Space-time Metric Round a Rotating Matter

Kim Mingi(xp2425@khu.ac.kr)

- Einstein field equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$, $R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$
- Conservation law of special relativity: $T^{\mu\nu}{}_{;\nu} = 0$ (continuity equation)
- Components in $T^{\mu\nu}$: $\begin{pmatrix} T^{00}: & \text{energy density} \\ T^{0k} & \text{flow of energy along } x^k \end{pmatrix}$, $\begin{pmatrix} T^{m0}: & \text{density of } m\text{ th comp. of momentum } (p^m) \\ T^{mn}: & \text{flow of } p^m \text{ along } x^n \end{pmatrix}$
- $T^{\mu\nu}$ is a symmetric tensor $\Leftrightarrow$ flow of energy is equivalent to density of momentum
- $T^{\mu\nu} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x{}^2/c^2 & v_x v_y/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & v_y{}^2/c^2 & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_z v_y/c^2 & v_z{}^2/c^2 \end{pmatrix}$

1. Weak Field Limit

For a weak gravitational field ($h_{\mu\nu} << 1$), the metric is near the Minkowski metric, $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,\ \ h_{\mu\nu} << 1 \tag{1}$$

Assume that $g^{\mu\nu} = \eta^{\mu\nu} + \chi^{\mu\nu}$, $\chi^{\mu\nu} << 1$. Then from $g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$,

$$\delta^\mu{}_\rho = (\eta^{\mu\nu} + \chi^{\mu\nu})\,(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^\mu{}_\rho + \chi^\mu{}_\rho + h^\mu{}_\rho + O$$

Namely, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ In short, $-\begin{pmatrix} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \end{pmatrix}$

Recall that Christoffel symbol and Ricci tensor are given by

$$\Gamma^\kappa{}_{\lambda\mu} = \frac{1}{2} g^{\kappa\rho}(g_{\rho\lambda,\mu} + g_{\rho\mu,\lambda} - g_{\lambda\mu,\rho}) \tag{2}$$

$$R_{\mu\nu} = \Gamma^\kappa{}_{\mu\nu,\kappa} - \Gamma^\kappa{}_{\mu\kappa,\nu} + \Gamma^\kappa{}_{\rho\kappa}\,\Gamma^\rho{}_{\mu\nu} - \Gamma^\kappa{}_{\rho\nu}\,\Gamma^\rho{}_{\mu\kappa} \tag{3}$$

In this case, they are given

$$\Gamma^\kappa{}_{\lambda\mu} = \frac{1}{2}(\eta^{\kappa\rho} - h^{\kappa\rho})\,(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) = \frac{1}{2}\eta^{\kappa\rho}(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) + O\left(h^2\right) \ = \frac{1}{2}\eta^{\kappa\rho}(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) \tag{4}$$

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\kappa{}_{\mu\nu,\kappa} - \Gamma^\kappa{}_{\mu\kappa,\nu} + \Gamma^\kappa{}_{\rho\kappa}\,\Gamma^\rho{}_{\mu\nu} - \Gamma^\kappa{}_{\rho\nu}\,\Gamma^\rho{}_{\mu\kappa} \\ &= \Gamma^\kappa{}_{\mu\nu,\kappa} - \Gamma^\kappa{}_{\mu\kappa,\nu} + O\left(h^2\right) \\ &= \tfrac{1}{2}\eta^{\kappa\rho}(h_{\rho\nu,\mu\kappa} + h_{\rho\mu,\nu\kappa} - h_{\mu\nu,\rho\kappa}) - \tfrac{1}{2}\eta^{\kappa\rho}(h_{\rho\mu,\kappa\nu} + h_{\rho\kappa,\mu\nu} - h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2}(\eta^{\kappa\rho}\,h_{\rho\nu,\mu\kappa} - \eta^{\kappa\rho}\,h_{\rho\kappa,\mu\nu} - \eta^{\kappa\rho}\,h_{\mu\nu,\rho\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2}(\eta^{\kappa\rho}\,h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \end{aligned}$$

$$R_{\mu\nu} = \frac{1}{2}(\eta^{\kappa\rho}\,h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \tag{5}$$

note that $\eta^{\mu\nu}\,\partial_\mu\,\partial_\nu$ is a D'Alembertian '$\Box$'

