Space-time Metric Round a Rotating Matter

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- Einstein field equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$, $R_{\mu\nu} = \frac{8\pi G}{c^2} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$
- Conservation law of special relativity: $T^{\mu\nu}{}_{;\nu} = 0$ (continuity equation)
- Components in $T^{\mu\nu}$: $\begin{cases} T^{00}: & \text{energy density} \\ T^{0k} & \text{flow of energy along } x^k \end{cases}$, $\begin{cases} T^{m0}/c: & \text{density of } m \text{ th comp. of momentum } (p^m) \\ T^{mn}: & \text{flow of } p^m \text{ along } x^n \end{cases}$
- Conservation equations

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{m0}}{\partial x^m} = 0 \ , \quad \frac{1}{c}\frac{\partial T^{m0}}{\partial t} + \frac{\partial T^{mn}}{\partial x^m} = 0$$

- $T^{\mu\nu}$ is a symmetric tensor $\Leftrightarrow$ flow of energy is equivalent to density of momentum
- $T^{\mu\nu} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & {v_x}^2/c^2 & v_x v_y/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & {v_y}^2/c^2 & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_z v_y/c^2 & {v_z}^2/c^2 \end{pmatrix}$

1. Weak Field Limit

For a weak gravitational field ($h_{\mu\nu} << 1$), the metric is near the Minkowski metric, $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \ \ h_{\mu\nu} << 1 \tag{1}$$

Assume that $g^{\mu\nu} = \eta^{\mu\nu} + \chi^{\mu\nu}$, $\chi^{\mu\nu} << 1$. Then from $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$,

$$\delta^{\mu}{}_{\rho} = (\eta^{\mu\nu} + \chi^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^{\mu}{}_{\rho} + \chi^{\mu}{}_{\rho} + h^{\mu}{}_{\rho} + O$$

Namely, $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ In short, $-\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \end{cases}$

Recall that Christoffel symbol and Ricci tensor are given by

$$\Gamma^{\kappa}{}_{\lambda\mu} = \frac{1}{2} g^{\kappa\rho}(g_{\rho\lambda,\mu} + g_{\rho\mu,\lambda} - g_{\lambda\mu,\rho}) \tag{2}$$

$$R_{\mu\nu} = \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + \Gamma^{\kappa}{}_{\rho\kappa}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\nu}\Gamma^{\rho}{}_{\mu\kappa} \tag{3}$$

In this case, they are given

$$\Gamma^{\kappa}{}_{\lambda\mu} = \frac{1}{2}(\eta^{\kappa\rho} - h^{\kappa\rho})(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) = \frac{1}{2}\eta^{\kappa\rho}(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) + O\left(h^2\right) \ = \frac{1}{2}\eta^{\kappa\rho}(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}) \tag{4}$$

$$\begin{aligned} R_{\mu\nu} &= \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + \Gamma^{\kappa}{}_{\rho\kappa}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\rho\nu}\Gamma^{\rho}{}_{\mu\kappa} \\ &= \Gamma^{\kappa}{}_{\mu\nu,\kappa} - \Gamma^{\kappa}{}_{\mu\kappa,\nu} + O\left(h^2\right) \\ &= \tfrac{1}{2}\eta^{\kappa\rho}(h_{\rho\nu,\mu\kappa} + h_{\rho\mu,\nu\kappa} - h_{\mu\nu,\rho\kappa}) - \tfrac{1}{2}\eta^{\kappa\rho}(h_{\rho\mu,\kappa\nu} + h_{\rho\kappa,\mu\nu} - h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2}(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \eta^{\kappa\rho} h_{\mu\nu,\rho\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu}) \\ &= \tfrac{1}{2}(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \end{aligned}$$

$$R_{\mu\nu} = \frac{1}{2}(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu}) \tag{5}$$

note that $\eta^{\mu\nu}\partial_\mu\partial_\nu$ is a D'Alembertian '$\Box$'

From Einstein field equation, ($S_{\mu\nu} = T_{\mu\nu} - 1/2\, g_{\mu\nu} T$)

$$R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = \frac{1}{2}(\eta^{\kappa\rho} h_{\rho\nu,\mu\kappa} + \eta^{\kappa\rho} h_{\mu\kappa,\rho\nu} - \eta^{\kappa\rho} h_{\rho\kappa,\mu\nu} - \Box h_{\mu\nu})$$

$$\frac{16\,\pi\mathrm{G}}{c^2}\,S_{\mu\nu} = \eta^{\kappa\rho}\,h_{\rho\nu\,,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa\,,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa\,,\mu\nu} - \Box\,h_{\mu\nu}$$

