Connection one-form

$\omega^k{}_\lambda = \Gamma^k{}_{\lambda\mu}\,\Theta^\mu$ Γ: Christoffel symbol or connection coefficient

Torsion two-form

$\Sigma^\mu = d\Theta^\mu + \omega^\mu{}_k \wedge \Theta^k$

The mainstream of GR is torsion-free!

Algorithm...

- Find Metric Tensor !
- Use Following Relation:

$d g_{\mu\nu} = \omega^k{}_\mu g_{k\nu} + \omega^k{}_\nu g_{k\mu} = \omega_{\mu\nu} + \omega_{\nu\mu}$
with above relation and connection one-form, find connection coefficient as possible

- Use Zero-Torsion Condition
- Now you can find all of connection coefficients

However, with computer, it's okay to use pesky definition;
$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$,

Ex. $\mathbb{E}^3$, polar coordinate

```
dim = 3;
coord = {r, θ, ϕ};
metric = {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[θ]^2}}
```

$\{\{1, 0, 0\}, \{0, r^2, 0\}, \{0, 0, r^2 \mathrm{Sin}[\theta]^2\}\}$

```
metric // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

```
inversemetric = Simplify[Inverse[metric]]
```

$\left\{\{1, 0, 0\}, \left\{0, \frac{1}{r^2}, 0\right\}, \left\{0, 0, \frac{\mathrm{Csc}[\theta]^2}{r^2}\right\}\right\}$

recall that the definition of connection coefficient;
$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$

```
affine := affine = Simplify[Table[(1/2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]] ] +
          D[metric[[s, k]], coord[[j]] ] - D[metric[[j, k]], coord[[s]] ]),
       {s, 1, dim}],
     {i, 1, dim}, {j, 1, dim}, {k, 1, dim}] ]
```

```
affine
```

$\left\{\left\{\{0, 0, 0\}, \{0, -r, 0\}, \{0, 0, -r\,\mathrm{Sin}[\theta]^2\}\right\},\right.$
$\left\{\left\{0, \frac{1}{r}, 0\right\}, \left\{\frac{1}{r}, 0, 0\right\}, \{0, 0, -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\}\right\},$
$\left.\left\{\left\{0, 0, \frac{1}{r}\right\}, \{0, 0, \mathrm{Cot}[\theta]\}, \left\{\frac{1}{r}, \mathrm{Cot}[\theta], 0\right\}\right\}\right\}$

Display it: Γ[1, 2, 3] stands for $\Gamma^1{}_{23}$

```
listaffine :=
 Table[{ToString[Γ[i, j, k]], affine[[i, j, k]]},
  {i, 1, dim}, {j, 1, dim}, {k, 1, dim}]
```

```
TableForm[listaffine, TableSpacing → {3, 3}]
```

Γ[1, 1, 1]	0	Γ[1, 2, 1]	0	Γ[1, 3, 1]	0
Γ[1, 1, 2]	0	Γ[1, 2, 2]	$-r$	Γ[1, 3, 2]	0
Γ[1, 1, 3]	0	Γ[1, 2, 3]	0	Γ[1, 3, 3]	$-r\,\mathrm{Sin}[\theta]^2$
Γ[2, 1, 1]	0	Γ[2, 2, 1]	$\frac{1}{r}$	Γ[2, 3, 1]	0
Γ[2, 1, 2]	$\frac{1}{r}$	Γ[2, 2, 2]	0	Γ[2, 3, 2]	0
Γ[2, 1, 3]	0	Γ[2, 2, 3]	0	Γ[2, 3, 3]	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
Γ[3, 1, 1]	0	Γ[3, 2, 1]	0	Γ[3, 3, 1]	$\frac{1}{r}$
Γ[3, 1, 2]	0	Γ[3, 2, 2]	0	Γ[3, 3, 2]	$\mathrm{Cot}[\theta]$
Γ[3, 1, 3]	$\frac{1}{r}$	Γ[3, 2, 3]	$\mathrm{Cot}[\theta]$	Γ[3, 3, 3]	0

```
listaffine :=
 Table[{Γ^ToString[i]_j,k, ToString["="], affine[[i, j, k]]},
  {i, 1, dim}, {j, 1, dim}, {k, 1, dim}]
```

```
listaffine // TableForm
```

$\Gamma^1{}_{1,1} = 0$	$\Gamma^1{}_{2,1} = 0$	$\Gamma^1{}_{3,1} = 0$
$\Gamma^1{}_{1,2} = 0$	$\Gamma^1{}_{2,2} = -r$	$\Gamma^1{}_{3,2} = 0$
$\Gamma^1{}_{1,3} = 0$	$\Gamma^1{}_{2,3} = 0$	$\Gamma^1{}_{3,3} = -r\,\mathrm{Sin}[\theta]^2$
$\Gamma^2{}_{1,1} = 0$	$\Gamma^2{}_{2,1} = \frac{1}{r}$	$\Gamma^2{}_{3,1} = 0$
$\Gamma^2{}_{1,2} = \frac{1}{r}$	$\Gamma^2{}_{2,2} = 0$	$\Gamma^2{}_{3,2} = 0$
$\Gamma^2{}_{1,3} = 0$	$\Gamma^2{}_{2,3} = 0$	$\Gamma^2{}_{3,3} = -\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
$\Gamma^3{}_{1,1} = 0$	$\Gamma^3{}_{2,1} = 0$	$\Gamma^3{}_{3,1} = \frac{1}{r}$
$\Gamma^3{}_{1,2} = 0$	$\Gamma^3{}_{2,2} = 0$	$\Gamma^3{}_{3,2} = \mathrm{Cot}[\theta]$
$\Gamma^3{}_{1,3} = \frac{1}{r}$	$\Gamma^3{}_{2,3} = \mathrm{Cot}[\theta]$	$\Gamma^3{}_{3,3} = 0$