From Einstein field equation, ($S_{\mu\nu} = T_{\mu\nu} - 1/2\, g_{\mu\nu}\, T$)

$$R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, T\right) = \frac{1}{2}(\eta^{\kappa\rho}\,h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu})$$

$$\frac{16\pi G}{c^2} S_{\mu\nu} = \eta^{\kappa\rho}\,h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}$$

To express in neater way define new quantities, $f^{\mu\nu}$

$$\sqrt{-g}\ g^{\mu\nu} = \eta^{\mu\nu} - f^{\mu\nu} \tag{6}$$

where $g = \det(g_{\mu\nu})$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

In[4]:=
```
η := DiagonalMatrix[{-1, 1, 1, 1}];
H := Table[h_{ToString[i],ToString[j]}, {i, 0, 3}, {j, 0, 3}];
η + H // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} -1 + h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} \\ h_{1,0} & 1 + h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,0} & h_{2,1} & 1 + h_{2,2} & h_{2,3} \\ h_{3,0} & h_{3,1} & h_{3,2} & 1 + h_{3,3} \end{pmatrix}$$

so,

$$\begin{aligned} g &= (-1 + h_{00})(1 + h_{11})(1 + h_{22})(1 + h_{22}) + O(h^2) \\ &= -1 + h_{00} - h_{11} - h_{22} - h_{33} + O(h^2) \\ &= -1 + \eta_{00}\, h^0{}_0 - \eta_{11}\, h^1{}_1 - \eta_{22}\, h^2{}_2 - \eta_{33}\, h^3{}_3 + O(h^2) \\ &= -1 - h^0{}_0 - h^1{}_1 - h^2{}_2 - h^3{}_3 + O(h^2) \\ &= -1 - h^\mu{}_\mu + O(h^2) \end{aligned}$$

hence,

$$\sqrt{-g} = (-g)^{1/2} = \left(1 + h^\lambda{}_\lambda + O(h^2)\right)^{1/2} = 1 + \frac{1}{2} h^\mu{}_\mu + O(h^2)$$

$$\sqrt{-g}\, g^{\mu\nu} = \left(1 + \frac{1}{2} h^\lambda{}_\lambda + O(h^2)\right)(\eta^{\mu\nu} - h^{\mu\nu}) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$\eta^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} h^\lambda{}_\lambda - h^{\mu\nu} + O(h^2) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\, \eta^{\mu\nu} h^\lambda{}_\lambda \tag{7}$$

$$\eta_{\mu\nu} f^{\mu\nu} = f^\mu{}_\mu = \eta_{\mu\nu}\left(h^{\mu\nu} - 1/2\, \eta^{\mu\nu} h^\lambda{}_\lambda\right) = h^\mu{}_\mu - 1/2\,(4)\, h^\lambda{}_\lambda = -h^\mu{}_\mu$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\, \eta^{\mu\nu}\left(-f^\lambda{}_\lambda\right) \longrightarrow h^{\mu\nu} = f^{\mu\nu} - 1/2\, \eta^{\mu\nu} f^\lambda{}_\lambda \tag{8}$$

$$h^\lambda{}_\nu = \eta_{\mu\nu} h^{\mu\lambda} = \eta_{\mu\nu}\left(f^{\mu\lambda} - 1/2\, \eta^{\mu\lambda} f^\rho{}_\rho\right) = f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho$$

$$h_{\mu\nu} = \eta_{\lambda\mu} h^\lambda{}_\nu = \eta_{\lambda\mu}\left(f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho\right) = f_{\mu\nu} - 1/2\, \eta_{\mu\nu} f^\rho{}_\rho$$

Now back to the field equation.

$$R_{\mu\nu} = \frac{1}{2}\left(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}\right) = \frac{8\pi G}{c^2} T_{\mu\nu}$$

$$\left(h^\lambda{}_{\nu,\mu\lambda} + h^\lambda{}_{\mu,\lambda\nu} - h^\lambda{}_{\lambda,\mu\nu} - \Box h_{\mu\nu}\right)$$

