§ Quantum Lorentz Transformation

$(\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1))$.

$\eta_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = \eta_{\mu\nu}\, dx^{\mu} dx^{\nu}$. (Line element 가 보존됨.)

$$\Downarrow$$

$$\eta_{\mu\nu} \frac{dx'^{\mu}}{dx^{\rho}} \frac{dx'^{\nu}}{dx^{\sigma}} = \eta_{\rho\sigma}.$$

Any coord. transformation

$$\underbrace{x'^{\mu} = \Lambda^{\mu}{}_{\nu}}_{\text{Homogeneous Lorentz transformation}} x^{\nu} + a^{\mu}. \rightarrow \text{Inhomogeneous Lorentz transformation}$$

$$\eta_{\mu\nu} \frac{dx'^{\mu}}{dx^{\rho}} \frac{dx'^{\nu}}{dx^{\sigma}} = \boxed{\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}.} \rightarrow$$

Lorentz transformation 의 성질! 밴드시 만족해야 함.

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$$

$$x''^{\mu} = \bar{\Lambda}^{\mu}{}_{\nu} x'^{\nu} + \bar{a}^{\mu}$$

$$= \bar{\Lambda}^{\mu}{}_{\nu} (\Lambda^{\nu}{}_{\rho} x^{\rho} + a^{\nu}) + \bar{a}^{\mu} = \bar{\Lambda}^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\rho} x^{\rho} + \bar{\Lambda}^{\mu}{}_{\nu} a^{\nu} + \bar{a}^{\mu}.$$

$$\therefore\ T(\bar{\Lambda}, \bar{a})\, T(\Lambda, a) = T(\bar{\Lambda}\Lambda,\ \bar{\Lambda} a + \bar{a})$$

$$\boxed{\begin{aligned} \rightarrow\ 1 &\equiv T(1, 0) \\ T^{-1}(\Lambda, a) &\equiv T(\Lambda^{-1}, -\Lambda^{-1} a) \end{aligned}}$$

한편, $\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$

$$\Downarrow$$

$$(\Lambda^{\rho}{}_{\mu})^{t} \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = \Lambda^{t} \eta \Lambda = \eta$$

$$\det(\Lambda^{t} \eta \Lambda) = \det(\Lambda)\, \cancel{\det(\eta)}\, \det(\Lambda) = \cancel{\det(\eta)}$$

$$|\det(\Lambda)|^2 = 1. \qquad \boxed{\det(\Lambda) = \pm 1}$$

$*(\Lambda^{-1})^{\rho}{}_{\nu}$?

$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$ 양변에 $(\Lambda^{-1})^{\rho}{}_{\nu}$ 를 곱하자.

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} (\Lambda^{-1})^{\rho}{}_{\nu} \Lambda^{\nu}{}_{\sigma} = (\Lambda^{-1})^{\rho}{}_{\nu}\, \eta_{\rho\sigma}$$

$$\eta_{\mu\nu} \delta^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\sigma} = \eta_{\mu\nu} \Lambda^{\mu}{}_{\sigma} = (\Lambda^{-1})^{\rho}{}_{\nu}\, \eta_{\rho\sigma}$$

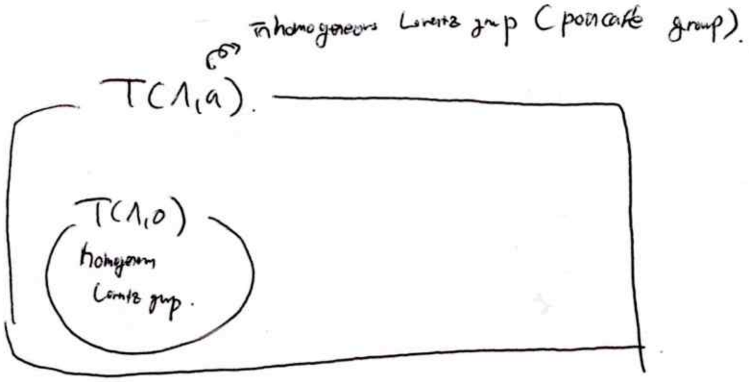
$$= \eta_{\nu\mu} \Lambda^{\mu}{}_{\sigma} \eta^{\rho\sigma} = (\Lambda^{-1})^{\rho}{}_{\nu}$$

$$\rightarrow (\Lambda^{-1})^{\rho}{}_{\nu} = \eta_{\nu\mu} \eta^{\rho\sigma} \Lambda^{\mu}{}_{\sigma} = \eta_{\nu\mu} \Lambda^{\mu\rho} = \Lambda_{\nu}{}^{\rho}.$$

$T(\Lambda,a)$ 의 Unitary representation 을 $U(\Lambda,a)$ 라고 하자.

