m - m_\rho, u_\rho\rangle = 0$ does not necessarily define a facet of P_D. Hence the intuition fails.

Proof (2) $\iff$ (3). This follows from the definition of Q_σ in (12.6).

(1) $\iff$ (2). Let $\rho_1, \ldots, \rho_r$ be the rays of σ. Up to replacing D by a translation, we may assume that $m_\sigma = 0$. Hence, we may take $m_{\rho_i} = 0$ for all i.

Let $\pi\colon Z \to X$ be the blow-up of X along Y. See [CLS11, Page 132] for the basic properties of the toric blow-up. Take the divisor π^*D on Z. We choose the pull-back metric π^*h on π^*L. Then $F_{\pi^*\theta}$ can be taken as π^*F_θ by (12.9). It follows $\Delta(\theta,\varphi) = \Delta(\pi^*\theta, \pi^*\varphi)$. On the other hand, the ray corresponding to the exceptional divisor E is generated by $u_{\rho_1} + \cdots + u_{\rho_r}$. Since X is smooth, this vector is primitive.

Recall that the support function of π^*D is the same as the support function of D, see [CLS11, Proposition 6.2.7]. In particular, we can take the Cartier datum $m_\rho = m_\sigma$ mod $M(\rho)$, where ρ is the ray corresponding to E.

It follows from Corollary 12.2.1 and Lemma 1.4.1 that

$$\nu(\varphi, Y) = \nu(\pi^*\varphi, E) = \inf\left\{(m - m_\sigma, u_{\rho_1} + \cdots + u_{\rho_r}) : m \in \Delta(\theta,\varphi)\right\}. \tag{12.18}$$

Our assertion follows in view of (12.4). □

It follows from (12.18) that

$$\nu(\varphi, Y) \geq \sum_{i=1}^{a} \nu(\varphi, E_i),$$

where the E_i's are the prime divisors corresponding to the rays of σ. This inequality seems to be new as well.

The following consequence of Theorem 12.2.1 is the key to the development of the toric pluripotential theory.

Theorem 12.2.2 *We have*

$$F_{V_\theta} \in \mathcal{E}^\infty(N_\mathbb{R}, P_D).^{4} \tag{12.19}$$

In particular,

$$\int_X \theta_{V_\theta}^n = n! \operatorname{vol} P_D. \tag{12.20}$$

Recall that $\mathcal{E}^\infty$ is defined in Definition A.3.1. The equation (12.19) says

$$F_{V_\theta} - \mathrm{Supp}_{P_D} \quad \text{is bounded.} \tag{12.21}$$

In particular,

$$\Delta(\theta, V_\theta) = P_D \tag{12.22}$$

and hence the Newton bodies $\Delta(\theta,\varphi)$ are all contained in P_D.

Proof Take $\varphi = V_\theta$ in Theorem 12.2.1, we find

$$\Phi_\mathbb{R}\left(\Delta(\theta, V_\theta)\right) = \Delta_{Y_\bullet}(\theta, V_\theta) = \Delta_{Y_\bullet}(L) = \Phi_\mathbb{R}(P_D),$$

[4] Initially I was only able to show $F_{V_\theta} \in \mathcal{E}(N_\mathbb{R}, P_D)$, the strengthened version was suggested by Robert Berman.

where we applied Proposition 12.2.2 in the last equality. Therefore, (12.22) follows. In particular, (12.20) follows from Proposition 12.2.1.

Next we prove (12.21). Take a smooth function $\chi\colon \mathbb{R} \to [0,1]$ so that

(1) $\operatorname{Supp}\chi \subseteq [-2,2]$;
(2) $\chi|_{[-1,1]} \equiv 1$.

Fix a T_c-invariant volume form μ on $T(\mathbb{C})$. Then it follows that $(\operatorname{Trop}^*\chi)\mu$ can be regarded as a T_c-invariant volume form on X. Take a suitable normalizing constant $C>0$, there is a unique solution

$$\theta_\varphi^n = C\,(\operatorname{Trop}^*\chi)\,\mu, \quad \sup_X \varphi = 0, \quad \varphi \in \mathcal{E}^\infty(X,\theta). \tag{12.23}$$

This is proved in [BEGZ10]. The uniqueness of φ further guarantees its T_c-invariance. Since φ has minimal singularities, $F_\varphi \sim F_{V_\theta}$. Next, using Proposition 12.2.1, the Monge–Ampère equation (12.23) implies

$$\mathrm{MA}_{\mathbb{R}}\left(F_\varphi\right) = C\chi \operatorname{Trop}_* \mu.$$

It follows from the regularity theorem [BB13, Theorem 2.19] and (12.22) that $F_\varphi \sim \operatorname{Supp}_{P_D}$. Hence, (12.21) follows. □

In particular, thanks to Corollary 12.2.1,

$$\nu(V_\theta, E) = \min\left\{\langle m - m_\rho, u_\rho\rangle : m \in P_D\right\}. \tag{12.24}$$

As an interesting consequence of (12.24), we have a geometric description of the divisorial Zariski decomposition of [Bou02b] in the toric setting: The negative part of D is given by

$$\sum_{\rho\in\Sigma(1)} \min\left\{\langle m - m_\rho, u_\rho\rangle : m \in P_D\right\} D_\rho.$$

A prime divisor D_ρ appears in the negative part if and only if the corresponding condition

$$\langle m, u_\rho\rangle \geq -a_\rho$$

is redundant in defining P_D by (12.1). The modified nef part of D is the $\mathbb{Q}$-divisor obtained after removing these redundancies.

12.3 The pluripotential theory

We continue to use the notations in Section 12.1.

Theorem 12.3.1 *There are canonical bijections*[5] *between the following sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$*;*
(2) *the set of* $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G|_{M_{\mathbb{R}} \setminus P_D} \equiv \infty.$$

The set $\mathcal{P}(N_{\mathbb{R}}, P_D)$ is defined in Definition A.3.1. As before, we write F_φ, G_φ for the functions determined by this construction.

Proof The proof is similar to that of Theorem 5.2.1, but due to its importance, we give the details. Again, the correspondence between (2) and (3) follows easily from Proposition A.2.5.

Given φ, we can construct F_φ in (2) as explained earlier in (12.10). Conversely, suppose that $F \in \mathcal{P}(N_{\mathbb{R}}, P_D)$, then $F \leq F_{V_\theta}$ by Theorem 12.2.2. Then

$$\mathrm{Trop}^* \left(F - F_\theta\right) \in \mathrm{PSH}\left(T(\mathbb{C}), \theta|_{T(\mathbb{C})}\right)$$

by Lemma 5.2.1. Since $F \leq F_{V_\theta}$, we see that $\mathrm{Trop}^*(F - F_\theta)$ is bounded from above. It follows that Grauert–Remmert's extension theorem Theorem 1.2.1 is applicable, and this function extends to a unique θ-psh function φ. The uniqueness of the extension guarantees that $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$.

The two maps are clearly inverse to each other. □

We fix a model potential $\phi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)_{>0}$ with Newton body $\Delta(\theta, \phi)$.

A similar argument guarantees the following:

Corollary 12.3.1 *There is a canonical bijection between the following sets:*

(1) *The set of* $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta; \phi)$*,*
(2) *the set of* $F \in \mathcal{P}(N_{\mathbb{R}}, \Delta(\theta, \phi))$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$G|_{M_{\mathbb{R}} \setminus \Delta(\theta, \phi)} = \infty.$$

Moreover, under these correspondences, we have the following bijections:

(1) *The set* $\mathcal{E}_{\mathrm{tor}}(X, \theta; \phi)$*,*
(2) *the set of* $F \in \mathcal{E}(N_{\mathbb{R}}, \Delta(\theta, \phi))$*, and*
(3) *the set of closed proper convex functions* $G \in \mathrm{Conv}(M_{\mathbb{R}})$ *satisfying*

$$\mathrm{Int}\{G < \infty\} = \Delta(\theta, \phi).$$

Here the notations are defined as follows:

[5] In the earlier version of this book, I required additional conditions in (2) and (3), namely $F \leq F_{V_\theta}$ and $G \geq G_{V_\theta}$ respectively. These conditions can be removed, since G_{V_θ} is always bounded, as suggested by Robert Berman. See Theorem 12.2.2.

$$\begin{aligned}\mathrm{PSH}_{\mathrm{tor}}(X,\theta;\phi) &:= \{\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta) : \varphi \leq \phi\},\\ \mathcal{E}_{\mathrm{tor}}(X,\theta;\phi) &:= \mathcal{E}(X,\theta;\phi) \cap \mathrm{PSH}_{\mathrm{tor}}(X,\theta).\end{aligned}$$

The proofs of the following results are similar to the ample case studied in Chapter 5. We omit the details.

Proposition 12.3.1 *Given $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and $C \in \mathbb{R}$. We have*

$$F_{\varphi+C} = F_\varphi + C, \quad G_{\varphi+C} = G_\varphi - C.$$

Proposition 12.3.2 *Given $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$, assume that $\varphi \wedge \psi \not\equiv -\infty$, then $\varphi \wedge \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and*

$$F_{\varphi\wedge\psi} = F_\varphi \wedge F_\psi, \quad G_{\varphi\wedge\psi} = G_\varphi \vee G_\psi.$$

Proposition 12.3.3 *Let $(\varphi_i)_{i\in I}$ be a family in $\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ uniformly bounded from above. Then $\sup_{i\in I}{}^*\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and*

$$F_{\sup_{i\in I}{}^*\varphi_i} = \bigvee_{i\in I} F_{\varphi_i}, \quad G_{\sup_{i\in I}{}^*\varphi_i} = \mathrm{cl} \bigwedge_{i\in I} G_{\varphi_i}.$$

Moreover, if I is finite, then

$$G_{\bigvee_{i\in I}\varphi_i} = \bigwedge_{i\in I} G_{\varphi_i}.$$

Similarly, if $\{\varphi_i\}_{i\in I}$ is a decreasing net in $\mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ such that $\inf_{i\in I}\varphi_i \not\equiv -\infty$, then $\inf_{i\in I}\varphi_i \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and

$$F_{\inf_{i\in I}\varphi_i} = \bigwedge_{i\in I} F_{\varphi_i}, \quad G_{\inf_{i\in I}\varphi_i} = \bigvee_{i\in I} G_{\varphi_i}.$$

Proposition 12.3.4 *Let $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$. Then $P_\theta[\varphi] \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and*

$$G_{P_\theta[\varphi]}(x) = \begin{cases} G_{V_\theta}(x), & \text{if } x \in \Delta(\theta,\varphi); \\ \infty, & \text{otherwise.}\end{cases} \tag{12.25}$$

As a consequence, we have

Corollary 12.3.2 *Let $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$. Then the following are equivalent:*

(1) $\varphi \preceq_P \psi$;
(2) $\varphi \preceq_{\mathcal{I}} \psi$;
(3) $\Delta(\theta,\varphi) \subseteq \Delta(\theta,\psi)$.

Proof (1) $\iff$ (2). This follows from the $\mathcal{I}$-goodness of φ and ψ, see Example 7.4.1.

(1) $\iff$ (3). When $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)_{>0}$, this follows from Proposition 12.3.4.

In general, fix an ample toric-invariant divisor H on X and a toric-invariant Hermitian metric h_H on $\mathcal{O}_X(H)$ with $\omega_H := \mathrm{dd^c} h_H$ being a Kähler form. We perturb L to $L + m^{-1}H$ for $m \in \mathbb{Z}_{>0}$, then thanks to Lemma A.3.1, we find that

$$\Delta(\theta + m^{-1}\omega_H, \varphi) := m^{-1}\Delta(m\theta + \omega_H, m\varphi) = \Delta(\theta, \varphi) + m^{-1}\Delta\left(\mathcal{O}_X(H)\right).$$

Then since $\varphi, \psi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta + m^{-1}\omega_H)_{>0}$, we know that $\varphi \preceq_P \psi$ if and only if

$$\Delta(\theta, \varphi) + m^{-1}\Delta\left(\mathcal{O}_X(H)\right) \subseteq \Delta(\theta, \psi) + m^{-1}\Delta\left(\mathcal{O}_X(H)\right)$$

for any $m \in \mathbb{Z}_{>0}$. The latter condition is equivalent to $\Delta(\theta, \varphi) \subseteq \Delta(\theta, \psi)$. □

Next we handle subgeodesics.

Proposition 12.3.5 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$. There is a canonical bijection between the following sets:*

(1) *The set of T_c-invariant subgeodesics from φ_0 to φ_1;*
(2) *the set of convex functions $F\colon N_{\mathbb{R}} \times (0,1) \to \mathbb{R}$ such that for each $r \in (0,1)$, the function*

$$F_r\colon N_{\mathbb{R}} \to \mathbb{R}, \quad n \mapsto F(n, r) \tag{12.26}$$

satisfies $F_r \to F_{\varphi_1}$ (resp. $F_r \to F_{\varphi_0}$) everywhere as $r \to 1-$ (resp. $r \to 0+$).

Proof We begin with a subgeodesic $(\varphi_t)_{t\in(0,1)}$ from φ_0 to φ_1. Then we define $F\colon N_{\mathbb{R}} \times (0,1) \to \mathbb{R}$ as follows:

$$F(n, t) = F_{\varphi_t}(n).$$

Define F_t as in (12.26), we have

$$\mathrm{Trop}^* F_t - \mathrm{Trop}^* F_\theta = \varphi_t, \quad t \in (0,1).$$

By definition, as $t \to 0+$, $\varphi_t \to \varphi_0$ almost everywhere. By Fubini's theorem, $F_t \to F_0$ almost everywhere, hence everywhere by Theorem A.1.2. Similarly, $F_t \to F_1$ everywhere as $t \to 1-$.

Next we show that F is convex. Let $p_1\colon X \times S \to X$ be the projection, where

$$S := \{z \in \mathbb{C} : \mathrm{e}^{-1} < |z|^2 < 1\}.$$

Since F is a subgeodesic, its complexification Φ is $p_1^*\theta$-psh. Recall that Φ is defined as

$$\Phi(x, z) = \mathrm{Trop}^*\left(F_{-\log|z|^2} - F_\theta\right)(x). \tag{12.27}$$

In particular, $\Psi\colon T(\mathbb{C}) \times S \to \mathbb{R}$ defined by

$$\Psi(x, z) := \Phi(x, z) + \mathrm{Trop}^* F_\theta(x) = \mathrm{Trop}^* F_{-\log|z|^2}(x)$$

is plurisubharmonic and $T_c \times S^1$-invariant. Fix a small enough $\epsilon > 0$, we could find a decreasing sequence of $T_c \times S^1$-invariant plurisubharmonic functions Ψ_i on $T(\mathbb{C}) \times S_\epsilon$ converging to Ψ everywhere, where

$$S_\epsilon := \{z \in \mathbb{C} : \mathrm{e}^{-1+\epsilon} < |z|^2 < \mathrm{e}{-}\epsilon\}.$$

Let us write

$$\Psi_i(x,z) = \mathrm{Trop}^* F_{i,-\log|z|^2}(x)$$

for some $F_i\colon X \times S_\epsilon \to \mathbb{R}$.

The same computation as in Lemma 5.2.1 shows that F_i is convex. It follows that F, as the decreasing limit of F_i, is also convex on $X \times (\epsilon, 1-\epsilon)$. Since $\epsilon > 0$ is arbitrary, we conclude that F is convex on $X \times (0,1)$.

Conversely, suppose that we are given F in (2). We define $\Phi\colon T(\mathbb{C}) \times S \to \mathbb{R}$ using (12.27). The arguments in the previous part can be reversed to show that Φ is $p_1^*\theta|_{T(\mathbb{C})\times S}$-psh.

By our assumption, for each $t \in (0,1)$, we have

$$F_t \leq tF_{\varphi_1} + (1-t)F_{\varphi_0} \leq F_{V_\theta} + C \tag{12.28}$$

for some constant $C \in \mathbb{R}$ independent of the choice of t. Therefore, Φ is bounded from above and hence by Theorem 1.2.1, we conclude that Φ admits a unique extension to a $p_1^*\theta$-psh extension to $X \times S$, which we still denote by Φ. We let

$$\varphi_t(x) = \Phi(x, \mathrm{e}^{-t/2})$$

for all $t \in (0,1)$ and $x \in X$. We claim that (φ_t) is a subgeodesic from φ_0 to φ_1.

For this purpose, we only need to show that $(\varphi_t)_{t\in(0,1)}$ has the correct boundary value. But from our assumption in (2), we know that as $t \to 0+$ (resp. $t \to 1-$), $\varphi_t \to \varphi_0$ (resp. $\varphi_t \to \varphi_1$) almost everywhere. In particular, $\sup_X \varphi_t \geq -C'$ for some large constant $C' > 0$ independent of $t \in (0,1)$. Therefore, together with (12.28), we deduce from Proposition 1.5.1 that $\{\varphi_t\}_{t\in(0,1)}$ is a relatively compact family with respect to the L^1-topology. We need to show that each cluster point ψ as $t \to 0+$ is equal to φ_0. But we already know that $\psi = \varphi_0$ almost everywhere. Hence we deduce $\psi = \varphi_0$ from Proposition 1.2.6. As $t \to 0+$, we have $\varphi_t \xrightarrow{L^1} \varphi_0$. Similarly, as $t \to 1-$, we have $\varphi_t \xrightarrow{L^1} \varphi_1$.

The two constructions are clearly inverse to each other.

Corollary 12.3.3 *Let $\varphi_0, \varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$. Then there is a canonical bijection between the following sets:*

(1) *The set of T_c-invariant subgeodesics from ψ_0 to ψ_1, where $\psi_0, \psi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ and $\psi_0 \leq \varphi_0$, $\psi_1 \leq \varphi_1$;*
(2) *the set of closed proper convex functions Ψ on $M_\mathbb{R} \times \mathbb{R}$ such that there is a closed proper convex function $G \in \mathrm{Conv}(M_\mathbb{R})$ satisfying*

$$G(m) + (s \vee 0) \geq \Psi(m,s) \geq G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right) \tag{12.29}$$

for all $m \in M_\mathbb{R}$ and $s \in \mathbb{R}$.

Proof Let us begin with a subgeodesic $(\psi_t)_{t\in(0,1)}$ as in (1). Let F be the convex function as in Proposition 12.3.5. We extend F to a function $F\colon N_\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows: For any $n \in N_\mathbb{R}$, we define

$$F(n,t)=\begin{cases}F_{\psi_0}(n), & \text{if } t=0,\\ F_{\psi_1}(n), & \text{if } t=1,\\ \infty, & \text{if } t>1 \text{ or } t<0.\end{cases}$$

Then F is a proper closed convex function on $N_{\mathbb{R}}\times\mathbb{R}$. Let Ψ be the Legendre transform of F. Then Ψ is a proper closed convex function on $M_{\mathbb{R}}\times\mathbb{R}$ by Theorem A.2.1. By (A.2), for any $m\in M_{\mathbb{R}}$ and $s\in\mathbb{R}$, we have

$$\begin{aligned}\Psi(m,s)=&\sup_{n\in N_{\mathbb{R}},t\in[0,1]}\left(\langle m,n\rangle+ts-F(n,t)\right)\\ =&\sup_{t\in[0,1]}\left(ts+F_t^*(m)\right).\end{aligned}$$

Therefore, the latter half of (12.29) follows. Next recall that

$$\eta\coloneqq\inf_{t\in(0,1)}\psi_t\in \mathrm{PSH}_{\mathrm{tor}}(X,\theta),$$

as follows from Proposition 4.1.2. Therefore,

$$\begin{aligned}\Psi(m,s)=&\sup_{n\in N_{\mathbb{R}},t\in[0,1]}\left(\langle m,n\rangle+ts-F(n,t)\right)\\ \leq&\sup_{n\in N_{\mathbb{R}},t\in[0,1]}\left(\langle m,n\rangle+ts-F_\eta\right)\\ =&\sup_{t\in[0,1]}ts+G_\eta(m)\\ =&(s\vee 0)+G_\eta(m).\end{aligned}$$

Conversely, let us begin with a function Ψ as in (2). Let F be the Legendre transform of Ψ. We first observe that $F(n,t)=\infty$ for all $n\in N_{\mathbb{R}}$ and $t\notin[0,1]$.

In fact,

$$\begin{aligned}F(n,t)=&\sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}}\left(\langle m,n\rangle+ts-\Psi(m,s)\right)\\ \leq&\sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}}\left(\langle m,n\rangle+ts-G_{\varphi_0}(m)\right)\\ =&\sup_{s\in\mathbb{R}}\left(ts+F_{\varphi_0}(n)\right)\\ =&\begin{cases}F_{\varphi_0}(n), & \text{if } t=0,\\ \infty, & \text{otherwise.}\end{cases}\end{aligned}$$

Similarly,

$$
\begin{aligned}
F(n,t) &= \sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}} \left(\langle m,n\rangle + ts - \Psi(m,s)\right)\\
&\leq \sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}} \left(\langle m,n\rangle + ts - G_{\varphi_1}(m) - s\right)\\
&= \sup_{s\in\mathbb{R}} \left(ts - s + F_{\varphi_1}(n)\right)\\
&= \begin{cases} F_{\varphi_0}(n), & \text{if } t=1,\\ \infty, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Therefore, we conclude that

$$
F(n,t) \leq tF_{\varphi_1} + (1-t)F_{\varphi_0}
$$

for all $t \in [0,1]$ and $n \in N_{\mathbb{R}}$. Let $(\psi_t)_{t\in(0,1)}$ be the subgeodesic defined by Proposition 12.3.5, then $(\psi_t)_{t\in(0,1)}$ satisfies (1). Next observe that

$$
\begin{aligned}
F(n,t) &= \sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}} \left(\langle m,n\rangle + ts - \Psi(m,s)\right)\\
&\geq \sup_{m\in M_{\mathbb{R}},s\in\mathbb{R}} \left(\langle m,n\rangle + ts - G(m) - s\vee 0\right)\\
&= G^*(n) + \sup_{s\in\mathbb{R}} \left(ts - (s\vee 0)\right)\\
&= \begin{cases} G^*(n), & \text{if } t\in[0,1],\\ \infty, & \text{otherwise.}\end{cases}
\end{aligned}
$$

The two operations are clearly inverse to each other. □

As an immediate corollary,

Corollary 12.3.4 *Let $\varphi_0,\varphi_1 \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)\cap\mathrm{PSH}(X,\theta)_{>0}$. Then the following are equivalent:*

(1) *$\varphi_0 \sim_P \varphi_1$;*
(2) *there is a subgeodesic from φ_0 to φ_1.*

If these conditions are satisfied, let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. Then $\varphi_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ for all $t\in(0,1)$ and

$$
G_{\varphi_t} = (1-t)G_{\varphi_1} + tG_{\varphi_0}. \tag{12.30}
$$

Proof The equivalence between (1) and (2) follows from a general fact Theorem 6.1.1. In the toric case, (2) $\implies$ (1) can also be argued more directly.

Assume that these conditions are satisfied. Let $(\varphi_t)_{t\in(0,1)}$ be the geodesic from φ_0 to φ_1. It is clear that $\varphi_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ for all $t\in(0,1)$. Let Ψ' be the proper convex function on $M_{\mathbb{R}}\times\mathbb{R}$ defined by Corollary 12.3.3. Then Ψ' is the minimum of all Ψ satisfying (12.29). We claim that

$$
\Psi'(m,s) = G_{\varphi_0}(m) \vee \left(G_{\varphi_1}(m) + s\right). \tag{12.31}
$$

It suffices to show that the right-hand side is proper, namely, $G_{\varphi_0} \vee G_{\varphi_1}$ is not identically ∞. But recall that by Proposition 4.1.2, we have $\varphi_0 \wedge \varphi_1 \in \mathrm{PSH}(X,\theta)$. Therefore, by Proposition 12.3.2,

$$G_{\varphi_0} \vee G_{\varphi_1} = G_{\varphi_0 \wedge \varphi_1} \not\equiv \infty.$$

In particular, (12.31) follows.

Now by construction,

$$G_{\varphi_t}(m) = \sup_{s\in\mathbb{R}} (st - \Psi'(m,s)) = (1-t)G_{\varphi_1}(m) + tG_{\varphi_0}(m)$$

for all $t \in (0,1)$. So (12.30) follows. □

Let $\phi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta) \cap \mathrm{PSH}(X,\theta)_{>0}$ be a model potential. We write

$$\mathcal{R}_{\mathrm{tor}}(X,\theta;\phi) \coloneqq \{\ell \in \mathcal{R}(X,\theta;\phi) : \ell_t \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta) \quad \forall t \geq 0\}.$$

Recall that $\mathcal{R}(X,\theta;\phi)$ is defined in Definition 4.2.2.

Corollary 12.3.5 *There is a canonical bijection between the following sets:*

(1) *The set of $\ell \in \mathcal{R}_{\mathrm{tor}}(X,\theta;\phi)$;*
(2) *The set of proper closed convex functions g on $M_{\mathbb{R}}$ with $\overline{\mathrm{Dom}\, g} = \Delta(\theta,\phi)$.*

Moreover, given $\ell \in \mathcal{R}_{\mathrm{tor}}(X,\theta;\phi)$, then

$$G_{\ell_t} = G_\phi + tg, \quad \forall t > 0. \tag{12.32}$$

Proof First observe that given g as in (2), (12.32) indeed induces a geodesic ray in $\mathcal{R}_{\mathrm{tor}}(X,\theta;\phi)$ thanks to Corollary 12.3.4. Therefore, we have a map from (2) to (1).

Conversely, given $\ell \in \mathcal{R}_{\mathrm{tor}}(X,\theta;\phi)$, it follows from Corollary 12.3.4 that G_{ℓ_t} is linear in $t \geq 0$ after restricted to $\mathrm{Int}\{G_\phi < \infty\}$. Let

$$g'(m) = \begin{cases} G_{\ell_1}(m) - G_\phi(m), & m \in \mathrm{Int}\{G_\phi < \infty\}; \\ \infty, & m \in M_{\mathbb{R}} \setminus \mathrm{Int}\{G_\phi < \infty\}. \end{cases}$$

Note that for $m \in \mathrm{Int}\{G_\phi < \infty\}$, we actually have

$$g'(m) = \lim_{t\to\infty} t^{-1} G_{\ell_t}(m).$$

Hence, g' is a proper convex function on $M_{\mathbb{R}}$. Define $g = \mathrm{cl}\, g'$. Then g is a proper closed convex function on $M_{\mathbb{R}}$, and $\overline{\mathrm{Dom}\, g} = \Delta(\theta,\phi)$. We have therefore a map from (1) to (2).

It remains to argue that this map is the converse of the proceeding map from (2) to (1). The non-trivial point is to verify (12.32) holds for the g we just constructed. Fix $t > 0$, we need to show that

$$G_{\ell_t} = G_\phi + tg.$$

This holds on $\mathrm{Int}\{G_\phi < \infty\}$ by Corollary 12.3.4 and the definition of g, but then it holds everywhere thanks to Proposition A.1.4. Our assertion follows. □

In particular, we can make the corresponding test curve more explicit.

Corollary 12.3.6 *Let $\ell \in \mathcal{R}_{\mathrm{tor}}(X, \theta; \phi)$ and g be the function as in Corollary 12.3.5, then*

$$\ell^*_{\max} = -\inf_{M_{\mathbb{R}}} g, \tag{12.33}$$

and for each $\tau < -\inf_{M_{\mathbb{R}}} g$, the function ℓ^*_τ is toric-invariant, and

$$G_{\ell^*_\tau}(m) = \begin{cases} G_\phi(m), & \text{if } g(m) \leq -\tau; \\ \infty, & \text{otherwise.} \end{cases}$$

In particular, for such τ,

$$\Delta\left(\theta, \ell^*_\tau\right) = \{g \leq -\tau\}.$$

In other words,

$$\left(\Delta\left(\theta, \ell^*_\tau\right)\right)_{\tau < -\inf_{M_{\mathbb{R}}} g}$$

is the inverse Legendre transform of $-g$ using the terminology of Definition 10.5.3.

As a consequence, in the toric setting, the Ross–Witt Nyström correspondence Theorem 9.2.1 reduces essentially to Theorem 10.5.1.

Proof Fix $\tau \in \mathbb{R}$, then

$$\ell^*_\tau = \inf_{t>0} (\ell_t - t\tau)$$

is clearly toric-invariant. Therefore,

$$F_{\ell^*_\tau} = \inf_{t>0} \left(F_{\ell_t} - t\tau\right).$$

Fix $m \in M_{\mathbb{R}}$, we compute

$$\begin{aligned} G_{\ell^*_\tau}(m) &= \sup_{n \in N_{\mathbb{R}}} \left(\langle m, n\rangle - F_{\ell^*_\tau}(n)\right) \\ &= \sup_{t>0} \sup_{n \in N_{\mathbb{R}}} \left(\langle m, n\rangle - F_{\ell_t}(n) + t\tau\right) \\ &= \sup_{t>0} \left(G_{\ell_t}(m) + t\tau\right) \\ &= G_\phi(m) + \sup_{t>0} t\left(g(m) + \tau\right) \\ &= \begin{cases} G_\phi(m), & \text{if } g(m) + \tau \leq 0; \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Our assertions follow. □

Next we consider the trace operator studied in Chapter 8. We wish to understand the trace operator in the toric situation. For this purpose, we will need to fix a

T-invariant subvariety $Y \subseteq X$. Let σ be the corresponding cone in Σ and Q be the corresponding face of P_D. The cocharacter lattice of Y is given by

$$N(\sigma) \coloneqq N/N \cap \langle \sigma \rangle,$$

where $\langle \sigma \rangle$ is the linear span of σ. See [CLS11, (3.2.6)]. In particular, we have a canonical identification of the character lattice $M(\sigma)$ of Y:

$$M(\sigma) = \sigma^{\perp} \cap M,$$

which is compatible with our previous notation (12.2). Let $i_\sigma \colon M(\sigma) \to M$ be the inclusion map. Let T_Y be the torus of Y. Then we have a natural surjection $q_T \colon T \to T_Y$. In particular, then tropicalization map

$$\mathrm{Trop} \colon T(\mathbb{C}) \to N_{\mathbb{R}}$$

descends to the tropicalization map of Y:

$$\mathrm{Trop}_Y \colon T_Y(\mathbb{C}) \to N(\sigma)_{\mathbb{R}}.$$

We let

$$D_Y = \sum_{\substack{\rho \in \Sigma(1) \\ \rho \not\leq \sigma}} a_\rho D_\rho|_Y,$$

where $\rho \not\leq \sigma$ means that ρ is not a face of σ. Then $\mathcal{O}_Y(D_Y) = L|_Y$.