$$\Rightarrow \left[\left(f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho\right)_{,\mu\lambda} + \left(f^\lambda{}_\mu - 1/2\, \eta^\lambda{}_\mu f^\rho{}_\rho\right)_{,\lambda\nu} - \left(-f^\rho{}_\rho\right)_{,\mu\nu} - \Box\left(f_{\mu\nu} - 1/2\, \eta_{\mu\nu} f^\rho{}_\rho\right)\right]$$

$$= \left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - (f^\rho{}_\rho)_{,\mu\nu} + f^\rho{}_{\rho,\mu\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right] = \left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right]$$

f has to be independent of time. Finally,

$$R_{\mu\nu} = \frac{1}{2}\left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right]$$

and

$$\begin{aligned} (1/2)\, \eta_{\mu\nu} R &= (1/2)\, \eta_{\mu\nu}(\eta^{\rho\sigma} R_{\rho\sigma}) \\ &= (1/2)\, \eta_{\mu\nu}\, \eta^{\rho\sigma}(1/2)\left[f^\lambda{}_{\sigma,\rho\lambda} + f^\lambda{}_{\rho,\sigma\nu} - \Box f_{\rho\sigma} + 1/2\, \eta_{\rho\sigma} \Box f^\lambda{}_\lambda\right] \\ &= (1/4)\, \eta_{\mu\nu}\left[f^{\lambda\rho}{}_{,\rho\lambda} + f^{\lambda\sigma}{}_{,\sigma\nu} - \Box f^\lambda{}_\lambda + 1/2\,(4)\, \Box f^\lambda{}_\lambda\right] \\ &= \tfrac{1}{4}\, \eta_{\mu\nu}\left[2\, f^{\lambda\rho}{}_{,\rho\lambda} + \Box f^\lambda{}_\lambda\right] = \tfrac{1}{2}\, \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} + \tfrac{1}{4}\, \eta_{\mu\nu} \Box f^\lambda{}_\lambda \end{aligned}$$

Then the field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$ give

$$\frac{1}{2}\left[f^{\lambda}{}_{\nu\,,\mu\lambda}+f^{\lambda}{}_{\mu\,,\lambda\nu}-\Box f_{\mu\nu}+1/2\,\eta_{\mu\nu}\Box f^{\rho}{}_{\rho}\right]-\left(\frac{1}{2}\eta_{\mu\nu}f^{\rho\sigma}{}_{,\rho\sigma}+\frac{1}{4}\eta_{\mu\nu}\Box f^{\lambda}{}_{\lambda}\right)$$

$$=\frac{1}{2}\left[f^{\lambda}{}_{\nu\,,\mu\lambda}+f^{\lambda}{}_{\mu\,,\lambda\nu}-\eta_{\mu\nu}f^{\rho\sigma}{}_{,\rho\sigma}-\Box f_{\mu\nu}\right]=\frac{8\,\pi\mathrm{G}}{c^2}T_{\mu\nu}$$

$$f^{\lambda}{}_{\nu\,,\mu\lambda}+f^{\lambda}{}_{\mu\,,\lambda\nu}-\eta_{\mu\nu}f^{\rho\sigma}{}_{,\rho\sigma}-\Box f_{\mu\nu}=\frac{16\,\pi\mathrm{G}}{c^2}T_{\mu\nu} \tag{9}$$

By choosing proper transformation, we can simplify above equation.

$$x^{\mu}\longrightarrow y^{\mu}=x^{\mu}+b^{\mu}(x)$$

Under given transformation,

$$\frac{\partial y^{\mu}}{\partial x^{\nu}}=\delta^{\mu}{}_{\nu}+b^{\mu}{}_{,\nu}$$

$$g^{\mu\nu}(x)\longrightarrow g'^{\,\mu\nu}(y)=\frac{\partial y^{\mu}}{\partial x^{\rho}}\frac{\partial y^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma}(x)=(\delta^{\mu}{}_{\rho}+b^{\mu}{}_{,\rho})(\delta^{\nu}{}_{\sigma}+b^{\nu}{}_{,\sigma})\,g^{\rho\sigma}$$

$$=(\delta^{\mu}{}_{\rho}+b^{\mu}{}_{,\rho})(g^{\rho\nu}+g^{\rho\sigma}b^{\nu}{}_{,\sigma})=g^{\mu\nu}+g^{\rho\nu}b^{\mu}{}_{,\rho}+g^{\mu\sigma}b^{\nu}{}_{,\sigma}+O\left(b^2\right)$$

Then the transformed metric tensor $g'^{\,\mu\nu}$ is given as below.

