Space-time Metric Round a Rotating Matter

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- Einstein field equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$, $R_{\mu\nu} = \frac{8\pi G}{c^2} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$
- Conservation law of special relativity: $T^{\mu\nu}{}_{;\nu} = 0$ (continuity equation)
- Components in $T^{\mu\nu}$: $\begin{pmatrix} T^{00}: & \text{energy density} \\ T^{0k} & \text{flow of energy along } x^k \end{pmatrix}$, $\begin{pmatrix} T^{m0}: & \text{density of } m \text{ th comp. of momentum } (p^m) \\ T^{mn}: & \text{flow of } p^m \text{ along } x^n \end{pmatrix}$
- $T^{\mu\nu}$ is a symmetric tensor $\Longleftrightarrow$ flow of energy is equivalent to density of momentum

1. Weak Field Limit

For a weak gravitational field ($h_{\mu\nu} << 1$), the metric is near the Minkowski metric, $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \;\; h_{\mu\nu} << 1 \tag{1}$$

Assume that $g^{\mu\nu} = \eta^{\mu\nu} + \chi^{\mu\nu}$, $\chi^{\mu\nu} << 1$. Then from $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$,

$$\delta^{\mu}{}_{\rho} = (\eta^{\mu\nu} + \chi^{\mu\nu}) (\eta_{\nu\rho} + h_{\nu\rho}) = \delta^{\mu}{}_{\rho} + \chi^{\mu}{}_{\rho} + h^{\mu}{}_{\rho} + O$$

Namely, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ In short, $-\begin{pmatrix} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \end{pmatrix}$

Recall that Christoffel symbol and Ricci tensor are given by

$$\Gamma^{\kappa}{}_{\lambda\mu} = \frac{1}{2} g^{\kappa\rho} (g_{\rho\lambda,\mu} + g_{\rho\mu,\lambda} - g_{\lambda\mu,\rho}) \tag{2}$$

$$R_{\mu\nu} = \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + \Gamma^{\kappa}{}_{\rho\kappa} \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\nu} \Gamma^{\rho}{}_{\mu\kappa} \tag{3}$$

In this case, they are given

$$\Gamma^{\kappa}{}_{\lambda\mu} = \frac{1}{2} (\eta^{\kappa\rho} - h^{\kappa\rho}) (h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) = \frac{1}{2} \eta^{\kappa\rho} (h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) + O\left(h^2\right) = \frac{1}{2} \eta^{\kappa\rho} (h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) \tag{4}$$

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + \Gamma^{\kappa}{}_{\rho\kappa} \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\nu} \Gamma^{\rho}{}_{\mu\kappa} \\ &= \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + O\left(h^2\right) \\ &= \tfrac{1}{2} \eta^{\kappa\rho} (h_{\rho\nu,\mu\kappa} + h_{\rho\mu,\nu\kappa} - h_{\mu\nu,\rho\kappa}) - \tfrac{1}{2} \eta^{\kappa\rho} (h_{\rho\mu,\kappa\nu} + h_{\rho\kappa,\mu\nu} - h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2} (\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \eta^{\kappa\rho} h_{\mu\nu,\rho\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2} (\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \end{aligned}$$

$$R_{\mu\nu} = \frac{1}{2} (\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \tag{5}$$

note that $\eta^{\mu\nu} \partial_\mu \partial_\nu$ is a D'Alembertian '$\Box$'

From Einstein field equation, ($S_{\mu\nu} = T_{\mu\nu} - 1/2\, g_{\mu\nu} T$)

$$R_{\mu\nu} = \frac{8\pi G}{c^2} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) = \frac{1}{2} (\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu})$$

$$\frac{16\pi G}{c^2} S_{\mu\nu} = \eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}$$