$U(\Lambda,a)$은 당연히 composition rule $U(\Lambda_1,a_1)U(\Lambda_2,a_2) = U(\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$

을 만족한다.



$T(\Lambda,a)$

$\det\Lambda = 1$ or $\det\Lambda = -1$.

이때가 1가 될때 subgroup이 된다.

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}.$$

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = -1$$

$$-(\Lambda^0{}_0)^2 + (\Lambda^i{}_0)(\Lambda^i{}_0) = -1.$$

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)(\Lambda^i{}_0).$$

$\therefore\ |\Lambda^0{}_0| \geq 1. \rightarrow \Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$.

특히 $\Lambda^0{}_0 \geq 1$ 조건부터 subgroup이 만들어진다.,

$\det\Lambda$	$+1$	-1
$\Lambda^0{}_0 \geq 1$	proper orthochronous Lorentz group. $SO(1,3)$	$\mathcal{P}$.
$\Lambda^0{}_0 \leq -1$	$\mathcal{T}$.	$\mathcal{P}\mathcal{T}$.

$\mathcal{P}$: space inversion

$$\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

$\mathcal{T}$: time-reversal

$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}.$$

§ Poincaré Algebra

$T(\Lambda, a) \to$ continuous group 이니까 infinitesimal transformation으로 모든 transform을 표현할 수 있다!

Near the identity : $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ $\qquad \omega^\mu{}_\nu \ll 1$

$a^\mu = \epsilon^\mu . \qquad \epsilon^\mu \ll 1 .$

$\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ 에 대입하면,

$$\eta_{\rho\sigma} = \eta_{\mu\nu}(\delta^\mu_\rho + \omega^\mu_\rho)(\delta^\nu_\sigma + \omega^\nu_\sigma)$$

$$= \eta_{\mu\nu}\delta^\mu_\rho \delta^\nu_\sigma + \eta_{\mu\nu}\omega^\mu_\rho \delta^\nu_\sigma + \eta_{\mu\nu}\delta^\mu{}_\rho \omega^\nu_\sigma + \eta_{\mu\nu}\omega^\mu_\rho \omega^\nu_\sigma$$

$$= \delta^\mu{}_\rho \eta_{\mu\sigma} + \eta_{\mu\sigma}\omega^\mu{}_\rho + \cancel{\eta_{\mu\nu}}\,\delta^\mu{}_\rho \omega_{\mu\sigma} + \mathcal{O}(\omega^2)$$

$$= \eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2)$$

$\therefore\ \omega_{\rho\sigma} + \omega_{\sigma\rho} = 0 . \qquad \omega_{\rho\sigma} = -\omega_{\sigma\rho} .$

$$\begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ -\omega^0{}_1 & 0 & \omega^1{}_2 & \omega^1{}_3 \\ -\omega^0{}_2 & -\omega^1{}_2 & 0 & \omega^2{}_3 \\ -\omega^0{}_3 & -\omega^1{}_3 & -\omega^2{}_3 & 0 \end{pmatrix}$$

Lorentz transformation의 infinitesimal rotation 부분 → antisymmetric!

Group theory..

* $\omega_{\mu\nu} = -\omega_{\nu\mu} \to \frac{4\times4-4}{2} = \frac{4(4-1)}{2} = 6$ comp.

$\epsilon^\mu \to 4$ comp.

$SO(1,3)$ 의 representation은 $6+4=10$ 개의 independent component 를 가진다!

이제 $T(\Lambda, a)$의 representation (unitary) 를 $U(\Lambda, a)$ 라고 써보자!

$$U(1+\omega, \epsilon) = 1 + \frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma} - i\epsilon_\rho P^\rho + \cdots \qquad UU^\dagger = 1$$

$$U^\dagger(1+\omega, \epsilon) = 1 - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\dagger + i\epsilon_\rho (P^\rho)^\dagger + \cdots \qquad \left(\begin{aligned} J^{\rho\sigma} &= (J^{\rho\sigma})^\dagger \\ P^\rho &= (P^\rho)^\dagger . \end{aligned} \right)$$

이때 $\omega_{\rho\sigma}$가 antisymmetric 하니 $J^{\rho\sigma}$ 도 antisymmetric 하게 선택할 수 있다.

$\ast\ (\underset{\text{Hamiltonian}}{P^0}, \underset{\text{momentum}}{\vec{P}})$ $\qquad J^{23} \to J_x \quad J^{31} \to J_y \quad J^{12} \to J_z$

Now examine the commutation relation of $J^{\rho\sigma}$ and P^ρ!

//

* adjoint?

