The Measure of the SO(3) group manifold

First, write $d\mu(g) = \sin\theta \, d\theta \, d\phi \, f(\psi) \, d\psi = d\Omega \, d\psi \, f(\psi)$

and suppose that $f(\psi)$ is proportional to ψ^2

In[1]:=
```
Rz[ψ] := {{Cos[ψ], -Sin[ψ], 0}, {Sin[ψ], Cos[ψ], 0}, {0, 0, 1}}
```

```
Rz[ψ] // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
IdentityMatrix[3] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Write an infinitesimal rotation as $R(\delta, \epsilon, \sigma) = I + \begin{pmatrix} 0 & -\delta & \sigma \\ \delta & 0 & -\epsilon \\ -\sigma & \epsilon & 0 \end{pmatrix} = I + A$

In[2]:=
```
A[δ, ∈, σ] := {{0, -δ, σ}, {`δ, 0, -∈}, {-σ, ∈, 0}};
A[δ, ∈, σ] // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 0 & -\delta & \sigma \\ \delta & 0 & -\epsilon \\ -\sigma & \epsilon & 0 \end{pmatrix}$$

$R(\vec{n}, \psi') = R(\vec{n}, \psi) \, R = R(\vec{n}, \psi) \, (I + A)$ fix $\vec{n}$ as z- axis

$= R(e_z, \psi) \, (I + A) =$

In[4]:=
```
R = Dot[Rz[ψ], (IdentityMatrix[3] + A[δ, ∈, σ])];
R // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\psi] - \delta\,\text{Sin}[\psi] & -\delta\,\text{Cos}[\psi] - \text{Sin}[\psi] & \sigma\,\text{Cos}[\psi] + \epsilon\,\text{Sin}[\psi] \\ \delta\,\text{Cos}[\psi] + \text{Sin}[\psi] & \text{Cos}[\psi] - \delta\,\text{Sin}[\psi] & -\epsilon\,\text{Cos}[\psi] + \sigma\,\text{Sin}[\psi] \\ -\sigma & \epsilon & 1 \end{pmatrix}$$

Remark
How can we determine rotation angle and direction for given rotation matrix R?

- Rotation Angle

Note that rotations of same angle are in same equivalence class regardless of its directions.

$R(\vec{n}, \theta) \sim R(\vec{m}, \theta) \sim R(\vec{e}_z, \theta)$

and, character is a function of class

$$\text{tr}(R(\vec{n}, \theta)) = \text{tr}(R(\vec{m}, \theta)) = \text{tr}(R(\vec{e}_z, \theta)) = \text{tr}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + \cos\theta$$

- Rotation direction

Use the fact that that $R\vec{n} = \vec{n}$. Then,

$R^T \vec{n} = R^T (R \vec{n}) = (R^T R) \vec{n} = \vec{n}$. Henceforth, $(R - R^T) \vec{n} = 0$

The solution of above homogenous equation is the direction of rotation! Let's see.

```
LinearSolve[R - Transpose[R], {0, 0, 0}]
```

```
{0, 0, 0}
```

Since above equation is homogeneous linear system, utilize NullSpace[] for instead.

In[6]:=
```
n := NullSpace[R - Transpose[R]];
n
```

Out[7]= $\left\{\left\{-\frac{-\epsilon - \epsilon \operatorname{Cos}[\psi] + \sigma \operatorname{Sin}[\psi]}{2 (\delta \operatorname{Cos}[\psi] + \operatorname{Sin}[\psi])}, -\frac{-\sigma - \sigma \operatorname{Cos}[\psi] - \epsilon \operatorname{Sin}[\psi]}{2 (\delta \operatorname{Cos}[\psi] + \operatorname{Sin}[\psi])}, 1\right\}\right\}$

In[8]:=
```
n = n /. δ → 0
```

Out[8]= $\left\{\left\{-\frac{1}{2} \operatorname{Csc}[\psi] (-\epsilon - \epsilon \operatorname{Cos}[\psi] + \sigma \operatorname{Sin}[\psi]), -\frac{1}{2} \operatorname{Csc}[\psi] (-\sigma - \sigma \operatorname{Cos}[\psi] - \epsilon \operatorname{Sin}[\psi]), 1\right\}\right\}$

Now we have determined the rotation axis. Obtaining rotation angle is more easy

