Schwarzschild Solution - Static, Spherical Symmetric and Vacuum

static: time independent!

$$ds^2 = -U(r)\,c^2\,dt^2 + V(r)\,dr^2 + W(r)\,r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \Longrightarrow -U(r)\,c^2\,dt^2 + V(r)\,dr^2 + W(r)\,r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

- let $U(r) \to e^{2\nu(r)}$, $V(r) \to e^{2\lambda(r)}$

Metric Tensor :

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2\lambda} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

In[1]:=
```
dim = 4;
coord = {ct, r, θ, ϕ};
metric = {{-e^(2 ν[r]), 0, 0, 0}, {0, e^(2 λ[r]), 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}};
```

```
metric // MatrixForm
```

$$\begin{pmatrix} -e^{2\nu[r]} & 0 & 0 & 0 \\ 0 & e^{2\lambda[r]} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

- $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \Gamma^{\rho}{}_{\mu\nu,\rho} - \Gamma^{\rho}{}_{\mu\rho,\nu} + \Gamma^{k}{}_{\mu\nu}\,\Gamma^{\rho}{}_{k\rho} - \Gamma^{k}{}_{\mu\rho}\,\Gamma^{\rho}{}_{k\nu}$

Let’s get the Christoffel symbol first!

In[4]:=
```
inversemetric = Simplify[Inverse[metric]];
affine := affine = Simplify[Table[(1/2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]] ] +
          D[metric[[s, k]], coord[[j]] ] - D[metric[[j, k]], coord[[s]] ]),
       {s, 1, dim}],
     {i, 1, dim}, {j, 1, dim}, {k, 1, dim}] ];
listaffine :=
  Table[{Γ^ToString[i]_j,k, ToString["="], affine[[i, j, k]]},
   {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];
TableForm[listaffine, TableSpacing → {3, 3}]
```

Out[7]//TableForm=

$\Gamma^1{}_{1,1} = 0$	$\Gamma^1{}_{2,1} = v'[r]$	$\Gamma^1{}_{3,1} = 0$	$\Gamma^1{}_{4,1} = 0$
$\Gamma^1{}_{1,2} = v'[r]$	$\Gamma^1{}_{2,2} = 0$	$\Gamma^1{}_{3,2} = 0$	$\Gamma^1{}_{4,2} = 0$
$\Gamma^1{}_{1,3} = 0$	$\Gamma^1{}_{2,3} = 0$	$\Gamma^1{}_{3,3} = 0$	$\Gamma^1{}_{4,3} = 0$
$\Gamma^1{}_{1,4} = 0$	$\Gamma^1{}_{2,4} = 0$	$\Gamma^1{}_{3,4} = 0$	$\Gamma^1{}_{4,4} = 0$
$\Gamma^2{}_{1,1} = e^{2v[r]-2\lambda[r]} v'[r]$	$\Gamma^2{}_{2,1} = 0$	$\Gamma^2{}_{3,1} = 0$	$\Gamma^2{}_{4,1} = 0$
$\Gamma^2{}_{1,2} = 0$	$\Gamma^2{}_{2,2} = \lambda'[r]$	$\Gamma^2{}_{3,2} = 0$	$\Gamma^2{}_{4,2} = 0$
$\Gamma^2{}_{1,3} = 0$	$\Gamma^2{}_{2,3} = 0$	$\Gamma^2{}_{3,3} = -e^{-2\lambda[r]} r$	$\Gamma^2{}_{4,3} = 0$
$\Gamma^2{}_{1,4} = 0$	$\Gamma^2{}_{2,4} = 0$	$\Gamma^2{}_{3,4} = 0$	$\Gamma^2{}_{4,4} = -e^{-2\lambda[r]} r \,\text{Sin}[\theta]^2$
$\Gamma^3{}_{1,1} = 0$	$\Gamma^3{}_{2,1} = 0$	$\Gamma^3{}_{3,1} = 0$	$\Gamma^3{}_{4,1} = 0$
$\Gamma^3{}_{1,2} = 0$	$\Gamma^3{}_{2,2} = 0$	$\Gamma^3{}_{3,2} = \frac{1}{r}$	$\Gamma^3{}_{4,2} = 0$
$\Gamma^3{}_{1,3} = 0$	$\Gamma^3{}_{2,3} = \frac{1}{r}$	$\Gamma^3{}_{3,3} = 0$	$\Gamma^3{}_{4,3} = 0$
$\Gamma^3{}_{1,4} = 0$	$\Gamma^3{}_{2,4} = 0$	$\Gamma^3{}_{3,4} = 0$	$\Gamma^3{}_{4,4} = -\text{Cos}[\theta]\,\text{Sin}[\theta]$
$\Gamma^4{}_{1,1} = 0$	$\Gamma^4{}_{2,1} = 0$	$\Gamma^4{}_{3,1} = 0$	$\Gamma^4{}_{4,1} = 0$
$\Gamma^4{}_{1,2} = 0$	$\Gamma^4{}_{2,2} = 0$	$\Gamma^4{}_{3,2} = 0$	$\Gamma^4{}_{4,2} = \frac{1}{r}$
$\Gamma^4{}_{1,3} = 0$	$\Gamma^4{}_{2,3} = 0$	$\Gamma^4{}_{3,3} = 0$	$\Gamma^4{}_{4,3} = \text{Cot}[\theta]$
$\Gamma^4{}_{1,4} = 0$	$\Gamma^4{}_{2,4} = \frac{1}{r}$	$\Gamma^4{}_{3,4} = \text{Cot}[\theta]$	$\Gamma^4{}_{4,4} = 0$

