Geodesic equation for Schwarzschild metric

From Schwarzschild metric, I had already derived connection coefficients (or Christoffel symbol) corresponds to the metric. Recall that in mind, we can write the geodesic equations with given coefficients.

$\ddot{x}^{\mu}+\Gamma^{\mu}{}_{\nu\sigma}\dot{x}^{\nu}\dot{x}^{\sigma}=0$, $\{c\,t,\ r,\ \theta,\ \phi\}$

- $\mu=0$

$\ddot{t}+\frac{2m}{r(r-2m)}\dot{t}\,\dot{r}=0 \implies \left(1-\frac{2m}{r}\right)\dot{t}=\text{const}=b$

- $\mu=2$

$\ddot{\theta}+\frac{2}{r}\dot{r}\,\dot{\theta}-\sin\theta\cos\theta\,\dot{\phi}^2=0$

- $\mu=3$

$\ddot{\phi}+\frac{2}{r}\dot{r}\,\dot{\phi}+2\cot\theta\,\dot{\theta}\,\dot{\phi}=0$

instead of $\mu=1$ case, handle the line element

$ds^2=-c^2\,d\tau^2=-\left(1-\frac{2m}{r}\right)c^2\,dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2\,d\Omega^2$ divide it by $ds^2=-c^2\,d\tau^2$

$1=\left(1-\frac{2m}{r}\right)\dot{t}^2-\left(1-\frac{2m}{r}\right)^{-1}\frac{\dot{r}^2}{c^2}-\frac{r^2}{c^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right).$

For light, on the other hand,

$0=\left(1-\frac{2m}{r}\right)\dot{t}^2-\left(1-\frac{2m}{r}\right)^{-1}\frac{\dot{r}^2}{c^2}-\frac{r^2}{c^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right).$

Now consider a geodesic passing through the equator, $\theta=\pi/2$, namely $\dot{\theta}=0$, then,

- $\mu=0$

$\ddot{t}+\frac{2m}{r(r-2m)}\dot{t}\,\dot{r}=0 \implies \left(1-\frac{2m}{r}\right)\dot{t}=\text{const}=b$

- $\mu=2$

$\ddot{\theta}=0$

- $\mu=3$

$\ddot{\phi}+\frac{2}{r}\dot{r}\,\dot{\phi}=0 \implies \left(r^2\,\dot{\phi}\right)=\text{const}=a$, $\dot{r}=\frac{dr}{d\phi}\frac{d\phi}{dt}=\frac{dr}{d\phi}\dot{\phi}=\frac{dr}{d\phi}\frac{a}{r^2}$

Substitute those into geodesic equation.

$1=\left(1-\frac{2m}{r}\right)\dot{t}^2-\left(1-\frac{2m}{r}\right)^{-1}\frac{\dot{r}^2}{c^2}-\frac{r^2}{c^2}\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right)$, $\left(\theta=\pi/2,\ \dot{\theta}=0\right)$

$1=\left(1-\frac{2m}{r}\right)^{-1}b^2-\left(1-\frac{2m}{r}\right)^{-1}\frac{a^2}{c^2}\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2-\frac{a^2}{c^2}\frac{1}{r^2}$

$\left(1-\frac{2m}{r}\right)\frac{1}{a^2}=\frac{b^2}{a^2}-\frac{1}{c^2}\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2-\frac{1}{r^2c^2}\left(1-\frac{2m}{r}\right)$

$\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2+\frac{1}{r^2}=\frac{c^2b^2}{a^2}+\left(\frac{2m}{r}-1\right)\frac{c^2}{a^2}+\frac{2m}{c^2r^3}$, or

$\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2+\frac{1}{r^2}=\frac{c^2b^2}{a^2}+\frac{2m}{c^2r^3}$ (for light)

differentiate with respect to ϕ

$2\frac{d}{d\phi}\left(\frac{1}{r}\right)\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)+\frac{2}{r}\frac{d}{d\phi}\left(\frac{1}{r}\right)=\frac{2mc^2}{a^2}\frac{d}{d\phi}\left(\frac{1}{r}\right)+2m\,3\left(\frac{1}{r}\right)^2\frac{d}{d\phi}\left(\frac{1}{r}\right)$