Theorem 12.3.2 *There is a canonical choice of the Cartier datum $m_\sigma \in M$ such that for any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X, \theta)$ with $\nu(\varphi, Y) = 0$, $\mathrm{Tr}_Y^\theta(\varphi)$ is defined and $\mathrm{vol}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) > 0$[6], we have*

$$\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) = \Delta(\theta, \varphi) \cap Q_\sigma - m_\sigma$$

as subsets of $M(\sigma)_{\mathbb{R}}$.

Observe that the condition $\nu(\varphi, Y) = 0$ means exactly that $\Delta(\theta, \varphi) \cap Q_\sigma \neq \emptyset$ by Corollary 12.2.2.

Since Y itself is a smooth toric variety, the proceeding constructions of X all apply to Y. We briefly summarize the situation in Table 12.1.

Recall that $\mathrm{Star}(\sigma)$ is the fan in $N(\sigma)_{\mathbb{R}}$ consisting of $\bar{\tau}$ for all faces $\tau \in \Sigma$ containing σ, where $\bar{\tau}$ is the image of τ in $N(\sigma)_{\mathbb{R}}$. See [CLS11, Proposition 3.2.7].

Proof The idea of the proof is that since we know how the partial Okounkov bodies behave under restrictions by Lemma 10.4.7 and Remark 10.4.1, and know how to compare partial Okounkov bodies and Newton bodies Theorem 12.2.1, we should be able to deduce the behavior of Newton bodies under restriction as well.

First we note that by our assumption, $L|_Y$ is a big line bundle. In particular, if we set $r = \dim \sigma$, then $\dim Y = n - r$.

[6] Note that $\mathrm{Tr}_Y^\theta \in \mathrm{PSH}_{\mathrm{tor}}(Y, \theta|_Y)$.

Notions for X	Notions for Y
N	$N(\sigma)$
M	$M(\sigma)$
Σ	$\mathrm{Star}(\sigma)$
D	D_Y
L	$L\vert_Y$
h	$h\vert_Y$
θ	$\theta\vert_Y$
Trop	Trop_Y
P_D	Q_σ
s_D	s_{D_Y}

Table 12.1 The correspondence between X and Y

For this purpose, let σ^0 be an n-dimensional face of Σ containing σ. The image $\overline{\sigma^0}$ in $N(\sigma)$ is then an r-dimensional face of $\mathrm{Star}(\sigma)$. We shall use these faces as the reference faces while defining the partial Okounkov bodies.

We list the rays in $\sigma^0(1)$ as follows:

$$\rho_1, \ldots, \rho_n, \tag{12.34}$$

where $\rho_1, \ldots, \rho_r \in \sigma(1)$ and hence $\rho_{r+1}, \ldots, \rho_n \notin \sigma(1)$. In particular, the images

$$\overline{\rho_{r+1}}, \ldots, \overline{\rho_n} \tag{12.35}$$

of the latter give a list of $\overline{\sigma^0}(1)$.

We construct the flag $Y_\bullet$ on X using the rays (12.34) and the flag $Z_\bullet$ on Y using the rays (12.35). Note that

$$Z_i = Y_{r+i},$$

where $i = 1, \ldots, n-r$.

Next we compute the Cartier data associated with $\overline{\sigma^0}$. By definition, $m_{\overline{\sigma_0}} \in M(\sigma)$ is the unique element satisfying

$$m_{\overline{\sigma_0}} \cdot u_{\rho_j} = -a_{\rho_j}$$

for all $j = r+1, \ldots, n$.

Let $\Phi\colon M \to \mathbb{Z}^n$ and $\Psi\colon M(\sigma) \to \mathbb{Z}^{n-r}$ be defined as

$$\begin{aligned}
\Phi(m) &= \left(\langle m - m_{\sigma^0}, u_{\rho_1}\rangle, \ldots, \langle m - m_{\sigma^0}, u_{\rho_n}\rangle\right) \\
&= \left(\langle m, u_{\rho_1}\rangle, \ldots, \langle m, u_{\rho_n}\rangle\right) + (a_{\rho_1}, \ldots, a_{\rho_n}), \\
\Psi(m) &= \left(\langle m - m_{\overline{\sigma_0}}, u_{\overline{\rho_1}}\rangle, \ldots, \langle m - m_{\overline{\sigma_0}}, u_{\overline{\rho_{n-r}}}\rangle\right)
\end{aligned}$$

Observe that for $i = r+1, \ldots, n$, we have

$$u_{\overline{\rho_i}} = u_{\rho_i} \mod N \cap \langle \sigma \rangle,$$

so

$$\Psi(m) = \left(\langle m, u_{\rho_{r+1}}\rangle, \ldots, \langle m, u_{\rho_n}\rangle\right) + (a_{\rho_{r+1}}, \ldots, a_{\rho_n})$$

for $m \in M(\sigma)$. Therefore, we have a commutative diagram

$$\begin{tikzcd} M_{\mathbb{R}} \arrow[r, "\Phi_{\mathbb{R}}"] & \mathbb{R}^n \arrow[d, "L"] \\ M(\sigma)_{\mathbb{R}} \arrow[u, hook] \arrow[r, "\Psi_{\mathbb{R}}"] & \mathbb{R}^{n-r}, \end{tikzcd}$$

where $L\colon \mathbb{R}^n \to \mathbb{R}^{n-r}$ is the map

$$(b_1, \ldots, b_n) \mapsto (b_{r+1}, \ldots, b_n).$$

By Theorem 12.2.1, we have

$$\Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi)\right) = \Delta_{Y_\bullet}(\theta,\varphi), \quad \Psi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi))\right) = \Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)).$$

The latter can be written as

$$\Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) = L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi))\right).$$

While by Lemma 10.4.7 and Remark 10.4.1,

$$\begin{aligned}\Delta_{Z_\bullet}(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) =& L\left(\Delta_{Y_\bullet}(\theta,\varphi) \cap (\{0\}^r \times \mathbb{R}^{n-r})\right)\\ =& L\left(\Phi_{\mathbb{R}}(\Delta(\theta),\varphi) \cap (\{0\}^r \times \mathbb{R}^{n-r})\right)\\ =& L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi) \cap Q_\sigma\right).\end{aligned}$$

Hence,

$$L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi))\right) = L \circ \Phi_{\mathbb{R}}\left(\Delta(\theta,\varphi) \cap Q_\sigma\right).$$

It follows that

$$\Delta(\theta|_Y, \mathrm{Tr}_Y^\theta(\varphi)) + m_{\sigma^0} - m_{\overline{\sigma^0}} = \Delta(\theta,\varphi) \cap Q_\sigma.$$

Finally, observe that $m_{\sigma^0} - m_{\overline{\sigma^0}}$ represents m_σ. Our assertion follows. □

Corollary 12.3.7 *For any $\varphi \in \mathrm{PSH}_{\mathrm{tor}}(X,\theta)$ with $\nu(\varphi, Y) = 0$, we have*

$$\mathrm{vol}\left(\theta + \mathrm{dd^c}\,\mathrm{Tr}_Y^\theta(\varphi)\right) = (\dim Y)!\,\mathrm{vol}_{\dim Y}\left(\Delta(\theta,\varphi) \cap Q_\sigma\right). \tag{12.36}$$

The left-hand side of (12.36) is understood as 0 if $\mathrm{Tr}_Y^\theta(\varphi)$ is not defined. On the right-hand side, $\mathrm{vol}_{\dim Y}$ is the $\dim Y$-dimensional Lebesgue measure on $M(\sigma)_{\mathbb{R}}$ normalized so that the unique cube in $M(\sigma)$ has volume 1.

Proof When $\mathrm{Tr}_Y^\theta(\varphi)$ is defined and has positive volume, this follows immediately from Theorem 12.3.2.

Next we consider the case where $\operatorname{Tr}_Y^{\theta}(\varphi)$ is defined by has 0-volume or is not defined. Take a toric-invariant ample line bundle H on X and a toric-invariant Kähler metric $\omega \in c_1(H)$. Then for any $\epsilon \in \mathbb{Q}_{>0}$, we have

$$\operatorname{vol}\left(\theta + \epsilon\omega + \mathrm{dd^c} \operatorname{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right) = (\dim Y)! \operatorname{vol}_{\dim Y}\left(\Delta(\theta+\epsilon\omega, \varphi) \cap Q_\sigma\right).$$

Thanks to Example 8.1.6, we have

$$\operatorname{vol}\left(\theta + \mathrm{dd^c} \operatorname{Tr}_Y^{\theta}(\varphi)\right) = \lim_{\mathbb{Q}\ni\epsilon\to 0+} \operatorname{vol}\left(\theta + \epsilon\omega + \mathrm{dd^c} \operatorname{Tr}_Y^{\theta+\epsilon\omega}(\varphi)\right) = 0.$$

Combining these equations, we find (12.36). □

As a corollary, we obtain an elegant characterization of the augmented base locus in the toric setting.

Corollary 12.3.8 *The augmented base locus (with the reduced complex structure) of D is a toric-invariant (possible reducible) subvariety given by the union of the $V(\tau)$'s, where τ runs over the elements in Σ such that*

$$\dim Q_\tau < n - \dim \tau\,^{7}.$$

Intuitively, we should think of $n - \dim \tau$ as the *expected* dimension of Q_τ. This corollary says that Q_τ fails to attain the expected dimension if and only if it corresponds to a subvariety in the augmented base locus.

With the help of some knowledge in convex geometry, we can also deduce this corollary from Corollary 12.1.1.

Proof We simply apply Corollary 12.3.7 applied to $\varphi = V_\theta$. This corollary follows from Nakamaye's theorem [ELM+09, Theorem 5.7] and Theorem 8.3.1. □

Example 12.3.1 Let us consider the example Example 12.1.2 again, we consider the subvariety corresponding to σ_7. We have

$$Q_{\sigma_7} = \{m \in P_D : \langle m, (1,1)\rangle = 0\}.$$

The situation is explained in Fig. 12.5. We find that $\dim Q_\sigma = 0$, but the expected dimension is $2 - 1 = 1$. So the corresponding divisor, namely the exceptional divisor on $\mathrm{Bl}_0\mathbb{P}^2$ is in the non-Kähler locus. Similarly, we can verify that the non-Kähler locus consists only of this divisor.

[7] Here we understand that $\dim \varnothing = -\infty$.

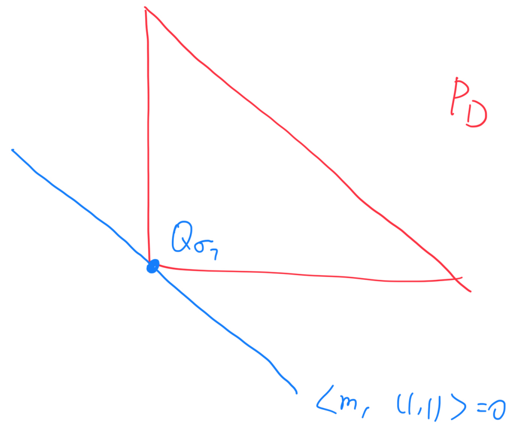

Fig. 12.5 The image of Q_{σ_7}.

Chapter 13
Non-Archimedean pluripotential theory

A good theorem lasts forever. Once proved, it will always stay proved, and other mathematicians are free to use it and build on it as they please, sometimes to great effect.
— John Tate[a]

[a] John Torrence Tate Jr. (1925–2019), the grandfather of Dustin Clausen, was one of the greatest minds in the whole history of America. However, his aversion to publishing papers arguably impeded the progress of the development of mathematics to some extent. For example, his foundational work on rigid non-Archimedean geometry was written in 1962, but was not available to the public until 1971.

In this chapter, we will establish the non-Archimedean pluripotential theory using the theory of $\mathcal{I}$-good singularities and test curves. We show that our theory extends the algebraic theory à la Boucksom–Jonsson in Section 13.4.

We also construct the Duistermaat–Heckman measure of a non-Archimedean metric in Section 13.3.

There is also a closely related theory developed by Mesquita-Piccione [MP24], where a Berkovich like analytification of a compact Kähler manifold is constructed. We refer to the well-written paper [MP24] for the details and the comparisons with the theory developed in the current chapter.

13.1 The definition of non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n. Let $\mathrm{Käh}(X)$ be the set of Kähler forms on X with the partial order given as follows: We say $\omega \leq \omega'$ if $\omega \geq \omega'$. Note that the partially ordered set $\mathrm{Käh}(X)$ is a directed set.

Let θ be a closed smooth real $(1,1)$-form.

Definition 13.1.1 We define

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) = \varprojlim_{\omega \in \mathrm{Käh}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}\,^{1}$$

in the category of sets, where the transition maps are given as follows: Suppose that $\omega, \omega' \in \mathrm{Käh}$ and $\omega \geq \omega'$, then the transition map is defined in Proposition 9.3.4:

[1] The annoying projective limit can be avoided if instead of relying the language of quasi-plurisubharmonic functions, we use that of augmented nef b-divisors developed in Definition 11.1.6 instead. But given the fact that there two formulations are completely equivalent to each other by Corollary 11.1.4, we just stick to the slightly more traditional language here.

$$P_{\theta+\omega}[\bullet]_{\mathcal{I}} : \mathrm{PSH}^{\mathrm{NA}}(X, \theta+\omega')_{>0} \to \mathrm{PSH}^{\mathrm{NA}}(X, \theta+\omega)_{>0}. \tag{13.1}$$

Recall that $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ is defined in Definition 9.3.1.

In general, when we denote an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ by Γ, its component in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}$ ($\omega \in \mathrm{K\ddot{a}h}(X)$) will be written as either Γ^{ω} or $P_{\theta+\omega'}[\Gamma]_{\mathcal{I}}$.

Note that $\Gamma^{\omega}_{\max}$ is independent of the choice of $\omega \in \mathrm{K\ddot{a}h}(X)$. We denote this common value by $\Gamma_{\max}$.

Remark 13.1.1 Thanks to Proposition 9.3.2, for any other θ' representing $[\theta]$, we have a canonical bijection

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(X,\theta').$$

Moreover, these bijections satisfy the cocycle condition. If we view the set of closed real smooth $(1,1)$-forms representing $[\theta]$ as a category with a unique morphism between any two objects, then we can define

$$\mathrm{PSH}^{\mathrm{NA}}(X,[\theta]) = \varprojlim_{\theta} \mathrm{PSH}^{\mathrm{NA}}(X,\theta).$$

This definition is independent of the choice of the explicit representative of the cohomology class $[\theta]$.

However, given the fact that our notations are already quite heavy, we decide to stick to the set $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. The readers should verify that all constructions below are independent of the choice of θ within its cohomology class.

Proposition 13.1.1 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Then given $\omega,\omega' \in \mathrm{K\ddot{a}h}(X)$ with $\omega \geq \omega'$, we have*

$$P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{-\infty}\right] = P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{-\infty}\right]_{\mathcal{I}} = \Gamma^{\theta+\omega}_{-\infty}.$$

Proof Since for any $\tau < \Gamma_{\max}$, the potential $\Gamma^{\theta+\omega'}_{\tau}$ is $\mathcal{I}$-good by Example 7.1.2, it follows that

$$P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{\tau}\right] = P_{\theta+\omega}\left[\Gamma^{\theta+\omega'}_{\tau}\right]_{\mathcal{I}} = \Gamma^{\theta+\omega}_{\tau}$$

for all $\tau < \Gamma_{\max}$. Our assertion follows from Proposition 3.1.11 and Proposition 3.2.14. □

Proposition 13.1.2 *There is a natural injective map*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \hookrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta), \quad \Gamma \mapsto \left(P_{\theta+\omega}[\Gamma]_{\mathcal{I}}\right)_{\omega\in \mathrm{K\ddot{a}h}(X)}.$$

In the sequel, we will not distinguish an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ with its image in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Note that given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$, the value of $\Gamma_{\max}$ does not depend on if we view it as an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ or in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$.

Proof It is obvious that this map is well-defined. It suffices to argue its injectivity. Suppose that $\Gamma,\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} = P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$$

for some Kähler form ω on X. Then $\Gamma_{\max} = \Gamma'_{\max}$ and for any $\tau < \Gamma_{\max}$, we have

$$\Gamma_\tau \sim_{\mathcal{I}} \Gamma'_\tau$$

by Proposition 6.1.3. It follows again from Proposition 6.1.3 that

$$\Gamma_\tau = \Gamma'_\tau.$$

Definition 13.1.2 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define its *volume* as follows:

$$\operatorname{vol}\Gamma := \lim_{\omega\in \mathrm{K\ddot{a}h}(X)} \int_X \left(\theta + \omega + \mathrm{dd^c}\Gamma_{-\infty}^{\theta+\omega}\right)^n \in [0,\infty).$$

Observe that the net is decreasing, so the limit exists.

Proposition 13.1.3 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. *Then*

$$\operatorname{vol}\Gamma = \int_X \left(\theta + \mathrm{dd^c}\Gamma_{-\infty}\right)^n .$$

Proof This follows from Proposition 3.1.10, Corollary 3.1.3 and Proposition 9.1.5.□

Lemma 13.1.1 *The image of the canonical injection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0} \hookrightarrow \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$$

is given by the set of $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *with positive volume.*

Proof By Proposition 13.1.3, it is clear that the image of an element in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ has positive volume.

Conversely, take $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with positive volume. We want to construct $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ representing Γ.

Fix a Kähler form ω on X. Define

$$\Gamma'_\tau := \lim_{k\to\infty} \Gamma_\tau^{\theta+k^{-1}\omega}, \quad \tau < \Gamma_{\max}. \tag{13.2}$$

We claim that it suffices to show

$$\lim_{k\to\infty} \int_X \left(\theta + k^{-1}\omega + \mathrm{dd^c}\Gamma_\tau^{\theta+k^{-1}\omega}\right)^n > 0 \tag{13.3}$$

for some $\tau < \Gamma_{\max}$. If this holds, then the argument of Lemma 9.1.1 implies that the same holds for all $\tau < \Gamma_{\max}$. Then Proposition 3.1.10 guarantees that $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and represents Γ.

It remains to argue (13.3). Let $\epsilon = \operatorname{vol}\Gamma > 0$. Take $\tau < \Gamma_{\max}$ so that

$$\int_X \left(\theta+\omega+\mathrm{dd^c}\Gamma^{\theta+\omega}_{-\infty}\right)^n - \int_X \left(\theta+\omega+\mathrm{dd^c}\Gamma^{\theta+\omega}_{\tau}\right)^n < \epsilon/2.$$

Expanding the left-hand side using the binomial expansion, in view of Proposition 9.1.5, we find that for any $k \geq 1$,

$$\int_X \left(\theta+k^{-1}\omega+\mathrm{dd^c}\Gamma^{\theta+k^{-1}\omega}_{-\infty}\right)^n - \int_X \left(\theta+k^{-1}\omega+\mathrm{dd^c}\Gamma^{\theta+k^{-1}\omega}_{\tau}\right)^n < \epsilon/2.$$

Therefore, (13.3) follows. □

Example 13.1.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we can define an associated non-Archimedean metric $\Gamma^\varphi \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as follows:

(1) $\Gamma^\varphi_{\max} = 0$;
(2) for any $\omega \in \mathrm{K\ddot{a}h}(X)$ and any $\tau < 0$, we set

$$\Gamma^{\varphi,\theta+\omega}_\tau = P_{\theta+\omega}[\varphi]_{\mathcal{I}}.$$

Such non-Archimedean metrics are called *homogeneous* non-Archimedean metrics.

Observe that

$$\mathrm{vol}\,\Gamma^\varphi = \mathrm{vol}\,\theta_\varphi.$$

See Proposition 7.3.1 and the footnote there.

Definition 13.1.3 Let ω be a closed real smooth positive $(1,1)$-form on X. We define the map

$$P_{\theta+\omega}[\bullet]_{\mathcal{I}} : \mathrm{PSH}^{\mathrm{NA}}(X,\theta) \to \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$$

as follows: Given $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, we define $P_{\theta+\omega}[\Gamma]_{\mathcal{I}}$ as the element such that for any $\omega' \in \mathrm{K\ddot{a}h}(X)$, we have

$$P_{\theta+\omega}[\Gamma]^{\theta+\omega+\omega'}_{\mathcal{I}} = \Gamma^{\theta+\omega+\omega'}.$$

It is straightforward to check that under the identification of Proposition 13.1.2, the map $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ extends the map (13.1).

Proposition 13.1.4 *The maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 together induce a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega\in\mathrm{K\ddot{a}h}(X)} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega). \tag{13.4}$$

Proof It is a tautology that the maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 are compatible with the transition maps. So the map (13.4) is well-defined. It is injective by the same argument as Proposition 13.1.2. We argue the surjectivity.

By unfolding the definitions, an object in the target of (13.4) is an assignment: With each $\omega \in \mathrm{K\ddot{a}h}(X)$, we associate a family $(\Gamma^{\omega,\omega'})_{\omega'\in\mathrm{K\ddot{a}h}(X)}$ satisfying:

(1) $\Gamma^{\omega,\omega'} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega+\omega')_{>0}$ for each $\omega,\omega' \in \mathrm{K\ddot{a}h}(X)$;

(2) for each $\omega, \omega', \omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega'' \geq \omega'$, we have

$$P_{\theta+\omega+\omega''}\left[\Gamma^{\omega,\omega'}\right]_{\mathcal{I}} = \Gamma^{\omega,\omega''};$$

(3) for each $\omega, \omega', \omega'' \in \mathrm{K\ddot{a}h}(X)$ satisfying $\omega \leq \omega'$, we have

$$P_{\theta+\omega'+\omega''}\left[\Gamma^{\omega,\omega''}\right]_{\mathcal{I}} = \Gamma^{\omega',\omega''}.$$

The preimage of such an object is given by the family $(\Gamma^{\omega})_{\omega\in\mathrm{K\ddot{a}h}(X)}$ given by

$$\Gamma^{\omega} = \Gamma^{\omega/2,\omega/2}.$$

The fact that the image of Γ is as expected is a tautology, which we leave to the readers. □

With an almost identical argument involving Proposition 3.1.10, we get

Proposition 13.1.5 *The maps $P_{\theta+\omega}[\bullet]_{\mathcal{I}}$ in Definition 13.1.3 and the injective maps Proposition 13.1.2 together induce bijections*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0} \xrightarrow{\sim} \varprojlim_{\omega} \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega), \tag{13.5}$$

where ω runs over either the partially ordered set of all smooth closed real positive $(1,1)$-forms with positive volume[2] *on X or* $\mathrm{K\ddot{a}h}(X)$.

Corollary 13.1.1 *Let $\pi\colon Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then π^* induces a bijection*

$$\mathrm{PSH}^{\mathrm{NA}}(X,\theta) \xrightarrow{\sim} \mathrm{PSH}^{\mathrm{NA}}(Y,\pi^*\theta).$$

Proof This follows immediately from Proposition 13.1.5. □

It is immediate to verify that π^* in Corollary 13.1.1 extends the map Proposition 9.3.3.

13.2 Operations on non-Archimedean metrics

Let X be a connected compact Kähler manifold of dimension n and $\theta, \theta', \theta''$ be closed real smooth $(1,1)$-forms on X representing big cohomology classes.

This section relies heavily on Section 9.4. We shall use the notions introduced in that section without further explanations.

Definition 13.2.1 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$. We say $\Gamma \leq \Gamma'$ if for some $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$\Gamma^{\theta+\omega} \geq \Gamma'^{\theta+\omega}.$$

[2] This partially ordered set is not directed.

This notion is independent of the choice of ω thanks to Lemma 9.4.1.

Moreover, we have the following:

Proposition 13.2.1 *Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and ω be a closed smooth positive $(1,1)$-form on X, then the following are equivalent:*

(1) $\Gamma \leq \Gamma'$;
(2) $P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \leq P_{\theta+\omega}[\Gamma']_{\mathcal{I}}$.

Proof This follows immediately from Lemma 9.4.1. □

Observe that this definition coincides with the corresponding definition in Definition 9.4.1 when $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$.

Proposition 13.2.2 *Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. Assume that $\Gamma \leq \Gamma'$, then*

$$\operatorname{vol} \Gamma \leq \operatorname{vol} \Gamma'.$$

Proof It suffices to show that for any Kähler form ω on X, we have

$$\operatorname{vol} \Gamma_{-\infty}^{\theta+\omega} \leq \operatorname{vol} \Gamma'^{\theta+\omega}_{-\infty},$$

which is an immediate consequence of Theorem 2.4.4. □

Definition 13.2.2 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta')$. Then we define $\Gamma + \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta + \theta')$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$(\Gamma + \Gamma')^{\theta+\theta'+2\omega} = \Gamma^{\theta+\omega} + \Gamma'^{\theta'+\omega}.$$

This definition yields an element in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta + \theta')$ by Lemma 9.4.3 and it extends the definition in Definition 9.4.2 by Lemma 9.4.3 as well.

Proposition 13.2.3 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta')$. Suppose that ω, ω' are two smooth closed positive $(1,1)$-forms on X. Then*

$$P_{\theta+\omega+\theta'+\omega'}[\Gamma + \Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + P_{\theta'+\omega'}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.3. □

Proposition 13.2.4 *The operation + is commutative and associative: For any $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta')$ and $\Gamma'' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta'')$, we have*

$$\Gamma + \Gamma' = \Gamma' + \Gamma, \quad (\Gamma + \Gamma') + \Gamma'' = \Gamma + (\Gamma' + \Gamma'').$$

Proof This is a direct consequence of Proposition 9.4.1. □

Definition 13.2.3 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $C \in \mathbb{R}$. We define $\Gamma + C \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(X)$, we have

$$(\Gamma + C)^{\theta+\omega} = \Gamma^{\theta+\omega} + C.$$

It is obvious from Definition 9.4.3 that $\Gamma + C \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. It is also obvious that this definition extends Definition 9.4.3.

Proposition 13.2.5 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ *and* $C \in \mathbb{R}$. *Suppose that* ω *is a smooth closed positive* $(1, 1)$*-form on* X. *Then*

$$P_{\theta+\omega}[\Gamma]_{\mathcal{I}} + C = P_{\theta+\omega}[\Gamma + C]_{\mathcal{I}}.$$

Proof This is clear by definition. □

Proposition 13.2.6 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta')$ *and* $C, C' \in \mathbb{R}$, *then*

(1) $(\Gamma + \Gamma') + C = \Gamma + (\Gamma' + C) = (\Gamma + C) + \Gamma'$;
(2) $\Gamma + (C + C') = (\Gamma + C) + C'$.

Proof This is a direct consequence of Proposition 9.4.2. □

Proposition 13.2.7 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ *and* $C \in \mathbb{R}$, *then*

$$\mathrm{vol}\, \Gamma = \mathrm{vol}(\Gamma + C).$$

Proof It suffices to show that for each Kähler form ω on X,

$$\mathrm{vol}\, \Gamma^{\theta+\omega}_{-\infty} = \mathrm{vol}(\Gamma + C)^{\theta+\omega}_{-\infty},$$

which is obvious. □

Definition 13.2.4 Let $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, we define $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ as the unique element such that for any $\omega \in \mathrm{Käh}(X)$, we have

$$(\Gamma \vee \Gamma')^{\theta+\omega} = \Gamma^{\theta+\omega} \vee \Gamma'^{\theta+\omega}.$$

It follows from Lemma 9.4.5 that $\Gamma \vee \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and this definition extends the corresponding definition in Definition 9.4.4.

Proposition 13.2.8 *Let* $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ *and* ω *be a closed smooth positive* $(1, 1)$*-form on* X. *Then*

$$P_{\theta+\omega}[\Gamma \vee \Gamma']_{\mathcal{I}} = P_{\theta+\omega}[\Gamma]_{\mathcal{I}} \vee P_{\theta+\omega}[\Gamma']_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.5. □

Proposition 13.2.9 *The operation* $\vee$ *is commutative and associative.*

In particular, given a finite non-empty family $(\Gamma^i)_{i \in I}$ in $\mathrm{PSH}^{\mathrm{NA}}(X, \theta)$, we then define $\bigvee_{i \in I} \Gamma^i$ in the obvious way.

Proof This is a direct consequence of Corollary 9.4.1. □

Definition 13.2.5 Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that

$$\sup_{i\in I}\Gamma^i_{\max}<\infty. \tag{13.6}$$

Then we define $\sup_{i\in I}{}^*\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for any $\omega\in\mathrm{K\ddot{a}h}(X)$, we have

$$\left(\sup_{i\in I}{}^*\Gamma^i\right)^{\theta+\omega}=\sup_{i\in I}{}^*\Gamma^{i,\theta+\omega}.$$

It follows immediately from Lemma 9.4.7 that $\sup_{i\in I}{}^*\Gamma^i\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and this definition extends Definition 9.4.6. Moreover, this definition clearly extends Definition 13.2.4 as well.

Proposition 13.2.10 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.6). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*

$$P_{\theta+\omega}\left[\sup_{i\in I}{}^*\Gamma^i\right]_{\mathcal{I}}=\sup_{i\in I}{}^*P_{\theta+\omega}\left[\Gamma^i\right]_{\mathcal{I}}.$$

Proof This is a direct consequence of Lemma 9.4.7. □

We also have a non-Archimedean version of Choquet's lemma.

Proposition 13.2.11 *Let $(\Gamma^i)_{i\in I}$ be a non-empty in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.6). Then there exists a countable subfamily $I'\subseteq I$ such that*

$$\sup_{i\in I}{}^*\Gamma^i=\sup_{i\in I'}{}^*\Gamma^i.$$

Proof For any fixed $\omega\in\mathrm{K\ddot{a}h}(X)$, thanks to Proposition 9.4.5, we could find a countable subfamily $I'\subseteq I$ such that

$$\sup_{i\in I}{}^*P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}}=\sup_{i\in I'}{}^*P_{\theta+\omega}[\Gamma^i]_{\mathcal{I}}.$$

It suffices to show that for any other $\omega'\in\mathrm{K\ddot{a}h}(X)$, we have

$$\sup_{i\in I}{}^*P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}}=\sup_{i\in I'}{}^*P_{\theta+\omega'}[\Gamma^i]_{\mathcal{I}}.$$

This is an immediate consequence of Proposition 6.1.6. □

Proposition 13.2.12 *Let $(\Gamma^i)_{i\in I}$ be a non-empty family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.6). Let $C\in\mathbb{R}$. Then*

$$\sup_{i\in I}{}^*(\Gamma^i+C)=\sup_{i\in I}{}^*\Gamma^i+C.$$

Suppose that $(\Gamma'^i)_{i\in I}$ is another family in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ satisfying (13.6). Suppose that $\Gamma^i\leq\Gamma'^i$ for all $i\in I$, then

$$\sup_{i\in I}{}^{*}\Gamma^{i} \leq \sup_{i\in I}{}^{*}\Gamma'^{i}.$$

Proof This is an immediate consequence of Proposition 9.4.6. □

Proposition 13.2.13 *Let* $(\Gamma^i)_{i\in I}$ *be an increasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.6)*. Then*

$$\operatorname{vol}\left(\sup_{i\in I}{}^{*}\Gamma^{i}\right) = \lim_{i\in I}\operatorname{vol}\Gamma^{i}. \tag{13.7}$$

Proof The $\geq$ direction in (13.7) is a direct consequence of Proposition 13.2.2. It remains to prove the reverse inequality.