In[8]:=
```
Riemann := Riemann =
   Simplify[Table[D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] +
      Sum[affine[[i, s, k]] affine[[s, j, l]] - affine[[i, s, l]] affine[[s, j, k]],
       {s, 1, dim}],
     {i, 1, dim}, {j, 1, dim}, {k, 1, dim}, {l, 1, dim}]];
Ricci := Table[Sum[Riemann[[u, a, u, b]], {u, 1, dim}], {a, 1, dim}, {b, 1, dim}];
RicciR := Sum[Ricci[[a, b]] inversemetric[[a, b]], {a, 1, dim}, {b, 1, dim}]
```

In[18]:=
```
For[i = 1, i < 5, i++ 1,
 For[j = 1, j < 5, j++ 1,
  Print[ToString[R[i, j]], ToString["="], Ricci[[i, j]]]]]
```

$$R[1,1]=\frac{2\,e^{2\nu[r]-2\lambda[r]}\,\nu'[r]}{r}+e^{2\nu[r]-2\lambda[r]}\left(\nu'[r]^2-\nu'[r]\,\lambda'[r]+\nu''[r]\right)$$

R[1, 2]=0

R[1, 3]=0

R[1, 4]=0

R[2, 1]=0

$$R[2,2]=-\nu'[r]^2+\frac{2\,\lambda'[r]}{r}+\nu'[r]\,\lambda'[r]-\nu''[r]$$

R[2, 3]=0

R[2, 4]=0

R[3, 1]=0

R[3, 2]=0

$$R[3,3]=1-e^{-2\lambda[r]}-e^{-2\lambda[r]}\,r\,\nu'[r]+e^{-2\lambda[r]}\,r\,\lambda'[r]$$

R[3, 4]=0

R[4, 1]=0

R[4, 2]=0

R[4, 3]=0

$$R[4,4]=\left(1-e^{-2\lambda[r]}\right)\mathrm{Sin}[\theta]^2-e^{-2\lambda[r]}\,r\,\mathrm{Sin}[\theta]^2\,\nu'[r]+e^{-2\lambda[r]}\,r\,\mathrm{Sin}[\theta]^2\,\lambda'[r]$$

Vacuum Einstein Field Equation:
$R_{\mu\nu}=0$

From $R_{00}=0$, $R_{11}=0$,
$\lambda'(r)+\nu'(r)=0$

From $R_{22}=0$,
$e^{2\lambda(r)}=1+2r\nu'(r)=e^{-2\nu(r)}$

$1=\left(e^{2\nu(r)}+2r\nu'(r)\,e^{2\nu(r)}\right)=\left(r\,e^{2\nu(r)}\right)'$
$r+C=r\,e^{2\nu(r)}$, let $C=-2m$
$e^{2\nu(r)}=1-\frac{2m}{r}$ Hence. $e^{2\lambda(r)}=\left(1-\frac{2m}{r}\right)^{-1}$

In[24]:=
```
metric = metric /. {e^(2ν[r]) → (1 - 2m/r), e^(2λ[r]) → (1 - 2m/r)^-1};
metric // MatrixForm
```

Out[25]//MatrixForm=

$$\begin{pmatrix} -1+\frac{2m}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Above metric is called “Schwarzschild metric”. ($m=GM/c^2$)
Note that g_{11} becomes singular at $r\to 2GM/c^2=r_s$: the emergence of Black Hole

Birkhoff’s theorem- For vacuum equation, any spherical symmetric solution of the field equation is static

Now suppose that metric evolves as a function of time. Then, unlike static case, here comes new friend, cross-term, $c\, dt\, dr$