To express in neater way define new quantities, $f^{\mu\nu}$

$$\sqrt{-g}\; g^{\mu\nu} = \eta^{\mu\nu} - f^{\mu\nu} \tag{6}$$

where $g = \det(g_{\mu\nu})$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

```
In[4]:= η := DiagonalMatrix[{-1, 1, 1, 1}];
        H := Table[h_{ToString[i],ToString[j]}, {i, 0, 3}, {j, 0, 3}];
        η + H // MatrixForm
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Out[6]//MatrixForm=

$$\begin{pmatrix} -1+h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} \\ h_{1,0} & 1+h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,0} & h_{2,1} & 1+h_{2,2} & h_{2,3} \\ h_{3,0} & h_{3,1} & h_{3,2} & 1+h_{3,3} \end{pmatrix}$$

so,

$$\begin{aligned} g &= (-1+h_{00})(1+h_{11})(1+h_{22})(1+h_{22}) + O(h^2) \\ &= -1 + h_{00} - h_{11} - h_{22} - h_{33} + O(h^2) \\ &= -1 + \eta_{00}\, h^0{}_0 - \eta_{11}\, h^1{}_1 - \eta_{22}\, h^2{}_2 - \eta_{33}\, h^3{}_3 + O(h^2) \\ &= -1 - h^0{}_0 - h^1{}_1 - h^2{}_2 - h^3{}_3 + O(h^2) \\ &= -1 - h^\mu{}_\mu + O(h^2) \end{aligned}$$

hence,

$$\sqrt{-g} = (-g)^{1/2} = \left(1 + h^\lambda{}_\lambda + O(h^2)\right)^{1/2} = 1 + \frac{1}{2} h^\mu{}_\mu + O(h^2)$$

$$\sqrt{-g}\; g^{\mu\nu} = \left(1 + \frac{1}{2} h^\lambda{}_\lambda + O(h^2)\right)(\eta^{\mu\nu} - h^{\mu\nu}) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$\eta^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu} h^\lambda{}_\lambda - h^{\mu\nu} + O(h^2) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\, \eta^{\mu\nu} h^\lambda{}_\lambda \tag{7}$$

$$\eta_{\mu\nu} f^{\mu\nu} = f^\mu{}_\mu = \eta_{\mu\nu}\left(h^{\mu\nu} - 1/2\, \eta^{\mu\nu} h^\lambda{}_\lambda\right) = h^\mu{}_\mu - 1/2\,(4)\, h^\lambda{}_\lambda = -h^\mu{}_\mu$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\, \eta^{\mu\nu}\left(-f^\lambda{}_\lambda\right) \longrightarrow h^{\mu\nu} = f^{\mu\nu} - 1/2\, \eta^{\mu\nu} f^\lambda{}_\lambda \tag{8}$$

$$h^\lambda{}_\nu = \eta_{\mu\nu} h^{\mu\lambda} = \eta_{\mu\nu}\left(f^{\mu\lambda} - 1/2\, \eta^{\mu\lambda} f^\rho{}_\rho\right) = f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho$$

$$h_{\mu\nu} = \eta_{\lambda\mu} h^\lambda{}_\nu = \eta_{\lambda\mu}\left(f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho\right) = f_{\mu\nu} - 1/2\, \eta_{\mu\nu} f^\rho{}_\rho$$

Now back to the field equation.

$$R_{\mu\nu} = \frac{1}{2}\left(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}\right) = \frac{8\pi G}{c^2} T_{\mu\nu}$$

$$\left(h^\lambda{}_{\nu,\mu\lambda} + h^\lambda{}_{\mu,\lambda\nu} - h^\lambda{}_{\lambda,\mu\nu} - \Box h_{\mu\nu}\right)$$

$$\Rightarrow \left[\left(f^\lambda{}_\nu - 1/2\, \eta^\lambda{}_\nu f^\rho{}_\rho\right)_{,\mu\lambda} + \left(f^\lambda{}_\mu - 1/2\, \eta^\lambda{}_\mu f^\rho{}_\rho\right)_{,\lambda\nu} - \left(-f^\rho{}_\rho\right)_{,\mu\nu} - \Box\left(f_{\mu\nu} - 1/2\, \eta_{\mu\nu} f^\rho{}_\rho\right)\right]$$

$$= \left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \left(f^\rho{}_\rho\right)_{,\mu\nu} + f^\rho{}_{\rho,\mu\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right] = \left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right]$$

f has to be independent of time. Finally,

$$R_{\mu\nu} = \frac{1}{2}\left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right]$$

and

$$\begin{aligned} (1/2)\, \eta_{\mu\nu} R &= (1/2)\, \eta_{\mu\nu}\left(\eta^{\rho\sigma} R_{\rho\sigma}\right) \\ &= (1/2)\, \eta_{\mu\nu}\, \eta^{\rho\sigma} (1/2)\left[f^\lambda{}_{\sigma,\rho\lambda} + f^\lambda{}_{\rho,\sigma\nu} - \Box f_{\rho\sigma} + 1/2\, \eta_{\rho\sigma} \Box f^\lambda{}_\lambda\right] \\ &= (1/4)\, \eta_{\mu\nu}\left[f^{\lambda\rho}{}_{,\rho\lambda} + f^{\lambda\sigma}{}_{,\sigma\nu} - \Box f^\lambda{}_\lambda + 1/2\,(4)\, \Box f^\lambda{}_\lambda\right] \\ &= \tfrac{1}{4}\, \eta_{\mu\nu}\left[2 f^{\lambda\rho}{}_{,\rho\lambda} + \Box f^\lambda{}_\lambda\right] = \tfrac{1}{2}\, \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} + \tfrac{1}{4}\, \eta_{\mu\nu} \Box f^\lambda{}_\lambda \end{aligned}$$

Then the field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$ give

$$\frac{1}{2}\left[f^\lambda{}_{\nu,\mu\lambda} + f^\lambda{}_{\mu,\lambda\nu} - \Box f_{\mu\nu} + 1/2\, \eta_{\mu\nu} \Box f^\rho{}_\rho\right] - \left(\frac{1}{2}\eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} + \frac{1}{4}\eta_{\mu\nu} \Box f^\lambda{}_\lambda\right)$$

$$= \frac{1}{2}\left[f^{\lambda}{}_{\nu,\mu\lambda} + f^{\lambda}{}_{\mu,\lambda\nu} - \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu}\right] = \frac{8\pi G}{c^2} T_{\mu\nu}$$

$$f^{\lambda}{}_{\nu,\mu\lambda} + f^{\lambda}{}_{\mu,\lambda\nu} - \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu} = \frac{16\pi G}{c^2} T_{\mu\nu} \tag{9}$$

By choosing proper transformation, we can simplify above equation.

$$x^{\mu} \longrightarrow y^{\mu} = x^{\mu} + b^{\mu}(x)$$

Under given transformation,

$$\frac{\partial y^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + b^{\mu}{}_{,\nu}$$

$$g^{\mu\nu}(x) \longrightarrow g'^{\mu\nu}(y) = \frac{\partial y^{\mu}}{\partial x^{\rho}} \frac{\partial y^{\nu}}{\partial x^{\sigma}} g^{\rho\sigma}(x) = (\delta^{\mu}{}_{\rho} + b^{\mu}{}_{,\rho})(\delta^{\nu}{}_{\sigma} + b^{\nu}{}_{,\sigma}) g^{\rho\sigma}$$

$$= (\delta^{\mu}{}_{\rho} + b^{\mu}{}_{,\rho})(g^{\rho\nu} + g^{\rho\sigma} b^{\nu}{}_{,\sigma}) = g^{\mu\nu} + g^{\rho\nu} b^{\mu}{}_{,\rho} + g^{\mu\sigma} b^{\nu}{}_{,\sigma} + O\left(b^2\right)$$

Then the transformed metric tensor $g'^{\mu\nu}$ is given as below.

In[35]:=
```
Table[g^ToString[{i,j}] + g^"ρ" ToString[j] (b^ToString[i])_",ρ" + g^ToString[i] "ρ" (b^ToString[j])_",ρ",
  {i, 0, 3}, {j, 0, 3}] // MatrixForm // TraditionalForm
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Out[35]//TraditionalForm=

