Sept 13th, 2016

Homework due Sept 21st!

- Madison math 375/376 friends.
  Office hours: Tue 3:50 - 4:50
  Thu 1:30 - 2:30.

- (In) dependence in a linear space.

  Definition: V - linear space. (real / complex numbers / scalars).
  A set S of V is called <u>independent</u> if...
  - $\forall$ <u>distinct</u> elements $x_1, x_2, \dots, x_n \in S$ and scalars.
  - $c_1, \dots, c_{\mathbb{R}}$, $\sum_{i=1}^{\mathbb{R}} c_i x_i = 0 \Rightarrow c_1 = c_2 = \dots = c_{\mathbb{R}} = 0$.

  (There is no non-trivial linear combination of elements of S).
  - S is called dependent if it is not independent.

Remark: Every subset of an independent set is independent. ①
Every subset of an dependent set is dependent.
(If a subset T of S is dependent, then S is dependent).

- If one element of S is a scalar multiple of another, ②
then S is dependent.

$x = ay$, where $c_1 = a$; $c_2 = -1$.

Then $c_1 y + c_2 x = ay - ay = 0$.

- If $\vec{0} \in S$, then S is dependent. ③

For any C, $C \cdot \emptyset = \emptyset$ ↘ zero vector component.

- The empty set $\emptyset$ is independent. ④

- $f_1(t) = \cos^2 t$

$f_2(t) = \sin^2 t$. $\qquad f_1 + f_2 - f_3 = 0$. ⑤

$f_3(t) = 1$.

- Let $f_k(t) = t^k$, where $k = 0, 1, 2, \dots$ ⑥

Such that, $S = \{f_0, f_1, f_2, \dots\}$ - independent. (claimed).

Show that for any k, then

$$C_0 t^0 + C_1 t^1 + \dots + C_k t^k = 0 \rightarrow h(t)$$

Which implies $C_1 = C_2 = \dots = C_k = 0$. ↓

$h(t) = 0$ for <u>all</u> t.

If $t=0$, then $\underbrace{h(0)}_{C_0} = 0$

Then $\underbrace{h'(t)}_{C_1 + 2C_2 t + \dots} = 0$, for all t.

$C_1 + 2C_2 t + \dots$, which when plugging in $t=0$, then $\underbrace{\cancel{S=0}}_{C_1=0}$. (and so on).

- If $a_1, \dots, a_n$ are DISTINCT real numbers.

Then $u_1(x) = e^{a_1 x}, \dots, u_n(x) = e^{a_n x}$ are independent ⑦

(⑦ Cont.)

Proof by induction. on n.

▲ (INTRODUCTION TO PROVE BY INDUCTION).

Prove $P(n)$ = property dependent on $n \in \mathbb{N}$

↳ Natural numbers.

By induction on n.

1. Check if $P(1)$ holds.
2. Assuming $P(n-1)$ holds, show $P(n)$ holds. $(n-1 \Rightarrow n)$.

Proof of ⑦ by induction on n.

1. $n=1$, is $e^{a_1 x}$ independent?

$C \cdot e^{a_1 x} = 0$ for all x.

Plug-in $x=0$. then $C \cdot 1 = 0$, therefore $C=0$.

2. $n-1 \Rightarrow n$.

If $C_1, C_2, \ldots, C_n$ are scalars so that

$C_1 u_1(x) + \ldots + C_n u_n(x) = 0$ for all x.

Let a_M = largest of $a_1, \ldots, a_n$ up to renumbering

Thus we may assume $M=n$.

$$C_1 e^{a_1 x} + \ldots + C_{(n-1)} e^{a_{(n-1)} \cdot x} + C_n e^{a_n x} = 0$$

Therefore $C_1 e^{(a_1 - a_n) \cdot x} + C_{(n-1)} e^{(a_{(n-1)} - a_n) \cdot x} + C_n \cdot 1 = 0$.

↓

$a_1 - a_n, \ldots, a_{(n-1)} - a_n < 0$, as $e^{-kx} \approx 0$ for very large x.

Thus, let $x \to \infty$, $C_n = 0$.

~~Assumped~~

Assumption:

as $C_1 u_1 + \ldots + C_{n-1} u_{n-1} + C_n = 0$, where $C_n = 0$.

Then the remainder of a C are equal to 0.

- Recall: Given a set S in V. where

$L(S)$ = span of S

↳ Linear ~~combination~~ combination. in V of element of S

Then let $S = \{x_1, x_2, \dots, x_k\}$ be independent. ①

Therefore all $k+1$ elements in $L(S)$ are dependent.

(Textbook Theorem 1.5).

