Symplectic Algebra

- Sp(2n,R)

$$\text{For real } 2\,n - \text{by} - 2\,n \text{ matrix } R,\ \text{ if } R^T J\,R = J,\ \text{ then } R \in \mathrm{Sp}(2\,n,\ R)$$

$$\text{where } J = I \otimes\ (i\,\sigma_2) \equiv \begin{pmatrix} O & | & I \\ - & - & - \\ -I & | & O \end{pmatrix}$$

$$\text{note that } J(-J) = I,\ \ J^{-1} = -J,\ \ (\det J)^2 = 1$$

- Sp(2n,C)

$$\text{For complex } 2\,n - \text{by} - 2\,n \text{ matrix } C,\ \text{ if } C^T J\,C = J,\ \text{ then } C \in \mathrm{Sp}(2\,n,\ C)$$

- det R=1

Proof

Define characteristic polynomial of R as P(λ)

$$P(\lambda) = \det(R - \lambda\,I) = \lambda^{2\,n} \det\left(\lambda^{-1}\,R - I\right)$$

$$= \lambda^{2\,n}\ \det R \det\left(\lambda^{-1}\,I - R^{-1}\right)$$

$$= \lambda^{2\,n} \det R \det\left(\lambda^{-1}\,I - J^{-1}\,R^T J\right)$$

$$= \lambda^{2\,n}\ \det R \det\left(\lambda^{-1}\,I + J\,R^T J\right)$$

$$= \lambda^{2\,n}(\det R)\,(\det J)\,(\det J)\det\left(\lambda^{-1}\,J^{-1}\,J^{-1} + R^T\right)$$

$$= \lambda^{2\,n}(\det R)\,(\det J)\,(\det J)\det\left(\lambda^{-1}\,J^{-1}(-J) + R^T\right)$$

$$= \lambda^{2\,n}(\det R)\,(\det J)\,(\det J)\det\left(-\lambda^{-1}\,I + R^T\right)$$

$$= \lambda^{2\,n}(\det R)\,(\det J)^2\det\left(R^T - \lambda^{-1}\,I\right)$$

$$= \lambda^{2\,n}(\det R)\det\left(R^T - \lambda^{-1}\,I\right)$$

$$= \lambda^{2\,n}(\det R)\,P\left(\lambda^{-1}\right)$$

$$P(\lambda) = \lambda^{2\,n}\ \det R\,P\left(\lambda^{-1}\right) \tag{1}$$

From (1), if $P(\lambda) = 0$, then $P(1/\lambda) = 0$, which is to say, if R has λ as *a* eigenvalue, then $1/\lambda$ is also its eigenvalue! Then, diagonalized R would be shown as below.

$$d^R = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/\lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n & 0 \\ 0 & 0 & \cdots & 0 & 1/\lambda_n \end{pmatrix}$$

Since $\det d^R = \det R$, $\left(\text{recall that } d^R \text{ is similarity transformation of } R\right)$,

$$\det R = \det d^R = 1$$