To express in neater way define new quantities, $f^{\mu\nu}$

$$\sqrt{-g}\;g^{\mu\nu} = \eta^{\mu\nu} - f^{\mu\nu} \tag{6}$$

where $g = \det(g_{\mu\nu})$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

```
In[8]:= η := DiagonalMatrix[{-1, 1, 1, 1}];
        H := Table[Subscript[h, i, j], {i, 0, 3}, {j, 0, 3}];
        η + H // MatrixForm // TraditionalForm
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Out[10]//TraditionalForm=

$$\begin{pmatrix} h_{0,0}-1 & h_{0,1} & h_{0,2} & h_{0,3} \\ h_{1,0} & h_{1,1}+1 & h_{1,2} & h_{1,3} \\ h_{2,0} & h_{2,1} & h_{2,2}+1 & h_{2,3} \\ h_{3,0} & h_{3,1} & h_{3,2} & h_{3,3}+1 \end{pmatrix}$$

so,

$$\begin{aligned} g &= (-1 + h_{00})\,(1 + h_{11})\,(1 + h_{22})\,(1 + h_{22}) + O\left(h^2\right) \\ &= -1 + h_{00} - h_{11} - h_{22} - h_{33} + O\left(h^2\right) \\ &= -1 + \eta_{00}\,h^0{}_0 - \eta_{11}\,h^1{}_1 - \eta_{22}\,h^2{}_2 - \eta_{33}\,h^3{}_3 + O\left(h^2\right) \\ &= -1 - h^0{}_0 - h^1{}_1 - h^2{}_2 - h^3{}_3 + O\left(h^2\right) \\ &= -1 - h^\mu{}_\mu + O\left(h^2\right) \end{aligned}$$

hence,

$$\sqrt{-g} = (-g)^{1/2} = \left(1 + h^\lambda{}_\lambda + O\left(h^2\right)\right)^{1/2} = 1 + \frac{1}{2}\,h^\mu{}_\mu + O\left(h^2\right)$$

$$\sqrt{-g}\;g^{\mu\nu} = \left(1 + \frac{1}{2}\,h^\lambda{}_\lambda + O\left(h^2\right)\right)(\eta^{\mu\nu} - h^{\mu\nu}) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$\eta^{\mu\nu} + \frac{1}{2}\,\eta^{\mu\nu}\,h^\lambda{}_\lambda - h^{\mu\nu} + O\left(h^2\right) = \eta^{\mu\nu} - f^{\mu\nu}$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\,h^\lambda{}_\lambda \tag{7}$$

$$\eta_{\mu\nu}\,f^{\mu\nu} = f^\mu{}_\mu = \eta_{\mu\nu}\left(h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\,h^\lambda{}_\lambda\right) = h^\mu{}_\mu - 1/2\,(4)\,h^\lambda{}_\lambda = -h^\mu{}_\mu$$

$$f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\left(-f^\lambda{}_\lambda\right) \longrightarrow\; h^{\mu\nu} = f^{\mu\nu} - 1/2\,\eta^{\mu\nu}\,f^\lambda{}_\lambda \tag{8}$$

$$h^\lambda{}_\nu = \eta_{\mu\nu}\,h^{\mu\lambda} = \eta_{\mu\nu}\left(f^{\mu\lambda} - 1/2\,\eta^{\mu\lambda}\,f^\rho{}_\rho\right) = f^\lambda{}_\nu - 1/2\,\eta^\lambda{}_\nu\,f^\rho{}_\rho$$

$$h_{\mu\nu} = \eta_{\lambda\mu}\,h^\lambda{}_\nu = \eta_{\lambda\mu}\left(f^\lambda{}_\nu - 1/2\,\eta^\lambda{}_\nu\,f^\rho{}_\rho\right) = f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\,f^\rho{}_\rho$$

Now back to the field equation.

$$R_{\mu\nu} = \frac{1}{2}\,(\eta^{\kappa\rho}\,h_{\rho\nu\,,\mu\kappa} + \eta^{\kappa\rho}\,h_{\mu\kappa\,,\rho\nu} - \eta^{\kappa\rho}\,h_{\rho\kappa\,,\mu\nu} - \Box\,h_{\mu\nu}) = \frac{8\,\pi\mathrm{G}}{c^2}\,T_{\mu\nu}$$

$$\left(h^\lambda{}_{\nu\,,\mu\lambda} + h^\lambda{}_{\mu\,,\lambda\nu} - h^\lambda{}_{\lambda\,,\mu\nu} - \Box\,h_{\mu\nu}\right)$$

$$\Rightarrow \left[\left(f^\lambda{}_\nu - 1/2\,\eta^\lambda{}_\nu\,f^\rho{}_\rho\right)_{,\mu\lambda} + \left(f^\lambda{}_\mu - 1/2\,\eta^\lambda{}_\mu\,f^\rho{}_\rho\right)_{,\lambda\nu} - (-f^\rho{}_\rho)_{,\mu\nu} - \Box(f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\,f^\rho{}_\rho)\right]$$