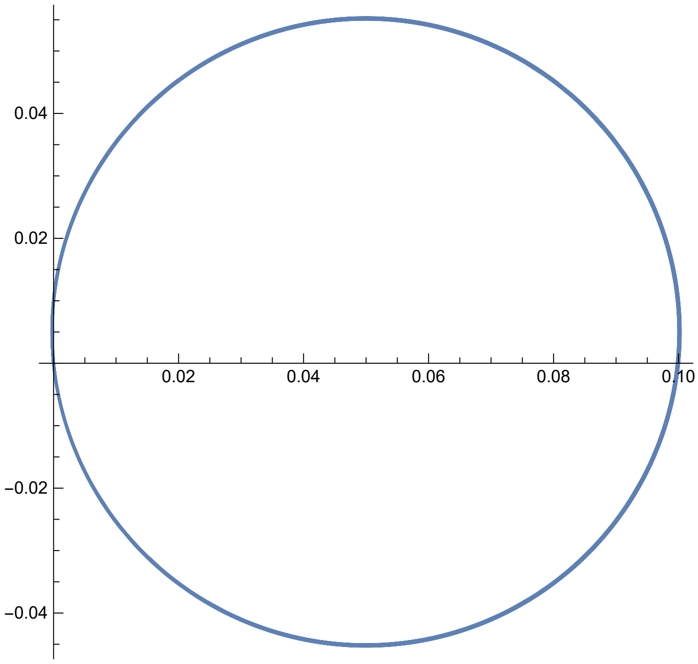
assume that $\frac{d}{d\phi}\left(\frac{1}{r}\right)\neq 0$, (rejecting circle trajectory case) then,

$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{m\,c^2}{a^2} + \frac{3\,m}{r^2}$ or,

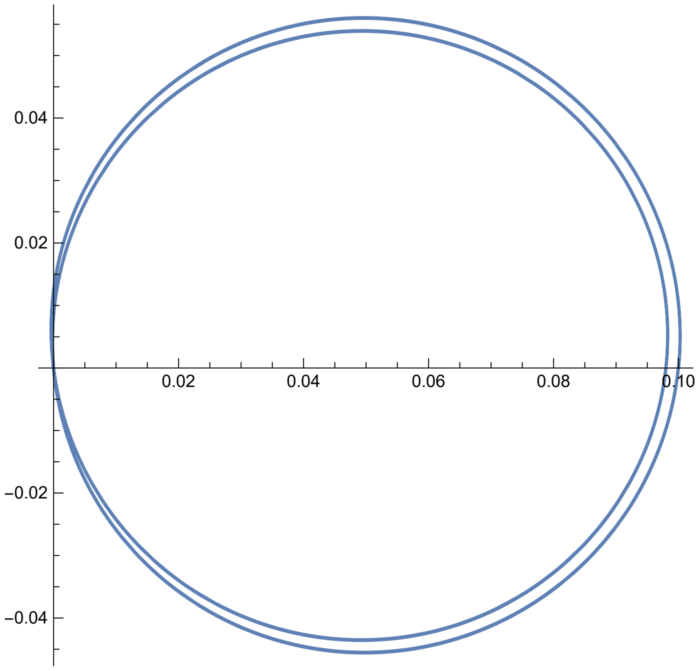
$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3\,m}{r^2}$ (light) where $m = G\,M/c^2$

let $G \to 1$ and $c \to 1$ and see what happens when light travel around the black hole!

In[130]:=
```
M = 0.01;
s =
  NDSolveValue[{w''[ϕ] + w[ϕ] == 3 M (w[ϕ])^2, w[0] == 0.1, w'[0] == 0.01}, w, {ϕ, 0, 4 π}];
PolarPlot[s[x], {x, 0.`, 4 π}, AspectRatio → Automatic]
```

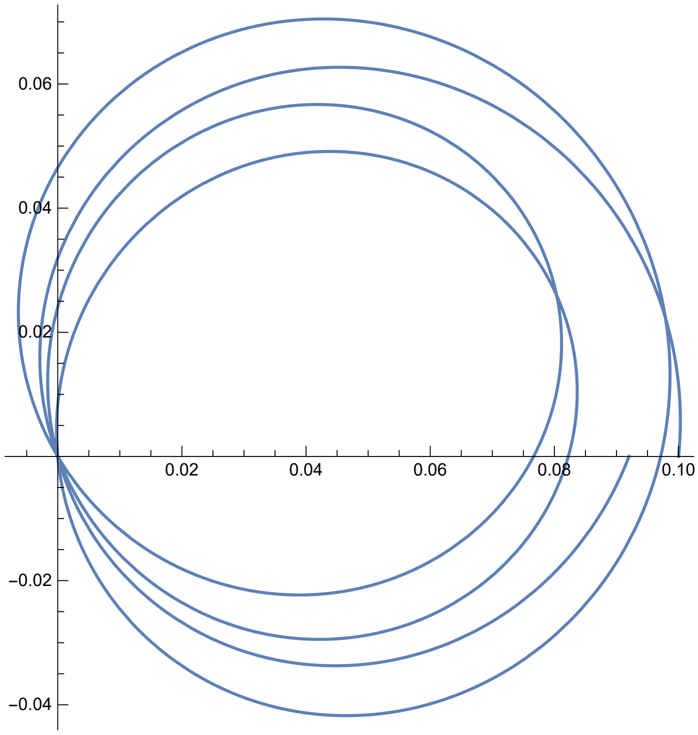
Out[132]=


In[133]:=
```
M = 0.1;
s =
  NDSolveValue[{w''[ϕ] + w[ϕ] == 3 M (w[ϕ])^2, w[0] == 0.1, w'[0] == 0.01}, w, {ϕ, 0, 4 π}];
PolarPlot[s[x], {x, 0, 4 π}]
```