```
In[35]:= Table[g^ToString[{i,j}] + g^"ρ" ToString[j] (b^ToString[i])_",ρ" + g^ToString[i] "ρ" (b^ToString[j])_",ρ",
    {i, 0, 3}, {j, 0, 3}] // MatrixForm // TraditionalForm
```

Out[35]//TraditionalForm=

$$\begin{pmatrix} 2\,b^0{}_{,\rho}\,g^{0\rho}+g^{\{0,0\}} & g^{0\rho}\,b^1{}_{,\rho}+b^0{}_{,\rho}\,g^{1\rho}+g^{\{0,1\}} & g^{0\rho}\,b^2{}_{,\rho}+b^0{}_{,\rho}\,g^{2\rho}+g^{\{0,2\}} & g^{0\rho}\,b^3{}_{,\rho}+b^0{}_{,\rho}\,g^{3\rho}+g^{\{0,3\}} \\ g^{0\rho}\,b^1{}_{,\rho}+b^0{}_{,\rho}\,g^{1\rho}+g^{\{1,0\}} & 2\,b^1{}_{,\rho}\,g^{1\rho}+g^{\{1,1\}} & g^{1\rho}\,b^2{}_{,\rho}+b^1{}_{,\rho}\,g^{2\rho}+g^{\{1,2\}} & g^{1\rho}\,b^3{}_{,\rho}+b^1{}_{,\rho}\,g^{3\rho}+g^{\{1,3\}} \\ g^{0\rho}\,b^2{}_{,\rho}+b^0{}_{,\rho}\,g^{2\rho}+g^{\{2,0\}} & g^{1\rho}\,b^2{}_{,\rho}+b^1{}_{,\rho}\,g^{2\rho}+g^{\{2,1\}} & 2\,b^2{}_{,\rho}\,g^{2\rho}+g^{\{2,2\}} & g^{2\rho}\,b^3{}_{,\rho}+b^2{}_{,\rho}\,g^{3\rho}+g^{\{2,3\}} \\ g^{0\rho}\,b^3{}_{,\rho}+b^0{}_{,\rho}\,g^{3\rho}+g^{\{3,0\}} & g^{1\rho}\,b^3{}_{,\rho}+b^1{}_{,\rho}\,g^{3\rho}+g^{\{3,1\}} & g^{2\rho}\,b^3{}_{,\rho}+b^2{}_{,\rho}\,g^{3\rho}+g^{\{3,2\}} & 2\,b^3{}_{,\rho}\,g^{3\rho}+g^{\{3,3\}} \end{pmatrix}$$

With $g'=\,|\,g'_{\mu\nu}\,|=\,|\,g'^{\,\mu\nu}\,|^{-1}$,

$$\begin{aligned}(g')^{-1}&=\,|\,g'^{\,\mu\nu}\,|=\left(g^{00}+2\,g^{0\rho}\,b^0{}_{,\rho}\right)\left(g^{11}+2\,g^{1\rho}\,b^1{}_{,\rho}\right)\left(g^{22}+2\,g^{2\rho}\,b^2{}_{,\rho}\right)\left(g^{33}+2\,g^{3\rho}\,b^3{}_{,\rho}\right)+O\left(b^2\right)\\&=g^{00}\,g^{11}\,g^{22}\,g^{33}+2\left(g^{\rho0}\,g^{11}\,g^{22}\,g^{33}\,b^0{}_{,\rho}+g^{00}\,g^{\rho1}\,g^{22}\,g^{33}\,b^1{}_{,\rho}+g^{00}\,g^{11}\,g^{\rho2}\,g^{33}\,b^2{}_{,\rho}+g^{00}\,g^{11}\,g^{22}\,g^{\rho3}\,b^3{}_{,\rho}\right)+O\left(b^2\right)\\&=g^{-1}+2\,g^{-1}\left(b^0{}_{,0}+b^1{}_{,1}+b^2{}_{,2}+b^3{}_{,3}\right)=g^{-1}\left(1+2\,b^{\lambda}{}_{,\lambda}\right)\end{aligned}$$

Namely,

$$g'=g\left(1-2\,b^{\lambda}{}_{,\lambda}\right),\quad \sqrt{-g'}=\sqrt{-g}\,\left(1-b^{\lambda}{}_{,\lambda}\right) \tag{10}$$

$$\sqrt{-g'}\;g'^{\,\mu\nu}=\sqrt{-g}\,\left(1-b^{\lambda}{}_{,\lambda}\right)(g^{\mu\nu}+g^{\rho\nu}b^{\mu}{}_{,\rho}+g^{\mu\sigma}b^{\nu}{}_{,\sigma})=\sqrt{-g}\,\left(g^{\mu\nu}+g^{\rho\nu}b^{\mu}{}_{,\rho}+g^{\mu\sigma}b^{\nu}{}_{,\sigma}-g^{\mu\nu}b^{\lambda}{}_{,\lambda}\right)+O\left(b^2\right)$$