$$= \left[f^\lambda{}_{\nu\,,\mu\lambda} + f^\lambda{}_{\mu\,,\lambda\nu} - (f^\rho{}_\rho)_{,\mu\nu} + f^\rho{}_{\rho\,,\mu\nu} - \Box f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\,\Box f^\rho{}_\rho\right] = \left[f^\lambda{}_{\nu\,,\mu\lambda} + f^\lambda{}_{\mu\,,\lambda\nu} - \Box f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\,\Box f^\rho{}_\rho\right]$$

f has to be independent of time. Finally,

$$R_{\mu\nu} = \frac{1}{2}\left[f^\lambda{}_{\nu\,,\mu\lambda} + f^\lambda{}_{\mu\,,\lambda\nu} - \Box f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\,\Box f^\rho{}_\rho\right]$$

and

$$\begin{aligned}(1/2)\,\eta_{\mu\nu}\,R &= (1/2)\,\eta_{\mu\nu}(\eta^{\rho\sigma}\,R_{\rho\sigma})\\ &= (1/2)\,\eta_{\mu\nu}\,\eta^{\rho\sigma}(1/2)\left[f^{\lambda}{}_{\sigma\,,\rho\lambda} + f^{\lambda}{}_{\rho\,,\sigma\nu} - \Box f_{\rho\sigma} + 1/2\,\eta_{\rho\sigma}\,\Box f^{\lambda}{}_{\lambda}\right]\\ &= (1/4)\,\eta_{\mu\nu}\left[f^{\lambda\rho}{}_{,\rho\lambda} + f^{\lambda\sigma}{}_{,\sigma\nu} - \Box f^{\lambda}{}_{\lambda} + 1/2\,(4)\,\Box f^{\lambda}{}_{\lambda}\right]\\ &= \tfrac{1}{4}\,\eta_{\mu\nu}\left[2\,f^{\lambda\rho}{}_{,\rho\lambda} + \Box f^{\lambda}{}_{\lambda}\right] = \tfrac{1}{2}\,\eta_{\mu\nu}\,f^{\rho\sigma}{}_{,\rho\sigma} + \tfrac{1}{4}\,\eta_{\mu\nu}\,\Box f^{\lambda}{}_{\lambda}\end{aligned}$$

Then the field equation $R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = \frac{8\pi G}{c^2}\,T_{\mu\nu}$ give

$$\frac{1}{2}\left[f^{\lambda}{}_{\nu\,,\mu\lambda} + f^{\lambda}{}_{\mu\,,\lambda\nu} - \Box f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\,\Box f^{\rho}{}_{\rho}\right] - \left(\frac{1}{2}\,\eta_{\mu\nu}\,f^{\rho\sigma}{}_{,\rho\sigma} + \frac{1}{4}\,\eta_{\mu\nu}\,\Box f^{\lambda}{}_{\lambda}\right)$$

$$= \frac{1}{2}\left[f^{\lambda}{}_{\nu\,,\mu\lambda} + f^{\lambda}{}_{\mu\,,\lambda\nu} - \eta_{\mu\nu}\,f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu}\right] = \frac{8\,\pi G}{c^2}\,T_{\mu\nu}$$

$$f^{\lambda}{}_{\nu\,,\mu\lambda} + f^{\lambda}{}_{\mu\,,\lambda\nu} - \eta_{\mu\nu}\,f^{\rho\sigma}{}_{,\rho\sigma} - \Box f_{\mu\nu} = \frac{16\,\pi G}{c^2}\,T_{\mu\nu} \tag{9}$$

By choosing proper transformation, we can simplify above equation.

$$x^{\mu} \longrightarrow y^{\mu} = x^{\mu} + b^{\mu}(x)$$

Under given transformation,

$$\frac{\partial y^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + b^{\mu}{}_{,\nu}$$

$$g^{\mu\nu}(x) \longrightarrow g'^{\,\mu\nu}(y) = \frac{\partial y^{\mu}}{\partial x^{\rho}}\,\frac{\partial y^{\nu}}{\partial x^{\sigma}}\,g^{\rho\sigma}(x) = (\delta^{\mu}{}_{\rho} + b^{\mu}{}_{,\rho})\,(\delta^{\nu}{}_{\sigma} + b^{\nu}{}_{,\sigma})\,g^{\rho\sigma}$$

$$= (\delta^{\mu}{}_{\rho} + b^{\mu}{}_{,\rho})\,(g^{\rho\nu} + g^{\rho\sigma}\,b^{\nu}{}_{,\sigma}) = g^{\mu\nu} + g^{\rho\nu}\,b^{\mu}{}_{,\rho} + g^{\mu\sigma}\,b^{\nu}{}_{,\sigma} + O\left(b^2\right)$$

Then the transformed metric tensor $g'^{\,\mu\nu}$ is given as below.