$g_1^{-1} g_2 g_1$ 모양. Lorentz trans. 의 경우,

$$T_1 T_2 T_1^{-1} \Rightarrow T(\Lambda, a)\, T(\bar\Lambda, \bar a)\, T(\Lambda, a)^{-1}$$
$$= T(\Lambda, a)\, T(\bar\Lambda, \bar a)\, T(\Lambda^{-1}, -\Lambda^{-1} a)$$
$$= T(\Lambda, a)\, T(\bar\Lambda \Lambda^{-1}, -\bar\Lambda \Lambda^{-1} a + \bar a)$$
$$= T(\Lambda \bar\Lambda \Lambda^{-1}, \Lambda(-\bar\Lambda \Lambda^{-1} a + \bar a) + a)$$
$$= T(\Lambda \bar\Lambda \Lambda^{-1}, -\Lambda \bar\Lambda \Lambda^{-1} a + \Lambda \bar a + a)$$

→ 이렇게 연산을 많이 하는 것 같음!

하려는 것: $U(1+\omega, \epsilon)$ 에 adjoint action 을 가해서 J, P 사이의 commutation relation 들을 구한다!

Lie's insight.

$$U(\Lambda, a)\, U(1+\omega, \epsilon)\, U^{-1}(\Lambda, a) = U(\Lambda, a)\left(1 + \frac{1}{2} i \omega_{\rho\sigma} J^{\rho\sigma} - i \epsilon_\rho P^\rho\right) U^{-1}(\Lambda, a)$$

$$U\left(\Lambda(1+\omega)\Lambda^{-1}, -\Lambda(1+\omega)\Lambda^{-1} a + \Lambda\epsilon + a\right)$$
$$= U\left(1 + \Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1} a + \Lambda\epsilon\right)$$

$$1 + \frac{1}{2} i \left[U(\Lambda, a)\, \omega_{\rho\sigma} J^{\rho\sigma} U^{-1}(\Lambda, a)\right] - i\left[U(\Lambda, a)\, \epsilon_\rho P^\rho U^{-1}(\Lambda, a)\right]$$

$$1 + \frac{1}{2} i [\Lambda\omega\Lambda^{-1}]_{\rho\sigma} J^{\rho\sigma} - i[-\Lambda\omega\Lambda^{-1} a + \Lambda\epsilon]_\rho P^\rho$$

i) $\frac{1}{2} U(\Lambda, a)\, \omega_{\rho\sigma} J^{\rho\sigma} U^{-1}(\Lambda, a) = \frac{1}{2}[\Lambda\omega\Lambda^{-1}]_{\rho\sigma} J^{\rho\sigma} + [\Lambda\omega\Lambda^{-1} a]_\rho P^\rho$

$$\frac{1}{2} U(\Lambda, a)\, \omega_{\rho\sigma} J^{\rho\sigma} U^{-1}(\Lambda, a) = \frac{1}{2}[\Lambda\omega\Lambda^{-1}]_{\mu\nu} J^{\mu\nu} + [\Lambda\omega\Lambda^{-1} a]_\mu P^\mu$$

$\Lambda_\mu{}^\rho \omega_{\rho\sigma} (\Lambda^{-1})^\sigma{}_\nu$ = $\omega_{\rho\sigma} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma$

$(\Lambda\omega\Lambda^{-1})_{\mu\nu} a^\nu P^\mu$
↓
$(\Lambda_\mu{}^\rho \omega_{\rho\sigma} \Lambda_\nu{}^\sigma)\, a^\nu P^\mu$
↓ antisymmetric

$$\frac{1}{2}\left((P^\mu \Lambda_\mu{}^\rho \omega_{\rho\sigma})(a^\nu \Lambda_\nu{}^\sigma) - (P^\nu \Lambda_\nu{}^\sigma \omega_{\sigma\rho})(a^\mu \Lambda_\mu{}^\rho)\right)$$
$$= \frac{1}{2}\left[\omega_{\rho\sigma} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma a^\nu P^\mu - \omega_{\sigma\rho} \Lambda_\nu{}^\sigma \Lambda_\mu{}^\rho a^\mu P^\nu\right]$$

※ $(\Lambda^{-1})^\sigma{}_\nu = \Lambda_\nu{}^\sigma$

$(\Lambda^{-1})^\sigma{}_\nu \Lambda^\nu{}_\rho$

$\Lambda_\nu{}^\sigma \Lambda^\nu{}_\rho$

$= g^\sigma{}_\rho$

$\Lambda_\nu{}^\sigma g_{\sigma\sigma} \Lambda^\nu{}_\rho$

$= g_{\chi\rho}$

$$\cancel{\tfrac{1}{2}} U(\Lambda,a)\, \cancel{\omega_{\rho\sigma}} J^{\rho\sigma} U^{-1}(\Lambda,a) = \cancel{\tfrac{1}{2}}\, \cancel{\omega_{\rho\sigma}} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu} + \cancel{\tfrac{1}{2}}\, \cancel{\omega_{\rho\sigma}} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \left(a^\nu p^\mu - a^\mu p^\nu \right)$$

