Scattering

1. Rutherford scattering

$$T = \frac{m}{2}\left(r'^2 + r^2\,\theta'^2\right),\;\; V = V(r),\;\; L = T - V$$

In[1]:=
```
Needs["VariationalMethods`"]

m = 1;
ϵ = 1;
q1 = -1;
q2 = 1;
b = 0.2;
V[r_] := 1/(4 π ϵ) q1 q2/r^2;
EOM = EulerEquations[m/2 (r'[t]^2 + r[t]^2 θ'[t]^2) - V[r[t]], {r[t], θ[t]}, t]
```

Out[28]=
$$\left\{-\frac{1}{2\,\pi\,r[t]^3} + r[t]\,\theta'[t]^2 - r''[t] == 0,\; -r[t]\,(2\,r'[t]\,\theta'[t] + r[t]\,\theta''[t]) == 0\right\}$$

In[33]:=
```
DSolve[Union[EOM, {r[0] == ∞, θ[0] == 0}], {r, θ}, t]
```

Out[33]=
$$\text{DSolve}\left[\left\{r[0] == \infty,\; \theta[0] == 0,\; -\frac{1}{2\,\pi\,r[t]^3} + r[t]\,\theta'[t]^2 - r''[t] == 0,\; -r[t]\,(2\,r'[t]\,\theta'[t] + r[t]\,\theta''[t]) == 0\right\},\; \{r, \theta\},\; t\right]$$

Mathematica doesn’t want to solve the Lagrangian equation of motion. Then let’s see the differential equation of the orbit of a particle moving under a central force.

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{m\,l^2\,u^2}\,f\left(u^{-1}\right) \tag{1}$$

where $u = 1/r$, $l = r^2\,\theta' =$ const. In case of Rutherford scattering, $f\left(u^{-1}\right) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q_1\,q_2}{r^2}$. Then differential equation of the orbit would be

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{m\,l^2}\,\frac{q_1\,q_2}{4\,\pi\,\epsilon_0} \tag{2}$$

where $|\,l\,| = |\,\boldsymbol{r}\times\boldsymbol{v}\,| = b\,v_0$

$$\frac{d^2 u}{d\theta^2} + u = -\frac{1}{m\,{v_0}^2\,b^2}\,\frac{q_1\,q_2}{4\,\pi\,\epsilon_0} = -\frac{q_1\,q_2}{8\,\pi\,\epsilon_0\,b^2\,E} \tag{3}$$

In[137]:=
```
Clear[m, ϵ, q1, q2, b, θ0]
```

In[138]:=
```
sol = DSolve[{u''[θ] + u[θ] == -q1 q2/(8 π ϵ b^2 E0), u[0] == 0, u[2 θ0] == 0}, u[θ], θ]
```

Out[138]=
$$\left\{\left\{u[\theta] \to \frac{-q1\,q2 + q1\,q2\,\text{Cos}[\theta] - q1\,q2\,\text{Cot}[2\,\theta0]\,\text{Sin}[\theta] + q1\,q2\,\text{Csc}[2\,\theta0]\,\text{Sin}[\theta]}{8\,b^2\,E0\,\pi\,\epsilon}\right\}\right\}$$

In[139]:=
```
r[θ] = FullSimplify[(u[θ] /. sol)^-1]
```

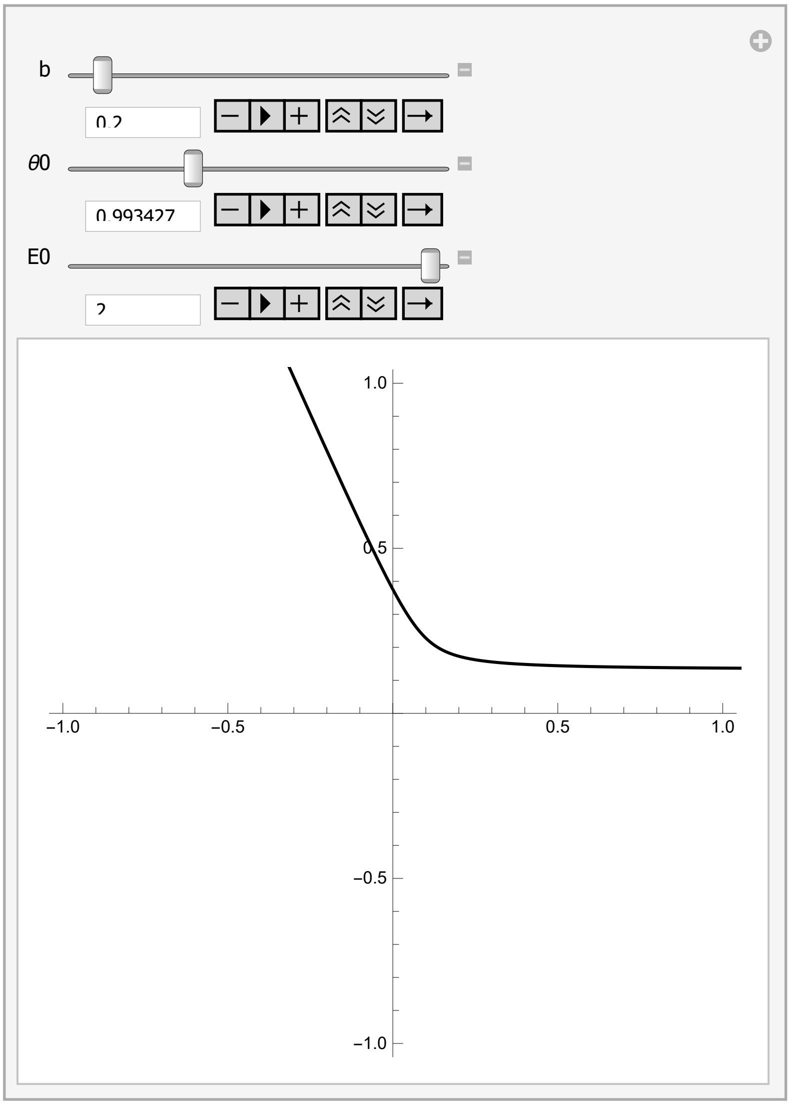
Out[139]=
$$\left\{\frac{8\,b^2\,E0\,\pi\,\epsilon\,\text{Cos}[\theta0]}{q1\,q2\,\text{Cos}[\theta-\theta0] - q1\,q2\,\text{Cos}[\theta0]}\right\}$$

In[168]:=
```
(r[θ] /. {ϵ → 0.1, q1 → 1, q2 → 1})
```

Out[168]= $\left\{\dfrac{2.51327\, b^2\, E0\, \mathrm{Cos}[\theta 0]}{\mathrm{Cos}[\theta - \theta 0] - \mathrm{Cos}[\theta 0]}\right\}$

In[174]:=
```
Manipulate[PolarPlot[2.5132741228718345` b^2 E0 Cos[θ0] / (Cos[θ - θ0] - Cos[θ0]), {θ, 0.0001, 2 θ0}, PlotRange → 1],
 {b, 0.1, 2, 0.1}, {θ0, 0.001, π}, {E0, 0, 2, 0.1}]
```

Out[174]=



Ma

M

Out[43]= u[θ]