```
In[4]:= Table[g^ToString[{i,j}] + g^"ρ" ToString[j] (b^ToString[i])",ρ" + g^ToString[i] "ρ" (b^ToString[j])",ρ",
          {i, 0, 3}, {j, 0, 3}] // MatrixForm // TraditionalForm
```

Out[4]//TraditionalForm=

$$\begin{pmatrix} 2\,b^0{}_{,\rho}\,g^{0\rho} + g^{\{0,0\}} & g^{0\rho}\,b^1{}_{,\rho} + b^0{}_{,\rho}\,g^{1\rho} + g^{\{0,1\}} & g^{0\rho}\,b^2{}_{,\rho} + b^0{}_{,\rho}\,g^{2\rho} + g^{\{0,2\}} & g^{0\rho}\,b^3{}_{,\rho} + b^0{}_{,\rho}\,g^{3\rho} + g^{\{0,3\}}\\ g^{0\rho}\,b^1{}_{,\rho} + b^0{}_{,\rho}\,g^{1\rho} + g^{\{1,0\}} & 2\,b^1{}_{,\rho}\,g^{1\rho} + g^{\{1,1\}} & g^{1\rho}\,b^2{}_{,\rho} + b^1{}_{,\rho}\,g^{2\rho} + g^{\{1,2\}} & g^{1\rho}\,b^3{}_{,\rho} + b^1{}_{,\rho}\,g^{3\rho} + g^{\{1,3\}}\\ g^{0\rho}\,b^2{}_{,\rho} + b^0{}_{,\rho}\,g^{2\rho} + g^{\{2,0\}} & g^{1\rho}\,b^2{}_{,\rho} + b^1{}_{,\rho}\,g^{2\rho} + g^{\{2,1\}} & 2\,b^2{}_{,\rho}\,g^{2\rho} + g^{\{2,2\}} & g^{2\rho}\,b^3{}_{,\rho} + b^2{}_{,\rho}\,g^{3\rho} + g^{\{2,3\}}\\ g^{0\rho}\,b^3{}_{,\rho} + b^0{}_{,\rho}\,g^{3\rho} + g^{\{3,0\}} & g^{1\rho}\,b^3{}_{,\rho} + b^1{}_{,\rho}\,g^{3\rho} + g^{\{3,1\}} & g^{2\rho}\,b^3{}_{,\rho} + b^2{}_{,\rho}\,g^{3\rho} + g^{\{3,2\}} & 2\,b^3{}_{,\rho}\,g^{3\rho} + g^{\{3,3\}} \end{pmatrix}$$

With $g' = |\,g'_{\mu\nu}\,| = |\,g'^{\,\mu\nu}\,|^{-1}$,

$$\begin{aligned}(g')^{-1} &= |\,g'^{\,\mu\nu}\,| = \left(g^{00} + 2\,g^{0\rho}\,b^0{}_{,\rho}\right)\left(g^{11} + 2\,g^{1\rho}\,b^1{}_{,\rho}\right)\left(g^{22} + 2\,g^{2\rho}\,b^2{}_{,\rho}\right)\left(g^{33} + 2\,g^{3\rho}\,b^3{}_{,\rho}\right) + O\left(b^2\right)\\ &= g^{00}\,g^{11}\,g^{22}\,g^{33} + 2\left(g^{\rho0}\,g^{11}\,g^{22}\,g^{33}\,b^0{}_{,\rho} + g^{00}\,g^{\rho1}\,g^{22}\,g^{33}\,b^1{}_{,\rho} + g^{00}\,g^{11}\,g^{\rho2}\,g^{33}\,b^2{}_{,\rho} + g^{00}\,g^{11}\,g^{22}\,g^{\rho3}\,b^3{}_{,\rho}\right) + O\left(b^2\right)\\ &= g^{-1} + 2\,g^{-1}\left(b^0{}_{,0} + b^1{}_{,1} + b^2{}_{,2} + b^3{}_{,3}\right) = g^{-1}\left(1 + 2\,b^{\lambda}{}_{,\lambda}\right)\end{aligned}$$

Namely,

$$g' = g\left(1 - 2\,b^{\lambda}{}_{,\lambda}\right),\quad \sqrt{-g'} = \sqrt{-g}\,\left(1 - b^{\lambda}{}_{,\lambda}\right) \tag{10}$$

$$\sqrt{-g'}\,g'^{\,\mu\nu} = \sqrt{-g}\,\left(1 - b^{\lambda}{}_{,\lambda}\right)\left(g^{\mu\nu} + g^{\rho\nu}\,b^{\mu}{}_{,\rho} + g^{\mu\sigma}\,b^{\nu}{}_{,\sigma}\right) = \sqrt{-g}\,\left(g^{\mu\nu} + g^{\rho\nu}\,b^{\mu}{}_{,\rho} + g^{\mu\sigma}\,b^{\nu}{}_{,\sigma} - g^{\mu\nu}\,b^{\lambda}{}_{,\lambda}\right) + O\left(b^2\right)$$

$$= \sqrt{-g}\,\left(g^{\mu\nu} + g^{\rho\nu}\,b^{\mu}{}_{,\rho} + g^{\mu\sigma}\,b^{\nu}{}_{,\sigma} - g^{\mu\nu}\,b^{\lambda}{}_{,\lambda}\right) = \eta^{\mu\nu} - f'^{\,\mu\nu}$$

last term come from $\sqrt{-g}\,g^{\mu\nu} = \eta^{\mu\nu} - f^{\mu\nu}$

