Reality Check for U(1)

- Reality checker for finite group

$$\eta^{(r)} = \frac{1}{N(G)} \sum_{g\in G} \chi^{(r)}(g^2) = \begin{cases} 1 & \text{Real} \\ -1 & \text{Pseudoreal} \\ 0 & \text{Complex} \end{cases} \tag{1}$$

- Reality checker for continuous group

$$\eta^{(r)} = \int d\mu(g)\, \chi^{(r)}(g^2) = \begin{cases} 1 & \text{Real} \\ -1 & \text{Pseudoreal} \\ 0 & \text{Complex} \end{cases} \tag{2}$$

Note that $U(1) \to e^{i\psi} I_1$ such that I_1 is 1-by-1 identity matrix. Hence, the character of $U(1)$ is

$$\chi(\psi) = e^{i\psi},\ \psi \text{ starts from 0 to } 2\pi$$

Moreover, in this case,

$$\int d\mu(g) = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2(\psi/2)$$

Check normalization constant.

```
In[13]:= Integrate[Sin[u / 2]^2, {u, 0, 2 π}]

Out[13]= π
```

Then the reality checker for $U(1)$ is

$$\eta^{(r)} = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2\left(\frac{\psi}{2}\right) \chi(2\psi) = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2\left(\frac{\psi}{2}\right) e^{i2\psi} \tag{3}$$

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In[12]:= Integrate[(Sin[ψ / 2])^2 Exp[2 ⅈ ψ] * 2 / π, {ψ, 0, 2 π}]

Out[12]= 0
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namely, $U(1)$ is complex.