Note that (13.7) holds when $\operatorname{vol}\Gamma^i > 0$ for each $i \in I$, as a consequence of Proposition 9.4.3, Corollary 6.2.3 and Theorem 6.2.5.

In particular, for each Kähler form ω on X, we have

$$\operatorname{vol}\left(\sup_{i\in I}{}^{*}\Gamma^{i,\theta+\omega}\right) = \lim_{i\in I}\operatorname{vol}\Gamma^{i,\theta+\omega}.$$

For our purpose, we need to show that for any $\epsilon > 0$, we can find ω so that

$$\sup_{i\in I}\operatorname{vol}\Gamma^{i,\theta+\omega} < \sup_{i\in I}\operatorname{vol}\Gamma^{i} + \epsilon.$$

We shall show that it is possible to choose ω so that the stronger statement holds:

$$\operatorname{vol}\Gamma^{i,\theta+\omega} < \operatorname{vol}\Gamma^{i} + \epsilon, \quad \forall i \in I.$$

Equivalently, we need to choose ω so that for any Kähler form ω' on X dominated by ω, we have

$$\operatorname{vol}\Gamma^{i,\theta+\omega}_{-\infty} < \operatorname{vol}\Gamma^{i,\theta+\omega'}_{-\infty} + \epsilon/2, \quad \forall i \in I.$$

Choose a Kähler form Ω on X so that $\Omega \geq \theta$, we compute

$$\begin{aligned}
\operatorname{vol}\Gamma^{i,\theta+\omega}_{-\infty} - \Gamma^{i,\theta+\omega'}_{-\infty} &= \int_X \left(\theta+\omega+\mathrm{dd^c}\Gamma^{i,\theta+\omega'}_{-\infty}\right)^n - \int_X \left(\theta+\omega'+\mathrm{dd^c}\Gamma^{i,\theta+\omega'}_{-\infty}\right)^n \\
&= \sum_{a=0}^{n-1}\binom{n}{a}\int_X \left(\theta+\mathrm{dd^c}\Gamma^{i,\theta+\omega'}_{-\infty}\right)^a \wedge \left(\omega^{n-a} - \omega'^{n-a}\right) \\
&\leq \sum_{a=0}^{n-1}\binom{n}{a}\int_X \Omega^a \wedge \omega^{n-a}.
\end{aligned}$$

It is clearly possible to choose ω so that the right-hand side is less than $\epsilon/2$. Our assertion then follows. □

Definition 13.2.6 Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Assume that

$$\inf_{i\in I}\Gamma^{i}_{\max} > -\infty, \tag{13.8}$$

then we define $\inf_{i\in I}\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ as the unique element such that for each $\omega\in \mathrm{Käh}(X)$, we have
$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega} = \inf_{i\in I}\Gamma^{i,\theta+\omega}. \tag{13.9}$$
We observe that
$$\left(\inf_{i\in I}\Gamma^i\right)^{\theta+\omega} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)_{>0}.$$
This follows from Proposition 9.4.9. Moreover, by Lemma 9.4.9, we have $\inf_{i\in I}\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, and this definition extends Definition 9.4.8.

In general,
$$\operatorname{vol}\left(\inf_{i\in I}\Gamma^i\right) \le \lim_{i\in I}\operatorname{vol}\Gamma^i$$
as a consequence of Proposition 13.2.2. But the reverse inequality fails in general.

Proposition 13.2.14 *Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.8). Assume that ω is a closed smooth positive $(1,1)$-form on X. Then*
$$P_{\theta+\omega}\left[\inf_{i\in I}\Gamma^i\right]_{\mathcal{I}} = \inf_{i\in I} P_{\theta+\omega}\left[\Gamma^i\right]_{\mathcal{I}}.$$

Proof This follows from Lemma 9.4.9. □

Proposition 13.2.15 *Let $(\Gamma^i)_{i\in I}$ be a decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ satisfying (13.8). Let $C\in\mathbb{R}$. Then*
$$\inf_{i\in I}(\Gamma^i + C) = \inf_{i\in I}\Gamma^i + C.$$
Suppose that $(\Gamma'^i)_{i\in I}$ is another decreasing net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta')$ satisfying (13.8). Suppose that $\Gamma^i\le\Gamma'^i$ for all $i\in I$, then
$$\inf_{i\in I}\Gamma^i \le \inf_{i\in I}\Gamma'^i.$$

Proof This is clear by definition. □

Definition 13.2.7 Let $\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\lambda\in\mathbb{R}_{>0}$, then we define $\lambda\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ as the unique element such that for any $\omega\in\mathrm{Käh}(X)$, we have
$$(\lambda\Gamma)^{\lambda\theta+\omega} = \lambda\Gamma^{\theta+\lambda^{-1}\omega}.$$
It follows immediately from Lemma 9.4.8 that $\lambda\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\lambda\theta)$ and this definition extends Definition 9.4.7.

Proposition 13.2.16 *Let $\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\lambda\in\mathbb{R}_{>0}$. Then for any closed smooth positive $(1,1)$-form ω on X, we have*
$$P_{\lambda\theta+\omega}[\lambda\Gamma]_{\mathcal{I}} = \lambda P_{\theta+\lambda^{-1}\omega}[\Gamma]_{\mathcal{I}}.$$

Proof This follows immediately from Lemma 9.4.8. □

Proposition 13.2.17 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$, $\Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta')$, $C \in \mathbb{R}$ *and* $\lambda, \lambda' > 0$, *we have*

$$\begin{aligned}\lambda(\Gamma+\Gamma') &= \lambda\Gamma + \lambda\Gamma',\\ (\lambda\lambda')\Gamma &= \lambda(\lambda'\Gamma),\\ \lambda(\Gamma + C) &= \lambda\Gamma + \lambda C.\end{aligned}$$

Suppose that $(\Gamma^i)_{i\in I}$ *is a non-empty family in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.6), *then*

$$\lambda\left(\sup_{i\in I}{}^{*}\Gamma^i\right) = \sup_{i\in I}{}^{*}(\lambda\Gamma^i).$$

If $(\Gamma^i)_{i\in I}$ *is a decreasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *satisfying* (13.8), *then*

$$\lambda\left(\inf_{i\in I}\Gamma^i\right) = \inf_{i\in I}(\lambda\Gamma^i).$$

Proof Everything except the last assertion follows from Proposition 9.4.8. The last assertion is obvious by definition. □

Proposition 13.2.18 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *and* $\lambda \in \mathbb{R}_{>0}$, *then*

$$\operatorname{vol}(\lambda\Gamma) = \lambda \operatorname{vol}\Gamma.$$

Proof This is clearly by definition. □

Definition 13.2.8 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. Let $Y \subseteq X$ be an irreducible analytic subset. We say that the trace operator of Γ along Y is *well-defined* if

$$\nu\left(\Gamma^{\theta+\omega}_{\tau}, Y\right) = 0$$

for small enough τ and any $\omega \in \mathrm{K\ddot{a}h}(X)$. We define

$$(\mathrm{Tr}_Y(\Gamma))_{\max} := \sup\left\{\tau < \Gamma_{\max} : \nu\left(\Gamma^{\theta+\omega}_{\tau}, Y\right) = 0\right\}.$$

In this case, we define $\mathrm{Tr}_Y(\Gamma) \in \mathrm{PSH}^{\mathrm{NA}}(\tilde{Y}, \theta|_{\tilde{Y}})$[3] as the unique element such that for any $\omega \in \mathrm{K\ddot{a}h}(\tilde{Y})$, the component

$$\mathrm{Tr}_Y(\Gamma)^{\theta|_{\tilde{Y}}+\omega} \in \mathrm{PSH}^{\mathrm{NA}}(Y, \theta|_{\tilde{Y}}+\omega)_{>0} \tag{13.10}$$

is defined as follows:

(1) We let

$$\left(\mathrm{Tr}_Y(\Gamma)^{\theta|_{\tilde{Y}}+\omega}\right)_{\max} = (\mathrm{Tr}_Y(\Gamma))_{\max}\,; \tag{13.11}$$

[3] Here $\tilde{Y} \to Y$ is the normalization of Y.

(2) for each $\tau \in \mathbb{R}$ less than $(\mathrm{Tr}_Y(\Gamma))_{\max}$, we define

$$\mathrm{Tr}_Y(\Gamma)_\tau^{\theta|_{\tilde{Y}}+\omega} := P_{\theta|_{\tilde{Y}}+\omega}\left[\mathrm{Tr}_Y^{\theta+\tilde{\omega}}\left(\Gamma_\tau^{\theta+\tilde{\omega}}\right)\right],$$

where $\tilde{\omega}$ is an arbitrary Kähler form on X such that $\omega \geq \tilde{\omega}|_{\tilde{Y}}$.

It follows from [GK20, Proposition 3.5] that $\tilde{Y}$ is a normal Kähler space and hence $\tilde{\omega}$ exists. We observe that the choice of the trace operator $\mathrm{Tr}_Y^{\theta+\tilde{\omega}}\left(\Gamma_\tau^{\theta+\tilde{\omega}}\right)$ is irrelevant since two different choice are $\mathcal{I}$-equivalent. Moreover, (13.10) holds as a consequence of Proposition 8.1.2 and Proposition 8.2.1. It is therefore clear that $\mathrm{Tr}_Y(\Gamma) \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$.

Proposition 13.2.19 *Let $\pi: Y \to X$ be a proper bimeromorphic morphism from a compact Kähler manifold Y. Then all definitions in this section are invariant under pulling-back to Y.*

The meaning is clear in most cases. In the case of the trace operator, this means the following: Suppose that $Z \subseteq X$ is an analytic subset and $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ has non-trivial restriction to Z. Suppose that Z is not contained in the non-isomorphism locus of π so that the strict transform W of Z is defined. If we write $\Pi: W \to Z$ for the restriction of π and $\tilde{\Pi}: \tilde{W} \to \tilde{Z}$ the strict transform of Π, then we have

$$\tilde{\Pi}^* \mathrm{Tr}_Z(\Gamma) = \mathrm{Tr}_W(\pi^*\Gamma).$$

The relevant notations are summarized in the following diagram:

$$\begin{tikzcd}
\tilde{W} \arrow[r] \arrow[d, "\tilde{\Pi}"] & W \arrow[r] \arrow[d, "\Pi"] & Y \arrow[d, "\pi"] \\
\tilde{Z} \arrow[r] & Z \arrow[r] & X.
\end{tikzcd}$$

Proof We only prove the assertion for the trace operator, as the other proofs are similar.

We shall use the notations above. Observe that for any closed positive smooth $(1,1)$-form ω on X with positive mass, we have

$$\left(\tilde{\Pi}^* \mathrm{Tr}_Z(\Gamma)\right)_{\max} = (\mathrm{Tr}_Z(\Gamma))_{\max} = \sup\left\{\tau < \Gamma_{\max} : \nu\left(\Gamma_\tau^{\theta+\omega}, Z\right) = 0\right\},$$

and

$$\begin{aligned}
(\mathrm{Tr}_W(\pi^*\Gamma))_{\max} &= \sup\left\{\tau < \Gamma_{\max} : \nu\left((\pi^*\Gamma_\tau)^{\pi^*\theta+\pi^*\omega}, W\right) = 0\right\} \\
&= \sup\left\{\tau < \Gamma_{\max} : \nu\left(\pi^*\Gamma_\tau^{\theta+\omega}, W\right) = 0\right\} \\
&= \sup\left\{\tau < \Gamma_{\max} : \nu\left(\Gamma_\tau^{\theta+\omega}, Z\right) = 0\right\}.
\end{aligned}$$

Here we applied implicitly Proposition 13.1.5. Therefore,

$$\left(\tilde{\Pi}^* \operatorname{Tr}_Z(\Gamma)\right)_{\max} = \left(\operatorname{Tr}_W(\pi^*\Gamma)\right)_{\max} .$$

Let $\tau \in \mathbb{R}$ be less than this common value. Take a Kähler form $\omega_{\tilde{Z}}$ (resp. $\omega_{\tilde{W}}$) on $\tilde{Z}$ (resp. $\tilde{W}$). Take a Kähler form ω_Y on Y (resp. ω_X on X) such that

$$\omega_{\tilde{W}} \geq \omega_X|_{\tilde{W}}, \quad \omega_{\tilde{Z}} \geq \omega_Y|_{\tilde{Z}}, \quad \omega_Y \geq \pi^*\omega_X.$$

We want to show that

$$\left(\tilde{\Pi}^* \operatorname{Tr}_Z(\Gamma)\right)_\tau^{\theta|_{\tilde{Z}}+\omega_{\tilde{Z}}} \sim_P \left(\operatorname{Tr}_W(\pi^*\Gamma)\right)_\tau^{(\pi^*\theta)|_{\tilde{W}}+\omega_{\tilde{W}}} .$$

Unfolding the definitions, we reduce to

$$\tilde{\Pi}^* \operatorname{Tr}_Z^{\theta+\omega_X}\left[\Gamma_\tau^{\theta+\omega_X}\right] \sim_P \operatorname{Tr}_W^{\pi^*\theta+\omega_Y}\left((\pi^*\Gamma)_\tau^{\pi^*\theta+\omega_Y}\right).$$

Using Proposition 8.2.1, this is equivalent to

$$\tilde{\Pi}^* \operatorname{Tr}_Z\left[\Gamma_\tau^{\theta+\omega_X}\right] \sim_P \operatorname{Tr}_W\left((\pi^*\Gamma)_\tau^{\pi^*\theta+\pi^*\omega_X}\right).$$

This is a consequence of Lemma 8.2.1. □

13.3 Duistermaat–Heckman measures

Let X be a connected compact Kähler manifold of dimension n and θ be a closed real smooth $(1,1)$-form on X representing a big cohomology class.

Definition 13.3.1 Assume that X admits a smooth flag $Y_\bullet$. Let $\Gamma \in \operatorname{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. The *Duistermaat–Heckman measure* $\operatorname{DH}(\Gamma)$ of Γ is defined as

$$\operatorname{DH}(\Gamma) := n! \cdot \operatorname{DH}\left(\Delta_{Y_\bullet}(\theta,\Gamma)\right) .$$

Recall that $\Delta_{Y_\bullet}(\theta,\Gamma) \in \operatorname{TC}(\Delta_{Y_\bullet}(\theta,\Gamma_{-\infty}))$ is the Okounkov test curve defined in Theorem 10.5.2. See Definition 10.5.4 for the definition of the Duistermaat–Heckman measure of an Okounkov test curve.

Theorem 13.3.1 *Assume that X admits a smooth flag $Y_\bullet$. The Duistermaat–Heckman measure* $\operatorname{DH}(\Gamma)$ *of* $\Gamma \in \operatorname{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ *in Definition 13.3.1 is independent of the choice of the smooth flag $Y_\bullet$. Furthermore, for any $m \in \mathbb{Z}_{>0}$, the m-th moment of* $\operatorname{DH}(\Gamma)$ *is given by*

$$\int_{\mathbb{R}} x^m \operatorname{DH}(\Gamma)(x) = \Gamma_{\max}^m \operatorname{vol}\Gamma + m\int_{-\infty}^{\Gamma_{\max}} \tau^{m-1}\left(\operatorname{vol}(\theta+\mathrm{dd^c}\Gamma_\tau) - \operatorname{vol}\Gamma\right) \mathrm{d}\tau, \tag{13.12}$$

and

$$\int_{\mathbb{R}} \operatorname{DH}(\Gamma) = \operatorname{vol}\Gamma. \tag{13.13}$$

Proof We observe that the moments of the random variable $G[\Delta_{Y_\bullet}(\theta,\Gamma)]$ as computed in Proposition 10.5.4 are independent of the choice of the flag: In fact, they are given by (13.12) and (13.13) thanks to Theorem 10.4.2(1).

Assume first that Γ is bounded. Since the Duistermaat–Heckman measure has bounded support in this case (c.f. Theorem 10.5.1), we conclude that $\mathrm{DH}(\Gamma)$ is uniquely determined.

In general, we may assume that $\Gamma_{\max}=0$. For each $\epsilon>0$, we define $\Gamma^\epsilon\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ as follows:

(1) Let $\Gamma^\epsilon_{\max}=0$, and
(2) we set

$$\Gamma^\epsilon_\tau=\begin{cases}\phi, \text{ if } \tau\leq -\epsilon^{-1},\\ P_\theta\left[(1+\epsilon\tau)\Gamma_\tau-\epsilon\tau\phi\right], \text{ if } \tau\in\left(-\epsilon^{-1},0\right).\end{cases}$$

Then it follows from the argument of Theorem 9.2.1 Step 3.3 that $\Delta_{Y_\bullet}(\Gamma)_\tau$ is the decreasing limit of $\Delta_{Y_\bullet}(\Gamma^\epsilon)_\tau$ for any $\tau<\Gamma_{\max}$ as $\epsilon\to 0+$. So $\mathrm{DH}(\Gamma^\epsilon)\rightharpoonup\mathrm{DH}(\Gamma)$ by Lemma 10.5.2. It follows that $\mathrm{DH}(\Gamma)$ is independent of the choice of the flag. □

More generally, when X does not admit a smooth flag, we could make a modification $\pi\colon Y\to X$ so that Y admits a flag. We define

$$\mathrm{DH}(\Gamma)\coloneqq\mathrm{DH}(\pi^*\Gamma). \tag{13.14}$$

It follows from Theorem 10.4.2(5) that this measure is independent of the choice of π.

Proposition 13.3.1 *Let $(\Gamma^i)_{i\in I}$ be a net in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ and $\Gamma\in\mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Assume one of the following conditions holds:*

(1) *The net $(\Gamma^i)_{i\in I}$ is decreasing and $\Gamma=\inf_{i\in I}\Gamma^i$. Assume that*

$$\operatorname{vol}\Gamma=\lim_{i\in I}\operatorname{vol}\Gamma^i. \tag{13.15}$$

(2) *The net $(\Gamma^i)_{i\in I}$ is increasing and $\Gamma=\sup_{i\in I}{}^*\Gamma^i$.*

Then

$$\mathrm{DH}(\Gamma^i)\rightharpoonup\mathrm{DH}(\Gamma). \tag{13.16}$$

Proof We may assume that X admits a smooth flag $Y_\bullet$.

Assume (1). Note that (13.15) implies that

$$\Gamma_{-\infty}=\inf_{i\in I}\Gamma^i_{-\infty}.$$

We want to derive (13.16) from Lemma 10.5.2. It boils down to prove the following: For any $\tau<\Gamma_{\max}$, we have

$$\Delta_{Y_\bullet}(\theta,\Gamma^i_\tau)\xrightarrow{d_{\mathrm{Haus}}}\Delta_{Y_\bullet}(\theta,\Gamma_\tau).$$

This follows immediately from Theorem 10.4.2(1) and Proposition 3.1.10.

The proof under the assumption (2) is similar. We only need to apply Lemma 10.5.3 instead of Lemma 10.5.2. □

Definition 13.3.2 When $[\theta]$ is a Hodge class and Γ is induced by a test configuration as in Example 9.3.1 and Remark 9.3.1, our Duistermaat–Heckman measure coincides with the more traditional definition of [BHJ17, Section 3.2]. This is explained in [Xia25b, Remark 7.17].

13.4 Comparison with Boucksom–Jonsson's theory

13.4.1 A brief recap of Boucksom–Jonsson's theory

In this section, we briefly recall the non-Archimedean global pluripotential theory à la Boucksom–Jonsson [BJ22a]. As our presentation is far from being complete, the readers are strongly recommended to read their original paper before reading the current section.

13.4.1.1 Valuations

Let X be an irreducible reduced variety over $\mathbb{C}$ of dimension n. We recall the notion of Berkovich analytification X^{An} of X with respect to the trivial valuation on $\mathbb{C}$.

Definition 13.4.1 A (real-valued) *valuation* on X (or a *valuation* of $\mathbb{C}(X)$) is a map $v\colon \mathbb{C}(X) \to (-\infty, \infty]$ satisfying the following conditions:

(1) For $f \in \mathbb{C}(X)$, $v(f) = \infty$ if and only if $f = 0$;
(2) For $f, g \in \mathbb{C}(X)$, $v(fg) = v(f) + v(g)$;
(3) For $f, g \in \mathbb{C}(X)$, $v(f + g) \geq v(f) \wedge v(g)$.

The set of valuations on X is denoted by X^{val}. The center of a valuation v is the scheme-theoretic point $c = c(v)$ of X such that $v \geq 0$ on $\mathcal{O}_{X,c}$ and $v > 0$ on the maximal ideal $\mathfrak{m}_{X,c}$ of $\mathcal{O}_{X,c}$. The center is unique if exists. It exists if X is proper.

In the remaining of this section, we assume that X is projective.

As a set, X^{An} is the set of *semi-valuations* on X, in other words, real-valued valuations v on irreducible reduced subvarieties Y in X that is trivial on $\mathbb{C}$. We call Y the *support* of the semi-valuation v. In other words,

$$X^{\mathrm{An}} = \coprod_{Y} Y^{\mathrm{val}}.$$

We will write $v_{\mathrm{triv}} \in X^{\mathrm{An}}$ for the trivial valuation on X: $v_{\mathrm{triv}}(f) = 0$ for any $f \in \mathbb{C}(X)^{\times}$.

We endow X^{An} with the coarsest topology such that

(1) for any Zariski open subset $U \subseteq X$, the subset U^{An} of X^{An} consisting of semi-valuations whose supports meet U is open;
(2) for each Zariski open subset $U \subseteq X$ and each $f \in \mathrm{H}^0(U, \mathcal{O}_X)$ (here $\mathcal{O}_X$ is the sheaf of regular functions), the map $|f| : U^{\mathrm{An}} \to \mathbb{R}$ sending v to $\exp(-v(f))$ is continuous.

See [Ber93] for more details.

We will be most interested in divisorial valuations. Recall that a *divisorial valuation* on X is a valuation of the form $t\,\mathrm{ord}_E$, where $t \in \mathbb{Q}_{>0}$ and E is a prime divisor over X. The set of divisorial valuations on X is denoted by X^{div}. When $\mathbb{Q}_{>0}$ is replaced by $\mathbb{R}_{>0}$, we can similarly define a space $X^{\mathrm{div}}_{\mathbb{R}}$.

Given any coherent ideal $\mathfrak{a}$ on X and any $v \in X^{\mathrm{An}}$, we define

$$v(\mathfrak{a}) := \min\{v(f) : f \in \mathfrak{a}_{c(v)}\} \in [0, \infty], \tag{13.17}$$

where $c(v)$ is the center of the valuation v on X.

Given any valuation v on X, the Gauss extension of v is a valuation $\sigma(v)$ on $X \times \mathbb{A}^1$:

$$\sigma(v)\left(\sum_i f_i t^i\right) := \min_i (v(f_i) + i). \tag{13.18}$$

Here t is the standard coordinate on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. The key property is that when v is a divisorial valuation, then so it $\sigma(v)$. See [BHJ17, Lemma 4.2].

13.4.1.2 Non-Archimedean plurisubharmonic functions

Let X be an irreducible complex projective variety of dimension n and L be a holomorphic pseudo-effective $\mathbb{Q}$-line bundle on X. Through the GAGA morphism $X^{\mathrm{An}} \to X$ of ringed spaces, L can be pulled-back to an analytic line bundle L^{An} on X. The purpose of this section is to study the psh metrics on L^{An}. We will follow the approach of [BJ22a], which avoids the direct treatment of L^{An} itself.

Following [BJ22a, Definition 2.18], we define $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L^{\mathrm{An}})$, the set of *(rational) generically finite Fubini–Study functions* $\phi : X^{\mathrm{An}} \to [-\infty, \infty)$, that are of the following form:

$$\phi = \frac{1}{m} \max_j \{\log |s_j| + \lambda_j\}. \tag{13.19}$$

Here $m \in \mathbb{Z}_{>0}$ is an integer such that L^m is a line bundle, the s_j's are a finite collection of non-vanishing sections in $\mathrm{H}^0(X, L^m)$, and $\lambda_j \in \mathbb{Q}$. We followed the convention of Boucksom–Jonsson by writing $\log |s_j|(v) = -v(s_j)$.

Definition 13.4.2 ([BJ22a, Definition 4.1]) A *plurisubharmonic metric* (or *psh metric* for short) on L^{An} is a function $\phi : X^{\mathrm{An}} \to [-\infty, \infty)$ that is not identically $-\infty$, and is the pointwise limit of a decreasing net $(\phi_i)_{i \in I}$, where $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L_i^{\mathrm{An}})$ for some $\mathbb{Q}$-line bundles L_i on X satisfying $c_1(L_i) \to c_1(L)$ in $\mathrm{NS}^1(X)_{\mathbb{R}}$.

The set of psh metrics on L^{An} is denoted by $\mathrm{PSH}(L^{\mathrm{An}})$. We endow $\mathrm{PSH}(L^{\mathrm{An}})$ with the topology of pointwise convergence on X^{div}. This topology is Hausdorff as functions in $\mathrm{PSH}(L^{\mathrm{An}})$ are completely determined by their restriction on X^{div}:

Theorem 13.4.1 ([BJ22a, Theorem 4.22]) *Let $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$ and $\psi : X^{\mathrm{An}} \to [-\infty, \infty)$ be an usc function. Assume that $\phi \leq \psi$ on X^{div}, then the same holds on X^{An}.*

Proposition 13.4.1 ([BJ22a, Theorem 4.7]) *Let $\phi, \phi' \in \mathrm{PSH}(L^{\mathrm{An}})$, then so is their pointwise maximum $\phi \vee \phi'$.*

Proposition 13.4.2 *Let H be an ample line bundle on X. Consider $\phi \in \mathrm{PSH}((L+H)^{\mathrm{An}})$. Assume that for each $m \in \mathbb{Z}_{>0}$, we have $\phi \in \mathrm{PSH}((L+m^{-1}H)^{\mathrm{An}})$, then $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$.*

This is a special case of [BJ22a, (PSH2)] on Page 45.

Next we note that we may use sequences instead of nets in the definition of $\mathrm{PSH}(L^{\mathrm{An}})$:

Theorem 13.4.2 ([BJ22a, Corollary 12.18]) *Let S be an ample line bundle on X. Let $\phi \in \mathrm{PSH}(L^{\mathrm{An}})$. Then there is a sequence of rational numbers $\varepsilon_i \searrow 0$ and a decreasing sequence $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L+\varepsilon_i S)^{\mathrm{An}})$ such that ϕ is the pointwise limit of ϕ_i, as $i \to \infty$.*

The space $\mathrm{PSH}(L^{\mathrm{An}})$ inherits most of the expected properties of (Archimedean) psh functions ([BJ22a, Theorem 4.7]). However, the following compactness result is not known:

Conjecture 13.4.1 ([BJ22a, §5]) Assume that X is unibranch, then every bounded from above increasing net of elements in $\mathrm{PSH}(L^{\mathrm{An}})$ converges in $\mathrm{PSH}(L^{\mathrm{An}})$.

This prediction is equivalent to so-called envelope conjecture [BJ22a, Conjecture 5.14]: the regularized supremum of a bounded from above family of functions in $\mathrm{PSH}(L^{\mathrm{An}})$ lies in $\mathrm{PSH}(L^{\mathrm{An}})$. See [BJ22a, Theorem 5.11] for the proof of the equivalence. This conjecture is proved when X is smooth and L is nef in [BJ22a]. More recently, in [BJ22b], Boucksom–Jonsson further established the case when X is smooth and L is pseudo-effective.

13.4.2 The analytifications

Let X be a connected projective manifold of dimension n. Let θ be a closed smooth real $(1,1)$-form on X representing a pseudo-effective cohomology class.

13.4.2.1 The transcendental setting

Definition 13.4.3 For $\varphi \in \mathrm{PSH}(X, \theta)$, we define the *analytification* $\varphi^{\mathrm{An}} \colon X^{\mathrm{An}} \to [-\infty, 0]$ as follows:

$$\varphi^{\mathrm{An}}(v) := -v(\varphi) = -\lim_{k\to\infty}\frac{1}{k}v\left(\mathcal{I}(k\varphi)\right). \tag{13.20}$$

By Theorem 1.4.2 and Fekete's lemma, the limit in (13.20) exists.

Note that we can also write

$$\varphi^{\mathrm{An}}(v) = \inf_{k\in\mathbb{Z}_{>0}} -2^{-k}v\left(\mathcal{I}(2^k\varphi)\right). \tag{13.21}$$

When $v = t\,\mathrm{ord}_E$ for some prime divisor E over X, $\varphi^{\mathrm{An}}(v) = -t\nu(\varphi, E)$ by Proposition 1.4.4.

Definition 13.4.4 Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$. We define the *analytification* $\Gamma^{\mathrm{An}}\colon X^{\mathrm{div}} \to [-\infty,\infty)$ of Γ as follows: For any $\omega \in \mathrm{Käh}(X)$, we define

$$\Gamma^{\mathrm{An}}(v) := \sup_{\tau\leq\Gamma_{\max}}\left(\Gamma_\tau^{\omega,\mathrm{An}}(v)+\tau\right). \tag{13.22}$$

Clearly, (13.22) is independent of the choice of ω.

Note that (13.22) can be equivalently written as

$$\Gamma^{\mathrm{An}}(v) = \sup_{\tau\leq\Gamma_{\max}}\left(\Gamma_\tau^{\omega,\mathrm{An}}(v)+\tau\right) = \sup_{\tau\in\mathbb{R}}\left(\Gamma_\tau^{\omega,\mathrm{An}}(v)+\tau\right)$$

with $(-\infty)^{\mathrm{An}}(v) = -\infty$ understood.

Proposition 13.4.3 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$ with $\Gamma_{\max} \leq 0$. Let Ψ be the complexification of Γ^*. Then*

$$\Gamma^{\mathrm{An}}(v) = -\sigma(v)(\Psi) \quad \forall v \in X^{\mathrm{div}}. \tag{13.23}$$

See Definition 4.1.2 for the definition of the complexification $\Psi \in \mathrm{QPSH}(X\times\Delta)$. Note that since $\Gamma_{\max} \leq 0$, by Corollary 9.2.3 and Theorem 1.2.1, Ψ extends uniquely to a quasi-psh function on $X \times \Delta$.