$$=\sqrt{-g}\,\left(g^{\mu\nu}+g^{\rho\nu}b^{\mu}{}_{,\rho}+g^{\mu\sigma}b^{\nu}{}_{,\sigma}-g^{\mu\nu}b^{\lambda}{}_{,\lambda}\right)=\eta^{\mu\nu}-f'^{\,\mu\nu}$$

last term come from $\sqrt{-g}\;g^{\mu\nu}=\eta^{\mu\nu}-f^{\mu\nu}$

$$\sqrt{-g}\,\left(g^{\mu\nu}+g^{\rho\nu}b^{\mu}{}_{,\rho}+g^{\mu\sigma}b^{\nu}{}_{,\sigma}-g^{\mu\nu}b^{\lambda}{}_{,\lambda}\right)=\eta^{\mu\nu}-f'^{\,\mu\nu}$$

$$\eta^{\mu\nu}-f^{\mu\nu}+(\eta^{\rho\nu}-f^{\rho\nu})\,b^{\mu}{}_{,\rho}+(\eta^{\mu\sigma}-f^{\mu\sigma})\,b^{\nu}{}_{,\sigma}-(\eta^{\mu\nu}-f^{\mu\nu})\,b^{\lambda}{}_{,\lambda}=\eta^{\mu\nu}-f'^{\,\mu\nu}$$

$$-f^{\mu\nu}+(\eta^{\rho\nu}-f^{\rho\nu})\,b^{\mu}{}_{,\rho}+(\eta^{\mu\sigma}-f^{\mu\sigma})\,b^{\nu}{}_{,\sigma}-(\eta^{\mu\nu}-f^{\mu\nu})\,b^{\lambda}{}_{,\lambda}=-f'^{\,\mu\nu}$$

Hence,

$$f'^{\,\mu\nu}=f^{\mu\nu}-\eta^{\rho\nu}\,b^{\mu}{}_{,\rho}-\eta^{\mu\sigma}\;b^{\nu}{}_{,\sigma}+\eta^{\mu\nu}\,b^{\lambda}{}_{,\lambda} \tag{11}$$

and,

$$f'^{\,\mu\nu}{}_{,\nu}=f^{\mu\nu}{}_{,\nu}-\eta^{\rho\nu}\,b^{\mu}{}_{,\rho\nu}-\eta^{\mu\sigma}\;b^{\nu}{}_{,\sigma\nu}+\eta^{\mu\nu}\,b^{\lambda}{}_{,\lambda\nu}=f^{\mu\nu}{}_{,\nu}-\eta^{\rho\nu}\,b^{\mu}{}_{,\rho\nu}=f^{\mu\nu}{}_{,\nu}-\Box\,b^{\mu} \tag{12}$$

where $\eta^{\rho\nu}\,\partial_{\rho}\,\partial_{\nu}$ is a D’Alembertian ‘□’. So we can properly choose b^{μ} such that $f^{\mu\nu}{}_{,\nu}=\Box\,b^{\mu}$, so that $f'^{\,\mu\nu}{}_{,\nu}=0$ or,

$$\left(\sqrt{-g'}\;g'^{\mu\nu}\right)_{,\nu}=0\quad\text{(harmonic condition)}$$

Under the harmonic condition,

$$f^{\lambda}{}_{\nu,\mu\lambda} + f^{\lambda}{}_{\mu,\lambda\nu} - \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu} = \frac{16\pi G}{c^2} T_{\mu\nu} \Longrightarrow -\Box f_{\mu\nu} = \frac{16\pi G}{c^2} T_{\mu\nu}$$

Finally the field equation reduced to the following Poisson equation

$$\Box f_{\mu\nu} = -\frac{16\pi G}{c^2} T_{\mu\nu} \tag{13}$$

with harmonic condition

$$f^{\mu\nu}{}_{,\nu} = \left(\sqrt{-g}\; g^{\mu\nu}\right)_{,\nu} = 0$$

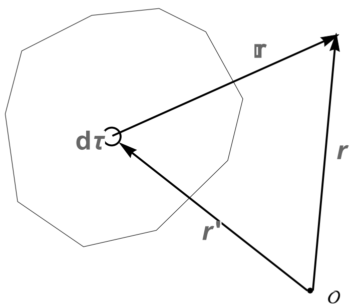
where