$$\sqrt{-g}\,\left(g^{\mu\nu} + g^{\rho\nu}\,b^{\mu}{}_{,\rho} + g^{\mu\sigma}\,b^{\nu}{}_{,\sigma} - g^{\mu\nu}\,b^{\lambda}{}_{,\lambda}\right) = \eta^{\mu\nu} - f'^{\,\mu\nu}$$

$$\eta^{\mu\nu} - f^{\mu\nu} + (\eta^{\rho\nu} - f^{\rho\nu})\,b^{\mu}{}_{,\rho} + (\eta^{\mu\sigma} - f^{\mu\sigma})\,b^{\nu}{}_{,\sigma} - (\eta^{\mu\nu} - f^{\mu\nu})\,b^{\lambda}{}_{,\lambda} = \eta^{\mu\nu} - f'^{\,\mu\nu}$$

$$-f^{\mu\nu} + (\eta^{\rho\nu} - f^{\rho\nu})\,b^{\mu}{}_{,\rho} + (\eta^{\mu\sigma} - f^{\mu\sigma})\,b^{\nu}{}_{,\sigma} - (\eta^{\mu\nu} - f^{\mu\nu})\,b^{\lambda}{}_{,\lambda} = -f'^{\,\mu\nu}$$

Hence,

$$f'^{\,\mu\nu} = f^{\mu\nu} - \eta^{\rho\nu}\,b^{\mu}{}_{,\rho} - \eta^{\mu\sigma}\;b^{\nu}{}_{,\sigma} + \eta^{\mu\nu}\,b^{\lambda}{}_{,\lambda} \tag{11}$$

and,

$$f'^{\mu\nu}{}_{,\nu} = f^{\mu\nu}{}_{,\nu} - \eta^{\rho\nu}\, b^{\mu}{}_{,\rho\nu} - \eta^{\mu\sigma}\ \ b^{\nu}{}_{,\sigma\nu} + \eta^{\mu\nu}\, b^{\lambda}{}_{,\lambda\nu} = f^{\mu\nu}{}_{,\nu} - \eta^{\rho\nu}\, b^{\mu}{}_{,\rho\nu} = f^{\mu\nu}{}_{,\nu} - \Box\, b^{\mu} \tag{12}$$

where $\eta^{\rho\nu}\,\partial_\rho\,\partial_\nu$ is a D'Alembertian '□'. So we can properly choose b^μ such that $f^{\mu\nu}{}_{,\nu} = \Box\, b^\mu$, so that $f'^{\mu\nu}{}_{,\nu} = 0$ or,

$$\left(\sqrt{-g'}\ g'^{\mu\nu}\right)_{,\nu} = 0 \quad \text{(harmonic condition)}$$

Under the harmonic condition,

$$f^{\lambda}{}_{\nu\,,\mu\lambda} + f^{\lambda}{}_{\mu\,,\lambda\nu} - \eta_{\mu\nu}\, f^{\rho\sigma}{}_{,\rho\sigma} - \Box\, f_{\mu\nu} = \frac{16\,\pi \mathrm{G}}{c^2}\, T_{\mu\nu} \Longrightarrow -\Box\, f_{\mu\nu} = \frac{16\,\pi \mathrm{G}}{c^2}\, T_{\mu\nu}$$

Finally the field equation reduced to the following Poisson equation

$$\Box\, f_{\mu\nu} = -\frac{16\,\pi \mathrm{G}}{c^2}\, T_{\mu\nu} \tag{13}$$

with harmonic condition

$$f^{\mu\nu}{}_{,\nu} = \left(\sqrt{-g}\ g^{\mu\nu}\right)_{,\nu} = 0$$

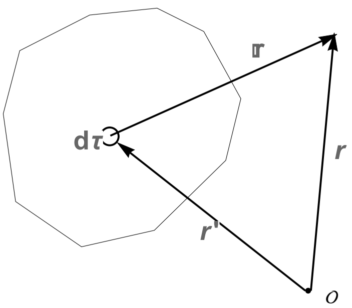
where

$$\left(\begin{array}{c} f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu}\, h^{\lambda}{}_{\lambda} \\ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\, f^{\lambda}{}_{\lambda} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} = \eta_{\mu\nu} - f_{\mu\nu} + 1/2\,\eta_{\mu\nu}\, f^{\lambda}{}_{\lambda} \\ f^{\lambda}{}_{\lambda} = \eta^{\mu\lambda}\, f_{\mu\lambda} = -f_{00} + f_{11} + f_{22} + f_{33} \end{array}\right.$$

We already dealt with the Poisson's equation at Electromagnetism course (see Griffith's book chap 10.2); solution of the equation (13) is given by

$$f_{\mu\nu}(\boldsymbol{r},\ t) = \frac{1}{4\,\pi}\ \frac{16\,\pi \mathrm{G}}{c^2} \int \frac{T_{\mu\nu}(\boldsymbol{r}',\ t_r)}{\mathfrak{r}}\, d\tau \tag{14}$$

where t_r (retarded time) $= t - \frac{\mathfrak{r}}{c}$ and $\boldsymbol{\mathfrak{r}} = \boldsymbol{r} - \boldsymbol{r}'$, $\mathfrak{r} = |\,\boldsymbol{r} - \boldsymbol{r}'\,|$