$$\begin{pmatrix} 2 b^0{}_{,\rho} g^{0\rho} + g^{\{0,0\}} & g^{0\rho} b^1{}_{,\rho} + b^0{}_{,\rho} g^{1\rho} + g^{\{0,1\}} & g^{0\rho} b^2{}_{,\rho} + b^0{}_{,\rho} g^{2\rho} + g^{\{0,2\}} & g^{0\rho} b^3{}_{,\rho} + b^0{}_{,\rho} g^{3\rho} + g^{\{0,3\}} \\ g^{0\rho} b^1{}_{,\rho} + b^0{}_{,\rho} g^{1\rho} + g^{\{1,0\}} & 2 b^1{}_{,\rho} g^{1\rho} + g^{\{1,1\}} & g^{1\rho} b^2{}_{,\rho} + b^1{}_{,\rho} g^{2\rho} + g^{\{1,2\}} & g^{1\rho} b^3{}_{,\rho} + b^1{}_{,\rho} g^{3\rho} + g^{\{1,3\}} \\ g^{0\rho} b^2{}_{,\rho} + b^0{}_{,\rho} g^{2\rho} + g^{\{2,0\}} & g^{1\rho} b^2{}_{,\rho} + b^1{}_{,\rho} g^{2\rho} + g^{\{2,1\}} & 2 b^2{}_{,\rho} g^{2\rho} + g^{\{2,2\}} & g^{2\rho} b^3{}_{,\rho} + b^2{}_{,\rho} g^{3\rho} + g^{\{2,3\}} \\ g^{0\rho} b^3{}_{,\rho} + b^0{}_{,\rho} g^{3\rho} + g^{\{3,0\}} & g^{1\rho} b^3{}_{,\rho} + b^1{}_{,\rho} g^{3\rho} + g^{\{3,1\}} & g^{2\rho} b^3{}_{,\rho} + b^2{}_{,\rho} g^{3\rho} + g^{\{3,2\}} & 2 b^3{}_{,\rho} g^{3\rho} + g^{\{3,3\}} \end{pmatrix}$$