$$= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \left(J^{\mu\nu} + a^\nu p^\mu - a^\mu p^\nu \right)$$

ii). $\cancel{[\Lambda\epsilon]_\rho p^\rho}$

$$U(\Lambda,a)\, \cancel{\epsilon_\rho} P^\rho U^{-1}(\Lambda,a) = [\Lambda\epsilon]_\mu p^\mu$$

$$= (\Lambda_\mu{}^\rho \cancel{\epsilon_\rho}) p^\mu$$

i), ii) 를 정리하면,

$$\begin{cases} \text{i)} \to U(\Lambda,a) J^{\rho\sigma} U^{-1}(\Lambda,a) \\ \quad = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \left(J^{\mu\nu} + a^\nu p^\mu - a^\mu p^\nu \right) \\ \text{ii)} \to U(\Lambda,a) P^\rho U^{-1}(\Lambda,a) \\ \quad = \Lambda_\mu{}^\rho p^\mu \end{cases}$$

★ Homogeneous Lorentz transformation $a=0$.

$U(\Lambda) J^{\rho\sigma} U^{-1}(\Lambda)$

$= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma J^{\mu\nu}$.

transform like a tensor!

★ Pure translation.

$$\begin{cases} U(1,a) J^{\rho\sigma} U^{-1}(1,a) \\ = \delta_\mu{}^\rho \delta_\nu{}^\sigma (J^{\mu\nu} + a^\nu p^\mu - a^\mu p^\nu) \\ = J^{\rho\sigma} + a^\sigma p^\rho - a^\rho p^\sigma \\ U(1,a) P^\rho U^{-1}(1,a) \\ = \delta_\mu{}^\rho p^\mu = p^\rho. \end{cases}$$

invariant over translation.

under $\begin{cases} \Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \\ a^\mu = \epsilon^\mu. \end{cases}$

i)에서,

$$\left(1 + \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right)\right) J^{\rho\sigma} \left(1 - \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right)\right)$$

$$= \left(\delta_\mu{}^\rho + \omega_\mu{}^\rho\right)\left(\delta_\nu{}^\sigma + \omega_\nu{}^\sigma\right)\left(J^{\mu\nu} + \epsilon^\nu p^\mu - \epsilon^\mu p^\nu\right)$$

$$\Rightarrow \left(J^{\rho\sigma} + \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right) J^{\rho\sigma} \right) \left(1 - \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right) \right)$$

$$= \cancel{J^{\rho\sigma}} + \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right) J^{\rho\sigma} - J^{\rho\sigma} \left(\tfrac{1}{2} i \omega_{\alpha\beta} J^{\alpha\beta} - i\epsilon_\alpha p^\alpha\right) + \cancel{\mathcal{O}(h^2)}$$

$$\Rightarrow \left(\delta_\mu{}^\rho \delta_\nu{}^\sigma + \omega_\mu{}^\rho \delta_\nu{}^\sigma + \delta_\mu{}^\rho \omega_\nu{}^\sigma \right) \left(J^{\mu\nu} - \epsilon^\mu p^\nu + \epsilon^\nu p^\mu \right)$$

$$= \cancel{J^{\rho\sigma}} + \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} - \epsilon^\rho p^\sigma + \epsilon^\sigma p^\rho + \cancel{\mathcal{O}(h^2)}$$

$$\hookrightarrow i\left[\tfrac{1}{2} \omega_{\alpha\beta} J^{\alpha\beta} - \epsilon_\alpha p^\alpha,\ J^{\rho\sigma} \right] = \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} - \epsilon^\rho p^\sigma + \epsilon^\sigma p^\rho$$

ii) 에서

$i\,[\![\ \tfrac{1}{2}\omega_{\alpha\beta}J^{\alpha\beta} - \epsilon_\alpha P^\alpha,\ P^\rho\ ]\!] = \omega_\mu{}^\rho P^\mu$.

∴ Poincaré Algebra.

i) $i\,[\![\ \tfrac{1}{2}\omega_{\alpha\beta}J^{\alpha\beta} - \epsilon_\alpha P^\alpha,\ J^{\rho\sigma}\ ]\!] = \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho$

ii) $i\,[\![\ \tfrac{1}{2}\omega_{\alpha\beta}J^{\alpha\beta} - \epsilon_\alpha P^\alpha,\ P^\rho\ ]\!] = \omega_\mu{}^\rho P^\mu$

i) 에서..