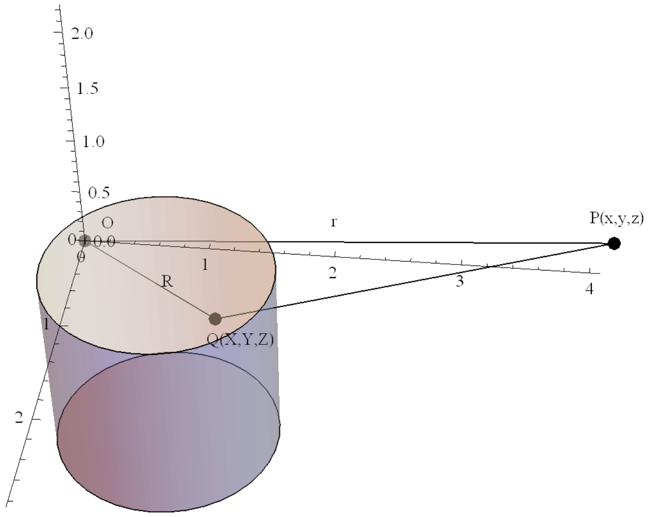


2. Rotating Body: Angular Momentum

Consider a body rotating with constant angular velocity $\omega = d\phi/dt$ about the x^3 − axis (z axis) and assume that $v << \mathrm{c}$. Then,

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x{}^2/c^2 & v_x\, v_y/c^2 & v_x\, v_z/c^2 \\ v_y/c & v_y\, v_x/c^2 & v_y{}^2/c^2 & v_y\, v_z/c^2 \\ v_z/c & v_z\, v_x/c^2 & v_z\, v_y/c^2 & v_z{}^2/c^2 \end{pmatrix} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & 0 \\ v_x/c & 0 & 0 & 0 \\ v_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_{\mu\nu} = \eta_{\mu\lambda}\,\eta_{\nu\rho}\, T^{\lambda\rho} = \rho \begin{pmatrix} 1 & -v_x/c & -v_y/c & 0 \\ -v_x/c & 0 & 0 & 0 \\ -v_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



As above figure, let's call the coordinates of a point Q (object) inside the rotating body by X^i and those of a point P (observer) outside the body by x^i then, $\left(\begin{array}{c} r^2 = x^i x_i \\ R^2 = X^i X_i \end{array}\right.$ with $R << \mathrm{r}$. Hence,

$$\mathbb{r} = |r - R| = \left(r^2 - 2\,\boldsymbol{r}\cdot\boldsymbol{R} + R^2\right)^{1/2} = r\left(1 - 2\frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2} + \frac{R^2}{r^2}\right)^{1/2} \simeq r\left(1 - \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) \tag{15}$$

$$1/\mathbb{r} = |r - R|^{-1} = \frac{1}{r}\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) \tag{16}$$

The field equation $\left(\Box f_{\mu\nu} = -\frac{16\,\pi\mathrm{G}}{c^2} T_{\mu\nu}\right)$ gives

$$\left(\begin{array}{l} \Box f_{00} = -\frac{16\pi\mathrm{G}}{c^2}\rho \\ \Box f_{01} = -\frac{16\pi\mathrm{G}}{c^2} T_{01} \\ \Box f_{02} = -\frac{16\pi\mathrm{G}}{c^2} T_{02} \end{array}\right. ,\ \text{other } f_{\mu\nu} = 0 \qquad \left(\text{recall that} \left(\begin{array}{c} f^{\mu\nu} = h^{\mu\nu} - 1/2\,\eta^{\mu\nu} h^{\lambda}{}_{\lambda} \\ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\,\eta_{\mu\nu} f^{\lambda}{}_{\lambda} \\ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} = \eta_{\mu\nu} - f_{\mu\nu} + 1/2\,\eta_{\mu\nu} f^{\lambda}{}_{\lambda} \\ f^{\lambda}{}_{\lambda} = \eta^{\mu\lambda} f_{\mu\lambda} = -f_{00} + f_{11} + f_{22} + f_{33} \end{array}\right)\right)$$

then $\Box f_{00} = -\frac{16\pi\mathrm{G}}{c^2}\rho$ is analog to Newton's equation: $\nabla^2\phi = -4\,\pi\mathrm{G}\rho \longrightarrow f_{00} = \frac{4\phi}{c^2},\quad f^{\lambda}{}_{\lambda} = -f_{00} = -\frac{4\phi}{c^2}$

$$g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu} - 1/2\,\eta_{\mu\nu}\left(-\frac{4\,\phi}{c^2}\right) = f_{\mu\nu} + \left(1 + \frac{2\,\phi}{c^2}\right)\eta_{\mu\nu}$$

$$g_{00} = \frac{4\,\phi}{c^2} - 1 - \frac{2\,\phi}{c^2} = -\left(1 - \frac{2\,\phi}{c^2}\right) \tag{17}$$

