Usefull Identities for General Relativity

$$\text{Note that} \quad \partial_v F^k \equiv F^k{}_{,v}$$

Connection coefficient or Christoffel symbol

$$\Gamma^{\mu}_{\lambda\nu} = \frac{1}{2} g^{\mu\kappa} \left(g_{\kappa\lambda,\nu} + g_{\kappa\nu,\lambda} - g_{\lambda\nu,\kappa} \right)$$

Note that Christoffel symbol is not a tensor!
Christoffel symbol은 basis vector 를 미분할 때 튀어나온다.

$$\mathbf{e}_{m,i} = \Gamma^{k}_{mi} \mathbf{e}_k$$

Inner product of basis vector and basis one-form

$$\langle \mathbf{e}_{\nu}, \theta^{\mu} \rangle = \delta^{\mu}{}_{\nu}$$

vector 와 one-form 은 서로 dual 이다. dual: inner product를 하면 scalar가 나옴 (ex. bra & ket vector, complex number와 그것의 complex conjugate 등등..)

Covariant derivative of vector

$$\mathcal{D}_{\mu} V^{\kappa} = V^{\kappa}{}_{;\mu} = \partial_{\mu} V^{\kappa} + \Gamma^{\kappa}_{\mu\lambda} V^{\lambda} = V^{\kappa}{}_{,\mu} + \Gamma^{\kappa}_{\mu\lambda} V^{\lambda}$$

$$\mathcal{D}_{\mu} V_{\kappa} = V_{\kappa;\mu} = \partial_{\mu} V_{\kappa} - \Gamma^{\lambda}_{\mu\kappa} V_{\lambda} = V_{\kappa,\mu} - \Gamma^{\lambda}_{\mu\kappa} V_{\lambda}$$

Covariant: 좌표가 변해도 형태가 변하지 않음, Covariant derivative는 tensor 임
Γ^{a}_{bc} : Connection coefficient 또는 Christoffel symbol (The term 'connection coefficient' comes about because this quantity connects the value of a vector field at one point with the value at another. It amounts to an additional structure possessed by the space.)
index 가 2개일 때

$$V^{mn}{}_{;s} = V^{mn}{}_{,s} + \Gamma^{m}_{st} V^{tn} + \Gamma^{n}_{st} V^{tm}$$

$$V_{mn;s} = V_{mn,s} - \Gamma^{t}_{sm} V_{tn} - \Gamma^{t}_{sn} V_{tm}$$

Covariant derivative of vector $\vec{V} = V^{\mu} \mathbf{e}_{\mu}$

$$\nabla \vec{V} = \left(\nabla_i \vec{V} \right) \theta^i = \left(V^{m}{}_{,i} \mathbf{e}_m + V^m \mathbf{e}_{m,i} \right) \theta^i$$

$$= \left(V^{m}{}_{,i} \mathbf{e}_m + V^m \Gamma^{k}_{mi} \mathbf{e}_k \right) \theta^i$$

$$= \left(V^k{}_{,i}\mathbf{e}_k + V^m \Gamma^k_{mi}\mathbf{e}_k\right)\theta^i$$

$$= \left(V^k{}_{,i} + \Gamma^k_{mi}V^m\right)\mathbf{e}_k \otimes \theta^i = \left(V^k{}_{;i}\right)\theta^i \otimes \mathbf{e}_k$$

Basis one-form 들을 dx^i 라 하면,

$$\nabla\vec{V} = V^k{}_{;i}dx^i \otimes \mathbf{e}_k$$

$\vec{U} = U^\mu \mathbf{e}_\mu$ 방향의 absolute derivative 는 둘의 inner product를 취하면 된다.

$$\nabla_{\mathbf{U}}\vec{V} = \left\langle \nabla\vec{V}, \vec{U} \right\rangle = \left\langle V^\kappa{}_{;\lambda}dx^\lambda \otimes \mathbf{e}_\kappa, U^\nu \mathbf{e}_\nu \right\rangle = V^\kappa{}_{;\lambda}U^\nu \left\langle dx^\lambda \otimes \mathbf{e}_\kappa, \mathbf{e}_\nu \right\rangle$$

$$= V^\kappa{}_{;\lambda}U^\nu \left\langle dx^\lambda \otimes \mathbf{e}_\kappa, \mathbf{e}_\nu \right\rangle = V^\kappa{}_{;\lambda}U^\nu \left\langle dx^\lambda, \mathbf{e}_\nu \right\rangle \otimes \mathbf{e}_\kappa$$

$$= V^\kappa{}_{;\lambda}U^\nu \delta^\lambda{}_\nu \mathbf{e}_\kappa = V^\kappa{}_{;\lambda}U^\lambda \mathbf{e}_\kappa$$

Geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

In geodesic coordinate, $g_{\mu\nu,\kappa} = 0, \Gamma^\mu_{\nu\kappa} = 0$

Connection one-form

$$\omega^\lambda{}_\nu = \Gamma^\lambda_{\nu\mu}\theta^\mu$$

Curvature two-form

$$\Omega^\mu{}_\nu = \mathbf{d}\omega^\mu{}_\nu + \omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu$$