$\rightsquigarrow\ \tfrac{1}{2}\omega_{\alpha\beta}\,[\![\ J^{\alpha\beta},\ J^{\rho\sigma}\ ]\!] = \omega_\mu{}^\rho J^{\mu\sigma} + \omega_\nu{}^\sigma J^{\rho\nu}$

$\omega_{\nu\mu}\eta^{\mu\sigma}J^{\rho\nu}$

$\omega_{\mu\nu}\eta^{\nu\rho}J^{\mu\sigma}$

$\tfrac{1}{2}(\omega_{\nu\mu}\eta^{\mu\sigma}J^{\rho\nu} + \omega_{\mu\nu}\eta^{\nu\sigma}J^{\rho\mu})$

$\tfrac{1}{2}(\omega_{\mu\nu}\eta^{\nu\rho}J^{\mu\sigma} + \omega_{\nu\mu}\eta^{\mu\rho}J^{\nu\sigma})$

$\tfrac{1}{2}\omega_{\mu\nu}\,(\eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu})$

$= \tfrac{1}{2}\omega_{\mu\nu}\,(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma})$

$\cancel{\tfrac{1}{2}\omega_{\mu\nu}}\,i\,[\![\ J^{\mu\nu},\ J^{\rho\sigma}\ ]\!] = \cancel{\tfrac{1}{2}\omega_{\mu\nu}}\left[\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu}\right]$

$i\,[\![\ J^{\mu\nu},\ J^{\rho\sigma}\ ]\!] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} + \eta^{\nu\sigma}J^{\rho\mu} - \eta^{\mu\sigma}J^{\rho\nu}$

$\rightsquigarrow\ -i\epsilon_\alpha\,[\![\ P^\alpha,\ J^{\rho\sigma}\ ]\!] = -\epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho$

$-i\cancel{\epsilon_\mu}\,[\![\ P^\mu,\ J^{\rho\sigma}\ ]\!] = -\cancel{\epsilon_\mu}\eta^{\mu\rho}P^\sigma + \cancel{\epsilon_\mu}\eta^{\mu\sigma}P^\rho$.

$i\,[\![\ P^\mu,\ J^{\rho\sigma}\ ]\!] = \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho$.

↓ translation 은 나름. ρ, σ 에 대해서 antisymmetric.

ii) 에서...

$i\,[\![\ P^\mu,\ P^\rho\ ]\!] = 0.$

↓ translation 들 commute.

$J^{\mu\nu},\ P^\mu\ \sim$ poincaré algebra 의 generator 들.

$\vec{P} = \{p^1, p^2, p^3\}$

$\vec{J} = \{J^{23}, J^{31}, J^{12}\}$

$\vec{K} = \{J^{01}, J^{02}, J^{03}\}$

$$J^{\mu\nu} \equiv \begin{pmatrix} 0 & J^{01} & J^{02} & J^{03} \\ -J^{01} & 0 & J^{12} & J^{13} \\ -J^{02} & -J^{12} & 0 & J^{23} \\ -J^{03} & -J^{13} & \cancel{J^{23}} & 0 \end{pmatrix} \qquad p^{\mu} \equiv \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$$i [\![J^{ij}, J^{kl}]\!] = \delta^{jk} J^{il} - \delta^{ik} J^{jl} + \delta^{jl} J^{ki} - \delta^{il} J^{kj}.$$

$\to$ $J^{ij} \to \epsilon_{ijk} J^k.$ $\qquad$ $\cancel{\epsilon_{ij}}$ $(\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}.)$

$J^{kl} \to \epsilon_{klm} J^m.$

$[\![J_i, J_i]\!] = 0.$

$\cancel{[\![J_k, J_m]\!]} = [\![\epsilon_{ijk} J^k, \epsilon_{ilm} J^m]\!]$

$$[\![J^{ij}, J^{ik}]\!] \to \cancel{\delta^{ji} J^{il}} - \delta^{ii} J^{jk} + \cancel{\delta^{jl} J^{ii}} - \cancel{\delta^{il} J^{ij}}$$

$j \neq l$

$$= -J^{jl}.$$

$$\cancel{\epsilon_{ijk}}\ \cancel{\epsilon_{ikm}}\ [\![J_k, J_m]\!] = \cancel{\ } J^{jk}. = \epsilon_{ijk} J_i$$

$i \ j \ k$ $\qquad$ $\boxed{\cancel{j \ l \ i}}$

$$[\![J_i, J_j]\!] = i \epsilon_{ijk} J_k.$$

$$i [\![J^{ij}, J^{0k}]\!] = \cancel{\eta^{j0} J^{ik}} - \cancel{\eta^{i0} J^{jk}} + \eta^{jk} J^{0i} - \eta^{ik} J^{0j}$$