$ds^2 = -P(r, t)\, c^2\, dt^2 + Q(r, t)\, dr^2 + 2\, R(r, t)\, c\, dt\, dr + S(r, t)\, r^2\, d\Omega^2$

let $f(r, t)\, (P(r, t)\, c\, dt) - R(r, t)\, dr) = c\, dF(r, t) = \partial_t F\, dt + \partial_r F\, dr$ → total derivative

Since $\partial_r \partial_t F = \partial_t \partial_r F$, $\partial_r (f(r, t)\, P(r, t)) = \partial_t (-f(r, t)\, R(r, t))$, namely,

$$-\frac{\partial f}{\partial t} = \frac{1}{R} \left(\frac{\partial f}{\partial r} P + f \frac{\partial P}{\partial r} + f \frac{\partial R}{\partial t} \right)$$

From $c^2\, dF^2 = f^2 (P\, c\, dt - R\, dr)^2 = f^2 (P^2\, c^2\, dt^2 - 2\, P\, R\, c\, dt\, dr + R^2\, dr^2)$,

$ds^2 = -\frac{c^2}{f^2 P} dF^2 + \left(\frac{R^2}{P} + Q \right) dr^2 + r^2\, d\Omega^2 = -U(r, t')\, c^2\, dt'^2 + V(r, t)\, dr^2 + r^2\, d\Omega^2$,(substitute t' for F)

Above form is pretty analogous to that of Schwarzschild metric! - the only difference is time dependency.

$ds^2 = -U(r, t')\, c^2\, dt'^2 + V(r, t)\, dr^2 + r^2\, d\Omega^2 = -e^{2\, v(r,t)}\, c^2\, dt^2 + e^{2\, \lambda(r,t)}\, dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$

Metric Tensor

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\, v(r,t)} & 0 & 0 & 0 \\ 0 & e^{2\, \lambda(r,t)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -e^{-2\, v(r,t)} & 0 & 0 & 0 \\ 0 & e^{-2\, \lambda(r,t)} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2 \sin^2\theta \end{pmatrix}$$

In[86]:=
```
dim = 4;
coord = {t , r, θ, ϕ};
metric = {{-ⅇ^(2 v[r,t]), 0, 0, 0}, {0, ⅇ^(2 λ[r,t]), 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}};
```

In[89]:=
```
inversemetric = Simplify[Inverse[metric]];
affine := affine = Simplify[Table[(1/2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]] ] +
          D[metric[[s, k]], coord[[j]] ] - D[metric[[j, k]], coord[[s]] ]),
       {s, 1, dim}],
     {i, 1, dim}, {j, 1, dim}, {k, 1, dim}] ];
listaffine :=
  Table[{Γ^ToString[i-1]_{j-1,k-1}, ToString["="], affine[[i, j, k]]},
   {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];
TableForm[listaffine, TableSpacing → {3, 3}]
```

Out[92]//TableForm=

$\Gamma^{0}{}_{0,0} = v^{(0,1)}[r, t]$	$\Gamma^{0}{}_{1,0} = v^{(1,0)}[r, t]$	$\Gamma^{0}{}_{2,0} = 0$
$\Gamma^{0}{}_{0,1} = v^{(1,0)}[r, t]$	$\Gamma^{0}{}_{1,1} = e^{-2 v[r,t]+2 \lambda[r,t]} \lambda^{(0,1)}[r, t]$	$\Gamma^{0}{}_{2,1} = 0$
$\Gamma^{0}{}_{0,2} = 0$	$\Gamma^{0}{}_{1,2} = 0$	$\Gamma^{0}{}_{2,2} = 0$
$\Gamma^{0}{}_{0,3} = 0$	$\Gamma^{0}{}_{1,3} = 0$	$\Gamma^{0}{}_{2,3} = 0$
$\Gamma^{1}{}_{0,0} = e^{2 v[r,t]-2 \lambda[r,t]} v^{(1,0)}[r, t]$	$\Gamma^{1}{}_{1,0} = \lambda^{(0,1)}[r, t]$	$\Gamma^{1}{}_{2,0} = 0$
$\Gamma^{1}{}_{0,1} = \lambda^{(0,1)}[r, t]$	$\Gamma^{1}{}_{1,1} = \lambda^{(1,0)}[r, t]$	$\Gamma^{1}{}_{2,1} = 0$
$\Gamma^{1}{}_{0,2} = 0$	$\Gamma^{1}{}_{1,2} = 0$	$\Gamma^{1}{}_{2,2} = - e^{-2 \lambda[r,t}$
$\Gamma^{1}{}_{0,3} = 0$	$\Gamma^{1}{}_{1,3} = 0$	$\Gamma^{1}{}_{2,3} = 0$
$\Gamma^{2}{}_{0,0} = 0$	$\Gamma^{2}{}_{1,0} = 0$	$\Gamma^{2}{}_{2,0} = 0$
$\Gamma^{2}{}_{0,1} = 0$	$\Gamma^{2}{}_{1,1} = 0$	$\Gamma^{2}{}_{2,1} = \frac{1}{r}$
$\Gamma^{2}{}_{0,2} = 0$	$\Gamma^{2}{}_{1,2} = \frac{1}{r}$	$\Gamma^{2}{}_{2,2} = 0$
$\Gamma^{2}{}_{0,3} = 0$	$\Gamma^{2}{}_{1,3} = 0$	$\Gamma^{2}{}_{2,3} = 0$
$\Gamma^{3}{}_{0,0} = 0$	$\Gamma^{3}{}_{1,0} = 0$	$\Gamma^{3}{}_{2,0} = 0$
$\Gamma^{3}{}_{0,1} = 0$	$\Gamma^{3}{}_{1,1} = 0$	$\Gamma^{3}{}_{2,1} = 0$
$\Gamma^{3}{}_{0,2} = 0$	$\Gamma^{3}{}_{1,2} = 0$	$\Gamma^{3}{}_{2,2} = 0$
$\Gamma^{3}{}_{0,3} = 0$	$\Gamma^{3}{}_{1,3} = \frac{1}{r}$	$\Gamma^{3}{}_{2,3} = \mathrm{Cot}[\theta]$