Proof Recall that

$$\Psi(x,\delta) = \sup_{\tau\leq\Gamma_{\max}}\left(\psi_\tau(x) - \log|\delta|^2\tau\right) \quad \text{for } x\in X, \delta\in\Delta^*.$$

By (13.18), we have $\sigma(v)(\log|\delta|^2) = 1$ and $\sigma(v)(\Gamma_\tau) = v(\Gamma_\tau)$ for all $\tau \leq \Gamma_{\max}$. So we have that

$$\sigma(v)(\Gamma_\tau(x) - \log|\delta|^2\tau) = v(\Gamma_\tau) - \tau.$$

Lastly, since $\sigma(v)$ is a divisorial valuation on $X\times\Delta$, by Corollary 1.4.1, we conclude (13.23). ☐

Definition 13.4.5 Let $N \in \mathbb{N}$, and $A_0,\dots,A_N$ be a finite collection of elements in $\mathrm{PSH}(X,\theta)$, and $\tau_0 > \tau_1 > \cdots > \tau_N$ be finitely many real numbers. Then the *piecewise linear curve* $A = (A_\tau)_{\tau\in\mathbb{R}}$ in $\mathrm{PSH}(X,\theta)\cup\{-\infty\}$ associated with these data is the affine interpolation of these data:

(1) $A_{\tau_i} = A_i$ for $i = 0, \dots, N$;
(2) $A_\tau = A_{\tau_N}$ for $\tau \le \tau_N$;
(3) for $t \in (0,1)$ and $i = 0, \dots, N-1$, we have

$$A_{(1-t)\tau_i + t\tau_{i+1}} = (1-t)A_{\tau_i} + tA_{\tau_{i+1}};$$

(4) $A_\tau \equiv -\infty$ for $\tau > \tau_0$.

The *analytification* of A is the function $A^{\mathrm{An}} \colon X^{\mathrm{An}} \to [-\infty, \infty)$ defined as follows:

$$A^{\mathrm{An}}(v) := \sup_{\tau \le \tau_0} (A^{\mathrm{An}}(v) + \tau) = \max_{i=0,\dots,N} \left(A^{\mathrm{An}}_{\tau_i}(v) + \tau_i \right) \quad \forall v \in X^{\mathrm{An}}. \tag{13.24}$$

We also say $A = (A_\tau)_{\tau \le \tau_0}$ is a piecewise linear curve in $\mathrm{PSH}(X, \theta)$.

Remark 13.4.1 Note that $\tau \mapsto A_\tau$ is upper semicontinuous, but not necessarily concave. Let $(A'_\tau)_{\tau \in \mathbb{R}}$ be the upper concave envelope of $\tau \mapsto A_\tau$. Then it can be inductively constructed as follows:

(1) For $\tau \in (\tau_0, \infty)$, we let $A'_\tau \equiv -\infty$;
(2) we set $A'_{\tau_0} = A_{\tau_0}$;
(3) define inductively for $j = 0, \dots, N-1$ the following: For $\tau \in [\tau_{j+1}, \tau_j)$, we set

$$A'_\tau = \max_{i=j+1,\dots,N} \left(\frac{\tau_j - \tau}{\tau_j - \tau_i} A_{\tau_i} + \frac{\tau_j - \tau}{\tau_j - \tau_i} A'_{\tau_j} \right) \vee A'_{\tau_j};$$

(4) for $\tau \in (-\infty, \tau_N)$, we set $A'_\tau = A_{\tau_N}$.

This construction is just a reformulation of the general formula Proposition A.1.2.

In particular, $A'_\tau \in \mathrm{PSH}(X, \theta)$ for all $\tau \le \tau_0$.

Note that A' is not necessarily piecewise linear.

Lemma 13.4.1 *Let A be a piecewise linear curve in $\mathrm{PSH}(X, \theta)$. Let $(A'_\tau)_{\tau \in \mathbb{R}}$ be the upper concave envelope of $\tau \mapsto A_\tau$. Then $\tilde{A} := (P_\theta[A'_\tau]_{\mathcal{I}})_{\tau < \tau_0} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. Moreover,*

$$A^{\mathrm{An}} = \tilde{A}^{\mathrm{An}} \quad \text{on } X^{\mathrm{div}}. \tag{13.25}$$

Here τ_0 is as in Definition 13.4.5.

Proof We continue to use the notations in Definition 13.4.5. The fact that $\tilde{A} \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ follows from Remark 13.4.1. In order to prove (13.25), we fix $v \in X^{\mathrm{div}}$. By Remark 13.4.1,

$$\tau \mapsto \left(P_\theta[A'_\tau]_{\mathcal{I}}\right)^{\mathrm{An}}(v) = \left(A'_\tau\right)^{\mathrm{An}}(v)$$

is just the upper concave envelope of

$$\tau \mapsto A^{\mathrm{An}}_\tau(v).$$

Therefore, (13.25) follows. □

13.4.2.2 The algebraic setting

Let L be a $\mathbb{Q}$-line bundle on X and h be a Hermitian metric on L with $\theta = c_1(L, \theta)$.

Lemma 13.4.2 *For any $\varphi \in \mathrm{PSH}(X, \theta)$ we have that $\varphi^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$.*

Proof After replacing L with a sufficiently high power, we may assume that L is a line bundle. Take a very ample line bundle H on X. By Siu's uniform global generation theorem [Siu98], [Dem12a, Theorem 6.27] there exists $b > 0$ large enough so that $H^b \otimes L^k \otimes \mathcal{I}(k\varphi)$ is globally generated for all $k > 0$. Let $\{s_i\}_i$ be a finite set of global sections that generate the sheaf $H^b \otimes L^k \otimes \mathcal{I}(k\varphi)$. Then

$$v(\mathcal{I}(k\varphi)) = \min_i v(s_i).$$

It follows that $v \mapsto -k^{-1} v\,(\mathcal{I}(k\varphi))$ lies in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L + \frac{b}{k} H)^{\mathrm{An}})$. Using (13.21), we conclude that $\varphi^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$. □

Lemma 13.4.3 *Let Γ be a piecewise linear curve in $\mathrm{PSH}(X, \theta)$. Then $\Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$.*

Proof The result follows from (13.24), Proposition 13.4.1 and Lemma 13.4.2. □

Lemma 13.4.4 *Let R be a commutative $\mathbb{C}$-algebra of finite type and I be an ideal of $R[t]$. If for any $a \in S^1$, $a^* I \subseteq I$, then I is stable under the $\mathbb{C}^*$-action. Moreover, there are ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m$ in R so that*

$$I = I_0 + I_1 t + \cdots + I_m (t^m), \tag{13.26}$$

Proof It suffices to argue that I can be expanded as in (13.26). To see this, assume that $a \in I$. We can write $a = a_0 + a_1 t + \cdots + a_m t^m$ with $a_i \in R$. Then our assumption implies that $\sum_i a_i \rho^i t^i \in I$ as well for all $\rho \in S^1$. So by the Lagrange interpolation formula, $a_i t^i \in I$ for all i. Therefore, we can write I as $I_0 + I_1 t + I_2 t^2 + \ldots$ for some ideals $I_0 \subseteq I_1 \subseteq \ldots$ in R. But as R is noetherian, there is $m \geq 0$ so that $I_{m'} = I_m$ for $m' > m$. (13.26) follows. □

Lemma 13.4.5 *Let X be a complex projective variety and $p : X \times \mathbb{C} \to X$ be the natural projection. Assume that $\mathcal{I}$ is an analytic coherent ideal sheaf on $X \times \mathbb{C}$. Assume that $\mathcal{I}|_{X \times \mathbb{C}^*} = p^* \mathcal{J}$ for some coherent ideal sheaf $\mathcal{J}$ on X. Then $\mathcal{I}$ is the analytification of an algebraic coherent ideal sheaf.*

Proof Let $q : X \times (\mathbb{P}^1 \setminus \{0\}) \to X$ be the natural projection. As $\mathbb{C}^* \subset \mathbb{P}^1 \setminus \{0\}$ we can glue $q^* \mathcal{J}$ with $\mathcal{I}$ to get an analytic coherent ideal sheaf on $X \times \mathbb{P}^1$. By the GAGA principle, this ideal sheaf is necessarily algebraic, hence so is its restriction to $X \times \mathbb{C}$. □

Next we point out a version of Siu's uniform global generatedness lemma [Siu88] that we will need in the proof of our next theorem:

Lemma 13.4.6 *Let L be a big line bundle on X such that $c_1(L) = \{\theta\}$ and $\Phi \in \mathrm{PSH}(X \times \Delta, p_1^*\theta)$, where Δ is the unit disk. Let G be an ample line bundle on X. Then there exists $k > 0$, only dependent on X and G such that $p_1^*(G^k \otimes L^m) \otimes \mathcal{I}(m\Phi)$ is globally generated for all $m \in \mathbb{N}$.*

Proof The argument for this is exactly the same as the one in [BBJ21, Lemma 5.6] with Nadal's vanishing replaced by the family version proved by Matsumura in [Mat22, Theorem 1.7]. □

Proposition 13.4.4 *Let $\phi \in \mathrm{PSH}(X, \theta)_{>0}$ be a model potential and $\ell \in \mathcal{R}(X, \theta; \phi)$ with $\sup_X \ell_1 \leq 0$. Let Φ be the complexification of ℓ. Then the function*

$$v \mapsto -\sigma(v)(\Phi) \quad \textit{for } v \in X^{\mathrm{div}}$$

admits a unique extension to an element in $\mathrm{PSH}(L^{\mathrm{An}})$.

Proof We may assume that L is a line bundle. Observe that the extension is unique if it exists by Theorem 13.4.1.

Let $p_1 : X \times \mathbb{C} \to X$ be the projection. Thanks to Proposition 1.4.5 and Lemma 8.5.3, for each $m \in \mathbb{Z}_{>0}$, we have

$$\mathcal{I}(m\Phi)|_{X \times \Delta^*} = p_1^* \mathcal{I}(m\phi)|_{X \times \Delta^*}.$$

In particular, $\mathcal{I}(m\Phi)$ admits a $\mathbb{C}^*$-invariant extension to a coherent ideal sheaf on $X \times \mathbb{C}$, namely $\mathcal{I}(mp_1^*\phi)$.

From Lemma 13.4.4 and Lemma 13.4.5, we get that

$$\mathcal{I}(m\Phi) = \mathfrak{a}_0 + \mathfrak{a}_1 t + \cdots + \mathfrak{a}_{N-1} t^{N-1} + \mathfrak{a}_N (t^N), \tag{13.27}$$

where the $\mathfrak{a}_i$'s are coherent ideal sheaves on X.

Using Lemma 13.4.6, there exists an ample line bundle T over X such that $p_1^* T \otimes L^m \otimes \mathcal{I}(m\Phi)$ is globally generated, which is equivalent to $T \otimes L^m \otimes \mathfrak{a}_i$ being globally generated for all i.[4]

We define

$$\varphi_m(v) := -\frac{1}{m}\sigma(v)(\mathcal{I}(m\Phi)) = -\frac{1}{m} \min_i (v(\mathfrak{a}_i) + i), \quad v \in X^{\mathrm{div}}.$$

From the right-hand side of the formula, φ_m can be extended to an element in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L + m^{-1}T)^{\mathrm{An}})$, which we denote by the same symbol.

For $v \in X^{\mathrm{div}}$,

$$-\sigma(v)(\Phi) = \lim_{m \to \infty} -\frac{1}{2^m}\sigma(v)(\mathcal{I}(2^m\Phi)) = \lim_{m \to \infty} \varphi_{2^m}(v)$$

and the right-hand side defines an element in $\mathrm{PSH}(L^{\mathrm{An}})$ by definition, since $\{\varphi_{2^m}\}_m$ is decreasing. □

[4] In contrast with the case where ϕ is bounded, explored in [BBJ21], $\mathfrak{a}_N \neq \mathcal{O}_X$ in general.

Corollary 13.4.1 *Let* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$. *Then* Γ^{An} *defined in Definition 13.4.4 admits a unique extension to* $\mathrm{PSH}(L^{\mathrm{An}})$.

The extension will be denoted by the same notation Γ^{An}.

Proof Observe that the extension is unique if it exists by Theorem 13.4.1. We may assume that $\Gamma_{\max} = 0$ without loss of generality.

When $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)_{>0}$, our assertion follows from Proposition 13.4.4 and Proposition 13.4.3.

In general, fix an ample line bundle H on X and a Kähler form $\omega \in c_1(H)$. Then we know that

$$\Gamma^{\mathrm{An}} = \left(\Gamma^{m^{-1}\omega}\right)^{\mathrm{An}} \in \mathrm{PSH}((L + m^{-1}H)^{\mathrm{An}})$$

for any $m \in \mathbb{Z}_{>0}$. Therefore, $\Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$ by Proposition 13.4.2. □

13.4.3 The comparison theorem

Let X be a connected projective manifold of dimension n. Let L be a pseudo-effective $\mathbb{Q}$-line bundle on X and h be a Hermitian metric on L with $\theta = c_1(L, h)$.

Thanks to Corollary 13.4.1, we already have a map

$$\mathrm{PSH}^{\mathrm{NA}}(X, \theta) \to \mathrm{PSH}(L^{\mathrm{An}}), \quad \Gamma \mapsto \Gamma^{\mathrm{An}}. \tag{13.28}$$

We observe that for $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and a Kähler form ω on X, we have

$$\left(P_{\theta+\omega}\left[\Gamma\right]_{\mathcal{I}}\right)^{\mathrm{An}} = \Gamma^{\mathrm{An}}.$$

Also observe that

$$\Gamma_{\max} = \Gamma^{\mathrm{An}}(v_{\mathrm{triv}}), \tag{13.29}$$

Lemma 13.4.7 *The map* (13.28) *is order preserving. Moreover, suppose that* $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ *satisfies that* $\Gamma^{\mathrm{An}} \leq \Gamma'^{\mathrm{An}}$, *then* $\Gamma \leq \Gamma'$.

In particular, the map (13.28) *is injective.*

Proof The map (13.28) is order preserving by definition. Let us take $\Gamma, \Gamma' \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ with $\Gamma^{\mathrm{An}} \leq \Gamma'^{\mathrm{An}}$. Fix a Kähler form ω on X.

Let $v \in X^{\mathrm{div}}$ and $t \in \mathbb{Q}_{>0}$. Then, using (13.22) we notice that

$$t\Gamma^{\mathrm{An}}\left(t^{-1}v\right) = \sup_{\tau \in \mathbb{R}} \left(\left(\Gamma^{\omega}_{\tau}\right)^{\mathrm{An}}(v) + t\tau\right). \tag{13.30}$$

A similar equality holds for Γ'. Therefore, by Corollary A.2.1, we have

$$\left(\Gamma^{\omega}_{\tau}\right)^{\mathrm{An}} \leq \left(\Gamma'^{\omega}_{\tau}\right)^{\mathrm{An}}$$

for all $\tau \in \mathbb{R}$. It follows that

$$\Gamma^{\omega}_{\tau} \leq \Gamma'^{\omega}_{\tau}$$

for all $\tau \in \mathbb{R}$. Our assertion follows. □

Lemma 13.4.8 *Let $\phi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L^{\mathrm{An}})$. Then there is a piecewise linear curve A in* $\mathrm{PSH}(X,\theta)$ *with $\phi = A^{\mathrm{An}}$. In particular, ϕ is in the image of* (13.28)*.*

Note that from the proof below, the test curve Γ corresponding to ϕ satisfies the following: For any $\tau \leq \Gamma_{\max}$, Γ_τ is elementary. See Definition 6.1.3 for the definition of elementary metrics.

Proof Let us write

$$\phi = \frac{1}{m}\max_{i=1,\dots,M}\left(\log|s_i| + \lambda_i\right), \tag{13.31}$$

where $m \in \mathbb{Z}_{>0}$, $s_1,\dots,s_M$ are a finite number of sections of L^m and $\lambda_1,\dots,\lambda_M \in \mathbb{Q}$.

Write I_λ for the set of i such that $\lambda_i = \lambda$. We denote the finitely many λ so that I_λ is non-empty as $\tau_0 > \cdots > \tau_N$. For each $i = 0,\dots,N$, we write

$$A_{\tau_i} = \frac{1}{m}\max_{j\in I_{\tau_i}}\left(\log|s_j|^2_{h^m} + \tau_i\right).$$

We define A as the piecewise linear curve associated with the A_{τ_i}'s and the τ_i's. Then clearly $\phi = A^{\mathrm{An}}$.

The final assertion follows from Lemma 13.4.1. □

Proposition 13.4.5 *Let $(\Gamma_i)_{i\in I}$ be a decreasing net in* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$*. Assume that* (13.8) *is satisfied. Then*

$$\left(\inf_{i\in I}\Gamma_i\right)^{\mathrm{An}} = \inf_{i\in I}\Gamma_i^{\mathrm{An}}.$$

Proof Take a Kähler form ω on X. We need to show that

$$\left(\inf_{i\in I}\Gamma_{i,\omega}\right)^{\mathrm{An}} = \inf_{i\in I}\Gamma_i^{\mathrm{An}}.$$

Therefore, after replacing θ by $\theta+\omega$, we may assume that $\Gamma_i \in \mathrm{PSH}(X,\theta)_{>0}$ for all $i \in I$ and $\inf_{i\in I}\Gamma_i \in \mathrm{PSH}(X,\theta)_{>0}$. Fix $\nu \in X^{\mathrm{div}}$. By Theorem 13.4.1, it suffices to prove that

$$\sup_{\tau\in\mathbb{R}}\left(\left(\inf_{i\in I}\Gamma_{i,\tau}\right)^{\mathrm{An}}(\nu) + \tau\right) = \inf_{i\in I}\sup_{\tau\in\mathbb{R}}\left(\Gamma^{\mathrm{An}}_{i,\tau}(\nu)+\tau\right). \tag{13.32}$$

But thanks to Proposition 3.1.10, we have

$$\left(\inf_{i\in I}\Gamma_{i,\tau}\right)^{\mathrm{An}}(\nu) = \inf_{i\in I}\Gamma^{\mathrm{An}}_{i,\tau}(\nu),$$

so (13.32) is a consequence of Proposition A.2.3. □

Theorem 13.4.3 *The map* (13.28) *is an order preserving bijection.*

Proof The map (13.28) is an order preserving injection by Lemma 13.4.7. It remains to prove that it is surjective. Let $\phi \in \mathrm{PSH}(L^{\mathrm{NA}})$. We want to construct $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ with $\Gamma^{\mathrm{An}} = \phi$.

Let H be an ample line bundle and $(\epsilon_i)_i$ be a decreasing sequence of rational numbers with limit 0, $\phi_i \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L+\epsilon_i H)^{\mathrm{An}})$ such that

$$\phi = \inf_{i>0} \phi_i .$$

The existence of these data is guaranteed by Theorem 13.4.2. Fix a Kähler form $\omega \in c_1(H)$,

Thanks to Lemma 13.4.8, we can find $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\epsilon_i\omega)$ with $(\Gamma^i)^{\mathrm{An}} = \phi_i$. It follows from Lemma 13.4.7 that

$$\Gamma^i \geq P_{\theta+\epsilon_i\omega}\left[\Gamma^{i+1}\right]_{\mathcal{I}} \geq \Gamma^{i+1}.$$

Therefore, for any $\omega' \in \mathrm{Käh}(X)$, the sequence $(P_{\theta+\omega'}\left[\Gamma^i\right]_{\mathcal{I}})_i$ is decreasing. We let

$$\Gamma^{\omega'} = \inf_{i>0} P_{\theta+\omega'}\left[\Gamma^i\right]_{\mathcal{I}} \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega').$$

Note that the infimum is defined thanks to (13.29). It follows from Proposition 13.4.5 that

$$\left(\Gamma^{\omega'}\right)^{\mathrm{An}} = \phi.$$

From this, it is clear that for $\omega',\omega'' \in \mathrm{Käh}(X)$ with $\omega' \leq \omega''$, we have

$$P_{\theta+\omega''}\left[\Gamma^{\omega'}\right]_{\mathcal{I}} = \Gamma^{\omega''}.$$

It follows that $(\Gamma^{\omega'})_{\omega'\in\mathrm{Käh}(X)}$ defines an element Γ in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ and $\Gamma^{\mathrm{An}} = \phi$.□

Theorem 13.4.4 *Under the bijection Lemma 13.4.7, the operations on* $\mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *defined in Section 13.2 all correspond to the corresponding operations on* $\mathrm{PSH}(L^{\mathrm{An}})$ *in Boucksom–Jonsson's theory.*

The meaning should be clear for all operations except for the trace operator, and the proofs are elementary, as we have seen in Proposition 13.4.5 in the case of infimum operator. We shall only restate and prove the case of trace operators, and leave the remaining arguments to the readers.[5]

Theorem 13.4.5 *Let $Y \subseteq X$ be an irreducible analytic subset. Consider an element* $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)$ *with well-defined restriction to Y. Then*

$$\mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}|_{Y^{\mathrm{div}}} = \Gamma^{\mathrm{An}}|_{Y^{\mathrm{div}}}. \tag{13.33}$$

Observe that there is a canonical identification $Y^{\mathrm{div}} = \tilde{Y}^{\mathrm{div}}$. Recall that a generalized Fubini–Study metric is defined in Definition 1.8.7.

[5] In case you find any of the arguments non-trivial, please refer to [Xia25a] for the full details.

Proof We may assume that $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta)_{>0}$. Let $\phi = \Gamma^{\mathrm{An}} \in \mathrm{PSH}(L^{\mathrm{An}})$. By Lemma 13.4.9, $\phi(v_{Y,\mathrm{triv}}) \neq -\infty$.

Take an ample line bundle S on X, a Kähler form ω in $c_1(S)$. Write ϕ as the decreasing limit of a sequence ϕ^i of elements in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}((L+i^{-1}S)^{\mathrm{An}})$ as in Theorem 13.4.2.

Take $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+i^{-1}\omega)$ such that $\Gamma^{i,\mathrm{An}} = \phi^i$. Note that by Lemma 13.1.1, $\Gamma^i \in \mathrm{PSH}^{\mathrm{NA}}(X,\theta+i^{-1}\omega)_{>0}$.

It follows from Proposition 13.4.5 (applied to the images of Γ^i in $\mathrm{PSH}^{\mathrm{NA}}(X,\theta+\omega)$) that for any $\tau < \Gamma_{\max}$, we have

$$\inf_{i\to\infty} \Gamma^i_\tau = \Gamma_\tau.$$

In particular, $\Gamma^i_\tau \xrightarrow{d_{S,\theta+\omega}} \Gamma_\tau$ for all $\tau < \Gamma_{\max}$.

By Lemma 13.4.9 again, each Γ^i has non-trivial restriction to E. By Proposition 8.2.2, for any Kähler form ω' on $\tilde{Y}$ satisfying $\omega' \geq \omega|_{\tilde{Y}}$ we have

$$\mathrm{Tr}_Y\left(\Gamma_\tau^{i,\theta|_{\tilde{Y}}+\omega'}\right) \xrightarrow{d_S} \mathrm{Tr}_Y\left(\Gamma_\tau^{\theta|_{\tilde{Y}}+\omega'}\right)$$

for any $\tau < (\mathrm{Tr}_Y(\Gamma))_{\max}$. Thanks to Theorem 6.2.4,

$$\mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}(c\,\mathrm{ord}_F) = \inf_{i\geq 1} \mathrm{Tr}_Y(\Gamma)^{i,\mathrm{An}}(c\,\mathrm{ord}_F)$$

for any $c\,\mathrm{ord}_F \in Y^{\mathrm{div}}$. In particular, it suffices to prove (13.33) with Γ^i in place of Γ.

In other words, we have reduced to the case where $\phi \in \mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(L)$ and L is big.

Let $\Gamma \in \mathrm{PSH}(X,\theta)_{>0}$ with $\Gamma^{\mathrm{An}} = \phi$. By Lemma 13.4.8, we can find a concave curve $(\Gamma'_\tau)_{\tau\leq\Gamma_{\max}}$ with Γ'_τ being a generalized Fubini–Study metric for each $\tau \leq \Gamma_{\max}$ and that

$$\Gamma_\tau = P_\theta[\Gamma'_\tau].$$

It follows that for any $c\,\mathrm{ord}_F \in E^{\mathrm{div}}$,

$$\begin{aligned}
\phi|_{Y^{\mathrm{An}}}(c\,\mathrm{ord}_F) &= \sup_{\tau<\Gamma_{\max}} \left(\Gamma'^{\,\mathrm{An}}_\tau(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\left(\Gamma'_\tau|_{\tilde{Y}}\right)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\mathrm{Tr}_Y(\Gamma'_\tau)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \sup_{\tau<\Gamma_{\max}} \left(\mathrm{Tr}_Y(\Gamma_\tau)^{\mathrm{An}}(c\,\mathrm{ord}_F) + \tau\right)\\
&= \mathrm{Tr}_Y(\Gamma)^{\mathrm{An}}(c\,\mathrm{ord}_F).
\end{aligned}$$

The third equality follows from Proposition 8.2.1. It remains to justify the second line. Namely, we want to show that for any generalized Fubini–Study metric φ, we have

$$\varphi^{\mathrm{An}}(c \operatorname{ord}_F) = \left(\varphi|_{\tilde{Y}}\right)(c \operatorname{ord}_F). \tag{13.34}$$

We could immediately reduce to the case where φ is a Fubini–Study metric, and then to the case

$$\varphi = \log |s|^2_{h_0},$$

where s is a holomorphic section of L, not vanishing identically on Y, in which case (13.34) is obvious. □

Lemma 13.4.9 *Let $\Gamma \in \mathrm{PSH}^{\mathrm{NA}}(X, \theta)$ and $Y \subseteq X$ be an irreducible analytic subset. Then the following are equivalent:*

(1) $\Gamma^{\mathrm{An}}(v_{Y,\mathrm{triv}}) \neq -\infty$;
(2) $\Gamma^{\mathrm{An}}|_{Y^{\mathrm{An}}} \not\equiv -\infty$;
(3) Γ *has a well-defined restriction to* Y.

Here $v_{Y,\mathrm{triv}}$ denotes the trivial valuation of $\mathbb{C}(Y)$.

Proof The equivalence between (1) and (2) is a simple consequence of the maximum principle [BJ22a, Lemma 1.4(i)].

To see the equivalence between (1) and (3), it suffices to observe that for any $\varphi \in \mathrm{PSH}(X, \theta)$,

$$\varphi^{\mathrm{An}}(v_{Y,\mathrm{triv}}) = \begin{cases} -\infty, & \text{if } \nu(\varphi, Y) > 0; \\ 0, & \text{if } \nu(\varphi, Y) = 0. \end{cases}$$

Chapter 14
Partial Bergman kernels

I speak twelve languages: English is the bestest.
— *Stefan Bergman*[a]

[a] Stefan Bergman (1895–1977), bearing a very Scandinavian name, was a Polish-American mathematician best known for his work in complex analysis, especially in several complex variables. He introduced the Bergman kernel, a fundamental concept in complex analysis that has influenced many areas of mathematics and theoretical physics.

Bergman was born in Poland (then part of the Russian Empire), and studied in Berlin. He fled Europe during World War II and eventually settled in the United States.

In this chapter, we prove the convergence of the partial Bergman kernels under very mild assumptions. The partial Bergman kernels are simply the Bergman kernels defined the L^2-integrable holomorphic sections of a line bundle with respect to a given psh weight. Our main result is Theorem 14.2.1, extending the celebrated result [BBWN11]. We strongly recommend that the readers read the well-written paper [BBWN11] before starting this chapter.

14.1 Partial envelopes

In this section, let X be a connected compact Kähler manifold of dimension n and $K \subseteq X$ be a closed non-pluripolar set. Let θ be a smooth closed real $(1,1)$-form on X representing a pseudo-effective cohomology class. Fix $\varphi \in \mathrm{PSH}(X,\theta)$.

Definition 14.1.1 Given a function $v\colon K \to [-\infty,\infty)$, we introduce the *relative P-envelope* of φ (with respect to K, v, θ) as

$$P_{\theta,K}[\varphi](v) := \sup{}^* \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \leq \varphi\}\,. \tag{14.1}$$

Similarly, we define the *relative $\mathcal{I}$-envelope* of φ (with respect to K, v, θ) as

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) := \sup{}^* \{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \preceq_{\mathcal{I}} \varphi\}\,. \tag{14.2}$$

Observe that when v is bounded, neither envelope is identically $-\infty$. When $K = X$ and $v = 0$, these definitions reduce to the usual P-envelope and $\mathcal{I}$-envelope of φ studied in Chapter 3.

It would be helpful to consider the following auxiliary functions:

$$P'_{\theta,K}[\varphi](v) := \sup\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \leq \varphi\},$$
$$P'_{\theta,K}[\varphi]_{\mathcal{I}}(v) := \sup\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v \text{ and } \eta \leq_{\mathcal{I}} \varphi\}.$$

We note the following maximum principles, that follow from the above definitions:

Lemma 14.1.1 *Let $v \in C^0(K)$. Let $\eta \in \mathrm{PSH}(X,\theta)$. Assume that $\eta \leq \varphi$, then*

$$\sup_K(\eta - v) = \sup_{\{\eta\neq-\infty\}}(\eta - P'_{\theta,K}[\varphi](v)) = \sup_{\{P'_{\theta,K}[\varphi](v)\neq-\infty\}}\left(\eta - P'_{\theta,K}[\varphi](v)\right). \tag{14.3}$$

Proof We prove the first equality at first. We write $S = \{\eta = -\infty\}$.

By definition, $P'_{\theta,K}[\varphi](v)|_K \leq v$, so

$$\left(h - P'_{\theta,K}[\varphi](v)\right)\Big|_{K\setminus S} \geq \eta|_{K\setminus S} - v|_{K\setminus S}.$$

This implies that

$$\sup_K(\eta - v) \leq \sup_{X\setminus S}(\eta - P'_{\theta,K}[\varphi](v)).$$

Conversely, observe that $\sup_K(\eta - v) > -\infty$ as K is non-pluripolar. Let $\eta' := \eta - \sup_K(\eta - v)$, then η' is a candidate in the definition of $P'_{\theta,K}[\varphi](v)$, hence $\eta' \leq P'_{\theta,K}[\varphi](v)$, namely,

$$\eta - \sup_K(\eta - v) \leq P'_{\theta,K}[\varphi](v),$$

the latter implies that

$$\sup_K(\eta - v) \geq \sup_{X\setminus S}(\eta - P'_{\theta,K}[\varphi](v)),$$

finishing the proof of the first identity.