$$= \eta^{jk} K_i - \eta^{ik} K_j$$

$$i [\![J^{ij}, K_k]\!] = \delta^{jk} K_i - \delta^{ik} K_j$$

$$i [\![\epsilon_{ijm} J_m, K_k]\!] = \delta^{jk} K_i - \delta^{ik} K_j$$

$$i \epsilon_{ijm} [\![J_m, K_k]\!] = \delta^{jk} K_i - \delta^{ik} K_j$$

$i \neq j$

$i [\![J^{ij}, K_j]\!] = K_i \qquad [\![J^{ij}, K_j]\!] = -i K_i$

$i [\![J^{ij}, K_i]\!] = -K_j \qquad [\![J^{ij}, K_i]\!] = i K_j$

$$\epsilon_{ijk} [\![J_k, K_j]\!] = -i K_i$$

$[\![J_i, P_j]\!]$

$i[\![J^{ij}, P^k]\!] = \eta^{kj} P^i - \eta^{ki} P^j$

$i \neq j.$ $\qquad = \delta^{kj} P^i - \delta^{ki} P^j.$

$i[\![J^{ij}, P_i]\!] = -P_j.$

$i[\![J^{ij}, P_j]\!] = P_i$

$i[\![J^{kj}, P_j]\!] = P_k$

$[\![J^{kj}, P_j]\!] = -iP_k$

$\epsilon_{kji} [\![J_i, P_j]\!] = -i P_k$

$-\epsilon_{ijk} [\![J_i, P_j]\!] = -i P_k$

$\underline{[\![J_i, P_j]\!] = i\epsilon_{ijk} P_k.}$

$J_i = \epsilon_{jki} J^{jk}$

$J_j = \epsilon_{kij} J^{ki}$

$i[\![J_i, J_j]\!]$

$= i\epsilon_{jki}\epsilon_{kij} \underbrace{[\![J^{jk}, J^{ki}]\!]}$

$\delta^{kk} J^{ji} = -J^{ij}$

$- \delta^{jk} J^{ki}$

$+ \cancel{\delta^{ki} J^{kj}}$

$- \cancel{\delta^{ji} J^{kk}}$ 0

$= \epsilon_{jki}\epsilon_{kij} J^{ji}$

$= \epsilon_{jki}\epsilon_{kij} (-J^{ij}) = -\epsilon_{ijk} J_k.$

$[\![K_i, P_j]\!]$

$i[\![J^{0i}, P_j]\!] = \eta^{ji} P^0 - \cancel{\eta^{j0} P^i}$

$= \delta^{ji} H - \cancel{0}$

$\underline{[\![K_i, P_j]\!] = -i\delta^{ij} H.}$

$[\![J_i, H]\!]$

$= i[\![\epsilon_{jki} J^{jk}, P^0]\!]$

$= i\epsilon_{ijk} [\![J^{jk}, P^0]\!] = i\epsilon_{ijk} (\eta^{0k} P^j - \eta^{0j} P^k) = 0.$

$[\![P_i, H]\!] = [\![H, H]\!] = 0.$

$[\![K_i, H]\!]$

$= i[\![J^{0i}, P^0]\!]$

$= \cancel{\eta^{0i} P^0} - \eta^{00} P^i$

$= P^i$

$[\![K_i, H]\!] = -iP_i$

* conserved operators

→ commute with energy operator $P^0 = H$.

e.g., $[\![\vec P, P^0]\!] = [\![\vec J, P^0]\!] = [\![H, H]\!] = 0$

simultaneous 하게 diagonalization 이 가능!

$\vec P, \vec J, H$는 conserved quantity.

$[\![K_i, H]\!] \neq 0.$

↳ K_i는 conserved 되지 않는다.

이유?

$i\frac{dQ}{dt} = [Q, H]$ 에서 Q, H가 commute 하면

§ One-particle state. ← According to their inhomogeneous L.T.

$\{T(\Lambda,a)\}$ ↔ $SO(1,3)$.

앞에서 봤듯, $\left([P^\mu, P^\nu] = 0 \right)$이기 때문에 P^μ들은 공통적인 eigenstate들을 갖는다.

따라서 $P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle$ 라고 정할 수 있다.

* Representation of pure translation

$$U(1,a) = e^{-i\vec{P}\cdot\vec{a}}$$

Representation of pure rotation

$$U(\Lambda) = e^{-i\vec{J}\cdot\vec{\theta}}$$

* $|p,\sigma\rangle$에 pure translation을 가했을 때.

$$U(1,a)|p,\sigma\rangle = e^{-iP^\mu a_\mu}|p,\sigma\rangle$$
$$= \left(1 - i a_\mu P^\mu - \frac{1}{2!}(a_\mu P^\mu)^2 + \cdots\right)|p,\sigma\rangle$$
$$= e^{-i\vec{p}\cdot\vec{a}}|p,\sigma\rangle.$$

$$\boxed{U(1,a)|p,\sigma\rangle = e^{-i\vec{p}\cdot\vec{a}}|p,\sigma\rangle}$$

* $|p,\sigma\rangle$에 pure rotation을 가했을 때.