Out[135]=


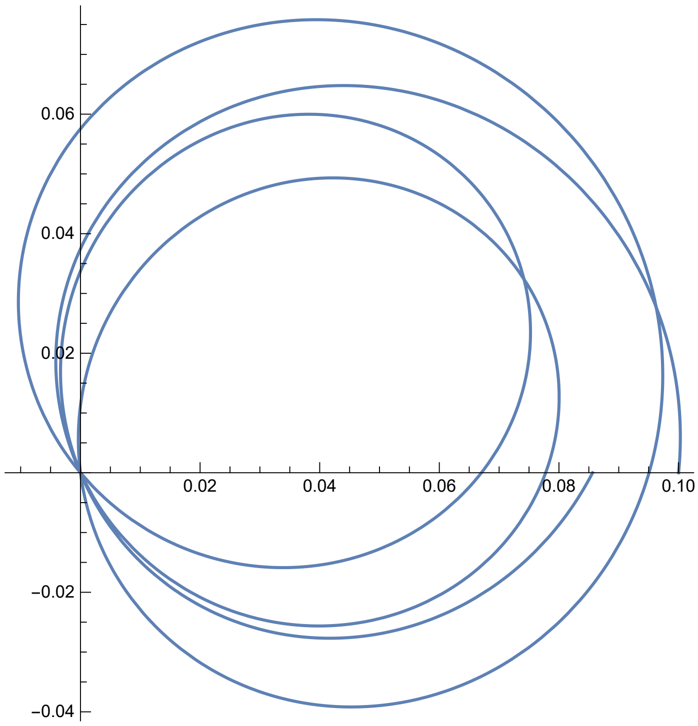
In[136]:=
```
M = 0.9;
s =
  NDSolveValue[{w''[ϕ] + w[ϕ] == 3 M (w[ϕ])^2, w[0] == 0.1, w'[0] == 0.01}, w, {ϕ, 0, 4 π}];
PolarPlot[s[x], {x, 0, 4 π}]
```

Out[138]=



In[139]:=
```
M = 1.1;
s =
  NDSolveValue[{w''[ϕ] + w[ϕ] == 3 M (w[ϕ])^2, w[0] == 0.1, w'[0] == 0.01}, w, {ϕ, 0, 4 π}];
PolarPlot[s[x], {x, 0, 4 π}]
```

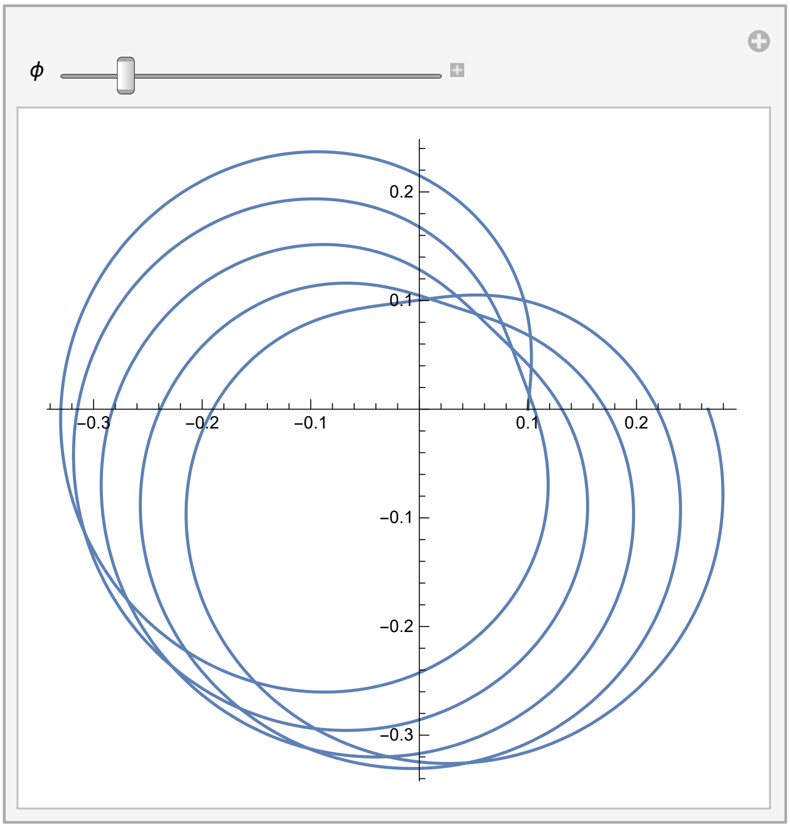
Out[141]=



Matter case ($1/a^2 = 2$)

In[181]:=
```
M = 0.1;
s = NDSolveValue[
    {w''[ϕ] + w[ϕ] == 3 M (w[ϕ])^2 + 2 M, w[0] == 0.1, w'[0] == 0.01}, w, {ϕ, 0, 40 π}];
Manipulate[PolarPlot[s[x], {x, 0, ϕ}], {ϕ, 5 π, 40 π, 5 π}]
```

Out[183]=



We can see the precession!