Suppose we have accurately reconstructed both the marginal and joint distributions of the observed variables. Under the assumption of conditional independence, we have the following identity:

$$\int_{\mathcal{X}_{<t}} P(\mathbf{x}_{>t}, \mathbf{x}_t|\mathbf{x}_{<t}) d\mathcal{X}_{<t} = \int_{\mathcal{Z}_t} \int_{\mathcal{X}_{<t}} P(\mathbf{x}_{>t}|\mathbf{z}_t) P(\mathbf{x}_t|\mathbf{z}_t) P(\mathbf{z}_t|\mathbf{x}_{<t}) d\mathcal{X}_{<t} d\mathcal{Z}_t.$$

In terms of the operator in the paper, we have:

$$L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}_{<t}} = \int_{\mathcal{X}_{<t}} P(\mathbf{x}_{>t}, \mathbf{x}_t|\mathbf{x}_{<t}) d\mathcal{X}_{<t},$$

$$L_{\mathbf{x}_{>t}|\mathbf{z}_t} = \int_{\mathcal{Z}_t} P(\mathbf{x}_{>t}|\mathbf{z}_t) d\mathcal{Z}_t,$$

$$L_{\mathbf{z}_t|\mathbf{x}<t} = \int_{\mathcal{X}_{<t}} P(\mathbf{z}_t|\mathbf{x}_{<t}) d\mathcal{X}_{<t},$$

$$D_{\mathbf{x}_t|\mathbf{z}_t} = P(\mathbf{x}_t|\mathbf{z}_t).$$

Consequently, the probability functions above can be represented as operators:

$$L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}_{<t}} = L_{\mathbf{x}_{>t}|\mathbf{z}_t} D_{\mathbf{x}_t|\mathbf{z}_t} L_{\mathbf{z}_t|\mathbf{x}<t}.$$

To obtain the marginal operator $L_{\mathbf{x}_{>t}|\mathbf{x}_{<t}}$, we integrate out $\mathbf{x}_t$ as follows:

$$\int_{\mathcal{X}_t} L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}_{<t}} d\mathbf{x}_t = \int_{\mathcal{X}_t} L_{\mathbf{x}_{>t}|\mathbf{z}_t} D_{\mathbf{x}_t|\mathbf{z}_t} L_{\mathbf{z}_t|\mathbf{x}_{<t}} d\mathbf{x}_t,$$

which yields:

$$L_{\mathbf{x}_{>t}|\mathbf{x}_{<t}} = L_{\mathbf{x}_{>t}|\mathbf{z}_t} L_{\mathbf{z}_t|\mathbf{x}<t}.$$

According to injectivity assumption (ii) in Theorem 3.2, the operator $L_{\mathbf{x}_{>t}|\mathbf{z}_t}$ is injective. Therefore, we can invert it and obtain:

$$L^{-1}_{\mathbf{x}>t|\mathbf{z}_t} L_{\mathbf{x}_{>t}|\mathbf{x}_{<t}} = L_{\mathbf{z}_t|\mathbf{x}<t}.$$

Substituting back into the original expression for the joint operator, we derive:

$$L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}<t} = L_{\mathbf{x}>t|\mathbf{z}_t} D_{\mathbf{x}_t|\mathbf{z}_t} L^{-1}_{\mathbf{x}>t|\mathbf{z}_t} L_{\mathbf{x}>t|\mathbf{x}<t}.$$

By Lemma A.4, if the operator $L_{\mathbf{x}<t|\mathbf{x}>t}$ is injective, then $L^{-1}_{\mathbf{x}>t|\mathbf{x}<t}$ exists. Consequently, we obtain:

$$L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}_{<t}} L^{-1}_{\mathbf{x}<t|\mathbf{x}>t} = L_{\mathbf{x}>t|\mathbf{z}_t} D_{\mathbf{x}_t|\mathbf{z}_t} L^{-1}_{\mathbf{x}>t|\mathbf{z}_t}.$$

Since both the marginal and conditional distributions of the observations are matched, the true model and the alternative model yield the same distribution over the observed variables. Therefore, we also have:

$$L_{\mathbf{x}_{>t},\mathbf{x}_t|\mathbf{x}_{<t}} L^{-1}_{\mathbf{x}<t|\mathbf{x}>t} = L_{\hat{\mathbf{x}}_{>t},\hat{\mathbf{x}}_t|\hat{\mathbf{x}}_{<t}} L^{-1}_{\hat{\mathbf{x}}<t|\hat{\mathbf{x}}>t},$$

where the L.H.S. corresponds to the true model and the R.H.S. to the alternative model. Thus, the spectral decomposition of the R.H.S. is:

$$L_{\hat{\mathbf{x}}_{>t},\hat{\mathbf{x}}_t|\hat{\mathbf{x}}_{<t}} L^{-1}_{\hat{\mathbf{x}}<t|\hat{\mathbf{x}}>t} = L_{\hat{\mathbf{x}}>t|\hat{\mathbf{z}}_t} D_{\hat{\mathbf{x}}_t|\hat{\mathbf{z}}_t} L^{-1}_{\mathbf{x}>t|\hat{\mathbf{z}}_t}.$$

Combining the aforementioned three equations, we have

$$L_{\mathbf{x}>t|\mathbf{z}_t} D_{\mathbf{x}_t|\mathbf{z}_t} L^{-1}_{\mathbf{x}>t|\mathbf{z}_t} = L_{\hat{\mathbf{x}}>t|\hat{\mathbf{z}}_t} D_{\hat{\mathbf{x}}_t|\hat{\mathbf{z}}_t} L^{-1}_{\mathbf{x}>t|\hat{\mathbf{z}}_t}.$$

Since the spectral decomposition is unique up to certain indeterminacies (Line 800-884), such as the ordering of eigenvalues and the choice of scaling for eigenvectors, it follows that $D_{\hat{\mathbf{x}}_t|\hat{\mathbf{z}}_t}$ corresponds to $D_{\mathbf{x}_t|\mathbf{z}_t}$ up to certain indeterminacies, as mentioned in Line 800-884.