```
In[76]:= F = Table[Subscript[f, i, j], {i, 0, 3}, {j, 0, 3}];
F[[1, 4]] = F[[4, 1]] = 0; F[[1, 1]] = 4 ϕ/c^2;
F[[2 ;; 4, 2 ;; 4]] = ConstantArray[0, {3, 3}];
F // MatrixForm // TraditionalForm
g := F + (1 + 2 ϕ/c^2) η;
g // MatrixForm // TraditionalForm
```

Out[78]//TraditionalForm=

$$\begin{pmatrix} \frac{4\phi}{c^2} & f_{0,1} & f_{0,2} & 0 \\ f_{1,0} & 0 & 0 & 0 \\ f_{2,0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[79]//TraditionalForm=

$$\begin{pmatrix} \frac{2\phi}{c^2} - 1 & f_{0,1} & f_{0,2} & 0 \\ f_{1,0} & \frac{2\phi}{c^2} + 1 & 0 & 0 \\ f_{2,0} & 0 & \frac{2\phi}{c^2} + 1 & 0 \\ 0 & 0 & 0 & \frac{2\phi}{c^2} + 1 \end{pmatrix}$$

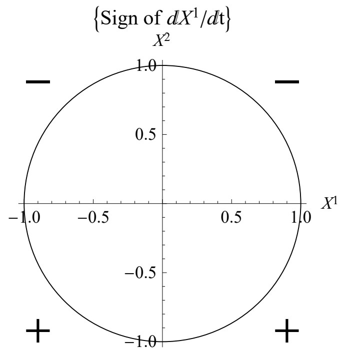
where $\phi = \frac{GM}{r} + \ldots$, As long as we get f_{01} and f_{02}, then we can figure out the whole metric tensor $g_{\mu\nu}$. To get $f_{0\,i}$, $(i = 1, 2)$ we have to utilize the solution of the Poisson's equation (eq.(14)) we've already met.

$$f_{\mu\nu}(\boldsymbol{r},\, t) = \frac{1}{4\pi}\frac{16\pi \mathrm{G}}{c^2}\int \frac{T_{\mu\nu}(\boldsymbol{r}',\, t_r)}{\mathfrak{r}}\, d\tau$$

$$\begin{cases} f_{01} = \frac{4G}{c^2}\int \frac{1}{\mathfrak{r}}\, T_{01}\, d^3X = \frac{4G}{c^2 r}\int\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) T_{01}\, d^3X = \frac{4G}{c^2 r}\left[\int T_{01}\, d^3X + \frac{x^i}{r^2}\int X_i\, T_{01}\, d^3X\right] \\ f_{02} = \frac{4G}{c^2}\int \frac{1}{\mathfrak{r}}\, T_{02}\, d^3X = \frac{4G}{c^2 r}\int\left(1 + \frac{\boldsymbol{r}\cdot\boldsymbol{R}}{r^2}\right) T_{02}\, d^3X = \frac{4G}{c^2 r}\left[\int T_{02}\, d^3X + \frac{x^i}{r^2}\int X_i\, T_{02}\, d^3X\right] \end{cases} \tag{18}$$

Evaluating the integral $\int T_{01}\, d^3X$ is not that difficult. Since $T_{01} = -\frac{\rho\, v_x}{c} = -\frac{\rho}{c}\frac{dX^1}{dt}$ and the quantity $\frac{dX^1}{dt}$ has two positive and two negative signs on circular motion, the value of integration reduced to zero.

Out[26]=

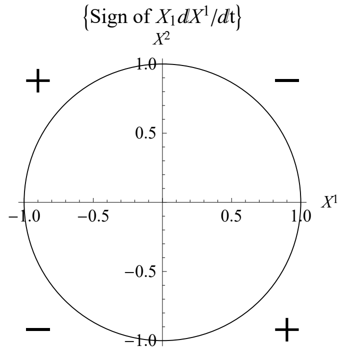


$$\begin{cases} f_{01} = \frac{4G}{c^2 r}\left[\frac{x^i}{r^2}\int X_i\, T_{01}\, d^3X\right] \\ f_{02} = \frac{4G}{c^2 r}\left[\frac{x^i}{r^2}\int X_i\, T_{02}\, d^3X\right] \end{cases} \tag{19}$$

Then let's consider the remaining integral $\int X_i\, T_{01}\, d^3X$. Similarly, we can check the sign of $X_i\, T_{01} = X_i\,\frac{dX^1}{dt}$ and kill some zeros.

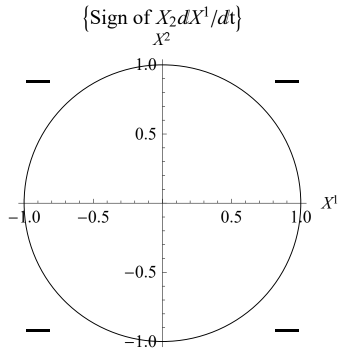
- $i = 1$; Sign of $X_1\,\frac{dX^1}{dt}$

Out[27]=



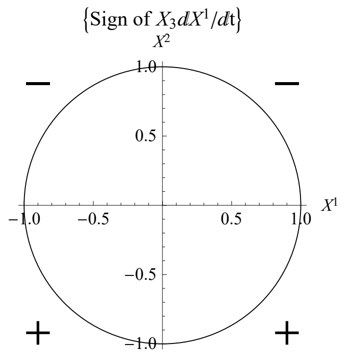
- $i = 2$; Sign of $X_2 \frac{dX^1}{dt}$

Out[28]=



- $i = 3$; Sign of $X_3 \frac{dX^1}{dt}$ is equal to sign of $\frac{dX^1}{dt}$, because $X_3 (= Z)$ is always positive in this situation.

Out[29]=



Thus, the only non-vanishing term is

$$\begin{cases} f_{01} = \frac{4G}{c^2 r}\left[\frac{x^2}{r^2}\int X_2\, T_{01}\, d^3X\right] = \frac{4Gy}{c^2 r^3}\int Y\, T_{01}\, d^3X = -\frac{4Gy}{c^2 r^3}\int Y\, T^{01}\, d^3X \\ f_{02} = \frac{4G}{c^2 r}\left[\frac{x^1}{r^2}\int X_1\, T_{02}\, d^3X\right] = \frac{4Gx}{c^2 r^3}\int X\, T_{02}\, d^3X = -\frac{4Gx}{c^2 r^3}\int X\, T^{02}\, d^3X \end{cases} \tag{20}$$

What is the meaning of above terms? Before we consider this, we need to talk about essential meaning of T^{0m}; do you remember?

$$T^{m0}/c: \quad \text{density of } m \text{ th comp. of momentum } (p^m)$$

So the volume integral of the T^{m0} must be the m the component of momentum, p^m itself. Moreover, in special relativity, the angular momentum is denoted by

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p}, \;\; J^3 = x^1 p^2 - x^2 p^1$$

more generally,

$$J^3 = \int \left(x^1\, dp^2 - x^2\, dp^1\right) = c \int \left(x^1\, T^{02} - x^2\, T^{01}\right) d\tau$$

In this case,

$$J^3 = c \int \left(X\, T^{02} - Y\, T^{01}\right) d^3 X \tag{21}$$

The conservation of angular momentum requires $\frac{\partial T^{ik}}{\partial x^k} = 0$ (in fact, it comes from the definition of energy-momentum tensor)

(Landau&Lifshitz) From the Lagrange's e.o.m (essentially, conservation of energy) of generalized coordinate q,

$$\frac{\partial}{\partial x^i}\frac{\partial \mathcal{L}}{\partial q_{,i}} - \frac{\partial \mathcal{L}}{\partial q} = 0, \quad \text{where } \mathcal{L} \text{ is a Lagrangian for this motion}$$

Then, the derivative of Lagrangian is given by

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{\partial \mathcal{L}}{\partial q}\frac{\partial q}{\partial x^i} + \frac{\partial \mathcal{L}}{\partial q_{,k}}\frac{\partial q_{,k}}{\partial x^i} = \left(\frac{\partial}{\partial x^k}\frac{\partial \mathcal{L}}{\partial q_{,k}}\right) q_{,i} + \frac{\partial \mathcal{L}}{\partial q_{,k}} q_{,k\,i} = \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial q_{,k}} q_{,i}\right)$$