In[93]:=
```
Riemann := Riemann =
   Simplify[Table[D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] +
      Sum[affine[[i, s, k]] affine[[s, j, l]] - affine[[i, s, l]] affine[[s, j, k]],
       {s, 1, dim}],
     {i, 1, dim}, {j, 1, dim}, {k, 1, dim}, {l, 1, dim}]];
Ricci := Table[Sum[Riemann[[u, a, u, b]], {u, 1, dim}], {a, 1, dim}, {b, 1, dim}];
RicciR := Sum[Ricci[[a, b]] inversemetric[[a, b]], {a, 1, dim}, {b, 1, dim}]
```

In[96]:=
```
For[i = 1, i < 5, i++ 1,
 For[j = 1, j < 5, j++ 1,
  Print[ToString[R[i - 1, j - 1]], ToString["="], Ricci[[i, j]]]]]
```

$$R[0,0]=v^{(0,1)}[r,t]\,\lambda^{(0,1)}[r,t]-\lambda^{(0,1)}[r,t]^2-\lambda^{(0,2)}[r,t]+\frac{2\,e^{2v[r,t]-2\lambda[r,t]}\,v^{(1,0)}[r,t]}{r}+e^{2v[r,t]-2\lambda[r,t]}\left(v^{(1,0)}[r,t]^2-v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r,t]+v^{(2,0)}[r,t]\right)$$

$$R[0,1]=\frac{2\,\lambda^{(0,1)}[r,t]}{r}$$

$R[0,2]=0$

$R[0,3]=0$

$$R[1,0]=\frac{2\,\lambda^{(0,1)}[r,t]}{r}$$

$$R[1,1]=-e^{-2v[r,t]+2\lambda[r,t]}\,v^{(0,1)}[r,t]\,\lambda^{(0,1)}[r,t]+e^{-2v[r,t]+2\lambda[r,t]}\,\lambda^{(0,1)}[r,t]^2+e^{-2v[r,t]+2\lambda[r,t]}\,\lambda^{(0,2)}[r,t]-v^{(1,0)}[r,t]^2+\frac{2\,\lambda^{(1,0)}[r,t]}{r}+v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r,t]-v^{(2,0)}[r,t]$$

$R[1,2]=0$

$R[1,3]=0$

$R[2,0]=0$

$R[2,1]=0$

$$R[2,2]=1-e^{-2\lambda[r,t]}-e^{-2\lambda[r,t]}\,r\,v^{(1,0)}[r,t]+e^{-2\lambda[r,t]}\,r\,\lambda^{(1,0)}[r,t]$$

$R[2,3]=0$

$R[3,0]=0$

$R[3,1]=0$

$R[3,2]=0$

$$R[3,3]=\left(1-e^{-2\lambda[r,t]}\right)\mathrm{Sin}[\theta]^2-e^{-2\lambda[r,t]}\,r\,\mathrm{Sin}[\theta]^2\,v^{(1,0)}[r,t]+e^{-2\lambda[r,t]}\,r\,\mathrm{Sin}[\theta]^2\,\lambda^{(1,0)}[r,t]$$