```
1 + 2 Cos[ψ + δ] == Tr[R]
```

$1 + 2 \operatorname{Cos}[\delta + \psi] == 1 + 2 \operatorname{Cos}[\psi] - 2 \delta \operatorname{Sin}[\psi]$

Note that $f(x + \delta x) \simeq f(x) + f'(x) \delta x$

Now call the Cartesian coordinate near ψ' by (x^1, x^2, x^3)

In[9]:=
```
coord := {x1, x2, x3};
coord = ψ Flatten[n, 2] + {0, 0, δ};
ψcoord = {ϵ, σ, δ};
{coord, ψcoord}
```

Out[12]= $\left\{\left\{-\frac{1}{2} \psi \operatorname{Csc}[\psi] (-\epsilon - \epsilon \operatorname{Cos}[\psi] + \sigma \operatorname{Sin}[\psi]), -\frac{1}{2} \psi \operatorname{Csc}[\psi] (-\sigma - \sigma \operatorname{Cos}[\psi] - \epsilon \operatorname{Sin}[\psi]), \delta + \psi\right\}, \{\epsilon, \sigma, \delta\}\right\}$

Then we can evaluate Jacobian for the transformation from the Cartesian coordinates $(\epsilon, \sigma, \delta)$ (near identity) to the Cartesian coordinates (x^1, x^2, x^3) (near ψ')

In[14]:=
```
JacobianMatrix[f_List?VectorQ, x_List] :=
  Outer[D, f, x] /; Equal @@ (Dimensions /@ {f, x});
JacobianDeterminant[f_List?VectorQ, x_List] :=
 Det[JacobianMatrix[f, x]] /; Equal @@ (Dimensions /@ {f, x})
```

In[16]:=
```
JacobianMatrix[coord, ψcoord] // MatrixForm
```

Out[16]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} \psi (-1 - \operatorname{Cos}[\psi]) \operatorname{Csc}[\psi] & -\frac{\psi}{2} & 0 \\ \frac{\psi}{2} & -\frac{1}{2} \psi (-1 - \operatorname{Cos}[\psi]) \operatorname{Csc}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[17]:=
```
J = JacobianDeterminant[coord, ψcoord]
```

Out[17]= $\frac{1}{4} \psi^2 \left(1 + \operatorname{Cot}[\psi]^2 + 2 \operatorname{Cot}[\psi] \operatorname{Csc}[\psi] + \operatorname{Csc}[\psi]^2\right)$

In[18]:=
```
J = Simplify[1/4 ψ^2 (1 + Cot[ψ]^2 + 2 Cot[ψ] Csc[ψ] + Csc[ψ]^2)]
```

Out[18]= $\frac{1}{4}\psi^2 \mathrm{Csc}\left[\frac{\psi}{2}\right]^2$

In[19]:=
```
4 J
```

Out[19]= $\psi^2 \mathrm{Csc}\left[\frac{\psi}{2}\right]^2$

From $J\, d\epsilon\, d\sigma\, d\delta = dx^1\, dx^2\, dx^3$,

$$d\epsilon\, d\sigma\, d\delta = dx^1\, dx^2\, dx^3 / J = \{(\psi^2\, d\psi)\,(\sin\theta\, d\theta\, d\phi)\}/(\psi^2 \csc^2(\psi/2))$$
$$= \psi^2\, d\psi\, d\Omega \sin^2(\psi/2)/\psi^2 = d\Omega\, d\psi \sin^2(\psi/2)$$

Notice that the factor ψ^2 canceled out. Finally $f(\psi) = \sin^2(\psi/2)$

```
Series[(Sin[ψ/2])^2, {ψ, 0, 5}]
```

$\frac{\psi^2}{4} - \frac{\psi^4}{48} + O[\psi]^6$

By construction, for very small angle ψ , $f(\psi)$ is proportional to ψ^2

Remark
The integrals of class functions over the SO(3) group manifold

$$\int_{SO(3)} d\mu(g)\, F(g) = \int_0^{\pi} d\psi (\sin^2(\psi/2))\, F(\psi)$$

- Example. Character Orthogonality

For finite group, we can confirm the character orthogonality by utilizing following;

$$\Sigma_{g\in G} (\chi^{(r)}(g))^* \chi^{(s)} * (g) = N(G)\, \delta_{rs}$$

When dealing with continuous group, substitute $\int d\mu(g)$ for Σ_g

$\int_{SO(3)} d\mu(g)\, (\chi(k,\, \psi))^* \chi(j,\, \psi)$ where character of irreducible representation of SO(3) χ is given by

$$\chi(j,\, \psi) = \frac{\sin(j+1/2)\, \psi}{\sin(\psi/2)}$$

In[20]:=
```
χ[j_, ψ_] := Sin[(j + 1/2) ψ] / Sin[ψ/2]
```

$$\int_{SO(3)} d\mu(g) \longrightarrow \int_0^{\pi} d\psi (\sin^2(\psi/2))$$

$$\int_{SO(3)} d\mu(g)\, (\chi(k,\, \psi))^* \chi(j,\, \psi) \to \int_0^{\pi} d\psi (\sin^2(\psi/2))\, (\chi(k,\, \psi))^* \chi(j,\, \psi)$$

```
HoldForm[Integrate[(Sin[ψ/2])^2 Conjugate[χ[k, ψ]] χ[j, ψ], {ψ, 0, π}]] //
 TraditionalForm
```

$$\int_0^\pi \sin^2\left(\frac{\psi}{2}\right) \chi(k,\psi)^* \, \chi(j,\psi)\, d\psi =$$

$$\int_0^\pi \frac{\sin^2\left(\frac{\psi}{2}\right) \left(\frac{\sin\left(\left(k+\frac{1}{2}\right)\psi\right)}{\sin\left(\frac{\psi}{2}\right)}\right)^* \sin\left(\left(j+\frac{1}{2}\right)\psi\right)}{\sin\left(\frac{\psi}{2}\right)} d\psi = \int_0^\pi d\psi \left(\sin\left(\left(k+\frac{1}{2}\right)\psi\right)\right)^* \sin\left(\left(j+\frac{1}{2}\right)\psi\right)$$

$$= \frac{1}{2}\int_0^\pi d\psi\, (\cos(j-k)\,\psi - \cos(j+k+1)\,\psi) = \frac{\pi}{2}\,\delta_{\mathrm{ij}}$$

Note that the volume of the group SO(3) is $\pi/2$

Reference

- A. Zee. Group Theory in a Nutshell for Physicists. Princeton University Press, 2016