where we can write

$$\frac{\partial \mathcal{L}}{\partial x^i} = \left(\delta^k{}_i \frac{\partial \mathcal{L}}{\partial x^k}\right)$$

Consequently,

$$\left(\delta^k{}_i \frac{\partial \mathcal{L}}{\partial x^k}\right) = \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial q_{,k}} q_{,i}\right), \quad \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial q_{,k}} q_{,i} - \delta^k{}_i \mathcal{L}\right) = 0$$

where $\frac{\partial \mathcal{L}}{\partial q_{,k}} q_{,i} - \delta^k{}_i \mathcal{L}$ *is called* $T^k{}_i$ *that* $T^k{}_{i,k} = 0$

Remind the given energy momentum tensor.

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & v_x/c & v_y/c & 0 \\ v_x/c & 0 & 0 & 0 \\ v_y/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T^{\mu\nu}{}_{,\nu} = 0$$

For a static distribution (=time independent) of matter, $T^{\mu 0}{}_{,0} = 0$, hence, with $\mu = 0$,

$$T^{0\,\nu}{}_{,\nu} = T^{00}{}_{,0} + T^{01}{}_{,1} + T^{02}{}_{,2} + T^{03}{}_{,3} = 0 \implies T^{01}{}_{,1} + T^{02}{}_{,2} = 0 \tag{22}$$

It follows that

$$\int X^k X^m \left(T^{01}{}_{,1} + T^{02}{}_{,2}\right) d^3 X = 0$$

$$\begin{aligned}
\int X^k X^m \left(T^{01}{}_{,1} + T^{02}{}_{,2}\right) d^3 X &= \int X^k X^m\, T^{0\,i}{}_{,i}\, d^3 X \\
&= \int \left[\partial_i\left(X^k X^m\, T^{0\,i}\right) - \partial_i\left(X^k\right) X^m\, T^{0\,i} - X^k\, \partial_i (X^m)\, T^{0\,i}\right] d^3 X \\
&= \int \left[\partial_i\left(X^k X^m\, T^{0\,i}\right) - \left(\delta^k{}_i\right) X^m\, T^{0\,i} - X^k (\delta^m{}_i)\, T^{0\,i}\right] d^3 X \\
&= \int \left[\partial_i\left(X^k X^m\, T^{0\,i}\right) - X^m\, T^{0\,k} - X^k\, T^{0\,m}\right] d^3 X = 0
\end{aligned}$$

The first term can be rewritten as $\int \boldsymbol{\nabla} \cdot \left(X^k X^m\, \boldsymbol{T}^0\right) d\tau = \oint \left(X^k X^m\, \boldsymbol{T}^0\right) \cdot d\boldsymbol{a}$, the divergence theorem. Note that the income is equal to the outcome momentum. Namely, the divergence term vanishes. Therefore,

$$\int \left(X^m\, T^{0\,k} + X^k\, T^{0\,m}\right) d^3 X = 0 \Longrightarrow \int \left(X\, T^{02} + Y\, T^{01}\right) d^3 X = 0 \tag{23}$$

From above relation, it turns out that

$$\int Y\, T^{01}\, d^3 X = -\int X\, T^{02}\, d^3 X = \frac{1}{2}\int \left(Y\, T^{01} - X\, T^{02}\right) d^3 X = -\frac{J^3}{2\,c} \tag{24}$$

Last equality comes from the eq.(21). Finally,

$$\begin{cases} f_{01} = -\frac{4\,G\,y}{c^2\,r^3}\int Y\, T^{01}\, d^3 X = -\frac{4\,G\,y}{c^2\,r^3}\left(-\frac{J^3}{2\,c}\right) = \frac{2\,G}{c^3}\,\frac{y}{r^3}\, J^3 \\ f_{02} = -\frac{4\,G\,x}{c^2\,r^3}\int X\, T^{02}\, d^3 X = -\frac{4\,G\,x}{c^2\,r^3}\left(\frac{J^3}{2\,c}\right) = -\frac{2\,G}{c^3}\,\frac{x}{r^3}\, J^3 \end{cases} \tag{25}$$

In[87]:=
```
F[[1, 2]] = F[[2, 1]] = (2 G/c^3) (y/r^3) J_z;
F[[1, 3]] = F[[3, 1]] = -(2 G/c^3) (x/r^3) J_z;
F // MatrixForm // TraditionalForm
g // MatrixForm // TraditionalForm
```

Out[88]//TraditionalForm=

$$\begin{pmatrix} \frac{4\phi}{c^2} & \frac{2 G y J_z}{c^3 r^3} & -\frac{2 G x J_z}{c^3 r^3} & 0 \\ \frac{2 G y J_z}{c^3 r^3} & 0 & 0 & 0 \\ -\frac{2 G x J_z}{c^3 r^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[89]//TraditionalForm=

$$\begin{pmatrix} \frac{2\phi}{c^2} - 1 & \frac{2 G y J_z}{c^3 r^3} & -\frac{2 G x J_z}{c^3 r^3} & 0 \\ \frac{2 G y J_z}{c^3 r^3} & \frac{2\phi}{c^2} + 1 & 0 & 0 \\ -\frac{2 G x J_z}{c^3 r^3} & 0 & \frac{2\phi}{c^2} + 1 & 0 \\ 0 & 0 & 0 & \frac{2\phi}{c^2} + 1 \end{pmatrix}$$

In linear approximation, $\left(\phi \simeq \frac{GM}{r}\right)$

In[90]:=
```
g /. ϕ → G M / r // MatrixForm // TraditionalForm
```

Out[90]//TraditionalForm=

$$\begin{pmatrix} \frac{2 G M}{c^2 r} - 1 & \frac{2 G y J_z}{c^3 r^3} & -\frac{2 G x J_z}{c^3 r^3} & 0 \\ \frac{2 G y J_z}{c^3 r^3} & \frac{2 G M}{c^2 r} + 1 & 0 & 0 \\ -\frac{2 G x J_z}{c^3 r^3} & 0 & \frac{2 G M}{c^2 r} + 1 & 0 \\ 0 & 0 & 0 & \frac{2 G M}{c^2 r} + 1 \end{pmatrix}$$

These are the components of the metric tensor outside a rotating body of mass M and angular momentum J_z in the linear approximation. That is to say, it is the approximation of the exact Kerr solution. Then come up with the Equivalence Principle, we can ask whether the gravitational effects of a rotating source are equivalent to a rotating frame of reference.

References

A. L. Ryder. (2009), Introduction to General Relativity, Cambridge University Press.

B. Landau, L.D. & Lifshitz, E.M. (1971), The Classical Theory of Fields, Oxford: Pergamon Press.