$$P^\mu\left(U(\Lambda)|p,\sigma\rangle\right) = U(\Lambda)\left[U^{-1}(\Lambda)P^\mu U(\Lambda)\right]|p,\sigma\rangle$$

adjoint 유도

Poincaré algebra 구할 때 봤음!

$$U(\Lambda,a)P^\rho U^{-1}(\Lambda,a) = \Lambda_\mu{}^\rho P^\mu$$
$$U(\Lambda)P^\rho = \Lambda_\mu{}^\rho P^\mu U(\Lambda)$$
$$P^\rho = \Lambda_\mu{}^\rho U^{-1}(\Lambda)P^\mu U(\Lambda)$$
$$\Lambda^\mu{}_\rho P^\rho = U^{-1}(\Lambda)P^\mu U(\Lambda).$$

$$= U(\Lambda)\left(\Lambda^\mu{}_\rho p^\rho\right)|p,\sigma\rangle$$

$$\boxed{P^\mu\left(U(\Lambda)|p,\sigma\rangle\right) = \Lambda^\mu{}_\rho p^\rho\left(U(\Lambda)|p,\sigma\rangle\right)}$$

$U(\Lambda)|p,\sigma\rangle$ 또한 P^μ의 eigenvector!

따라서 $|\Lambda p,\sigma'\rangle$들의 linear comb.로 표시할 수 있다.

$$\rightarrow U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)|\Lambda p,\sigma'\rangle.$$

$U(\Lambda)$의 representation.

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda,p)|\Lambda p,\sigma'\rangle.$$

$U(\Lambda)$의 representation.

$$C_{\sigma'\sigma} \equiv \langle \Lambda p,\sigma' | U(\Lambda) | p,\sigma\rangle.$$

적당히 state들을 잘 배열하면 $C_{\sigma'\sigma}$를 Block-diagonal 하게 만들 수 있다.

⇒ 즉, $|p,\sigma\rangle$ with σ within any one block by themselves furnish a representation of the inhomogeneous Lorentz group.

⇒ representation C 끼리의 composition behavior를 관찰하면 더 많은 정보를 얻을 수 있다.

$$U(\Lambda_1\Lambda_2)|p,\sigma\rangle = U(\Lambda_1)U(\Lambda_2)|p,\sigma\rangle$$

$$\sum_{\sigma''} C_{\sigma''\sigma}(\Lambda_1\Lambda_2,p)|\Lambda_1\Lambda_2 p,\sigma''\rangle \qquad U(\Lambda_1)\sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2,p)|\Lambda_2 p,\sigma'\rangle$$

$$\sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2,p)\,U(\Lambda_1)|\Lambda_2 p,\sigma'\rangle$$

$$\sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2,p)\sum_{\sigma''} C_{\sigma''\sigma'}(\Lambda_1\Lambda_2,p)|\Lambda_1\Lambda_2 p,\sigma''\rangle \qquad \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2,p)\sum_{\sigma''} C_{\sigma''\sigma'}(\Lambda_1,\Lambda_2 p)|\Lambda_1\Lambda_2 p,\sigma''\rangle$$

$$\sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2, p) \sum_{\sigma''} C_{\sigma''\sigma'}(\Lambda_1, \Lambda_2 p) \,|\, \Lambda_1\Lambda_2 p, \sigma'' \rangle$$

$$= \sum_{\sigma''} C_{\sigma''\sigma}(\Lambda_1\Lambda_2, p) \,|\, \Lambda_1\Lambda_2 p, \sigma'' \rangle .$$

$$C_{\lambda\sigma}(\Lambda_1\Lambda_2, p) = \langle \Lambda_1\Lambda_2 p, \lambda \,|\, \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2, p) \sum_{\sigma''} C_{\sigma''\sigma'}(\Lambda_1, \Lambda_2 p) \,|\, \Lambda_1\Lambda_2 p, \sigma'' \rangle$$

$$= \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda_2, p)\, C_{\lambda\sigma'}(\Lambda_1, \Lambda_2 p)$$

$$\boxed{C_{\lambda\sigma}(\Lambda_1\Lambda_2, p) = \sum_{\sigma'} C_{\lambda\sigma'}(\Lambda_1, \Lambda_2 p)\, C_{\sigma'\sigma}(\Lambda_2, p)}$$ → $U(\Lambda)$의 representation

C 의 composition rule !

In general p 에 대해 $\Lambda p \neq p$ 이다.

하지만 어떤 p 에 대해 $\exists\, \Lambda$ such that $\Lambda p = p$. (p^μ left invariant)

(따라서 이 Λ는 p-dependency 가 있다.)