We have $\{P'_{\theta,K}[\varphi](v) = -\infty\} \subseteq S$, and we notice that points in $S \setminus \{P'_{\theta,K}[\varphi](v) = -\infty\}$ do not contribute to the supremum of $\eta - P'_{\theta,K}[\varphi](v)$ on $X\setminus\{P'_{\theta,K}[\varphi](v) = -\infty\}$, hence the last equality of (14.3) also follows. □

Next, we make the following observations about the singularity types of our envelopes:

Lemma 14.1.2 *For any $v \in C^0(K)$ we have*

$$P_{\theta,K}[\varphi](v) \sim P_\theta[\varphi], \quad P_{\theta,K}[\varphi]_{\mathcal{I}}(v) \sim P_\theta[\varphi]_{\mathcal{I}}.$$

If φ has analytic singularities, we have

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[\varphi]_{\mathcal{I}}(v). \tag{14.4}$$

Proof Let $C > 0$ such that $-C \leq v \leq C$. Then

$$P_\theta[\varphi] - C \leq P_{\theta,K}[\varphi](v).$$

Since K is non-pluripolar, for $\eta \in \mathrm{PSH}(X,\theta)$ the condition $\eta|_K \leq v \leq C$ implies that $\eta \leq \tilde{C}$ on X for some $\tilde{C} := \tilde{C}(C,K) > 0$ by Remark 1.5.2. This implies that

$$P_{\theta,K}[\varphi](v) \leq P_\theta[\varphi] + \tilde{C},$$

giving

$$P_{\theta,K}[\varphi](v) \sim P_\theta[\varphi].$$

The exact same argument applies in case of the relative $\mathcal{I}$-envelope.

Next assume that φ has analytic singularities, then we have that

$$\varphi \sim P_\theta[\varphi]_{\mathcal{I}}$$

by Proposition 3.2.10. In particular, for $\eta \in \mathrm{PSH}(X,\theta)$, $\eta \leq \varphi$ if and only if $\eta \leq P_\theta[\varphi]_{\mathcal{I}}$. So (14.4) follows. □

Corollary 14.1.1 *Let $v \in C^0(X)$. Then*

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) = P_{\theta,X}\left[P_{\theta,K}[\varphi]_{\mathcal{I}}(v)\right]_{\mathcal{I}}(v).$$

Proof By definition, we have

$$\begin{aligned}
&P_{\theta,X}[P_{\theta,K}[\varphi]_{\mathcal{I}}(v)]_{\mathcal{I}}(v)\\
=&\sup{}^*\left\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v, \eta \leq_{\mathcal{I}} P_{\theta,K}[\varphi]_{\mathcal{I}}(v)\right\}\\
=&\sup{}^*\{\eta \in \mathrm{PSH}(X,\theta) : \eta|_K \leq v, \eta \leq_{\mathcal{I}} \varphi\}\\
=&P_{\theta,K}[\varphi]_{\mathcal{I}}(v),
\end{aligned}$$

where we applied Lemma 14.1.2 on the third line. □

Lemma 14.1.3 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Let $v \in C^0(K)$. Let $S \subseteq X$ be a pluripolar set and $\eta \in \mathrm{PSH}(X,\theta)_{>0}$ with $\eta \leq \varphi$. Assume that $\eta|_{K\setminus S} \leq v|_{K\setminus S}$, then $\eta \leq P_{\theta,K}[\varphi](v)$.*

Proof By Theorem 1.1.5, there is $\chi \in \mathrm{PSH}(X,\theta)$, such that $\chi|_S \equiv -\infty$. We claim that we can choose χ so that

$$\chi \leq \eta.$$

In fact, since $\int_X \theta_\eta^n > 0$, fixing some χ and $\epsilon \in (0,1)$ small enough, we have

$$\int_X \theta^n_{\epsilon\chi+(1-\epsilon)V_\theta} + \int_X \theta^n_\eta > \int_X \theta^n_{V_\theta}.$$

Thus, by Proposition 3.1.5, we have

$$(\epsilon\chi + (1-\epsilon)V_\theta) \wedge \eta \in \mathrm{PSH}(X,\theta).$$

It suffices to replace χ by $(\epsilon\chi + (1-\epsilon)V_\theta) \wedge \eta$.

Fix $\chi \leq \eta$ as above. For any $\delta \in (0,1)$, we have

$$(1-\delta)\eta|_K + \delta\chi|_K \leq v, \quad (1-\delta)\eta + \delta\chi \leq \varphi.$$

Hence,

$$(1-\delta)\eta + \delta\chi \leq P_{\theta,K}[\varphi](v).$$

Letting $\delta \to 0+$, we conclude that $\eta \leq P_{\theta,K}[\varphi](v)$. □

Corollary 14.1.2 *Assume that $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Let $v \in C^0(K)$. Then*

$$P_{\theta,K}[\varphi](v) = P_{\theta,X}[\varphi]\left(P_{\theta,K}[V_\theta](v)\right).$$

Proof It is clear that

$$P_{\theta,K}[\varphi](v) \leq P_{\theta,X}[\varphi]\left(P_{\theta,K}[V_\theta](v)\right).$$

For the reverse direction, it suffices to prove that any $\eta \in \mathrm{PSH}(X,\theta)$ such that

$$\eta \leq \varphi, \quad \eta \leq P_{\theta,K}[V_\theta](v),$$

we have

$$\eta \leq P_{\theta,K}[\varphi](v). \tag{14.5}$$

As φ has positive mass, we can assume that η has positive mass as well. Let

$$S = \left\{P_{\theta,K}[V_\theta](v) > P'_{\theta,K}[V_\theta](v)\right\}.$$

By Proposition 1.2.5, S is a pluripolar set. Observe that

$$\eta|_{K\setminus S} \leq v|_{K\setminus S}.$$

Hence, (14.5) follows from Lemma 14.1.3. □

The next result motivates our terminology to call the measures $\theta^n_{P_{\theta,K}[\varphi](v)}$ the *partial equilibrium measures* of our context:

Lemma 14.1.4 *Let $v \in C^0(K)$. Then*

$$\int_{X\setminus K} \theta^n_{P_{\theta,K}[\varphi](v)} = 0.$$

Moreover, $P_{\theta,K}[\varphi](v)|_K = v$ almost everywhere with respect to $\theta^n_{P_{\theta,K}[\varphi](v)}$. More precisely, we have

$$\theta^n_{P_{\theta,K}[\varphi](v)} \leq \mathbb{1}_{K\cap\{P_{\theta,K}[\varphi](v)=P_{\theta,K}[V_\theta](v)=v\}}\,\theta^n_{P_{\theta,K}[V_\theta](v)}. \tag{14.6}$$

Proof **Step 1**. We address the case where $\varphi = V_\theta$.

Let $S \subseteq X$ be a closed pluripolar set, such that V_θ is locally bounded on $X \setminus S$. This is possible because we can always find a Kähler current with analytic singularities in the cohomology class $[\theta]$, as a consequence of Theorem 1.6.2.

For the first assertion, it suffices to show that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge any open ball $B \Subset X \setminus (S \cup K)$.

By Proposition 1.2.2, we can take an increasing sequence $(\eta_j)_j$ in $\mathrm{PSH}(X,\theta)$ such that

$$\eta_j \to P_{\theta,K}[V_\theta](v) \text{ almost everywhere}, \quad \eta_j|_K \leq v \text{ for all } j \geq 1.$$

By [BT82, Proposition 9.1], for each $j \geq 1$, we can find $\gamma_j \in \mathrm{PSH}(X,\theta)$, such that $(\theta + \mathrm{dd^c}\gamma_j|_B)^n = 0$ and w_j agrees with η_j outside B. Note that $(\gamma_j)_j$ is clearly increasing and

$$\gamma_j \geq \eta_j, \quad \gamma_j|_K \leq v.$$

for all $j \geq 1$.

It follows that γ_j converges to $P_{\theta,K}[V_\theta](v)$ almost everywhere as well. By Theorem 2.4.3, we find that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge B, as desired.

For the second assertion, let $x \in (X \setminus S) \cap K$ be a point such that $P_{\theta,K}[V_\theta](v)(x) < v(x) - \epsilon$ for some $\epsilon > 0$. Let B be a ball centered at x, small enough so that θ has a local potential on B, allowing us to identify θ-psh functions with psh functions (on B). By shrinking B, we can further guarantee

(1) $\overline{B} \subseteq X \setminus S$.
(2) $P_{\theta,K}[V_\theta](v)|_{\overline{B}} < v(x) - \epsilon$.
(3) $v|_{\overline{B}\cap K} > v(x) - \epsilon$.

Construct the sequences η_j, γ_j as above. On B, by choosing a local potential of θ, we may identify η_j, γ_j with the corresponding psh functions in a neighborhood of $\overline{B}$. By (2), we have $\gamma_j \leq v(x) - \epsilon$ on ∂B, hence by the comparison principle, $\gamma_j|_B \leq v(x) - \epsilon$. By (3), we have $\gamma_j|_{B\cap K} \leq v|_{B\cap K}$. Thus, we conclude that $\theta^n_{P_{\theta,K}[V_\theta](v)}$ does not charge B, as in the previous paragraph.

Step 2. We handle the general case. We can assume $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Indeed, due to Lemma 14.1.2 and Theorem 2.4.4, we have that

$$\int_X \theta^n_{P_{\theta,K}[\varphi](v)} = \int_X \theta^n_\varphi.$$

Hence, there is nothing to prove if $\int_X \theta^n_\varphi = 0$.

By Corollary 14.1.2,

$$P_{\theta,K}[\varphi](v) = P_{\theta,X}[\varphi](P_{\theta,K}[V_\theta](v)).$$

Now Corollary 3.1.1 gives

$$\begin{aligned}\theta^n_{P_{\theta,K}[\varphi](v)} \leq & \mathbb{1}_{\{P_{\theta,K}[\varphi](v) = P_{\theta,K}[V_\theta](v)\}} \theta^n_{P_{\theta,K}[V_\theta](v)} \\ \leq & \mathbb{1}_{\{P_{\theta,K}[\varphi](v) = v\}} \theta^n_{P_{\theta,K}[V_\theta](v)},\end{aligned}$$

where in the second inequality we have used Step 1. □

Corollary 14.1.3 *Let $v \in C^0(K)$. Then*

$$\begin{aligned} \int_{(X\setminus K)\cup\{P_{\theta,K}[\varphi](v)<v\}} \theta^n_{P_{\theta,K}[\varphi](v)} &= 0, \\ \int_{(X\setminus K)\cup\{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)<v\}} \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} &= 0. \end{aligned} \tag{14.7}$$

Proof The first equation in (14.7) follows from Lemma 14.1.4. For the second, we can assume that

$$\int_X \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} > 0, \tag{14.8}$$

otherwise there is nothing to prove. By definition, we have

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) = P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v).$$

Next we show that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) = P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}](v).$$

The $\geq$ direction is trivial. It remains to prove the reverse inequality. By Lemma 14.1.2, we get that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \sim P_\theta[\varphi]_{\mathcal{I}}.$$

Due to Proposition 1.2.5, we get that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \leq v$$

on $K \setminus S$, where $S \subseteq X$ is a pluripolar set. As a result, due to (14.8), Lemma 14.1.3 allows to conclude that

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) \leq P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}](v).$$

Since

$$P_{\theta,K}[P_\theta[\varphi]_{\mathcal{I}}]_{\mathcal{I}}(v) = P_{\theta,K}[\varphi]_{\mathcal{I}}(v),$$

we get that the second equation in (14.7), using the first. □

Proposition 14.1.1 *Assume that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$. Let $v \in C^0(K)$. Then*

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_\theta[\varphi]](v). \tag{14.9}$$

In particular,

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_{\theta,K}[\varphi](v)](v).$$

Proof The $\leq$ direction in (14.9) is obvious. We to prove the reverse inequality. As $P_{\theta,K}[\varphi](v)$ and $P_{\theta,K}[P_\theta[\varphi]](v)$ have the same singularity types by Lemma 14.1.2, by the domination principle Theorem 2.4.6, it suffices to show that

$$P_{\theta,K}[\varphi](v) \geq P_{\theta,K}[P_\theta[\varphi]](v) \text{ almost everywhere with respect to } \theta^n_{P_{\theta,K}[\varphi](v)}. \tag{14.10}$$

By (14.6),

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[V_\theta](v) = v$$

almost everywhere with respect to $\theta^n_{P_{\theta,K}[\varphi](v)}$. Hence,

$$P_{\theta,K}[P_\theta[\varphi]](v) = v$$

almost everywhere with respect to $\theta^n_{P_{\theta,K}[\varphi](v)}$. We conclude that

$$P_{\theta,K}[\varphi](v) = P_{\theta,K}[P_\theta[\varphi]](v).$$

Finally, (14.10) follows from Lemma 14.1.2 and (14.9). □

Definition 14.1.2 Given $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$, the *partial equilibrium energy* functional $\mathcal{E}^\theta_{[\varphi],K} \colon C^0(K) \to \mathbb{R}$ of $v \in C^0(K)$ as follows

$$\mathcal{E}^\varphi_{\theta,K}(v) := E_\theta^{P_\theta[\varphi]_{\mathcal{I}}}(P_{\theta,K}[\varphi]_{\mathcal{I}}(v)). \tag{14.11}$$

Recall that the energy $E_\theta^{P_\theta[\varphi]_{\mathcal{I}}}$ functional is defined in Definition 3.1.5.

Note that by Lemma 14.1.2, we have

$$P_{\theta,K}[\varphi]_{\mathcal{I}}(v) \in \mathcal{E}^\infty(X,\theta;P_\theta[\varphi]_{\mathcal{I}}),$$

so $\mathcal{E}^\varphi_{\theta,K}(v) \in \mathbb{R}$.

Proposition 14.1.2 *Let $K \subseteq X$ be a closed non-pluripolar set, $v, f \in C^0(K)$ and $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$. Then $\mathbb{R} \ni t \mapsto \mathcal{E}^\varphi_{\theta,K}(v+tf)$ is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^\varphi_{\theta,K}(v+tf) = \int_K f\,\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)} \tag{14.12}$$

for all $t \in \mathbb{R}$.

Proof We may assume that φ is $\mathcal{I}$-model by replacing φ by $P_\theta[\varphi]_{\mathcal{I}}$.

Note that it suffices to prove (14.12) at $t = 0$, which is equivalent to

$$\lim_{t\to 0} \frac{E^\varphi_\theta(P_{\theta,K}[\varphi]_{\mathcal{I}}(v+tf)) - E^\varphi_\theta(P_{\theta,K}[\varphi]_{\mathcal{I}}(v))}{t} = \int_K f\,\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}. \tag{14.13}$$

By switching f to $-f$, we may assume that $t > 0$ in the above limit.

By the comparison principle (3.24) and Proposition 3.1.15, we find

$$\begin{aligned}
&E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)) - E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v))\\
=&\frac{1}{n+1}\sum_{i=0}^n \int_X (P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\, \theta^i_{P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)} \wedge \theta^{n-i}_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}\\
\leq& \int_X (P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}.
\end{aligned}$$

By Lemma 14.1.4,

$$\int_X (P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi]_\mathcal{I}(v))\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)} \leq t\int_K f\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}.$$

Thus, we get the inequality,

$$\varlimsup_{t\to 0+} \frac{E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)) - E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v))}{t} \leq \int_K f\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v)}.$$

Similarly, we have

$$\begin{aligned}
&E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)) - E_\theta^\varphi(P_{\theta,K}[\varphi]_\mathcal{I}(v))\\
\geq& \int_X (P_{\theta,K}[\varphi]_\mathcal{I}(v+tf) - P_{\theta,K}[\varphi](v))\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)}\\
\geq& t\int_K f\, \theta^n_{P_{\theta,K}[\varphi]_\mathcal{I}(v+tf)}.
\end{aligned}$$

Together with the above, this implies (14.13). □

Lemma 14.1.5 *Fix a Kähler form ω on X. For $v \in C^0(K)$ there exists an increasing bounded sequence $(v_j^-)_j$ in $C^\infty(X)$ and a decreasing bounded sequence $(v_j^+)_j$ in $C^\infty(X)$, such that for all $\varphi \in \mathrm{PSH}(X,\theta)_{>0}$ and $\delta \in [0,1]$ we have*

(1) $P_{\theta+\delta\omega,X}[\varphi](v_j^+) \searrow P_{\theta+\delta\omega,K}[\varphi](v)$,
(2) $P_{\theta+\delta\omega,X}[\varphi](v_j^-) \nearrow P_{\theta+\delta\omega,K}[\varphi](v)$ *almost everywhere,*
(3) $\sup_X |v_j^-| \leq C$, $\sup_X |v_j^+| \leq C$ *for some constant C depending only on $\|v\|_{C^0(K)}$, K and $\theta+\omega$, and*
(4)
$$\lim_{j\to\infty} \mathcal{E}^\varphi_{\theta,K}(v_j^-) = \mathcal{E}^\varphi_{\theta,K}(v), \quad \lim_{j\to\infty} \mathcal{E}^\varphi_{\theta,K}(v_j^+) = \mathcal{E}^\varphi_{\theta,K}(v).$$

Proof We fix $\delta \in [0,1]$. First we prove the existence of $(v_j^-)_j$. Let

$$C_{K,v} := \sup\left\{\sup_X \eta : \eta \in \mathrm{PSH}(X,\theta+\omega), \eta|_K \leq v\right\}.$$

Since K is non-pluripolar, we have that $C_{K,v} \in \mathbb{R}$. Now define $\tilde{v}\colon X \to \mathbb{R}$ as

$$\tilde{v}(x) = \begin{cases} v(x), & x \in K;\\ C_{k,v}+1, & x \in X\setminus K.\end{cases}$$

Since $\tilde{v}$ is lower semicontinuous, there exists an increasing and uniformly bounded sequence $(v_j^-)_j$ in $C^\infty(X)$, such that $v_j^- \nearrow \tilde{v}$.

Observe that $P_{\theta+\delta\omega,X}[\varphi](v_j^-)$ is increasing in $j \geq 1$, and

$$P_{\theta+\delta\omega,X}[\varphi](v_j^-) \leq P_{\theta+\delta\omega,K}[\varphi](v).$$

To prove that

$$P_{\theta+\delta\omega,X}[\varphi](v_j^-) \nearrow P_{\theta+\delta\omega,K}[\varphi](v)$$

almost everywhere, let η be a candidate for $P_{\theta+\delta\omega,K}[\varphi](v)$ such that $\sup_K(\eta - v) < 0$. Then, since η is upper semicontinuous and $\eta < \tilde{v}$, by Dini's lemma there exists $j_0 > 0$ such that $\eta < v_j^-$ for $j \geq j_0$, i.e.

$$\eta \leq P_{\theta+\delta\omega,X}[\varphi](v_j^-),$$

proving existence of $(v_j^-)_j$.

Next, we prove the existence of $(v_j^+)_j$. Since

$$h \coloneqq P_{\theta+\omega,K}[V_{\theta+\omega}](v) \vee (\inf_K v - 1)$$

is usc, there exists a decreasing and uniformly bounded sequence $(v_j^+)_j$ in $C^\infty(X)$, such that $v_j^+ \searrow h$. Trivially,

$$\chi \coloneqq \lim_{j\to\infty} P_{\theta+\delta\omega,X}[\varphi](v_j^+) \geq P_{\theta+\delta\omega,K}[\varphi](v).$$

In particular, χ has positive mass, since it has the same singularity types as $P_{\theta+\delta\omega,K}[\varphi](v)$ by Lemma 14.1.2. We introduce

$$S \coloneqq \left\{P'_{\theta+\omega,K}[V_{\theta+\omega}](v) < P_{\theta+\omega,K}[V_{\theta+\omega}](v)\right\}.$$

By Proposition 1.2.5, S is a pluripolar set. Observe that

$$P_{\theta+\delta\omega,X}[\varphi](v_j^+) \leq v_j^+$$

for all $j \geq 1$. Thus, $\chi \leq h$. On the other hand, $h \leq v$ on $K \setminus S$. So in particular, $\chi|_{K\setminus S} \leq v|_{K\setminus S}$. By Lemma 14.1.2 we also have that $\chi \sim P_{\theta+\delta\omega,K}[\varphi](v)$. Hence, by Lemma 14.1.3,

$$\chi \leq P_{\theta+\delta\omega,K}[P_{\theta+\delta\omega,K}[\varphi](v)](v) = P_{\theta+\delta\omega,K}[\varphi](v),$$

where we also used the last statement of Proposition 14.1.1.

Finally observe that (4) follows from Lemma 14.1.2, Lemma 14.1.5 and Theorem 2.4.3. □

Proposition 14.1.3 *Let $K \subseteq X$ be a compact and non-pluripolar subset. Let $v \in C^0(K)$. Let $\varphi_j, \varphi \in \mathrm{PSH}(X,\theta)_{>0}$ ($j \geq 1$) with $\varphi_j \xrightarrow{d_S} \varphi$. Then the following hold:*

(1) *If* $\varphi_j \searrow \varphi$, *then* $P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) \searrow P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$ *and* $P_{\theta,K}[\varphi_j](v) \searrow P_{\theta,K}[u](v)$.

(2) *If* $\varphi_j \nearrow \varphi$ *almost everywhere then* $P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) \nearrow P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$ *almost everywhere, and* $P_{\theta,K}[\varphi_j](v) \nearrow P_{\theta,K}[\varphi](v)$ *almost everywhere.*

Proof (1) By Theorem 6.2.1, we have

$$\lim_{j\to\infty} \int_X \theta^n_{\varphi_j} = \int_X \theta^n_{\varphi}.$$

It follows from Lemma 2.4.2 that there is a decreasing sequence $\epsilon_j \searrow 0$ with $\epsilon_j \in (0,1)$ and $\eta_j \in \mathrm{PSH}(X,\theta)$ such that

$$(1-\epsilon_j)\varphi_j + \epsilon_j \eta_j \leq \varphi.$$

By the concavity similar to Proposition 3.2.11, we get

$$\begin{aligned}(1-\epsilon_j)P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) + \epsilon_j P_{\theta,K}[\eta_j]_{\mathcal{I}}(v) \leq & P_{\theta,K}[(1-\epsilon_j)\varphi_j + \epsilon_j\eta_j]_{\mathcal{I}}(v)\\ \leq & P_{\theta,K}[\varphi]_{\mathcal{I}}(v).\end{aligned}$$

Since $(\varphi_j)_j$ is decreasing, so is $(P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v))_j$, hence

$$\psi := \lim_{j\to\infty} P_\theta[\varphi_j]_{\mathcal{I}}(v) \geq P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$$

exists. Since $\epsilon_j \to 0$ and $\sup_X P_{\theta,K}[\eta_j]_{\mathcal{I}}(v)$ is bounded, we can let $j \to \infty$ in the above estimate to conclude that

$$\psi = P_{\theta,K}[\varphi]_{\mathcal{I}}(v).$$

The same ideas yield that

$$P_{\theta,K}[\varphi_j](v) \searrow P_{\theta,K}[\varphi](v).$$

The proof of (2) is similar and is left to the readers. □

14.2 Quantization of partial equilibrium measures

Let X be a connected compact Kähler manifold of dimension n and L be a pseudo-effective line bundle on X. Let h be a Hermitian metric on L and set $\theta = c_1(L,h)$. Let (T, h_T) be a Hermitian line bundle on X. Take a Kähler form ω on X so that

$$\int_X \omega^n = 1.$$

14.2.1 Bernstein–Markov measures

Let $K \subseteq X$ be a closed non-pluripolar subset. Let v be a measurable function on K and let μ be a positive Borel probability measure on K. We introduce the following functions on $\mathrm{H}^0(X, L^k \otimes T)$ ($k \geq 1$), with values possibly equaling ∞:

$$N^k_{v,\upsilon}(s) := \left(\int_K h^k \otimes h_T(s,s)\mathrm{e}^{-kv}\, \mathrm{d}\upsilon \right)^{1/2},$$

$$N^k_{v,K}(s) := \sup_{K \setminus \{v=-\infty\}} \left(h^k \otimes h_T(s,s)\mathrm{e}^{-kv} \right)^{1/2}.$$

We start with the following elementary observation:

Lemma 14.2.1 *Let $v_1 \leq v_2$ be two measurable functions on X. Assume that $\{v_1 = -\infty\} = \{v_2 = -\infty\}$. Then for any $s \in \mathrm{H}^0(X, L^k \otimes T)$ ($k \geq 1$), we have*

$$N^k_{v_1,K}(s) \geq N^k_{v_2,K}(s).$$

If υ puts no mass on $\{v = -\infty\}$ then we always have

$$N^k_{v,\upsilon}(s) \leq N^k_{v,K}(s). \tag{14.14}$$

Definition 14.2.1 A *weighted subset* of X is a pair (K, v) consisting of a closed non-pluripolar subset $K \subseteq X$ and a function $v \in C^0(K)$.

Definition 14.2.2 Let (K, v) be a weighted subset of X. A positive Borel probability measure υ on K is *Bernstein–Markov* with respect to (K, v) if for each $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$N^k_{v,K}(s) \leq C_\epsilon \mathrm{e}^{\epsilon k} N^k_{v,\upsilon}(s) \tag{14.15}$$

for any $s \in \mathrm{H}^0(X, L^k \otimes T)$ and any $k \in \mathbb{N}$. We write $\mathrm{BM}(K, v)$ for the set of Bernstein–Markov measures with respect to (K, v).

As pointed out in [BBWN11], any volume form on X is Bernstein–Markov with respect to (X, v), with $v \in C^\infty(X)$.

Proposition 14.2.1 *Assume that (K, v) is a weighted subset of X, then*

(1) $N^k_{v,K}$ *is a norm on* $\mathrm{H}^0(X, L^k \otimes T)$.
(2) *For any $\upsilon \in \mathrm{BM}(K, v)$, $N^k_{v,\upsilon}$ is a norm on* $\mathrm{H}^0(X, L^k \otimes T)$.

Proof (1) As v is bounded, $N^k_{v,K}$ is clearly finite on $\mathrm{H}^0(X, L^k \otimes T)$. In order to show that it is a norm, it suffices to show that for any $s \in \mathrm{H}^0(X, L^k \otimes T)$, $N^k_{v,K}(s) = 0$ implies that $s = 0$. In fact, we have $s|_K = 0$, hence $s = 0$ by the connectedness of X.

(2) As v is bounded, clearly $N^k_{v,\upsilon}$ is finite and satisfies the triangle inequality. Non-degeneracy follows from the fact that $N^k_{v,K}$ is a norm and (14.15). □

14.2.2 Partial Bergman kernels

In this section, we fix a weighted subset (K, ν) of X and $\upsilon \in \mathrm{BM}(K, \nu)$.

Definition 14.2.3 For any $\varphi \in \mathrm{PSH}(X, \theta)$, we introduce the *partial Bergman kernels* of φ (with respect to (K, ν)) as follows: For any $k \geq 0$, we introduce

$$\begin{aligned} B^k_{\nu,\varphi,\upsilon}(x) \coloneqq \sup\big\{ & h^k \otimes h_T(s,s)\mathrm{e}^{-k\nu}(x) : N^k_{\nu,\upsilon}(s,s) \leq 1, \\ & s \in \mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))\big\}, \quad x \in K. \end{aligned} \tag{14.16}$$

We extend $B^k_{\nu,\varphi,\upsilon}$ to the whole X by setting it to be 0 outside K.

The *partial Bergman measures* of φ (with respect to (K, ν)) are defined as

$$\beta^k_{\nu,\varphi,\upsilon} \coloneqq \frac{n!}{k^n} B^k_{\nu,\varphi,\upsilon}\,\mathrm{d}\upsilon \tag{14.17}$$

for each $k \geq 0$.

Observe that

$$\int_K \beta^k_{\nu,\varphi,\upsilon} = \frac{n!}{k^n} h^0(X, T \otimes L^k \otimes \mathcal{I}(k\varphi)). \tag{14.18}$$

The goal of this section is to prove the following theorem:

Theorem 14.2.1 *Suppose that $\varphi \in \mathrm{PSH}(X, \theta)_{>0}$. Let (K, ν) be a weighed subset of X, let $\upsilon \in \mathrm{BM}(K, \nu)$. Then*

$$\beta^k_{\nu,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(\nu)} \tag{14.19}$$

as $k \to \infty$.

Proposition 14.2.2 *Let $\varphi \in \mathrm{PSH}(X, \theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. If $\nu \in C^\infty(X)$, then*

$$\beta^k_{\nu,\varphi,\omega^n} \rightharpoonup \theta^n_{P_{\theta,X}[\varphi]_{\mathcal{I}}(\nu)} = \theta^n_{P_{\theta,X}[\varphi](\nu)} \tag{14.20}$$

as $k \to \infty$.

Proof The equality part in (14.20) follows from Lemma 14.1.2. We start with noticing that as $k \to \infty$,

$$\beta^k_{\nu,\varphi,\omega^n} \leq \beta^k_{\nu,V_\theta,\omega^n} \rightharpoonup \theta^n_{P_{\theta,X}[V_\theta](\nu)} = \mathbb{1}_{\{\nu = P_{\theta,X}[V_\theta](\nu)\}}\theta^n_\nu,$$

where the convergence follows from [Ber09, Theorem 1.2], and the last identity is due to [DNT21, Corollary 3.4]. Let μ be the weak limit of a subsequence of $\beta^k_{\nu,\varphi,\omega^n}$, then we obtain that

$$\mu \leq \mathbb{1}_{\{\nu = P_{\theta,X}[V_\theta](\nu)\}}\theta^n_\nu\,. \tag{14.21}$$

Let $k \geq 0$, $s \in \mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))$ be a section such that $N^k_{\nu,\omega^n}(s,s) \leq 1$. Then by [Ber09, Lemma 4.1], there exists $C > 0$ such that

$$h^k \otimes h_T(s,s)\mathrm{e}^{-kv} \leq B^k_{v,\varphi,\omega^n} \leq B^k_{v,V_\theta,\omega^n} \leq k^n C.$$

This implies that

$$\frac{1}{k}\log h^k \otimes h_T(s,s) \leq v + \frac{\log C}{k} + n\frac{\log k}{k}.$$

We define φ_k as in Proposition 1.8.2. Take $\alpha_k \nearrow 1$ as in Proposition 1.8.2. Then

$$\frac{1}{k}\log h^k \otimes h_T(s,s) \leq \varphi_k \leq \alpha_k\varphi.$$

Let $\epsilon > 0$. We notice that $\frac{1}{k}\log h^k \otimes h_T(s,s) \in \mathrm{PSH}(X, \theta + \epsilon\omega)$ for all $k \geq k_0(\epsilon)$. In particular,

$$\frac{1}{k}\log h^k \otimes h_T(s,s) - \frac{\log C}{k} - n\frac{\log k}{k} \leq P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](v).$$

Now taking supremum over all candidates s, we obtain that

$$B^k_{v,\varphi,\omega^n} \leq Ck^n\mathrm{e}^{k(P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](v)-v)}, \quad k \geq k_0. \tag{14.22}$$

We claim that μ does not put mass on $\{P_{\theta+\epsilon\omega,X}[\varphi](v) < v\}$ for any $\epsilon > 0$. Since

$$P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](v) \searrow P_{\theta+\epsilon\omega,X}[\varphi](v)$$

by Proposition 14.1.3, we get that

$$\{P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](v) < v\} \nearrow \{P_{\theta+\epsilon\omega,X}[\varphi](v) < v\}.$$

As a result, to argue the claim, it suffices to show that μ does not put mass on the set $\{P_{\theta+\epsilon\omega,X}[\alpha_k\varphi](v) < v\}$ for any k. Note that the latter set is open, hence (14.22) implies our claim.