$$\left(\begin{array}{c} f^{\mu\nu} = h^{\mu\nu} - 1/2\, \eta^{\mu\nu} h^{\lambda}{}_{\lambda} \\ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\, \eta_{\mu\nu} f^{\lambda}{}_{\lambda} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} = \eta_{\mu\nu} - f_{\mu\nu} + 1/2\, \eta_{\mu\nu} f^{\lambda}{}_{\lambda} \\ f^{\lambda}{}_{\lambda} = \eta^{\mu\lambda} f_{\mu\lambda} = -f_{00} + f_{11} + f_{22} + f_{33} \end{array}\right.$$

We already dealt with the Poisson's equation at Electromagnetism course (see Griffith's book chap 10.2); solution of the equation (13) is given by

$$f_{\mu\nu}(\boldsymbol{r}, t) = \frac{1}{4\pi} \frac{16\pi G}{c^2} \int \frac{T_{\mu\nu}(\boldsymbol{r}', t_r)}{\mathfrak{r}} d\tau \tag{14}$$

where t_r (retarded time) $= t - \frac{\mathfrak{r}}{c}$ and $\boldsymbol{\mathfrak{r}} = \boldsymbol{r} - \boldsymbol{r}'$, $\mathfrak{r} = |\boldsymbol{r} - \boldsymbol{r}'|$



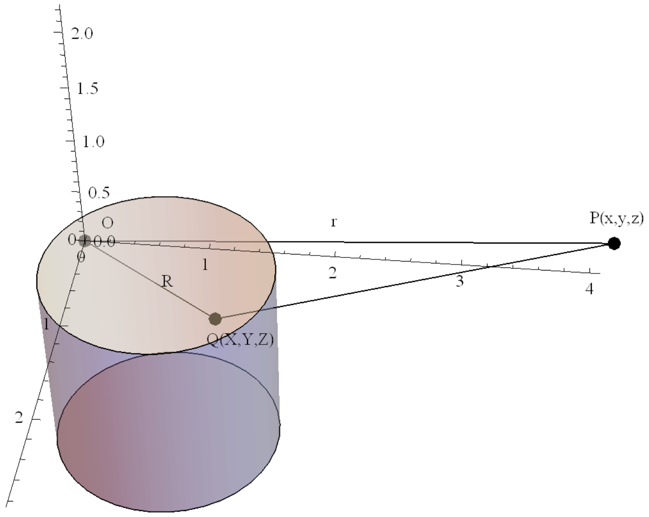
2. Rotating Body: Angular Momentum

Consider a body rotating with constant angular velocity $\omega = d\phi/dt$ about the x^3 − axis (z axis) and assume that $v << c$. Then,

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x^2/c^2 & v_x v_y/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & v_y^2/c^2 & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_z v_y/c^2 & v_z^2/c^2 \end{pmatrix} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & 0 \\ v_x/c & 0 & 0 & 0 \\ v_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{\mu\nu} = \eta_{\mu\lambda} \eta_{\nu\rho} T^{\lambda\rho} = \rho \begin{pmatrix} 1 & -v_x/c & -v_y/c & 0 \\ -v_x/c & 0 & 0 & 0 \\ -v_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[70]=



As above figure, let's call the coordinates of a point Q (object) inside the rotating body by X^i and those of a point P (observer) outside the body by x^i then, $\begin{pmatrix} r^2 = x^i x_i \\ R^2 = X^i X_i \end{pmatrix}$ with $R << \text{r}$. Hence,

$$\mathbb{r} = |r - R| = \left(r^2 - 2\,\boldsymbol{r}\cdot\boldsymbol{R} + R^2\right)^{1/2} = r\left(1 - 2\,\frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2} + \frac{R^2}{r^2}\right)^{1/2} \simeq r\left(1 - \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) \tag{15}$$

$$1/\mathbb{r} = |r - R|^{-1} = \frac{1}{r}\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) \tag{16}$$

The field equation $\left(\Box f_{\mu\nu} = -\dfrac{16\,\pi\text{G}}{c^2}\,T_{\mu\nu}\right)$ gives