With $g' = |g'_{\mu\nu}| = |g'^{\mu\nu}|^{-1}$,

$$\begin{aligned}(g')^{-1} = |g'^{\mu\nu}| &= \left(g^{00} + 2 g^{0\rho} b^0{}_{,\rho}\right)\left(g^{11} + 2 g^{1\rho} b^1{}_{,\rho}\right)\left(g^{22} + 2 g^{2\rho} b^2{}_{,\rho}\right)\left(g^{33} + 2 g^{3\rho} b^3{}_{,\rho}\right) + O\left(b^2\right) \\ &= g^{00} g^{11} g^{22} g^{33} + 2\left(g^{\rho 0} g^{11} g^{22} g^{33} b^0{}_{,\rho} + g^{00} g^{\rho 1} g^{22} g^{33} b^1{}_{,\rho} + g^{00} g^{11} g^{\rho 2} g^{33} b^2{}_{,\rho} + g^{00} g^{11} g^{22} g^{\rho 3} b^3{}_{,\rho}\right) + O\left(b^2\right) \\ &= g^{-1} + 2 g^{-1}\left(b^0{}_{,0} + b^1{}_{,1} + b^2{}_{,2} + b^3{}_{,3}\right) = g^{-1}\left(1 + 2 b^{\lambda}{}_{,\lambda}\right)\end{aligned}$$

Namely,

$$g' = g\left(1 - 2 b^{\lambda}{}_{,\lambda}\right), \quad \sqrt{-g'} = \sqrt{-g}\left(1 - b^{\lambda}{}_{,\lambda}\right) \tag{10}$$

$$\sqrt{-g'}\, g'^{\mu\nu} = \sqrt{-g}\left(1 - b^{\lambda}{}_{,\lambda}\right)(g^{\mu\nu} + g^{\rho\nu} b^{\mu}{}_{,\rho} + g^{\mu\sigma} b^{\nu}{}_{,\sigma}) = \sqrt{-g}\left(g^{\mu\nu} + g^{\rho\nu} b^{\mu}{}_{,\rho} + g^{\mu\sigma} b^{\nu}{}_{,\sigma} - g^{\mu\nu} b^{\lambda}{}_{,\lambda}\right) + O\left(b^2\right)$$

$$= \sqrt{-g}\left(g^{\mu\nu} + g^{\rho\nu} b^{\mu}{}_{,\rho} + g^{\mu\sigma} b^{\nu}{}_{,\sigma} - g^{\mu\nu} b^{\lambda}{}_{,\lambda}\right) = \eta^{\mu\nu} - f'^{\mu\nu}$$

last term come from $\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - f^{\mu\nu}$