Since φ has analytic singularities, we have that

$$P_{\theta+\epsilon\omega,X}[\varphi](v) \sim \varphi$$

for all $\epsilon \geq 0$ by Lemma 14.1.2 and Proposition 3.2.10. As a result,

$$P_{\theta+\epsilon\omega,X}[\varphi](v) \searrow P_{\theta,X}[\varphi](v),$$

and we can let $\epsilon \searrow 0$ to conclude that μ does not put mass on $\{P_{\theta,X}[\varphi](v) < v\} = \bigcup_{\epsilon>0}\{P_{\theta+\epsilon\omega,X}[\varphi](v) < v\}$. Putting this together with (14.21), we obtain that

$$\mu \leq \mathbb{1}_{\{P_{\theta,X}[\varphi](v)=v\}}\theta_v^n = \theta^n_{P_{\theta,X}[\varphi](v)},$$

where the last equality is due to [DNT21, Corollary 3.4]. Comparing total masses via (14.18) and Theorem 7.4.1, we conclude that $\mu = \theta^n_{P_{\theta,X}[\varphi](v)}$. As μ is an arbitrary cluster point of $\beta^k_{v,\varphi,\omega^n}$, we conclude that $\beta^k_{v,\varphi,\omega^n}$ converges weakly to $\theta^n_{P_{\theta,X}[\varphi](v)}$, as $k \to \infty$. □

Definition 14.2.4 Take $k \geq 0$ and $\varphi \in \mathrm{PSH}(X,\theta)$, let $\mathrm{Norm}(\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi)))$ be the space of Hermitian norms on the vector space $\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))$.

Let $\mathcal{L}_{k,\varphi}\colon \mathrm{Norm}(\mathrm{H}^0(X, L^k \otimes T \otimes \mathcal{I}(k\varphi))) \to \mathbb{R}$ be the *partial Donaldson functional*:

$$\mathcal{L}_{k,\varphi}(H) = \frac{n!}{k^{n+1}} \log \frac{\mathrm{vol}\{s : H(s) \leq 1\}}{\mathrm{vol}\{s : N^k_{0,\omega^n}(s) \leq 1\}}, \tag{14.23}$$

where vol is simply the Euclidean volume.

Proposition 14.2.3 *Let $w, w' \in C^0(X)$ and $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current, then*

$$\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n}) \right) = \mathcal{E}^{\varphi}_{\theta,X}(w) - \mathcal{E}^{\varphi}_{\theta,X}(w'). \tag{14.24}$$

In particular,

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) = \mathcal{E}^{\varphi}_{\theta,X}(w)\,. \tag{14.25}$$

Proof First observe that by Proposition 14.2.1, for any $k \geq 0$, N^k_{w,ω^n} and N^k_{w',ω^n} are both norms, hence the expressions inside the limit in (14.24) make sense.

To start, we make the following observation:

$$\begin{aligned}\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{k,\varphi}(N^k_{w+t(w'-w),\omega^n})\,\mathrm{d}t \\ &= \int_0^1 \int_X (w'-w)\,\beta^k_{w+t(w'-w),\varphi,\omega^n}\,\mathrm{d}t.\end{aligned}$$

By Proposition 14.2.2, we have

$$\lim_{k\to\infty} \int_X (w'-w)\,\beta^k_{w+t(w'-w),\varphi,\omega^n} = \int_X (w'-w)\,\theta^n_{P_{\theta,X}[\varphi](w+t(w'-w))}.$$

By Theorem 7.4.1, we have $|\int_X (w'-w)\beta^k_{w+t(w'-w),u,\omega^n}| \leq C \sup_X |w-w'|$. Hence, by the dominated convergence theorem we obtain that

$$\begin{aligned}\lim_{k\to\infty} \left(\mathcal{L}_{k,\varphi}(N^k_{w,\omega^n}) - \mathcal{L}_{k,\varphi}(N^k_{w',\omega^n}) \right) &= \int_0^1 \int_X (w'-w)\,\theta^n_{P_{\theta,X}[\varphi](w+t(w'-w))}\,\mathrm{d}t \\ &= \mathcal{E}^{\varphi}_{\theta,X}(w) - \mathcal{E}^{\varphi}_{\theta,X}(w'),\end{aligned}$$

where in the last line we have used Proposition 14.1.2.

Finally, (14.25) is just a special case of (14.24) with $w' = 0$. □

Lemma 14.2.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$. Let (K,ν) be a weighted subset of X. Let $\upsilon \in \mathrm{BM}(K,\nu)$. Then*

$$\lim_{k\to\infty}\left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{v,\nu})\right) = 0\,. \tag{14.26}$$

Proof This is a direct consequence of the definition of Bernstein–Markov measures (14.15). □

Corollary 14.2.1 *Let $w \in C^0(X)$, $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Then*

$$\lim_{k\to\infty}\mathcal{L}_{k,\varphi}(N^k_{w,X}) = \mathcal{E}^{\varphi}_{\theta,X}(w).$$

Proof This follows from Lemma 14.2.2 and Proposition 14.2.3 and the fact that $\omega^n \in \mathrm{BM}(X,0)$. □

Proposition 14.2.4 *Let $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K,v), (K',v') be two weighted subsets of X. Then*

$$\lim_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{v',K'})\Big) = \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K'}(v'). \tag{14.27}$$

In particular,

$$\lim_{k\to\infty}\mathcal{L}_{k,\varphi}(N^k_{v,K}) = \mathcal{E}^{\varphi}_{\theta,K}(v). \tag{14.28}$$

Proof First observe that by Proposition 14.2.1, for any $k>0$, $N^k_{v,K}$ and $N^k_{v',K'}$ are both norms, hence the expressions inside the limit in (14.27) make sense. Moreover, (14.28) is just a special case of (14.27) for $K'=X$ and $v'=0$.

To prove (14.27) it is enough to show that for any fixed $w \in C^\infty(X)$ we have

$$\lim_{k\to\infty}\left(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,\omega^n})\right) = \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,X}(w)\,. \tag{14.29}$$

For $\epsilon \in (0,1)$ small enough we have that $\theta_{(1-\epsilon)\varphi}$ is still a Kähler current. Let us fix such ϵ, along with an arbitrary $\epsilon' \in (0,1)$.

Let $(v_j^-)_j$, $(v_j^+)_j$ be the sequences of smooth functions constructed in Lemma 14.1.5 for the data (K,v).

By Proposition 1.8.2 there exists $k_0(\epsilon,\epsilon') \in \mathbb{N}$ such that

$$\frac{1}{k}\log h^k \otimes h_T(s,s) \le (1-\epsilon)u,$$

and $\frac{1}{k}\log h^k\otimes h_T(s,s) \in \mathrm{PSH}(X,\theta+\epsilon'\omega)$ for any $s \in \mathrm{H}^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi))$, as long as $k \ge k_0(\epsilon,\epsilon')$.

In particular, Lemma 14.1.1 gives that

$$\begin{aligned}
N^k_{P'_{\theta+\epsilon'\omega,K}[(1-\epsilon)\varphi](v),X}(s) &= N^k_{v,K}(s),\\
N^k_{P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^-),X}(s) &= N^k_{v_j^-,X}(s),\\
N^k_{P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^+),X}(s) &= N^k_{v_j^+,X}(s).
\end{aligned}$$

As

$$P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^-) \le P'_{\theta+\epsilon'\omega,K}[(1-\epsilon)\varphi](v) \le P'_{\theta+\epsilon'\omega,X}[(1-\epsilon)\varphi](v_j^+),$$

by Lemma 14.2.1 we have

$$N^k_{v_j^+,X}(s) \le N^k_{v,K}(s) \le N^k_{v_j^-,X}(s), \quad s \in \mathrm{H}^0(X, T\otimes L^k \otimes \mathcal{I}(k\varphi)), k \ge k_0(\epsilon,\epsilon').$$

Composing with $\mathcal{L}_{k,\varphi}$ we arrive at

$$\mathcal{L}_{k,\varphi}(N^k_{v_j^-,X}) \le \mathcal{L}_{k,\varphi}(N^k_{v,K}) \le \mathcal{L}_{k,\varphi}(N^k_{v_j^+,X})\,,\ k \ge k_0(\epsilon,\epsilon').$$

For any $j > 0$, by Corollary 14.2.1 we get

$$\begin{aligned}
\mathcal{E}^{\varphi}_{\theta,X}(v_j^-) - \mathcal{E}^{\varphi}_{\theta,X}(w) &= \lim_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v_j^+,X}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&\le \varliminf_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&\le \varlimsup_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&\le \lim_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v_j^-,X}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&= \mathcal{E}^{\varphi}_{\theta,X}(v_j^+) - \mathcal{E}^{\varphi}_{\theta,X}(w)\,.
\end{aligned}$$

Using Lemma 14.1.5, we can let $j \to \infty$ to arrive at

$$\begin{aligned}
\mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K}(w) &\le \varliminf_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&\le \varlimsup_{k\to\infty}\Big(\mathcal{L}_{k,\varphi}(N^k_{v,K}) - \mathcal{L}_{k,\varphi}(N^k_{w,X})\Big)\\
&\le \mathcal{E}^{\varphi}_{\theta,K}(v) - \mathcal{E}^{\varphi}_{\theta,K}(w)\,.
\end{aligned}$$

Hence, (14.29) follows. □

Corollary 14.2.2 *Let $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K, v) be a weighted subset of X. Assume that $\upsilon \in \mathrm{BM}(K, v)$. Then*

$$\lim_{k\to\infty} \mathcal{L}_{k,\varphi}(N^k_{v,\upsilon}) = \mathcal{E}^{\varphi}_{\theta,K}(v).$$

Proof Our claim follows from Proposition 14.2.4 and Lemma 14.2.2. □

Proposition 14.2.5 *Suppose that $\varphi \in \mathrm{PSH}(X,\theta)$ be a potential with analytic singularities such that θ_φ is a Kähler current. Let (K, v) be a weighted subset of X. Let $\upsilon \in \mathrm{BM}(K, v)$. Then*

$$\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} = \theta^n_{P_{\theta,K}[\varphi](v)}$$

weakly as $k \to \infty$.

Proof For $w \in C^0(X)$, let

$$f_k(t) := \mathcal{L}_{k,\varphi}(N^k_{v+tw,\upsilon}), \quad g(t) := \mathcal{E}^{\varphi}_{\theta,K}(v+tw).$$

By Corollary 14.2.2 $\underline{\lim}_{k\to\infty} f_k(t) = g(t)$. Note that f_k is concave by Hölder's inequality (see [BBWN11, Proposition 2.4]), so by [BB10, Lemma 7.6], $\lim_{k\to\infty} f'_k(0) = g'(0)$, which is equivalent to $\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi](v)}$, by Proposition 14.1.2. □

Proposition 14.2.6 *Suppose that* $\varphi \in \mathrm{PSH}(X,\theta)$ *such that* θ_φ *is a Kähler current. Let* (K,v) *be a weighted subset of* X *and* $\upsilon \in \mathrm{BM}(K,v)$. *Then*

$$\beta^k_{v,\varphi,\upsilon} \rightharpoonup \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} \tag{14.30}$$

as $k \to \infty$.

Proof Let μ be the weak limit of a subsequence of $\beta^k_{v,\varphi,\upsilon}$. We claim that

$$\mu \leq \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}. \tag{14.31}$$

Observe that this claim implies the conclusion. In fact, by Theorem 7.4.1, we have equality of the total masses, so equality holds in (14.31). As μ is an arbitrary cluster point of the sequence $(\beta^k_{v,\varphi,\upsilon})_k$, we get (14.30).

It remains to prove (14.31). Let (φ_j) be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We may assume that θ_{φ_j} is a Kähler current for all $j \geq 1$. By Lemma 14.1.2, Corollary 7.1.2, we know that

$$\varphi_j \xrightarrow{d_S} P_{\theta,K}[\varphi]_{\mathcal{I}}(v).$$

In particular,

$$\lim_{j\to\infty} \int_X \theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)} = \int_X \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}. \tag{14.32}$$

Observe that

$$\beta^k_{v,\varphi,\upsilon} \leq \beta^k_{v,\varphi_j,\upsilon}$$

for any $k \geq 1$. As $\upsilon \in \mathrm{BM}(K,v)$, by Proposition 14.2.5,

$$\mu \leq \theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)},$$

for any $j \geq 1$ fixed. By Proposition 14.1.3,

$$P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v) \searrow P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$$

as $j \to \infty$. Hence, by (14.32) and Theorem 2.4.3, (14.31) follows. □

Proof (Proof of Theorem 14.2.1) By Lemma 14.1.2, we have that

$$\begin{aligned}\mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(k\varphi)\right) &= \mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(kP_\theta[\varphi]_{\mathcal{I}})\right)\\ &= \mathrm{H}^0\left(X, L^k \otimes T \otimes \mathcal{I}(kP_{\theta,K}[\varphi]_{\mathcal{I}}(v))\right).\end{aligned}$$

This allows us to replace φ with $P_{\theta,K}[\varphi]_{\mathcal{I}}(v)$.

By Lemma 2.4.3, there exists $\varphi_j \in \mathrm{PSH}(X,\theta)$, such that $\varphi_j \nearrow \varphi$ a.e. and θ_{φ_j} is a Kähler current for each $j \geq 1$. This gives

$$\beta^k_{v,\varphi_j,\upsilon} \leq \beta^k_{v,\varphi,\upsilon}.$$

Let μ be the weak limit of a subsequence of $(\beta^k_{v,\varphi,\upsilon})_k$. Then by Proposition 14.2.6,

$$\theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)} \leq \mu.$$

By Proposition 14.1.3 and Theorem 2.4.3 we have that

$$\theta^n_{P_{\theta,K}[\varphi_j]_{\mathcal{I}}(v)} \nearrow \theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)}.$$

Hence,

$$\theta^n_{P_{\theta,K}[\varphi]_{\mathcal{I}}(v)} \leq \mu. \tag{14.33}$$

A comparison of total masses using (14.18) and Theorem 7.4.1 gives that equality holds in (14.33). As μ is an arbitrary cluster limit of the weak compact sequence $(\beta^k_{v,\varphi,\mu})_k$, we obtain (14.19). □

Remark 14.2.1 The results in this chapter could also be reformulated as the large deviation principle of a determinantal point process on X using the Gärtner–Ellis theorem exactly as in [Ber14]. We leave the details to the readers.

Comments

A brief history

Here we recall the origin of various results.

Chapter 1.

The notion of plurisubharmonic functions was introduced by Lelong [Lel45], based on F. Riesz's theory of subharmonic functions [Rie26]. See [Bre72] for an excellent introduction to the early history of the subject. We refer to [Bre65] for the foundations of potential theory and [GZ17] for the pluripotential theory.

The global Josefson theorem Theorem 1.1.5 was due to Vu [Vu19]. In the projective setting, it was due to Dinh–Sibony [DS06] and in the Kähler setting, it was established by Guedj–Zeriahi [GZ05].

The extension theorem Theorem 1.2.1 was proved in [GR56]. In fact, they proved a more general version for complex spaces, see Theorem B.2.2. For some related important extension theorems, see [Shi72, Wan24].

Proposition 1.2.8 was due to Kiselman [Kis78].

The plurifine topology was introduced by Fuglede during the Séminaire d'analyse de Lelong–Dolbeault–Skoda of the year 1983/1984 [LDS86] based on H. Cartan's works on the fine topology. The key result Theorem 1.3.2 was claimed in Bedford–Taylor's work [BT87, Theorem 2.3] without proof. The first rigorous proof was given by El Marzguioui–Wiegerinck [EMW06]. A weaker result was proved earlier in [Kli91, Theorem 4.8.7].

Results in Section 1.3.2 are certainly well-known and are already implicitly used in the literature. I could not find the proofs in the literature and hence all details are presented.

The semicontinuity theorem Theorem 1.4.1 was due to Siu [Siu74].

The idea of Theorem 1.4.3 first appeared in the ground-breaking work of Boucksom–Favre–Jonsson [BFJ08].

The strong openness Theorem 1.4.4 was first established by Guan–Zhou [GZ15]. A more elegant proof was due to Hiep [Hie14].

Lemma 1.6.3 was due to [Dem15, Proposition 4.1.6].

Chapter 2

The Monge–Ampère operators for bound plurisubharmonic functions were introduced by Bedford–Taylor [BT76, BT82]. The non-pluripolar product is due to Bedford–Taylor [BT87], Guedj–Zeriahi [GZ07] and Boucksom–Eyssidieux–Guedj–Zeriahi [BEGZ10].

The key lemma Lemma 2.4.2 was proved in [DDNL21b]. Theorem 2.4.5 was due to [DDNL23].

Chapter 3

Lemma 3.1.1 was proved in [DDNL18b, Lemma 3.7].

The notion of the P-envelope is due to Ross–Witt Nyström [RWN14] based on the ideas of Rashkovskii–Sigurdsson [RS05].

Theorem 3.1.1 is due to [DDNL18b, Theorem 3.8]. The diamond inequality Theorem 3.1.3 and Proposition 3.1.5 are due to [DDNL21b, Theorem 5.4]. Most results in Section 3.1.3 are simple generalizations of the corresponding results in [DDNL18a, DDNL18c].

The $\mathcal{I}$-envelope was introduced by Darvas–Xia [DX22], inspired by the works of Dano Kim [Kim15] and Boucksom–Favre–Jonsson [BFJ08]. The notion of $\mathcal{I}$-model singularities was first formulated in the explicit way in [DX22] in 2020, although it was already essentially known in Boucksom–Jonsson's work. In fact, they correspond exactly to the homogeneous non-Archimedean potentials assuming that the relevant masses do not vanish. A less explicit equivalent formulation of $\mathcal{I}$-model potentials also appeared in [Dem15]. A few months later, the same notion was rediscovered by Trusiani [Tru22].

Chapter 4

The notion of weak geodesics was studied in detail by Darvas [Dar17] in the Kähler case.

The case of general big classes was partly handled in [DDNL18c], [DDNL18a]. However, the key fact that the geodesics between two full mass potentials have the correct limit at the end points does not seem to have been proved in any references. We give a proof in Proposition 4.2.1. We also extend the relevant results to the relative setting.

Previously, Proposition 4.2.2 and Proposition 4.2.4 were only known in the Kähler case.

Most results in Section 4.3 are simple extensions of [DDNL18a].

Chapter 5

The toric framework was first written down by Berman–Berndtsson [BB13] and Coman–Guedj–Sahin–Zeriahi in [CGSZ19].

The beautiful theorem Theorem 5.2.2 was first proved by Yi Yao, who did not publish the result. Later on, a new proof was found by Botero–Burgos Gil–Holmes–de Jong [BBGHdJ22]. We chose to present the approach of Yao, which integrates naturally with our framework.

Chapter 6

The notion of P-partial order is new, as well as most results in Section 6.1.

The d_S-pseudometric was introduced in [DDNL21b]. The basic properties are proved in [DDNL21b] and [Xia25b].

Example 6.1.3 was due to Berman–Boucksom–Jonsson [BBJ21].

Theorem 6.2.4 is proved in [Xia22b]. Theorem 6.2.6 and Theorem 6.2.5 appear to be new. These results appeared previously in the form of lecture notes.

Chapter 7

The notion of $\mathcal{I}$-good singularities was due to [DX24b]. The name *$\mathcal{I}$-good* was chosen in [Xia22b].

Theorem 7.1.1 and Theorem 7.4.1 are due to [DX24b, DX22].

There are some further examples of $\mathcal{I}$-good singularities provided by [BBGHdJ22] with applications in the theory of modular forms in [BBGHdJ24].

Chapter 8

The trace operator was introduced in [DX24a]. Here we present a different point of view. Theorem 8.4.1 was proved in [DX24a].

The analytic Bertini theorem Theorem 8.5.1 was proved in [Xia22a], based on the works of Matsumura–Fujino [FM21] and [Fuj23]. A weaker result was established by Meng–Zhou [MZ23].

Chapter 9

The technique of test curves originates from [RWN14]. It was generalized by Darvas–Di Nezza–Lu [DDNL18a], [DX24b], [DZ24] and [DXZ25]. We give the full details of the proofs.

Test curves in Definition 9.1.1 are called *maximal test curves* in the literature, a terminology which I do not like. I prefer to call the usual notion of test curves in the literature *sub-test curves*.

Proposition 9.2.2 was first proved by He–Testorf–Wang in [HTW23]. Proposition 9.2.3 was due to Hisamoto [His16].

Remark 9.3.2 was a folklore result. I am unaware of any written proof in the literature before our paper [DX22]. Finski [Fin22b] also gave a different proof with different techniques later on.

Definition 9.3.3 was not the original definition of maximal geodesic rays of Berman–Boucksom–Jonsson in [BBJ21]. One of the first major applications of our theory was this pluripotential-theoretical characterization of maximal geodesic rays, as proved in our very first paper [DX22].

Results in Section 9.4 are easy generalizations of the results proved in [Xia25a].

Chapter 10

The algebraic theory of partial Okounkov bodies was developed in [Xia25b]. The transcendental Okounkov body was first defined by Deng [Den17] as suggested by Demailly. The volume identity was proved in [DRWN+23]. The transcendental theory of partial Okounkov bodies is new. Results in Section 11.3 are also new.

Chapter 11

The applications of b-divisors in pluripotential theory began with [BFJ09]. The intersection theory of nef b-divisors was introduced by Dang–Favre [DF22]. The technique of singularity b-divisors was introduced in [Xia23b] in 2020. The general form first appeared in [Xia22b]. One year later, a special case was rediscovered in [BBGHdJ22].

The current chapter reproduces a large part of [Xia25c].

Chapter 12

The whole chapter appears to be new. The study of toric pluripotential theory on big line bundles was made possible by the development of partial Okounkov bodies. The key result is Theorem 12.2.2.

Most results in this chapter resulted from discussions with Yi Yao.

Chapter 13

Most results from this chapter are from [Xia25a]. Results from Section 13.3 are new, although the main idea was already contained in [Xia25b].

Theorem 13.4.3 is due to [DXZ25]. An alternative approach to the transcendental theory is due to Mesquita-Piccione [MP24].

Special cases of the results in this section have been applied to study K-stability, see [Xia23b], [DZ24], [DXZ25] and [DR22]. In [DX22], we established the bijective correspondence between a class of $\mathcal{I}$-model test curves with the maximal geodesic rays in the sense of [BBJ21].

Chapter 14

The special case of Theorem 14.2.1 without the prescribed singularity φ was due to Berman–Boucksom–Witt Nyström, see [BB10], [BBWN11]. The general case is due to [DX24b].

Open problems

We give a list of important open problem in this theory.

We do not repeat the conjectures mentioned in the main text.

Conjecture 14.2.1 Let X be a connected compact Kähler manifold and Y be a submanifold. Fix a Kähler class α on X. For each Kähler current $S \in \alpha|_Y$, we can find a Kähler current $T \in \alpha$ such that

$$\mathrm{Tr}_Y(T) \sim_{\mathcal{I}} S.$$

If we formally view Tr_Y as an analogue of the trace operator in the theory of Sobolev spaces, then this conjecture corresponds exactly to the Dirichlet problem.

Using Proposition 8.2.2, one could also reduce this conjecture to a strong version of the extension theorem Theorem 1.6.3.

Conjecture 14.2.2 Let X be a connected compact Kähler manifold and Y be a submanifold. Fix a Kähler class α on X. Consider Kähler currents $R \in \alpha$, $S \in \alpha|_Y$ with gentle analytic singularities such that $S \leq R|_Y$. Then there is a Kähler current $T \in \alpha$ with analytic singularities such that

$$\mathrm{Tr}_Y(T) \sim_{\mathcal{I}} S, \quad T \leq R.$$

This conjecture was also proposed by Darvas for different purposes.

Conjecture 14.2.3 Let X be a connected smooth projective variety of dimension n. Assume that (L_i, h_i) is a Hermitian big line bundle on X for each $i = 1, \ldots, n$ with the h_i's being $\mathcal{I}$-good. Then

$$\int_X c_1(L_1, h_1) \wedge \cdots \wedge c_1(L_n, h_n) = \sup_\nu \operatorname{vol}\left(\Delta_\nu(L_1, h_1), \dots, \Delta_\nu(L_n, h_n)\right),$$

where $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ runs over all (surjective) valuation of rank n.

See [Sch93, Section 5.1] for the notion of mixed volumes.

This conjecture seems reasonable in view of Corollary 10.3.3 and Corollary 10.3.2.

Even when $h_1, \dots, h_n$ have minimal singularities, this conjecture remains open:

Conjecture 14.2.4 Let X be a connected smooth projective variety of dimension n. Assume that $L_1, \dots, L_n$ are big line bundles on X. Then

$$\langle L_1, \dots, L_n \rangle = \sup_\nu \operatorname{vol}\left(\Delta_\nu(L_1), \dots, \Delta_\nu(L_n)\right), \tag{14.34}$$

where $\nu \colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ runs over all (surjective) valuation of rank n.

Here on the left-hand side, we are using the movable intersection theory [BDPP13].

In [Wil25], Wilms proved the $\leq$ direction of (14.34).

Problem 14.2.1 Is it possible to extend the definition of the trace operator Tr_Y to the case where the ambient variety is only unibranch?

The difficulty lies in the lack of Demailly type regularization theorems.

Problem 14.2.2 Is there a natural definition of the transcendental Okounkov body of a closed positive $(1,1)$-current T with 0-mass so that its dimension is equal to the numerical dimension of T?

See [Cao14] for the definition of the numerical dimension of a current.

The following two problems are proposed by Witt Nyström.

Problem 14.2.3 Consider a compact Kähler manifold X and a connected submanifold Y. We have defined the trace operator Tr_Y from a subset of $\operatorname{QPSH}(X)/{\sim_\mathcal{I}}$ to $\operatorname{QPSH}(Y)/{\sim_\mathcal{I}}$. Is it possible to refine this operator to one from a subset of $\operatorname{QPSH}(X)/{\sim_P}$ to $\operatorname{QPSH}(Y)/{\sim_P}$?

Problem 14.2.4 Consider a connected compact Kähler manifold X of dimension n and a smooth flag $Y_\bullet$ on X. Consider closed smooth real $(1,1)$-form θ on X representing a big cohomology class and $\varphi \in \operatorname{PSH}(X, \theta)$ with $\int_X \theta_\varphi^n > 0$.

Can one define a refined notion of partial Okounkov bodies $\Delta'_{Y_\bullet}(\theta + \mathrm{dd^c}\varphi)$ contained in $\Delta_{Y_\bullet}(\theta + \mathrm{dd^c}\varphi)$ with volume given by $\frac{1}{n!}\int_X \theta_\varphi^n$?

Note that a satisfactory solution to the latter problem is not very likely, as can be easily seen from examples on $\mathbb{P}^1$.

We also look for generalizations of our theory to more general settings.

Problem 14.2.5 To what extent can the results in the current book be generalized to the non-Kähler setting?

The non-pluripolar products in the non-Kähler setting was recently studied by Boucksom–Guedj–Lu in [BGL24]. See also the references therein.

Problem 14.2.6 To what extent can the results in the current book about closed positive $(1,1)$-currents be generalized to closed positive currents of higher bidegree?

A fundamental issue is the lack of a strong enough Demailly type approximation for general currents. The regularization theorem of Dinh–Sibony [DS04] seems too weak for our purposes.

Appendix A
Convex functions and convex bodies

We recall some basic facts about convex functions in this section. Our basic reference is [Roc70]. The results in this appendix can be applied to concave functions after considering their negatives.

A.1 The notion of convex functions

Let N be a real vector space of finite dimension.

Definition A.1.1 Let $F\colon N \to [-\infty, \infty]$ be a function. The *epigraph* of F is defined as the following set

$$\operatorname{epi} F := \{(n, r) \in N \times \mathbb{R} : r \geq F(n)\}\,.$$

Definition A.1.2 A *convex function* on N is a function $F\colon N \to [-\infty, \infty]$ such that the epigraph $\operatorname{epi} F$ is a convex subset of $N \times \mathbb{R}$.

The *effective domain* of F is the set

$$\operatorname{Dom} F := \{n \in N : F(n) < \infty\}\,.$$

A convex function F on N such that $\operatorname{Dom} F \neq \varnothing$ and $F(n) \neq -\infty$ for all $n \in N$ is said to be *proper*.

The set of convex functions on N is denoted by $\operatorname{Conv}(N)$. The subset set of proper convex functions is denoted by $\operatorname{Conv}^{\mathrm{prop}}(N)$.

The following characterization of convex functions is well-known.

Lemma A.1.1 *Let $F\colon N \to [-\infty, \infty]$. Then F is convex if and only if the following condition holds: suppose that $n, r \in N$ and $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$, then for any $t \in (0, 1)$, we have*

$$F(tn + (1-t)r) < ta + (1-t)b.$$

See [Roc70, Theorem 4.2] for the proof.

Example A.1.1 Let $A \subseteq N$ be a convex subset. Then the *characteristic function* $\chi_A \colon N \to \{0, \infty\}$ of A is defined by

$$\chi_A(n) := \begin{cases} 0, & n \in A; \\ \infty, & n \notin A. \end{cases}$$

The function χ_A lies in $\mathrm{Conv}(N)$.

Example A.1.2 Let M be the dual vector space of N and $P \subseteq M$ be a convex subset. The *support function* $\mathrm{Supp}_P \in \mathrm{Conv}(N)$ of P is defined as follows:

$$\mathrm{Supp}_P(n) := \sup\{\langle m, n\rangle : m \in P\}.$$

It is well-known that convexity is preserved by a number of natural operations. We recall a few to fix the notation.

Definition A.1.3 Let $F_1, \dots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$ $(m \in \mathbb{Z}_{>0})$. We define their *infimal convolution* $F_1 \square \cdots \square F_m \in \mathrm{Conv}(N)$ as follows:

$$F_1 \square \cdots \square F_m(n) := \inf\left\{\sum_{i=1}^m F_i(n_i) : n_i \in N, \sum_{i=1}^m n_i = n\right\}.$$

The fact $F_1 \square \cdots \square F_m \in \mathrm{Conv}(N)$ is proved in [Roc70, Theorem 5.4]. One should note that $F_1 \square \cdots \square F_m$ is not always proper.

Proposition A.1.1 *Let* $\{F_i\}_{i\in I}$ *be a non-empty family in* $\mathrm{Conv}(N)$. *Then* $\sup_{i\in I} F_i \in \mathrm{Conv}(N)$.

This follows from [Roc70, Theorem 5.5]. In particular, this allows us to introduce

Definition A.1.4 Let $f \colon N \to [-\infty, \infty]$. The *lower convex envelope* of f is defined as

$$\mathrm{CE}\, f := \sup\{F \in \mathrm{Conv}(N) : F \le f\}.$$

It follows from Proposition A.1.1 that $\mathrm{CE}\, f \in \mathrm{Conv}(N)$.