$$\begin{pmatrix} \Box f_{00} = -\frac{16\pi\text{G}}{c^2}\,\rho \\ \Box f_{01} = -\frac{16\pi\text{G}}{c^2}\,T_{01} \\ \Box f_{02} = -\frac{16\pi\text{G}}{c^2}\,T_{02} \end{pmatrix}, \text{ other } f_{\mu\nu} = 0 \qquad \left(\text{recall that} \begin{pmatrix} f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\,h^\lambda{}_\lambda \\ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\,f^\lambda{}_\lambda \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} = \eta_{\mu\nu} - f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\,f^\lambda{}_\lambda \\ f^\lambda{}_\lambda = \eta^{\mu\lambda}\,f_{\mu\lambda} = -f_{00} + f_{11} + f_{22} + f_{33} \end{pmatrix}\right)$$

then $\Box f_{00} = -\frac{16\pi\text{G}}{c^2}\,\rho$ is analog to Newton's equation: $\nabla^2\,\phi = -4\,\pi\text{G}\rho \longrightarrow f_{00} = \frac{4\,\phi}{c^2}, \quad f^\lambda{}_\lambda = -f_{00} = -\frac{4\,\phi}{c^2}$

$$g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\left(-\frac{4\,\phi}{c^2}\right) = f_{\mu\nu} + \left(1 + \frac{2\,\phi}{c^2}\right)\eta_{\mu\nu}$$

$$g_{00} = \frac{4\,\phi}{c^2} - 1 - \frac{2\,\phi}{c^2} = -\left(1 - \frac{2\,\phi}{c^2}\right) \tag{17}$$

In[144]:=
```
F = Table[Subscript[f, i, j], {i, 0, 3}, {j, 0, 3}];
F[[1, 4]] = 0; F[[4, 1]] = 0;
F[[2 ;; 4, 2 ;; 4]] = ConstantArray[0, {3, 3}];
F // MatrixForm // TraditionalForm
g = F + (1 + 2 ϕ/c^2) η; g /. {F[[1, 1]] → 4 ϕ/c^2} // MatrixForm // TraditionalForm
```

Out[146]//TraditionalForm=

$$\begin{pmatrix} f_{0,0} & f_{0,1} & f_{0,2} & 0 \\ f_{1,0} & 0 & 0 & 0 \\ f_{2,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[147]//TraditionalForm=

$$\begin{pmatrix} \frac{2\,\phi}{c^2} - 1 & f_{0,1} & f_{0,2} & 0 \\ f_{1,0} & \frac{2\,\phi}{c^2} + 1 & 0 & 0 \\ f_{2,0} & 0 & \frac{2\,\phi}{c^2} + 1 & 0 \\ 0 & 0 & 0 & \frac{2\,\phi}{c^2} + 1 \end{pmatrix}$$

where $\phi = \frac{GM}{r} + \ldots,$ As long as we get f_{01} and f_{02}, then we can figure out the whole metric tensor $g_{\mu\nu}$. To get $f_{0\,i}$, $(i = 1, 2)$ we have to utilize the solution of the Poisson's equation (eq.(14)) we've already met.

$$f_{\mu\nu}(\boldsymbol{r},\ t) = \frac{1}{4\,\pi} \frac{16\,\pi\mathrm{G}}{c^2} \int \frac{T_{\mu\nu}(\boldsymbol{r}',\ t_r)}{\mathfrak{r}} \,d\tau$$