Note that $\lambda^{(0,1)}[r,t]$ means time derivative of $\lambda[r,t]$

Einstein field equation of vacuum condition : $R_{\mu\nu}=0$

$$v^{(0,1)}[r,t]\,\lambda^{(0,1)}[r,t]-\lambda^{(0,1)}[r,t]^2-\lambda^{(0,2)}[r,t]+\frac{2\,e^{2v[r,t]-2\lambda[r,t]}\,v^{(1,0)}[r,t]}{r}+ \text{----(A)}$$
$$e^{2v[r,t]-2\lambda[r,t]}\left(v^{(1,0)}[r,t]^2-v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r,t]+v^{(2,0)}[r,t]\right)=0$$

$\frac{2\lambda^{(0,1)}[r,t]}{r}=0$ -----(B) tells us that λ is independent of time. $\partial_t\lambda=0$

$\frac{2\lambda^{(0,1)}[r,t]}{r}=0$ ----(C), (B)=(C)

$$-e^{-2v[r,t]+2\lambda[r,t]}\,v^{(0,1)}[r,t]\,\lambda^{(0,1)}[r,t]+e^{-2v[r,t]+2\lambda[r,t]}\,\lambda^{(0,1)}[r,t]^2+ \text{-----(D)}$$
$$e^{-2v[r,t]+2\lambda[r,t]}\,\lambda^{(0,2)}[r,t]-v^{(1,0)}[r,t]^2+\frac{2\lambda^{(1,0)}[r,t]}{r}+v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r,t]-v^{(2,0)}[r,t]=0$$

$1-e^{-2\lambda[r,t]}-e^{-2\lambda[r,t]}\,r\,v^{(1,0)}[r,t]+e^{-2\lambda[r,t]}\,r\,\lambda^{(1,0)}[r,t]=0$ -----(E)

$\left(1-e^{-2\lambda[r,t]}\right)\mathrm{Sin}[\theta]^2-e^{-2\lambda[r,t]}\,r\,\mathrm{Sin}[\theta]^2\,v^{(1,0)}[r,t]+e^{-2\lambda[r,t]}\,r\,\mathrm{Sin}[\theta]^2\,\lambda^{(1,0)}[r,t]=0$ ----(F) ~ (E)

In other words,

$v^{(1,0)}[r,t]^2+2\,v^{(1,0)}[r,t]/r-v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r]+v^{(2,0)}[r,t]=0$ ----(A)

$-v^{(1,0)}[r,t]^2+2\,\lambda^{(1,0)}[r]/r+v^{(1,0)}[r,t]\,\lambda^{(1,0)}[r]-v^{(2,0)}[r,t]=0$ -----(D)

$1-e^{-2\lambda[r]}\left(1+r\,v^{(1,0)}[r,t]-r\,\lambda^{(1,0)}[r]\right)=0$ -----(E)

$(A)+(D)\Rightarrow\partial_r v(r,t)+\partial_r\lambda(r)=0\,,\;\; v(r,t)+\lambda(r)=\kappa(t)$

$(E)\;\Rightarrow\;1-e^{-2\lambda(r)}(1-2\,r\,\partial_r\lambda(r))=0,\;\; e^{-2\lambda(r)}(1-2\,r\,\partial_r\lambda(r))=1$

$\left(r\,e^{-2\lambda(r)}\right)'=1,\;\left(r\,e^{-2\lambda(r)}\right)=r+C\to e^{-2\lambda(r)}=1+C/r$ (let $C=-2\,m$) $=1-2\,m/r$

$2\,\nu(r,\,t) = 2\,\kappa(t) - 2\,\lambda(r),\;\; e^{2\,\nu} = e^{2\,\kappa}\,e^{-2\,\lambda} = e^{2\,\kappa}\left(1 - \frac{2\,m}{r}\right)$

Then,

$ds^2 = -e^{2\,\kappa}\left(1 - \frac{2\,m}{r}\right)c^2\,dt^2 + \left(1 - \frac{2\,m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$ with $m = \frac{M\,G}{c^2}$

Finally, redefining the time coordinate as $e^{-\kappa(t)}\,dt \to dt'$,

$ds^2 = -\left(1 - \frac{2\,m}{r}\right)c^2\,dt'^2 + \left(1 - \frac{2\,m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$ just as ‘static’ Schwarzschild solution!

Birkhoff’s theorem tells us that any spherically symmetric solution of the field equations is necessarily static.

e.g. A star pulsating radially has the same external field as a star at rest (that is to say, a radially pulsating star emits no gravitational radiation.)

Reference

- Lewis Ryder (2009), *Introduction to General Relativity*, New York: Cambridge University Press.