Definition A.1.5 Given a non-empty family $\{F_i\}_{i\in I}$ in $\mathrm{Conv}(N)$, we define

$$\bigwedge_{i\in I} F_i := \mathrm{CE}\left(\inf_{i\in I} F_i\right).$$

When the family I is finite, say $I = \{1, \dots, m\}$, we also write

$$F_1 \wedge \cdots \wedge F_m = \bigwedge_{i\in I} F_i.$$

Definition A.1.6 Given a non-empty family $\{F_i\}_{i\in I}$ in $\mathrm{Conv}(N)$, we define

$$\bigvee_{i\in I} F_i := \sup_{i\in I} F_i.$$

When the family I is finite, say $I = \{1, \dots, m\}$, we also write

$$F_1 \vee \dots \vee F_m = \bigvee_{i\in I} F_i.$$

Recall that $\bigvee_{i\in I} F_i \in \mathrm{Conv}(N)$ by Proposition A.1.1.

Proposition A.1.2 *Let* $F_1, \dots, F_m \in \mathrm{Conv}^{\mathrm{prop}}(N)$*, then*

$$\begin{aligned} F_1 \wedge \dots \wedge F_m(x) = \inf \Bigg\{ & \sum_{i=1}^m \lambda_i F_i(x_i) : x_i \in \mathrm{Dom}(F_i), \\ & \lambda_i \in [0,1], \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \Bigg\}. \end{aligned}$$

See [Roc70, Theorem 5.6] for the more general result.

Lemma A.1.2 *Let* $\{F_i\}_{i\in I}$ *be a decreasing net in* $\mathrm{Conv}(N)$*. Then* $\inf_{i\in I} F_i \in \mathrm{Conv}(N)$.

Proof Write $F = \inf_{i\in I} F_i$. We shall apply the characterization in Lemma A.1.1. Take $n, r \in N$, $a, b \in \mathbb{R}$ such that $a > F(n)$, $b > F(r)$ and $t \in (0,1)$. We need to show that

$$F(tn + (1-t)r) < ta + (1-t)b. \tag{A.1}$$

By definition, there exists $j \in I$ such that for any $i \geq I$ with $i \geq j$, we have

$$a > F_i(n), \quad b > F_i(r).$$

It follows from Lemma A.1.1 that

$$F_i(tn + (1-t)r) < ta + (1-t)b$$

for any $i \geq j$. Since F_i is decreasing in i, we conclude (A.1). □

Definition A.1.7 Let $F \in \mathrm{Conv}(N)$. The *closure* $\mathrm{cl}\, F \in \mathrm{Conv}(N)$ of F is defined as follows: If $F(n) = -\infty$ for some $n \in N$, then $\mathrm{cl}\, F := -\infty$. Otherwise, we define $\mathrm{cl}\, F$ as the lower semicontinuity regularization of F.

A convex function $F \in \mathrm{Conv}(N)$ is *closed* if $F = \mathrm{cl}\, F$. In other words, $F \in \mathrm{Conv}(N)$ if one of the following conditions hold:

(1) $F \equiv -\infty$;
(2) $F \equiv \infty$;
(3) F is proper and lower semi-continuous.

Proposition A.1.3 *Let $F \in \mathrm{Conv}(N)$ be a closed convex function. Then F is the supremum of all affine functions lying below F.*

See [Roc70, Theorem 12.1].

Theorem A.1.1 *Let $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. Then $\mathrm{cl}\, F$ is a closed proper convex function. Moreover, $\mathrm{cl}\, F$ agrees with F except possibly on the relative boundary of $\mathrm{Dom}\, F$.*

See [Roc70, Theorem 7.4].

Proposition A.1.4 *Let $F, F' \in \mathrm{Conv}(N)$ be closed convex functions. Assume that*

(1) $\mathrm{RelInt}\,\mathrm{Dom}\, F = \mathrm{RelInt}\,\mathrm{Dom}\, F'$, *and*
(2) $F = F'$ *on* $\mathrm{RelInt}\,\mathrm{Dom}\, F$.

Then $F = F'$.

This is a special case of [Roc70, Corollary 7.3.4].

Definition A.1.8 Given $F, F' \in \mathrm{Conv}(N)$, we write $F \preceq F'$ if there is $C \in \mathbb{R}$ such that

$$F \leq F' + C.$$

We say $F \sim F'$ if $F \preceq F'$ and $F' \preceq F$ both hold.

Theorem A.1.2 *Let $C \subseteq N$ be an open subset. Let $(f_i)_{i>0}$ be a sequence of real-valued convex functions on C. Suppose that the sequence converges on a dense subset of C and the limit is finite, then the limit*

$$f(x) \coloneqq \lim_{i\to\infty} f_i(x)$$

exists for all $x \in C$ and is convex on C. Moreover, the sequence $(f_i)_i$ converges uniformly to f on each compact subset of C.

This is a special case of [Roc70, Theorem 10.8].

A.2 Legendre transform

Let N be a real vector space of finite dimension and M be the dual vector space. The pairing $M \times N \to \mathbb{R}$ will be denoted by $\langle \bullet, \bullet \rangle$.

Definition A.2.1 Let $F \in \mathrm{Conv}(N)$ be a convex function. We define the *Legendre transform* of F as the function $F^* \in \mathrm{Conv}(M)$:

$$F^*(m) \coloneqq \sup_{n\in N} (\langle m, n\rangle - F(n)) = \sup_{n\in \mathrm{RelInt}\,\mathrm{Dom}\, F} (\langle m, n\rangle - F(n)) . \tag{A.2}$$

The latter equality follows from [Roc70, Corollary 12.2.2].

Recall the well-known Legendre–Fenchel duality [Roc70, Theorem 12.2].

Theorem A.2.1 *Let $F \in \mathrm{Conv}(N)$. Then F^* is a closed convex function. The function F^* is proper if and only if F is.*

Moreover, we have $(\mathrm{cl}\, F)^* = F^*$ and

$$F^{**} = \mathrm{cl}\, F.$$

Example A.2.1 Let $P \subseteq M$ be a closed convex subset. Then

$$\mathrm{Supp}_P^* = \chi_P, \quad \chi_P^* = \mathrm{Supp}_P .$$

See [Roc70, Theorem 13.2].

The following special case will be useful to us in the sequel.

Corollary A.2.1 *Let $F\colon (0,\infty) \to [-\infty,\infty)$ be a convex function. If we define $G\colon \mathbb{R} \to (-\infty,\infty]$ by*

$$G(\tau) = \sup_{t>0} (t\tau - F(t)),$$

then G is a convex function and

$$F(t) = G^*(t), \quad \forall t > 0. \tag{A.3}$$

Moreover,

$$G(\tau) = \sup_{t \in \mathbb{Q}_{>0}} (t\tau - F(t)). \tag{A.4}$$

Proof We distinguish two cases.

First suppose that $F(t) = -\infty$ for some $t > 0$. Then $F(t) = -\infty$ for all $t > 0$ by the convexity of F. Our assertions are clear in this case.

Next assume that $F(t) \neq -\infty$ for all $t > 0$. In this case, Theorem A.1.1 guarantees that F admits a closed proper extension $\tilde{F} \in \mathrm{Conv}(\mathbb{R})$ with

$$\tilde{F}(t) = \infty, \quad \forall t < 0.$$

It follows from (A.2) that

$$G(\tau) = \tilde{F}^*(\tau), \quad \forall \tau \in \mathbb{R}.$$

Now Theorem A.2.1 implies (A.3). Finally (A.4) follows from the continuity of F. □

Proposition A.2.1 *Let $F\colon N \to [-\infty,\infty]$, then the function $F^*\colon M \to [-\infty,\infty]$ defined by*

$$F^*(m) := \sup_{n \in N} (\langle m, n\rangle - F(n)).$$

Then

$$F^* = (\mathrm{cl}\,\mathrm{CE}\, f)^*.$$

See [Roc70, Corollary 12.1.1].

Definition A.2.2 Let $F \in \mathrm{Conv}(N)$ and $n \in N$. An element $m \in M$ is a *subgradient* of F at n if

$$F(n') \geq F(n) + \langle n' - n, m\rangle, \quad \forall n' \in N. \tag{A.5}$$

The set of subgradients of F at n is denoted by $\nabla F(n)$.

More generally, for any subset $E \subseteq N$, we write

$$\nabla F(E) = \bigcup_{n \in E} \nabla F(n).$$

Definition A.2.3 Given $F, F' \in \mathrm{Conv}(N)$, we write $F \preceq_P F'$ if

$$\overline{\nabla F(N)} \subseteq \overline{\nabla F'(N)}.$$

We write $F \sim_P F'$ if $F \preceq_P F'$ and $F' \preceq_P F$.

Theorem A.2.2 *Suppose that* $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. *Then the following hold:*

(1) *For any* $n \notin \mathrm{Dom}\, F$, $\nabla F(n) = \varnothing$;
(2) *for any* $n \in \mathrm{RelInt}\,\mathrm{Dom}\, F$, $\nabla F(n) \neq \varnothing$; *Moreover, for any* $n' \in N$, *we have*

$$\partial_{n'} F(n) = \sup\left\{\langle n', m\rangle : m \in \nabla F(n)\right\};$$

(3) *for* $n \in N$, *the set* $\nabla F(n)$ *is bounded if and only if* $n \in \mathrm{Int}\,\mathrm{Dom}\, F$.

For the proof, we refer to [Roc70, Theorem 23.4].

Proposition A.2.2 *Let* $F \in \mathrm{Conv}^{\mathrm{prop}}(N)$. *Then*

$$\nabla F(N) \subseteq \mathrm{Dom}\, F^*.$$

If moreover F *is closed, we have*

$$\mathrm{RelInt}\,\mathrm{Dom}\, F^* \subseteq \nabla F(N). \tag{A.6}$$

In particular, if F *is a proper closed convex function on* N, *then*

$$\overline{\nabla F(N)} = \overline{\mathrm{Dom}\, F^*}.$$

Proof Suppose that $m \in \nabla F(n)$ for some $n \in N$, it follows that (A.5) holds. In particular,

$$\langle m, n'\rangle - F(n') \leq \langle m, n\rangle - F(n).$$

It follows that

$$F^*(m) \leq \langle m, n\rangle - F(n) < \infty.$$

(A.6) is proved in [Roc70, Corollary 23.5.1]. For the last assertion, it suffices to observe that $\overline{\mathrm{RelInt}\,\mathrm{Dom}\, F^*} = \overline{\mathrm{Dom}\, F^*}$. □

Proposition A.2.3 *Let* $\{F_i\}_{i \in I}$ *be a non-empty family in* $\mathrm{Conv}^{\mathrm{prop}}(N)$. *Then*

$$\left(\bigwedge_{i\in I} F_i\right)^* = \bigvee_{i\in I} F_i^*, \quad \left(\bigvee_{i\in I} \operatorname{cl} F_i\right)^* = \operatorname{cl}\bigwedge_{i\in I} F_i^*.$$

If I is finite and $\overline{\operatorname{Dom} F_i}$ is independent of the choice of $i \in I$, then

$$\left(\bigvee_{i\in I} F_i\right)^* = \bigwedge_{i\in I} F_i^*.$$

Recall that $\wedge$ is defined in Definition A.1.5 and $\vee$ in Definition A.1.6. See [Roc70, Theorem 16.5] for the proof.

Proposition A.2.4 *Let $F_1, \ldots, F_r \in \operatorname{Conv}^{\mathrm{prop}}(N)$ ($r \in \mathbb{Z}_{>0}$). Assume that*

$$\bigcap_{i=1}^r \operatorname{RelInt}\operatorname{Dom}(F_i) \neq \varnothing,$$

then for any $m \in M$,

$$\left(\sum_{i=1}^r F_i\right)^*(m) = \inf\left\{\sum_{i=1}^r F_i^*(m_i) : m_1, \ldots, m_r \in M, \sum_{i=1}^r m_i = m\right\}.$$

Proposition A.2.5 *Let $P \subseteq M$ be a convex body*[1] *and $F \in \operatorname{Conv}^{\mathrm{prop}}(N)$. The following are equivalent:*

(1) *$F \preceq \operatorname{Supp}_P$;*
(2) *$\operatorname{Dom} F = N$ and $F^*|_{M\setminus P} \equiv \infty$;*
(3) *$\operatorname{Dom} F = N$ and $\nabla F(N) \subseteq P$.*

Moreover, under these conditions,

$$F(n) - \operatorname{Supp}_P(n) \leq F(0), \quad \forall n \in N. \tag{A.7}$$

Proof (1) $\implies$ (2). It is clear that $\operatorname{Dom} F = N$ since $\operatorname{Dom}\operatorname{Supp}_P = N$. From $F \preceq \operatorname{Supp}_P$ and Example A.2.1, we know that

$$\chi_P = \operatorname{Supp}_P^* \preceq F^*.$$

So ii follows.

(2) $\implies$ (3). This follows from Proposition A.2.2.

(3) $\implies$ (1). Taken $n \in N$, we know that F is locally Lipschitz [Roc70, Theorem 10.4], so we can compute

[1] Here a convex body refers to a non-empty closed convex subset, not necessarily having non-empty interior.

$$F(n) - F(0) = \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(tn)\,\mathrm{d}t = \int_0^1 \langle \nabla F(tn), n\rangle\,\mathrm{d}t$$
$$\leq \int_0^1 \mathrm{Supp}_P(n)\,\mathrm{d}t = \mathrm{Supp}_P(n).$$

In particular, (A.7) also follows. □

A.3 Classes of convex functions

Let N be a real vector space of finite dimension and M be the dual vector space.

We shall fix a convex body $P \subseteq M$.

The following classes are introduced in [BB13].

Definition A.3.1 We define the set $\mathcal{P}(N,P)$ as the set of proper convex functions $F \in \mathrm{Conv}(N)$ such that $F \preceq \mathrm{Supp}_P$.

We define the set $\mathcal{E}^\infty(N,P)$ as the set of closed convex functions $F \in \mathrm{Conv}(N)$ such that $F \sim \mathrm{Supp}_P$.

We define the set $\mathcal{E}(N,P)$ as follows: Suppose that $\mathrm{Int}\,P = \emptyset$, then $\mathcal{E}(N,P) := \mathcal{P}(N,P)$; otherwise, let

$$\mathcal{E}(N,P) = \left\{F \in \mathcal{P}(N,P) : P = \overline{\nabla F(N)}\right\}.$$

We define the set $\mathcal{E}^1(N,P)$ as the subset of $\mathcal{E}(N,P)$ consisting of $F \in \mathcal{E}(N,P)$ with

$$\int_P F^*\,\mathrm{d}\,\mathrm{vol} < \infty,$$

where $\mathrm{d}\,\mathrm{vol}$ is any Lebesgue measure on N.

Observe that for any $F \in \mathcal{P}(N,P)$, we have $\mathrm{Dom}\,F = N$ and F is necessarily closed.

Proposition A.3.1 *We have*

$$\mathcal{E}^\infty(N,P) \subseteq \mathcal{E}^1(N,P) \subseteq \mathcal{E}(N,P) \subseteq \mathcal{P}(N,P).$$

Proof When $\mathrm{Int}\,P = \emptyset$, the assertion is clear. We assume that $\mathrm{Int}\,P \neq \emptyset$. The second inclusion follows from definition. We only hand the first inequality. Take $F \in \mathcal{E}^\infty(N,P)$. By definition, $F \sim \mathrm{Supp}_P$ and hence $F^* \sim \chi_P$. It follows that $P = \mathrm{Dom}\,F^*$.

By Proposition A.2.5, we already know that

$$\nabla F(N) \subseteq P = \mathrm{Dom}\,F^*.$$

On the other hand, by Proposition A.2.2, we have

$$\mathrm{Int}\,P \subseteq \nabla F(N).$$

So it follows that

$$P = \overline{\nabla F(N)}.$$

It is clear that $F^* \sim \chi_P$ is integrable. □

Proposition A.3.2 *For any $F \in \mathcal{E}^\infty(N, P)$, we have $F^*|_{M \setminus P} \equiv \infty$ and F^* is bounded on P.*

Proof From $F \sim \mathrm{Supp}_P$, we take the Legendre transform to get $F^* \sim \mathrm{Supp}_P^* = \chi_P$, where we applied Example A.2.1. □

Definition A.3.2 We endow the topology of pointwise convergence on $\mathcal{P}(N, P)$. Note that this topology coincides with the compact-open topology.

Proposition A.3.3 *Let $F \in \mathcal{P}(N, P)$. Then there is a decreasing sequence $F_j \in \mathcal{E}^\infty(N, P) \cap C^\infty(N)$ converging to F.*

See [BB13, Lemma 2.2].

We observe that the point $0 \in N$ plays a special role since it does in the definition of the support function.

Proposition A.3.4 *For any $F \in \mathrm{Conv}(N, P)$, we have*

$$\max_N (F - \mathrm{Supp}_P) = F(0).$$

Proof It follows from (A.7) that

$$\sup_N (F - \mathrm{Supp}_P) \leq F(0).$$

The equality is clearly obtained at $0 \in N$. □

Lemma A.3.1 *Let $P' \subseteq M$ be another convex body. Then for any $F \in \mathcal{P}(N, P)$ and $F' \in \mathcal{P}(N, P')$, we have*

$$F + F' \in \mathcal{P}(N, P + P').$$

Similarly, if $F \in \mathcal{E}(N, P)$ and $F' \in \mathcal{E}(N, P')$, we have

$$F + F' \in \mathcal{E}(N, P + P').$$

Proof The former assertion follows immediately from the observation

$$\mathrm{Supp}_{P+P'} = \mathrm{Supp}_P + \mathrm{Supp}_{P'}.$$

As for the latter, it suffices to prove the following more general statement:

$$\overline{\nabla F} + \overline{\nabla F'} = \overline{\nabla (F + F')}$$

for any real-valued convex functions F and F' on N with $\mathrm{Dom}\, F^*$ bounded. In view of Proposition A.2.2, this means

$$\overline{\mathrm{Dom}(F+F')^*} = \overline{\mathrm{Dom}\,F^*} + \overline{\mathrm{Dom}\,F'^*}. \tag{A.8}$$

It follows from Proposition A.2.4 that

$$\mathrm{Dom}(F+F')^* = \mathrm{Dom}\,F^* + \mathrm{Dom}\,F'^*.$$

Since $\overline{\mathrm{Dom}\,F^*}$ is compact, (A.8) follows[2]. □

A.4 Monge–Ampère measures

Let N be a free Abelian group of finite rank (i.e. a lattice) and M be its dual lattice. There is a canonical Lebesgue type measure on $M_{\mathbb{R}}$, denoted by $\mathrm{d}\,\mathrm{vol}$, normalized so that the smallest cubes in M have volume 1. Similarly, the canonical measure on $N_{\mathbb{R}}$ is normalized in the same way and is denoted by $\mathrm{d}\,\mathrm{vol}$ as well.

We will write

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}.$$

Definition A.4.1 Let $F \in \mathrm{Conv}(N_{\mathbb{R}})$, we define the *real Monge–Ampère measure* $\mathrm{MA}_{\mathbb{R}}\,F$ as the Borel measure on $N_{\mathbb{R}}$ given as follows: for each Borel measurable set $E \subseteq N_{\mathbb{R}}$, define

$$\mathrm{MA}_{\mathbb{R}}\,F(E) := n! \int_{\nabla F(E)} \mathrm{d}\,\mathrm{vol}\,.$$

Proposition A.4.1 *Suppose that $F \in C^{1,1}(N_{\mathbb{R}}) \cap \mathrm{Conv}(N_{\mathbb{R}})$, fix an identification $N = \mathbb{Z}^n$, then*

$$\mathrm{MA}_{\mathbb{R}}\,F = n! \cdot \det \nabla^2 F \,\mathrm{d}\,\mathrm{vol}\,.$$

See [Fig17, Example 2.2].

Proposition A.4.2 *Let $P \in M_{\mathbb{R}}$ be a convex body and $F \in \mathcal{P}(N_{\mathbb{R}}, P)$. Then $F \in \mathcal{E}(N_{\mathbb{R}}, P)$ if and only if*

$$\int_{M_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}\,F = n!\,\mathrm{vol}\,P. \tag{A.9}$$

Proof By definition of $\mathrm{MA}_{\mathbb{R}}$, (A.9) is equivalent to

$$\mathrm{vol}\,\overline{\nabla F(N_{\mathbb{R}})} = \mathrm{vol}\,P.$$

We first handle the case where $\mathrm{Int}\,P \neq \emptyset$. By Proposition A.2.5, the latter is equivalent to

$$\overline{\nabla F(N_{\mathbb{R}})} = P.$$

Now assume that $\mathrm{Int}\,P = \emptyset$, then $\mathrm{vol}\,\overline{\nabla F(N)} = \mathrm{vol}\,P = 0$ by Proposition A.2.5. The assertion is clear. □

[2] In general, the Minkowski sum does not commute with the closure.

Theorem A.4.1 *Let $F, F_j \in \mathcal{P}(N_{\mathbb{R}}, P)$ ($j \in \mathbb{Z}_{>0}$). Assume that $F_j \to F$, then $\mathrm{MA}_{\mathbb{R}}(F_j)$ converges to $\mathrm{MA}_{\mathbb{R}}(F)$ weakly.*

See [Fig17, Proposition 2.6].

There is a well-known comparison principle.

Theorem A.4.2 *Let $F, F' \in \mathcal{P}(N_{\mathbb{R}}, P)$. Assume that $F \preceq F'$, then*

$$\overline{\nabla F(N_{\mathbb{R}})} \subseteq \overline{\nabla F'(N_{\mathbb{R}})}, \qquad \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F) \leq \int_{N_{\mathbb{R}}} \mathrm{MA}_{\mathbb{R}}(F').$$

Proof It suffices to observe that $G'^* \leq G^*$, and hence the first assertion follows from Proposition A.2.2. The second assertion follows from the first. □

A.5 Separation lemmata

Lemma A.5.1 *Let $\alpha, \beta_1, \ldots, \beta_m \in \mathbb{Z}^n$. Let Δ be the polytope generated by $\beta_1, \ldots, \beta_m$. Then the following are equivalent:*

(1)
$$|z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \tag{A.10}$$

is a bounded function on $\mathbb{C}^{*n}$.

(2) $\alpha \in \Delta$.

Proof (2) $\Longrightarrow$ (1). Write $\alpha = \sum_i t_i \beta_i$, where $t_i \in [0,1]$, $\sum_i t_i = 1$. Then

$$\begin{aligned} |z^\alpha|^2 \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} &= \prod_i |z^{\beta_i}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \\ &\leq \prod_i \sum_j |z^{\beta_j}|^{2t_i} \left(\sum_{i=1}^m |z^{\beta_i}|^2 \right)^{-1} \leq 1. \end{aligned}$$

(1) $\Longrightarrow$ (2). Assume that $\alpha \notin \Delta$. Let H be a hyperplane that separates α and Δ. Say H is defined by $a_1 x_1 + \cdots + a_n x_n = C$. Set

$$z(t) := (t^{a_1}, \ldots, t^{a_n}).$$

Then clearly (A.10) evaluated at $z(t)$ is not bounded. □

Lemma A.5.2 *Let $\beta_1, \ldots, \beta_m \in \mathbb{N}^n$ and $\beta \in \mathbb{R}^n$. Then the following are equivalent*

(1) $\log \sum_{i=1}^m \mathrm{e}^{x \cdot \beta_i} - (x, \beta)$ *is bounded from below.*

(2) *β is in the convex hull of the β_i's.*

Proof The proof follows the same pattern as Lemma A.5.1. □

Appendix B
Pluripotential theory on unibranch spaces

In this appendix, we extend the theory in the book to compact unibranch Kähler spaces.

B.1 Complex spaces

A complex space is assumed to be reduced, Hausdorff and paracompact in the whole book.

Definition B.1.1 A *prime divisor* over an irreducible complex space Z is a connected smooth hypersurface $E \subseteq X'$, where $X' \to Z$ is a proper bimeromorphic morphism with X' smooth. Such a morphism $X' \to Z$ is also called a *resolution* of Z. The *center* of the prime divisor is defined as the image of E in Z.

Two prime divisors $E_1 \subseteq X'_1$ and $E_2 \subseteq X'_2$ over Z are *equivalent* if there is a common resolution $X'' \to X$ dominating both X'_1 and X'_2 such that the strict transforms of E_1 and E_2 coincide.

The set Z^{div} is the set of pairs (c, E), where $c \in \mathbb{Q}_{>0}$ and E is an equivalence class of a prime divisor over Z. For simplicity, we will denote the pair (c, E) by $c \operatorname{ord}_E$, although one should not really think of this object as a valuation unless Z is projective and irreducible.

Note that a prime divisor on Z does not always define a prime divisor over Z if Z is singular.

Definition B.1.2 A complex space X is *unibranch* if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is unibranch.

It is shown in the arXiv version of [Xia23a, Remark 2.7] that when X is a projective variety, this notion coincides with the corresponding algebraic notion of unibranchness.

Theorem B.1.1 (Zariski's main theorem) *Let $\pi: Y \to X$ be a proper bimeromorphic morphism between complex spaces. Assume that X is unibranch, then π has connected fibers.*

We refer to [Dem85, Proof of Théorème 1.7].

Definition B.1.3 A *modification* of a compact complex space X is a finite composition of blow-ups with smooth centers.

We say a modification $\pi': Z \to X$ *dominates* another $\pi: Y \to X$ if there is a morphism $g: Z \to Y$ making the following diagram commutative:

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "\pi'"'] & & Y \arrow[dl, "\pi"] \\ & X. & \end{tikzcd} \tag{B.1}$$

The modifications of X together with the domination relation form a directed set.

Theorem B.1.2 (Hironaka's Chow lemma) *Suppose that X is a compact complex space. Then every proper bimeromorphic morphism to X can be dominated by a modification.*

This follows from the proof of [Hir75, Corollary 2].

Theorem B.1.3 *Let X be a compact complex space. Then there is a modification $\pi: Y \to X$ such that Y is smooth.*

See [BM97, Wł09].

Corollary B.1.1 *Let X be a compact complex space and E be a prime divisor over X. Then there is a modification $\pi: Y \to X$ such that Y is smooth and E can be realized as a prime divisor on Y.*

B.2 Plurisubharmonic functions

Let X be a complex space.

Definition B.2.1 A function $\varphi: X \to [-\infty, \infty)$ is *plurisubharmonic* if

(1) φ is not identically $-\infty$ on any irreducible component of X, and
(2) for any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a plurisubharmonic function $\tilde{\varphi} \in \mathrm{PSH}(\Omega)$ such that $\varphi|_{\Omega\cap V} = \tilde{\varphi}|_{\Omega\cap V}$.

The set of plurisubharmonic functions on X is denoted by $\mathrm{PSH}(X)$.

Similarly, if θ is a smooth closed[1] real $(1,1)$-form on X, then a function $\varphi\colon X \to [-\infty,\infty)$ is *θ-plurisubharmonic* if for any $x \in X$, there is an open neighbourhood V of x in X, a domain $\Omega \subseteq \mathbb{C}^N$, a closed immersion $V \hookrightarrow \Omega$ and a smooth function g on Ω such that $\theta = (\mathrm{dd^c} g)|_{V\cap\Omega}$ and $g + \varphi|_V \in \mathrm{PSH}(V)$.

Theorem B.2.1 (Fornaess–Narasimhan) *Let $\varphi\colon X \to [-\infty,\infty)$ be a function. Assume that φ is not identically $-\infty$ on any irreducible component of X, then the following are equivalent:*

(1) *φ is psh;*
(2) *φ is usc and for any morphism $f : \Delta \to X$ from the open unit disk Δ in $\mathbb{C}$ to X such that $f^*\varphi$ is not identically $-\infty$, the pull-back $f^*\varphi$ is psh.*

See [FsN80].

Theorem B.2.2 (Grauert–Remmert) *Let X be a unibranch[2] complex space and Z be an analytic subset in X and $\varphi \in \mathrm{PSH}(X \setminus Z)$. Then the function φ admits an extension to $\mathrm{PSH}(X)$ in the following two cases:*

(1) *The set Z has codimension at least 2 everywhere.*
(2) *The set Z has codimension at least 1 everywhere and is locally bounded from above on an open neighborhood of Z.*

In both cases, the extension is unique and is given by

$$\varphi(x) = \varlimsup_{X\setminus Z \ni y \to x} \varphi(y), \quad x \in X. \tag{B.2}$$

The proof given below combines [Dem85, Théorème 1.7] and [GR56]. [3]

Proof We first prove the uniqueness, which is a local problem on X. Let ψ denote the function defined by the right-hand side of (B.2). Since any extension φ has to be upper semicontinuous, we know that $\varphi \geq \psi$. Conversely, take $z \in Z$, we take a holomorphic map $f : \Delta \to X$ such that $f(0) = z$ and $f(\Delta) \not\subset Z$. From the subharmonicity of $f^*\varphi$ and (1.2), we find that

[1] Here *closed* means that locally θ is defined by a closed form under a local embedding.

[2] Unibranchness is very important here. Otherwise, consider the case where X is the union of two copies of $\mathbb{C}$ intersecting only at their origins, Z is the common origin. If we set $\varphi \equiv 0$ on one punctured plane and $\varphi \equiv 1$ on the other, then it is clear that φ cannot be extended to X. This leads to a few misleading statements in the modern literature. The problem is that in the early German literature, *komplexer Raum* is assumed to be either normal or unibranch!

[3] This theorem has a quite entangled history. The corresponding results for subharmonic functions are generally attributed to Brelot. In [GR56], they cited a paper of Brelot written in 1934, which I cannot find. But in 1949, on the very first issue of *Annales de l'Institut Fourier*, Brelot published a paper [Bre49] with a very similar title, studying the behavior of a subharmonic function on the punctured neighborhood of a point. The general theorem was due to Grauert and Remmert, see [GR56]. Their original proof was quite complicated, due to the fact that resolution of singularities was not available at that time. Later on, in 1985, Demailly published the influential paper [Dem85] and gave a simpler proof. Oddly enough, Demailly did not cite either Grauert–Remmert or Brelot. He did not even mention that this result was already proved by Grauert–Remmert. The paper [Dem85] is so influential that in France few people know the existence of [GR56].

$$\varphi(z) = f^*\varphi(0) = \varlimsup_{\Delta^* \ni w \to 0} f^*\varphi(w) \leq \varlimsup_{X \setminus Z \ni y \to x} \varphi(y).$$

So (B.2) follows.

Having established the uniqueness of the extension, the existence also becomes a local problem. So we are going to use the same descriptions as in the first paragraph above.

(2) Let $\pi \colon Y \to X$ be a resolution of singularities. By Theorem 1.2.1, we know that $\pi^*\varphi$ admits a unique extension to a psh function on Y, which we denote by η. Note that all fibers of π are connected since X is unibranch. Hence η is constant along the fibers of π. It therefore descends to an upper semicontinuous function η on X.