따라서

$$C_{\lambda\sigma}(\Lambda_1\Lambda_2, p) = \sum_{\sigma'} C_{\lambda\sigma'}(\Lambda_1, \Lambda_2 p)\, C_{\sigma'\sigma}(\Lambda_2, p) \text{ 에서}$$

$\Lambda_2 p = p$ 인 $\Lambda_2(p)$를 고르면

$$C_{\lambda\sigma}(\Lambda_1\Lambda_2, p) = C_{\lambda\sigma'}(\Lambda_1, p)\, C_{\sigma'\sigma}(\Lambda_2, p)$$ summed over repeated index.

* $G_p = \{ \Lambda \,|\, \Lambda p = p \}$ ← $\boxed{\text{L.G 의 subgroup.}}$

어떤 p 에 대해서만 만족하는 Λ 들의 모임.

이때 이 Λ 를 standard Lorentz Transformation 이라고 부르고

p 에 의존한다. $\Lambda \rightsquigarrow L(p)$.

$U(\Lambda, a) \to U(1, a)\, U(\Lambda)$

$$U(\Lambda, a)|p, \sigma\rangle = U(1, a)\, U(\Lambda)\,|p, \sigma\rangle$$

$$= U(1, a) \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\,|\Lambda p, \sigma'\rangle$$

$$= e^{-i\Lambda p \cdot a} \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\,|\Lambda p, \sigma'\rangle .$$

↓ 이제,

적절한 k 에 대해

$p^\mu = L^\mu{}_\nu(p)\, k^\nu = (L(p)k)^\mu$ 라고 하자. $\left(L(p) \in G_p = \{ \Lambda \,|\, \Lambda p = p \} \subset SO(1,3) \right.$

이때 k 를 standard choice of basis 라고 부름.

다음으로

Normalization factor

$|p,\sigma\rangle = N(p)\, U(L(p))\, |k,\sigma\rangle$ 라고 정의하자.

여기에 Lorentz transformation $U(\Lambda)$를 가하면,

$$U(\Lambda)|p,\sigma\rangle = U(\Lambda)\left(N(p)\, U(L(p))\,|k,\sigma\rangle\right)$$

$$= N(p)\, U(\Lambda L(p))\,|k,\sigma\rangle$$

$$\cancel{= N(p)\, U(\Lambda L(p))\, U(}$$

$$= N(p)\, U(L(\Lambda p))\, U(L^{-1}(\Lambda p)\Lambda L(p))\,|k,\sigma\rangle.$$

$\ast$ $U(L^{-1}(\Lambda p)\Lambda L(p))|k,\sigma\rangle.$

$U(L^{-1}(\Lambda p))\, U(\Lambda)\, U(L(p))|k,\sigma\rangle.$

$\hat{P}[U(\Lambda)|p,\sigma\rangle]$
$= \Lambda p\,[U(\Lambda)|p,\sigma\rangle]$

$\hat{P}(U(L(p))|k,\sigma\rangle) = L(p)k\,[U(L(p))|k,\sigma\rangle]$
$= p\,[U(L(p))|k,\sigma\rangle]$
$\therefore U(L(p))|k,\sigma\rangle \sim |p,\sigma\rangle.$

$U(\Lambda)|p,\sigma\rangle \sim |\Lambda p,\sigma\rangle$

$U(L^{-1}(\Lambda p))|\Lambda p,\sigma\rangle \approx |k,\sigma\rangle.$

$L(\Lambda p)k = \Lambda p$

$\therefore U(L^{-1}(\Lambda p)\Lambda L(p))|k,\sigma\rangle \sim |k,\sigma\rangle.$

Invariant transformation. $L^{-1}(\Lambda p)\Lambda L(p) = W(\Lambda, p).$

$W(\Lambda,p)k = W^{\mu}{}_{\nu}(\Lambda,p)k^{\nu}$
$= (L^{-1}(\Lambda p)\Lambda L(p))^{\mu}{}_{\nu}\, k^{\nu}$
$= k$

$W^{\mu}{}_{\nu}k^{\nu} = k^{\mu}.$
Invariant!

W
$\downarrow$
little group을 형성함.

Λ
W

$$= N(p)\, U(L(\Lambda p))\, U(W)\,|k,\sigma\rangle.$$

$W \subset \Lambda$ 이니까
$U(\Lambda)|p,\sigma\rangle = \sum C_{\sigma\sigma'}(\Lambda)|\Lambda p,\sigma\rangle$ 가 그대로 적용된다.

$$= N(p)\, U(L(\Lambda p)) \sum_{\sigma'} \underbrace{D_{\sigma'\sigma}(W(\Lambda,p))}_{\text{little group의 representation}}\, |Wk,\sigma'\rangle$$

$$= N(p)\, U(L(\Lambda p)) \sum_{\sigma'} D_{\sigma'\sigma}(W)\,|k,\sigma'\rangle$$

$$\boxed{U(\Lambda)|p,\sigma\rangle = N(p)\, U(L(\Lambda p))\sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle}$$