$$\sqrt{-g}\left(g^{\mu\nu} + g^{\rho\nu} b^{\mu}{}_{,\rho} + g^{\mu\sigma} b^{\nu}{}_{,\sigma} - g^{\mu\nu} b^{\lambda}{}_{,\lambda}\right) = \eta^{\mu\nu} - f'^{\mu\nu}$$

$$\eta^{\mu\nu} - f^{\mu\nu} + (\eta^{\rho\nu} - f^{\rho\nu}) b^{\mu}{}_{,\rho} + (\eta^{\mu\sigma} - f^{\mu\sigma}) b^{\nu}{}_{,\sigma} - (\eta^{\mu\nu} - f^{\mu\nu}) b^{\lambda}{}_{,\lambda} = \eta^{\mu\nu} - f'^{\mu\nu}$$

$$-f^{\mu\nu} + (\eta^{\rho\nu} - f^{\rho\nu}) b^{\mu}{}_{,\rho} + (\eta^{\mu\sigma} - f^{\mu\sigma}) b^{\nu}{}_{,\sigma} - (\eta^{\mu\nu} - f^{\mu\nu}) b^{\lambda}{}_{,\lambda} = -f'^{\mu\nu}$$

Hence,

$$f'^{\mu\nu} = f^{\mu\nu} - \eta^{\rho\nu} b^{\mu}{}_{,\rho} - \eta^{\mu\sigma}\, b^{\nu}{}_{,\sigma} + \eta^{\mu\nu} b^{\lambda}{}_{,\lambda} \tag{11}$$

and,

$$f'^{\mu\nu}{}_{,\nu} = f^{\mu\nu}{}_{,\nu} - \eta^{\rho\nu} b^{\mu}{}_{,\rho\nu} - \eta^{\mu\sigma}\, b^{\nu}{}_{,\sigma\nu} + \eta^{\mu\nu} b^{\lambda}{}_{,\lambda\nu} = f^{\mu\nu}{}_{,\nu} - \eta^{\rho\nu} b^{\mu}{}_{,\rho\nu} = f^{\mu\nu}{}_{,\nu} - \Box b^{\mu} \tag{12}$$

where $\eta^{\rho\nu}\partial_{\rho}\partial_{\nu}$ is a D'Alembertian '□'. So we can properly choose b^{μ} such that $f^{\mu\nu}{}_{,\nu} = \Box b^{\mu}$, so that $f'^{\mu\nu}{}_{,\nu} = 0$ or,

$$\left(\sqrt{-g'}\, g'^{\mu\nu}\right)_{,\nu} = 0 \quad \text{(harmonic condition)}$$

Under the harmonic condition,

$$f^{\lambda}{}_{\nu,\mu\lambda} + f^{\lambda}{}_{\mu,\lambda\nu} - \eta_{\mu\nu} f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu} = \frac{16\pi G}{c^2} T_{\mu\nu} \Longrightarrow -\Box f_{\mu\nu} = \frac{16\pi G}{c^2} T_{\mu\nu}$$

Finally the field equation reduced to the following Poisson equation

$$\Box f_{\mu\nu} = -\frac{16\,\pi G}{c^2}\,T_{\mu\nu} \tag{13}$$

with harmonic condition

$$f^{\mu\nu}{}_{,\nu} = \left(\sqrt{-g}\;g^{\mu\nu}\right)_{,\nu} = 0$$

where

$$\begin{cases} f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\,h^{\lambda}{}_{\lambda} \\ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \end{cases}$$

We already dealt with the Poisson's equation at Electromagnetism course (see Griffith's book chap 10.2); solution of the equation (13) is given by

$$f_{\mu\nu}(\boldsymbol{r},\,t) = \frac{1}{4\,\pi}\,\frac{16\,\pi G}{c^2}\int\frac{T_{\mu\nu}(\boldsymbol{r}',\,t_r)}{\mathfrak{r}}\,d\tau \tag{14}$$

where t_r (retarded time) $= t - \frac{\mathfrak{r}}{c}$ and $\boldsymbol{\mathfrak{r}} = \boldsymbol{r} - \boldsymbol{r}'$, $\mathfrak{r} = |\,\boldsymbol{r} - \boldsymbol{r}'\,|$