We verify that φ is plurisubharmonic using Theorem B.2.1. Let $f \colon \Delta \to X$ be a holomorphic map. We assume that $f^*\varphi \not\equiv -\infty$. It suffices to show that $f^*\varphi$ it is subharmonic at $0 \in \Delta$. The germ of f lifts to Y, say represented by $f' \colon \Delta \to Y$ so that

$$f(t^k) = \pi(f'(t))$$

for all t close to 0, where k is an integer. Therefore, $\psi(f(t^k)) = \eta(f(t))$ near 0. It follows that $f^*\varphi$ is subharmonic near 0.

(1) By the local description of complex spaces [GR84, Section 3.4], we may assume that there is a domain $\Omega \subseteq \mathbb{C}^n$, a finite s-sheet branched covering $\Phi \colon X \to \Omega$ with branched locus contained in a proper analytic subset $V \subseteq \Omega$. We may assume that X is connected, $n \geq 1$ and $Z \subseteq \Phi^{-1}(V)$.

If suffices to show that φ is locally bounded from above near Z. Suppose that this fails. Then by (2) we can find $z \in Z$ and $x_i \in X \setminus (\Phi^{-1}(\Phi(Z) \cup V))$ $(i \geq 1)$ converging to z such that

$$\lim_{i \to \infty} \varphi(x_i) = \infty.$$

Let L be a complex line passing through $\Phi(z)$ intersecting $(\Phi(Z) \cup V) \cap \bar{B}$ only at $\Phi(z)$, where $B \Subset B'$ are two small open balls centered at $\Phi(z)$. We can find a sequence of lines L_i passing through $\Phi(x_i)$ converging to L such that $L_i \cap (B' \cap \Phi(Z)) = \varnothing$[4] while $L_i \cap (B' \cap V)$ is discrete. Then Φ restricts to a branched covering over $B' \cap L_i$ for all $i \geq 1$. Adding a constant to φ, we may assume that $\varphi|_{\Phi^{-1}(L \cap \partial B)} < 0$. We can then find an open neighborhood U of $\Phi^{-1}(L \cap \partial B)$ so that $\varphi|_U < 0$. For large i we have $\Phi^{-1}(L_i \cap \partial B) \subseteq U$, it follows from the maximum principle that $\varphi(x_i) \leq 0$, which is a contradiction. □

Corollary B.2.1 *Let $\pi \colon Y \to X$ be a proper bimeromorphic morphism between compact Kähler spaces. Let θ be a smooth closed real $(1,1)$-form on X. Assume that X is unibranch, then the pull-back induces a bijection*

$$\pi^* \colon \mathrm{PSH}(X, \theta) \xrightarrow{\sim} \mathrm{PSH}(Y, \pi^*\theta).$$

[4] Here we need the assumption that Z has codimension at least 2.

B.3 Extensions of the results in the smooth setting

Let X be an irreducible unibranch compact Kähler space of dimension n. Let θ be a closed real smooth $(1,1)$-form on X. We say *the cohomology class* $[\theta]$ is big if for any proper bimeromorphic morphism $\pi\colon Y \to X$ from a compact Kähler manifold Y, $[\pi^*\theta]$ is big.

The non-pluripolar products can be defined exactly as in Chapter 2 and the results in that chapter hold *mutadis mutandis*.

The results in Chapter 3 can be also be easily extended. The definition of the P-envelope remains unchanged. As for the $\mathcal{I}$-envelope, we define

Definition B.3.1 Given $\varphi \in \mathrm{PSH}(X,\theta)$, we define $P_\theta[\varphi]_\mathcal{I} \in \mathrm{PSH}(X,\theta)$ as the unique element with the following property: If $\pi\colon Y \to X$ is a proper bimeromorphic morphism from a compact Kähler manifold Y, then

$$\pi^* P_\theta[\varphi]_\mathcal{I} = P_{\pi^*\theta}[\pi^*\varphi]_\mathcal{I}.$$

It follows from Corollary B.2.1 and Proposition 3.2.5 that $P_\theta[\varphi]_\mathcal{I}$ is independent of the choice of π and is well-defined. The other results can be easily extended.

Chapter 4 and Chapter 6 can be extended without big changes. The only exception is Theorem 6.2.6, where we do not have the notion of multiplier ideal sheaves. So we do not know how to extend this theorem.

Chapter 7 can be extended except for Section 7.4 for the same reason as above.

The trace operator defined in Chapter 8 can be extended as long as Y is not contained in X^{Sing} using the embedded resolution. In general, due to the lack of Demailly regularization, we do not know how to define the trace operator.

Chapter 9 can be easily extended.

Chapter 10 is easy to extend since the partial Okounkov bodies are bimeromorphically invariant in the sense of Theorem 10.4.2.

Chapter 11 is unchanged, since we always take projective limits with respect to all models in that section.

Chapter 13 can be extended except for the parts involving the trace operator.

Chapter 14 can be easily extended by considering a resolution.

I do not know how to extend the results in Chapter 5 and Chapter 12 to the singular setting.

Appendix C
Almost semigroups

We introduce and study almost semigroups. In particular, we will define the Okounkov bodies of almost semigroups.

C.1 Convex bodies

Fix $n \in \mathbb{N}$.

Definition C.1.1 A *convex body* in $\mathbb{R}^n$ is a non-empty compact convex set.

We allow a convex body to have empty interior.

We write $\mathcal{K}_n$ for the set of convex bodies in $\mathbb{R}^n$.

Definition C.1.2 The *Hausdorff metric* between $K_1, K_2 \in \mathcal{K}_n$ is given by

$$d_{\mathrm{Haus}}(K_1, K_2) := \max\left\{ \sup_{x_1 \in K_1} \inf_{x_2 \in K_2} |x_1 - x_2|, \sup_{x_2 \in K_2} \inf_{x_1 \in K_1} |x_1 - x_2| \right\}.$$

It is well-known that the metric space $(\mathcal{K}_n, d_{\mathrm{Haus}})$ is complete. We will need the following fundamental theorem:

Theorem C.1.1 (Blaschke selection theorem) *The metric space* $(\mathcal{K}_n, d_{\mathrm{Haus}})$ *is locally compact.*

We refer to [Sch93, Theorem 1.8.7] for details.

Theorem C.1.2 *The Lebesgue volume* $\mathrm{vol}\colon \mathcal{K}_n \to \mathbb{R}_{\geq 0}$ *is continuous.*

See [Sch93, Theorem 1.8.20].

Theorem C.1.3 *Let* $K_i, K \in \mathcal{K}_n$ *(*$i \in \mathbb{N}$*). Then* $K_i \xrightarrow{d_{\mathrm{Haus}}} K$ *if and only if the following conditions hold:*

(1) *each point* $x \in K$ *is the limit of a sequence* $x_i \in K_i$*, and*

(2) *the limit of any convergent sequence* $(x_{i_j})_{j\in\mathbb{N}}$ *with* $x_{i_j} \in K_{i_j}$ *lies in* K*, where* i_j *is a strictly increasing sequence in* $\mathbb{Z}_{>0}$.

See [Sch93, Theorem 1.8.8].

Lemma C.1.1 *Let* $K \in \mathcal{K}_n$ *be a convex body with positive volume and* $K' \in \mathcal{K}_n$*. Assume that for some large enough* $k \in \mathbb{Z}_{>0}$*,* K' *contains* $K \cap (k^{-1}\mathbb{Z})^n$*, then* $K' \supseteq K^{n^{1/2}k^{-1}}$.

Proof Let $x \in K^{n^{1/2}k^{-1}}$, by assumption, the closed ball B with center x and radius $n^{1/2}k^{-1}$ is contained in K. Observe that x can be written as a convex combination of points in $B \cap (k^{-1}\mathbb{Z})^n$, which are contained in K' by assumption. It follows that $x \in K'$. □

Given a sequence of convex bodies K_i ($i \in \mathbb{N}$), we set

$$\varliminf_{i\to\infty} K_i = \overline{\bigcup_{i=0}^{\infty} \bigcap_{j\geq i} K_j}.$$

Suppose K is the limit of a subsequence of K_i, we have

$$\varliminf_{i\to\infty} K_i \subseteq K. \tag{C.1}$$

This is a simple consequence of Theorem C.1.3.

Lemma C.1.2 *Let* $K \subseteq \mathbb{R}^n$ *be a convex body. Let*

$$t_{\min} := \min\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}, \quad t_{\max} := \max\{t \in \mathbb{R} : \{x_1 = t\} \cap K \neq \varnothing\}.$$

Then for $t \in [t_{\min}, t_{\max}]$*, the map*

$$t \mapsto \{x_1 = t\} \cap K$$

is continuous with respect to the Hausdorff metric.

Here x_1 denotes the first coordinate in $\mathbb{R}^n$.

Proof We may assume that $t_{\min} < t_{\max}$ as otherwise there is nothing to prove.

For each $t \in [t_{\min}, t_{\max}]$, we write $K_t = \{x_1 = t\} \cap K$. Let $t_j \to t$ be a convergent sequence in $[t_{\min}, t_{\max}]$, we want to show that K_{t_j} converges to K_t with respect to the Hausdorff metric. Recall that this amounts to the following two assertions:

(1) For each convergent sequence $x_j \in K_{t_j}$ with limit x, we have $x \in K_t$;
(2) Given any $x \in K_t$, up to replacing t_j by a subsequence, we can find $x_j \in K_{t_j}$ converging to x. □

The first assertion is obvious. Let us prove the second. Take $x = (t, x') \in K_t$. Up to replacing t_j by a subsequence and taking the symmetry into account, we may assume that $t_j > t$ for all t. In particular, $t < t_{\max}$.

We can find a point $y = (y^1, y') \in K$ such that $y^1 > t$ (for example, there is always such a point with $y^1 = t_{\max}$). Replacing t_j by a subsequence, we may assume that $t_j \in (t, y^1)$ for all j. Then it suffices to take

$$x_j = \frac{y^1 - t_j}{y^1 - t} x + \frac{t_j - t}{y^1 - t} y.$$

Lemma C.1.3 *Let $D_j \subseteq \mathbb{R}^n$ $(j \geq 1)$ be a decreasing sequence of convex sets. Assume that* $\operatorname{vol} \bigcap_j D_j > 0$*, then*

$$\overline{\bigcap_{j=1}^{\infty} D_j} = \bigcap_{j=1}^{\infty} \overline{D_j}.$$

Proof The $\subseteq$ direction is clear. By convexity, it suffices to show that both sides have the same positive volume. As the boundary of convex sets has zero Lebesgue measure, it follows that the volumes of both sides are equal to $\lim_{j\to\infty} \operatorname{vol} D_j$. □

Definition C.1.3 Let $K, K' \in \mathcal{K}_n$, their *Minkowski sum* is given by

$$K + K' := \{x + x' : x \in K, x' \in K'\}.$$

Proposition C.1.1 *The Minkowski sum $\mathcal{K}_n \times \mathcal{K}_n \to \mathcal{K}_n$ is continuous.*

See [Sch93, Page 139].

Theorem C.1.4 (Brunn–Minkowski) *Let $K, K' \in \mathcal{K}_n$, then for any $t \in (0, 1)$, we have*

$$\operatorname{vol}((1-t)K' + tK) \geq (\operatorname{vol} K')^{(1-t)} (\operatorname{vol} K)^t.$$

In other words, the volume is log concave. See [Sch93, Page 372].

C.2 The Okounkov bodies of almost semigroups

Fix an integer $n \geq 0$. Fix a closed convex cone $C \subseteq \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ such that $C \cap \{x_{n+1} = 0\} = \{0\}$. Here x_{n+1} is the last coordinate of $\mathbb{R}^{n+1}$.

C.2.1 Generalities on semigroups

Write $\hat{\mathcal{S}}(C)$ for the set of subsets of $C \cap \mathbb{Z}^{n+1}$ and $\mathcal{S}(C)$ for the set of sub-semigroups $S \subseteq C \cap \mathbb{Z}^{n+1}$. For each $k \in \mathbb{N}$ and $S \in \hat{\mathcal{S}}(C)$, we write

$$S_k := \{x \in \mathbb{Z}^n : (x, k) \in S\}.$$

Note that S_k is a finite set by our assumption on C.

We introduce a pseudometric on $\hat{\mathcal{S}}(C)$ as follows:

$$d_{\mathrm{sg}}(S,S') \coloneqq \varlimsup_{k\to\infty} k^{-n}\left(|S_k|+|S'_k|-2|(S\cap S')_k|\right). \tag{C.2}$$

Here $|\bullet|$ denotes the cardinality of a finite set.

Lemma C.2.1 *The above defined d_{sg} is a pseudometric on $\hat{\mathcal{S}}(C)$.*

Proof Only the triangle inequality needs to be argued. Take $S,S',S''\in\hat{\mathcal{S}}(C)$. We claim that for any $k\in\mathbb{N}$,

$$|S_k|+|S'_k|-2|S_k\cap S'_k|+|S''_k|+|S'_k|-2|S''_k\cap S'_k|\geq |S_k|+|S''_k|-2|S_k\cap S''_k|.$$

From this the triangle inequality follows. To argue the claim, we rearrange it to the following form:

$$|S'_k|-|S_k\cap S'_k|\geq |S'_k\cap S''_k|-|S_k\cap S''_k|,$$

which is obvious. □

Given $S,S'\in\hat{\mathcal{S}}(C)$, we say S is equivalent to S' and write $S\sim S'$ if $d_{\mathrm{sg}}(S,S')=0$. This is an equivalence relation by Lemma C.2.1.

Lemma C.2.2 *Given $S,S',S''\in\hat{\mathcal{S}}(C)$, we have*

$$d_{\mathrm{sg}}(S\cap S'',S'\cap S'')\leq d_{\mathrm{sg}}(S,S').$$

In particular, if $S^i,S'^i\in\hat{\mathcal{S}}(C)$ ($i\in\mathbb{N}$) and $S^i\to S$, $S'^i\to S'$, then

$$S^i\cap S'^i\to S\cap S'.$$

Proof Observe that for any $k\in\mathbb{N}$,

$$|S_k\cap S''_k|-|S_k\cap S'_k\cap S''_k|\leq |S_k|-|S_k\cap S'_k|.$$

The same holds if we interchange S with S'. It follows that

$$|S_k\cap S''_k|+|S'_k\cap S''_k|-2|S_k\cap S'_k\cap S''_k|\leq |S_k|+|S'_k|-2|S_k\cap S'_k|.$$

The first assertion follows.

Next we compute

$$\begin{aligned}d_{\mathrm{sg}}(S^i\cap S'^i,S\cap S')\leq& d_{\mathrm{sg}}(S^i\cap S'^i,S^i\cap S')+d_{\mathrm{sg}}(S^i\cap S',S\cap S')\\ \leq& d_{\mathrm{sg}}(S'^i,S')+d_{\mathrm{sg}}(S^i,S)\end{aligned}$$

and the second assertion follows. □

The volume of $S\in\mathcal{S}(C)$ is defined as

$$\operatorname{vol} S\coloneqq \lim_{k\to\infty}(ka)^{-n}|S_{ka}|=\varlimsup_{k\to\infty}k^{-n}|S_k|,$$

where a is a sufficiently divisible positive integer. The existence of the limit and its independence from a both follow from the more precise result [KK12, Theorem 2].

Lemma C.2.3 *Let $S, S' \in \mathcal{S}(C)$, then*

$$|\operatorname{vol} S - \operatorname{vol} S'| \leq d_{\mathrm{sg}}(S, S').$$

Proof By definition, we have

$$d_{\mathrm{sg}}(S, S') \geq \operatorname{vol} S + \operatorname{vol} S' - 2\operatorname{vol}(S \cap S').$$

It follows that $\operatorname{vol} S - \operatorname{vol} S' \leq d_{\mathrm{sg}}(S, S')$ and $\operatorname{vol} S' - \operatorname{vol} S \leq d_{\mathrm{sg}}(S, S')$. □

We define $\overline{\mathcal{S}}(C)$ as the closure of $\mathcal{S}(C)$ in $\hat{\mathcal{S}}(C)$ with respect to the topology defined by the pseudometric d. By Lemma C.2.3, $\operatorname{vol}\colon \mathcal{S}(C) \to \mathbb{R}$ admits a unique 1-Lipschitz extension to

$$\operatorname{vol}\colon \overline{\mathcal{S}}(C) \to \mathbb{R}. \tag{C.3}$$

Lemma C.2.4 *Suppose that $S, S' \in \overline{\mathcal{S}}(C)$ and $S \subseteq S'$. Then*

$$\operatorname{vol} S \leq \operatorname{vol} S'.$$

Proof Take sequences S^j, S'^j in $\mathcal{S}(C)$ such that $S^j \to S$, $S'^j \to S'$. By Lemma C.2.2, after replacing S^j by $S^j \cap S'^j$, we may assume that $S^j \subseteq S'^j$ for each j. Then our assertion follows easily. □

C.2.2 Okounkov bodies of semigroups

Given $S \in \hat{\mathcal{S}}(C)$, we will write $C(S) \subseteq C$ for the closed convex cone generated by $S \cup \{0\}$. Moreover, for each $k \in \mathbb{Z}_{>0}$, we define

$$\Delta_k(S) \coloneqq \operatorname{Conv}\left\{k^{-1}x \in \mathbb{R}^n : x \in S_k\right\} \subseteq \mathbb{R}^n.$$

Here Conv denotes the convex hull.

Definition C.2.1 Let $\mathcal{S}'(C)$ be the subset of $\mathcal{S}(C)$ consisting of semigroups S such that S generates $\mathbb{Z}^{n+1}$ (as an Abelian group).

Note that for any $S \in \mathcal{S}'(C)$, the cone $C(S)$ has full dimension (i.e. the topological interior is non-empty). Given a full-dimensional subcone $C' \subseteq C$, it is clear that $C' \cap \mathbb{Z}^{n+1} \in \mathcal{S}'(C)$.

This class behaves well under intersections:

Lemma C.2.5 *Let $S, S' \in \mathcal{S}'(C)$. Assume that* $\operatorname{vol}(S \cap S') > 0$*, then $S \cap S' \in \mathcal{S}'(C)$.*

The lemma obviously fails if $\operatorname{vol}(S \cap S') = 0$.

Proof We first observe that the cone $C(S) \cap C(S')$ has full dimension since otherwise $\operatorname{vol}(S \cap S') = 0$. Take a full-dimensional subcone C' in $C(S) \cap C(S')$ such that C' intersects the boundary of $C(S) \cap C(S')$ only at 0. It follows from [KK12, Theorem 1] that there is an integer $N > 0$ such that for any $x \in \mathbb{Z}^{n+1} \cap C'$ with Euclidean norm no less than N lies in $S \cap S'$. Therefore, $S \cap S' \in \mathcal{S}'(C)$. □

We recall the following definition from [KK12].

Definition C.2.2 Given $S \in \mathcal{S}'(C)$, its *Okounkov body* is defined as follows

$$\Delta(S) := \{x \in \mathbb{R}^n : (x, 1) \in C(S)\} .$$

Theorem C.2.1 *For each $S \in \mathcal{S}'(C)$, we have*

$$\operatorname{vol} S = \lim_{k \to \infty} k^{-n} |S_k| = \operatorname{vol} \Delta(S) > 0. \tag{C.4}$$

Moreover, as $k \to \infty$,

$$\Delta_k(S) \xrightarrow{d_{\text{Haus}}} \Delta(S). \tag{C.5}$$

This is essentially proved in [WN14, Lemma 4.8], which itself follows from a theorem of Khovanskii [Kho92]. We remind the readers that (C.4) fails for a general $W \in \mathcal{S}(C)$, see [KK12, Theorem 2].

Proof The equalities (C.4) follow from the general theorem [KK12, Theorem 2].

It remains to prove (C.5). By the argument of [WN14, Lemma 4.8], for any compact set $K \subseteq \operatorname{Int} \Delta(S)$, there is $k_0 > 0$ such that for any $k \geq k_0$, $\alpha \in K \cap (k^{-1}\mathbb{Z})^n$ implies that $\alpha \in \Delta_k(S)$.

In particular, taking $K = \Delta(S)^{\delta}$ for any $\delta > 0$ and applying Lemma C.1.1, we find

$$d_{\text{Haus}}(\Delta(S), \Delta_k(S)) \leq n^{1/2} k^{-1} + \delta$$

when k is large enough. This implies (C.5). □

Corollary C.2.1 *Let $S, S' \in \mathcal{S}'(C)$. Assume that* $\operatorname{vol}(S \cap S') > 0$*, then we have*

$$d_{\text{sg}}(S, S') = \operatorname{vol}(S) + \operatorname{vol}(S') - 2 \operatorname{vol}(S \cap S').$$

Proof This is a direct consequence of Lemma C.2.5 and (C.4). □

Lemma C.2.6 *Given $S \in \mathcal{S}'(C)$, we have $S \sim$* $\operatorname{Reg}(S)$.

Recall that the regularization $\operatorname{Reg}(S)$ of S is defined as $C(S) \cap \mathbb{Z}^{n+1}$.

Proof Since S and $\operatorname{Reg}(S)$ have the same Okounkov body, we have $\operatorname{vol} S = \operatorname{vol} \operatorname{Reg}(S)$ by Theorem C.2.1. By Corollary C.2.1 again,

$$d_{\text{sg}}(\operatorname{Reg}(S), S) = \operatorname{vol} \operatorname{Reg}(S) - \operatorname{vol} S = 0.$$

Lemma C.2.7 *Let $S, S' \in \mathcal{S}'(C)$. Assume that $d_{\text{sg}}(S, S') = 0$, then $\Delta(S) = \Delta(S')$.*

Proof Observe that $\mathrm{vol}(S \cap S') > 0$, as otherwise

$$d_{\mathrm{sg}}(S, S') \geq \mathrm{vol}\, S + \mathrm{vol}\, S' > 0,$$

which is a contradiction.

It follows from Lemma C.2.5 that $S \cap S' \in \mathcal{S}'(C)$. It suffices to show that $\Delta(S) = \Delta(S \cap S')$. In fact, suppose that this holds, since $\mathrm{vol}\, \Delta(S') = \mathrm{vol}\, S' = \mathrm{vol}\, S = \mathrm{vol}\, \Delta(S)$, the inclusion $\Delta(S') \supseteq \Delta(S \cap S') = \Delta(S)$ is an equality.

By Lemma C.2.2, we can therefore replace S' by $S \cap S'$ and assume that $S \supseteq S'$. Then clearly $\Delta(S) \supseteq \Delta(S')$. By (C.4),

$$\mathrm{vol}\, \Delta(S) = \mathrm{vol}\, \Delta(S') > 0.$$

Thus, $\Delta(S) = \Delta(S')$. □

Lemma C.2.8 *Suppose that $S^i \in \mathcal{S}'(C)$ is a decreasing sequence such that*

$$\lim_{i \to \infty} \mathrm{vol}\, S^i > 0.$$

Then there is $S \in \mathcal{S}'(C)$ such that $S^i \to S$.

In general, one cannot simply take $S = \bigcap_i S^i$. For example, consider the sequence $S^i = S^1 \cap \{x_{n+1} \geq i\}$.

Proof By Lemma C.2.6, we may replace S^i by its regularization and assume that $S^i = C(S^i) \cap \mathbb{Z}^{n+1}$. We define

$$S = \left(\bigcap_{i=1}^{\infty} C(S^i) \right) \cap \mathbb{Z}^{n+1}.$$

Since $\bigcap_{i=1}^{\infty} C(S^i)$ is a full-dimensional cone by assumption, we have $S \in \mathcal{S}'(C)$. By Corollary C.2.1 and Theorem C.2.1, we can compute the distance

$$d_{\mathrm{sg}}(S, S^i) = \mathrm{vol}\, S^i - \mathrm{vol}\, S = \mathrm{vol}\, \Delta(S^i) - \mathrm{vol}\, \Delta(S),$$

which tends to 0 by construction. □

C.2.3 Okounkov bodies of almost semigroups

Definition C.2.3 We define $\overline{\mathcal{S}'(C)}_{>0}$ as elements in the closure of $\mathcal{S}'(C)$ in $\hat{\mathcal{S}}(C)$ with positive volume. An element in $\overline{\mathcal{S}'(C)}_{>0}$ is called an *almost semigroup* in C.

Recall that the volume here is defined in (C.3).

Our goal is to prove the following theorem:

Theorem C.2.2 *The Okounkov body map* $\Delta\colon \mathcal{S}'(C) \to \mathcal{K}_n$ *as defined in Definition C.2.2 admits a unique continuous extension*

$$\Delta\colon \overline{\mathcal{S}'(C)}_{>0} \to \mathcal{K}_n. \tag{C.6}$$

Moreover, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, *we have*

$$\operatorname{vol} S = \operatorname{vol} \Delta(S). \tag{C.7}$$

Proof The uniqueness of the extension is clear as long as it exists. Moreover, (C.7) follows easily from Theorem C.2.1 and Theorem C.1.2 by continuity. It remains to argue the existence of the continuous extension. We first construct an extension and prove its continuity.

Step 1. We construct the desired map (C.6). Let $S \in \overline{\mathcal{S}'(C)}_{>0}$. We wish to construct a convex body $\Delta(S) \in \mathcal{K}_n$.

Let $S^i \in \mathcal{S}'(C)$ be a sequence that converges to S such that

$$d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}.$$

For each $i, j \geq 0$, we introduce

$$S^{i,j} = S^i \cap S^{i+1} \cdots \cap S^{i+j}.$$

Then by Lemma C.2.2,

$$d_{\mathrm{sg}}(S^{i,j}, S^{i,j+1}) \leq 2^{-i-j}.$$

Take $i_0 > 0$ large enough so that for $i \geq i_0$, $\operatorname{vol} S^i > 2^{-1} \operatorname{vol} S$ and $2^{2-i} < \operatorname{vol} S$ and hence

$$\operatorname{vol} S^i - \operatorname{vol} S^{i,j} \leq d_{\mathrm{sg}}(S^{i,0}, S^{i,1}) + d_{\mathrm{sg}}(S^{i,1}, S^{i,2}) + \cdots + d_{\mathrm{sg}}(S^{i,j-1}, S^{i,j}) \leq 2^{1-i}.$$

It follows that $\operatorname{vol} S^{i,j} > 2^{-1} \operatorname{vol} S - 2^{1-i} > 0$ whenever $i \geq i_0$. In particular, by Lemma C.2.5, $S^{i,j} \in \mathcal{S}'(C)$ for $i \geq i_0$.

By Lemma C.2.8, for $i \geq i_0$, there exists $T^i \in \mathcal{S}'(C)$ such that $S^{i,j} \to T^i$ as $j \to \infty$. Moreover,

$$d_{\mathrm{sg}}(T^i, S) = \lim_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S) \leq \lim_{j\to\infty} d_{\mathrm{sg}}(S^{i,j}, S^i) + d_{\mathrm{sg}}(S^i, S) \leq 2^{1-i} + d_{\mathrm{sg}}(S^i, S).$$

Therefore, $T^i \to S$. We then define

$$\Delta(S) \coloneqq \overline{\bigcup_{i=i_0}^{\infty} \Delta(T^i)}.$$

In other words, we have defined

$$\Delta(S) \coloneqq \varliminf_{i\to\infty} \Delta(S^i).$$

This is an honest limit: if Δ is the limit of a subsequence of $\Delta(S^i)$, then $\Delta(S) \subseteq \Delta$ by (C.1). Comparing the volumes, we find that equality holds. So by Theorem C.1.1,

$$\Delta(S) = \lim_{i\to\infty} \Delta(S^i). \tag{C.8}$$

Next we claim that $\Delta(S)$ as defined above does not depend on the choice of the sequence S^i. In fact, suppose that $S'^i \in \mathcal{S}'(C)$ is another sequence satisfying the same conditions as S^i. The same holds for $R^i := S^{i+1} \cap S'^{i+1}$. It follows that

$$\lim_{i\to\infty} \Delta(R^i) \subseteq \lim_{i\to\infty} \Delta(S^i).$$

Comparing the volumes, we find that equality holds. The same is true with S'^i in place of S^i. So we conclude that $\Delta(S)$ as in (C.8) does not depend on the choices we made.

Step 2. It remains to prove the continuity of Δ defined in Step 1. Suppose that $S^i \in \overline{\mathcal{S}'(C)}_{>0}$ is a sequence with limit $S \in \overline{\mathcal{S}'(C)}_{>0}$. We want to show that

$$\Delta(S^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S). \tag{C.9}$$

We first reduce to the case where $S^i \in \mathcal{S}'(C)$. By (C.8), for each i, we can choose $T^i \in \mathcal{S}'(C)$ such that $d_{\mathrm{sg}}(S^i, T^i) < 2^{-i}$ and $d_{\mathrm{Haus}}(\Delta(S^i), \Delta(T^i)) < 2^{-i}$. If we have shown $\Delta(T^i) \xrightarrow{d_{\mathrm{Haus}}} \Delta(S)$, then (C.9) follows immediately.

Next we reduce to the case where $d_{\mathrm{sg}}(S^i, S^{i+1}) \leq 2^{-i}$. In fact, thanks to Theorem C.1.1, in order to prove (C.9), it suffices to show that each subsequence of $\Delta(S^i)$ admits a subsequence that converges to $\Delta(S)$. Hence, we easily reduce to the required case.

After these reductions, (C.9) is nothing but (C.8). □

Remark C.2.1 As the readers can easily verify from the proof, for any $S \in \overline{\mathcal{S}'(C)}_{>0}$, there is $S' \in \mathcal{S}'(C)$ such that $S \sim S'$.

Corollary C.2.2 *Suppose that $S, S' \in \overline{\mathcal{S}'(C)}_{>0}$ with $S \subseteq S'$, then*

$$\Delta(S) \subseteq \Delta(S'). \tag{C.10}$$

Proof Let $S^j, S'^j \in \mathcal{S}'(C)$ be elements such that $S^j \to S$, $S'^j \to S'$. Then it follows from Lemma C.2.2 that $S^j \cap S'^j \to S$. Since vol is continuous, for large j, $S^j \cap S'^j$ has positive volume and hence lies in $\mathcal{S}'(C)$ by Lemma C.2.5. We may therefore replace S^j by $S^j \cap S'^j$ and assume that $S^j \subseteq S'^j$. Hence, (C.10) follows from the continuity of Δ proved in Theorem C.2.2. □

Remark C.2.2 As the readers can easily verify, the construction of Δ is independent of the choice of C in the following sense: Suppose that C' is another cone satisfying the same assumptions as C and $C' \supseteq C$, then the Okounkov body map $\Delta\colon \overline{\mathcal{S}'(C')}_{>0} \to \mathcal{K}_n$ is an extension of the corresponding map (C.6). We will constantly use this fact without further explanations.

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