- Bases Dimension

Definition: A FINITE SET ~~of~~ S of elements in a linear space V is called a FINITE BASIS of V.

If:

1. S is independent.
2. S spans V, $L(S) = V$.

- V is FINITE-DIMENSIONAL iff it has a FINITE-BASIS. GIVEN a INFINITE-BASIS will deem otherwise.

Theorem: Let V be a finite dimensional linear space. Then every basis for V has the same number of elements. ②

↳ Which is called a "dimension of V"

Proof: Let S and T be two finite bases for V. Suppose S has k elements, and T has l elements.

Show $k = l$.

(② Cont.).

S is a basis, therefore independent by definition.
and $L(S) = V$.

By Theorem ①, every $k+1$ element in $L(S) = V$ is dependent.
∴ Any every set of $> k$ elements must be dependent.

But T is a set of l independent elements.

$l \leq k$, as T is independent.

- Reverse the roles of S and T to get $k \leq l$.
  (?).

- Examples. ①

$$V = \mathbb{R}^n = \{ (x_1, x_2, \dots, x_n),\ x_1, x_2, \dots, x_n \in \mathbb{R} \}.$$

$$\left.\begin{array}{ll} a. & e_1 = (1, 0, 0, \dots, 0) \\ b. & e_2 = (0, 1, 0, \dots, 0) \\ c. & e_3 = (0, 0, 1, \dots, 0). \\ d & e_n = (0, 0, 0, \dots, 1) \end{array}\right\} \text{Bases of } \mathbb{R}^n.$$

Span $(x_1, \dots, x_n) = x_1 e_1 + \dots + x_n e_n$.

$\mathbb{R}$ ① ② ③ ⓝ.

Given example of $\mathbb{R}^2$.

$$(x, y) = x(1,0) + y(0,1)$$
$$= x e_1 + y e_2.$$

- Example ②.

P_n = polynomial of degree $\leq n$. w/ real coefficient. ②

Claim basis $\{1, t, t^2, \dots, t^n\}$

Span: $a_0 + a_1 t + \dots + a_n t^n = p(t)$

(② cont.)

Let V be a linear space of dimension n ($\dim V = n$).

Thm:

1. Any set of independent element in V is a subset of some basis of V
2. Any set of $\underline{\underline{n}}$ independent element is a basis

$\downarrow$ There cannot be any more elements in a basis than V. dimensions.

Proof. a. Let $S = \{x_1, x_2, \dots, x_n\}$ be an independent set in V.

- If $L(S) = V$, then S is a basis.
- If $L(S) \neq V$, then $\exists\, y \in V$, which $y \notin L(S)$.

Then, let $S' = \{x_1, x_2, \dots, x_k, y\}$.

Claim: S' is independent.

$c_1 x_1 + \dots + c_k x_k + c_{k+1} \cdot y = 0$.

show $c_1 = \dots = c_{k+1} = 0$.

If $c_{k+1} \neq 0$, then

$y = -c_1/c_{k+1} - \dots - c_k/c_{k+1} \in L(S)$

Which contradicts with $y \notin L(S)$

Therefore $c_{k+1} = 0$.

With theorem 1.5, $c_1 x_1 = \dots = c_k x_k = 0$

Given S is independent,

$c_1 = \dots = c_k = 0$.

- If $L(S') = V$, then S' is a basis of V containing S.
- If $L(S') \neq 0$, repeat the previous argument to get a S'' of V.

(② Cont.)

... with $k+2$ elements.

Keep going until you reach a set with n independent elements (There cannot exist $n+1$ independent elements in V).

b. Let S be a set with n independent elements in V.

By proof in **a**, S is contained in a basis B in V.

But B has $\dim(V) = n$ elements.

$$\underbrace{S \subseteq B}_{\text{of equal number of elements}} \Rightarrow S = B. \text{ Therefore}$$

$\boxed{S \text{ is a basis}}$

- Components

① V is a linear space. $\dim(V) = n$.

② $B = \{e_1, \dots, e_n\}$ basis of V.

$L(B) = V$, $\exists$ scalars $c_1, \dots, c_n$ that $x = c_1 e_1 + \dots + c_n e_n$.

$(c_1, \dots, c_n)$ = components of x relative to the basis B.

$d_1 e_1 + \dots + d_n e_n = x = c_1 e_1 + \dots + c_n e_n$.

$\Rightarrow (c_1 - d_1) e_1 + \dots + (c_n - d_n) e_n = 0$.

$\xRightarrow{\text{indep.}}$ ① $c_1 - d_1 = 0$

② $c_n - d_n = 0$.

Therefore...

$c_k = d_k$, where $k = 1, \dots, n$.